MODULATIONAL INSTABILITY IN SOME SHALLOW WATER WAVE MODELS

BY

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DISSERTATION

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Abstract

Modulational or Benjamin-Feir instability is a well known phenomenon of Stokes' periodic waves on the water surface. In this dissertation, we study this phenomenon for periodic traveling wave solutions of various shallow water wave models. We study the spectral stability or instability with respect to long wave length perturbations of small amplitude periodic traveling waves of shallow water wave models like Benjamin-Bona-Mahony and Camassa-Holm equations. We propose a bi-directional shallow water model which generalizes Whitham equation to contain the nonlinearities of nonlinear shallow water equations. The analysis yields a modulational instability index for each model which is solely determined by the wavenumber of underlying periodic traveling wave. For a fixed wavenumber, the sign of the index determines modulational instability. We also includes the effects of surface tension in full-dispersion shallow water models and study its effects on modulational instability.

To Gajendra Bhaia

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LIST OF SYMBOLS

x	Spatial variable with values in $\mathbb{R}$.
t	Time variable with values in $\mathbb{R}^+$.
k	Wavenumber.
ξ	Floquet exponent.
b, b_1 or b_2	Constants of integration.
a	An amplitude parameter.
λ	An eigenvalue.
T	Surface tension.
$L^2(\mathbb{R})$	Space of square integrable functions on $\mathbb{R}$.
$L^2(\mathbb{T})$	Space of 2π-periodic square integrable functions.
$u(x,t)$	Typically a velocity depending on x and t.
$\eta(x,t)$	Typically a surface displacement depending on x and t.
$v(x,t)$	Perturbation to a solution depending on x and t.
$\mathcal{M}$	A multiplier operator.
$\mathcal{M}_k$	A multiplier operator restricted to space of periodic functions.
$m(k)$	Symbol of multiplier operator $\mathcal{M}$.
$\mathcal{L}_{\xi,a}$ or $\mathcal{L}(\xi,a)$	Bloch operators.
$\omega_{n,\xi}$ or $\omega(n,\xi)$	Eigenvalues of Bloch operators for $a = 0$.
ϕ_j	Eigenfunctions.
Δ	Modulational instability index.

Chapter 1

Introduction

A fundamental question in the study of partial differential equations (PDEs) is the stability of solutions. At its core, a solution of a PDE is deemed stable if it is not "affected" by a "small disturbance" added to it initially. To make the terms "affected" and "small disturbance" precise, we work in a suitable function space with a suitable notion of distance. These choices give rise to different notions of stability. In this dissertation, we will be mainly concerned with spectral or linear stability.

We study systems which take the form

$$u_t = Lu + N(u), \tag{1.0.1}$$

where L is some linear operator and N is a nonlinear term depending on u. Here, u can be a scalar or vector-valued function depending on a spatial variable $x \in \mathbb{R}$ and a time variable $t \in \mathbb{R}^+$. All the systems discussed in this dissertation arise as models for water waves in shallow water wave theory. Probably, the most famous example of such a system is the Korteweg-de Vries (KdV) equation,

$$u_t + u_x + u_{xxx} + uu_x = 0.$$

Here, u typically represents the average horizontal velocity of the fluid. This equation was first formulated by Boussinesq [Bou77] and later studied by Diederik Korteweg and Gustav de Vries [KdV95]. A major drawback of the KdV equation is that the linear phase velocity for the KdV equation,

$$c_{\text{KdV}}(k) = 1 - k^2$$

becomes negative for $|k| > 1$, where k is the wave number; thereby contradicting the assumption of uni-directional propagation. A couple of alternatives to the KdV equation have been proposed. The Benjamin-Bona-Mahony (BBM) equation or regularized long-wave equation

$$u_t + u_x - u_{xxt} + uu_x = 0,$$

was introduced in 1966 by Peregrine [Per66] in the study of undular bores. It was studied by Benjamin, Bona and Mahony [BBM72] later in 1972 as an improvement of the Korteweg-de Vries (KdV) equation. The linear phase velocity for the BBM equation,

$$c_{\text{BBM}}(k) = \frac{1}{1+k^2},$$

is bounded for all k, and unlike KdV, it is a good model even for high wave numbers. An improvement over the BBM equation is the Camassa-Holm (CH) equation [CH93, CHH94]

$$u_t + u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

which extend the BBM equation to include higher order nonlinearities. There is an entire family of the CH equations both for the velocity u and surface displacement η which we discuss in detail in Chapter 3.

The linear phase velocity for the Euler's equations describing surface gravity waves is

$$c_{\text{ww}}(k) := \sqrt{\frac{\tanh k}{k}}, \tag{1.0.2}$$

where "ww" in the subscript stands for Water Waves. Note that the linear phase velocities of the KdV, BBM or CH equations agree with (1.0.2) only for small wave numbers k. It was Whitham who realized that the full-dispersion of water waves is necessary to observe the phenomenon of wave breaking and proposed the Whitham equation [Whi74]

$$u_t + c_{\text{ww}}(|\partial_x|)u_x + uu_x = 0,$$

which has same linear phase velocity as in (1.0.2). Here, $c_{\text{ww}}(|\partial_x|)$ is a Fourier multiplier operator given by

$$\widehat{c_{\text{ww}}(|\partial_x|)f}(k) = c_{\text{ww}}(k)\widehat{f}(k) = \sqrt{\frac{\tanh k}{k}}\widehat{f}(k).$$

The Whitham equation can be thought as a full-dispersion generalization of the KdV equation as both have same nonlinearity. Likewise, a full-dispersion generalization of the CH equation can also be defined as

$$u_t + c_{\text{ww}}(|\partial_x|)u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

In Chapter 3, we introduce another version of full-dispersion generalization of the CH equation which not only includes full-dispersion of water waves but also improves upon the nonlinearity of the CH equation.

All the models we introduced so far only describe uni-directional propagation but Euler's equations describing surface gravity waves are bi-directional. The nonlinear shallow water equations,

$$\eta_t + u_x + (u\eta)_x = 0,$$
$$u_t + \eta_x + uu_x = 0,$$

arise from the direct approximation of the Euler's equations in shallow water regime. Here, η is the fluid surface displacement. The linear phase velocity for the nonlinear shallow water equations is constant. The nonlinear shallow water equations can be generalized to include full-dispersion of water waves in (1.0.2). Two such possible systems are

$$\begin{aligned} &\eta_t + u_x + (u\eta)_x = 0, \\ &u_t + c_{\text{ww}}^2(|\partial_x|)\eta_x + uu_x = 0, \end{aligned} \tag{1.0.3}$$

and

$$\begin{aligned} &\eta_t + c_{\text{ww}}^2(|\partial_x|)u_x + (u\eta)_x = 0, \\ &u_t + \eta_x + uu_x = 0, \end{aligned} \tag{1.0.4}$$

where $c_{\text{ww}}^2(|\partial_x|)$ is the Fourier multiplier operator with symbol $c_{\text{ww}}^2(k)$, where $c_{\text{ww}}(k)$ is defined in (1.0.2). The system (1.0.3) was proposed in [HJ15a, HT18] while (1.0.4) was proposed in [MKD15].

We look for periodic traveling wave solutions of these systems. A periodic traveling wave solution is of the form, abusing notation, $u(x,t) = u(x - ct)$, where u is a periodic function of its argument. For all speeds $c > 0$, $u \equiv 0$, is a solution. To prove the existence of nontrivial solutions bifurcating from the trivial solution, we use Lyapunov-Schmidt reduction (see [Nir01, Section 2.7.6], for instance). At some specific values of the speed c, the linearized operator turns out to be a Fredholm operator. Then the original problem can be written as an equivalent system of two equations. The first equation can be solved by implicit function theorem as the linearized operator is invertible on the reduced space. Since the linearized operator is Fredholm, the second equation is finite-dimensional and can be solved easily. We carry out the Lyapunov-Schmidt reduction in a suitable Sobolev space and then establish the smoothness of the solution using some bootstrapping argument. We also calculate small amplitude expansion of these solutions.

Assume that there is a periodic traveling wave solution, $u_*(x,t) = u(x - ct)$ of (1.0.1). Formally applying the ansatz $u = u_* + v$ and expanding the function N in a Taylor series, we obtain,

$$v_t = Lv + DN(u_*)v + O(v^2),$$

where D is the derivative operator. When v is small, which corresponds to the original solution u being near the particular solution u_*, the $O(v^2)$ terms are small when compared with the linear term $Lv + DN(u_*)v$. Therefore, a reasonable approximation to the equation $u_t = Lu + N(u)$ near the solution u_* is

$$v_t = Lv + DN(u_*)v =: \mathcal{L}v, \tag{1.0.5}$$

which is referred to as the linearization of the PDE at the solution u_*. We look for solutions of the form $v(z,t) = e^{\lambda t}v(z)$ of (1.0.5), which reduces (1.0.5) to a spectral problem:

$$\lambda v = \mathcal{L}v.$$

We say that u_* is *spectrally unstable* if the $L^2(\mathbb{R})$-spectrum (or $L^2(\mathbb{R}) \times L^2(\mathbb{R})$-spectrum, if u is vector-valued) of $\mathcal{L}$ intersects the open, right half plane of $\mathbb{C}$ and it is *stable* otherwise. For all the systems in this dissertation, $\mathcal{L}$ is symmetric with respect to the reflections about the real and imaginary axes and therefore, u_* is spectrally unstable if and only if the $L^2(\mathbb{R})$-spectrum of $\mathcal{L}$ is not contained in the imaginary axis. If the solution is spectrally stable or unstable then we can hope that in many situations this will imply the same is true (locally) for the full, nonlinear equation. The goal of this dissertation is to study the spectral stability or instability of periodic traveling wave solutions of some aforementioned shallow water wave models. In Chapter 2, we remark that how the spectral instability result obtained for the BBM equation in this dissertation has been used to obtain a nonlinear instability in [JLL17].

