CONTROL OF LINEAR SWITCHED SYSTEMS USING STATE FEEDBACK WITH SATURATION CONSTRAINTS

BY

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THESIS

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ABSTRACT

This thesis considers the stabilization of discrete time switched systems using average output variance as the performance criterion, incorporating actuator saturation constraints into this optimal synthesis. Necessary and sufficient conditions are presented for the existence of a stabilizing static state feedback controller subject to saturation constraints, together with a constructive method to find this controller. These are presented as semi-definite optimization problems.

To my family, for their love and support.

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CHAPTER 1

INTRODUCTION

This thesis considers satisfying a desired performance metric subject to the uniform stabilization of discrete-time switched linear systems with input constraints. These systems contain many discrete-time subsystems with a rule that governs the switching between the different subsystems or modes [1]. The mode at any given time is chosen independently and can be represented by a non-deterministic signal [2]. The study of these types of systems is prevalent in both research and industry to represent and investigate the properties of a large class of plants. These plants are modeled as switched systems and can be thought of as simplified continuous hybrid models governed by discrete events [3]. Switched systems have applications in control of mechanical systems, automotive systems, air traffic control, communications networks, modeling bio-chemical reaction, networks control, and many other diverse areas [1].

Work in stabilizing switched systems using receding horizon control has been published in [3], [4], [5]. These methods solve for the optimal control action over a future window subject to input constraints, implement only the first step, allow the system to move forward, then recalculate the optimal control action. This type of model predictive control (MPC) is heuristic for it offers no guarantees of performance metric or stabilization. It is assumed that for a horizon long enough the system is stable, but the solution is not exact. Further work has been done to expand MPC for linear switched systems to give exact conditions to satisfy a specified performance metric and guarantee stabilization of the system [6]. However, this exact solution does not allow for the inclusion of saturation constraints into the controller design.

A lot of work has also been done in the field of input constraint for non-switched systems. Many articles consider the min-max approach, where the performance criterion used is minimized over the worst-case disturbance realization accounting for saturation constraints [7], [8], [9]. Although this approach can be used for a wide variety of systems, it does not produce an exact solution, is often computationally demanding, and the resulting control laws can be extremely conservative. Further work included the stochastic properties of the disturbance to get less conservative control laws [10], [7].

The work presented in this thesis expands the work on MPC for linear switched systems satisfying a desired performance metric subject to stabilization in [6] by adding conditions for saturation constraints in the form of a semi definite program. The system considered in this thesis is a linear switched system that relies on a finite number of past parameters and has knowledge of a finite future horizon. The performance metric considered is the average output variance over a finite forward window. The stability of switched linear systems with a full state feedback controller is considered in [6], [11]. This thesis will limit the controller type to a static state feedback controller for simplification and provide conditions in the form of linear matrix inequalities for stability and performance incorporating saturation constraints.

This thesis is organized in the following way. Chapter 2 presents mathematical equations and identities necessary for the understanding of further concepts in this thesis. Additionally, the chapter covers using linear matrix inequalities to provide sufficient conditions for the existence of a controller subject to input constraints, and includes a section on the stability of autonomous switched linear time varying systems. Chapter 3 introduces the performance criterion used to judge the performance of switched systems and applies the stability results from chapter 2. The results are then extended to the closed loop case using a static state feedback controller with knowledge of a finite horizon of future modes. Finally, input constraints are added as sufficient linear matrix inequalities to the closed loop system. Conclusions and suggestions for future work are outlined in chapter 4.

CHAPTER 2

BACKGROUND MATERIAL

This chapter presents background material necessary for the understanding of the concepts presented in this thesis. The first sections will detail well-known mathematical identities. Next, a sufficient linear matrix inequality will be presented to guarantee that input constraints on a system are met. The final section will detail basic knowledge on the stability of autonomous discrete-time linear switched systems.

Throughout the sections of this thesis, the following notation will be used uniformly. For any matrix $X \in \mathbb{R}^{n \times m}$, where $\mathbb{R}$ is the set of real numbers, X^* is the Hermitian conjugate of X. The set of complex numbers is denoted by $\mathbb{C}$. A Hermitian matrix is any matrix X that satisfies $X = X^*$ and will be denoted as $X \in \mathbb{H}^n$. A Hermitian matrix X, is positive definite if it satisfies $x^* X x \succ 0$ for all $x \neq 0$, this is denoted by $X \succ 0$. The trace of X is represented by Tr{X}. For any vector $x \in \mathbb{R}^n$, the norm $\|x\|$ denotes the Euclidean norm $\|x\| = \sqrt{x^* x}$. For any Hermitian matrix $X \succ 0$, the corresponding norm is $\|y\|_X = \sqrt{y^* X y}$. The space $\ell_2(\mathbb{R}^n)$ represents sequences $x = (x_0, x_1, x_2, ...)$ where each $x_k \in \mathbb{R}^n$ and

$$\sum_{k=0}^{\infty} \|x_k\|^2 < \infty$$

2.1 Mathematical Analysis

A Linear Matrix Inequality (LMI) is a matrix inequality that has the following form

$$F(x) := F_0 + x_1 F_1 + x_2 F_2 + ... + x_n F_n \succeq 0$$

where $F_0, F_1, ..., F_n \in \mathbb{H}$ and $x_i \in \mathbb{R}$. Multiple LMIs $F_0(x), F_1(x)..., F_n(x) \succ 0$ can be expressed as a single LMI of the form

$$\begin{bmatrix} F_0 & 0 & \dots & 0 \\ 0 & F_1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & F_n \end{bmatrix} \succ 0$$

Since this expression always holds for any set of LMIs, no distinction will be made between a set of LMIs and a single LMI in the subsequent sections. However, it is worth noting that for computational purposes, there can be a significant difference.

Proposition 2.1 Consider a partitioned matrix given by

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}$$

where X_{11}, X_{12} and X_{22} are matrices and X_{11}, X_{12} are self-adjoint. Then the following statements are equivalent:

$$X \succ 0 \tag{2.1 a}$$

$$X_{22} \succ 0 \text{ and } X_{11} - X_{12}X_{22}^{-1}X_{12}^* \succ 0 \tag{2.1 b}$$

$$X_{11} \succ 0 \text{ and } X_{22} - X_{21}^*X_{11}^{-1}X_{12} \succ 0 \tag{2.1 c}$$

Proof: Since multiple LMIs can be expressed as a single LMI, equation (2.1 b) can be written as

$$\begin{bmatrix} X_{11} - X_{12}X_{22}^{-1}X_{12}^* & 0 \\ 0 & X_{22} \end{bmatrix} \succ 0$$

Now multiplying from the left by

$$\begin{bmatrix} I & X_{12}X_{22}^{-1} \\ 0 & I \end{bmatrix} \succ 0$$

and from the right by its adjoint, gives the equation

$$\begin{bmatrix} I & X_{12}X_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} X_{11} - X_{12}X_{22}^{-1}X_{12}^* & 0 \\ 0 & X_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ X_{22}^{-1}X_{12}^* & I \end{bmatrix} \succ 0$$

Multiplying through gives equation (2.1 a) as desired.

