COLORING PROBLEMS IN COMBINATORICS AND DESCRIPTIVE SET THEORY

BY

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DISSERTATION

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Abstract

In this dissertation we study problems related to colorings of combinatorial structures both in the "classical" finite context and in the framework of descriptive set theory, with applications to topological dynamics and ergodic theory. This work consists of two parts, each of which is in turn split into a number of chapters. Although the individual chapters are largely independent from each other (with the exception of Chapters 4 and 6, which partially rely on some of the results obtained in Chapter 3), certain common themes feature throughout—most prominently, the use of probabilistic techniques.

In Chapter 1, we establish a generalization of the Lovász Local Lemma (a powerful tool in probabilistic combinatorics), which we call the *Local Cut Lemma*, and apply it to a variety of problems in graph coloring.

In Chapter 2, we study *DP-coloring* (also known as *correspondence coloring*)—an extension of list coloring that was recently introduced by Dvořák and Postle. The goal of that chapter is to gain some understanding of the similarities and the differences between DP-coloring and list coloring, and we find many instances of both.

In Chapter 3, we adapt the Lovász Local Lemma for the needs of descriptive set theory and use it to establish new bounds on measurable chromatic numbers of graphs induced by group actions.

In Chapter 4, we study shift actions of countable groups Γ on spaces of the form A^Γ, where A is a finite set, and apply the Lovász Local Lemma to find "large" closed shift-invariant subsets $X \subseteq A^\Gamma$ on which the induced action of Γ is free.

In Chapter 5, we establish precise connections between certain problems in graph theory and in descriptive set theory. As a corollary of our general result, we obtain new upper bounds on Baire measurable chromatic numbers from known results in finite combinatorics.

Finally, in Chapter 6, we consider the notions of *weak containment* and *weak equivalence* of probability measure-preserving actions of a countable group—relations introduced by Kechris that are combinatorial in spirit and involve the way the action interacts with finite colorings of the underlying probability space.

This work is based on the following papers and preprints: [Ber16a; Ber16b; Ber16c; Ber17a; Ber17b; Ber17c; Ber18a; Ber18b], [BK16; BK17a] (with Alexandr Kostochka), [BKP17] (with Alexandr Kostochka and Sergei Pron), and [BKZ17; BKZ18] (with Alexandr Kostochka and Xuding Zhu).

Acknowledgments

> Which leaves open the question of what is the author's contribution to the paper.
>
> *Shmuel Safra*, "On the complexity of ω-automata"

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Contents

Introduction ond overview

A typical combinatorial problem is that of *coloring*, i.e., assigning to each element of a given structure (for instance, a graph) an element of some (usually finite or at most countable) set—a "color"—in a way that fulfills a specific family of constraints. For example, one might require the colors of adjacent vertices in a graph to be distinct. This setting is quite flexible and can take many different forms depending on the choice of the underlying structure and the type of constraints that must be met. In this dissertation, we will encounter coloring problems that appear in such diverse areas as (*hyper*)*graph theory*, *probability theory*, *descriptive set theory*, *ergodic theory*, and *topological dynamics*.

When coloring problems are considered purely combinatorially, there is usually no loss of generality in restricting one's attention to *finite* structures. Indeed, coloring an infinite structure can often be reduced, using a compactness argument, to coloring each of its finite substructures. The following is a prototypical instance of this phenomenon:

Theorem (de Bruijn–Erdős [BE51, Theorem 1])**.** *An infinite graph G is k-colorable if and only if every finite subgraph of G is k-colorable.*

Consequently, classical combinatorics mostly studies colorings of finite structures, such as finite graphs and hypergraphs. Our contributions to this area constitute Part I of this dissertation.

Nevertheless, coloring infinite structures can also present unique challenges that are absent in the finite setting. The source of these challenges is that infinite sets are often equipped with some extra data, such as a topology or a measure. It is then reasonable (and, for some applications, necessary) to consider colorings which not only satisfy the combinatorial constraints, but also behave well topologically or measure-theoretically. With these additional requirements, compactness can no longer be used to directly reduce the problem to the finite case. Here is perhaps the simplest concrete example:

Example. Fix some irrational $\alpha \in (0;1)$ and consider the graph G_α with vertex set the half-open interval $[0;1)$ whose edges connect the pairs of vertices x, y with $y - x = \pm\alpha \pmod 1$. Combinatorially, G_α is simply a disjoint union of (continuumly many) paths, infinite in both directions. In particular, G_α is 2-colorable. Note, however, that in order to properly color G_α using two colors, one must select one of the two possible colorings for each connected component of G_α. This can be easily done with the help of the Axiom of Choice, but the Axiom of Choice itself gives no control over the regularity properties of the resulting coloring. Indeed, a simple ergodicity argument shows that a proper 2-coloring of G_α can neither be Lebesgue measurable nor Baire measurable (see, e.g., [CMT16, p. 2]). On the other hand, G_α has a Borel proper coloring with three colors (see Fig. 1).

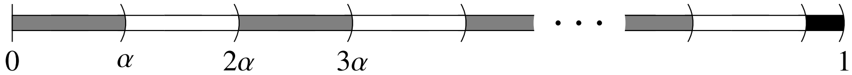

Figure 1 – A Borel proper 3-coloring of G_α.

Questions regarding colorings that are both combinatorially nice and topologically or measure-theoretically well-behaved are studied in **descriptive combinatorics**, a recently emerged area at the interface of descriptive set theory and combinatorics that has deep connections to other fields such as ergodic theory and probability theory. A comprehensive state-of-the-art survey of this subject can be found in [KM16]. It is remarkable that, even though the problems in descriptive combinatorics possess a flavor distinctly different from those in finite combinatorics, results in descriptive combinatorics are often proved by adapting some of the methods known to work in the finite setting. Our contributions to descriptive combinatorics and related areas form Part II of this dissertation.

Even though Parts I and II are self-contained and independent from each other, there are some common themes that bind them together. Most prominently, throughout both parts we heavily rely on the so-called **probabilistic method**, a set of combinatorial techniques that was pioneered by Erdős in the mid-1940s and has since become absolutely indispensable. The classic introduction to this method is [AS00], and a large selection of its applications to graph coloring can be found in [MR02]. The general premise of the probabilistic method is that to prove the existence of an object with some properties, it suffices to verify that a *random* object has the desired properties *with positive probability*. This approach has two important advantages. First, it alleviates the burden of explicit construction, making it possible to work where there is little useful structure to exploit. Second, it brings in a variety of powerful tools from probability theory, such as *concentration of measure*, *correlation inequalities*, and the *Lovász Local Lemma*, to name a few. (The Lovász Local Lemma, or the *LLL* for short, plays a particularly important role in this work: The only chapter where the LLL does not make an appearance is Chapter 5, while Chapters 1 and 3 are specifically dedicated to strengthening and extending the scope of the LLL.)

In the remainder of this introduction we give a brief chapter-by-chapter overview of this work.

Part I: The finite

Part I is based on the following papers and preprints: [Ber16b; Ber16c; Ber17b; Ber17c], [BK16; BK17a] (with Alexandr Kostochka), [BKP17] (with Alexandr Kostochka and Sergei Pron), and [BKZ17; BKZ18] (with Alexandr Kostochka and Xuding Zhu).

Chapter 1: The Local Cut Lemma

As we have already mentioned above, an important tool in probabilistic combinatorics is the so-called **Lovász Local Lemma** (the **LLL** for short), which was introduced by Erdős and Lovász in their seminal paper [EL75]. Let $\mathscr{B}$ be a finite family of "bad" random events in a probability space, and suppose we wish to show that, with positive probability, all the events in $\mathscr{B}$ can be avoided. This would be easy if the events in $\mathscr{B}$ were mutually independent—but that usually does not happen in practice. However, often the dependencies between the events in $\mathscr{B}$ can be somehow controlled, and if that is the case, then the LLL may be applied. For the precise statement, see, e.g., [AS00, Lemma 5.1.1] or Theorem 1.1.1 below. (Throughout this dissertation, the LLL is stated, in different forms, a total of at least five times.)

Recently, there has been a burst of activity surrounding the LLL and related topics, prompted by the groundbreaking work of Moser and Tardos [MT10], who devised and *algorithmic* approach to the LLL. It was observed, first by Grytczuk, Kozik, and Micek in their study of nonrepetitive sequences [GKM13], that the Moser–Tardos method, originally developed as an alternative way of proving the LLL, often outperforms the LLL when applied to concrete problems. This technique has become known as the **entropy compression method**; see, e.g., [GKM13; Duj+15; EP13; GMP14].

Each time the entropy compression method is applied, a series of somewhat technical ad hoc calculations is necessary. It is natural to wonder if there is a single "master theorem" that can replace these calculations. Indeed, such a general theorem, called the **Local Cut Lemma** (the **LCL** for short), is the main result presented in Chapter 1 (see Theorem 1.2.4). Interestingly, the proof of the LCL itself does not rely on the entropy compression method; in other words, although the entropy compression method was necessary to *discover* the new combinatorial results, it is not, in principle, needed to *prove* them.

The general statement of the LCL involves random cuts in directed graphs (hence the name), but for most applications the following weaker version is sufficient. Let I be a finite set. A family $A \subseteq \mathrm{Pow}(I)$ of subsets of I is **downwards-closed** if for each $S \in A$, $\mathrm{Pow}(S) \subseteq A$. Let the **boundary** of A be

$$\partial A \coloneqq \{i \in I \,:\, S \in A \text{ and } S \cup \{i\} \notin A \text{ for some } S \subseteq I \setminus \{i\}\}.$$

Let Ω be a probability space and let $A \colon \Omega \to \mathrm{Pow}(\mathrm{Pow}(I))$ be a random variable. Usually, A is taken to be the family of all sets $S \subseteq I$ that are colored "properly" in a random coloring (according to some notion of "properness"), and the goal is to prove that, with positive probability, $I \in A$, i.e., the entire coloring is "proper." For a random event B, a function $\tau \colon I \to [1; +\infty)$, and an element $i \in I$, define

$$\sigma_\tau^A(B, i) \coloneqq \min_{i \in X \subseteq I} \max_{Z \subseteq I \setminus X} \left[\mathbb{P}[B|Z \in A] \cdot \prod_{i \in X} \tau(i)\right].$$

Theorem 1.3.1. *Let I be a finite set, let Ω be a probability space, and let $A \colon \Omega \to \mathrm{Pow}(\mathrm{Pow}(I))$ be a random variable such that, with probability 1, A is a nonempty downwards-closed family of subsets of I. For each $i \in I$, let $\mathscr{B}(i)$ be a finite collection of random events in Ω such that whenever $i \in \partial A$, at least one of the events in $\mathscr{B}(i)$ holds. If there is a function $\tau \colon I \to [1; +\infty)$ such that for all $i \in I$,*

$$\tau(i) \geqslant 1 + \sum_{B \in \mathscr{B}(i)} \sigma_\tau^A(B, i),$$

then $I \in A$ with positive probability.

The LLL becomes a special case of Theorem 1.3.1 under the assumptions that (1) the set A always contains an inclusion-maximum element; and (2) each of the sets $\mathscr{B}(i)$ is a singleton, containing only one "bad" event. In typical applications, however, none of these assumptions are satisfied. In Chapter 1, after stating and proving the LCL in full generality, we present short LCL-based proofs of several results that were originally derived using the entropy compression method. We also present some new results that were first obtained using the LCL: an improved lower bound on the number of edges in color-critical hypergraphs (see §1.7), an interesting probabilistic statement regarding choice functions (see §1.8), and new upper bounds on the acyclic chromatic index (see §1.9).

Chapter 2: DP-Coloring

In the 1970s, Vizing [Viz76] and independently Erdős, Rubin, and Taylor [ERT79] introduced an important generalization of graph coloring called **list coloring**. In this setting, each vertex $u \in V(G)$ is assigned a **list** $L(u)$ of colors that are available for that vertex. An **L-coloring** of G is a vertex coloring f of G that is **proper** (i.e., adjacent vertices receive distinct colors) and satisfies $f(u) \in L(u)$ for all $u \in V(G)$. The **list-chromatic number** $\chi_\ell(G)$ is the smallest k such that G has an L-coloring whenever $|L(u)| \geqslant k$ for all $u \in V(G)$.

It is clear that ordinary graph coloring is a special case of list coloring (just make all lists equal to each other); in particular, $\chi_\ell(G) \geqslant \chi(G)$ for any graph G. On the other hand, $\chi_\ell(G)$ is not bounded above by any function of $\chi(G)$. For instance, the list-chromatic number of the balanced complete bipartite graph $K_{n,n}$ is $(1 + o(1)) \log_2(n) \to \infty$ as $n \to \infty$ (while $\chi(K_{n,n}) = 2$ for all n).

Recently, Dvořák and Postle [DP15] generalized list coloring further by allowing the identifications between the colors in the lists to vary from edge to edge. They called it **correspondence coloring**, here referred to as **DP-coloring** for short.

Definition 2.1.1. Let G be a graph. A **cover** of G is a pair $\mathscr{H} = (L, H)$, consisting of a graph H and a function $L \colon V(G) \to \mathrm{Pow}(V(H))$, satisfying the following requirements:

(C1) the sets $\{L(u) \,:\, u \in V(G)\}$ form a partition of $V(H)$;

(C2) for every $u \in V(G)$, the graph $H[L(u)]$ is complete;

(C3) if $E_H(L(u), L(v)) \neq \varnothing$, then either $u = v$ or $uv \in E(G)$;

(C4) if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching.

A cover $\mathscr{H} = (L, H)$ of G is **k-fold** if $|L(u)| = k$ for all $u \in V(G)$.

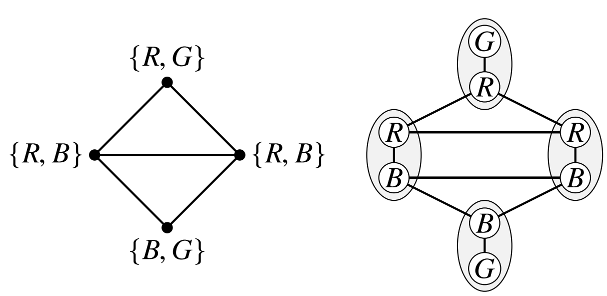

Figure 2 – A list assignment and the corresponding 2-fold cover.

Recall that a set $I \subseteq V(H)$ is **independent** if no edge joins two elements of I. Let $\mathscr{H} = (L, H)$ be a cover of a graph G. An $\mathscr{H}$**-coloring** is an independent set in H of size $|V(G)|$. The **DP-chromatic number** $\chi_{DP}(G)$ is the smallest k such that G admits an $\mathscr{H}$-coloring for every k-fold cover $\mathscr{H}$.

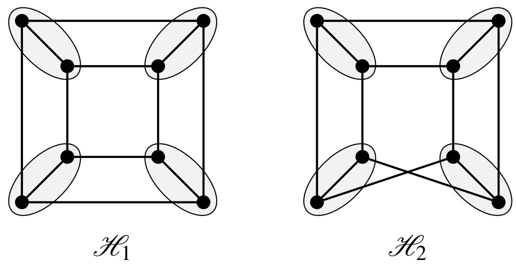

Figure 3 – The 4-cycle is $\mathscr{H}_1$-colorable but not $\mathscr{H}_2$-colorable.

List coloring is a special case of DP-coloring (see Fig. 2), and, in particular, $\chi_{DP}(G) \geqslant \chi_\ell(G)$ for any graph G. Some upper bounds on the list-chromatic number hold for the DP-chromatic number as well. For instance, it is easy to see that $\chi_{DP}(G) \leqslant \Delta + 1$ for any graph G of maximum degree Δ. On the other hand, the only connected graphs G with $\chi_\ell(G) = \Delta + 1$ are cliques and odd cycles, while for DP-coloring, even cycles also become sharpness examples (see Fig. 3).

In Chapter 2 we investigate how much of the classical theory of list coloring can be transferred to the setting of DP-coloring. It turns out that, in general, DP-chromatic numbers are rather large: the DP-chromatic number of *any* graph with average degree d is $\Omega(d/\log(d))$, i.e., close to linear in d (see Theorem 2.1.5). In spite of this, known upper bounds on list-chromatic numbers often extend to DP-chromatic numbers. Notably, by Johansson's theorem [Joh96], triangle-free graphs G of maximum degree Δ satisfy $\chi_\ell(G) = O(\Delta/\log\Delta)$. The same asymptotic upper bound holds for $\chi_{DP}(G)$. Recently, Molloy [Mol17] refined Johansson's result to $\chi_\ell(G) \leqslant (1 + o(1))\Delta/\ln\Delta$, and this improved bound, including the constant factor, also generalizes to DP-colorings (see §2.3). Together, the above results have the following curious corollary:

Corollary 2.3.4. *For every Δ-regular triangle-free graph G,*

$$(1/2 - o(1))\frac{\Delta}{\ln\Delta} \leqslant \chi_{DP}(G) \leqslant (1 + o(1))\frac{\Delta}{\ln\Delta}.$$

Furthermore, there exist statements about list coloring whose *only* known proofs involve DP-coloring in essential ways. For example, Dvořák and Postle originally introduced DP-coloring in order to show that every planar graph without cycles of lengths 4 to 8 is 3-list-colorable [DP15, Theorem 1], answering a long-standing open question. In §2.4, we extend Dirac's lower bound on the number of edges in critical graphs [Dir57; Dir74] to the framework of DP-colorings and completely classify the graphs that satisfy Dirac's bound with equality. This classification was only conjectural even in the list-coloring setting [KS02].

Yet, DP-coloring is remarkably different from list coloring in some respects. We only mention two examples here, and several other ones can be found in Chapter 2. A classical result of Alon and Tarsi [AT92, Corollary 3.4] asserts that every planar bipartite graph is 3-list-colorable, but this is not true for DP-colorings—see Theorem 2.2.2. The Noel–Reed–Wu theorem [NRW15] states that if G is an n-vertex graph with $\chi(G) \geqslant (n-1)/2$, then $\chi_\ell(G) = \chi(G)$; Corollary 2.5.5 in §2.5 asserts that to obtain the same conclusion for $\chi_{DP}(G)$, it is necessary to require $\chi(G) \geqslant n - \Theta(\sqrt{n})$.

Part II: The infinite

Part II is based on the following preprints: [Ber16a; Ber17a; Ber18a; Ber18b].

Chapter 3: Measurable versions of the Lovász Local Lemma

Along with developments in finite combinatorics described in Chapter 1, the Moser–Tardos algorithmic approach to the Lovász Local Lemma spurred the study of various "constructive" aspects of the LLL. A salient example is the *computable* LLL of Rumyantsev and Shen [RS14]. In Chapter 3, we investigate the behavior of the LLL in the *measurable* setting. To motivate this work, let us first describe a specific application of our general results.

Let X be a standard Borel space, equipped with a Polish topology τ and a probability measure μ, and let G be a graph with vertex set X. (In all cases of interest, the vertex set of G has cardinality continuum.) Suppose that we wish to find a proper coloring $f \colon X \to C$ of G with colors in a set C. If C is *also* a standard Borel space (for instance, if it is countable), then we may restrict our attention to colorings $f \colon X \to C$ that are—as functions—Borel, μ-measurable, or Baire measurable with respect to τ, leading to the definitions of **Borel**, **measurable**, and **Baire measurable chromatic numbers** of G, denoted by $\chi_{\mathrm{B}}(G)$, $\chi_{\mathrm{M}}(G)$, and $\chi_{\mathrm{BM}}(G)$ respectively.

An ample supply of examples in descriptive combinatorics is provided by actions of countable groups. Given a group action $\Gamma \curvearrowright X$ and a symmetric generating set $S \subseteq \Gamma$, define the corresponding "Cayley" graph $G_S(X)$ with vertex set X by

$$E(G_S(X)) \coloneqq \{(x, \gamma \cdot x) \,:\, x \in X \text{ and } \gamma \in S\}.$$

Of particular interest is the situation when X is the free part of the shift action $\Gamma \curvearrowright [0;1]^\Gamma$, in which case the graph $G_S(X)$ is denoted by $G_\infty(\Gamma, S)$.

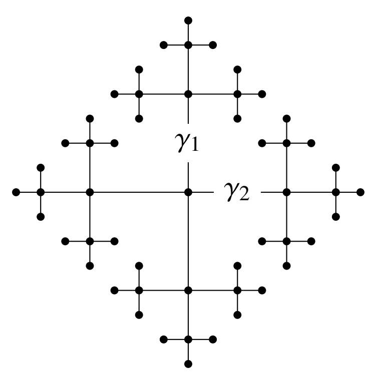

Figure 4 – A fragment of the Cayley graph of $\mathbb{F}_2$.

Now consider the free group $\mathbb{F}_n$ on n generators $\gamma_1, \ldots, \gamma_n$. Let $S_n \coloneqq \{\gamma_1^{\pm 1}, \ldots, \gamma_n^{\pm 1}\}$ be the standard symmetric generating set for $\mathbb{F}_n$ and let $G_n := G_\infty(\mathbb{F}_n, S_n)$. Remarkably, for large n, the values $\chi(G_n)$, $\chi_{\mathrm{B}}(G_n)$, $\chi_{\mathrm{M}}(G_n)$, and $\chi_{\mathrm{BM}}(G_n)$ are all distinct:

$\chi(G_n)$	2	Easily seen as G_n is acyclic.
$\chi_{\mathrm{BM}}(G_n)$	3	Conley–Miller [CM16].
$\chi_{\mathrm{M}}(G_n)$	$\geqslant \dfrac{n}{\ln(2n)}$; $\leqslant 2n$	Lyons–Nazarov [LN11] (lower bound); Conley–Marks–Tucker-Drob [CMT16] (upper bound)
$\chi_{\mathrm{B}}(G_n)$	$2n+1$	Kechris–Solecki–Todorcevic [KST99] (upper bound); Marks [Mar16] (lower bound)

Table 1 – Chromatic numbers of the graph G_n.

Notice the gap in the third line of the above table. Lyons and Nazarov [LN11] posed the problem of determining the correct order of magnitude of $\chi_{\mathrm{M}}(G_n)$ as a function of n. A potential approach to answering their question is as follows. By a result of Kim [Kim95], any *finite* graph G with no cycles of lengths 3 and 4 and of maximum degree Δ satisfies $\chi(G) \leqslant (1 + o(1))\Delta/\ln \Delta$. Since the (infinite) graph G_n is acyclic and $2n$-regular (see Fig. 4), if Kim's theorem worked in the measurable setting, it would yield $\chi_{\mathrm{M}}(G_n) \leqslant (2 + o(1))n/\ln(2n)$, which is asymptotically within a factor of 2 from the known lower bound. In Chapter 3, we successfully implement this strategy:

Theorem (see Theorem 3.1.1)**.** *Let Γ be a countable group with a symmetric generating set S of size d. If the Cayley graph of Γ with respect to S contains no cycles of lengths 3 and 4, then $\chi_{\mathrm{M}}(G_\infty(\Gamma, S)) \leqslant (1+o(1))d/\ln d$.*

Corollary 3.1.2. *We have*

$$\frac{n}{\ln(2n)} \leqslant \chi_{\mathrm{M}}(G_n) \leqslant (2 + o(1))\frac{n}{\ln(2n)}.$$

The LLL plays a crucial role in the proof of Kim's theorem, so it is desirable to have a *measurable version* of the LLL that can be used to transfer existing LLL-based arguments to the measurable setting. In Chapter 3, we accomplish this goal in the case when the underlying structure is, in a certain sense, induced by the $[0; 1]$-shift action of a countable group—see Theorem 3.5.6. Our general result is so powerful that Theorem 3.1.1 follows simply by replacing the LLL with Theorem 3.5.6 everywhere in Kim's original proof, essentially without any further modifications. In a similar fashion one obtains bounds on various other measurable coloring parameters, such as the measurable chromatic index (see Theorem 3.1.3). The statement of our measurable LLL requires a few definitions, so we will not reproduce it in this introduction. However, it is worth pointing out that it allows the LLL to be *iterated*, which is necessary to prove measurable analogs of sophisticated results such as Kim's theorem (and hence Theorem 3.1.1).

It is natural to wonder if the measurable LLL can be extended to probability measure-preserving actions beyond the shift action. In this direction, we prove that, at least for amenable groups, the restriction to shift actions is sharp: a probability measure-preserving action of a countably infinite amenable group satisfies the measurable version of the LLL *if and only if* it admits a factor map to the $[0; 1]$-shift (see Theorems 3.6.1 and 3.6.1′). The proof combines the tools of the Ornstein–Weiss theory of entropy for actions of amenable groups with concepts from computability theory, specifically, *Kolmogorov complexity*.

Chapter 4: Building large free subshifts using the LLL

In Chapter 4 we apply the LLL to certain problems in topological dynamics. Let Γ be a countably infinite group and let $k \in \mathbb{N}$. A **subshift** is a closed shift-invariant subset of k^Γ. It has been a matter of interest to determine which groups Γ admit a nonempty subshift X that is **free**, i.e., such that the stabilizer of every point $x \in X$ under the shift action of Γ is trivial. It has been known at least since the early 1900s that such a subshift exists for $\Gamma = \mathbb{Z}$ (the set of all cube-free[1] sequences $X \subseteq 2^\mathbb{Z}$ is a nonempty free subshift, for example). After some partial results by Dranishnikov and Schroeder [DS07] and Glasner and Uspenskij [GU09], the problem was fully resolved by Gao, Jackson, and Seward [GJS09; GJS16], who showed that not only do nonempty free subshifts exist for all groups, but they are rather numerous: For any $k \geqslant 2$, every nonempty shift-invariant open subset $U \subseteq k^\Gamma$ contains continuumly many pairwise disjoint nonempty free subshifts [GJS16, Theorem 1.4.1].

The elements of k^Γ can be thought of as colorings of Γ with k colors. If one defines a free subshift $X \subseteq k^\Gamma$ as the set of all colorings that avoid a certain family of "bad" events, then the LLL may be used to prove that X is nonempty. This approach was employed by Aubrun, Barbieri, and Thomassé [ABT16] to give a simple alternative construction of a nonempty free subshift $X \subseteq 2^\Gamma$ for an arbitrary group Γ (while the original proof of [GJS16, Theorem 1.4.1] due to Gao, Jackson, and Seward was quite technical).

The aim of Chapter 4 is to use the LLL in order to prove the existence of free subshifts that are not only nonempty, but in fact rather "large" in various senses. Specifically, we investigate the following questions, which are attributed by Gao, Jackson, and Seward to Juan Souto:

> *For a given group Γ, what is the largest possible Hausdorff dimension of a free subshift $X \subseteq k^\Gamma$?*
>
> *For groups Γ in which a notion of entropy exists, what is the largest possible entropy of a free subshift $X \subseteq k^\Gamma$?*

Note that neither the Hausdorff dimension nor the entropy of a subshift $X \subseteq k^\Gamma$ can exceed $\log_2 k$. We answer the above questions by showing that the Hausdorff dimension and the entropy of a free subshift can be made *arbitrarily close* to this upper bound:

Theorem 4.1.3. *Let $U \subseteq k^\Gamma$ be a nonempty shift-invariant open set. Then, for any $h < \log_2 k$:*

(i) *there exists a free minimal subshift $X \subseteq U$ of Hausdorff dimension at least h;*

(ii) *if Γ is amenable, then there exists a free minimal subshift $X \subseteq U$ of entropy at least h;*

(iii) *if Γ is sofic, then there exists a free subshift $X \subseteq U$ whose entropy with respect to any sofic approximation is at least h.*

In fact, we define a novel LLL-inspired notion of size for a subshift, which we call **breadth** (see §4.3), and show that it serves as a lower bound for various other notions of size and that a subshift whose breadth is large enough must contain a free subshift—see Theorem 4.3.4. In the proof of Theorem 4.3.4, we use both the usual LLL and one of its measurable versions established in Chapter 3, namely Corollary 3.5.7.

[1]A sequence $w = \dots a_{-2}a_{-1}a_0a_1a_2 \dots$ is **cube-free** if no finite subsequence is repeated in w three times in a row.

Chapter 5: Baire measurable colorings of group actions

It is common for results in descriptive combinatorics to be based, in one way or another, on a known theorem or a method that works in the finite setting (the results in Chapter 3 serve as an example). This suggests that there might exist some precise correspondences between the finite and the infinite; the existence of a well-behaved coloring of a certain kind could, perhaps, be equivalent to a finitary statement such as the existence of an algorithm of a specific form to find such a coloring.

The goal of Chapter 5 is to confirm this suspicion for the problem of coloring the shift action of a countable group Γ in a Baire measurable way. If the set of available colors is countable, for concreteness $\mathbb{N}$, then a coloring problem over Γ can be identified with a **subshift** $\Omega \subseteq \mathbb{N}^\Gamma$, i.e., a closed shift-invariant set whose elements are the "good" colorings of Γ (note that here the notion of a subshift is different from the one used in Chapter 4; in particular, for the purposes of Chapter 5, a subshift need not be compact). Given a continuous action $\Gamma \curvearrowright X$ of Γ on a Polish space, any coloring $f \colon X \to \mathbb{N}$ gives rise to the Γ-equivariant map $\pi_f \colon X \to \mathbb{N}^\Gamma$ called the **coding map** (or the **symbolic representation**) and defined by

$$\pi_f(x)(\gamma) \coloneqq f(\gamma \cdot x) \quad \text{for all } x \in X \text{ and } \gamma \in \Gamma.$$

A **Baire measurable Ω-coloring** is a Baire measurable function $f \colon X \to \mathbb{N}$ such that $\pi_f(x) \in \Omega$ for comeagerly many $x \in X$, i.e., the coloring of a generic orbit is "good." Let $\mathbf{Sh}_{\mathrm{BM}}(X)$ denote the set of all subshifts Ω such that X admits a Baire measurable Ω-coloring. It turns out that, apart from some trivial cases, the set $\mathbf{Sh}_{\mathrm{BM}}(X)$ is *complete analytic* (see Theorem 5.2.3); in particular, it is not Borel. Intuitively, this means that there is no "easy" way to decide if a given coloring problem has a Baire measurable solution.

Nevertheless, we are able to describe the set $\mathbf{Sh}_{\mathrm{BM}}([0;1]^\Gamma)$ purely combinatorially (where the action on $[0;1]^\Gamma$ is by shift). Roughly speaking, determining whether $\Omega \in \mathbf{Sh}_{\mathrm{BM}}([0;1]^\Gamma)$ is equivalent to settling a question of the following form:

> "Is it possible to decide, only using 'local' information, if a given coloring $\varphi \colon S \to \mathbb{N}$ of a finite subset $S \subseteq \Gamma$ can be extended to a coloring $\omega \in \Omega$ of the entire group?" $(*)$

Some interpretations of $(*)$ have already been investigated in graph theory, especially in relation with the problem of jointly extending given partial colorings of finite subsets that are "sufficiently far apart" from each other (see, e.g., [AKW05; Dvo+17; PT16]). We formalize this notion as the **join property** of subshifts (see Definition 5.2.7), and the following result is obtained:

Theorem (see Theorem 5.2.10)**.** *The following are equivalent for a subshift* $\Omega \subseteq \mathbb{N}^\Gamma$*:*

(i) *the shift action of* Γ *admits a Baire measurable* Ω*-coloring;*

(ii) $\Omega \supseteq \Omega'$ *for some subshift* Ω' *with the join property.*

Implication (ii) $\Rightarrow$ (i) of the above theorem can be used to derive bounds on Baire measurable chromatic numbers from known results in finite combinatorics. For instance, a theorem of Postle and Thomas [PT16, Theorem 8.10] yields the following:

Corollary 5.2.11. *Suppose that Γ is generated by a finite symmetric set $S \subseteq \Gamma$ such that the corresponding Cayley graph is planar. Let $G := G_\infty(\Gamma, S)$. Then*

$$\chi_{\mathrm{BM}}(G) \leqslant \begin{cases} 3 & \text{if } G \text{ contains no cycles of lengths } 3 \text{ and } 4; \\ 4 & \text{if } G \text{ contains a cycle of length } 4 \text{ but not of length } 3; \\ 5 & \text{otherwise.} \end{cases}$$

Note that the best upper bounds on $\chi_{\mathrm{BM}}(G)$ under the assumptions of Corollary 5.2.11 that follow from previously known results are $\chi_{\mathrm{BM}}(G) \leqslant 7$ in general and $\chi_{\mathrm{BM}}(G) \leqslant 5$ if G contains no cycles of length 3; these are implied by the work of Conley and Miller [CM16, Theorem B].

Chapter 6: Results on weak containment of probability measure-preserving actions

Let Γ be a countable group. In ergodic theory, one aims to classify measure-preserving actions $\Gamma \curvearrowright (X, \mu)$ of Γ on probability spaces. Unfortunately, the isomorphism relation between such actions is extremely complicated, and so it is impossible to classify them up to isomorphism in any "reasonable" or "explicit" way. Hence, it is natural to consider some coarser and, hopefully, better behaved equivalence relations. One such relation—*weak equivalence*—was proposed by Kechris in [Kec10, Section 10(C)]. The definition of weak equivalence is combinatorial and involves the way the action interacts with finite colorings of the underlying probability space.

To define weak equivalence, we need to fix some notation first. Let $\alpha \colon \Gamma \curvearrowright X$ be an action of Γ on a set X and let $f \colon X \to k \in \mathbb{N}$ be a finite coloring of X. Define an equivariant map $\pi_f \colon X \to k^\Gamma$ by

$$\pi_f(x)(\gamma) := f(\gamma \cdot x) \qquad \text{for all } x \in X \text{ and } \gamma \in \Gamma.$$

Given a measure-preserving action $\alpha \colon \Gamma \curvearrowright (X, \mu)$ on a standard probability space (X, μ), an integer $k \in \mathbb{N}$, a Borel function $f \colon X \to k$, a finite set $S \subseteq \Gamma$, and a map $w \colon S \to k$, the **frequency** $\Phi_\mu(\alpha, f, w)$ of w in (α, f) with respect to μ is defined by

$$\Phi_\mu(\alpha, f, w) := \mu(\{x \in X \,:\, \pi_f(x) \supseteq w\}).$$

Definition 6.1.1. Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ and $\beta \colon \Gamma \curvearrowright (Y, \nu)$ be probability measure-preserving actions of Γ. We say that α is **weakly contained** in β, in symbols $\alpha \preccurlyeq \beta$, if for every finite set $S \subseteq \Gamma$ and for all $\varepsilon > 0$, the following holds: Let $k \in \mathbb{N}$ and let $f \colon X \to k$ be a Borel function. Then there exists a Borel map $g \colon Y \to k$ satisfying

$$|\Phi_\nu(\beta, g, w) - \Phi_\mu(\alpha, f, w)| < \varepsilon \qquad \text{for all } w \colon S \to k.$$

If simultaneously $\alpha \preccurlyeq \beta$ and $\beta \preccurlyeq \alpha$, then α and β are said to be **weakly equivalent**, in symbols $\alpha \simeq \beta$.

Due to the combinatorial nature of weak containment and weak equivalence, it is not surprising that an essentially equivalent notion of *local-global convergence* was introduced independently in the theory of graph limits [HLS14].

Abért and Weiss [AW13] proved that the shift action $\Gamma \curvearrowright [0;1]^\Gamma$ is weakly contained in every free probability measure-preserving action of Γ. In §6.2, we strengthen this result by replacing the frequencies $\Phi_\mu(\alpha, f, w)$ with certain pointwise averages. In particular, we prove a *purely Borel* version of the Abért–Weiss theorem in the case when Γ is a finitely generated group of subexponential growth (see Theorem 6.2.4). The results in that section are derived using three different measurable versions of the LLL, two of which are proved in Chapter 3, while the other one was established by Csóka, Grabowski, Máthé, Pikhurko, and Tyros [Csó+16].

Let $\mathcal{W}_\Gamma$ denote the set of all weak equivalence classes of measure-preserving actions of Γ on atomless standard probability spaces and let $\mathcal{FW}_\Gamma \subseteq \mathcal{W}_\Gamma$ be the subset of all weak equivalence classes of free actions. A useful feature of weak equivalence is that $\mathcal{W}_\Gamma$ carries a natural compact metrizable topology, introduced by Abért and Elek [AE11], and $\mathcal{FW}_\Gamma$ is closed in this topology. There is a natural multiplication operation on $\mathcal{W}_\Gamma$ (induced by taking products of actions) that makes $\mathcal{W}_\Gamma$ an Abelian semigroup. Burton, Kechris, and Tamuz [BK17b, Theorem 10.37] showed that if Γ is *amenable*, then $\mathcal{W}_\Gamma$ is a topological semigroup, i.e., the product map $\mathcal{W}_\Gamma \times \mathcal{W}_\Gamma \to \mathcal{W}_\Gamma \colon (\mathfrak{a}, \mathfrak{b}) \mapsto \mathfrak{a} \times \mathfrak{b}$ is continuous. It is natural to wonder if the same is true for every countable group Γ [BK17b, Problem 10.36]; however, in §6.3 we show that that is not the case for a certain class of nonamenable groups Γ, including the non-Abelian free groups:

Theorem 6.3.7. *Let $d \geqslant 2$ and let $\Gamma \leqslant \mathrm{SL}_d(\mathbb{Z})$ be a subgroup that is Zariski dense in $\mathrm{SL}_d(\mathbb{R})$.*

(1) *The map $\mathcal{FW}_\Gamma \to \mathcal{FW}_\Gamma \colon \mathfrak{a} \mapsto \mathfrak{a} \times \mathfrak{a}$ is discontinuous.*

(2) *There is $\mathfrak{b} \in \mathcal{FW}_\Gamma$ such that the map $\mathcal{FW}_\Gamma \to \mathcal{FW}_\Gamma \colon \mathfrak{a} \mapsto \mathfrak{a} \times \mathfrak{b}$ is discontinuous.*

In view of the above theorem and the result of Burton, Kechris, and Tamuz, it is tempting to conjecture that $\mathcal{W}_\Gamma$ is a topological semigroup *if and only if* Γ is amenable. However, at this point we do not even know whether multiplication of weak equivalence classes is discontinuous for every countable group that contains a non-Abelian free subgroup.

Notation and terminology

In this chapter we collect all of the basic notation and terminology that will be used throughout this dissertation.

Integers

We use $\mathbb{N} := \{0, 1, \ldots\}$ to denote the set of all nonnegative integers and let $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. As is common in set theory, each integer $k \in \mathbb{N}$ is identified with the k-element set $\{i \in \mathbb{N} \,:\, i < k\}$. We view $\mathbb{N}$ and k as discrete topological spaces.

Sets

For a set X, $\mathrm{Pow}(X)$ denotes the set of all subsets of X and $[X]^{<\infty}$ denotes the set of all finite subsets of X. For a set A, we write A^{c} to indicate the complement of A, i.e., the set $U \setminus A$ where U is some ambient set that is understood from context. We use the expressions $|S|$ and $\#S$ for the cardinality of a set S interchangeably. (Most of the time, we only use $|S|$; the expression $\#S$ usually suggests that it is a random variable.) For sets X, Y, elements $x \in X$, $y \in Y$, and a subset $A \subseteq X \times Y$, we use the following notation:

$$A_x := \{y \in Y \,:\, (x, y) \in A\} \qquad \text{and} \qquad A^y := \{x \in X \,:\, (x, y) \in A\}.$$

Functions

We identify every function f with its graph, i.e., with the set $\{(x, y) \,:\, f(x) = y\}$. This enables the use of set-theoretic notation, such as $\cap$, $\subseteq$, etc., for functions. The symbol $\varnothing$ denotes the empty function as well as the empty set. For a function f and a set $S \subseteq \mathrm{dom}(f)$, we use $f|S$ to denote the restriction of f to S.

For sets X and Y, the set of all functions $f \colon X \to Y$ is denoted by Y^X. We write $f \colon X \rightharpoonup Y$ to indicate that f is a partial function from X to Y, i.e., a function of the form $f \colon X' \to Y$ with $X' \subseteq X$. The expression $[X \to Y]^{<\infty}$ denotes the set of all partial maps $f \colon X \rightharpoonup Y$ with $\mathrm{dom}(f) \in [X]^{<\infty}$.

Graphs

Unless explicitly stated otherwise, by a **graph** we mean an undirected simple graph. A **multigraph** is allowed to have parallel edges but not loops. Directed graphs are referred to as **digraphs** and may or may not contain parallel edges depending on the context. In Part I, all graphs are additionally assumed to be finite.

For a graph G, its vertex and edge sets are denoted by $V(G)$ and $E(G)$ respectively. For a subset $U \subseteq V(G)$, $U^{\mathsf{c}} := V(G) \setminus U$ denotes the complement of U and $G[U]$ is the subgraph of G induced by U. Let

$G - U := G[U^c]$. For $u \in V(G)$, let $G - u := G - \{u\}$. For a pair of subsets $U_1, U_2 \subseteq V(G)$, $E_G(U_1, U_2)$ is the set of all edges of G with one endpoint in U_1 and the other one in U_2, and $G[U_1, U_2]$ is the subgraph of G with vertex set $U_1 \cup U_2$ and edge set $E_G(U_1, U_2)$.

For $u \in V(G)$, $N_G(u)$ denotes the set of all neighbors of u in G, and $\deg_G(u) := |N_G(u)|$ is the degree of u in G. Let $N_G[u] := N_G(u) \cup \{u\}$ denote the closed neighborhood of u. For $r \in \mathbb{N}$, $N_G^r(u)$ (resp. $N_G^r[u]$) denotes the set of all vertices whose distance from u is in $\{1, \ldots, r\}$ (resp. $\{0, 1, \ldots, r\}$). For a subset $U \subseteq V(G)$, let $N_G(U) := \bigcup_{u \in U} N_G(u)$ and $N_G[U] := \bigcup_{u \in U} N_G[u]$. The sets $N_G^r(U)$ and $N_G^r[U]$ are defined similarly. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum and the minimum degree of G, respectively.

Similarly to the above, if D is a digraph and $u \in V(D)$, then we write

$$N_D^+(u) := \{v \in V(D) : uv \in E(D)\} \qquad \text{and} \qquad N_D^-(u) := \{v \in V(D) : vu \in E(D)\}$$

for the out- and the in-neighborhood of u respectively, and all the other expressions such as $N_D^+[u]$, $\deg_D^-(u)$, $\Delta^+(D)$, $N_D^+(U)$ for $U \subseteq V(D)$, etc. are defined accordingly.

Multigraphs

Let G be a multigraph. For vertices u, $v \in V(G)$, we write $e_G(u, v) := |E_G(u, v)|$ to indicate the number of edges joining u and v. The degree of a vertex $u \in V(G)$ is defined to be $\deg_G(u) := \sum_{v \in V(G)} e_G(u, v)$.

Maximum average degree and degeneracy

A graph G is **d-degenerate** if $\delta(H) \leqslant d$ for every nonempty subgraph H of G. A digraph D is **acyclic** if it does not contain a directed cycle. A finite graph G is d-degenerate if and only if there is an acyclic orientation D of G with $\Delta^+(D) \leqslant d$. The **maximum average degree** of a finite graph G is defined by

$$\mathrm{mad}(G) := \max_H \frac{2|E(H)|}{|V(H)|},$$

where the maximum is taken over all nonempty subgraphs H of G. If $\mathrm{mad}(G) \leqslant d$, then G is d-degenerate. On the other hand, if G is d-degenerate, then $\mathrm{mad}(G) \leqslant 2d$.

Independent sets and colorings

Let G be a graph. A set $I \subseteq V(G)$ is **independent** if $I \cap N_G(I) = \varnothing$, i.e., if there are no u, $v \in I$ with $uv \in E(G)$. A **proper** (**vertex**) **coloring** of G if a function $f \colon V(G) \to C$, where C is a set whose elements are referred to as **colors**, such that $f(u) \neq f(v)$ for all $uv \in E(G)$, i.e., the preimage of every color is an independent set. The smallest cardinality of a set C such that there is a proper coloring $f \colon V(G) \to C$ is called the **chromatic number** of G and is denoted by $\chi(G)$.

Similarly, a **proper edge coloring** of G is a function $f \colon E(G) \to C$ such that $f(e) \neq f(h)$ whenever e, $h \in E(G)$ are two adjacent edges (we say that two edges in a graph are **adjacent** if they share an endpoint). The smallest cardinality of a set C such that there is a proper edge coloring $f \colon E(G) \to C$ is called the **chromatic index** of G and is denoted by $\chi'(G)$.

Cycles

Let G be a graph. The **girth** $g(G)$ of G is the length of the shortest cycle in G (if G is acyclic, then $g(G) = +\infty$). Depending on the context, we think of a cycle in G as either a subgraph of G, a set (or a sequence) of edges, or a set (or a sequence) of vertices.

Special graphs

The complete graph on n vertices in denoted by K_n. The complete bipartite graph with parts of size n and m is denoted by $K_{n,m}$. The cycle on n vertices is denoted by C_n.

Sums of graphs

If $G_1, \ldots, G_k$ are graphs, then $G_1 + \cdots + G_k$ is the graph with vertex set $V(G_1) \cup \ldots \cup V(G_k)$ and edge set $E(G_1) \cup \ldots \cup E(G_k)$.

Blocks

A **cut vertex** in a connected graph G is a vertex $u \in V(G)$ such that the graph $G - u$ has at least two vertices and is disconnected. We say that a connected graph G is 2**-connected** if $|V(G)| \geqslant 3$ and G has no cut vertices. A **block** in a graph G is a maximal connected subgraph H of G that has no cut vertices. A block H in G is a **leaf block** if $V(H)$ contains at most one cut vertex of G.

Probability

We write $\mathbb{P}[A]$ to denote the probability of a random event A, and $\mathbb{E}[\xi]$ to denote the expectation of a random variable ξ. If A and B are random events (in the same probability space), then $\mathbb{P}[A|B]$ denotes the conditional probability of A given B. We adopt the convention that $\mathbb{P}[A|B] = 0$ whenever $\mathbb{P}[B] = 0$ (note that this way the equality $\mathbb{P}[A \wedge B] = \mathbb{P}[A|B] \cdot \mathbb{P}[B]$ is preserved).

Standard Borel spaces

Our references for descriptive set theory are [Kec95] and [Tse16].

A separable topological space is called **Polish** if its topology is induced by a complete metric. A **standard Borel space** is a set X equipped with a σ-algebra $\mathfrak{B}$ of **Borel sets** such that there is a Polish topology τ on X whose Borel σ-algebra is $\mathfrak{B}$. Due to the **Borel isomorphism theorem** [Tse16, Theorem 13.10], all countable standard Borel spaces are discrete and all uncountable ones are isomorphic to each other. A function $f \colon X \to Y$ between standard Borel spaces is **Borel** if f-preimages of Borel subsets of Y are Borel in X.

A subset A of a standard Borel space X is **analytic** if it is the image of a Borel set under a Borel function. Somewhat informally, a set is analytic if it can be defined using existential (but not universal) quantifiers ranging over Borel sets. The complement of an analytic set is said to be **co-analytic**. If a set is both analytic and co-analytic, then it is Borel [Tse16, Corollary 12.7].

The following fundamental result will be used without mention:

Theorem (**Luzin–Novikov theorem**; [Kec95, Theorem 18.10])**.** *Let X and Y be standard Borel spaces and let $A \subseteq X \times Y$ be a Borel set such that for all $x \in X$, the set A_x is countable. Then A can be written as a countable union $A = \bigcup_{n=0}^{\infty} A_n$, where the sets $(A_n)_{n=0}^{\infty}$ are pairwise disjoint and for each $n \in \mathbb{N}$ and $x \in X$, $|(A_n)_x| \leqslant 1$. In particular, the set $\mathrm{proj}_X(A) \coloneqq \{x \in X \,:\, A_x \neq \varnothing\}$ is Borel.*

Informally, the Luzin–Novikov theorem implies that if a set is defined only using quantifiers ranging over countable sets, then it is Borel.

Spaces of finite sets

If X is a standard Borel space, then $[X]^{<\infty}$ is also naturally equipped with a standard Borel structure. For any standard Borel space X, there exists a Borel map $f \colon [X]^{<\infty} \setminus \{\varnothing\} \to X$ such that $f(S) \in S$ for all $S \in [X]^{<\infty} \setminus \{\varnothing\}$; for example, if $<$ is a Borel linear ordering of X (which exists as X is Borel isomorphic to $\mathbb{R}$), then the function $S \mapsto \min_< S$ is Borel. If X and Y are standard Borel spaces, then $[X \to Y]^{<\infty}$ is also a standard Borel space, which can be identified with a Borel subset of $[X \times Y]^{<\infty}$.

Measures

We use $\mathbf{Prob}(X)$ to denote the set of all probability Borel measures on a standard Borel space X. If $\mu \in \mathbf{Prob}(X)$, then the pair (X, μ) is called a **standard probability space**. A measure $\mu \in \mathbf{Prob}(X)$ is **atomless** if $\mu(\{x\}) = 0$ for all $x \in X$. The **measure isomorphism theorem** [Tse16, Theorem 10.6] asserts that all atomless standard probability spaces (X, μ) are Borel isomorphic to each other.

If X is a standard Borel space and $X' \subseteq X$ is a Borel set, then we identify $\mathbf{Prob}(X')$ with a subset of $\mathbf{Prob}(X)$ in the natural way. In particular, given $\mu \in \mathbf{Prob}(X')$, we also use μ to denote the extension of μ to X (i.e., the pushforward $\iota_*(\mu)$ of μ under the inclusion map $\iota \colon X' \to X$). Similarly, if $\mu \in \mathbf{Prob}(X)$ and X' is μ-conull, then we use μ to denote the restriction of μ to X'.

Analytic subsets of a standard Borel space X are universally measurable, i.e., μ-measurable for every $\mu \in \mathbf{Prob}(X)$ [Tse16, Corollary 14.10].

The Lebesgue probability measure on the unit interval $[0; 1]$ is denoted by λ.

Group actions

Let Γ be a countable group. The identity element of Γ is denoted by $\mathbf{1}_\Gamma$, or, if Γ is understood, simply by $\mathbf{1}$. All group actions, unless explicitly stated otherwise, are from the left. An action $\Gamma \curvearrowright X$ is **free** if the stabilizer of every point $x \in X$ is trivial:

$$\gamma \cdot x = x \implies \gamma = \mathbf{1} \qquad \text{for all } \gamma \in \Gamma \text{ and } x \in X.$$

Let $\alpha \colon \Gamma \curvearrowright X$ and $\beta \colon \Gamma \curvearrowright Y$ be actions of Γ. A map $\pi \colon X \to Y$ is **equivariant** if it intertwines the actions, i.e., if $\gamma \cdot \pi(x) = \pi(\gamma \cdot x)$ for all $\gamma \in \Gamma$ and $x \in X$.

Measure-preserving actions

A **probability measure-preserving (p.m.p.) action** of a countable group Γ is an action of the form $\alpha\colon \Gamma \curvearrowright (X, \mu)$, where (X, μ) is a standard probability space and the measure μ is α-invariant. For p.m.p. actions, we use the word "free" to mean "free almost everywhere." In other words, we call a p.m.p. action $\alpha\colon \Gamma \curvearrowright (X, \mu)$ **free** if

$$\mu(\{x \in X \,:\, \gamma \cdot x \neq x \text{ for all } \mathbf{1} \neq \gamma \in \Gamma\}) = 1.$$

Shift actions

Let Γ be a countable group and let A be a set. The **shift action** $\sigma_A\colon \Gamma \curvearrowright A^\Gamma$ is defined via

$$(\gamma \cdot x)(\delta) \coloneqq x(\delta\gamma) \qquad \text{for all } \gamma,\ \delta \in \Gamma \text{ and } x \in A^\Gamma.$$

Occasionally, we will also have to consider the **right shift action** $A^\Gamma \curvearrowleft \Gamma$, defined similarly by

$$(x \cdot \gamma)(\delta) \coloneqq x(\gamma\delta) \qquad \text{for all } \gamma,\ \delta \in \Gamma \text{ and } x \in A^\Gamma.$$

Whenever we refer to the shift action, for instance when talking about shift-invariant sets, the *left* shift action must be understood, unless explicitly stated otherwise. If A is a topological space, then the shift actions $\Gamma \curvearrowright A^\Gamma$ and $A^\Gamma \curvearrowleft \Gamma$ are continuous with respect to the product topology on A^Γ.

The shift action $\Gamma \curvearrowright A^\Gamma$ naturally extends to an action $\Gamma \curvearrowright [\Gamma \to A]^{<\infty}$. Namely, for all $w \in [\Gamma \to A]^{<\infty}$ and $\gamma \in \Gamma$, let

$$\mathrm{dom}(\gamma \cdot w) \coloneqq \mathrm{dom}(w)\gamma^{-1} \qquad \text{and} \qquad (\gamma \cdot w)(\delta) \coloneqq w(\delta\gamma) \text{ for all } \delta \in \mathrm{dom}(\gamma \cdot w).$$

The right shift action $[\Gamma \to A]^{<\infty} \curvearrowleft \Gamma$ is defined similarly in the obvious way.

Equivalence relations

We identify an equivalence relation E on set X with the set of pairs $\{(x, y) \,:\, x \mathrel{E} y\}$. In particular, if X is a standard Borel space, then E is **Borel** if it is a Borel subset of $X \times X$. We use X/E to denote the set of all E-classes. A set $X' \subseteq X$ is **E-invariant** if it is a union of E-classes; i.e., for all $x \in X'$ and $y \in X$ with $x \mathrel{E} y$, we have $y \in X'$. For $S \subseteq X$, we use $[S]_E$ to denote the **E-saturation** of S, i.e., the smallest E-invariant subset of X that contains S. For brevity, given $x \in X$, we write $[x]_E$ instead of $[\{x\}]_E$.

We say that an equivalence relation E is **countable** if every E-class is countable. It follows from the Luzin–Novikov theorem that if E is a countable Borel equivalence relation on a standard Borel space X, then the E-saturation of every Borel subset of X is Borel.

Part I

The finite

1 | The Local Cut Lemma

1.1 Introduction

One of the most useful tools in probabilistic combinatorics is the so-called **Lovász Local Lemma** (the **LLL** for short), which was proved by Erdős and Lovász in their seminal paper [EL75]. Roughly speaking, the LLL asserts that, given a family $\mathscr{B}$ of random events whose individual probabilities are small and whose dependency is somehow limited, there is a positive probability that none of the events in $\mathscr{B}$ happen. More precisely:

Theorem 1.1.1 (**Lovász Local Lemma**, [AS00, Lemma 5.1.1])**.** *Let $B_1, \ldots, B_n$ be random events in a probability space Ω. For each $1 \leqslant i \leqslant n$, let $\Gamma(i)$ be a subset of $\{1, \ldots, n\} \setminus \{i\}$ such that the event B_i is independent from the algebra generated by $\{B_j \,:\, j \notin \Gamma(i) \cup \{i\}\}$. Suppose that there exists a function $\mu \colon \{1, \ldots, n\} \to [0; 1)$ such that for every $1 \leqslant i \leqslant n$,*

$$\mathbb{P}[B_i] \leqslant \mu(i) \prod_{j \in \Gamma(i)} (1 - \mu(j)).$$

Then

$$\mathbb{P}\left[\bigwedge_{i=1}^{n} \neg B_i\right] \geqslant \prod_{i=1}^{n} (1 - \mu(i)) > 0.$$

Note that the probability $\mathbb{P}\left[\bigwedge_{i\in I} \neg B_i\right]$, which the LLL bounds from below, is usually exponentially small (in the parameter n). This contrasts with the more common situation when the probability of interest is not only positive, but separated from zero. Although this property of the LLL makes it an indispensable tool in proving combinatorial existence results, it also makes these results seemingly nonconstructive, since sampling the probability space to find an object with the desired properties would usually take an exponentially long expected time. A major breakthrough was made by Moser and Tardos [MT10], who showed that, in a special framework for the LLL called the **variable version** (the name is due to Kolipaka and Szegedy [KS11]), there exists a simple Las Vegas algorithm with expected polynomial runtime that searches the probability space for a point which avoids all the events in $\mathscr{B}$. Their algorithm was subsequently refined and extended to other situations by several authors; see, e.g., [Peg14; KS11; AI14; CGH13].

This chapter is based on [Ber16b; Ber17c].

The key ingredient of Moser and Tardos's proof is the so-called **entropy compression method** (the name is due to Tao [Tao09]). The idea of this method is to encode the execution process of the algorithm in such a way that the original sequence of random inputs can be uniquely recovered from the resulting encoding. One then proceeds to show that if the algorithm runs for too long, the space of possible codes becomes smaller than the space of inputs, which leads to a contradiction.

Later it was discovered that applying the entropy compression method directly to a specific problem can lead to better results than simply using the LLL. This approach, first introduced by Grytczuk, Kozik, and Micek in their study of nonrepetitive sequences [GKM13], consists in constructing a randomized procedure that solves a given combinatorial problem and then using an entropy compression argument to show that the procedure runs in expected finite time. A wealth of new results have been obtained using this paradigm; see, e.g., [Duj+15; EP13; GMP14]. Some of these examples are discussed in more detail in §1.3.

Note that the entropy compression method is indeed a "method" that one can use to attack a problem rather than a general theorem that contains various combinatorial results as its special cases. It is natural to ask if such a theorem exists, i.e., if there is a generalization of the LLL that implies the new combinatorial results obtained using the entropy compression method. The goal of this chapter is to describe such a generalization, which we call the **Local Cut Lemma** (the **LCL** for short). It is important to note that this result is purely probabilistic and similar to the LLL in flavor. In particular, its short and simple probabilistic proof does not use the entropy compression method. Instead, it estimates certain probabilities explicitly, in much the same way as the original (nonconstructive) proof of the LLL does. We state and prove the LCL in §1.2. In §1.3 we introduce a simplified special case of the LCL (namely Theorem 1.3.1), which turns out to be sufficient for most applications. In fact, Theorem 1.3.1 already implies the classical LLL, as we show in §1.4. In §1.5 we discuss one simple example (hypergraph coloring) which provides the intuition behind the LCL and serves as a model for the more substantial applications described later. In §1.6 we show how to use the LCL to prove upper bounds on the nonrepetitive chromatic number that were previously obtained using the entropy compression method. In §1.7 we present an improved lower bound on the number of edges in color-critical hypergraphs that was first derived using the LCL. In §1.8 we discuss a curious probabilistic corollary of the LCL. Finally, we present some applications of the LCL to the acyclic edge coloring problem in §1.9.

1.2 The Local Cut Lemma: statement and proof

1.2.1 Statement of the LCL

To state the main result of this chapter, we need to fix some notation and terminology. In what follows, a **digraph** always means a finite directed multigraph. Let D be a digraph with vertex set V and edge set E. For $x, y \in V$, let $E(x, y) \subseteq E$ denote the set of all edges with tail x and head y.

A digraph D is **simple** if for all $x, y \in V$, $|E(x, y)| \leqslant 1$. If D is simple and $|E(x, y)| = 1$, then the unique edge with tail x and head y is denoted by xy (or sometimes (x, y)). For an arbitrary digraph D, let D^s denote its **underlying simple digraph**, i.e., the simple digraph with vertex set V in which xy is an edge if and only if $E(x, y) \neq \varnothing$. The edge set of D^s is denoted by E^s. For a subset $F \subseteq E$, let $F^s \subseteq E^s$ be the set of all edges $xy \in E^s$ such that $F \cap E(x, y) \neq \varnothing$. A set $A \subseteq V$ is **out-closed** if for all $xy \in E^s$, $x \in A$ implies $y \in A$.

Definition 1.2.1. Let D be a digraph with vertex set V and edge set E and let $A \subseteq V$ be an out-closed set of vertices. A set $F \subseteq E$ of edges is an A**-cut** if if F contains at least one edge $e \in E(x, y)$ for each $xy \in E^s$ such that $x \notin A$ and $y \in A$ (see Fig. 5).

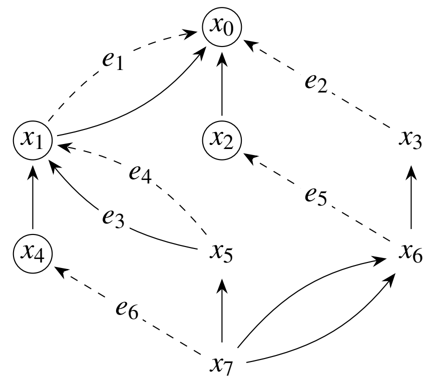

Figure 5 – A digraph D with an out-closed set $A = \{x_0, x_1, x_2, x_4\}$. Any A-cut must contain the edges $\{e_2, e_5, e_6\}$ and at least one of $\{e_3, e_4\}$. For example, the set $F = \{e_1, e_2, e_4, e_5, e_6\}$ consisting of the dashed edges forms an A-cut.

We say that a vertex $z \in V$ is **reachable** from $x \in V$ if D (or, equivalently, D^s) contains a directed xz-path. The set of all vertices reachable from x is denoted by $R_D(x)$.

Definition 1.2.2. Let D be a digraph with vertex set V and edge set E. For a function $\omega \colon E^s \to [1; +\infty)$ and vertices $x \in V$ and $z \in R_D(x)$, define

$$\underline{\omega}(x, z) := \min \left\{ \prod_{i=1}^{k} \omega(z_{i-1} z_i) \ :\ x = z_0 \to z_1 \to \ldots \to z_k = z \text{ is a directed } xz\text{-path} \right\}.$$

Definition 1.2.3. Let D be a digraph with vertex set V and edge set E. Let Ω be a probability space and let $A \colon \Omega \to \mathrm{Pow}(V)$ and $F \colon \Omega \to \mathrm{Pow}(E)$ be random variables such that with probability 1, A is an out-closed set of vertices and F is an A-cut. Fix a function $\omega \colon E^s \to [1; +\infty)$. For $xy \in E^s$, $e \in E(x, y)$, and $z \in R_D(y)$, let

$$\rho_\omega^{A,F}(e, z) := \mathbb{P}[e \in F | z \in A] \cdot \underline{\omega}(x, z).$$

For $e \in E(x, y)$, define the **risk** to e as

$$\rho_\omega^{A,F}(e) := \min_{z \in R_D(y)} \rho_\omega^{A,F}(e, z).$$

We are now ready to state the main result of this chapter.

Theorem 1.2.4 (**Local Cut Lemma**). *Let D be a digraph with vertex set V and edge set E. Let Ω be a probability space and let $A \colon \Omega \to \mathrm{Pow}(V)$ and $F \colon \Omega \to \mathrm{Pow}(E)$ be random variables such that with probability 1, A is an out-closed set of vertices and F is an A-cut. If a function $\omega \colon E^s \to [1; +\infty)$ satisfies the* *following inequality for all $xy \in E^s$:*

$$\omega(xy) \geqslant 1 + \sum_{e \in E(x,y)} \rho_{\omega}^{A,F}(e), \tag{1.2.1}$$

then for all $xy \in E^s$,

$$\mathbb{P}[y \in A] \leqslant \mathbb{P}[x \in A] \cdot \omega(xy).$$

The following immediate corollary is the main tool used in combinatorial applications of Theorem 1.2.4:

Corollary 1.2.5. *Let D, A, F, ω be as in Theorem 1.2.4. Let $x \in V$, $z \in R_D(x)$, and suppose that $\mathbb{P}[z \in A] > 0$. Then*

$$\mathbb{P}[x \in A] \geqslant \frac{\mathbb{P}[z \in A]}{\underline{\omega}(x,z)} > 0.$$

1.2.2 Proof of the LCL

In this section we prove Theorem 1.2.4. Let D, A, F be as in the statement of Theorem 1.2.4 and assume that a function $\omega \colon E^s \to [1;+\infty)$ satisfies

$$\omega(xy) \geqslant 1 + \sum_{e \in E(x,y)} \rho_{\omega}^{A,F}(e) \tag{1.2.1}$$

for all $xy \in E^s$. For each $\upsilon \colon E^s \to [1;+\infty)$, let $f(\upsilon) \colon E^s \to [1;+\infty)$ be defined by

$$f(\upsilon)(xy) \coloneqq 1 + \sum_{e \in E(x,y)} \rho_{\upsilon}^{A,F}(e).$$

Also, let $f(\mathbb{0}) \coloneqq \mathbb{1}$, where $\mathbb{0}$ and $\mathbb{1}$ denote the constant 0 and 1 functions respectively. Then (1.2.1) is equivalent to

$$\omega(xy) \geqslant f(\omega)(xy). \tag{1.2.2}$$

Note that the map f is monotone increasing, i.e., if $\upsilon(xy) \leqslant \upsilon'(xy)$ for all $xy \in E^s$, then $f(\upsilon)(xy) \leqslant f(\upsilon')(xy)$ for all $xy \in E^s$ as well.

Let $\omega_0 \coloneqq \mathbb{0}$ and let $\omega_{n+1} \coloneqq f(\omega_n)$ for all $n \in \mathbb{N}$. To simplify the notation, let $\rho_n \coloneqq \rho_{\omega_n}^{A,F}$.

Claim 1.2.6. *For all $n \in \mathbb{N}$ and $xy \in E^s$,*

$$\omega_n(xy) \leqslant \omega_{n+1}(xy). \tag{1.2.3}$$

Proof. Induction on n. If $n = 0$, then (1.2.3) asserts that $0 \leqslant 1$. Now suppose that (1.2.3) holds for some $n \in \mathbb{N}$. Then we have

$$\omega_{n+1}(xy) = f(\omega_n)(xy) \leqslant f(\omega_{n+1})(xy) = \omega_{n+2}(xy),$$

as desired. ∎

Claim 1.2.7. *For all $n \in \mathbb{N}$ and $xy \in E^s$,*

$$\omega_n(xy) \leqslant \omega(xy). \tag{1.2.4}$$

Proof. Proof is again by induction on n. If $n = 0$, then (1.2.4) says that $0 \leqslant \omega(xy)$. Now suppose that (1.2.4) holds for some $n \in \mathbb{N}$. Then, using (1.2.2), we get

$$\omega_{n+1}(xy) = f(\omega_n)(xy) \leqslant f(\omega)(xy) \leqslant \omega(xy),$$

as desired. ■

Since the sequence $\{\omega_n(xy)\}_{n=0}^{\infty}$ is monotone increasing and bounded by $\omega(xy)$, it has a limit, so let

$$\omega_\infty(xy) \coloneqq \lim_{n\to\infty} \omega_n(xy).$$

We still have $\omega_\infty(xy) \leqslant \omega(xy)$ for all $xy \in E^s$, so it is enough to prove that for all $xy \in E^s$,

$$\mathbb{P}[y \in A] \leqslant \mathbb{P}[x \in A] \cdot \omega_\infty(xy). \tag{1.2.5}$$

We will derive (1.2.5) from the following lemma.

Lemma 1.2.8. *For every $n \in \mathbb{N}$ and $xy \in E^s$,*

$$\mathbb{P}[y \in A] \leqslant \mathbb{P}[x \in A] \cdot \omega_n(xy) + \omega_{n+1}(xy) - \omega_n(xy). \tag{1.2.6}$$

If Lemma 1.2.8 holds, then we are done, since it implies that

$$\mathbb{P}[y \in A] \leqslant \lim_{n\to\infty} (\mathbb{P}[x \in A] \cdot \omega_n(xy) + \omega_{n+1}(xy) - \omega_n(xy)) = \mathbb{P}[x \in A] \cdot \omega_\infty(xy),$$

as desired.

To establish Lemma 1.2.8, we need the following claim.

Claim 1.2.9. *Let $n \in \mathbb{N}$ and suppose that for all $xy \in E^s$,* (1.2.6) *holds. Then for all $x \in V$ and $z \in R_D(x)$,*

$$\mathbb{P}[z \in A] \leqslant \mathbb{P}[x \in A] \cdot \underline{\omega_n}(x, z) + \underline{\omega_{n+1}}(x, z) - \underline{\omega_n}(x, z). \tag{1.2.7}$$

The proof of Claim 1.2.9 uses the following simple algebraic inequality.

Claim 1.2.10. *Let $a_1, \dots, a_k, b_1, \dots, b_k$ be nonnegative real numbers with $b_i \geqslant \max\{a_i, 1\}$ for all $1 \leqslant i \leqslant k$. Then*

$$\sum_{i=1}^{k} \left(\prod_{j=1}^{i-1} a_j \right) (b_i - a_i) \leqslant \prod_{i=1}^{k} b_i - \prod_{i=1}^{k} a_i. \tag{1.2.8}$$

Proof. Proof is by induction on k. If $k = 1$, then both sides of (1.2.8) are equal to $b_1 - a_1$. If the claim is established for some k, then for $k+1$ we get

$$\begin{aligned}
\sum_{i=1}^{k+1}\left(\prod_{j=1}^{i-1} a_j\right)(b_i - a_i) &= \sum_{i=1}^{k}\left(\prod_{j=1}^{i-1} a_j\right)(b_i - a_i) + \left(\prod_{i=1}^{k} a_i\right) b_{k+1} - \prod_{i=1}^{k+1} a_i \\
&\leqslant \prod_{i=1}^{k} b_i - \prod_{i=1}^{k} a_i + \left(\prod_{i=1}^{k} a_i\right) b_{k+1} - \prod_{i=1}^{k+1} a_i \\
&= \prod_{i=1}^{k+1} b_i - \prod_{i=1}^{k+1} a_i - \left(\prod_{i=1}^{k} b_i - \prod_{i=1}^{k} a_i\right)(b_{k+1} - 1) \\
&\leqslant \prod_{i=1}^{k+1} b_i - \prod_{i=1}^{k+1} a_i,
\end{aligned}$$

as desired. ■

Proof of Claim 1.2.9. Let $x = z_0 \to z_1 \to \ldots \to z_k = z$ be some directed xz-path in D^s. For $1 \leqslant i \leqslant k$, let

$$a_i \coloneqq \omega_n(z_{k-i}z_{k-i+1}) \qquad \text{and} \qquad b_i \coloneqq \omega_{n+1}(z_{k-i}z_{k-i+1}).$$

Note that $b_i \geqslant \max\{a_i, 1\}$. Due to (1.2.6), we have

$$\mathbb{P}[z \in A] \leqslant \mathbb{P}[z_{k-1} \in A] \cdot a_1 + b_1 - a_1.$$

Similarly,

$$\mathbb{P}[z_{k-1} \in A] \leqslant \mathbb{P}[z_{k-2} \in A] \cdot a_2 + b_2 - a_2,$$

so

$$\mathbb{P}[z \in A] \leqslant \mathbb{P}[z_{k-2} \in A] \cdot a_1 a_2 + b_1 - a_1 + a_1(b_2 - a_2).$$

Continuing such substitutions, we finally obtain

$$\mathbb{P}[z \in A] \leqslant \mathbb{P}[x \in A] \cdot \prod_{i=1}^{k} a_i + \sum_{i=1}^{k}\left(\prod_{j=1}^{i-1} a_j\right)(b_i - a_i).$$

Using Claim 1.2.10, we get

$$\mathbb{P}[z \in A] \leqslant \mathbb{P}[x \in A] \cdot \prod_{i=1}^{k} a_i + \prod_{i=1}^{k} b_i - \prod_{i=1}^{k} a_i.$$

Note that

$$\prod_{i=1}^{k} a_i = \prod_{i=1}^{k} \omega_n(z_{i-1}z_i) \geqslant \underline{\omega_n}(x, z).$$

Since $\mathbb{P}[x \in A] \leqslant 1$, this implies

$$\mathbb{P}[z \in A] \leqslant \mathbb{P}[x \in A] \cdot \underline{\omega_n}(x, z) + \prod_{i=1}^{k} b_i - \underline{\omega_n}(x, z). \tag{1.2.9}$$

It remains to observe that inequality (1.2.9) holds for all directed xz-paths, so we can replace $\prod_{i=1}^{k} b_i$ by $\underline{\omega_{n+1}}(x, z)$, obtaining

$$\mathbb{P}[z \in A] \leqslant \mathbb{P}[x \in A] \cdot \underline{\omega_n}(x, z) + \underline{\omega_{n+1}}(x, z) - \underline{\omega_n}(x, z),$$

as desired. ■

Proof of Lemma 1.2.8. Induction on n. For $n = 0$, the lemma asserts that $\mathbb{P}[y \in A] \leqslant 1$. Now assume that (1.2.6) holds for some $n \in \mathbb{N}$ and consider an edge $xy \in E^s$. Since A is out-closed, $x \in A$ implies $y \in A$, so

$$\mathbb{P}[y \in A] = \mathbb{P}[x \in A] + \mathbb{P}[x \notin A \text{ and } y \in A].$$

Since F is an A-cut, it contains at least one edge $e \in E(x, y)$ whenever $x \notin A$ and $y \in A$. Using the union bound, we obtain

$$\mathbb{P}[x \notin A \text{ and } y \in A] \leqslant \sum_{e \in E(x,y)} \mathbb{P}[e \in F \text{ and } y \in A].$$

Thus,

$$\mathbb{P}[y \in A] \leqslant \mathbb{P}[x \in A] + \sum_{e \in E(x,y)} \mathbb{P}[e \in F \text{ and } y \in A]. \tag{1.2.10}$$

Let us now estimate $\mathbb{P}[e \in F \text{ and } y \in A]$ for each $e \in E(x, y)$. Consider any $z \in R_D(y)$. Since A is out-closed, $y \in A$ implies $z \in A$, so

$$\mathbb{P}[e \in F \text{ and } y \in A] \leqslant \mathbb{P}[e \in F \text{ and } z \in A] = \mathbb{P}[e \in F | z \in A] \cdot \mathbb{P}[z \in A].$$

Due to Claim 1.2.9,

$$\mathbb{P}[z \in A] \leqslant \mathbb{P}[x \in A] \cdot \underline{\omega_n}(x, z) + \underline{\omega_{n+1}}(x, z) - \underline{\omega_n}(x, z),$$

so

$$\begin{aligned}\mathbb{P}[e \in F \text{ and } y \in A] &\leqslant \mathbb{P}[e \in F | z \in A] \cdot \Big(\mathbb{P}[x \in A] \cdot \underline{\omega_n}(x, z) + \underline{\omega_{n+1}}(x, z) - \underline{\omega_n}(x, z)\Big) \\ &= \mathbb{P}[x \in A] \cdot \rho_n(e, z) + \rho_{n+1}(e, z) - \rho_n(e, z).\end{aligned}$$

Since $\mathbb{P}[x \in A] \leqslant 1$ and $\rho_n(e, z) \geqslant \rho_n(e)$, we get

$$\mathbb{P}[e \in F \text{ and } y \in A] \leqslant \mathbb{P}[x \in A] \cdot \rho_n(e) + \rho_{n+1}(e, z) - \rho_n(e).$$

The last inequality holds for every $z \in R_D(y)$, so we can replace $\rho_{n+1}(e, z)$ by $\rho_{n+1}(e)$, obtaining

$$\mathbb{P}[e \in F \text{ and } y \in A] \leqslant \mathbb{P}[x \in A] \cdot \rho_n(e) + \rho_{n+1}(e) - \rho_n(e). \tag{1.2.11}$$

Plugging (1.2.11) into (1.2.10), we get

$$\mathbb{P}[y \in A] \leqslant \mathbb{P}[x \in A] + \sum_{e \in E(x,y)} (\mathbb{P}[x \in A] \cdot \rho_n(e) + \rho_{n+1}(e) - \rho_n(e)).$$

The right-hand side of the last inequality can be rewritten as

$$\begin{aligned} &\mathbb{P}[x \in A] \cdot \left(1 + \sum_{e \in E(x,y)} \rho_n(e)\right) + \sum_{e \in E(x,y)} \rho_{n+1}(e) - \sum_{e \in E(x,y)} \rho_n(e) \\ &= \mathbb{P}[x \in A] \cdot f(\omega_n)(xy) + f(\omega_{n+1})(xy) - f(\omega_n)(xy) \\ &= \mathbb{P}[x \in A] \cdot \omega_{n+1}(xy) + \omega_{n+2}(xy) - \omega_{n+1}(xy), \end{aligned}$$

as desired. ■

1.3 A special version of the LCL

In this section we introduce a particular and perhaps more intuitive set-up for the LCL that will be sufficient for almost all applications discussed later.

Let I be a finite set. A family $A \in \mathrm{Pow}(\mathrm{Pow}(I))$ of subsets of I is **downwards-closed** if for each $S \in A$, $\mathrm{Pow}(S) \subseteq A$. The **boundary** ∂A of a downwards-closed family is defined to be

$$\partial A \coloneqq \{i \in I \,:\, S \in A \text{ and } S \cup \{i\} \notin A \text{ for some } S \subseteq I \setminus \{i\}\}.$$

Suppose that Ω is a probability space and $A \colon \Omega \to \mathrm{Pow}(\mathrm{Pow}(I))$ is a random variable such that A is downwards-closed with probability 1. Let B be a random event and let $\tau \colon I \to [1; +\infty)$ be a function. For a subset $X \subseteq I$, let

$$\tau(X) \coloneqq \prod_{i \in X} \tau(i),$$

and

$$\sigma_\tau^A(B, X) \coloneqq \max_{Z \subseteq I \setminus X} \mathbb{P}[B | Z \in A] \cdot \tau(X).$$

Finally, for an element $i \in I$, let

$$\sigma_\tau^A(B, i) \coloneqq \min_{i \in X \subseteq I} \sigma_\tau^A(B, X).$$

Theorem 1.3.1. *Let I be a finite set. Let Ω be a probability space and let $A \colon \Omega \to \mathrm{Pow}(\mathrm{Pow}(I))$ be a random variable such that with probability 1, A is a nonempty downwards-closed family of subsets of I. For each $i \in I$, let $\mathscr{B}(i)$ be a finite collection of random events such that whenever $i \in \partial A$, at least one of the events in $\mathscr{B}(i)$ holds. Suppose that there is a function $\tau \colon I \to [1; +\infty)$ such that for all $i \in I$, we have*

$$\tau(i) \geqslant 1 + \sum_{B \in \mathscr{B}(i)} \sigma_\tau^A(B, i). \tag{1.3.1}$$

Then $\mathbb{P}[I \in A] \geqslant 1/\tau(I) > 0$.

Proof. For convenience, we may assume that for each $i \in I$, the set $\mathscr{B}(i)$ is nonempty (we can arrange that by adding the empty event to each $\mathscr{B}(i)$). Let D be the digraph with vertex set $\mathrm{Pow}(I)$ and edge set

$$E \coloneqq \{e_{i,S,B} \,:\, i \in I,\ S \subseteq I \setminus \{i\},\ B \in \mathscr{B}(i)\},$$

where the edge $e_{i,S,B}$ goes from $S \cup \{i\}$ to S. Thus, we have

$$E^s = \{(S \cup \{i\}, S) \,:\, i \in I,\ S \subseteq I \setminus \{i\}\},$$

which implies that for $S, Z \subseteq I$,

$$Z \in R_D(S) \iff Z \subseteq S.$$

Moreover, if $Z \subseteq S \subseteq I$, then all directed (S, Z)-paths have length exactly $|S \setminus Z|$.

Since A is downwards-closed, it is out-closed in D. Let $F \colon \Omega \to \mathrm{Pow}(E)$ be the random set of edges defined by

$$e_{i,S,B} \in F \iff B \text{ holds.}$$

We claim that F is an A-cut. Indeed, consider any edge $(S \cup \{i\}, S) \in E^s$ and suppose that we have $S \cup \{i\} \notin A$ and $S \in A$. By definition, this means that $i \in \partial A$, so at least one event $B \in \mathscr{B}(i)$ holds. But then $e_{i,S,B} \in F \cap E(S \cup \{i\}, S)$, as desired.

Let $\tau \colon I \to [1; +\infty)$ be a function satisfying (1.3.1) and let $\omega \colon E^s \to [1; +\infty)$ be given by $\omega((S \cup \{i\}, S)) \coloneqq \tau(i)$. Note that for any $Z \subseteq S \subseteq I$, we have $\underline{\omega}(S, Z) = \tau(S \setminus Z)$.

Claim (A). *Let $i \in I$, $S \subseteq I \setminus \{i\}$, and $B \in \mathscr{B}(i)$. Then*

$$\rho_\omega^{A,F}(e_{i,S,B}) \leqslant \sigma_\tau^A(B, i).$$

Proof. Let X be a set with $i \in X \subseteq I$ such that $\sigma_\tau^A(B, i) = \sigma_\tau^A(B, X)$ and let $Z \coloneqq S \setminus X$. We have

$$\begin{aligned} \rho_\omega^{A,F}(e_{i,S,B}) \leqslant \rho_\omega^{A,F}(e_{i,S,B}, Z) &= \mathbb{P}[e_{i,S,B} \in F | Z \in A] \cdot \underline{\omega}(S \cup \{i\}, Z) \\ &= \mathbb{P}[B | Z \in A] \cdot \tau((S \cup \{i\}) \setminus Z). \end{aligned}$$

Since $(S \cup \{i\}) \setminus Z \subseteq X$ and τ takes values in $[1; +\infty)$, we have $\tau((S \cup \{i\}) \setminus Z) \leqslant \tau(X)$, so

$$\mathbb{P}[B | Z \in A] \cdot \tau((S \cup \{i\}) \setminus Z) \leqslant \mathbb{P}[B | Z \in A] \cdot \tau(X) \leqslant \sigma_\tau^A(B, X) = \sigma_\tau^A(B, i). \qquad \dashv$$

Let $(S \cup \{i\}, S) \in E^s$. Using (1.3.1) and Claim (A), we obtain

$$\begin{aligned} \omega((S \cup \{i\}, S)) = \tau(i) &\geqslant 1 + \sum_{B \in \mathscr{B}(i)} \sigma_\tau^A(B, i) \\ &\geqslant 1 + \sum_{B \in \mathscr{B}(i)} \rho_\omega^{A,F}(e_{i,S,B}) = 1 + \sum_{e \in E(S \cup \{i\}, S)} \rho_\omega^{A,F}(e), \end{aligned}$$

i.e., ω satisfies (1.2.1). Thus, by Corollary 1.2.5,

$$\mathbb{P}[I \in A] \geqslant \frac{\mathbb{P}[\varnothing \in A]}{\underline{\omega}(I, \varnothing)} = \frac{1}{\tau(I)} > 0,$$

as desired. (Here we are using that $\mathbb{P}[\varnothing \in A] = 1$, which follows from the fact that with probability 1, A is nonempty and downwards-closed.) ∎

1.4 The LCL implies the Lopsided LLL

In this section we use the LCL to prove the Lopsided LLL, which is a strengthening of the standard LLL.

Theorem 1.4.1 (**Lopsided Lovász Local Lemma** [ES91])**.** *Let $B_1, \ldots, B_n$ be random events in a probability space Ω. For each $1 \leqslant i \leqslant n$, let $\Gamma(i)$ be a subset of $\{1, \ldots, n\} \setminus \{i\}$ such that for all $Z \subseteq \{1, \ldots, n\} \setminus (\Gamma(i) \cup \{i\})$, we have*

$$\mathbb{P}\left[B_i \middle| \bigwedge_{j \in Z} \neg B_j\right] \leqslant \mathbb{P}[B_i]. \tag{1.4.1}$$

Suppose that there exists a function $\mu \colon \{1, \ldots, n\} \to [0;1)$ such that for every $1 \leqslant i \leqslant n$,

$$\mathbb{P}[B_i] \leqslant \mu(i) \prod_{j \in \Gamma(i)} (1 - \mu(j)). \tag{1.4.2}$$

Then

$$\mathbb{P}\left[\bigwedge_{i=1}^{n} \neg B_i\right] \geqslant \prod_{i=1}^{n} (1 - \mu(i)) > 0.$$

Proof. We will use Theorem 1.3.1. Set $I \coloneqq \{1, \ldots, n\}$ and let $I_0 \colon \Omega \to \mathrm{Pow}(I)$ and $I_1 \colon \Omega \to \mathrm{Pow}(I)$ be random variables defined by

$$I_1 \coloneqq \{i \in I \,:\, B_i \text{ holds}\} \qquad \text{and} \qquad I_0 \coloneqq I \setminus I_1.$$

Set $A \coloneqq \mathrm{Pow}(I_0)$. In other words, a set $S \subseteq I$ belongs A if and only if $\bigwedge_{i \in S} \neg B_i$ holds. It follows that A is a nonempty downwards-closed family of subsets of I and $\partial A = I_1$ (i.e., $i \in \partial A$ if and only if B_i holds). Therefore, we can apply Theorem 1.3.1 with $\mathscr{B}(i) \coloneqq \{B_i\}$ for each $i \in I$.

By (1.4.1), if $i \in I$ and $Z \subseteq I \setminus (\Gamma(i) \cup \{i\})$, then

$$\mathbb{P}[B_i | Z \in A] = \mathbb{P}\left[B_i \middle| \bigwedge_{j \in Z} \neg B_j\right] \leqslant \mathbb{P}[B_i].$$

Thus, for any $i \in I$ and $\tau \colon I \to [1; +\infty)$, we have

$$\sigma_\tau^A(B_i, i) \leqslant \sigma_\tau^A(B_i, \Gamma(i) \cup \{i\}) = \max_{Z \subseteq I \setminus (\Gamma(i) \cup \{i\})} \mathbb{P}[B_i | Z \in A] \cdot \tau(\Gamma(i) \cup \{i\}) \leqslant \mathbb{P}[B_i] \cdot \tau(\Gamma(i) \cup \{i\}).$$

Therefore, (1.3.1) holds as long as for each $i \in I$, we have

$$\tau(i) \geqslant 1 + \mathbb{P}[B_i] \cdot \tau(\Gamma(i) \cup \{i\}). \tag{1.4.3}$$

Suppose that $\mu \colon I \to [0; 1)$ satisfies (1.4.2). Then $\tau(i) \coloneqq 1/(1 - \mu(i))$ satisfies (1.4.3). Indeed,

$$\begin{aligned}
1 + \mathbb{P}[B_i] \cdot \tau(\Gamma(i) \cup \{i\}) &= 1 + \mathbb{P}[B_i] \cdot \prod_{j \in \Gamma(i) \cup \{i\}} \tau(j) \\
&= 1 + \frac{\mathbb{P}[B_i]}{\prod_{j \in \Gamma(i) \cup \{i\}} (1 - \mu(j))} \\
\text{[by (1.4.2)]} \qquad &\leqslant 1 + \frac{\mu(B_i) \prod_{j \in \Gamma(i)} (1 - \mu(j))}{\prod_{j \in \Gamma(i) \cup \{i\}} (1 - \mu(j))} \\
&= 1 + \frac{\mu(i)}{1 - \mu(i)} = \frac{1}{1 - \mu(i)} = \tau(i),
\end{aligned}$$

Theorem 1.3.1 now yields

$$\mathbb{P}\left[\bigwedge_{i=1}^{n} \neg B_i\right] = \mathbb{P}[I \in A] \geqslant \frac{1}{\tau(I)} = \frac{1}{\prod_{i=1}^{n} \tau(i)} = \prod_{i=1}^{n} (1 - \mu(i)),$$

as desired. ∎

The above derivation of the Lopsided LLL from Theorem 1.3.1 clarifies the precise relationship between the two statements. Essentially, Theorem 1.3.1 reduces to the classical LLL under the following two main assumptions: (1) the set A contains an inclusion-maximum element; and (2) each of the sets $\mathscr{B}(i)$ is a singleton, containing only one "bad" event. Neither of these assumptions is satisfied in the applications discussed later, where the LCL outperforms the LLL.

1.5 First example: hypergraph coloring

In this section we provide some intuition behind the LCL using a very basic example: coloring uniform hypergraphs with 2 colors.

Let $\mathcal{H}$ be a d-regular k-uniform hypergraph with vertex set V and edge set E, and suppose we want to establish a relation between d and k that guarantees that $\mathcal{H}$ is 2-colorable. A straightforward application of the LLL gives the bound

$$\frac{e}{2^{k-1}}((d-1)k + 1) \leqslant 1,$$

which is equivalent to

$$d \leqslant \frac{2^{k-1}}{ek} + 1 - \frac{1}{k}. \tag{1.5.1}$$

Let us now explain how to apply the LCL (in the simplified form of Theorem 1.3.1) to this problem.

Choose a coloring $\varphi\colon V \to 2$ uniformly at random. Define $A \subseteq \mathrm{Pow}(V)$ by

$$A \coloneqq \{S \subseteq V \,:\, \text{there is no } \varphi\text{-monochromatic edge } h \subseteq S\}.$$

Clearly, A is downwards-closed, and, since we always have $\varnothing \in A$, A is nonempty. Moreover, $V \in A$ if and only if φ is a proper coloring of $\mathcal{H}$. Therefore, if we can apply Theorem 1.3.1 to show that $\mathbb{P}[V \in A] > 0$, then $\mathcal{H}$ is 2-colorable.

In order to apply Theorem 1.3.1, we have to specify, for each $v \in V$, a finite family $\mathscr{B}(v)$ of "bad" random events such that whenever $v \in \partial A$, at least one of the events in $\mathscr{B}(v)$ holds. Notice that if $v \in \partial A$, i.e., for some $S \subseteq V \setminus \{v\}$, we have $S \in A$ and $S \cup \{v\} \notin A$, then there must exist at least one φ-monochromatic edge $h \ni v$. Thus, we can set

$$\mathscr{B}(v) \coloneqq \{B_h \,:\, v \in h \in E\},$$

where the event B_h happens is and only if h is φ-monochromatic. Since $\mathcal{H}$ is d-regular, $|\mathscr{B}(v)| = d$.

We will assume that $\tau(v) = \tau \in [1; +\infty)$ is a constant function. In that case, for any $S \subseteq V$, $\tau(S) = \tau^{|S|}$. Let $v \in V$ and let $h \in E$ be such that $h \ni v$. To verify (1.3.1), we require an upper bound on the quantity $\sigma_\tau^A(B_h, v)$. By definition,

$$\sigma_\tau^A(B_h, v) = \min_{v \in X \subseteq V} \sigma_\tau^A(B_h, X),$$

so it is sufficient to upper bound $\sigma_\tau^A(B_h, X)$ for some set $X \ni v$. Since

$$\sigma_\tau^A(B_h, X) = \max_{Z \subseteq V \setminus X} \mathbb{P}[B_h | Z \in A] \cdot \tau^{|X|},$$

we just need to find a set $X \ni v$ such that the conditional probability $\mathbb{P}[B_h | Z \in A]$ for $Z \subseteq V \setminus X$ is easy to bound. Moreover, we would like $|X|$ to be as small as possible (to minimize the factor $\tau^{|X|}$).

Since the colors of distinct vertices are independent, the events B_h and "$Z \in A$" are independent whenever $Z \cap h = \varnothing$. Therefore, for $Z \subseteq V \setminus h$,

$$\mathbb{P}[B_h | Z \in A] \leqslant \mathbb{P}[B_h] = \frac{1}{2^{k-1}}. \tag{1.5.2}$$

(The inequality might be strict if $\mathbb{P}[Z \in A] = 0$, in which case $\mathbb{P}[B_h | Z \in A] = 0$ as well.) Thus, it is natural to take $X = h$, which gives

$$\sigma_\tau^A(B_h, v) \leqslant \sigma_\tau^A(B_h, h) = \max_{Z \subseteq V \setminus h} \mathbb{P}[B_h | Z \in A] \cdot \tau^{|h|} \leqslant \frac{\tau^k}{2^{k-1}}.$$

Hence it is enough to ensure that τ satisfies

$$\tau \geqslant 1 + \frac{d\tau^k}{2^{k-1}}.$$

A straightforward calculation shows that the following condition is sufficient:

$$d \leqslant \frac{2^{k-1}}{k}\left(1-\frac{1}{k}\right)^{k-1}, \tag{1.5.3}$$

or, a bit more crudely,

$$d \leqslant \frac{2^{k-1}}{ek}, \tag{1.5.4}$$

which is almost identical to (1.5.1). The precise bound (1.5.3) is, in fact, better than (1.5.1) for $k \geqslant 10$.

We can improve (1.5.4) slightly by estimating $\sigma_\tau^A(B_h, v)$ more carefully. Observe that the inequality (1.5.2) holds even if $|Z \cap h| = 1$ (because fixing the color of one of the vertices in h does not change the probability that h is monochromatic). Therefore, upon choosing any vertex $u \in h \setminus \{v\}$ and taking $X = h \setminus \{u\}$, we obtain

$$\sigma_\tau^A(B_h, v) \leqslant \sigma_\tau^A(B_h, h \setminus \{u\}) = \max_{Z \subseteq (V\setminus h)\cup\{u\}} \mathbb{P}[B_h | Z \in A] \cdot \tau^{|h\setminus\{u\}|} \leqslant \frac{\tau^{k-1}}{2^{k-1}}.$$

Thus, it is enough to ensure that

$$\tau \geqslant 1 + \frac{d\tau^{k-1}}{2^{k-1}},$$

which can be satisfied as long as

$$d \leqslant \frac{2^{k-1}}{e(k-1)}. \tag{1.5.5}$$

The bound (1.5.5) is better than (1.5.4) by a quantity of order $\Omega\left(2^k/k^2\right)$. This is, of course, not a significant improvement (and the bound is still considerably weaker than the best known result due to Radhakrishnan and Srinivasan [RS00], namely $d \leqslant \varepsilon 2^k/\sqrt{k \log k}$ for some constant $\varepsilon > 0$). However, the observation that helped us improve (1.5.4) to (1.5.5) highlights one of the important strengths of the LCL. The fact that $\mathbb{P}[B_h | Z \in A] \leqslant 1/2^{k-1}$ for all Z such that $|Z \cap h| \leqslant 1$ (and not only when $Z \cap h = \varnothing$) contains information beyond the individual probabilities of “bad” events and their dependencies, and the LCL has a mechanism for putting that additional information to use. Similar ideas will reappear several times in later applications.

1.6 Nonrepetitive sequences and nonrepetitive colorings

A finite sequence $a_1a_2\ldots a_n$ is **nonrepetitive** if it does not contain the same nonempty substring twice in a row, i.e., if there are no s, $1 \leqslant s \leqslant n-1$, and t, $1 \leqslant t \leqslant \lfloor (n-s+1)/2 \rfloor$, such that $a_k = a_{k+t}$ for all $s \leqslant k \leqslant s+t-1$. A well-known result by Thue [Thu06] asserts that there exist arbitrarily long nonrepetitive sequences of elements from $\{0, 1, 2\}$. The next theorem is a choosability version of Thue's result. It was the first example of a new combinatorial bound obtained using the entropy compression method that surpasses the analogous bound provided by a direct application of the LLL.

Theorem 1.6.1 (Grytczuk–Przybyło–Zhu [GPZ11]; Grytczuk–Kozik–Micek [GKM13])**.** *Let $L_1, L_2, \ldots, L_n$ be a sequence of sets with $|L_i| \geqslant 4$ for all $1 \leqslant i \leqslant n$. Then there exists a nonrepetitive sequence $a_1a_2\ldots a_n$ such that $a_i \in L_i$ for all $1 \leqslant i \leqslant n$.*

Note that it is an open problem whether the same result holds for $|L_i| \geqslant 3$.

Proof. This is the only example in this chapter where the LCL is applied directly, without reducing it to Theorem 1.3.1. Let P be the directed path of length n with vertex set $V := \{v_1, \ldots, v_n\}$ and with edges of the form (v_{i+1}, v_i) for all $1 \leqslant i \leqslant n-1$. Choose a random sequence $a_1 a_2 \ldots a_n$ by selecting each $a_i \in L_i$ uniformly and independently from each other. Define a set $A \subseteq V$ as follows:

$$v_i \in A \iff a_1 a_2 \ldots a_i \text{ is a nonrepetitive sequence.}$$

Note that A is out-closed, $\mathbb{P}[v_1 \in A] = 1$, and $v_n \in A$ if and only if $a_1 a_2 \ldots a_n$ is a nonrepetitive sequence.

Consider an edge (v_{i+1}, v_i) of P. If $v_i \in A$ but $v_{i+1} \notin A$, then there exist s and t such that

$$s + 2t - 1 = i + 1$$

and $a_k = a_{k+t}$ for all $s \leqslant k \leqslant s + t - 1$ (i.e., $a_s a_{s+1} \ldots a_{i+1}$ is a repetition). This observation motivates the following construction. Let D be the digraph such that $D^s = P$, and for each $(v_{i+1}, v_i) \in E(P)$ and s, t with $s + 2t - 1 = i + 1$, there is a corresponding edge $e_{s,t} \in E(D)$ going from v_{i+1} to v_i. Let

$$e_{s,t} \in F \iff a_k = a_{k+t} \text{ for all } s \leqslant k \leqslant s + t - 1.$$

Then F is an A-cut (see Fig. 6). Note that for each fixed $t \geqslant 1$, there exists at most one s such that $s + 2t - 1 = i + 1$, so there is at most one edge of the form $e_{s,t} \in E(v_{i+1}, v_i)$, where E is the edge set of D.

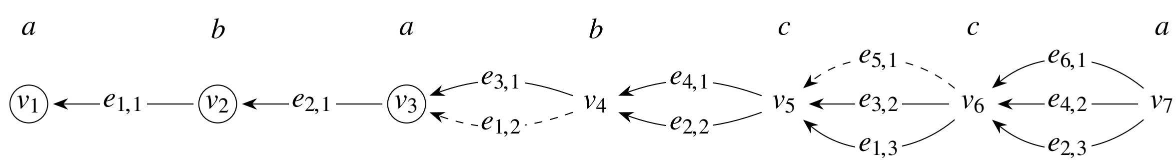

Figure 6 – For $n = 7$ and a sequence $a_1a_2a_3a_4a_5a_6a_7 = ababcca$, we have $A = \{v_1, v_2, v_3\}$ (since the first 4 letters contain a repetition) and $F = \{e_{1,2}, e_{5,1}\}$ (due to the repetitions ***abab****cca* and *abab****cc****a*).

A vertex v_j is reachable from v_i if and only if $j \leqslant i$. In particular, if $s + 2t - 1 = i + 1$, then v_{s+t-1} is reachable from v_i. Observe that the probability of $a_k = a_{k+t}$ is at most $1/|L_{k+t}|$, even if the value of a_k is fixed. Therefore, for $e_{s,t} \in E(v_{i+1}, v_i)$, we have

$$\begin{aligned} \mathbb{P}\left[e_{s,t} \in F \middle| v_{s+t-1} \in A\right] &= \mathbb{P}\left[a_k = a_{k+t} \text{ for all } s \leqslant k \leqslant s+t-1 | v_{s+t-1} \in A\right] \\ &\leqslant \prod_{k=s}^{s+t-1} \frac{1}{|L_{k+t}|} \leqslant \frac{1}{4^t}. \end{aligned}$$

If $\omega(v_{i+1}, v_i) = \omega \in [1; +\infty)$ is a fixed constant, then for all $i \geqslant j$, $\underline{\omega}(v_i, v_j) = \omega^{i-j}$. In particular, if $s + 2t - 1 = i + 1$, then

$$\underline{\omega}(v_{i+1}, v_{s+t-1}) = \omega^t.$$

Thus,

$$\rho_{\omega}^{A,F}(e_{s,t}) \leqslant \rho_{\omega}^{A,F}(e_{s,t}, v_{s+t-1}) = \mathbb{P}\left[e_{s,t} \in F \middle| v_{s+t-1} \in A\right] \cdot \underline{\omega}(v_{i+1}, v_{s+t-1}) \leqslant \frac{\omega^t}{4^t}.$$

Hence, it is enough to find a constant $\omega \in [1; +\infty)$ such that

$$\omega \geqslant 1 + \sum_{t=1}^{\infty} \frac{\omega^t}{4^t} = \frac{1}{1 - \omega/4},$$

where the last equality is subject to $\omega < 4$. Setting $\omega = 2$ completes the proof. ■

A vertex coloring φ of a graph G is **nonrepetitive** if there is no path P in G with an even number of vertices such that the first half of P receives the same sequence of colors as the second half of P, i.e., if there is no path $v_1, v_2, \ldots, v_{2t}$ of length $2t$ such that $\varphi(v_k) = \varphi(v_{k+t})$ for all $1 \leqslant k \leqslant t$. The least number of colors that is needed for a nonrepetitive coloring of G is called the **nonrepetitive chromatic number** of G and is denoted by $\pi(G)$.

The first upper bound on $\pi(G)$ in terms of the maximum degree $\Delta(G)$ was given by Alon, Grytczuk, Hałuszczak, and Riordan [Alo+02], who proved that there is a constant c such that $\pi(G) \leqslant c\Delta(G)^2$. Originally this result was obtained with $c = 2e^{16}$. The constant was improved to $c = 16$ by Grytczuk [Gry07], and then to $c = 12.92$ by Harant and Jendrol' [HJ12]. All these results were based on the LLL.

Dujmović, Joret, Kozik, and Wood [Duj+15] managed to decrease the value of the aforementioned constant c drastically using the entropy compression method. Namely, they lowered the constant to 1, or, to be precise, they showed that $\pi(G) \leqslant (1 + o(1))\Delta(G)^2$ (assuming $\Delta(G) \to \infty$).

The currently best known bound is given by the following theorem.

Theorem 1.6.2 (Gonçalves–Montassier–Pinlou [GMP14]). *For a graph G with maximum degree Δ,*

$$\pi(G) \leqslant \left\lceil \Delta^2 + \frac{3}{2^{2/3}}\Delta^{5/3} + \frac{2^{2/3}\Delta^{5/3}}{\Delta^{1/3} - 2^{1/3}} \right\rceil.$$

Proof. Suppose that

$$k \geqslant \Delta^2 + \frac{3}{2^{2/3}}\Delta^{5/3} + \frac{2^{2/3}\Delta^{5/3}}{\Delta^{1/3} - 2^{1/3}}. \tag{1.6.1}$$

We will use Theorem 1.3.1 to show that G has a nonrepetitive k-coloring.

For brevity, let $V := V(G)$ and $E := E(G)$. Choose a k-coloring φ of G uniformly at random. Define a set $A \subseteq \mathrm{Pow}(V)$ by

$$A := \{S \subseteq V \,:\, \varphi \text{ is a nonrepetitive coloring of } G[S]\},$$

where $G[S]$ denotes the induced subgraph of G with vertex set S. Note that A is downwards-closed and nonempty with probability 1, and $V \in A$ if and only if φ is a nonrepetitive coloring of G.

Consider any $v \in V$. If $v \in \partial A$, then there exists a path $P \ni v$ of even length that is colored repetitively by φ. Thus, we can set

$$\mathscr{B}(v) := \{B_P \,:\, P \ni v \text{ is a path of even length}\},$$

where the event B_P happens if and only if P is colored repetitively by φ.

The number of events in $\mathscr{B}(v)$ corresponding to paths of some fixed length $2t$ is equal to the number of all paths P of length $2t$ passing through v, which does not exceed $t\Delta^{2t-1}$. Indeed, if $P = v_1, v_2, \ldots, v_{2t}$, then we can assume v is one of the vertices $v_1, v_2, \ldots, v_t$, so there are t ways to choose the position of v on P. After the position of v has been determined, we can select all other vertices one by one so that each time we are choosing only from the neighbors of one of the previous vertices. Since the maximum degree of G is Δ, we get the bound $t\Delta^{2t-1}$, as desired.