The $L^2(\mathbb{R})$-spectrum of $\mathcal{L}$ is continuous since the coefficients of $\mathcal{L}$ are periodic. This makes the spectral stability of periodic traveling waves a delicate issue. The problem is much simpler if we look at $L^2(\mathbb{T})$-spectrum, in which case the spectrum is discrete and only consists of eigenvalues. Then the question is: is there a way to connect $L^2(\mathbb{R})$-spectrum with $L^2(\mathbb{T})$-spectrum? Fortunately, Floquet-Bloch theory comes to the rescue. It follows (see [Joh13], for a proof) that $\lambda \in \mathbb{C}$ belongs to the $L^2(\mathbb{R})$-spectrum of $\mathcal{L}$ if and only if

$$\lambda\phi = e^{-i\xi z}\mathcal{L}e^{i\xi z}\phi =: \mathcal{L}_\xi\phi \tag{1.0.6}$$

for some $\xi \in [-1/2, 1/2)$ and $\phi \in L^2(\mathbb{T})$. For each $\xi \in [-1/2, 1/2)$, the $L^2(\mathbb{T})$-spectrum of

$\mathcal{L}_\xi$ comprises of discrete eigenvalues of finite multiplicities. Moreover

$$\text{spec}_{L^2(\mathbb{R})}(\mathcal{L}) = \bigcup_{\xi\in[-1/2,1/2)} \text{spec}_{L^2(\mathbb{T})}(\mathcal{L}_\xi).$$

In other words, the continuous $L^2(\mathbb{R})$-spectrum of $\mathcal{L}$ is parametrized by the family of discrete $L^2(\mathbb{T})$-spectra of $\mathcal{L}_\xi$'s.

If $\phi \in L^2(\mathbb{T})$ is an eigenfunction of $\mathcal{L}_\xi$, then from (1.0.6), $e^{i\xi z}\phi$ is an eigenfunction of $\mathcal{L}$. Note that $e^{i\xi z}\phi$ is $2\pi n$-periodic in z if $\xi = 1/n$ for some $n \in \mathbb{N}$. In this dissertation, we analyze the $L^2(\mathbb{T})$-spectrum of $\mathcal{L}_\xi$ for $|\xi|$ small that is close to zero. If $\mathcal{L}_\xi$ has an unstable eigenvalue that is an eigenvalue $\lambda(\xi)$ with $\Re(\lambda(\xi)) > 0$ for all $|\xi|$ sufficiently small, then $\mathcal{L}$ is spectrally unstable to perturbations which have large periods or in other words, long wavelengths. The long wavelength perturbations create small changes or modulations in the periodic wave and therefore, the resulting instability is called *modulational instability*.

The modulational instability in periodic water waves was first observed in 1960s by Benjamin and Feir [BF67, BH67] experimentally and theoretically using some formal expansions. They created uniform trains of periodic waves on the water surface using a wave maker but the wave train got disintegrated after certain time; see Figure 1.1. During the same time, independently, Whitham [Whi67] discovered that a periodic wave on the water surface would be unstable to long wavelength perturbations, namely, the *Benjamin-Feir* or modulational instability provided that

$$kh > 1.363\ldots,$$

where k denotes the carrier wave number, and h is the undisturbed water depth. Results in the support of the Benjamin-Feir instability arrived about the same time, but independently, by Lighthill [Lig65] and Zakharov [Zak68], among others. For full account of the early history, see [ZO09]. The theories developed by Benjamin and Feir or Whitham are difficult to justify in a functional analytic setting. In the 1990s, Bridges and Mielke [BM95] addressed the corresponding spectral instability in a rigorous manner. But the proof leaves some important issues open, such as the stability and instability away from the origin in the spectral plane. The governing equations of the water wave problem are complicated, and they as a rule prevent a detailed account. One may resort to approximate models to gain insights.

Whitham in [Whi65, Whi67] (see also [Whi74]) developed a formal asymptotic approach to study the effects of slow modulations in nonlinear dispersive waves. Since then, there has

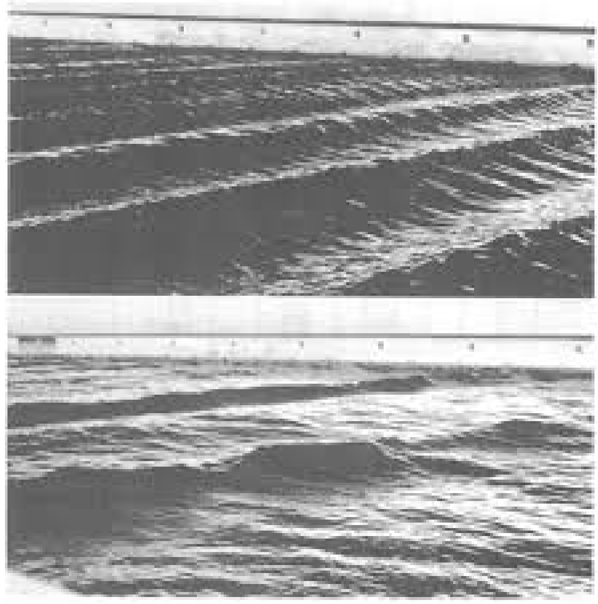


Figure 1.1: The Benjamin-Feir's laboratory experiment

been considerable interest in the mathematical community in rigorously justifying predictions from Whitham's modulation theory. Recently in [BH14, Joh13, HJ15a, HJ15b, HP16b, HP16a, HP17, Pan17] (see also [BHJ16]), long wavelength perturbations were carried out analytically for a class of Hamiltonian systems permitting nonlocal dispersion, for which Evans function techniques and other ODE methods may not be applicable. Specifically, modulational instability indices were derived either with the help of variational structure (see [BH14]) or using asymptotic expansions of the solution (see [Joh13, HJ15a, HJ15b, HP16b, HP16a, HP17, Pan17]). This dissertation is a collection of works of the author in [HP16b, HP16a, HP17, Pan17].

In Chapter 2, we work with a general equation with multiplier operator. For a specific multiplier, this equation reduces to the Benjamin-Bona-Mahony equation. Unfortunately, there is no other known physical example of this general equation but it makes the exposition simple. The equations we study in Chapter 3 includes the Camassa-Holm equation and full-dispersion Camassa-Holm equation. The analysis for both the equations are slightly different and we only produce the details for the full-dispersion Camassa-Holm equation and only hit the main points for the Camassa-Holm equation. Chapter 4 is devoted to the study of modulational instability of small amplitude periodic traveling waves in full-dispersion shallow water models (1.0.3) and (1.0.4).

The modulational instability changes immensely when the effects of surface tension is added. It has been investigated for the capillary-gravity waves using formal asymptotic expansions in [Kaw75] and [DR77], for example. These studies show that for large surface tensions, the modulational instability is similar to gravity waves but for small surface tensions, the wave numbers get partitioned into three intervals of stability and three intervals of instability. Recently, in [HJ15b], the authors determined the modulational stability and instability of a sufficiently small and periodic traveling wave of the Whitham equation with $c_{\rm ww}(|\partial_x|)$ replaced by $c_{\rm ww}(|\partial_x|;T)$ where

$$c_{\rm ww}\widehat{(|\partial_x|;T)f}(k) = \sqrt{(1+Tk^2)\frac{\tanh k}{k}}\widehat{f}(k),$$

where T is the coefficient of surface tension. The result agrees by and large with those in [Kaw75, DR77], for instance, from formal asymptotic expansions of the physical problem. But it fails to predict the limit of "strong surface tension". In a joint work, the author studied the effects of surface tension on modulational instability in various aforementioned shallow water wave models in [HP16a, HP17, Pan17]. In Chapter 5, we present the effects of surface tension on modulational instability in some shallow water models.

Chapter 2

The Benjamin-Bona-Mahony equation

The Benjamin-Bona-Mahony (BBM) equation or regularized long-wave equation

$$u_t - u_{xxt} + u_x + (u^2)_x = 0, \tag{2.0.1}$$

was introduced in 1966 by Peregrine [Per66] in the study of undular bores. It was studied by Benjamin, Bona and Mahony [BBM72] later in 1972 as an improvement of the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + (u^2)_x = 0$$

for modeling long surface gravity waves of small amplitude.

In this chapter, we prove the existence of periodic traveling waves of the BBM equation using Lyapunov-Schmidt reduction method. As a by-product, we also obtain the small amplitude asymptotics of periodic traveling waves. Then, we study the stability and instability of periodic traveling waves in the vicinity of the origin in the spectral plane. We derive modulational instability index as a function of the wave number of the small underlying wave. We show that a sufficiently small, periodic traveling wave of the BBM equation is modulationally unstable to long wavelength perturbations if the wave number is greater than a critical value much like Benjamin-Feir instability of Stokes' waves.