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \succ 0$$

The proof from equation (2.1 c) to equation (2.1 a) follows a similar procedure using

$$\begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} - X_{21}^* X_{11}^{-1} X_{12} \end{bmatrix} \succ 0$$

Multiplying from the left by

$$\begin{bmatrix} I & 0 \\ X_{12}^* X_{11}^{-1} & I \end{bmatrix} \succ 0$$

and from the right by the adjoint gives the desired result. □

This is the well-known Schur compliment formula. The same proof can be found in [12], [13].

2.2 Analysis of Input Constraints Using LMI

This section details a method to include input constraints on systems in the form of a linear matrix inequality (LMI). The system considered is a linear time-varying (LTV) system

$$x_{t+1} = A_t x_t + B_t u_t \tag{2.2}$$

$$z_t = C x_t$$

where $u_t \in \mathbb{R}^{n_u}$ is the control input, $x_t \in \mathbb{R}^{n_x}$ is the state of the plant, and $z_t \in \mathbb{R}^{n_y}$ is the plant output at time t.

In the following lemma, it is assumed that X_t is in the gramian at time t, so the system subject to the constraint always lies in the set of reachable states.

Now desired limits on the control signal will be expressed as sufficient LMI constraints. The basic idea presented below can be found for continuous time systems in [14], and the discrete time version using the same notation is presented in [13]. For clarity, the proof is included below.

Lemma 2.3 The input constraint

$$\|u_t\| \leq u_{max} \tag{2.3}$$

can be represented by the equivalent LMI constraint

$$\begin{bmatrix} u_{max}^2 I & Q \\ Q^* & X \end{bmatrix} \succ 0 \tag{2.4}$$

Proof: First, start with the desired constraint on the input u. Squaring both sides of the equation gives

$$\|u_t\|^2 \leq u_{max}^2$$

This can be represented as the equivalent equation

$$\max_{i \geq 0} \|u_t\|^2 \leq u_{max}^2$$

Assuming the system has a static state feedback controller with $u_t = Fx_t$ and $F = QX^{-1}$ gives

$$\max_{i \geq 0} \|QX^{-1}x_t\|^2 \leq u_{max}^2$$

Now, change the set such that

$$\max_{\|v\| \leq 1} \|QX^{-\frac{1}{2}}v\|^2 \leq u_{max}^2$$

The induced matrix norm removes the maximization and leaves

$$\|QX^{-\frac{1}{2}}\|^2 \leq u_{max}^2$$

Applying a well-known result from spectral theory gives

$$QX^{-1}Q^* \preceq u_{max}^2 I$$

Then using the Schur compliment formula, the above inequality becomes

$$\begin{bmatrix} u_{max}^2 I & Q \\ Q^* & X \end{bmatrix} \succ 0$$

which is an LMI that is linear in both X and Q. □

This LMI represents a sufficient constraint that guarantees the control input constraints will be satisfied.

2.3 Switched Systems

Switched systems are prolific in modern control contexts. The type of switched system considered in this thesis exhibit non-deterministic switching between nodes. The stability of the following systems have been analyzed in [6], [11] and the proofs have been included here for completeness. For a time-valued sequence, $\theta(t)$, $\theta_{(a:b)}$ will be used to denote the sequence $(\theta(a), ..., \theta(b))$. $[N]$ will be used to denote the set of indicies $\{1, ..., N\}$. The system has finitely many such modes, each of which possesses a state space model. This can be represented by the system

$$x_{t+1} = A_{\theta(t)}x_t + B_{\theta(t)}w_t \tag{2.5}$$

$$z_t = C_{\theta(t)}x_t + D_{\theta(t)}w_t$$

where each $\theta(t) : \mathbb{Z}^+ \to [N]$ represents the current value at time t of an admissible switching sequence between modes (N). The allowable switching paths are determined by an adjacency matrix $Q \in \{0, 1\}^{N \times N}$ where the switching dynamics are controlled by an automaton. When discussing LTV systems, it is often useful to utilize operator notation. The unilateral shift operator, Z, is defined so that for any sequence $x = (x_0, x_1, x_2, ...)$,

$$Zx = (0, x_0, x_1, ...)$$

Additionally, any bounded operator $Q : \ell_2(\mathbb{R}^n) \to \ell_2(\mathbb{R}^m)$ is called block diagonal if a sequence of operators $Q_k : \mathbb{R}^n \to \mathbb{R}^m$ such that if $y = Qx$ then $y_k = Q_k x_k$, for $k \in \{0, 1, 2, ...\}$. Block diagonal operator Q has the form

$$Q = \begin{bmatrix} Q_0 & 0 & \dots \\ 0 & Q_1 & \\ \vdots & & \ddots \end{bmatrix}$$

Using block diagonal operators to solve LTV systems is covered in [15] and [6].

The discussion of stability for switched linear systems without any performance measure is presented to provide results needed for future sections. The autonomous system is represented by

$$x_{t+1} = A_t x_t \tag{2.6}$$

where $A(t) \in \mathbb{R}^{n\times n}$ for all $t \geq 0$. In this section, $A(t)$ is assumed to be uniformly bounded. The equivalent system written in operator notation using the shift operator, Z, is represented by

$$x = ZAx \tag{2.7}$$

Definition 2.4 An LTV system is uniformly exponentially stable if there exist constants $c \geq 1$ and $\lambda \in (0, 1)$ such that for all $k \geq 0$

$$\|(ZA)^k\| \leq c\lambda^k \tag{2.8}$$

Lemma 2.5 An LTV system is uniformly exponentially stable if and only if there exists a matrix $X \succ 0$ such that

$$(ZA)X(ZA)^* - X \prec 0 \tag{2.9}$$

where X is a block diagonal matrix

Proof: This lemma is the operator version of the well-known discrete time Lyapunov equation. A full proof using operator notation can be found in [15]. □

Lemma 2.6 If X is a solution to the discrete time Lyapunov equation and there exist positive constants α, β, γ such that

$$\alpha I \preceq X \preceq \beta I$$

$$(ZA)X(ZA)^* - X \preceq -\gamma I \tag{2.10}$$

Then the system is uniformly exponentially stable and the constants c, λ are given by

$$c = \sqrt{\frac{\beta}{\alpha}};\ \lambda = \sqrt{1 - \frac{\gamma}{\beta}}$$

Proof: Using the inequality above gives

$$\|(ZA)x\|_X^2 - \|x\|_X^2 \leq -\gamma\|x\|^2$$

The norm $\|x\|_X$ is bounded by

$$\alpha\|x\|^2 \leq \|x\|_X^2 \leq \beta\|x\|^2$$

Using the upper bound shows that $\gamma < \beta$ and results in

$$\|(ZA)x\|_X^2 \leq (1 - \frac{\gamma}{\beta})\|x\|_X^2$$

Using the submultiplicative property of the norm

$$\|(ZA)^k\|_X^2 \leq (1 - \frac{\gamma}{\beta})^k$$

Again, using the bounds on X gives

$$\|(ZA)^k x\|^2 \leq \frac{\beta}{\alpha}(1 - \frac{\gamma}{\beta})^k \|x\|^2$$

Taking the square root and rearranging the terms gives

$$\frac{\|(ZA)^k x\|}{\|x\|} \leq \sqrt{\frac{\beta}{\alpha}} \left(\sqrt{1 - \frac{\gamma}{\beta}}\right)^k$$

which is the desired result. □

Expanding the stability results above to switched systems requires looking at a solution to the discrete time Lyupanov equation that is dependent on a finite number of past parameters.