We will assume $\tau(v) = \tau \in [1; +\infty)$ is a constant. We need to upper bound $\sigma_\tau^A(B_P, v)$ for each $v \in V$ and a path $P \ni v$ of length $2t$. Let P' be the half of P that contains v. Note that if $Z \subseteq V \setminus P'$, then $\mathbb{P}[B_P|Z \in A] \leqslant 1/k^t$, since the coloring of P' is independent from the coloring of Z. Therefore,

$$\sigma_\tau^A(B_P, v) \leqslant \sigma_\tau^A(B_P, P') = \max_{Z \subseteq V \setminus P'} \mathbb{P}[B_P|Z \in A] \cdot \tau^{|P'|} \leqslant \frac{\tau^t}{k^t}.$$

Hence, it is enough to ensure that there exists $\tau \in [1; +\infty)$ such that

$$\tau \geqslant 1 + \sum_{t=1}^{\infty} t\Delta^{2t-1} \cdot \frac{\tau^t}{k^t} = 1 + \frac{\Delta\tau/k}{(1 - \Delta^2\tau/k)^2}, \tag{1.6.2}$$

where the last equality is subject to $\Delta^2\tau/k < 1$. Setting $y := \Delta^2\tau/k$, we can rewrite (1.6.2) as

$$\frac{k}{\Delta^2} \geqslant \frac{1}{y} + \frac{1}{\Delta(1-y)^2}. \tag{1.6.3}$$

Following [GMP14], we take $y = 1 - (2/\Delta)^{1/3}$, and (1.6.3) becomes

$$\frac{k}{\Delta^2} \geqslant 1 + \frac{3}{2^{2/3}\Delta^{1/3}} + \frac{2^{2/3}}{\Delta^{2/3} - (2\Delta)^{1/3}},$$

which is true by (1.6.1). ∎

1.7 Color-critical hypergraphs

A hypergraph $\mathcal{H}$ is $(k+1)$**-critical** if it is not k-colorable, but each of its proper subhypergraphs is. Call a hypergraph $\mathcal{H}$ **true** if all its edges have size at least 3. It is interesting to know what the least possible number of edges in a $(k+1)$-critical true hypergraph on n vertices is. The best known constructions due to Abbott and Hare [AH89] and Abbott, Hare, and Zhou [AHZ94] contain roughly $(k-1)n$ edges. This bound is asymptotically tight for $k \to \infty$, as the following theorem due to Kostochka and Stiebitz asserts:

Theorem 1.7.1 (Kostochka–Stiebitz [KS00])**.** *Every $(k+1)$-critical true hypergraph with n vertices contains at least $(k - 3k^{2/3})n$ edges.*

Here we improve this result, obtaining the following new bound:

Theorem 1.7.2. *Every $(k+1)$-critical true hypergraph with n vertices contains at least $(k - 4\sqrt{k})n$ edges.*

Proof. Our proof is essentially the same as the proof of Theorem 1.7.1 given in [KS00]. The only difference is that we replace the application of the LLL by an application of the LCL.

Let $\mathcal{H}$ be a $(k+1)$-critical true hypergraph with n vertices. Denote $V \coloneqq V(\mathcal{H})$ and $E \coloneqq E(\mathcal{H})$. Let $c \coloneqq 4\sqrt{k}$. Fix some positive constant z (to be determined later). Let $g \colon \mathbb{N}^+ \to \mathbb{R}$ be given by

$$g(t) \coloneqq \begin{cases} 1 - z^{-1} \text{ if } t = 1; \\ 2^{1-t} z^{-1} \text{ if } t > 1. \end{cases}$$

Inductively construct a sequence $(V_i)_{i=0}^m$, where $0 \leqslant m \leqslant n$, of subsets of V according to the following rule. Let $V_0 \coloneqq V$. If there is a vertex $v \in V_i$ such that

$$\sum_{\substack{h \in E: \\ h \ni v}} g(|h \cap V_i|) \geqslant k - c, \tag{1.7.1}$$

then select one such vertex, denote it by v_i, and let $V_{i+1} \coloneqq V_i \setminus \{v_i\}$. Otherwise let $m \coloneqq i$ and stop.

If $m = n$, then

$$|E| = \sum_{h \in E} 1 > \sum_{h \in E} \sum_{j=1}^{|h|} g(j) = \sum_{i=0}^{n-1} \sum_{\substack{h \in E: \\ h \ni v_i}} g(|h \cap V_i|) \geqslant (k-c)n,$$

as desired. Now suppose that $m < n$. Let $V' \coloneqq V_m$. Since V' is nonempty, the hypergraph $\mathcal{H} - V'$ obtained from $\mathcal{H}$ by deleting the vertices in V' is k-colorable. Fix a proper k-coloring ψ of $\mathcal{H} - V'$ and extend it to a k-coloring φ of $\mathcal{H}$ by choosing a color for each vertex in V' uniformly and independently from all other vertices. Let $A \subseteq \mathrm{Pow}(V')$ be given by

$$A \coloneqq \{S \subseteq V' \,:\, \text{there is no } \varphi\text{-monochromatic edge } h \subseteq (V \setminus V') \cup S\}.$$

Note that A is downwards-closed and $\mathbb{P}[\varnothing \in A] = 1$ (because the coloring ψ of $V \setminus V'$ is proper). We will use Theorem 1.3.1 to prove that $\mathbb{P}[V' \in A] > 0$, which will be a contradiction since $\mathcal{H}$ is not k-colorable.

For $v \in V'$, let

$$\mathscr{B}(v) \coloneqq \{B_h \,:\, v \in h \in E\},$$

where the event B_h happens if and only if h is φ-monochromatic. Clearly, if $v \in \partial A$, then at least one of the events $B_h \in \mathscr{B}(v)$ holds.

Let $\tau(v) = \tau \in [1; +\infty)$ be a constant function. Consider some $B_h \in \mathscr{B}(v)$. There are two cases. First suppose that $h \nsubseteq V'$. Note that such h is φ-monochromatic if and only if $h \setminus V'$ is ψ-monochromatic and $\varphi(u) = \psi(w)$ for all $u \in h \cap V'$ and $w \in h \setminus V'$. Therefore, for each such h and for $Z \subseteq V' \setminus h$, $\mathbb{P}[B_h | Z \in A] \leqslant \mathbb{P}[B_h] \leqslant 1/k^{|h \cap V'|}$. Thus,

$$\sigma_\tau^A(B_h, v) \leqslant \sigma_\tau^A(B_h, h \cap V') = \max_{Z \subseteq V' \setminus h} \mathbb{P}[B_h | Z \in A] \cdot \tau^{|h \cap V'|} \leqslant \frac{\tau^{|h \cap V'|}}{k^{|h \cap V'|}}.$$

If, on the other hand, $h \subseteq V'$, then choose an arbitrary vertex $u \in h \setminus \{v\}$ and consider $Z \subseteq (V' \setminus h) \cup \{u\}$.

(This idea is analogous to the one we discussed in §1.5.) Since fixing the color of u does not change the probability that h is monochromatic, we have $\mathbb{P}[B_h|Z \in A] \leqslant 1/k^{|h|-1}$, so

$$\sigma_\tau^A(B_h, v) \leqslant \sigma_\tau^A(B_h, E \setminus \{u\}) = \max_{Z \subseteq (V' \setminus h) \cup \{u\}} \mathbb{P}[B_h|Z \in A] \cdot \tau^{|h \setminus \{u\}|} \leqslant \frac{\tau^{|h|-1}}{k^{|h|-1}}.$$

For a vertex $v \in V'$, let

$$a_t(v) := |\{h \in E \,:\, v \in h \nsubseteq V',\ |h \cap V'| = t\}|;$$

$$b_t(v) := |\{h \in E \,:\, v \in h \subseteq V',\ |h| = t\}|.$$

To apply Theorem 1.3.1, it is enough to guarantee that there exists a constant $\tau \in [1; +\infty)$ such that for all $v \in V'$,

$$\tau \geqslant 1 + \sum_{t=1}^{\infty} a_t(v) \frac{\tau^t}{k^t} + \sum_{t=3}^{\infty} b_t(v) \frac{\tau^{t-1}}{k^{t-1}}. \tag{1.7.2}$$

Since V' is the last set in the sequence $(V_i)_{i=0}^m$, no vertex in V' satisfies (1.7.1). In other words, for all $v \in V'$,

$$\sum_{t=1}^{\infty} a_t(v) g(t) + \sum_{t=3}^{\infty} b_t(v) g(t) < k - c. \tag{1.7.3}$$

Let

$$\alpha_t(v) := a_t(v) g(t);$$

$$\beta_t(v) := b_t(v) g(t).$$

Then (1.7.3) can be rewritten as

$$\gamma(v) := \sum_{t=1}^{\infty} \alpha_t(v) + \sum_{t=3}^{\infty} \beta_t(v) < k - c,$$

and (1.7.2) turns into

$$\tau \geqslant 1 + \sum_{t=1}^{\infty} \alpha_t(v) \cdot \frac{1}{g(t)} \left(\frac{\tau}{k}\right)^t + \sum_{t=3}^{\infty} \beta_t(v) \cdot \frac{1}{g(t)} \left(\frac{\tau}{k}\right)^{t-1},$$

which, after substituting the actual values for g, becomes

$$\tau \geqslant 1 + \alpha_1(v) \cdot \frac{z}{z-1} \frac{\tau}{k} + \sum_{t=2}^{\infty} \alpha_t(v) \cdot \frac{1}{2} z \left(\frac{2\tau}{k}\right)^t + \sum_{t=3}^{\infty} \beta_t(v) \cdot z \left(\frac{2\tau}{k}\right)^{t-1}. \tag{1.7.4}$$

We can view the right-hand side of (1.7.4) as a linear combination of variables $\alpha_t(v)$, $\beta_t(v)$. If we assume that

$$\frac{4\tau}{k} \geqslant \frac{1}{z-1},$$

then the largest coefficient in this linear combination is $z\,(2\tau/k)^2$ (the coefficient of $\beta_3(v)$). Thus, it is enough to find τ, z satisfying the following two inequalities:

$$\frac{4\tau}{k} \geqslant \frac{1}{z-1}; \tag{1.7.5}$$

$$\tau \geqslant 1 + \frac{4z\tau^2(k-c)}{k^2}. \tag{1.7.6}$$

(Inequality (1.7.6) is obtained by replacing all coefficients on the right hand side of (1.7.4) by the largest one and using the fact that $\gamma(v) < k - c$.) If we choose

$$z = \frac{k}{4\tau} + 1,$$

then (1.7.5) is satisfied, while (1.7.6) becomes

$$\tau \geqslant 1 + \frac{4\tau^2(k-c)}{k^2}\left(\frac{k}{4\tau} + 1\right) = 1 + \frac{k-c}{k}\tau + \frac{4(k-c)}{k^2}\tau^2.$$

Thus, we just have to make sure that the following inequality has a solution τ:

$$\frac{4(k-c)}{k^2}\tau^2 - \frac{c}{k}\tau + 1 \leqslant 0.$$

This is true if and only if $c^2 \geqslant 16(k-c)$; in particular, $c = 4\sqrt{k}$ works. Therefore, φ is a proper k-coloring of $\mathcal{H}$ with positive probability. This contradiction completes the proof. ∎

1.8 Choice functions

Let $U_1, \ldots, U_n$ be a collection of pairwise disjoint nonempty finite sets. A **choice function** F is a subset of $\bigcup_{i=1}^n U_i$ such that for all $1 \leqslant i \leqslant n$, $|F \cap U_i| = 1$. A **partial choice function** P is a subset of $\bigcup_{i=1}^n U_i$ such that for all $1 \leqslant i \leqslant n$, $|P \cap U_i| \leqslant 1$. For a partial choice function P, let

$$\mathrm{dom}(P) \coloneqq \{i \,:\, P \cap U_i \neq \varnothing\}.$$

Thus, a choice function F is a partial choice function with $\mathrm{dom}(F) = \{1, \ldots, n\}$.

Let F be a choice function and let P be a partial choice function. We say that P **occurs** in F if $P \subseteq F$, and we say that F **avoids** P if P does not occur in F. Many natural combinatorial problems (especially ones related to coloring) can be stated using the language of choice functions. For instance, consider a graph G with vertex set $\{1, \ldots, n\}$. Fix a positive integer k and let $U_i \coloneqq \{(i, c) \,:\, 1 \leqslant c \leqslant k\}$ for each $1 \leqslant i \leqslant n$. For each edge $ij \in E(G)$ and $1 \leqslant c \leqslant k$, define a partial choice function $P_{ij}^c \coloneqq \{(i, c), (j, c)\}$. Then a proper vertex k-coloring of G can be identified with a choice function F such that none of $\{P_{ij}^c\}_{ij \in E(G), 1 \leqslant c \leqslant k}$ occur in F. Another problem that has a straightforward formulation using choice functions is the k-SAT (which also serves as a standard example of a problem that can be approached with the LLL).

A **multichoice function** M is simply a subset of $\bigcup_{i=1}^n U_i$ (one should think of it as a generalized choice function where one is allowed to choose multiple or zero elements from each set). For a multichoice function M, let $M_i \coloneqq M \cap U_i$. Again, we say that a partial choice function P **occurs** in a multichoice function M if $P \subseteq M$. Suppose that we are given a family $P_1, \ldots, P_m$ of nonempty "forbidden" partial choice functions. For a multichoice function M, the ***i*-th defect** of M (notation: $\mathrm{def}_i(M)$) is the number of indices j such that $i \in \mathrm{dom}(P_j)$ and P_j occurs in M. Observe that there exists a choice function F that avoids all of $P_1, \ldots, P_m$ if and only if there exists a multichoice function M such that for all $1 \leqslant i \leqslant n$,

$$|M_i| \geqslant 1 + \mathrm{def}_i(M). \tag{1.8.1}$$

Indeed, if F avoids all of $P_1, \ldots, P_m$, then F itself satisfies (1.8.1). On the other hand, if M satisfies (1.8.1), then, for every i, there is an element $x_i \in M_i$ that does not belong to any P_j occurring in M. Therefore, $\{x_i\}_{i=1}^n$ is a choice function that avoids all of $P_1, \ldots, P_m$, as desired.

The main result of this section is that, in fact, it is enough to establish (1.8.1) *on average* for some random multichoice function M.

Theorem 1.8.1. *Let $U_1, \ldots, U_n$ be a collection of pairwise disjoint nonempty finite sets and let $P_1, \ldots, P_m$ be a family of nonempty partial choice functions. Let Ω be a probability space and let $M_i \colon \Omega \to \mathrm{Pow}(U_i)$, $1 \leqslant i \leqslant n$, be a collection of mutually independent random variables. Set $M \coloneqq \bigcup_{i=1}^n M_i$. If for all $1 \leqslant i \leqslant n$,*

$$\mathbb{E}|M_i| \geqslant 1 + \mathbb{E}\,\mathrm{def}_i(M), \tag{1.8.2}$$

then there exists a choice function F that avoids all of $P_1, \ldots, P_m$.

Proof. For $x \in \bigcup_{i=1}^n U_i$, let $p(x) \coloneqq \mathbb{P}[x \in M]$. Then

$$\mathbb{E}|M_i| = \sum_{x \in U_i} p(x).$$

Since the variables M_i, $1 \leqslant i \leqslant n$, are independent,

$$\mathbb{P}[P_j \subseteq M] = \prod_{x \in P_j} p(x).$$

Therefore, if $N_i \coloneqq \{j \,:\, i \in \mathrm{dom}(P_j)\}$,

$$\mathbb{E}\,\mathrm{def}_i(M) = \sum_{j \in N_i} \mathbb{P}[P_j \subseteq M] = \sum_{j \in N_i} \prod_{x \in P_j} p(x).$$

Thus, (1.8.2) is equivalent to

$$\sum_{x \in U_i} p(x) \geqslant 1 + \sum_{j \in N_i} \prod_{x \in P_j} p(x). \tag{1.8.3}$$

Let $\tau(i) \coloneqq \sum_{x\in U_i} p(x)$ and let $q(x) \coloneqq p(x)/\tau(i)$ for all $x \in U_i$. Then (1.8.3) can be rewritten as

$$\tau(i) \geqslant 1 + \sum_{j\in N_i} \prod_{x\in P_j} q(x) \cdot \tau(\mathrm{dom}(P_j)). \tag{1.8.4}$$

We will only use the numerical condition (1.8.4), ignoring its probabilistic meaning. Construct a random choice function F (in a new probability space) as follows: Choose an element $x \in U_i$ with probability $q(x)$, making the choices for different U_i's independently (this definition is correct, since $\sum_{x\in U_i} q(x) = 1$). Set $I \coloneqq \{1, \ldots, n\}$ and define a random subset $A \subseteq \mathrm{Pow}(I)$ as follows:

$$A \coloneqq \{S \subseteq I \,:\, \text{no } P_j \text{ with } \mathrm{dom}(P_j) \subseteq S \text{ occurs in } F\}.$$

Then A is a nonempty downwards-closed family of subsets of I, and $I \in A$ if and only if F avoids all of P_1, $\ldots$, P_m. For $i \in I$, let

$$\mathscr{B}(i) \coloneqq \{B_j \,:\, j \in N_i\},$$

where the event B_j happens if and only if $P_j \subseteq F$. Clearly, if $i \in \partial A$, then there is some $j \in N_i$ such that $P_j \subseteq F$, so we can apply Theorem 1.3.1.

Consider any $i \in I$ and $j \in N_i$. Since $\mathbb{P}[B_j] = \prod_{x\in P_j} q(x)$, we have

$$\begin{aligned}\sigma_\tau^A(B_j, i) \leqslant \sigma_\tau^A(B_j, \mathrm{dom}(P_j)) &= \max_{Z\subseteq I\setminus \mathrm{dom}(P_j)} \mathbb{P}[B_j | Z \in A] \cdot \tau(\mathrm{dom}(P_j)) \\ &\leqslant \mathbb{P}[B_j] \cdot \tau(\mathrm{dom}(P_j)) = \prod_{x\in P_j} q(x) \cdot \tau(\mathrm{dom}(P_j)).\end{aligned}$$

Therefore, in this case (1.8.4) implies (1.3.1), yielding $\mathbb{P}[I \in A] > 0$, as desired. ∎

1.9 New bounds for the acyclic chromatic index

1.9.1 Acyclic edge coloring: definitions and results

An edge coloring of a graph G is called an **acyclic edge coloring** if it is proper (i.e. adjacent edges receive different colors) and every cycle in G contains edges of at least three different colors (there are no *bichromatic cycles* in G). The least number of colors needed for an acyclic edge coloring of G is called the **acyclic chromatic index** of G and is denoted by $a'(G)$. The notion of acyclic (vertex) coloring was first introduced by Grünbaum [Grü73]. The edge version was first considered by Fiamčik [Fia78], and independently by Alon, McDiarmid, and Reed [AMR91].

As in the case of nonrepetitive colorings, it is quite natural to ask for an upper bound on the acyclic chromatic index of a graph G in terms of its maximum degree $\Delta(G)$. Since $a'(G) \geqslant \chi'(G) \geqslant \Delta(G)$, where $\chi'(G)$ denotes the ordinary chromatic index of G, this bound must be at least linear in $\Delta(G)$. The first linear bound was given by Alon et al. [AMR91], who showed that $a'(G) \leqslant 64\Delta(G)$. Although it resolved the problem of determining the order of growth of $a'(G)$ in terms of $\Delta(G)$, it was conjectured that the sharp bound should be much lower.

Conjecture 1.9.1 (Fiamčik [Fia78]; Alon–Sudakov–Zaks [ASZ01])**.** *For every graph G, $a'(G) \leqslant \Delta(G) + 2$.*

Note that the bound in Conjecture 1.9.1 is only one more than Vizing's bound on the chromatic index of G. However, this elegant conjecture is still far from being proven.

The first major improvement to the bound $a'(G) \leqslant 64\Delta(G)$ was made by Molloy and Reed [MR98], who proved that $a'(G) \leqslant 16\Delta(G)$. This bound remained the best for a while, until Ndreca, Procacci, and Scoppola [NPS12] managed to improve it to $a'(G) \leqslant \lceil 9.62(\Delta(G) - 1)\rceil$. Again, first bounds for $a'(G)$ were obtained using the LLL. The bound $a'(G) \leqslant \lceil 9.62(\Delta(G) - 1)\rceil$ by Ndreca et al. used an improved version of the LLL due to Bissacot, Fernández, Procacci, and Scoppola [Bis+11].

The best current bound for $a'(G)$ in terms of $\Delta(G)$ was obtained by Esperet and Parreau via the entropy compression method.

Theorem 1.9.2 (Esperet–Parreau [EP13])**.** *For every graph G with maximum degree Δ, $a'(G) \leqslant 4(\Delta - 1)$.*

We present a proof of Theorem 1.9.2 using the LCL in §1.9.2.

The probability that a cycle would become bichromatic in a random coloring is less if the cycle is longer. Thus, it should be easier to establish better bounds on the acyclic chromatic index for graphs with high enough girth. Indeed, Alon et al. [ASZ01] showed that if $g(G) \geqslant c_1\Delta(G)\log\Delta(G)$, where c_1 is some universal constant, then $a'(G) \leqslant \Delta(G) + 2$. They also proved that if $g(G) \geqslant c_2 \log\Delta(G)$, then $a'(G) \leqslant 2\Delta(G) + 2$. This was later improved by Muthu, Narayanan, and Subramanian [MNS07] in the following way: For every $\varepsilon > 0$, there exists a constant c such that if $g(G) \geqslant c\log\Delta(G)$, then $a'(G) \leqslant (1 + \varepsilon)\Delta(G) + o(\Delta(G))$.

We shall consider the case when $g(G)$ is bounded below by some constant independent of $\Delta(G)$. The first bounds of such type were given by Muthu et al. [MNS07], who proved that $a'(G) \leqslant 9\Delta(G)$ if $g(G) \geqslant 9$, and $a'(G) \leqslant 4.52\Delta(G)$ if $g(G) \geqslant 220$. Esperet and Parreau [EP13] not only improved both these estimates even in the case of arbitrary $g(G)$, but they also showed that $a'(G) \leqslant \lceil 3.74(\Delta(G) - 1)\rceil$ if $g(G) \geqslant 7$, $a'(G) \leqslant \lceil 3.14(\Delta(G) - 1)\rceil$ if $g(G) \geqslant 53$, and, in fact, for every $\varepsilon > 0$, there exists a constant c such that if $g(G) \geqslant c$, then $a'(G) \leqslant (3 + \varepsilon)\Delta(G) + o(\Delta(G))$.

Using the LCL, we improve these bounds further. Namely, we establish the following:

Theorem 1.9.3. *Let G be a graph with maximum degree Δ and let H be some bipartite graph. If G does not contain H as a subgraph, then $a'(G) \leqslant 3\Delta + o(\Delta)$.*

Remark. We originally established Theorem 1.9.3 in the case when H is the 4-cycle. We are grateful to Louis Esperet and Rémi de Verclos for pointing out that essentially the same proof works for any bipartite H.

Remark. The $o(\Delta)$ term in the statement of Theorem 1.9.3 depends on H. In fact, our proof shows that for the complete bipartite graph $K_{k,k}$, it is of the order $O\left(\Delta^{1-1/2k}\right)$.

Theorem 1.9.4. *For every $\varepsilon > 0$, there exists a constant c such that for every graph G with maximum degree Δ and $g(G) \geqslant c$, we have $a'(G) \leqslant (2 + \varepsilon)\Delta + o(\Delta)$.*

Remark. The bound of the last theorem was recently improved to $a'(G) \leqslant (1 + \varepsilon)\Delta + o(\Delta)$ by Cai, Perarnau, Reed, and Watts [Cai+17] using a different (and much more sophisticated) argument.

We prove Theorems 1.9.3 and 1.9.4 in §§1.9.3 and 1.9.4 respectively.

1.9.2 Proof of the Esperet–Parreau bound

Here we prove Theorem 1.9.2. Let G be a graph of maximum degree Δ. For brevity, let $E := E(G)$. Choose a $4(\Delta - 1)$-edge coloring φ of G uniformly at random. Call a cycle C of length $2t$ **φ-bichromatic** if $C = e_1, e_2, \ldots, e_{2t}$ and $\varphi(e_{2i-1}) = \varphi(e_{2t-1})$, $\varphi(e_{2i}) = \varphi(e_{2t})$ for all $1 \leqslant i \leqslant t - 1$. Let

$$A := \{S \subseteq E \ : \ \varphi \text{ is an acyclic edge coloring of } G[S]\},$$

where $G[S]$ is the graph obtained from G by removing all the edges outside S. Note that with probability 1, A is a nonempty downwards-closed family of subsets of E, and $E \in A$ if and only if φ is an acyclic edge coloring of G.

Consider any $e \in E$. If $e \in \partial A$, then either there exists an edge e' adjacent to e such that $\varphi(e) = \varphi(e')$, or there exists a φ-bichromatic cycle $C \ni e$ of even length. The crucial idea of [EP13] (which is credited to Jakub Kozik by the authors) is to handle 4-cycles and cycles of length at least 6 separately. Set

$$\mathscr{B}(e) := \{B_C \ : \ C \ni e \text{ is a cycle of length } 2t \geqslant 6\} \cup \{B_e\},$$

where

1. B_C happens if and only if the cycle C is φ-bichromatic;
2. B_e happens if and only if either there exists an edge e' adjacent to e such that $\varphi(e) = \varphi(e')$, or there exists a φ-bichromatic 4-cycle $C \ni e$.

Again, we will assume that $\tau(e) = \tau \in [1; +\infty)$ is a constant. Consider the event $B_e \in \mathscr{B}(e)$ of the second kind. We will estimate the probability $\mathbb{P}[B_e | Z \in A]$ for $Z \subseteq E \setminus \{e\}$ using the following claim, which also plays a crucial role in the original proof by Esperet and Parreau.

Claim 1.9.5. *Suppose that some edges of G are properly colored. If $e \in E$ is uncolored, then there exist at most $2(\Delta - 1)$ ways to color e so that the resulting coloring either is not proper, or contains a bichromatic 4-cycle going through e.*

Proof. Denote the given proper partial coloring by ψ and let $e = uv$. Let L_1 (resp. L_2) be the set of colors appearing on the edges incident to u (resp. v). The coloring becomes not proper if e is colored using a color from $L_1 \cup L_2$, so there are $|L_1 \cup L_2|$ such options. Suppose that coloring e with color c creates a bichromatic 4-cycle $uvxy$. Then $c = \psi(xy)$ and $\psi(vx) = \psi(uy)$. Hence, the number of such colors c is at most the number of pairs of edges vx, uy such that $\psi(vx) = \psi(uy)$. Note that, since ψ is proper, there can be at most one pair vx, uy such that $\psi(vx) = \psi(uy) = c'$ for a particular color c'. Therefore, the total number of such pairs is exactly $|L_1 \cap L_2|$. Thus, there are at most $|L_1 \cup L_2| + |L_1 \cap L_2| = |L_1| + |L_2| \leqslant 2(\Delta - 1)$ "forbidden" colors for e, as desired. ∎

Using Claim 1.9.5, we obtain

$$\mathbb{P}[B_e | Z \in A] \leqslant \frac{2(\Delta - 1)}{4(\Delta - 1)} = \frac{1}{2}$$

for all $Z \subseteq E \setminus \{e\}$. Therefore,

$$\sigma_\tau^A(B_e, e) \leqslant \sigma_\tau^A(B_e, \{e\}) = \max_{Z \subseteq E \setminus \{e\}} \mathbb{P}[B_e | Z \in A] \cdot \tau^{|\{e\}|} \leqslant \frac{\tau}{2}.$$

Now we need to deal with the events of the form $B_C \in \mathscr{B}(e)$. Note that there are at most $(\Delta - 1)^{2t-2}$ cycles of length $2t$ passing through e. Therefore, the number of events in $\mathscr{B}(e)$ corresponding to cycles of length $2t$ is at most $(\Delta - 1)^{2t-2}$. Consider any such event B_C. Suppose that $C = e_1, e_2, \ldots, e_{2t}$, where $e_1 = e$. Then B_C happens if and only if $\varphi(e_{2i-1}) = \varphi(e_{2t-1})$ and $\varphi(e_{2i}) = \varphi(e_{2t})$ for all $1 \leqslant i \leqslant t - 1$. Even if the colors of e_{2t-1} and e_{2t} are fixed, the probability of this happening is $1/(4(\Delta - 1))^{2t-2}$. Due to this observation, if $C' \coloneqq \{e_1, e_2, \ldots, e_{2t-2}\}$ and $Z \subseteq E \setminus C'$, then $\mathbb{P}[B_C | Z \in A] \leqslant 1/(4(\Delta - 1))^{2t-2}$. Therefore,

$$\sigma_\tau^A(B_C, e) \leqslant \sigma_\tau^A(B_C, C') = \max_{Z \subseteq E \setminus C'} \mathbb{P}[B_C | Z \in A] \cdot \tau^{|C'|} \leqslant \frac{\tau^{2t-2}}{(4(\Delta - 1))^{2t-2}}.$$

Putting everything together, it is enough to find a constant $\tau \in [1; +\infty)$ such that

$$\tau \geqslant 1 + \sum_{t=3}^{\infty} (\Delta - 1)^{2t-2} \cdot \frac{\tau^{2t-2}}{(4(\Delta - 1))^{2t-2}} + \frac{\tau}{2} = 1 + \frac{(\tau/4)^4}{1 - (\tau/4)^2} + \frac{\tau}{2},$$

where the last equality is valid if $\tau/4 < 1$. Setting $\tau = 2(\sqrt{5} - 1)$ completes the proof.

1.9.3 Graphs with a forbidden bipartite subgraph

Combinatorial lemmas

For this section we assume that a bipartite graph H is fixed. In particular, all constants that we mention depend on H. We will use the following version of the Kővari–Sós–Turán theorem.

Theorem 1.9.6 (Kővari, Sós, Turán [KST54])**.** *Let G be a graph with n vertices and m edges that does not contain the complete bipartite graph $K_{k,k}$ as a subgraph. Then $m \leqslant O(n^{2-1/k})$ (assuming that $n \to \infty$).*

Corollary 1.9.7. *There exist positive constants α and δ such that if a graph G with n vertices and m edges does not contain H as a subgraph, then $m \leqslant \alpha n^{2-\delta}$.*

In what follows we fix the constants α and δ from the statement of Corollary 1.9.7.

Lemma 1.9.8. *There is a positive constant β such that the following holds. Let G be a graph with maximum degree Δ that does not contain H as a subgraph. Then for any two vertices u, $v \in V(G)$, the number of uv-paths of length 3 in G is at most $\beta\Delta^{2-\delta}$.*

Proof. Suppose that $uxyv$ is a uv-path of length 3 in G. Then $x \in N_G(u)$ and $y \in N_G(v)$, and hence $xy \in E(G[N_G(u) \cup N_G(v)])$. Note that any edge $xy \in E(G[N_G(u) \cup N_G(v)])$ can possibly give rise to at most two different uv-paths of length 3 (namely $uxyv$ and $uyxv$). Therefore, the number of uv-paths of length 3 in G cannot exceed $2|E(G[N_G(u) \cup N_G(v)])|$. Since $|V(G[N_G(u) \cup N_G(v)])| \leqslant 2\Delta$, by Corollary 1.9.7 we have that $|E(G[N_G(u) \cup N_G(v)])| \leqslant \alpha(2\Delta)^{2-\delta}$, so the number of uv-paths of length 3 in G is at most $\left(2^{3-\delta}\alpha\right) \Delta^{2-\delta}$. ∎

In what follows we fix the constant β from the statement of Lemma 1.9.8.

Lemma 1.9.9. *Let G be a graph with maximum degree Δ that does not contain H as a subgraph. Then for any edge $e \in E(G)$ and for any integer $k \geqslant 4$, the number of cycles of length k in G that contain e is at most $\beta\Delta^{k-2-\delta}$.*

Proof. Suppose that $e = uv \in E(G)$. Note that the number of cycles of length k that contain e is not greater than the number of uv-paths of length $k-1$. Consider any uv-path $ux_1 \ldots x_{k-2}v$ of length $k-1$. Then $ux_1 \ldots x_{k-4}$ is a path of length $k-4$, and $x_{k-4}x_{k-3}x_{k-2}v$ is a path of length 3. There are at most Δ^{k-4} paths of length $k-4$ starting at u, and, given a path $ux_1 \ldots x_{k-4}$, the number of $x_{k-4}v$-paths of length 3 is at most $\beta\Delta^{2-\delta}$. Hence the number of uv-paths of length $k-1$ is at most $\Delta^{k-4} \cdot \beta\Delta^{2-\delta} = \beta\Delta^{k-2-\delta}$. ∎

Probabilistic set-up

Let G be a graph with maximum degree Δ that does not contain H as a subgraph. Let $E \coloneqq E(G)$. Fix some constant c and let φ be a $(2+c)\Delta$-edge coloring of G chosen uniformly at random. As in §1.9.2, we let

$$A \coloneqq \{S \subseteq E \,:\, \varphi \text{ is an acyclic edge coloring of } G[S]\},$$

and set

$$\mathscr{B}(e) \coloneqq \{B_C \,:\, C \ni e \text{ is a cycle of even length}\} \cup \{B_e\},$$

where

1. B_C happens if and only if the cycle C is φ-bichromatic;
2. B_e happens if and only if there exists an edge e' adjacent to e such that $\varphi(e) = \varphi(e')$.

Let $\tau(e) = \tau \in [1; +\infty)$ be a constant. Since the colors of different edges are independent, for any $Z \subseteq E \setminus \{e\}$, we have

$$\mathbb{P}[B_e | Z \in A] \leqslant 2\Delta \cdot \frac{1}{(2+c)\Delta} = \frac{2}{2+c}.$$

Hence,

$$\sigma_\tau^A(B_e, e) \leqslant \sigma_\tau^A(B_e, \{e\}) = \max_{Z \subseteq E \setminus \{e\}} \mathbb{P}[B_e | Z \in A] \cdot \tau^{|\{e\}|} \leqslant \frac{2\tau}{2+c}.$$

The same analysis as in §1.9.2 shows that for any cycle $C \ni e$ of length $2t$, we have

$$\sigma_\tau^A(B_C, e) \leqslant \frac{\tau^{2t-2}}{((2+c)\Delta)^{2t-2}}.$$

Using Lemma 1.9.9, we see that to apply Theorem 1.3.1, it suffices to find a constant $\tau \in [1; +\infty)$ such that

$$\tau \geqslant 1 + \sum_{t=2}^{\infty} \frac{\beta\Delta^{2t-2-\delta}\tau^{2t-2}}{((2+c)\Delta)^{2t-2}} + \frac{2\tau}{2+c} = 1 + \beta\Delta^{-\delta}\frac{(\tau/(2+c))^2}{1-(\tau/(2+c))^2} + \frac{2\tau}{2+c}, \tag{1.9.1}$$

where the last equality holds whenever $\tau/(2+c) < 1$. If we denote $y = \tau/(2+c)$, then (1.9.1) turns into

$$c \geqslant \frac{1}{y} + \beta\Delta^{-\delta}\frac{y^2}{1-y^2}. \tag{1.9.2}$$

Now if $c = 1+\varepsilon$ for any given $\varepsilon > 0$, then we can take $1/y = 1+\varepsilon/2$. For this particular value of y, we have

$$\beta\Delta^{-\delta}\frac{y^2}{1-y^2} \xrightarrow[\Delta\to\infty]{} 0,$$

so for Δ large enough, $\beta\Delta^{-\delta}y^2/(1-y^2) \leqslant \varepsilon/2$, and (1.9.2) is satisfied. This observation completes the proof of Theorem 1.9.3. A more precise calculation shows that (1.9.2) can be satisfied for $c = 1+\varepsilon$ as long as $\varepsilon \geqslant \gamma\Delta^{-\delta/2}$ for some absolute constant γ.

1.9.4 Graphs with large girth

Breaking short cycles

The proof of Theorem 1.9.4 proceeds in two steps. Assuming that the girth of G is large enough, we first show that there is a proper edge coloring of G using $(2+\varepsilon/2)\Delta$ colors with no "short" bichromatic cycles (where "short" means of length roughly $\log\Delta$). Then we use the remaining $\varepsilon\Delta/2$ colors to break all the "long" bichromatic cycles.

We start with the following observations analogous to Lemmas 1.9.8 and 1.9.9.

Lemma 1.9.10. *Let G be a graph with maximum degree Δ and girth $g > 2r$, where $r \geqslant 2$. Then for any two vertices $u, v \in V(G)$, the number of uv-paths of length r in G is at most* 1.

Proof. If there are two uv-paths of length r, then their union forms a closed walk of length $2r$, which means that G contains a cycle of length at most $2r$. ∎

Lemma 1.9.11. *Let G be a graph with maximum degree Δ and girth $g > 2r$, where $r \geqslant 2$. Then for any edge $e \in E(G)$ and for any integer $k \geqslant 4$, the number of cycles of length k in G that contain e is at most Δ^{k-r-1}.*

Proof. Suppose that $e = uv \in E(G)$. Note that the number of cycles of length k that contain e is not greater than the number of uv-paths of length $k-1$. Consider any uv-path $ux_1 \ldots x_{k-2}v$ of length $k-1$. Then $ux_1 \ldots x_{k-r-1}$ is a path of length $k-r-1$, and $x_{k-r-1}x_{k-r} \ldots x_{k-2}v$ is a path of length r. There are at most Δ^{k-r-1} paths of length $k-r-1$ starting at u, and, given a path $ux_1 \ldots x_{k-r-1}$, the number of $x_{k-r-1}v$-paths of length r is at most 1. Hence the number of uv-paths of length $k-1$ is at most Δ^{k-r-1}. ∎

Lemma 1.9.12. *For every $\varepsilon > 0$, there exists a positive constant a_ε such that the following holds. Let G be a graph with maximum degree Δ and girth $g > 2r$, where $r \geqslant 2$. Then there is a proper edge coloring of G using at most $(2+\varepsilon)\Delta + o(\Delta)$ colors that contains no bichromatic cycles of length at most $2L$, where $L := a_\varepsilon(r-2)\log\Delta + 1$.*

Proof. We work in a probabilistic setting similar to the one used in the proof of Theorem 1.9.3 (see §1.9.3 for the notation used), but this time

$$A \coloneqq \{S \subseteq E(G) \,:\, \varphi \text{ is a proper edge coloring of } G[S] \text{ with no bichromatic cycles of length at most } 2L\}.$$

Then, taking into account Lemma 1.9.11, (1.9.1) is replaced by

$$\tau \geqslant 1 + \sum_{t=r+1}^{L} \frac{\Delta^{2t-r-1}\tau^{2t-2}}{((2+c)\Delta)^{2t-2}} + \frac{2\tau}{2+c} = 1 + \Delta^{-r+1} \sum_{t=r+1}^{L} \left(\frac{\tau}{2+c}\right)^{2t-2} + \frac{2\tau}{2+c}. \tag{1.9.3}$$

If $y \coloneqq \tau/(2+c)$, then (1.9.3) becomes

$$c \geqslant \frac{1}{y} + \Delta^{-r+1} \sum_{t=r+1}^{L} y^{2t-3}.$$

Note that if $y > 1$ and $L \leqslant \Delta$, then we have

$$\sum_{t=r+1}^{L} y^{2t-3} \leqslant \sum_{t=r+1}^{L} y^{2L-3} = (L-r)y^{2L-3} \leqslant \Delta y^{2L-3},$$

so it is enough to get

$$c \geqslant \frac{1}{y} + \Delta^{-r+2} y^{2L-3}.$$

Now take $y = 2/\varepsilon$ and $c = \varepsilon$. We need

$$\frac{\varepsilon}{2} \geqslant \Delta^{-r+2} \left(\frac{2}{\varepsilon}\right)^{2L-3},$$

i.e.,

$$L \leqslant \left(2 \log \frac{2}{\varepsilon}\right)^{-1} (r-2) \log \Delta + 1,$$

and we are done. ∎

Breaking long cycles

To deal with "long" cycles we need a different random procedure. A similar procedure was analyzed in [MNS07] using the LLL.

Lemma 1.9.13. *For every $\varepsilon > 0$, there exist positive constants b_ε and d_ε such that the following holds. Let G be a graph with maximum degree Δ and let $\psi : E \to \mathscr{C}$ be a proper edge coloring of G. Then there is a proper edge coloring $\varphi : E \to \mathscr{C} \cup \mathscr{C}'$ such that*

- $|\mathscr{C}'| = \varepsilon\Delta + o(\Delta)$*;*

– *if a cycle is φ-bichromatic, then it was ψ-bichromatic;*

– *there are no φ-bichromatic cycles of length at least L, where $L \coloneqq b_\varepsilon \log \Delta + d_\varepsilon$.*

Proof. Let $\mathscr{C}'$ be a set of colors disjoint from $\mathscr{C}$ with $|\mathscr{C}'| = c\Delta$. Set $E \coloneqq E(G)$. Fix some $0 < p < 1$ and construct a random edge coloring φ in the following way: For each edge $e \in E$ either do not change its color with probability $1 - p$, or choose for it one of the new colors, each with probability $p/|\mathscr{C}'| = p/(c\Delta)$. Let

$$A \coloneqq \{S \subseteq E \,:\, \varphi|S \text{ satisfies the conditions of the lemma}\},$$

where $\varphi|S$ denotes the restriction of φ to S. The set A is out-closed, $\varnothing \in A$ with probability 1, and $E \in A$ if and only if φ satisfies the conditions of the lemma.

For each $e \in E$, let

$$\begin{aligned}\mathscr{B}(e) \coloneqq \quad & \{B^{\mathrm{I}}_{e,e'} \,:\, e' \text{ is an edge adjacent to } e\} \\ & \cup \{B^{\mathrm{II}}_{C} \,:\, C \ni e \text{ is a } \psi\text{-bichromatic cycle of length at least } L\} \\ & \cup \{B^{\mathrm{III}}_{C} \,:\, C \ni e \text{ is a cycle of even length}\} \\ & \cup \{B^{\mathrm{IV}}_{e,C} \,:\, C \text{ is a cycle, } C = e_1, e_2, \ldots, e_{2t} \text{ with } e_1 = e \text{ and } \psi(e_1) = \psi(e_3) = \ldots = \psi(e_{2t-1})\} \\ & \cup \{B^{\mathrm{V}}_{e,C} \,:\, C \text{ is a cycle, } C = e_1, e_2, \ldots, e_{2t} \text{ with } e_1 = e \text{ and } \psi(e_2) = \psi(e_4) = \ldots = \psi(e_{2t})\},\end{aligned}$$

where

1. $B^{\mathrm{I}}_{e,e'}$ happens if and only if $\varphi(e) = \varphi(e') \in \mathscr{C}'$;

2. B^{II}_{C} happens if and only if $\varphi(e') = \psi(e')$ for all $e' \in C$;

3. B^{III}_{C} happens if and only if C is φ-bichromatic and $\varphi(e') \in \mathscr{C}'$ for all $e' \in C$;

4. $B^{\mathrm{IV}}_{e,C}$ happens if and only if C is φ-bichromatic and $\varphi(e_{2k-1}) = \psi(e_{2k-1})$, $\varphi(e_{2k}) \in \mathscr{C}'$ for all $1 \leqslant k \leqslant t$;

5. $B^{\mathrm{V}}_{e,C}$ happens if and only if C is φ-bichromatic and $\varphi(e_{2k-1}) \in \mathscr{C}'$, $\varphi(e_{2k}) = \psi(e_{2k})$ for all $1 \leqslant k \leqslant t$.

It is easy to see that the situations described above exhaust all possible circumstances under which $e \in \partial A$.

Let us proceed to estimate the contributions of the events of different types to the right-hand side of (1.3.1). From now on, we assume that $\tau \in [1; +\infty)$ is a constant.

Type I. For $Z \subseteq E \setminus \{e, e'\}$, we have

$$\mathbb{P}[B^{\mathrm{I}}_{e,e'} | Z \in A] \;=\; \mathbb{P}[\varphi(e) = \varphi(e') \in \mathscr{C}' | Z \in A] \;\leqslant\; \frac{p^2}{c\Delta},$$

so

$$\sigma^A_\tau(B^{\mathrm{I}}_{e,e'}, e) \;\leqslant\; \sigma^A_\tau(B^{\mathrm{I}}_{e,e'}, \{e, e'\}) \;\leqslant\; \frac{p^2\tau^2}{c\Delta}.$$

Since there are fewer than 2Δ edges adjacent to any given edge e, the events of the first type contribute at most

$$2\Delta \cdot \frac{p^2\tau^2}{c\Delta} = 2c\left(\frac{p\tau}{c}\right)^2 \tag{1.9.4}$$

to the right-hand side of (1.3.1).

Type II. For $Z \subseteq E \setminus C$, we have

$$\mathbb{P}[B_C^{\mathrm{II}} | Z \in A] \; = \; \mathbb{P}\left[\varphi(e') = \psi(e') \text{ for all } e' \in C \,|\, Z \in A\right] \; \leqslant \; (1-p)^{|C|}.$$

Therefore,

$$\sigma_\tau^A(B_C^{\mathrm{II}}, e) \leqslant \sigma_\tau^A(B_C^{\mathrm{II}}, C) \leqslant (1-p)^{|C|}\tau^{|C|}.$$

If we further assume that $(1-p)\tau < 1$, then

$$\sigma_\tau^A(B_C^{\mathrm{II}}, e) \leqslant ((1-p)\tau)^L.$$

Finally, note that there are fewer than Δ cycles that contain a given edge e and are ψ-bichromatic (because the second edge on such a cycle determines it uniquely). Therefore, the events of this type contribute at most

$$\Delta((1-p)\tau)^L \tag{1.9.5}$$

to the right-hand side of (1.3.1).

Type III. For $Z \subseteq E \setminus C$, we have

$$\mathbb{P}[B_C^{\mathrm{III}} | Z \in A] = \mathbb{P}\left[C \text{ is } \varphi\text{-bichromatic and } \varphi(e') \in \mathscr{C}' \text{ for all } e' \in C \,|\, Z \in A\right] \; \leqslant \; \frac{p^{|C|}}{(c\Delta)^{|C|-2}}.$$

Therefore,

$$\sigma_\tau^A(B_C^{\mathrm{III}}, e) \leqslant \sigma_\tau^A(B_C^{\mathrm{III}}, C) \leqslant \frac{p^{|C|}\tau^{|C|}}{(c\Delta)^{|C|-2}}.$$

There can be at most Δ^{2t-2} cycles of length $2t$ containing a given edge e. Hence, if we assume that $p\tau/c < 1$, then the events of the third type contribute at most

$$\sum_{t=2}^{\infty} \Delta^{2t-2} \cdot \frac{p^{2t}\tau^{2t}}{(c\Delta)^{2t-2}} = c^2 \sum_{t=2}^{\infty} \left(\frac{p\tau}{c}\right)^{2t} = c^2 \frac{(p\tau/c)^4}{1-(p\tau/c)^2} \tag{1.9.6}$$

to the right-hand side of (1.3.1).

Type IV. For $Z \subseteq E \setminus C$, we have

$$\mathbb{P}[B_{e,C}^{\mathrm{IV}} | Z \in A] \leqslant \frac{p^{|C|/2}(1-p)^{|C|/2}}{(c\Delta)^{|C|/2-1}}.$$

Therefore,

$$\sigma_\tau^A(B_{e,C}^{\mathrm{IV}}, e) \leqslant \sigma_\tau^A(B_{e,C}^{\mathrm{IV}}, C) \leqslant \frac{p^{|C|/2}(1-p)^{|C|/2}}{(c\Delta)^{|C|/2-1}} \cdot \tau^{|C|}.$$

If we further assume that $(1-p)\tau < 1$, then

$$\sigma_\tau^A(B_{e,C}^{\mathrm{IV}}, e) \leqslant \frac{p^{|C|/2}\tau^{|C|/2}}{(c\Delta)^{|C|/2-1}}.$$

There can be at most Δ^{t-1} cycles C of length $2t$ containing a given edge e such that every second edge in C is colored the same by ψ. (See Fig. 7. Solid edges retain their color from ψ, which must be the same for all of them. The arrows indicate the $t-1$ edges that must be specified in order to fully determine the cycle). Hence, if we assume that $p\tau/c < 1$, then the events of the fourth type contribute at most

$$\sum_{t=2}^{\infty} \Delta^{t-1} \cdot \frac{p^t \tau^t}{(c\Delta)^{t-1}} \leqslant c \sum_{t=2}^{\infty} \left(\frac{p\tau}{c}\right)^t = c\frac{(p\tau/c)^2}{1-p\tau/c} \tag{1.9.7}$$

to the right-hand side of (1.3.1).

Type V. The same analysis as for Type IV (see Fig. 8) shows that the contribution of the events of this type to the right-hand side of (1.3.1) is at most

$$c\frac{(p\tau/c)^2}{1-p\tau/c}, \tag{1.9.8}$$

provided that $(1-p)\tau < 1$ and $p\tau/c < 1$.

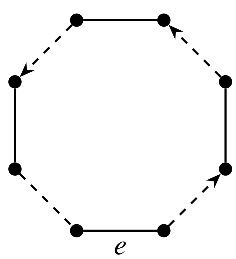

Figure 7 – Type IV

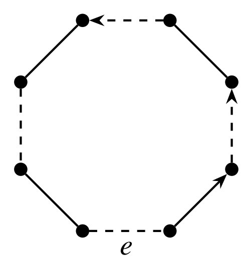

Figure 8 – Type V

Adding together (1.9.4), (1.9.5), (1.9.6), (1.9.7), and (1.9.8), it is enough to have the following inequality:

$$\tau \geqslant 1 + 2c\,(p\tau/c)^2 + \Delta((1-p)\tau)^L + c^2\frac{(p\tau/c)^4}{1-(p\tau/c)^2} + 2c\frac{(p\tau/c)^2}{1-p\tau/c}, \tag{1.9.9}$$

under the assumptions that $(1-p)\tau < 1$ and $p\tau/c < 1$. Denote $y \coloneqq p\tau/c$. Then (1.9.9) turns into

$$\frac{c}{p} \geqslant \frac{1}{y} + 2cy + c^2\frac{y^3}{1-y^2} + 2c\frac{y}{1-y} + \frac{\Delta}{y}\left(\frac{c(1-p)}{p}y\right)^L,$$

and we have the conditions $y < 1$ and $y < p/(c(1-p))$. Let $c = \varepsilon$. We can assume that ε satisfies

$$2\varepsilon^2 + \frac{\varepsilon^5}{1-\varepsilon^2} + \frac{2\varepsilon^2}{1-\varepsilon} \leqslant \frac{\varepsilon}{4}.$$

Take $y = \varepsilon$. Then it is enough to have

$$\frac{\varepsilon}{p} \geqslant \frac{1}{\varepsilon} + \frac{\varepsilon}{4} + \frac{\Delta}{\varepsilon}\left(\frac{\varepsilon^2(1-p)}{p}\right)^L. \tag{1.9.10}$$

Let $p_\varepsilon \coloneqq \varepsilon/(\varepsilon/2 + 1/\varepsilon)$. Note that

$$\frac{p_\varepsilon}{\varepsilon(1-p_\varepsilon)} = \frac{1}{\left(\frac{\varepsilon}{2}+\frac{1}{\varepsilon}\right)\left(1-\varepsilon\Big/\left(\frac{\varepsilon}{2}+\frac{1}{\varepsilon}\right)\right)} = \frac{1}{\frac{1}{\varepsilon}-\frac{\varepsilon}{2}} > \varepsilon,$$

so this choice of p_ε does not contradict our assumptions. Then (1.9.10) becomes

$$\frac{\varepsilon}{4} \geqslant \frac{\Delta}{\varepsilon}\left(\frac{\varepsilon^2(1-p_\varepsilon)}{p_\varepsilon}\right)^L,$$

which is true provided that

$$L \geqslant \left(\log\left(\frac{p_\varepsilon}{\varepsilon^2(1-p_\varepsilon)}\right)\right)^{-1}\left(\log\Delta + \log\frac{4}{\varepsilon^2}\right),$$

and we are done. ∎

Finishing the proof

To finish the proof of Theorem 1.9.4, fix $\varepsilon > 0$. By Lemma 1.9.12, if $g(G) > 2r$, where $r \geqslant 2$, then there is a proper edge coloring ψ of G using at most $(2+\varepsilon/2)\Delta + o(\Delta)$ colors that contains no bichromatic cycles of length at most $L_1 \coloneqq 2a_{\varepsilon/2}(r-2)\log\Delta + 2$. Applying Lemma 1.9.13 to this coloring gives a new coloring φ that uses at most $(2+\varepsilon)\Delta + o(\Delta)$ colors and contains no bichromatic cycles of length at most L_1 (because there were no such cycles in ψ) and at least $L_2 \coloneqq b_{\varepsilon/2}\log\Delta + d_{\varepsilon/2}$. If $r - 2 > b_{\varepsilon/2}/(2a_{\varepsilon/2})$ and Δ is large enough, then $L_1 > L_2$, and φ must be acyclic. This observation completes the proof.

Concluding remarks

We conclude with some remarks on why it seems difficult to get closer to the desired bound $a'(G) \leqslant \Delta(G) + 2$ using the same approach as in the proof of Theorem 1.9.4. Observe that in the proof of Theorem 1.9.4 (specifically in the proof of Lemma 1.9.12) we reserve 2Δ colors for making a coloring proper and use only $c\Delta$ "free" colors to make this coloring acyclic. Essentially, Theorem 1.9.4 asserts that c can be made as small as $\varepsilon + o(1)$, provided that $g(G)$ is large enough. It means that the only way to improve the linear term in our bound is to reduce the number of reserved colors, in other words, to implement in the proof some Vizing-like argument. Unfortunately, we do not know how to prove Vizing's theorem by a relatively straightforward application of the LLL (or any analog of it). On the other hand, using a more sophisticated technique (similar to the one used by Kahn [Kah00] in his celebrated proof that every graph is $(1+o(1))\Delta$-edge-list-colorable), Cai et al. [Cai+17] managed to obtain the bound $a'(G) \leqslant (1+\varepsilon)\Delta + o(\Delta)$, which is very close to the desired $a'(G) \leqslant \Delta(G) + 2$.

2 | DP-Coloring

2.1 DP-coloring: definitions and overview

List coloring is a generalization of ordinary graph coloring that was introduced independently by Vizing [Viz76] and Erdős, Rubin, and Taylor [ERT79]. Let C be a set of **colors**. A **list assignment** for a graph G is a function $L\colon V(G) \to \mathrm{Pow}(C)$. If $|L(u)| = k$ for every vertex $u \in V(G)$, then L is called a **k-list assignment**. A proper coloring $f\colon V(G) \to C$ is called an **L-coloring** if $f(u) \in L(u)$ for each $u \in V(G)$. The **list-chromatic number** $\chi_\ell(G)$ of G is the smallest $k \in \mathbb{N}$ such that G admits an L-coloring for every k-list assignment L for G. An immediate consequence of this definition is that $\chi_\ell(G) \geqslant \chi(G)$ for all graphs G, since ordinary coloring is the same as L-coloring with $L(u) = C$ for all $u \in V(G)$. On the other hand, it is well-known that the gap between $\chi(G)$ and $\chi_\ell(G)$ can be arbitrarily large; for instance, $\chi(K_{n,n}) = 2$, while $\chi_\ell(K_{n,n}) = (1 + o(1)) \log_2(n) \to \infty$ as $n \to \infty$.

In this chapter we study a further generalization of list coloring that was recently introduced by Dvořák and Postle [DP15]; they called it **correspondence coloring**, and we call it **DP-coloring** for short. In the setting of DP-coloring, not only does each vertex get its own list of available colors, but also the identifications between the colors in the lists can vary from edge to edge.

Definition 2.1.1. Let G be a graph. A **cover** of G is a pair $\mathscr{H} = (L, H)$, consisting of a graph H and a function $L\colon V(G) \to \mathrm{Pow}(V(H))$, satisfying the following requirements:

(C1) the sets $\{L(u) \,:\, u \in V(G)\}$ form a partition of $V(H)$;

(C2) for every $u \in V(G)$, the graph $H[L(u)]$ is complete;

(C3) if $E_H(L(u), L(v)) \neq \varnothing$, then either $u = v$ or $uv \in E(G)$;

(C4) if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching.

A cover $\mathscr{H} = (L, H)$ of G is **k-fold** if $|L(u)| = k$ for all $u \in V(G)$.

Remark. The matching $E_H(L(u), L(v))$ in Definition 2.1.1(C4) does not have to be perfect and, in particular, is allowed to be empty.

This chapter is based on [Ber16c; Ber17b; BK16; BK17a; BKP17; BKZ17; BKZ18]. Parts of this work are joint with Alexandr Kostochka, Sergei Pron, and Xuding Zhu.

Definition 2.1.2. Let G be a graph and let $\mathscr{H} = (L, H)$ be a cover of G. An $\mathscr{H}$**-coloring** of G is an independent set in H of size $|V(G)|$.

Remark. By definition, if $\mathscr{H} = (L, H)$ is a cover of G, then $\{L(u) : u \in V(G)\}$ is a partition of H into $|V(G)|$ cliques. Therefore, an independent set $I \subseteq V(H)$ is an $\mathscr{H}$-coloring of G if and only if $|I \cap L(u)| = 1$ for all $u \in V(G)$.

Remark. Suppose that G is a graph, $\mathscr{H} = (L, H)$ is a cover of G, and G' is a subgraph of G. In such situations, we will allow a slight abuse of terminology and speak of $\mathscr{H}$-colorings of G' (even though, strictly speaking, $\mathscr{H}$ is not a cover of G'), meaning $\mathscr{H}'$-colorings of G', where $\mathscr{H}'$ is the cover of G' obtained by removing from H all the vertices that are in $L(u)$ for some $u \notin V(G')$ and all the edges that are in $E_H(L(u), L(v))$ for some $u \neq v$, $uv \notin E(G')$.

Definition 2.1.3. Let G be a graph. The **DP-chromatic number** $\chi_{DP}(G)$ of G is the smallest $k \in \mathbb{N}$ such that G admits an $\mathscr{H}$-coloring for every k-fold cover $\mathscr{H}$ of G.

Example 2.1.4. Figure 9 shows two distinct 2-fold covers of the 4-cycle C_4. Note that C_4 admits an $\mathscr{H}_1$-coloring but not an $\mathscr{H}_2$-coloring. In particular, $\chi_{DP}(C_4) \geqslant 3$. On the other hand, it can be easily seen that $\chi_{DP}(G) \leqslant \Delta(G) + 1$ for any graph G, and so we have $\chi_{DP}(C_4) = 3$. A similar argument demonstrates that $\chi_{DP}(C_n) = 3$ for any cycle C_n of length $n \geqslant 3$.

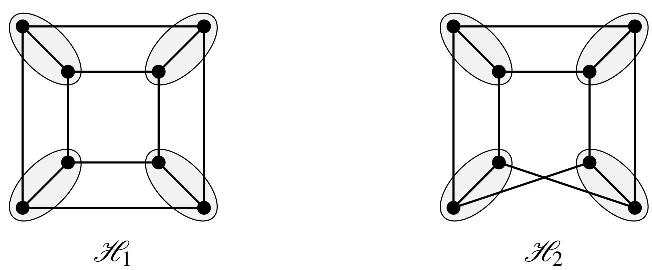

Figure 9 – Two distinct 2-fold covers of a 4-cycle.

Plesnevič and Vizing [PV65] proved that a graph G admits a proper k-coloring if and only if the Cartesian product $G \square K_k$ contains an independent set of size $|V(G)|$. A version of their construction shows that list coloring is a special case of DP-coloring and, in particular, $\chi_{DP}(G) \geqslant \chi_\ell(G)$ for all graphs G.

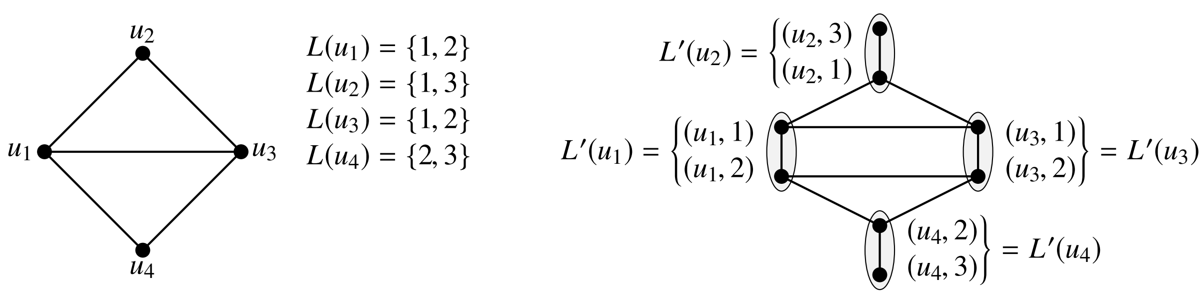

Figure 10 – A graph with a 2-list assignment and the corresponding 2-fold cover.

More precisely, let G be a graph and suppose that $L\colon V(G) \to \mathrm{Pow}(C)$ is a list assignment for G, where C is a set of colors. Let H be the graph with vertex set

$$V(H) := \{(u, c) \,:\, u \in V(G) \text{ and } c \in L(u)\},$$

in which two distinct vertices (u, c) and (v, d) are adjacent if and only if either $u = v$, or else, $uv \in E(G)$ and $c = d$. For each $u \in V(G)$, set

$$L'(u) := \{(u, c) \,:\, c \in L(u)\}.$$

Then $\mathscr{H} := (L', H)$ is a cover of G, and there is a natural bijective correspondence between the L-colorings and the $\mathscr{H}$-colorings of G: If $f\colon V(G) \to C$ is an L-coloring of G, then the set $I_f := \{(u, f(u)) \,:\, u \in V(G)\}$ is an $\mathscr{H}$-coloring of G. Conversely, given an $\mathscr{H}$-coloring I of G, $|I \cap L'(u)| = 1$ for all $u \in V(G)$, so we can define an L-coloring $f_I\colon V(G) \to C$ by the property $(u, f_I(u)) \in I \cap L'(u)$ for all $u \in V(G)$.

Some upper bounds on list-chromatic number hold for DP-chromatic number as well. For instance, it is easy to see that $\chi_{DP}(G) \leqslant d + 1$ for any d-degenerate graph G. Dvořák and Postle [DP15] observed that for any planar graph G, $\chi_{DP}(G) \leqslant 5$ and, moreover, $\chi_{DP}(G) \leqslant 3$ if G is a planar graph of girth at least 5 (these statements are extensions of classical results of Thomassen [Tho94; Tho95] on list colorings).

Furthermore, there are statements about list coloring whose only known proofs involve DP-coloring in essential ways. For example, the reason why Dvořák and Postle originally introduced DP-coloring was to prove that every planar graph without cycles of lengths 4 to 8 is 3-list-colorable [DP15, Theorem 1], thus answering a long-standing question of Borodin [Bor13, Problem 8.1]. Another example is discussed in §2.4, where Dirac's theorem on the minimum number of edges in critical graphs [Dir57; Dir74] is extended to the framework of DP-colorings, yielding a solution to the problem, posed by Kostochka and Stiebitz [KS02], of classifying list-critical graphs that satisfy Dirac's bound with equality.

On the other hand, DP-coloring and list coloring are also strikingly different in some respects. Importantly, the DP-chromatic number of a graph cannot be too small:

Theorem 2.1.5. *If G is a graph of maximum average degree $d \geqslant 2$, then $\chi_{DP}(G) \geqslant d/(2 \ln d)$.*

Proof. After passing to a subgraph, we may assume that the average degree of G itself is d. Set $n := |V(G)|$ and $m := |E(G)|$. Then we have $m = dn/2$. Fix an arbitrary integer $k \leqslant d/(2 \ln d)$. Let $\{L(u) \,:\, u \in V(G)\}$ be a collection of pairwise disjoint sets of size k. Define $X := \bigcup_{u \in V(G)} L(u)$, and build a random graph H with vertex set X by making each $L(u)$ a clique and putting, independently for each $uv \in E(G)$, a uniformly random perfect matching between $L(u)$ and $L(v)$. Let $\mathscr{H} := (L, H)$ denote the resulting random k-fold cover of G.

Consider an arbitrary set $I \subseteq X$ with $|I \cap L(u)| = 1$ for all $u \in V(G)$. Since the matchings corresponding to different edges of G are drawn independently from each other, we have

$$\mathbb{P}\,[I \text{ is independent in } H] = (1 - 1/k)^m \leqslant \exp(-m/k).$$

There are k^n possible choices for I, so

$$\mathbb{P}\,[G \text{ is } \mathscr{H}\text{-colorable}] \leqslant \exp(-m/k) \cdot k^n = (\exp(-d/(2k)) \cdot k)^n.$$

It remains to notice that

$$\exp(-d/(2k)) \cdot k \leqslant k/d < 1,$$

as long as $d > \sqrt{e} \approx 1.64$. ■

Theorem 2.1.5 shows that the DP-chromatic number of a graph grows very quickly with the average degree and is only a logarithmic factor away from the trivial upper bound $\chi_{DP}(G) \leqslant d + 1$ for d-degenerate G. Theorem 2.1.5 must be compared with the celebrated result of Alon [Alo00] that the list-chromatic number of a graph G with maximum average degree d is $\Omega(\log d)$.

In spite of this, known upper bounds on list-chromatic numbers often have the same order of magnitude as in the DP-coloring setting. Notably, by Johansson's theorem [Joh96], triangle-free graphs G of maximum degree Δ satisfy $\chi_\ell(G) = O(\Delta/\log \Delta)$. The same asymptotic upper bound holds for $\chi_{DP}(G)$; see §2.3 for further discussion.

The goal of this chapter is to study the properties of DP-coloring, and in particular to gain some understanding of the similarities and the differences between DP-coloring and list coloring. We start in §2.2 by showing that certain classical results on list coloring do not generalize to the DP-coloring setting. Then, in §2.3, we prove a version of Johansson's theorem (in the strengthened form established recently by Molloy [Mol17]) for DP-coloring. The proof approach described there is interesting in its own right, even for ordinary or list coloring. In §2.4 we investigate lower bounds on the number of edges in DP-critical graphs. Some of the results there are new even for list-critical graphs (but we do not know how to prove them without using DP-coloring). In §2.5 we show that for every n-vertex graph G whose chromatic number $\chi(G)$ is "close" to n, the DP-chromatic number of G equals $\chi(G)$ (this is a DP-version of the Noel–Reed–Wu theorem for list coloring [NRW15]). Finally, in §2.6 we consider the fractional analog of DP-coloring and show that for a fairly large class of graphs, fractional DP-chromatic number is very tightly controlled by maximum average degree.

2.2 Some differences between DP-coloring and list coloring

2.2.1 Statements of results

Important tools in the study of list coloring that do not generalize to the framework of DP-coloring are the orientation theorems of Alon and Tarsi [AT92] and the closely related Bondy–Boppana–Siegel lemma (see [AT92]). Indeed, they can be used to prove that even cycles are 2-list-colorable, while the DP-chromatic number of any cycle is 3, regardless of its length (see Example 2.1.4). In this section we demonstrate the failure in the context of DP-coloring of two other list-coloring results whose proofs rely on either the Alon–Tarsi method or the Bondy–Boppana–Siegel lemma.

A well-known application of the orientation method is the following result:

Theorem 2.2.1 (Alon–Tarsi [AT92, Corollary 3.4])**.** *Every planar bipartite graph is* 3*-list-colorable.*

We show that Theorem 2.2.1 does not hold for DP-colorings (note that every planar triangle-free graph is 3-degenerate, hence 4-DP-colorable):

Theorem 2.2.2. *There exists a planar bipartite graph G with $\chi_{DP}(G) = 4$.*

This answers a question of Grytczuk (personal communication, 2016). We prove Theorem 2.2.2 in §2.2.2.

Our second result concerns edge colorings. Recall that the **line graph** $\mathsf{Line}(G)$ of a graph G is the graph with vertex set $E(G)$ such that two vertices of $\mathsf{Line}(G)$ are adjacent if and only if the corresponding edges of G share an endpoint. The chromatic number, the list-chromatic number, and the DP-chromatic number of $\mathsf{Line}(G)$ are called the **chromatic index**, the **list-chromatic index**, and the **DP-chromatic index** of G and are denoted by $\chi'(G)$, $\chi'_\ell(G)$, and $\chi'_{DP}(G)$ respectively. The following hypothesis is known as the **Edge List Coloring Conjecture** and is a major open problem in graph theory:

Conjecture 2.2.3 (**Edge List Coloring Conjecture**, see [BM08, Conjecture 17.8])**.** *For every graph G, $\chi'_\ell(G) = \chi'(G)$.*

In an elegant application of the orientation method, Galvin [Gal95] verified the Edge List Coloring Conjecture for bipartite graphs:

Theorem 2.2.4 (Galvin [Gal95])**.** *For every bipartite graph G, $\chi'_\ell(G) = \chi'(G) = \Delta(G)$.*

We show that this famous result fails for DP-coloring; in fact, it is impossible for a d-regular graph G with $d \geqslant 2$ to have DP-chromatic index d:

Theorem 2.2.5. *If $d \geqslant 2$, then every d-regular graph G satisfies $\chi'_{DP}(G) \geqslant d + 1$.*

We prove Theorem 2.2.5 in §2.2.3.

Vizing [Viz64] proved that the inequality $\chi'(G) \leqslant \Delta(G) + 1$ holds for all graphs G. He also conjectured the following weakening of the Edge List Coloring Conjecture:

Conjecture 2.2.6 (Vizing)**.** *For every graph G, $\chi'_\ell(G) \leqslant \Delta(G) + 1$.*

We do not know if Conjecture 2.2.6 can be extended to DP-colorings:

Problem 2.2.7. *Do there exist graphs G with $\chi'_{DP}(G) \geqslant \Delta(G) + 2$?*

In §2.2.4 we discuss two natural ways to define edge-DP-colorings for multigraphs. According to one of them, the DP-chromatic index of the multigraph K_2^d with two vertices joined by d parallel edges is $2d$.

2.2.2 Proof of Theorem 2.2.2

In this subsection we construct a planar bipartite graph G with DP-chromatic number 4. The main building block of our construction is the graph Q shown in Figure 11 on the left, i.e., the skeleton of the 3-dimensional cube. Let $\mathscr{F} = (L, F)$ denote the cover of Q shown in Figure 11 on the right.

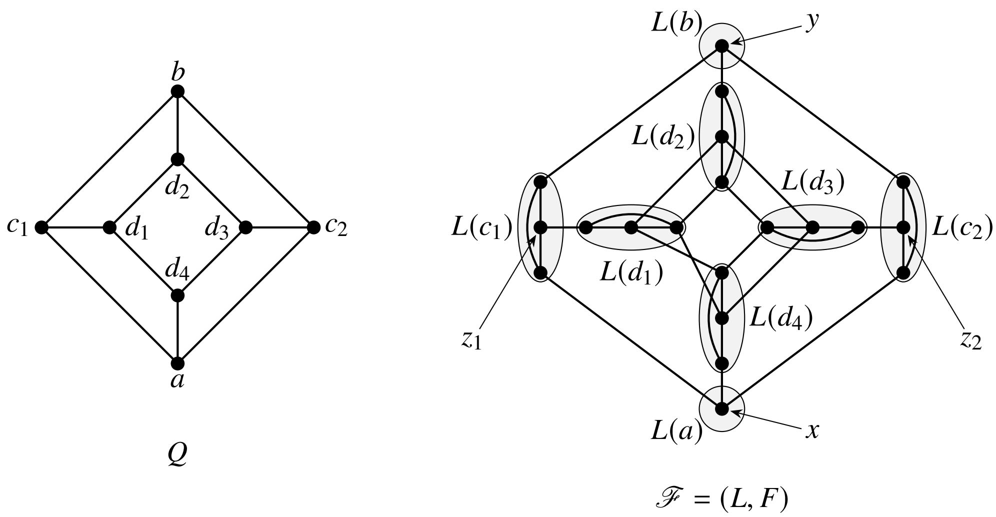

Figure 11 – The graph Q (left) and its cover $\mathscr{F}$ (right).

Lemma 2.2.8. *The graph Q is not $\mathscr{F}$-colorable.*

Proof. Suppose, towards a contradiction, that I is an $\mathscr{F}$-coloring of Q. Since $L(a) = \{x\}$, we have $x \in I$, and, similarly, $y \in I$. Since z_1 is the only vertex in $L(c_1)$ that is not adjacent to x or y, we also have $z_1 \in I$, and, similarly, $z_2 \in I$. This leaves only 2 vertices available in each of $L(d_1)$, $L(d_2)$, $L(d_3)$, and $L(d_4)$, and it is easy to see that these 8 vertices do not contain an independent set of size 4 (cf. the cover $\mathscr{H}_2$ of the 4-cycle shown in Figure 9 on the right). ∎

Consider 9 pairwise disjoint copies of Q, labeled Q_{ij} for $1 \leqslant i, j \leqslant 3$. For each vertex $u \in V(Q)$, its copy in Q_{ij} is denoted by u_{ij}. Let $\mathscr{F}_{ij} = (L_{ij}, F_{ij})$ be a cover of Q_{ij} isomorphic to $\mathscr{F}$. Again, we assume that the graphs F_{ij} are pairwise disjoint and use u_{ij} to denote the copy of a vertex $u \in V(F)$ in F_{ij}. Let G be the graph obtained from the (disjoint) union of the graphs Q_{ij} by identifying the vertices $a_{11}, \ldots, a_{33}$ to a new vertex a^* and the vertices $b_{11}, \ldots, b_{33}$ to a new vertex b^*. Let H be the graph obtained from the union of the graphs F_{ij} by identifying, for each $1 \leqslant i, j \leqslant 3$, the vertices x_{i1}, x_{i2}, x_{i3} to a new vertex x_i and the vertices y_{1j}, y_{2j}, y_{3j} to a new vertex y_j. Define the map $L^* \colon V(G) \to \mathrm{Pow}(V(H))$ as follows:

$$L^*(u) := \begin{cases} L_{ij}(u) & \text{if } u \in V(Q_{ij}); \\ \{x_1, x_2, x_3\} & \text{if } u = a^*; \\ \{y_1, y_2, y_3\} & \text{if } u = b^*. \end{cases}$$

Then $\mathscr{H} := (L^*, H)$ is a 3-fold cover of G. We claim that G is not $\mathscr{H}$-colorable. Indeed, suppose that I is an $\mathscr{H}$-coloring of G and let i and j be the indices such that $\{x_i, y_j\} \subset I$. Then I induces an $\mathscr{F}_{ij}$-coloring of Q_{ij}, which cannot exist by Lemma 2.2.8. Since G is evidently planar and bipartite, the proof of Theorem 2.2.2 is complete.

2.2.3 Proof of Theorem 2.2.5

Let $d \geqslant 2$ and let G be an n-vertex d-regular graph. If $\chi'(G) = d + 1$, then $\chi'_{DP}(G) \geqslant d + 1$. Thus, from now on we will assume that $\chi'(G) = d$. In particular, n is even. Indeed, a proper coloring of $\mathsf{Line}(G)$ is the same as a partition of $E(G)$ into matchings, and if n is odd, then d matchings can cover at most $d(n-1)/2 < dn/2 = |E(G)|$ edges of G.

Let $uv \in E(G)$ and let $G' := G - uv$. Our argument hinges on the following simple observation:

Lemma 2.2.9. *Let C be a set of size d and let $f : E(G') \to C$ be a proper coloring of $\mathsf{Line}(G')$. For each $w \in \{u, v\}$, let f_w denote the unique color in C not used in coloring the edges incident to w. Then $f_u = f_v$.*

Proof. For each $c \in C$, let $M_c \subseteq E(G')$ denote the matching formed by the edges e with $f(e) = c$. Then $|M_c| \leqslant n/2$ for all $c \in C$. Moreover, by definition, $\max\{|M_{f_u}|, |M_{f_v}|\} \leqslant n/2 - 1$. Thus, if $f_u \neq f_v$, then

$$\frac{dn}{2} - 1 = |E(G')| = \sum_{c \in C} |M_c| \leqslant \frac{dn}{2} - 2;$$

a contradiction. ■

Let $\mathbb{Z}/d\mathbb{Z}$ denote the additive group of integers modulo d and let H be the graph with vertex set

$$V(H) := E(G) \times (\mathbb{Z}/d\mathbb{Z}),$$

in which the following pairs of vertices are adjacent:

- (e, i) and (e, j) for $e \in E(G)$ and $i, j \in \mathbb{Z}/d\mathbb{Z}$ with $i \neq j$,
- (e, i) and (h, i) for $eh \in E(\mathsf{Line(G')})$ and $i \in \mathbb{Z}/d\mathbb{Z}$,
- (uv, i) and (uv', i) for $uv' \in E(G')$ and $i \in \mathbb{Z}/d\mathbb{Z}$;
- (uv, i) and $(u'v, i + 1)$ for $u'v \in E(G')$ and $i \in \mathbb{Z}/d\mathbb{Z}$.

For each $e \in E(G)$, let $L(e) := \{e\} \times (\mathbb{Z}/d\mathbb{Z})$. Then $\mathscr{H} := (L, H)$ is a d-fold cover of $\mathsf{Line}(G)$. We claim that $\mathsf{Line}(G)$ is not $\mathscr{H}$-colorable (which proves Theorem 2.2.5). Indeed, suppose that I is an $\mathscr{H}$-coloring of $\mathsf{Line}(G)$. For each $e \in E(G')$, let $f(e)$ denote the unique element of $\mathbb{Z}/d\mathbb{Z}$ such that $(e, f(e)) \in I$. Then f is a proper coloring of $\mathsf{Line}(G')$ with $\mathbb{Z}/d\mathbb{Z}$ as its set of colors. Let i be the unique element of $\mathbb{Z}/d\mathbb{Z}$ that is not used in coloring the edges incident to u. Then the only element of $L(uv)$ that can, and therefore must, belong to I is (uv, i). On the other hand, Lemma 2.2.9 implies that i is also the unique element of $\mathbb{Z}/d\mathbb{Z}$ that is not used in coloring the edges incident to v, and, in particular, for some $u'v \in E(G')$, $f(u'v) = i + 1$. Since (uv, i) and $(u'v, i + 1)$ are adjacent vertices of H, I is not an independent set, which is a contradiction.

2.2.4 Edge-DP-colorings of multigraphs

One can extend the notion of DP-coloring to loopless multigraphs. The definitions are almost identical; the only difference is that in Definition 2.1.1, (C4) is replaced by the following:

(C4′) If u and v are connected by $t \geqslant 1$ edges in G, then $E_H(L(u), L(v))$ is a union of t matchings.

An interesting property of DP-coloring of multigraphs is that the DP-chromatic number of a multigraph may be larger than its number of vertices. For example, the multigraph K_k^t obtained from the complete graph K_k by replacing each edge with t parallel edges satisfies

$$\chi_{DP}(K_k^t) = \Delta(K_k^t) + 1 = tk - t + 1.$$

Similarly to the case of simple graphs, the **line graph** $\mathsf{Line}(G)$ of a multigraph G is the graph with vertex set $E(G)$ such that two vertices of $\mathsf{Line}(G)$ are adjacent if and only if the corresponding edges of G share at least one endpoint. Notice that, in particular, $\mathsf{Line}(G)$ is always a simple graph. Sometimes, instead of $\mathsf{Line}(G)$, it is more natural to consider the **line multigraph** $\mathsf{MLine}(G)$, where if two edges of G share both endpoints, then the corresponding vertices of $\mathsf{MLine}(G)$ are joined by a pair edges. Line multigraphs were used, e.g., in the seminal paper by Galvin [Gal95] and also in [BKW97; BKW98].

Somewhat surprisingly, Shannon's bound $\chi'(G) \leqslant 3\Delta(G)/2$ [Sha49] on the chromatic index of a multigraph G does not extend to $\chi_{DP}(\mathsf{MLine}(G))$. Indeed, if $G \cong K_2^d$, i.e., if G is the 2-vertex multigraph with d parallel edges, then $\mathsf{MLine}(G) \cong K_d^2$, so

$$\chi_{DP}(\mathsf{MLine}(G)) = \chi_{DP}(K_d^2) = 2d - 1 = 2\Delta(G) - 1.$$

This is in contrast with the result in [BKW97] that $\chi'_\ell(G) \leqslant 3\Delta(G)/2$ for every multigraph G. However, we conjecture that the analog of Shannon's theorem holds for line *graphs*:

Conjecture 2.2.10. *For every multigraph G, $\chi_{DP}(\mathsf{Line}(G)) \leqslant 3\Delta(G)/2$.*

2.3 The Johansson–Molloy theorem for DP-coloring

2.3.1 Statements of results

The starting point of this section is the following is a celebrated result of Johansson [Joh96]:

Theorem 2.3.1 (Johansson [Joh96])**.** *There exists a positive constant C such that for every triangle-free graph G with maximum degree Δ,*

$$\chi_\ell(G) \leqslant (C + o(1))\frac{\Delta}{\ln \Delta}.$$

Remark. Throughout this section, we use $o(1)$ to indicate a function of Δ that approaches 0 as $\Delta \to \infty$.

Johansson originally proved Theorem 2.3.1 with $C = 9$. Subsequently, Pettie and Su [PS15] improved the bound to $C = 4$. Very recently, Molloy [Mol17] reduced the constant to $C = 1$:

Theorem 2.3.2 (Molloy [Mol17, Theorem 1])**.** *For every triangle-free graph G with maximum degree Δ,*

$$\chi_\ell(G) \leqslant (1 + o(1))\frac{\Delta}{\ln \Delta}.$$

The two main new ideas that allowed Molloy to dramatically simplify Johansson's proof and establish Theorem 2.3.2 are:

– a new coupon collector-type result [Mol17, Lemma 6] with elements drawn uniformly at random from possibly distinct sets; and

– the use of the entropy compression method instead of iterated applications of the Lovász Local Lemma.

The entropy compression method was already discussed in some detail in Chapter 1; see §1.1 for references. It is an algorithmic approach to the Lovász Local Lemma that was developed by Moser and Tardos [MT10] and has since found many applications, especially in the study of graph coloring. One may wonder however why the entropy compression method should be significantly superior to the Local Lemma when applied specifically to the problem of coloring triangle-free graphs. Indeed, there is a lot of "slackness" in the way the Local Lemma is used in Johansson's proof of Theorem 2.3.1: certain events happen with exponentially small probabilities, even though a polynomial upper bound would have sufficed. In other words, the "bottleneck" in the proof is not the Local Lemma *per se*, but rather some expectation/concentration details. Thus, it may appear surprising that using a better alternative to the Local Lemma leads to improvements in this particular case.

The goal of this section is to show that the intuition outlined in the previous paragraph is, in fact, accurate: one can replace the entropy compression method in Molloy's proof of Theorem 2.3.2 by the standard Local Lemma. This makes the argument particularly short and straightforward, as it removes the need for the technical analysis of a randomized recoloring procedure.

The main novelty in our version of the proof consists in choosing a partial proper coloring f of G *uniformly at random* (see Lemma 2.3.9). Note that the colors of individual vertices under f are highly dependent, so understanding the behavior of f at first appears rather difficult. That is why one usually assigns colors to the vertices of G *independently* from each other. But independence comes at a price: It is impossible to ensure that the resulting coloring is proper away from a very small part of the graph. This necessitates an iterative approach, forcing one to repeat the procedure several times until a sufficiently large proportion of the vertices has been colored. Our main observation is that, despite the dependencies, it is still possible to use the Local Lemma to directly analyze a uniformly random partial proper coloring, thus obviating the need for iteration.

Using the Local Lemma instead of the entropy compression is the only significant difference between our argument and the original proof of Theorem 2.3.2 due to Molloy. In particular, we need a coupon collector-type lemma (Lemma 2.3.10), which is, essentially, a rephrasing of [Mol17, Lemma 6]. Nevertheless, to make the presentation self-contained, we include all (or most of) the details.

In addition to simplifying Molloy's proof, we also verify the conclusion of Theorem 2.3.2 for DP-coloring:

Theorem 2.3.3. *For every triangle-free graph G with maximum degree Δ,*

$$\chi_{DP}(G) \leqslant (1+o(1))\frac{\Delta}{\ln \Delta}.$$

Theorem 2.3.3 combined with Theorem 2.1.5 bounds rather tightly the DP-chromatic number of triangle-free regular graphs:

Corollary 2.3.4. *For every Δ-regular triangle-free graph G,*

$$(1/2 - o(1))\frac{\Delta}{\ln \Delta} \leqslant \chi_{DP}(G) \leqslant (1 + o(1))\frac{\Delta}{\ln \Delta}.$$

Another result of Johansson asserts that $\chi_\ell(G) = O(\Delta(G) \ln\ln \Delta(G)/\ln \Delta(G))$ if G is K_r-free for some fixed $r \geqslant 4$. Molloy [Mol17, Theorem 2] also gave a new short proof of this bound with explicit dependence on r. In §2.3.3, we extend it to DP-coloring:

Theorem 2.3.5. *There exists a positive constant C such that for any $r \geqslant 4$ and for every K_r-free graph G with maximum degree Δ,*

$$\chi_{DP}(G) \leqslant C\frac{r\Delta \ln\ln \Delta}{\ln \Delta}.$$

We make no attempt to optimize the constant factor in Theorem 2.3.5. It is conjectured [AKS99, Conjecture 3.1] that the correct upper bound for fixed r should be of the order $O(\Delta/\ln \Delta)$.

The main technical step in the proof of Theorem 2.3.5 is Lemma 2.3.12. It is similar to [Mol17, Lemma 12]; however, it is necessary to modify the proof of [Mol17, Lemma 12] somewhat in order to adapt it for the DP-coloring framework, since, in contrast to list coloring, a DP-coloring of a graph cannot be naturally represented as a partition of its vertex set into independent subsets.

2.3.2 Proof of Theorem 2.3.3

Probabilistic tools

We use the following "lopsided" version of the Symmetric LLL:

Lemma 2.3.6 (**Lovász Local Lemma**; see [AS00, p. 65])**.** *Let I be a finite set. For each $i \in I$, let B_i be a random event. Suppose that for every $i \in I$, there is a set $\Gamma(i) \subseteq I$ such that $|\Gamma(i)| \leqslant d$ and for all $Z \subseteq I \setminus \Gamma(i)$,*

$$\mathbb{P}\left[B_i \middle| \bigwedge_{j \in Z} \neg B_j\right] \leqslant p.$$

If $4pd \leqslant 1$, then $\mathbb{P}\left[\bigwedge_{i \in I} \neg B_i\right] > 0$.

In a more commonly used version of the LLL, each event B_i is mutually independent from the events B_j with $j \notin \Gamma(i) \cup \{i\}$ (cf. Theorem 1.1.1). Lemma 2.3.6 has exactly the same proof, but it makes no independence requirements. This will be crucial for its application in the proof of Lemma 2.3.9.

We will need a version of Chernoff bounds for negatively correlated random variables, introduced by Panconesi and Srinivasan [PS97]. We say that $\{0, 1\}$-valued random variables $X_1, \ldots, X_n$ are **negatively correlated** if for all $S \subseteq \{1, \ldots, n\}$,

$$\mathbb{P}[X_i = 1 \text{ for all } i \in S] \leqslant \prod_{i \in S} \mathbb{P}[X_i = 1].$$

Lemma 2.3.7 (**Chernoff bounds**; see [PS97] and [Mol17, Lemma 3])**.** *Let $X_1, \ldots, X_n$ be $\{0,1\}$-valued random variables and let $Y_i := 1 - X_i$. Set $X := \sum_{i=1}^n X_i$. If the variables $Y_1, \ldots, Y_n$ are negatively correlated, then*

$$\mathbb{P}\left[X \leqslant (1-\delta)\mathbb{E}[X]\right] \leqslant \exp\left(-\delta^2\mathbb{E}[X]/2\right) \text{ for any } 0 < \delta < 1.$$

If the variables $X_1, \ldots, X_n$ are negatively correlated, then

$$\mathbb{P}\left[X \geqslant (1+\delta)\mathbb{E}[X]\right] \leqslant \exp\left(-\delta\mathbb{E}[X]/3\right) \text{ for any } \delta > 1.$$

Additional notation

Let G be a graph and let $\mathscr{H} = (L, H)$ be a cover of G. For $U \subseteq V(G)$, let $L(U) := \bigcup_{u \in U} L(u)$. Define H^* to be the spanning subgraph of H such that an edge $xy \in E(H)$ belongs to $E(H^*)$ if and only if x and y are in different parts of the partition $\{L(u) \,:\, u \in V(G)\}$. For clarity, and to emphasize the dependence on L, we write $\deg^*_{\mathscr{H}}(x)$ instead of $\deg_{H^*}(x)$. The **domain** of an independent set I in H is $\mathrm{dom}(I) := \{u \in V(G) \,:\, I \cap L(u) \neq \varnothing\}$. Let $G_I := G - \mathrm{dom}(I)$ and let $\mathscr{H}_I = (L_I, H_I)$ denote the cover of G_I defined by

$$H_I := H - N_H[I] \qquad \text{and} \qquad L_I(u) := L(u) \setminus N_H(I) \text{ for all } u \in V(G_I).$$

By definition, if I' is an $\mathscr{H}_I$-coloring of G_I, then $I \cup I'$ is an $\mathscr{H}$-coloring of G. Recall that for $U \subseteq V(G)$, we write $U^{\mathsf{c}} := V(G) \setminus U$ to indicate the complement of U.

The proof

We will reduce Theorem 2.3.3 to the following result of Haxell [Hax01]:

Lemma 2.3.8 (Haxell [Hax01])**.** *Let $\mathscr{H} = (L, H)$ be a cover of a graph G. If there is a positive integer ℓ such that $|L(u)| \geqslant \ell$ for all $u \in V(G)$ and $\deg^*_{\mathscr{H}}(x) \leqslant \ell/2$ for all $x \in V(H)$, then G is $\mathscr{H}$-colorable.*

Under the stronger assumption $\deg^*_{\mathscr{H}}(x) \leqslant \ell/8$ for all $x \in V(H)$, Lemma 2.3.8 can be proved using a standard LLL-based argument (this weaker version of Lemma 2.3.8 is also sufficient for our purposes; see [MR02, Theorem 4.3] and [Mol17, Lemma 4] for two of its incarnations in the list coloring setting).

Standing assumptions. For the rest of the proof, fix $0 < \varepsilon < 1$, a triangle-free graph G of sufficiently large maximum degree Δ, and a k-fold cover $\mathscr{H} = (L, H)$ of G with $k = (1+\varepsilon)\Delta/\ln\Delta$. Set $\ell := \Delta^{\varepsilon/2}$.

In view of Lemma 2.3.8, it suffices to establish the following:

Lemma 2.3.9. *The graph H contains an independent set I such that:*

(i) $|L_I(u)| \geqslant \ell$ *for all $u \in V(G_I)$; and*

(ii) $\deg^*_{\mathscr{H}_I}(x) \leqslant \ell/2$ *for all $x \in V(H_I)$.*

To prove Lemma 2.3.9, we need a variant of [Mol17, Lemma 6]:

Lemma 2.3.10. *Fix a vertex $u \in V(G)$ and an independent set $J \subseteq L(N_G[u]^{\mathsf{c}})$. Let I' be a uniformly random independent subset of $L_J(N_G(u))$ and let $I := J \cup I'$. Then:*

(*a*) $\mathbb{P}\left[|L_I(u)| < \ell\right] \leqslant \Delta^{-3}/8$*; and*

(*b*) $\mathbb{P}\left[\text{there is } x \in L_I(u) \text{ with } \deg^*_{\mathscr{H}_I}(x) > \ell/2\right] \leqslant \Delta^{-3}/8.$

The proof of Lemma 2.3.10 is virtually identical to that of [Mol17, Lemma 6], so we first show how to derive Lemma 2.3.9 from Lemma 2.3.10 (this is the new ingredient in our version of Molloy's argument).

Proof of Lemma 2.3.9 (*assuming Lemma 2.3.10*). Choose an independent set I in H uniformly at random. (Since the domain of I may be a proper subset of $V(G)$, in the context of list coloring this is equivalent to choosing a uniformly random partial proper coloring.) The following immediate observation plays a key role in the proof:

Fix $U \subseteq V(G)$ and an independent set $J \subseteq L(U^{\mathsf{c}})$. Then the random variable $I \cap L(U)$, conditioned on the event $\{I \cap L(U^{\mathsf{c}}) = J\}$, is uniformly distributed over the independent subsets of $L_J(U)$. (2.3.1)

For each $u \in V(G)$, let B_u denote the event

$$B_u := \{u \notin \mathrm{dom}(I) \text{ and either } |L_I(u)| < \ell \text{ or there is } x \in L_I(u) \text{ with } \deg^*_{\mathscr{H}_I}(x) > \ell/2\}.$$

Clearly, if none of the events B_u happen, then I satisfies the conclusion of Lemma 2.3.9.

For each $u \in V(G)$, set $\Gamma(u) := N_G^3[u]$. Since $|\Gamma(u)| \leqslant \Delta^3$, to apply the LLL, it remains to verify that for all $Z \subseteq \Gamma(u)^{\mathsf{c}}$,

$$\mathbb{P}\left[B_u \middle| \bigwedge_{v \in Z} \neg B_v\right] \leqslant \Delta^{-3}/4.$$

By definition, the outcome of any B_v is determined by the set $I \cap L(N_G^2[v])$. If $v \notin \Gamma(u)$, then the distance between u and v is at least 4, so $N_G^2[v] \subset N_G(u)^{\mathsf{c}}$. Therefore, the set $I \cap L(N_G(u)^{\mathsf{c}})$ determines the outcome of every B_v with $v \notin \Gamma(u)$. Hence, it suffices to prove that

$$\mathbb{P}\left[B_u \middle| I \cap L(N_G(u)^{\mathsf{c}}) = J\right] \leqslant \Delta^{-3}/4 \quad \text{for all independent } J \subseteq L(N_G(u)^{\mathsf{c}}).$$

To that end, fix a vertex $u \in V(G)$ and an independent set $J \subseteq L(N_G(u)^{\mathsf{c}})$. We may assume that $u \notin \mathrm{dom}(J)$, i.e., $J \subseteq L(N_G[u]^{\mathsf{c}})$ (otherwise the event B_u is incompatible with $\{I \cap L(N_G(u)^{\mathsf{c}}) = J\}$). Let $I' := I \cap L(N_G(u))$. By (2.3.1), the variable I', under the condition $\{I \cap L(N_G(u)^{\mathsf{c}}) = J\}$, is uniformly distributed over the independent subsets of $L_J(N_G(u))$. Thus, we are in the situation described by Lemma 2.3.10, and so

$$\mathbb{P}\left[B_u \middle| I \cap L(N_G(u)^{\mathsf{c}}) = J\right] \leqslant \Delta^{-3}/8 + \Delta^{-3}/8 = \Delta^{-3}/4,$$

as desired. ∎

Proof of Lemma 2.3.10. Define

$$p_0 \coloneqq \mathbb{P}\left[|L_I(u)| < \ell\right] \qquad \text{and} \qquad p_1 \coloneqq \mathbb{P}\left[\text{there is } x \in L_I(u) \text{ with } \deg^*_{H_I}(x) > \ell/2\right].$$

Let $\bullet$ be a special symbol distinct from all the elements of $V(H)$. Since G is triangle-free, the set $N_G(u)$ is independent, so $E_H(L(v), L(w)) = \varnothing$ for any two distinct $v, w \in N_G(u)$. Hence, I' can be constructed via the following procedure:

> For each $v \in N_G(u)$, uniformly at random select an element x_v from $L_J(v) \cup \{\bullet\}$.
>
> – If $x_v = \bullet$, then leave $I' \cap L(v)$ empty;
>
> – otherwise, set $I' \cap L(v) \coloneqq \{x_v\}$.

For $x \in L(u)$, let $\tilde{N}(x)$ denote the set of all vertices $v \in N_G(u)$ such that $N_H(x) \cap L_J(v) \neq \varnothing$. Using this notation, we obtain

$$\mathbb{P}\left[x \in L_I(u)\right] = \mathbb{P}\left[I' \cap N_H(x) = \varnothing\right] = \prod_{v \in \tilde{N}(x)} \left(1 - \frac{1}{|L_J(v)| + 1}\right).$$

Since $|L_J(v)| \geqslant 1$ for all $v \in \tilde{N}(x)$ and $\exp(-1/\alpha) \leqslant 1 - 1/(\alpha+1) \leqslant \exp(-1/(\alpha+1))$ for all $\alpha > 0$,

$$\exp\left(-\sum_{v \in \tilde{N}(x)} \frac{1}{|L_J(v)|}\right) \leqslant \mathbb{P}\left[x \in L_I(u)\right] \leqslant \exp\left(-\sum_{v \in \tilde{N}(x)} \frac{1}{|L_J(v)| + 1}\right). \tag{2.3.2}$$

Now we can conclude

$$\mathbb{E}\left[|L_I(u)|\right] = \sum_{x \in L(u)} \mathbb{P}\left[x \in L_I(u)\right] \geqslant \sum_{x \in L(u)} \exp\left(-\sum_{v \in \tilde{N}(x)} \frac{1}{|L_J(v)|}\right).$$

Notice that

$$\sum_{x \in L(u)} \sum_{v \in \tilde{N}(x)} \frac{1}{|L_J(v)|} \leqslant \sum_{\substack{v \in N_G(u):\\ L_J(v) \neq \varnothing}} \sum_{y \in L_J(v)} \frac{1}{|L_J(v)|} \leqslant \deg_G(u) \leqslant \Delta,$$

so, by the convexity of the exponential function,

$$\sum_{x \in L(u)} \exp\left(-\sum_{v \in \tilde{N}(x)} \frac{1}{|L_J(v)|}\right) \geqslant k \exp\left(-\frac{\Delta}{k}\right) = \frac{(1+\varepsilon)\Delta}{\ln \Delta} \cdot \Delta^{-1/(1+\varepsilon)} \geqslant 2\ell,$$

provided Δ is large enough. Putting everything together, we obtain $\mathbb{E}\left[|L_I(u)|\right] \geqslant 2\ell$. Since the indicator random variables of the events $\{x \notin L_I(u)\}$ for $x \in L(u)$ are easily seen to be negatively correlated, Lemma 2.3.7 gives

$$p_0 \leqslant \mathbb{P}\left[|L_I(u)| < \frac{1}{2}\mathbb{E}\left[|L_I(u)|\right]\right] \leqslant \exp\left(-\frac{1}{8}\mathbb{E}\left[|L_I(u)|\right]\right) \leqslant \exp\left(-\ell/4\right) < \Delta^{-3}/8,$$

for large enough Δ. This proves (a).

To prove (b), we will show that for all $x \in L(u)$,

$$p_x := \mathbb{P}\left[x \in L_I(u) \text{ and } \deg^*_{\mathscr{H}_I}(x) > \ell/2\right] \leqslant \Delta^{-4}.$$

This is enough, as $p_1 \leqslant \sum_{x \in L(u)} p_x$ and $|L(u)| = k < \Delta/8$ for large enough Δ. Let $x \in L(u)$. The second inequality in (2.3.2) implies

$$p_x \leqslant \mathbb{P}\left[x \in L_I(u)\right] \leqslant \exp\left(-\sum_{v \in \tilde{N}(x)} \frac{1}{|L_J(v)|+1}\right),$$

so we may assume

$$\exp\left(-\sum_{v \in \tilde{N}(x)} \frac{1}{|L_J(v)|+1}\right) \geqslant \Delta^{-4}, \qquad \text{i.e.,} \qquad \sum_{v \in \tilde{N}(x)} \frac{1}{|L_J(v)|+1} \leqslant 4 \ln \Delta.$$

Then

$$\mathbb{E}\left[\deg^*_{\mathscr{H}_I}(x)\right] = \sum_{v \in \tilde{N}(x)} \mathbb{P}\left[v \notin \mathrm{dom}(I')\right] = \sum_{v \in \tilde{N}(x)} \frac{1}{|L_J(v)|+1} \leqslant 4 \ln \Delta \leqslant \ell/4,$$

for large enough Δ. The events $\{v \notin \mathrm{dom}(I')\}$ for $v \in \tilde{N}(x)$ are mutually independent, so the Chernoff bound for independent $\{0,1\}$-valued random variables yields

$$p_x \leqslant \mathbb{P}\left[\deg^*_{\mathscr{H}_I}(x) > \ell/2\right] \leqslant \mathbb{P}\left[\deg^*_{\mathscr{H}_I}(x) > \mathbb{E}\left[\deg^*_{\mathscr{H}_I}(x)\right] + \ell/4\right] \leqslant \exp\left(-\ell/12\right) \leqslant \Delta^{-4},$$

for large enough Δ, as desired. ∎

2.3.3 Proof of Theorem 2.3.5

The general scheme of the argument is similar to that of the proof of Theorem 2.3.3.

Standing assumptions. Fix an integer $r \geqslant 4$, a K_r-free graph G of large maximum degree Δ, and a k-fold cover $\mathscr{H} = (L, H)$ of G with $k \geqslant 200r\Delta \log_2 \log_2 \Delta / \log_2 \Delta$. Set $\ell := \Delta^{9/10}$.

The role of Lemma 2.3.9 is played by the following statement:

Lemma 2.3.11. *The graph H contains an independent set I such that*

$$|L_I(u)| \geqslant \ell \text{ for all } u \in V(G_I) \qquad \text{and} \qquad \Delta(G_I) < \ell.$$

Note that Lemma 2.3.11 readily implies Theorem 2.3.5, since the DP-chromatic number of a graph is always at most one plus its maximum degree. Lemma 2.3.11 in turn follows from an analog of [Mol17, Lemma 12] for DP-coloring:

Lemma 2.3.12. *Fix a vertex $u \in V(G)$ and an independent set $J \subseteq L(N_G[u]^{\mathsf{c}})$. Let I' be a uniformly random independent subset of $L_J(N_G(u))$ and let $I := J \cup I'$. Then:*

(*a*) $\mathbb{P}\left[|L_I(u)| < \ell\right] \leqslant \Delta^{-3}/8$; *and*

(*b*) $\mathbb{P}\left[\deg_{G_I}(u) \geqslant \ell \text{ and } |L_I(v)| \geqslant \ell \text{ for all } v \in N_{G_I}(u)\right] \leqslant \Delta^{-3}/8$.

The derivation of Lemma 2.3.11 from Lemma 2.3.12 is almost verbatim identical to that of Lemma 2.3.9 from Lemma 2.3.10, and we do not spell it out here. To prove Lemma 2.3.12, we need a variant of a result due to Shearer [She95] that was established by Molloy [Mol17, Lemma 11]. For a graph F, let $\mathrm{ind}(F)$ denote the number of independent sets in F and let $\overline{\alpha}(F)$ denote the **median** size of an independent set in F, i.e., the supremum of all $\alpha \geqslant 0$ such that F contains at least $\mathrm{ind}(F)/2$ independent sets of size at least α. For $\lambda > 0$, define

$$f(\lambda) \coloneqq \frac{\log_2 \lambda}{2r \log_2 \log_2 \lambda}.$$

Lemma 2.3.13 ([Mol17, Lemma 11])**.** *If F is a nonempty K_r-free graph, then $\overline{\alpha}(F) \geqslant f(\mathrm{ind}(F))$.*

Proof of Lemma 2.3.12. We start with the proof of (*a*). For each $x \in L(u)$, the **layer** of x is the set

$$\Lambda(x) \coloneqq L_J(N_G(u)) \cap N_H(x).$$

The layer of x intersects each list $L(v)$ for $v \in N_G(u)$ in at most one element. For distinct $x, x' \in L(u)$, the layers $\Lambda(x)$ and $\Lambda(x')$ are disjoint. However, in contrast to the situation in list coloring, H may contain edges between $\Lambda(x)$ and $\Lambda(x')$ that are not covered by the cliques $H[L(v)]$.

Let $x \in L(u)$. For an independent set $Q \subseteq L_J(N_G(u)) \setminus \Lambda(x)$, let $F(x, Q)$ denote the subgraph of H induced by the vertices in $\Lambda(x)$ with no neighbors in Q. The following observation is similar to (2.3.1) from the proof of Lemma 2.3.9:

> *Fix $x \in L(u)$ and an independent set $Q \subseteq L_J(N_G(u)) \setminus \Lambda(x)$. Then the random variable $I' \cap \Lambda(x)$, conditioned on the event $\{I' \setminus \Lambda(x) = Q\}$, is uniformly distributed over the independent sets in $F(x, Q)$.* (2.3.3)

From (2.3.3), it follows that I' can be constructed via the following randomized procedure. Let $x_1, \ldots, x_k$ be an arbitrary ordering of the set $L(u)$. For each $1 \leqslant i \leqslant k$, let $\Lambda_i \coloneqq \Lambda(x_i)$.