2.1 The equation

We do our analysis on a more general equation

$$u_t + \mathcal{M}(u + u^2)_x = 0. \tag{2.1.1}$$

Here, $\mathcal{M}$ is a Fourier multiplier, defined via its symbol as

$$\widehat{\mathcal{M}f}(k) = m(k)\widehat{f}(k)$$

and characterizing dispersion in the linear limit. Note that

$$m(k) = \text{the phase speed} \quad \text{and} \quad (km(k))' = \text{the group speed}. \tag{2.1.2}$$

Assumption 2.1.1. We assume that:

(M1) m is real valued and twice continuously differentiable,

(M2) m is even and, without loss of generality, $m(0) = 1$,

(M3) $C_1|k|^\alpha < m(k) < C_2|k|^\alpha$ for $|k| \gg 1$ for some $\alpha \geqslant -1$ and $C_1, C_2 > 0$,

(M4) $m(k) \neq m(nk)$ for all $k > 0$ and $n = 2, 3, \dots$.

In the case of $\mathcal{M} = (1 - \partial_x^2)^{-1}$, note that (2.1.1) reduces to the BBM equation.

2.2 Periodic traveling waves

By a traveling wave of (2.1.1) we mean a solution of the form, abusing notation, $u(x, t) = u(x - ct)$ which progresses at a constant velocity $c > 0$ without change of form. Then u satisfies by quadrature that

$$\mathcal{M}(u + u^2) - cu = (1 - c)^2 b$$

for some $b \in \mathbb{R}$. We seek a $2\pi/k$-periodic traveling wave, i.e. $k > 0$ is the wave number and, abusing notation, u is a 2π-periodic function of $z := kx$, satisfying that

$$\mathcal{M}_k(u + u^2) - cu = (1 - c)^2 b. \tag{2.2.1}$$

Here and elsewhere,

$$\mathcal{M}_k e^{inz} = m(kn)e^{inz} \quad \text{for} \quad n \in \mathbb{Z} \tag{2.2.2}$$

and it is extended by linearity and continuity. Note from (M2) of Assumption 2.1.1 that $\mathcal{M}_k$ maps even functions to even functions. Note from (M3) of Assumption 2.1.1 that

$$\mathcal{M}_k : H^s_{2\pi} \to H^{s-\alpha}_{2\pi} \qquad \text{for all } k > 0 \quad \text{for all } s \geqslant 0$$

is bounded.

We begin by proving regularity of solutions of (2.2.1).

Lemma 2.2.1 (Regularity)**.** *If $u \in H^1_{2\pi}$ solves* (2.2.1) *for some $c > 0$, $k > 0$ and $b \in \mathbb{R}$ and if $|1 + u(z)| > 0$ for all z then $u \in H^\infty_{2\pi}$.*

Proof. In the case of $\alpha < 0$, indeed, it follows from the Sobolev inequality that

$$cu = \mathcal{M}_k(u + u^2) - (1 - c)^2 b \in H^{1-\alpha}_{2\pi}.$$

In the case of $\alpha > 0$, similarly,

$$u = \frac{1}{1+u}(c\mathcal{M}_k^{-1}u + (1 - c)^2 b) \in H^{1+\alpha}_{2\pi}.$$

The claim then follows from a bootstrapping argument. □

Note that, since we are only going to consider small amplitude solutions, the condition $|1 + u(z)| > 0$ for all z in Lemma 2.2.1 is redundant. We are now ready to prove existence of a solution of (2.2.1) in $H^1_{2\pi}$ which will be in $H^\infty_{2\pi}$ by Lemma 2.2.1.

Lemma 2.2.2 (Existence)**.** *Under Assumption 2.1.1, for each $k > 0$ and $|b|$ sufficiently small, a one-parameter family of $2\pi/k$-periodic traveling waves of* (2.1.1) *exists and, abusing notation,*

$$u(x, t) = u(a, b)(k(x - c(k, a, b)t)) =: u(k, a, b)(z)$$

for $|a|$ sufficiently small; u and c depend analytically on k, a, b, and u is smooth, even and 2π-periodic in z, and c is even in a. Furthermore

$$\begin{aligned} u(k, a, b)(z) =&b(m(k) - 1) + a\cos z \\ &+ \frac{1}{2}a^2\Big(\frac{1}{m(k) - 1} + \frac{m(2k)}{m(k) - m(2k)}\cos 2z\Big) + O(a(a^2 + b)), \end{aligned} \tag{2.2.3}$$

$$\begin{aligned} c(k, a, b)(z) =&m(k) + 2bm(k)(m(k) - 1) \\ &+ a^2 m(k)\Big(\frac{1}{m(k) - 1} + \frac{1}{2}\frac{m(2k)}{m(k) - m(2k)}\Big) + O(a(a^2 + b)) \end{aligned} \tag{2.2.4}$$

as $a, b \to 0$.

Proof. The proof follows along the same line as the arguments in [Joh13, Appendix A], for instance. For an arbitrary $k > 0$, a straightforward calculation reveals that

$$u_0(k, c, b) = b(c - 1) + O(b^2)$$

makes a constant solution of (2.2.1) for all $c > 0$ and $|b|$ sufficiently small. (Another constant solution is $u = (1 - b)(c - 1) + O(b^2)$, which we discard for the sake of near-zero solutions.)

We are interested in determining at which value of c there bifurcates a family of non-constant $H^1_{2\pi}$-solutions, and hence smooth solutions of (2.2.1) by Lemma 2.2.1. A necessary condition, it turns out, is that the linearized operator of (2.2.1) about u_0 allows a nontrivial kernel. This is not in general a sufficient condition. But bifurcation does take place if the kernel and co-kernel is one dimensional. Under (M4) of Assumption 2.1.1, a straightforward calculation reveals that

$$\ker(\mathcal{M}_k(1+2u_0)-c_0)=\mathrm{span}\{\cos z\}$$

in the sector of even functions in $H^1_{2\pi}$, provided that

$$c_0(k,b):=m(k)(1+2u_0)=m(k)+2bm(k)(m(k)-1)+O(b^2). \tag{2.2.5}$$

Therefore

$$u_0(k,b):=u_0(k,c_0,b)=b(m(k)-1)+O(b^2). \tag{2.2.6}$$

For arbitrary $k>0$ and $|b|$ sufficiently small, one may then employ a Lyapunov-Schmidt reduction and construct a one-parameter family of non-constant, even and smooth solutions of (2.2.1) near $u=u_0(k,b)$ and $c=c_0(k,b)$. We may assume that $\alpha<0$ in (M3) in Assumption 2.1.1. Let

$$F(u;k,c,b)=\mathcal{M}_k(u+u^2)-cu-(1-c)^2b$$

and note from (M3) of Assumption 2.1.1 and the Sobolev inequality that $F:H^1_{2\pi}\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\to H^1_{2\pi}$ is well defined. In the case of $\alpha>0$ in (M3) of Assumption 2.1.1, let

$$F(u;k,c,b)=u+u^2-c\mathcal{M}_k^{-1}u-(1-c)^2b,$$

instead, and the proof is nearly identical. Note that

$$\partial_uF(u;k,c,b)v=(\mathcal{M}_k(1+2u)-c)v\in L^2_{2\pi},\qquad v\in H^1_{2\pi},$$

and $\partial_kF(u;k,c,b)\delta:=\mathcal{M}'_\delta(u+u^2)$, $\delta\in\mathbb{R}$, are continuous, where a straightforward calculation reveals that

$$\mathcal{M}'_\delta e^{inz}=\delta nm'(kn)e^{inz}\quad\text{for}\quad n\in\mathbb{Z}.$$

Since

$$\partial_cF(u;k,c,b)=-u-2(1-c)b\quad\text{and}\quad\partial_bF(u;k,c,b)=-(1-c)^2$$

are continuous, we deduce that $F:H^1_{2\pi}\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\to H^1_{2\pi}$ is C^1. Recall that u_0 and c_0, in (2.2.6) and (2.2.5), satisfy that

$$Le^{\pm iz} =: (\mathcal{M}_k(1+2u_0) - c_0)e^{\pm iz} = 0.$$

For arbitrary $k > 0$ and $|b|$ sufficiently small, we seek a non-constant solution $u \in H^1_{2\pi}$ near u_0 of

$$F(u;k,c,b) = 0 \tag{2.2.7}$$

for some c near c_0. Let

$$u(z) = u_0(k,b) + \frac{1}{2}ae^{iz} + \frac{1}{2}\bar{a}e^{-iz} + v(z) \quad \text{and} \quad c = c_0 + r,$$

where $a \in \mathbb{C}$ and $v \in H^1_{2\pi}$ satisfying that

$$\int_{-\pi}^{\pi} v(z)e^{\pm iz}\, dz = 0,$$

and $r \in \mathbb{R}$. Substituting these into (2.2.7) and using $Le^{\pm iz} = 0$, we arrive at that

$$Lv =: g(a,\bar{a},v,r,b), \tag{2.2.8}$$

where g is analytic in its argument and $g(0,0,0,r,b) = 0$ for all $r, b \in \mathbb{R}$. We define $\Pi : L^2_{2\pi} \to \ker L$ as

$$\Pi f(z) = \widehat{f}(1)e^{iz} + \widehat{f}(-1)e^{-iz}.$$

Since $\Pi v = 0$, we may write (2.2.8) as

$$Lv = (I - \Pi)g(a,\bar{a},v,r,b) \quad \text{and} \quad 0 = \Pi g(a,\bar{a},v,r,b). \tag{2.2.9}$$

Note that

$$(L_{|(I-\Pi)H^1_{2\pi}})^{-1}f(z) = \sum_{n\neq\pm1} \frac{\widehat{f}(n)}{(1+2u_0)m(kn) - c_0}e^{inz}.$$

Consequently, we may rewrite (2.2.9) as

$$v = L^{-1}(I-\Pi)g(a,\bar{a},v,r,b) \quad \text{and} \quad 0 = \Pi g(a,\bar{a},v,r,b). \tag{2.2.10}$$

Clearly $(L_{|(I-\Pi)H^1_{2\pi}})^{-1}$ depends analytically on its arguments.