Lemma 2.7 Suppose the system is uniformly exponentially stable, and that the solution $X \succ 0$ is a block diagonal matrix whose blocks depend only on finitely many past parameters.

Proof: Suppose c and λ are constants such that the system is uniformly exponentially stable. Pick M so that $c\lambda^M < 1$. Now let

$$X^{(M)} = \sum_{k=0}^{M-1} (ZA)^k [(ZA)^*]^k \tag{2.11}$$

It is clear from this definition that X^M is positive definite. Substituting X^M for X in equation (2.9) gives

$$(ZA)X^{(M)}(ZA)^* - X^{(M)}$$

$$= (ZA)^M [(ZA)^*]^M - I$$

$$\preceq -(1 - c^2\lambda^{2M})I$$

Therefore, $X^{(M)}$ satisfies equation (2.9). The individual blocks $X_k^{(M)}$ are of the form

$$X_k^{(M)} = I + \sum_{s=max\{0,k-M\}}^{k-1} (A_s \cdot ... \cdot A_{k+1})(A_s \cdot ... \cdot A_{k+1})^*$$

From this form it is shown that $X^{(M)}$ depends on at most M past parameters as desired. □

For the last part of the discussion on autonomous switched linear systems, the stability of the following system will be considered

$$x_{t+1} = A_{\theta(t)} x_t \tag{2.12}$$

Definition 2.8 A switched linear system is uniformly exponentially stable if there exists constants $c \geq 1$ and $\lambda \in (0, 1)$ so that the system is uniformly exponentially using Definition 2.4 for every admissible switching sequence $\theta = (\theta(1), \theta(1), ...)$.

The parameters of the system depend only on the current mode. The following function $\phi : [N]^{L+H+1} \to [N]$ will be used to represent this dependence.

$$\phi(\theta_{(t-L:t+H)}) = \theta(t), \phi((i_{-L}, ..., i_0, ..., i_H)) = i_0 \tag{2.13}$$

where L is the controller memory, and H is the controller horizon.

Lemma 2.9 For $H \geq 0$ and $L \geq 0$, the switched system above is uniformly exponentially stable if and only if for all admissible $i_{-L:H}$ and ϕ as defined in equation (2.13) there exists $M \geq 0$ and matrices $X_j \succ 0$ for $j \in [N]^{L+M+H}$ such that

$$A^*_{\phi(i_{(-L:H)})} X_{i_{(-L-M+1:H)}} A_{\phi(i_{(-L:H)})} - X_{i_{(-L-M+1:H-1)}} \prec 0 \tag{2.14}$$

Proof: Suppose there exists an M and a set of matrices X_j that satisfy equation (2.14). There are finitely many inequalities, so there exist positive constants α, β so that

$$\alpha I \preceq X_j \preceq \beta I$$

$$A^*_{\phi(\theta_{(-L:H)})} X_{i_{(-L-M+1:H)}} A_{\phi(\theta_{(-L:H)})} - X_{i_{(-L-M+1:H-1)}} \preceq -\alpha I \tag{2.15}$$

Let $\theta(t)$ be an admissible switching sequence. Now, pick modes $\psi_{-L-M}, ..., \psi_{-1}$ so that the resulting switching sequence $(\psi_{-L-M}, ..., \psi_{-1}, \theta(0))$ is admissible. Define $\theta(-L - M) = \psi_{-L-M}, ..., \theta(-1) = \psi_{-1}$ which ensures that $\theta(t)$ is defined for $t \geq -L-M$. Finally, construct the block diagonal operator X with $X_t = X_{\theta_{t-L:t+H}}$ and the block diagonal operator $(A_\theta)_t = A_{\phi(\theta_{(t-L:t+H)})}$. Substituting these two block diagonal operators into the equations yields

$$\alpha I \preceq X \preceq \beta I$$

$$(ZA_\theta)^* X (ZA_\theta) - X \preceq -\alpha I \tag{2.16}$$

Using Lemma 2.5, the above system is exponentially stable and by Lemma 2.6, the bounds on the function rely on constants α and β instead of the switching sequence, proving uniform exponential stability. □

CHAPTER 3

PERFORMANCE OF SWITCHED LINEAR SYSTEMS WITH INPUT CONSTRAINTS

This chapter provides average output variance as a performance criterion for discrete time switched linear systems. Both the open loop and closed loop configurations will be considered. The system has knowledge of a finite horizon of future switching modes. Previous work has been completed using the same performance to study the stability of these systems using a full state feedback controller. The controller has access to finitely many future modes but the input is unconstrained. The results presented extend the previous work to add sufficient LMI input constraints to the controlled system.

This chapter is broken into sections that cover the following topics. First, discrete time switched systems are examined using average output variance in the open loop case. Conditions for stability are quantified and sufficient conditions to guarantee performance are presented. The second section extends the first section to the closed loop case considering a static state feedback controller, K, that has knowledge of finitely many future modes. This section also provides sufficient conditions for stability and performance objectives as LMI constraints. The final section gives input constraints to switched systems and applies those constraints to the closed loop switched system. Again, sufficiency conditions are presented as LMI constraints.

3.1 Windowed Output Regulation for Open Loop Systems

In this section, the performance criterion of the system utilized is the average output variance over a finite window. A full analysis of this type of performance measure for switched systems can be found in [6]. The system considered is

$$x_{t+1} = A_t x_t + B_t w_t \tag{3.1}$$

$$z_t = C_t x_t + D_t w_t$$

where $A_t \in \mathbb{R}^{n\times n}$, $B_t \in \mathbb{R}^{n\times m}$, $C_t \in \mathbb{R}^{l\times n}$, $D_t \in \mathbb{R}^{l\times m}$ for $t > 0$. For simplicity, the feed-through matrix, D_t, is assumed to be zero in this thesis. The parameters of the system are assumed to be bounded uniformly and can be expressed in the operator notation seen in chapter 2. In this section, the disturbance input, w_t is an independent, identically distributed (i.i.d.) sequence satisfying

$$\mathbb{E}[w_t] = 0 \tag{3.2}$$

$$\mathbb{E}[w_t w_s^*] = \begin{cases} I & \text{for } t = s \\ 0 & \text{for } t \neq s \end{cases}$$

For all s, $t \geq 0$. Assuming the length of the finite window is $T \geq 0$, the system performance is quantified as follows.