> Let I_0 be a uniformly random independent subset of $L_J(N_G(u))$. Set $s_0 \coloneqq 0$ and $t_0 \coloneqq 0$.
> Repeat the next steps for each $1 \leqslant i \leqslant k$:
>
> – Let $F_i \coloneqq F(x_i, I_{i-1} \setminus \Lambda_i)$. Define s_i and t_i as follows:
>
> $$\begin{array}{llllll} \text{if} & \mathrm{ind}(F_i) > \Delta^{1/20}, & \text{then} & s_i \coloneqq s_{i-1} + 1 & \text{and} & t_i \coloneqq t_{i-1}, \quad \text{while} \\ \text{if} & \mathrm{ind}(F_i) \leqslant \Delta^{1/20}, & \text{then} & s_i \coloneqq s_{i-1} & \text{and} & t_i \coloneqq t_{i-1} + 1. \end{array}$$
>
> – Let S_i be a uniformly random independent set in F_i and let $I_i \coloneqq (I_{i-1} \setminus \Lambda_i) \cup S_i$.
>
> Set $I' \coloneqq I_k$.

It is clear from (2.3.3) that the set I' constructed by the above procedure is uniformly distributed over the independent subsets of $L_J(N_G(u))$ (see also [Mol17, Lemma 12, Claim 1]).

Let $a(1)$, $a(2)$, ... and $b(1)$, $b(2)$, ... be two infinite random sequences of zeros and ones drawn independently from each other, such that for all s and t, we have

$$\mathbb{P}[a(s) = 1] = 1/2 \qquad \text{and} \qquad \mathbb{P}[b(t) = 1] = \Delta^{-1/20}.$$

If the values $I_0, S_1, \ldots, S_{i-1}$ are fixed, then the corresponding conditional probability of $\{|S_i| \geqslant \overline{\alpha}(F_i)\}$ is at least $1/2$, while the conditional probability of $\{S_i = \varnothing\}$ is precisely $1/\mathrm{ind}(F_i)$ (here we are using the fact that the sets $I_0, S_1, \ldots, S_{i-1}$ fully determine F_i). Therefore, we can couple the distributions of the sequences $a(1)$, $a(2), \ldots$ and $b(1), b(2), \ldots$ with the randomized procedure described above in such a way that

$$\begin{array}{llllll} \text{if} & \mathrm{ind}(F_i) > \Delta^{1/20} & \text{and} & a(s_i) = 1, & \text{then} & |S_i| \geqslant \overline{\alpha}(F_i), \quad \text{while} \\ \text{if} & \mathrm{ind}(F_i) \leqslant \Delta^{1/20} & \text{and} & b(t_i) = 1, & \text{then} & S_i = \varnothing. \end{array} \tag{2.3.4}$$

The Chernoff bound for independent random variables implies that, with probability at least $1 - \Delta^{-3}/8$,

$$|\{1 \leqslant s \leqslant k/2 \,:\, a(s) = 1\}| \geqslant k/5 \qquad \text{and} \qquad |\{1 \leqslant t \leqslant k/2 \,:\, b(t) = 1\}| \geqslant \ell. \tag{2.3.5}$$

We claim that $|L_I(u)| \geqslant \ell$ whenever (2.3.5) holds. Since $s_k + t_k = k$, we always have either $s_k \geqslant k/2$ or $t_k \geqslant k/2$. If $s_k \geqslant k/2$, then (2.3.4) and the first part of (2.3.5) imply that there are at least $k/5$ indices i such that $\mathrm{ind}(F_i) > \Delta^{1/20}$ and $|S_i| \geqslant \overline{\alpha}(F_i)$. By Lemma 2.3.13, any such i satisfies

$$|S_i| \geqslant \overline{\alpha}(F_i) \geqslant f(\mathrm{ind}(F_i)) \geqslant f(\Delta^{1/20}) > \frac{\log_2 \Delta}{40r \log_2 \log_2 \Delta},$$

so in this case

$$|I'| = \sum_{i=1}^{k} |S_i| > \frac{k}{5} \cdot \frac{\log_2 \Delta}{40r \log_2 \log_2 \Delta} \geqslant \Delta.$$

This is a contradiction, as $|I'| \leqslant \deg_G(u) \leqslant \Delta$. Thus, we must have $t_k \geqslant k/2$. From (2.3.4) and the second part of (2.3.5), we obtain that there are at least ℓ indices i such that $\mathrm{ind}(F_i) \leqslant \Delta^{1/20}$ and $S_i = \varnothing$. But $x_i \in L_I(u)$ for any such i, so $|L_I(u)| \geqslant \ell$, as desired. This completes the proof of (a).

To prove (b), consider any collection $v_1, \ldots, v_{\lceil \ell \rceil}$ of $\lceil \ell \rceil$ distinct elements of $N_G(u)$. We claim that

$$\mathbb{P}\left[v_t \notin \mathrm{dom}(I') \text{ and } |L_I(v_t)| \geqslant \ell \text{ for all } 1 \leqslant t \leqslant \lceil \ell \rceil\right] \leqslant \frac{1}{\lceil \ell \rceil !}, \tag{2.3.6}$$

which is enough as $\binom{\Delta}{\lceil \ell \rceil}/\lceil \ell \rceil! < \Delta^{-3}/8$ for large Δ. To show (2.3.6), consider an arbitrary independent set $Q \subseteq L_J(N_G(u))$ disjoint from $L(v_t)$ for all $1 \leqslant t \leqslant \lceil \ell \rceil$. We either have $|L_J(v_t) \setminus N_H(Q)| < \ell$ for some t, or else, there exist at least $\lceil \ell \rceil!$ ways to greedily choose elements $x_t \in L_J(v_t)$ so that $Q \cup \{x_1, \ldots, x_{\lceil \ell \rceil}\}$ is an independent set. Therefore,

$$\mathbb{P}\left[v_t \notin \mathrm{dom}(I') \text{ and } |L_I(v_t)| \geqslant \ell \text{ for all } 1 \leqslant t \leqslant \lceil \ell \rceil \,\middle|\, I' \setminus L(\{v_1, \ldots, v_{\lceil \ell \rceil}\}) = Q\right] \leqslant \frac{1}{\lceil \ell \rceil !}.$$

Since Q is arbitrary, this yields (2.3.6). ■

2.4 DP-critical graphs

2.4.1 Introduction

Critical graphs and the theorems of Brooks, Dirac, and Gallai

A graph G is said to be $(k+1)$**-vertex-critical** if $\chi(G) = k+1$ but $\chi(G-u) \leqslant k$ for all $u \in V(G)$. We will only consider vertex-critical graphs, so for brevity we will call them simply **critical**. Since every graph G with $\chi(G) > k$ contains a $(k+1)$-critical subgraph, understanding the structure of critical graphs is crucial for the study of graph coloring. We will only consider $k \geqslant 3$, the case $k \leqslant 2$ being trivial (the only 1-critical graph is K_1, the only 2-critical graph is K_2, and the only 3-critical graphs are odd cycles).

Let $k \geqslant 3$ and suppose that G is a $(k+1)$-critical graph with n vertices and m edges. A classical problem in the study of critical graphs is to understand how small m can be depending on n and k. Evidently, $\delta(G) \geqslant k$; in particular, $2m \geqslant kn$. Brooks's Theorem is equivalent to the assertion that the only situation in which $2m = kn$ is when $G \cong K_{k+1}$:

Theorem 2.4.1 (Brooks [Die00, Theorem 5.2.4])**.** *Let $k \geqslant 3$ and let G be a $(k+1)$-critical graph distinct from K_{k+1}. Set $n := |V(G)|$ and $m := |E(G)|$. Then $2m > kn$.*

Brooks's theorem was subsequently sharpened by Dirac, who established a linear in k lower bound on the difference $2m - kn$:

Theorem 2.4.2 (Dirac [Dir57, Theorem 15])**.** *Let $k \geqslant 3$ and let G be a $(k+1)$-critical graph distinct from K_{k+1}. Set $n := |V(G)|$ and $m := |E(G)|$. Then*

$$2m \geqslant kn + k - 2. \tag{2.4.1}$$

Bound (2.4.1) is sharp in the sense that for every $k \geqslant 3$, there exist $(k+1)$-critical graphs that satisfy $2m = kn + k - 2$. However, for each k, there are only finitely many such graphs; in fact, they admit a simple characterization, which we present below.

Definition 2.4.3. Let $k \geqslant 3$. A graph G is k**-Dirac** if its vertex set can be partitioned into three subsets V_1, V_2, V_3 such that:

- $|V_1| = k$, $|V_2| = k-1$, and $|V_3| = 2$;
- the graphs $G[V_1]$ and $G[V_2]$ are complete;
- each vertex in V_1 is adjacent to exactly one vertex in V_3;

– each vertex in V_3 is adjacent to at least one vertex in V_1;

– each vertex in V_2 is adjacent to both vertices in V_3; and

– G has no other edges.

We denote the family of all k-Dirac graphs by $\mathbf{Dir}_k$.

A typical member of $\mathbf{Dir}_k$ is shown in Fig. 12.

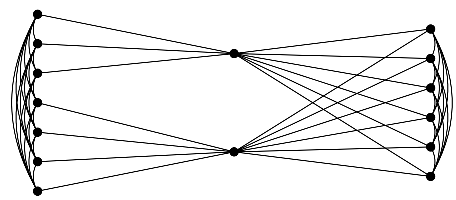

Figure 12 – A typical k-Dirac graph.

Theorem 2.4.4 (Dirac [Dir74, Theorem, p. 152])**.** *Let $k \geqslant 3$ and let G be a $(k+1)$-critical graph distinct from K_{k+1}. Set $n \coloneqq |V(G)|$ and $m \coloneqq |E(G)|$. Then*

$$2m = kn + k - 2 \iff G \in \mathbf{Dir}_k.$$

As n goes to infinity, the gap between Dirac's lower bound and the sharp bound increases. In fact, Gallai [Gal63] observed that the asymptotic density of large $(k+1)$-critical graphs distinct from K_{k+1} is strictly greater than $k/2$:

Theorem 2.4.5 (Gallai [Gal63])**.** *Let $k \geqslant 3$ and let G be a $(k+1)$-critical graph distinct from K_{k+1}. Set $n \coloneqq |V(G)|$ and $m \coloneqq |E(G)|$. Then*

$$2m \geqslant \left(k + \frac{k-2}{k^2+2k-2}\right) n. \tag{2.4.2}$$

Note, however, that Gallai's bound (2.4.2) is stronger than (2.4.1) only for n at least quadratic in k.

List-critical graphs

A list assignment L for a graph G is called a **degree list assignment** if $|L(u)| \geqslant \deg_G(u)$ for all $u \in V(G)$. A fundamental result of Borodin [Bor79] and Erdős–Rubin–Taylor [ERT79], which can be seen as a generalization of Brooks's theorem to list colorings, provides a complete characterization of all graphs G that are not L-colorable with respect to some degree list assignment L.

Definition 2.4.6. A **Gallai tree** is a connected graph in which every block is either a clique or an odd cycle. A **Gallai forest** is a graph in which every connected component is a Gallai tree.

Theorem 2.4.7 (Borodin [Bor79]; Erdős–Rubin–Taylor [ERT79, Theorem, p. 142])**.** *Let G be a connected graph and let L be a degree list assignment for G. If G is not L-colorable, then G is a Gallai tree; furthermore, $|L(u)| = \deg_G(u)$ for all $u \in V(G)$, and if $u, v \in V(G)$ are two adjacent non-cut vertices, then $L(u) = L(v)$.*

Theorem 2.4.7 provides some useful information about the structure of critical graphs:

Corollary 2.4.8. *Let $k \geqslant 3$ and let G be a $(k+1)$-critical graph. Set*

$$D \coloneqq \{u \in V(G) \,:\, \deg_G(u) = k\}.$$

Then $G[D]$ is a Gallai forest.

Corollary 2.4.8 was originally proved by Gallai [Gal63] using a different method. It is crucial for the proof of Gallai's Theorem 2.4.5.

The definition of critical graphs can be naturally extended to list colorings. A graph G is said to be L**-critical**, where L is a list assignment for G, if G is not L-colorable but for any $u \in V(G)$, the graph $G - u$ is L-colorable. Note that if we set $L(u) \coloneqq \{0, 1, \ldots, k-1\}$ for all $u \in V(G)$, then G being L-critical is equivalent to it being $(k+1)$-critical. Repeating the argument used to prove Corollary 2.4.8, we obtain the following more general statement:

Corollary 2.4.9 (Kostochka–Stiebitz–Wirth [KSW96, Theorem 5])**.** *Let $k \geqslant 3$ and let G be a graph. Suppose that L is a k-list assignment for G such that G is L-critical. Set*

$$D \coloneqq \{u \in V(G) \,:\, \deg_G(u) = k\}.$$

Then $G[D]$ is a Gallai forest.

Corollary 2.4.9 can be used to prove a version of Gallai's theorem for list-critical graphs:

Theorem 2.4.10 (Kostochka–Stiebitz–Wirth [KSW96, Theorem 6])**.** *Let $k \geqslant 3$. Let G be a graph distinct from K_{k+1} and let L be a k-list assignment for G such that G is L-critical. Set $n \coloneqq |V(G)|$ and $m \coloneqq |E(G)|$. Then*

$$2m \geqslant \left(k + \frac{k-2}{k^2+2k-2}\right) n.$$

On the other hand, list-critical graphs distinct from K_{k+1} do not, in general, admit a nontrivial lower bound on the difference $2m - kn$ that only depends on k (analogous to the one given by Dirac's Theorem 2.4.2 for $(k+1)$-critical graphs). Consider the following example, presented in [KS02, p. 167]. Fix $k \in \mathbb{N}$ and let G be the graph with vertex set $\{a_0, \ldots, a_k, b_0, \ldots, b_k\}$ of size $2(k+1)$ and edge set $\{a_i a_j, b_i b_j \,:\, i \neq j\} \cup \{a_0 b_0\}$. For each $i \in [k]$, let $L(a_i) = L(b_i) \coloneqq \{0, 1, \ldots, k-1\}$, and let $L(a_0) = L(b_0) \coloneqq \{1, 2, \ldots, k\}$. Then G is L-critical; however, $2|E(G)| - k|V(G)| = 2$.

Nonetheless, Theorem 2.4.2 can be extended to the list coloring framework if we restrict our attention to graphs that do not contain K_{k+1} as a *subgraph*:

Theorem 2.4.11 (Kostochka–Stiebitz [KS02, Theorem 2])**.** *Let $k \geqslant 3$. Let G be a graph and let L be a k-list assignment for G such that G is L-critical. Suppose that G does not contain a clique of size $k+1$. Set $n := |V(G)|$ and $m := |E(G)|$. Then*

$$2m \geqslant kn + k - 2.$$

Kostochka and Stiebitz [KS02, Section 4] asked whether the conclusion of Theorem 2.4.4 also holds for list critical graphs with no K_{k+1} as a subgraph. We answer this question in the affirmative; see Corollary 2.4.17.

DP-critical graphs and the results of this section

A cover $\mathscr{H} = (L, H)$ of a graph G is a **degree cover** if $|L(u)| \geqslant \deg_G(u)$ for all $u \in V(G)$. In §2.4.2 we establish the following generalization of Theorem 2.4.7 (see Theorem 2.4.21):

Definition 2.4.12. A **GDP-tree** is a connected graph in which every block is either a clique or a cycle. A **GDP-forest** is a graph in which every connected component is a GDP-tree.

Theorem 2.4.13. *Let G be a connected graph and let $\mathscr{H} = (L, H)$ be a degree cover of G. If G is not $\mathscr{H}$-colorable, then G is a GDP-tree; furthermore, $|L(u)| = \deg_G(u)$ for all $u \in V(G)$, and if $u, v \in V(G)$ are two adjacent non-cut vertices, then $E_H(L(u), L(v))$ is a perfect matching.*

Let G be a graph and let $\mathscr{H} = (L, H)$ be a cover of G. We say that G is $\mathscr{H}$**-critical** if G is not $\mathscr{H}$-colorable but for any $u \in V(G)$, the graph $G - u$ is $\mathscr{H}$-colorable, i.e., there exists an independent set $I \subseteq V(H)$ such that $I \cap L(v) \neq \varnothing$ for all $v \neq u$. Theorem 2.4.13 implies the following:

Corollary 2.4.14. *Let $k \geqslant 3$ and let G be a graph. Suppose that $\mathscr{H}$ is a k-fold cover of G such that G is $\mathscr{H}$-critical. Set*

$$D := \{u \in V(G) \,:\, \deg_G(u) = k\}.$$

Then $G[D]$ is a GDP-forest.

Corollary 2.4.14 yields an extension of Gallai's theorem to DP-critical graphs:

Theorem 2.4.15. *Let $k \geqslant 3$. Let G be a graph distinct from K_{k+1} and let $\mathscr{H}$ be a k-fold cover of G such that G is $\mathscr{H}$-critical. Set $n := |V(G)|$ and $m := |E(G)|$. Then*

$$2m \geqslant \left(k + \frac{k-2}{k^2 + 2k - 2}\right) n.$$

The derivation of Theorem 2.4.15 from Corollary 2.4.14 is essentially the same as the derivation of Theorem 2.4.5 from Corollary 2.4.8. For completeness, we include the proof of Theorem 2.4.15 in §2.4.3.

The main result of this section is a generalization of Theorem 2.4.11 to DP-critical graphs. In fact, we establish a sharp version that also generalizes Theorem 2.4.4:

Theorem 2.4.16. *Let $k \geqslant 3$. Let G be a graph and let $\mathscr{H}$ be a k-fold cover of G such that G is $\mathscr{H}$-critical. Suppose that G does not contain a clique of size $k+1$. Set $n := |V(G)|$ and $m := |E(G)|$. If $G \notin \mathbf{Dir}_k$, then*

$$2m > kn + k - 2.$$

An immediate corollary of Theorem 2.4.16 is the following version of Theorem 2.4.4 for list colorings:

Corollary 2.4.17. *Let $k \geqslant 3$. Let G be a graph and let L be a k-list assignment for G such that G is L-critical. Suppose that G does not contain a clique of size $k+1$. Set $n \coloneqq |V(G)|$ and $m \coloneqq |E(G)|$. If $G \notin \mathbf{Dir}_k$, then*

$$2m > kn + k - 2.$$

We prove Theorem 2.4.16 in §2.4.4. Our proof is inductive (we consider a smallest counterexample). As often is the case, having a stronger inductive assumption (due to working with DP-critical and not just list-critical graphs) allows for more flexibility in the proof. In particular, we do not know if our argument can be adapted to give a "DP-free" proof of Corollary 2.4.17.

2.4.2 DP-degree-colorable multigraphs

In this subsection we prove Theorem 2.4.13. In fact, we do more and establish a version of Theorem 2.4.13 for *multigraphs*. It has already been noted in §2.2.4 that the notion of DP-coloring can be naturally extended to loopless multigraphs by replacing (C4) in Definition 2.1.1 by the following:

(C4′) If u and v are connected by $t \geqslant 1$ edges in G, then $E_H(L(u), L(v))$ is a union of t matchings.

We say that a multigraph G is **DP-degree-colorable** if G is $\mathscr{H}$-colorable whenever $\mathscr{H}$ is a degree cover of G. For a positive integer k and a multigraph G, let G^k denote the multigraph obtained from G by replacing each edge in G with a set of k parallel edges (so $G^1 = G$ for every G). The next two lemmas demonstrate two classes of multigraphs that are not DP-degree-colorable. The first of them exhibits multigraphs whose DP-chromatic number exceeds the number of vertices. In particular, for each $k \geqslant 2$, the 2-vertex multigraph K_2^k has DP-chromatic number $k+1$.

Lemma 2.4.18. *The multigraph K_n^k is not DP-degree-colorable.*

Proof. Let $G \coloneqq K_n^k$. For each $v \in V(G)$, let

$$L(v) \coloneqq \{(v, i, j) \,:\, i < n-1,\ j < k\},$$

and let

$$(v_1, i_1, j_1)(v_2, i_2, j_2) \in E(H) \;:\Longleftrightarrow\; v_1 = v_2 \text{ or } i_1 = i_2.$$

Then $\mathscr{H} \coloneqq (L, H)$ is a cover of G and $|L(v)| = k(n-1) = \deg_G(v)$ for all $v \in V(G)$. We claim that G is not $\mathscr{H}$-colorable. Indeed, if $I \subseteq V(H)$ is such that $|I \cap L(v)| = 1$ for all $v \in V$, then for some distinct (v_1, i_1, j_1), $(v_2, i_2, j_2) \in I$, we have $i_1 = i_2$. Thus, I is not an independent set. ∎

Lemma 2.4.19. *The multigraph C_n^k is not DP-degree-colorable.*

Proof. Let $G \coloneqq C_n^k$. Without loss of generality, assume that $V(G) = \{0, 1, \ldots, n-1\}$ and u and v are adjacent if and only if $|u - v| = 1$ or $\{u, v\} = \{0, n-1\}$. For each $v \in V(G)$, let

$$L(v) \coloneqq \{(v, i, j) \,:\, i < 2,\ j < k\},$$

and let

$$(v_1, i_1, j_1)(v_2, i_2, j_2) \in E(H) \ :\Longleftrightarrow \begin{cases} v_1 = v_2; \text{ or} \\ |v_1 - v_2| = 1 \text{ and } i_1 = i_2; \text{ or} \\ \{v_1, v_2\} = \{0, n-1\} \text{ and } i_1 = i_2 + n - 1 \pmod 2. \end{cases}$$

Then $\mathscr{H} := (L, H)$ is a cover of G and $|L(v)| = 2k = \deg_G(v)$ for all $v \in V(G)$. We claim that G is not $\mathscr{H}$-colorable. Indeed, suppose that $I = \{(v, i_v, j_v)\}_{v=0}^{n-1} \subset V(H)$ is an $\mathscr{H}$-coloring of G. Without loss of generality, we may assume that $i_0 = 0$. Then for each $v \in V(G)$, we have $i_v = v \pmod 2$. Thus,

$$i_0 = i_{n-1} + n - 1 \pmod 2,$$

and so $(1, i_0, j_0)(n-1, i_{n-1}, j_{n-1}) \in E(H)$. Therefore, I is not independent. ∎

The main result of this subsection is that the above lemmas describe all 2-connected multigraphs that are not DP-degree-colorable (this is a generalization of Theorem 2.4.13):

Definition 2.4.20. A **multi-GDP-tree** is a connected multigraph in which every block is isomorphic to one of the graphs K_n^k, C_n^k for some n and k.

Theorem 2.4.21. *Let G be a connected multigraph and let $\mathscr{H} = (L, H)$ be a degree cover of G. If G is not $\mathscr{H}$-colorable, then G is a multi-GDP-tree; furthermore, $|L(u)| = \deg_G(u)$ for all $u \in V(G)$, and if u, $v \in V(G)$ are two adjacent non-cut vertices, then $E_H(L(u), L(v))$ is a union of $e_G(u, v)$ perfect matchings.*

Proof of Theorem 2.4.21

We proceed via a series of lemmas.

Lemma 2.4.22. *Suppose that G is a regular n-vertex multigraph whose underlying simple graph is a cycle. Then G is not DP-degree-colorable if and only if $G \cong C_n^k$ for some k.*

Proof. Without loss of generality, assume that $V(G) = \{0, \ldots, n-1\}$ and and u and v are adjacent if and only if $|u - v| = 1$ or $\{u, v\} = \{0, n-1\}$. Suppose that $G \not\cong C_n^k$. Since G is regular, this implies that n is even and for some distinct positive $r, s \in \mathbb{N}$, we have $e_G(v, v+1) = r$ for all even $v < n$ and $e_G(0, n-1) = e_G(v, v+1) = s$ for all odd $v < n-1$. Without loss of generality, assume $s > r$.

Let $\mathscr{H} = (L, H)$ be a cover of G such that $|L(v)| = \deg_G(v) = r + s$ for all $v \in V(G)$. We will show that G is $\mathscr{H}$-colorable. For $x \in L(0)$, say that a color $y \in L(v)$ is ***x*-admissible** if there exists a set $I \subseteq V(H)$ that is independent in $H - E_H(L(0), L(n-1))$ such that $|I \cap L(u)| = 1$ for all $u \leqslant v$ and $\{x, y\} \subseteq I$. Let $A_x(v) \subseteq L(v)$ denote the set of all x-admissible colors in $L(v)$. Clearly, for each $x \in L(0)$, $|A_x(1)| \geqslant s$ and $|A_x(2)| \geqslant r$. Suppose that for some $x \in L(0)$, $|A_x(2)| > r$. Since each color in $L(3)$ has at most r neighbors in $L(2)$, $A_x(3) = L(3)$. Similarly, $A_x(v) = L(v)$ for all $v \geqslant 3$. In particular, $A_x(n-1) = L(n-1)$. Take any $y \in L(n-1) \setminus N_H(x)$. Since $y \in A_x(n-1)$, there exists a set $I \subseteq V(H)$ that is independent in $H - E_H(L(0), L(n-1))$ such that $|I \cap L(u)| = 1$ for all $u \in V(G)$ and $\{x, y\} \subseteq I$. But then I is independent in

H, so I is an $\mathscr{H}$-coloring of G. Thus, we may assume that $|A_x(2)| = r$ for all $x \in L(0)$. Note that

$$L(2) \setminus A_x(2) = L(2) \cap \bigcap_{y \in A_x(1)} N_H(y).$$

Therefore, $L(2) \cap N_H(y)$ is the same set of size s for all $y \in A_x(1)$. Since each vertex in $L(2)$ has at most s neighbors in $L(1)$, the graph $H[A_x(1) \cup (L(2) \setminus A_x(2))]$ is a complete $2s$-vertex graph. Since every vertex in $L(1)$ is x-admissible for some $x \in L(0)$, $H[L(1) \cup L(2)]$ contains a disjoint union of at least two complete $2s$-vertex graphs. Therefore, $|L(1) \cup L(2)| \geqslant 4s$. But $|L(1)| = |L(2)| = r + s < 2s$; a contradiction. ∎

Lemma 2.4.23. *Let G be a connected multigraph and suppose $\mathscr{H} = (L, H)$ is a degree cover of G such that $|L(v_0)| > \deg_G(v_0)$ for some $v_0 \in V(G)$. Then G is $\mathscr{H}$-colorable.*

Proof. If $|V(G)| = 1$, the statement is clear. Now suppose G is a counterexample with the fewest vertices. Consider the multigraph $G' := G - v_0$. For each $v \in V(G')$, let $L'(v) := L(v)$, and let $H' := H - L(v_0)$. By construction, $\mathscr{H}' := (L', H')$ is a degree cover of G'. Moreover, since G is connected, each connected component of G' contains a vertex u adjacent in G to v_0 and thus satisfying $\deg_{G'}(u) < \deg_G(u)$. Hence, by the minimality assumption, G' is $\mathscr{H}'$-colorable. Let $I' \subseteq V(H')$ be an $\mathscr{H}'$-coloring of G'. Then $|N_G(I') \cap L(v_0)| \leqslant \deg_G(v_0)$, so $L(v_0) \setminus N_G(I') \neq \varnothing$. Thus, I' can be extended to an $\mathscr{H}$-coloring I of G. ∎

Lemma 2.4.24. *Let G be a connected multigraph and let $\mathscr{H} = (L, H)$ be a degree cover of G. Suppose that there exist a vertex $v_1 \in V(G)$ and a color $x_1 \in L(v_1)$ such that $G - v_1$ is connected and for some $v_2 \in V(G) \setminus \{v_1\}$, x_1 has fewer than $e_G(v_1, v_2)$ neighbors in $L(v_2)$. Then G is $\mathscr{H}$-colorable.*

Proof. Let $G' := G - v_1$. For each $v \in V(G')$, let $L'(v) := L(v) \setminus N_H(x_1)$ and $H' := H - L(v_1) - N_H(x_1)$. Then $\mathscr{H}' := (L', H')$ is a cover of G'. Moreover, for each $v \in V(G')$,

$$|L'(v)| = |L(v)| - |L(v) \cap N_H(x_1)| \geqslant \deg_G(v) - e_G(v, v_1) = \deg_{G'}(v),$$

and

$$|L'(v_2)| = |L(v_2)| - |L(v_2) \cap N_H(x_1)| > \deg_G(v_2) - e_G(v_2, v_1) = \deg_{G'}(v_2).$$

Since G' is connected, Lemma 2.4.23 implies that G' is $\mathscr{H}'$-colorable. But if $I' \subseteq V(H')$ is an $\mathscr{H}'$-coloring of G', then $I' \cup \{x_1\}$ is an $\mathscr{H}$-coloring of G, as desired. ∎

Lemma 2.4.25. *Suppose that G is a 2-connected multigraph and $\mathscr{H} = (L, H)$ is a degree cover of G. If G is not (L, H)-colorable, then G is regular and for each pair of adjacent vertices v_1, $v_2 \in V(G)$, the bipartite graph $H[L(v_1), L(v_2)]$ is $e_G(v_1, v_2)$-regular.*

Proof. Consider any two adjacent $v_1, v_2 \in V(G)$. By Lemma 2.4.24, $H[L(v_1), L(v_2)]$ is an $e_G(v_1, v_2)$-regular bipartite graph with parts $L(v_1)$, $L(v_2)$. Therefore, $|L(v_1)| = |L(v_2)|$, so $\deg_G(v_1) = \deg_G(v_2)$, as desired. Since G is connected and v_1, v_2 are arbitrary adjacent vertices in G, this means that G is regular. ∎

Lemma 2.4.26. *Let G be a 2-connected multigraph. Suppose that u_1, u_2, $w \in V(G)$ are distinct vertices such that $G - u_1 - u_2$ is connected, $e_G(u_1, u_2) < e_G(u_1, w)$, and $e_G(u_2, w) \geqslant 1$. Then G is DP-degree-colorable.*

Proof. Suppose G is not $\mathscr{H}$-colorable for some degree cover $\mathscr{H} = (L, H)$. First we show that

$$\text{there are nonadjacent } x_1 \in L(u_1),\ x_2 \in L(u_2) \text{ with } N_H(x_1) \cap N_H(x_2) \cap L(w) \neq \varnothing. \tag{2.4.3}$$

Consider any $x_2 \in L(u_2)$. By Lemma 2.4.25, $|L(w) \cap N_H(x_2)| = e_G(u_2, w) \geqslant 1$. Similarly, for each $y \in L(w) \cap N_H(x_2)$,

$$|L(u_1) \cap N_H(y)| = e_G(u_1, w) > e_G(u_1, u_2) = |L(u_1) \cap N_H(x_2)|.$$

Thus, there exists some $x_1 \in (L(u_1) \cap N_H(y)) \setminus (L(u_1) \cap N_H(x_2))$. By the choice, x_1 and x_2 are nonadjacent and $y \in N_H(x_1) \cap N_H(x_2) \cap L(w)$. This proves (2.4.3).

Let x_1 and x_2 satisfy (2.4.3). Let $G' \coloneqq G - u_1 - u_2$. For each $v \in V(G')$, let

$$L'(v) \coloneqq L(v) \setminus (N_H(x_1) \cup N_H(x_2)),$$

and

$$H' \coloneqq H - L(u_1) - L(u_2) - N_H(x_1) - N_H(x_2).$$

Then G' is connected and $\mathscr{H}' \coloneqq (L', H')$ is a cover of G' satisfying the conditions of Lemma 2.4.23 with w in the role of v_0. Thus G' is $\mathscr{H}'$-colorable, and hence G is $\mathscr{H}$-colorable, which is a contradiction. ∎

Lemma 2.4.27. *Suppose that G is an n-vertex 2-connected multigraph that contains a vertex adjacent to all the other vertices. Then either $G \cong K_n^k$ for some k, or else, G is DP-degree-colorable.*

Proof. Suppose that G is an n-vertex multigraph that is not DP-degree-colorable and assume that $w \in V(G)$ is adjacent to all the other vertices. If some distinct $u_1, u_2 \in V(G) \setminus \{w\}$ are nonadjacent, then the triple u_1, u_2, w satisfies the conditions of Lemma 2.4.26, so G is DP-degree-colorable. Hence any two vertices in G are adjacent; in other words, the underlying simple graph of G is K_n. It remains to show that any two vertices in G are connected by the same number of edges. Indeed, if $u_1, u_2, u_3 \in V(G)$ are such that $e_G(u_1, u_2) < e_G(u_1, u_3)$, then, by Lemma 2.4.26 again, G is DP-degree-colorable. ∎

Lemma 2.4.28. *Suppose that G is a 2-connected n-vertex multigraph in which each vertex has at most 2 neighbors. Then either $G \cong C_n^k$ for some k, or else, G is DP-degree-colorable.*

Proof. Suppose that G is a 2-connected n-vertex multigraph in which each vertex has at most 2 neighbors and that is not DP-degree-colorable. Then the underlying simple graph of G is a cycle and Lemma 2.4.25 implies that G is regular, so $G \cong C_n^k$ by Lemma 2.4.22. ∎

Lemma 2.4.29. *Suppose that G is a 2-connected n-vertex multigraph that is not DP-degree-colorable. Then $G \cong K_n^k$ or C_n^k for some k.*

Proof. By Lemmas 2.4.27 and 2.4.28, we may assume that G contains a vertex u such that $3 \leqslant |N_G(u)| \leqslant n-2$. Since G is 2-connected, $G - u$ is connected. However, $G - u$ is not 2-connected. Indeed, let v_1 be any vertex in

$V(G) \setminus (\{u\} \cup N_G(u))$ that shares a neighbor w with u. Due to Lemma 2.4.26 with u in place of v_2, $G - v_1 - u$ is disconnected, so v_1 is a cut vertex in $G - u$.

Therefore, $G - u$ contains at least two leaf blocks, say B_1 and B_2. For $i \in \{1, 2\}$, let x_i be the cut vertex of $G - u$ contained in B_i. Since G itself is 2-connected, u has a neighbor $v_i \in B_i - x_i$ for each $i \in \{1, 2\}$. Then v_1 and v_2 are nonadjacent and $G - u - v_1 - v_2$ is connected. Since u has at least 3 neighbors, $G - v_1 - v_2$ is also connected. Hence, we are done by Lemma 2.4.26 with u in the role of w. ■

Lemma 2.4.30. *Suppose that $w \in V(G)$, $G = G_1 + G_2$, and $V(G_1) \cap V(G_2) = \{w\}$. If G_1 and G_2 are not DP-degree-colorable, then G is also not DP-degree-colorable.*

Proof. For each $i \in \{1, 2\}$, let $\mathscr{H}_i = (L_i, H_i)$ be a degree cover of G_i such that G_i is not $\mathscr{H}_i$-colorable. Without loss of generality, assume that $L_1(v_1) \cap L_2(v_2) = \varnothing$ for all $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. For each $v \in V(G)$, let

$$L(v) \coloneqq \begin{cases} L_1(v) & \text{if } v \in V(G_1) \setminus \{w\}; \\ L_2(v) & \text{if } v \in V(G_2) \setminus \{w\}; \\ L_1(w) \cup L_2(w) & \text{if } v = w, \end{cases}$$

and let $H \coloneqq H_1 + H_2 + K(L(w))$, where $K(L(w))$ denotes the complete graph with vertex set $L(w)$. Then $\mathscr{H} \coloneqq (L, H)$ is a degree cover of G. Suppose that G is $\mathscr{H}$-colorable and let I be an $\mathscr{H}$-coloring of G. Without loss of generality, assume $I \cap L(w) \subseteq L_1(w)$. Then $I \cap V(H_1)$ is an $\mathscr{H}_1$-coloring of G_1; a contradiction. ■

Proof of Theorem 2.4.21. Lemmas 2.4.18, 2.4.19, and 2.4.30 show that if G is a multi-GDP-tree, then G is not DP-degree-colorable.

Now assume that G is a connected multigraph that is not DP-degree-colorable. If G is 2-connected, then we are done by Lemma 2.4.29. Therefore, we may assume that G has a cut vertex $w \in V(G)$. Let G_1 and G_2 be nontrivial connected subgraphs of G such that $G = G_1 + G_2$ and $V(G_1) \cap V(G_2) = \{w\}$. It remains to show that neither G_1 nor G_2 is DP-degree-colorable, since then we will be done by induction. Suppose towards a contradiction that G_1 is DP-degree-colorable. Let $\mathscr{H} = (L, H)$ be a degree cover of G. Due to Lemma 2.4.23 applied to the connected components of $G_2 - w$, there exists an $\mathscr{H}$-coloring I_2 of $G_2 - w$. For each $v \in V(G_1)$, let

$$L_1(v) \coloneqq L(v) \setminus N_H(I_2).$$

(Note that $L_1(v) = L(v)$ for all $v \in V(G_1) \setminus \{w\}$.) Also, let

$$H_1 \coloneqq H\left[\bigcup_{v \in V(G_1)} L_1(v)\right].$$

Then $\mathscr{H}_1 \coloneqq (L_1, H_1)$ is a degree cover of G_1. Since G_1 is DP-degree-colorable, it is $\mathscr{H}_1$-colorable. But if I_1 is an $\mathscr{H}_1$-coloring of G_1, then $I_1 \cup I_2$ is an $\mathscr{H}$-coloring of G.

The "furthermore" part of the theorem is a direct consequence of Lemmas 2.4.23 and 2.4.24. ■

2.4.3 Gallai's theorem for DP-critical graphs

In this subsection we prove Theorem 2.4.15. The lemma below was proved for Gallai trees by Gallai himself [Gal63]; the same proof works for GDP-trees as well.

Lemma 2.4.31 (ess. Gallai [Gal63])**.** *Let $k \geqslant 3$ and let T be an n-vertex GDP-tree of maximum degree at most k not containing K_{k+1}. Set $n \coloneqq |V(T)|$ and $m \coloneqq |E(T)|$. Then*

$$2m \leqslant \left(k - 1 + \frac{2}{k}\right) n. \tag{2.4.4}$$

To establish Theorem 2.4.15, we use discharging. Let G be an n-vertex graph with m edges distinct from K_{k+1} and let $\mathscr{H}$ be a k-fold cover of G such that G is $\mathscr{H}$-critical. Note that the minimum degree of G is at least k. The initial charge of each vertex $v \in V(G)$ is $\mathrm{ch}(v) \coloneqq \deg_G(v)$. The only discharging rule is this:

Each vertex $v \in V(G)$ with $\deg_G(v) \geqslant k+1$ sends to each neighbor the charge $k/(k^2+2k-2)$.

Denote the new charge of each vertex v by $\mathrm{ch}^*(v)$. We will show that

$$\sum_{v \in V(G)} \mathrm{ch}^*(v) \geqslant \left(k + \frac{k-2}{k^2+2k-2}\right) n. \tag{2.4.5}$$

Indeed, if $\deg_G(v) \geqslant k+1$, then

$$\mathrm{ch}^*(v) \geqslant \deg_G(v) - \frac{k}{k^2+2k-2} \cdot \deg_G(v) \geqslant (k+1)\left(1 - \frac{k}{k^2+2k-2}\right) = k + \frac{k-2}{k^2+2k-2}. \tag{2.4.6}$$

Also, if T is any component of the subgraph G' of G induced by the vertices of degree k, then

$$\sum_{v \in V(T)} \mathrm{ch}^*(v) \geqslant k|V(T)| + \frac{k}{k^2+2k-2} \left|E_G(V(T), V(G) \setminus V(T))\right|.$$

Since T is a GDP-tree that does not contain K_{k+1}, by Lemma 2.4.31,

$$|E_G(V(T), V(G) \setminus V(T))| \geqslant k|V(T)| - \left(k - 1 + \frac{2}{k}\right)|V(T)| = \left(1 - \frac{2}{k}\right)|V(T)|.$$

Thus, for every component T of G', we have

$$\sum_{v \in V(T)} \mathrm{ch}^*(v) \geqslant k|V(T)| + \frac{k}{k^2+2k-2} \cdot \left(1 - \frac{2}{k}\right) \cdot |V(T)| = \left(k + \frac{k-2}{k^2+2k-2}\right)|V(T)|.$$

Together with (2.4.6), this implies (2.4.5).

2.4.4 Sharp Dirac's theorem for DP-critical graphs

In this subsection we prove Theorem 2.4.16.

First observations

Set-up and notation From now on, we fix a counterexample to Theorem 2.4.16; more precisely, we fix the following data:

- an integer $k \geqslant 3$;
- a graph G with n vertices and m edges such that:
  - $G \notin \mathbf{Dir}_k$;
  - G does not contain a clique of size $k+1$; and
  - G satisfies the inequality
$$2m \leqslant kn + k - 2; \tag{2.4.7}$$
- a k-fold cover $\mathscr{H} = (L, H)$ of G such that G is $\mathscr{H}$-critical.

Furthermore, we assume that G is a counterexample with the fewest vertices.

For brevity, we denote $V := V(G)$ and $E := E(G)$. As usual, for a subset $U \subseteq V$, we use U^{c} to denote the complement of U in V, i.e., $U^{\mathsf{c}} := V \setminus U$. For $u \in V$ and $U \subseteq V$, set

$$\deg(u) := \deg_G(u) \qquad \text{and} \qquad \deg_U(u) := |U \cap N_G(u)|.$$

For $u \in V$, set
$$\varepsilon(u) := \deg(u) - k,$$
and for $U \subseteq V$, define
$$\varepsilon(U) := \sum_{u \in U} \varepsilon(u).$$
Note that (2.4.7) is equivalent to
$$\varepsilon(V) \leqslant k - 2. \tag{2.4.8}$$
Since G is $\mathscr{H}$-critical, we have $\delta(G) \geqslant k$, i.e., $\varepsilon(u) \geqslant 0$ for all $u \in V$. Let

$$D := \{u \in V \,:\, \deg(u) = k\} = \{u \in V \,:\, \varepsilon(u) = 0\}.$$

Since $\varepsilon(u) \geqslant 1$ for every $u \in D^{\mathsf{c}}$, (2.4.8) yields

$$|D^{\mathsf{c}}| \leqslant k - 2.$$

By Corollary 2.4.14, $G[D]$ is a GDP-forest. Furthermore, since $n \geqslant k+1$, $D \neq \varnothing$.

From now on, we refer to the vertices of H as **colors** and to the independent sets in H as **colorings**. The set of all independent sets in H is denoted by $\mathbf{Ind}(H)$. For I, $I' \in \mathbf{Ind}(H)$, we say that I' **extends** I if $I' \supseteq I$. For $I \in \mathbf{Ind}(H)$, let
$$\operatorname{dom}(I) := \{u \in V \,:\, I \cap L(u) \neq \varnothing\}.$$

Since G is $\mathscr{H}$-critical, there is no coloring I with $\mathrm{dom}(I) = V$, but for every proper subset $U \subset V$, there exists a coloring I with $\mathrm{dom}(I) = U$.

For $I \in \mathbf{Ind}(H)$ and $u \in (\mathrm{dom}(I))^{\mathsf{c}}$, let

$$L_I(u) := L(u) \setminus N_H(I).$$

In other words, $L_I(u)$ is the set of all colors available for u in a coloring extending I. For $u \in V$ and $U \subseteq V$, let

$$\varphi_U(u) := \deg_U(u) - \varepsilon(u).$$

In particular, if $u \in D$, then $\varphi_U(u) = \deg_U(u)$. Note that

$$\varphi_U(u) = \deg_U(u) - (\deg(u) - k) = k - (\deg(u) - \deg_U(u)) = k - \deg_{U^{\mathsf{c}}}(u).$$

Therefore, if I is a coloring such that $\mathrm{dom}(I) = U^{\mathsf{c}}$, then for all $u \in U$,

$$|L_I(u)| \geqslant \varphi_U(u). \tag{2.4.9}$$

A property of GDP-forests The following simple general property of GDP-forests will be quite useful:

Proposition 2.4.32. *Let F be a nonempty GDP-forest of maximum degree at most k not containing a clique of size $k+1$. Then*

$$\sum_{u \in V(F)} (k - \deg_F(u)) \geqslant k, \tag{2.4.10}$$

with equality only if $F \cong K_1$ or $F \cong K_k$.

Proof. It suffices to establish the proposition for the case when F is connected, i.e., a GDP-tree. If F is 2-connected, i.e., a clique or a cycle, then the statement follows via a simple calculation. It remains to notice that adding a leaf block to a GDP-tree of maximum degree at most k cannot decrease the quantity on the left-hand side of (2.4.10). ■

Corollary 2.4.33. *Let $U \subseteq D$ be the vertex set of a connected component of $G[D]$. Then*

$$|E_G(U, D^{\mathsf{c}})| \geqslant k,$$

with equality only if $G[U] \cong K_k$.

Proof. We have

$$|E_G(U, D^{\mathsf{c}})| = \sum_{u \in U} \deg_{D^{\mathsf{c}}}(u) = \sum_{u \in U} \deg_{U^{\mathsf{c}}}(u) = \sum_{u \in U}(k - \deg_U(u)).$$

By Proposition 2.4.32 applied to $G[U]$, the latter quantity is at least k, with equality only if $G[U] \cong K_1$ or $G[U] \cong K_k$. It remains to notice that $G[U] \not\cong K_1$, since $\deg(u) = k$ for each $u \in U$, while $|D^{\mathsf{c}}| \leqslant k - 2$. ■

Enhanced vertices The following definition will play a crucial role in our argument:

Definition 2.4.34. Let I be a coloring and let $U \coloneqq (\mathrm{dom}(I))^{\mathsf{c}}$. A vertex $u \in U \cap D$ is **enhanced** by I, or I **enhances** u, if $|L_I(u)| > \deg_U(u)$.

Remark. Note that, in the context of Definition 2.4.34, we always have $|L_I(u)| \geqslant \deg_U(u)$.

The importance of Definition 2.4.34 stems from the following lemma:

Lemma 2.4.35. *Let I be a coloring and let $U \coloneqq (\mathrm{dom}(I))^{\mathsf{c}}$.*

(i) *Suppose that I' is a coloring extending I. Let $u \in (\mathrm{dom}(I'))^{\mathsf{c}} \cap D$. If u is enhanced by I, then it is also enhanced by I'.*

(ii) *Let $U' \subseteq U \cap D$ be a subset such that the graph $G[U']$ is connected. Suppose that U' contains a vertex enhanced by I. Then I can be extended to a coloring I' with $\mathrm{dom}(I') = U^{\mathsf{c}} \cup U'$.*

(iii) *Suppose that I enhances at least one vertex in each component of $G[U \cap D]$. Then I cannot be extended to a coloring I' with $\mathrm{dom}(I') \supseteq D^{\mathsf{c}}$.*

Proof. Since (i) is an immediate corollary of the definition and (ii) follows from Theorem 2.4.13, it only remains to prove (iii). To that end, suppose, under the assumptions of (iii), that I' is a coloring extending I with $\mathrm{dom}(I') \supseteq D^{\mathsf{c}}$. Reducing I' if necessary, we may arrange that $\mathrm{dom}(I') = U^{\mathsf{c}} \cup D^{\mathsf{c}}$. Then, by (i), I' enhances at least one vertex in each component of $G[U \cap D]$. Applying (ii) to each connected component of $G[U \cap D]$, we can extend I' to a coloring of the entire graph G; a contradiction. ■

The next lemma gives a convenient sufficient condition under which a given coloring can be extended so that the resulting coloring enhances a particular vertex:

Lemma 2.4.36. *Let I be a coloring and let $U \coloneqq (\mathrm{dom}(I))^{\mathsf{c}}$. Let $u \in U \cap D$ and suppose that $A \subseteq U \cap N_G(u)$ is an independent set in G. Moreover, suppose that*

$$\min\{\varphi_U(v) \,:\, v \in A\} > 0 \qquad \text{and} \qquad \sum_{v \in A} \varphi_U(v) > \deg_U(u).$$

Then there is a coloring I' with $\mathrm{dom}(I') = U^{\mathsf{c}} \cup A$ that extends I and enhances u.

Proof. Since A is independent and for all $v \in A$, we have $\varphi_U(v) > 0$ (and hence, by (2.4.9), $|L_I(v)| > 0$), any coloring I' with $\mathrm{dom}(I') \subseteq U^{\mathsf{c}} \cup A$ can be extended to a coloring with domain $U^{\mathsf{c}} \cup A$. Therefore, it suffices to find a coloring that extends I and enhances u and whose domain is contained in $U^{\mathsf{c}} \cup A$.

If u is enhanced by I itself, then we are done, so assume that $|L_I(u)| = \deg_U(u)$. If for some $v \in A$, there is $x \in L_I(v)$ with no neighbor in $L_I(u)$, then u is enhanced by $I \cup \{x\}$, and we are done again. Thus, we may assume that for every $v \in A$, the matching $E_H(L_I(v), L_I(u))$ saturates $L_I(v)$. For each $v \in A$ and $x \in L_I(v)$, let $f(x)$ denote the neighbor of x in $L_I(u)$. Since $\sum_{v \in A} \varphi_U(v) > \deg_U(u)$, and hence, by (2.4.9), $\sum_{v \in A} |L_I(v)| > |L_I(u)|$, there exist distinct vertices $v, w \in A$ and colors $x \in L_I(v)$, $y \in L_I(w)$ such that $f(x) = f(y)$. Then u is enhanced by the coloring $I \cup \{x, y\}$, and the proof is complete. ■

Corollary 2.4.37. *Suppose that u, u_1, $u_2 \in D$ are distinct vertices such that uu_1, $uu_2 \in E$, while $u_1u_2 \notin E$. Then the graph $G[D] - u_1 - u_2$ is disconnected.*

Proof. Note that, since $u, u_1, u_2 \in D$, we have

$$\varphi_V(u_1) = \varphi_V(u_2) = k \qquad \text{and} \qquad \deg(u) = k,$$

so, by Lemma 2.4.36, there exist $x_1 \in L(u_1)$ and $x_2 \in L(u_2)$ such that u is enhanced by the coloring $\{x_1, x_2\}$. Since for all $v \in D^{\mathsf{c}}$,

$$|L_{\{x_1,x_2\}}(v)| \geqslant |L(v)| - |\{x_1, x_2\}| = k - 2 \geqslant |D^{\mathsf{c}}|,$$

we can extend $\{x_1, x_2\}$ to a coloring I with $\mathrm{dom}(I) = \{u_1, u_2\} \cup D^{\mathsf{c}}$. Due to Lemma 2.4.35(iii), at least one connected component of the graph $G[D] - u_1 - u_2$ contains no vertices enhanced by I. Since, by Lemma 2.4.35(i), I enhances u, $G[D] - u_1 - u_2$ is disconnected, as desired. ■

We will often apply Lemma 2.4.36 in the form of the following corollary:

Corollary 2.4.38. *Suppose that $u \in D$ and let v_1, $v_2 \in D^{\mathsf{c}} \cap N_G(u)$ be distinct vertices such that $v_1v_2 \notin E$. Let $U \subseteq D$ be any set such that $u \in U$ and the graph $G[U]$ is connected. Then*

$$\begin{array}{rll} \textit{either} & \min\{\varphi_U(v_1), \varphi_U(v_2)\} & \leqslant 0, \\ \textit{or} & \varphi_U(v_1) + \varphi_U(v_2) & \leqslant \deg_U(u) + 2. \end{array}$$

Proof. Notice that

$$\varphi_{U\cup\{v_1,v_2\}}(v_i) = \varphi_U(v_i) \text{ for each } i \in \{1, 2\}, \qquad \text{and} \qquad \deg_{U\cup\{v_1,v_2\}}(u) = \deg_U(u) + 2.$$

Therefore, is the claim fails, then we can first fix any coloring I with $\mathrm{dom}(I) = (U \cup \{v_1, v_2\})^{\mathsf{c}}$, and then apply Lemma 2.4.36 to extend it to a coloring I' with $\mathrm{dom}(I') = U^{\mathsf{c}}$ that enhances u. Since $G[U]$ is connected, such a coloring cannot exist by Lemma 2.4.35(iii). ■

The following observation can be viewed as an analog of Lemma 2.4.35(ii) for edges instead of vertices:

Lemma 2.4.39. *Let I be a coloring and let $U := (\mathrm{dom}(I))^{\mathsf{c}}$. Let $U' \subseteq U \cap D$ be a subset such that the graph $G[U']$ is connected and let u_1, $u_2 \in U'$ be adjacent non-cut vertices in $G[U']$. Suppose that the matching $E_H(L_I(u_1), L_I(u_2))$ is not perfect. Then I can be extended to a coloring I' with* $\mathrm{dom}(I') = U^{\mathsf{c}} \cup U'$.

Proof. Follows from Theorem 2.4.13. ■

Vertices of small degree Now we establish some structural properties that G must possess if the minimum degree of the graph $G[D]$ is "small" (namely at most 2).

Lemma 2.4.40. (i) *The minimum degree of $G[D]$ is at least* 2.

(ii) *If there is a vertex $u \in D$ such that* $\deg_D(u) = 2$, *then $|D^{\mathsf{c}}| = k - 2$, u is adjacent to every vertex in D^{c}, and $\varepsilon(v) = 1$ for all $v \in D^{\mathsf{c}}$.*

(iii) *If the graph $G[D]$ has a connected component with at least 3 vertices of degree 2, then $G[D^{\mathsf{c}}]$ is a disjoint union of cliques.*

(iv) *If the graph $G[D]$ has a connected component with at least 4 vertices of degree 2, then $G[D^{\mathsf{c}}] \cong K_{k-2}$.*

Proof. (i) For each $u \in D$, we have

$$k - 2 \geqslant |D^{\mathsf{c}}| \geqslant \deg_{D^{\mathsf{c}}}(u) = k - \deg_D(u),$$

so $\deg_D(u) \geqslant 2$.

(ii) If $u \in D$ and $\deg_D(u) = 2$, then u has exactly $k-2$ neighbors in D^{c}. Thus,

$$\varepsilon(D^{\mathsf{c}}) = |D^{\mathsf{c}}| = k - 2,$$

which implies all the statements in (ii).

(iii) Let $U \subseteq D$ be the vertex set of a connected component of $G[D]$ such that $G[U]$ contains at least 3 vertices of degree 2. Suppose, towards a contradiction, that $G[D^{\mathsf{c}}]$ is not a disjoint union of cliques, i.e., there exist distinct vertices $v_0, v_1, v_2 \in D^{\mathsf{c}}$ such that $v_0v_1, v_0v_2 \in E$, while $v_1v_2 \notin E$. By (ii), each vertex in D^{c} is adjacent to every vertex of degree 2 in $G[D]$, $|D^{\mathsf{c}}| = k - 2$, and $\varepsilon(v) = 1$ for all $v \in D^{\mathsf{c}}$. Thus,

$$\varphi_{U\cup\{v_0,v_1,v_2\}}(v_i) = \deg_{U\cup\{v_0,v_1,v_2\}}(v_i) - \varepsilon(v_i) \geqslant (3+1) - 1 = 3 \quad \text{for each } i \in \{1,2\}.$$

Fix any vertex $u \in U$ such that $\deg_U(u) = 2$. Then

$$\deg_{U\cup\{v_0,v_1,v_2\}}(u) = 2 + 3 = 5.$$

Therefore, by Lemma 2.4.36, there exists a coloring I with domain

$$\mathrm{dom}(I) = (U \cup \{v_0, v_1, v_2\})^{\mathsf{c}} \cup \{v_1, v_2\} = (U \cup \{v_0\})^{\mathsf{c}}$$

that enhances u. By (2.4.9),

$$|L_I(v_0)| \geqslant \varphi_U(v_0) = \deg_U(v_0) - \varepsilon(v_0) \geqslant 3 - 1 = 2 > 0,$$

so I can be extended to a coloring I' with $\mathrm{dom}(I') = U^{\mathsf{c}}$. This contradicts Lemma 2.4.35(iii).

(iv) If $U \subseteq D$ is the vertex set of a connected component of $G[D]$ with at least 4 vertices of degree 2 and $v_1, v_2 \in D^{\mathsf{c}}$ are distinct nonadjacent vertices, then we have

$$\varphi_U(v_i) = \deg_U(v_i) - \varepsilon(v_i) \geqslant 4 - 1 = 3 \quad \text{for each } i \in \{1,2\},$$

so for every vertex $u \in U$ with $\deg_U(u) = 2$, we have

$$\varphi_U(v_1) + \varphi_U(v_2) \geqslant 3 + 3 > 4 = \deg_U(u) + 2;$$

a contradiction to Corollary 2.4.38. ∎

Terminal sets The following definitions will be used throughout the rest of the proof.

Definition 2.4.41. A **terminal set** is a subset $B \subseteq D$ such that $G[B]$ is a leaf block in a connected component of $G[D]$. For a terminal set B, $C_B \supseteq B$ denotes the vertex set of the connected component of $G[D]$ that contains B. A vertex $u \in D$ is **terminal** if it belongs to some terminal set B and is not a cut-vertex in $G[C_B]$.

By definition, a terminal set contains at most one non-terminal vertex. Since $G[D]$ is a GDP-forest, if B is a terminal set, then $G[B]$ is either a cycle or a clique. By Lemma 2.4.40(i), the cardinality of a terminal set is at least 3.

Definition 2.4.42. A terminal set B is **dense** if $G[B]$ is not a cycle; otherwise, B is **sparse**.

By definition, the cardinality of a dense terminal set is at least 4.

Our proof hinges on the following key fact:

Lemma 2.4.43. *There exists a dense terminal set.*

Proof. Suppose that every terminal set is sparse. Since every terminal set induces a cycle, each component of $G[D]$ contains at least 3 vertices of degree 2, and a component of $G[D]$ with exactly 3 vertices of degree 2 must be isomorphic to a triangle. By Lemma 2.4.40(ii), each vertex in D^{c} is adjacent to every vertex of degree 2 in $G[D]$, $|D^{\mathsf{c}}| = k - 2$, and $\varepsilon(v) = 1$ for all $v \in D^{\mathsf{c}}$. Furthermore, by Lemma 2.4.40(iii)(iv), $G[D^{\mathsf{c}}]$ is a disjoint union of cliques and, unless every component of $G[D]$ is isomorphic to a triangle, $G[D^{\mathsf{c}}] \cong K_{k-2}$.

Claim (A). $G[D^{\mathsf{c}}] \not\cong K_{k-2}$.

Proof. Assume, towards a contradiction, that $G[D^{\mathsf{c}}] \cong K_{k-2}$. Then every vertex in D^{c} has exactly $(k+1) - (k-3) = 4$ neighbors in D. Therefore, the number of vertices of degree 2 in $G[D]$ is at most 4. Since every component of $G[D]$ contains at least 3 vertices of degree 2, the graph $G[D]$ is connected. Since $|D| \geqslant 4$, $G[D]$ is not a triangle. Thus, it contains precisely 4 terminal vertices of degree 2; i.e., it either is a 4-cycle, or contains exactly two leaf blocks, both of which are triangles.

Case 1: $G[D]$ *is a* 4-*cycle*. We will show that in this case G is $\mathscr{H}$-colorable. First, we make the following observation:

$$\text{Let } W_4 \text{ denote the 4-wheel. Then } \chi_{DP}(W_4) = 3. \tag{2.4.11}$$

Indeed, let $\mathscr{F} = (M, F)$ be a 3-fold cover of W_4 and suppose that W_4 is not $\mathscr{F}$-colorable. Let $v \in V(W_4)$ be the center of W_4 and let $U := V(W_4) \setminus \{v\}$ (so $W_4[U]$ is a 4-cycle). Define a function $f \colon V(F) \to M(v)$ by

$$f(x) = y \;:\Longleftrightarrow\; (x = y) \text{ or } (x \notin M(v) \text{ and } xy \in E(F)).$$

Since $\deg_{W_4}(u) = 3$ for all $u \in U$, Theorem 2.4.13 implies that f is well-defined. Since W_4 is 3-colorable (in the sense of ordinary graph coloring), there exist an edge $u_1u_2 \in E(W_4)$ and a pair of colors $x_1 \in M(u_1)$,

$x_2 \in M(u_2)$ such that $x_1 x_2 \in E(F)$ and $f(x_1) \neq f(x_2)$. Note that $u_1 \neq v$ since otherwise $f(x_1) = x_1 = f(x_2)$ by definition. Similarly, $u_2 \neq v$, so $\{u_1, u_2\} \subset U$. Let $y \coloneqq f(x_2)$. Then x_1 has no neighbor in $M(u_2) \setminus N_F(y)$, so $\{y\}$ can be extended to an $\mathscr{F}$-coloring of W_4; a contradiction.

Let us now return to the graph G. Choose any vertex $v \in D^{\mathsf{c}}$ and let $W \coloneqq G[\{v\} \cup D]$. Note that W is a 4-wheel. Fix an arbitrary coloring $I \in \mathbf{Ind}(H)$ with $\mathrm{dom}(I) = (\{v\} \cup D)^{\mathsf{c}}$. For all $u \in \{v\} \cup D$, we have $|L_I(u)| \geqslant k - (k-3) = 3$, so by (2.4.11), I can be extended to an $\mathscr{H}$-coloring of the entire graph G.

Case 2: *$G[D]$ contains exactly two leaf blocks, both of which are triangles.* Since each vertex in D^{c} has only 4 neighbors in D, every non-terminal vertex in D has degree k in $G[D]$. Notice that every vertex of degree k in $G[D]$ is a cut-vertex. Indeed, if a vertex $u \in D$ is not a cut-vertex in $G[D]$, then the degree of any cut-vertex in the same block as u strictly exceeds the degree of u (since the blocks of the GDP-tree $G[D]$ are regular graphs). Thus, either the two terminal triangles share a cut-vertex (and, in particular, $k = 4$), or else, their cut-vertices are joined by an edge (and $k = 3$). The former option contradicts Corollary 2.4.37; the latter one implies $G \in \mathbf{Dir}_3$. ⊣

By Claim (A), $G[D^{\mathsf{c}}]$ is a disjoint union of at least 2 cliques. In particular, every connected component of $G[D]$ is isomorphic to a triangle. Suppose that $G[D]$ has ℓ connected components (so $|D| = 3\ell$). If a vertex $v \in D^{\mathsf{c}}$ belongs to a component of $G[D^{\mathsf{c}}]$ of size r, then its degree in G is precisely $(r-1) + 3\ell$. On the other hand, $\deg(v) = k+1$. Thus, $k + 1 = (r-1) + 3\ell$, i.e., $r = k - 3\ell + 2$. In particular, $|D^{\mathsf{c}}| = k - 2$ is divisible by $k - 3\ell + 2$, so $\ell \geqslant 2$.

Case 1: *The set D^{c} is not independent, i.e., $k - 3\ell + 2 \geqslant 2$.* Let $T_1, T_2 \subset D$ (resp. $C_1, C_2 \subset D^{\mathsf{c}}$) be the vertex sets of any two distinct connected components of $G[D]$ (resp. $G[D^{\mathsf{c}}]$). For each $i \in \{1, 2\}$, fix a vertex $u_i \in T_i$ and a pair of distinct vertices $v_{i1}, v_{i2} \in C_i$. Set $U \coloneqq T_1 \cup T_2 \cup \{v_{11}, v_{12}, v_{21}, v_{22}\}$ and let $I \in \mathbf{Ind}(H)$ be such that $\mathrm{dom}(I) = U^{\mathsf{c}}$. Note that

$$\varphi_U(v_{11}) = \varphi_U(v_{21}) = 7 - 1 = 6,$$

while $\deg_U(u_1) = 6$, so, by Lemma 2.4.36, there exist $x_{11} \in L_I(v_{11})$ and $x_{21} \in L_I(v_{21})$ such that

$$I' \coloneqq I \cup \{x_{11}, x_{21}\}$$

is a coloring that enhances u_1. Now, upon setting $U' \coloneqq U \setminus \{v_{11}, v_{21}\}$, we obtain

$$\varphi_{U'}(v_{12}) = \varphi_{U'}(v_{22}) = 6 - 1 = 5,$$

while $\deg_{U'}(u_2) = 4$, so, by Lemma 2.4.36 again, we can choose $x_{12} \in L_{I'}(v_{12})$ and $x_{22} \in L_{I'}(v_{22})$ so that

$$I'' \coloneqq I' \cup \{x_{12}, x_{22}\}$$

is a coloring that enhances both u_1 and u_2. However, the existence of such I'' contradicts Lemma 2.4.35(iii).

Case 2: *The set D^{c} is independent, i.e., $k - 3\ell + 2 = 1$.* In other words, we have $k = 3\ell - 1$. Since $\ell \geqslant 2$, we get $k \geqslant 6 - 1 = 5$, so $|D^{\mathsf{c}}| = k - 2 \geqslant 3$. Let v_1, v_2, $v_3 \in D^{\mathsf{c}}$ be any three distinct vertices in D^{c} and let

$T \subset D$ be the vertex set of any connected component of $G[D]$. Fix a vertex $u \in T$, set $U \coloneqq T \cup \{v_1, v_2, v_3\}$, and let $I \in \mathbf{Ind}(H)$ be such that $\mathrm{dom}(I) = U^{\mathsf{c}}$. Note that

$$\varphi_U(v_1) = \varphi_U(v_2) = \varphi_U(v_3) = 3 - 1 = 2,$$

while $\deg_U(u) = 5$. Therefore, by Lemma 2.4.36, we can choose $x_1 \in L_I(v_1)$, $x_2 \in L_I(v_2)$, and $x_3 \in L_I(v_3)$ so that

$$I' \coloneqq I \cup \{x_1, x_2, x_3\}$$

enhances u. This observation contradicts Lemma 2.4.35(iii) and finishes the proof. ∎

Dense terminal sets and their neighborhoods

Outline of the proof Lemma 2.4.43 asserts that at least one terminal set is dense. In the remainder of the proof of Theorem 2.4.16 we will explore the structural consequences of this assertion and eventually arrive at a contradiction.

Definition 2.4.44. Let B be a terminal set. Let S_B denote the set of all vertices in B^{c} that are adjacent to every vertex in B and let $T_B \coloneqq N_G(B) \setminus (B \cup S_B)$.

By definition, $S_B \subseteq D^{\mathsf{c}}$; however, if $B \neq C_B$, then $T_B \cap D \neq \varnothing$.

Lemma 2.4.45. *Let B be a dense terminal set and let $v \in T_B$. Then v has at least $k - 1$ neighbors outside of B. If, moreover, there exist terminal vertices u_0, $u_1 \in B$ such that $u_0v \notin E$, $u_1v \in E$, then v has at least $k - 1$ neighbors outside of C_B.*

Proof. Let u_0, $u_1 \in B$ be such that $u_0v \notin E$ and $u_1v \in E$. If one of u_0, u_1 is not terminal, then set $U \coloneqq B$; otherwise, set $U \coloneqq C_B$. Our goal is to show that v has at least $k - 1$ neighbors outside of U. Assume, towards a contradiction, that $\deg_{U^{\mathsf{c}}}(v) \leqslant k - 2$. Let $I \in \mathbf{Ind}(H)$ be such that $\mathrm{dom}(I) = (U \cup \{v\})^{\mathsf{c}}$. By (2.4.9), we have

$$|L_I(v)| \geqslant \varphi_U(v) \geqslant k - (k - 2) = 2,$$

so let x_1, x_2 be any two distinct elements of $L_I(v)$. Since $u_0v \notin E$, we have

$$L_{I \cup \{x_1\}}(u_0) = L_{I \cup \{x_2\}}(u_0) = L_I(u_0),$$

so, by Lemma 2.4.39, the matching $E_H(L_I(u_0), L_{I \cup \{x_i\}}(u_1))$ is perfect for each $i \in \{1, 2\}$. This implies that the unique vertex in $L_I(u_1)$ that has no neighbor in $L_I(u_0)$ is adjacent to both x_1 and x_2, which is impossible. ∎

The rest of the proof of Theorem 2.4.16 proceeds as follows. Consider a dense terminal set B. Roughly speaking, Lemma 2.4.45 asserts that the vertices in T_B must have "many" neighbors outside of B. Since the degrees of the vertices in D^{c} cannot be too big, the vertices in T_B should only have "very few" neighbors in B. This implies that "most" edges between B and D^{c} actually connect B with S_B. This intuition guides the proof of Corollary 2.4.50, which asserts that $G[B \cup S_B]$ is a clique of size k (however, the proof of Lemma 2.4.49, the main step towards Corollary 2.4.50, is somewhat lengthy and technical).

The fact that G is a *minimum* counterexample to Theorem 2.4.16 is only used once during the course of the proof, namely in establishing Lemma 2.4.54, which claims that for a dense terminal set B, the graph $G[T_B]$ is a clique. The proof of Lemma 2.4.54 is also the only time when it is important to work in the more general setting of DP-colorings rather than just with list colorings. The proof proceeds by assuming, towards a contradiction, that there exist two nonadjacent vertices $v_1, v_2 \in T_B$, and letting G^* be the graph obtained from G by removing B and adding an edge between v_1 and v_2. Since G^* has fewer vertices than G, it cannot contain a counterexample to Theorem 2.4.16 as a subgraph. This fact can be used to eventually arrive at a contradiction. En route to that goal we study the properties of a certain cover $\mathscr{H}^*$ of G^*, and that cover is not necessarily induced by a list assignment, even if $\mathscr{H}$ is.

With Lemma 2.4.54 at hand, we can pin down the structure of $G[S_B \cup T_B]$ very precisely, which is done in Lemmas 2.4.56 and 2.4.57 and in Corollary 2.4.58. The restrictiveness of these results precludes having "too many" dense terminal sets; this is made precise by Lemma 2.4.59, which asserts that at least one terminal set is sparse. However, due to Lemma 2.4.40, having a sparse terminal set leads to its own restrictions on the structure of $G[D^c]$, which finally yield a contradiction that finishes the proof of Theorem 2.4.16.

The set S_B is large Here we prove that for any dense terminal set B, $|S_B| \geqslant k - |B|$ (see Lemma 2.4.48).

Lemma 2.4.46. *Let B be a dense terminal set. If $|S_B| \leqslant k - |B| - 1$, then the following statements hold:*

(i) $|S_B| = k - |B| - 1$*;*

(ii) $D^c = S_B \cup (T_B \cap D^c)$*;*

(iii) $|D^c| = |S_B| + |T_B \cap D^c| = k - 2$*, and thus $\varepsilon(v) = 1$ for every $v \in D^c$;*

(iv) *every vertex in $T_B \cap D^c$ has exactly $k - 1$ neighbors outside of B; and*

(v) *$B \neq C_B$, and the cut vertex $u_0 \in B$ of $G[C_B]$ has no neighbors in $T_B \cap D^c$.*

Proof. Let $S := S_B$ and let $T := T_B \cap D^c$. Set $b := |B|$, $s := |S|$, and $t := |T|$. Suppose that $s \leqslant k - b - 1$. Since each terminal vertex in B has exactly $k - (b - 1) - s$ neighbors in T, the number of edges between B and T is at least $(b - 1)(k - (b - 1) - s)$. Also, by Lemma 2.4.45, each vertex in T has at least $k - 1$ neighbors in B^c. Hence,

$$\begin{aligned}\varepsilon(D^c) &\geqslant \varepsilon(S) + \varepsilon(T)\\ &\geqslant s + (b-1)(k-(b-1)-s) + (k-1)t - kt\\ &= s + (b-1)(k-(b-1)-s) - t.\end{aligned}$$

Note that $s + t \leqslant |D^c| \leqslant k - 2$, so $t \leqslant k - 2 - s$. Therefore,

$$s + (b-1)(k-(b-1)-s) - t \geqslant 2s + (b-1)(k-(b-1)-s) - k + 2.$$

Since $b \geqslant 4$, the last expression is decreasing in s, and hence

$$\begin{aligned}
&2s + (b-1)(k-(b-1)-s) - k + 2 \\
\geqslant\ &2(k-b-1) + (b-1)(k-(b-1)-(k-b-1)) - k + 2 \\
=\ &k-2.
\end{aligned}$$

On the other hand, $\varepsilon(D^{\mathrm{c}}) \leqslant k-2$. Hence, none of the above inequalities can be strict, yielding (i)–(v). ■

Lemma 2.4.47. *Let B be a dense terminal set. Suppose that $v_1,\ v_2 \in S_B$ are distinct vertices such that $v_1v_2 \notin E$. Then the following statements hold:*

(i) $|D^{\mathrm{c}}| = k - |B| + 1$*;*

(ii) $\varepsilon(v_1) + \varepsilon(v_2) = |B| - 1$*; and*

(iii) $\varepsilon(v) = 1$ *for every* $v \in D^{\mathrm{c}} \setminus \{v_1, v_2\}$.

Proof. Let $b := |B|$. Each terminal vertex $u \in B$ has exactly $k-b+1$ neighbors in D^{c}; in particular, $|D^{\mathrm{c}}| \geqslant k-b+1$. By Corollary 2.4.38, we have

$$\begin{array}{rll}
\text{either} & \min\{\varphi_B(v_1), \varphi_B(v_2)\} & \leqslant 0, \\
\text{or} & \varphi_B(v_1) + \varphi_B(v_2) & \leqslant b+1.
\end{array}$$

In the case when $\varphi_B(v_i) \leqslant 0$ for some $i \in \{1,2\}$, we have $\varepsilon(v_i) = b - \varphi_B(v_i) \geqslant b$, so

$$\varepsilon(v_1) + \varepsilon(v_2) \geqslant b. \tag{2.4.12}$$

In the other case, i.e., when $\varphi_B(v_1) + \varphi_B(v_2) \leqslant b+1$, we get

$$\varepsilon(v_1) + \varepsilon(v_2) = (b - \varphi_B(v_1)) + (b - \varphi_B(v_2)) \geqslant b-1.$$

Hence

$$\varepsilon(D^{\mathrm{c}}) \geqslant \varepsilon(v_1) + \varepsilon(v_2) + |D^{\mathrm{c}} \setminus \{v_1, v_2\}| \geqslant (b-1) + (k-b-1) = k-2. \tag{2.4.13}$$

Since $\varepsilon(D^{\mathrm{c}}) \leqslant k-2$, (2.4.12) fails and none of the inequalities in (2.4.13) can be strict, yielding (i)–(iii). ■

Lemma 2.4.48. *Let B be a dense terminal set. Then $|S_B| \geqslant k - |B|$.*

Proof. Let $S := S_B$ and let $T := T_B \cap D^{\mathrm{c}}$. Set $b := |B|$, $s := |S|$, and $t := |T|$. Suppose that $s \leqslant k-b-1$. Then, by Lemma 2.4.46(i), $s = k-b-1$. We claim that $G[S]$ is a clique. Indeed, otherwise, by Lemma 2.4.47(i), $|D^{\mathrm{c}}| = k-b+1$; on the other hand, by Lemma 2.4.46(iii), $|D^{\mathrm{c}}| = k-2$, so we get $k-2 = k-b+1$, i.e., $b = 3$, which contradicts the fact that B is dense.

By Lemma 2.4.46(iii), the degree of every vertex in D^{c} is exactly $k+1$. Since each vertex in S has b neighbors in B and $s-1 = k-b-2$ neighbors in S, it has exactly $(k+1)-b-(k-b-2) = 3$ neighbors in $(B \cup S)^{\mathrm{c}}$.

By Lemma 2.4.46(v), $B \neq C_B$. Let u_0 denote the cut vertex in B and let B' be any terminal subset of C_B distinct from B. Set $b' \coloneqq |B'|$.

By Lemma 2.4.46(ii), $t = |D^c| - s = (k-2) - (k-b-1) = b - 1 \geqslant 3$; in particular, $T \neq \varnothing$. Due to Lemma 2.4.46(iv)(v), every vertex in T has exactly $k-1$ neighbors in B^c and is not adjacent to u_0. Together with Lemma 2.4.46(iii), this implies that each vertex in T has exactly $(k+1)-(k-1) = 2$ neighbors in $B \setminus \{u_0\}$. We have $|B \setminus \{u_0\}| = b - 1 \geqslant 3$, so, by Lemma 2.4.45, every vertex in T has $k-1$ neighbors outside of C_B. Therefore, there are no edges between T and $C_B \setminus B$; in particular, there are no edges connecting T to the terminal vertices in B'.

Consider any terminal vertex $u \in B'$. Since $T \neq \varnothing$ and no edges connect u and T, $\deg(u) > 2$; therefore, B' is a dense terminal set. By Lemma 2.4.46(ii), $D^c = S \cup T$, so u has exactly $k - b' + 1$ neighbors in S. Thus, $k - b - 1 = s \geqslant k - b' + 1$, i.e., $b' \geqslant b + 2 \geqslant 6$. Let v be any neighbor of u in S. Since v has only 3 neighbors in $(B \cup S)^c$ and $b' > 4$, there exists another terminal vertex $u' \in B'$ such that $u'v \notin E$. By Lemma 2.4.45, v has at least $k-1$ neighbors outside of $C_{B'} = C_B$. Of those, $s-1$ belong to S; since v has only 3 neighbors outside of $B \cup S$ and is adjacent to u, it has at most $3 - 1 = 2$ neighbors in $(C_B \cup S)^c$. Hence, $k - 1 \leqslant (s-1) + 2 = (k-b-2) + 2 = k - b$, i.e., $b \leqslant 1$, which is impossible. ■

The graph $G[S_B]$

Lemma 2.4.49. *Let B be a dense terminal set. Then $G[S_B]$ is a clique.*

Proof. Let $S \coloneqq S_B$ and suppose that $G[S]$ is not a clique, i.e., there exist distinct $v_1, v_2 \in S$ such that $v_1 v_2 \notin E$. Without loss of generality, we may assume that $\deg(v_1) \geqslant \deg(v_2)$. We will proceed via a series of claims, establishing a precise structure of $G[D^c]$, which will eventually lead to a contradiction. For the rest of the proof, we set $b \coloneqq |B|$ and $s \coloneqq |S|$. Recall that, by Lemma 2.4.47, we have the following:

(i) $|D^c| = k - b + 1$;

(ii) $\varepsilon(v_1) + \varepsilon(v_2) = b - 1$; and

(iii) $\varepsilon(v) = 1$ for every $v \in D^c \setminus \{v_1, v_2\}$.

Claim (A). *$D^c = S$ and $B = C_B$.*

Proof. Suppose, towards a contradiction, that there is a vertex $v \in D^c \setminus S$. Since, by Lemma 2.4.47(i), $|D^c| = k - b + 1$, each terminal vertex in B is adjacent to every vertex in D^c. Therefore, $\deg_B(v) = b - 1$ and, due to Lemma 2.4.45, $\deg_{B^c}(v) \geqslant k - 1$. Then

$$\varepsilon(v) = \deg(v) - k \geqslant (b-1) + (k-1) - k = b - 2 > 1;$$

a contradiction to Lemma 2.4.47(iii).

Since $|S| = |D^c| = k - b + 1$, every vertex in B has $(b-1) + (k-b+1) = k$ neighbors in $B \cup S$, so there are no edges between B and $D \setminus B$; therefore, $B = C_B$. ⊣

Claim (B). *The graph $G[D]$ has no vertices of degree 2.*

Proof. Indeed, otherwise Lemma 2.4.40 would yield $|D^{c}| = k - 2$. Since $|D^{c}| = k - b + 1$, this implies $b = 3$, contradicting the denseness of B. ⊣

Claim (C). *$s \geqslant 3$, i.e., $b \leqslant k - 2$.*

Proof. Suppose, towards a contradiction, that $s = 2$, i.e., $S = \{v_1, v_2\}$. We will argue that in this case $G \in \mathbf{Dir}_k$. Since, by Lemma 2.4.47(i), $s = k - b + 1$, we have $b = k - 1$. In particular, since $b \geqslant 4$, we have $k \geqslant 5$. By Lemma 2.4.47(ii), $\varepsilon(S) = b - 1 = k - 2$, so there are exactly $(k - 2) + 2k - 2(k - 1) = k$ edges between S and $D \setminus B$. Let U be any connected component of $G[D]$ distinct from B. By Corollary 2.4.33, the number of edges between U and S is at least k, with equality only if $G[U] \cong K_k$; therefore, $D \setminus B = U$, and we indeed have $G[U] \cong K_k$. Then every vertex in U has exactly one neighbor in S and each vertex in S has at least two neighbors in U (for its degree is at least $k + 1$), yielding $G \in \mathbf{Dir}_k$, as desired. ⊣

Claim (D). *$G[S \setminus \{v_1\}]$ is a clique.*

Proof. Suppose that for some distinct $w_1, w_2 \in S \setminus \{v_1\}$, we have $w_1 w_2 \notin E$. Applying Lemma 2.4.47(iii) with w_1 and w_2 in place of v_1 and v_2, we obtain $\varepsilon(v_1) = 1$. Since, by our choice, $\deg(v_1) \geqslant \deg(v_2)$, and thus $\varepsilon(v_1) \geqslant \varepsilon(v_2)$, we get $\varepsilon(v_2) = 1$ as well. But then $2 = \varepsilon(v_1) + \varepsilon(v_2) = b - 1$, i.e., $b = 3$; a contradiction. ⊣

Claim (E). $\deg_S(v_1) = 0$.

Proof. Suppose that $v \in S \setminus \{v_1, v_2\}$ is adjacent to v_1. Note that by Claim (D), v is also adjacent to v_2. Let $U \coloneqq B \cup \{v_1, v_2, v\}$ and let u be any vertex in B. Note that

$$\deg_U(u) = (b - 1) + 3 = b + 2.$$

On the other hand, since $\varepsilon(v_1) + \varepsilon(v_2) = b - 1$, for each $i \in \{1, 2\}$, we have $\varepsilon(v_i) \leqslant b - 2$, so

$$\varphi_U(v_i) = (b + 1) - \varepsilon(v_i) \geqslant (b + 1) - (b - 2) = 3 > 0;$$

moreover,

$$\varphi_U(v_1) + \varphi_U(v_2) = 2(b + 1) - (b - 1) = b + 3 > b + 2.$$

Therefore, by Lemma 2.4.36, for any $I \in \mathbf{Ind}(H)$ with $\mathrm{dom}(I) = U^{c}$, we can find $x_1 \in L_I(v_1)$ and $x_2 \in L_I(v_2)$ such that u is enhanced by $I' \coloneqq I \cup \{x_1, x_2\}$. Note that

$$|L_{I'}(v)| \geqslant \varphi_B(v) = b - 1 > 0,$$

so I' can be extended to a coloring with domain B^{c}, which contradicts Lemma 2.4.35(iii). ⊣

Claim (F). *$\varepsilon(v_1) = b - 2$ and $\varepsilon(v) = 1$ for all $v \in S \setminus \{v_1\}$.*

Proof. Consider any $v \in S \setminus \{v_1\}$. By Claim (C), we can choose some $v' \in S \setminus \{v_1, v\}$. Due to Claim (E), $v_1 v' \notin E$, so we can apply Lemma 2.4.47(iii) with v' in place of v_2 to obtain $\varepsilon(v) = 1$. In particular, $\varepsilon(v_2) = 1$, so $\varepsilon(v_1) = (b - 1) - \varepsilon(v_2) = b - 2$. ⊣

Claim (G). *Every terminal set distinct from B induces a clique of size k.*

Proof. Suppose that B' is a terminal set distinct from B and $b' := |B'| \leqslant k - 1$. By Claim (B), B' is dense. Thus, by Lemma 2.4.48, $|S_{B'}| \geqslant k - b'$, i.e., S contains at least $k - b'$ vertices that are adjacent to every vertex in B'. Consider $v \in S \setminus \{v_1\}$. By definition, v has b neighbors in B; due to Claim (D), v also has $s - 2 = (k - b + 1) - 2 = k - b - 1$ neighbors in S. On the other hand, by Claim (F), $\deg(v) = k + 1$. Therefore,

$$\deg_{D \setminus B}(v) = (k + 1) - b - (k - b - 1) = 2.$$

In particular, v cannot be adjacent to all the vertices in B'. Thus, $S_{B'} = \{v_1\}$ and $|B'| = k - 1$. But

$$\deg_{D \setminus B}(v_1) = \varepsilon(v_1) + k - \deg_B(v_1) - \deg_S(v_1) = (b - 2) + k - b - 0 = k - 2 < k - 1;$$

a contradiction. ⊣

Claim (H). *There are exactly two terminal sets distinct from B.*

Proof. Suppose $D \setminus B$ contains ℓ terminal sets. By Claim (G), the number of edges between S and the terminal vertices of any terminal set B' distinct from B is at least $k - 1$ and at most k. On the other hand, the number of edges between S and $D \setminus B$ is exactly $(k - 2) + 2(k - b) = 3k - 2b - 2$. Therefore,

$$\ell(k - 1) \leqslant 3k - 2b - 2 \leqslant \ell k,$$

so $1 \leqslant \ell \leqslant 2$. However, if $\ell = 1$, then $3k - 2b - 2 \leqslant k$, so $b \geqslant k - 1$, which contradicts Claim (C). Thus, $\ell = 2$, as desired. ⊣

Now we are ready to finish the argument. Let B_1 and B_2 denote the only two terminal sets in $D \setminus B$, which, by Claim (G), induce cliques of size k. We have $D \setminus B = C_{B_1} \cup C_{B_2}$. Notice that v_1 is adjacent to at least one terminal vertex in $B_1 \cup B_2$. Indeed, there are at least $2(k - 1)$ edges between S and the terminal vertices in $B_1 \cup B_2$, while each vertex in $S \setminus \{v_1\}$ has 2 neighbors in $D \setminus B$, providing in total only $2(k - b)$ edges.

Without loss of generality, assume that v_1 is adjacent to at least one terminal vertex in B_1. Since v_1 has only $k - 2$ neighbors in $D \setminus B$, Lemma 2.4.45 implies that v_1 has at least $k - 1$ neighbors outside of C_{B_1}. Since v_1 has only $b \leqslant k - 2$ neighbors outside of $C_{B_1} \cup C_{B_2}$, we see that $C_{B_1} \neq C_{B_2}$ and v_1 has a neighbor in C_{B_2}. Since B_1 and B_2 are the unique terminal sets in C_{B_1} and C_{B_2} respectively, we have $B_1 = C_{B_1}$ and $B_2 = C_{B_2}$. Therefore, v_1 is also adjacent to at least one terminal vertex in B_2 and, hence, has at least $k - 1$ neighbors outside of B_2.

Notice that $2k = |E_G(B_1 \cup B_2, S)| = 3k - 2b - 2$, i.e., $k = 2b + 2$. Let $d_i := \deg_{B_i}(v_1)$. Then for each $i \in \{1, 2\}$, $d_i \geqslant k - 1 - b$. Since

$$b + d_1 + d_2 = \deg(v_1) = k + b - 2,$$

we obtain that $k + b - 2 \geqslant b + 2(k - 1 - b)$, i.e., $2b \geqslant k$, contradicting $k = 2b + 2$. ■

Corollary 2.4.50. *Let B be a dense terminal set. Then $G[B \cup S_B]$ is a clique of size k.*

Proof. By Lemma 2.4.48, $|B \cup S_B| \geqslant k$; on the other hand, by Lemma 2.4.49, $G[B \cup S_B]$ is a clique, so $|B \cup S_B| \leqslant k$. ∎

Corollary 2.4.51. *There does not exist a subset $U \subseteq V$ of size $k+1$ such that $G[U]$ is a complete graph minus an edge with the two nonadjacent vertices in D^{c}.*

Proof. Suppose, towards a contradiction, that U is such a set and let $v_1, v_2 \in U \cap D^{\mathsf{c}}$ be the two nonadjacent vertices in U. Set $B := U \cap D$. Note that $|B| \geqslant |U| - |D^{\mathsf{c}}| \geqslant (k+1) - (k-2) = 3$.

Since for each $u \in B$, $\deg_U(u) = k$, there are no edges between B and U^{c}. In particular, $B = C_B$. If $|B| \geqslant 4$, then B is a dense set and $U = B \cup S_B$, which is impossible due to Corollary 2.4.50. Therefore, $|B| = 3$. Thus, $|U \setminus B| = (k+1) - 3 = k-2$, so $D^{\mathsf{c}} = U \setminus B$. By Lemma 2.4.40(iii), $G[D^{\mathsf{c}}]$ is a disjoint union of cliques. On the other hand, $G[D^{\mathsf{c}}]$ is a complete graph minus the edge $v_1 v_2$. The only possibility then is that $|D^{\mathsf{c}}| = 2$, i.e., $k = 4$. Each vertex in D^{c} is of degree 5 and, therefore, has exactly 2 neighbors in $D \setminus B$. By Lemma 2.4.43, there exists a dense terminal set $B' \subseteq D \setminus B$. Since $k = 4$, we must have $|B'| = 4$, so there are 4 edges between B' and D^{c}. This implies that $D \setminus B = B'$ and each vertex in D^{c} has exactly 2 neighbors in B'. But then $G \in \mathbf{Dir}_4$. ∎

The graph $G[T_B]$ In this section we show that if B is a dense terminal set, then $G[T_B]$ is a clique. However, in order for some of our arguments to go through, we need to establish some of the results for the more general case when B is a terminal set such that $G[B \cup S_B]$ is a clique of size k (i.e., $G[B]$ can also be isomorphic to a triangle).

Lemma 2.4.52. *Let B be a terminal set such that $G[B \cup S_B]$ is a clique of size k. Then every vertex in S_B has at most $|B| - 1$ neighbors outside of $B \cup S_B$.*

Proof. Set $S := S_B$. Let $v \in S$ and suppose that v has d neighbors outside of $B \cup S$. Then

$$\varepsilon(v) = \deg_{B \cup S}(v) + \deg_{(B \cup S)^{\mathsf{c}}}(v) - k = (k-1) + d - k = d - 1,$$

so, using that $|S| = k - |B|$, we obtain

$$k - 2 \geqslant \varepsilon(D^{\mathsf{c}}) = \varepsilon(S) + \varepsilon(D^{\mathsf{c}} \setminus S) \geqslant (d-1) + (k - |B| - 1) + |D^{\mathsf{c}} \setminus S|,$$

i.e., $d \leqslant |B| - |D^{\mathsf{c}} \setminus S|$. But $D^{\mathsf{c}} \setminus S \neq \varnothing$, since each terminal vertex in B has a neighbor in $D^{\mathsf{c}} \setminus S$. ∎

Lemma 2.4.53. *Let B be a terminal set such that $G[B \cup S_B]$ is a clique of size k. Let I be a coloring with* $\mathrm{dom}(I) = (B \cup S_B)^{\mathsf{c}}$*. Then for any $u \in B$, $|L_I(u)| = k - 1$, and for any two distinct $u_1, u_2 \in B$, the matching $E_H(L_I(u_1), L_I(u_2))$ is perfect.*

Proof. Set $S := S_B$. By Lemma 2.4.52, $|L_I(v)| \geqslant k - |B| + 1$ for all $v \in S$. Since $|S| = k - |B|$, I can be extended to a coloring I' with $\mathrm{dom}(I') = B^{\mathsf{c}}$. Therefore, due to Lemma 2.4.35(iii) and since $G[B]$ is connected, I does not enhance any $u \in B$, i.e., $|L_I(u)| = k - 1$, as claimed. Now, let u_1, u_2 be two distinct vertices in B and suppose, towards a contradiction, that $x \in L_I(u_1)$ has no neighbor in $L_I(u_2)$. For each $v \in S$, let $L'(v) := L_I(v) \setminus N_H(x)$. Then $|L'(v)| \geqslant k - |B| = |S|$ for all $v \in S$, so there is $I' \in \mathbf{Ind}(H)$ with $\mathrm{dom}(I') = S$ such that $I' \subseteq \bigcup_{v \in S} L'(v)$. Then $I \cup I'$ is a coloring with domain B^{c}; moreover, $x \in L_{I \cup I'}(u_1)$, which implies that the matching $E_H(L_{I\cup I'}(u_1), L_{I \cup I'}(u_2))$ is not perfect. Due to Lemma 2.4.39, $I \cup I'$ can be extended to an $\mathscr{H}$-coloring of G; a contradiction. ∎

Lemma 2.4.54. *Let B be a terminal set such that $G[B \cup S_B]$ is a clique of size k. Then $G[T_B]$ is a clique of size at least* 2.

Proof. Set $S := S_B$ and $T := T_B$. First, observe that $|T| \geqslant 2$: Each vertex in B has a (unique) neighbor in T; thus, if $|T| = 1$, then the only vertex in T has to be adjacent to all the vertices in B, which contradicts the way T is defined.

Now suppose that $v_1, v_2 \in T$ are two distinct nonadjacent vertices. For each $i \in \{1, 2\}$, choose a neighbor $u_i \in B$ of v_i. Since every vertex in B has only one neighbor outside of $B \cup S$, $u_1v_2, u_2v_1 \notin E$. Note that, by Lemma 2.4.53, there are at least $k - 1$ edges between $L(u_1)$ and $L(u_2)$. Let H' be the graph obtained from H by adding, if necessary, a single edge between $L(u_1)$ and $L(u_2)$ that completes a perfect matching between those two sets. Let H^* be the graph obtained from H by adding a matching M between $L(v_1)$ and $L(v_2)$ in which $x_1 \in L(v_1)$ is adjacent to $x_2 \in L(v_2)$ if and only if there exist $y_1 \in L(u_1)$, $y_2 \in L(u_2)$ such that $x_1y_1y_2x_2$ is a path in H'. Then $\mathscr{H}^* := (L, H^*)$ is a cover of the graph G^* obtained from G by adding the edge v_1v_2.

Claim (A). *There is no independent set I in H^* with* $\mathrm{dom}(I) = (B \cup S)^{\mathsf{c}}$.

Proof. Assume, towards a contradiction, that I is an independent set in H^* such that $\mathrm{dom}(I) = (B \cup S)^{\mathsf{c}}$. Since, in particular, $I \in \mathbf{Ind}(H)$, Lemma 2.4.53 guarantees that the edges of H between $L_I(u_1)$ and $L_I(u_2)$ form a perfect matching of size $k - 1$. For each $i \in \{1, 2\}$, let y_i be the unique element of $L(u_i) \setminus L_I(u_i)$. Then y_1y_2 is an edge in H'. However, since $y_i \notin L_I(u_i)$, the unique element of $I \cap L(v_i)$, which we denote by x_i, is adjacent to y_i in H. Therefore, $x_1y_1y_2x_2$ is a path in H', so x_1x_2 is an edge in H^*. This contradicts the independence of I in H^*. ⊣

Let $W \subseteq (B \cup S)^{\mathsf{c}}$ be an inclusion-minimal subset for which there is no independent set I in H^* with $\mathrm{dom}(I) = W$. Since G is $\mathscr{H}$-critical, $G^*[W]$ is not a subgraph of G, so $\{v_1, v_2\} \subseteq W$. Since for all $v \in W$, $\deg(v) \geqslant \deg_{G^*[W]}(v)$, we have

$$\varepsilon(W) \geqslant \sum_{v \in W} (\deg_{G^*[W]}(v) - k).$$

In particular,

$$\sum_{v \in W} (\deg_{G^*[W]}(v) - k) \leqslant k - 2.$$

By the minimality of G, either $G^*[W] \in \mathbf{Dir}_k$, or else, $G^*[W]$ contains a clique of size $k + 1$.

If $G^*[W] \in \mathbf{Dir}_k$, then

$$\sum_{v \in W} (\deg_{G^*[W]}(v) - k) = k - 2.$$

Therefore, $D^{\mathsf{c}} \subseteq W$ and $\deg(v) = \deg_{G^*[W]}(v)$ for all $v \in W$. The latter condition implies that the only vertices in W that are adjacent to a vertex in B are v_1 and v_2; moreover, the only neighbor of v_1 in B is u_1

and the only neighbor of v_2 in B is u_2. Since $D^c \subseteq W$, this implies $S = \varnothing$ and $T \cap D^c \subseteq \{v_1, v_2\}$. Therefore, $|B| = k - |S| = k$. Each terminal vertex in B has a neighbor in $T \cap D^c$, so the set of all terminal vertices in B is a subset of $\{u_1, u_2\}$. Since $k \geqslant 3$, this implies $k = |B| = 3$ and u_1, u_2 are indeed the terminal vertices in B. But then every vertex in D^c is adjacent both to u_1 and to u_2, contradicting the fact that v_1 and v_2 each have only one neighbor among u_1, u_2.

Thus, $G^*[W]$ contains a clique of size $k + 1$. Since G does not contain such a clique, there exists a set $U \subseteq (B \cup S \cup \{v_1, v_2\})^c$ of size $k - 1$ such that the graph $G[U \cup \{v_1, v_2\}]$ is isomorphic to K_{k+1} minus the edge $v_1 v_2$. Note that $U \nsubseteq D^c$, since $|D^c| \leqslant k - 2$. Thus, the set $B' := U \cap D$ is nonempty. Let $S' := U \setminus B'$. Each vertex in B' has k neighbors in $U \cup \{v_1, v_2\}$, so there are no edges between B' and $(U \cup \{v_1, v_2\})^c$. Due to Corollary 2.4.51, $\{v_1, v_2\} \nsubseteq D^c$, so we may assume, without loss of generality, that $v_2 \in D$ and let $B^* := B' \cup \{v_2\}$. Then $G[B \cup B^*]$ is a connected component of $G[D]$, with $u_2 v_2$ being a unique edge between terminal sets B and B^*. Note that $S' = S_{B^*}$ and $G[B^* \cup S']$ is a clique of size k. Moreover, $v_1 u_2 \notin E$ and $\{v_1, u_2\} \in T_{B^*}$. Thus, we can apply the above reasoning to B^* in place of B and v_1, u_2 in place of v_1, v_2. As a result, we see that $G[B \cup S \cup \{v_1\}]$ is isomorphic to K_{k+1} minus the edge $v_1 u_2$. Therefore,

$$\varepsilon(v_1) \geqslant \deg_{B \cup S}(v_1) + \deg_{B^* \cup S'}(v_1) - k = (k-1) + (k-1) - k = k - 2.$$

Thus, $D^c = \{v_1\}$, $S = S' = \varnothing$, and $|B| = |B^*| = k$. This implies that $G \in \mathbf{Dir}_k$. ■

Corollary 2.4.55. *Let B be a dense terminal set. Then $G[T_B]$ is a clique of size at least* 2.

Proof. Follows from Corollary 2.4.50 and Lemma 2.4.54. ■

The graph $G[S_B \cup T_B]$

Lemma 2.4.56. *Let B be a dense terminal set. Then:*

(i) $|T_B| = 2$*;*

(ii) $D^c = S_B \cup (T_B \cap D^c)$*;*

(iii) *each vertex in T_B has exactly $k - 1$ neighbors outside of B; and*

(iv) $\varepsilon(v) = 1$ *for all $v \in S_B$.*

Proof. Let $S := S_B$ and $T := T_B$. By Corollaries 2.4.50 and 2.4.55, $G[S \cup B]$ is a clique of size k and $G[T]$ is a clique of size at least 2.

Suppose that (i) does not hold, i.e., $|T| \geqslant 3$. Recall that, by Lemma 2.4.45, each vertex in T has at least $k - 1$ neighbors outside of B. If T contains at most one vertex with exactly $k - 1$ neighbors outside of B, then

$$\varepsilon(S) + \varepsilon(T) \geqslant |S| + \sum_{v \in T} \deg(v) - k|T| \geqslant (k - |B|) + (|B| + k|T| - 1) - k|T| = k - 1;$$

a contradiction. Thus, there exist two distinct vertices $v_1, v_2 \in T$ such that

$$\deg_{B^c}(v_1) = \deg_{B^c}(v_2) = k - 1.$$

Since $|T| \geqslant 3$ and every vertex in B has exactly one neighbor in T, there exists a vertex $u_0 \in B$ such that u_0v_1, $u_0v_2 \notin E$. Also, we can choose a vertex $u_1 \in B$ with $u_1v_1 \in E$; note that $u_1v_2 \notin E$. Let $I \in \mathbf{Ind}(H)$ be such that $\mathrm{dom}(I) = (B \cup \{v_1, v_2\})^{\mathsf{c}}$. Then

$$\varphi_{B\cup\{v_1,v_2\}}(v_1) = \varphi_{B\cup\{v_1,v_2\}}(v_2) = k - (k-2) = 2.$$

(Here we use that v_1 and v_2 are adjacent to each other.) Let x_1, x_2 be any two distinct elements of $L_I(v_1)$ and choose y_1, $y_2 \in L_I(v_2)$ so that x_1y_1, $x_2y_2 \notin E(H)$. Since

$$L_{I\cup\{x_1,y_1\}}(u_0) = L_{I\cup\{x_2,y_2\}}(u_0) = L_I(u_0),$$

and for each $i \in \{1, 2\}$,

$$L_{I\cup\{x_i,y_i\}}(u_1) = L_{I\cup\{x_i\}}(u_1),$$

Lemma 2.4.39 implies that for each $i \in \{1, 2\}$, the matching $E_H(L_I(u_0), L_{I\cup\{x_i\}}(u_1))$ is perfect. But then the unique vertex in $L_I(u_1)$ that has no neighbor in $L_I(u_0)$ is adjacent to both x_1 and x_2, which is impossible. This contradiction proves (i).

In view of (i), we now have

$$\varepsilon(D^{\mathsf{c}}) \geqslant \varepsilon(S) + \varepsilon(T) \geqslant (k - |B|) + (|B| + 2(k-1)) - 2k = k - 2, \tag{2.4.14}$$

so none of the inequalities in (2.4.14) can be strict. This yields (ii), (iii), and (iv). ∎

Lemma 2.4.57. *Let B be a dense terminal set. Then $B = C_B$.*

Proof. Suppose, towards a contradiction, that $B \neq C_B$. Then $T_B \cap D \neq \varnothing$. On the other hand, every terminal vertex in B has a neighbor in $T_B \cap D^{\mathsf{c}}$, so we also have $T_B \cap D^{\mathsf{c}} \neq \varnothing$. By Lemma 2.4.56(i), $|T_B| = 2$, so $T_B =: \{v, u\}$, where $v \in D^{\mathsf{c}}$ and $u \in D$, with v adjacent to all the terminal vertices in B. By Lemma 2.4.56(ii), $D^{\mathsf{c}} = S_B \cup \{v\}$. By Corollary 2.4.50, $G[B \cup S_B] \cong K_k$; in particular, $|S_B| = k - |B|$. Therefore,

$$|D^{\mathsf{c}}| = k - |B| + 1. \tag{2.4.15}$$

Let B' be any other terminal set such that $C_{B'} = C_B$. Note that B' is dense, since, otherwise, by Lemma 2.4.40(ii), $|D^{\mathsf{c}}| = k-2$, contradicting (2.4.15). Therefore, the above reasoning can be applied to B' in place of B. In particular, $T_{B'} =: \{v', u'\}$, where $v' \in D^{\mathsf{c}}$ and $u' \in D$, with v' adjacent to all the terminal vertices in B'. Moreover, $D^{\mathsf{c}} = S_B \cup \{v\} = S_{B'} \cup \{v'\}$. Consider any vertex $w \in D^{\mathsf{c}}$. If $w \in S_{B'}$, then, by definition, w is adjacent to every vertex in B'. If, on the other hand, $w = v'$, then w is adjacent to all the terminal vertices in B'. In either case, w has at least $|B'| - 1$ neighbors in B'. However, if $w \in S_B$, then due to Lemma 2.4.56(iv), w has exactly $(k+1) - (k-1) = 2$ neighbors outside of $B \cup S_B$. This implies that $S_B = \varnothing$, and similarly $S_{B'} = \varnothing$. Thus, $v = v'$ and $|B| = |B'| = k$. Then v is adjacent to $k-1$ terminal vertices in B' and to u, contradicting Lemma 2.4.56(iii). ∎

Corollary 2.4.58. *Let B be a dense terminal set. Then $D^{\mathsf{c}} = S_B \cup T_B$.*

Proof. Follows immediately by Lemma 2.4.57 and Lemma 2.4.56(ii). ∎

Finishing the proof of Theorem 2.4.16

Lemma 2.4.59. *There exists a sparse terminal set.*

Proof. Suppose, towards a contradiction, that every terminal set is dense. Lemma 2.4.57 implies that in such case every connected component of $G[D]$ is a clique of size at least 4. Moreover, due to Corollary 2.4.58, Corollary 2.4.50, and Lemma 2.4.56(i), the size of every connected component of $G[D]$ is precisely $k - |D^{\mathsf{c}}| + 2 =: b$. Note that due to Lemma 2.4.56(iii), the graph $G[D]$ is disconnected.