It follows from the implicit function theorem that a unique solution

$$v = V(a,\bar{a},r,b) \in (I-\Pi)H^1_{2\pi}$$

exists to the former equation in (2.2.10) in the vicinity of $(a, \bar{a}, r, b) = (0, 0, 0, b)$, which depends analytically on its argument. By uniqueness, moreover,

$$V(0, 0, r, b) = 0 \qquad \text{for all } r \in \mathbb{R} \text{ and } |b| \text{ sufficiently small.} \tag{2.2.11}$$

Since (2.2.1) remains invariant under $z \to z + z_0$ and $z \to -z$, it follows that

$$V(a, \bar{a}, r, b)(z + z_0) = V(ae^{iz_0}, \bar{a}e^{-iz_0}, r, b) \quad \text{and} \quad V(a, \bar{a}, r, b)(-z) = V(a, \bar{a}, r, b)(z) \tag{2.2.12}$$

for any $z_0 \in \mathbb{R}$. To proceed, we rewrite the latter equation in (2.2.10) as

$$\Pi g(a, \bar{a}, V(a, \bar{a}, r, b), r, b) = 0,$$

which is solvable provided that

$$Q_{\pm}(a, \bar{a}, r, b) := \int_{-\pi}^{\pi} \frac{1}{2}(ae^{iz} \pm \bar{a}e^{-iz})g(a, \bar{a}, V(a, \bar{a}, r, b), r, b)\; dz = 0.$$

Taking $z_0 = -2\arg(a)$ in (2.2.12) we find that

$$Q_{-}(\bar{a}, a, r, b) = Q_{-}(a, \bar{a}, r, b) = -Q_{-}(\bar{a}, a, r, b).$$

Therefore $Q_{-}(a, \bar{a}, r, b) = 0$, which is trivial. Taking $z_0 = -\arg(a)$ in (2.2.12), similarly,

$$Q_{+}(a, \bar{a}, r, b) = Q_{+}(|a|, |a|, r, b).$$

Therefore $Q_{+}(a, a, r, b) = 0$ for any $a \in \mathbb{R}$. Since (2.2.11) implies that $a^{-1}V(a, a, r, b)$ is analytic in a for $|a|$ sufficiently small, we arrive at that

$$Q_{+}(a, a, r, b) = \int_{-\pi}^{\pi} a(\cos z)g(a, \bar{a}, V(a, \bar{a}, r, b)(z), r, b)\; dz =: a^2(\pi r + R(a, r, b)),$$

where R is analytic in its argument, even in a and $R(0, 0, b) = \partial_r R(0, 0, b) = 0$. It then follows from the implicit function theorem that a unique solution to

$$\pi r(a, b) + R(a, r(a, b), b) = 0$$

exists for $|a|$ sufficiently small, which is real analytic for $|a|$ sufficiently small and even in a.

To summarize,

$$(v, r) = (V(a, \bar{a}, r, b) \quad \text{and} \quad r(|a|, b))$$

uniquely solve (2.2.10) for $|a|$, $|b|$ sufficiently small. Consequently,

$$u(z) = u_0 + a\cos z + V(a, a, r(|a|, b), b)(z) \quad \text{and} \quad c = c_0 + r(|a|, b)$$

solve (2.2.7) for $|a|$, $|b|$ sufficiently small.

It remains to show (2.2.3) and (2.2.4). Let $k > 0$ be fixed and suppressed to simplify the exposition. We assume that $b = 0$. Since u and c depend analytically on a for $|a|, |b|$ sufficiently small and since c is even in a, we write that

$$u(k, a, b)(z) := u_0(k, b) + a\cos z + a^2 u_2(z) + a^3 u_3(z) + O(a^4)$$

and

$$c(k, a, b) := c_0(k, b) + a^2 c_2 + O(a^4)$$

as $a \to 0$, where u_2, $u_3, \dots$ are even and 2π-periodic in z. Substituting these into (2.2.1), at the order of a^2, we gather that

$$\mathcal{M}_k(u_2(z) + \cos^2 z) - m(k)u_2(z) = 0.$$

A straightforward calculation then reveals that

$$u_2(z) = \frac{1}{2}\Big(\frac{1}{m(k) - 1} + \frac{m(2k)\cos(2z)}{m(k) - m(2k)}\Big).$$

Continuing, at the order of a^3,

$$\mathcal{M}_k(u_3(z) + 2u_2(z)\cos z) - m(k)u_3(z) - c_2\cos z = 0,$$

whence

$$c_2 = m(k)\Big(\frac{1}{m(k) - 1} + \frac{1}{2}\frac{m(2k)}{m(k) - m(2k)}\Big).$$

This completes the proof. □

In the remainder of the chapter we assume that $b = 0$; loosely speaking, the wave height is small. A small amplitude, but not necessarily small height, periodic traveling wave of (2.1.1) may be studied in like manner. But expressions become quite complicated. Hence we do not pursue here. Let $u = u(k, a, 0)$ and $c = c(k, a, 0)$ for $k > 0$ and $|a|$ sufficiently small, be as in Lemma 2.2.2. We are interested in its stability and instability.

2.3 The spectral problem

Linearizing (2.1.1) about u in the coordinate frame moving at the speed c, we arrive at that

$$v_t + k\partial_z(\mathcal{M}_k(1 + 2u) - c)v = 0.$$

Seeking a solution of the form $v(z, t) = e^{\lambda kt}v(z)$, $\lambda \in \mathbb{C}$ and $v \in L^2(\mathbb{R})$, moreover, we arrive at that

$$\lambda v = \partial_z(-\mathcal{M}_k(1 + 2u) + c)v =: \mathcal{L}(k, a)v. \tag{2.3.1}$$

We employ Floquet theory to study the $L^2(\mathbb{R})$-spectrum of $\mathcal{L}(k, a)$. The corresponding Bloch operators are given by

$$\mathcal{L}(\xi)(k, a) := e^{-i\xi z}\mathcal{L}(k, a)e^{i\xi z} \tag{2.3.2}$$

for some $\xi \in (-1/2, 1/2]$. Since

$$\mathrm{spec}_{L^2_{2\pi}}(\mathcal{L}(\xi)) = \overline{\mathrm{spec}_{L^2_{2\pi}}(\mathcal{L}(-\xi))},$$

it suffices to take $\xi \in [0, 1/2]$.

Notation. In the remainder of the section, $k > 0$ is fixed and suppressed to simplify the exposition, unless specified otherwise. Let

$$\mathcal{L}_{\xi,a} = \mathcal{L}(\xi)(k, a).$$

In the case of $a = 0$, namely the zero solution, a straightforward calculation reveals that

$$\mathcal{L}_{\xi,0}e^{inz} = i\omega_{n,\xi}e^{inz} \quad \text{for all } n \in \mathbb{Z} \quad \text{for all } \xi \in [0, 1/2], \tag{2.3.3}$$

where

$$\omega_{n,\xi} = (\xi + n)(m(k) - m(k(\xi + n))). \tag{2.3.4}$$

We pause to remark that (2.3.3)-(2.3.4) imply that the zero solution of (2.1.1) is spectrally stable to square integrable perturbations. Observe that

$$\omega_{1,0} = \omega_{-1,0} = \omega_{0,0} = 0,$$

and $\omega_{n,0} \neq 0$ otherwise. As a matter of fact, zero is an $L^2_{2\pi}$-eigenvalue of $\mathcal{L}_{0,0}$ with algebraic and geometric multiplicity three, and

$$\cos z, \qquad \sin z \quad \text{and} \quad 1 \tag{2.3.5}$$

form a (real-valued) orthogonal basis of the corresponding eigenspace. For ξ small (and $a = 0$), furthermore, they form an orthogonal basis of the spectral subspace associated with eigenvalues $i\omega_{1,\xi}$, $i\omega_{-1,\xi}$, $i\omega_{0,\xi}$ of $\mathcal{L}_{\xi,0}$.

For $|a|$ small but $\xi = 0$, on the other hand, zero is a generalized $L^2_{2\pi}$-eigenvalue of $\mathcal{L}_{0,a}$ with algebraic multiplicity three and geometric multiplicity two, and

$$\begin{aligned}\phi_1(z) =&: \frac{1}{2m(k)(m(k)-1)}((\partial_b c)(\partial_a u) - (\partial_a c)(\partial_b u))(k,a,0)(z) \\ &= \cos z - \frac{1}{2}a\frac{m(2k)}{m(k)-m(2k)} + a\frac{m(2k)}{m(k)-m(2k)}\cos 2z + O(a^2) \end{aligned} \tag{2.3.6}$$

$$\phi_2(z) =: -\frac{1}{a}\partial_z u(k,a,0)(z) = \sin z + a\frac{m(2k)}{m(k)-m(2k)}\sin 2z + O(a^2) \tag{2.3.7}$$

$$\phi_3(z) =: \frac{1}{m(k)-1}\partial_b u(k,a,0)(z) = 1 + O(a^2) \tag{2.3.8}$$

form a basis of the corresponding generalized eigenspace. Indeed, differentiating (2.2.1) with respect to z, a, b, we find that

$$\mathcal{L}_{0,a}(\partial_z u) = 0, \qquad \mathcal{L}_{0,a}(\partial_a u) = (\partial_a c)(\partial_z u), \qquad \mathcal{L}_{0,a}(\partial_b u) = (\partial_b c)(\partial_z u),$$

respectively, and (2.3.6)-(2.3.8) follows at once; see [HJ15a, Lemma 3.1] for details. In the case of $a = 0$, note that (2.3.6)-(2.3.8) reduce to (2.3.5).