Definition 3.1 The LTV system above satisfies the *T-step uniform performance level* γ if for $\gamma > 0$, input w_t defined above, and initial state $x_0 = 0$, the system output z_t satisfies

$$\frac{1}{T+1}\sum_{s=0}^{t+T} \mathbb{E}[\|z_s\|^2] \leq \gamma^2 \tag{3.3}$$

for all $t \geq 0$.

Using operator notation, a discrete time LTV system is uniformly exponentially stable if there exists a unique solution, $X_0 \succ 0$ that satisfies the Lyapunov equation

$$(ZA)X_0(ZA)^* - X_0 + (ZB)(ZB)^* = 0 \tag{3.4}$$

where X_0 is a block diagonal matrix.

The structure of X_0 is given by

$$(X_0)_t = \sum_{s=0}^{t} \Phi_{t+1,s+1} B_s B_s^* \Phi_{t+1,s+1}^* \tag{3.5}$$

where $\Phi_{t,s}$ is defined as

$$\Phi_{t,s} = \begin{cases} I & \text{for } t = s \\ A_{t-1} \cdot \ldots \cdot A_s & \text{for } t \geq s \end{cases} \tag{3.6}$$

Remark 3.2 Let $X \succ 0$ be a block diagonal operator satisfying

$$(ZA)X(ZA)^* - X + (ZB)(ZB)^* \preceq 0 \tag{3.7}$$

From the definition of X_0 in equation (3.4) it is clear that for any X satisfying equation (3.7), $X \succeq X_0$.

Proposition 3.3 Using the LTV system defined in equation (3.1) and the definition of w_t given in equation (3.2) yields

$$\mathbb{E}[\|z_t\|^2] = \mathrm{Tr}\{\mathrm{CX_0C^*}\} \tag{3.8}$$

Proof: Begin by considering

$$z_t z_t^* = C_t x_t x_t^* C_t^*$$

Substitute in the definition of x_t from equation (3.1) in operator notation to get

$$z_t z_t^* = C_t (I - ZA)^{-1} (ZB) w_t w_t^* (ZB)^* (I - ZA)^* C_t^*$$

Taking the trace of both sides gives

$$\mathrm{Tr}\{\mathrm{z_t z_t^*}\} = \mathrm{Tr}\{\mathrm{C_t (I - ZA)^{-1} (ZB) w_t w_t^* (ZB)^* (I - ZA)^* C_t^*}\}$$

Now, take the expected value of each side and using the definition of $\mathbb{E}[w_t w_s^*]$ in equation (3.2) leaves

$$\mathbb{E}[\|z_t\|^2] = \mathrm{Tr}\{\mathrm{C_t (I - ZA)^{-1} (ZB)(ZB)^* (I - ZA)^* C_t^*}\}$$

Finally substituting in X_0 completes the proof. □

Definition 3.4 Define a *windowed trace* for a block diagonal operator as

$$\mathrm{Tr_{(t,T)}}\{\mathrm{X}\} = \frac{1}{\mathrm{T}+1} \sum_{\mathrm{s=t}}^{\mathrm{t+T}} \mathrm{Tr}\{\mathrm{X_t}\} \tag{3.9}$$

Using equations (3.3) and (3.9), it is clear that the T-step uniform performance level can be written as

$$\mathrm{Tr_{(t,T)}}\{\mathrm{CX_0C^*}\} < \gamma^2 \tag{3.10}$$

for $t \geq 0$.

Theorem 3.5 The LTV system described in equation (3.1) is uniformly exponentially stable and satisfies T-step uniform performance level γ if and only if there exists a block diagonal operator $X \succ 0$ with blocks dependent on a finite number of past parameters so that

$$(ZA)X(ZA)^* - X + (ZB)(ZB)^* \preceq 0 \tag{3.11}$$

$$\mathrm{Tr}_{(\mathrm{t,T})}\{\mathrm{CXC}^*\} < \gamma^2 \tag{3.12}$$

Proof: Examining the forward direction, assume a solution to equations (3.11) and (3.12) exists. By observation, it is clear that equation (3.11) is a stricter inequality than equation (2.8). Therefore, any X satisfying equation (3.11) also satisfies equation (2.8). Then, using Lemma 2.5, the system is uniformly exponentially stable.

Now, construct X_0 by solving equation (3.7) to obtain the following equation for the T-step performance

$$\mathrm{Tr}_{(\mathrm{t,T})}\{\mathrm{CX_0C}^*\} < \gamma^2$$

Combining Remark 3.2 and using the fact that the windowed trace preserves order (if $X \succeq Y$, then $\mathrm{Tr}_{(\mathrm{t,T})}\{\mathrm{X}\} \geq \mathrm{Tr}_{(\mathrm{t,T})}\{\mathrm{Y}\}$) yields

$$\mathrm{Tr}_{(\mathrm{t,T})}\{\mathrm{CXC}^*\} < \gamma^2$$

Finally, using Proposition 3.3 shows that the system satisfies performance level γ.

Now looking at the reverse direction, define a constant $\epsilon \geq 0$ and look at the sequence

$$X^{(\epsilon,M+1)} = (ZA)X^{(\epsilon,M)}(ZA)^* + (ZB)(ZB)^* + \epsilon I \tag{3.13}$$

The starting term of the sequence is $X^{(\epsilon,0)} = \epsilon I$. Each individual block of operator $X_t^{(\epsilon,M)}$ depends on at most M past parameters. Note that for any $t \geq 0$ the blocks $X_t^{(\epsilon,M)} = X_t^{(\epsilon,M+1)}$ for all $M \geq t$. Defining $X^{(\epsilon)}$ be the weak limit of the blocks in this sequence,

$$X^{(\epsilon)} = (ZA)X^{(\epsilon)}(ZA)^* + (ZB)(ZB)^* + \epsilon I \tag{3.14}$$

then for $M \geq 0$,

$$\epsilon I \preceq X^{(\epsilon,M)} \preceq X^{(\epsilon,M+1)} \preceq X^{(\epsilon)}$$

Let $\widetilde{X}$ be the solution to the Lyapunov equation

$$(ZA)\widetilde{X}(ZA)^* - \widetilde{X} = -I$$

Using the boundedness of X_0 and $\widetilde{X}$, there exists a constant $\beta > 0$ so that $X^{(\epsilon)} \preceq \beta I$. Therefore,

$$\mathrm{Tr}_{(\mathrm{t,T})}\{\mathrm{CX^{(\epsilon)}C}^*\} = \mathrm{Tr}_{(\mathrm{t,T})}\{\mathrm{CX_0C}^*\} + \epsilon\mathrm{Tr}_{(\mathrm{t,T})}\{\mathrm{C\widetilde{X}C}^*\}$$