Let B_1 and B_2 be the vertex sets of any two distinct connected components of $G[D]$. Lemma 2.4.56(iv) implies that $S_{B_1} \cap S_{B_2} = \varnothing$, since every vertex in S_{B_1} has only 2 neighbors outside of $B_1 \cup S_{B_1}$. Since, by Corollary 2.4.58,

$$D^{\mathsf{c}} = S_{B_1} \cup T_{B_1} = S_{B_2} \cup T_{B_2},$$

it follows that $S_{B_1} \subseteq T_{B_2}$ and $S_{B_2} \subseteq T_{B_1}$. Therefore, $|S_{B_1}| \leqslant |T_{B_2}|$, i.e., $k - b \leqslant 2$, which implies

$$b \in \{k-2, k-1, k\}.$$

Now it remains to consider the three possibilities.

Case 1: $b = k - 2$. Let B be the vertex set of any connected component of $G[D]$. Set $T_B =: \{v_1, v_2\}$ and let $u_1, u_2 \in B$ be such that $u_1 v_1, u_2 v_2 \in E$. Choose any $x \in L(v_1)$. Note that $|L_{\{x\}}(v_2)| \geqslant 2$, so we can choose $y \in L_{\{x\}}(v_2)$ in such a way that $E_H(L_{\{x,y\}}(u_1), L_{\{x,y\}}(u_2))$ is not a perfect matching. For all $u \in D \setminus B$, we have $|L_{\{x,y\}}(u)| \geqslant k - 2$ and the size of every connected component of $G[D \setminus B]$ is $k - 2$. Therefore, there exists a coloring I with $\mathrm{dom}(I) = D \setminus B$ such that $I \cup \{x, y\} \in \mathbf{Ind}(H)$. But then

$$\mathrm{dom}(I \cup \{x, y\}) = (B \cup S_B)^{\mathsf{c}},$$

and the matching $E_H(L_{I\cup\{x,y\}}(u_1), L_{I\cup\{x,y\}}(u_1))$ is not perfect, contradicting Lemma 2.4.53.

Case 2: $b = k - 1$. Let B_1 and B_2 be the vertex sets of any two distinct connected components of $G[D]$. Let v be the unique vertex in S_{B_1}. Then $v \in T_{B_2}$ (recall that $S_{B_1} \cap S_{B_2} = \varnothing$), so, by Lemma 2.4.56(iii), v has exactly $k - 1$ neighbors outside of B_2. By Corollary 2.4.55, one of the neighbors of v is the other vertex in T_{B_2}. Therefore, v can have at most $k - 2$ neighbors in B_1; a contradiction with the choice of v.

Case 3: $b = k$. In this case, $G[D^{\mathsf{c}}] \cong K_2$ and every vertex in D has exactly one neighbor in D^{c}, so there are exactly k edges between D^{c} and every connected component of $G[D]$. On the other hand, if B is the vertex set of a connected component of $G[D]$, then, by Lemma 2.4.56(iii), there are exactly $2(k-2) < 2k$ edges between D^{c} and $D \setminus B$. Thus, the graph $G[D \setminus B]$ is connected. Moreover, $k \geqslant 4$, for $2 \cdot (3 - 2) = 2 < 3$. Let $B' := D \setminus B$ (so $G[B']$ is a clique of size k). Set $D^{\mathsf{c}} =: \{v_1, v_2\}$ an let $u_1, u_2 \in B$, $u_1', u_2' \in B'$ be such that $u_1 v_1, u_2 v_2, u_1' v_1, u_2' v_2 \in E$. Choose any $x \in L(v_1)$. Note that $|L_{\{x\}}(v_2)| \geqslant 3$. There is at most one element $y \in L_{\{x\}}(v_2)$ such that $E_H(L_{\{x,y\}}(u_1), L_{\{x,y\}}(u_2))$ is a perfect matching; similarly for u_1' and u_2'. Therefore, there exists $z \in L_{\{x\}}(v_2)$ such that neither $E_H(L_{\{x,z\}}(u_1), L_{\{x,z\}}(u_2))$ nor $E_H(L_{\{x,z\}}(u_1'), L_{\{x,z\}}(u_2'))$ are perfect matchings. Thus, by Lemma 2.4.39, $\{x, z\}$ can be extended to an $\mathscr{H}$-coloring of G; a contradiction. ∎

Now we are ready to finish the proof of Theorem 2.4.16. Let B be a dense terminal set (which exists by Lemma 2.4.43) and let B' be a sparse terminal set (which exists by Lemma 2.4.59). By Lemma 2.4.57, $B = C_B$ and every terminal set in $C_{B'}$ is sparse. In particular, $G[C_{B'}]$ contains at least 3 vertices of degree 2. Thus, by Lemma 2.4.40(ii), every vertex in D^{c} has at least 3 neighbors in $C_{B'}$. On the other hand, by Lemma 2.4.56(iv), a vertex in S_B has only 2 neighbors in $(B \cup S_B)^{\mathsf{c}}$. Therefore, $S_B = \varnothing$. Due to Corollary 2.4.58, we obtain $D^{\mathsf{c}} = T_B$; thus, by Corollary 2.4.55 and Lemma 2.4.56(i), $G[D^{\mathsf{c}}] \cong K_2$. On the other hand, by Lemma 2.4.40(ii), $|D^{\mathsf{c}}| = k - 2$, so $k = 4$. But each vertex in T_B has at least 4 neighbors outside of B (1 in T_B by Corollary 2.4.55 and 3 in $C_{B'}$ by Lemma 2.4.40(ii)), which contradicts Lemma 2.4.56(iii).

2.4.5 Concluding remarks

Theorem 2.4.16 applies to DP-critical simple graphs. Meanwhile, bounding the difference $2|E(G)| - k|V(G)|$ for DP-critical *multigraphs* G appears to be a challenging problem.

Definition 2.4.60. For $k \geqslant 3$, a k**-brick** is a k-regular multigraph whose underlying simple graph is either a clique or a cycle and in which the multiplicities of all edges are the same.

Note that for a k-brick G, $2|E(G)| = k|V(G)|$. According to Theorem 2.4.13, k-bricks are the only k-DP-critical multigraphs with this property.

Theorem 2.4.16 fails for multigraphs, as the following example demonstrates. Fix an integer $k \in \mathbb{N}$ divisible by 3 and let G be the multigraph with vertex set $\{0, 1, 2\}$ such that $e_G(0, 1) = k/3$ and $e_G(0, 2) = e_G(1, 2) = 2k/3$, so we have $2|E(G)| - k|V(G)| = k/3$. Let H be the graph with vertex set

$$\{0, 1, 2\} \times \{0, 1, 2\} \times \{0, 1, \ldots, k/3 - 1\}$$

in which two distinct vertices (i_1, j_1, a_1) and (i_2, j_2, a_2) are adjacent if and only if one of the following three (mutually exclusive) situations occurs:

1. $\{i_1, i_2\} = \{0, 1\}$ and $j_1 = j_2$;
2. $\{i_1, i_2\} \neq \{0, 1\}$ and $j_1 \neq j_2$; or
3. $(i_1, j_1) = (i_2, j_2)$.

For each $i \in \{0, 1, 2\}$, let $L(i) \coloneqq \{i\} \times \{0, 1, 2\} \times \{0, 1, \ldots, k/3 - 1\}$. Then $\mathscr{H} \coloneqq (L, H)$ is a k-fold cover of G. We claim that G is not $\mathscr{H}$-colorable. Indeed, suppose that I is an $\mathscr{H}$-coloring of G and for each $i \in \{0, 1, 2\}$, let $I \cap L(i) =: \{(i, j_i, a_i)\}$. By the definition of H, we have $j_0 \neq j_1$, while also $j_0 = j_2 = j_1$, which is a contradiction. It is also easy to check that G is $\mathscr{H}$-critical and that it does not contain any k-brick as a subgraph.

In light of the above example, we propose the following problem:

Problem 2.4.61. *Let $k \geqslant 3$. Let G be a multigraph and let $\mathscr{H}$ be a k-fold cover of G such that G is $\mathscr{H}$-critical. Suppose that G does not contain any k-brick as a subgraph. What is the minimum possible value of the difference $2|E(G)| - k|V(G)|$, as a function of k?*

2.5 DP-colorings of graphs with high chromatic number

2.5.1 Introduction

It is well-known that the list chromatic number of a graph can significantly exceed its ordinary chromatic number. On the other hand, Noel, Reed, and Wu [NRW15] established the following result, which was conjectured by Ohba [Ohb02, Conjecture 1.3]:

Theorem 2.5.1 (Noel–Reed–Wu [NRW15])**.** *Let G be an n-vertex graph with $\chi(G) \geqslant (n-1)/2$. Then $\chi_\ell(G) = \chi(G)$.*

For a graph G and $s \in \mathbb{N}$, let $\mathbf{J}(G, s)$ denote the **join** of G and a copy of K_s, i.e., the graph obtained from G by adding s new vertices that are adjacent to every vertex in $V(G)$ and to each other. It is clear from the definition that for all G and s, $\chi(\mathbf{J}(G, s)) = \chi(G) + s$. Moreover, we have $\chi_\ell(\mathbf{J}(G, s)) \leqslant \chi_\ell(G) + s$; however, this inequality can be strict. Indeed, Theorem 2.5.1 implies that for every graph G and every $s \geqslant |V(G)| - 2\chi(G) - 1$,

$$\chi_\ell(\mathbf{J}(G, s)) = \chi(\mathbf{J}(G, s)),$$

even if $\chi_\ell(G)$ is much larger than $\chi(G)$. In view of this observation, it is interesting to consider the following parameter:

$$Z_\ell(G) \coloneqq \min\{s \in \mathbb{N} \,:\, \chi_\ell(\mathbf{J}(G, s)) = \chi(\mathbf{J}(G, s))\}, \tag{2.5.1}$$

i.e., the smallest $s \in \mathbb{N}$ such that the list and the ordinary chromatic numbers of $\mathbf{J}(G, s)$ coincide. The parameter $Z_\ell(G)$ was explicitly defined by Enomoto, Ohba, Ota, and Sakamoto in [Eno+02, page 65] (they denoted it $\psi(G)$). Recently, Kim, Park, and Zhu (personal communication, 2016) obtained new lower bounds on $Z_\ell(K_{2,n})$, $Z_\ell(K_{n,n})$, and $Z_\ell(K_{n,n,n})$. One can also consider, for $n \in \mathbb{N}$,

$$Z_\ell(n) \coloneqq \max\{Z_\ell(G) \,:\, |V(G)| = n\}. \tag{2.5.2}$$

The parameter $Z_\ell(n)$ is closely related to the Noel–Reed–Wu Theorem 2.5.1, since, by definition, there exists a graph G on $n + Z_\ell(n) - 1$ vertices whose ordinary chromatic number is at least $Z_\ell(n)$ and whose list and ordinary chromatic numbers are distinct. The finiteness of $Z_\ell(n)$ for all $n \in \mathbb{N}$ was first established by Ohba [Ohb02, Theorem 1.3]. Theorem 2.5.1 yields an upper bound $Z_\ell(n) \leqslant n - 5$ for all $n \geqslant 5$; on the other hand, a result of Enomoto, Ohba, Ota, and Sakamoto [Eno+02, Proposition 6] implies that $Z_\ell(n) \geqslant n - O(\sqrt{n})$.

By analogy with (2.5.1) and (2.5.2), we consider the parameters

$$Z_{DP}(G) \coloneqq \min\{s \in \mathbb{N} \,:\, \chi_{DP}(\mathbf{J}(G, s)) = \chi(\mathbf{J}(G, s))\},$$

and

$$Z_{DP}(n) \coloneqq \max\{Z_{DP}(G) \,:\, |V(G)| = n\}.$$

The main result of this subsection is that for all graphs G, $Z_{DP}(G)$ is finite:

Theorem 2.5.2. *Let G be a graph with n vertices, m edges, and chromatic number k. Then $Z_{DP}(G) \leqslant 3m$. Moreover, if $\delta(G) \geqslant k-1$, then*

$$Z_{DP}(G) \leqslant 3m - \frac{3}{2}(k-1)n.$$

Corollary 2.5.3. *For all $n \in \mathbb{N}$, $Z_{DP}(n) \leqslant 3n^2/2$.*

Note that the upper bound on $Z_{DP}(n)$ given by Corollary 2.5.3 is quadratic in n, in contrast to the linear upper bound on $Z_\ell(n)$ implied by Theorem 2.5.1. The next result shows that the order of magnitude of $Z_{DP}(n)$ is indeed quadratic:

Theorem 2.5.4. *For all $n \in \mathbb{N}$, $Z_{DP}(n) \geqslant n^2/4 - O(n)$.*

Corollary 2.5.3 and Theorem 2.5.4 also yield the following analog of Theorem 2.5.1 for DP-coloring:

Corollary 2.5.5. *For $n \in \mathbb{N}$, let $r(n)$ denote the minimum $r \in \mathbb{N}$ such that for every n-vertex graph G with $\chi(G) \geqslant r$, we have $\chi_{DP}(G) = \chi(G)$. Then*

$$n - r(n) = \Theta(\sqrt{n}).$$

We prove Theorem 2.5.2 in §2.5.2 and Theorem 2.5.4 in §2.5.3. The derivation of Corollary 2.5.5 from Corollary 2.5.3 and Theorem 2.5.4 is straightforward; for completeness, we include it at the end of §2.5.3.

2.5.2 Proof of Theorem 2.5.2

For a graph G and a finite set A disjoint from $V(G)$, let $\mathbf{J}(G, A)$ denote the graph with vertex set $V(G) \cup A$ obtained from G be adding all edges with at least one endpoint in A (i.e., $\mathbf{J}(G, A)$ is a concrete representative of the isomorphism type of $\mathbf{J}(G, |A|)$).

First we prove the following more technical version of Theorem 2.5.2:

Theorem 2.5.6. *Let G be a k-colorable graph. Let A be a finite set disjoint from $V(G)$ and let $\mathscr{H} = (L, H)$ be a cover of $\mathbf{J}(G, A)$ such that for all $a \in A$, $|L(a)| \geqslant |A| + k$. Suppose that*

$$|A| \geqslant \frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) + |A| - |L(v)| + 1,\ 0\}. \tag{2.5.3}$$

Then $\mathbf{J}(G, A)$ is $\mathscr{H}$-colorable.

Proof. For a graph G, a set A disjoint from $V(G)$, a cover $\mathscr{H} = (L, H)$ of $\mathbf{J}(G, A)$, and a vertex $v \in V(G)$, let

$$\sigma(G, A, \mathscr{H}, v) := \max\{\deg_G(v) + |A| - |L(v)| + 1,\ 0\}$$

and

$$\sigma(G, A, \mathscr{H}) := \sum_{v \in V(G)} \sigma(G, A, \mathscr{H}, v).$$

Assume, towards a contradiction, that a tuple $(k, G, A, \mathscr{H})$ forms a counterexample which minimizes k, then $|V(G)|$, and then $|A|$. For brevity, we will use the following shortcuts:

$$\sigma(v) \coloneqq \sigma(G, A, \mathscr{H}, v); \qquad \sigma \coloneqq \sigma(G, A, \mathscr{H}).$$

Thus, (2.5.3) is equivalent to

$$|A| \geqslant \frac{3\sigma}{2}.$$

Note that $|V(G)|$ and $|A|$ are both positive. Indeed, if $V(G) = \varnothing$, then $\mathbf{J}(G, A)$ is just a clique with vertex set A, so its DP-chromatic number is $|A|$. If, on the other hand, $A = \varnothing$, then (2.5.3) implies that $|L(v)| \geqslant \deg_G(v) + 1$ for all $v \in V(G)$, so an $\mathscr{H}$-coloring of G can be constructed greedily. Furthermore, $\chi(G) = k$, since otherwise we could have used the same $(G, A, \mathscr{H})$ with a smaller value of k.

For brevity, we write $L(U) \coloneqq \bigcup_{u \in U} L(u)$ for $U \subseteq V(G) \cup A$.

Claim (A). *For every $v \in V(G)$, the graph $\mathbf{J}(G - v, A)$ is $\mathscr{H}$-colorable.*

Proof. Consider any $v_0 \in V(G)$ and let $G' \coloneqq G - v_0$. For all $v \in V(G')$, $\deg_{G'}(v) \leqslant \deg_G(v)$, and thus $\sigma(G', A, \mathscr{H}, v) \leqslant \sigma(v)$. Therefore,

$$\frac{3}{2}\sigma(G', A, \mathscr{H}) \leqslant \frac{3\sigma}{2} \leqslant |A|.$$

By the minimality of $|V(G)|$, the conclusion of Theorem 2.5.6 holds for $(k, G', A, \mathscr{H})$. In other words, $\mathbf{J}(G', A)$ is $\mathscr{H}$-colorable, as claimed. ⊣

Corollary (B). *For every $v \in V(G)$,*

$$\sigma(v) = \deg_G(v) + |A| - |L(v)| + 1 > 0.$$

Proof. Suppose that for some $v_0 \in V(G)$,

$$\deg_G(v_0) + |A| - |L(v_0)| + 1 \leqslant 0,$$

i.e.,

$$|L(v_0)| \geqslant \deg_G(v_0) + |A| + 1.$$

Using Claim (A), fix any $\mathscr{H}$-coloring I of $\mathbf{J}(G - v_0, A)$. Since v_0 still has at least

$$|L(v_0)| - (\deg_G(v_0) + |A|) \geqslant 1$$

available colors, I can be extended to an $\mathscr{H}$-coloring of $\mathbf{J}(G, A)$ greedily; a contradiction. ⊣

Claim (C). *For every $v \in V(G)$ and $x \in L(A)$, there is $y \in L(v)$ such that $xy \in E(H)$.*

Proof. Suppose that for some $a_0 \in A$, $x_0 \in L(a_0)$, and $v_0 \in V(G)$, we have $L(v_0) \cap N_H(x_0) = \varnothing$. Let $A' \coloneqq A \setminus \{a_0\}$, and for every $w \in V(G) \cup A'$, let $L'(w) \coloneqq L(w) \setminus N_H(x_0)$. Set $\mathscr{H}' \coloneqq (L', H')$, where H'

is the subgraph of H induced by the union of the sets $L'(w)$ over $w \in V(G) \cup A'$. Note that for all $a \in A'$, we have $|L'(a)| \geqslant |A'| + k$, and for all $v \in V(G)$, we have $\sigma(G, A', \mathscr{H}', v) \leqslant \sigma(v)$. Moreover, by the choice of x_0, $|L'(v_0)| = |L(v_0)|$, which, due to Corollary (B), yields $\sigma(G, A', \mathscr{H}', v_0) \leqslant \sigma(v_0) - 1$. This implies $\sigma(G, A', \mathscr{H}') \leqslant \sigma - 1$, and thus

$$\frac{3}{2}\sigma(G, A', \mathscr{H}') \leqslant \frac{3(\sigma - 1)}{2} \leqslant |A| - \frac{3}{2} < |A'|.$$

By the minimality of $|A|$, the conclusion of Theorem 2.5.6 holds for $(k, G, A', \mathscr{H}')$, i.e., the graph $\mathbf{J}(G, A')$ is $\mathscr{H}'$-colorable. By the definition of L', for any $\mathscr{H}'$-coloring I of $\mathbf{J}(G, A')$, $I \cup \{x_0\}$ is an $\mathscr{H}$-coloring of $\mathbf{J}(G, A)$. This is a contradiction. ⊣

Corollary (D). $k \geqslant 2$.

Proof. Let $v \in V(G)$ and consider any $a \in A$. Since, by Claim (C), each $x \in L(a)$ has a neighbor in $L(v)$, we have

$$|L(v)| \geqslant |L(a)| \geqslant |A| + k.$$

Using Corollary (B), we obtain

$$0 \leqslant \deg_G(v) + |A| - |L(v)| \leqslant \deg_G(v) - k,$$

i.e., $\deg_G(v) \geqslant k$. Since $V(G) \neq \varnothing$, $k \geqslant 1$, which implies $\deg_G(v) \geqslant 1$. But then $\chi(G) \geqslant 2$, as desired. ⊣

Claim (E). *H does not contain a walk of the form $x_0 y_0 x_1 y_1 x_2$, where*

- *x_0, x_1, $x_2 \in L(A)$;*
- *y_0, $y_1 \in L(V(G))$;*
- *$x_0 \neq x_1 \neq x_2$ and $y_0 \neq y_1$ (but it is possible that $x_0 = x_2$);*
- *the set $\{x_0, x_1, x_2\}$ is independent in H.*

Proof. Suppose that such a walk exists and let a_0, a_1, $a_2 \in A$ and v_0, $v_1 \in V(G)$ be such that $x_0 \in L(a_0)$, $y_0 \in L(v_0)$, $x_1 \in L(a_1)$, $y_1 \in L(v_1)$, and $x_2 \in L(a_2)$. Let $A' := A \setminus \{a_0, a_1, a_2\}$, and for every $w \in V(G) \cup A'$, let $L'(w) := L(w) \setminus N_H(x_0, x_1, x_2)$. Set $\mathscr{H}' := (L', H')$, where H' is the subgraph of H induced by the union of the sets $L'(w)$ over $w \in V(G) \cup A'$. Since $\{x_0, x_1, x_2\}$ is an independent set, for all $a \in A'$, we have $|L'(a)| \geqslant |A'| + k$, while for all $v \in V(G)$, we have $\sigma(G, A', \mathscr{H}', v) \leqslant \sigma(v)$. Moreover, since for each $i \in \{0, 1\}$, the set $\{x_0, x_1, x_2\}$ contains two distinct neighbors of y_i, we have $\sigma(G, A', \mathscr{H}', v_i) \leqslant \sigma(v_i) - 1$. Therefore, $\sigma(G, A', \mathscr{H}') \leqslant \sigma - 2$, and thus

$$\frac{3}{2}\sigma(G, A', \mathscr{H}') \leqslant \frac{3(\sigma - 2)}{2} \leqslant |A| - 3 \leqslant |A'|.$$

By the minimality of $|A|$, the conclusion of Theorem 2.5.6 holds for $(k, G, A', \mathscr{H}')$, i.e., the graph $\mathbf{J}(G, A')$ is $\mathscr{H}'$-colorable. By the definition of L', for any $\mathscr{H}'$-coloring I of $\mathbf{J}(G, A')$, $I \cup \{x_0, x_1, x_2\}$ is an $\mathscr{H}$-coloring of $\mathbf{J}(G, A)$. This is a contradiction. ⊣

Due to Corollary (D), we can choose a pair of disjoint independent sets U_0, $U_1 \subset V(G)$ such that $\chi(G - U_0) = \chi(G - U_1) = k - 1$. Choose arbitrary elements $a_1 \in A$ and $x_1 \in L(a_1)$. By Claim (C), for each $u \in U_0 \cup U_1$, there is a unique element $y(u) \in L(u)$ adjacent to x_1 in H (the uniqueness of $y(u)$ follows from the definition of a cover). Let

$$I_0 := \{y(u) \,:\, u \in U_0\} \qquad \text{and} \qquad I_1 := \{y(u) \,:\, u \in U_1\}.$$

Since U_0 and U_1 are independent sets in G, I_0 and I_1 are independent sets in H.

Claim (F). *There exists an element $a_0 \in A \setminus \{a_1\}$ such that $L(a_0) \cap N_H(I_0) \nsubseteq N_H(x_1)$.*

Proof. Assume that for all $a \in A \setminus \{a_1\}$, we have $L(a) \cap N_H(I_0) \subseteq N_H(x_1)$. Let $G' := G - U_0$, and for each $w \in V(G') \cup A$, let $L'(w) := L(w) \setminus N_H(I_0)$. By the definition of I_0, $L'(a_1) = L(a_1) \setminus \{x_1\}$, so

$$|L'(a_1)| = |L(a_1)| - 1 \geqslant |A| + (k - 1).$$

On the other hand, by our assumption, for each $a \in A \setminus \{a_1\}$, we have

$$|L'(a)| = |L(a) \setminus N_H(I_0)| \geqslant |L(a) \setminus N_H(x_1)| \geqslant |L(a)| - 1 \geqslant |A| + (k - 1).$$

Set $\mathscr{H}' := (L', H')$, where H' is the subgraph of H induced by the union of the sets $L'(w)$ over $w \in V(G') \cup A$. Since for all $v \in V(G)$, $\sigma(G', A, \mathscr{H}', v) \leqslant \sigma(v)$, the minimality of k implies the conclusion of Theorem 2.5.6 for $(k - 1, G', A, \mathscr{H}')$; in other words, the graph $\mathbf{J}(G', A)$ is $\mathscr{H}'$-colorable. By the definition of L', for any $\mathscr{H}'$-coloring I of $\mathbf{J}(G', A)$, $I \cup I_0$ is an $\mathscr{H}$-coloring of $\mathbf{J}(G, A)$; this is a contradiction. ⊣

Using Claim (F), fix some $a_0 \in A \setminus \{a_1\}$ satisfying $L(a_0) \cap N_H(I_0) \nsubseteq N_H(x_1)$, and choose any

$$x_0 \in (L(a_0) \cap N_H(I_0)) \setminus N_H(x_1).$$

Since $x_0 \in N_H(I_0)$, we can also choose $y_0 \in I_0$ so that $x_0 y_0 \in E(H)$.

Claim (G). $x_0 \notin N_H(I_1)$.

Proof. If there were $y_1 \in I_1$ such that $x_0 y_1 \in E(H)$, then $x_0 y_0 x_1 y_1 x_0$ would be a walk in H whose existence is ruled out by Claim (E). ⊣

Claim (H). *There is an element $a_2 \in A \setminus \{a_0, a_1\}$ such that $L(a_2) \cap N_H(I_1) \nsubseteq N_H(x_0, x_1)$.*

Proof. The proof is almost identical to the proof of Claim (F). Assume that for all $a \in A \setminus \{a_0, a_1\}$, we have $L(a) \cap N_H(I_1) \subseteq N_H(x_0, x_1)$. Let $G' := G - U_1$, $A' := A \setminus \{a_0\}$, and for each $w \in V(G') \cup A'$, let

$L'(w) := L(w) \setminus N_H(\{x_0\} \cup I_1)$. By the definition of I_1, $L(a_1) \cap N_H(I_1) = \{x_1\}$, so

$$|L'(a_1)| \geqslant |L(a_1)| - 2 \geqslant |A| + k - 2 = |A'| + (k-1).$$

On the other hand, by our assumption, for each $a \in A \setminus \{a_0, a_1\}$, we have

$$|L'(a)| \geqslant |L(a) \setminus N_H(x_0, x_1)| \geqslant |L(a)| - 2 \geqslant |A| + k - 2 = |A'| + (k-1).$$

Set $\mathscr{H}' := (L', H')$, where H' is the subgraph of H induced by the union of the sets $L'(w)$ over $w \in V(G') \cup A'$. Since for all $v \in V(G)$, $\sigma(G', A', \mathscr{H}', v) \leqslant \sigma(v)$, the minimality of k implies the conclusion of Theorem 2.5.6 for $(k-1, G', A', \mathscr{H}')$; in other words, the graph $\mathbf{J}(G', A')$ is $\mathscr{H}'$-colorable. By the definition of L', for any $\mathscr{H}'$-coloring I of $\mathbf{J}(G', A)$, $I \cup \{x_0\} \cup I_1$ is an $\mathscr{H}$-coloring of $\mathbf{J}(G, A)$. This is a contradiction. $\dashv$

Now we are ready to finish the proof of Theorem 2.5.6. Fix some $a_2 \in A \setminus \{a_0, a_1\}$ satisfying $L(a_2) \cap N_H(I_1) \nsubseteq N_H(x_0, x_1)$, and choose any

$$x_2 \in (L(a_2) \cap N_H(I_1)) \setminus N_H(x_0, x_1).$$

Since $x_2 \in N_H(I_1)$, there is $y_1 \in I_1$ such that $x_2 y_1 \in E(H)$. Then $x_0 y_0 x_1 y_1 x_2$ is a walk in H contradicting the conclusion of Claim (E). ■

Now it is easy to derive Theorem 2.5.2. Indeed, let G be a graph with n vertices, m edges, and chromatic number k, let A be a finite set disjoint from $V(G)$, and let $\mathscr{H} = (L, H)$ be a cover of $\mathbf{J}(G, A)$ such that for all $v \in V(G)$ and $a \in A$, $|L(v)| = |L(a)| = \chi(\mathbf{J}(G, A)) = |A| + k$. Note that

$$\frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) - |L(v)| + |A| + 1,\ 0\} = \frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) - k + 1,\ 0\}.$$

If $|A| \geqslant 3m$, then

$$\frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) - k + 1,\ 0\} \leqslant \frac{3}{2} \sum_{v \in V(G)} \deg_G(v) = 3m \leqslant |A|,$$

so Theorem 2.5.6 implies that $\mathbf{J}(G, A)$ is $\mathscr{H}$-colorable, and hence $Z_{DP}(G) \leqslant 3m$. Moreover, if $\delta(G) \geqslant k-1$, then

$$\frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) - k + 1,\ 0\} = \frac{3}{2} \sum_{v \in V(G)} (\deg_G(v) - k + 1) = 3m - \frac{3}{2}(k-1)n,$$

so $Z_{DP}(G) \leqslant 3m - \frac{3}{2}(k-1)n$, as desired. Finally, Corollary 2.5.3 follows from Theorem 2.5.2 and the fact that an n-vertex graph can have at most $\binom{n}{2} \leqslant n^2/2$ edges.

2.5.3 Proof of Theorem 2.5.4

We will prove the following precise version of Theorem 2.5.4:

Theorem 2.5.7. *For all even $n \in \mathbb{N}$, $Z_{DP}(n) \geqslant n^2/4 - n$.*

Proof. Let $n \in \mathbb{N}$ be even and let $k := n/2 - 1$. Note that $n^2/4 - n = k^2 - 1$. Thus, it is enough to exhibit an n-vertex bipartite graph G and a k^2-fold cover $\mathscr{H}$ of $\mathbf{J}(G, k^2 - 2)$ such that $\mathbf{J}(G, k^2 - 2)$ is not $\mathscr{H}$-colorable.

Let $G \cong K_{n/2,n/2}$ be an n-vertex complete bipartite graph with parts $X = \{x, x_0, \ldots, x_{k-1}\}$ and $Y = \{y, y_0, \ldots, y_{k-1}\}$, where the indices $0, \ldots, k-1$ are viewed as elements of the additive group $\mathbb{Z}/k\mathbb{Z}$ of integers modulo k. Let A be a set of size $k^2 - 2$ disjoint from $X \cup Y$. For each $u \in X \cup Y \cup A$, let

$$L(u) := \{u\} \times (\mathbb{Z}/k\mathbb{Z}) \times (\mathbb{Z}/k\mathbb{Z}).$$

Let H be the graph with vertex set

$$(X \cup Y \cup A) \times (\mathbb{Z}/k\mathbb{Z}) \times (\mathbb{Z}/k\mathbb{Z})$$

in which the following pairs of vertices are adjacent:

- (u, i, j) and (u, i', j') for all $u \in X \cup Y \cup A$ and $i, j, i', j' \in \mathbb{Z}/k\mathbb{Z}$ such that $(i, j) \neq (i', j')$;
- (u, i, j) and (v, i, j) for all $u \in \{x, y\} \cup A$, $v \in N_{\mathbf{J}(G,A)}(u)$, and $i, j \in \mathbb{Z}/k\mathbb{Z}$;
- (x_s, i, j) and $(y_t, i + s, j + t)$ for all $s, t, i, j \in \mathbb{Z}/k\mathbb{Z}$.

It is easy to see that $\mathscr{H} := (L, H)$ is a cover of $\mathbf{J}(G, A)$. We claim that $\mathbf{J}(G, A)$ is not $\mathscr{H}$-colorable. Indeed, suppose that I is an $\mathscr{H}$-coloring of $\mathbf{J}(G, A)$. For each $u \in X \cup Y \cup A$, let $i(u)$ and $j(u)$ be the unique elements of $\mathbb{Z}/k\mathbb{Z}$ such that $(u, i(u), j(u)) \in I$. By the construction of H and since I is an independent set, we have

$$(i(u), j(u)) \neq (i(a), j(a))$$

for all $u \in X \cup Y$ and $a \in A$. Since all the $k^2 - 2$ pairs $(i(a), j(a))$ for $a \in A$ are pairwise distinct, $(i(u), j(u))$ can take at most 2 distinct values as u is ranging over $X \cup Y$. One of those 2 values is $(i(y), j(y))$, and if $u \in X$, then

$$(i(u), j(u)) \neq (i(y), j(y)),$$

so the value of $(i(u), j(u))$ must be the same for all $u \in X$; let us denote it by (i, j). Similarly, the value of $(i(u), j(u))$ is the same for all $u \in Y$, and we denote it by (i', j'). It remains to notice that the vertices $(x_{i'-i}, i, j)$ and $(y_{j'-j}, i', j')$ are adjacent in H, so I is not independent. ■

Now we can prove Corollary 2.5.5:

Proof of Corollary 2.5.5. First, suppose that G is an n-vertex graph with $\chi(G) = r$ that maximizes the difference $\chi_{DP}(G) - \chi(G)$. Adding edges to G if necessary, we may arrange G to be a complete r-partite graph. Assuming $2r > n$, at least $2r - n$ of the parts must be of size 1, i.e., G is of the form $\mathbf{J}(G', 2r - n)$ for some $2(n - r)$-vertex graph G'. By Corollary 2.5.3, we have $\chi_{DP}(G) = \chi(G)$ as long as $2r - n \geqslant 6(n - r)^2$, which holds for all $r \geqslant n - (1/\sqrt{6} - o(1))\sqrt{n}$. This establishes the upper bound $r(n) \leqslant n - \Omega(\sqrt{n})$.

On the other hand, due to Theorem 2.5.4, for each n, we can find a graph G with s vertices, where $s \leqslant (2+o(1))\sqrt{n}$, such that $\chi_{DP}(\mathbf{J}(G, n-s)) > \chi(\mathbf{J}(G, n-s))$. Since $\mathbf{J}(G, n-s)$ is an n-vertex graph, we get

$$r(n) > \chi(\mathbf{J}(G, n-s)) = \chi(G) + n - s \geqslant n - (2+o(1))\sqrt{n} = n - O(\sqrt{n}). \qquad \blacksquare$$

2.6 Fractional DP-coloring

2.6.1 Introduction

In this section we introduce and study the fractional version of DP-coloring. We start with a brief review of the classical concepts of fractional coloring and fractional list coloring. For a survey of the topic, see, e.g., [SU97, Chapter 3].

Let G be a graph. An (η, k)**-coloring** of G, where $\eta \in [0; 1]$ and $k \in \mathbb{N}$, is a map $f \colon V(G) \to \mathrm{Pow}(k)$ with the following properties:

(F1) for every vertex $u \in V(G)$, we have $|f(u)| \geqslant \eta k$;

(F2) for every edge $uv \in E(G)$, we have $f(u) \cap f(v) = \varnothing$.

For given $k \in \mathbb{N}$, let

$$\vartheta(G, k) \coloneqq \max\{\eta \in [0; 1] \,:\, G \text{ admits an } (\eta, k)\text{-coloring}\}.$$

(The maximum is attained, as only the values of the form ℓ/k for integer ℓ are relevant.) The **fractional chromatic number** $\chi^*(G)$ of G is defined by

$$\chi^*(G) \coloneqq \inf\{\vartheta(G, k)^{-1} \,:\, k \in \mathbb{N}\}. \tag{2.6.1}$$

It is well-known [SU97, §3.1] that the infimum in (2.6.1) is actually a minimum: For every graph G, there is some $k \in \mathbb{N}$ such that $\chi^*(G) = \vartheta(G, k)^{-1}$. In particular, $\chi^*(G)$ is always a rational number.

Fractional coloring allows a natural list-version. Let G be a graph and let L be a list assignment for G. An (η, L)**-coloring** of G, where $\eta \in [0; 1]$, is a map f that associates to each $u \in V(G)$ a subset $f(u) \subseteq L(u)$ with the following properties:

(FL1) for every vertex $u \in V(G)$, we have $|f(u)| \geqslant \eta |L(u)|$;

(FL2) for every edge $uv \in E(G)$, we have $f(u) \cap f(v) = \varnothing$.

We say that L is a k**-list assignment** if $|L(u)| = k$ for all $u \in V(G)$. For given $k \in \mathbb{N}$, let

$$\vartheta_\ell(G, k) \coloneqq \max\{\eta \in [0; 1] \,:\, G \text{ admits an } (\eta, L)\text{-coloring for every } k\text{-list assignment } L \text{ for } G\}.$$

The **fractional list-chromatic number** $\chi^*_\ell(G)$ of G is defined by

$$\chi^*_\ell(G) \coloneqq \inf\{\vartheta_\ell(G, k)^{-1} \,:\, k \in \mathbb{N}\}.$$

Somewhat surprisingly, Alon, Tuza, and Voigt [ATV97] showed that $\chi^*_\ell(G) = \chi^*(G)$ for all graphs G and, in fact, for each G, there is $k \in \mathbb{N}$ such that

$$\chi^*_\ell(G) = \chi^*(G) = \vartheta_\ell(G,k)^{-1} = \vartheta(G,k)^{-1}.$$

(Recall that the list-chromatic number of a graph cannot be bounded above by any function of its ordinary chromatic number.)

Now we proceed with our main definitions. Given a cover $\mathscr{H} = (L, H)$ of a graph G, we refer to the edges of H connecting distinct parts of the partition $\{L(u) : u \in V(G)\}$ as **cross-edges**. A subset $S \subseteq V(H)$ is **quasi-independent** if it spans no cross-edges.

Definition 2.6.1. Let $\mathscr{H} = (L, H)$ be a cover of a graph G and let $\eta \in [0; 1]$. An $(\eta, \mathscr{H})$**-coloring** of G is a quasi-independent set $S \subseteq V(H)$ such that $|S \cap L(u)| \geqslant \eta|L(u)|$ for all $u \in V(G)$.

Definition 2.6.2. Let G be a graph. For $k \in \mathbb{N}$, let

$$\vartheta_{DP}(G,k) := \max\{\eta \in [0;1] : G \text{ admits an } (\eta,\mathscr{H})\text{-coloring for every } k\text{-fold cover } \mathscr{H} \text{ of } G\}.$$

The **fractional DP-chromatic number** $\chi^*_{DP}(G)$ is defined by

$$\chi^*_{DP}(G) := \inf\{\vartheta_{DP}(G,k)^{-1} : k \in \mathbb{N}\}. \tag{2.6.2}$$

Clearly, for any graph G, we have $\chi^*(G) \leqslant \chi^*_{DP}(G) \leqslant \chi_{DP}(G)$. Our results described below imply that both inequalities can be strict.

Since $\chi_{DP}(C_n) = 3$ for any cycle C_n, a connected graph G satisfies $\chi_{DP}(G) \leqslant 2$ if and only if G is a tree. The first result of this section is the characterization of graphs G with $\chi^*_{DP}(G) \leqslant 2$:

Theorem 2.6.3. *Let G be a connected graph. Then $\chi^*_{DP}(G) \leqslant 2$ if and only if G contains no odd cycles and at most one even cycle. Furthermore, if G contains no odd cycles and exactly one even cycle, then $\chi^*_{DP}(G) = 2$, even though $\vartheta_{DP}(G,k)^{-1} > 2$ for all $k \in \mathbb{N}$ (i.e., the infimum in* (2.6.2) *is not attained).*

Theorem 2.6.3 shows that the Alon–Tuza–Voigt theorem does not extend to fractional DP-coloring, as every connected bipartite graph G with $|E(G)| \geqslant |V(G)| + 1$ satisfies $\chi^*(G) = \chi(G) = 2$, while $\chi^*_{DP}(G) > 2$. Theorem 2.6.3 also provides examples of graphs for which the infimum in (2.6.2) is not attained. However, the following natural question remains open:

Problem 2.6.4. *Do there exist graphs G for which $\chi^*_{DP}(G)$ is irrational?*

Recall that, by Theorem 2.1.5, $\chi_{DP}(G) = \Omega(d/\ln d)$, where d is the maximum average degree of G. Using a similar argument, we can extend this asymptotic lower bound to the fractional setting:

Theorem 2.6.5. *If G is a graph of maximum average degree $d \geqslant 4$, then $\chi^*_{DP}(G) \geqslant d/(2\ln d)$.*

From Theorem 2.6.5, it follows that $\chi^*_{DP}(G)$ cannot be bounded above by any function of $\chi^*(G)$, since there exist bipartite graphs of arbitrarily high average degree.

Note that every graph G with maximum average degree d is d-degenerate, i.e., it has an acyclic orientation D with $\Delta^+(D) \leqslant d$. Our next result describes additional conditions on such an orientation D under which the lower bound given by Theorem 2.6.5 is asymptotically tight.

Theorem 2.6.6. *Suppose that a graph G has an acyclic orientation D such that*

(D1) $\Delta^+(D) \leqslant d$*; and*

(D2) *for all $uv \in E(D)$, there is no directed uv-path of even length in D.*

Then $\chi^*_{DP}(G) \leqslant (1 + o(1))d/\ln d$.

Obviously, every orientation D of a *bipartite* graph G satisfies condition (D2) of Theorem 2.6.6. Hence, we obtain the following:

Corollary 2.6.7. *If G is a d-degenerate bipartite graph, then $\chi^*_{DP}(G) \leqslant (1 + o(1))d/\ln d$.*

The conclusion of Theorem 2.6.6 is interesting even for the ordinary fractional chromatic number, especially since its requirements are satisfied by several known constructions of graphs with high girth and high chromatic number. For example, consider the following scheme analyzed in [KN99] (based on the Blanche Descartes construction of triangle-free graphs with high chromatic number). Start by setting $G_1 := K_2$ and let D_1 be an orientation of G_1. When G_i and D_i are defined for some i, take an $|V(G_i)|$-uniform non-$(i + 1)$-colorable hypergraph H_i. Build G_{i+1} by making $V(H_i)$ an independent set, adding $|E(H_i)|$ disjoint copies of G_i, establishing a bijection between the copies of G_i and the edges of H_i, and joining each copy to its corresponding edge via a perfect matching. Finally, let D_{i+1} be an orientation of G_{i+1} obtained by orienting each copy of G_i according to D_i and directing every remaining edge towards its endpoint in $V(H_i)$. It is easy to show [KN99, Property 1] that $\chi(G_i) \geqslant i + 1$ for all i, and it is clear from the construction that the orientation D_i is acyclic and the out-degree of every vertex in D_i is at most i. Furthermore, the (undirected) subgraph of G_i induced by the vertices reachable in D_i from any given vertex $u \in V(G_i)$, including u itself, is acyclic; in particular, for all $uv \in E(D_i)$, the only directed uv-path is the single edge $u \to v$. Therefore, condition (D2) of Theorem 2.6.6 holds and we can conclude $\chi^*_{DP}(G_i) \leqslant (1 + o(1))i/\ln i$. Note that the girth of G_i can be made arbitrarily large by using hypergraphs of large girth in the construction.

Another related family of graphs of high chromatic number that falls under the conditions of Theorem 2.6.6 is described in [Alo+16, Theorem 3.4].

The above examples yield the following corollary:

Corollary 2.6.8. *For all d, $g \in \mathbb{N}$, there exists a graph $G_{d,g}$ with chromatic number at least d, girth at least g, and $\chi^*_{DP}(G_{d,g}) \leqslant (1 + o(1))d/\ln d$.*

Theorem 2.6.6 is related to the following open problem posed by David Harris:

Problem 2.6.9 (Harris [Har16])**.** *Is it true that $\chi^*(G) = O(d/\ln d)$ for any triangle-free graph G of maximum average degree d?*

The remainder of this section is organized as follows. First, we prove Theorem 2.6.5 in §2.6.2. Then, in §2.6.3, we establish Theorem 2.6.3. Finally, §2.6.4 is dedicated to the proof of Theorem 2.6.6.

2.6.2 Proof of Theorem 2.6.5

What follows is a slight modification of the proof of Theorem 2.1.5. Let G be a graph of maximum average degree d. After passing to a subgraph, we may assume that the average degree of G itself is d. Set $n \coloneqq |V(G)|$ and $m \coloneqq |E(G)|$. Then we have $m = dn/2$. Let $\eta_0 \coloneqq 2\ln d/d$. Our goal is to show that $\vartheta_{DP}(G,k) < \eta_0$ for all $k \in \mathbb{N}$. To that end, fix arbitrary $k \in \mathbb{N}$ and let $\eta \coloneqq \lceil \eta_0 k\rceil/k$. It is enough to prove $\vartheta_{DP}(G,k) < \eta$. Let $\{L(u) \,:\, u \in V(G)\}$ be a collection of pairwise disjoint sets of size k. Define $X \coloneqq \bigcup_{u\in V(G)} L(u)$, and build a random graph H with vertex set X by making each $L(u)$ a clique and putting, independently for each $uv \in E(G)$, a uniformly random perfect matching between $L(u)$ and $L(v)$. Let $\mathscr{H} \coloneqq (L,H)$ denote the resulting random k-fold cover of G. Consider an arbitrary set $S \subseteq X$ with $|S \cap L(u)| = \eta k$ for all $u \in V(G)$. Since the matchings corresponding to different edges of G are drawn independently from each other, we have

$$\begin{aligned}
&\mathbb{P}\left[S \text{ is quasi-independent in } H\right] \\
&= \prod_{uv\in E(G)} \mathbb{P}\left[\text{there are no cross-edges between } S\cap L(u) \text{ and } S \cap L(v)\right] = \left(\frac{\binom{(1-\eta)k}{\eta k}}{\binom{k}{\eta k}}\right)^m.
\end{aligned}$$

There are $\binom{k}{\eta k}^n$ possible choices for S, so

$$\mathbb{P}\left[G \text{ is } (\eta, \mathscr{H})\text{-colorable}\right] \leqslant \left(\frac{\binom{(1-\eta)k}{\eta k}}{\binom{k}{\eta k}}\right)^m \binom{k}{\eta k}^n = \left(\binom{(1-\eta)k}{\eta k}^{d/2}\binom{k}{\eta k}^{-(d/2-1)}\right)^n.$$

Thus, we only need to show that

$$\binom{(1-\eta)k}{\eta k}^{d/2}\binom{k}{\eta k}^{-(d/2-1)} < 1.$$

Notice that

$$\binom{(1-\eta)k}{\eta k}\binom{k}{\eta k}^{-1} = \prod_{i=0}^{\eta k-1}\frac{(1-\eta)k-i}{k-i} \leqslant (1-\eta)^{\eta k}.$$

Additionally,

$$\binom{k}{\eta k} \leqslant \left(\frac{e}{\eta}\right)^{\eta k}.$$

Therefore,

$$\binom{(1-\eta)k}{\eta k}^{d/2}\binom{k}{\eta k}^{-(d/2-1)} \leqslant \left(\frac{e(1-\eta)^{d/2}}{\eta}\right)^{\eta k},$$

so it is enough to establish

$$e(1-\eta)^{d/2} < \eta.$$

Since $1-\eta \leqslant \exp(-\eta)$, we have

$$e(1-\eta)^{d/2} \leqslant e\cdot\exp(-\eta d/2) \leqslant ed^{-1} < \eta,$$

as long as $d > e^{e/2} \approx 3.89$, as desired.

2.6.3 Proof of Theorem 2.6.3

Lemma 2.6.10. *If G is a graph such that $|E(G)| \geqslant |V(G)| + 1$, then $\chi^*_{DP}(G) > 2$.*

Proof. Set $n := |V(G)|$. Without loss of generality, we may assume that $|E(G)| = n + 1$. Let $\eta_0 \in (0; 1/2)$ be a number close to $1/2$ (it will be clear from the rest of the proof what value η_0 should take). Our aim is to show that for all $k \in \mathbb{N}$, $\vartheta_{DP}(G, k) < \eta_0$. Fix $k \in \mathbb{N}$ and let $\eta := \lceil \eta_0 k \rceil / k$, so it suffices to show that $\vartheta_{DP}(G, k) < \eta$. We use the same approach and notation as in the proof of Theorem 2.6.5 (see §2.6.2). Thus, $\mathscr{H} = (L, H)$ is a random k-fold cover of G, where $V(H) = X$, and if $S \subseteq X$ is a set with $|S \cap L(u)| = \eta k$ for all $u \in V(G)$, then

$$\mathbb{P}\left[S \text{ is quasi-independent in } H\right] = \left(\frac{\binom{(1-\eta)k}{\eta k}}{\binom{k}{\eta k}}\right)^{n+1},$$

so the probability that G is $(\eta, \mathscr{H})$-colorable is at most

$$\left(\frac{\binom{(1-\eta)k}{\eta k}}{\binom{k}{\eta k}}\right)^{n+1} \binom{k}{\eta k}^{n} = \binom{(1-\eta)k}{\eta k}^{n+1} \binom{k}{\eta k}^{-1}.$$

Note that

$$\binom{(1-\eta)k}{\eta k} = \binom{(1-\eta)k}{(1-2\eta)k} \leqslant \left(\frac{e(1-\eta)}{1-2\eta}\right)^{(1-2\eta)k},$$

Additionally,

$$\binom{k}{\eta k} \geqslant \left(\frac{1}{\eta}\right)^{\eta k}.$$

Therefore, the probability that G is $(\eta, \mathscr{H})$-colorable is less than 1 provided that

$$\left(\frac{e(1-\eta)}{1-2\eta}\right)^{(1-2\eta)(n+1)} < \left(\frac{1}{\eta}\right)^{\eta}.$$

It remains to notice that, as $\eta \to 1/2$, we have

$$\left(\frac{e(1-\eta)}{1-2\eta}\right)^{(1-2\eta)(n+1)} \to 1, \qquad \text{while} \qquad \left(\frac{1}{\eta}\right)^{\eta} \to \sqrt{2}. \qquad \blacksquare$$

Lemma 2.6.11. *If G is a cycle of even length, then $\chi^*_{DP}(G) = 2$, while $\vartheta_{DP}(G, k)^{-1} > 2$ for all $k \in \mathbb{N}$.*

Proof. Let the vertex and the edge sets of G be $\{v_1, \ldots, v_n\}$ and $\{v_1v_2, v_2v_3, \ldots, v_nv_1\}$. Given $k \in \mathbb{N}$ and a permutation $\sigma \colon \{1, \ldots, k\} \to \{1, \ldots, k\}$, we define a k-fold cover $\mathscr{H}_\sigma = (L_\sigma, H_\sigma)$ of G as follows. First, for each $1 \leqslant i \leqslant k$, let

$$L_\sigma(v_i) := \{i\} \times \{1, \ldots, k\}.$$

Then, for each $1 \leqslant i < n$, define

$$E_{H_\sigma}(L_\sigma(v_i), L_\sigma(v_{i+1})) := \{\{(i, j), (i+1, j)\} \,:\, 1 \leqslant j \leqslant k\}.$$

Finally, let

$$E_{H_\sigma}(L_\sigma(v_1), L_f(v_n)) \coloneqq \{\{(1, j), (n, \sigma(j))\} \,:\, 1 \leqslant j \leqslant k\}.$$

It is clear that to determine $\vartheta_{DP}(G, k)$ it is enough to consider k-fold covers of the form $\mathscr{H}_\sigma$ for some σ.

Suppose that $\vartheta_{DP}(G, k) = 1/2$ for some $k \in \mathbb{N}$. Consider a permutation $\sigma \colon \{1, \ldots, k\} \to \{1, \ldots, k\}$ that consists of a single cycle. Note that if $X \subseteq \{1, \ldots, k\}$ satisfies $\sigma(X) = X$, then $X \in \{\varnothing, \{1, \ldots, k\}\}$. Let S be a $(1/2, \mathscr{H}_\sigma)$-coloring of G. For each $1 \leqslant i \leqslant k$, let

$$S_i \coloneqq \{j \,:\, (i, j) \in S\}.$$

Since S is quasi-independent, $S_i \cap S_{i+1} = \varnothing$ for all $1 \leqslant i < n$. But we also have $|S_i| = |S_{i+1}| = k/2$, so $S_{i+1} = \{1, \ldots, k\} \setminus S_i$. Since n is even, we conclude that $S_n = \{1, \ldots, k\} \setminus S_1$. For every $j \in S_1$, we have $\sigma(j) \notin S_n$, which yields $\sigma(j) \in S_1$. In other words, $\sigma(S_1) = S_1$. But then $S_1 \in \{\varnothing, \{1, \ldots, k\}\}$; a contradiction.

It remains to prove that for any $\eta < 1/2$, there is $k \in \mathbb{N}$ such that $\vartheta_{DP}(G, k) \geqslant \eta$. Take a large odd integer k and let $\sigma \colon \{1, \ldots, k\} \to \{1, \ldots, k\}$ be a permutation. Write σ as a product of disjoint cycles:

$$\sigma = \pi_1 \cdots \pi_m.$$

We may rearrange the set $\{1, \ldots, k\}$ so that the support of each cycle π_i is an interval $\{\ell_i, \ldots, r_i\}$, and

$$\pi_i(\ell_i) = \ell_i + 1, \quad \pi_i(\ell_i + 1) = \ell_i + 2, \quad \ldots, \quad \pi_i(r_i) = \ell_i.$$

Then $\sigma(i) \leqslant i + 1$ for all $1 \leqslant i \leqslant k$. Now let

$$X \coloneqq \{1, \ldots, (k-1)/2\} \quad \text{and} \quad Y \coloneqq \{(k+3)/2, \ldots, k\}.$$

Note that $|X| = |Y| = (k-1)/2$, $X \cap Y = \varnothing$, and $\sigma(X) \cap Y = \varnothing$. Hence, if we define

$$S \coloneqq \{(i, j) \,:\, 1 \leqslant i \leqslant n,\ j \in X \text{ if } i \text{ is odd and } j \in Y \text{ if } i \text{ is even}\},$$

then S is a $((1 - 1/k)/2, \mathscr{H}_\sigma)$-coloring of G, and we are done. ∎

Proof of Theorem 2.6.3. Let G be a connected graph and suppose that $\chi^*_{DP}(G) \leqslant 2$. Even the ordinary fractional chromatic number of any odd cycle exceeds 2 (see [SU97, Proposition 3.1.2]), so G must be bipartite. Furthermore, by Lemma 2.6.10, $|E(G)| \leqslant |V(G)|$, so G contains at most one even cycle. Conversely, suppose that G contains no odd cycles and at most one even cycle. If G is acyclic, then $\chi^*_{DP}(G) = \chi_{DP}(G) \leqslant 2$. It remains to consider the case when G contains a single even cycle. On the one hand, Lemma 2.6.11 shows that $\vartheta_{DP}(G, k)^{-1} > 2$ for all $k \in \mathbb{N}$. On the other hand, G is obtained from an even cycle by repeatedly adding vertices of degree 1, so we can combine the result of Lemma 2.6.11 with the following obvious observation to conclude that $\chi^*_{DP}(G) = 2$:

Observation. *Let G be a graph and let $u \in V(G)$. Suppose that*

$$\deg_G(u) \leqslant \chi^*_{DP}(G - u) - 1.$$

Then $\chi^*_{DP}(G) = \chi^*_{DP}(G - u)$. ■

2.6.4 Proof of Theorem 2.6.6

Let D be a digraph. We use $R^+_D(u)$ to denote the set of all vertices $v \in V(D)$ that are reachable from u via a directed path of positive length (so D is acyclic if and only if $u \notin R^+_D(u)$ for all $u \in V(D)$). Let $R^-_D(u)$ denote the set of all $v \in V(D)$ such that $u \in R^+_D(v)$. We write

$$R^+_D[u] \coloneqq R^+_D(u) \cup \{u\} \qquad \text{and} \qquad R^-_D[u] \coloneqq R^-_D(u) \cup \{u\}.$$

For a subset $U \subseteq V(D)$, let

$$R^+_D(U) \coloneqq \bigcup_{u \in U} R^+_D(u); \qquad R^+_D[U] \coloneqq \bigcup_{u \in U} R^+_D[u],$$

and $R^-_D(U)$ and $R^-_D[U]$ are defined similarly. We use expressions $|S|$ and $\#S$ for the cardinality of a set S interchangeably (usually, $\#S$ suggests that it is a random variable).

Now we can begin the proof. Let G, D, and d be as in the statement of Theorem 2.6.6. For brevity, we set $V \coloneqq V(G)$ and omit subscripts G and D in expressions such as $N_G(u)$, $R^-_D[u]$, $\deg^+_D(u)$, etc. We will often use the acyclicity of D to make inductive definitions or arguments by describing how to deal with a vertex u provided that all v reachable from u have already been considered.

Fix $\varepsilon \in (0; 1)$ and define $\eta \coloneqq (1 - \varepsilon) \ln d/d$. We will show that $\chi^*_{DP}(G) \leqslant \eta^{-1}$ if d is large enough (as a function of ε).

Let $\mathscr{H} = (L, H)$ be a k-fold cover of G. Our aim is to show that if k is sufficiently large (where the lower bound may depend on the entire graph G), then G has an $(\eta, \mathscr{H})$-coloring. For a set $U \subseteq V$, let $L(U) \coloneqq \bigcup_{u \in U} L(u)$ and let $\mathbf{QI}(U)$ denote the set of all quasi-independent sets contained in $L(U)$.

Let F be the orientation of the cross-edges of H in which a cross-edge xy is directed from x to y if and only if the vertices $u, v \in V$ such that $x \in L(u)$ and $y \in L(v)$ satisfy $uv \in E(D)$. Again, we omit subscripts H and F in expressions such as $N_H[x]$, $R^+_F(x)$, etc.

Given a set of probabilities $p(u) \in [0; 1]$ for $u \in V$, we define random subsets $S(u) \subseteq L(u)$ inductively as follows. Consider $u \in V$ and suppose that the sets $S(v)$ for all v reachable from u have already been defined. Independently for each $x \in L(u)$, set

$$\xi(x) \coloneqq \begin{cases} 1 & \text{with probability } p(u); \\ 0 & \text{with probability } 1 - p(u), \end{cases} \tag{2.6.3}$$

Define

$$L'(u) \coloneqq \{x \in L(u) \,:\, N^+(x) \cap S(v) = \varnothing \text{ for all } v \in N^+(u)\},$$

and then

$$S(u) := \{x \in L'(u) \,:\, \xi(x) = 1\}.$$

Note that for every $u \in V$, the set $S(u)$ only depends on the random choices associated with the elements of $L(R^+[u])$. For each $U \subseteq V$, write $S(U) := \bigcup_{u \in U} S(u)$ and set $S := S(V)$. By construction, S is always a quasi-independent set. We will argue that, for a suitable choice of $\{p(u) \,:\, u \in V\}$ and sufficiently large k, $|S(u)| \geqslant \eta k$ for all $u \in V$ with high probability.

We start with a positive correlation inequality.

Lemma 2.6.12. *Let $u \in V$ and define*

$$A := R^+[u] \setminus R^-[N^+[u]].$$

Let $Q \in \mathbf{QI}(A)$ and let $Y \subseteq L(N^+(u))$. Then

$$\mathbb{P}\left[y \notin S \textit{ for all } y \in Y \,|\, S(A) = Q\right] \geqslant \prod_{y \in Y} \mathbb{P}\left[y \notin S \,|\, S(A) = Q\right].$$

Proof. Since nothing in the statement of the lemma depends on the vertices outside of $R^+[u]$, we may pass to a subgraph and assume that $V = R^+[u]$. Set

$$B := R^-[N^+[u]],$$

so $B = V \setminus A$. The lemma is trivially true if $Y = \varnothing$, so we may assume $Y \neq \varnothing$, and hence $N^+(u) \neq \varnothing$.

Notice that the graph $G[B]$ is bipartite. Indeed, consider any $v \in B$. Then, on the one hand, v is reachable from u, and, on the other hand, there is a vertex $w \in N^+(u)$ reachable from v. We claim that if P_1 and P_2 are two directed uv-paths, then $\mathrm{length}(P_1) \equiv \mathrm{length}(P_2) \pmod 2$. Indeed, let P_3 be any directed vw-path. If $\mathrm{length}(P_1) \not\equiv \mathrm{length}(P_2) \pmod 2$, then either $P_1 + P_3$ or $P_2 + P_3$ is a directed uw-path of even length, which contradicts assumption (D2). Thus, we can 2-color the vertices in B based on the parity of the directed paths leading from u to them.

Let $\{U_1, U_2\}$ be a partition of B into two independent sets such that $u \in U_1$. Define a random subset $X_\xi \subseteq L(B)$ as follows:

$$X_\xi := \{x \in L(U_1) \,:\, \xi(x) = 1\} \cup \{x \in L(U_2) \,:\, \xi(x) = 0\}.$$

(Recall that ξ is defined in (2.6.3).) The set X_ξ is obtained by independently selecting each element $x \in L(B)$ with probability $q(x)$ given by

$$q(x) := \begin{cases} p(v) & \text{if } x \in L(v) \text{ for } v \in U_1; \\ 1 - p(v) & \text{if } x \in L(v) \text{ for } v \in U_2. \end{cases}$$

To complete the construction of the set S, given that $S(A) = Q$, we only need to know the values $\xi(x)$ for all $x \in L(B)$. Since all of them are determined by the set X_ξ, we may, for fixed $X \subseteq L(B)$, denote by S_X the value S would take under the assumptions $S(A) = Q$ and $X_\xi = X$. For each $x \in L(B)$, let

$$\mathcal{F}_x \coloneqq \{X \subseteq L(B) \,:\, x \in S_X\}.$$

Recall that a family $\mathcal{F}$ of sets is **increasing** if whenever $X_1 \supseteq X_2 \in \mathcal{F}$, we also have $X_1 \in \mathcal{F}$; similarly, $\mathcal{F}$ is **decreasing** if $X_1 \subseteq X_2 \in \mathcal{F}$ implies $X_1 \in \mathcal{F}$.

Claim (A). *For each $x \in L(U_1)$, the family $\mathcal{F}_x$ is increasing; while for each $x \in L(U_2)$, the family $\mathcal{F}_x$ is decreasing.*

Proof. We argue inductively. Let $v \in B$ and suppose that the claim has been verified for all $x \in L(w)$ with $w \in B$ reachable from v. Consider any $x \in L(v)$. We will give the proof for the case $v \in U_1$, as the case $v \in U_2$ is analogous. By definition,

$$x \in S \iff \xi(x) = 1 \text{ and } y \notin S \text{ for all } y \in N^+(x). \tag{2.6.4}$$

If $x \in N^-(Q)$, then $\mathcal{F}_x = \varnothing$ and there is nothing to prove. If, on the other hand, $x \notin N^-(Q)$, then (2.6.4) yields

$$\mathcal{F}_x = \{X \subseteq L(B) \,:\, x \in X\} \cap \bigcap_{y \in N^+(x) \cap L(B)} \mathcal{F}_y^{\mathsf{c}},$$

where $\mathcal{F}_y^{\mathsf{c}}$ denotes the complement of $\mathcal{F}_y$. Each $y \in N^+(x) \cap L(B)$ belongs to $L(U_2)$, so, by the inductive assumption, the families $\mathcal{F}_y$ are decreasing, while their complements $\mathcal{F}_y^{\mathsf{c}}$ are increasing. Therefore, $\mathcal{F}_x$ is an intersection of increasing families, so it is itself increasing. ⊣

With Claim (A) in hand, the conclusion of the lemma follows from the fact that $Y \subseteq L(U_2)$ and a form of the FKG inequality, tracing back to Kleitman [Kle66]:

Theorem 2.6.13 ([AS00, Theorem 6.3.2])**.** *Let X be a random subset of a finite set I obtained by selecting each $i \in I$ independently with probability $q(i) \in [0; 1]$. If $\mathcal{F}$ and $\mathcal{G}$ are increasing families of subsets of I, then*

$$\mathbb{P}\left[X \in \mathcal{F} \text{ and } X \in \mathcal{G}\right] \geqslant \mathbb{P}\left[X \in \mathcal{F}\right] \cdot \mathbb{P}\left[X \in \mathcal{G}\right].$$

The same conclusion holds if $\mathcal{F}$ and $\mathcal{G}$ are decreasing. ■

The next lemma gives a lower bound on the *expected* sizes of the sets $S(u)$.

Lemma 2.6.14. *Let α be a positive real number such that*

$$(1 + \alpha)^2(1 - \varepsilon) < 1.$$

Then there exists a choice of $\{p(u) \,:\, u \in V\}$ such that for all $u \in V$,

$$\mathbb{E}[\#S(u)] = (1 + \alpha)\eta k.$$

Proof. Let $\beta \in (0; 1)$ be such that

$$1 - \lambda \geqslant \exp\left(-(1+\alpha)\lambda\right) \text{ for all } 0 < \lambda \leqslant \beta. \tag{2.6.5}$$

We will frequently use the following form of the inequality of arithmetic and geometric means: Given nonnegative real numbers $\lambda_1, \ldots, \lambda_m$ and nonnegative weights $w_1, \ldots, w_m$ satisfying $\sum_{i=1}^m w_i = 1$,

$$\sum_{i=1}^m w_i \lambda_i \geqslant \prod_{i=1}^m \lambda_i^{w_i}. \tag{2.6.6}$$

We define the values $p(u)$ inductively. Let $u \in V$ and assume that we have already defined $p(v)$ for all v reachable from u so that

$$\mathbb{E}[\#S(v)] = (1+\alpha)\eta k \quad \text{and} \quad p(v) \leqslant \beta \quad \text{for all} \quad v \in R^+(u).$$

We will show that in that case

$$\mathbb{E}[\#L'(u)] \geqslant \beta^{-1}(1+\alpha)\eta k. \tag{2.6.7}$$

After (2.6.7) is established, we can define

$$p(u) \coloneqq \frac{(1+\alpha)\eta k}{\mathbb{E}[\#L'(u)]},$$

which gives

$$\mathbb{E}[\#S(u)] = p(u)\mathbb{E}[\#L'(u)] = (1+\alpha)\eta k,$$

as desired, and, furthermore, $p(u) \leqslant \beta$, allowing the induction to continue.

Let

$$A \coloneqq R^+[u] \setminus R^-[N^+[u]].$$

We have

$$\mathbb{E}[\#L'(u)] = \sum_{Q \in \mathbf{QI}(A)} \mathbb{E}\left[\#L'(u) \,|\, S(A) = Q\right] \cdot \mathbb{P}\left[S(A) = Q\right]. \tag{2.6.8}$$

Consider any $Q \in \mathbf{QI}(A)$. By the linearity of expectation,

$$\begin{aligned} \mathbb{E}\left[\#L'(u) \,|\, S(A) = Q\right] &= \sum_{x \in L(u)} \mathbb{P}\left[x \in L'(u) \,|\, S(A) = Q\right] \\ &= \sum_{x \in L(u)} \mathbb{P}\left[y \notin S \text{ for all } y \in N^+(x) \,\middle|\, S(A) = Q\right]. \end{aligned}$$

From Lemma 2.6.12 we derive

$$\sum_{x \in L(u)} \mathbb{P}\left[y \notin S \text{ for all } y \in N^+(x) \,\middle|\, S(A) = Q\right] \geqslant \sum_{x \in L(u)} \prod_{y \in N^+(x)} \mathbb{P}\left[y \notin S \,|\, S(A) = Q\right],$$

which, by (2.6.6), is at least

$$k\left(\prod_{x\in L(u)}\prod_{y\in N^+(x)}\mathbb{P}\left[y\notin S\,|\,S(A)=Q\right]\right)^{1/k}.$$

After changing the order of multiplication, we get

$$\prod_{x\in L(u)}\prod_{y\in N^+(x)}\mathbb{P}\left[y\notin S\,|\,S(A)=Q\right]\geqslant\prod_{v\in N^+(u)}\prod_{y\in L(v)}\mathbb{P}\left[y\notin S\,|\,S(A)=Q\right].$$

Now consider any $v\in N^+(u)$. Let

$$A_v\coloneqq A\cup R^+(v).$$

Since $A\subseteq A_v$, the set $S(A)$ is determined by $S(A_v)$, and hence

$$\prod_{y\in L(v)}\mathbb{P}\left[y\notin S\,|\,S(A)=Q\right]=\prod_{y\in L(v)}\sum_{R\in\mathbf{QI}(A_v)}\mathbb{P}\left[y\notin S\,|\,S(A_v)=R\right]\cdot\mathbb{P}\left[S(A_v)=R\,|\,S(A)=Q\right].$$

Applying (2.6.6) again, we see that the last expression is at least

$$\begin{aligned}&\prod_{y\in L(v)}\prod_{R\in\mathbf{QI}(A_v)}\left(\mathbb{P}\left[y\notin S\,|\,S(A_v)=R\right]\right)^{\mathbb{P}[S(A_v)=R\,|\,S(A)=Q]}\\&=\prod_{R\in\mathbf{QI}(A_v)}\left(\prod_{y\in L(v)}\mathbb{P}\left[y\notin S\,|\,S(A_v)=R\right]\right)^{\mathbb{P}[S(A_v)=R\,|\,S(A)=Q]}.\end{aligned}\tag{2.6.9}$$

Note that the set $L'(v)$ is completely determined by $S(A_v)$. This allows us to introduce notation $L'_R(v)$ for the value of $L'(v)$ under the assumption $S(A_v)=R$; or, explicitly,

$$L'_R(v)\coloneqq L(v)\setminus N^-(R).$$

Since $v\notin A_v$, for fixed $R\in\mathbf{QI}(A_v)$ and $y\in L(v)$, we have

$$\mathbb{P}\left[y\notin S\,|\,S(A_v)=R\right]=\begin{cases}1-p(v) & \text{if } y\in L'_R(v);\\ 1 & \text{otherwise.}\end{cases}$$

Therefore,

$$\prod_{y\in L(v)}\mathbb{P}\left[y\notin S\,|\,S(A_v)=R\right]=(1-p(v))^{|L'_R(v)|}.$$

Plugging this into (2.6.9), we obtain

$$\begin{aligned}\prod_{y\in L(v)} \mathbb{P}\left[y \notin S \mid S(A)=Q\right] &\geqslant \prod_{R\in\mathbf{QI}(A_v)} (1-p(v))^{|L'_R(v)|\cdot\mathbb{P}[S(A_v)=R \,|\, S(A)=Q]} \\ &= (1-p(v))^{\sum_{R\in\mathbf{QI}(A_v)}|L'_R(v)|\cdot\mathbb{P}[S(A_v)=R \,|\, S(A)=Q]} \\ &= (1-p(v))^{\mathbb{E}[\#L'(v) \,|\, S(A)=Q]}\end{aligned}$$

Since, by our assumption, $p(v) \leqslant \beta$, inequality (2.6.5) yields

$$\begin{aligned}(1-p(v))^{\mathbb{E}[\#L'(v) \,|\, S(A)=Q]} &\geqslant \exp\left(-(1+\alpha)p(v)\mathbb{E}\left[\#L'(v) \mid S(A)=Q\right]\right) \\ &= \exp\left(-(1+\alpha)\mathbb{E}\left[\#S(v) \mid S(A)=Q\right]\right).\end{aligned}$$

This allows us to lower bound $\mathbb{E}\left[\#L'(u) \mid S(A)=Q\right]$ as

$$\begin{aligned}\mathbb{E}\left[\#L'(u) \mid S(A)=Q\right] &\geqslant k\left(\prod_{v\in N^+(u)} \exp\left(-(1+\alpha)\mathbb{E}\left[\#S(v) \mid S(A)=Q\right]\right)\right)^{1/k} \\ &= k\exp\left(-\frac{1+\alpha}{k}\sum_{v\in N^+(u)}\mathbb{E}\left[\#S(v) \mid S(A)=Q\right]\right).\end{aligned}$$

Returning to (2.6.8), we conclude

$$\mathbb{E}[\#L'(u)] \geqslant k\sum_{Q\in\mathbf{QI}(A)}\exp\left(-\frac{1+\alpha}{k}\sum_{v\in N^+(u)}\mathbb{E}\left[\#S(v) \mid S(A)=Q\right]\right)\cdot\mathbb{P}\left[S(A)=Q\right].$$

Due to the convexity of the exponential function (or by (2.6.6) again), the last expression is at least

$$k\exp\left(-\frac{1+\alpha}{k}\sum_{v\in N^+(u)}\mathbb{E}\left[\#S(v)\right]\right),$$

which, since $\mathbb{E}[\#S(v)] = (1+\alpha)\eta k$ for all $v \in N^+(u)$ by assumption, finally becomes

$$k\exp\left(-(1+\alpha)^2\eta\deg^+(u)\right) \geqslant k\exp\left(-(1+\alpha)^2\eta d\right) = kd^{-(1+\alpha)^2(1-\varepsilon)}.$$

It remains to notice that, since $(1+\alpha)^2(1-\varepsilon) < 1$, the quantity $d^{-(1+\alpha)^2(1-\varepsilon)}$ is asymptotically bigger than $\beta^{-1}(1+\alpha)\eta = \Theta(\ln d/d)$. This finishes the proof of (2.6.7). ∎

Finally, we show that the sizes of the sets $S(u)$ are highly concentrated.

Lemma 2.6.15. *There is $C > 0$, depending on G but not on k, such that for all $\alpha > 0$ and $u \in V$,*

$$\mathbb{P}\left[|\#S(u) - \mathbb{E}[\#S(u)]| \geqslant \alpha k\right] \leqslant 2\exp\left(-C\alpha^2 k\right).$$

Proof. We use the following concentration result:

Theorem 2.6.16 (**Simple Concentration Bound** [MR02, p. 79])**.** *Let ζ be a random variable determined by s independent trials such that changing the outcome of any one trial can affect ζ at most by c. Then*

$$\mathbb{P}[|\zeta - \mathbb{E}\zeta| > t] \leqslant 2\exp\left(-\frac{t^2}{2c^2 s}\right).$$

The value $\#S(u)$ is determined by $k|V|$ independent trials, namely by the values $\xi(x)$ for $x \in V(H)$, so, to apply Theorem 2.6.16, we only need to establish the following:

Claim (A). *Changing the value $\xi(x)$ for some $x \in V(H)$ can affect $\#S(u)$ at most by some amount c that depends on G but not on k.*

Proof. Suppose that $x \in L(v)$ for some $v \in V$. The value $\xi(x)$ can only affect $y \in R^-[x]$, so it suffices to upper bound $|R^-[x] \cap L(u)|$. Let $y = z_1 \to \cdots \to z_\ell = x$ be a directed yx-path for some $y \in L(u)$. For each $1 \leqslant i \leqslant \ell$, choose $v_i \in V$ so that $z_i \in L(v_i)$. Then $u = v_1 \to \cdots \to v_\ell = v$ is a directed uv-path in D. Notice that the uv-path $v_1 \to \cdots \to v_\ell$ uniquely identifies $y = z_1$. Indeed, by definition, $z_\ell = x$, so $z_{\ell-1}$ must be the unique neighbor of x in $L(v_{\ell-1})$. Then $z_{\ell-2}$ must be the unique neighbor of $z_{\ell-1}$ in $L(v_{\ell-2})$; and so on. Thus, $|R^-[x] \cap L(u)|$ does not exceed the number of directed uv-paths, which is independent of k. ⊣

The conclusion of the lemma is now immediate. ■

Now we can easily finish the proof of Theorem 2.6.6. Pick some $\alpha > 0$ so that $(1+\alpha)^2(1-\varepsilon) < 1$ and apply Lemma 2.6.14 to obtain $\{p(u) \,:\, u \in V\}$ such that for all $u \in V$,

$$\mathbb{E}[\#S(u)] = (1+\alpha)\eta k.$$

Then

$$\mathbb{P}\left[\#S(u) < \eta k\right] \leqslant 2\exp\left(-C\alpha^2\eta^2 k\right),$$

where C is the constant from Lemma 2.6.15. Therefore,

$$\mathbb{P}\left[S \text{ is not an } (\eta, \mathscr{H})\text{-coloring of } G\right] \leqslant 2n\exp\left(-C\alpha^2\eta^2 k\right) \xrightarrow[k\to\infty]{} 0,$$

as desired.

Part II

The infinite

3 | Measurable versions of the Lovász Local Lemma

3.1 Introduction

3.1.1 Graph colorings in the Borel and measurable settings

In this chapter we investigate the extent to which some classical results in finite combinatorics can be transferred to the measurable setting. Our main object of study will be the Lovász Local Lemma, which will be discussed in some detail in the next subsection. We start with a "preview" of some particular applications that our general techniques can provide.

We will be interested in the properties of Borel graphs; see [KM16] for a comprehensive survey of the topic. A graph G on a standard Borel space X is **Borel** if its **edge relation**, i.e., the set $\{(x, y) \in X^2 \,:\, xy \in E(G)\}$, is a Borel subset of X^2. An important source of Borel graphs are Borel group actions. Let Γ be a countable group acting by Borel automorphisms on a standard Borel space X. Denote this action by $\alpha \colon \Gamma \curvearrowright X$. Let $S \subseteq \Gamma$ be a generating set and define the graph $G(\alpha, S)$ on X via

$$xy \in E(G(\alpha, S)) \;:\Longleftrightarrow\; x \neq y \text{ and } \gamma \cdot x = y \text{ for some } \gamma \in S \cup S^{-1}.$$

Then $G(\alpha, S)$ is locally countable and Borel.

For a Borel graph G on X, its **Borel chromatic number** (notation: $\chi_{\mathrm{B}}(G)$) is the smallest cardinality of a standard Borel space Y such that G admits a Borel proper coloring $f \colon X \to Y$. Borel chromatic numbers were first introduced and systematically studied by Kechris, Solecki, and Todorcevic [KST99]. Clearly, $\chi(G) \leqslant \chi_{\mathrm{B}}(G)$. One of the starting points of Borel combinatorics is the observation that this inequality can be strict. In fact, Kechris, Solecki, and Todorcevic [KST99, Example 3.1] gave an example of an *acyclic* locally countable Borel graph G such that $\chi_{\mathrm{B}}(G) = 2^{\aleph_0}$ (note that if G is acyclic, then $\chi(G) \leqslant 2$). On the other hand, they showed [KST99, Proposition 4.6] that if $\Delta(G)$ is finite, then $\chi_{\mathrm{B}}(G) \leqslant \Delta(G) + 1$, in analogy with the finite case.

The bound $\chi(G) \leqslant \Delta(G) + 1$ is rather weak: recall that, by Brooks's theorem, $\chi(G) \leqslant \Delta(G)$ for all G apart from a few natural exceptions [Die00, Theorem 5.2.4]. As it turns out, there is no hope for any result along these lines in the Borel setting: Marks [Mar16, Theorem 1.3] showed that the Borel chromatic number

This chapter is based on [Ber16a].

of an acyclic Borel graph G with maximum degree $d \in \mathbb{N}$ can attain the value $d + 1$ (and, in fact, *any* value between 2 and $d + 1$).

Marks's results indicate that the Borelness requirement is too restrictive to allow any interesting analogs of classical coloring results. It is reasonable, therefore, to try asking for somewhat less. For instance, we can only require that "most" of the graph should be colored, in an appropriate sense of the word "most." Natural candidates for such a notion of largeness are Baire category and measure. We say that a set $X' \subseteq X$ is **G-invariant**, where G is a graph on X, if $G[X']$ is a union of connected components of G. If τ is a Polish topology on X that is compatible with the Borel structure on X, then the **τ-Baire-measurable chromatic number** of G is defined as follows:

$$\chi_\tau(G) \coloneqq \min\{\chi_{\mathrm{B}}(G[X']) \,:\, X' \text{ is a } \tau\text{-comeager } G\text{-invariant Borel subset of } X\}.$$

Similarly, if μ is a probability Borel measure on X, then the **μ-measurable chromatic number** of G is

$$\chi_\mu(G) \coloneqq \min\{\chi_{\mathrm{B}}(G[X']) \,:\, X' \text{ is a } \mu\text{-conull } G\text{-invariant Borel subset of } X\}.$$

Like $\chi_{\mathrm{B}}(G)$, both $\chi_\tau(G)$ and $\chi_\mu(G)$ can exceed $\chi(G)$, even for locally finite acyclic graphs. A simple example is the graph $G \coloneqq G(\alpha, \{1\})$, where $\alpha \colon \mathbb{Z} \curvearrowright S^1$ is an irrational rotation action of $\mathbb{Z}$ on the unit circle S^1. Each component of G is a bi-infinite path, so G is acyclic; but an easy ergodicity argument shows that $\chi_\tau(G)$, $\chi_\mu(G) > 2$, where τ is the usual topology and μ is the Lebesgue probability measure on S^1. (Since $\Delta(G) = 2$, [KST99, Proposition 4.6] yields $\chi_\tau(G) = \chi_\mu(G) = \chi_{\mathrm{B}}(G) = 3$.)

Nevertheless, Conley and Miller [CM16, Theorem B] showed that $\chi_\tau(G)$ cannot differ from $\chi(G)$ "too much"; specifically, they proved that for a locally finite Borel graph G on a standard Borel space X, if $\chi(G)$ is finite, then $\chi_\tau(G) \leqslant 2\chi(G) - 1$ with respect to any compatible Polish topology τ on X. In particular, if G is acyclic (or, more generally, $\chi(G) \leqslant 2$), then $\chi_\tau(G) \leqslant 3$.

Our main focus in this chapter will be on μ-measurable chromatic numbers and μ-measurable analogs of other combinatorial parameters (while some results pertaining to Baire measurable colorings will be presented in Chapter 5). Conley, Marks, and Tucker-Drob [CMT16, Theorem 1.2] proved a μ-measurable analog of Brooks's theorem for graphs with maximum degree at least 3 (the example of an irrational rotation action shows that Brooks's theorem for graphs with maximum degree 2 does not hold in the measurable setting). In particular, $\chi_\mu(G)$ can be *strictly less* than $\chi_{\mathrm{B}}(G)$. On the other hand, in contrast to Baire measurable chromatic numbers, $\chi_\mu(G)$ cannot be bounded above by *any* function of $\chi(G)$, as we explain below.

Let S be a finite set and let $\mathbb{F}(S)$ be the free group over S. Let $\alpha \colon \mathbb{F}(S) \curvearrowright [0;1]^{\mathbb{F}(S)}$ be the shift action of $\mathbb{F}(S)$ on $[0;1]^{\mathbb{F}(S)}$ and set $G \coloneqq G(\alpha, S)$. Let λ denote the Lebesgue measure on $[0;1]$ (we will use this notation throughout) and set $\mu \coloneqq \lambda^{\mathbb{F}(S)}$. Off of a μ-null set, the action α is free, so every connected component of G is an infinite $2|S|$-regular tree and hence is 2-colorable. However, as Lyons and Nazarov [LN11] observed, a result of Frieze and Łuczak [FŁ92] implies that $\chi_\mu(G) \geqslant |S|/\ln(2|S|)$ (see also [KM16, Theorem 5.44]). In particular, $\chi_\mu(G) \to \infty$ as $|S| \to \infty$. Note that the group $\mathbb{F}(S)$ for $|S| \geqslant 2$ is nonamenable; in fact, Conley and Kechris [CK13] mention that there are no known examples of graphs G induced by probability measure-preserving actions of amenable groups such that $\chi_\mu(G) > \chi(G) + 1$ (see [KM16, Problem 5.19]).

Note that the best known upper bound on $\chi_\mu(G)$ is $2|S|$ (given by the measurable Brooks's theorem of Conley–Marks–Tucker-Drob), so the orders of magnitude of the lower and upper bounds are different. Lyons and Nazarov [LN11] asked what the correct value of $\chi_\mu(G)$ should be. As an immediate corollary of one of our general results (namely Theorem 3.5.6), we can show that $|S|/\ln(|S|)$ is the right order. In fact, we have the following general theorem:

Theorem 3.1.1. *Let Γ be a countable group with a finite generating set $S \subseteq \Gamma$. Denote $d := |S \cup S^{-1}|$. Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of Γ and set $G := G(\alpha, S)$. Suppose that α factors to the shift action $\Gamma \curvearrowright ([0;1]^\Gamma, \lambda^\Gamma)$. If $g(G) \geqslant 4$, then $\chi_\mu(G) = O(d/\ln d)$; furthermore, if $g(G) \geqslant 5$, then $\chi_\mu(G) \leqslant (1 + o(1))d/\ln d$.*

Corollary 3.1.2. *Let S be a finite set of size k, let $\alpha \colon \mathbb{F}(S) \curvearrowright [0;1]^{\mathbb{F}(S)}$ be the $[0;1]$-shift action of the free group $\mathbb{F}(S)$, and let $G := G(\alpha, S)$. Then*

$$(1 - o(1))\frac{k}{\ln k} \leqslant \chi_{\lambda^{\mathbb{F}(S)}}(G) \leqslant (2 + o(1))\frac{k}{\ln k}. \tag{3.1.1}$$

Note that, by a result of Bowen [Bow11, Theorem 1.1], any two nontrivial[1] shift actions of $\mathbb{F}(S)$, where $|S| \geqslant 2$, admit factor maps to each other, so (3.1.1) holds for any such action as well.

One can also consider edge colorings in the Borel or measurable setting. Naturally, for a Borel graph G on a standard Borel space X, its **Borel chromatic index** $\chi'_{\mathrm{B}}(G)$ is the smallest cardinality of a standard Borel space Y such that G admits a Borel proper edge coloring $f \colon E(G) \to Y$. Clearly, $\chi'(G) \leqslant \chi'_{\mathrm{B}}(G)$. Marks [Mar16, Theorem 1.4] showed that the Borel chromatic index of an acyclic Borel graph G with maximum degree $d \in \mathbb{N}$ can be as large as $2d - 1$ (and this bound is tight—finding a proper edge coloring of a graph with maximum degree d is equivalent to finding a proper *vertex* coloring of an auxiliary graph with maximum degree $2d - 2$).

One can define the μ-measurable chromatic index of a Borel graph G by analogy with its μ-measurable chromatic number; namely,

$$\chi'_\mu(G) := \min\{\chi'_{\mathrm{B}}(G[X']) \,:\, X' \text{ is a } \mu\text{-conull } G\text{-invariant Borel subset of } X\}.$$

Csóka, Lippner, and Pikhurko [CLP16, Theorem 1.4] proved that Vizing's theorem holds measurably for locally finite bipartite graphs and that $\chi_\mu(G) \leqslant \Delta(G) + o(\Delta(G))$ in general, provided that the measure μ is G-invariant. Theorem 3.5.6 gives a different proof of the second part of this result for graphs induced by shift actions (with a slightly worse lower order term); moreover, it implies the following "list version":

Theorem 3.1.3. *For every $d \in \mathbb{N}$, there exists $k = d + o(d)$ such that the following holds. Let Γ be a countable group with a finite generating set $S \subseteq \Gamma$ such that $|S \cup S^{-1}| = d$. For each $\gamma \in S \cup S^{-1}$, let $L(\gamma)$ be a finite set such that $L(\gamma) = L(\gamma^{-1})$ and $|L(\gamma)| \geqslant k$ for all $\gamma \in S \cup S^{-1}$. Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of Γ and let $G := G(\alpha, S)$. Suppose that α factors to the shift action $\Gamma \curvearrowright ([0;1]^\Gamma, \lambda^\Gamma)$. Then there exists a Γ-invariant μ-conull Borel subset $X' \subseteq X$ and a Borel proper edge coloring f of $G[X']$ such that for all $x \in X'$, $f(x, \gamma \cdot x) \in L(\gamma)$.*

[1]Here, a probability measure ν is said to be **nontrivial** if it is not concentrated on a single point.

One can further relax the conditions on a coloring to allow a small (but positive) margin of error. Let G be a graph with vertex set X. For a map $f\colon X \to Y$, define the **defect** set $\mathbf{Def}(f) \subseteq X$ by

$$x \in \mathbf{Def}(f) \;:\Longleftrightarrow\; f(x) = f(y) \text{ for some } y \in N_G(x).$$

In other words, a vertex x belongs to $\mathbf{Def}(f)$ if and only if it shares a color with a neighbor. If the graph G is Borel, then a Borel map $f\colon X \to Y$ is a **(μ, ε)-approximately proper Borel coloring** of G if $\mu(\mathbf{Def}(f)) \leqslant \varepsilon$. The **$\mu$-approximate chromatic number** of G (notation: ${}^{\mathrm{ap}}\chi_\mu(G)$) is the smallest cardinality of a standard Borel space Y such that for every $\varepsilon > 0$, there is a (μ, ε)-approximately proper Borel coloring $f\colon X \to Y$ of G. Approximate chromatic numbers were studied extensively by Conley and Kechris [CK13]. In particular, they proved that if G is induced by a measure-preserving action of a countable *amenable* group, then its μ-approximate chromatic number is essentially determined by the ordinary chromatic number; more precisely, for such G,

$${}^{\mathrm{ap}}\chi_\mu(G) = \min\{\chi(G[X']) \,:\, X' \text{ is a } \mu\text{-conull } G\text{-invariant Borel subset of } X\}.$$

However, the lower bound ${}^{\mathrm{ap}}\chi_{\lambda^{\mathbb{F}(S)}}(G(\alpha, S)) \geqslant |S|/\ln(2|S|)$, where $\alpha\colon \mathbb{F}(S) \curvearrowright [0;1]^{\mathbb{F}(S)}$ is the shift action of the free group $\mathbb{F}(S)$ over a finite set S, still holds.

For an edge coloring $f\colon E(G) \to Y$, let $\mathbf{Def}'(f) \subseteq X$ be given by

$$x \in \mathbf{Def}'(f) \;:\Longleftrightarrow\; \begin{array}{l} \exists y \in N_G(x)\, \exists z \in N_G(y)\, (z \neq x \text{ and } f(xy) = f(yz)); \quad \text{or} \\ \exists y \in N_G(x)\, \exists z \in N_G(x)\, (z \neq y \text{ and } f(xy) = f(xz)). \end{array}$$

In other words, $x \in \mathbf{Def}'(f)$ if and only if x is incident to an edge that shares an endpoint with another edge of the same color. The **μ-approximate chromatic index** ${}^{\mathrm{ap}}\chi'_\mu(G)$ of a Borel graph G is defined similarly to ${}^{\mathrm{ap}}\chi_\mu(G)$. As a corollary of our other general result (namely Theorem 3.4.1), Theorems 3.1.1 and 3.1.3 can be generalized to arbitrary locally finite Borel graphs in the context of approximate colorings.

Theorem 3.1.4. *Let G be a Borel graph on a standard Borel space X and suppose that $\Delta(G) = d \in \mathbb{N}$. Let μ be a probability Borel measure on X. If $g(G) \geqslant 4$, then ${}^{\mathrm{ap}}\chi_\mu(G) = O(d/\ln d)$; furthermore, if $g(G) \geqslant 5$, then ${}^{\mathrm{ap}}\chi_\mu(G) \leqslant (1 + o(1))d/\ln d$.*

Theorem 3.1.5. *Let G be a Borel graph on a standard Borel space X and suppose that $\Delta(G) = d \in \mathbb{N}$. Let μ be a probability Borel measure on X. Then ${}^{\mathrm{ap}}\chi'_\mu(G) = d + o(d)$.*

3.1.2 The Lovász Local Lemma and its applications

The Lovász Local Lemma, or the LLL, is a powerful probabilistic tool developed by Erdős and Lovász [EL75]. We refer to [AS00, Chapter 5] for background on the Lovász Local Lemma and its applications in combinatorics; several other classical applications can be found, e.g., in [MR02]. In this chapter we will only use the LLL in a somewhat restricted set-up that is described below. (For the full statement of the LLL, see Theorem 1.1.1.)

Let X be a set and consider any $S \in [X]^{<\infty}$. Even though X itself is just a set with no additional structure, $[0;1]^S$ is a standard Borel space equipped with the Lebesgue probability measure λ^S. We refer to the Borel subsets $B \subseteq [0;1]^S$ as **bad events** over X. Every bad event is a subset of $[X \to [0;1]]^{<\infty}$. If $B \subseteq [0;1]^S$ is a nonempty bad event, then we call S the **domain** of B and write $\mathrm{dom}(B) \coloneqq S$; since B is nonempty, S is determined uniquely. Set $\mathrm{dom}(\varnothing) \coloneqq \varnothing$. The **probability** of a bad event B is

$$\mathbb{P}[B] \coloneqq \lambda^{\mathrm{dom}(B)}(B).$$

A function $f \colon X \to [0;1]$ **avoids** a bad event B if there is no $w \in B$ with $w \subseteq f$. An **instance** (**of the LLL**) over X is a set $\mathscr{B}$ of bad events over X. A **solution** to an instance $\mathscr{B}$ is a map $f \colon X \to [0;1]$ that avoids all $B \in \mathscr{B}$. For an instance $\mathscr{B}$ and a bad event $B \in \mathscr{B}$, the **neighborhood** of B in $\mathscr{B}$ is

$$N_{\mathscr{B}}(B) \coloneqq \{B' \in \mathscr{B} \setminus \{B\} \;:\; \mathrm{dom}(B') \cap \mathrm{dom}(B) \neq \varnothing\}.$$

The **degree** of B in $\mathscr{B}$ is

$$\deg_{\mathscr{B}}(B) \coloneqq |N_{\mathscr{B}}(B)|.$$

Let

$$p(\mathscr{B}) \coloneqq \sup_{B \in \mathscr{B}} \mathbb{P}[B] \qquad \text{and} \qquad d(\mathscr{B}) \coloneqq \sup_{B \in \mathscr{B}} \deg_{\mathscr{B}}(B).$$

An instance $\mathscr{B}$ is **correct for the Symmetric LLL** (the SLLL for short) if

$$e \cdot p(\mathscr{B}) \cdot (d(\mathscr{B}) + 1) < 1,$$

where $e = 2.71\ldots$ denotes the base of the natural logarithm.

Theorem 3.1.6 (Erdős–Lovász [EL75]; **Symmetric Lovász Local Lemma—finite case**)**.** *Let $\mathscr{B}$ be an instance of the LLL over a finite set X. If $\mathscr{B}$ is correct for the SLLL, then $\mathscr{B}$ has a solution.*

The Symmetric LLL was introduced by Erdős and Lovász (with 4 in place of e) in their seminal paper [EL75]; the constant was later improved by Lovász (the sharpened version first appeared in [Spe77]). Theorem 3.1.6 is a special case of the SLLL in the so-called *variable framework* (the name is due to Kolipaka and Szegedy [KS11]), which encompasses most typical applications (with a notable exception of the ones concerning random permutations, see, e.g., [ES91]). For the full statement of the SLLL, see Lemma 2.3.6 or [AS00, Corollary 5.1.2] (deducing Theorem 3.1.6 from that is routine; see, e.g., [MR02, p. 41]).

Theorem 3.1.6 can be also extended to instances $\mathscr{B}$ with $d(\mathscr{B}) = \infty$, provided that for $B \in \mathscr{B}$, $\mathbb{P}[B]$ decays sufficiently fast as $|\mathrm{dom}(B)|$ increases. An instance $\mathscr{B}$ is **correct for the General LLL** (the GLLL for short), or simply **correct**, if the neighborhood of each $B \in \mathscr{B}$ is countable, and there exists a function $\omega \colon \mathscr{B} \to [0;1)$ such that for all $B \in \mathscr{B}$,

$$\mathbb{P}[B] \leqslant \omega(B) \prod_{B' \in N_{\mathscr{B}}(B)} (1 - \omega(B')).$$

Theorem 3.1.7 (**General Lovász Local Lemma—finite case**; [AS00, Lemma 5.1.1])**.** *Let $\mathscr{B}$ be an instance of the LLL over a finite set X. If $\mathscr{B}$ is correct for the GLLL, then $\mathscr{B}$ has a solution.*

A standard calculation (see [AS00, proof of Corollary 5.1.2]) shows that if an instance $\mathscr{B}$ is correct for the SLLL, then it is also correct for the GLLL, hence the name "General LLL."

Remark 3.1.8. If $\mathscr{B}$ is a correct instance of the LLL, then we may assume that $\mathrm{dom}(B) \neq \varnothing$ for all $B \in \mathscr{B}$. Indeed, there are only two bad events with empty domain: $\varnothing$ and $\{\varnothing\}$. The event $\varnothing$ is always avoided, so it does not matter if $\varnothing \in \mathscr{B}$ or not. On the other hand, $\{\varnothing\}$ cannot be avoided; in particular, if $\mathscr{B}$ is correct, then $\{\varnothing\} \notin \mathscr{B}$.

Remark 3.1.9. The definition of bad events can be naturally extended to include subsets of $[X \to Y]^{<\infty}$ for standard probability spaces (Y, ν) other than $([0;1], \lambda)$. Indeed, in standard combinatorial applications, Y is often a finite set. However, any standard probability space (Y, ν) can be "simulated" by $([0;1], \lambda)$, in the sense that there exists a Borel map $\varphi \colon [0;1] \to Y$ such that $\varphi_*(\lambda) = \nu$. As far as the LLL is concerned, a set $B \subseteq [X \to Y]^{<\infty}$ can be replaced by its "pullback" $\varphi^*(B) \subseteq [X \to [0;1]]^{<\infty}$ defined via

$$w \in \varphi^*(B) \;:\Longleftrightarrow\; \varphi \circ w \in B.$$

Therefore, no generality is lost when only working with subsets of $[X \to [0;1]]^{<\infty}$.

Theorems 3.1.6 and 3.1.7 also hold in the case when the ground set X is infinite. In most applications, one may assume that each bad event B is an open subset of $[0;1]^{\mathrm{dom}(B)}$ and obtain infinitary analogs of the LLL through standard compactness arguments (see, e.g., [AS00, Theorem 5.2.2]). Yet, a different proof is required in general. Kun [Kun13, Lemma 13] showed that the infinite version of the LLL can be derived using the effective approach developed by Moser and Tardos [MT10].

Theorem 3.1.10 (Kun [Kun13, Lemma 13]; **General Lovász Local Lemma—infinite version**). *Let $\mathscr{B}$ be an instance of the LLL over an arbitrary set X. If $\mathscr{B}$ is correct for the GLLL, then $\mathscr{B}$ has a solution.*

Since the Moser–Tardos theory will play a crucial role in our investigation, we present its main tools, including a proof of Theorem 3.1.10, in §3.2.

As a simple example, let H be a k-uniform hypergraph with vertex set X, and recall that a **proper 2-coloring** of H is a map $f \colon X \to 2$ such that every edge $e \in E(H)$ contains vertices of both colors. For $e \in E(H)$, let $w_{e,0}, w_{e,1} \colon e \to 2$ denote the constant 0 and 1 functions respectively and define $B_e \coloneqq \{w_{e,0}, w_{e,1}\}$. Set

$$\mathscr{B} \coloneqq \{B_e \,:\, e \in E(H)\}.$$

As explained in Remark 3.1.9, $\mathscr{B}$ can be viewed as an instance over X. The proper 2-colorings of H are precisely the solutions to $\mathscr{B}$. It is straightforward to check the conditions under which $\mathscr{B}$ is correct for the SLLL, and, after an easy calculation, one recovers the following theorem due to Erdős and Lovász, which historically was the first application of the LLL:

Theorem 3.1.11 (Erdős–Lovász [EL75]). *Let H be a k-uniform hypergraph and suppose that every edge of H intersects at most d other edges. If $e(d+1) \leqslant 2^{k-1}$, then H is 2-colorable.* [2]

[2]The best currently known bound that guarantees 2-colorability of H is $d \leqslant c(k/\ln k)^{1/2}2^k$ for some positive absolute constant c, due to Radhakrishnan and Srinivasan [RS00, Theorem 4.2]. Their proof also relies on the LLL.

To illustrate the types of results one can obtain using the LLL, we describe a few other applications below.

Kim's and Johansson's theorems

Let G be a "sparse" graph, in that it does not contain any "short" cycles. Can one show that $\chi(G)$ is much smaller than $\Delta(G)$, the bound given by Brooks's theorem? It is well-known that there exist d-regular graphs with arbitrarily large girth and with chromatic number at least $(1/2 - o(1))d/\ln d$. After a series of partial results by a number of researchers (see [JT95, Section 4.6] for a survey), Kim [Kim95] proved an upper bound that (asymptotically) exceeds the lower bound only by a factor of 2:

Theorem 3.1.12 (Kim [Kim95]; see also [MR02, Chapter 12])**.** *Let G be a graph with maximum degree $d \in \mathbb{N}$. If $g(G) \geqslant 5$, then $\chi(G) \leqslant (1 + o(1))d/\ln d$.*

Shortly after, Johansson [Joh96] reduced the girth requirement and extended Kim's result (modulo a constant factor) to triangle-free graphs.

Theorem 3.1.13 (Johansson [Joh96]; see also [MR02, Chapter 13])**.** *Let G be a graph with maximum degree $d \in \mathbb{N}$. If $g(G) \geqslant 4$, then $\chi(G) = O(d/\ln d)$.*

The proofs of Theorems 3.1.12 and 3.1.13 are examples of a particular general approach to coloring problems. The key idea is to iterate applications of the LLL so that on each stage, the LLL produces only a partial coloring of G—but this coloring is also made to satisfy some additional requirements. These requirements allow the process to be repeated, until finally the uncolored part of the graph becomes so sparse that a single application of the LLL (or a basic greedy algorithm) can finish the proof. Dealing with such iterated applications of the LLL will be one of the major difficulties we will have to face in §3.5. An interested reader is referred to [MR02] for an excellent exposition of both proofs.[3]

Kahn's theorem

As mentioned in §3.1.1, Vizing's theorem asserts that if $\Delta(G)$ is finite, then $\chi'(G) \leqslant \Delta(G) + 1$. There are several known proofs of Vizing's theorem, none of them using the LLL.

An important generalization of graph coloring is so-called **list coloring**; see §2.1 for its definition. The following outstanding open problem concerns *edge* list colorings:

Conjecture 3.1.14 (**Edge List Coloring Conjecture**; see Conjecture 2.2.3 and [BM08, Conjecture 17.8])**.** *For every finite graph G, $\chi'_\ell(G) = \chi'(G)$, where $\chi'_\ell(G)$ is the list chromatic index of G.*

As a step towards settling Conjecture 3.1.14, Kahn [Kah00] proved the following asymptotic version of Vizing's theorem for list colorings:

Theorem 3.1.15 (Kahn [Kah00]; see also [MR02, Chapter 14])**.** *Let G be a graph with maximum degree $d \in \mathbb{N}$. Then $\chi'_\ell(G) = d + o(d)$.*

[3]Recently, Molloy [Mol17] showed that the bound $\chi(G) \leqslant (1 + o(1))\Delta(G)/\ln \Delta(G)$ from Theorem 3.1.12 holds for triangle-free graphs as well. Unfortunately, the proof techniques used in [Mol17] cannot be adapted using our machinery. See §2.3 for more details.

Note that, in contrast to Vizing's theorem, Kahn's proof is based on the LLL; in fact, it is similar to the proofs of Kim's and Johansson's theorems in that it uses iterated applications of the LLL to produce partial colorings with some additional properties. Note that Kahn's theorem yields an LLL-based proof of the bound $\chi'(G) = d + o(d)$ for *ordinary* edge colorings as well.

Nonrepetitive and acyclic colorings

The LLL can also be applied to produce upper bounds on more "exotic" types of chromatic numbers. Here we only mention two examples (both of which have already been looked at in Chapter 1). A vertex coloring f of a graph G is **nonrepetitive** if there is no path P in G with an even number of vertices such that the first half of P receives the same sequence of colors as the second half of P, i.e., if there is no path v_1, v_2, ..., v_{2t} of length $2t$ such that $f(v_k) = f(v_{k+t})$ for all $1 \leqslant k \leqslant t$. The least number of colors that is needed for a nonrepetitive coloring of G is called the **nonrepetitive chromatic number** of G and is denoted by $\pi(G)$. The following theorem of Alon, Grytczuk, Hałuszczak, and Riordan [Alo+02] gives an upper bound on $\pi(G)$ in terms of $\Delta(G)$:

Theorem 3.1.16 (Alon–Grytczuk–Hałuszczak–Riordan [Alo+02, Theorem 1])**.** *Let G be a graph with maximum degree $d \in \mathbb{N}$. Then $\pi(G) = O(d^2)$.*

A proper (vertex) coloring f of a graph G is **acyclic** if every cycle in G receives at least three different colors. The least number of colors needed for an acyclic proper coloring of G is called the **acyclic chromatic number** of G and is denoted by $a(G)$. In 1976, Erdős conjectured that $a(G) = o(\Delta(G)^2)$; 15 years later, Alon, McDiarmid, and Reed [AMR91] confirmed Erdős's hypothesis.