2.4 The perturbation analysis

To recapitulate, in the case of ξ small and $a = 0$, $\mathcal{L}_{\xi,0}$ possesses three purely imaginary eigenvalues near the origin and functions in (2.3.5) form an orthogonal basis of the associated spectral subspace. In the case of $\xi = 0$ and a small, moreover, $\mathcal{L}_{0,a}$ possesses three eigenvalues at the origin and functions in (2.3.6)-(2.3.8) form a bases of the associated eigenspace. In order to study how three eigenvalues at the origin vary with ξ and $|a|$ small, we proceed as in [HJ15a] and compute 3×3 matrices

$$\mathbf{B}_{\xi,a}=\left(\frac{\langle \mathcal{L}_{\xi,a}\phi_j,\phi_k\rangle}{\langle\phi_j,\phi_j\rangle}\right)_{j,k=1,2,3} \quad\text{and}\quad \mathbf{I}_a=\left(\frac{\langle\phi_j,\phi_k\rangle}{\langle\phi_j,\phi_j\rangle}\right)_{j,k=1,2,3}, \tag{2.4.1}$$

where ϕ_j's, $j=1,2,3$, are in (2.3.6)-(2.3.8) and $\langle\,,\rangle=\langle\,,\rangle_{L^2_{2\pi}}$ is inner product in $L^2(\mathbb{T})$. Note that $\mathbf{B}_{\xi,a}$ and $\mathbf{I}_a$, respectively, represent actions of $\mathcal{L}_{\xi,a}$ and the identity on the spectral subspace associated with three eigenvalues at the origin. For ξ and $|a|$ sufficiently small, eigenvalues of $\mathcal{L}_{\xi,a}$ agree in location and multiplicity with the roots of the characteristic equation $\det(\mathbf{B}_{\xi,a}-\lambda\mathbf{I}_a)=0$; see [Kat76, Section 4.3.5], for instance, for details.

Using (2.3.1), (2.3.2) and (2.2.3), (2.2.4), we make a Baker-Campbell-Hausdorff expansion to write that

$$\begin{aligned}\mathcal{L}_{\xi,a}=&\mathcal{L}_{0,0}+i\xi[\mathcal{L}_{0,0},z]-\frac{\xi^2}{2}[[\mathcal{L}_{0,0},z],z]\\ &-2a\mathcal{M}_k\partial_z(\cos z)-2i\xi a[\partial_z\mathcal{M}_k,z]\cos z+O(\xi^3+\xi^2a+a^2)\\ =:&L-2a\mathcal{M}_k\partial_z(\cos z)-2i\xi aM_1\cos z+O(\xi^3+\xi^2a+a^2)\end{aligned} \tag{2.4.2}$$

as $\xi,a\to 0$. Note that $M_1=[\mathcal{L}_{0,0},z]$ and $[[\mathcal{L}_{0,0},z],z]$ are well defined in $L^2_{2\pi}$ even though z is not. Note moreover that $L=\mathcal{L}_{\xi,0}$ up to second order for $\xi\ll 1$ and M_1 is the $O(\xi)$ term in the asymptotic expansion of $\mathcal{L}_{\xi,0}$ for $\xi\ll 1$.

We use (2.3.3) and (2.3.4), or its Taylor expansion (see (M1) of Assumption 2.1.1), to compute that

$$\begin{aligned}\mathcal{L}_{\xi,0}e^{\pm inz}=\pm in(m(k)-m(kn))e^{\pm inz}&+i\xi(m(k)-m(kn)-km'(kn))e^{\pm inz}\\ &\mp\frac{1}{2}\xi^2(2km'(kn)+k^2m''(kn))e^{\pm inz}+O(\xi^3)\end{aligned}$$

as $\xi\to 0$. Therefore we infer that

$$L1=i\xi(m(k)-1)\qquad\text{and}\quad M_11=m(k)-1.$$

Similarly,

$$\begin{aligned}L\begin{Bmatrix}\cos z\\ \sin z\end{Bmatrix}=&-i\xi km'(k)\begin{Bmatrix}\cos z\\ \sin z\end{Bmatrix}\pm\frac{1}{2}\xi^2(2km'(k)+k^2m''(k))\begin{Bmatrix}\sin z\\ \cos z\end{Bmatrix},\\ M_1\begin{Bmatrix}\cos z\\ \sin z\end{Bmatrix}=&-km'(k)\begin{Bmatrix}\cos z\\ \sin z\end{Bmatrix}\end{aligned}$$

and

$$L\begin{Bmatrix}\cos 2z\\ \sin 2z\end{Bmatrix} = \mp 2(m(k)-m(2k))\begin{Bmatrix}\sin 2z\\ \cos 2z\end{Bmatrix} + i\xi(m(k)-m(2k)-km'(2k))\begin{Bmatrix}\cos 2z\\ \sin 2z\end{Bmatrix}$$
$$\pm \frac{1}{2}\xi^2(2km'(2k)+k^2m''(2k))\begin{Bmatrix}\sin 2z\\ \cos 2z\end{Bmatrix},$$
$$M_1\begin{Bmatrix}\cos 2z\\ \sin 2z\end{Bmatrix} = (m(k)-m(2k)-km'(2k))\begin{Bmatrix}\cos 2z\\ \sin 2z\end{Bmatrix}.$$

Substituting (2.3.6)-(2.3.8) into (2.4.2), and using the above and (2.3.3), we make a lengthy but straightforward calculation to find that

$$\begin{aligned}\mathcal{L}_{\xi,a}\phi_1 =& -i\xi km'(k)\cos z\\ & - i\xi a\Big(1+\frac{m(2k)(m(k)-1)}{2(m(k)-m(2k))}\Big)\\ & - i\xi a\Big(m(2k)+2km'(2k)-\frac{m(2k)(m(k)-m(2k)-2km'(2k))}{m(k)-m(2k)}\Big)\cos 2z\\ & + \frac{1}{2}\xi^2(2km'(k)+k^2m''(k))\sin z + O(\xi^3+a^2)\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_{\xi,a}\phi_2 =& -i\xi km'(k)\sin z\\ & - i\xi a\Big(m(2k)+2km'(2k)-\frac{m(2k)(m(k)-m(2k)-2km'(2k))}{m(k)-m(2k)}\Big)\sin 2z\\ & - \frac{1}{2}\xi^2(2km'(k)+k^2m''(k))\cos z + O(\xi^3+a^2),\\ \mathcal{L}_{\xi,a}\phi_3 =& 2am(k)\sin z + i\xi(m(k)-1) - 2i\xi a(m(k)+km'(k))\cos z + O(\xi^3+a^2)\end{aligned}$$

as $\xi, a \to 0$. Using the above and (2.3.6)-(2.3.8), we make another lengthy but straightforward calculation to find that

$$\begin{aligned}\langle \mathcal{L}_{\xi,a}\phi_1,\phi_1\rangle &= \langle \mathcal{L}_{\xi,a}\phi_2,\phi_2\rangle = -\frac{1}{2}i\xi km'(k) + O(\xi^3+a^2),\\ \langle \mathcal{L}_{\xi,a}\phi_1,\phi_2\rangle &= -\langle \mathcal{L}_{\xi,a}\phi_2,\phi_1\rangle = \frac{1}{4}\xi^2(2km'(k)+k^2m''(k)) + O(\xi^3+a^2),\\ \langle \mathcal{L}_{\xi,a}\phi_1,\phi_3\rangle &= -i\xi a\Big(1+\frac{1}{2}\frac{m(2k)(m(k)-1)}{m(k)-m(2k)}\Big) + O(\xi^3+a^2)\end{aligned}$$

and

$$
\begin{aligned}
\langle \mathcal{L}_{\xi,a}\phi_2, \phi_3\rangle &= 0 + O(\xi^3 + a^2),\\
\langle \mathcal{L}_{\xi,a}\phi_3, \phi_1\rangle &= -i\xi a\Big(m(k) + km'(k) + \frac{1}{2}\frac{m(2k)(m(k)-1)}{m(k)-m(2k)}\Big) + O(\xi^3 + a^2),\\
\langle \mathcal{L}_{\xi,a}\phi_3, \phi_2\rangle &= am(k) + O(\xi^3 + a^2),\\
\langle \mathcal{L}_{\xi,a}\phi_3, \phi_3\rangle &= i\xi(m(k)-1) + O(\xi^3 + a^2)
\end{aligned}
$$

as $\xi, a \to 0$. Moreover we use (2.3.6)-(2.3.8) to compute that

$$
\begin{aligned}
\langle \phi_1, \phi_1\rangle = \langle \phi_2, \phi_2\rangle &= \frac{1}{2} + O(\xi^3 + a^2),\\
\langle \phi_1, \phi_2\rangle &= 0 + O(\xi^3 + a^2),\\
\langle \phi_1, \phi_3\rangle &= -a\frac{1}{2}\frac{m(2k)}{m(k)-m(2k)} + O(\xi^3 + a^2),\\
\langle \phi_2, \phi_2\rangle &= 0 + O(\xi^3 + a^2),\\
\langle \phi_3, \phi_3\rangle &= 1 + O(\xi^3 + a^2)
\end{aligned}
$$

as $\xi, a \to 0$. To summarize, (2.4.1) becomes

$$
\begin{aligned}
\mathbf{B}_{\xi,a} = {} & am(k)\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix} \qquad (2.4.3)\\
& + i\xi\begin{pmatrix} -km'(k) & 0 & 0\\ 0 & -km'(k) & 0\\ 0 & 0 & m(k)-1\end{pmatrix}\\
& - i\xi a\begin{pmatrix} 0 & 0 & 2 + \dfrac{m(2k)(m(k)-1)}{m(k)-m(2k)}\\ 0 & 0 & 0\\ m(k) + km'(k) + \dfrac{1}{2}\dfrac{m(2k)(m(k)-1)}{m(k)-m(2k)} & 0 & 0\end{pmatrix}\\
& + \xi^2(km'(k) + \tfrac{1}{2}k^2m''(k))\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + O(\xi^3 + a^2)
\end{aligned}
$$

and

$$\mathbf{I}_a = \mathbf{I} - a\frac{m(2k)}{2(m(k)-m(2k))}\begin{pmatrix}0 & 0 & 2\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix} + O(a^2) \tag{2.4.4}$$

as $\xi, a \to 0$. Here $\mathbf{I}$ denotes the 3×3 identity matrix.