Since operators $\widetilde{X}$ and C are uniformly bounded, $\mathrm{Tr}_{(\mathrm{t,T})}\{\mathrm{C\widetilde{X}C}^*\}$ is also bounded. Therefore choosing ϵ arbitrarily small will allow $\epsilon\mathrm{Tr}_{(\mathrm{t,T})}\{\mathrm{C\widetilde{X}C}^*\}$ to also be arbitrarily small. Since the system satisfies T-step performance level γ,

$$\mathrm{Tr}_{(\mathrm{t},\mathrm{T})}\{\mathrm{C}\mathrm{X}_0\mathrm{C}^*\} < \gamma^2$$

Choosing ϵ small enough gives

$$\mathrm{Tr}_{(\mathrm{t},\mathrm{T})}\{\mathrm{C}\mathrm{X}^{(\epsilon)}\mathrm{C}^*\} < \gamma^2$$

Using the uniform exponential stability defined in equation (2.10) and Definition 2.4, there exists constants c and λ. Choose M so that

$$c^2\lambda^{2M} < \frac{\alpha}{(\beta-\alpha)} \tag{3.15}$$

Then starting with the equation

$$(ZA)X^{(\epsilon,M)}(ZA)^* - X^{(\epsilon,M)}$$

and using the relationship in equation (3.13) gives

$$= (ZA)[X^{(\epsilon,M)} - X^{(\epsilon,M-1)}](ZA)^* - \epsilon I$$

Now, iterating and noticing that at each iteration, $-\epsilon I$ and ϵI cancel leaves

$$= (ZA)^M[X^{(\epsilon,1)} - X^{(\epsilon,0)}]((ZA)^*)^M - \epsilon I$$

Accounting for the bounds in equation (3.15) yields

$$\preceq (c^2\lambda^{2M})(\beta-\alpha)I - \epsilon I$$

Finally, using the fact that $X^{(\epsilon,M+1)} \preceq X^{(\epsilon)}$ gives

$$\mathrm{Tr}_{(\mathrm{t},\mathrm{T})}\{\mathrm{C}\mathrm{X}^{(\epsilon,\mathrm{M})}\mathrm{C}^*\} < \gamma^2 \tag{3.16}$$

which gives the desired result. □

Use the results for the LTV system, it is now possible to extend these derived formulas to the switched case.

Theorem 3.6 The system described in equation (3.1) is uniformly exponentially stable and satisfies the T-step performance level γ if and only if for $H \geq 0$ and $L \geq 0$ there exists an integer $M \geq 0$ and matrices $X_j \succ 0$ for $j \in [N]^{L+M+H}$ so that for all admissible, $i_{(-L-M:H)}$ and $\hat{i}_{(-L-M:H+T)}$

$$\begin{aligned} A_{\phi(i_{(-L:H)})} X_{i_{(-L-M:H)}} A^*_{\phi(i_{(-L:H)})} - X_{i_{(-L-M+1:H-1)}} \\ + B_{\phi(i_{(-L:H)})} B^*_{\phi(i_{(-L:H)})} \prec 0 \end{aligned} \tag{3.17}$$

$$\mathrm{Tr}_{(t,T)}\{\mathrm{C}_{\phi(\hat{\mathrm{i}}_{(t-L:t+H)})} \mathrm{X}_{\hat{\mathrm{i}}_{(t-L-M:t+H-1)}} \mathrm{C}^*_{\phi(\hat{\mathrm{i}}_{(t-L:t+H)})}\} < \gamma^2 \tag{3.18}$$

Proof: The proof follows the same format as the proof from Lemma 2.9 and including the conditions used in Theorem 3.5. □

This result concludes the section on windowed output regulation for an uncontrolled switched system. Now, the same performance measure will be used to analyze the closed loop system.

3.2 Windowed Output Regulation for Closed Loop Systems

Sufficient conditions for a uniformly stabilizing controller for a switched linear system are now examined. The state feedback arrangement considered in this section is shown in figure 3.1. Where the system in question is the same as equation (3.1) where, for simplification, $D_{\theta(t)}$ is assumed to be zero. This results in the following state space equations

$$x_{t+1} = A_{\theta(t)} x_t + B_{\theta(t)} w_t \tag{3.19}$$

$$z_{t+1} = C_{\theta(t)} x_t \tag{3.20}$$

where w_t is defined as in the previous section. This system is connected via state feedback of the form

$$w_t = K_{\theta(t)} x_t \tag{3.21}$$

For the purpose of this thesis, the controller K is assumed to be static with $K = F$.

Using the definition of controller K and the open loop system, the closed loop equations are as follows

$$x_{t+1} = (A_{\theta(t)} - B_{2,\theta(t)} F_{\theta(t)}) x_t + B_{1,\theta(t)} w_t \tag{3.22}$$

$$z_t = C_{1,\theta(t)} x_t \tag{3.23}$$

Uniform exponential stability of the closed loop system is the same as equation (2.8) for the open loop LTV system.

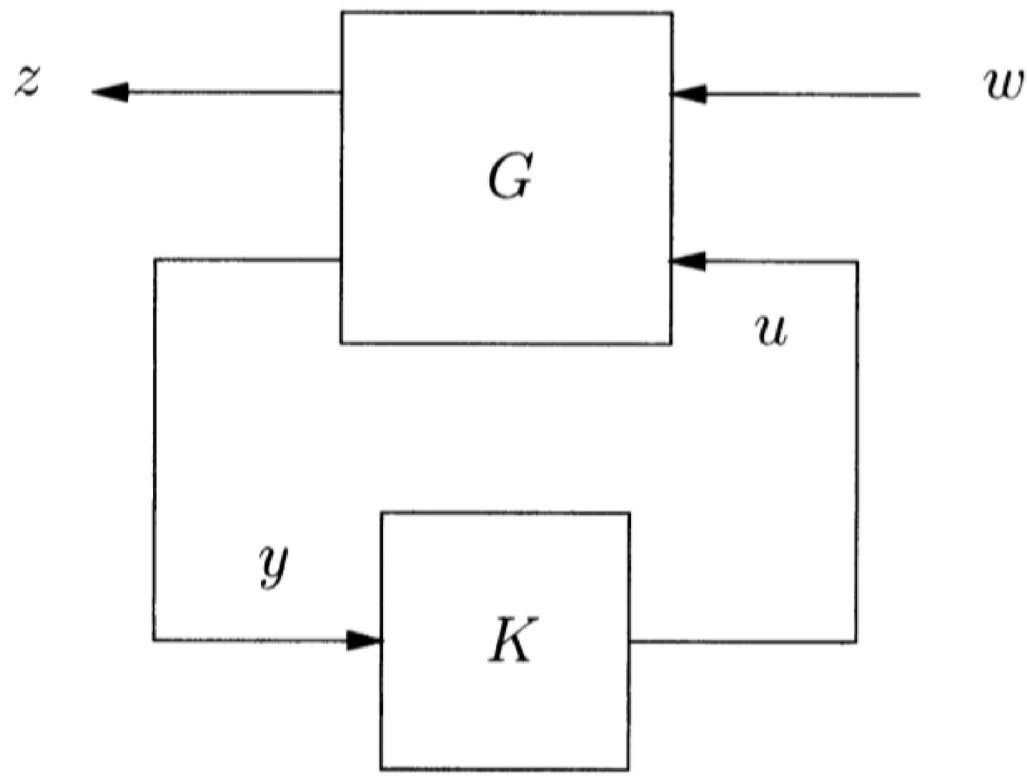

Figure 3.1: Closed Loop State Feedback Arrangement [12]