Theorem 3.1.17 (Alon–McDiarmid–Reed [AMR91, Theorem 1.1])**.** *Let G be a graph with maximum degree $d \in \mathbb{N}$. Then $a(G) = O(d^{4/3})$.*

Each of Theorems 3.1.16 and 3.1.17 is proved via a single application of the LLL to a carefully constructed correct instance.

3.1.3 Overview of the main results of this chapter

Let X be a standard Borel space. An instance $\mathscr{B}$ over X is **Borel** if

$$\bigcup \mathscr{B} := \{w \in [X \to [0;1]]^{<\infty} \ : \ w \in B \text{ for some } B \in \mathscr{B}\}$$

is a Borel subset of $[X \to [0;1]]^{<\infty}$. In general, given a correct Borel instance $\mathscr{B}$ over X, one cannot guarantee the existence of a Borel solution [Con+16, Theorem 1.6]. Suppose, however, that μ is a probability Borel measure on X. When can one ensure that there is a "large" (in terms of μ) Borel subset of X on which $\mathscr{B}$ admits a Borel solution?

The Moser–Tardos theory

In our investigation, we rely heavily on the algorithmic approach to the LLL due to Moser and Tardos [MT10]. The original motivation behind Moser and Tardos's work was to develop a randomized algorithm which, given a correct instance $\mathscr{B}$ over a finite set X, quickly finds a solution to $\mathscr{B}$. It turns out that the Moser–Tardos method naturally extends to the case when X is infinite, leading to the possibility of analogs of the LLL that are "constructive" in various senses; a notable example is the computable version of the LLL due to Rumyantsev and Shen [RS14]. In §3.2 we describe (a generalized version of) the Moser–Tardos algorithm and consider its behavior in the Borel setting. The Moser–Tardos technique was first used in the measurable framework in [Kun13].

A universal combinatorial structure—hereditarily finite sets

By definition, an instance of the LLL over a set X puts a set of constraints on a map $f\colon X \to [0;1]$. For example, if X is the vertex set of a graph G, then by solving instances over X one finds vertex colorings of G with desired properties. However, sometimes we want to consider *edge* colorings instead, or maybe maps defined on some other combinatorial structures "built" from G, such as, say, paths of length 2, or cycles, etc. Additionally, even when looking for vertex colorings, it is sometimes necessary to assign to each vertex several colors at once, which can be viewed as replacing every element of X by finitely many "copies" of it and coloring each "copy" independently. In order to cover all potential combinatorial applications, we enlarge the set X, adding points for various combinatorial data that can be built from the elements of X. We call the resulting "universal" combinatorial structure the **amplification** of X and denote it by $\mathbf{HF}(X)$ (here the letters "HF" stand for "hereditarily finite"). Roughly speaking, the points of $\mathbf{HF}(X)$ correspond to all sets that can be obtained from X by repeatedly taking finite subsets. The precise construction of $\mathbf{HF}(X)$ is described in §3.3. All our results are stated for instances over $\mathbf{HF}(X)$; however, to simplify the current discussion, we will be only talking about instances over X in this subsection.

Approximate LLL

The first main result of this chapter is the approximate LLL, which we state and prove in §3.4. Let X be a set. For an instance $\mathscr{B}$ over X and a map $f\colon X \to [0;1]$, the **defect** $\mathbf{Def}_{\mathscr{B}}(f)$ of f with respect to $\mathscr{B}$ is the set of all $x \in X$ such that $x \in \mathrm{dom}(w)$ for some $w \in B \in \mathscr{B}$ with $w \subseteq f$. Thus, f is a solution to $\mathscr{B}$ if and only if $\mathbf{Def}_{\mathscr{B}}(f) = \varnothing$. An instance $\mathscr{B}$ is **locally finite** if $\deg_{\mathscr{B}}(B) < \infty$ for all $B \in \mathscr{B}$. For locally finite instances, we prove the following:

Theorem 3.4.1 (**Approximate LLL**). *Let $\mathscr{B}$ be a correct locally finite Borel instance over a standard probability space (X, μ). Then for any $\varepsilon > 0$, there exists a Borel function $f\colon X \to [0;1]$ with $\mu(\mathbf{Def}_{\mathscr{B}}(f)) \leqslant \varepsilon$.*

Most (but not all) standard applications of the LLL only consider locally finite instances; for example, any instance that is correct for the SLLL is locally finite. Among the examples listed in §3.1.2, Theorems 3.1.11, 3.1.12, 3.1.13, and 3.1.15 only use locally finite instances; in particular, Theorem 3.4.1 immediately yields Theorems 3.1.4 and 3.1.5 on approximate chromatic numbers of Borel graphs. On the other hand, Theorems 3.1.16 and 3.1.17 apply the LLL to instances that are in general *not* locally finite, as there can be infinitely many paths or cycles passing through a given vertex in a locally finite graph.

We point out that in their recent study [Csó+16], carried out independently from this work, Csóka, Grabowski, Máthé, Pikhurko, and Tyros use an approach similar to ours in order to establish a purely Borel version of the LLL for a class of instances satisfying stronger boundedness assumptions (namely having uniformly subexponential growth).

Measure-preserving group actions

Our second main result is the measurable version of the LLL for probability measure-preserving actions of countable groups, which we present in §3.5. It shows that under certain additional restrictions on the correct instance $\mathscr{B}$, one can find a Borel function that solves it on a conull subset—even when $\mathscr{B}$ is not locally finite. To motivate these restrictions, consider a graph G on a set X. Combinatorial problems related to G usually require solving instances of the LLL that possess the following two properties:

- the correctness of a solution can be verified separately within each component of G;
- the instance only depends on the graph structure of G, in other words, it is invariant under the (combinatorial/abstract) automorphisms of G.

These two properties are captured in the following definition: Let $\alpha\colon \Gamma \curvearrowright (X,\mu)$ be a p.m.p. action of a countable group Γ and let $\mathcal{I}_\alpha$ denote the set of all equivariant bijections $\varphi\colon O \to O'$ between α-orbits. An **instance** (**of the LLL**) over α is a Borel instance $\mathscr{B}$ over X such that:

- for all $B \in \mathscr{B}$, $\mathrm{dom}(B)$ is contained within a single orbit of α; and
- the set $\mathscr{B}$ is (μ-almost everywhere) invariant under the functions $\varphi \in \mathcal{I}_\alpha$.

A basic measurable version of the LLL for probability measure-preserving group actions is as follows:

Corollary 3.5.7. *Let $\alpha\colon \Gamma \curvearrowright (X,\mu)$ be a p.m.p. action of a countable group Γ. Suppose that α factors to the shift action $\Gamma \curvearrowright ([0;1]^\Gamma, \lambda^\Gamma)$ and let $\mathscr{B}$ be a correct instance over α. Then there exists a Borel function $f\colon X \to [0;1]$ with $\mu(\mathbf{Def}_{\mathscr{B}}(f)) = 0$.*

Corollary 3.5.7 is sufficient for many applications; for instance, it yields measurable analogs of Theorems 3.1.16 and 3.1.17. However, a more general result is required to derive Theorems 3.1.1 and 3.1.3. As mentioned in §3.1.2, to establish their combinatorial counterparts (namely Theorems 3.1.12, 3.1.13, and 3.1.15) the LLL is applied iteratively to a series of instances, with each next instance defined *using the solutions to the previous ones*: Even though the very first instance $\mathscr{B}_0$ is invariant under *all* functions $\varphi \in \mathcal{I}_\alpha$, as soon as a solution f_0 to $\mathscr{B}_0$ is fixed, the next instance $\mathscr{B}_1$ is only guaranteed to be invariant under those $\varphi \in \mathcal{I}_\alpha$ that additionally *preserve* the value of f_0, so Corollary 3.5.7 can no longer be used.

To formalize this complication, we define a game between two players, called the **LLL Game**. A run of the LLL Game over an action $\alpha\colon \Gamma \curvearrowright (X,\mu)$ looks like this:

Player I	$\mathscr{B}_0$		$\mathscr{B}_1$		...	$\mathscr{B}_n$		...
Player II		f_0		f_1	...		f_n	...

On his first turn, Player I chooses a correct instance $\mathscr{B}_0$ over α. Player II responds by choosing a μ-measurable solution f_0 to $\mathscr{B}_0$. Player I then picks a new correct Borel instance $\mathscr{B}_1$, this time only invariant under the functions $\varphi \in \mathcal{I}_\alpha$ that preserve f_0. Player II must respond by finding a μ-measurable solution f_1 to $\mathscr{B}_1$. On the next step, Player I selects a correct Borel instance $\mathscr{B}_2$ invariant under the functions $\varphi \in \mathcal{I}_\alpha$ that preserve both f_0 and f_1; and so on. Player II wins if the game continues indefinitely and loses if at any step, she is presented with an instance that has no μ-measurable solution. Our result, Theorem 3.5.6, asserts that Player II has a winning strategy in this game:

Theorem 3.5.6 (Measurable LLL for group actions). *Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of a countable group Γ. If α factors to the shift action $\Gamma \curvearrowright ([0;1]^\Gamma, \lambda^\Gamma)$, then Player II has a winning strategy in the LLL Game over α.*

A partial converse

Finally, we turn to the following natural question:

> Is it necessary to assume that α admits a factor map to the $[0;1]$-shift action in order to establish Theorem 3.5.6 and Corollary 3.5.7, or is this assumption just an artifact of our proof?

In §3.6, we demonstrate that, at least for amenable groups, this assumption is indeed necessary; furthermore, a probability measure-preserving free ergodic action α of a countably infinite amenable group Γ factors to the $[0;1]$-shift action *if and only if* it satisfies the conclusion of Corollary 3.5.7. In fact, a much weaker version of the LLL than Corollary 3.5.7 already yields a factor map to the $[0;1]$-shift, which, in particular, shows that Theorem 3.4.1 fails for instances that are not locally finite.

To establish these results, we combine the tools of the Ornstein–Weiss theory of entropy for actions of amenable groups with concepts from computability theory. By a theorem of Ornstein and Weiss, a free ergodic probability measure-preserving action $\alpha \colon \Gamma \curvearrowright (X, \mu)$ of a countably infinite amenable group Γ factors to the $[0;1]$-shift action if and only if $H_\mu(\alpha) = \infty$, where $H_\mu(\alpha)$ is the so-called *Kolmogorov–Sinai entropy* of α. Intuitively, $H_\mu(\alpha)$ measures how "unpredictable" or "random" the interaction of α with a Borel map $f \colon X \to k \in \mathbb{N}$ can be. Therefore, in proving a converse to Theorem 3.5.6, we have to apply the LLL in order to exhibit Borel functions f whose behavior is highly "random." Notice that entropy is a "global" parameter that depends on f as a whole, while the LLL can only constrain a function "locally." In other words, we require a way to certify high entropy in a "local," or "pointwise," manner. To that end, we use *Kolmogorov complexity*—a deterministic alternative to entropy defined in the language of computability theory—to measure the "randomness" of a given Borel function *at each point*. The crux of our argument is Lemma 3.6.8, which is of independent interest. It gives a lower bound on the Kolmogorov–Sinai entropy of a Borel function in terms of the average value of its pointwise Kolmogorov complexity. The proof of Lemma 3.6.8 invokes the result of Ornstein and Weiss concerning the existence of quasi-tilings in amenable groups and is inspired by previous work of Brudno [Bru82] in the case of $\mathbb{Z}$-actions.

3.2 Moser–Tardos theory

As mentioned in the previous section, a major role in our arguments is played by ideas stemming from the algorithmic proof of the LLL due to Moser and Tardos [MT10]. In this section we review their method and introduce some convenient notation and terminology. Most results of this section are essentially present in [MT10]; nevertheless, we include a fair amount of detail for completeness.

For the rest of this section, fix a set X and a correct instance $\mathscr{B}$ over X. Motivated by algorithmic applications, Moser and Tardos only consider the case when the ground set X is finite; however, their technique naturally extends to the case of infinite X.

Let $\mathrm{dom}(\mathscr{B}) \coloneqq \{\mathrm{dom}(B) \,:\, B \in \mathscr{B}\}$. For the reasons explained in Remark 3.1.8, we may assume that $\varnothing \notin \mathrm{dom}(\mathscr{B})$. For $S \in \mathrm{dom}(\mathscr{B})$, define

$$\mathscr{B}_S \coloneqq \bigcup \{B \in \mathscr{B} \,:\, \mathrm{dom}(B) = S\} = \{w \colon S \to [0;1] \,:\, w \in B \text{ for some } B \in \mathscr{B}\}.$$

The correctness of $\mathscr{B}$ implies that the set $\{B \in \mathscr{B} \,:\, \mathrm{dom}(B) = S\}$ is countable. Therefore, $\mathscr{B}_S$ is a Borel subset of $[0;1]^S$. For brevity, we write

$$\mathbb{P}[S] \coloneqq \lambda^S(\mathscr{B}_S).$$

(Note that this notation implicitly depends on $\mathscr{B}$.)

We say that a family A of sets is **disjoint** if the elements of A are pairwise disjoint.

Definition 3.2.1 (**Moser–Tardos process**). A **table** is a map $\vartheta \colon X \times \mathbb{N} \to [0;1]$. Fix a table ϑ and consider the following inductive construction:

> Set $t_0(x) \coloneqq 0$ for all $x \in X$.
>
> Step $n \in \mathbb{N}$: Define
>
> $$f_n(x) \coloneqq \vartheta(x, t_n(x)) \text{ for all } x \in X \qquad \text{and} \qquad A'_n \coloneqq \{S \in \mathrm{dom}(\mathscr{B}) \,:\, f_n \supseteq w \text{ for some } w \in \mathscr{B}_S\}.$$
>
> Choose A_n to be an arbitrary maximal disjoint subset of A'_n and let
>
> $$t_{n+1}(x) \coloneqq \begin{cases} t_n(x) + 1 & \text{if } x \in S \text{ for some } S \in A_n; \\ t_n(x) & \text{otherwise.} \end{cases}$$

A sequence $\mathcal{A} = (A_n)_{n=0}^{\infty}$ of subsets of $\mathrm{dom}(\mathscr{B})$ obtained via the above procedure is called a **Moser–Tardos process** with input ϑ.

Remark. Since each set A_n in a Moser–Tardos process is disjoint, for every $x \in X$ with $t_{n+1}(x) > t_n(x)$, there is a *unique* set $S \in A_n$ such that $x \in S$.

Proposition 3.2.2. *Let $\mathcal{A} = (A_n)_{n=0}^{\infty}$ be a Moser–Tardos process. For $n \in \mathbb{N}$, let*

$$X_n \coloneqq \{x \in X \,:\, x \in S \textit{ for some } S \in A_n\}.$$

Then f_n avoids all bad events $B \in \mathscr{B}$ with $\operatorname{dom}(B) \cap X_n = \varnothing$.

Proof. If $\operatorname{dom}(B) \cap X_n = \varnothing$, then $\operatorname{dom}(B)$ is disjoint from all $S \in A_n$. Since we assume $\operatorname{dom}(B) \neq \varnothing$, this implies $\operatorname{dom}(B) \notin A_n$. By the choice of A_n, we then get $\operatorname{dom}(B) \notin A'_n$, as desired. ■

Suppose that $\mathcal{A}$ is a Moser–Tardos process. By definition, the sequence $t_0(x), t_1(x), \ldots$ is non-decreasing for all $x \in X$. We say that an element $x \in X$ is **$\mathcal{A}$-stable** if the sequence $t_0(x), t_1(x), \ldots$ is eventually constant. Let $\mathbf{Stab}(\mathcal{A}) \subseteq X$ denote the set of all $\mathcal{A}$-stable elements of X. For $x \in \mathbf{Stab}(\mathcal{A})$, define

$$t(x) \coloneqq \lim_{n\to\infty} t_n(x) \qquad \text{and} \qquad f(x) \coloneqq \vartheta(x, t(x)).$$

We have the following limit analog of Proposition 3.2.2:

Proposition 3.2.3. *Let $\mathcal{A} = (A_n)_{n=0}^{\infty}$ be a Moser–Tardos process. Then f avoids all bad events $B \in \mathscr{B}$ with* $\operatorname{dom}(B) \subseteq \mathbf{Stab}(\mathcal{A})$.

Proof. Fix $B \in \mathscr{B}$ with $\operatorname{dom}(B) \subseteq \mathbf{Stab}(\mathcal{A})$ and choose $n \in \mathbb{N}$ so large that for all $x \in \operatorname{dom}(B)$, we have $t(x) = t_n(x)$. Then $f|\operatorname{dom}(B) = f_n|\operatorname{dom}(B)$, and thus it remains to show that f_n avoids B. Notice that $\operatorname{dom}(B)$ is disjoint from all $S \in A_n$; indeed, if $x \in \operatorname{dom}(B) \cap S$ for some $S \in A_n$, then $t_{n+1}(x) = t_n(x) + 1$, which contradicts the choice of n. Now we are done by Proposition 3.2.2. ■

For each $S \in \operatorname{dom}(\mathscr{B})$, define the **index** $\mathbf{Ind}(S, \mathcal{A}) \in \mathbb{N} \cup \{\infty\}$ of S in $\mathcal{A}$ by

$$\mathbf{Ind}(S, \mathcal{A}) \coloneqq |\{n \in \mathbb{N} \,:\, S \in A_n\}|.$$

Note that for all $x \in X$,

$$\lim_{n\to\infty} t_n(x) = \sum_{S \in \operatorname{dom}(\mathscr{B})\,:\, S \ni x} \mathbf{Ind}(S, \mathcal{A}), \tag{3.2.1}$$

so $x \in \mathbf{Stab}(\mathcal{A})$ if and only if the expression on the right hand side of (3.2.1) is finite. Our goal therefore is to obtain good upper bounds on the numbers $\mathbf{Ind}(S, \mathcal{A})$. To that end, we look at certain patterns in the table ϑ.

A **pile** is a nonempty finite set $\mathscr{P}$ of functions of the form $\tau \colon S \to \mathbb{N}$ with $S \in \operatorname{dom}(\mathscr{B})$, satisfying the following requirements:

- the graphs of the elements of $\mathscr{P}$ are pairwise disjoint; in other words, for every pair of distinct functions $\tau, \tau' \in \mathscr{P}$ and for each $x \in \operatorname{dom}(\tau) \cap \operatorname{dom}(\tau')$, we have $\tau(x) \neq \tau'(x)$;
- for every $\tau \in \mathscr{P}$ and $x \in \operatorname{dom}(\tau)$, either $\tau(x) = 0$, or else, there is $\tau' \in \mathscr{P}$ with $x \in \operatorname{dom}(\tau')$ and $\tau'(x) = \tau(x) - 1$.

The **support** of a pile $\mathscr{P}$ is the set

$$\operatorname{supp}(\mathscr{P}) \coloneqq \bigcup_{\tau \in \mathscr{P}} \operatorname{dom}(\tau).$$

Note that $\operatorname{supp}(\mathscr{P})$ is a finite subset of X.

Let $\mathscr{P}$ be a pile and let $\tau, \tau' \in \mathscr{P}$. We say that τ' **supports** τ, in symbols $\tau' \prec \tau$, if there is an element $x \in \operatorname{dom}(\tau) \cap \operatorname{dom}(\tau')$ such that $\tau'(x) = \tau(x) - 1$. A pile $\mathscr{P}$ is **neat** if there does not exist a sequence of

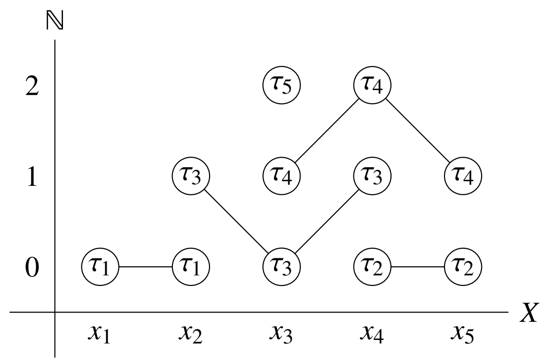

Figure 13 – $\mathscr{P} = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ is a neat pile of height 4 with $\mathrm{supp}(\mathscr{P}) = \{x_1, x_2, x_3, x_4, x_5\}$ and $\mathbf{Top}(\mathscr{P}) = \{\tau_5\}$.

functions $\tau_1, \tau_2, \ldots, \tau_k \in \mathscr{P}$ with $k \geqslant 2$ such that $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_k \prec \tau_1$. Equivalently, $\mathscr{P}$ is neat if the transitive closure of the relation $\prec$ on $\mathscr{P}$ is a (strict) partial order.

A **top element** in a pile $\mathscr{P}$ is any $\tau \in \mathscr{P}$ for which there is no $\tau' \in \mathscr{P}$ with $\tau \prec \tau'$. The set of all top elements in $\mathscr{P}$ is denoted $\mathbf{Top}(\mathscr{P})$. Notice that if $\mathscr{P}$ is a neat pile, then $\mathbf{Top}(\mathscr{P}) \neq \varnothing$. The **height** $h(\mathscr{P})$ of a neat pile $\mathscr{P}$ is the largest $k \in \mathbb{N}$ such that there is a sequence $\tau_1, \ldots, \tau_k \in \mathscr{P}$ with $\tau_1 \prec \ldots \prec \tau_k$ (so necessarily $h(\mathscr{P}) \geqslant 1$).

We say that a pile $\mathscr{P}$ **appears** in a table $\vartheta \colon X \times \mathbb{N} \to [0;1]$ if for all $\tau \in \mathscr{P}$, the map

$$\mathrm{dom}(\tau) \to [0;1] \colon x \mapsto \vartheta(x, \tau(x))$$

belongs to $\mathscr{B}_{\mathrm{dom}(\tau)}$. For $S \in \mathrm{dom}(\mathscr{B})$, let $\mathbf{Piles}(S)$ denote the set of all neat piles $\mathscr{P}$ with $\mathbf{Top}(\mathscr{P}) = \{\tau\}$ such that the unique top element τ of $\mathscr{P}$ satisfies $\mathrm{dom}(\tau) = S$. The **index** $\mathbf{Ind}(S, \vartheta) \in \mathbb{N} \cup \{\infty\}$ of S in ϑ is

$$\mathbf{Ind}(S, \vartheta) \coloneqq |\{\mathscr{P} \in \mathbf{Piles}(S) \,:\, \mathscr{P} \text{ appears in } \vartheta\}|\,.$$

The next proposition asserts that $\mathbf{Ind}(S, \vartheta) \geqslant \mathbf{Ind}(S, \mathcal{A})$ for any Moser–Tardos process $\mathcal{A}$ with input ϑ:

Proposition 3.2.4. *Let $\mathcal{A} = (A_n)_{n=0}^{\infty}$ be a Moser–Tardos process with input ϑ and let $S \in \mathrm{dom}(\mathscr{B})$. If $n \in \mathbb{N}$ is such that $S \in A'_n$, then there exists a neat pile $\mathscr{P} \in \mathbf{Piles}(S)$ of height precisely $n+1$ that appears in ϑ. In particular, $\mathbf{Ind}(S, \mathcal{A}) \leqslant \mathbf{Ind}(S, \vartheta)$.*

Proof. The "in particular" part follows, since for different n with $S \in A'_n$, the neat piles given by the first part of the proposition are distinct (they have distinct heights).

To prove the main statement, fix $S \in \mathrm{dom}(\mathscr{B})$ and $n \in \mathbb{N}$ with $S \in A'_n$. Build $\mathscr{P}$ by "tracing back" the steps of the Moser–Tardos process as follows. Start by setting $\mathscr{P}_0$ to be the one-element set $\{t_n|S\}$ and let $R_0 \coloneqq S$. If $k < n$, then, after $R_k \subseteq X$ is determined, define $\mathscr{P}_{k+1}$ as the family of all maps of the form $t_{n-k-1}|S'$, where S' is an element of A_{n-k-1} such that $S' \cap R_k \neq \varnothing$, and let $R_{k+1} \coloneqq R_k \cup \bigcup \mathscr{P}_{k+1}$. Finally, let $\mathscr{P} \coloneqq \mathscr{P}_0 \cup \ldots \cup \mathscr{P}_n$. It is straightforward to check that $\mathscr{P}$ is a neat pile with support R_n that has all the desired properties. ∎

Given a table $\vartheta \colon X \times \mathbb{N} \to [0;1]$, we say that an element $x \in X$ is **ϑ-stable** if

$$\sum_{S \in \mathrm{dom}(\mathscr{B}) \,:\, S \ni x} \mathbf{Ind}(S, \vartheta) < \infty.$$

The set of all ϑ-stable elements is denoted $\mathbf{Stab}(\vartheta)$. Due to Proposition 3.2.4, $\mathbf{Stab}(\vartheta) \subseteq \mathbf{Stab}(\mathcal{A})$ for every Moser–Tardos process $\mathcal{A}$ with input ϑ.

Now the strategy is to switch the order of summation and, instead of counting how many piles from $\mathbf{Piles}(S)$ appear in a particular table ϑ, fix a pile $\mathscr{P}$ and estimate the probability that $\mathscr{P}$ appears in a table ϑ chosen *at random*. For a given pile $\mathscr{P}$, the restriction of ϑ to $\mathrm{supp}(\mathscr{P}) \times \mathbb{N}$ fully determines whether $\mathscr{P}$ appears in ϑ or not. Thus, we may let $\mathbf{App}(\mathscr{P}) \subseteq [0;1]^{\mathrm{supp}(\mathscr{P}) \times \mathbb{N}}$ be the set such that

$$\mathscr{P} \text{ appears in } \vartheta \iff \vartheta|(\mathrm{supp}(\mathscr{P}) \times \mathbb{N}) \in \mathbf{App}(\mathscr{P}).$$

It is easy to see that the set $\mathbf{App}(\mathscr{P})$ is Borel. Since the graphs of the elements of $\mathscr{P}$ are pairwise disjoint, there is a simple expression for the Lebesgue measure of $\mathbf{App}(\mathscr{P})$; namely, we have

$$\lambda^{\mathrm{supp}(\mathscr{P}) \times \mathbb{N}}(\mathbf{App}(\mathscr{P})) = \prod_{\tau \in \mathscr{P}} \mathbb{P}[\mathrm{dom}(\tau)].$$

Now we are ready to state the cornerstone result of Moser–Tardos theory:

Theorem 3.2.5 (Moser–Tardos [MT10])**.** *Let $\omega \colon \mathscr{B} \to [0;1)$ be a function witnessing the correctness of $\mathscr{B}$ and let $S \in \mathrm{dom}(\mathscr{B})$. Then*

$$\sum_{\mathscr{P} \in \mathbf{Piles}(S)} \lambda^{\mathrm{supp}(\mathscr{P}) \times \mathbb{N}}(\mathbf{App}(\mathscr{P})) \leqslant \sum_{\substack{B \in \mathscr{B}: \\ \mathrm{dom}(B) = S}} \frac{\omega(B)}{1 - \omega(B)}. \tag{3.2.2}$$

The proof of Theorem 3.2.5 using our notation can be found in [Ber16a, Appendix A]. The following corollary is immediate:

Corollary 3.2.6. *For all $x \in X$, we have*

$$\sum_{\substack{S \in \mathrm{dom}(\mathscr{B}): \\ S \ni x}} \sum_{\mathscr{P} \in \mathbf{Piles}(S)} \lambda^{\mathrm{supp}(\mathscr{P}) \times \mathbb{N}}(\mathbf{App}(\mathscr{P})) < \infty.$$

Proof. Let $\omega \colon \mathscr{B} \to [0;1)$ witness the correctness of $\mathscr{B}$. By Theorem 3.2.5, it suffices to check that the sum

$$\sum_{\substack{S \in \mathrm{dom}(\mathscr{B}): \\ S \ni x}} \sum_{\substack{B \in \mathscr{B}: \\ \mathrm{dom}(B) = S}} \frac{\omega(B)}{1 - \omega(B)} = \sum_{\substack{B \in \mathscr{B}: \\ \mathrm{dom}(B) \ni x}} \frac{\omega(B)}{1 - \omega(B)} \tag{3.2.3}$$

is finite. We may assume that $\omega(B) = 0$ whenever $\mathbb{P}[B] = 0$. If for all $B \in \mathscr{B}$ with $x \in \mathrm{dom}(B)$, we have $\mathbb{P}[B] = 0$, then the sum (3.2.3) is 0 (hence finite). Otherwise, for some $B_0 \in \mathscr{B}$ with $x \in \mathrm{dom}(B)$, we have

$\mathbb{P}[B_0] > 0$, and thus the correctness of $\mathscr{B}$ implies

$$\prod_{B \in N_{\mathscr{B}}(B_0)} (1 - \omega(B_0)) > 0.$$

Therefore,

$$\sum_{\substack{B \in \mathscr{B}: \\ \mathrm{dom}(B) \ni x}} \omega(B) \leqslant \sum_{B \in N_{\mathscr{B}}(B_0)} \omega(B) < \infty. \tag{3.2.4}$$

In particular, for all but finitely many events $B \in \mathscr{B}$ with $x \in \mathrm{dom}(B)$, we have $\omega(B) \leqslant 1/2$, so

$$\frac{\omega(B)}{1 - \omega(B)} \leqslant 2\omega(B).$$

Together with (3.2.4), this shows that the sum (3.2.3) is finite, as desired. ■

The next corollary considers the case when the table ϑ is chosen randomly from $[0;1]^{X \times \mathbb{N}}$. (Note that the product probability space $([0;1]^{X \times \mathbb{N}}, \lambda^{X \times \mathbb{N}})$ is standard only if X is countable.)

Corollary 3.2.7. *For each $x \in X$, we have*

$$\int_{[0;1]^{X \times \mathbb{N}}} \sum_{\substack{S \in \mathrm{dom}(\mathscr{B}): \\ S \ni x}} \mathbf{Ind}(S, \vartheta) \, \mathrm{d}\,\lambda^{X \times \mathbb{N}}(\vartheta) < \infty.$$

In particular,

$$\lambda^{X \times \mathbb{N}}(\{\vartheta \in [0;1]^{X \times \mathbb{N}} \,:\, x \in \mathbf{Stab}(\vartheta)\}) = 1.$$

Proof. Corollary 3.2.6 yields

$$\begin{aligned}
\int_{[0;1]^{X \times \mathbb{N}}} \sum_{\substack{S \in \mathrm{dom}(\mathscr{B}): \\ S \ni x}} \mathbf{Ind}(S, \vartheta) \, \mathrm{d}\,\lambda^{X \times \mathbb{N}}(\vartheta) &= \sum_{\substack{S \in \mathrm{dom}(\mathscr{B}): \\ S \ni x}} \int_{[0;1]^{X \times \mathbb{N}}} \mathbf{Ind}(S, \vartheta) \, \mathrm{d}\,\lambda^{X \times \mathbb{N}}(\vartheta) \\
&= \sum_{\substack{S \in \mathrm{dom}(\mathscr{B}): \\ S \ni x}} \sum_{\mathscr{P} \in \mathbf{Piles}(S)} \lambda^{\mathrm{supp}(\mathscr{P}) \times \mathbb{N}} (\mathbf{App}(\mathscr{P})) < \infty.
\end{aligned}$$

■

We can now deduce the LLL in the form of Theorem 3.1.10. Since the set $N_{\mathscr{B}}(B)$ is countable for each $B \in \mathscr{B}$, we may assume that X is countable. By Corollary 3.2.7, each $x \in X$ satisfies

$$\lambda^{X \times \mathbb{N}}(\{\vartheta \in [0;1]^{X \times \mathbb{N}} \,:\, x \in \mathbf{Stab}(\vartheta)\}) = 1.$$

As X is countable, we obtain

$$\lambda^{X \times \mathbb{N}}(\{\vartheta \in [0;1]^{X \times \mathbb{N}} \,:\, X = \mathbf{Stab}(\vartheta)\}) = 1.$$

Choose any ϑ such that $X = \mathbf{Stab}(\vartheta)$ and let $\mathcal{A}$ be any Moser–Tardos process with input ϑ. Then $\mathbf{Stab}(\mathcal{A}) = X$ and Theorem 3.1.10 follows from Proposition 3.2.3.

3.2.1 Moser–Tardos theory in the Borel setting

Let X be a standard Borel space. Recall that an instance $\mathscr{B}$ over X is **Borel** if $\bigcup \mathscr{B}$ is a Borel subset of $[X \to [0;1]]^{<\infty}$. Notice that if $\mathscr{B}$ is a Borel instance over X, then $\mathrm{dom}(\mathscr{B})$ is an analytic subset of $[X]^{<\infty}$.[4] A Moser–Tardos process $\mathcal{A} = (A_n)_{n=0}^{\infty}$ with Borel input $\vartheta \colon X \times \mathbb{N} \to [0;1]$ is **Borel** if each A_n is a Borel subset of $[X]^{<\infty}$. Note that if $\mathcal{A}$ is a Borel Moser–Tardos process, then the associated maps $t_n \colon X \to \mathbb{N}$ and $f_n \colon X \to [0;1]$ are Borel.

Proposition 3.2.8 (**Borel Moser–Tardos processes**). *Let X be a standard Borel space and let $\mathscr{B}$ be a correct Borel instance over X. Let $\vartheta \colon X \times \mathbb{N} \to [0;1]$ be a Borel table. Then there exists a Borel Moser–Tardos process $\mathcal{A}$ with input ϑ.*

Proof. We use the following result of Kechris and Miller:

Lemma 3.2.9 (Kechris–Miller [KM04, Lemma 7.3]; **maximal disjoint subfamilies**). *Let X be a standard Borel space and let $A \subseteq [X]^{<\infty}$ be a Borel set such that for every $x \in X$, the set $\{S \in A \,:\, x \in S\}$ is countable. Then there is a Borel maximal disjoint subset $A_0 \subseteq A$.*

On Step n of the Moser–Tardos process, we are given a Borel map $f_n \colon X \to [0;1]$, so the set

$$A'_n \coloneqq \{S \in \mathrm{dom}(\mathscr{B}) \,:\, f_n \supseteq w \text{ for some } w \in \mathscr{B}_S\} = \{S \in \mathrm{dom}(\mathscr{B}) \,:\, f_n|S \in \textstyle\bigcup \mathscr{B}\}$$

is Borel. Hence, we can use Lemma 3.2.9 to pick a Borel maximal disjoint subset $A_n \subseteq A'_n$. ■

3.3 Hereditarily finite sets

In this section we describe the construction of a "universal" combinatorial structure over a space X, whose points encode various combinatorial data that can be built from the elements of X.

The set $\mathbf{HF}_{\varnothing}(X)$ of all **hereditarily finite sets** over X is defined inductively as follows[5]:

- $\mathbf{HF}^{(0)}(X) \coloneqq X$;
- $\mathbf{HF}^{(n+1)}(X) \coloneqq \mathbf{HF}^{(n)}(X) \cup [\mathbf{HF}^{(n)}(X)]^{<\infty}$ for all $n \in \mathbb{N}$;
- $\mathbf{HF}_{\varnothing}(X) \coloneqq \bigcup_{n=0}^{\infty} \mathbf{HF}^{(n)}(X)$ (note that this union is increasing).

In other words, $\mathbf{HF}_{\varnothing}(X)$ is the smallest set containing X that is closed under taking finite subsets. For $h \in \mathbf{HF}_{\varnothing}(X)$, the **underlying set** of h, in symbols $\mathbf{Set}(h)$, is defined inductively by:

- for $x \in X$, $\mathbf{Set}(x) \coloneqq \{x\}$;

[4] In most applications, each bad event $B \in \mathscr{B}$ has positive probability. If that is the case, then $\mathrm{dom}(\mathscr{B})$ is actually a Borel subset of $[X]^{<\infty}$ due to the "large section" uniformization theorem [Kec95, Corollary 18.7].

[5] Here we treat the points of X as *urelements*, i.e., not sets. Formally, we can replace X with, say, the diagonal

$$\Delta_X^{\mathbb{N}} \coloneqq \{(x, x, x, \ldots) \,:\, x \in X\} \subseteq X^{\mathbb{N}},$$

ensuring that no point in X is a finite set.

– for $h \in \mathbf{HF}^{(n+1)}(X) \setminus \mathbf{HF}^{(n)}(X)$, $\mathbf{Set}(h) \coloneqq \bigcup_{h' \in h} \mathbf{Set}(h')$.

Equivalently, $\mathbf{Set}(h)$ is the smallest subset S of X such that $h \in \mathbf{HF}_\varnothing(S)$. The **amplification** of X is

$$\mathbf{HF}(X) \coloneqq \{h \in \mathbf{HF}_\varnothing(X) \,:\, \mathbf{Set}(h) \neq \varnothing\}.$$

If X is a standard Borel space, then so are $\mathbf{HF}_\varnothing(X)$ and $\mathbf{HF}(X)$. The space $\mathbf{HF}(X)$ encodes the "combinatorics" of X. For instance, $\mathbf{HF}(X)$ contains (as Borel subsets) the space $X^{<\infty}$ of all nonempty finite sequences of elements of X and the space $X \times \mathbb{N}$, i.e., the union of countably many disjoint copies of X.[6] In fact, $\mathbf{HF}(X) \supseteq \mathbf{HF}(X) \times \mathbb{N}$, i.e., $\mathbf{HF}(X)$ contains "countably many disjoint copies of itself." If G is a Borel graph on X, then the edge set of G, viewed as a set of 2-element subsets of X, is also a Borel subset of $\mathbf{HF}(X)$. So are other, more complicated, objects associated with G. For instance, the set of all cycles in G, i.e., the set of all finite subsets $C \subseteq E(G)$ whose elements form a cycle, is a Borel subset of $\mathbf{HF}(X)$.

If X' is a Borel subset of $\mathbf{HF}(X)$, then the inclusions

$$[X']^{<\infty} \subseteq [\mathbf{HF}(X)]^{<\infty} \quad \text{and} \quad [X' \to [0;1]]^{<\infty} \subseteq [\mathbf{HF}(X) \to [0;1]]^{<\infty}$$

are Borel as well. Therefore, a Borel instance of the LLL over X' is also a Borel instance over $\mathbf{HF}(X)$. Because of that, we will restrict our attention to instances over $\mathbf{HF}(X)$, and this will include various combinatorial applications such as vertex coloring or edge coloring.

Functions between sets naturally lift to functions between their amplifications. Namely, given a map $\varphi\colon X \to Y$, define $\tilde{\varphi}_\varnothing\colon \mathbf{HF}_\varnothing(X) \to \mathbf{HF}_\varnothing(Y)$ inductively via:

– for $x \in X$, $\tilde{\varphi}_\varnothing(x) \coloneqq \varphi(x)$;

– for $h \in \mathbf{HF}^{(n+1)}(X) \setminus \mathbf{HF}^{(n)}(X)$, $\tilde{\varphi}_\varnothing(h) \coloneqq \{\tilde{\varphi}_\varnothing(h') \,:\, h' \in h\}$.

The **amplification** of φ is the map $\tilde{\varphi}\colon \mathbf{HF}(X) \to \mathbf{HF}(Y)$ given by

$$\tilde{\varphi} \coloneqq \tilde{\varphi}_\varnothing|\mathbf{HF}(X).$$

For $S \in [X]^{<\infty} \setminus \{\varnothing\}$, we have $\tilde{\varphi}(S) = \varphi(S)$ (where $\varphi(S)$ denotes, as usual, the image of S under φ). If φ is injective (resp. surjective), then $\tilde{\varphi}$ is also injective (resp. surjective).

3.4 Approximate LLL

In this section we state and prove the first main result of this chapter: the approximate LLL for Borel instances.

Let (X, μ) be a standard probability space. Suppose that $\mathscr{B}$ is a Borel instance over $\mathbf{HF}(X)$. For each $x \in X$, consider the following set:

$$\partial_x(\mathscr{B}) \coloneqq \{S \in \mathrm{dom}(\mathscr{B}) \,:\, x \in \mathbf{Set}(h) \text{ for some } h \in S\}.$$

[6] To embed $\mathbb{N}$ in $\mathbf{HF}_\varnothing(X)$, we use the standard von Neumann convention $0 = \varnothing$, $1 = \{\varnothing\}$, $2 = \{\varnothing, \{\varnothing\}\}$, etc.

We call $\partial_x(\mathscr{B})$ the **shadow** of $\mathscr{B}$ over x. We say that $\mathscr{B}$ is **hereditarily locally finite** if $\partial_x(\mathscr{B})$ is finite for all $x \in X$. For a Borel map $f : \mathbf{HF}(X) \to [0;1]$, its **defect** with respect to $\mathscr{B}$ is the set

$$\mathbf{Def}_{\mathscr{B}}(f) \coloneqq \{x \in X \ : \ f|S \in \bigcup \mathscr{B} \text{ for some } S \in \partial_x(\mathscr{B})\}.$$

Note that if B is hereditarily locally finite, then $\mathbf{Def}_{\mathscr{B}}(f)$ is a Borel subset of X.

Theorem 3.4.1 (**Approximate LLL**). *Let (X, μ) be a standard probability space and let $\mathscr{B}$ be a hereditarily locally finite correct Borel instance over $\mathbf{HF}(X)$. Then for any $\varepsilon > 0$, there is a Borel map $f : \mathbf{HF}(X) \to [0;1]$ with $\mu(\mathbf{Def}_{\mathscr{B}}(f)) \leqslant \varepsilon$.*

3.4.1 Proof of Theorem 3.4.1

Let (X, μ) be a standard probability space and let $\mathscr{B}$ be a hereditarily locally finite correct Borel instance over $\mathbf{HF}(X)$. Fix $\varepsilon > 0$. For $S \in \mathrm{dom}(\mathscr{B})$ and $n \in \mathbb{N}$, let $\mathbf{Piles}_n(S)$ denote the set of all neat piles $\mathscr{P} \in \mathbf{Piles}(S)$ of height precisely $n + 1$. In particular, we have

$$\mathbf{Piles}(S) = \bigcup_{n=0}^{\infty} \mathbf{Piles}_n(S),$$

and the above union is disjoint. For $n \in \mathbb{N}$, let D_n denote the set of all $x \in X$ such that

$$\sum_{S \in \partial_x(\mathscr{B})} \ \sum_{\mathscr{P} \in \mathbf{Piles}_n(S)} \lambda^{\mathrm{supp}(\mathscr{P}) \times \mathbb{N}}(\mathbf{App}(\mathscr{P})) > \varepsilon/2.$$

It is clear from the definition that the set D_n is analytic; in particular, it is μ-measurable.[7] Due to Corollary 3.2.6 and the fact that $\mathscr{B}$ is hereditarily locally finite, each $x \in X$ satisfies

$$\sum_{S \in \partial_x(\mathscr{B})} \ \sum_{\mathscr{P} \in \mathbf{Piles}(S)} \lambda^{\mathrm{supp}(\mathscr{P}) \times \mathbb{N}}(\mathbf{App}(\mathscr{P})) < \infty.$$

Hence we can choose $N \in \mathbb{N}$ so large that $\mu(D_N) \leqslant \varepsilon/2$.

Let G be the graph on $\mathbf{HF}(X)$ given by

$$h_1 h_2 \in E(G) \ :\Longleftrightarrow \ h_1 \neq h_2 \text{ and } \{h_1, h_2\} \subseteq S \text{ for some } S \in \mathrm{dom}(\mathscr{B}).$$

Clearly, G is analytic. Since $\mathscr{B}$ is hereditarily locally finite, G is locally finite. For $n \in \mathbb{N}$, let G^n denote the analytic graph on $\mathbf{HF}(X)$ in which distinct elements $h_1, h_2 \in \mathbf{HF}(X)$ are adjacent if and only if G contains a path of length at most n joining h_1 and h_2 (in particular, $G^1 = G$). Since G is locally finite, so is G^n for each $n \in \mathbb{N}$. Therefore, $\chi_{\mathrm{B}}(G^n) \leqslant \aleph_0$ for all $n \in \mathbb{N}$, so let $c : \mathbf{HF}(X) \to \mathbb{N}$ be a Borel proper coloring of $G^{2(N+1)}$.

[7] In fact, D_n is Borel. Indeed, if there is $\tau \in \mathscr{P}$ with $\mathbb{P}[\mathrm{dom}(\tau)] = 0$, then $\lambda^{\mathrm{supp}(\mathscr{P}) \times \mathbb{N}}(\mathbf{App}(\mathscr{P})) = 0$; and the set $\{S \in [\mathbf{HF}(X)]^{<\infty} \ : \ \mathbb{P}[S] > 0\}$ is Borel due to the "large section" uniformization theorem [Kec95, Corollary 18.7].

For a function $\vartheta\colon \mathbb{N}\times\mathbb{N}\to[0;1]$, define a map ϑ_c by

$$\vartheta_c\colon \mathbf{HF}(X)\times\mathbb{N}\to[0;1]\colon (x,n)\mapsto\vartheta(c(x),n).$$

Note that ϑ_c is a Borel table in the sense of the Moser–Tardos algorithm on $\mathbf{HF}(X)$. Let Q be the set of all pairs (x,ϑ) with $x\in X$ and $\vartheta\colon\mathbb{N}\times\mathbb{N}\to[0;1]$ such that

$$\text{there exist } S\in\partial_x(\mathscr{B}) \text{ and } \mathscr{P}\in\mathbf{Piles}_N(S) \text{ such that } \mathscr{P} \text{ appears in } \vartheta_c.$$

By definition, Q is an analytic subset of $X\times[0;1]^{\mathbb{N}\times\mathbb{N}}$.[8] Recall that for $x\in X$ and $\vartheta\colon\mathbb{N}\times\mathbb{N}\to[0;1]$, we use Q_x and Q^ϑ to denote the corresponding fibers of Q.

Lemma 3.4.2. *For all $x\in X\setminus D_N$, we have $\lambda^{\mathbb{N}\times\mathbb{N}}(Q_x)\leqslant\varepsilon/2$.*

Proof. If $\mathscr{P}$ is a neat pile with a unique top element τ, then for every $\tau'\in\mathscr{P}$, there exists a sequence τ_1, $\ldots$, $\tau_k\in\mathscr{P}$ such that $\tau_1=\tau'$, $\tau_k=\tau$, and $\tau_1\prec\ldots\prec\tau_k$. In particular, $\mathrm{dom}(\tau_i)\cap\mathrm{dom}(\tau_{i+1})\neq\varnothing$ for all $1\leqslant i<k$, so the distance in G between any element of $\mathrm{dom}(\tau')$ and any element of $\mathrm{dom}(\tau)$ is at most $k\leqslant h(\mathscr{P})$. Therefore, the distance in G between any two elements of $\mathrm{supp}(\mathscr{P})$ is at most $2h(\mathscr{P})$.

Fix any $x\in X\setminus D_N$ and let $S\in\partial_x(\mathscr{B})$ and $\mathscr{P}\in\mathbf{Piles}_N(S)$. Since $h(\mathscr{P})=N+1$, the distance in G between any two elements of $\mathrm{supp}(\mathscr{P})$ is at most $2(N+1)$; in other words, any two distinct elements of $\mathrm{supp}(\mathscr{P})$ are adjacent in $G^{2(N+1)}$. Therefore, the coloring c is injective on $\mathrm{supp}(\mathscr{P})$. Hence, the map

$$[0;1]^{\mathbb{N}\times\mathbb{N}}\to[0;1]^{\mathrm{supp}(\mathscr{P})\times\mathbb{N}}\colon \vartheta\mapsto\vartheta_c|(\mathrm{supp}(\mathscr{P})\times\mathbb{N})$$

is measure-preserving. Since

$$\mathscr{P}\text{ appears in }\vartheta_c \iff \vartheta_c|(\mathrm{supp}(\mathscr{P})\times\mathbb{N})\in\mathbf{App}(\mathscr{P}),$$

we may conclude

$$\lambda^{\mathbb{N}\times\mathbb{N}}(\{\vartheta\in[0;1]^{\mathbb{N}\times\mathbb{N}}\,:\,\mathscr{P}\text{ appears in }\vartheta_c\})=\lambda^{\mathrm{supp}(\mathscr{P})\times\mathbb{N}}(\mathbf{App}(\mathscr{P})).$$

Therefore,

$$\begin{aligned}\lambda^{\mathbb{N}\times\mathbb{N}}(Q_x)&\leqslant\sum_{S\in\partial_x(\mathscr{B})}\ \sum_{\mathscr{P}\in\mathbf{Piles}_N(S)}\lambda^{\mathbb{N}\times\mathbb{N}}(\{\vartheta\in[0;1]^{\mathbb{N}\times\mathbb{N}}\,:\,\mathscr{P}\text{ appears in }\vartheta_c\})\\&=\sum_{S\in\partial_x(\mathscr{B})}\ \sum_{\mathscr{P}\in\mathbf{Piles}_N(S)}\lambda^{\mathrm{supp}(\mathscr{P})\times\mathbb{N}}(\mathbf{App}(\mathscr{P}))\ \leqslant\ \varepsilon/2,\end{aligned}$$

by the definition of D_N. ■

[8] Again, one can show that Q is actually Borel.

Using Fubini's theorem and Lemma 3.4.2, we get

$$(\mu \times \lambda^{\mathbb{N}\times\mathbb{N}})(Q) = \int_X \lambda^{\mathbb{N}\times\mathbb{N}}(Q_x)\,\mathrm{d}\,\mu(x) \leqslant \mu(D_N) + (1 - \mu(D_N)) \cdot \varepsilon/2 \leqslant \varepsilon.$$

Therefore, Fubini's theorem yields some $\vartheta \colon \mathbb{N} \times \mathbb{N} \to [0;1]$ with $\mu(Q^\vartheta) \leqslant \varepsilon$. Fix any such ϑ and let $\mathcal{A} = (A_n)_{n=0}^\infty$ be any Borel Moser–Tardos process with input ϑ_c. Let t_n and f_n denote the associated maps.

Lemma 3.4.3. $\mathbf{Def}_{\mathscr{B}}(f_N) \subseteq Q^\vartheta$.

Proof. If $x \in \mathbf{Def}_{\mathscr{B}}(f_N)$, then, by definition, there is $S \in \partial_x(\mathscr{B})$ such that $f_N|S \in \mathscr{B}_S$, i.e., $S \in A'_N$. By Proposition 3.2.4, there is $\mathscr{P} \in \mathbf{Piles}_N(S)$ that appears in ϑ_c. Therefore, $(x, \vartheta) \in Q$, as desired. ∎

Finally, we obtain $\mu(\mathbf{Def}_{\mathscr{B}}(f_N)) \leqslant \mu(Q^\vartheta) \leqslant \varepsilon$, and the proof of Theorem 3.4.1 is complete.

3.5 The LLL for probability measure-preserving group actions

3.5.1 Definitions and the statement of the theorem

As discussed in this chapter's introduction, we would like to establish a measurable version of the LLL for Borel instances that, in a certain sense, "respect" some additional structure on the space X, specifically, an action of a countable group Γ. To make this idea precise, we introduce L-systems—objects consisting of a standard probability space equipped with a family of functions ("partial isomorphisms") under which any instance of the LLL that we might consider must be invariant. We then define the LLL Game over an L-system, which captures the need for iterated applications of the LLL.

Equivalence relations

Given an equivalence relation E on a set X, we write (somewhat ambiguously)

$$\begin{aligned} [E]^{<\infty} &:= \{S \in [X]^{<\infty} \,:\, S \text{ is contained in a single } E\text{-class}\} \\ \text{and} \qquad [E \to Y]^{<\infty} &:= \{w \in [X \to Y]^{<\infty} \,:\, \mathrm{dom}(w) \in [E]^{<\infty}\}. \end{aligned}$$

An **instance** (**of the LLL**) over E is an instance $\mathscr{B}$ over X such that $\mathrm{dom}(\mathscr{B}) \subseteq [E]^{<\infty}$.

Example 3.5.1 (**Equivalence relations induced by graphs**)**.** Let G be a graph on a set X. We use E_G to denote the equivalence relation on X whose classes are the connected components of G.

Example 3.5.2 (**Equivalence relations induced by group actions**)**.** Let $\alpha \colon \Gamma \curvearrowright X$ be an action of a group Γ on a set X. Then E_α denotes the corresponding **orbit equivalence relation**, i.e., the equivalence relation whose classes are the orbits of α. Notice that if $S \subseteq \Gamma$ is a generating set, then $E_\alpha = E_{G(\alpha,S)}$.

Isomorphism structures

An **isomorphism structure** on an equivalence relation E on a set X is a family $\mathcal{I}$ of bijections between E-classes which forms a groupoid[9] whose set of objects is X/E; more precisely, the following conditions must be fulfilled:

- for each $C \in X/E$, the identity map $\mathrm{id}_C \colon C \to C$ belongs to $\mathcal{I}$;
- for each $\varphi \in \mathcal{I}$, we have $\varphi^{-1} \in \mathcal{I}$;
- for all $\varphi, \psi \in \mathcal{I}$, if $\mathrm{im}(\varphi) = \mathrm{dom}(\psi)$, then $\psi \circ \varphi \in \mathcal{I}$.

The following are the main examples of isomorphism structures we will be considering.

Example 3.5.3 (**Isomorphism structures induced by graphs**)**.** Let G be a graph on a set X. Define the isomorphism structure $\mathcal{I}_G$ on E_G as follows: A bijection $\varphi \colon C_1 \to C_2$ between components C_1 and C_2 belongs to $\mathcal{I}_G$ if and only if it is an isomorphism between the graphs $G[C_1]$ and $G[C_2]$.

Example 3.5.4 (**Isomorphism structures induced by group actions**)**.** Let $\alpha \colon \Gamma \curvearrowright X$ be an action of a group Γ on a set X. The isomorphism structure $\mathcal{I}_\alpha$ on E_α is defined as follows: A bijection $\varphi \colon O_1 \to O_2$ between orbits O_1 and O_2 belongs to $\mathcal{I}_\alpha$ if and only if it is Γ**-equivariant**, i.e., $\varphi(\gamma \cdot x) = \gamma \cdot \varphi(x)$ for all $x \in O_1$ and $\gamma \in \Gamma$. Notice that if $S \subseteq \Gamma$ is a generating set, then $\mathcal{I}_\alpha \subseteq \mathcal{I}_{G(\alpha,S)}$.

Let E be a Borel equivalence relation on a standard probability space (X, μ) and let $\mathcal{I}$ be an isomorphism structure on E. We say that an instance $\mathscr{B}$ over E is $\mathcal{I}$**-invariant** on a set $X' \subseteq X$ if for all $\varphi \in \mathcal{I}$ with $\mathrm{dom}(\varphi) \cup \mathrm{im}(\varphi) \subseteq X'$ and for all $B \in \mathscr{B}$ with $\mathrm{dom}(B) \subseteq \mathrm{im}(\varphi)$, we have

$$\{w \circ \varphi \,:\, w \in B\} \in \mathscr{B}.$$

An instance $\mathscr{B}$ is μ**-almost everywhere** $\mathcal{I}$**-invariant** if it is $\mathcal{I}$-invariant on an E-invariant μ-conull Borel subset $X' \subseteq X$.

L-Systems and instances of the LLL over them

An **L-system**[10] is a tuple $\mathcal{L} = (X_\mathcal{L}, E_\mathcal{L}, \mathcal{I}_\mathcal{L}, \mu_\mathcal{L})$, where

- $(X_\mathcal{L}, \mu_\mathcal{L})$ is a standard probability space;
- $E_\mathcal{L}$ is a countable Borel equivalence relation on $X_\mathcal{L}$;
- $\mathcal{I}_\mathcal{L}$ is an isomorphism structure on $E_\mathcal{L}$.

An **instance** (**of the LLL**) over an L-system $\mathcal{L}$ is a $\mu_\mathcal{L}$-almost everywhere $\mathcal{I}_\mathcal{L}$-invariant Borel instance over $E_\mathcal{L}$. A Borel map $f \colon X_\mathcal{L} \to [0;1]$ is a **measurable solution** to an instance $\mathscr{B}$ over $\mathcal{L}$ if $\mathbf{Def}_\mathscr{B}(f)$ is contained in an $E_\mathcal{L}$-invariant $\mu_\mathcal{L}$-null Borel subset of $X_\mathcal{L}$.

[9] A **groupoid** is a category in which every morphism has an inverse.
[10] "L" is for "Lovász."

For a p.m.p. action $\alpha\colon \Gamma \curvearrowright (X,\mu)$, let $\mathcal{L}(\alpha,\mu)$ denote the L-system $(X, E_\alpha, \mathcal{I}_\alpha, \mu)$ **induced** by α. An instance over $\mathcal{L}(\alpha,\mu)$ is simply a Borel instance over X such that the domain of each bad event $B \in \mathscr{B}$ is contained within a single α-orbit and $\mathscr{B}$ is (μ-almost everywhere) invariant under the Γ-equivariant bijections between the orbits of α.

Amplifications and expansions

Before we can state the main result of this section, we need a few more definitions describing how to build new L-systems from old ones.

Let E be an equivalence relation on a set X. Define (somewhat ambiguously)

$$\mathbf{HF}(E) \coloneqq \{h \in \mathbf{HF}(X) \,:\, \mathbf{Set}(h) \in [E]^{<\infty}\}.$$

The **amplification** of E is the equivalence relation $\tilde{E}$ on $\mathbf{HF}(E)$ defined by

$$h_1 \,\tilde{E}\, h_2 \;:\Longleftrightarrow\; [\mathbf{Set}(h_1)]_E = [\mathbf{Set}(h_2)]_E.$$

In other words, $\tilde{E}$ is the equivalence relation on $\mathbf{HF}(E)$ whose classes are the sets $\mathbf{HF}(C)$ with $C \in X/E$. For a bijection $\varphi\colon C_1 \to C_2$ between E-classes, we may extend it to a bijection $\tilde{\varphi}\colon \mathbf{HF}(C_1) \to \mathbf{HF}(C_2)$ between the corresponding $\tilde{E}$-classes. The **amplification** of an isomorphism structure $\mathcal{I}$ on E is the isomorphism structure $\tilde{\mathcal{I}}$ on $\tilde{E}$ given by

$$\tilde{\mathcal{I}} \coloneqq \{\tilde{\varphi} \,:\, \varphi \in \mathcal{I}\}.$$

Given an L-system $\mathcal{L} = (X, E, \mathcal{I}, \mu)$, its **amplification** is the L-system

$$\mathbf{HF}(\mathcal{L}) \coloneqq (\mathbf{HF}(E), \tilde{E}, \tilde{\mathcal{I}}, \mu).$$

Notice that the measure in $\mathbf{HF}(\mathcal{L})$ is the same as in $\mathcal{L}$ and is concentrated on $X \subseteq \mathbf{HF}(X)$.

Another way of obtaining new L-systems is via *expansions*. Let $\mathcal{I}$ be an isomorphism structure on an equivalence relation E on a set X. Given a partial map $f\colon X \rightharpoonup Y$, the **expansion** of $\mathcal{I}$ by f is the subset $\mathcal{I}[f] \subseteq \mathcal{I}$ defined as follows:

$$\mathcal{I}[f] \coloneqq \{\varphi \in \mathcal{I} \,:\, f(x) = f(\varphi(x)) \text{ for all } x \in \mathrm{dom}(\varphi)\}.$$

Here the equality "$f(x) = f(\varphi(x))$" should be interpreted as a shorthand for:

"Either $\{x, \varphi(x)\} \subseteq \mathrm{dom}(f)$ and $f(x) = f(\varphi(x))$, or else, $\{x, \varphi(x)\} \cap \mathrm{dom}(f) = \varnothing$."

For an L-system $\mathcal{L} = (X, E, \mathcal{I}, \mu)$ and a Borel map $f\colon X \rightharpoonup Y$, the **expansion** of $\mathcal{L}$ by f is the L-system

$$\mathcal{L}[f] \coloneqq (X, E, \mathcal{I}[f], \mu).$$

The term "expansion" conveys the following intuition: If $\mathcal{I}$ is thought of as a family of isomorphisms between certain substructures of X, then expanding $\mathcal{I}$ by f corresponds to adding f to X as a new "predicate" whose values must be preserved by isomorphisms.

The LLL game

As we mentioned previously, many combinatorial arguments contain iterated applications of the LLL, where the output of a previous iteration can be used to create an instance for the next one. To accommodate such arguments, we introduce the following definition.

Definition 3.5.5 (**LLL Game**). The **LLL Game** over an L-system $\mathcal{L}$ is played as follows. Set $\mathcal{L}_0 \coloneqq \mathcal{L}$. On Step $n \in \mathbb{N}$, Player I chooses a correct instance $\mathscr{B}_n$ over $\mathcal{L}_n$. Player II must respond by playing a measurable solution f_n to $\mathscr{B}_n$ and setting $\mathcal{L}_{n+1} \coloneqq \mathcal{L}_n[f_n]$. Player I wins if Player II does not have an available move on some finite stage of the game; Player II wins if the game continues indefinitely. A run of the LLL Game looks like this:

Player I	$\mathscr{B}_0$		$\mathscr{B}_1$		$\ldots$	$\mathscr{B}_n$		$\ldots$
Player II		f_0		f_1	$\ldots$		f_n	$\ldots$

One can think of the LLL Game as a struggle between a malevolent combinatorial proof (Player I) and a descriptive set theorist (Player II), who wants to adapt this proof to the measurable setting. The proof consists of a series of steps, each of which is an application of the LLL. The goal of Player II is to perform these steps measurably; however, she might not know what the steps are in advance, and each time she solves an instance of the LLL, her solution may be "used against her" in creating new instances.

With Definition 3.5.5 at hand, we are finally ready to state the main result of this section:

Theorem 3.5.6 (**Measurable LLL for group actions**). *Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of a countable group Γ. If α factors to the shift action $\Gamma \curvearrowright ([0;1]^\Gamma, \lambda^\Gamma)$, then Player II has a winning strategy in the LLL Game over $\mathbf{HF}(\mathcal{L}(\alpha, \mu))$.*

A very specific case of Theorem 3.5.6 is given by the following immediate corollary:

Corollary 3.5.7. *Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of a countable group Γ. Suppose that α factors to the shift action $\Gamma \curvearrowright ([0;1]^\Gamma, \lambda^\Gamma)$ and let $\mathscr{B}$ be a correct instance over $\mathcal{L}(\alpha, \mu)$. Then there exists a Borel function $f \colon X \to [0;1]$ with $\mu(\mathbf{Def}_{\mathscr{B}}(f)) = 0$.*

3.5.2 Outline of the proof

Let $\mathscr{G}$ denote the class of all L-systems of the form $\mathcal{L}(\alpha, \mu)$, where $\alpha \colon \Gamma \curvearrowright (X, \mu)$ is a measure-preserving action of a countable group Γ on a standard probability space (X, μ) that factors to the $[0;1]$-shift action of Γ. Let $\mathscr{L}$ be the class of all L-systems such that Player II has a winning strategy in the LLL Game over $\mathbf{HF}(\mathcal{L})$. Our goal is to show $\mathscr{G} \subseteq \mathscr{L}$. To that end, we will introduce an intermediate class $\mathscr{C}$ such that $\mathscr{G} \subseteq \mathscr{C} \subseteq \mathscr{L}$.

Our strategy for showing that $\mathscr{C} \subseteq \mathscr{L}$ will be to ensure that $\mathscr{C}$ has the following two properties:

(A1) if $\mathcal{L} \in \mathscr{C}$, then $\mathbf{HF}(\mathcal{L}) \in \mathscr{C}$;

(A2) if $\mathcal{L} \in \mathscr{C}$ and $\mathscr{B}$ is a correct instance over $\mathcal{L}$, then there exists a measurable solution f to $\mathscr{B}$ such that $\mathcal{L}[f] \in \mathscr{C}$.

The above conditions imply that $\mathscr{C} \subseteq \mathscr{L}$. Indeed, due to Property (A1), it is enough to show that for every $\mathcal{L} \in \mathscr{C}$, Player II has a winning strategy in the LLL Game over $\mathcal{L}$. The existence of such strategy is guaranteed by Property (A2), since, provided that $\mathcal{L}_n \in \mathscr{C}$, Player II can always find a measurable solution f_n to $\mathscr{B}_n$ such that $\mathcal{L}_{n+1} = \mathcal{L}_n[f_n] \in \mathscr{C}$.

It is easy to see that Property (A1) fails for $\mathscr{G}$. For instance, if $\mathcal{L} = (X, E, \mathcal{I}, \mu) \in \mathscr{G}$, then the measure μ is E-invariant, while it is not even $\tilde{E}$-quasi-invariant. To overcome this complication, we will introduce *countable Borel groupoids*—algebraic structures more general than countable groups—and their actions on standard Borel spaces. Every Borel action of a countable Borel groupoid on a standard probability space induces an L-system. We will also define shift actions of countable Borel groupoids, generalizing shift actions of countable groups. Our choice for $\mathscr{C}$ will be the class of all L-systems that admit factor maps to L-systems induced by shift actions of countable Borel groupoids (we define what a factor map between two general L-systems is in §3.5.3).

3.5.3 Factors of L-systems

In this section we introduce the notion of a factor map between two L-systems. It will allow us to transfer instances of the LLL from a given L-system to a simpler or better-behaved one.

Definition 3.5.8 (Factors). Let $\mathcal{L}_1 = (X_1, E_1, \mathcal{I}_1, \mu_1)$ and $\mathcal{L}_2 = (X_2, E_2, \mathcal{I}_2, \mu_2)$ be L-systems. A Borel partial map $\pi \colon X_1 \rightharpoonup X_2$, defined on an E_1-invariant μ_1-conull Borel subset of X_1, is called a **factor map** (notation: $\pi \colon \mathcal{L}_1 \to \mathcal{L}_2$) if the following requirements are fulfilled:

(i) $\pi_*(\mu_1) = \mu_2$;

(ii) the map π is **class-bijective**, i.e., for each E_1-class $C \subseteq \mathrm{dom}(\pi)$, its image $\pi(C)$ is an E_2-class and the restriction $\pi|C \colon C \to \pi(C)$ is a bijection;

(iii) for all E_1-classes $C_1, C_2 \subseteq \mathrm{dom}(\pi)$, whenever $\varphi_2 \in \mathcal{I}_2$ is a bijection between $\pi(C_1)$ and $\pi(C_2)$, there is a bijection $\varphi_1 \in \mathcal{I}_1$ between C_1 and C_2 that makes the following diagram commute:

$$\begin{tikzcd} C_1 \arrow[r, dashed, "\varphi_1"] \arrow[d, "\pi"] & C_2 \arrow[d, "\pi"] \\ \pi(C_1) \arrow[r, "\varphi_2"] & \pi(C_2). \end{tikzcd}$$

Proposition 3.5.9. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be L-systems with a factor map $\pi \colon \mathcal{L}_1 \to \mathcal{L}_2$ between them. Then there exists a factor map from* $\mathbf{HF}(\mathcal{L}_1)$ *to* $\mathbf{HF}(\mathcal{L}_2)$.

Proof. Let $\tilde{\pi} \colon \mathbf{HF}(\mathrm{dom}(\pi)) \to \mathbf{HF}(X_{\mathcal{L}_2})$ be the amplification of π. Then the restriction of $\tilde{\pi}$ to the set $\mathbf{HF}(E_{\mathcal{L}_1}) \cap \mathrm{dom}(\tilde{\pi})$ is a factor map from $\mathbf{HF}(\mathcal{L}_1)$ to $\mathbf{HF}(\mathcal{L}_2)$. ■

Lemma 3.5.10. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be L-systems with a factor map $\pi\colon \mathcal{L}_1 \to \mathcal{L}_2$ between them. Then for every correct instance $\mathscr{B}$ over $\mathcal{L}_1$, there exists a correct instance $\pi(\mathscr{B})$ over $\mathcal{L}_2$ such that whenever f is a measurable solution to $\pi(\mathscr{B})$, the composition $f \circ \pi$, possibly restricted to a smaller invariant conull Borel subset, is a measurable solution to $\mathscr{B}$.*

Proof. For $i \in \{1, 2\}$, let $\mathcal{L}_i =: (X_i, E_i, \mathcal{I}_i, \mu_i)$. Suppose that $\mathscr{B}$ is a correct instance over $\mathcal{L}_1$. Restricting π to a smaller E_1-invariant μ_1-conull Borel subset of X_1 if necessary, we arrange that $\mathscr{B}_1$ is $\mathcal{I}_1$-invariant on $\mathrm{dom}(\pi)$ and $\mathrm{im}(\pi)$ is a Borel subset of X_2. Then we replace X_1 and X_2 by their invariant conull Borel subsets $\mathrm{dom}(\pi)$ and $\mathrm{im}(\pi)$ respectively. Thus, we now assume that $\pi\colon X_1 \to X_2$ is defined everywhere and is surjective.

Consider any $B \in \mathscr{B}$. Since $\mathrm{dom}(B)$ is contained within a single E_1-class, the restriction

$$\pi|\mathrm{dom}(B)\colon \mathrm{dom}(B) \to \pi(\mathrm{dom}(B))$$

is bijective; in particular, the inverse

$$(\pi|\mathrm{dom}(B))^{-1}\colon \pi(\mathrm{dom}(B)) \to \mathrm{dom}(B)$$

is well-defined. Let

$$\pi(B) \coloneqq \{w \circ (\pi|\mathrm{dom}(B))^{-1} \;:\; w \in B\}.$$

Then $\pi(B)$ is a bad event over X_2 with domain $\pi(\mathrm{dom}(B))$. Define

$$\pi(\mathscr{B}) \coloneqq \{\pi(B) \;:\; B \in \mathscr{B}\}.$$

It is routine to check that $\pi(\mathscr{B})$ is as desired. The only non-trivial step is to show that $\pi(\mathscr{B})$ is Borel; to that end, observe that the set $\bigcup \pi(\mathscr{B})$ is both analytic and co-analytic, as for $w \in [E_1 \to [0;1]]^{<\infty}$,

$$\begin{aligned} w \in \bigcup \pi(\mathscr{B}) &\iff \exists S \in [E_1]^{<\infty}\ (\pi(S) = \mathrm{dom}(w) \text{ and } w \circ (\pi|S) \in \bigcup \mathscr{B}) \\ &\iff \forall S \in [E_1]^{<\infty}\ (\pi(S) = \mathrm{dom}(w) \implies w \circ (\pi|S) \in \bigcup \mathscr{B}). \end{aligned}$$ ∎

For a class $\mathscr{C}$ of L-systems, define the class $\mathscr{C}^*$ by

$$\mathcal{L} \in \mathscr{C}^* \;:\!\iff\; \mathcal{L} \text{ admits a factor map to } \mathcal{L}' \text{ for some } \mathcal{L}' \in \mathscr{C},$$

so $\mathscr{C}^* \supseteq \mathscr{C}$ and $(\mathscr{C}^*)^* = \mathscr{C}^*$. Let $\mathscr{C}$ be a class of L-systems satisfying the following two conditions:

(B1) if $\mathcal{L} \in \mathscr{C}$, then $\mathbf{HF}(\mathcal{L}) \in \mathscr{C}^*$;

(B2) if $\mathcal{L} \in \mathscr{C}$ and $\mathscr{B}$ is a correct instance over $\mathcal{L}$, then there exists a measurable solution f to $\mathscr{B}$ such that $\mathcal{L}[f] \in \mathscr{C}^*$.

Due to Proposition 3.5.9 and Lemma 3.5.10, if $\mathscr{C}$ satisfies the above conditions, then $\mathscr{C}^*$ has Properties (A1) and (A2) from §3.5.2, and hence $\mathscr{C} \subseteq \mathscr{C}^* \subseteq \mathscr{L}$.

3.5.4 Countable Borel groupoids and their actions

Definition 3.5.11 (Countable Borel groupoids). A **countable Borel groupoid** (R, Γ) is a structure consisting of a standard Borel space R together with a countable set Γ and Borel maps

$$\begin{array}{lllll} & \mathbf{a}\colon \Gamma \times R \to R & : (\gamma, r) \mapsto \gamma \cdot r & (\textit{action}); \\ & \mathbf{c}\colon \Gamma^2 \times R \to \Gamma & : (\gamma, \delta, r) \mapsto \gamma \circ_r \delta & (\textit{composition}); \\ & \mathbf{id}\colon R \to \Gamma & : r \mapsto \mathbf{1}_r & (\textit{identity}); \\ \text{and} & \mathbf{inv}\colon \Gamma \times R \to \Gamma & : (\gamma, r) \mapsto \gamma_r^{-1} & (\textit{inverse}), \end{array}$$

satisfying the following axioms:

– *consistency*: for all $\gamma, \delta \in \Gamma$ and $r \in R$, $\gamma \cdot (\delta \cdot r) = (\gamma \circ_r \delta) \cdot r$;
– *associativity*: for all $\gamma, \delta, \varepsilon \in \Gamma$ and $r \in R$, $\gamma \circ_r (\delta \circ_r \varepsilon) = (\gamma \circ_{\varepsilon \cdot r} \delta) \circ_r \varepsilon$;
– *identity*: for all $r \in R$ and $\gamma \in \Gamma$, $\mathbf{1}_r \cdot r = r$ and $\mathbf{1}_{\gamma \cdot r} \circ_r \gamma = \gamma \circ_r \mathbf{1}_r = \gamma$;
– *inverse*: for all $r \in R$ and $\gamma \in \Gamma$, $\gamma_r^{-1} \circ_r \gamma = \mathbf{1}_r$ and $\gamma \circ_{\gamma \cdot r} \gamma_r^{-1} = \mathbf{1}_{\gamma \cdot r}$.

$$\begin{tikzcd} \varepsilon \cdot r \arrow[r, "\delta"] & \delta \cdot (\varepsilon \cdot r) \arrow[d, "\gamma"] \\ r \arrow[u, "\varepsilon"] \arrow[ur, "\delta \circ_r \varepsilon"'] \arrow[r, dashed, "\gamma \circ_r (\delta \circ_r \varepsilon)"'] & \gamma \cdot (\delta \cdot (\varepsilon \cdot r)) \end{tikzcd} \qquad \begin{tikzcd} \varepsilon \cdot r \arrow[r, "\delta"] \arrow[dr, "\gamma \circ_{\varepsilon \cdot r} \delta"] & \delta \cdot (\varepsilon \cdot r) \arrow[d, "\gamma"] \\ r \arrow[u, "\varepsilon"] \arrow[r, dashed, "(\gamma \circ_{\varepsilon \cdot r} \delta) \circ_r \varepsilon"'] & \gamma \cdot (\delta \cdot (\varepsilon \cdot r)) \end{tikzcd}$$

Figure 14 – Associativity: the dashed arrows must coincide.

Any countable group Γ can be canonically viewed as a countable Borel groupoid in the following way. Let $R := \{r\}$ be a single point. For each $\gamma \in \Gamma$, set $\gamma \cdot r := r$. Now we just transfer compositions, the identity, and inverses directly from the group (we use $\mathbf{1}_\Gamma$ to denote the identity element of Γ):

$$\gamma \circ_r \delta := \gamma\delta; \qquad \mathbf{1}_r := \mathbf{1}_\Gamma; \qquad \text{and} \qquad \gamma_r^{-1} := \gamma^{-1}. \tag{3.5.1}$$

A more general class of examples is given by Borel actions of countable groups. Let $\alpha\colon \Gamma \curvearrowright R$ be a Borel action of a countable group Γ on a standard Borel space R. Then (R, Γ) can be endowed with the structure of a countable Borel groupoid as follows: Set $\gamma \cdot r := \gamma \cdot_\alpha r$ for all $\gamma \in \Gamma$, $r \in R$, and define compositions, identities, and inverses via (3.5.1) (i.e., in a way that does not depend on $r \in R$).

An interesting example of a countable Borel groupoid is produced by "bundling" all countable groups into a single algebraic structure. Let $\mathcal{G}$ be the standard Borel space of all countably infinite groups with ground set $\mathbb{N}$ (which can be viewed as a Borel subset of the Cantor space $2^{\mathbb{N}^3}$). Define a countable Borel groupoid $(\mathcal{G}, \mathbb{N})$ as follows: For each $n \in \mathbb{N}$ and $\Gamma \in \mathcal{G}$, let $n \cdot \Gamma := \Gamma$. Now set

$$\begin{array}{ll} n \circ_\Gamma m & \text{to be the product of } n \text{ and } m \text{ as elements of } \Gamma; \\ \mathbf{1}_\Gamma & \text{to be the identity element of } \Gamma; \\ n_\Gamma^{-1} & \text{to be the inverse of } n \text{ in } \Gamma. \end{array}$$

The following proposition is a useful and easy-to-check condition that guarantees that a certain structure is a countable Borel groupoid.

Proposition 3.5.12. *Let R be a standard Borel space and let E be a countable Borel equivalence relation on R. Let Γ be a countable set and let $\mathbf{a}\colon \Gamma \times R \to R\colon (\gamma, r) \mapsto \gamma \cdot r$ be a Borel function. Suppose that for each $r \in R$, the map $\gamma \mapsto \gamma \cdot r$ is a bijection between Γ and $[r]_E$. Then there is a unique countable Borel groupoid structure on (R, Γ) with $\mathbf{a}$ as its action map.*

Proof. For $r_1, r_2 \in R$ with $r_1 \mathrel{E} r_2$, let $\varepsilon(r_1, r_2)$ denote the unique element $\varepsilon \in \Gamma$ such that $r_2 = \varepsilon \cdot r_1$. The only consistent way to turn (R, Γ) into a countable Borel groupoid is as follows:

$$\gamma \circ_r \delta \coloneqq \varepsilon(r, \gamma \cdot (\delta \cdot r)); \qquad \mathbf{1}_r \coloneqq \varepsilon(r, r); \qquad \text{and} \qquad \gamma_r^{-1} \coloneqq \varepsilon(\gamma \cdot r, r).$$

A straightforward verification shows that the above definition satisfies all the axioms. ■

Now we proceed to the definition of Borel actions of countable Borel groupoids.

Definition 3.5.13 (Actions). Let (R, Γ) be a countable Borel groupoid. A (Borel) **action** (ρ, α) of (R, Γ) on a standard Borel space X is a pair of Borel maps $\rho\colon X \to R$ and $\alpha\colon \Gamma \times X \to X\colon (\gamma, x) \mapsto \gamma \cdot_\alpha x$ satisfying the following conditions:

– *equivariance*: for all $x \in X$ and $\gamma \in \Gamma$, $\qquad \rho(\gamma \cdot_\alpha x) = \gamma \cdot \rho(x)$;
– *identity*: for all $x \in X$, $\qquad \mathbf{1}_{\rho(x)} \cdot_\alpha x = x$;
– *compatibility*: for all $x \in X$ and $\gamma, \delta \in \Gamma$ $\qquad \gamma \cdot_\alpha (\delta \cdot_\alpha x) = (\gamma \circ_{\rho(x)} \delta) \cdot_\alpha x$.

As with group actions, we will usually simply write $\gamma \cdot x$ for $\gamma \cdot_\alpha x$.

Clearly, a (left) group action $\Gamma \curvearrowright X$ is also a countable Borel groupoid action if Γ is understood as a countable Borel groupoid. Now suppose that a countable group Γ acts (in a Borel way) on a standard Borel space R. Viewing (R, Γ) as a countable Borel groupoid, consider an action (ρ, α) of (R, Γ) on some space X. By the identity and the compatibility conditions in Definition 3.5.13, α is an action of Γ on X, while the equivariance condition stipulates that the map $\rho\colon X \to R$ must be Γ-equivariant. Thus, a Borel action of (R, Γ) is the same as a Γ-space equipped with a Borel Γ-equivariant map to R. If $(\mathcal{G}, \mathbb{N})$ is the countable Borel groupoid of all countable groups, then an action of $(\mathcal{G}, \mathbb{N})$ on X consists of a Borel map $\rho\colon X \to \mathcal{G}$ and a Γ-action on $\rho^{-1}(\Gamma)$ for each $\Gamma \in \mathcal{G}$.

Definition 3.5.14 (Shift actions). Let (R, Γ) be a countable Borel groupoid and let Y be a standard Borel space. The ***Y*-shift action** $(\rho, \alpha)\colon (R, \Gamma) \curvearrowright R \times Y^\Gamma$ is defined as follows: For each $(r, \vartheta) \in R \times Y^\Gamma$, set $\rho(r, \vartheta) \coloneqq r$, and for $\gamma \in \Gamma$, define

$$\gamma \cdot_\alpha (r, \vartheta) \coloneqq (\gamma \cdot r, \vartheta'), \qquad \text{where} \qquad \vartheta'(\delta) \coloneqq \vartheta(\delta \circ_r \gamma) \text{ for all } \delta \in \Gamma.$$

It is routine to check that the Y-shift action as defined above is indeed an action of (R, Γ). We give the proof here to help the reader get familiar with the definitions. The equivariance condition is satisfied trivially.

For the identity condition, observe that if $x = (r, \vartheta) \in R \times Y^\Gamma$, then

$$\mathbf{1}_{\rho(x)} \cdot x = \mathbf{1}_r \cdot (r, \vartheta) = (\mathbf{1}_r \cdot r, \vartheta') = (r, \vartheta'),$$

where for each $\delta \in \Gamma$,

$$\vartheta'(\delta) = \vartheta(\delta \circ_r \mathbf{1}_r) = \vartheta(\delta),$$

so $\vartheta' = \vartheta$, as desired. Finally, for the compatibility condition, we have

$$\gamma \cdot (\delta \cdot x) = \gamma \cdot (\delta \cdot (r, \vartheta)) = \gamma \cdot (\delta \cdot r, \vartheta') = (\gamma \cdot (\delta \cdot r), \vartheta'') = ((\gamma \circ_r \delta) \cdot r, \vartheta''),$$

where for each $\varepsilon \in \Gamma$,

$$\vartheta''(\varepsilon) = \vartheta'(\varepsilon \circ_{\delta \cdot r} \gamma) = \vartheta((\varepsilon \circ_{\delta \cdot r} \gamma) \circ_r \delta) = \vartheta(\varepsilon \circ_r (\gamma \circ_r \delta)),$$

so $\gamma \cdot (\delta \cdot x) = (\gamma \circ_r \delta) \cdot x$, as desired.

Note that for a countable group Γ, Definition 3.5.14 is equivalent to the usual definition of the Y-shift action of Γ.

By analogy with group actions, we can define L-systems corresponding to actions of countable Borel groupoids. Namely, let $(\rho, \alpha) \colon (R, \Gamma) \curvearrowright X$ be a Borel action of a countable Borel groupoid (R, Γ) on a standard Borel space X. Let E_α be the corresponding **orbit equivalence relation** on X, defined by

$$x \, E_\alpha \, y \; :\Longleftrightarrow \; \gamma \cdot x = y \text{ for some } \gamma \in \Gamma.$$

This is clearly a countable Borel equivalence relation. Note that E_α does not depend on ρ. Let $\mathcal{I}_{(\rho,\alpha)}$ denote the isomorphism structure on E_α such that a bijection $\varphi \colon C_1 \to C_2$ between E_α-classes C_1, C_2 belongs to $\mathcal{I}_{(\rho,\alpha)}$ if and only if φ is (R, Γ)**-equivariant**, i.e., for all $x \in C_1$ and $\gamma \in \Gamma$,

$$\rho(\varphi(x)) = \rho(x) \qquad \text{and} \qquad \gamma \cdot \varphi(x) = \varphi(\gamma \cdot x).$$

For $\mu \in \mathbf{Prob}(X)$, let $\mathcal{L}(\rho, \alpha, \mu)$ denote the L-system $(X, E_\alpha, \mathcal{I}_{(\rho,\alpha)}, \mu)$. In the case when $|R| = 1$, i.e., (R, Γ) is a group, this definition coincides with the one given previously for group actions.

We will be mostly interested in the properties of L-systems induced by shift actions of countable Borel groupoids. More precisely:

Definition 3.5.15 (**Shift L-systems**)**.** A **shift L-system** is any L-system of the form $\mathcal{L}(\rho, \alpha, \mu \times \nu^\Gamma)$, where $(\rho, \alpha) \colon (R, \Gamma) \curvearrowright R \times Y^\Gamma$ is the Y-shift action of a countable Borel groupoid (R, Γ) for some standard Borel space Y, $\mu \in \mathbf{Prob}(R)$, and $\nu \in \mathbf{Prob}(Y)$ is atomless.

Thanks to the measure isomorphism theorem, it is enough to consider shift L-systems induced by the $[0; 1]$-shift action of (R, Γ) with $\nu = \lambda$. However, sometimes it will be more convenient to use other choices for Y and ν; in particular, we will often assume that $Y = [0; 1]^S$ and $\nu = \lambda^S$ for some countable set S.

3.5.5 Factors of L-systems induced by actions of countable Borel groupoids

Let (R, Γ) be a countable Borel groupoid and let $(\rho, \alpha)\colon (R, \Gamma) \curvearrowright X$ be a Borel action of (R, Γ) on a standard Borel space X. The action (ρ, α) is **free** if for all $x \in X$ and $\gamma \in \Gamma$,

$$\gamma \cdot x = x \iff \gamma = \mathbf{1}_{\rho(x)}.$$

The **free part** of (ρ, α) (notation: $\mathbf{Free}(\rho, \alpha)$ or $\mathbf{Free}(X)$ if the action is clear from the context) is the largest E_α-invariant subset of X on which the action is free. The free part of an action is always an invariant Borel set. For $\mu \in \mathbf{Prob}(X)$, an action is **free μ-almost everywhere** if its free part is μ-conull. By definition, if $x \in \mathbf{Free}(X)$, then the map $\gamma \mapsto \gamma \cdot x$ is a bijection between Γ and the orbit of x.

Proposition 3.5.16. *Let (R, Γ) be a countable Borel groupoid and let $\mu \in \mathbf{Prob}(R)$. Let Y be a standard Borel space and let $\nu \in \mathbf{Prob}(Y)$ be atomless. Then the Y-shift action of (R, Γ) is free $(\mu \times \nu^\Gamma)$-almost everywhere.*

Proof. It is enough to notice that $\mathbf{Free}(R \times Y^\Gamma) \supseteq R \times F$, where $F \coloneqq \{\vartheta \in Y^\Gamma \,:\, \vartheta\colon \Gamma \to Y \text{ is injective}\}$. ∎

The next lemma will be useful in verifying that certain maps between L-systems induced by actions of countable Borel groupoids are factor maps.

Lemma 3.5.17. *Let (R, Γ) be a countable Borel groupoid and let*

$$(\rho_1, \alpha_1)\colon (R, \Gamma) \curvearrowright X_1 \qquad \text{and} \qquad (\rho_2, \alpha_2)\colon (R, \Gamma) \curvearrowright X_2$$

be two Borel actions of (R, Γ). Let $\mu_1 \in \mathbf{Prob}(X_1)$ and $\mu_2 \in \mathbf{Prob}(X_2)$. Suppose that (ρ_2, α_2) is μ_2-almost everywhere free. Let $\pi\colon X_1 \rightharpoonup X_2$ be a measure-preserving (R, Γ)-equivariant Borel map defined on an E_{α_1}-invariant μ_1-conull Borel subset of X_1. Then π, possibly restricted to a smaller invariant conull Borel subset of X_1, is a factor map from $\mathcal{L}(\rho_1, \alpha_1, \mu_1)$ to $\mathcal{L}(\rho_2, \alpha_2, \mu_2)$.

Proof. For $i \in \{1, 2\}$, let $E_i \coloneqq E_{\alpha_i}$ and $\mathcal{I}_i \coloneqq \mathcal{I}_{(\rho_i, \alpha_i)}$. Let $C \subseteq \mathrm{dom}(\pi)$ be an E_1-class. The equivariance of π implies that $\pi(C)$ is an E_2-class. Since (ρ_2, α_2) is free μ_2-almost everywhere, we may assume that $\pi(C) \subseteq \mathbf{Free}(X_2)$, in which case the map $\pi|C\colon C \to \pi(C)$ is a bijection.

It remains to check the existence of $\varphi_1 \in \mathcal{I}_1$ that closes the following diagram:

$$\begin{tikzcd} C_1 \arrow[r, dashed, "\varphi_1"] \arrow[d, "\pi"] & C_2 \arrow[d, "\pi"] \\ \pi(C_1) \arrow[r, "\varphi_2"] & \pi(C_2). \end{tikzcd}$$

Again, since (ρ_2, α_2) is free μ_2-almost everywhere, we may assume that the maps

$$\pi|C_1\colon C_1 \to \pi(C_1) \qquad \text{and} \qquad \pi|C_2\colon C_2 \to \pi(C_2)$$

are bijections. Since φ_2 is (R, Γ)-equivariant,

$$\varphi_1 \coloneqq (\pi|C_2)^{-1} \circ \varphi_2 \circ (\pi|C_1)$$

is an equivariant bijection from C_1 to C_2; in other words, $\varphi_1 \in \mathcal{I}_1$, as desired. ■

Let Γ be a countable group and let $\alpha_1 \colon \Gamma \curvearrowright (X_1, \mu_1)$ and $\alpha_2 \colon \Gamma \curvearrowright (X_2, \mu_2)$ be two probability measure-preserving actions of Γ. If α_2 is free μ_2-almost everywhere, then, by Lemma 3.5.17, a factor map $\pi \colon (X_1, \mu_1) \to (X_2, \mu_2)$ in the usual ergodic theory sense induces a factor map between the L-systems $\mathcal{L}(\alpha_1, \mu_1)$ and $\mathcal{L}(\alpha_2, \mu_2)$.

3.5.6 Closure properties of the class of shift L-systems

In this subsection we show that the class of shift L-systems is closed under (certain) expansions and under amplifications.

Lemma 3.5.18. *Let (R, Γ) be a countable Borel groupoid and let $\mathcal{L} = \mathcal{L}(\rho, \alpha, \mu \times (\lambda^2)^\Gamma)$, where $\mu \in \mathbf{Prob}(R)$, be the shift L-system induced by the $[0;1]^2$-shift action of (R, Γ). Let Y be a standard Borel space and let*

$$f \colon R \times ([0;1]^2)^\Gamma \to Y$$

be a Borel function that does not depend on the third coordinate, i.e., for all $r \in R$ and ϑ, ω, $\omega' \in [0;1]^\Gamma$,

$$f(r, \vartheta, \omega) = f(r, \vartheta, \omega').$$

Then $\mathcal{L}[f]$ admits a factor map to a shift L-system.

Proof. Set $Q \coloneqq R \times [0;1]^\Gamma$. The $[0;1]$-shift action of (R, Γ) on Q turns (Q, Γ) into a countable Borel groupoid via

$$\gamma \circ_{(r,\vartheta)} \delta \coloneqq \gamma \circ_r \delta; \qquad \mathbf{1}_{(r,\vartheta)} \coloneqq \mathbf{1}_r; \qquad \text{and} \qquad \gamma^{-1}_{(r,\vartheta)} \coloneqq \gamma^{-1}_r.$$

Let (σ, α') denote the $[0;1]$-shift action of (Q, Γ). If we identify $([0;1]^2)^\Gamma$ with $[0;1]^\Gamma \times [0;1]^\Gamma$ in the natural way, then

$$R \times ([0;1]^2)^\Gamma = R \times [0;1]^\Gamma \times [0;1]^\Gamma = Q \times [0;1]^\Gamma,$$

and, in fact, $\alpha' = \alpha$. By definition, for all $r \in R$ and $\vartheta, \omega \in [0;1]^\Gamma$,

$$\sigma(r, \vartheta, \omega) = (r, \vartheta),$$

so the value $f(x)$ is determined by $\sigma(x)$ for all x. Therefore, the identity function

$$\mathrm{id} \colon R \times ([0;1]^2)^\Gamma \to Q \times [0;1]^\Gamma$$

is a factor map from $\mathcal{L}[f]$ to the shift L-system $\mathcal{L}' \coloneqq \mathcal{L}(\sigma, \alpha, \mu \times \lambda^\Gamma \times \lambda^\Gamma)$ induced by (σ, α). ■

Lemma 3.5.19. *If $\mathcal{L}$ is a shift L-system, then $\mathbf{HF}(\mathcal{L})$ factors to a shift L-system.*

Proof. Suppose that $\mathcal{L}$ is induced by a shift action of a countable Borel groupoid (R, Γ). We will proceed in three steps. First, we will construct a countable Borel groupoid (Q, Δ), where $\Delta = \mathbf{HF}(\Gamma)$. Then we will show that $\mathbf{HF}(\mathcal{L})$ is induced by an (almost everywhere) free action of (Q, Δ). Finally, we will define a measure-preserving (Q, Δ)-equivariant Borel map from this action to the $[0;1]$-shift action of (Q, Δ), which will give us a desired factor map, thanks to Lemma 3.5.17.

STEP 1. Let $\mu \in \mathbf{Prob}(R)$ and consider the $[0;1]$-shift action $(R, \Gamma) \curvearrowright R \times [0;1]^{\Gamma}$. Let E denote the induced orbit equivalence relation. Define

$$Q := \mathbf{HF}(E|\mathbf{Free}(R \times [0;1]^{\Gamma})) \qquad \text{and} \qquad \Delta := \mathbf{HF}(\Gamma).$$

Note that for each $x \in \mathbf{Free}(R \times [0;1]^{\Gamma})$, the following map is a bijection between Γ and $[x]_E$:

$$\varphi_x \colon \Gamma \to [x]_E \colon \gamma \mapsto \gamma \cdot x.$$

Therefore, its amplification

$$\tilde{\varphi}_x \colon \mathbf{HF}(\Gamma) = \Delta \to \mathbf{HF}([x]_E) = [x]_{\tilde{E}}$$

is a bijection between Δ and $[x]_{\tilde{E}}$. Fix a Borel map $x_0 \colon Q \to \mathbf{Free}(R \times [0;1]^{\Gamma})$ such that $x_0(q) \in \mathbf{Set}(q)$ for all $q \in Q$, and let

$$\tilde{\varphi}_q := \tilde{\varphi}_{x_0(q)}.$$

Then for each $q \in Q$, the map $\tilde{\varphi}_q$ is a bijection from Δ to $[x_0(q)]_{\tilde{E}} = [q]_{\tilde{E}}$. For $q \in Q$ and $\delta \in \Delta$, define

$$\delta \cdot q := \tilde{\varphi}_q(\delta).$$

Since $\tilde{\varphi}_q \colon \Delta \to [q]_{\tilde{E}}$ is a bijection for each $q \in Q$, by Proposition 3.5.12, (Q, Δ) is equipped with a unique countable Borel groupoid structure. It is useful to observe that

$$\delta \cdot q = \tilde{\varphi}_q(\delta) = \tilde{\varphi}_{x_0(q)}(\delta) = \delta \cdot x_0(q).$$

STEP 2. Now we turn to the shift L-system $\mathcal{L}$. Suppose that $\mathcal{L} = \mathcal{L}(\rho, \alpha, \mu \times (\lambda \times \nu)^{\Gamma})$, where

$$(\rho, \alpha) \colon (R, \Gamma) \curvearrowright R \times ([0;1] \times Y)^{\Gamma}$$

is the $([0;1] \times Y)$-shift action of (R, Γ), $\mu \in \mathbf{Prob}(R)$, and $\nu \in \mathbf{Prob}(Y)$ is atomless. Here Y is an arbitrary standard Borel space; we will specify a concrete choice for Y later. Let

$$F := \mathbf{Free}(R \times [0;1]^{\Gamma}) \times Y^{\Gamma}.$$

Then F is a conull E_α-invariant Borel subset of $\mathbf{Free}(\rho, \alpha)$. We will now define a free action of (Q, Δ) on $H := \mathbf{HF}(E_\alpha|F)$ (which is a conull $\tilde{E}_\alpha$-invariant Borel subset of $\mathbf{HF}(E_\alpha)$).

The construction is analogous to the one from Step 1. For each $x \in F$, define $\varphi_x \colon \Gamma \to [x]_{E_\alpha}$ by

$$\varphi_x \colon \Gamma \to [x]_{E_\alpha} \colon \gamma \mapsto \gamma \cdot x.$$

Then φ_x is a bijection between Γ and $[x]_{E_\alpha}$. Therefore, $\tilde{\varphi}_x \colon \Delta \to [x]_{\tilde{E}_\alpha}$ is a bijection from Δ to $[x]_{\tilde{E}_\alpha}$. Let $\sigma \colon F \to \mathbf{Free}(R \times [0;1]^\Gamma)$ denote the projection on the first two coordinates, i.e.,

$$\sigma(r, \vartheta, y) \coloneqq (r, \vartheta) \qquad \text{for all } (r, \vartheta) \in \mathbf{Free}(R \times [0;1]^\Gamma) \text{ and } y \in Y^\Gamma.$$

For every $h \in H$, we have $\tilde{\sigma}(h) \in Q$ and the map $\sigma|\mathbf{Set}(h) \colon \mathbf{Set}(h) \to \mathbf{Set}(\tilde{\sigma}(h))$ is a bijection. Let $x_0(h)$ be the unique element of $\mathbf{Set}(h)$ such that

$$\sigma(x_0(h)) = x_0(\tilde{\sigma}(h)).$$

Define $\tilde{\varphi}_h \coloneqq \tilde{\varphi}_{x_0(h)}$. Then $\tilde{\varphi}_h \colon \Delta \to [h]_{\tilde{E}_\alpha}$ is a bijection. Hence, if we let

$$\beta \colon \Delta \times H \colon (\delta, h) \mapsto \delta \cdot h \coloneqq \tilde{\varphi}_h(\delta),$$

then $(\tilde{\sigma}, \beta)$ is a free action of (Q, Δ) on H. Note that we again have

$$\delta \cdot h = \tilde{\varphi}_h(\delta) = \tilde{\varphi}_{x_0(h)}(\delta) = \delta \cdot x_0(h).$$

It is clear that the restriction of $\mathbf{HF}(\mathcal{L})$ to H coincides with $\mathcal{L}(\tilde{\sigma}, \beta, \mu \times (\lambda \times \nu)^\Gamma)$.

Step 3. So far we have constructed a countable Borel groupoid (Q, Δ) and a free action $(\tilde{\sigma}, \beta)$ of (Q, Δ) that essentially (i.e., up to an invariant null set) induces the L-system $\mathbf{HF}(\mathcal{L})$. It remains to define a factor map from that action to the L-system induced by the $[0;1]$-shift action of (Q, Δ).

Choose Y to be $[0;1]^\Delta$ and ν to be λ^Δ. Consider any $h \in H$. Suppose that $x_0(h) = (x, y)$, where $x \in \mathbf{Free}(R \times [0;1]^\Gamma)$ and $y \in Y^\Gamma = ([0;1]^\Delta)^\Gamma = [0;1]^{\Gamma \times \Delta}$. Define $\xi(h) \in [0;1]$ by

$$\xi(h) \coloneqq y(\mathbf{1}_{\rho(x)}, \mathbf{1}_{\tilde{\sigma}(h)}).$$

Here $\rho(x) \in R$ and $\tilde{\sigma}(h) \in Q$, so $\mathbf{1}_{\rho(x)} \in \Gamma$ and $\mathbf{1}_{\tilde{\sigma}(h)} \in \Delta$. Now define $\xi^\Delta \colon H \to [0;1]^\Delta$ by setting, for all $h \in H$ and $\delta \in \Delta$,

$$\xi^\Delta(h)(\delta) \coloneqq \xi(\delta \cdot h).$$

By construction, the map

$$(\tilde{\sigma}, \xi^\Delta) \colon H \to Q \times [0;1]^\Delta$$

is (Q, Δ)-equivariant. Due to Lemma 3.5.17, we only need to check that this map is measure-preserving.

Since we have the freedom to choose the measure on Q, we can take it to be

$$\tilde{\sigma}_*(\mu \times \lambda^\Gamma \times \lambda^{\Gamma \times \Delta}),$$

so we only have to show that

$$\xi^\Delta_*(\mu \times \lambda^\Gamma \times \lambda^{\Gamma \times \Delta}) = \lambda^\Delta.$$

Since $\mu \times \lambda^{\Gamma} \times \lambda^{\Gamma\times\Delta}$ is concentrated on F, it is enough to verify that

$$(\xi^{\Delta}|F)_*(\mu \times \lambda^{\Gamma} \times \lambda^{\Gamma\times\Delta}) = \lambda^{\Delta}.$$

To this end, we will show that for each $x \in \mathbf{Free}(R \times [0;1]^{\Gamma})$, the map

$$\xi^{\Delta}_x \colon [0;1]^{\Gamma\times\Delta} \to [0;1]^{\Delta} \colon y \mapsto \xi^{\Delta}(x, y)$$

satisfies $(\xi^{\Delta}_x)_*(\lambda^{\Gamma\times\Delta}) = \lambda^{\Delta}$; an application of Fubini's theorem then completes the proof.

Fix some $x \in \mathbf{Free}(R \times [0;1]^{\Gamma})$. For each $\delta \in \Delta$, let $\gamma_{x,\delta}$ be the unique element of Γ such that

$$x_0(\delta \cdot x) = \gamma_{x,\delta} \cdot x.$$

Observe that the map

$$\Delta \to \Gamma \times \Delta \colon \delta \mapsto (\gamma_{x,\delta}, \mathbf{1}_{\delta\cdot x})$$

is injective. Indeed, we have

$$\tilde{\varphi}_x(\delta) = \delta \cdot x = \mathbf{1}_{\delta\cdot x} \cdot (\delta \cdot x) = \mathbf{1}_{\delta\cdot x} \cdot x_0(\delta \cdot x) = \mathbf{1}_{\delta\cdot x} \cdot (\gamma_{x,\delta} \cdot x),$$

and the map $\tilde{\varphi}_x$ is injective. Let $y \in [0;1]^{\Gamma\times\Delta}$. We have

$$x_0(\delta \cdot (x, y)) = \gamma_{x,\delta} \cdot (x, y) = (\gamma_{x,\delta} \cdot x, y') \qquad \text{and} \qquad \tilde{\sigma}(\delta \cdot (x, y)) = \delta \cdot x,$$

where y' is a particular element of $[0;1]^{\Gamma\times\Delta}$. Therefore,

$$\xi^{\Delta}_x(y)(\delta) = \xi(\delta \cdot (x, y)) = y'(\mathbf{1}_{\rho(\gamma_{x,\delta}\cdot x)}, \mathbf{1}_{\delta\cdot x}) = y'(\mathbf{1}_{\gamma_{x,\delta}\cdot\rho(x)}, \mathbf{1}_{\delta\cdot x}).$$

Since

$$\mathbf{1}_{\gamma_{x,\delta}\cdot\rho(x)} \circ_{\rho(x)} \gamma_{x,\delta} = \gamma_{x,\delta},$$

by the definition of the shift action, we get

$$y'(\mathbf{1}_{\gamma_{x,\delta}\cdot\rho(x)}, \mathbf{1}_{\delta\cdot x}) = y(\gamma_{x,\delta}, \mathbf{1}_{\delta\cdot x}).$$

To summarize,

$$\xi^{\Delta}_x(y)(\delta) = y(\gamma_{x,\delta}, \mathbf{1}_{\delta\cdot x}).$$

In other words, ξ^{Δ}_x acts as the projection on the set of coordinates $\{(\gamma_{x,\delta}, \mathbf{1}_{\delta\cdot x}) \,:\, \delta \in \Delta\}$. Therefore, it pushes $\lambda^{\Gamma\times\Delta}$ forward to λ^{Δ}, as desired. ∎

3.5.7 The Moser–Tardos algorithm for shift L-systems

In this subsection we use Moser–Tardos theory to show that any correct instance $\mathscr{B}$ over a shift L-system $\mathcal{L}$ admits a measurable solution. To do this, we will reduce $\mathscr{B}$ to a family $(\mathscr{B}_r)_{r\in R}$ of correct instances over the (countable) set Γ indexed by the elements of R, where (R, Γ) is the countable Borel groupoid whose shift action induces $\mathcal{L}$.