2.5 The modulational instability index

Now, we are ready to prove the following theorem.

Theorem 2.5.1 (Modulational instability index for (2.1.1)). *Under Assumption 2.1.1, a sufficiently small, $2\pi/k$-periodic traveling wave of* (2.1.1) *is spectrally unstable to long wavelength perturbations if*

$$\Delta(k) := \frac{i_1(k)i_2(k)}{i_3(k)} i_4(k) < 0, \tag{2.5.1}$$

where

$$\begin{aligned} i_1(k) =&(km(k))'', \\ i_2(k) =&((km(k))') - 1, \\ i_3(k) =&m(k) - m(2k), \end{aligned} \tag{2.5.2a}$$

and

$$i_4(k) =2i_3(k) + m(2k)i_2(k). \tag{2.5.2b}$$

Otherwise, it is stable to square integrable perturbations in the vicinity of the origin in the spectral plane.

Proof. We turn the attention to the roots of the characteristic polynomial

$$\det(\mathbf{B}_{\xi,a} - \lambda\mathbf{I}_a) = D_3(\xi,a)\lambda^3 + iD_2(\xi,a)\lambda^2 + D_1(\xi,a)\lambda + iD_0(\xi,a)$$

for ξ and $|a|$ sufficiently small, where $\mathbf{B}_{\xi,a}$ and $\mathbf{I}_a$ are in (2.4.3) and (2.4.4). Details are found in [HJ15a, Section 3.3]. Hence we merely hit the main points.

Observe that $D_j = \xi^{3-j}d_j$, $j = 0,1,2,3$, for some real d_j's. We may therefore write that

$$\det(\mathbf{B}_{\xi,a} - (-i\xi)\lambda\mathbf{I}_a) = i\xi^3(d_3(\xi,a)\lambda^3 - d_2(\xi,a)\lambda^2 - d_1(\xi,a)\lambda + d_0(\xi,a)).$$

The underlying, periodic traveling wave of (2.1.1) is then modulationally unstable if $\det(\mathbf{B}_{\xi,a} - (-i\xi)\lambda\mathbf{I}_a)$ admits a pair of complex roots, or equivalently, the discriminant of the cubic polynomial

$$\mathrm{disc}(k;\xi,a) := 18d_3d_2d_1d_0 + d_2^2d_1^2 + 4d_2^3d_0 + 4d_3d_1^3 - 27d_3^2d_0^2 < 0$$

for ξ and $|a|$ sufficiently small, while it is modulationally stable if $\mathrm{disc}(\xi,a) > 0$. Observe that $\mathrm{disc}(\xi,a)$ is even in ξ and a, whereby we write that

$$\mathrm{disc}(k;\xi,a) := \mathrm{disc}(k;\xi,0) + \Delta(k)a^2 + O(a^2(a^2+\xi^2))$$

as $\xi, a \to 0$. It is readily seen from (2.4.3) and (2.4.4) that $\mathrm{disc}(k;\xi,0) > 0$ for all $k > 0$. Specifically, a Mathematica calculation reveals that

$$\mathrm{disc}(k;\xi,0) = \frac{1}{16}\xi^2(ki_1(k)(ki_1(k)\xi - 4i_2(k))(ki_1(k)\xi + 4i_2(k)))^2.$$

Therefore, the sign of $\Delta(k)$ determines modulational stability and instability. As a matter of fact, if $\Delta(k) < 0$ then $\mathrm{disc}(k;\xi,a) < 0$ for ξ sufficiently small, depending on a sufficiently small but fixed, implying modulational instability, whereas if $\Delta(k) > 0$ then $\mathrm{disc}(k;\xi,a) > 0$ for all k and $\xi, |a|$ sufficiently small, implying modulational stability. Recalling (2.4.3) and (2.4.4), a Mathematica calculation then reveals that the sign of $\Delta(k)$ agrees with that of (2.5.1). This completes the proof of Theorem 2.5.1. □

2.6 Results

We illustrate the results in Theorem 2.5.1 for the BBM equation. Note that

$$m(k) = \frac{1}{1+k^2}$$

satisfies Assumption 2.1.1 and it reduces (2.1.1) to (2.0.1). For an arbitrary $k > 0$, note from Lemma 2.2.2 that

$$\begin{cases} u(x,t;k,a) = a\cos(k(x-ct)) + a^2\dfrac{1+k^2}{6k^2}(\cos(2k(x-ct)) - 3) + O(a^3), \\ c(k,a) = \dfrac{1}{1+k^2} - a^2\dfrac{5}{6k^2} + O(a^4), \end{cases} \tag{2.6.1}$$

for $|a| \ll 1$, make a sufficiently small, $2\pi/k$-periodic wave of (2.0.1) traveling at the speed $c(k,a)$. A straightforward calculation reveals that

$$i_1(k) = \frac{2k(k^2-3)}{(1+k^2)^3} > 0$$

if and only if $k > \sqrt{3}$,

$$i_2(k) = -\frac{k^2(3+k^2)}{(1+k^2)^2} < 0 \quad \text{and} \quad i_3(k) = \frac{3k^2}{1+5k^2+4k^4} > 0$$

for all $k > 0$, where i_1, i_2, i_3 are in (2.5.2a). Moreover,

$$i_4(k) = \frac{k^2(3+5k^2)}{(1+k^2)^2(1+4k^2)} > 0$$

for all $k > 0$, where i_4 is in (2.5.2b). Collectively, $\Delta(k) < 0$ if and only if $k > \sqrt{3}$, where Δ is in (2.5.1). It then follows from Theorem 2.5.1 that (2.6.1) is modulationally unstable if $k > \sqrt{3}$ and it is stable in the vicinity of the origin in the spectral plane, otherwise.

Away from the origin in the spectral plane, since the $L^2_{2\pi}$-spectrum of $\mathcal{L}_{\xi,a}$ associated with (2.0.1) is symmetric about the imaginary axis, its eigenvalues may leave the imaginary axis, leading to instability, as ξ and a vary, only through collisions with other purely imaginary eigenvalues. Recall (2.3.3) and (2.3.4). Since $m(k)$ decreases in k, we deduce that

$$\cdots < \omega_{-3,\xi} < \omega_{-2,\xi} < 0 < \omega_{1,\xi} < \omega_{2,\xi} < \omega_{3,\xi} < \dots$$

for each $\xi \in [0, 1/2]$. Moreover it is readily seen that $\omega_{0,\xi} < 0$ and $\omega_{-1,\xi} > 0$ for all $\xi \in [0, 1/2]$. A straightforward calculation reveals that if $\omega_{-1,\xi}$ and $\omega_{n,\xi}$ collide for some $n \geqslant 1$ an integer and $\xi \in [0, 1/2]$ then $n = 1$, whence

$$k = \sqrt{\frac{3}{1-\xi^2}} \geqslant \sqrt{3}.$$

But the underlying wave is modulationally unstable in the range. Similarly if $\omega_{0,\xi}$ and $\omega_{n,\xi}$ collide for some $n \leqslant -2$ an integer and $\xi \in [0, 1/2]$ then

$$k = \sqrt{\frac{1-n^2+3n\xi-3\xi^2}{\xi^4-2n\xi^3+(n^2+1)\xi^2-n\xi}} \geqslant 2\sqrt{3/5}.$$

For $k < 2\sqrt{3/5}$, therefore, eigenvalue collide only at the origin, which incidentally does not lead to instability since $\Delta(k) > 0$. In other words, the underlying wave is spectrally stable. Below we summarize the conclusion.

Corollary 2.6.1 (Modulational instability and spectral stability for (2.0.1))**.** *A sufficiently small, $2\pi/k$-periodic traveling wave of* (2.0.1) *is spectrally unstable to long wavelength perturbations if $k > \sqrt{3}$, and it is spectrally stable to square integrable perturbations if* $0 < k < 2\sqrt{3/5}$.

For $2\sqrt{3/5} < k \leqslant \sqrt{3}$, a Krein signature calculation may be made to determine the stability and instability. But we do not pursue here.

Corollary 2.6.1 agrees with that in [Joh10], where the author proved that periodic traveling waves of (2.0.1) of sufficiently large period, or conversely sufficiently small wave number, (but not necessarily small amplitude) are modulationally stable.

We remark that in [Har08], the author employed a similar method to show that a periodic traveling wave of (2.0.1) of sufficiently small amplitude near $u = c-1$, the non-zero constant solution of (2.2.1) when $b = 0$, is modulationally stable, for which the wave number $k < 1$ incidentally.

In [JLL17], authors using the linear modulational instability result in Corollary 2.6.1, prove a version of nonlinear modulational instability.