Theorem 3.7 For $H \geq 0$ and $L \geq 0$ the closed loop system outlined in equations (3.22) and (3.23) is uniformly exponentially stable and satisfies T-step performance measure γ if and only if there exists an integer $M \geq 0$ and a collection a matrices $X_j \succ 0$ for $j \in [N]^{L+M+H}$ such that for all admissible $i_{(-L-M:H)}$ and $\hat{i}_{(-L-M:H+T)}$

$$\begin{aligned}(A_{\phi(i_{(-L:H)})} + B_{2,\phi(i_{(-L:H)})}F_{\phi(i_{(-L:H)})})X_{i_{(-L-M:H)}}(A_{\phi(i_{(-L:H)})} + B_{2,\phi(i_{(-L:H)})}F_{\phi(i_{(-L:H)})})^* \\ -X_{i_{(-L-M+1:H-1)}} - B_{1,\phi(i_{(-L:H)})}B^*_{1,\phi(i_{(-L:H)})} \prec 0\end{aligned} \tag{3.24}$$

$$\mathrm{Tr}_{(t,T)}\{\mathrm{C}_{1,\phi(\hat{\mathrm{i}}_{(t-L:t+H)})}\mathrm{X}_{\hat{\mathrm{i}}_{(t-L-M:t+H)}}\mathrm{C}^*_{1,\phi(\hat{\mathrm{i}}_{(t-L:t+H)})}\} < \gamma^2 \tag{3.25}$$

Proof: The proof follows from the proof in Theorem 3.6 using the closed loop system. $\square$

Using the controller relationship

$$F_{i(-L:H)} = Q_{i(-L-M:H)}X^{-1}_{i(-L-M:H)}$$

the Lyapunov equation in equation (3.24) becomes

$$\begin{aligned}&(A_{\phi(i_{(-L:H)})} + B_{2,\phi(i_{(-L:H)})}Q_{i(-L-M:H)}X^{-1}_{i(-L-M:H)})X_{i_{(-L-M:H)}} \\ &\quad (A_{\phi(i_{(-L:H)})} + B_{2,\phi(i_{(-L:H)})}Q_{i(-L-M:H)}X^{-1}_{i(-L-M:H)})^* \\ &\quad -X_{i_{(-L-M+1:H-1)}} - B_{1,\phi(i_{(-L:H)})}B^*_{1,\phi(i_{(-L:H)})} \prec 0\end{aligned}$$

which is clearly not linear in X and requires manipulation in order to be expressed as an LMI. Multiplying through and rearranging the terms gives

$$\begin{aligned}&-(A_{\phi(i_{(-L:H)})}X_{i_{(-L-M:H)}}A^*_{\phi(i_{(-L:H)})} + B_{2,\phi(i_{(-L:H)})}Q_{i(-L-M:H)}A^*_{\phi(i_{(-L:H)})} \\ &+A_{\phi(i_{(-L:H)})}Q^*_{i(-L-M:H)}B_{2,\phi(i_{(-L:H)})} - X_{i_{(-L-M+1:H-1)}} + B_{1,\phi(i_{(-L:H)})}B^*_{1,\phi(i_{(-L:H)})}) \\ &-(B_{2,\phi(i_{(-L:H)})}Q_{i(-L-M:H)})X^{-1}_{i_{(-L-M:H)}}(B_{2,\phi(i_{(-L:H)})}Q_{i(-L-M:H)})^* \succeq 0\end{aligned}$$

Taking the Schur compliment gives

$$\begin{bmatrix} V(X,Q) & (B_{2,\phi(i_{(-L:H)})}Q_{i(-L-M:H)})^* \\ (B_{2,\phi(i_{(-L:H)})}Q_{i(-L-M:H)}) & X_{i(-L-M:H)} \end{bmatrix} \succ 0$$

$$\begin{aligned}V(X,Q) = &-A_{\phi(i_{(-L:H)})}X_{i(-L-M:H)}A^*_{\phi(i_{(-L:H)})} - B_{2,\phi(i_{(-L:H)})}Q_{i(-L-M:H)}A^*_{\phi(i_{(-L:H)})} \\ &-A_{\phi(i_{(-L:H)})}Q^*_{i(-L-M:H)}B^*_{2,\phi(i_{(-L:H)})} + X_{i(-L-M+1:H-1)} - B_{1,\phi(i_{(-L:H)})}B^*_{1,\phi(i_{(-L:H)})}\end{aligned}$$

which is clearly linear in both X and Q. Therefore, Theorem 3.7 can be represented using LMI constraints.

Theorem 3.8 For $H \geq 0$ and $L \geq 0$ the closed loop system outlined above is uniformly exponentially stable and satisfies T-step performance measure γ if and only if there exists an integer $M \geq 0$ and a collection of matrices $X_j \succ 0$ for $j \in [N]^{L+M+H}$ such that for all admissible $i_{(-L-M:H)}$ and $\hat{i}_{(-L-M:H+T)}$

$$F_{i(-L:H)} = Q_{i(-L-M:H)} X^{-1}_{i(-L-M:H)} \tag{3.26}$$

$$\begin{bmatrix} V(X,Q) & (B_{2,\phi(i_{(-L:H)})} Q_{i(-L-M:H)})^* \\ (B_{2,\phi(i_{(-L:H)})} Q_{i(-L-M:H)}) & X_{i(-L-M:H)} \end{bmatrix} \succ 0 \tag{3.27}$$

$$\begin{aligned} V(X,Q) = &-A_{\phi(i_{(-L:H)})} X_{i(-L-M:H)} A^*_{\phi(i_{(-L:H)})} - B_{2,\phi(i_{(-L:H)})} Q_{i(-L-M:H)} A^*_{\phi(i_{(-L:H)})} \\ &-A_{\phi(i_{(-L:H)})} Q^*_{i(-L-M:H)} B^*_{2,\phi(i_{(-L:H)})} + X_{i(-L-M+1:H-1)} - B_{1,\phi(i_{(-L:H)})} B^*_{1,\phi(i_{(-L:H)})} \end{aligned}$$

$$\mathrm{Tr}_{(\mathrm{t,T})}\{\mathrm{C}_{1,\phi(\hat{\mathrm{i}}_{(\mathrm{t-L:t+H})})} \mathrm{X}_{\hat{\mathrm{i}}_{(\mathrm{t-L-M:t+H})}} \mathrm{C}^*_{1,\phi(\hat{\mathrm{i}}_{(\mathrm{t-L:t+H})})}\} < \gamma^2 \tag{3.28}$$

Proof: The proof follows exactly from the proof of Theorem 3.7 and the substitutions outlined above. □

3.3 Reachable Ellipsoid of the Controllability Gramian

The feasibility for the saturation constraints presented in Section 2.2 will now be proved for the LTV case and extended to the switched case. The following analysis for the linear case is found in [12]. There exists a block diagonal $X_0 \succ 0$, which satisfies equation (3.4). The structure of $(X_0)_t$ is as stated in equation (3.5).