Lemma 3.5.20. *Let $\mathcal{L}$ be a shift L-system. Then every correct instance over $\mathcal{L}$ has a measurable solution.*

Proof. Let (R, Γ) be a countable Borel groupoid, let $\mu \in \mathbf{Prob}(R)$, and let $(\rho, \alpha)\colon (R, \Gamma) \curvearrowright R \times [0;1]^{\Gamma\times\mathbb{N}}$ be the $[0;1]^{\mathbb{N}}$-shift action of (R, Γ). Let $\mathcal{L} := \mathcal{L}(\rho, \alpha, \mu \times \lambda^{\Gamma\times\mathbb{N}})$. We use the following notation:

$$X := R \times [0;1]^{\Gamma\times\mathbb{N}}, \qquad E := E_\alpha, \qquad \text{and} \qquad \mathcal{I} := \mathcal{I}_{(\rho,\alpha)}.$$

Suppose $\mathscr{B}$ is a correct instance over $\mathcal{L}$. Due to Propositions 3.2.8 and 3.2.3, it is enough to show that there exists a Borel table $\xi\colon X \times \mathbb{N} \to [0;1]$ such that

$$(\mu \times \lambda^{\Gamma\times\mathbb{N}})(\{x \in X \,:\, \gamma \cdot x \in \mathbf{Stab}(\xi) \text{ for all } \gamma \in \Gamma\}) = 1. \tag{3.5.2}$$

We claim that the map

$$\xi\colon X \times \mathbb{N} \to [0;1]\colon ((r, \vartheta), n) \mapsto \vartheta(\mathbf{1}_r)(n)$$

satisfies (3.5.2). Note that for every $\gamma \in \Gamma$,

$$\xi(\gamma \cdot (r, \vartheta), n) = \vartheta(\gamma)(n).$$

For each $x \in X$, there is a surjection

$$\varphi_x\colon \Gamma \to [x]_E\colon \gamma \mapsto \gamma \cdot x$$

from Γ onto $[x]_E$. Since the action (ρ, α) is free almost everywhere, φ_x is bijective for almost all $x \in X$. Hence, for almost every $x \in X$, the map φ_x can be used to define a correct instance $\mathscr{B}_x$ over Γ by "pulling back" the restriction of $\mathscr{B}$ to $[x]_E$. Formally, we set

$$\mathscr{B}_x := \{\{f \circ \varphi_x \,:\, f \in B\} \,:\, B \in \mathscr{B}\}.$$

Note that whenever $r \in R$ and ϑ, $\omega \in [0;1]^{\Gamma\times\mathbb{N}}$ and both (r, ϑ) and (r, ω) belong to the free part of the action (ρ, α), the map $\gamma \cdot (r, \vartheta) \mapsto \gamma \cdot (r, \omega)$ is a well-defined (R, Γ)-equivariant bijection between $[(r, \vartheta)]_E$ and $[(r, \omega)]_E$. Therefore, since $\mathscr{B}$ is almost everywhere $\mathcal{I}$-invariant, the following definition makes sense for almost all $r \in R$:

$$\mathscr{B}_r := \mathscr{B}_{(r,\vartheta)} \text{ for almost all } \vartheta \in [0;1]^{\Gamma\times\mathbb{N}}.$$

Now, for almost every $r \in R$ and for all $\gamma \in \Gamma$, using Corollary 3.2.7, we obtain

$$\begin{aligned} &\lambda^{\Gamma\times\mathbb{N}}(\{\vartheta \in [0;1]^{\Gamma\times\mathbb{N}} \,:\, \gamma\cdot(r,\vartheta) \text{ is } \xi\text{-stable with respect to } \mathscr{B}\}) \\ &= \lambda^{\Gamma\times\mathbb{N}}(\{\vartheta \in [0;1]^{\Gamma\times\mathbb{N}} \,:\, \gamma \text{ is } \vartheta\text{-stable with respect to } \mathscr{B}_r\}) \;=\; 1. \end{aligned}$$

An application of Fubini's theorem yields (3.5.2). ∎

3.5.8 Completing the proof of Theorem 3.5.6

Now we have all the necessary ingredients to prove the following generalization of Theorem 3.5.6:

Theorem 3.5.21 (**Measurable LLL for shift L-systems**). *Let $\mathcal{L}$ be an L-system that admits a factor map to a shift L-system. Then Player II has a winning strategy in the LLL Game over* $\mathbf{HF}(\mathcal{L})$.

Proof. We need to verify that the class $\mathscr{C}$ of shift L-systems satisfies conditions (B1) and (B2) from §3.5.3. Condition (B1) is given by Lemma 3.5.19. It remains to show that if $\mathcal{L}$ is a shift L-system and $\mathscr{B}$ is a correct instance over $\mathcal{L}$, then there is a measurable solution f to $\mathscr{B}$ such that $\mathcal{L}[f]$ factors to another shift L-system.

To that end, suppose that $\mathcal{L}$ is induced by the $[0;1]^2$-shift action of a countable Borel groupoid (R,Γ) with measure $\mu \times (\lambda^2)^\Gamma$, where $\mu \in \mathbf{Prob}(R)$. Consider the L-system $\mathcal{L}'$ induced by the $[0;1]$-shift action of (R,Γ) with measure $\mu \times \lambda^\Gamma$. The projection onto the first two coordinates, i.e., the map

$$\pi\colon\, R \times [0;1]^\Gamma \times [0;1]^\Gamma \to R \times [0;1]^\Gamma \colon (r,\vartheta,\omega) \mapsto (r,\vartheta),$$

is (R,Γ)-equivariant and measure-preserving, so, by Lemma 3.5.17, it is a factor map from $\mathcal{L}$ to $\mathcal{L}'$. Due to Lemma 3.5.10, there is a correct instance $\pi(B)$ over $\mathcal{L}'$ such that whenever f' is a measurable solution to $\pi(\mathscr{B})$, then $f' \circ \pi$ is a measurable solution to $\mathscr{B}$ (modulo an invariant null set). Lemma 3.5.20 does indeed provide a measurable solution f' to $\pi(\mathscr{B})$, so let $f \coloneqq f' \circ \pi$. By definition, f does not depend on the third coordinate. Therefore, by Lemma 3.5.18, $\mathcal{L}[f]$ factors to a shift L-system, as desired. ∎

3.6 The converse of Theorem 3.5.6 for actions of amenable groups

Corollary 3.5.7 asserts that if a probability measure-preserving action $\alpha\colon \Gamma \curvearrowright (X,\mu)$ of a countable group Γ factors to the $[0;1]$-shift action, then every correct instance $\mathscr{B}$ over α admits a Borel solution μ-almost everywhere. In this section we show that if Γ is amenable, then the converse also holds. In fact, we will prove that even (seemingly) much weaker assumptions already imply the existence of a factor map to the $[0;1]$-shift.

To articulate these weaker assumptions, we need a few definitions. An instance $\mathscr{B}$ over a set X is **ε-correct**, where $0 < \varepsilon \leqslant 1$, if the neighborhood of each $B \in \mathscr{B}$ is countable, and there exists a function $\omega\colon \mathscr{B} \to [0;1)$ such that for all $B \in \mathscr{B}$,

$$\mathbb{P}[B] \leqslant \varepsilon^{|\mathrm{dom}(B)|}\omega(B) \prod_{B' \in N_{\mathscr{B}}(B)} (1 - \omega(B')).$$

Hence, correct is the same as 1-correct, and

$$B \text{ is } \varepsilon\text{-correct} \implies B \text{ is } \varepsilon'\text{-correct} \quad \text{whenever } 0 < \varepsilon \leqslant \varepsilon' \leqslant 1.$$

An instance $\mathscr{B}$ over a set X is **discrete** if there exist a finite set S and a Borel function $\varphi \colon [0; 1] \to S$ such that for all $B \in \mathscr{B}$ and $w, w' \colon \mathrm{dom}(B) \to [0; 1]$ with $\varphi \circ w = \varphi \circ w'$, we have

$$w \in B \iff w' \in B.$$

In other words, $\mathscr{B}$ is discrete if the bad events in $\mathscr{B}$ can be identified with subsets of $[X \to S]^{<\infty}$, where S is equipped with the probability measure $\varphi_*(\lambda)$ (see Remark 3.1.9). Most instances of the LLL that appear in combinatorial applications are discrete. If $\varphi_*(\lambda)$ is the uniform probability measure on S, then $\mathscr{B}$ is said to be **uniformly discrete**.

Given a graph G on a set X, an **instance** (**or the LLL**) over G is an instance $\mathscr{B}$ over X such that:

– for each $B \in \mathscr{B}$, the (finite) graph $G[\mathrm{dom}(B)]$ is connected;

– if $B \in \mathscr{B}$, $S \subseteq X$, and $\varphi \colon S \to \mathrm{dom}(B)$ is an isomorphism between $G[S]$ and $G[\mathrm{dom}(B)]$, then

$$\{w \circ \varphi \,:\, w \in B\} \in \mathscr{B}.$$

Note that if $\alpha \colon \Gamma \curvearrowright X$ is an action of a countable group Γ generated by a set $S \subseteq \Gamma$, then every instance over $G(\alpha, S)$ is in particular an instance over α.

Now we are ready to state the first version of the converse theorem.

Theorem 3.6.1. *Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action of a countably infinite amenable group Γ. Suppose that $S \subseteq \Gamma$ is a finite generating set and let $G \coloneqq G(\alpha, S)$. The following statements are equivalent:*

(i) *there exists $\varepsilon \in (0; 1]$ such that for every ε-correct uniformly discrete Borel instance $\mathscr{B}$ over G, there is a Borel map $f \colon X \to [0; 1]$ with $\mu(\mathbf{Def}_{\mathscr{B}}(f)) < 1$;*

(ii) *α factors to the shift action $\Gamma \curvearrowright ([0; 1]^{\Gamma}, \lambda^{\Gamma})$.*

In general, the conclusion of Theorem 3.6.1 fails for infinite S. To see this, consider any free ergodic p.m.p. action $\alpha \colon \Gamma \curvearrowright (X, \mu)$ of a countably infinite amenable group Γ and set $G \coloneqq G(\alpha, \Gamma)$. We claim that for every correct Borel instance $\mathscr{B}$ over G, there is a Borel map $f \colon X \to [0; 1]$ with $\mu(\mathbf{Def}_{\mathscr{B}}(f)) = 0$, regardless of the choice of α. Indeed,

$$E(G) = \{xy \,:\, x \, E_{\alpha} \, y \text{ and } x \neq y\},$$

so G only depends on the orbit equivalence relation E_{α} and not on the action α itself. Since, by a theorem of Dye and Ornstein–Weiss [KM04, Theorem 10.7], all free ergodic p.m.p. actions of countable amenable groups are orbit-equivalent, we may replace α by the $[0; 1]$-shift action and apply Corollary 3.5.7.

However, by keeping track of slightly more information than just the graph $G(\alpha, S)$, one can still establish an analog of Theorem 3.6.1 for infinite S (and in particular for groups that are not finitely generated). An (S-)**labeled graph** on X is a family $G = (G_\gamma)_{\gamma \in S}$ of graphs on X indexed by the elements of a given countable set S. Note that the edge sets $E(G_\gamma)$ are not required to be disjoint, i.e., the same edge can receive more than one label. A labeled graph G on a standard Borel space is **Borel** if each G_γ is Borel. An isomorphism between labeled graphs G_1 and G_2 must preserve the labeling, i.e., it has to be an isomorphism between each $(G_1)_\gamma$ and $(G_2)_\gamma$ individually. For an S-labeled graph G on X and a subset $X' \subseteq X$, let $G[X']$ denote the S-labeled graph on X' given by $(G[X'])_\gamma \coloneqq G_\gamma[X']$. For an S-labeled graph G, its **underlying graph** is the graph with the same vertex set as G and edge set $\bigcup_{\gamma \in S} E(G_\gamma)$. A labeled graph G is **connected** if its underlying graph is connected. The definition of an instance over G extends verbatim to the case when G is labeled. If $\alpha \colon \Gamma \curvearrowright X$ is an action of a countable group Γ on a set X and $S \subseteq \Gamma$ is a generating set, then $G_\ell(\alpha, S)$ denotes the S-labeled graph on X given by

$$xy \in E((G_\ell(\alpha, S))_\gamma) \;:\Longleftrightarrow\; x \neq y \text{ and } (\gamma \cdot x = y \text{ or } \gamma \cdot y = x).$$

Thus, the underlying graph of $G_\ell(\alpha, S)$ is $G(\alpha, S)$. Now we have the following:

Theorem 3.6.1′. *Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action of a countably infinite amenable group Γ. Let $S \subseteq \Gamma$ be a generating set and let $G \coloneqq G_\ell(\alpha, S)$. The following statements are equivalent:*

(i) *there exists $\varepsilon \in (0; 1]$ such that for every ε-correct uniformly discrete Borel instance $\mathscr{B}$ over G, there is a Borel map $f \colon X \to [0; 1]$ with $\mu(\mathbf{Def}_{\mathscr{B}}(f)) < 1$;*

(ii) *α factors to the shift action $\Gamma \curvearrowright ([0; 1]^\Gamma, \lambda^\Gamma)$.*

Notice that Theorems 3.6.1 and 3.6.1′ also demonstrate that the local finiteness requirement in the statement of Theorem 3.4.1 is necessary.

3.6.1 Outline of the proof

The proofs of Theorems 3.6.1 and 3.6.1′ are almost identical, so we will present them simultaneously. We only have to show the forward implication in both statements (the other direction is handled by Corollary 3.5.7). Here we briefly sketch our plan of attack.

For simplicity, assume that $\Gamma = \mathbb{Z}$ and let $\alpha \colon \mathbb{Z} \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action of $\mathbb{Z}$. There is a simple criterion, called *Sinai's factor theorem*, that determines whether there is a factor map $\pi \colon (X, \mu) \to ([0; 1]^\mathbb{Z}, \lambda^\mathbb{Z})$: Such π exists if and only if α has infinite *Kolmogorov–Sinai entropy*. The Kolmogorov–Sinai entropy of α is defined as follows. Consider any Borel function $f \colon X \to I$ to a finite set I. The **Shannon entropy** of f measures how "uncertain" the value $f(x)$ is when $x \in X$ is chosen randomly with respect to μ; formally,

$$h_\mu(f) \coloneqq -\sum_{i \in I} \mu(f^{-1}(i)) \log_2 \mu(f^{-1}(i)).$$

Now the action comes into play: Given $x \in X$ and $n \in \mathbb{N}$, we record the sequence of values

$$f((-n) \cdot x),\ f((-n+1) \cdot x),\ \ldots,\ f(n \cdot x);$$

this gives us a tuple of elements of I of length $2n+1$. Let $f_n\colon X \to I^{2n+1}$ be the corresponding function. We can compute the *average* amount of uncertainty in $f_n(x)$ per symbol; in other words, we can look at the quantity $h_\mu(f_n(x))/(2n+1)$. It turns out that, as n grows, this quantity decreases, so there exists a limit

$$H_\mu(\alpha, f) \coloneqq \lim_{n\to\infty} \frac{h_\mu(f_n)}{2n+1}.$$

This limit is called the **Kolmogorov–Sinai entropy** of f with respect to α. The **Kolmogorov–Sinai entropy** of the action α itself measures the "maximum level of uncertainty" that can be achieved with respect to α; formally, it is defined as

$$H_\mu(\alpha) \coloneqq \sup_f H_\mu(\alpha, f),$$

where f is ranging over all Borel functions from X to a finite set. As mentioned previously, α factors to the $[0;1]$-shift action if and only if $H_\mu(\alpha) = \infty$.

How can we use the LLL to prove that $H_\mu(\alpha) = \infty$? By definition, we have to exhibit Borel functions f with arbitrarily large values of $H_\mu(\alpha, f)$. But $H_\mu(\alpha, f)$ is, in some sense, a "global" parameter—it is defined in terms of the measures of certain subsets of X—while instances of the LLL can only put "local" constraints on the function f. However, high value of $H_\mu(\alpha, f)$ indicates that the functions f_n behave very "randomly" or "unpredictably." Thus, what we need is a way to measure "randomness" or "unpredictability" deterministically, which we can then apply to the values of f_n at each point instead of looking at the function f_n as a whole.

There is indeed a convenient deterministic analog of Shannon's entropy, namely the so-called *Kolmogorov complexity*. Roughly speaking, a finite sequence w of symbols has high Kolmogorov complexity if there is no way to encode it by a significantly shorter sequence. Our instance of the LLL will require $f_n(x)$ to have high Kolmogorov complexity for all $n \in \mathbb{N}$ and $x \in X$. We will show that solving this instance, even partially, guarantees that $H_\mu(\alpha, f)$ must also be high.

The structure of the rest of this section is as follows. In §3.6.2 we list the necessary definitions and preliminary results regarding the structure of amenable groups, Kolmogorov–Sinai entropy of their actions (including the version of Sinai's factor theorem with a general amenable group in place of $\mathbb{Z}$), and Kolmogorov complexity. In §3.6.3 we prove the main lemma that connects Kolmogorov complexity and Kolmogorov–Sinai entropy. Finally, §3.6.4 completes the proof by constructing a series of instances of the LLL whose solutions necessarily have high Kolmogorov complexity and hence high Kolmogorov–Sinai entropy.

3.6.2 Preliminaries

Background on amenable groups

Recall that a countable group Γ is **amenable** if it admits a **Følner sequence**, i.e., a sequence $(F_n)_{n=0}^\infty$ of nonempty finite subsets of Γ such that for all $\gamma \in \Gamma$,

$$\lim_{n\to\infty} \frac{|\gamma F_n \triangle F_n|}{|F_n|} = 0, \tag{3.6.1}$$

where $\triangle$ denotes symmetric difference of sets. Note that if $S \subseteq \Gamma$ is a generating set and (3.6.1) holds for all $\gamma \in S$, then $(F_n)_{n=0}^{\infty}$ is a Følner sequence (see [KM04, Remark 5.12]).

Proposition 3.6.2. *Let Γ be a countably infinite amenable group and let $S \subseteq \Gamma$ be a generating set. Let $G := \mathrm{Cay}(\Gamma, S)$ denote the corresponding Cayley graph. Then Γ admits a Følner sequence $(F_n)_{n=0}^{\infty}$ such that every (finite) graph $G[F_n]$ is connected.*

Proof. Let $\gamma_0, \gamma_1, \ldots$ be a list of all the elements of S in an arbitrary order, possibly with repetitions (so the list is infinite even if S is finite) and let $(F_n)_{n=0}^{\infty}$ be a Følner sequence for Γ. By passing to a subsequence if necessary, we can arrange that for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} \frac{|\gamma_i F_n \triangle F_n|}{|F_n|} \leqslant \frac{1}{n}. \tag{3.6.2}$$

Suppose $G[F_n]$ has k_n connected components and let $F_{n,1}, \ldots, F_{n,k_n} \subseteq F_n$ denote their vertex sets. For all $i \in \mathbb{N}$ and $1 \leqslant j_1 < j_2 \leqslant k_n$, we have $\gamma_i F_{n,j_1} \cap F_{n,j_2} = \varnothing$, so

$$\gamma_i F_n \triangle F_n = \bigcup_{j=1}^{k_n} (\gamma_i F_{n,j} \triangle F_{n,j}), \tag{3.6.3}$$

and the union on the right-hand side of (3.6.3) is disjoint. Therefore,

$$\sum_{i=0}^{n} \frac{|\gamma_i F_n \triangle F_n|}{|F_n|} = \frac{\sum_{j=1}^{k_n} \sum_{i=0}^{n} |\gamma_i F_{n,j} \triangle F_{n,j}|}{\sum_{j=1}^{k_n} |F_{n,j}|}.$$

If for all $1 \leqslant j \leqslant k_n$, we have

$$\sum_{i=0}^{n} \frac{|\gamma_i F_{n,j} \triangle F_{n,j}|}{|F_{n,j}|} > \frac{1}{n},$$

then

$$\frac{\sum_{j=1}^{k_n} \sum_{i=0}^{n} |\gamma_i F_{n,j} \triangle F_{n,j}|}{\sum_{j=1}^{k_n} |F_{n,j}|} > \frac{\sum_{j=1}^{k_n} \frac{1}{n} |F_{n,j}|}{\sum_{j=1}^{k_n} |F_{n,j}|} = \frac{1}{n},$$

which contradicts (3.6.2). Hence, there is some $1 \leqslant j_n \leqslant k_n$ such that

$$\sum_{i=0}^{n} \frac{|\gamma_i F_{n,j} \triangle F_{n,j}|}{|F_{n,j}|} \leqslant \frac{1}{n}.$$

Then $(F_{n,j_n})_{n=0}^{\infty}$ is a desired Følner sequence consisting of connected sets. ■

Corollary 3.6.3. *Let Γ be a countably infinite amenable group and let $S \subseteq \Gamma$ be a generating set. Let $G := \mathrm{Cay}(\Gamma, S)$ denote the corresponding Cayley graph. Then Γ admits a Følner sequence $(F_n)_{n=0}^{\infty}$ such that:*

- *for each $n \in \mathbb{N}$, the graph $\mathrm{Cay}(\Gamma, S)[F_n]$ is connected;*

- *$\mathbf{1} \in F_0 \subset F_1 \subset \ldots$, where $\mathbf{1}$ is the identity element of Γ;*

– $\bigcup_{n=0}^{\infty} F_n = \Gamma$;

– $\lim_{n\to\infty} |F_n|/\log_2 n = \infty$.

Proof. Proposition 3.6.2 gives a Følner sequence $(F_n)_{n=0}^{\infty}$ satisfying the first condition. Since Γ is infinite, we have $|F_n| \to \infty$ as $n \to \infty$. If $\mathbf{1} \notin F_n$ for some $n \in \mathbb{N}$, then choose any $\gamma \in F_n$ and replace F_n with $F_n\gamma^{-1}$. Now we construct a new sequence $(F'_n)_{n=0}^{\infty}$ inductively. Let $\gamma_0, \gamma_1, \ldots$ be a list of all the elements of S in an arbitrary order, possibly with repetitions. For $n \in \mathbb{N}$, set $S_n \coloneqq \{\gamma_0, \ldots, \gamma_n\}$. Let B_n denote the collection of all the elements of Γ that can be expressed as products of at most n elements of S_n. Note that $\mathbf{1} \in B_n$ and the graph $\mathrm{Cay}(\Gamma, S)[B_n]$ is connected. Let $F'_0 \coloneqq F_0$. On step $n+1$, choose N large enough so that

$$|F_N| > n \cdot \left(\sum_{i=0}^{n} |F'_i| + |B_n| + \log_2 n \right),$$

and define

$$F'_{n+1} \coloneqq F_N \cup \bigcup_{i=0}^{n} F'_i \cup B_n.$$

Clearly, $(F'_n)_{n=0}^{\infty}$ is a Følner sequence satisfying all the requirements. ■

We will need a result of Ornstein and Weiss on the existence of quasi-tilings in amenable groups. A family $A_1, \ldots, A_k$ of finite sets is said to be ε**-disjoint**, $\varepsilon > 0$, if there exist pairwise disjoint subsets $B_1 \subseteq A_1, \ldots, B_k \subseteq A_k$ such that for all $1 \leqslant i \leqslant k$,

$$|B_i| \geqslant (1-\varepsilon)|A_i|.$$

A finite set A is $(1-\varepsilon)$**-covered** by $A_1, \ldots, A_k$ if

$$\left| A \cap \bigcup_{i=1}^{k} A_i \right| \geqslant (1-\varepsilon)|A|.$$

Let Γ be a countable group and let $A, A_1, \ldots, A_k$ be finite subsets of Γ. An ε**-quasi-tiling** of A by the sets $A_1, \ldots, A_k$ is a collection $C_1, \ldots, C_k$ of finite subsets of Γ such that:

– for each $1 \leqslant i \leqslant k$, we have $A_iC_i \subseteq A$ and the family of sets sets $(A_i\gamma)_{\gamma \in C_i}$ is ε-disjoint;

– the sets $A_1C_1, \ldots, A_kC_k$ are pairwise disjoint;

– A is $(1-\varepsilon)$-covered by the sets $A_1C_1, \ldots, A_kC_k$.

Theorem 3.6.4 (Ornstein–Weiss [OW87]; see also [WZ92, Theorem 2.6] and [ZCY16, Proposition 2.3])**.** *Let Γ be a countable amenable group and let $(F_n)_{n=0}^{\infty}$ be a Følner sequence in Γ. Then for all $\varepsilon > 0$ and for all $n \in \mathbb{N}$, there exist $k, \ell_1, \ldots, \ell_k, m_0 \in \mathbb{N}$ with $n \leqslant \ell_1 < \ell_2 < \ldots < \ell_k$ such that for each $m \geqslant m_0$, there exists an ε-quasi-tiling of F_m by $F_{\ell_1}, \ldots, F_{\ell_k}$.*

Background on Kolmogorov–Sinai entropy

An important invariant of an amenable probability measure-preserving system is its Kolmogorov–Sinai entropy. It is usually defined in terms of finite Borel partitions; however, for our purposes it will be more convenient to define it in terms of Borel functions to a finite set (the two notions are, of course, equivalent).

Let (X, μ) be a standard probability space. A (**finite**) **coloring** of X is a function $f \colon X \to I$, where I is a finite set. The **Shannon entropy** of a Borel finite coloring $f \colon X \to I$ is defined to be

$$h_\mu(f) := -\sum_{i \in I} \mu(f^{-1}(i)) \log_2 \mu(f^{-1}(i)).$$

Here we adopt the convention that $0 \cdot \log_2 0 = 0$. Note that $0 \leqslant h_\mu(f) \leqslant \log_2 |I|$.

Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of a countable amenable group Γ. For a finite coloring $f \colon X \to I$ and a set $F \in [\Gamma]^{<\infty}$, let $f^F : X \to I^F$ denote the finite coloring defined by setting, for all $x \in X$ and $\gamma \in F$,

$$f^F(x)(\gamma) := f(\gamma \cdot x).$$

The **Kolmogorov–Sinai entropy** of a Borel finite coloring f with respect to α is given by

$$H_\mu(\alpha, f) := \lim_{n \to \infty} \frac{h_\mu(f^{F_n})}{|F_n|}, \tag{3.6.4}$$

where $(F_n)_{n=0}^\infty$ is a Følner sequence in Γ. Due to a fundamental result of Ornstein and Weiss [OW87], the limit in (3.6.4) always exists and is independent of the choice of $(F_n)_{n=0}^\infty$. Note that we again have $0 \leqslant H_\mu(\alpha, f) \leqslant \log_2 |I|$, where I is the range of f. The **Kolmogorov–Sinai entropy** of α is defined by

$$H_\mu(\alpha) := \sup\{H_\mu(\alpha, f) \,:\, f \text{ is a Borel finite coloring of } X\}.$$

We will use the following special case of a generalization of Sinai's factor theorem to actions of arbitrary amenable groups proven by Ornstein and Weiss:

Theorem 3.6.5 (Ornstein–Weiss [OW87]). *Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action of a countably infinite amenable group Γ. Suppose that $H_\mu(\alpha) = \infty$. Then there exists a factor map $\pi \colon (X, \mu) \to ([0;1]^\Gamma, \lambda^\Gamma)$ to the $[0;1]$-shift action of Γ.*

Background on Kolmogorov complexity

We will use some basic properties of Kolmogorov complexity. Let 2^* denote the set of all finite sequences of zeroes and ones (including the empty sequence). For $w \in 2^*$, let $|w|$ denote the length of w. For a partial function $D \colon 2^* \rightharpoonup 2^*$, define the map $K_D \colon 2^* \to \mathbb{N} \cup \{\infty\}$ via

$$K_D(x) := \inf\{|w| \,:\, D(w) = x\}.$$

Given two partial functions $D_1, D_2 \colon 2^* \rightharpoonup 2^*$, we say that D_1 **minorizes** D_2 (notation: $D_1 \leqslant_K D_2$) if there is a constant $c \in \mathbb{N}$ such that for all $x \in 2^*$,

$$K_{D_1}(x) \leqslant K_{D_2}(x) + c.$$

Clearly, $\leqslant_K$ is a preorder. If $\mathscr{C}$ is a class of partial functions $2^* \rightharpoonup 2^*$, then $D \in \mathscr{C}$ is **optimal** in $\mathscr{C}$ if for all $D' \in \mathscr{C}$, $D \leqslant_K D'$.

We restrict our attention to the class of all partial maps $D \colon 2^* \rightharpoonup 2^*$ that are computable relative to a fixed oracle $\mathbb{O}$ (denote this class by $\mathscr{C}_{\mathbb{O}}$). A cornerstone of the theory of Kolmogorov complexity is the following observation:

Theorem 3.6.6 (Solomonoff–Kolmogorov; see [LV08, Lemma 2.1.1] and [UVS10, Theorem 1])**.** *Fix an oracle $\mathbb{O}$. There exists a map $D \in \mathscr{C}_{\mathbb{O}}$ that is optimal in $\mathscr{C}_{\mathbb{O}}$.*

In the light of Theorem 3.6.6, we can define the **Kolmogorov complexity** of a word $x \in 2^*$ relative to an oracle $\mathbb{O}$ to be

$$K_{\mathbb{O}}(x) \coloneqq K_D(x),$$

for some fixed optimal $D \in \mathscr{C}_{\mathbb{O}}$. Note that if $D, D' \in \mathscr{C}_{\mathbb{O}}$ are two optimal functions, then there is a constant $c \in \mathbb{N}$ such that $|K_D(x) - K_{D'}(x)| \leqslant c$ for all $x \in 2^*$; in this sense, the value $K_{\mathbb{O}}(x)$ is defined up to an additive constant.

The following property of Kolmogorov complexity will play a crucial role in our argument.

Proposition 3.6.7. *Fix an oracle $\mathbb{O}$. Let $c, n \in \mathbb{N}$ and let ν_n denote the uniform probability measure on 2^n. Then*

$$\nu_n(\{x \in 2^n \,:\, K_{\mathbb{O}}(x) \leqslant n - c\}) < 2^{-c+1}.$$

Proof. Let $D \in \mathscr{C}_{\mathbb{O}}$ be the optimal function used in the definition of Kolmogorov complexity relative to $\mathbb{O}$. There are exactly $2^{n-c+1} - 1$ sequences of zeroes and ones of length at most $n - c$, so there can be at most $2^{n-c+1} - 1$ words $x \in 2^*$ with $K_{\mathbb{O}}(x) = K_D(x) \leqslant n - c$. Therefore,

$$\nu_n(\{x \in 2^n \,:\, K_{\mathbb{O}}(x) \leqslant n - c\}) \leqslant \frac{2^{n-c+1} - 1}{2^n} = 2^{-c+1} - 2^{-n} < 2^{-c+1},$$

as desired. ∎

3.6.3 Kolmogorov complexity vs. Kolmogorov–Sinai entropy

For the rest of this section, we fix a countably infinite amenable group Γ, a generating set $S \subseteq \Gamma$, a standard probability space (X, μ), and a free ergodic measure-preserving action $\alpha \colon \Gamma \curvearrowright X$. We also fix a Følner sequence $(F_n)_{n=0}^{\infty}$ in Γ satisfying the requirements of Corollary 3.6.3, i.e., such that:

- for each $n \in \mathbb{N}$, the graph $\mathrm{Cay}(\Gamma, S)[F_n]$ is connected;
- $\mathbf{1} \in F_0 \subset F_1 \subset \ldots$, where $\mathbf{1}$ is the identity element of Γ;

– $\bigcup_{n=0}^{\infty} F_n = \Gamma$;

– $\lim_{n\to\infty} |F_n|/\log_2 n = \infty$.

Let $\mathbb{O}$ be an oracle relative to which the following data are computable:

– the group structure of Γ and a fixed linear ordering $<$ on Γ (we may assume, for instance, that the ground set of Γ is $\mathbb{N}$);

– the sequence $(F_n)_{n=0}^{\infty}$ (meaning that the set $\{(\gamma, n) \in \Gamma \times \mathbb{N} \,:\, \gamma \in F_n\}$ and the sequence $(|F_n|)_{n=0}^{\infty}$ are decidable relative to $\mathbb{O}$).

Given a set $F \in [\Gamma]^{<\infty}$ and a function $w \colon F \to 2^s$, we can use the ordering on Γ to identify w with a sequence of zeroes and ones of length $s|F|$. This identification enables us to talk about the Kolmogorov complexity $K_{\mathbb{O}}(w)$ of w. For a Borel coloring $f \colon X \to 2^s$, a point $x \in X$, and $n \in \mathbb{N}$, let

$$f_n(x) \coloneqq f^{F_n}(x).$$

Note that the map $X \times \mathbb{N} \to \mathbb{N} \colon (x, n) \mapsto K_{\mathbb{O}}(f_n(x))$ is Borel.

The following lemma connects Kolmogorov complexity and Kolmogorov–Sinai entropy:

Lemma 3.6.8 (**High complexity $\Longrightarrow$ high entropy**). *Let $s \in \mathbb{N}$ and let $f \colon X \to 2^s$ be a Borel coloring of X. Then*

$$\limsup_{m\to\infty} \int_X \frac{K_{\mathbb{O}}(f_m(x))}{|F_m|} \,\mathrm{d}\,\mu(x) \leqslant H_\mu(\alpha, f).$$

Proof. Our argument is inspired by the work of Brudno [Bru82], who established a close relationship between Kolmogorov complexity and Kolmogorov–Sinai entropy in the case of $\mathbb{Z}$-actions (see also [Mor15] for an extension of Brudno's theory to a wider class of amenable groups).

Fix $\varepsilon \in (0; 1)$. Choose $n \in \mathbb{N}$ large enough so that for all $\ell \geqslant n$,

$$\frac{h_\mu(f_\ell)}{|F_\ell|} \leqslant H_\mu(\alpha, f) + \varepsilon.$$

For most of the proof, n and s will be treated as fixed constants. In particular, the implied constants in asymptotic notation may depend on n and s.

Using Theorem 3.6.4, choose $k, \ell_1, \ldots, \ell_k, m_0 \in \mathbb{N}$ so that $n \leqslant \ell_1 < \ldots < \ell_k$ and for every $m \geqslant m_0$, there exists an ε-quasi-tiling of F_m by $F_{\ell_1}, \ldots, F_{\ell_k}$. For each $m \geqslant m_0$, let $C_{m,1}, \ldots, C_{m,k}$ be an ε-quasi-tiling of F_m by $F_{\ell_1}, \ldots, F_{\ell_k}$, chosen in such a way that the map

$$m \mapsto (C_{m,i})_{i=1}^{k}$$

is computable relative to $\mathbb{O}$. (For instance, we can choose the sequence of finite sets $C_{m,1}, \ldots, C_{m,k}$ to be the first in some computable ordering.)

We will now devise a binary code for pairs of the form (m, w), where $m \geqslant m_0$ and $w \colon F_m \to 2^s$. The decoding procedure for this code will be computable relative to $\mathbb{O}$, so the length of the code will provide an upper bound on the Kolmogorov complexity of w (modulo an additive constant).

Let $c_0(m)$ be the sequence of $\lceil \log_2 m \rceil$ ones followed by a single zero and let $c_1(m)$ be any fixed binary code for the integer m of length exactly $\lceil \log_2 m \rceil$. Note that for any $c \in 2^*$, the pair (m, c) is uniquely determined by $c_0(m)^\frown c_1(m)^\frown c$. Also note that

$$|c_0(m)^\frown c_1(m)| \leqslant 2 \log_2 m + O(1) = o_{m\to\infty}(|F_m|).$$

Consider the set

$$\Lambda_m \coloneqq F_m \setminus \bigcup_{i=1}^{k} F_{\ell_i} C_{m,i}.$$

We can view $w|\Lambda_m$ as a binary word of length $s|\Lambda_m|$, which we denote by $c_2(m, w)$. Note that the length of $c_2(m, w)$ is determined by m and satisfies

$$|c_2(m, w)| = s|\Lambda_m| \leqslant \varepsilon s|F_m|,$$

since F_m is $(1 - \varepsilon)$-covered by the family $(F_{\ell_i} C_{m,i})_{i=1}^k$.

For each $1 \leqslant i \leqslant k$ and $\gamma \in C_{m,i}$, we can view

$$w_{i,\gamma} \coloneqq w|(F_{\ell_i}\gamma)$$

as a binary word of length $s|F_{\ell_i}|$. For each binary word u of length $s|F_{\ell_i}|$, let $\eta_{i,u}(m, w)$ be the frequency of u among the words of the form $w_{i,\gamma}$, i.e., let

$$\eta_{i,u}(m, w) \coloneqq |\{\gamma \in C_{m,i} \ : \ w_{i,\gamma} = u\}|.$$

By definition,

$$\sum_{u} \eta_{i,u}(m, w) = |C_{m,i}|, \tag{3.6.5}$$

where the summation is over all binary words of length $s|F_{\ell_i}|$. Since $0 \leqslant \eta_{i,u}(m, w) \leqslant |C_{m,i}|$, we can encode $\eta_{i,u}(m, w)$ by a binary word $c_3(m, w, i, u)$ of length exactly

$$\lceil \log_2(|C_{m,i}| + 1) \rceil \leqslant \log_2 |C_{m,i}| + O(1) \leqslant \log_2 |F_m| + O(1),$$

so the length of $c_3(m, w, i, u)$ is determined by m. Let $c_3(m, w, i)$ denote the concatenation of all the words of the form $c_3(m, w, i, u)$ with u ranging over the binary words of length $s|F_{\ell_i}|$, and let

$$c_3(m, w) \coloneqq c_3(m, w, 1)^\frown \dots {}^\frown c_3(m, w, k).$$

The length of $c_3(m, w)$ is at most $O(\log_2 |F_m|) = o_{m\to\infty}(|F_m|)$.

Now we consider the word

$$w_i \coloneqq w|(F_{\ell_i} C_{m,i}).$$

Since w_i is determined by the family $(w_{i,\gamma} \,:\, \gamma \in C_{m,i})$, there are at most

$$\frac{|C_{m,i}|!}{\prod_u \eta_{i,u}(m,w)!}$$

options for w_i, where the product is over all binary words of length $s|F_{\ell_i}|$. Due to Stirling's formula and equation (3.6.5), we have

$$\log_2\left(\frac{|C_{m,i}|!}{\prod_u \eta_{i,u}(m,w)!}\right) \leqslant -|C_{m,i}| \sum_u \frac{\eta_{i,u}(m,w)}{|C_{m,i}|} \log_2\left(\frac{\eta_{i,u}(m,w)}{|C_{m,i}|}\right).$$

Thus, provided that m and all the $\eta_{i,u}(m,w)$'s are given, w_i can be encoded by a binary word $c_4(m,w,i)$ of length

$$\left\lceil \log_2\left(\frac{|C_{m,i}|!}{\prod_u \eta_{i,u}(m,w)!}\right)\right\rceil \leqslant -|C_{m,i}| \sum_u \frac{\eta_{i,u}(m,w)}{|C_{m,i}|} \log_2\left(\frac{\eta_{i,u}(m,w)}{|C_{m,i}|}\right) + O(1).$$

Let

$$c_4(m,w) \coloneqq c_4(m,w,1)^\frown \ldots {}^\frown c_4(m,w,k).$$

The length of $c_4(m,w)$ is at most

$$-\sum_{i=1}^{k} |C_{m,i}| \sum_u \frac{\eta_{i,u}(m,w)}{|C_{m,i}|} \log_2\left(\frac{\eta_{i,u}(m,w)}{|C_{m,i}|}\right) + o_{m\to\infty}(|F_m|).$$

Our code for (m,w) is the concatenation

$$\mathrm{code}(m,w) \coloneqq c_0(m)^\frown c_1(m)^\frown c_2(m,w)^\frown c_3(m,w)^\frown c_4(m,w).$$

It is clear that $\mathrm{code}(m,w)$ uniquely determines m and w and, moreover, the map

$$\mathrm{code}(m,w) \mapsto (m,w)$$

is computable relative to $\mathbb{O}$. Combining the above upper bounds for the lengths of $c_0(m)$, $c_1(m)$, $c_2(m,w)$, $c_3(m,w)$, and $c_4(m,w)$, we get

$$|\mathrm{code}(m,w)| \leqslant \varepsilon s|F_m| - \sum_{i=1}^{k} |C_{m,i}| \sum_u \frac{\eta_{i,u}(m,w)}{|C_{m,i}|} \log_2\left(\frac{\eta_{i,u}(m,w)}{|C_{m,i}|}\right) + o_{m\to\infty}(|F_m|).$$

Since $K_{\mathbb{O}}(w) \leqslant |\mathrm{code}(m,w)| + O(1)$, the same asymptotic upper bound holds for $K_{\mathbb{O}}(w)$ as well.

Applying this analysis to a point $x \in X$, we obtain

$$\frac{K_{\mathbb{O}}(f_m(x))}{|F_m|} \leqslant \varepsilon s - \sum_{i=1}^{k} \frac{|C_{m,i}|}{|F_m|} \sum_u \frac{\eta_{i,u}(m,f_m(x))}{|C_{m,i}|} \log_2\left(\frac{\eta_{i,u}(m,f_m(x))}{|C_{m,i}|}\right) + o_{m\to\infty}(1). \tag{3.6.6}$$

Claim (A). *For each $m \geqslant m_0$, $1 \leqslant i \leqslant k$, and a binary word u of length $s|F_{\ell_i}|$, we have*

$$\int_X \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} \,\mathrm{d}\,\mu(x) = \mu(f_{\ell_i}^{-1}(u)).$$

Proof. Recall that, by definition,

$$\eta_{i,u}(m, w) = |\{\gamma \in C_{m,i} \,:\, w_{i,\gamma} = u\}| = |\{\gamma \in C_{m,i} \,:\, w|(F_{\ell_i}\gamma) = u\}|.$$

Notice that $f_m(x)|(F_{\ell_i}\gamma) = f_{\ell_i}(\gamma \cdot x)$, so we have

$$\begin{aligned}\eta_{i,u}(m, f_m(x)) &= |\{\gamma \in C_{m,i} \,:\, f_m(x)|(F_{\ell_i}\gamma) = u\}| \\ &= |\{\gamma \in C_{m,i} \,:\, f_{\ell_i}(\gamma \cdot x) = u\}|.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_X \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} \,\mathrm{d}\,\mu(x) &= \frac{1}{|C_{m,i}|}\int_X |\{\gamma \in C_{m,i} \,:\, f_{\ell_i}(\gamma \cdot x) = u\}| \,\mathrm{d}\,\mu(x) \\ &= \frac{1}{|C_{m,i}|}\int_X |\{\gamma \in C_{m,i} \,:\, f_{\ell_i}(x) = u\}| \,\mathrm{d}\,\mu(x) \\ &= \frac{1}{|C_{m,i}|} \cdot |C_{m,i}| \cdot \mu(f_{\ell_i}^{-1}(u)) = \mu(f_{\ell_i}^{-1}(u)),\end{aligned}$$

where the second equality holds since μ is α-invariant. ⊣

The function $\alpha \mapsto -\alpha \log_2 \alpha$ is concave for $0 \leqslant \alpha \leqslant 1$, so, by Claim (A), for $1 \leqslant i \leqslant k$, we have

$$\begin{aligned}&-\int_X \sum_u \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} \log_2\left(\frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|}\right) \mathrm{d}\,\mu(x) \\ &\leqslant -\sum_u \int_X \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} \,\mathrm{d}\,\mu(x) \log_2\left(\int_X \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} \,\mathrm{d}\,\mu(x)\right) \\ &= -\sum_u \mu(f_{\ell_i}^{-1}(u)) \log_2 \mu(f_{\ell_i}^{-1}(u)) \\ &= h_\mu(f_{\ell_i}).\end{aligned}$$

Combining this with (3.6.6) gives

$$\int_X \frac{K_{\mathbb{O}}(f_m(x))}{|F_m|} \,\mathrm{d}\,\mu(x) \leqslant \varepsilon s + \sum_{i=1}^{k} \frac{|C_{m,i}|}{|F_m|} h_\mu(f_{\ell_i}) + o_{m\to\infty}(1).$$

Recall that, by the choice of n,

$$\frac{h_\mu(f_\ell)}{|F_\ell|} \leqslant H_\mu(\alpha, f) + \varepsilon$$

for all $\ell \geqslant n$. Since $n \leqslant \ell_1 < \ldots < \ell_k$, we get

$$\int_X \frac{K_{\mathbb{O}}(f_m(x))}{|F_m|} \,\mathrm{d}\,\mu(x) \leqslant \varepsilon s + \sum_{i=1}^{k} \frac{|C_{m,i}|}{|F_m|} \cdot |F_{\ell_i}|(H_\mu(\alpha, f) + \varepsilon) + o_{m\to\infty}(1).$$

Since the sets $(F_{\ell_i} C_{m,i})_{i=1}^k$ are pairwise disjoint, and for each $1 \leqslant i \leqslant k$, the family of sets $(F_{\ell_i}\gamma)_{\gamma \in C_{m,i}}$ is ε-disjoint, we have

$$|F_m| \geqslant \sum_{i=1}^{k} |F_{\ell_i} C_{m,i}| \geqslant (1-\varepsilon) \sum_{i=1}^{k} |F_{\ell_i}||C_{m,i}|,$$

so

$$\int_X \frac{K_{\mathbb{O}}(f_m(x))}{|F_m|} \,\mathrm{d}\,\mu(x) \leqslant \varepsilon s + \frac{1}{1-\varepsilon}(H_\mu(\alpha, f) + \varepsilon) + o_{m\to\infty}(1). \tag{3.6.7}$$

Since (3.6.7) holds for every $\varepsilon \in (0; 1)$ and for every sufficiently large m, we finally obtain

$$\int_X \frac{K_{\mathbb{O}}(f_m(x))}{|F_m|} \,\mathrm{d}\,\mu(x) \leqslant H_\mu(\alpha, f) + o_{m\to\infty}(1),$$

as desired. ■

3.6.4 Building the instances

Define

$$G \coloneqq \begin{cases} G(a, S) & \text{if } S \text{ is finite;} \\ G_\ell(a, S) & \text{if } S \text{ is infinite.} \end{cases}$$

If S is finite, let $\mathrm{Cay}(\Gamma, S)$ denote the corresponding (unlabeled) Cayley graph; otherwise, assume that $\mathrm{Cay}(\Gamma, S)$ is S-labeled. Fix $\varepsilon \in (0; 1]$ such that for every ε-correct uniformly discrete Borel instance $\mathscr{B}$ over G, there is a Borel map $f \colon X \to [0; 1]$ with $\mu(\mathbf{Def}_{\mathscr{B}}(f)) < 1$. Our goal is to show that $H_\mu(\alpha) = \infty$.

For each pair of nonnegative integers $s, t \in \mathbb{N}$ with $s \geqslant t$, we will construct a uniformly discrete Borel instance $\mathscr{B}(s, t)$ over G. For convenience, we will view each bad event $B \in \mathscr{B}(s, t)$ as a set of partial maps in $[X \to 2^s]^{<\infty}$ instead of the usual $[X \to [0; 1]]^{<\infty}$.

For $n \in \mathbb{N}$, let $G_n \coloneqq \mathrm{Cay}(\Gamma, S)[F_n]$. By the choice of the Følner sequence $(F_n)_{n=0}^\infty$, each graph G_n is connected. Given an isomorphism $\varphi \colon F_n \to X$ between the graphs G_n and $G[\mathrm{im}(\varphi)]$, let $B_\varphi(s, t)$ denote the bad event with domain $\mathrm{im}(\varphi)$ consisting of all maps $w \colon \mathrm{im}(\varphi) \to 2^s$ such that

$$K_{\mathbb{O}}(w \circ \varphi) \leqslant (s - t)|F_n|.$$

Let $\mathscr{B}(s, t)$ denote the collection of all bad events $B_\varphi(s, t)$ defined above. It is clear that $\mathscr{B}(s, t)$ is a uniformly discrete Borel instance of the LLL over G. We will show that there is some $t \in \mathbb{N}$ such that for all $s \geqslant t$, the instance $\mathscr{B}(s, t)$ is also ε-correct.

Set

$$d \coloneqq \begin{cases} |S \cup S^{-1}| & \text{if } S \text{ is finite;} \\ 2 & \text{if } S \text{ is infinite.} \end{cases}$$

Lemma 3.6.9. *For all $\gamma \in F_n$ and $x \in X$, the number of isomorphic embeddings $\varphi\colon F_n \to X$ of G_n into G with $\varphi(\gamma) = x$ does not exceed $d^{|F_n|}$.*

Proof. If S is finite, then $\Delta(G) \leqslant |S \cup S^{-1}| = d$; if S is infinite, then for any given $\delta \in S$ and any $y \in X$, the graph G can contain at most 2 edges labeled by δ that are incident to y. Now the statement follows from the connectedness of G_n. ■

Lemma 3.6.10. *Let $B \in \mathscr{B}(s,t)$ and $k \in \mathbb{N}$. Then*

$$|\{B' \in N_{\mathscr{B}(s,t)}(B) \,:\, |\mathrm{dom}(B')| = k\}| \leqslant |\mathrm{dom}(B)| \cdot kd^k.$$

Proof. If there is no $n \in \mathbb{N}$ with $|F_n| = k$, then there is nothing to prove, so suppose that $|F_n| = k$ (such n is unique since the sequence $(F_n)_{n=0}^{\infty}$ is strictly increasing). Consider any $B' \in N_{\mathscr{B}(s,t)}(B)$ with $|\mathrm{dom}(B')| = k$. Then $B' = B_\varphi(s,t)$ for some embedding $\varphi\colon F_n \to X$ of G_n into G. As $B' \in N_{\mathscr{B}(s,t)}(B)$, we have $\mathrm{im}(\varphi) \cap \mathrm{dom}(B) \neq \varnothing$, i.e., there exist $\gamma \in F_n$ and $x \in \mathrm{dom}(B)$ such that $\varphi(\gamma) = x$. Now we have $|\mathrm{dom}(B)|$ choices for x, k choices for γ, and, by Lemma 3.6.9, at most d^k choices for φ given γ and x. ■

By Proposition 3.6.7, if $B \in \mathscr{B}(s,t)$ and $|\mathrm{dom}(B)| =: n$, then

$$\mathbb{P}[B] < 2^{-tn+1}.$$

In the light of Lemma 3.6.10, to show that $\mathscr{B}(s,t)$ is ε-correct, it suffices to find a sequence $(\omega_n)_{n=1}^{\infty}$ with each $\omega_n \in [0;1)$ such that for all $n \in \mathbb{N}$,

$$2^{-tn+1} \leqslant \varepsilon^n \omega_n \prod_{k=1}^{\infty} (1-\omega_k)^{nkd^k}. \tag{3.6.8}$$

Note that inequality (3.6.8) does not mention s; in other words, if it holds for some $(\omega_n)_{n=1}^{\infty}$ and for all $n \in \mathbb{N}$, then $\mathscr{B}(s,t)$ is ε-correct for *all* $s \geqslant t$.

To solve (3.6.8), let $\delta > 0$ be sufficiently small and set $\omega_n := \delta^n$. For every $n \in \mathbb{N}$, we have

$$1 - \omega_n > e^{-2\omega_n}.$$

Moreover, if we choose $\delta < 1/d$, then the series

$$\sum_{k=1}^{\infty} \omega_k kd^k = \sum_{k=1}^{\infty} k(\delta d)^k$$

converges; denote its sum by c. Now we have

$$\prod_{k=1}^{\infty} (1-\omega_k)^{nkd^k} > e^{-2cn},$$

so (3.6.8) holds as long as

$$2^{-tn+1} \leqslant \varepsilon^n \cdot \delta^n \cdot e^{-2cn},$$

for which it suffices to have

$$t \geqslant 1 - \log_2(\varepsilon\delta e^{-2c}). \tag{3.6.9}$$

Choose any $t \in \mathbb{N}$ that satisfies (3.6.9); for all $s \geqslant t$, the instance $\mathscr{B}(s,t)$ is ε-correct.

If $\mathscr{B}(s,t)$ is ε-correct, then there must exist a Borel map $f \colon X \to 2^s$ with $\mu(\mathbf{Def}_{\mathscr{B}}(f)) < 1$. Since the action α is ergodic, the set $[X \setminus \mathbf{Def}_{\mathscr{B}}(f)]_{E_\alpha}$ is conull (recall that for $A \subseteq X$, $[A]_{E_\alpha}$ is the E_α-saturation of A, i.e., the smallest α-invariant subset of X that contains A). We claim that for all $x \in [X \setminus \mathbf{Def}_{\mathscr{B}}(f)]_{E_\alpha}$,

$$\liminf_{m\to\infty} \frac{K_{\mathbb{O}}(f_m(x))}{|F_m|} \geqslant s - t.$$

Indeed, let $x \in X$ and let $y \in X \setminus \mathbf{Def}_{\mathscr{B}}(f)$ be such that $x \, E_\alpha \, y$. The sequence $(F_n)_{n\in\mathbb{N}}$ is increasing and exhaustive, so there is $m_0 \in \mathbb{N}$ such that $y \in F_m \cdot x$ for all $m \geqslant m_0$. Since $y \notin \mathbf{Def}_{\mathscr{B}}(f)$, for each $m \geqslant m_0$, the restriction of f to $F_m \cdot x$ satisfies the constraints laid down by $\mathscr{B}(s,t)$. By definition, this means that $K_{\mathbb{O}}(f_m(x)) > (s-t)|F_m|$, as claimed.

Using Fatou's lemma together with Lemma 3.6.8, we obtain

$$\begin{aligned} s - t &\leqslant \int_X \liminf_{m\to\infty} \frac{K_{\mathbb{O}}(f_m(x))}{|F_m|} \,\mathrm{d}\,\mu(x) \\ &\leqslant \liminf_{m\to\infty} \int_X \frac{K_{\mathbb{O}}(f_m(x))}{|F_m|} \,\mathrm{d}\,\mu(x) \\ &\leqslant \limsup_{m\to\infty} \int_X \frac{K_{\mathbb{O}}(f_m(x))}{|F_m|} \,\mathrm{d}\,\mu(x) \leqslant H_\mu(\alpha, f). \end{aligned} \tag{3.6.10}$$

Since for any $s \geqslant t$, we can find f such that (3.6.10) holds, $H_\mu(\alpha) = \infty$, as desired.

4 | Building large free subshifts using the LLL

4.1 Introduction

In this chapter we apply the Lovász Local Lemma to some problems in topological dynamics. Throughout this chapter, Γ is a countably infinite group with identity element $\mathbf{1}$.

Let $k \in \mathbb{N}$. Then k^Γ is a totally disconnected compact metrizable space equipped with the shift action of Γ. A shift-invariant closed subset $X \subseteq k^\Gamma$ is called a **subshift**. A subshift X is **minimal** if $X \neq \varnothing$ and there is no subshift Y such that $\varnothing \neq Y \subsetneq X$. A subshift X is **free** if the induced action $\Gamma \curvearrowright X$ is free, i.e., if the stabilizer of every point $x \in X$ is trivial. Glasner and Uspenskij [GU09, Problem 6.2] asked if every countable group admits a nonempty free subshift and gave a positive answer for groups that are either Abelian or residually finite [GU09, Theorem 5.1]. Somewhat earlier, Dranishnikov and Schroeder [DS07, Theorem 2] reached the same conclusion for torsion-free hyperbolic groups. The problem was finally resolved in a tour de force by Gao, Jackson, and Seward [GJS09; GJS16], who showed that not only do nonempty free subshifts exist for all groups, but they are rather numerous: For any $k \geqslant 2$, every nonempty shift-invariant open subset $U \subseteq k^\Gamma$ contains continuumly many pairwise disjoint nonempty free subshifts [GJS16, Theorem 1.4.1].

Seward and Tucker-Drob [ST16] further developed the techniques of [GJS09; GJS16] in order to establish the following very strong result: If $\Gamma \curvearrowright X$ is a free Borel action of Γ on a standard Borel space X, then there exists an equivariant Borel map $\pi \colon X \to 2^\Gamma$ such that $\overline{\pi(X)}$ is a free subshift [ST16, Theorem 1.1]. (Here, and in what follows, a horizontal line indicates topological closure.) This in particular implies that every countable group admits a free subshift *with an invariant probability measure*, which answers a question raised by Gao, Jackson, and Seward [GJS16, Problem 11.2.6]. Indeed, if the action $\Gamma \curvearrowright X$ preserves a probability measure μ, then the pushforward $\pi_*(\mu)$ is an invariant probability measure on $\overline{\pi(X)}$.

For the rest of this chapter, fix an integer $k \geqslant 2$. We study how "large," in various senses, a free subshift $X \subseteq k^\Gamma$ can be. Specifically, we investigate the following questions, which are attributed by Gao, Jackson, and Seward to Juan Souto:

Question 4.1.1 ([GJS16, Problem 11.2.5])**.** For a given group Γ, what is the largest possible Hausdorff dimension of a free subshift $X \subseteq k^\Gamma$?

Question 4.1.2 ([GJS16, Problem 11.2.4])**.** For groups Γ in which a notion of entropy exists, what is the largest possible entropy of a free subshift $X \subseteq k^\Gamma$?

This chapter is based on [Ber18a].

The notions of Hausdorff dimension and (topological) entropy are reviewed in §4.2. To date, the largest class of groups for which a well-developed theory of entropy exists is formed by the so-called *sofic* groups. Entropy for measure-preserving actions of sofic groups was introduced by Bowen [Bow10] and then extended to the topological setting by Kerr and Li [KL11]. For the smaller class of *amenable* groups, entropy was introduced earlier by Keiffer [Kei75] (with important further developments by Ornstein and Weiss [OW87]) and is somewhat better behaved. Both the Hausdorff dimension and, if Γ is sofic, the entropy of k^Γ are equal to $\log_2 k$. We answer Questions 4.1.1 and 4.1.2 by showing that the Hausdorff dimension and, if Γ is sofic, the entropy of a free subshift can be made *arbitrarily close* to this upper bound:

Theorem 4.1.3. *Let $U \subseteq k^\Gamma$ be a nonempty shift-invariant open set. Then, for any $h < \log_2 k$:*

(i) *there exists a free minimal subshift $X \subseteq U$ of Hausdorff dimension at least h;*

(ii) *if Γ is amenable, then there exists a free minimal subshift $X \subseteq U$ of entropy at least h;*

(iii) *if Γ is sofic, then there exists a free subshift $X \subseteq U$ whose entropy with respect to any sofic approximation is at least h.*

The main ingredient in our proof of Theorem 4.1.3 is the Lovász Local Lemma, or the LLL. For the reader's convenience, we give a brief review of the LLL in §4.4.2. Although the original proof of [GJS16, Theorem 1.4.1] due to Gao, Jackson, and Seward is quite technical, Aubrun, Barbieri, and Thomassé [ABT16] later employed the LLL to find a simple alternative construction of a nonempty free subshift $X \subseteq 2^\Gamma$ for an arbitrary group Γ. Elek [Ele17] followed an approach based on nonrepetitive graph colorings and inspired by [Alo+02] to obtain a new proof that there exist free subshifts with invariant probability measures under the assumptions that Γ is finitely generated and sofic; Elek's argument also relies heavily on the LLL.

The main result of this chapter is Theorem 4.3.4, of which Theorem 4.1.3 is a simple special case. We state Theorem 4.3.4 in §4.3 after introducing some necessary definitions. In the remainder of this section we give a brief informal overview of the statement of Theorem 4.3.4 without being precise about the technical details.

In this chapter, we work with five notions of size for subshifts: Hausdorff dimension, entropy (with the cases of amenable and general sofic groups treated somewhat differently), width, pointwise width, and breadth. The former two are standard and reviewed in §4.2, while the latter three are defined in §4.3 and are crucial for the statement of Theorem 4.3.4.

The *width* $\mathfrak{w}(X)$ of a subset $X \subseteq k^\Gamma$ is defined in a way that is quite similar to the definition of Hausdorff dimension, with the advantage of not requiring to choose a metric. It is not hard to see that for a subshift X, $\mathfrak{w}(X)$ is a lower bound for the Hausdorff dimension and, if Γ is amenable, the entropy of X (see Proposition 4.3.3(i),(iii)). The *pointwise width* of a set X is defined via

$$\mathfrak{w}^*(X) \coloneqq \inf_{x \in X} \min\{\mathfrak{w}(\overline{\Gamma \cdot x}),\, \mathfrak{w}(\overline{x \cdot \Gamma})\}.$$

If X is a nonempty subshift with $\mathfrak{w}^*(X) \geqslant h$, then we have $\mathfrak{w}(X) \geqslant h$, and, moreover, $\mathfrak{w}(Y) \geqslant h$ for all nonempty subshifts $Y \subseteq X$. Furthermore, if $\mathfrak{w}^*(X)$ is sufficiently large, namely strictly higher than $(1/2)\log_2 k$, then X must be *free* (see Proposition 4.3.3(iv)). In view of these considerations, finding subshifts of large pointwise width becomes our primary objective.

The last notion of size for a subshift X that we introduce (and the last ingredient needed for the statement of our main result) is its *breadth* $\mathfrak{b}(X)$. The definition of breadth is directly informed by the requirements of the LLL. In contrast to the other notions, it estimates the size of X in a somewhat roundabout way: by measuring how "small" one can make a family of open sets whose translates cover the *complement* of X. The main advantage of breadth is that it is usually easy to bound from below, since a lower bound on $\mathfrak{b}(X)$ can be witnessed by a *single* family $\mathcal{U}$ of open sets.

Now we can state the main result of this chapter:

Theorem 4.3.4. *Let $X \subseteq k^\Gamma$ be a subshift such that $\mathfrak{b}(X) > 0$. Then $\mathfrak{w}(X) \geqslant \mathfrak{b}(X)$; moreover, for any $h < \mathfrak{b}(X)$, there exists a nonempty subshift $X' \subseteq X$ with the following properties:*

(i) *the pointwise width of X' is at least h;*

(ii) *if Γ is sofic, then the entropy of X' with respect to any sofic approximation is at least h;*

(iii) *there exist an invariant probability measure μ on X' and a factor map*

$$\pi\colon ([0;1]^\Gamma, \lambda^\Gamma) \to (X', \mu).$$

Theorem 4.1.3 easily follows from Theorem 4.3.4 since $\mathfrak{b}(k^\Gamma) = \log_2 k$; the details are given in §4.3.

4.2 Preliminaries

Basic open sets

The topology on the space k^Γ is generated by the **basic** open sets of the form

$$U_\varphi \coloneqq \{x \in k^\Gamma \,:\, x \supset \varphi\}, \qquad \text{where} \qquad \varphi \in [\Gamma \to k]^{<\infty} \setminus \{\varnothing\}.$$

Observe that each basic open set is also closed. Note that the space k^Γ itself is *not* a basic open set (this convention will simplify some of our definitions later). For $X \subseteq k^\Gamma$ and $F \in [\Gamma]^{<\infty} \setminus \{\varnothing\}$, let

$$X_F \coloneqq \{\varphi \in k^F \,:\, X \cap U_\varphi \neq \varnothing\} = \{x|F \,:\, x \in X\}.$$

Hausdorff dimension

To define Hausdorff dimension, we must first fix a metric on k^Γ. To that end, let $\gamma_0, \gamma_1, \ldots$ be an arbitrary enumeration of the elements of Γ. For distinct $x, y \in k^\Gamma$, let

$$\mathrm{dist}(x, y) \coloneqq 2^{-n}, \text{ where } n \coloneqq \min\{i \in \mathbb{N} \,:\, x(\gamma_i) \neq y(\gamma_i)\}.$$

Of course, if $x = y$, then $\mathrm{dist}(x, y) \coloneqq 0$. Note that this metric is not shift-invariant and depends on the choice of the enumeration $\gamma_0, \gamma_1, \ldots$ (in fact, the topology on k^Γ is not induced by any invariant metric). For

$h \in [0; +\infty)$, the **h-dimensional Hausdorff content** $C_h(X)$ of a set $X \subseteq k^\Gamma$ is the infimum of all $\varepsilon \in [0; +\infty)$ such that there is a cover $\mathscr{B}$ of X by open balls with

$$\sum_{B \in \mathscr{B}} \operatorname{diam}(B)^h = \varepsilon.$$

The **Hausdorff dimension** of X, denoted $\dim_H(X)$, is given by

$$\dim_H(X) \coloneqq \inf\{h \in [0; +\infty) \,:\, C_h(X) = 0\}.$$

The following observation is easy:

Proposition 4.2.1. *We have* $\dim_H(k^\Gamma) = \log_2 k$ *and* $C_{\log_2 k}(X) = 0$ *for any subshift* $X \subsetneq k^\Gamma$.

Entropy for amenable groups

Recall that a group Γ is called **amenable** if it admits a **Følner sequence**, i.e., a sequence of nonempty finite subsets $(F_n)_{n=0}^\infty$ such that

$$\lim_{n\to\infty} \frac{|\gamma F_n \vartriangle F_n|}{|F_n|} \;=\; 0 \qquad \text{for all } \gamma \in \Gamma.$$

If Γ is amenable, then the (**topological**) **entropy** $h(X)$ of a nonempty subshift $X \subseteq k^\Gamma$ is given by

$$h(X) \;\coloneqq\; \lim_{n\to\infty} \frac{\log_2 |X_{F_n}|}{|F_n|},$$

where $(F_n)_{n=0}^\infty$ is a Følner sequence in Γ. By a fundamental result of Ornstein and Weiss [OW87], the above limit always exists and is independent of the choice of the Følner sequence $(F_n)_{n=0}^\infty$. The entropy of a subshift obeys the following bounds:

Proposition 4.2.2. *If* Γ *is amenable, then* $h(k^\Gamma) = \log_2 k$ *and* $h(X) < \log_2 k$ *for any subshift* $X \subsetneq k^\Gamma$.

Entropy for sofic groups

A **pseudo-action** of Γ on a set V is a map $\alpha \colon \Gamma \times V \to V \colon (\gamma, v) \mapsto \gamma \cdot_\alpha v$. We write $\alpha \colon \Gamma \mathrel{\tilde{\curvearrowright}} V$ to indicate that α is a pseudo-action of Γ on V. When α is understood, we usually simply write $\gamma \cdot v$ instead of $\gamma \cdot_\alpha v$.

Let $\alpha \colon \Gamma \mathrel{\tilde{\curvearrowright}} V$ be a pseudo-action of Γ on a set V and let $F \subseteq \Gamma$. An element $v \in V$ is **F-proper** (with respect to α) if the following conditions are satisfied:

- *identity*: $\mathbf{1} \cdot v = v$;
- *F-equivariance*: $\gamma \cdot (\delta \cdot v) = (\gamma\delta) \cdot v$ for all $\gamma, \delta \in F$ such that $\gamma\delta \in F$;
- *F-freeness*: $\gamma \cdot v = \delta \cdot v \implies \gamma = \delta$ for all $\gamma, \delta \in F$.

Let $\operatorname{Prop}_F(\alpha)$ denote the set of all F-proper elements $v \in V$. Note that α is a free action if and only if $\operatorname{Prop}_\Gamma(\alpha) = V$. If V is a finite set, then α is **(ε, F)-faithful** for $\varepsilon > 0$ and $F \subseteq \Gamma$ if

$$|\operatorname{Prop}_F(\alpha)| \geqslant (1 - \varepsilon)|V|.$$

A group Γ is called **sofic** if it admits a **sofic approximation**, i.e., a sequence $(\alpha_n)_{n=0}^{\infty}$ of pseudo-actions on nonempty finite sets such that for all $\varepsilon > 0$ and $F \in [\Gamma]^{<\infty}$, all but finitely many of the pseudo-actions α_n are (ε, F)-faithful.

Sofic groups were introduced by Gromov [Gro99] as a common generalization of amenable and residually finite groups (the term "sofic" was coined somewhat later by Weiss [Wei00b]). In a major breakthrough, Bowen [Bow10] generalized the notion of entropy from amenable to all sofic groups. Bowen's work was further extended by Kerr and Li [KL11], who, in particular, introduced sofic entropy to the topological setting and proved the variational principle for actions of sofic groups. The presentation below is a slight modification of [Bow17, Section 7].

Let $\alpha \colon \Gamma \mathrel{\tilde{\curvearrowright}} V$ be a pseudo-action of Γ. For a function $f \colon V \to k$, define the map $\pi_f \colon V \to k^\Gamma$ via

$$\pi_f(v)(\gamma) \coloneqq f(\gamma \cdot v) \qquad \text{for all } v \in V \text{ and } \gamma \in \Gamma.$$

Note that if α is an *action*, then the map $\pi_f \colon V \to k^\Gamma$ is equivariant; in general, we have

$$(\gamma \cdot \pi_f(v))(\delta) = \pi_f(\gamma \cdot v)(\delta) \qquad \text{whenever } \gamma, \delta \in \Gamma \text{ and } v \text{ is } \{\gamma, \delta, \delta\gamma\}\text{-proper.}$$

Let $X \subseteq k^\Gamma$ be a subshift and suppose that the set V is finite. An (ε, F)**-approximate** X**-coloring** of α, where $\varepsilon > 0$ and $F \in [\Gamma]^{<\infty}$, is a function $f \colon V \to k$ such that

$$|\{v \in V \,:\, \pi_f(v)|F \in X_F\}| \geqslant (1 - \varepsilon)|V|.$$

The set of all (ε, F)-approximate X-colorings of α is denoted by $\mathbf{Col}_{\varepsilon,F}(X, \alpha)$. Let

$$h_{\varepsilon,F}(X, \alpha) \;\coloneqq\; \frac{\log_2 |\mathbf{Col}_{\varepsilon,F}(X, \alpha)|}{|V|}.$$

If $\mathbf{Col}_{\varepsilon,F}(X, \alpha) = \varnothing$, then, by definition, $h_{\varepsilon,F}(X, \alpha) \coloneqq -\infty$.

Now assume that Γ is sofic and let $\Sigma = (\alpha_n)_{n=0}^{\infty}$ be a sofic approximation to Γ. The (**topological**) **entropy** $h(X, \Sigma)$ of a nonempty subshift $X \subseteq k^\Gamma$ with respect to Σ is given by

$$h(X, \Sigma) \;\coloneqq\; \inf_{\varepsilon, F} \limsup_{n \to \infty} \; h_{\varepsilon,F}(X, \alpha_n),$$

where ε ranges over the positive reals and F—over the finite subsets of Γ. If Γ is amenable, then we have $h(X, \Sigma) = h(X)$ for any sofic approximation Σ [Bow12; KL13]. In general, however, the value of $h(X, \Sigma)$ may depend on Σ. Nevertheless, we have the following:

Proposition 4.2.3 ([KL17, Propositions 10.28 and 10.29])**.** *If Γ is sofic and Σ is a sofic approximation to Γ, then $h(k^\Gamma, \Sigma) = \log_2 k$ and $h(X, \Sigma) < \log_2 k$ for any subshift $X \subsetneq k^\Gamma$.*

Note that Proposition 4.2.2 is a special case of Proposition 4.2.3.

4.3 Main definitions and results

Width and pointwise width

For $\varphi \in [\Gamma \to k]^{<\infty} \setminus \{\varnothing\}$, let

$$\mathfrak{d}(U_\varphi) := 2^{-|\varphi|}.$$

Note that we have $0 < \mathfrak{d}(U) \leqslant 1/2$ for every basic open set U. The value $\mathfrak{d}(U)$ is preserved by the shift action and thus can be viewed as a shift-invariant alternative to the diameter $\operatorname{diam}(U)$. For a family $\mathcal{U}$ of basic open sets and a parameter $h \in [0; +\infty)$, let $\rho_h(\mathcal{U}) := \sum_{U \in \mathcal{U}} \mathfrak{d}(U)^h$ and define

$$\mathfrak{w}(\mathcal{U}) := \inf\{h \in [0; +\infty) \,:\, \rho_h(\mathcal{U}) \leqslant 1\}.$$

If the family $\mathcal{U}$ is finite and nonempty, then $\rho_h(\mathcal{U})$, viewed as a function of h, is continuous and strictly decreasing. Thus, $\mathfrak{w}(\mathcal{U})$ for such $\mathcal{U}$ is equal to the unique $h \in [0; +\infty)$ with $\rho_h(\mathcal{U}) = 1$.

A **cover** of a set $X \subseteq k^\Gamma$ is a family $\mathcal{U}$ of basic open sets such that $X \subseteq \bigcup \mathcal{U}$. The **width** of X, denoted $\mathfrak{w}(X)$, is defined via

$$\mathfrak{w}(X) := \inf\{\mathfrak{w}(\mathcal{U}) \,:\, \mathcal{U} \text{ is a cover of } X\}.$$

Notice the close analogy between this definition and that of $\dim_H(X)$ (see also Proposition 4.3.3(i)). We will frequently use the fact that, since the space k^Γ is compact, to determine $\mathfrak{w}(X)$ for a closed subset $X \subseteq k^\Gamma$ it is enough to only consider finite covers of X.

The **pointwise width** of a set $X \subseteq k^\Gamma$, denoted $\mathfrak{w}^*(X)$, is given by

$$\mathfrak{w}^*(X) := \inf_{x \in X} \min\{\mathfrak{w}(\overline{\Gamma \cdot x}),\, \mathfrak{w}(\overline{x \cdot \Gamma})\}.$$

Technically, we have $\mathfrak{w}^*(\varnothing) = +\infty$ (even though $\mathfrak{w}(\varnothing) = 0$).

Proposition 4.3.1. *The following statements are valid:*

(i) *If $Y \subseteq X \subseteq k^\Gamma$, then $\mathfrak{w}(Y) \leqslant \mathfrak{w}(X)$ and $\mathfrak{w}^*(Y) \geqslant \mathfrak{w}^*(X)$.*

(ii) *If $X \subseteq k^\Gamma$ is a nonempty subshift, then $\mathfrak{w}^*(X) \leqslant \mathfrak{w}(X)$.*

Proof. Part (i) is clear, and for part (ii), notice that for every point $x \in X$, we have $\overline{\Gamma \cdot x} \subseteq X$, hence $\mathfrak{w}^*(X) \leqslant \mathfrak{w}(\overline{\Gamma \cdot x}) \leqslant \mathfrak{w}(X)$. ∎

Proposition 4.3.2. *We have $\mathfrak{w}(k^\Gamma) = \log_2 k$ and $\mathfrak{w}(X) < \log_2 k$ for any closed set $X \subsetneq k^\Gamma$.*

Proof. Let ν denote the uniform probability measure on k and let $X \subseteq k^\Gamma$ be a closed set. Since the product measure ν^Γ on k^Γ is regular, we have

$$\begin{aligned}\nu^\Gamma(X) &= \inf\{\nu^\Gamma(U) \,:\, U \subseteq k^\Gamma \text{ is an open set with } U \supseteq X\} \\ &= \inf\{\nu^\Gamma(\textstyle\bigcup \mathcal{U}) \,:\, \mathcal{U} \text{ is a finite cover of } X\}.\end{aligned}$$

Since every finite family of basic open subsets of k^Γ admits a finite refinement consisting of pairwise disjoint basic open sets, we conclude that

$$\begin{aligned}\nu^\Gamma(X) &= \inf\{\textstyle\sum_{U\in\mathcal{U}} \nu^\Gamma(U) \,:\, \mathcal{U} \text{ is a finite cover of } X\}\\ &= \inf\{\textstyle\sum_{U\in\mathcal{U}} \mathfrak{d}(U)^{\log_2 k} \,:\, \mathcal{U} \text{ is a finite cover of } X\}\\ &= \inf\{\rho_{\log_2 k}(\mathcal{U}) \,:\, \mathcal{U} \text{ is a finite cover of } X\}. \end{aligned} \tag{4.3.1}$$

The desired conclusion now follows since $\nu^\Gamma(k^\Gamma) = 1$ and $\nu^\Gamma(X) < 1$ if $X \neq k^\Gamma$. ∎

The next proposition confirms the importance of width and pointwise width as notions of size:

Proposition 4.3.3. *If $X \subseteq k^\Gamma$ is a subshift, then:*

(i) *the Hausdorff dimension of X is at least $\mathfrak{w}(X)$;*

(ii) *for every set $F \in [\Gamma]^{<\infty} \setminus \{\varnothing\}$, we have $\log_2 |X_F|/|F| \geqslant \mathfrak{w}(X)$;*

(iii) *if Γ is amenable, then the entropy of X is a least $\mathfrak{w}(X)$;*

(iv) *if $\mathfrak{w}^*(X) > (1/2)\log_2 k$, then X is free;*

(v) *if $U \subseteq k^\Gamma$ is a shift-invariant open set and $\mathfrak{w}^*(X) > \mathfrak{w}(k^\Gamma \setminus U)$, then $X \subseteq U$.*

Proof. (i) Let $\mathbf{Ball}(x, r)$ denote the open ball of radius $r > 0$ centered at a point $x \in k^\Gamma$. If $n \in \mathbb{N}$ is such that $2^{-n-1} < r \leqslant 2^{-n}$, then $\mathbf{Ball}(x, r)$ is a basic open set with $\mathfrak{d}(\mathbf{Ball}(x, r)) = \mathrm{diam}(\mathbf{Ball}(x, r)) = 2^{-n-1}$, and the desired result follows.

(ii) The family $\{U_\varphi \,:\, \varphi \in X_F\}$ is a cover of X with $\mathfrak{w}(\{U_\varphi \,:\, \varphi \in X_F\}) = \log_2 |X_F|/|F|$.

(iii) Follows from (ii).

(iv) It is enough to prove that $\mathfrak{w}(\overline{x \cdot \Gamma}) \leqslant (1/2)\log_2 k$ for every point $x \in k^\Gamma$ with $\mathrm{St}_\Gamma(x) \neq \{\mathbf{1}\}$. To that end, suppose that $\mathbf{1} \neq \gamma \in \mathrm{St}_\Gamma(x)$. For each $i < k$, let $\varphi_i \colon \{\mathbf{1}, \gamma\} \to k$ be the map given by $\varphi_i(\mathbf{1}) = \varphi_i(\gamma) \coloneqq i$. Then $\{U_{\varphi_i} \,:\, i < k\}$ is a cover of $\overline{x \cdot \Gamma}$ and $\mathfrak{w}(\{U_{\varphi_i} \,:\, i < k\}) = (1/2)\log_2 k$.

(v) For any $x \in k^\Gamma \setminus U$, we have $\overline{\Gamma \cdot x} \subseteq k^\Gamma \setminus U$, and hence $\mathfrak{w}(\overline{\Gamma \cdot x}) \leqslant \mathfrak{w}(k^\Gamma \setminus U)$. ∎

Breadth

For a basic open set U and a parameter $h \in (0; +\infty)$, let

$$\sigma_h(U) \coloneqq \log_2 \mathfrak{d}(U) \cdot \log_2(1 - \mathfrak{d}(U)^h).$$

Note that both $\log_2 \mathfrak{d}(U)$ and $\log_2(1 - \mathfrak{d}(U)^h)$ are negative, so $\sigma_h(U) > 0$. It is often useful to keep in mind that, when $\mathfrak{d}(U)$ is small, we have

$$\sigma_h(U) \,\approx\, \log_2 e \cdot |\log_2 \mathfrak{d}(U)| \cdot \mathfrak{d}(U)^h. \tag{4.3.2}$$

If $U = U_\varphi$ for $\varphi \in [\Gamma \to k]^{<\infty} \setminus \{\varnothing\}$, then (4.3.2) can be rewritten as

$$\sigma_h(U) \,\approx\, \log_2 e \cdot |\varphi| \cdot 2^{-h|\varphi|}.$$

For large $|\varphi|$, the "main" term in the above expression is $2^{-h|\varphi|}$, which is equal to $\mathfrak{d}(U)^h$. In other words, it is usually safe to think of $\sigma_h(U)$ as "almost" equal to $\mathfrak{d}(U)^h$, modulo a small perturbation.

For a family $\mathcal{U}$ of basic open sets and $h \in (0; +\infty)$, let $\sigma_h(\mathcal{U}) \coloneqq \sum_{U \in \mathcal{U}} \sigma_h(U)$ and define

$$\mathfrak{b}(\mathcal{U}) \coloneqq \sup\{h \in (0; +\infty) \,:\, h + \sigma_h(\mathcal{U}) < \log_2 k\}. \tag{4.3.3}$$

The value $\sigma_h(\mathcal{U})$ is non-increasing as a function of h (we cannot say that it is *strictly* decreasing, but only because it may be infinite). Due to this fact, the expression $h + \sigma_h(\mathcal{U})$ appearing in (4.3.3) is not, in general, a monotone function of h. By definition, if $h + \sigma_h(\mathcal{U}) \geqslant \log_2 k$ for all $h \in (0; +\infty)$, then $\mathfrak{b}(\mathcal{U}) = 0$.

An **action-cover** of a set $W \subseteq k^\Gamma$ is a family $\mathcal{U}$ of basic open sets such that $W \subseteq \bigcup(\Gamma \cdot \mathcal{U})$, i.e., the translates of the sets in $\mathcal{U}$ cover W. The **breadth** of a set $X \subseteq k^\Gamma$, denoted $\mathfrak{b}(X)$, is given by

$$\mathfrak{b}(X) \coloneqq \sup\{\mathfrak{b}(\mathcal{U}) \,:\, \mathcal{U} \text{ is an action-cover of } k^\Gamma \setminus X\}.$$

In contrast to $\mathfrak{w}(X)$, to determine $\mathfrak{b}(X)$ for a subshift X we typically have to allow infinite families $\mathcal{U}$. As mentioned in the introduction, the notion of breadth is made useful by the fact that a lower bound on $\mathfrak{b}(X)$ can be witnessed by a *single* action-cover $\mathcal{U}$ of $k^\Gamma \setminus X$. On the other hand, obtaining upper bounds on $\mathfrak{b}(X)$ can be more difficult. Indeed, *a priori* it is not even obvious that $\mathfrak{b}(\varnothing) = 0$. (However, this statement is true and is part of our main result.)

The main result

At this point, after all the necessary definitions have been introduced, we restate our main result, for the reader's convenience:

Theorem 4.3.4. *Let $X \subseteq k^\Gamma$ be a subshift such that $\mathfrak{b}(X) > 0$. Then $\mathfrak{w}(X) \geqslant \mathfrak{b}(X)$; moreover, for any $h < \mathfrak{b}(X)$, there exists a nonempty subshift $X' \subseteq X$ with the following properties:*

(i) *the pointwise width of X' is at least h;*

(ii) *if Γ is sofic, then the entropy of X' with respect to any sofic approximation is at least h;*

(iii) *there exist an invariant probability measure μ on X' and a factor map*

$$\pi \colon ([0;1]^\Gamma, \lambda^\Gamma) \to (X', \mu).$$

With Theorem 4.3.4 in hand, it is easy to derive Theorem 4.1.3:

Proof. Fix a nonempty shift-invariant open set $U \subseteq k^\Gamma$ and let $h < \log_2 k$. Without loss of generality, we may assume that

$$h > \max\{(1/2)\log_2 k,\ \mathfrak{w}(k^\Gamma \setminus U)\}. \tag{4.3.4}$$

Since, trivially, $\mathfrak{b}(k^\Gamma) = \log_2 k$, Theorem 4.3.4 applied to k^Γ yields a nonempty subshift $X \subseteq k^\Gamma$ such that $\mathfrak{w}^*(X) \geqslant h$ and, if Γ is sofic, the entropy of X with respect to any sofic approximation is at least h. From

Proposition 4.3.3 and (4.3.4), it follows that X is free and $X \subseteq U$. Let $Y \subseteq X$ be an arbitrary minimal subshift. Since we also have $\mathfrak{w}^*(Y) \geqslant h$, Proposition 4.3.3 implies that the Hausdorff dimension and, if Γ is amenable, the entropy of Y are at least h. ∎

4.4 Proof of Theorem 4.3.4

For the purposes of the proof, we split Theorem 4.3.4 into two parts.

Lemma 4.4.1. *Let $X \subseteq k^\Gamma$ be a subshift such that $\mathfrak{b}(X) > 0$. Then, for any $h < \mathfrak{b}(X)$, there exists a subshift $X' \subseteq X$ such that $\mathfrak{b}(X') \geqslant h$ and $\mathfrak{w}^*(X') \geqslant h$.*

Note that Lemma 4.4.1 does not yet guarantee that $X' \neq \varnothing$ (as $\mathfrak{w}^*(\varnothing) = +\infty$). This is taken care of in Lemma 4.4.2:

Lemma 4.4.2. *Let $X \subseteq k^\Gamma$ be a subshift such that $\mathfrak{b}(X) > 0$. Then $X \neq \varnothing$; moreover,*

(i) *if Γ is sofic, then the entropy of X with respect to any sofic approximation is at least $\mathfrak{b}(X)$;*

(ii) *there exist an invariant probability measure μ on X and a factor map*

$$\pi \colon ([0;1]^\Gamma, \lambda^\Gamma) \to (X, \mu).$$

It is clear that Lemmas 4.4.1 and 4.4.2 combined yield Theorem 4.3.4. We prove Lemma 4.4.1 in §4.4.1 by constructing the required subshift X' explicitly. The proof of Lemma 4.4.2 crucially relies on the LLL. We briefly review the required version of the LLL in §4.4.2 and then prove Lemma 4.4.2 in §4.4.3.

4.4.1 Proof of Lemma 4.4.1

Claim 4.4.3. *Let $\mathcal{F}$ be a finite family of basic open sets with $\mathfrak{w}(\mathcal{F}) < h$. Then, for any $\varepsilon > 0$, there exist families $\mathcal{V}$ and $\mathcal{W}$ of basic open sets such that*

$$\sigma_h(\mathcal{V}) < \varepsilon \qquad \text{and} \qquad \sigma_h(\mathcal{W}) < \varepsilon,$$

and for all $x \in k^\Gamma \setminus \bigcup \mathcal{V}$ and $y \in k^\Gamma \setminus \bigcup \mathcal{W}$, we have

$$\Gamma \cdot x \not\subseteq \bigcup \mathcal{F} \qquad \text{and} \qquad y \cdot \Gamma \not\subseteq \bigcup \mathcal{F}.$$

Proof. Below we only describe the construction of the family $\mathcal{V}$, as the family $\mathcal{W}$ is built in virtually the same way, the only difference being the use of the right instead of the left shift action.

Let $\Phi \subset [\Gamma \to k]^{<\infty}$ be the (finite) set such that $\mathcal{F} = \{U_\varphi \,:\, \varphi \in \Phi\}$ and let $\varepsilon > 0$. Let $N \in \mathbb{N}$ be a large integer (to be chosen later). Since Γ is infinite, we can find N elements $\gamma_1, \ldots, \gamma_N \in \Gamma$ such that for all φ, $\psi \in \Phi$ and $1 \leqslant i < j \leqslant N$, we have $\mathrm{dom}(\varphi)\gamma_i \cap \mathrm{dom}(\psi)\gamma_j = \varnothing$. This implies that for all $U_1, \ldots, U_N \in \mathcal{F}$, the set $\bigcap_{i=1}^{N}(\gamma_i^{-1} \cdot U_i)$ is basic open with

$$\mathfrak{d}\left(\bigcap_{i=1}^{N}(\gamma_i^{-1} \cdot U_i)\right) = \prod_{i=1}^{N} \mathfrak{d}(U_i). \tag{4.4.1}$$

We claim that the family $\mathcal{V} \coloneqq \{\bigcap_{i=1}^{N}(\gamma_i^{-1} \cdot U_i) : U_1, \ldots, U_N \in \mathcal{F}\}$ is as desired.

Suppose that some $x \in k^\Gamma \setminus \bigcup \mathcal{V}$ satisfies $\Gamma \cdot x \subseteq \bigcup \mathcal{F}$. Then we can choose $U_i \in \mathcal{F}$ for each $1 \leqslant i \leqslant N$ so that $\gamma_i \cdot x \in U_i$. But this yields $x \in \bigcap_{i=1}^{N}(\gamma_i^{-1} \cdot U_i) \in \mathcal{V}$, which is a contradiction. Hence, it only remains to show that, if N is large enough, then $\sigma_h(\mathcal{V}) < \varepsilon$. To that end, consider an arbitrary sequence $U_1, \ldots,$ $U_N \in \mathcal{F}$. By (4.4.1), we have

$$\sigma_h\left(\bigcap_{i=1}^{N}(\gamma_i^{-1} \cdot U_i)\right) = \log_2 \prod_{i=1}^{N} \mathfrak{d}(U_i) \cdot \log_2\left(1 - \prod_{i=1}^{N} \mathfrak{d}(U_i)^h\right).$$

Let $c_1 \coloneqq \max\{|\log_2 \mathfrak{d}(U)| : U \in \mathcal{F}\}$ and $c_2 \coloneqq 2^h |\log_2(1 - 2^{-h})|$. (Note that the values c_1 and c_2 do not depend on N.) We have

$$\left|\log_2 \prod_{i=1}^{N} \mathfrak{d}(U_i)\right| = \left|\sum_{i=1}^{N} \log_2 \mathfrak{d}(U_i)\right| \leqslant c_1 N,$$

and, since $|\log_2(1-a)| \leqslant c_2 \cdot a$ for all $a \in [0; 2^{-h}]$, we also have

$$\log_2\left(1 - \prod_{i=1}^{N} \mathfrak{d}(U_i)^h\right) \leqslant c_2 \cdot \prod_{i=1}^{N} \mathfrak{d}(U_i)^h.$$

Therefore,

$$\sigma_h\left(\bigcap_{i=1}^{N}(\gamma_i^{-1} \cdot U_i)\right) \leqslant c_1 c_2 \cdot N \cdot \prod_{i=1}^{N} \mathfrak{d}(U_i)^h.$$

Since $\mathfrak{w}(\mathcal{F}) < h$, we have $\rho_h(\mathcal{F}) < 1$, and hence

$$\sigma_h(\mathcal{V}) \leqslant c_1 c_2 \cdot N \cdot \sum_{U_1,\ldots,U_N \in \mathcal{F}} \prod_{i=1}^{N} \mathfrak{d}(U_i)^h = c_1 c_2 \cdot N \cdot \rho_h(\mathcal{F})^N \xrightarrow[N \to \infty]{} 0. \qquad \blacksquare$$

Let $X \subseteq k^\Gamma$ be a subshift such that $\mathfrak{b}(X) > 0$ and let $h < \mathfrak{b}(X)$. We may assume that $h > 0$ and that there exists an action-cover $\mathcal{U}$ of $k^\Gamma \setminus X$ such that $h + \sigma_h(\mathcal{U}) < \log_2 k$. Let $\mathcal{F}_0, \mathcal{F}_1, \ldots$ be an arbitrary enumeration of all the finite families $\mathcal{F}$ of basic open sets satisfying $\mathfrak{w}(\mathcal{F}) < h$. For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ and $\mathcal{W}_n$ be the families given by Claim 4.4.3 applied to the family $\mathcal{F}_n$ with

$$\varepsilon_n \coloneqq \frac{\log_2 k - h - \sigma_h(\mathcal{U})}{2^{n+2}}.$$

Set $\mathcal{U}' \coloneqq \mathcal{U} \cup \bigcup_{n=0}^{\infty}(\mathcal{V}_n \cup \mathcal{W}_n)$. Then the subshift $X' \coloneqq k^\Gamma \setminus (\Gamma \cdot \mathcal{U}')$ is as desired. Indeed, since $\mathcal{U}$ is an action-cover of $k^\Gamma \setminus X$, we have $X' \subseteq k^\Gamma \setminus (\Gamma \cdot \mathcal{U}) \subseteq X$. Since

$$\begin{aligned} &h + \sigma_h(\mathcal{U}) + \sum_{n=0}^{\infty}(\sigma_h(\mathcal{V}_n) + \sigma_h(\mathcal{W}_n)) \\ &< h + \sigma_h(\mathcal{U}) + \sum_{n=0}^{\infty} \frac{\log_2 k - h - \sigma_h(\mathcal{U})}{2^{n+1}} = \log_2 k, \end{aligned}$$

we conclude that $\mathfrak{b}(X') \geqslant h$. Finally, if $x \in k^\Gamma$ satisfies $\mathfrak{w}(\overline{\Gamma \cdot x}) < h$ or $\mathfrak{w}(\overline{x \cdot \Gamma}) < h$, then there exists an index $n \in \mathbb{N}$ such that $\Gamma \cdot x \subseteq \bigcup \mathcal{F}_n$ or $x \cdot \Gamma \subseteq \bigcup \mathcal{F}_n$. By the choice of $\mathcal{V}_n$ and $\mathcal{W}_n$, such x cannot belong to X', and hence $\mathfrak{w}^*(X') \geqslant h$. The proof of Lemma 4.4.1 is complete.

4.4.2 The Lovász Local Lemma

For the full statement of the LLL, see Theorem 1.1.1 or [AS00, Lemma 5.1.1]. We will apply the LLL in the framework similar to that described in Chapter 3, which we briefly review below for the reader's convenience.

Let X be a set and let $\Phi \subseteq [X \to k]^{<\infty}$. Let **Forb**$(\Phi)$ denote the set of all maps $f \colon X \to k$ such that $f \not\supseteq \varphi$ for all $\varphi \in \Phi$. For each $\varphi \in \Phi$, let

$$N(\varphi, \Phi) \coloneqq \{\psi \in \Phi \,:\, \mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi) \neq \varnothing\}.$$

We say that Φ is **correct** (**for the LLL**) if there is a function $\omega \colon \Phi \to [0; 1)$ such that

$$k^{-|\varphi|} \leqslant \omega(\varphi) \prod_{\psi \in N(\varphi,\Phi)} (1 - \omega(\psi)) \qquad \text{for all } \varphi \in \Phi.$$

In this case ω is called a **witness** to the correctness of Φ.

Lemma 4.4.4 (**Lovász Local Lemma**). *Let X be a set and let $\Phi \subseteq [X \to k]^{<\infty}$. If the set Φ is correct, then* **Forb**$(\Phi) \neq \varnothing$. *Furthermore, if X is finite and $\omega \colon \Phi \to [0; 1)$ is a witness to the correctness of Φ, then*

$$|\mathbf{Forb}(\Phi)| \geqslant k^{|X|} \prod_{\varphi \in \Phi} (1 - \omega(\varphi)).$$

For finite X, deducing Lemma 4.4.4 from Theorem 1.1.1 or [AS00, Lemma 5.1.1] is routine (see, e.g., [MR02, p. 41]). The infinite case is derived from the finite one via a straightforward compactness argument. (Cf. Theorems 3.1.7 and 3.1.10.)

In order to establish (ii), we will use one of the measurable versions of the LLL proved in Chapter 3. Let $\Phi \subseteq [\Gamma \to k]^{<\infty}$. Then we have **Forb**$(\Phi) = k^\Gamma \setminus \bigcup_{\varphi \in \Phi} U_\varphi$. In particular, if the set Φ is shift-invariant, then **Forb**(Φ) is a subshift. Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ be a measure-preserving action of Γ on a probability space (X, μ). Given a shift-invariant set $\Phi \subseteq [\Gamma \to k]^{<\infty}$, a **measurable solution** to Φ over α is a measurable function $f \colon X \to k$ such that for μ-almost all $x \in X$, the map

$$\pi_f(x) \colon \Gamma \to k : \gamma \mapsto f(\gamma \cdot x)$$

belongs to **Forb**(Φ).

Theorem 4.4.5 (see Corollary 3.5.7)**.** *Let $\Phi \subseteq [\Gamma \to k]^{<\infty}$ be a correct shift-invariant set. Then the shift action $\Gamma \curvearrowright ([0;1]^\Gamma, \lambda^\Gamma)$ admits a measurable solution to Φ.*

Corollary 4.4.6. *Let $\Phi \subseteq [\Gamma \to k]^{<\infty}$ be a correct shift-invariant set. Then there exist an invariant probability measure μ on* **Forb**(Φ) *and a factor map*

$$\pi \colon ([0;1]^\Gamma, \lambda^\Gamma) \to (\mathbf{Forb}(\Phi), \mu).$$

Proof. Let $f \colon [0;1]^\Gamma \to k$ be a measurable solution to Φ given by Theorem 4.4.5. We may then take $\pi \coloneqq \pi_f$ and $\mu \coloneqq (\pi_f)_*(\lambda^\Gamma)$. ∎

4.4.3 Proof of Lemma 4.4.2

Claim 4.4.7. *Let $\mathcal{U}$ be a family of basic open sets and let $h \in (0;+\infty)$ be such that*

$$h + \sigma_h(\mathcal{U}) < \log_2 k. \tag{4.4.2}$$

Let $\Phi \subseteq [\Gamma \to k]^{<\infty}$ be the set such that $\mathcal{U} = \{U_\varphi \,:\, \varphi \in \Phi\}$. For each $\varphi \in \Gamma \cdot \Phi$, define $\omega(\varphi) \coloneqq 2^{-h|\varphi|}$. Then ω is a witness to the correctness of $\Gamma \cdot \Phi$.

Proof. Since ω is invariant under the shift action $\Gamma \curvearrowright \Gamma \cdot \Phi$, we only have to verify that

$$k^{-|\varphi|} \leqslant \omega(\varphi) \prod_{\psi \in N(\varphi, \Gamma\cdot\Phi)} (1 - \omega(\psi)) \qquad \text{for all } \varphi \in \Phi.$$

Let $\varphi \in \Phi$. By definition, $N(\varphi, \Gamma \cdot \Phi)$ is the set of all products of the form $\delta \cdot \psi$, where $\psi \in \Phi$ and $\delta \in \Gamma$, with the property that $\mathrm{dom}(\varphi) \cap \mathrm{dom}(\delta \cdot \psi) \neq \varnothing$. This is equivalent to $\delta \in \mathrm{dom}(\varphi)^{-1}\mathrm{dom}(\psi)$, so, for each choice of $\psi \in \Phi$, there are at most $|\mathrm{dom}(\varphi)^{-1}\mathrm{dom}(\psi)| \leqslant |\varphi||\psi|$ possible choices for $\delta \in \Gamma$. Using this observation together with the shift-invariance of ω, we obtain

$$\omega(\varphi) \prod_{\psi \in N(\varphi, \Gamma\cdot\Phi)} (1 - \omega(\psi)) \geqslant \omega(\varphi) \prod_{\psi \in \Phi} (1 - \omega(\psi))^{|\varphi||\psi|}.$$

It remains to show that

$$k^{-|\varphi|} \leqslant \omega(\varphi) \prod_{\psi \in \Phi} (1 - \omega(\psi))^{|\varphi||\psi|}. \tag{4.4.3}$$

Plugging the definition of ω into (4.4.3), we get

$$k^{-|\varphi|} \leqslant 2^{-h|\varphi|} \prod_{\psi \in \Phi} (1 - 2^{-h|\psi|})^{|\varphi||\psi|},$$

which is equivalent to

$$k^{-1} \leqslant 2^{-h} \prod_{\psi \in \Phi} (1 - 2^{-h|\psi|})^{|\psi|}.$$

Taking the logarithm on both sides turns the last inequality into

$$\log_2 k \;\geqslant\; h - \sum_{\psi \in \Phi} |\psi| \cdot \log_2(1 - 2^{-h|\psi|}).$$

But $|\psi| = -\log_2(\mathfrak{d}(U_\psi))$ and $2^{-h|\psi|} = \mathfrak{d}(U_\psi)^h$, so

$$-\sum_{\psi \in \Phi} |\psi| \cdot \log_2(1 - 2^{-h|\psi|}) \;=\; \sum_{U \in \mathcal{U}} \log_2(\mathfrak{d}(U)) \cdot \log_2(1 - \mathfrak{d}(U)^h) \;=\; \sigma_h(\mathcal{U}),$$

and we are done by (4.4.2). ∎

Let $X \subseteq k^\Gamma$ be a subshift with $\mathfrak{b}(X) > 0$ and consider any action-cover $\mathcal{U}$ of $k^\Gamma \setminus X$ with $\mathfrak{b}(\mathcal{U}) > 0$. Let $h \in (0; +\infty)$ be such that $h + \sigma_h(\mathcal{U}) < \log_2 k$. Note that $\mathcal{U}$ and h can be chosen so that h is as close to $\mathfrak{b}(X)$ as desired. Let $\Phi \subseteq [\Gamma \to k]^{<\infty}$ be the set such that $\mathcal{U} = \{U_\varphi \,:\, \varphi \in \Phi\}$. According to Claim 4.4.7, the set $\Gamma \cdot \Phi$ is correct for the LLL. Lemma 4.4.4 then implies that $\mathbf{Forb}(\Gamma \cdot \Phi) \neq \varnothing$; furthermore, according to Corollary 4.4.6, there exist an invariant probability measure μ on $\mathbf{Forb}(\Gamma \cdot \Phi)$ and a factor map $\pi \colon ([0;1]^\Gamma, \lambda^\Gamma) \to (\mathbf{Forb}(\Gamma \cdot \Phi), \mu)$. Since $\mathbf{Forb}(\Gamma \cdot \Phi) = k^\Gamma \setminus \bigcup(\Gamma \cdot \mathcal{U}) \subseteq X$, we conclude that $X \neq \varnothing$ and part (ii) of Lemma 4.4.2 holds.

It remains to verify that if Γ is sofic, then the entropy of X with respect to any sofic approximation is at least $\mathfrak{b}(X)$. In fact, we will show that the entropy of $\mathbf{Forb}(\Gamma \cdot \Phi)$ is at least h, which will yield the desired result as $\mathbf{Forb}(\Gamma \cdot \Phi) \subseteq X$ and h can be made arbitrarily close to $\mathfrak{b}(X)$. The idea is simple: Given a pseudo-action $\alpha \colon \Gamma \mathrel{\tilde{\curvearrowright}} V$ on a finite set V, we "copy" $\Gamma \cdot \Phi$ over to V and build a set $\Phi_\alpha \subseteq [V \to k]^{<\infty}$ such that every map in $\mathbf{Forb}(\Phi_\alpha)$ is an approximate $(\Gamma \cdot \Phi)$-coloring of α; then we apply the LLL to obtain a lower bound on $|\mathbf{Forb}(\Phi_\alpha)|$. In the remainder of the proof, we work out the technical details of this approach.

Let $\varepsilon > 0$ and let $F \in [\Gamma]^{<\infty} \setminus \{\varnothing\}$. Recall that for a subshift $Y \subseteq k^\Gamma$, the set Y_F is defined by

$$Y_F \coloneqq \{\varphi \in k^F \,:\, Y \cap U_\varphi \neq \varnothing\} = \{y|F \,:\, y \in Y\}.$$

By compactness, we can find a finite set $S \in [\Gamma]^{<\infty}$ such that

$$\mathbf{Forb}(\Gamma \cdot \Phi)_F \;=\; \mathbf{Forb}(S \cdot (\Phi \cap [S \to k]^{<\infty}))_F. \tag{4.4.4}$$

We may assume that the set S is symmetric and contains $\mathbf{1}$. For each $n \in \mathbb{N}$, let

$$S^n \coloneqq \{\gamma_1 \cdots \gamma_n \,:\, \gamma_1, \ldots, \gamma_n \in S\}.$$

Let $\alpha \colon \Gamma \mathrel{\tilde{\curvearrowright}} V$ be an (ε, S^4)-faithful pseudo-action of Γ on a finite set V. We will show that

$$h_{\varepsilon,F}(\mathbf{Forb}(\Gamma \cdot \Phi), \alpha) \geqslant h.$$

For each $\varphi \in [\Gamma \to k]^{<\infty}$ and $v \in \mathrm{Prop}_{\mathrm{dom}(\varphi)}(\alpha)$, define the map $\varphi_v \in [V \to k]^{<\infty}$ by

$$\mathrm{dom}(\varphi_v) \coloneqq \mathrm{dom}(\varphi) \cdot v \qquad \text{and} \qquad \varphi_v(\gamma \cdot v) \coloneqq \varphi(\gamma) \text{ for all } \gamma \in \mathrm{dom}(\varphi),$$

and let

$$\Phi_\alpha \coloneqq \{\varphi_v \,:\, \varphi \in \Phi \cap [S \to k]^{<\infty},\ v \in \mathrm{Prop}_{S^3}(\alpha)\}.$$

Claim 4.4.8. *We have* $\mathbf{Forb}(\Phi_\alpha) \subseteq \mathbf{Col}_{\varepsilon,F}(\mathbf{Forb}(\Gamma \cdot \Phi), \alpha)$.