Chapter 3

The Camassa-Holm equation

We determine the stability and instability of sufficiently small periodic traveling waves to long wavelength perturbations, for a nonlinear dispersive equation which extends a Camassa-Holm equation to include the full-dispersion of water waves and the Whitham equation to include nonlinearities of medium amplitude waves. The result qualitatively agrees with the Benjamin-Feir instability of a Stokes' wave. We discuss the modulational stability and instability in the Camassa-Holm equation.

3.1 The equation

In the 1960s, Whitham (see [Whi74], for instance) proposed

$$\eta_t + c_{\rm ww}(\sqrt{\beta}|\partial_x|)\eta_x + (3\sqrt{(1+\alpha\eta)} - 3)\eta_x = 0, \tag{3.1.1}$$

to argue for wave breaking in shallow water. That is, the solution remains bounded but its slope becomes unbounded in finite time. Here $t \in \mathbb{R}$ is proportional to elapsed time, and $x \in \mathbb{R}$ is the spatial variable in the primary direction of wave propagation; $\eta = \eta(x,t)$ is the fluid surface displacement from the undisturbed depth,

$$\alpha = \frac{\text{a typical amplitude}}{\text{the undisturbed fluid depth}} \quad \text{and} \quad \beta = \frac{(\text{the undisturbed fluid depth})^2}{(\text{a typical wavelength})^2}.$$

Moreover, $c_{\rm ww}(|\partial_x|)$ is a Fourier multiplier operator, defined as

$$\widehat{c_{\rm ww}(|\partial_x|)f}(k) = \sqrt{\frac{\tanh k}{k}}\widehat{f}(k). \tag{3.1.2}$$

Note that $c_{\rm ww}(k)$ means the phase speed in the linear theory of water waves. For small amplitude waves satisfying $\alpha \ll 1$, we may expand the nonlinearity of (3.1.1) up to terms of order α to arrive at

$$\eta_t + c_{\rm ww}(\sqrt{\beta}|\partial_x|)\eta_x + \frac{3}{2}\alpha\eta\eta_x = 0. \tag{3.1.3}$$

For relatively shallow water or, equivalently, relatively long waves satisfying $\beta \ll 1$, we may expand the right side of (3.1.2) up to terms of order β to find

$$c_{\mathrm{ww}}(\sqrt{\beta}k) = 1 - \frac{1}{6}\beta k^2 + O(\beta^2).$$

Therefore, for small amplitude and long waves satisfying $\alpha = O(\beta)$ and $\beta \ll 1$, we arrive at the famous Korteweg-de Vries equation

$$\eta_t + \eta_x + \frac{1}{6}\beta\eta_{xxx} + \frac{3}{2}\alpha\eta\eta_x = 0. \tag{3.1.4}$$

As a matter of fact, for well-prepared initial data, the solutions of the Whitham equation and the Korteweg-de Vries equation differ from those of the water wave problem merely by higher order terms over the relevant time scale; see [Lan13], for instance, for details. But (3.1.3) and (3.1.2) offer improvements over (3.1.4) for short waves. Whitham conjectured wave breaking for (3.1.3) and (3.1.2). It was recently proved in [Hur17]. In stark contrast, no solutions of (3.1.4) break.

Moreover, Johnson and Hur [HJ15a] showed that a sufficiently small and $2\pi/k$-periodic traveling wave of the Whitham equation be spectrally unstable to long wavelength perturbations, provided that $k > 1.145\dots$. In other words, (3.1.3) (or (3.1.1)) and (3.1.2) predict the Benjamin-Feir instability of a Stokes' wave; see [BF67, BH67, Whi67] and [BM95], for instance. In contrast, periodic traveling waves of the Korteweg-de Vries equation are all modulationally stable. By the way, under the assumption that $\eta_t + \eta_x$ is small, we may modify (3.1.4) to arrive at the Benjamin-Bona-Mahony equation

$$\eta_t + \eta_x - \frac{1}{6}\beta\eta_{xxt} + \frac{3}{2}\alpha\eta\eta_x = 0. \tag{3.1.5}$$

It agrees with (3.1.4) for long waves but is preferable for short waves. Note that the phase speed for (3.1.5) is bounded for all frequencies. We show in Chapter 2 that a sufficiently small and $2\pi/k$ periodic traveling wave of (3.1.5) be modulationally unstable if $k > \sqrt{3}$. Hence the Benjamin-Bona-Mahony equation seems to predict the Benjamin-Feir instability of a Stokes wave. But the instability mechanism is different from that in the Whitham equation or the water wave problem; see [HP16b] for details.

As a matter of fact, for medium amplitude and long waves satisfying $\alpha = O(\sqrt{\beta})$ and $\beta \ll 1$, the Camassa-Holm equations for the fluid surface displacement

$$\eta_t + \eta_x + \beta(a\eta_{xxx} + b\eta_{xxt}) + \frac{3}{2}\alpha\eta\eta_x - \frac{3}{8}\alpha^2\eta^2\eta_x + \frac{3}{16}\alpha^3\eta^3\eta_x = -\alpha\beta(c\eta\eta_{xxx} + d\eta_x\eta_{xx}) \tag{3.1.6}$$

and for the average horizontal velocity

$$u_t + u_x + \beta(au_{xxx} + bu_{xxt}) + \frac{3}{2}\alpha uu_x = -\alpha\beta(cuu_{xxx} + du_xu_{xx}), \tag{3.1.7}$$

where

$$0 \leqslant a \leqslant \frac{1}{6}, \qquad b = a - \frac{1}{6}, \qquad c = \frac{3}{2}a + \frac{1}{6}, \quad \text{and} \quad d = \frac{9}{2}a + \frac{5}{24},$$

extend the Korteweg-de Vries equation to include higher order nonlinearities, and they approximate the physical problem; see [Lan13], for instance, for details. In the case of $a = 1/12$, (3.1.6) reads

$$\eta_t + \eta_x + \frac{1}{12}\beta(\eta_{xxx} - \eta_{xxt}) + \frac{3}{2}\alpha\eta\eta_x - \frac{3}{8}\alpha^2\eta^2\eta_x + \frac{3}{16}\alpha^3\eta^3\eta_x = -\frac{7}{24}\alpha\beta(\eta\eta_{xxx} + 2\eta_x\eta_{xx}),$$

which is particularly interesting because it predicts wave breaking; see [Lan13] and references therein. Note that

$$\frac{3\alpha\eta}{1 + \sqrt{1 + \alpha\eta}} = \frac{3}{2}\alpha\eta - \frac{3}{8}\alpha^2\eta^2 + \frac{3}{16}\alpha^3\eta^3 + O(\alpha^4).$$

Lannes [Lan13] combined the dispersion relation of water waves and a Camassa-Holm equation, to propose the *full-dispersion Camassa-Holm* (FDCH) equation for the fluid surface displacement

$$\eta_t + c_{\text{ww}}(\sqrt{\beta}|\partial_x|)\eta_x + \frac{3\alpha\eta}{1 + \sqrt{1 + \alpha\eta}}\eta_x = -\alpha\beta\Big(\frac{5}{12}\eta\eta_{xxx} + \frac{23}{24}\eta_x\eta_{xx}\Big), \tag{3.1.8}$$

where $c_{\text{ww}}(|\partial_x|)$ is in (3.1.2). For relatively long waves satisfying $\beta \ll 1$, (3.1.8) and (3.1.2) agree with (3.1.6), where $a = 1/6$, up to terms of order β. But, including all the dispersion of water waves, (3.1.8) and (3.1.2) may offer an improvement over (3.1.6) for short waves. For small amplitude waves satisfying $\alpha \ll 1$, (3.1.8) agrees with (3.1.3) up to terms of order α. But, including higher order nonlinearities, (3.1.8) may offer an improvement over (3.1.3) for medium amplitude waves. For the average horizontal velocity, we may combine (3.1.2) and (3.1.7) to introduce

$$u_t + c_{\text{ww}}(\sqrt{\beta}|\partial_x|)u_x + \frac{3}{2}\alpha uu_x = -\alpha\beta\Big(\frac{5}{12}uu_{xxx} + \frac{23}{24}u_xu_{xx}\Big). \tag{3.1.9}$$

We follow along the same line as the arguments in [HJ15a,HJ15b,HP16b] (see also [BHJ16]) and investigate the modulational stability and instability in the FDCH equation. A main difference lies in that the nonlinearities of (3.1.8) involve higher order derivatives and, hence, a periodic traveling wave is not a priori smooth. We examine the mapping properties of various operators to construct a smooth solution.

3.2 Periodic traveling waves

We determine periodic traveling waves of the FDCH equation, after normalization of parameters,

$$\eta_t + c_{\rm ww}(|\partial_x|)\eta_x + \frac{3\eta}{1+\sqrt{1+\eta}}\eta_x = -\Big(\frac{5}{12}\eta\eta_{xxx} + \frac{23}{24}\eta_x\eta_{xx}\Big), \tag{3.2.1}$$

where $c_{\rm ww}(|\partial_x|)$ is in (3.1.2), and we calculate their small amplitude expansion.