Proposition 3.9 Given $X_0 \succ 0$, the following are equivalent.

(a) the matrix $(\Psi_c)_{t+1}(\Psi_c)^*_{t+1} = (X_0)_t$ is nonsingular

(b) for any $(x_0)_{t+1} \in \mathbb{C}^n$ the input $u_{opt} = (\Psi_c)^*_{t+1}(X_0)^{-1}_t(x_0)_{t+1}$ is an element of minimum norm in the set $\{u_t \in \ell_2(-\infty, 0], (\Psi_c)_{t+1}u_t = (x_0)_{t+1}\}$

Proof: First, the expression for $(\Psi_c)_{t+1}$ is given by

$$(\Psi_c)_{t+1} = \Phi_{t,s} B_s \tag{3.29}$$

where $\Phi_{t,s}$ is as defined in equation (3.6). It is therefore clear from equations (3.29) and

(3.5) that

$$(X_0)_t = (\Psi_c)_{t+1}(\Psi_c)^*_{t+1}$$

which is defined as the controllability gramian. Using the definition for u_{opt} above and multiplying from the right by $(\Psi_c)_{t+1}$ gives the equation

$$(\Psi_c)_{t+1}u_{opt} = (\Psi_c)_{t+1}(\Psi_c)^*_{t+1}(X_0)_t^{-1}(x_0)_{t+1}$$

Simplifying gives

$$(x_0)_{t+1} = (\Psi_c)_{t+1}u_{opt} \tag{3.30}$$

which shows that $u_t = u_{opt}$ belongs to the allowable set of inputs. Now, it must be shown that for any u_t in this set, $\|u_t\| \geq \|u_{opt}\|$. Define the operator

$$P = (\Psi_c)^*_{t+1}(X_0)_t^{-1}(\Psi_c)_{t+1}$$

which is an orthogonal projection satisfying $P = P^2 = P^*$. Using this identity gives

$$\|u_t\| = \|Pu_t\|^2 + \|(I-P)u_t\|^2 \geq \|Pu_t\|^2 \tag{3.31}$$

for any $u_t \in \ell_2(-\infty, 0]$. Let u_t satisfy $(x_0)_{t+1} = (\Psi_c)_{t+1}u_t$.

$$Pu_t = (\Psi_c)^*_{t+1}(X_0)_t^{-1}(\Psi_c)_{t+1}u_t = (\Psi_c)^*_{t+1}(X_0)_t^{-1}(x_0)_{t+1} = u_{opt}$$

Using the relationship in equation (3.31) leaves

$$\|u_t\|^2 \geq \|u_{opt}\|^2$$

as desired. □

Proposition 3.9 shows that the optimal way to reach any state $(x_0)_{t+1}$ is given by $u_{opt} = (\Psi_c)^*_{t+1}(X_0)_t^{-1}(x_0)_{t+1}$. Now, the final reachable states defined in equation (3.30) that can be reached with an input of unit norm are presented.

Proposition 3.10 The following sets are equivalent

$$\{(\Psi_c)_{t+1}u_t : u_t \in \ell_2(-\infty, 0) \text{ and } \|u_t\| \leq 1\} \tag{3.32 a}$$

$$(X_0)_t^{\frac{1}{2}}(x_c)_{t+1} : (x_c)_{t+1} \in \mathbb{C} \text{ and } |(x_c)_{t+1}| \leq 1\} \tag{3.32 b}$$

Proof: To prove that the set in equation (3.32 a) is contained in the set of equation (3.32

b), choose any u_t with $\|u_t\| \leq 1$. Let

$$(x_c)_{t+1} = (X_0)_t^{\frac{1}{2}}(\Psi_c)_{t+1}u_t$$

Notice that the norm of $(x_c)_{t+1}$ is given by

$$|(x_c)_{t+1}|^2 = \langle (X_0)_t^{\frac{1}{2}}(\Psi_c)_{t+1}u_t, (X_0)_t^{\frac{1}{2}}(\Psi_c)_{t+1}u_t\rangle$$

$$= \langle u_t, (\Psi_c)_{t+1}^*(X_0)_t^{-1}(\Psi_c)_{t+1}u_t\rangle$$

Substituting in the projection operator, P gives

$$\langle u_t, Pu_t\rangle = \|Pu_t\|^2$$

Using equation (3.31) shows

$$\|Pu_t\|^2 \leq \|u_t\|^2 \leq 1$$

Proving that $(\Psi_c)_{t+1}u_t = (X_0)_t^{\frac{1}{2}}(x_c)_{t+1}$ is in the set in equation (3.32 b). Looking now at the reverse direction, let $(x_c)_{t+1}$ be a vector such that $|(x_c)_{t+1}| \leq 1$. Choose the input of minimum norm that satisfies $(\Psi_c)_{t+1}u_t = (X_0)_t^{\frac{1}{2}}(x_c)_{t+1}$. The input is u_{opt} and its norm is given by

$$\|u_{opt}\|^2 = ((X_0)_t^{\frac{1}{2}}(x_c)_{t+1})^*(X_0)_t^{-1}((X_0)_t^{\frac{1}{2}}(x_c)_{t+1}) = |(x_c)_{t+1}| \leq 1$$

This shows that $(X_0)_t^{\frac{1}{2}}(x_c)_{t+1}$ is in the set defined in equation (3.32 a), completing the proof. □

Proposition 3.10 showed that all states reachable with input u_t satisfying $\|u_t\| \leq 1$ are given by equation (3.32 b). Geometrically, define the controllability ellipsoid as

$$\mathcal{E}_c = \{(X_0)_t^{\frac{1}{2}}(x_c)_{t+1} : (x_c)_{t+1} \in \mathbb{C}^n \text{ and } |(x_c)_{t+1}| \leq 1\} \tag{3.33}$$

which defines the boundary of the set of reachable states. Now let $X \succ 0$ be a block diagonal matrix satisfying equation (3.37). Notice that for any X satisfying equation (3.37) and any X_0 satisfying equation (3.34), $X \succeq X_0$. Therefore define the ellipsoid

$$\mathcal{E} = \{(X)_t^{\frac{1}{2}}(x_c)_{t+1} : (x_c)_{t+1} \in \mathbb{C}^n \text{ and } |(x_c)_{t+1}| \leq 1\} \tag{3.34}$$

It is clear that any $\mathcal{E}$ satisfying equation (3.34) contains $\mathcal{E}_c$ satisfying equation (3.33). Therefore using X to define input constrains guarantees that the resulting solution is feasible. It is now possible to extend the LTV result to the switched case.