Proof. Let $f \in \mathbf{Forb}(\Phi_\alpha)$. Note that for any $v \in \mathrm{Prop}_{S^4}(\alpha)$, we have $S \cdot v \subseteq \mathrm{Prop}_{S^3}(\alpha)$, and therefore $\pi_f(v) \in \mathbf{Forb}(S \cdot (\Phi \cap [S \to k]^{<\infty}))$. From (4.4.4) we conclude

$$|\{v \in V \,:\, \pi_f(v)|F \in \mathbf{Forb}(\Gamma \cdot \Phi)_F\}| \geqslant |\mathrm{Prop}_{S^4}(\alpha)| \geqslant (1-\varepsilon)|V|. \qquad \blacksquare$$

Recall that, according to Claim 4.4.7, the map

$$\omega \colon \Gamma \cdot \Phi \to [0;1) \colon \varphi \mapsto 2^{-h|\varphi|}$$

is a witness to the correctness of $\Gamma \cdot \Phi$. Define

$$\omega_\alpha \colon \Phi_\alpha \to [0;1) \colon \psi \mapsto 2^{-h|\psi|}.$$

Claim 4.4.9. *The map ω_α is a witness to the correctness of Φ_α.*

Proof. Consider any $v \in \mathrm{Prop}_{S^3}(\alpha)$ and $\varphi \in \Phi \cap [S \to k]^{<\infty}$. We will define an injective map

$$\iota \colon N(\varphi_v, \Phi_\alpha) \to N(\varphi, \Gamma \cdot \Phi)$$

such that for all $\psi \in N(\varphi_v, \Phi_\alpha)$, we have $|\psi| = |\iota(\psi)|$. Since then we also have $\omega_\alpha(\psi) = \omega(\iota(\psi))$, the desired conclusion follows by Claim 4.4.7.

Suppose that $\psi_u \in N(\varphi_v, \Phi_\alpha)$ for some $u \in \mathrm{Prop}_{S^3}(\alpha)$ and $\psi \in \Phi \cap [S \to k]^{<\infty}$. Choose arbitrary $\gamma \in \mathrm{dom}(\varphi)$ and $\delta \in \mathrm{dom}(\psi)$ such that $\gamma \cdot v = \delta \cdot u$ and define

$$\iota(\psi_u) \coloneqq (\gamma^{-1}\delta) \cdot \psi.$$

Clearly, $\iota(\psi_u) \in N(\varphi, \Gamma \cdot \Phi)$ since $\gamma \in \mathrm{dom}(\iota(\psi_u))$. Also, we have $|\psi_u| = |\psi| = |\iota(\psi_u)|$. Finally, the map ι is injective, since it is invertible: $\psi_u = (\iota(\psi_u))_v$. Indeed, as v and u are both S^3-proper, for every $\zeta \in \mathrm{dom}(\psi)$, we have

$$(\zeta\delta^{-1}\gamma) \cdot v = \zeta \cdot (\delta^{-1} \cdot (\gamma \cdot v)) = \zeta \cdot (\delta^{-1} \cdot (\delta \cdot u)) = \zeta \cdot u,$$

so $\psi_u = (\iota(\psi_u))_v$, as claimed. $\blacksquare$

From Claim 4.4.9 and Lemma 4.4.4, we obtain

$$|\mathbf{Forb}(\Phi_\alpha)| \geqslant k^{|V|} \prod_{\psi \in \Phi_\alpha} (1 - \omega_\alpha(\psi)) \geqslant k^{|V|} \prod_{\varphi \in \Phi \cap [S \to k]^{<\infty}} \prod_{v \in \mathrm{Prop}_{S^3}(\alpha)} (1 - \omega_\alpha(\varphi_v))$$
$$\geqslant k^{|V|} \prod_{\varphi \in \Phi} (1 - 2^{-h|\varphi|})^{|V|}.$$

Therefore, by Claim 4.4.8,

$$h_{\varepsilon,F}(\mathbf{Forb}(\Gamma \cdot \Phi), \alpha) = \frac{\log_2 |\mathbf{Col}_{\varepsilon,F}(\mathbf{Forb}(\Gamma \cdot \Phi), \alpha)|}{|V|}$$
$$\geqslant \frac{\log_2 |\mathbf{Forb}(\Phi_\alpha)|}{|V|} \geqslant \log_2 k + \sum_{\varphi \in \Phi} \log_2(1 - 2^{-h|\varphi|}).$$

But $2^{-h|\varphi|} = \mathfrak{d}(U_\varphi)^h$ and $-\log_2(\mathfrak{d}(U_\varphi)) = |\varphi| \geqslant 1$, so

$$\log_2 k + \sum_{\varphi \in \Phi} \log_2(1 - 2^{-h|\varphi|}) = \log_2 k + \sum_{U \in \mathcal{U}} \log_2(1 - \mathfrak{d}(U)^h)$$
$$\geqslant \log_2 k - \sum_{U \in \mathcal{U}} \log_2(\mathfrak{d}(U)) \cdot \log_2(1 - \mathfrak{d}(U)^h)$$
$$= \log_2 k - \sigma_h(\mathcal{U}) > h,$$

as desired.

5 | Baire measurable colorings of group actions

5.1 Introduction

In Chapter 3 we encountered a family of results in descriptive combinatorics that follow by adapting techniques from finite combinatorics. Such examples are numerous: in fact, most upper bounds in descriptive combinatorics are, in one way or another, based on a known theorem or a method that works in the finite setting. For example, in their seminal paper [KST99], Kechris, Solecki, and Todorcevic established [KST99, Proposition 4.6] that a Borel graph G with finite maximum degree d admits a Borel proper coloring using at most $d + 1$ colors. For a *finite* graph G, such a coloring can be found "greedily": One simply considers the vertices of G one by one and assigns to each vertex the first color not yet used on any of its neighbors; since the total number of colors exceeds the maximum number of neighbors a vertex can have, there is always at least one color available. In their proof of [KST99, Proposition 4.6], Kechris, Solecki, and Todorcevic devised a Borel analog of this "greedy" algorithm.

Some techniques in finite combinatorics are more amenable to descriptive generalizations (more *constructive*, one could say) than others. For instance, of the ways of obtaining matchings in graphs, the arguments based on augmenting paths appear to be especially well-suited for the purposes of descriptive combinatorics (see, e.g., [EL10; LN11]). In Chapter 3 we studied measurable versions of the Lovász Local Lemma, and our arguments relied crucially on the algorithmic approach to the Local Lemma that was developed by Moser and Tardos in the finite setting [MT10].

The above examples suggest that some precise correspondences between results in finite and descriptive combinatorics might still be present; the existence of a well-behaved coloring of a certain kind could, perhaps, be equivalent to a purely combinatorial statement such as the existence of a "greedy"-like algorithm to find it. One of the main results of this chapter is Theorem 5.2.10, which confirms this suspicion for a particular class of coloring problems and a specific notion of well-behavedness, namely Baire measurability (see Definition 5.2.2).

An ample supply of examples in descriptive combinatorics is provided by actions of countable groups, and that is the convenient framework in which we perform our investigation. This chapter therefore can be considered a contribution to *generic dynamics*; see [Wei00a; SW15] for an introduction to the subject.

This chapter is based on [Ber17a].

Basics of Baire category

Recall that a separable topological space is **Polish** if its topology is generated by a complete metric. Note that a compact space is Polish if and only if it is metrizable. Most notions related to Baire category make sense for a wider class of topological spaces (the so-called *Baire spaces*); however, to simplify the matters, we will only talk about Polish spaces here. A subset of a Polish space is **meager** if it can be covered by countably many nowhere dense sets; **nonmeager** if it is not meager; and **comeager** if its complement is meager. We say that two sets A and B are **equal modulo a meager set**, or **$*$-equal**, in symbols $A =^* B$, if their symmetric difference $A \triangle B$ is meager. A set is **Baire measurable** if it is $*$-equal to an open set.[1] The meager sets form a σ-ideal (i.e., meagerness is a notion of smallness), and the Baire measurable sets form a σ-algebra, which contains all Borel sets (and much more). The cornerstone result of the Baire category theory is the **Baire category theorem**, which asserts that a nonempty open subset of a Polish space is nonmeager; equivalently, the intersection of countably many dense open subsets of a Polish space is dense. For a Baire measurable set A and a nonempty open set U, we say that U **forces** A, or A is **comeager in** U, in symbols $U \Vdash A$, if the difference $U \setminus A$ is meager. The following way of phrasing the Baire category theorem is rather useful:

Proposition 5.1.1 (**Baire alternative** [Tse16, Proposition 9.8])**.** *A Baire measurable subset of a Polish space is either meager, or else, comeager in some nonempty open set.*

A function $f\colon X \to Y$ from a Polish space X to a standard Borel space Y is **Baire measurable** if for all Borel $B \subseteq Y$, the preimage $f^{-1}(B)$ is Baire measurable (as a subset of X). For more background on Baire category, see [Kec95, Section 8] and [Tse16, Sections 6 and 9].

5.2 Main definitions and statements of results

5.2.1 Groups, group actions, and their colorings

Throughout this chapter, Γ denotes a countably infinite discrete group with identity element $\mathbf{1}$. We fix an arbitrary proper[2] right-invariant metric dist on Γ. Note that such a metric always exists. Indeed, if Γ is finitely generated, then dist could be the word metric with respect to any finite generating set; in general, one can take

$$\mathrm{dist}(\gamma, \delta) \coloneqq \min\{i_1 + \ldots + i_k \,:\, \varepsilon_{i_1}^{\pm 1} \cdots \varepsilon_{i_k}^{\pm 1}\gamma = \delta\},$$

where $\{\varepsilon_1, \varepsilon_2, \ldots\} = \Gamma$ is an enumeration of the elements of Γ in some order. Any two proper right-invariant metrics on Γ are coarsely equivalent, so the specific choice of the metric will be irrelevant for our purposes. We use $\mathbf{Ball}(\gamma, r)$ to denote the (closed) ball of radius $r \in [0; +\infty)$ around $\gamma \in \Gamma$. For $S \subseteq \Gamma$, let $\mathbf{Ball}(S, r) \coloneqq \bigcup_{\gamma \in S} \mathbf{Ball}(\gamma, r)$. For $S, T \subseteq \Gamma$, define

$$\mathrm{dist}(S, T) \coloneqq \inf\{\mathrm{dist}(\gamma, \delta) \,:\, \gamma \in S,\ \delta \in T\}.$$

[1] Sometimes meager sets are referred to as sets **of first category**; nonmeager—as **of second category**; comeager—as **residual**; and Baire measurable—as having the **property of Baire**.

[2] Recall that a metric space is **proper** if every closed and bounded subset of it is compact. For discrete spaces, this is equivalent to saying that every ball of finite radius is a finite set.

Note that if $\alpha \colon \Gamma \curvearrowright X$ is a continuous action of Γ on a Polish space X and $A \subseteq X$ is comeager, then there is a further comeager subset $A' \subseteq A$ that is α-invariant, namely $A' \coloneqq \bigcap_{\gamma \in \Gamma}(\gamma \cdot A)$.

To discourse about colorings we need to fix a set of "colors"; for concreteness, we will use the discrete space $\mathbb{N}$ in that role (although sometimes it might be more convenient to use a different countable discrete space instead; for instance, we use $\mathbb{N} \times \mathbb{N}$ in the proof of Lemma 5.5.2). By a **coloring** of a set S we simply mean a map $\omega \colon S \to \mathbb{N}$. A combinatorial coloring problem over Γ is meant to specify which colorings of Γ are considered "nice" or "acceptable." We identify such coloring problems with *subshifts*:

Definition 5.2.1. A **subshift** is a subset of $\mathbb{N}^\Gamma$ that is closed (in the product topology) and invariant under the shift action. The set of all subshifts is denoted by $\mathbf{Sh}_0(\Gamma, \mathbb{N})$, and the set of all nonempty subshifts is denoted by $\mathbf{Sh}(\Gamma, \mathbb{N})$.

Remark. Note that the notion of a subshift given by Definition 5.2.1 is different from the one used in Chapter 4. In particular, for the purposes of this chapter, a subshift need not be compact.

Let $\alpha \colon \Gamma \curvearrowright X$ be an action of Γ on a set X. Each coloring of X then gives rise to a family of colorings of Γ parameterized by the elements of X. Specifically, given $f \colon X \to \mathbb{N}$ and $x \in X$, we define $\pi_f(x) \colon \Gamma \to \mathbb{N}$ by

$$\pi_f(x)(\gamma) \coloneqq f(\gamma \cdot x).$$

It is clear that the map $\pi_f \colon X \to \mathbb{N}^\Gamma$ is equivariant. Conversely, for each equivariant function $\pi \colon X \to \mathbb{N}^\Gamma$, there is a unique coloring $f \colon X \to \mathbb{N}$ such that $\pi = \pi_f$, namely the one given by $f(x) \coloneqq \pi(x)(\mathbf{1})$ for all $x \in X$. The map π_f is called the **symbolic representation**, or the **coding map**, for the dynamical system (X, Γ, α, f).

The following definition identifies our main objects of study:

Definition 5.2.2. Let $\alpha \colon \Gamma \curvearrowright X$ be a continuous action of Γ on a Polish space X. Given a subshift $\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$, a **Baire measurable Ω-coloring** of α (or of X, if α is clear from the context) is a Baire measurable function $f \colon X \to \mathbb{N}$ such that the preimage of Ω under π_f is comeager. The set of all $\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$ such that α admits a Baire measurable Ω-coloring is denoted by $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$.

Remark. Clearly, $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N}) \subseteq \mathbf{Sh}(\Gamma, \mathbb{N})$, unless $X = \varnothing$.

Remark. In view of the bijective correspondence $f \longleftrightarrow \pi_f$ between colorings and equivariant functions, Definition 5.2.2 can be equivalently restated in purely dynamical terms as follows:

> A continuous action $\alpha \colon \Gamma \curvearrowright X$ admits a Baire measurable Ω-coloring if and only if there exists a Baire measurable map $\pi \colon X \to \Omega$ which is equivariant on a comeager set.

5.2.2 Example: proper graph colorings

Even though our framework concerns groups and group actions rather than graphs, there is a standard way of associating a graph to a (finitely generated) group and to each of its free actions (and we have already used it in Chapter 3). Namely, assume that Γ is generated by a finite symmetric subset S with $\mathbf{1} \notin S$. For a free continuous action $\alpha\colon \Gamma \curvearrowright X$ on a Polish space X, let $G(\alpha, S)$ denote the graph **induced** by α, i.e., the graph with vertex set X and edge set

$$\{(x, \delta \cdot x) \,:\, x \in X \text{ and } \delta \in S\}.$$

Since α is free, every connected component of $G(\alpha, S)$ is isomorphic to the Cayley graph $\mathrm{Cay}(\Gamma, S)$; specifically, for $x \in X$, the map $\gamma \mapsto \gamma \cdot x$ is an isomorphism from $\mathrm{Cay}(\Gamma, S)$ onto the connected component of $G(\alpha, S)$ containing x (which coincides with the α-orbit of x).

For $k \in \mathbb{N}$, let $\mathbf{PrCol}(k, S)$ denote the set of all **proper k-colorings** of $\mathrm{Cay}(\Gamma, S)$, i.e., all functions $\omega\colon \Gamma \to k$ such that $\omega(\gamma) \neq \omega(\delta\gamma)$ whenever $\gamma \in \Gamma$ and $\delta \in S$. It is clear that $\mathbf{PrCol}(k, S)$ is a subshift. The smallest k such that $\mathbf{PrCol}(k, S) \neq \varnothing$ is called the **chromatic number** of $\mathrm{Cay}(\Gamma, S)$ and is denoted by $\chi(\mathrm{Cay}(\Gamma, S))$, or simply $\chi(\Gamma, S)$.

Since every vertex in $\mathrm{Cay}(\Gamma, S)$ has exactly $|S|$ neighbors, it is immediate that

$$\chi(\Gamma, S) \leqslant |S| + 1.$$

Furthermore, by Brooks's theorem [Die00, Theorem 5.2.4], we have

$$\chi(\Gamma, S) \leqslant |S| \quad \text{for} \quad |S| \geqslant 2.$$

For a free continuous action $\alpha\colon \Gamma \curvearrowright X$ on a Polish space X, the smallest k such that $\mathbf{PrCol}(k, S) \in \mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$ is called the **Baire measurable chromatic number** of $G(\alpha, S)$ and is denoted by $\chi_{\mathrm{BM}}(G(\alpha, S))$, or simply $\chi_{\mathrm{BM}}(\alpha, S)$. Clearly,

$$\chi_{\mathrm{BM}}(\alpha, S) \geqslant \chi(\Gamma, S) \quad \text{for} \quad X \neq \varnothing.$$

A somewhat surprising result of Conley and Miller [CM16, Theorem B] implies that $\chi_{\mathrm{BM}}(\alpha, S)$ is also *upper* bounded by a function of $\chi(\Gamma, S)$; more precisely,

$$\chi_{\mathrm{BM}}(\alpha, S) \leqslant 2\chi(\Gamma, S) - 1. \tag{5.2.1}$$

Another important result concerning Baire measurable chromatic numbers is a (Baire) measurable version of Brooks's theorem due to Conley, Marks, and Tucker-Drob [CMT16, Theorem 1.2(2)], which implies that, similarly to the situation with ordinary chromatic numbers,

$$\chi_{\mathrm{BM}}(\alpha, S) \leqslant |S| \quad \text{for} \quad |S| \geqslant 2.$$

5.2.3 A completeness result

The aim of this chapter is to make progress towards the understanding of the structure of the sets $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$. The first natural question to ask is, how complex, in descriptive set-theoretic terms, is $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$, as a subset of $\mathbf{Sh}_0(\Gamma, \mathbb{N})$?

First, we have to make $\mathbf{Sh}_0(\Gamma, \mathbb{N})$ a Polish or, at least, a standard Borel space. It is straightforward to check that $\mathbf{Sh}_0(\Gamma, \mathbb{N})$ is a Borel subset of the *Effros standard Borel space* $\mathcal{F}(\mathbb{N}^\Gamma)$ and as such is itself standard Borel (for more details on the Effros space see [Kec95, §12.C] and [Tse16, §13.D]). Furthermore, in §5.3 we put a natural Polish topology on $\mathbf{Sh}_0(\Gamma, \mathbb{N})$ (which results in the same Borel σ-algebra).

Let $\alpha \colon \Gamma \curvearrowright X$ be a free continuous action of Γ on a nonempty Polish space X. We say that α is **generically smooth** if there is a Baire measurable map $f \colon X \to \mathbb{R}$ such that for all $x, y \in X$,

$$f(x) = f(y) \iff y = \gamma \cdot x \text{ for some } \gamma \in \Gamma.$$

For smooth actions, descriptive and finite combinatorics essentially coincide; in particular, it is easy to show that if α is generically smooth, then $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N}) = \mathbf{Sh}(\Gamma, \mathbb{N})$ (see Lemma 5.4.7). In other words, from the point of view of descriptive combinatorics, smooth actions are trivial and it is only interesting to consider non-smooth ones.

We show that in the non-smooth case, $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$ is a complete analytic subset of $\mathbf{Sh}_0(\Gamma, \mathbb{N})$; in particular, it is not Borel. Informally, this means that there is no hope for an "explicit" description of the subshifts Ω for which a given non-smooth action α admits a Baire measurable Ω-coloring.

Theorem 5.2.3. *Let α be a free continuous action of Γ on a nonempty Polish space. Then*

– *either α is generically smooth, in which case $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N}) = \mathbf{Sh}(\Gamma, \mathbb{N})$;*

– *or else, the set $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$ is complete analytic.*

We prove Theorem 5.2.3 in §5.4. En route to proving Theorem 5.2.3, we show that the set of all Baire measurable maps between two Polish spaces, taken modulo the equivalence relation of equality on a comeager set, can be naturally turned into a standard Borel space (see §5.4.1); this construction appears to be new and of independent interest.

5.2.4 A combinatorial characterization of $\mathbf{Sh}_{\mathrm{BM}}(\sigma, \mathbb{N})$

The following result was first established by Keane for the 2- and the 3-shift and subsequently generalized by Weiss [Wei00a]:

Theorem 5.2.4 (Keane–Weiss [Wei00a, Theorem 2])**.** *Let X, Y be Polish spaces of cardinality at least 2. Then the shift actions $\sigma_X \colon \Gamma \curvearrowright X^\Gamma$ and $\sigma_Y \colon \Gamma \curvearrowright Y^\Gamma$ are generically isomorphic; i.e., there exist comeager shift-invariant subsets $X' \subseteq X^\Gamma$, $Y' \subseteq Y^\Gamma$ with an equivariant homeomorphism $\pi \colon X' \to Y'$ between them.*

Remark. Theorem 2 in [Wei00a] is stated for zero-dimensional spaces only. The result for general Polish spaces follows since every Polish space contains a comeager zero-dimensional subspace.

Theorem 5.2.4 allows us to refer, when meager sets may be ignored, to *the* shift action σ, meaning *any* shift action $\sigma_X \colon \Gamma \curvearrowright X^\Gamma$ for a Polish space X of cardinality at least 2. Note that this is in striking contrast to the situation in measurable dynamics.

We associate with each subshift a certain countable object, which we call a Γ-*ideal*.

Definition 5.2.5. A subset $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ is called a **Γ-ideal** if it is invariant under the action of Γ on $[\Gamma \to \mathbb{N}]^{<\infty}$ and closed under restrictions (i.e., if $\varphi \subseteq \varphi' \in \mathfrak{I}$, then $\varphi \in \mathfrak{I}$).

Definition 5.2.6. For a subshift $\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$, a map $\varphi \in [\Gamma \to \mathbb{N}]^{<\infty}$ is called a **finite Ω-coloring** if there exists a coloring $\omega \in \Omega$ extending φ. The set of all finite Ω-colorings is denoted by $\mathbf{Fin}(\Omega)$.

Clearly, for any $\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$, the set $\mathbf{Fin}(\Omega)$ is a Γ-ideal. However, not every Γ-ideal arises in this way. We call the Γ-ideals of the form $\mathbf{Fin}(\Omega)$ **extendable** and characterize them combinatorially in §5.3. There we also assemble a "dictionary" of some correspondences between subshifts and extendable Γ-ideals. They are useful, for example, in defining the topology on $\mathbf{Sh}_0(\Gamma, \mathbb{N})$.

The second main result of this article is a purely combinatorial description of the set $\mathbf{Sh}_{\mathrm{BM}}(\sigma, \mathbb{N})$. Roughly speaking, we show that determining whether there exists a Baire measurable Ω-coloring of σ is equivalent to settling a question of the following form:

"Is it possible to decide whether a given partial coloring $\varphi \in [\Gamma \to \mathbb{N}]^{<\infty}$ belongs to $\mathbf{Fin}(\Omega)$ only using 'local' information?" $(*)$

This question is rather natural, and some of its versions have already been studied in finite combinatorics with no connection to descriptive set theory. One particular interpretation of $(*)$, which is of special interest in graph theory, is the problem of jointly extending given pre-colorings of substructures that are sufficiently far apart from each other. There is an extensive literature on this subject; see [Alb96; AKW05; Dvo+17; PT16] for a small sample. We formalize this idea in Definition 5.2.7 as the *join property* of subshifts. Definition 5.2.8 isolates the class of *local* subshifts; locality is stronger than the join property (see Remark after Definition 5.2.8).

Let $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ be a Γ-ideal. A function $R \colon \mathfrak{I} \to [0; +\infty)$ is **invariant** if $R(\gamma \cdot \varphi) = R(\varphi)$ for all $\varphi \in \mathfrak{I}$ and $\gamma \in \Gamma$. We say that $\varphi, \psi \in \mathfrak{I}$ are ***R*-separated** if

$$\mathrm{dist}(\mathrm{dom}(\varphi), \mathrm{dom}(\psi)) \ > \ R(\varphi) + R(\psi).$$

Definition 5.2.7. Let $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ be a Γ-ideal. We say that $\mathfrak{I}$ has the **join property** if there is an invariant function $R \colon \mathfrak{I} \to [0; +\infty)$ such that for all $k \in \mathbb{N}$, if $\varphi_1, \ldots, \varphi_k \in \mathfrak{I}$ are pairwise R-separated, then $\varphi_1 \cup \ldots \cup \varphi_k \in \mathfrak{I}$. A subshift $\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$ has the join property if so does the Γ-ideal $\mathbf{Fin}(\Omega)$.

Remark. For $k = 0$, we interpret the above definition to mean that $\varnothing \in \mathfrak{I}$; in other words, a Γ-ideal with the join property is necessarily nonempty.

Given $\varphi \in [\Gamma \to \mathbb{N}]^{<\infty}$, an element $\gamma \in \Gamma$, and a radius $r \in [0; +\infty)$, define

$$\varphi[\gamma, r] := \varphi|(\mathrm{dom}(\varphi) \cap \mathbf{Ball}(\gamma, r)).$$

Let $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ be a Γ-ideal. Given a function $r \colon \mathbb{N} \to [0; +\infty)$, we say that $\varphi \colon [\Gamma \to \mathbb{N}]^{<\infty}$ is ***r*-locally in** $\mathfrak{I}$ if for each $\gamma \in \mathrm{dom}(\varphi)$,

$$\varphi[\gamma, r(\varphi(\gamma))] \in \mathfrak{I}.$$

The set of all $\varphi \in [\Gamma \to \mathbb{N}]^{<\infty}$ that are r-locally in $\mathfrak{I}$ is denoted by $\mathbf{Loc}_r(\mathfrak{I})$. Note that since $\mathfrak{I}$ is closed under restrictions, we have $\mathbf{Loc}_r(\mathfrak{I}) \supseteq \mathfrak{I}$ for all $r \colon \mathbb{N} \to [0; +\infty)$.

Definition 5.2.8. Let $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ be a Γ-ideal. We say that $\mathfrak{I}$ is **local** if $\mathfrak{I} = \mathbf{Loc}_r(\mathfrak{I})$ for some function $r \colon \mathbb{N} \to [0; +\infty)$. A subshift $\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$ is local if so is the Γ-ideal $\mathbf{Fin}(\Omega)$.

Remark. Notice that every local Γ-ideal has the join property. Indeed, suppose that $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ is a local Γ-ideal and let $r \colon \mathbb{N} \to [0; +\infty)$ be a function such that $\mathfrak{I} = \mathbf{Loc}_r(\mathfrak{I})$. Define an invariant map $R \colon \mathfrak{I} \to [0; +\infty)$ by

$$R(\varphi) \coloneqq \sup\{r(\varphi(\gamma)) \,:\, \gamma \in \mathrm{dom}(\varphi)\}.$$

Let $\varphi_1, \ldots, \varphi_k \in \mathfrak{I}$ be pairwise R-separated and set $\varphi \coloneqq \varphi_1 \cup \ldots \cup \varphi_k$. Consider an arbitrary element $\gamma \in \mathrm{dom}(\varphi)$. Then $\gamma \in \mathrm{dom}(\varphi_i)$ for a unique $1 \leqslant i \leqslant k$. Since $\varphi_1, \ldots, \varphi_k$ are pairwise R-separated, for each $j \neq i$, we have

$$\mathbf{Ball}(\gamma, R(\varphi_i)) \cap \mathrm{dom}(\varphi_j) = \varnothing.$$

Since $r(\varphi(\gamma)) = r(\varphi_i(\gamma)) \leqslant R(\varphi_i)$, we conclude that

$$\varphi[\gamma, r(\varphi(\gamma))] \subseteq \varphi[\gamma, R(\varphi_i)] = \varphi_i[\gamma, R(\varphi_i)] \subseteq \varphi_i \in \mathfrak{I}.$$

Therefore, $\varphi \in \mathbf{Loc}_r(\mathfrak{I})$. As $\mathbf{Loc}_r(\mathfrak{I}) = \mathfrak{I}$, we obtain $\varphi \in \mathfrak{I}$, as desired.

We need one last definition:

Definition 5.2.9. If $\Omega, \Omega' \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$ are subshifts, then Ω is **reducible** to Ω', in symbols $\Omega \succcurlyeq \Omega'$, if there is a map $\rho \colon \mathbb{N} \to \mathbb{N}$, called a **reduction**, such that for all $\omega \in \Omega'$, we have $\rho \circ \omega \in \Omega$.

Remark. A special case of reducibility is when $\Omega \supseteq \Omega'$. Indeed, if $\Omega \supseteq \Omega'$, then the identity map $\mathrm{id}_{\mathbb{N}} \colon \mathbb{N} \to \mathbb{N}$ is a reduction from Ω to Ω'. This explains the orientation of the symbol "$\succcurlyeq$."

Remark. If $\Omega \succcurlyeq \Omega'$ and $\Omega' \in \mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$ for some continuous action $\alpha \colon \Gamma \curvearrowright X$ on a Polish space X, then $\Omega \in \mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$ as well. Indeed, if $\rho \colon \mathbb{N} \to \mathbb{N}$ is a reduction from Ω to Ω' and $f \colon X \to \mathbb{N}$ is a Baire measurable Ω'-coloring of α, then $\rho \circ f$ is a Baire measurable Ω-coloring of α.

Finally, we are ready to state our result:

Theorem 5.2.10. *The following statements are equivalent for a subshift* $\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$*:*

(i) $\Omega \in \mathbf{Sh}_{\mathrm{BM}}(\sigma, \mathbb{N})$*;*

(ii) $\Omega \supseteq \Omega'$ *for some subshift* Ω' *with the join property;*

(iii) $\Omega \succcurlyeq \Omega'$ *for some local subshift* Ω'*.*

We prove Theorem 5.2.10 in §5.5.

5.2.5 Some corollaries

As mentioned previously, the join property and its analogs have been an object of study in graph theory (although Definition 5.2.7 does not appear to have been explicitly articulated before). In particular, implication

(ii) $\implies$ (i) of Theorem 5.2.10 can be used to derive bounds on Baire measurable chromatic numbers from known results in finite combinatorics. For instance, deep results of Postle and Thomas [PT16] yield the following:

Corollary 5.2.11. *Suppose that* Γ *is generated by a finite symmetric set* $S \subset \Gamma$ *with* $\mathbf{1} \notin S$ *such that the corresponding Cayley graph* $G \coloneqq \mathrm{Cay}(\Gamma, S)$ *is planar. Then*

$$\chi_{\mathrm{BM}}(\sigma, S) \leqslant \begin{cases} 3 & \text{if } G \text{ contains no cycles of lengths } 3 \text{ and } 4; \\ 4 & \text{if } G \text{ contains a cycle of length } 4 \text{ but not of length } 3; \\ 5 & \text{otherwise}. \end{cases} \tag{5.2.2}$$

Proof. Assume that Γ and S satisfy the above assumptions and let k denote the quantity on the right hand side of (5.2.2). The fact that $\mathbf{PrCol}(S, k)$ is a subshift with the join property is a consequence of [PT16, Theorem 8.10]. ∎

Note that the best upper bounds for $\chi_{\mathrm{BM}}(\sigma, S)$ under the assumptions of Corollary 5.2.11 that follow from previously known results are $\chi_{\mathrm{BM}}(\sigma, S) \leqslant 7$ in general and $\chi_{\mathrm{BM}}(\sigma, S) \leqslant 5$ if $\mathrm{Cay}(\Gamma, S)$ contains no cycles of length 3; these follow from combining [CM16, Theorem B] (see (5.2.1) above) with the Four Color Theorem [Die00, Theorem 5.1.1] and Grötzsch's theorem [Die00, Theorem 5.1.3] respectively. The proof of [PT16, Theorem 8.10] due to Postle and Thomas is quite difficult.

Locality of a subshift is often rather easy to check, which makes condition (iii) of Theorem 5.2.10 a convenient tool for constructing subshifts in $\mathbf{Sh}_{\mathrm{BM}}(\sigma, \mathbb{N})$ with additional interesting properties. To illustrate this, in §5.5.4 we prove the following:

Corollary 5.2.12. *There exists a free continuous action* α *of* Γ *on a Polish space such that*

$$\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N}) \not\supseteq \mathbf{Sh}_{\mathrm{BM}}(\sigma, \mathbb{N}).$$

We find Corollary 5.2.12 somewhat surprising. Indeed, due to Theorem 5.2.4, all non-trivial shift actions of Γ admit exactly the same types of Baire measurable colorings. Analogous statements hold in the purely Borel context and in the context of approximate measure colorings; the former follows from a result of Seward and Tucker-Drob [ST16, Theorem 1.1], the latter—from the Abért–Weiss theorem on weak containment of Bernoulli shifts [AW13, Theorem 1]. However, both in the Borel and in the approximate measure frameworks, the shift actions are actually the *hardest* ones to color (which also follows from [ST16, Theorem 1.1] and [AW13, Theorem 1]), whereas, as Corollary 5.2.12 asserts, that is *not* the case in the Baire category setting.

5.3 Extendable Γ-ideals

Due to their combinatorial nature, we sometimes find working with Γ-ideals more convenient than referring to subshifts directly. In this section we summarize some useful correspondences between the two kinds of objects. Most statements made here follow readily from definitions.

Given a Γ-ideal $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$, an **$\mathfrak{I}$-coloring** is a map $\omega \colon \Gamma \to \mathbb{N}$ such that

$$\omega|S \in \mathfrak{I} \quad \text{for all} \quad S \in [\Gamma]^{<\infty}.$$

The set of all $\mathfrak{I}$-colorings is denoted $\mathbf{Col}(\mathfrak{I})$. It is clear that $\mathbf{Col}(\mathfrak{I}) \subseteq \mathbb{N}^\Gamma$ is a subshift.

Definition 5.3.1. A Γ-ideal $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ is **extendable** if for every $\varphi \in \mathfrak{I}$ and $\gamma \in \Gamma \setminus \mathrm{dom}(\varphi)$, there is a color $c \in \mathbb{N}$ such that $\varphi \cup \{(\gamma, c)\} \in \mathfrak{I}$.

Proposition 5.3.2. *Let $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ be a Γ-ideal. The following statements are equivalent:*

- $\mathfrak{I} = \mathbf{Fin}(\Omega)$ *for some* $\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$*;*
- $\mathfrak{I}$ *is extendable.*

If $\mathfrak{I}$ is extendable, then the subshift $\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$ such that $\mathfrak{I} = \mathbf{Fin}(\Omega)$ is unique, namely $\Omega = \mathbf{Col}(\mathfrak{I})$.

The proof of Proposition 5.3.2 is straightforward, and we do not spell it out here.

Note that the set $\mathbf{Ext}(\Gamma, \mathbb{N})$ of all extendable Γ-ideals is a G_δ subset of the power set of $[\Gamma \to \mathbb{N}]^{<\infty}$. By Alexandrov's theorem [Kec95, Theorem 3.11], $\mathbf{Ext}(\Gamma, \mathbb{N})$ is Polish in its relative topology. The bijection between $\mathbf{Ext}(\Gamma, \mathbb{N})$ and $\mathbf{Sh}_0(\Gamma, \mathbb{N})$, given by the maps $\mathbf{Col} \colon \mathbf{Ext}(\Gamma, \mathbb{N}) \to \mathbf{Sh}_0(\Gamma, \mathbb{N})$ and $\mathbf{Fin} \colon \mathbf{Sh}_0(\Gamma, \mathbb{N}) \to \mathbf{Ext}(\Gamma, \mathbb{N})$, allows us to transfer the Polish topology from $\mathbf{Ext}(\Gamma, \mathbb{N})$ to $\mathbf{Sh}_0(\Gamma, \mathbb{N})$, thus turning $\mathbf{Sh}_0(\Gamma, \mathbb{N})$ into a Polish space. Explicitly, the Polish topology on $\mathbf{Sh}_0(\Gamma, \mathbb{N})$ is generated by the open sets of the form

$$\{\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N}) \,:\, \varphi \in \mathbf{Fin}(\Omega)\} \quad \text{and} \quad \{\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N}) \,:\, \varphi \notin \mathbf{Fin}(\Omega)\},$$

where φ is ranging over $[\Gamma \to \mathbb{N}]^{<\infty}$.

The next definition is the analog of Definition 5.2.9 for Γ-ideals:

Definition 5.3.3. If $\mathfrak{I}, \mathfrak{I}' \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ are Γ-ideals, then $\mathfrak{I}$ is **reducible** to $\mathfrak{I}'$, in symbols $\mathfrak{I} \succcurlyeq \mathfrak{I}'$, if there is a map $\rho \colon \mathbb{N} \to \mathbb{N}$, called a **reduction**, such that for all $\varphi \in \mathfrak{I}'$, we have $\rho \circ \varphi \in \mathfrak{I}$.

The following statements are also straightforward:

Proposition 5.3.4. *Let $\Omega, \Omega' \in S_0(\Gamma, \mathbb{N})$. Then*

- $\Omega \supseteq \Omega'$ *if and only if* $\mathbf{Fin}(\Omega) \supseteq \mathbf{Fin}(\Omega')$*; and*
- $\Omega \succcurlyeq \Omega'$ *if and only if* $\mathbf{Fin}(\Omega) \succcurlyeq \mathbf{Fin}(\Omega')$*.*

Finally, given a Γ-ideal $\mathfrak{I}$ and a continuous action $\alpha \colon \Gamma \curvearrowright X$ on a Polish space X, a **Baire measurable $\mathfrak{I}$-coloring** of α is the same as a Baire measurable $\mathbf{Col}(\mathfrak{I})$-coloring of α.

5.4 Proof of Theorem 5.2.3

5.4.1 The space of Baire measurable functions

For the rest of this subsection (save Corollary 5.4.5), we fix a Polish space X and a standard Borel space Y. Two Baire measurable functions $f, g \colon X \to Y$ are **equal on a comeager set**, or **$*$-equal**, in symbols $f =^* g$, if the set $\{x \in X \,:\, f(x) = g(x)\}$ is comeager. The set of all Baire measurable functions from X to Y, taken modulo the equivalence relation of $*$-equality, is denoted by $[\![X, Y]\!]$. For a nonempty open set $U \subseteq X$ and a Borel subset $A \subseteq Y$, let

$$[\![U, A]\!] \coloneqq \{f \in [\![X, Y]\!] \,:\, U \Vdash f^{-1}(A)\}.$$

Let $\mathfrak{Baire}$ denote the σ-algebra on $[\![X, Y]\!]$ generated by the sets of the form $[\![U, A]\!]$ for all nonempty open $U \subseteq X$ and Borel $A \subseteq Y$.

Theorem 5.4.1. *The measurable space* $([\![X, Y]\!], \mathfrak{Baire})$ *is standard Borel.*

A σ-algebra $\mathfrak{S}$ on a set Z **separates points** if for all $z, z' \in Z$, if $z \neq z'$, then there exists $A \in \mathfrak{S}$ such that $z \in A$ and $z' \notin A$.

Lemma 5.4.2. *The σ-algebra* $\mathfrak{Baire}$ *separates points.*

Proof. Suppose that $f, g \in [\![X, Y]\!]$ are not $*$-equal, i.e., the set $\{x \in X \,:\, f(x) \neq g(x)\}$ is nonmeager. Fix an arbitrary Polish topology on Y that generates its Borel σ-algebra. By [Kec95, Theorem 8.38], there is a comeager subset $X' \subseteq X$ such that the restricted functions $f|X'$, $g|X'$ are continuous. Then the set $\{x \in X' \,:\, f(x) \neq g(x)\}$ is also nonmeager, and hence nonempty. Consider any $x_0 \in X'$ with $f(x_0) \neq g(x_0)$ and let V, $W \subset Y$ be disjoint open neighborhoods of $f(x_0)$ and $g(x_0)$ respectively. By the continuity of $f|X'$ and $g|X'$, there exists an open neighborhood $U \subseteq X$ of x_0 such that $f(x) \in V$ and $g(x) \in W$ for all $x \in U \cap X'$. This yields $f \in [\![U, V]\!]$ and $g \in [\![U, W]\!]$. As $[\![U, V]\!] \cap [\![U, W]\!] = \varnothing$, and so $g \notin [\![U, V]\!]$, the proof is complete. ∎

Lemma 5.4.3. *Let $\mathcal{U}$ be a countable basis for the topology on X consisting of nonempty open sets, and let $\mathcal{A}$ be a countable basis for the Borel σ-algebra on Y. Then* $\mathfrak{Baire}$ *is generated by the family of sets*

$$[\![\mathcal{U}, \mathcal{A}]\!] \coloneqq \{[\![U, A]\!] \,:\, U \in \mathcal{U},\ A \in \mathcal{A}\}.$$

Proof. Since for $U_0, U_1, \ldots \in \mathcal{U}$ and $A \in \mathcal{A}$, we have

$$[\![\bigcup_{i=0}^{\infty} U_i, A]\!] = \bigcap_{i=0}^{\infty} [\![U_i, A]\!],$$

the σ-algebra $\mathfrak{Baire}$ is generated by the sets $[\![U, A]\!]$ with $U \in \mathcal{U}$. Since we also have

$$[\![U, A^{\mathrm{c}}]\!] = \bigcap\{[\![V, A]\!]^{\mathrm{c}} \,:\, V \in \mathcal{U},\ V \subseteq U\},$$

where $(\cdot)^{\mathsf{c}}$ denotes set complement, and

$$[\![U, \bigcap_{i=0}^{\infty} A_i]\!] = \bigcap_{i=0}^{\infty} [\![U, A_i]\!],$$

we conclude that $\mathfrak{Baire}$ is indeed generated by the sets $[\![U, A]\!]$ with $U \in \mathcal{U}$ and $A \in \mathcal{A}$. ∎

Proposition 5.4.4. *Let $(Z, \mathfrak{S})$ be a measurable space such that the σ-algebra $\mathfrak{S}$ separates points. Let $\mathcal{A} \subseteq \mathfrak{S}$ be a countable generating set for $\mathfrak{S}$. For each $z \in Z$, define $\vartheta_z \colon \mathcal{A} \to 2$ as follows:*

$$\vartheta_z(A) \coloneqq \begin{cases} 1 & \text{if } z \in A; \\ 0 & \text{if } z \notin A. \end{cases}$$

Let Θ denote the image of Z under the map $z \mapsto \vartheta_z$. Then $(Z, \mathfrak{S})$ is standard Borel if and only if Θ is a Borel subset of the product space $2^{\mathcal{A}}$.

Proof. Let $\mathfrak{B} \coloneqq \mathfrak{B}(2^{\mathcal{A}})$ denote the Borel σ-algebra on $2^{\mathcal{A}}$. Since $\mathfrak{S}$ separates points, the map $z \mapsto \vartheta_z$ is injective; by construction, it is therefore an isomorphism of measurable spaces $(Z, \mathfrak{S})$ and $(\Theta, \mathfrak{B}|\Theta)$, where $\mathfrak{B}|\Theta$ is the relative σ-algebra on Θ. Thus, $(Z, \mathfrak{S})$ is standard Borel if and only if so is $(\Theta, \mathfrak{B}|\Theta)$; by the Luzin–Suslin theorem [Kec95, Theorem 15.1], the latter condition is equivalent to Θ being a Borel subset of $2^{\mathcal{A}}$. ∎

Proof of Theorem 5.4.1. Let $\mathcal{U}$ be a countable basis for the topology X consisting of nonempty open sets. Using the Borel isomorphism theorem [Kec95, Theorem 15.6], we can choose a zero-dimensional compact metrizable topology on Y that generates its Borel σ-algebra; let $\mathcal{A}$ be a countable basis for that topology consisting of sets that are simultaneously open and closed.

For each $f \in [\![X, Y]\!]$, define $\vartheta_f \colon \mathcal{U} \times \mathcal{A} \to 2$ as follows:

$$\vartheta_f(U, A) \coloneqq \begin{cases} 1 & \text{if } U \Vdash f^{-1}(A); \\ 0 & \text{if } U \nVdash f^{-1}(A), \end{cases}$$

and let Θ denote the image of $[\![X, Y]\!]$ under the map $f \mapsto \vartheta_f$. In view of Lemmas 5.4.2, 5.4.3, and Proposition 5.4.4, we only need to show that Θ is a Borel subset of the product space $2^{\mathcal{U} \times \mathcal{A}}$.

Let Θ' denote the set of all functions $\vartheta \colon \mathcal{U} \times \mathcal{A} \to 2$ satisfying the following two requirements:

(1) for all $k \in \mathbb{N}$, $U_1, \ldots, U_k \in \mathcal{U}$, and $A_1, \ldots, A_k \in \mathcal{A}$,

$$\begin{array}{lll} \text{if} & U_1 \cap \ldots \cap U_k \neq \varnothing \quad \text{and} \quad \vartheta(U_1, A_1) = \ldots = \vartheta(U_k, A_k) = 1, \\ \text{then} & A_1 \cap \ldots \cap A_k \neq \varnothing; \end{array}$$

(2) for all $U \in \mathcal{U}$ and $A \in \mathcal{A}$, if $\vartheta(U, A) = 0$, then there exist $V \in \mathcal{U}$ and $B \in \mathcal{A}$ such that

$$V \subseteq U, \qquad B \cap A = \varnothing, \qquad \text{and} \qquad \vartheta(V, B) = 1.$$

Note that Θ' is evidently a Borel (in fact, G_δ) subset of $2^{\mathcal{U}\times\mathcal{A}}$.

Claim (A). $\Theta \subseteq \Theta'$.

Proof. Let $f \in [\![X, Y]\!]$. We need to show that ϑ_f satisfies conditions (1) and (2).

(1) If $U_1, \ldots, U_k \in \mathcal{U}$ and $A_1, \ldots, A_k \in \mathcal{A}$ are such that

$$U_1 \cap \ldots \cap U_k \neq \varnothing \quad \text{and} \quad \vartheta_f(U_1, A_1) = \ldots = \vartheta_f(U_k, A_k) = 1,$$

then $U_1 \cap \ldots \cap U_k$ is nonempty open and

$$U_1 \cap \ldots \cap U_k \Vdash f^{-1}(A_1) \cap \ldots \cap f^{-1}(A_k) \;=\; f^{-1}(A_1 \cap \ldots \cap A_k),$$

implying that $f^{-1}(A_1 \cap \ldots \cap A_k)$ is nonmeager, and hence $A_1 \cap \ldots \cap A_k \neq \varnothing$.

(2) Let $U \in \mathcal{U}$ and $A \in \mathcal{A}$ be such that $\vartheta_f(U, A) = 0$, i.e., $U \nVdash f^{-1}(A)$. The sets in $\mathcal{A}$ are simultaneously open and closed; in particular, the complement of A is open and hence equal to the union of all $B \in \mathcal{A}$ with $B \cap A = \varnothing$. Therefore, for some $B \in \mathcal{A}$ with $B \cap A = \varnothing$, the set $U \cap f^{-1}(B)$ is nonmeager. By the Baire alternative, there is $V \in \mathcal{U}$ such that $V \subseteq U$ and $V \Vdash f^{-1}(B)$, i.e., $\vartheta_f(V, B) = 1$. ⊣

Claim (B). $\Theta' \subseteq \Theta$.

Proof. Let $\vartheta \in \Theta'$. We need to find a function $f \in [\![X, Y]\!]$ such that $\vartheta_f = \vartheta$. For each $x \in X$, let

$$\mathcal{A}_x \coloneqq \{A \in \mathcal{A} \,:\, \vartheta(U, A) = 1 \text{ for some } U \in \mathcal{U} \text{ with } U \ni x\}.$$

Note that $\mathcal{A}_x$ is a family of closed subsets of the compact space Y, and condition (1) implies that it has the finite intersection property; therefore, $R_x \coloneqq \bigcap \mathcal{A}_x$ is a nonempty compact set. The set

$$R \coloneqq \{(x, y) \in X \times Y \,:\, y \in R_x\}$$

is Borel (in fact, closed) in $X \times Y$, so by [Kec95, Theorem 28.8], there exists a Borel map $f \colon X \to Y$ such that $f(x) \in R_x$ for all $x \in X$. We claim that $\vartheta_f = \vartheta$ for any such f. Indeed, let $U \in \mathcal{U}$ and $A \in \mathcal{A}$. If $\vartheta(U, A) = 1$, then for all $x \in U$,

$$f(x) \in R_x \subseteq A,$$

so $U \subseteq f^{-1}(A)$, and thus $\vartheta_f(U, A) = 1$. On the other hand, if $\vartheta(U, A) = 0$, then, by (2), there exist sets $V \in \mathcal{U}$ and $B \in \mathcal{A}$ such that

$$V \subseteq U, \qquad B \cap A = \varnothing, \qquad \text{and} \qquad \vartheta(V, B) = 1.$$

By the previous argument, $\vartheta_f(V, B) = 1$, i.e., $V \Vdash f^{-1}(B)$. Since $B \cap A = \varnothing$, this implies $U \nVdash f^{-1}(A)$, and so $\vartheta_f(U, A) = 0$. ⊣

Together, Claims (A) and (B) yield $\Theta = \Theta'$; in particular, Θ is Borel. ■

Corollary 5.4.5. *Let $\alpha \colon \Gamma \curvearrowright X$ be a continuous action of Γ on a Polish space X. Then the set $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$ is analytic.*

Proof. It is routine to check that the set

$$\mathbf{Hom}(\alpha, \sigma_{\mathbb{N}}) \coloneqq \{\pi \in [\![X, \mathbb{N}^\Gamma]\!] \,:\, \pi \text{ is equivariant on a comeager set}\}$$

is a Borel subset of $[\![X, \mathbb{N}^\Gamma]\!]$; furthermore, the set

$$\{(\pi, \Omega) \in \mathbf{Hom}(\alpha, \sigma_{\mathbb{N}}) \times \mathbf{Sh}_0(\Gamma, \mathbb{N}) \,:\, \pi^{-1}(\Omega) \text{ is comeager}\}$$

is a Borel subset of $\mathbf{Hom}(\alpha, \sigma_{\mathbb{N}}) \times \mathbf{Sh}_0(\Gamma, \mathbb{N})$. As

$$\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N}) = \{\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N}) \,:\, \pi^{-1}(\Omega) \text{ is comeager for some } \pi \in \mathbf{Hom}(\alpha, \sigma_{\mathbb{N}})\},$$

we see that $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$ is analytic. ∎

5.4.2 Smoothness

Let X be a Polish (or, more generally, standard Borel) space. A Borel equivalence relation E on X is **smooth** if there is a Borel function $f \colon X \to \mathbb{R}$ such that for all $x, y \in X$,

$$f(x) = f(y) \iff x \text{ and } y \text{ are } E\text{-equivalent.}$$

A set $T \subseteq X$ is a **transversal** for E if every E-class intersects T in exactly one point. The following useful proposition follows from the Luzin–Novikov theorem:

Proposition 5.4.6 ([Tse16, Proposition 20.6])**.** *Let E be a Borel equivalence relation on a Polish space X. Suppose that every E-class is countable. Then the following statements are equivalent:*

- *E is smooth;*
- *there exists a Borel transversal $T \subseteq X$ for E.*

Given a continuous action $\alpha \colon \Gamma \curvearrowright X$ on a Polish space X, let E_α denote the induced **orbit equivalence relation**, i.e., the equivalence relation on X whose classes are precisely the orbits of α. The definition of generic smoothness for actions of Γ from §5.2.3 then can be phrased as follows:

> A continuous action $\alpha \colon \Gamma \curvearrowright X$ is generically smooth if and only if there exists a comeager α-invariant Borel subset $X' \subseteq X$ such that the relation E_α restricted to X' is smooth.

Lemma 5.4.7. *If α is a generically smooth free continuous action of Γ on a nonempty Polish space, then $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N}) = \mathbf{Sh}(\Gamma, \mathbb{N})$.*

Proof. Let $\alpha \colon \Gamma \curvearrowright X$ be as in the statement of the lemma. The inclusion $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N}) \subseteq \mathbf{Sh}(\Gamma, \mathbb{N})$ is clear as $X \neq \varnothing$. To prove the other inclusion, consider any nonempty subshift $\Omega \in \mathbf{Sh}(\Gamma, \mathbb{N})$ and let $\omega \in \Omega$

be an arbitrary coloring. After discarding a meager set if necessary, we may assume that E_α is smooth; Proposition 5.4.6 then gives a Borel transversal $T \subseteq X$ for E_α. Since α is free, for each $x \in X$, there is a unique element $\gamma_x \in \Gamma$ such that $(\gamma_x)^{-1} \cdot x \in T$. Set $f(x) \coloneqq \omega(\gamma_x)$ for all $x \in X$. Then for all $x \in X$,

$$\pi_f(x) = \gamma_x \cdot \omega \in \Omega,$$

i.e., f is a desired Baire measurable Ω-coloring of α. ∎

The remainder of this section is dedicated to proving the "hard" part of Theorem 5.2.3: the completeness of the set $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$ for generically non-smooth α.

Let $\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$ be a subshift. We say that Ω is **easy** if $\Omega \in \mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$ for every free continuous action α of Γ on a Polish space; we say that Ω is **hard** if $\Omega \in \mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$ only for generically smooth α.

Lemma 5.4.8. *Let $\Omega \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$ be a subshift. Suppose that for each $\omega \in \Omega$, there is some $c \in \mathbb{N}$ such that the set $\{\gamma \in \Gamma \,:\, \omega(\gamma) = c\}$ contains precisely one element. Then Ω is hard.*

Proof. For each $\omega \in \Omega$, let

$$c_\omega \coloneqq \min\{c \in \mathbb{N} \,:\, |\{\gamma \in \Gamma \,:\, \omega(\gamma) = c\}| = 1\}.$$

Define a Borel set $T \subseteq \Omega$ by

$$T \coloneqq \{\omega \in \Omega \,:\, c_\omega = \omega(\mathbf{1})\}.$$

Let $\alpha \colon \Gamma \curvearrowright X$ be a continuous action of α on a Polish space X and suppose that $f \colon X \to \mathbb{N}$ is a Baire measurable Ω-coloring of α. After passing to a comeager subset if necessary, we may assume that the map f is Borel and $X = (\pi_f)^{-1}(\Omega)$. Then $(\pi_f)^{-1}(T)$ is a Borel transversal for E_α. ∎

Lemma 5.4.9. *Let $\Omega_0, \Omega_1, \ldots \in \mathbf{Sh}_0(\Gamma, \mathbb{N})$ be a countable sequence of hard subshifts. If $\Omega \coloneqq \bigcup_{i=0}^\infty \Omega_i$ is a subshift, then Ω is also hard.*

Proof. Let $\alpha \colon \Gamma \curvearrowright X$ be a continuous action of α on a Polish space X and suppose that $f \colon X \to \mathbb{N}$ is a Baire measurable Ω-coloring of α. Set $X_i \coloneqq (\pi_f)^{-1}(\Omega_i)$. After discarding a meager subset if necessary, we may assume that the map f is Borel and $X = (\pi_f)^{-1}(\Omega) = \bigcup_{i=0}^\infty X_i$. Passing to an even further comeager subset, we may assume that the relation E_α restricted to each X_i is smooth. Using Proposition 5.4.6, we obtain Borel transversals $T_i \subseteq X_i$ for the restricted relations. Let

$$\begin{aligned} S_0 &\coloneqq T_0; \\ S_{i+1} &\coloneqq T_{i+1} \setminus \textstyle\bigcup_{j=0}^i X_j \quad \text{for all } i \in \mathbb{N}, \end{aligned}$$

and set $S \coloneqq \bigcup_{i=0}^\infty S_i$. Then S a Borel transversal for E_α. ∎

5.4.3 Combinatorial lemmas

In this subsection we describe the main combinatorial construction behind our proof of Theorem 5.2.3.

Lemma 5.4.10. *Let $(d_0, d_1, \dots) \in [0; +\infty)^{\mathbb{N}}$ be a sequence such that a ball of radius d_0 in Γ contains at least 2 elements, and for each $c \in \mathbb{N}$, a ball of radius d_{c+1} in Γ contains two disjoint balls of radius d_c. Suppose that $\omega \colon \Gamma \to \mathbb{N}$ is a coloring such that for all $c \in \mathbb{N}$,*

$$\inf\{\mathrm{dist}(\gamma, \delta) \,:\, \gamma, \delta \in \Gamma,\ \gamma \neq \delta,\ \omega(\gamma) = \omega(\delta) = c\} \,>\, 2d_c.$$

Then ω uses infinitely many colors, i.e., the set $\{\omega(\gamma) \,:\, \gamma \in \Gamma\}$ is infinite.

Proof. We use induction on c to show that any ball of radius d_c in Γ contains an element γ with $\omega(\gamma) > c$. For $c = 0$, the assertion follows from the fact that each ball of radius d_0 contains at least 2 elements, and it is impossible for both of them to have color 0, since the distance between any two distinct elements γ, δ with $\omega(\gamma) = \omega(\delta) = 0$ is strictly greater than $2d_0$. Now assume that the assertion has been verified for some c and consider any ball of radius d_{c+1}. It contains two disjoint balls of radius d_c, so it must, by the inductive hypothesis, contain two distinct elements γ, δ with $\omega(\gamma), \omega(\delta) > c$. As $\mathrm{dist}(\gamma, \delta) \leqslant 2d_{c+1}$, it is impossible to have $\omega(\gamma) = \omega(\delta) = c + 1$, so at least one of $\omega(\gamma)$, $\omega(\delta)$ exceeds $c + 1$. ∎

Remark. Besides its application in the proof of Theorem 5.2.3, Lemma 5.4.10 will be used once more in the proof of Corollary 5.2.12.

Let $\alpha \colon \Gamma \curvearrowright X$ be a free action of Γ. For x, $y \in X$, write

$$\mathrm{dist}(x, y) \coloneqq \begin{cases} \mathrm{dist}(\mathbf{1}, \gamma) & \text{if } \gamma \in \Gamma \text{ is such that } \gamma \cdot x = y; \\ \infty & \text{if } x \text{ and } y \text{ are in different } \alpha\text{-orbits.} \end{cases}$$

Due to the right-invariance of the metric dist, for all $x \in X$ and γ, $\delta \in \Gamma$, we have

$$\mathrm{dist}(\gamma \cdot x, \delta \cdot x) \;=\; \mathrm{dist}(\gamma, \delta).$$

The next lemma is essentially a restatement of [MU16, Lemma 3.1]; we include its proof here for completeness.

Lemma 5.4.11 (ess. Marks–Unger [MU16, Lemma 3.1])**.** *Let $\alpha \colon \Gamma \curvearrowright X$ be a free continuous action of Γ on a nonempty Polish space X. Then for every sequence $(d_0, d_1, \dots) \in [0; +\infty)^{\mathbb{N}}$, there exists a Baire measurable coloring $f \colon X \to \mathbb{N}$ such that for all $c \in \mathbb{N}$,*

$$\inf\{\mathrm{dist}(x, y) \,:\, x, y \in X,\ x \neq y,\ f(x) = f(y) = c\} \,>\, d_c.$$

Proof. It suffices to show that there exists a partial Baire measurable map $f \colon X \rightharpoonup \mathbb{N}$ defined on a comeager subset of X and such that for all $c \in \mathbb{N}$,

$$\inf\{\mathrm{dist}(x, y) \,:\, x, y \in \mathrm{dom}(f),\ x \neq y,\ f(x) = f(y) = c\} \,>\, d_c. \tag{5.4.1}$$

For $c \in \mathbb{N}$, let G_c denote the graph with vertex set X and edge set

$$\{(x, y) \in X \times X \,:\, x \neq y \text{ and } \mathrm{dist}(x, y) \leqslant d_c\}.$$

The graph G_c is Borel (closed, in fact), and the neighborhood of every vertex in G_c is finite, so G_c admits a Borel proper $\mathbb{N}$-coloring. For each $c \in \mathbb{N}$, we fix one such coloring $\eta_c \colon X \to \mathbb{N}$.

Given a sequence $s = (s_0, s_1, \ldots) \in \mathbb{N}^{\mathbb{N}}$, define a partial function $f_s \colon X \rightharpoonup \mathbb{N}$ as follows:

$$f_s(x) \coloneqq \text{the smallest } c \in \mathbb{N} \text{ such that } \eta_c(x) = s_c, \text{ if such exists.}$$

Note that for any $s \in \mathbb{N}^{\mathbb{N}}$, (5.4.1) is satisfied with $f = f_s$. Indeed, if x, $y \in X$ are distinct and such that $f_s(x) = f_s(y) = c$, then, by definition, $\eta_c(x) = \eta_c(y) = s_c$, so $xy \notin E(G_c)$, i.e., $\mathrm{dist}(x, y) > d_c$. As the map f_s is Borel, it remains to prove that for some $s \in \mathbb{N}^{\mathbb{N}}$, the set

$$\{x \in X \,:\, f_s(x) \text{ is defined}\}$$

is comeager. Due to the Kuratowski–Ulam theorem [Kec95, Theorem 8.41], it suffices to show that for all $x \in X$, the set

$$\{s \in \mathbb{N}^{\mathbb{N}} \,:\, f_s(x) \text{ is defined}\} \;=\; \{s \in \mathbb{N}^{\mathbb{N}} \,:\, s_c = \eta_c(x) \text{ for some } c \in \mathbb{N}\}$$

is comeager in $\mathbb{N}^{\mathbb{N}}$, which is indeed the case as it is open and dense. ■

Now we combine Lemmas 5.4.10 and 5.4.11 to prove the main technical result of this subsection:

Lemma 5.4.12. *There exist a nonempty compact metrizable space H with no isolated points, a dense countable subset $H_0 \subset H$, and a continuous map $H \to \mathbf{Sh}_0(\Gamma, \mathbb{N}) \colon h \mapsto \Omega_h$ such that*

- *for all $h \in H_0$, the subshift Ω_h is hard; and*
- *for all $h \in H \setminus H_0$, the subshift Ω_h is easy.*

Proof. Let $\mathbb{N} \cup \{\infty\}$ be the compactification of $\mathbb{N}$ obtained by adding the point ∞ so that the neighborhood filter of ∞ is generated by the sets $\{n \in \mathbb{N} \cup \{\infty\} \,:\, n \geqslant m\}$ with m ranging over $\mathbb{N}$. Let H denote the set of all *nondecreasing* sequences in $(\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$, which we write as $h = (h_0, h_1, \ldots)$. Being a closed subset of a compact metrizable space, H itself is compact and metrizable, and it is easy to see that H contains no isolated points. Let

$$H_0 \coloneqq \{h \in H \,:\, h_c = \infty \text{ for some } c \in \mathbb{N}\}.$$

Evidently, H_0 is a dense countable subset of H.

Fix a sequence $(d_0, d_1, \ldots) \in [0; +\infty)^{\mathbb{N}}$ such that a ball of radius d_0 in Γ contains at least 2 elements, and for each $c \in \mathbb{N}$, a ball of radius d_{c+1} in Γ contains two disjoint balls of radius d_c (such a sequence exists since Γ is infinite, while every ball of finite radius in Γ is finite). For $h \in H$, let Ω_h denote the set of all colorings $\omega \colon \Gamma \to \mathbb{N}$ such that for all $c \in \mathbb{N}$,

$$\inf\{\mathrm{dist}(\gamma, \delta) \,:\, \gamma, \delta \in \Gamma,\, \gamma \neq \delta,\, \omega(\gamma) = \omega(\delta) = c\} \;\geqslant\; \max\{2d_c + 1, h_c\}.$$

The set Ω_h is a subshift; furthermore, the map $h \mapsto \Omega_h$ is continuous, since determining whether given $\varphi \in [\Gamma \to \mathbb{N}]^{<\infty}$ belongs to $\mathbf{Fin}(\Omega_h)$ only involves checking bounds on h_c for finitely many colors $c \in \mathbb{N}$.

If $h \in H \setminus H_0$, then Ω_h is easy by Lemma 5.4.11. Now suppose $h \in H_0$ and consider any $\omega \in \Omega_h$. By Lemma 5.4.10, the set $\{\omega(\gamma) : \gamma \in \Gamma\}$ is infinite. As $h \in H_0$, all but finitely many entries in h are equal to ∞; therefore, there is some $c \in \mathbb{N}$ such that $h_c = \infty$ and $\omega(\gamma) = c$ for some $\gamma \in \Gamma$. Since $\omega \in \Omega_h$, if $h_c = \infty$, then there is at most a single element $\gamma \in \Gamma$ with $\omega(\gamma) = c$. Therefore, Ω_h is hard by Lemma 5.4.8. ∎

5.4.4 The space of compact sets and a final reduction

The last step in our argument is inspired by the dichotomy theorem for co-analytic σ-ideals of compact sets due to Kechris, Louveau, and Woodin [Kec95, Theorem 33.3], which asserts that such a σ-ideal is either G_δ, or else, complete co-analytic. The Kechris–Louveau–Woodin dichotomy theorem is proved using a result of Hurewicz (see Theorem 5.4.13 below), which we will utilize in much the same way in our proof of Theorem 5.2.3.

Before stating Hurewicz's theorem, we need to introduce some notation and terminology. Let X be a Polish space. We use $\mathcal{K}(X)$ to denote the set of all compact subsets of X. The set $\mathcal{K}(X)$ is equipped with the **Vietoris topology**, which is generated by the open sets of the form

$$\{C \in \mathcal{K}(X) : C \cap U \neq \varnothing\} \quad \text{and} \quad \{C \in \mathcal{K}(X) : C \subseteq U\},$$

where U is ranging over the open subsets of X. The space $\mathcal{K}(X)$ is itself Polish [Kec95, Theorem 4.25]. For more background on the Vietoris topology and related concepts, see [Kec95, Section 4.F] and [Tse16, Section 3.D].

Theorem 5.4.13 (Hurewicz [Kec95, Exercise 27.4(ii)])**.** *Let X be a Polish space and let $A \subseteq X$ be a subset which is G_δ but not F_σ. Then the set $\{C \in \mathcal{K}(X) : C \cap A \neq \varnothing\}$ is complete analytic.*

Lemma 5.4.14. *If $C \subseteq \mathbf{Sh}_0(\Gamma, \mathbb{N})$ is a compact set, then $\bigcup_{\Omega \in C} \Omega$ is a subshift. Furthermore, the map*

$$\mathcal{K}(\mathbf{Sh}_0(\Gamma, \mathbb{N})) \to \mathbf{Sh}_0(\Gamma, \mathbb{N}) \colon C \mapsto \bigcup_{\Omega \in C} \Omega$$

is continuous.

Proof. Let $C \in \mathcal{K}(\mathbf{Sh}_0(\Gamma, \mathbb{N}))$. The set $\bigcup_{\Omega \in C} \Omega$ is clearly shift-invariant. Consider any $\omega \in \overline{\bigcup_{\Omega \in C} \Omega}$ (the bar denotes topological closure). There exist a sequence of subshifts $\Omega_0, \Omega_1, \ldots \in C$ and a sequence of colorings $\omega_0 \in \Omega_0$, $\omega_1 \in \Omega_1, \ldots$ such that $\lim_{i \to \infty} \omega_i = \omega$. Since C is compact, we may pass to a subsequence and assume that the sequence $\Omega_0, \Omega_1, \ldots$ converges to a limit $\Omega_\infty \in C$. Consider any $S \in [\Gamma]^{<\infty}$ and let $\varphi := \omega|S$. As $\omega = \lim_{i \to \infty} \omega_i$, we have

$$\varphi = \omega_i|S \text{ for all sufficiently large } i \in \mathbb{N}.$$

This implies

$$\varphi \in \mathbf{Fin}(\Omega_i) \text{ for all sufficiently large } i \in \mathbb{N},$$

and thus, $\varphi \in \mathbf{Fin}(\Omega_\infty)$. Therefore, $\omega \in \Omega_\infty$, and hence,

$$\bigcup_{\Omega \in C} \Omega = \overline{\bigcup_{\Omega \in C} \Omega},$$

i.e., the set $\bigcup_{\Omega \in C} \Omega$ is closed, and hence, it is a subshift. The continuity of the map $C \mapsto \bigcup_{\Omega \in C} \Omega$ then follows immediately from the definitions of the topologies on $\mathbf{Sh}_0(\Gamma, \mathbb{N})$ and $\mathcal{K}(\mathbf{Sh}_0(\Gamma, \mathbb{N}))$. ■

Now we have all the necessary tools to finish the proof of Theorem 5.2.3.

Proof of Theorem 5.2.3. Let $\alpha \colon \Gamma \curvearrowright X$ be a free continuous action of Γ on a nonempty Polish space X. As observed in Corollary 5.4.5, the set $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$ is analytic. The case of generically smooth α is handled in Lemma 5.4.7, so it remains to show that if α is not generically smooth, then $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$ is complete.

Let H and H_0 be as in Lemma 5.4.12 and let $H \to \mathbf{Sh}_0(\Gamma, \mathbb{N}) \colon h \mapsto \Omega_h$ be a continuous function such that

- for all $h \in H_0$, the subshift Ω_h is hard; and
- for all $h \in H \setminus H_0$, the subshift Ω_h is easy.

Since continuous images of compact spaces are compact, for each $C \in \mathcal{K}(H)$, we have

$$\{\Omega_h \ : \ h \in C\} \in \mathcal{K}(\mathbf{Sh}_0(\Gamma, \mathbb{N}));$$

moreover, the map

$$\mathcal{K}(H) \to \mathcal{K}(\mathbf{Sh}_0(\Gamma, \mathbb{N})) \colon C \mapsto \{\Omega_h \ : \ h \in C\}$$

is continuous. Using Lemma 5.4.14, we can then define a continuous function $\mathcal{K}(H) \to \mathbf{Sh}_0(\Gamma, \mathbb{N})$ by sending each $C \in \mathcal{K}(H)$ to the subshift $\Omega_C \coloneqq \bigcup_{h \in C} \Omega_h$. Notice that if $C \cap (H \setminus H_0) \neq \varnothing$, then $\Omega_C \supseteq \Omega_h$ for some $h \in H \setminus H_0$, so Ω_C is easy and, in particular, $\Omega_C \in \mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$. On the other hand, if $C \cap (H \setminus H_0) = \varnothing$, i.e., if $C \subseteq H_0$, then Ω_C is a union of countably many hard subshifts, so, by Lemma 5.4.9, it is itself hard; since α is not generically smooth, this implies $\Omega_C \notin \mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$. Therefore,

$$C \cap (H \setminus H_0) \neq \varnothing \iff \Omega_C \in \mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N}). \tag{5.4.2}$$

Since $H \setminus H_0$ is the complement of a dense countable subset of a nonempty Polish space H with no isolated points, it is G_δ but not F_σ; thus, by Theorem 5.4.13, the set

$$\{C \in \mathcal{K}(H) \ : \ C \cap (H \setminus H_0) \neq \varnothing\} \tag{5.4.3}$$

is complete analytic. It remains to notice that, by (5.4.2), the map $C \mapsto \Omega_C$ is a continuous reduction of the complete analytic set (5.4.3) to $\mathbf{Sh}_{\mathrm{BM}}(\alpha, \mathbb{N})$. ■

5.5 Proof of Theorem 5.2.10

We break proving Theorem 5.2.10 up into three steps, each corresponting to one of the implications

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(i)}.$$

Using observations made in §5.3, we phrase and prove each implication in terms of Γ-ideals rather than subshifts. Finally, we deduce Corollary 5.2.12 in §5.5.4.

5.5.1 Extendable Γ-ideals with the join property from Baire measurable colorings

Lemma 5.5.1. *Let $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ be a Γ-ideal. If σ admits a Baire measurable $\mathfrak{I}$-coloring, then there is an extendable Γ-ideal $\mathfrak{I}' \subseteq \mathfrak{I}$ with the join property.*

Proof. Using Theorem 5.2.4, identify σ with the shift action $\sigma_{\mathbb{N}} \colon \Gamma \curvearrowright \mathbb{N}^\Gamma$. For $\varphi \in [\Gamma \to \mathbb{N}]^{<\infty}$, let

$$U_\varphi := \{\omega \in \mathbb{N}^\Gamma \,:\, \omega \supset \varphi\}.$$

Note that $\{U_\varphi \,:\, \varphi \in [\Gamma \to \mathbb{N}]^{<\infty}\}$ is a basis for the topology on $\mathbb{N}^\Gamma$ consisting of nonempty clopen sets.

Let $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ be a Γ-ideal and let $f \colon \mathbb{N}^\Gamma \to \mathbb{N}$ be a Baire measurable $\mathfrak{I}$-coloring of $\mathbb{N}^\Gamma$. Set $\pi := \pi_f$ and define

$$\mathfrak{I}' := \{\varphi \in [\Gamma \to \mathbb{N}]^{<\infty} \,:\, \text{the set } \pi^{-1}(U_\varphi) \text{ is nonmeager}\}.$$

It is clear that $\mathfrak{I}'$ is a Γ-ideal and, by the choice of f, we have $\mathfrak{I}' \subseteq \mathfrak{I}$.

We claim that $\mathfrak{I}'$ is extendable. Indeed, let $\varphi \in \mathfrak{I}'$ and $\gamma \in \Gamma \setminus \mathrm{dom}(\varphi)$. For each $c \in \mathbb{N}$, set $\varphi_c := \varphi \cup \{(\gamma, c)\}$. We need to show that $\varphi_c \in \mathfrak{I}'$ for some $c \in \mathbb{N}$. To that end, notice that

$$\pi^{-1}(U_\varphi) = \bigcup_{c \in \mathbb{N}} \pi^{-1}(U_{\varphi_c}). \tag{5.5.1}$$

Since $\varphi \in \mathfrak{I}'$, the set on the left-hand side of (5.5.1) is nonmeager; thus, at least one of the sets whose union is taken on the right-hand side of (5.5.1) must also be nonmeager, as desired.

To finish the proof of the lemma, it remains to show that $\mathfrak{I}'$ has the join property. Define an invariant map $R \colon \mathfrak{I}' \to [0; +\infty)$ as follows: For each $\varphi \in \mathfrak{I}'$, set $R(\varphi)$ to be the smallest $R \in \mathbb{N}$ such that there is a map $\psi \colon \mathbf{Ball}(\mathrm{dom}(\varphi), R) \to \mathbb{N}$ with $U_\psi \Vdash \pi^{-1}(U_\varphi)$. (Such R exists since the set $\pi^{-1}(U_\varphi)$ is nonmeager.) Suppose that $\varphi_1, \ldots, \varphi_k \in \mathfrak{I}'$ are pairwise R-separated and let $\varphi := \varphi_1 \cup \ldots \cup \varphi_k$. For each $1 \leqslant i \leqslant k$, choose $\psi_i \colon \mathbf{Ball}(\mathrm{dom}(\varphi_i), R(\varphi_i)) \to \mathbb{N}$ so that $U_{\psi_i} \Vdash \pi^{-1}(U_{\varphi_i})$. Since $\varphi_1, \ldots, \varphi_k$ are pairwise R-separated, for all $i \neq j$, we have

$$\mathrm{dom}(\psi_i) \cap \mathrm{dom}(\psi_j) = \mathbf{Ball}(\mathrm{dom}(\varphi_i), R(\varphi_i)) \cap \mathbf{Ball}(\mathrm{dom}(\varphi_j), R(\varphi_j)) = \varnothing,$$

so $\psi := \psi_1 \cup \ldots \cup \psi_k$ is a function in $[\Gamma \to \mathbb{N}]^{<\infty}$. Then

$$U_\psi = U_{\psi_1} \cap \ldots \cap U_{\psi_k} \Vdash \pi^{-1}(U_{\varphi_1}) \cap \ldots \cap \pi^{-1}(U_{\varphi_k}) = \pi^{-1}(U_\varphi).$$

Therefore, the set U_φ is nonmeager, i.e., $\varphi \in \mathfrak{I}'$, as desired. ∎

5.5.2 Reducing extendable Γ-ideals with the join property to local ones

Lemma 5.5.2. *Every extendable Γ-ideal with the join property is reducible to a local extendable Γ-ideal.*

Proof. Let $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ be an extendable Γ-ideal with the join property and let $R \colon \mathfrak{I} \to [0; +\infty)$ be an invariant function such that whenever $k \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_k \in \mathfrak{I}$ are pairwise R-separated, we have

$\varphi_1 \cup \ldots \cup \varphi_k \in \mathfrak{I}$. We may assume that R is monotone increasing, i.e., for all $\varphi, \varphi' \in \mathfrak{I}$,

$$\varphi \subseteq \varphi' \implies R(\varphi) \leqslant R(\varphi').$$

Otherwise we can replace R with the map $\tilde{R} \colon \mathfrak{I} \to [0; +\infty)$ defined by

$$\tilde{R}(\varphi) \coloneqq \sup\{R(\varphi') \,:\, \varphi' \subseteq \varphi\}.$$

We will explicitly construct a local extendable Γ-ideal $\mathfrak{I}'$ such that $\mathfrak{I} \succcurlyeq \mathfrak{I}'$. It will be more convenient to view $\mathfrak{I}'$ as a subset of $[\Gamma \to (\mathbb{N} \times \mathbb{N})]^{<\infty}$ rather than $[\Gamma \to \mathbb{N}]^{<\infty}$ (of course, we can turn it into a subset of $[\Gamma \to \mathbb{N}]^{<\infty}$ using a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$). Let $\pi_1, \pi_2 \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ denote the projection maps:

$$\pi_1(h, c) \coloneqq h \quad \text{and} \quad \pi_2(h, c) \coloneqq c \quad \text{for all} \quad (h, c) \in \mathbb{N} \times \mathbb{N}.$$

Given $\varphi \in [\Gamma \to (\mathbb{N} \times \mathbb{N})]^{<\infty}$, an element $\gamma \in \Gamma$, a radius $r \in [0; +\infty)$, and a threshold $h \in \mathbb{N}$, define

$$\varphi[\gamma, r; h] \coloneqq \varphi|\{\delta \in \mathrm{dom}(\varphi) \cap \mathbf{Ball}(\gamma, r) \,:\, (\pi_1 \circ \varphi)(\delta) \leqslant h\}.$$

By definition, $\varphi[\gamma, r; h] \subseteq \varphi[\gamma, r]$. Let $\mathfrak{I}'$ denote the set of all partial maps $\varphi \in [\Gamma \to (\mathbb{N} \times \mathbb{N})]^{<\infty}$ such that the following holds for all $\gamma \in \mathrm{dom}(\varphi)$: If we let $h \coloneqq (\pi_1 \circ \varphi)(\gamma)$ and $\psi \coloneqq \pi_2 \circ (\varphi[\gamma, 3h; h])$, then

$$\mathrm{dom}(\psi) \subseteq \mathbf{Ball}(\gamma, h); \qquad \psi \in \mathfrak{I}; \qquad \text{and} \qquad R(\psi) \leqslant h.$$

Evidently, $\mathfrak{I}'$ is invariant under the action $\Gamma \curvearrowright [\Gamma \to (\mathbb{N} \times \mathbb{N})]^{<\infty}$. Moreover, since the map R is monotone increasing, $\mathfrak{I}'$ is closed under restrictions; in other words, $\mathfrak{I}'$ is a Γ-ideal. By definition,

$$\mathfrak{I}' = \mathbf{Loc}_r(\mathfrak{I}') \quad \text{for} \quad r \colon \mathbb{N} \times \mathbb{N} \to [0; +\infty) \colon (h, c) \mapsto 3h.$$

It remains to verify that $\mathfrak{I}'$ is extendable and $\mathfrak{I} \succcurlyeq \mathfrak{I}'$.

Claim (A). *Let $\varphi \in \mathfrak{I}'$ and let*

$$h \coloneqq \sup\{(\pi_1 \circ \varphi)(\gamma) \,:\, \gamma \in \mathrm{dom}(\varphi)\}.$$

Then φ can be written as a union $\varphi = \varphi_1 \cup \ldots \cup \varphi_k$ for some $k \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_k \in \mathfrak{I}'$ with the following properties:

– *for each $1 \leqslant i \leqslant k$, the map $\psi_i \coloneqq \pi_2 \circ \varphi_i$ belongs to $\mathfrak{I}$;*

– *for each $1 \leqslant i \leqslant k$, we have $R(\psi_i) \leqslant h$;*

– *the maps $\psi_1, \ldots, \psi_k$ are pairwise R-separated.*

Proof. The proof is by induction on $|\mathrm{dom}(\varphi)|$. If $\varphi = \varnothing$, then the claim holds vacuously with $k = 0$. Now suppose that $\varphi \neq \varnothing$. Then there is $\gamma_0 \in \mathrm{dom}(\varphi)$ such that $(\pi_1 \circ \varphi)(\gamma_0) = h$. Set

$$\varphi_0 \coloneqq \varphi[\gamma_0, 3h]; \qquad \psi_0 \coloneqq \pi_2 \circ \varphi_0; \qquad \text{and} \qquad \varphi' \coloneqq \varphi \setminus \varphi_0.$$

By the choice of h, we have $\varphi[\gamma_0, 3h; h] = \varphi_0$. Thus, by the definition of $\mathfrak{I}'$,

$$\mathrm{dom}(\psi_0) \subseteq \mathbf{Ball}(\gamma_0, h); \qquad \psi_0 \in \mathfrak{I}; \qquad \text{and} \qquad R(\psi_0) \leqslant h.$$

Applying the inductive hypothesis to φ', we can write $\varphi' = \varphi_1 \cup \ldots \cup \varphi_k$ for some $\varphi_1, \ldots, \varphi_k \in \mathfrak{I}'$ with the following properties:

- for each $1 \leqslant i \leqslant k$, the map $\psi_i \coloneqq \pi_2 \circ \varphi_i$ belongs to $\mathfrak{I}$;
- for each $1 \leqslant i \leqslant k$, we have $R(\psi_i) \leqslant h$;
- the maps $\psi_1, \ldots, \psi_k$ are pairwise R-separated.

It remains to show that ψ_0 is R-separated from each ψ_i with $1 \leqslant i \leqslant k$. Suppose, towards a contradiction, that for some $1 \leqslant i \leqslant k$,

$$\mathrm{dist}(\mathrm{dom}(\psi_0), \mathrm{dom}(\psi_i)) \;\leqslant\; R(\psi_0) + R(\psi_i) \;\leqslant\; 2h.$$

Let $\gamma \in \mathrm{dom}(\psi_0)$ be such that $\mathrm{dist}(\gamma, \mathrm{dom}(\psi_i)) \leqslant 2h$. Since $\mathrm{dom}(\psi_0) \subseteq \mathbf{Ball}(\gamma_0, h)$, we obtain

$$\mathrm{dist}(\gamma_0, \mathrm{dom}(\psi_i)) \;\leqslant\; \mathrm{dist}(\gamma_0, \gamma) + \mathrm{dist}(\gamma, \mathrm{dom}(\psi_i)) \;\leqslant\; h + 2h \;=\; 3h.$$

On the other hand, by construction, $\mathrm{dom}(\psi_i) \cap \mathbf{Ball}(\gamma_0, 3h) = \varnothing$. This contradiction completes the proof. ⊣

Consider any $\varphi \in \mathfrak{I}'$ and let $\varphi = \varphi_1 \cup \ldots \cup \varphi_k$ be a decomposition of φ given by Claim (A). For each $1 \leqslant i \leqslant k$, let $\psi_i \coloneqq \pi_2 \circ \varphi_i$. Then every ψ_i belongs to $\mathfrak{I}$ and $\psi_1, \ldots, \psi_k$ are pairwise R-separated. By the choice of R, this yields

$$\pi_2 \circ \varphi \;=\; \psi_1 \cup \ldots \cup \psi_k \;\in\; \mathfrak{I}.$$

Therefore, π_2 is a reduction of $\mathfrak{I}$ to $\mathfrak{I}'$.

Finally, to see that $\mathfrak{I}'$ is extendable, let $\varphi \in \mathfrak{I}'$ and let $\gamma \in \Gamma \setminus \mathrm{dom}(\varphi)$. Set $\psi \coloneqq \pi_2 \circ \varphi$. We already know that $\psi \in \mathfrak{I}$. Since $\mathfrak{I}$ is extendable, there is $c \in \mathbb{N}$ such that $\psi' \coloneqq \psi \cup \{(\gamma, c)\} \in \mathfrak{I}$. Choose $h \in \mathbb{N}$ so large that the following statements are true:

$$h \geqslant R(\psi'); \qquad h > (\pi_1 \circ \varphi)(\delta) \text{ for all } \delta \in \mathrm{dom}(\varphi); \qquad \text{and} \qquad \mathbf{Ball}(\gamma, h) \supseteq \mathrm{dom}(\psi').$$

Then $\varphi \cup \{(\gamma, (h, c))\} \in \mathfrak{I}'$, as desired. ■

5.5.3 Baire measurable colorings from locality and extendability

Before proceeding with the last part of the proof of Theorem 5.2.10, we introduce some terminology and notation related to partial (but not necessarily finite) maps $\varphi \colon \Gamma \rightharpoonup \mathbb{N}$. The set of all such maps is denoted by $[\Gamma \to \mathbb{N}]$. A partial map $\varphi \colon \Gamma \rightharpoonup \mathbb{N}$ can be viewed as a (total) function $\varphi \colon \Gamma \to \mathbb{N} \cup \{\text{undefined}\}$, where "undefined" is a special symbol distinct from all the elements of $\mathbb{N}$. In that way,

$$[\Gamma \to \mathbb{N}] \quad \text{is the same as} \quad (\mathbb{N} \cup \{\text{undefined}\})^\Gamma,$$

and the latter set carries the product topology (the topology on $\mathbb{N} \cup \{\text{undefined}\}$ is discrete) and is equipped with the shift action of Γ. Note that $[\Gamma \to \mathbb{N}]^{<\infty}$ is a countable dense subset of $[\Gamma \to \mathbb{N}]$ and $\mathbb{N}^\Gamma$ is a closed subset of $[\Gamma \to \mathbb{N}]$.

Similarly to the notation we use for finite partial functions, given $\varphi \in [\Gamma \to \mathbb{N}]$, an element $\gamma \in \Gamma$, and a radius $r \in [0; +\infty)$, let

$$\varphi[\gamma, r] \coloneqq \varphi|(\mathrm{dom}(\varphi) \cap \mathbf{Ball}(\gamma, r)).$$

By definition, $\varphi[\gamma, r] \in [\Gamma \to \mathbb{N}]^{<\infty}$.

Let $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ be a Γ-ideal. A **partial $\mathfrak{I}$-coloring** is a map $\varphi \in [\Gamma \to \mathbb{N}]$ such that

$$\varphi|S \in \mathfrak{I} \quad \text{for all} \quad S \in [\mathrm{dom}(\varphi)]^{<\infty}.$$

The set of all partial $\mathfrak{I}$-colorings is denoted by $[\Gamma \to \mathbb{N}]^{\mathfrak{I}}$. Note that

$$[\Gamma \to \mathbb{N}]^{\mathfrak{I}} \cap [\Gamma \to \mathbb{N}]^{<\infty} = \mathfrak{I} \quad \text{and} \quad [\Gamma \to \mathbb{N}]^{\mathfrak{I}} \cap \mathbb{N}^\Gamma = \mathbf{Col}(\mathfrak{I}).$$

If $\mathfrak{I}$ is local and $r \colon \mathbb{N} \to [0; +\infty)$ is a function such that $\mathfrak{I} = \mathbf{Loc}_r(\mathfrak{I})$, then for all $\varphi \in [\Gamma \to \mathbb{N}]$,

$$\varphi \in [\Gamma \to \mathbb{N}]^{\mathfrak{I}} \iff \varphi[\gamma, r(\varphi(\gamma))] \in \mathfrak{I} \text{ for all } \gamma \in \mathrm{dom}(\varphi). \tag{5.5.2}$$

Lemma 5.5.3. *If $\mathfrak{I}$ is a local extendable Γ-ideal, then σ admits a Baire measurable $\mathfrak{I}$-coloring.*

Proof. Let $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ be a local extendable Γ-ideal and let $r \colon \mathbb{N} \to [0; +\infty)$ be a function such that $\mathfrak{I} = \mathbf{Loc}_r(\mathfrak{I})$.

Using Theorem 5.2.4, identify σ with the shift action $\sigma_{2^\mathbb{N}} \colon \Gamma \curvearrowright (2^\mathbb{N})^\Gamma$ and then replace it by the product action $(\sigma_2)^\mathbb{N} \colon \Gamma \curvearrowright (2^\Gamma)^\mathbb{N}$ (the spaces $(2^\mathbb{N})^\Gamma$ and $(2^\Gamma)^\mathbb{N}$ are equivariantly homeomorphic). We will find an equivariant Borel map $\pi \colon (2^\Gamma)^\mathbb{N} \to [\Gamma \to \mathbb{N}]^{\mathfrak{I}}$ such that the set $\pi^{-1}(\mathbb{N}^\Gamma)$ is comeager. This will yield the desired result since given such π, any function $f \colon (2^\Gamma)^\mathbb{N} \to \mathbb{N}$ with $f(x) = \pi(x)(\mathbf{1})$ for all $x \in \pi^{-1}(\mathbb{N}^\Gamma)$ is a Baire measurable $\mathfrak{I}$-coloring of $(2^\Gamma)^\mathbb{N}$.

For $x \in 2^\Gamma$, the **support** of x is the set

$$\mathrm{supp}(x) \coloneqq \{\gamma \in \Gamma \,:\, x(\gamma) = 1\}.$$

Set $X \coloneqq (2^\Gamma)^\mathbb{N}$. We write the elements of X as sequences of the form $x = (x_0, x_1, \ldots)$.

Fix a sequence $(c_0, c_1, \ldots) \in \mathbb{N}^\mathbb{N}$ in which every $c \in \mathbb{N}$ appears infinitely many times and set

$$R_i \coloneqq \sup\{r(c_0), \ldots, r(c_{i-1})\}.$$

For each $x \in X$, define a sequence of partial maps $\pi_i(x) \in [\Gamma \to \mathbb{N}]$ inductively as follows:

> STEP 0: Set $\pi_0(x) \coloneqq \varnothing$.
> STEP $i + 1$: Let $S_i(x)$ denote the set of all $\gamma \in \Gamma$ such that
>
> – $\pi_i(x)(\gamma)$ is not defined;
>
> – $\mathbf{Ball}(\gamma, 2R_i) \cap \mathrm{supp}(x_i) = \{\gamma\}$; and
>
> – $\pi_i(x)[\gamma, 2R_i] \cup \{(\gamma, c_i)\} \in \mathfrak{I}$.
>
> For all $\gamma \in \Gamma$, set
> $$\pi_{i+1}(x)(\gamma) \coloneqq \begin{cases} \pi_i(x)(\gamma) & \text{if } \pi_i(x)(\gamma) \text{ is defined;} \\ c_i & \text{if } \gamma \in S_i(x); \\ \text{undefined} & \text{otherwise.} \end{cases}$$

By construction, for all $x \in X$, we have

$$\varnothing = \pi_0(x) \subseteq \pi_1(x) \subseteq \ldots,$$

so we can define $\pi_\infty(x) \in [\Gamma \to \mathbb{N}]$ via

$$\pi_\infty(x) \coloneqq \bigcup_{i=0}^{\infty} \pi_i(x).$$

It is clear that the maps $\pi_i \colon X \to [\Gamma \to \mathbb{N}]$ are equivariant. Notice that they are also continuous. Indeed, the value $\pi_i(x)(\gamma)$—including whether or not it is defined—is determined by the restrictions of the first i functions $x_0, \ldots, x_{i-1}$ to the finite set $\mathbf{Ball}(\gamma, 2R_0 + \cdots + 2R_{i-1})$. Being a pointwise limit of equivariant continuous functions, the map $\pi_\infty \colon X \to [\Gamma \to \mathbb{N}]$ is equivariant and Borel.

Claim (A). *For all $x \in X$, we have $\pi_\infty(x) \in [\Gamma \to \mathbb{N}]^{\mathfrak{I}}$.*

Proof. Let $x \in X$. Since $\pi_\infty(x)$ is the union of the increasing sequence $\pi_0(x) \subseteq \pi_1(x) \subseteq \ldots$, it is sufficient (and necessary) to establish that $\pi_i(x) \in [\Gamma \to \mathbb{N}]^{\mathfrak{I}}$ for all $i \in \mathbb{N}$. We proceed by induction on i. The base case is trivial since $\pi_0(x) = \varnothing \in \mathfrak{I}$ by definition (recall that $\mathfrak{I}$ is local, hence nonempty). Now suppose that $\pi_i(x) \in [\Gamma \to \mathbb{N}]^{\mathfrak{I}}$ and consider the partial map $\pi_{i+1}(x)$. By (5.5.2), it is enough to show that for all $\gamma \in \mathrm{dom}(\pi_{i+1}(x))$,

$$\pi_{i+1}(x)[\gamma, r(\pi_{i+1}(x)(\gamma))] \in \mathfrak{I}.$$

By construction, $\pi_{i+1}(x)$ takes values in the set $\{c_0, \ldots, c_i\}$. Therefore,

$$r(\pi_{i+1}(x)(\gamma)) \leqslant R_i \quad \text{for all} \quad \gamma \in \mathrm{dom}(\pi_{i+1}(x)).$$

Thus, it suffices to prove that for all $\gamma \in \mathrm{dom}(\pi_{i+1}(x))$,

$$\pi_{i+1}(x)[\gamma, R_i] \in \mathfrak{I}.$$

Consider any $\gamma \in \mathrm{dom}(\pi_{i+1}(x))$. If $\mathbf{Ball}(\gamma, R_i) \cap S_i(x) = \varnothing$, then

$$\pi_{i+1}(x)[\gamma, R_i] = \pi_i(x)[\gamma, R_i] \in \mathfrak{I}$$

by the inductive hypothesis. Now assume that $\delta \in \mathbf{Ball}(\gamma, R_i) \cap S_i(x)$. Then $\mathbf{Ball}(\gamma, R_i) \subseteq \mathbf{Ball}(\delta, 2R_i)$, so it is enough to show

$$\pi_{i+1}(x)[\delta, 2R_i] \in \mathfrak{I}.$$

As $\delta \in S_i(x)$, we have $\pi_{i+1}(x)(\delta) = c_i$ and

$$\delta \in \mathbf{Ball}(\delta, 2R_i) \cap S_i(x) \subseteq \mathbf{Ball}(\delta, 2R_i) \cap \mathrm{supp}(x_i) = \{\delta\},$$

which implies

$$\pi_{i+1}(x)[\delta, 2R_i] = \pi_i(x)[\delta, 2R_i] \cup \{(\delta, c_i)\} \in \mathfrak{I}. \qquad \dashv$$

Thus, the above construction produces an equivariant Borel map $\pi_\infty \colon X \to [\Gamma \to \mathbb{N}]^{\mathfrak{I}}$. To finish the argument, it remains to show that the set $(\pi_\infty)^{-1}(\mathbb{N}^\Gamma)$ is comeager. We have

$$\begin{aligned}(\pi_\infty)^{-1}(\mathbb{N}^\Gamma) &= \{x \in X \,:\, \pi_\infty(x)(\gamma) \text{ is defined for all } \gamma \in \Gamma\} \\ &= \bigcap_{\gamma \in \Gamma} \{x \in X \,:\, \pi_\infty(x)(\gamma) \text{ is defined}\},\end{aligned}$$

so we only need to verify that for each $\gamma \in \Gamma$, the set $\{x \in X \,:\, \pi_\infty(x)(\gamma) \text{ is defined}\}$ is comeager. To that end, consider any $\gamma \in \Gamma$ and write

$$\{x \in X \,:\, \pi_\infty(x)(\gamma) \text{ is defined}\} = \bigcup_{i=0}^{\infty} \{x \in X \,:\, \pi_i(x)(\gamma) \text{ is defined}\}.$$

By the continuity of π_i for all $i \in \mathbb{N}$, the sets $\{x \in X \,:\, \pi_i(x)(\gamma) \text{ is defined}\}$ are open. Therefore, their union is open as well; it remains to show that it is dense. Let $U \subseteq X$ be a nonempty open subset. We need to find an element $x \in U$ such that $\pi_\infty(x)(\gamma)$ is defined. By passing to a smaller open subset if necessary, we may assume that U is of the form

$$U = U_0 \times \cdots \times U_{i-1} \times 2^\Gamma \times 2^\Gamma \cdots,$$

for some nonempty open subsets $U_0, \ldots, U_{i-1} \subseteq 2^\Gamma$. Notice that the set of all functions $\Gamma \to 2$ with finite support is dense in 2^Γ; therefore, for each $0 \leqslant k < i$, we can choose $y_k \in U_k$ so that $\mathrm{supp}(y_k)$ is finite. Let

$$A \coloneqq \{x \in X \,:\, x_k = y_k \text{ for all } 0 \leqslant k < i\}.$$

By the choice of $y_0, \ldots, y_{i-1}$, we have $\varnothing \neq A \subseteq U$. Since for all $x \in X$, the value $\pi_i(x)$ is determined by the first i functions $x_0, \ldots, x_{i-1}$, we can define $\varphi \in [\Gamma \to \mathbb{N}]^{\mathfrak{I}}$ by

$$\varphi := \pi_i(x) \quad \text{for some (hence all)} \quad x \in A.$$

If $\gamma \in \mathrm{dom}(\varphi)$, then $\pi_\infty(x)(\gamma)$ is defined for all $x \in A$ and we are done, so assume that $\gamma \notin \mathrm{dom}(\varphi)$. Since

$$\mathrm{dom}(\varphi) \subseteq \mathrm{supp}(x_0) \cup \ldots \cup \mathrm{supp}(x_{i-1}),$$

the domain of φ is finite, i.e., $\varphi \in \mathfrak{I}$. The Γ-ideal $\mathfrak{I}$ is extendable, so there is $c \in \mathbb{N}$ such that

$$\psi := \varphi \cup \{(\gamma, c)\} \in \mathfrak{I}.$$

By the choice of the sequence $(c_0, c_1, \ldots)$, there is some index $j \geqslant i$ such that $c_j = c$. For all $i \leqslant k < j$, set $y_k \colon \Gamma \to 2$ to be the constant 0 function, and set $y_j \colon \Gamma \to 2$ to be the characteristic function of the one-element set $\{\gamma\}$. Let

$$B := \{x \in X \,:\, x_k = y_k \text{ for all } 0 \leqslant k \leqslant j\}.$$

Then $\varnothing \neq B \subseteq A$, and for all $x \in B$, we have $\pi_{j+1}(x) = \psi$, in particular, $\pi_\infty(x)(\gamma)$ is defined. ■

5.5.4 Proof of Corollary 5.2.12

A continuous action α of Γ on a compact space is **minimal** if every α-orbit is dense. It follows from a result of Gao, Jackson, and Seward [GJS16, Theorem 1.4.1] that Γ admits a free minimal action on a nonempty compact metrizable space. (For more information on the Gao–Jackson–Seward theorem, see Chapter 4.)

Recall the following notation, introduced in §5.4.3: For a free action $\alpha \colon \Gamma \curvearrowright X$ and $x, y \in X$, write

$$\mathrm{dist}(x, y) := \begin{cases} \mathrm{dist}(\mathbf{1}, \gamma) & \text{if } \gamma \in \Gamma \text{ is such that } \gamma \cdot x = y; \\ \infty & \text{if } x \text{ and } y \text{ are in different } \alpha\text{-orbits.} \end{cases}$$

Given $x \in X$ and $A \subseteq X$, let

$$\mathrm{dist}(x, A) := \inf\{\mathrm{dist}(x, y) \,:\, y \in A\}.$$

Lemma 5.5.4. *Let $\alpha \colon \Gamma \curvearrowright X$ be a free minimal action of Γ on a nonempty compact metrizable space X. Suppose that $A \subseteq X$ is a nonmeager Baire measurable set. Then there exists a radius $R \in [0; +\infty)$ such that the set $\{x \in X \,:\, \mathrm{dist}(x, A) \leqslant R\}$ is comeager.*

Proof. By the Baire alternative, there is nonempty open $U \subseteq X$ such that $U \Vdash A$. Since α is minimal, we have $X = \bigcup_{\gamma \in \Gamma} (\gamma \cdot U)$. As X is compact, there is a radius $R \in [0; +\infty)$ such that

$$X = \bigcup_{\gamma \in \mathbf{Ball}(\mathbf{1}, R)} (\gamma \cdot U).$$

Therefore, the set

$$X' := \bigcup_{\gamma \in \mathbf{Ball}(\mathbf{1}, R)} (\gamma \cdot A)$$

is comeager. It remains to notice that $\operatorname{dist}(x, A) \leqslant R$ for all $x \in X'$. ■

Proof of Corollary 5.2.12. Let $\alpha \colon \Gamma \curvearrowright X$ be any free minimal action of Γ on a nonempty compact metrizable space X. We will explicitly construct a Γ-ideal $\mathfrak{I} \subseteq [\Gamma \to \mathbb{N}]^{<\infty}$ such that σ admits a Baire measurable $\mathfrak{I}$-coloring, while α does not.

Fix a sequence $(d_0, d_1, \ldots) \in [0; +\infty)^{\mathbb{N}}$ such that a ball of radius d_0 in Γ contains at least 2 elements, and for each $c \in \mathbb{N}$, a ball of radius d_{c+1} in Γ contains two disjoint balls of radius d_c (such a sequence exists since Γ is infinite, while every ball of finite radius in Γ is finite). For each $c \in \mathbb{N}$, choose $D_c \in [0; +\infty)$ so that the set $\{\gamma \in \Gamma \,:\, 2d_c < \operatorname{dist}(\mathbf{1}, \gamma) \leqslant D_c\}$ contains a ball of radius d_c.

Let $\mathfrak{I}$ denote the set of all partial maps $\varphi \in [\Gamma \to \mathbb{N}]^{<\infty}$ such that the following holds for all $\gamma \in \operatorname{dom}(\varphi)$: If we let $c \coloneqq \varphi(\gamma)$, then for all $\delta \in \operatorname{dom}(\varphi)$,

(1) if $\operatorname{dist}(\gamma, \delta) \leqslant 2d_c$, then $\varphi(\delta) \neq c$;

(2) if $2d_c < \operatorname{dist}(\gamma, \delta) \leqslant D_c$, then $\varphi(\delta) > c$.

Clearly, $\mathfrak{I}$ is a Γ-ideal. By definition, we have

$$\mathfrak{I} = \mathbf{Loc}_r(\mathfrak{I}) \quad \text{for} \quad r \colon \mathbb{N} \to [0; +\infty) \colon c \mapsto D_c,$$

so $\mathfrak{I}$ is local. Consider any $\varphi \in \mathfrak{I}$ and $\gamma \in \Gamma \setminus \operatorname{dom}(\varphi)$. Choose $c \in \mathbb{N}$ so large that the following statements are true:

$$c > \varphi(\delta) \text{ for all } \delta \in \operatorname{dom}(\varphi) \qquad \text{and} \qquad \mathbf{Ball}(\gamma, 2d_c) \supseteq \operatorname{dom}(\varphi).$$

Then $\varphi \cup \{(\gamma, c)\} \in \mathfrak{I}$. This shows that $\mathfrak{I}$ is extendable. Using Theorem 5.2.10, we then conclude that σ admits a Baire measurable $\mathfrak{I}$-coloring.