Proposition 3.11 Suppose there exist matrices $X_j \succ 0$ and $(X_c)_j \succ 0$ for $j \in [N]^{L+M+H}$ and for all admissible $i_{(-L-M:H)}$ and $\hat{i}_{(-L-M:H+T)}$. Let X_j be the solution to equation (3.15), and $(X_c)_j$ be the solution to

$$A_{\phi(i_{(-L:H)})}(X_c)_{i_{(-L-M:H)}}A^*_{\phi(i_{(-L:H)})} - (X_c)_{i_{(-L-M+1:H-1)}} + B_{\phi(i_{(-L:H)})}B^*_{\phi(i_{(-L:H)})} = 0$$

Then the ellipsoid defined by

$$\mathcal{E} = \{X^{\frac{1}{2}}_{i_{(-L-M:H)}}(x_c)_{t+1} : (x_c)_{t+1} \in \mathbb{C}^n \text{ and } |(x_c)_{t+1}| \leq 1\} \tag{3.35}$$

contains the reachable states defined by the controllability ellipsoid $\mathcal{E}$ where

$$\mathcal{E} = \{(X_c)^{\frac{1}{2}}_{i_{(-L-M:H)}}(x_c)_{t+1} : (x_c)_{t+1} \in \mathbb{C}^n \text{ and } |(x_c)_{t+1}| \leq 1\} \tag{3.36}$$

Proof: The proof follows the same format as the proofs from Proposition 3.9 and Proposition 3.10 where X_j and $(X_c)_j$ are block diagonal matrices for every admissible switching sequence θ. $\square$

3.4 Input Constrained Windowed Output Regulation

Now, the sufficient LMI conditions in equation (2.3) will be examined for linear switched systems. X_j is the solution to equation (3.15), and the ellipsoid defined by equation (3.35) contains the set of all reachable states.

Lemma 3.12 The input constraint

$$\|u_t\| \leq u_{max} \tag{3.37}$$

can be represented by the equivalent LMI constraint

$$\begin{bmatrix} u^2_{max}I & Q_{i(-L-M:H)} \\ Q^*_{i(-L-M:H)} & X_{i(-L-M:H)} \end{bmatrix} \succ 0 \tag{3.38}$$

Proof: First, start with the desired constraint on the input u. Squaring both sides of the equation gives

$$\|u_t\|^2 \leq u^2_{max}$$

This can be represented as the equivalent equation

$$\max_{i\geq 0}\|u_t\|^2 \leq u_{max}^2$$

Assuming the system has a static state feedback controller with $u_t = F_{\theta(t)}x_t$ and $F_{i(-L:H)} = Q_{i(-L-M:H)}X_{i(-L-M:H)}^{-1}$ gives

$$\max_{i\geq 0}\|Q_{i(-L-M:H)}X_{i(-L-M:H)}^{-1}x_t\|^2 \leq u_{max}^2$$

Now, change the set such that

$$\max_{\|v\|\leq 1}\|Q_{i(-L-M:H)}X_{i(-L-M:H)}^{-\frac{1}{2}}v\|^2 \leq u_{max}^2$$

The induced matrix norm removes the maximization and leaves

$$\|Q_{i(-L-M:H)}X_{i(-L-M:H)}^{-\frac{1}{2}}\|^2 \leq u_{max}^2$$

Applying a well-known result from spectral theory gives

$$Q_{i(-L-M:H)}X_{i(-L-M:H)}^{-1}Q_{i(-L-M:H)}^{*} \preceq u_{max}^2 I$$

Then using the Schur compliment formula on the above inequality becomes

$$\begin{bmatrix} u_{max}^2 I & Q_{i(-L-M:H)} \\ Q_{i(-L-M:H)}^{*} & X_{i(-L-M:H)} \end{bmatrix} \succ 0$$

which is an LMI representing a sufficient condition for constraining the input of a switched system. □

Theorem 3.13 There exists a path-dependent controller with $H \geq 0$ and $T \geq 0$ such that the system given in equations (3.22) and (3.23) is uniformly exponentially stable and satisfies the T-step uniform performance level γ if and only if there exists a collection of matrices $X_j \succ 0$ for $j \in [N]$ such that for all admissible paths $i(-L:H)$ and $\hat{i}(-L:H+T)$

$$F_{i(-L:H)} = Q_{i(-L-M:H)}X_{i(-L-M:H)}^{-1}$$

$$\begin{bmatrix} V(X,Q) & (B_{2,\phi(i_{(-L:H)})}Q_{i(-L-M:H)})^{*} \\ (B_{2,\phi(i_{(-L:H)})}Q_{i(-L-M:H)}) & X_{i(-L-M:H)} \end{bmatrix} \succ 0 \tag{3.39}$$

$$V(X,Q) = -A_{\phi(i_{(-L:H)})}X_{i(-L-M:H)}A^*_{\phi(i_{(-L:H)})} - B_{2,\phi(i_{(-L:H)})}Q_{i(-L-M:H)}A^*_{\phi(i_{(-L:H)})}$$
$$-A_{\phi(i_{(-L:H)})}Q^*_{i(-L-M:H)}B^*_{2,\phi(i_{(-L:H)})} + X_{i(-L-M+1:H-1)} - B_{1,\phi(i_{(-L:H)})}B^*_{1,\phi(i_{(-L:H)})}$$
$$\mathrm{Tr}_{(t,T)}(\mathrm{C}_{1,\hat{i}(t-L:t+H)}\mathrm{X}_{\hat{i}(t-L-M:t+H)}\mathrm{C}_{1,\hat{i}(t-L:t+H)}) > \gamma^2 \tag{3.40}$$
$$\begin{bmatrix} u^2_{max}I & Q_{i(-L-M:H)} \\ Q^*_{i(-L-M:H)} & X_{i(-L-M:H)} \end{bmatrix} \succ 0 \tag{3.41}$$

Proof: The proof follows directly from the proof of Theorem 3.8 adding the input constraint from Lemma 3.12. □

Remark 3.14 The solution for $K_{\theta(t)}$ is dependent on the controller in equation (3.26). Notice that the switching modes of matrices X and Q depend on a path of past parameters of length $-L-M$. Therefore, $-L-M$ represents the minimum number of past parameters necessary for the controller. Although controllers reliant on shorter paths may exist, this theorem does not guarantee their existence.

CHAPTER 4

CONCLUSIONS

This thesis considered satisfying a performance metric subject to stabilization of discrete-time switched linear systems with a dependence on a finite number of past parameters and a finite future horizon. The performance metric considered for the closed loop system was the average output variance over a finite forward window using a random process as the disturbance input. Conditions for the existence of a static state feedback controller subject to saturation constraints were quantified using linear matrix inequalities. The main result of this thesis is Theorem 3.13 which states that a static state feedback controller exists if and only if the semi-definite program can be solved.

Some potential future work includes expanding the controller type to a full state feedback controller subject to saturation constraints. Additionally, the simplifying condition that the feed-through matrix $D = 0$ could be relaxed and the results presented here could be expanded to cover a wider array of systems.

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