Now suppose, towards a contradiction, that $f \colon X \to \mathbb{N}$ is a Baire measurable $\mathfrak{I}$-coloring of α. Let $c_0 \in \mathbb{N}$ be any color such that the set $A \coloneqq f^{-1}(c_0)$ is nonmeager. By Lemma 5.5.4, there is a radius $R \in [0; +\infty)$ such that the set $\{x \in X \,:\, \operatorname{dist}(x, A) \leqslant R\}$ is comeager. Since the set $(\pi_f)^{-1}(\mathbf{Col}(\mathfrak{I}))$ is also comeager, we can choose $x \in X$ so that

$$\pi_f(x) \in \mathbf{Col}(\mathfrak{I}) \qquad \text{and} \qquad \operatorname{dist}(\gamma \cdot x, A) \leqslant R \text{ for all } \gamma \in \Gamma.$$

Let $\omega \coloneqq \pi_f(x)$. Since $\omega \in \mathbf{Col}(\mathfrak{I})$, Lemma 5.4.10 implies that the set $\{\omega(\gamma) \,:\, \gamma \in \Gamma\}$ is infinite; in particular, it contains an element c such that $c \geqslant c_0$ and $d_c \geqslant R$. Take any $\gamma \in \Gamma$ with $\omega(\gamma) = c$. By the choice of D_c, there is some $\delta \in \Gamma$ satisfying

$$\mathbf{Ball}(\delta, R) \subseteq \mathbf{Ball}(\delta, d_c) \subseteq \{\varepsilon \in \Gamma \,:\, 2d_c < \operatorname{dist}(\gamma, \varepsilon) \leqslant D_c\}.$$

Since $\omega \in \mathbf{Col}(\mathfrak{I})$, we have $\omega(\varepsilon) > c$ for all $\varepsilon \in \mathbf{Ball}(\delta, R)$; in particular, there is no $\varepsilon \in \mathbf{Ball}(\delta, R)$ with $\omega(\varepsilon) = c_0$. But then $\operatorname{dist}(\delta \cdot x, A) > R$; a contradiction. ■

6 | Results on weak containment of probability measure-preserving actions

6.1 Weak containment and weak equivalence

Throughout this chapter, Γ denotes a countably infinite group with identity element $\mathbf{1}$. The concepts of *weak containment* and *weak equivalence* of p.m.p. actions of a countable group Γ were introduced by Kechris in [Kec10, Section 10(C)]. They were inspired by the analogous notions for unitary representations and are closely related to the so-called *local-global convergence* in the theory of graph limits [HLS14]. Roughly speaking, a p.m.p. action $\alpha \colon \Gamma \curvearrowright (X, \mu)$ is weakly contained in another p.m.p. action $\beta \colon \Gamma \curvearrowright (Y, \nu)$, in symbols $\alpha \preccurlyeq \beta$, if the interaction between any finite measurable coloring of X and a finite collection of elements of Γ can be simulated, with arbitrarily small error, by a measurable coloring of Y (see Definition 6.1.1). If both $\alpha \preccurlyeq \beta$ and $\beta \preccurlyeq \alpha$, then α and β are said to be weakly equivalent, in symbols $\alpha \simeq \beta$.

The relation of weak equivalence is much coarser than the conjugacy relation, which makes it relatively well-behaved. On the other hand, several interesting parameters associated with p.m.p. actions—such as their cost, type, etc.—turn out to be invariants of weak equivalence. Due to these favorable properties, the relations of weak containment and weak equivalence have attracted a considerable amount of attention in recent years. For a survey of the topic, see [BK17b].

A number of equivalent definitions of weak containment exist, and several of them can be found in [BK17b, §§2.1, 2.2]. We use a slight variation of the characterization given in [BK17b, Theorem 2.5(iv)], due to Abért and Weiss [AW13, Lemma 8] (see also [Tuc15, Proposition 3.6]). Let $\alpha \colon \Gamma \curvearrowright X$ be an action of Γ on a set X and let $f \colon X \to k \in \mathbb{N}$ be a finite coloring of X. Define an equivariant map $\pi_f \colon X \to k^\Gamma$ by

$$\pi_f(x)(\gamma) \coloneqq f(\gamma \cdot x) \qquad \text{for all } x \in X \text{ and } \gamma \in \Gamma.$$

Given a p.m.p. action $\alpha \colon \Gamma \curvearrowright (X, \mu)$, $k \in \mathbb{N}$, a Borel function $f \colon X \to k$, and a map $w \in [\Gamma \to k]^{<\infty}$, the (**global**) **frequency** $\Phi_\mu(\alpha, f, w)$ of w in (α, f) with respect to μ is defined by

$$\Phi_\mu(\alpha, f, w) \coloneqq \mu(\{x \in X \,:\, \pi_f(x) \supseteq w\}). \tag{6.1.1}$$

This chapter is partially based on [Ber18b].

Definition 6.1.1. Let $\alpha\colon \Gamma \curvearrowright (X, \mu)$ and $\beta\colon \Gamma \curvearrowright (Y, \nu)$ be p.m.p. actions of Γ. We say that α is **weakly contained** in β, in symbols $\alpha \preccurlyeq \beta$, if for every $S \in [\Gamma]^{<\infty}$ and for all $\varepsilon > 0$, the following holds: Let $k \in \mathbb{N}$ and let $f\colon X \to k$ be a Borel function. Then there exists a Borel map $g\colon Y \to k$ satisfying

$$|\Phi_\nu(\beta, g, w) - \Phi_\mu(\alpha, f, w)| < \varepsilon \qquad \text{for all } w\colon S \to k.$$

If simultaneously $\alpha \preccurlyeq \beta$ and $\beta \preccurlyeq \alpha$, then α and β are said to be **weakly equivalent**, in symbols $\alpha \simeq \beta$.

Burton [Bur16, Corollary 4.2] (see also [BK17b, Theorem 3.3]) proved that if Γ is infinite, then there exist continuumly many distinct weak equivalence classes. Glasner, Thouvenot, and Weiss [GTW06] and independently Hjorth (unpublished) proved that the pre-order of weak containment has a maximum element (see [BK17b, Theorem 3.1]). A complementary result of Abért and Weiss [AW13, Theorem 1] (see also [BK17b, Theorem 3.5]) asserts that the shift action $\sigma \coloneqq \sigma_{[0;1]}\colon \Gamma \curvearrowright ([0;1]^\Gamma, \lambda^\Gamma)$ is minimum among all free p.m.p. actions of Γ:

Theorem 6.1.2 (Abért–Weiss [AW13, Theorem 1])**.** *If $\alpha\colon \Gamma \curvearrowright (X, \mu)$ is a free p.m.p. action of Γ, then $\sigma \preccurlyeq \alpha$.*

In §6.2 we strengthen and generalize Theorem 6.1.2 by replacing the frequencies defined in (6.1.1) by certain pointwise averages. Our results in that section serve as further applications of the Lovász Local Lemma and its measurable analogs in ergodic theory.

A useful feature of weak equivalence is that the set of all weak equivalence classes carries a natural compact metrizable topology, introduced by Abért and Elek [AE11]. In §6.3 we study how this topology interacts with taking products of actions and show that for a certain family of groups, including the non-Abelian free groups, multiplication of weak equivalence classes is a discontinuous operation.

6.2 Pointwise analogs of the Abért–Weiss theorem

6.2.1 Definitions and results

Let $\alpha\colon \Gamma \curvearrowright X$ be a Borel action of Γ on a standard Borel space X. Let $k \in \mathbb{N}$ and let $f\colon X \to k$ be a Borel function. Fix $x \in X$ and $\varnothing \neq D \in [\Gamma]^{<\infty}$. For a map $w \in [\Gamma \to k]^{<\infty}$, the **local frequency** $\Phi_{x,D}(\alpha, f, w)$ of w in (α, f) with respect to (x, D) is defined by

$$\Phi_{x,D}(\alpha, f, w) \coloneqq \frac{|\{\delta \in D \,:\, \pi_f(\delta \cdot x) \supseteq w\}|}{|D|}. \tag{6.2.1}$$

There is an obvious relationship between local and global frequencies:

Proposition 6.2.1. *Let $\alpha\colon \Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of Γ. Let $k \in \mathbb{N}$ and let $f\colon X \to k$ be a Borel function. If $\varnothing \neq D \in [\Gamma]^{<\infty}$, then for all $w \in [\Gamma \to k]^{<\infty}$, we have*

$$\Phi_\mu(\alpha, f, w) = \int_X \Phi_{x,D}(\alpha, f, w)\, \mathrm{d}\,\mu(x).$$

Proof. Follows from the α-invariance of μ. ■

Given the following data:

- Borel actions $\alpha \colon \Gamma \curvearrowright X$ and $\beta \colon \Gamma \curvearrowright Y$;
- Borel maps $f \colon X \to k$ and $g \colon Y \to k$;
- $S \in [\Gamma]^{<\infty}$ and $\varepsilon > 0$;
- $\mu \in \mathbf{Prob}(X)$ and $\varnothing \neq D \in [\Gamma]^{<\infty}$,

let $\mathcal{A}_{S,\varepsilon}(\alpha, f, \mu; \beta, g, D)$ denote the set of all points $y \in Y$ satisfying

$$|\Phi_{y,D}(\beta, g, w) - \Phi_\mu(\alpha, f, w)| < \varepsilon \qquad \text{for all } w \colon S \to k.$$

Due to Proposition 6.2.1, if ν is a β-invariant probability Borel measure on Y, then for any $w \colon S \to k$,

$$|\Phi_\nu(\beta, g, w) - \Phi_\mu(\alpha, f, w)| < \varepsilon + 1 - \nu(\mathcal{A}_{S,\varepsilon}(\alpha, f, \mu; \beta, g, D)),$$

so if the measure of the set $\mathcal{A}_{S,\varepsilon}(\alpha, f, \mu; \beta, g, D)$ is close to 1, then g approximates well the statistics of f and thus can be used as a witness to the weak containment $\alpha \preccurlyeq \beta$.

We need one last definition before stating our first result. An action $\alpha \colon \Gamma \curvearrowright X$ is S-**free**, where $S \subseteq \Gamma$, if

$$\gamma \cdot x = \delta \cdot x \implies \gamma = \delta \qquad \text{for all } \gamma,\ \delta \in S \text{ and } x \in X.$$

Thus, "free" is the same as "Γ-free." The following statement is a natural pointwise refinement of the Abért–Weiss Theorem 6.1.2 (recall that $\sigma \coloneqq \sigma_{[0;1]} \colon \Gamma \curvearrowright ([0;1]^\Gamma, \lambda^\Gamma)$ denotes the $[0;1]$-shift action of Γ):

Theorem 6.2.2. *Let $k \in \mathbb{N}$ and let $f \colon [0;1]^\Gamma \to k$ be a Borel function. Fix $S \in [\Gamma]^{<\infty}$ and $\varepsilon > 0$. There exist $S' \in [\Gamma]^{<\infty}$ and $n \in \mathbb{N}$ such that for all $D \in [\Gamma]^{<\infty}$ with $|D| > n$, the following holds: Let $\alpha \colon \Gamma \curvearrowright X$ be an $(S' \cup D)$-free Borel action of Γ on a standard Borel space X and let $\mu \in \mathbf{Prob}(X)$. Then for any $\delta > 0$, there exists a Borel map $g \colon X \to k$ satisfying*

$$\mu(\mathcal{A}_{S,\varepsilon}(\sigma, f, \lambda^\Gamma; \alpha, g, D)) \geqslant 1 - \delta.$$

The measure μ in the statement of Theorem 6.2.2 is not required to be α-invariant (or even α-quasi-invariant). We also emphasize that the averaging set D in Theorem 6.2.2 does not depend on the choice of δ. We do not know if, in general, Theorem 6.2.2 also holds with $\delta = 0$. However, it does so under certain conditions described in our next two results.

Notice that Theorem 6.2.2 is meaningful even when it is applied with $\alpha = \sigma$. In that case, it can be further strengthened by taking $\delta = 0$ and, moreover, by ensuring that the map g can be obtained by adjusting the given function f on a set of arbitrarily small measure. Given a standard probability space (X, μ) and Borel functions $f, g \colon X \to k$, define

$$\mathrm{dist}_\mu(f, g) \coloneqq \mu(\{x \in X \ :\ f(x) \neq g(x)\}).$$

Theorem 6.2.3. *Let $k \in \mathbb{N}$ and let $f\colon [0;1]^\Gamma \to k$ be a Borel function. Fix $S \in [\Gamma]^{<\infty}$ and $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that for every $D \in [\Gamma]^{<\infty}$ with $|D| > n$, there is a Borel map $g\colon [0;1]^\Gamma \to k$ satisfying*

$$\mathrm{dist}_{\lambda^\Gamma}(f, g) < \varepsilon \qquad \textit{and} \qquad \lambda^\Gamma(\mathcal{A}_{S,\varepsilon}(\sigma, f, \lambda^\Gamma; \sigma, g, D)) = 1.$$

Remark. The usual Abért–Weiss Theorem 6.1.2 can be combined with Theorem 6.2.3 to derive the case of Theorem 6.2.2 for actions that are free and measure-preserving.

Expression (6.2.1) makes sense for an arbitrary Borel action $\alpha\colon \Gamma \curvearrowright X$ and does not require fixing a probability measure on X. It is therefore natural to ask for a purely *Borel* pointwise version of Theorem 6.1.2, and indeed, we establish such a version for finitely generated groups of subexponential growth and, more generally, for uniformly subexponential Borel actions. Let $\alpha\colon \Gamma \curvearrowright X$ be a Borel action of Γ on a standard Borel space X. We say that α is **uniformly subexponential** if for every $S \in [\Gamma]^{<\infty}$ and for all $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and for all $x \in X$, $|S^n \cdot x| \leqslant (1+\varepsilon)^n$, where $S^n \coloneqq \{\gamma_1 \cdots \gamma_n \,:\, \gamma_i \in S \text{ for all } 1 \leqslant i \leqslant n\}$. For example, if Γ is a finitely generated group of subexponential growth, then every action of Γ is uniformly subexponential.

Theorem 6.2.4. *Let $k \in \mathbb{N}$ and let $f\colon [0;1]^\Gamma \to k$ be a Borel function. Fix $S \in [\Gamma]^{<\infty}$ and $\varepsilon > 0$. There exist $S' \in [\Gamma]^{<\infty}$ and $n \in \mathbb{N}$ such that for all $D \in [\Gamma]^{<\infty}$ with $|D| > n$, the following holds: Let $\alpha\colon \Gamma \curvearrowright X$ be a uniformly subexponential $(S' \cup D)$-free Borel action of Γ on a standard Borel space X. Then there exists a Borel map $g\colon X \to k$ satisfying*

$$\mathcal{A}_{S,\varepsilon}(\sigma, f, \lambda^\Gamma; \alpha, g, D) = X.$$

It is important to point out that, even though groups of subexponential growth are amenable, the averaging set D in the statement of Theorem 6.2.4 is *not* assumed to be a Følner set.

We derive Theorems 6.2.2, 6.2.3, and 6.2.4 from a single combinatorial statement, namely Lemma 6.2.10, combined with three different measurable versions of the Lovász Local Lemma, two of which were proved in Chapter 3 (see Theorem 3.4.1 and Corollary 3.5.7) and the other one was established by Csóka, Grabowski, Máthé, Pikhurko, and Tyros in [Csó+16].

6.2.2 Review of the LLL and its measurable analogs

We will use the Symmetric LLL, see [AS00, Corollary 5.1.2] and Lemma 2.3.6. We will apply the LLL in the framework similar to that described in Chapter 3 and in §4.4.2, which we review below for the reader's convenience.

The Symmetric LLL

Let X be a set and let $k \in \mathbb{N}$. Consider any $S \in [X]^{<\infty}$. A subset $B \subseteq k^S$ is called a **bad event** over X with **domain** $\mathrm{dom}(B) \coloneqq S$ (by convention, $\mathrm{dom}(\varnothing) \coloneqq \varnothing$). By definition, every bad event is a subset of $[X \to k]^{<\infty}$. The **probability** of a bad event B is

$$\mathbb{P}[B] \coloneqq \frac{|B|}{k^{|S|}}.$$

A function $f\colon X \to k$ **avoids** a bad event B if there is no $w \in B$ with $w \subseteq f$. An **instance** (**of the LLL**) over X is a set $\mathscr{B}$ of bad events over X. A **solution** to an instance $\mathscr{B}$ is a map $f\colon X \to k$ that avoids all $B \in \mathscr{B}$. For an instance $\mathscr{B}$ and a bad event $B \in \mathscr{B}$, the **neighborhood** of B in $\mathscr{B}$ is

$$N_{\mathscr{B}}(B) \coloneqq \{B' \in \mathscr{B} \setminus \{B\} \ : \ \mathrm{dom}(B') \cap \mathrm{dom}(B) \neq \varnothing\}.$$

The **degree** of B in $\mathscr{B}$ is $\deg_{\mathscr{B}}(B) \coloneqq |N_{\mathscr{B}}(B)|$. Let

$$p(\mathscr{B}) \coloneqq \sup_{B\in\mathscr{B}} \mathbb{P}[B] \qquad \text{and} \qquad d(\mathscr{B}) \coloneqq \sup_{B\in\mathscr{B}} \deg_{\mathscr{B}}(B).$$

An instance $\mathscr{B}$ is **correct for the Symmetric LLL** (the SLLL for short) if

$$e \cdot p(\mathscr{B}) \cdot (d(\mathscr{B}) + 1) < 1,$$

where $e = 2.71\ldots$ denotes the base of the natural logarithm.

Theorem 6.2.5 (**Symmetric Lovász Local Lemma**; cf. Theorem 3.1.6)**.** *Let $\mathscr{B}$ be an instance of the LLL over a set X. If $\mathscr{B}$ is correct for the SLLL, then $\mathscr{B}$ has a solution.*

For finite X, deducing Theorem 6.2.5 from the usual [AS00, Corollary 5.1.2] is routine (see, e.g., [MR02, p. 41]). The infinite case is derived from the finite one via a straightforward compactness argument.

Measurable versions of the LLL for group actions

Now we describe the measurable analogs of the LLL that we will need to prove Theorems 6.2.2, 6.2.3, and 6.2.4. For simplicity, we will confine the current presentation to the case when the underlying combinatorial structure is induced by a Borel group action.

Let $\alpha\colon \Gamma \curvearrowright X$ be a Borel action of Γ on a standard Borel space X and let $k \in \mathbb{N}$. For $f\colon X \rightharpoonup k$, write

$$\pi_f(x)(\gamma) \coloneqq f(\gamma \cdot x) \qquad \text{for all } x \in X \text{ and } \gamma \in \Gamma \text{ such that } \gamma \cdot x \in \mathrm{dom}(f),$$

thus extending the same notation for total functions $f\colon X \to k$. Let $\Phi \subseteq [\Gamma \to k]^{<\infty}$ be a bad event over Γ. For $x \in X$, define

$$B_x(\Phi, \alpha) \coloneqq \{w\colon (\mathrm{dom}(\Phi) \cdot x) \to k \ : \ \pi_w(x) \in \Phi\}.$$

Then $B_x(\Phi, \alpha)$ is a bad event with domain $\mathrm{dom}(\Phi) \cdot x$ (assuming it is nonempty; otherwise the domain of $B_x(\Phi, \alpha)$ is $\varnothing$). Define the instance $\mathscr{B}(\Phi, \alpha)$ over X as follows:

$$\mathscr{B}(\Phi, \alpha) \coloneqq \{B_x(\Phi, \alpha) \ : \ x \in X\}.$$

By definition, a map $f\colon X \to k$ is a solution to $\mathscr{B}(\Phi, \alpha)$ if and only if $\pi_f(x)$ avoids Φ for all $x \in X$. Given a function $f\colon X \to k$, its **defect** $\mathbf{Def}_\Phi(f, \alpha)$ with respect to Φ is the set of all $x \in X$ such that $\pi_f(x)$ does *not* avoid Φ. Thus, f is a solution to $\mathscr{B}(\Phi, \alpha)$ if and only if $\mathbf{Def}_\Phi(f, \alpha) = \varnothing$.

The following result is a corollary of Theorem 3.4.1:

Theorem 6.2.6 (see Theorem 3.4.1)**.** *Let $\alpha\colon \Gamma \curvearrowright X$ be a Borel action of Γ on a standard Borel space X and let $k \in \mathbb{N}$. Let $\Phi \subseteq [\Gamma \to k]^{<\infty}$ be a bad event over Γ and suppose that the instance $\mathscr{B}(\Phi, \alpha)$ is correct for the SLLL. Then, for any $\mu \in \mathbf{Prob}(X)$ and $\delta > 0$, there exists a Borel function $f\colon X \to k$ with $\mu(\mathbf{Def}_\Phi(f, \alpha)) < \delta$.*

If $\alpha = \sigma$, then we can say more:

Theorem 6.2.7 (see Corollary 3.5.7)**.** *Let $k \in \mathbb{N}$ and let $\Phi \subseteq [\Gamma \to k]^{<\infty}$ be a bad event over Γ. Suppose that the instance $\mathscr{B}(\Phi, \sigma)$, restricted to the free part of σ, is correct for the SLLL. Then there is a Borel function $f\colon [0;1]^\Gamma \to k$ with $\lambda^\Gamma(\mathbf{Def}_\Phi(f, \sigma)) = 0$.*

Finally, the following purely Borel version of the LLL was proved by Csóka, Grabowski, Máthé, Pikhurko, and Tyros [Csó+16]:

Theorem 6.2.8 (Csóka–Grabowski–Máthé–Pikhurko–Tyros [Csó+16])**.** *Let $\alpha\colon \Gamma \curvearrowright X$ be a uniformly subexponential Borel action of Γ on a standard Borel space X and let $k \in \mathbb{N}$. Let $\Phi \subseteq [\Gamma \to k]^{<\infty}$ be a bad event over Γ and suppose that the instance $\mathscr{B}(\Phi, \alpha)$ is correct for the SLLL. Then $\mathscr{B}(\Phi, \alpha)$ has a Borel solution $f\colon X \to k$.*

6.2.3 Proofs of Theorems 6.2.2, 6.2.3, and 6.2.4

Let us start with some notation. For $k \in \mathbb{N}$, define a map $f_k\colon k^{\Gamma\times\mathbb{N}} \to k$ by

$$f_k(x) \coloneqq x(\mathbf{1}, 0) \qquad \text{for all } x\colon \Gamma \times \mathbb{N} \to k.$$

The space $k^{\Gamma\times\mathbb{N}}$ can be identified with $(k^{\mathbb{N}})^\Gamma$, and so it is equipped with the shift action $\tilde{\sigma}_k \coloneqq \sigma_{k^{\mathbb{N}}}$ of Γ. Let μ_k be the measure on $k^{\Gamma\times\mathbb{N}}$ obtained as the power of the uniform probability measure on k.

Lemma 6.2.9. *Let $k \in \mathbb{N}$. Fix $S \in [\Gamma]^{<\infty}$ and $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that for all $D \in [\Gamma]^{<\infty}$ with $|D| > n$, the following statements hold:*

(i) *Let $\alpha\colon \Gamma \curvearrowright X$ be an $(S \cup D)$-free Borel action of Γ on a standard Borel space X and let $\mu \in \mathbf{Prob}(X)$. Then for any $\delta > 0$, there exists a Borel map $g\colon X \to k$ satisfying*

$$\mu(\mathcal{A}_{S,\varepsilon}(\tilde{\sigma}_k, f_k, \mu_k; \alpha, g, D)) \geqslant 1 - \delta.$$

(ii) *There is a Borel map $g\colon k^{\Gamma\times\mathbb{N}} \to k$ satisfying*

$$\mathrm{dist}_{\mu_k}(f_k, g) < \varepsilon \qquad \text{and} \qquad \mu_k(\mathcal{A}_{S,\varepsilon}(\tilde{\sigma}_k, f_k, \mu_k; \tilde{\sigma}_k, g, D)) = 1.$$

(iii) *Let $\alpha\colon \Gamma \curvearrowright X$ be a uniformly subexponential $(S \cup D)$-free Borel action of Γ on a standard Borel space X. Then there exists a Borel map $g\colon X \to k$ satisfying*

$$\mathcal{A}_{S,\varepsilon}(\tilde{\sigma}_k, f_k, \mu_k; \alpha, g, D) = X.$$

Proofs of Theorems 6.2.2, 6.2.3, and 6.2.4 (*assuming Lemma 6.2.9*). Let $k \in \mathbb{N}$ and let $f : [0;1]^\Gamma \to k$ be a Borel function. Fix $S \in [\Gamma]^{<\infty}$. Owing to the measure isomorphism theorem, we may replace $[0;1]$ by $2^{\mathbb{N}}$ and assume that f is a function from $2^{\Gamma\times\mathbb{N}}$ to k. Upon changing f on a set of arbitrarily small measure, we can arrange f to be continuous (this uses the fact that the space $2^{\Gamma\times\mathbb{N}}$ is zero-dimensional). This means that there exist $R \in [\Gamma]^{<\infty}$ and $\ell \in \mathbb{N}$ such that the value $f(x)$ for $x : \Gamma \times \mathbb{N} \to 2$ only depends on the restriction of x to $R \times \ell$. Set $m \coloneqq 2^\ell$. Consider the equivariant bijection $2^{\Gamma\times\mathbb{N}} \to m^{\Gamma\times\mathbb{N}}$ that maps each $x : \Gamma \times \mathbb{N} \to 2$ to the function $x' : \Gamma \times \mathbb{N} \to m$ given by

$$x'(\gamma, i) \coloneqq (x(\gamma, im),\ x(\gamma, im+1),\ \ldots,\ x(\gamma, im+m-1)) \qquad \text{for all } \gamma \in \Gamma \text{ and } i \in \mathbb{N}.$$

We use this bijection to replace the underlying space $2^{\Gamma\times\mathbb{N}}$ by $m^{\Gamma\times\mathbb{N}}$, after which for every $x : \Gamma \times \mathbb{N} \to m$, the value $f(x)$ is fully determined by the restriction of $\pi_{f_m}(x)$ to R. It remains to set $S' \coloneqq RS$ and apply Lemma 6.2.9 to the function f_m with S' in place of S and with a small enough ε. ■

In the remainder of this subsection, we prove Lemma 6.2.9. Fix $k \in \mathbb{N}$, $\varnothing \neq S \in [\Gamma]^{<\infty}$, and $\varepsilon > 0$. For $\varnothing \neq D \in [\Gamma]^{<\infty}$, let $\Phi(D) \subseteq [\Gamma \to k]^{<\infty}$ denote the bad event over Γ consisting of all functions $w : SD \to k$ such that for some $u : S \to k$,

$$\left| \frac{|\{\delta \in D \,:\, (\delta \cdot w) \supseteq u\}|}{|D|} - \frac{1}{k^{|S|}} \right| \geqslant \varepsilon.$$

By definition, if $\alpha : \Gamma \curvearrowright X$ is a Borel action of Γ on a standard Borel space X and $g : X \to k$ is a Borel map, then we have

$$\mathcal{A}_{S,\varepsilon}(\tilde{\sigma}_k, f_k, \mu_k; \alpha, g, D) = X \setminus \mathbf{Def}_{\Phi(D)}(g, \alpha). \tag{6.2.2}$$

Lemma 6.2.10. *There exists $n \in \mathbb{N}$ such that for all $D \in [\Gamma]^{<\infty}$ with $|D| > n$ and for every $(S \cup D)$-free action $\alpha : \Gamma \curvearrowright X$, the instance $\mathscr{B}(\Phi(D), \alpha)$ is correct for the SLLL.*

Proof. Let $\varnothing \neq D \in [\Gamma]^{<\infty}$ and let $\alpha : \Gamma \curvearrowright X$ be an $(S \cup D)$-free action of Γ. Set $\Phi \coloneqq \Phi(D)$, $\mathscr{B} \coloneqq \mathscr{B}(\Phi, \alpha)$, and $B_x \coloneqq B_x(\Phi, \alpha)$ for all $x \in X$. By definition,

$$N_{\mathscr{B}}(B_x) = \{B_y \in \mathscr{B} \setminus \{B_x\} \,:\, (SD \cdot y) \cap (SD \cdot x) \neq \varnothing\}.$$

Since $(SD \cdot y) \cap (SD \cdot x) \neq \varnothing$ if and only if $y \in (SD)^{-1}SD \cdot x$, we obtain

$$\deg_{\mathscr{B}}(B_x) \leqslant |(SD)^{-1}SD| - 1 \leqslant |S|^2|D|^2 - 1.$$

(We subtracted 1 since y cannot be equal to x.) Hence $d(\mathscr{B}) \leqslant |S|^2|D|^2 - 1$.

Now we need to bound $p(\mathscr{B})$. To that end, we will use the following concentration result:

Theorem 6.2.11 (**Simple Concentration Bound**; [MR02, p. 79]). *Let ζ be a random variable determined by s independent trials such that changing the outcome of any one trial can affect ζ at most by c. Then*

$$\mathbb{P}[|\zeta - \mathbb{E}\zeta| > t] \leqslant 2\exp\left(-\frac{t^2}{2c^2 s}\right).$$

Consider any $x \in X$. Fix $u\colon S \to k$ and choose $w\colon (SD \cdot x) \to k$ uniformly at random. Since α is S-free, for each $y \in D \cdot x$, we have

$$\mathbb{P}[\pi_w(y) \supseteq u] = \frac{1}{k^{|S|}}.$$

Therefore, since α is also D-free,

$$\mathbb{E}[|\{y \in D \cdot x \,:\, \pi_w(y) \supseteq u\}|] = \sum_{y \in D \cdot x} \mathbb{P}[\pi_w(y) \supseteq u] = \frac{|D|}{k^{|S|}}.$$

For any $z \in SD \cdot x$, if $w, v\colon (SD \cdot x) \to k$ agree on $(SD \cdot x) \setminus \{z\}$, then

$$\{y \in D \cdot x \,:\, \pi_w(y) \supseteq u\} \vartriangle \{y \in D \cdot x \,:\, \pi_v(y) \supseteq u\} \subseteq S^{-1} \cdot z.$$

Since $|S \cdot z| = |S|$, we may apply the Simple Concentration Bound with parameters $s \coloneqq |SD \cdot x| \leqslant |S||D|$, $c \coloneqq |S|$, and $t \coloneqq \varepsilon|D|$ to obtain

$$\mathbb{P}\left[\left||\{y \in D \,:\, \pi_w(y) \supseteq u\}| - \frac{|D|}{k^{|S|}}\right| > \varepsilon|D|\right] \leqslant 2\exp\left(-\varepsilon^2 \frac{|D|}{2|S|^3}\right),$$

and hence,

$$\mathbb{P}[B_x] \leqslant 2k^{|S|}\exp\left(-\varepsilon^2 \frac{|D|}{2|S|^3}\right),$$

and the same upper bound is satisfied by $p(\mathscr{B})$. Therefore, $\mathscr{B}$ is correct for the SLLL as long as

$$e \cdot 2k^{|S|}\exp\left(-\varepsilon^2 \frac{|D|}{2|S|^3}\right) \cdot |S|^2|D|^2 < 1,$$

which holds whenever $|D|$ is sufficiently large. ∎

The combination of (6.2.2) and Lemma 6.2.10 with Theorems 6.2.6, 6.2.7, and 6.2.8 immediately yields most of Lemma 6.2.9. The only claim that remains to be verified is that in part (ii), the function $g\colon k^{\Gamma\times\mathbb{N}} \to k$ can be chosen so that $\mathrm{dist}_{\mu_k}(f_k, g) < \varepsilon$. The argument for this is somewhat more difficult and involves reviewing the *proof* of the measurable LLL in the form of Theorem 6.2.7.

The key tool used to prove Theorem 6.2.7 is the *Moser–Tardos algorithm* that was developed by Moser and Tardos in [MT10]. In §3.2, we gave a thorough presentation of their approach. Here we outline, very briefly, only the most relevant details of Moser–Tardos theory as it applies to our current situation.

Let $\Phi \subseteq [\Gamma \to k]^{<\infty}$ be a bad event. For $x \in k^{\Gamma\times\mathbb{N}}$, let $B_x \coloneqq B_x(\Phi, \tilde{\sigma}_k)$ and let $\mathscr{B}$ denote the instance $\mathscr{B}(\Phi, \tilde{\sigma}_k)$ restricted to the free part of $\tilde{\sigma}_k$. It is not hard to see [AS00, proof of Corollary 5.1.2] that if $\mathscr{B}$ is correct for the SLLL, then

$$p(\mathscr{B}) \leqslant \frac{1}{d(\mathscr{B})+1}\left(1 - \frac{1}{d(\mathscr{B})+1}\right)^{d(\mathscr{B})}. \tag{6.2.3}$$

We say that a number $\omega \in [0;1)$ is a **witness** to the correctness of $\mathscr{B}$ if

$$p(\mathscr{B}) \leqslant \omega(1-\omega)^{d(\mathscr{B})}. \tag{6.2.4}$$

In particular, from (6.2.3) we see that $1/(d(\mathscr{B})+1)$ is a witness to the correctness of $\mathscr{B}$.

Consider the following inductive construction:

> Set $t_0(x) \coloneqq 0$ for all $x \in k^{\Gamma\times\mathbb{N}}$.
>
> STEP $i \in \mathbb{N}$: Define
>
> $$g_i(x) \coloneqq x(\mathbf{1}, t_i(x)) \text{ for all } x \in k^{\Gamma\times\mathbb{N}} \qquad \text{and} \qquad A_i' \coloneqq \{x \in k^{\Gamma\times\mathbb{N}} \,:\, g_i \supseteq w \text{ for some } w \in B_x\}.$$
>
> Choose A_i to be an arbitrary Borel maximal subset of A_i' with the property that
>
> $$(\mathrm{dom}(\Phi)\cdot x) \cap (\mathrm{dom}(\Phi)\cdot y) = \varnothing \qquad \text{for all distinct } x,\ y \in A_i.$$
>
> (Such A_i exists by [KST99, Proposition 4.2].) Let
>
> $$t_{i+1}(x) \coloneqq \begin{cases} t_i(x)+1 & \text{if } x \in \mathrm{dom}(\Phi)\cdot y \text{ for some } y \in A_i; \\ t_i(x) & \text{otherwise.} \end{cases}$$

A sequence $\mathcal{A} = (A_i)_{i=0}^{\infty}$ obtained via the above procedure is called a **Borel Moser–Tardos process**. Note that, by definition, the map g_0 always coincides with f_k.

Let $\mathcal{A} = (A_i)_{i=0}^{\infty}$ be a Borel Moser–Tardos process. The sequence $t_0(x), t_1(x), \ldots$ is non-decreasing for all $x \in k^{\Gamma\times\mathbb{N}}$. We say that $x \in k^{\Gamma\times\mathbb{N}}$ is **$\mathcal{A}$-stable** if the sequence $t_0(x), t_1(x), \ldots$ is eventually constant. Let $\mathbf{Stab}(\mathcal{A}) \subseteq k^{\Gamma\times\mathbb{N}}$ denote the set of all $\mathcal{A}$-stable elements. For $x \in \mathbf{Stab}(\mathcal{A})$, define

$$t(x) \coloneqq \lim_{n\to\infty} t_n(x) \qquad \text{and} \qquad g(x) \coloneqq x(\mathbf{1}, t(x)). \tag{6.2.5}$$

In is easy to verify (see Proposition 3.2.3) that if $\mathrm{dom}(\Phi)\cdot x \subseteq \mathbf{Stab}(\mathcal{A})$, then $x \notin \mathbf{Def}_\Phi(g, \tilde{\sigma}_k)$. Define the **index** $\mathbf{Ind}(x, \mathcal{A}) \in \mathbb{N}\cup\{\infty\}$ of $x \in k^{\Gamma\times\mathbb{N}}$ in $\mathcal{A}$ by

$$\mathbf{Ind}(x, \mathcal{A}) \coloneqq |\{i \in \mathbb{N} \,:\, x \in A_i\}|.$$

Note that for all $x \in k^{\Gamma\times\mathbb{N}}$,

$$\lim_{i\to\infty} t_i(x) = \sum_{y \in (\mathrm{dom}(\Phi))^{-1}\cdot x} \mathbf{Ind}(y, \mathcal{A}), \tag{6.2.6}$$

so $x \in \mathbf{Stab}(\mathcal{A})$ if and only if the expression on the right hand side of (6.2.6) is finite.

Theorem 6.2.12 (Moser–Tardos [MT10]; cf. Theorem 3.2.5)**.** *Let $\omega \in [0;1)$ be a witness to the correctness of $\mathscr{B}$. Then, for any Borel Moser–Tardos process $\mathcal{A}$,*

$$\int_{k^{\Gamma\times\mathbb{N}}} \mathbf{Ind}(x, \mathcal{A})\, \mathrm{d}\,\mu_k(x) \leqslant \frac{\omega}{1-\omega}. \tag{6.2.7}$$

To establish Lemma 6.2.9(ii), we will use the following corollary of Theorem 6.2.12:

Corollary 6.2.13. *Let $\omega \in [0;1)$ be a witness to the correctness of $\mathscr{B}$. There exists a Borel map $g \colon k^{\Gamma\times\mathbb{N}} \to k$ satisfying*

$$\mathrm{dist}_{\mu_k}(f_k, g) \leqslant |\mathrm{dom}(\Phi)|\frac{\omega}{1-\omega} \qquad \text{and} \qquad \mu_k(\mathbf{Def}_\Phi(g, \tilde{\sigma}_k)) = 0.$$

Proof. Let $\mathcal{A} = (A_i)_{i=0}^\infty$ be an arbitrary Borel Moser–Tardos process and let $g \colon k^{\Gamma\times\mathbb{N}} \to k$ be given by (6.2.5). From (6.2.6) and Theorem 6.2.12, we get

$$\begin{aligned}
\int_{k^{\Gamma\times\mathbb{N}}} \lim_{i\to\infty} t_i(x)\,\mathrm{d}\,\mu_k(x) &= \int_{k^{\Gamma\times\mathbb{N}}} \sum_{y\in(\mathrm{dom}(\Phi))^{-1}\cdot x} \mathbf{Ind}(y,\mathcal{A})\,\mathrm{d}\,\mu_k(x) \\
\text{[since the measure } \mu_k \text{ is shift-invariant]} \quad &= |\mathrm{dom}(\Phi)| \int_{k^{\Gamma\times\mathbb{N}}} \mathbf{Ind}(x,\mathcal{A})\,\mathrm{d}\,\mu_k(x) \\
&\leqslant |\mathrm{dom}(\Phi)|\frac{\omega}{1-\omega} < \infty.
\end{aligned}$$

In particular, $\mu_k(\mathbf{Stab}(\mathcal{A})) = 1$. This implies that $\mu_k(\mathbf{Def}_\Phi(g, \tilde{\sigma}_k)) = 0$. Furthermore, if $x \in \mathbf{Stab}(\mathcal{A})$ and $g(x) \neq f_k(x)$, then $t(x) > 0$. Thus,

$$\mathrm{dist}_{\mu_k}(f_k, g) \leqslant \mu_k(\{x \in \mathbf{Stab}(\mathcal{A}) \,:\, t(x) > 0\}) \leqslant \int_{\mathbf{Stab}(\mathcal{A})} t(x)\,\mathrm{d}\,\mu_k(x) \leqslant |\mathrm{dom}(\Phi)|\frac{\omega}{1-\omega}. \qquad \blacksquare$$

With Corollary 6.2.13 in hand, we can finish the proof of Lemma 6.2.9:

Proof of Lemma 6.2.9(ii). Let $\varnothing \neq D \in [\Gamma]^{<\infty}$. Set $\Phi \coloneqq \Phi(D)$. Let $B_x \coloneqq B_x(\Phi, \tilde{\sigma}_k)$ for all $x \in k^{\Gamma\times\mathbb{N}}$ and let $\mathscr{B}$ denote the instance $\mathscr{B}(\Phi, \tilde{\sigma}_k)$ restricted to the free part of $\tilde{\sigma}_k$. In the light of Corollary 6.2.13, we just need to argue that if $|D|$ is sufficiently large, then there is a witness $\omega \in [0;1)$ to the correctness of $\mathscr{B}$ such that

$$|S||D|\frac{\omega}{1-\omega} < \varepsilon, \tag{6.2.8}$$

where ε is a given positive number. Take $\omega \coloneqq |S|^{-2}|D|^{-2}$. If $|D|$ is large enough, then this choice of ω satisfies (6.2.8), so it is only left to make sure that ω is a witness to the correctness of $\mathscr{B}$. Recall that from the proof of Lemma 6.2.10 we have

$$p(\mathscr{B}) \leqslant 2k^{|S|}\exp\left(-\varepsilon^2\frac{|D|}{2|S|^3}\right) \qquad \text{and} \qquad d(\mathscr{B}) \leqslant |S|^2|D|^2 - 1.$$

Therefore, to establish (6.2.4), it suffices to prove

$$2k^{|S|}\exp\left(-\varepsilon^2\frac{|D|}{2|S|^3}\right) \leqslant \frac{1}{|S|^2|D|^2}\left(1 - \frac{1}{|S|^2|D|^2}\right)^{|S|^2|D|^2-1}. \tag{6.2.9}$$

Note that, since $|S|^2|D|^2 \geqslant 2$, we have $(1 - |S|^{-2}|D|^{-2})^{|S|^2|D|^2-1} \geqslant e^{-1}$, and hence (6.2.9) is implied by

$$2k^{|S|}\exp\left(-\varepsilon^2\frac{|D|}{2|S|^3}\right) \leqslant \frac{1}{e|S|^2|D|^2},$$

which holds whenever $|D|$ is sufficiently large. $\blacksquare$

6.3 Multiplication of weak equivalence classes

6.3.1 The space of weak equivalence classes

For a standard probability space (X, μ) and $k \in \mathbb{N}^+$, let $\mathrm{Meas}_k(X, \mu)$ denote the space of all measurable maps $f : X \to k$, equipped with the pseudometric

$$\mathrm{dist}_\mu(f, g) \coloneqq \mu(\{x \in X \,:\, f(x) \neq g(x)\}).$$

Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of Γ. For $S \in [\Gamma]^{<\infty}$, $k \in \mathbb{N}^+$, and $f \in \mathrm{Meas}_k(X, \mu)$, define

$$\vartheta_{S,k}(\alpha, f) : S \times k \times k \to [0; 1]$$

by setting, for all $\gamma \in S$ and $i, j < k$,

$$\vartheta_{S,k}(\alpha, f)(\gamma, i, j) \coloneqq \mu(\{x \in X \,:\, f(x) = i,\ f(\gamma \cdot x) = j\}).$$

Thus, $\vartheta_{S,k}(\alpha, f)$ is a vector in the unit cube $Q_{S,k} \coloneqq [0; 1]^{S\times k\times k}$. For $F \subseteq \mathrm{Meas}_k(X, \mu)$, let

$$\vartheta_{S,k}(\alpha, F) \coloneqq \{\vartheta_{S,k}(\alpha, f) \,:\, f \in F\},$$

and define $\vartheta_{S,k}(\alpha)$ to be the closure of the set $\vartheta_{S,k}(\alpha, \mathrm{Meas}_k(X, \mu))$ in $Q_{S,k}$. Using this notation, we have the following characterization of weak containment:

Proposition 6.3.1 ([BK17b, §2.2(1)]). *Let α and β be p.m.p. actions of Γ. Then $\alpha \preccurlyeq \beta$ if and only if for all $S \in [\Gamma]^{<\infty}$ and $k \in \mathbb{N}^+$, $\vartheta_{S,k}(\alpha) \subseteq \vartheta_{S,k}(\beta)$. Hence, $\alpha \simeq \beta$ if and only if for all $S \in [\Gamma]^{<\infty}$ and $k \in \mathbb{N}^+$, $\vartheta_{S,k}(\alpha) = \vartheta_{S,k}(\beta)$.*

In view of Proposition 6.3.1, we refer to the sequence

$$[\alpha] \coloneqq (\vartheta_{S,k}(\alpha))_{S,k},$$

where S and k run over $[\Gamma]^{<\infty}$ and $\mathbb{N}^+$ respectively, as the **weak equivalence class** of α. Let

$$\mathcal{W}_\Gamma \coloneqq \{[\alpha] \,:\, \alpha : \Gamma \curvearrowright (X, \mu),\ \text{where } (X, \mu) \text{ is atomless}\}.$$

A weak equivalence class $\mathfrak{a} \in \mathcal{W}_\Gamma$ is **free** if $\mathfrak{a} = [\alpha]$ for some free p.m.p. action α (recall that for p.m.p. actions, "free" means "free almost everywhere"). Define

$$\mathcal{FW}_\Gamma \coloneqq \{\mathfrak{a} \in \mathcal{W}_\Gamma \,:\, \mathfrak{a} \text{ is free}\}.$$

Theorem 6.3.2 ([BK17b, Theorem 3.4]). *Let α and β be p.m.p. actions of Γ. If α is free and $\alpha \preccurlyeq \beta$, then β is also free. In particular, if $\mathfrak{a} \in \mathcal{FW}_\Gamma$, then all p.m.p. actions α with $[\alpha] = \mathfrak{a}$ are free.*

We now proceed to define the topology on $\mathcal{W}_\Gamma$. For $S \in [\Gamma]^{<\infty}$ and $k \in \mathbb{N}^+$, the cube $Q_{S,k} = [0;1]^{S \times k \times k}$ is equipped with the **∞-metric**:

$$\mathrm{dist}_\infty(u, v) = \|u - v\|_\infty := \max_{\gamma,i,j} |u(\gamma, i, j) - v(\gamma, i, j)|.$$

Let $\mathcal{K}(Q_{S,k})$ denote the set of all nonempty compact subsets of $Q_{S,k}$. For $C \in \mathcal{K}(Q_{S,k})$, let

$$\mathbf{Ball}_\varepsilon(C) := \{u \in Q_{S,k} \,:\, \mathrm{dist}_\infty(u, C) < \varepsilon\}.$$

Define the **Hausdorff metric** on $\mathcal{K}(Q_{S,k})$ by

$$\mathrm{dist}_H(C_1, C_2) := \inf\{\varepsilon > 0 \,:\, C_1 \subseteq \mathbf{Ball}_\varepsilon(C_2) \text{ and } C_2 \subseteq \mathbf{Ball}_\varepsilon(C_1)\}.$$

This metric makes $\mathcal{K}(Q_{S,k})$ into a compact space [Kec95, Theorem 4.26]. By definition, for any p.m.p. action α, we have $\vartheta_{S,k}(\alpha) \in \mathcal{K}(Q_{S,k})$, so $\mathcal{W}_\Gamma$ is a subset of the compact metrizable space

$$\prod_{S,k} \mathcal{K}(Q_{S,k}),$$

where the product is over all $S \in [\Gamma]^{<\infty}$ and $k \in \mathbb{N}^+$, and as such, $\mathcal{W}_\Gamma$ inherits a relative topology. The following fundamental result is due to Abért and Elek:

Theorem 6.3.3 (Abért–Elek [AE11, Theorem 1]; see also [BK17b, Theorem 10.1])**.** *The set $\mathcal{W}_\Gamma$ is closed in $\prod_{S,k} \mathcal{K}(Q_{S,k})$. In other words, the space $\mathcal{W}_\Gamma$ is compact.*

The subspace $\mathcal{F}\mathcal{W}_\Gamma$ is also compact:

Theorem 6.3.4 ([BK17b, Corollary 10.7])**.** *The set $\mathcal{F}\mathcal{W}_\Gamma$ is closed in $\mathcal{W}_\Gamma$.*

It is useful to note that the map $\vartheta_{S,k}(\alpha, -)$ is Lipschitz:

Proposition 6.3.5. *Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of Γ. If $k \in \mathbb{N}^+$ and $f, g \in \mathrm{Meas}_k(X, \mu)$, then, for any $S \in [\Gamma]^{<\infty}$,*

$$\mathrm{dist}_\infty(\vartheta_{S,k}(\alpha, f), \vartheta_{S,k}(\alpha, g)) \leqslant 2 \cdot \mathrm{dist}_\mu(f, g).$$

Proof. Take any $\gamma \in S$ and $i, j < k$ and let

$$A := \{x \in X \,:\, f(x) = i,\ f(\gamma \cdot x) = j\} \qquad \text{and} \qquad B := \{x \in X \,:\, g(x) = i,\ g(\gamma \cdot x) = j\}.$$

Then, by definition,

$$|\vartheta_{S,k}(\alpha, f)(\gamma, i, j) - \vartheta_{S,k}(\alpha, g)(\gamma, i, j)| = |\mu(A) - \mu(B)| \leqslant \mu(A \triangle B).$$

If $x \in A \triangle B$, then $f(x) \neq g(x)$ or $f(\gamma \cdot x) \neq g(\gamma \cdot x)$, so $\mu(A \triangle B) \leqslant 2 \cdot \mathrm{dist}_\mu(f, g)$, as desired. ■

6.3.2 The semigroup structure on $\mathcal{W}_\Gamma$

The **product** of two p.m.p. actions $\alpha \colon \Gamma \curvearrowright (X, \mu)$ and $\beta \colon \Gamma \curvearrowright (Y, \nu)$ is the action

$$\alpha \times \beta \colon \Gamma \curvearrowright (X \times Y, \mu \times \nu), \qquad \text{given by} \qquad \gamma \cdot (x, y) \coloneqq (\gamma \cdot x, \gamma \cdot y).$$

It can be easily seen that the weak equivalence class of $\alpha \times \beta$ is determined by the weak equivalence classes of α and β (for completeness, we include a proof of this fact—see Corollary 6.3.12), hence there is a well-defined multiplication operation on $\mathcal{W}_\Gamma$, namely

$$[\alpha] \times [\beta] \coloneqq [\alpha \times \beta].$$

Equipped with this operation, $\mathcal{W}_\Gamma$ is an Abelian semigroup and $\mathcal{FW}_\Gamma$ is a subsemigroup (in fact, an ideal) in $\mathcal{W}_\Gamma$. We are interested in the following natural question:

Question 6.3.6 ([BK17b, Problem 10.36])**.** Is $\mathcal{W}_\Gamma$ a topological semigroup? In other words, is the map $\mathcal{W}_\Gamma \times \mathcal{W}_\Gamma \to \mathcal{W}_\Gamma \colon (\mathfrak{a}, \mathfrak{b}) \mapsto \mathfrak{a} \times \mathfrak{b}$ continuous?

Burton, Kechris, and Tamuz answered Question 6.3.6 positively when the group Γ is amenable [BK17b, Theorem 10.37]. A crucial role in their argument is played by the identification of the space $\mathcal{W}_\Gamma$ for amenable Γ with the space of the so-called *invariant random subgroups* of Γ [BK17b, Theorem 10.6]. Note that the continuity of multiplication on the subspace $\mathcal{FW}_\Gamma$ for amenable Γ is a triviality, since if Γ is amenable, then $\mathcal{FW}_\Gamma$ contains only a single point [BK17b, p. 15]. On the other hand, if Γ is nonamenable, then $\mathcal{FW}_\Gamma$ has cardinality continuum [Tuc15, Remark 4.3].

In this section we give a *negative* answer to Question 6.3.6 for a certain class of nonamenable groups Γ, including the non-Abelian free groups:

Theorem 6.3.7. *Let $d \geqslant 2$ and let $\Gamma \leqslant \mathrm{SL}_d(\mathbb{Z})$ be a subgroup that is Zariski dense in $\mathrm{SL}_d(\mathbb{R})$.*

(1) *The map $\mathcal{FW}_\Gamma \to \mathcal{FW}_\Gamma \colon \mathfrak{a} \mapsto \mathfrak{a} \times \mathfrak{a}$ is discontinuous.*

(2) *There is $\mathfrak{b} \in \mathcal{FW}_\Gamma$ such that the map $\mathcal{FW}_\Gamma \to \mathcal{FW}_\Gamma \colon \mathfrak{a} \mapsto \mathfrak{a} \times \mathfrak{b}$ is discontinuous.*

As observed in [BK17b, §10.2], part (1) of Theorem 6.3.7 yields the following corollary:

Corollary 6.3.8. *There exists a countable group Δ with a normal subgroup $\Gamma \lhd \Delta$ of index 2 such that the co-induction map $\mathcal{W}_\Gamma \to \mathcal{W}_\Delta$ is discontinuous.*

Proof. Let $d \geqslant 2$ and let $\Gamma \leqslant \mathrm{SL}_d(\mathbb{Z})$ be any Zariski dense subgroup. Set $\Delta \coloneqq \Gamma \times (\mathbb{Z}/2\mathbb{Z})$ and identify Γ with a normal subgroup of Δ of index 2 in the obvious way. Then, for any p.m.p. action α of Γ, the restriction of the co-induced action $\mathrm{CInd}_\Gamma^\Delta(\alpha)$ back to Γ is isomorphic to $\alpha \times \alpha$. Since the restriction map $\mathcal{W}_\Delta \to \mathcal{W}_\Gamma$ is continuous [BK17b, Proposition 10.10], Theorem 6.3.7(1) forces the co-induction map $\mathcal{W}_\Gamma \to \mathcal{W}_\Delta$ to be discontinuous. For details, see [BK17b, §10.2]. ∎

In view of Theorem 6.3.7 and the result of Burton, Kechris, and Tamuz, it is tempting to conjecture that $\mathcal{W}_\Gamma$ is a topological semigroup *if and only if* Γ is amenable. However, at this point we do not even know whether multiplication of weak equivalence classes is discontinuous for every countable group that contains a non-Abelian free subgroup.

Our proof of Theorem 6.3.7 provides explicit examples of sequences of p.m.p. actions that witness the discontinuity of multiplication on $\mathcal{W}_\Gamma$. We describe one such example here. Let $d \geqslant 2$ and let $\Gamma \leqslant \mathrm{SL}_d(\mathbb{Z})$ be a Zariski dense subgroup. For a prime p, let $\mathbb{Z}_p$ denote the ring of p-adic integers. Then $\mathrm{SL}_d(\mathbb{Z}_p)$ is an infinite profinite group. Since $\mathrm{SL}_d(\mathbb{Z})$ naturally embeds in $\mathrm{SL}_d(\mathbb{Z}_p)$, we may identify Γ with a subgroup of $\mathrm{SL}_d(\mathbb{Z}_p)$ and consider the left multiplication action $\alpha_p \colon \Gamma \curvearrowright \mathrm{SL}_d(\mathbb{Z}_p)$, which we view as a p.m.p. action by putting the Haar probability measure on $\mathrm{SL}_d(\mathbb{Z}_p)$. Let $\mathfrak{a}_p$ denote the weak equivalence class of α_p. Using the compactness of $\mathcal{W}_\Gamma$, we can pick an increasing sequence of primes $p_0, p_1, \ldots$ such that the sequence $(\mathfrak{a}_{p_i})_{i\in\mathbb{N}}$ converges in $\mathcal{W}_\Gamma$ to some weak equivalence class $\mathfrak{a}$. Then it follows from our results that the sequence $(\mathfrak{a}_{p_i} \times \mathfrak{a}_{p_i})_{i\in\mathbb{N}}$ does *not* converge to $\mathfrak{a} \times \mathfrak{a}$, thus demonstrating that multiplication on $\mathcal{W}_\Gamma$ is discontinuous.

The main tools that we use to prove Theorem 6.3.7 come from the study of expansion properties in finite groups of Lie type, specifically the groups $\mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z})$ for $n \in \mathbb{N}^+$. Our primary reference for this subject is the book [Tao15].

The rest of this section is organized as follows. In §6.3.3, we introduce the terminology pertaining to step functions and use it in §6.3.4 to prove Theorem 6.3.14, an explicit criterion for continuity of multiplication, which is of some independent interest. The proof of Theorem 6.3.7 is presented in §6.3.5.

6.3.3 Step functions

In this subsection we establish some basic facts pertaining to step functions on products of probability spaces. In particular, we show that multiplication is a well-defined operation on $\mathcal{W}_\Gamma$.

To begin with, we need a few definitions. Let (X, μ) and (Y, ν) be standard probability spaces and let k, $N \in \mathbb{N}^+$. We call a map $f \in \mathrm{Meas}_k(X \times Y, \mu \times \nu)$ an **N-step function** if there exist

$$g \in \mathrm{Meas}_N(X, \mu), \qquad h \in \mathrm{Meas}_N(Y, \nu), \qquad \text{and} \qquad \varphi \colon N \times N \to k,$$

such that $f = \varphi \circ (g, h)$, i.e., we have

$$f(x, y) = \varphi(g(x), h(y)) \qquad \text{for all } x \in X \text{ and } y \in Y.$$

Let $\mathrm{Step}_{k,N}(X, \mu; Y, \nu) \subseteq \mathrm{Meas}_k(X \times Y, \mu \times \nu)$ denote the set of all N-step functions and let

$$\mathrm{Step}_k(X, \mu; Y, \nu) \coloneqq \bigcup_{N\in\mathbb{N}^+} \mathrm{Step}_{k,N}(X, \mu; Y, \nu). \tag{6.3.1}$$

The maps in $\mathrm{Step}_k(X, \mu; Y, \nu)$ are called **step functions**. Note that the union in (6.3.1) is increasing. It is a basic fact in measure theory that the set $\mathrm{Step}_k(X, \mu; Y, \nu)$ is dense in $\mathrm{Meas}_k(X \times Y, \mu \times \nu)$.

It will be useful to have a concrete description of the vectors of the form $\vartheta_{S,k}(\alpha \times \beta, f)$, where f is a step function. To that end, we introduce the following operation:

Definition 6.3.9. Let k, $N \in \mathbb{N}^+$ and $\varphi \colon N \times N \to k$. Given $S \in [\Gamma]^{<\infty}$ and vectors u, $v \in \mathbb{R}^{S \times N \times N}$, the **$\varphi$-convolution** $u *_\varphi v \in \mathbb{R}^{S \times k \times k}$ of u and v is given by the formula

$$(u *_\varphi v)(\gamma, i, j) \coloneqq \sum_{(a,b) \in \varphi^{-1}(i)} \sum_{(c,d) \in \varphi^{-1}(j)} u(\gamma, a, c) \cdot v(\gamma, b, d).$$

The next proposition is an immediate consequence of the definitions:

Proposition 6.3.10. *Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ and $\beta \colon \Gamma \curvearrowright (Y, \nu)$ be p.m.p. actions of Γ. Let k, $N \in \mathbb{N}^+$ and*

$$g \in \mathrm{Meas}_N(X, \mu), \qquad h \in \mathrm{Meas}_N(Y, \nu), \qquad \textit{and} \qquad \varphi \colon N \times N \to k.$$

Set $f \coloneqq \varphi \circ (g, h)$. Then, for any $S \in [\Gamma]^{<\infty}$,

$$\vartheta_{S,k}(\alpha \times \beta, f) \ = \ \vartheta_{S,N}(\alpha, g) *_\varphi \vartheta_{S,N}(\beta, h).$$

It is useful to note that the φ-convolution operation is Lipschitz on $Q_{S,N}$:

Proposition 6.3.11. *Let k, $N \in \mathbb{N}^+$ and $\varphi \colon N \times N \to k$. For all $S \in [\Gamma]^{<\infty}$ and u, v, $\tilde{u}$, $\tilde{v} \in Q_{S,N}$,*

$$\mathrm{dist}_\infty(u *_\varphi v,\ \tilde{u} *_\varphi \tilde{v}) \ \leqslant \ N^4 \cdot (\mathrm{dist}_\infty(u, \tilde{u}) + \mathrm{dist}_\infty(v, \tilde{v})).$$

Proof. Since

$$\mathrm{dist}_\infty(u *_\varphi v,\ \tilde{u} *_\varphi \tilde{v}) \ \leqslant \ \mathrm{dist}_\infty(u *_\varphi v,\ \tilde{u} *_\varphi v) + \mathrm{dist}_\infty(\tilde{u} *_\varphi v,\ \tilde{u} *_\varphi \tilde{v}),$$

it suffices to prove the inequality when, say, $v = \tilde{v}$. To that end, take $\gamma \in S$ and i, $j < k$. We have

$$\begin{aligned} |(u *_\varphi v)(\gamma, i, j) \ - \ (\tilde{u} *_\varphi v)(\gamma, i, j)| \ &\leqslant \sum_{(a,b) \in \varphi^{-1}(i)} \sum_{(c,d) \in \varphi^{-1}(j)} |u(\gamma, a, c) \ - \ \tilde{u}(\gamma, a, c)| \cdot v(\gamma, b, d) \\ &\leqslant \ |\varphi^{-1}(i)| \cdot |\varphi^{-1}(j)| \cdot \mathrm{dist}_\infty(u, \tilde{u}) \ \leqslant \ N^4 \cdot \mathrm{dist}_\infty(u, \tilde{u}). \end{aligned}$$

■

Corollary 6.3.12. *If α, $\tilde{\alpha}$, and β are p.m.p. actions of Γ and $\alpha \preccurlyeq \tilde{\alpha}$, then $\alpha \times \beta \preccurlyeq \tilde{\alpha} \times \beta$. In particular, the multiplication operation on $\mathcal{W}_\Gamma$ is well-defined.*

Proof. Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$, $\tilde{\alpha} \colon \Gamma \curvearrowright (\tilde{X}, \tilde{\mu})$, and $\beta \colon \Gamma \curvearrowright (Y, \nu)$ be p.m.p. actions of Γ and suppose that $\alpha \preccurlyeq \tilde{\alpha}$. Take any $S \in [\Gamma]^{<\infty}$ and $k \in \mathbb{N}^+$. By Proposition 6.3.5 and since $\mathrm{Step}_k(X, \mu; Y, \nu)$ is dense in $\mathrm{Meas}_k(X \times Y, \mu \times \nu)$, it suffices to show that for all $f \in \mathrm{Step}_k(X, \mu; Y, \nu)$ and $\varepsilon > 0$, there is $\tilde{f} \in \mathrm{Meas}_k(\tilde{X} \times Y, \tilde{\mu} \times \nu)$ such that

$$\mathrm{dist}_\infty(\vartheta_{S,k}(\alpha \times \beta, f),\ \vartheta_{S,k}(\tilde{\alpha} \times \beta, \tilde{f})) \ < \ \varepsilon.$$

Let $N \in \mathbb{N}^+$, $g \in \mathrm{Meas}_N(X, \mu)$, $h \in \mathrm{Meas}_N(Y, \nu)$, and $\varphi \colon N \times N \to k$ be such that $f = \varphi \circ (g, h)$. Since $\alpha \preccurlyeq \tilde{\alpha}$, there is a map $\tilde{g} \in \mathrm{Meas}_N(\tilde{X}, \tilde{\mu})$ with

$$\mathrm{dist}_\infty(\vartheta_{S,N}(\alpha, g),\ \vartheta_{S,N}(\tilde{\alpha}, \tilde{g})) \ < \ \varepsilon N^{-4}.$$

From Propositions 6.3.10 and 6.3.11, it follows that the map $\tilde{f} \coloneqq \varphi \circ (\tilde{g}, h)$ is as desired. ∎

6.3.4 A criterion of continuity

The purpose of this subsection is to establish an explicit necessary and sufficient condition for the continuity of multiplication on the space of weak equivalence classes.

Recall that a subset Y of a metric space X is called an **ε-net** if for every $x \in X$, there is $y \in Y$ such that the distance between x and y is less than ε.

Lemma/Definition 6.3.13. *Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ and $\beta \colon \Gamma \curvearrowright (Y, \nu)$ be p.m.p. actions of Γ. For any $S \in [\Gamma]^{<\infty}$, $k \in \mathbb{N}^+$, and $\varepsilon > 0$, there exists $N \in \mathbb{N}^+$ such that the set*

$$\vartheta_{S,k}(\alpha \times \beta, \mathrm{Step}_{k,N}(X, \mu; Y, \nu))$$

is an ε-net in $\vartheta_{S,k}(\alpha \times \beta)$. We denote the smallest such N by $N_{S,k}(\alpha, \beta, \varepsilon)$.

Furthermore, the value $N_{S,k}(\alpha, \beta, \varepsilon)$ is determined by the weak equivalence classes of α and β, so we can define $N_{S,k}([\alpha], [\beta], \varepsilon) \coloneqq N_{S,k}(\alpha, \beta, \varepsilon)$.

Proof. By Proposition 6.3.5 and since $\mathrm{Step}_k(X, \mu; Y, \nu)$ is dense in $\mathrm{Meas}_k(X \times Y, \mu \times \nu)$, the set

$$\vartheta_{S,k}(\alpha \times \beta, \mathrm{Step}_k(X, \mu; Y, \nu))$$

is dense in $\vartheta_{S,k}(\alpha \times \beta)$. The existence of $N_{S,k}(\alpha, \beta, \varepsilon)$ then follows since $\vartheta_{S,k}(\alpha \times \beta)$ is compact.

To prove the "furthermore" part, let $\tilde{\alpha} \colon \Gamma \curvearrowright (\tilde{X}, \tilde{\mu})$ and $\tilde{\beta} \colon \Gamma \curvearrowright (\tilde{Y}, \tilde{\nu})$ be p.m.p. actions of Γ such that $\alpha \simeq \tilde{\alpha}$ and $\beta \simeq \tilde{\beta}$. Set $N \coloneqq N_{S,k}(\alpha, \beta, \varepsilon)$. We have to show that

$$N_{S,k}(\tilde{\alpha}, \tilde{\beta}, \varepsilon) \leqslant N.$$

Take any $u \in \vartheta_{S,k}(\tilde{\alpha} \times \tilde{\beta})$. Corollary 6.3.12 implies that $\vartheta_{S,k}(\tilde{\alpha} \times \tilde{\beta}) = \vartheta_{S,k}(\alpha \times \beta)$, so, by the choice of N, there is $f \in \mathrm{Step}_{k,N}(X, \mu; Y, \nu)$ such that

$$\delta \coloneqq \varepsilon - \mathrm{dist}_\infty(u, \vartheta_{S,k}(\alpha \times \beta, f)) \ > \ 0.$$

Let $g \in \mathrm{Meas}_N(X, \mu)$, $h \in \mathrm{Meas}_N(Y, \nu)$, and $\varphi \colon N \times N \to k$ be such that $f = \varphi \circ (g, h)$. Since we have $\alpha \simeq \tilde{\alpha}$ and $\beta \simeq \tilde{\beta}$, there exist maps $\tilde{g} \in \mathrm{Meas}_N(\tilde{X}, \tilde{\mu})$ and $\tilde{h} \in \mathrm{Meas}_N(\tilde{Y}, \tilde{\nu})$ with

$$\mathrm{dist}_\infty(\vartheta_{S,N}(\tilde{\alpha}, \tilde{g}),\ \vartheta_{S,N}(\alpha, g)) + \mathrm{dist}_\infty(\vartheta_{S,N}(\tilde{\beta}, \tilde{h}),\ \vartheta_{S,N}(\beta, h)) \ < \ \delta N^{-4}.$$

Set $\tilde{f} \coloneqq \varphi \circ (\tilde{g}, \tilde{h})$. Then $\tilde{f} \in \mathrm{Step}_{k,N}(\tilde{X}, \tilde{\mu}; \tilde{Y}, \tilde{\nu})$, and, from Propositions 6.3.10 and 6.3.11, it follows that

$$\mathrm{dist}_\infty(u, \vartheta_{S,k}(\tilde{\alpha} \times \tilde{\beta}, \tilde{f})) \ < \ \varepsilon.$$

Since u was chosen arbitrarily, this concludes the proof. ∎

Now we can state the main result of this subsection:

Theorem 6.3.14. *Let $C \subseteq \mathcal{W}_\Gamma \times \mathcal{W}_\Gamma$ be a closed set. The following statements are equivalent:*

(1) *the map $C \to \mathcal{W}_\Gamma \colon (\mathfrak{a}, \mathfrak{b}) \mapsto \mathfrak{a} \times \mathfrak{b}$ is continuous;*

(2) *for all $S \in [\Gamma]^{<\infty}$, $k \in \mathbb{N}^+$, and $\varepsilon > 0$, there is $N \in \mathbb{N}^+$ such that for all $(\mathfrak{a}, \mathfrak{b}) \in C$,*

$$N_{S,k}(\mathfrak{a}, \mathfrak{b}, \varepsilon) \leqslant N.$$

Proof. We start with the implication (1) $\Longrightarrow$ (2). Suppose that (1) holds and assume that for some $S \in [\Gamma]^{<\infty}$, $k \in \mathbb{N}^+$, and $\varepsilon > 0$, there is a sequence of pairs $(\mathfrak{a}_n, \mathfrak{b}_n) \in C$ with $N_{S,k}(\mathfrak{a}_n, \mathfrak{b}_n, \varepsilon) \longrightarrow \infty$. Since C is compact, we may pass to a subsequence so that $(\mathfrak{a}_n, \mathfrak{b}_n) \longrightarrow (\mathfrak{a}, \mathfrak{b}) \in C$. By (1), we then also have $\mathfrak{a}_n \times \mathfrak{b}_n \longrightarrow \mathfrak{a} \times \mathfrak{b}$. Set $N \coloneqq N_{S,k}(\mathfrak{a}, \mathfrak{b}, \varepsilon/3)$.

Let $\alpha_n \colon \Gamma \curvearrowright (X_n, \mu_n)$, $\beta_n \colon \Gamma \curvearrowright (Y_n, \nu_n)$, $\alpha \colon \Gamma \curvearrowright (X, \mu)$, and $\beta \colon \Gamma \curvearrowright (Y, \nu)$ be representatives of the weak equivalence classes $\mathfrak{a}_n$, $\mathfrak{b}_n$, $\mathfrak{a}$, and $\mathfrak{b}$ respectively. We claim that $N_{S,k}(\alpha_n, \beta_n, \varepsilon) \leqslant N$ for all sufficiently large $n \in \mathbb{N}$, contradicting the choice of $(\mathfrak{a}_n, \mathfrak{b}_n)$. Indeed, take any $u \in \vartheta_{S,k}(\alpha_n \times \beta_n)$. If n is large enough, then there is $v \in \vartheta_{S,k}(\alpha \times \beta)$ such that

$$\mathrm{dist}_\infty(u, v) \,<\, \varepsilon/3.$$

By the choice of N, there is a step function $f \in \mathrm{Step}_{k,N}(X, \mu; Y, \nu)$ such that

$$\mathrm{dist}_\infty(v, \vartheta_{S,k}(\alpha \times \beta, f)) \,<\, \varepsilon/3.$$

Let $g \in \mathrm{Meas}_N(X, \mu)$, $h \in \mathrm{Meas}_N(Y, \nu)$, and $\varphi \colon N \times N \to k$ be such that $f = \varphi \circ (g, h)$. If n is large enough, then there exist maps $\tilde{g} \in \mathrm{Meas}_N(X_n, \mu_n)$ and $\tilde{h} \in \mathrm{Meas}_N(Y_n, \nu_n)$ satisfying

$$\mathrm{dist}_\infty(\vartheta_{S,N}(\alpha_n, \tilde{g}), \vartheta_{S,N}(\alpha, g)) + \mathrm{dist}_\infty(\vartheta_{S,N}(\beta_n, \tilde{h}), \vartheta_{S,N}(\beta, h)) \,<\, \varepsilon N^{-4}/3.$$

Let $\tilde{f} \coloneqq \varphi \circ (\tilde{g}, \tilde{h})$. From Propositions 6.3.10 and 6.3.11, it follows that

$$\mathrm{dist}_\infty(u, \vartheta_{S,k}(\alpha_n \times \beta_n, \tilde{f})) \,<\, \varepsilon/3 + \varepsilon/3 + N^4 \cdot (\varepsilon N^{-4}/3) \,=\, \varepsilon,$$

as desired.

Now we proceed to the implication (2) $\Longrightarrow$ (1). Suppose that (2) holds and let $(\mathfrak{a}_n, \mathfrak{b}_n)$, $(\mathfrak{a}, \mathfrak{b}) \in C$ be such that $(\mathfrak{a}_n, \mathfrak{b}_n) \longrightarrow (\mathfrak{a}, \mathfrak{b})$. We have to show that $\mathfrak{a}_n \times \mathfrak{b}_n \longrightarrow \mathfrak{a} \times \mathfrak{b}$. Let $\alpha_n \colon \Gamma \curvearrowright (X_n, \mu_n)$, $\beta_n \colon \Gamma \curvearrowright (Y_n, \nu_n)$, $\alpha \colon \Gamma \curvearrowright (X, \mu)$, and $\beta \colon \Gamma \curvearrowright (Y, \nu)$ be representatives of the weak equivalence classes $\mathfrak{a}_n$, $\mathfrak{b}_n$, $\mathfrak{a}$, and $\mathfrak{b}$ respectively. We must argue that for any $S \in [\Gamma]^{<\infty}$, $k \in \mathbb{N}^+$, and $\varepsilon > 0$ and for all sufficiently large $n \in \mathbb{N}$,

$$\vartheta_{S,k}(\alpha \times \beta) \,\subseteq\, \mathbf{Ball}_\varepsilon(\vartheta_{S,k}(\alpha_n \times \beta_n)); \tag{6.3.2}$$

$$\vartheta_{S,k}(\alpha_n \times \beta_n) \,\subseteq\, \mathbf{Ball}_\varepsilon(\vartheta_{S,k}(\alpha \times \beta)). \tag{6.3.3}$$

To prove (6.3.2), let $N \coloneqq N_{S,k}(\alpha, \beta, \varepsilon/2)$ and consider any $u \in \vartheta_{S,k}(\alpha \times \beta)$. By the choice of N, there is a step function $f \in \mathrm{Step}_{k,N}(X, \mu; Y, \nu)$ such that

$$\mathrm{dist}_\infty(u, \vartheta_{S,k}(\alpha \times \beta, f)) < \varepsilon/2.$$

Let $g \in \mathrm{Meas}_N(X, \mu)$, $h \in \mathrm{Meas}_N(Y, \nu)$, and $\varphi\colon N \times N \to k$ be such that $f = \varphi \circ (g, h)$. If n is large enough, then there exist maps $\tilde{g} \in \mathrm{Meas}_N(X_n, \mu_n)$ and $\tilde{h} \in \mathrm{Meas}_N(Y_n, \nu_n)$ satisfying

$$\mathrm{dist}_\infty(\vartheta_{S,N}(\alpha_n, \tilde{g}), \vartheta_{S,N}(\alpha, g)) + \mathrm{dist}_\infty(\vartheta_{S,N}(\beta_n, \tilde{h}), \vartheta_{S,N}(\beta, h)) < \varepsilon N^{-4}/2.$$

Let $\tilde{f} \coloneqq \varphi \circ (\tilde{g}, \tilde{h})$. From Propositions 6.3.10 and 6.3.11, it follows that

$$\mathrm{dist}_\infty(u, \vartheta_{S,k}(\alpha_n \times \beta_n, \tilde{f})) < \varepsilon/2 + N^4 \cdot (\varepsilon N^{-4}/2) = \varepsilon,$$

i.e., $u \in \mathbf{Ball}_\varepsilon(\vartheta_{S,k}(\alpha_n \times \beta_n))$, as desired. Notice that this argument did not involve assumption (2).

To prove (6.3.3), we use (2) and choose N so that $N_{S,k}(\alpha_n, \beta_n, \varepsilon) \leqslant N$ for all $n \in \mathbb{N}$. Consider any $u \in \vartheta_{S,k}(\alpha_n \times \beta_n)$. Then there is a step function $f \in \mathrm{Step}_{k,N}(X_n, \mu_n; Y_n, \nu_n)$ such that

$$\mathrm{dist}_\infty(u, \vartheta_{S,k}(\alpha_n \times \beta_n, f)) < \varepsilon/2.$$

Let $g \in \mathrm{Meas}_N(X_n, \mu_n)$, $h \in \mathrm{Meas}_N(Y_n, \nu_n)$, and $\varphi\colon N \times N \to k$ be such that $f = \varphi \circ (g, h)$. If n is large enough, then there exist maps $\tilde{g} \in \mathrm{Meas}_N(X, \mu)$ and $\tilde{h} \in \mathrm{Meas}_N(Y, \nu)$ satisfying

$$\mathrm{dist}_\infty(\vartheta_{S,N}(\alpha, \tilde{g}), \vartheta_{S,N}(\alpha_n, g)) + \mathrm{dist}_\infty(\vartheta_{S,N}(\beta, \tilde{h}), \vartheta_{S,N}(\beta_n, h)) < \varepsilon N^{-4}/2.$$

Let $\tilde{f} \coloneqq \varphi \circ (\tilde{g}, \tilde{h})$. From Propositions 6.3.10 and 6.3.11, it follows that

$$\mathrm{dist}_\infty(u, \vartheta_{S,k}(\alpha \times \beta, \tilde{f})) < \varepsilon/2 + N^4 \cdot (\varepsilon N^{-4}/2) = \varepsilon,$$

i.e., $u \in \mathbf{Ball}_\varepsilon(\vartheta_{S,k}(\alpha \times \beta))$, and we are done. ■

6.3.5 Proof of Theorem 6.3.7

Expansion in $\mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z})$

For $n \in \mathbb{N}^+$, we use π_n to indicate reduction modulo n in various contexts. That is, we slightly abuse notation and give the same name to the residue maps

$$\pi_n\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}, \qquad \pi_n\colon \mathrm{SL}_d(\mathbb{Z}) \to \mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z}), \qquad \text{etc.}$$

Let G be a nontrivial finite group. For $A, S \subseteq G$, the **boundary**[1] of A with respect to S is

$$\partial(A, S) \coloneqq \{a \in A \,:\, Sa \nsubseteq A\}.$$

[1]For our purposes it will be more convenient to consider the vertex rather than the edge boundary.

The **Cheeger constant** $h(G, S)$ of G with respect to S is given by

$$h(G, S) \coloneqq \min_{A} \frac{|\partial(A, S)|}{|A|},$$

where the minimum is taken over all nonempty subsets $A \subseteq G$ of size at most $|G|/2$. Notice that we have $h(G, S) > 0$ if and only if S generates G. Indeed, let $\langle S \rangle$ be the subgroup of G generated by S. If $\langle S \rangle \neq G$, then $|\langle S \rangle| \leqslant |G|/2$, while $\partial(\langle S \rangle, S) = \varnothing$, hence $h(G, S) = 0$. Conversely, if $h(G, S) = 0$, then there is a nonempty proper subset $A \subsetneq G$ closed under left multiplication by the elements of S. This means that A is a union of right cosets of $\langle S \rangle$, and thus $\langle S \rangle \neq G$.

Theorem 6.3.15 (Bourgain–Varjú [BV12, Theorem 1])**.** *Let $d \geqslant 2$ and let $S \in [\mathrm{SL}_d(\mathbb{Z})]^{<\infty}$ be a finite symmetric subset such that the subgroup $\langle S \rangle$ of $\mathrm{SL}_d(\mathbb{Z})$ generated by S is Zariski dense in $\mathrm{SL}_d(\mathbb{R})$. Then there exist $n_0 \in \mathbb{N}^+$ and $\varepsilon > 0$ such that for all $n \geqslant 2$, if $\gcd(n, n_0) = 1$, then*

$$h(\mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z}), \pi_n(S)) \geqslant \varepsilon.$$

Theorem 6.3.15 is an outcome of a long series of contributions by a number of researchers; for more background, see [BV12; Tao15] and the references therein.

A finite group G is called **D-quasirandom**, where $D \geqslant 1$, if every nontrivial unitary representation of G has dimension at least D (a representation ρ of G is nontrivial if $\rho(a) \neq 1$ for some $a \in G$). This notion was introduced by Gowers [Gow08]. For a map $\zeta \colon G \to \mathbb{C}$, we write

$$\mathbb{E}\zeta \coloneqq \frac{1}{|G|} \sum_{x \in G} \zeta(x), \qquad \|\zeta\|_\infty \coloneqq \max_{x \in G} |\zeta(x)|, \qquad \text{and} \qquad \|\zeta\|_2 \coloneqq \sqrt{\sum_{x \in G} |\zeta(x)|^2}.$$

Given $\zeta, \eta \colon G \to \mathbb{C}$, define the **convolution** $\zeta * \eta \colon G \to \mathbb{C}$ of ζ and η by the formula

$$(\zeta * \eta)(x) \coloneqq \sum_{ab = x} \zeta(a)\eta(b),$$

where the sum is taken over all pairs of $a, b \in G$ such that $ab = x$.

Theorem 6.3.16 ([Tao15, Proposition 1.3.7])**.** *Let G be a finite group and let $\zeta, \eta \colon G \to \mathbb{C}$. Suppose that G is D-quasirandom. If $\mathbb{E}\zeta = \mathbb{E}\eta = 0$, then*

$$\|\zeta * \eta\|_2 \leqslant \sqrt{\frac{|G|}{D}} \|\zeta\|_2 \|\eta\|_2.$$

In order to apply Theorem 6.3.16, we will need the following variation of Frobenius's lemma:

Proposition 6.3.17 (cf. [Tao15, Lemma 1.3.3])**.** *Let d, $n \geqslant 2$ and let p be the smallest prime divisor of n. Then the group $\mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z})$ is $(p-1)/2$-quasirandom.*

Proof. The statement is trivial for $p = 2$, so assume that p is odd. Write n as a product of powers of distinct primes: $n = p_1^{k_1} \cdots p_r^{k_r}$. Then, by the Chinese remainder theorem,

$$\mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z}) \;\cong\; \mathrm{SL}_d(\mathbb{Z}/p_1^{k_1}\mathbb{Z}) \times \cdots \times \mathrm{SL}_d(\mathbb{Z}/p_r^{k_r}\mathbb{Z}).$$

Since the product of D-quasirandom groups is again D-quasirandom [Tao15, Exercise 1.3.2], it is enough to consider the case when $r = 1$ and $n = p^k$.

Let ρ be a nontrivial finite-dimensional unitary representation of $\mathrm{SL}_d(\mathbb{Z}/p^k\mathbb{Z})$. By [HO89, Theorem 4.3.9], the group $\mathrm{SL}_d(\mathbb{Z}/p^k\mathbb{Z})$ is generated by the **elementary** matrices, i.e., those that differ from the identity matrix in precisely one off-diagonal entry. Thus, there exists an elementary matrix $e \in \mathrm{SL}_d(\mathbb{Z}/p^k\mathbb{Z})$ such that $\rho(e) \neq 1$. Without loss of generality, we may assume that e is of the form

$$e = \begin{pmatrix} 1 & a & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

where $0 \neq a \in \mathbb{Z}/p^k\mathbb{Z}$. Choose e so as to maximize the power of p that divides a. Let λ be an arbitrary eigenvalue of $\rho(e)$ not equal to 1 (such λ exists since $\rho(e) \neq 1$ and is unitary). We have

$$e^p = \begin{pmatrix} 1 & pa & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

so, by the choice of e, $\rho(e)^p = \rho(e^p) = 1$. Hence, $\lambda^p = 1$, so the values $\lambda, \lambda^2, \ldots, \lambda^{p-1}$ are pairwise distinct. Let $b \in \mathbb{N}^+$ be an integer coprime to p and let $c := b^2$. Since b is invertible in $\mathbb{Z}/p^k\mathbb{Z}$, we can form a diagonal matrix $h \in \mathrm{SL}_d(\mathbb{Z}/p^k\mathbb{Z})$ with entries $(b, b^{-1}, 1 \ldots, 1)$. Then h^{-1} is the diagonal matrix with entries $(b^{-1}, b, 1, \ldots, 1)$, and we have

$$heh^{-1} = \begin{pmatrix} 1 & ca & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = e^c.$$

This shows that e and e^c are conjugate in $\mathrm{SL}_d(\mathbb{Z}/p^k\mathbb{Z})$, and hence $\rho(e)$ and $\rho(e)^c$ are conjugate as well. Since λ^c is an eigenvalue of $\rho(e)^c$, it must also be an eigenvalue of $\rho(e)$. It remains to notice that there exist $(p-1)/2$ choices for c that are distinct modulo p (corresponding to the $(p-1)/2$ nonzero quadratic residues modulo p), so $\rho(e)$ must have at least $(p-1)/2$ distinct eigenvalues, which is only possible if the dimension of ρ is at least $(p-1)/2$. ■

The main lemma

For the rest of §6.3.5, fix $d \geqslant 2$ and let Γ be a subgroup of $\mathrm{SL}_d(\mathbb{Z})$ that is Zariski dense in $\mathrm{SL}_d(\mathbb{R})$.

For $n \geqslant 2$, define $G_n \coloneqq \mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z})$. Let $\alpha_n \colon \Gamma \curvearrowright G_n$ be the action given by

$$\gamma \cdot x \coloneqq \pi_n(\gamma)x \qquad \text{for all } \gamma \in \Gamma \text{ and } x \in G_n.$$

We view α_n as a p.m.p. action by equipping G_n with the uniform probability measure (to simplify notation, we will avoid mentioning this measure explicitly).

The group Γ has a Zariski dense finitely generated subgroup (by Tits's theorem [Tit72, Theorem 3], such a subgroup can be chosen to be free of rank 2), so fix an arbitrary finite symmetric set $S \in [\Gamma]^{<\infty}$ such that the group $\langle S \rangle$ is Zariski dense in $\mathrm{SL}_d(\mathbb{R})$. Fix $n_0 \in \mathbb{N}^+$ and $\varepsilon > 0$ provided by Theorem 6.3.15 applied to S and let

$$\delta \coloneqq \frac{\varepsilon}{32|S|}.$$

Define $u \in [0;1]^{S \times 2 \times 2}$ by setting, for all $\gamma \in S$ and $i, j < 2$,

$$u(\gamma, i, j) \coloneqq \begin{cases} 1/2 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

The heart of the proof of Theorem 6.3.7 lies in the following lemma:

Lemma 6.3.18. *Let $n, m \geqslant 2$ be such that n divides m and $\gcd(m, n_0) = 1$. Let p be the smallest prime divisor of n and let*

$$N \coloneqq \left\lfloor \frac{1}{25}\sqrt{p-1} \right\rfloor.$$

Assume that $N \geqslant 1$. Then $u \in \vartheta_{S,2}(\alpha_n \times \alpha_m)$, yet for all $f \in \mathrm{Step}_{2,N}(G_n; G_m)$, we have

$$\mathrm{dist}_\infty(u, \vartheta_{S,2}(\alpha_n \times \alpha_m, f)) \geqslant \delta.$$

In particular, $N_{S,2}(\alpha_n, \alpha_m, \delta) > N$.

Proof. Let $\mathrm{proj}_2 \colon G_n \times G_m \to G_m$ denote the projection on the second coordinate. Note that, by definition, the map proj_2 is equivariant. Since n divides m, there is a well-defined reduction modulo n map $\pi_n \colon G_m \to G_n$, and it is surjective. For $z \in G_n$, define

$$O_z \coloneqq \{(x, y) \in G_n \times G_m \ : \ x = \pi_n(y)z\}.$$

Evidently, the set O_z is $(\alpha_n \times \alpha_m)$-invariant. Furthermore, the map proj_2 establishes an equivariant bijection between O_z and G_m. Since $\gcd(m, n_0) = 1$, Theorem 6.3.15 implies that the action α_m is transitive, and hence so is the restriction of the action $\alpha_n \times \alpha_m$ to O_z. Thus, the orbits of $\alpha_n \times \alpha_m$ are precisely the sets O_z for $z \in G_n$.

Given a subset $Z \subseteq G_n$, define $f_Z \colon G_n \times G_m \to 2$ by

$$f_Z(x, y) \coloneqq \begin{cases} 0 & \text{if } \pi_n(y)^{-1}x \notin Z; \\ 1 & \text{if } \pi_n(y)^{-1}x \in Z. \end{cases}$$

The functions of the form f_Z for $Z \subseteq G_n$ are precisely the $(\alpha_n \times \alpha_m)$-invariant maps $G_n \times G_m \to 2$.

Now we can show that $u \in \vartheta_{S,2}(\alpha_n \times \alpha_m)$. The group G_n contains an element of order 2, namely the diagonal matrix with entries $(-1, -1, 1, \ldots, 1)$, so $|G_n|$ is even. Hence, for any set $Z \subset G_n$ of size exactly $|G_n|/2$, we have $\vartheta_{S,2}(\alpha_n \times \alpha_m, f_Z) = u$, as desired.

For $z \in G_n$ and $A \subseteq O_z$, define the **boundary** of A by

$$\partial A \coloneqq \{(x, y) \in A \;:\; S \cdot (x, y) \nsubseteq A\}.$$

Suppose that $|A| \leqslant |G_m|/2$ (note that $|G_m| = |O_z|$). Then, since proj_2 establishes an equivariant bijection between O_z and G_m, Theorem 6.3.15 yields

$$|\partial A| \geqslant \varepsilon |A|. \tag{6.3.4}$$

Claim (A). *Let $f \colon G_n \times G_m \to 2$ be such that*

$$\mathrm{dist}_\infty(u, \vartheta_{S,2}(\alpha_n \times \alpha_m, f)) < \delta. \tag{6.3.5}$$

Then there is a set $Z \subseteq G_n$ such that $\mathrm{dist}(f, f_Z) < 1/16$.

Proof. For each $\gamma \in S$, let

$$B_\gamma \coloneqq \{(x, y) \in G_n \times G_m \;:\; f(x, y) \neq f(\gamma \cdot x, \gamma \cdot y)\},$$

and define $B \coloneqq \bigcup_{\gamma \in S} B_\gamma$. By (6.3.5), for any $\gamma \in S$, we have

$$\frac{|B_\gamma|}{|G_n||G_m|} \;=\; \vartheta_{S,2}(\alpha_n \times \alpha_m, f)(\gamma, 0, 1) + \vartheta_{S,2}(\alpha_n \times \alpha_m, f)(\gamma, 1, 0) \;<\; 2\delta,$$

and therefore

$$|B| \;<\; 2\delta |S||G_n||G_m| \;=\; \frac{\varepsilon}{16}|G_n||G_m|.$$

We will show that the set

$$Z \coloneqq \{z \in G_n \;:\; f(x, y) = 1 \text{ for at least } |G_m|/2 \text{ pairs } (x, y) \in O_z\}$$

is as desired. Define

$$A \coloneqq \{(x, y) \in G_n \times G_m \;:\; f(x, y) \neq f_Z(x, y)\},$$

so $\mathrm{dist}(f, f_Z) = |A|/(|G_n||G_m|)$. Take any $z \in G_n$. By the definition of Z, we have $|A \cap O_z| \leqslant |G_m|/2$, and hence, by (6.3.4),

$$|\partial(A \cap O_z)| \geqslant \varepsilon |A \cap O_z|.$$

Note that $\partial(A \cap O_z) \subseteq B \cap O_z$, so we have

$$|B \cap O_z| \geqslant |\partial(A \cap O_z)| \geqslant \varepsilon |A \cap O_z|.$$

Hence,

$$|A| \;=\; \sum_{z \in G_n} |A \cap O_z| \;\leqslant\; \sum_{z \in G_n} \varepsilon^{-1} |B \cap O_z| \;=\; \varepsilon^{-1}|B| \;<\; \frac{1}{16}|G_n||G_m|.$$

In other words, $\mathrm{dist}(f, f_Z) < 1/16$, as claimed. ⊣

For $\zeta \colon G_n \to \mathbb{C}$ and $\xi \colon G_m \to \mathbb{C}$, define $\zeta \circledast \xi \colon G_n \to \mathbb{C}$ by the formula

$$(\zeta \circledast \xi)(x) \coloneqq \sum_{a\pi_n(b) = x} \zeta(a)\xi(b),$$

where the sum is taken over all pairs of $a \in G_n$ and $b \in G_m$ such that $a\pi_n(b) = x$. We will need the following corollary of Theorem 6.3.16 and Proposition 6.3.17:

Claim (B). *Let $\zeta, \eta \colon G_n \to \mathbb{C}$ and $\xi \colon G_m \to \mathbb{C}$. Then*

$$\|(\zeta * \eta) \circledast \xi \;-\; (\mathbb{E}\zeta)(\mathbb{E}\eta)(\mathbb{E}\xi)|G_n||G_m|\|_\infty \;\leqslant\; \sqrt{\frac{2|G_m|}{p-1}}\|\zeta\|_2\|\eta\|_2\|\xi\|_2.$$

Proof. This is a variant of [Tao15, Exercise 1.3.12]. After subtracting its expectation from each function, we may assume that $\mathbb{E}\zeta = \mathbb{E}\eta = \mathbb{E}\xi = 0$. By the Cauchy–Schwarz inequality, we have

$$\|(\zeta * \eta) \circledast \xi\|_\infty \;\leqslant\; \sqrt{\frac{|G_m|}{|G_n|}}\|\zeta * \eta\|_2\|\xi\|_2,$$

while Theorem 6.3.16 and Proposition 6.3.17 yield

$$\|\zeta * \eta\|_2 \;\leqslant\; \sqrt{\frac{2|G_n|}{p-1}}\|\zeta\|_2\|\eta\|_2.$$ ⊣

We use Claim (B) to prove that invariant maps are hard to approximate by step functions:

Claim (C). *Let $f \in \mathrm{Step}_{2,N}(G_n; G_m)$ and $Z \subseteq G_n$. Suppose that $\min\{|Z|, |G_n| - |Z|\} \geqslant |G_n|/4$. Then $\mathrm{dist}(f, f_Z) \geqslant 1/8$.*

Proof. Let $g \colon G_n \to N$, $h \colon G_m \to N$, and $\varphi \colon N \times N \to 2$ be such that $f = \varphi \circ (g, h)$. For $i < N$, set $X_i \coloneqq g^{-1}(i)$. Thus, $\{X_i \,:\, i < N\}$ is a partition of G_n into N pieces. Given $i < N$ and $j < 2$, let

$$Y_{i,j} \;\coloneqq\; \{y \in G_m \,:\, \varphi(i, h(y)) = j\} \;=\; \{y \in G_m \,:\, f(x, y) = j \text{ for all } x \in X_i\}.$$

Note that $Y_{i,0} \cup Y_{i,1} = G_m$. Define

$$A := \{(x, y) \in G_n \times G_m \ : \ f(x, y) \neq f_Z(x, y)\},$$

so $\mathrm{dist}(f, f_Z) = |A|/(|G_n||G_m|)$. Let $\mathbf{1}_{G_n}$ be the identity element of G_n, and for each set $F \subseteq G_n$, let $\mathbb{1}_F \colon G_n \to 2$ denote the indicator function of F. Then, for any $i < N$, we have

$$\begin{aligned} |(X_i \times Y_{i,0}) \cap A| &= |\{(x, y) \in X_i \times Y_{i,0} \ : \ f_Z(x, y) = 1\}| \\ &= |\{(z, x, y) \in Z \times X_i \times Y_{i,0} \ : \ zx^{-1}\pi_n(y) = \mathbf{1}_{G_n}\}| \\ &= ((\mathbb{1}_Z * \mathbb{1}_{X_i^{-1}}) \circledast \mathbb{1}_{Y_{i,0}})(\mathbf{1}_{G_n}). \end{aligned}$$

By Claim (B), the last expression is at least

$$\frac{|Z||X_i||Y_{i,0}|}{|G_n|} - \sqrt{\frac{2|G_m||Z||X_i||Y_{i,0}|}{p-1}} \geqslant \frac{|X_i||Y_{i,0}|}{4} - \sqrt{\frac{2}{p-1}}|G_n||G_m|.$$

Similarly, we have

$$|(X_i \times Y_{i,1}) \cap A| \geqslant \frac{|X_i||Y_{i,1}|}{4} - \sqrt{\frac{2}{p-1}}|G_n||G_m|,$$

and hence

$$|(X_i \times G_m) \cap A| \geqslant \frac{|X_i||G_m|}{4} - \sqrt{\frac{8}{p-1}}|G_n||G_m|.$$

Therefore,

$$|A| = \sum_{i<N} |(X_i \times G_m) \cap A| \geqslant \left(\frac{1}{4} - N\sqrt{\frac{8}{p-1}}\right)|G_n||G_m| > \frac{1}{8}|G_n||G_m|,$$

and thus $\mathrm{dist}(f, f_Z) > 1/8$, as desired. ⊣

It remains to combine Claims (A) and (C). Suppose that $f \in \mathrm{Step}_{2,N}(G_n; G_m)$ satisfies

$$\mathrm{dist}_\infty(u, \vartheta_{S,2}(\alpha_n \times \alpha_m, f)) < \delta.$$

By Claim (A), there is a set $Z \subseteq G_n$ such that $\mathrm{dist}(f, f_Z) < 1/16$. By Proposition 6.3.5, we have

$$\mathrm{dist}_\infty(u, \vartheta_{S,2}(\alpha_n \times \alpha_m, f_Z)) < \delta + 2 \cdot (1/16) < 1/4.$$

In particular, for any $\gamma \in S$,

$$\frac{|Z|}{|G_n|} = \vartheta_{S,2}(\alpha_n \times \alpha_m, f_Z)(\gamma, 1, 1) > u(\gamma, 1, 1) - 1/4 = 1/4,$$

i.e., $|Z| \geqslant |G_n|/4$, and, similarly, $|G_n| - |Z| \geqslant |G_n|/4$. Therefore, by Claim (C), $\mathrm{dist}(f, f_Z) \geqslant 1/8$, which is a contradiction. The proof of Lemma 6.3.18 is complete. ∎

Finishing the proof

We say that $\mathcal{N} \subseteq \mathbb{N}^+$ is a **directed set** if $\mathcal{N}$ is infinite and for any two elements $n_1, n_2 \in \mathcal{N}$, there is some $m \in \mathcal{N}$ divisible by both n_1 and n_2. Each directed set $\mathcal{N} \subseteq \mathbb{N}^+$ gives rise to an inverse system consisting of the groups $(G_n)_{n \in \mathcal{N}}$ together with the homomorphisms $\pi_n \colon G_m \to G_n$ for every pair of $n, m \in \mathcal{N}$ such that n divides m. The inverse limit of this system is an infinite profinite group, which we denote by $G_{\mathcal{N}}$. For example, if we let

$$\mathcal{N}(p) \coloneqq \{p, p^2, p^3, \ldots\}$$

for some prime p, then $G_{\mathcal{N}(p)} \cong \mathrm{SL}_d(\mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the ring of p-adic integers.

If $\mathcal{N} \subseteq \mathbb{N}^+$ is a directed set, then $\mathrm{SL}_d(\mathbb{Z})$ naturally embeds into $G_{\mathcal{N}}$, so we can identify Γ with a subgroup of $G_{\mathcal{N}}$. This allows us to consider the left multiplication action $\alpha_{\mathcal{N}} \colon \Gamma \curvearrowright G_{\mathcal{N}}$. As the group $G_{\mathcal{N}}$ is compact, we can equip $G_{\mathcal{N}}$ with the Haar probability measure and view $\alpha_{\mathcal{N}}$ as a p.m.p. action. Clearly, the action $\alpha_{\mathcal{N}}$ is free. Note that for each $n \in \mathcal{N}$, there is a well-defined reduction modulo n map $\pi_n \colon G_{\mathcal{N}} \to G_n$, which is equivariant and pushes the Haar measure on $G_{\mathcal{N}}$ forward to the uniform probability measure on G_n. In particular, α_n is a factor of $\alpha_{\mathcal{N}}$, and hence $\alpha_n \preccurlyeq \alpha_{\mathcal{N}}$.

The following is a direct consequence of Lemma 6.3.18:

Lemma 6.3.19. *Let $\mathcal{N}$, $\mathcal{M} \subseteq \mathbb{N}^+$ be directed sets such that $\mathcal{N} \subseteq \mathcal{M}$ and* $\gcd(m, n_0) = 1$ *for all $m \in \mathcal{M}$. Let p be the smallest prime number that divides an element of $\mathcal{N}$ and let*

$$N \coloneqq \left\lfloor \frac{1}{25}\sqrt{p-1} \right\rfloor.$$

Assume that $N \geqslant 1$. Then $u \in \vartheta_{S,2}(\alpha_{\mathcal{N}} \times \alpha_{\mathcal{M}})$, yet for all $f \in \mathrm{Step}_{2,N}(G_{\mathcal{N}}; G_{\mathcal{M}})$, we have

$$\mathrm{dist}_\infty(u, \vartheta_{S,2}(\alpha_{\mathcal{N}} \times \alpha_{\mathcal{M}}, f)) \geqslant \delta.$$

In particular, $N_{S,2}(\alpha_{\mathcal{N}}, \alpha_{\mathcal{M}}, \delta) > N$.

Proof. To prove that $u \in \vartheta_{S,2}(\alpha_{\mathcal{N}} \times \alpha_{\mathcal{M}})$, take any $n \in \mathcal{N}$, $n \geqslant 2$. Since $\alpha_n \preccurlyeq \alpha_{\mathcal{N}}, \alpha_{\mathcal{M}}$, it follows from Corollary 6.3.12 that $\alpha_n \times \alpha_n \preccurlyeq \alpha_{\mathcal{N}} \times \alpha_{\mathcal{M}}$, and, by Lemma 6.3.18, we obtain

$$u \in \vartheta_{S,2}(\alpha_n \times \alpha_n) \subseteq \vartheta_{S,2}(\alpha_{\mathcal{N}} \times \alpha_{\mathcal{M}}).$$

Now suppose that some $f \in \mathrm{Step}_{2,N}(G_{\mathcal{N}}; G_{\mathcal{M}})$ satisfies

$$\mathrm{dist}_\infty(u, \vartheta_{S,2}(\alpha_{\mathcal{N}} \times \alpha_{\mathcal{M}}, f)) < \delta. \tag{6.3.6}$$

Let $g \in \mathrm{Meas}_N(G_{\mathcal{N}})$, $h \in \mathrm{Meas}_N(G_{\mathcal{M}})$, and $\varphi \colon N \times N \to 2$ be such that $f = \varphi \circ (g, h)$. After modifying the maps g and h on sets of arbitrarily small measure, we can arrange that there exist integers $n \in \mathcal{N}$ and $m \in \mathcal{M}$ and functions $\tilde{g} \colon G_n \to N$ and $\tilde{h} \colon G_m \to N$ such that

$$g = \tilde{g} \circ \pi_n \qquad \text{and} \qquad h = \tilde{h} \circ \pi_m.$$

Using Propositions 6.3.10 and 6.3.11, we can ensure that inequality (6.3.6) is still valid after this modification. We may furthermore assume that $n \geqslant 2$ and n divides m (the last part uses that $\mathcal{N} \subseteq \mathcal{M}$ and $\mathcal{M}$ is a directed set). Let $\tilde{f} \coloneqq \varphi \circ (\tilde{g}, \tilde{h})$. Then $\tilde{f} \in \mathrm{Step}_{2,N}(G_n; G_m)$ and

$$\vartheta_{S,2}(\alpha_n \times \alpha_m, \tilde{f}) = \vartheta_{S,2}(\alpha_{\mathcal{N}} \times \alpha_{\mathcal{M}}, f).$$

But the existence of such $\tilde{f}$ contradicts Lemma 6.3.18. ∎

Now we can complete the proof of Theorem 6.3.7:

Proof of Theorem 6.3.7. Recall that Γ is a Zariski dense subgroup of $\mathrm{SL}_d(\mathbb{Z})$ with $d \geqslant 2$; S is a finite symmetric subset of Γ such that the group $\langle S \rangle$ is still Zariski dense; $n_0 \in \mathbb{N}^+$ and $\varepsilon > 0$ are given by Theorem 6.3.15 applied to S; and $\delta = \varepsilon/(32|S|)$.

(1) By Lemma 6.3.19, we have

$$\lim_{p \text{ prime}} N_{S,2}(\alpha_{\mathcal{N}(p)}, \alpha_{\mathcal{N}(p)}, \delta) = \infty.$$

The desired conclusion follows by applying Theorem 6.3.14 to the set $C \coloneqq \{(\mathfrak{a}, \mathfrak{a}) : \mathfrak{a} \in \mathcal{FW}_\Gamma\}$.

(2) Let $\mathcal{M} \coloneqq \{m \in \mathbb{N}^+ : \gcd(m, n_0) = 1\}$. Then $\mathcal{M}$ is a directed set, and we claim that $\mathfrak{b} \coloneqq [\alpha_{\mathcal{M}}]$ is as desired. Indeed, by Lemma 6.3.19, we have

$$\lim_{p \text{ prime}} N_{S,2}(\alpha_{\mathcal{N}(p)}, \alpha_{\mathcal{M}}, \delta) = \infty,$$

so it remains to apply Theorem 6.3.14 to the set $C \coloneqq \{(\mathfrak{a}, \mathfrak{b}) : \mathfrak{a} \in \mathcal{FW}_\Gamma\}$. ∎

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