By a traveling wave of (3.2.1) and (3.1.2), we mean a solution of the form $\eta(x,t)=\eta(x-ct)$ for some $c>0$, the wave speed, where η satisfies by quadrature

$$(c_{\rm ww}(|\partial_x|) - c - 3)\eta + 2(1+\eta)^{3/2} - 2 + \frac{5}{12}\eta\eta_{xx} + \frac{13}{48}\eta_x^2 = (1-c)^2 b$$

for some $b\in\mathbb{R}$. We seek a periodic traveling wave of (3.2.1) and (3.1.2). That is, η is a 2π periodic function of $z:=kx$ for some $k>0$, the wave number, and it satisfies

$$(c_{\rm ww}(k|\partial_z|) - c - 3)\eta + 2(1+\eta)^{3/2} - 2 + \frac{5}{12}k^2\eta\eta_{zz} + \frac{13}{48}k^2\eta_z^2 = (1-c)^2 b. \tag{3.2.2}$$

Note that

$$c_{\rm ww}(k|\partial_z|) : H^s(\mathbb{T}) \to H^{s+1/2}(\mathbb{T}) \tag{3.2.3}$$

for any $k>0$ and $s\in\mathbb{R}$. Note that

$$c_{\rm ww}(k|\partial_z|)e^{inz} = c_{\rm ww}(nk)e^{inz} \quad \text{for } n\in\mathbb{Z}. \tag{3.2.4}$$

Note that (3.2.2) remains invariant under

$$z\mapsto z+z_0 \quad\text{and}\quad z\mapsto -z \tag{3.2.5}$$

for any $z_0\in\mathbb{R}$. Hence we may assume that η is even. But (3.2.2) does not possess scaling invariance. Hence we may not a priori assume that $k=1$. Rather, the (in)stability result herein depends on the carrier wave number. Moreover, (3.2.2) does not possess Galilean invariance. Hence we may not a priori assume that $b=0$. Rather, we exploit the variation of (3.2.2) in the b variable in the instability proof.

For any integer $s\geqslant 0$, let

$$F : H^{s+2}(\mathbb{T}) \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \to H^s(\mathbb{T})$$

denote

$$F(\eta,c;b,k)=(c_{\mathrm{ww}}(k|\partial_z|)-c-3)\eta+2(1+\eta)^{3/2}-2 \\ +\frac{5}{12}k^2\eta\eta_{zz}+\frac{13}{48}k^2\eta_z^2-(1-c)^2b. \quad (3.2.6)$$

It is well defined by (3.2.3) and a Sobolev inequality. We seek a solution $\eta\in H^{s+2}(\mathbb{T})$, $c>0$, and $b\in\mathbb{R}$ of

$$F(\eta,c;b,k)=0. \quad (3.2.7)$$

Since s is arbitrary, $\eta\in H^\infty(\mathbb{T})$. Note that F is invariant under (3.2.5). Hence we may assume that η is even.

For any $c>0$, $b\in\mathbb{R}$, $k>0$, note that

$$F_\eta(\eta,c;b,k)\zeta=\Big(c_{\mathrm{ww}}(k|\partial_z|)-c-3+3(1+\eta)^{1/2} \\ +k^2\Big(\frac{5}{12}(\eta_{zz}+\eta\partial_z^2)+\frac{13}{24}\eta_z\partial_z\Big)\Big)\zeta:H^{s+2}(\mathbb{T})\to H^s(\mathbb{T})$$

is continuous by (3.2.3) and a Sobolev inequality. Here a subscript means Fréchet differentiation. Moreover, $\eta\in H^{s+2}(\mathbb{T})$, $k>0$, $b\in\mathbb{R}$, note that $F_c(\eta;k,c,b)=-\eta+2(1-c)b:\mathbb{R}\to H^s(\mathbb{T})$ is continuous. Since $F_b(\eta;k,c,b)=-(1-c)^2$ and

$$F_k(\eta;k,c,b):=c'_{\mathrm{ww}}(k|\partial_z|)\eta+\frac{5}{6}k\eta\eta_{zz}+\frac{13}{24}k\eta_z^2$$

are continuous likewise, F depends continuously differentiably on its arguments. Furthermore, since the Fréchet derivatives of F with respect to η, and c, b of all orders $\geqslant 3$ are zero everywhere by brutal force, and since c_{ww} is a real analytic function, F is a real analytic operator.

For any $k>0$, for any $c>0$, $b\in\mathbb{R}$ and $|b|$ sufficiently small, note that

$$\eta_0(c;b,k)=b(1-c)+O(b^2) \quad (3.2.8)$$

makes a constant solution of (3.2.6)-(3.2.7) and, hence, (3.2.2). It follows from the implicit function theorem that if non-constant solutions of (3.2.6)-(3.2.7) and, hence, (3.2.2) bifurcate from $\eta=\eta_0$ for some $c=c_0$ then, necessarily,

$$L_0:=F_\eta(\eta_0,c_0;b,k):H^{s+2}(\mathbb{T})\to H^s(\mathbb{T}),$$

where

$$L_0 = c_{\text{ww}}(k|\partial_z|) - c_0 - 3 + 3(1+\eta_0)^{1/2} + \frac{5}{12}k^2\eta_0\partial_z^2, \tag{3.2.9}$$

is not an isomorphism. Here η_0 depends on c_0. But we suppress it for simplicity of notation. A straightforward calculation reveals that $L_0 e^{inz} = 0$, $n \in \mathbb{Z}$, if and only if

$$c_0 = c_{\text{ww}}(nk) - 3 + 3(1+\eta_0)^{1/2} - \frac{5}{12}k^2 n^2 \eta_0. \tag{3.2.10}$$

For $b = 0$ and, hence, $\eta_0 = 0$ by (3.2.8), it simplifies to $c_0 = c_{\text{ww}}(nk)$. Without loss of generality, we restrict the attention to $n = 1$. For $|b|$ sufficiently small, (3.2.10) and (3.2.8) become

$$c_0(b,k) = c_{\text{ww}}(k) + b\Big(\frac{3}{2} - \frac{5}{12}k^2\Big)(1 - c_{\text{ww}}(k)) + O(b^2) \tag{3.2.11}$$

and

$$\eta_0(b,k) = b(1 - c_{\text{ww}}(k)) + O(b^2). \tag{3.2.12}$$

Since $c_{\text{ww}}(k) > c_{\text{ww}}(nk)$ for $n = 2, 3, \dots$ everywhere in $\mathbb{R}$, it is straightforward to verify that for any $k > 0$, $b \in \mathbb{R}$ and $|b|$ sufficiently small, the kernel of $L_0 : H^{s+2}(\mathbb{T}) \to H^s(\mathbb{T})$ is two dimensional and spanned by $e^{\pm iz}$. Moreover, the co-kernel of L_0 is two dimensional. Therefore, L_0 is a Fredholm operator of index zero.

For any $k > 0$, $b \in \mathbb{R}$ and $|b|$ sufficiently small, we employ a Lyapunov-Schmidt procedure to construct non-constant solutions of (3.2.6)-(3.2.7) and, hence, (3.2.2) bifurcating from $\eta = \eta_0$ and $c = c_0$, where η_0 and c_0 are in (3.2.12) and (3.2.11). Throughout the proof, k, and b are fixed and suppressed for simplicity of notation.

Recall that $F(\eta_0, c_0) = 0$, where F is in (3.2.6), and $L_0 e^{\pm iz} = 0$, where L_0 is in (3.2.9). We write that

$$\eta(z) = \eta_0 + \frac{1}{2}(ae^{iz} + a^* e^{-iz}) + \eta_r(z) \quad \text{and} \quad c = c_0 + c_r, \tag{3.2.13}$$

and we require that $a \in \mathbb{C}$, $\eta_r \in H^{k+2}(\mathbb{T})$ be even and

$$\langle \eta_r, e^{\pm iz} \rangle_{L^2(\mathbb{T})} = 0, \tag{3.2.14}$$

and $c_r \in \mathbb{R}$. Substituting (3.2.13) into (3.2.6)-(3.2.7), we use $F(\eta_0, c_0) = 0$, $L_0 e^{\pm iz} = 0$, and we make an explicit calculation to arrive at

$$\begin{aligned}
L_0\eta_r =&(3(1+\eta_0)^{1/2}+c_r)\Big(\frac{1}{2}(ae^{iz}+a^*e^{-iz})+\eta_r\Big)\\
&-2\Big(1+\eta_0+\frac{1}{2}(ae^{iz}+a^*e^{-iz})+\eta_r\Big)^{3/2}\\
&-\frac{13}{48}k^2\Big(\frac{i}{2}(ae^{iz}-a^*e^{-iz})+\eta_r'\Big)^2\\
&-\frac{5}{12}k^2\Big(\frac{1}{2}(ae^{iz}+a^*e^{-iz})+\eta_r\Big)\Big(-\frac{1}{2}(ae^{iz}+a^*e^{-iz})+\eta_r''\Big)\\
=:&g(\eta_r;a,a^*,c_r).
\end{aligned} \tag{3.2.15}$$

Here and elsewhere, the prime means ordinary differentiation. Note that

$$g: H^{s+2}(\mathbb{T})\times\mathbb{C}\times\mathbb{C}\times\mathbb{R}\to H^s(\mathbb{T}).$$

Recall that F is a real analytic operator. Hence g depends analytically on its arguments. Clearly, $g(0;0,0,c_r)=0$ for all $c_r\in\mathbb{R}$.

Let $\Pi: L^2(\mathbb{T})\to\ker L_0$ denote the spectral projection, defined as

$$\Pi f(z)=\widehat{f}(1)e^{iz}+\widehat{f}(-1)e^{-iz}.$$

Since $\Pi\eta_r=0$ by (3.2.14), we may rewrite (3.2.15) as

$$L_0\eta_r=(1-\Pi)g(\eta_r;a,a^*,c_r)\quad\text{and}\quad 0=\Pi g(\eta_r;a,a^*,c_r). \tag{3.2.16}$$

Moreover, note that L_0 is invertible on $(1-\Pi)H^k(\mathbb{T})$. Specifically,

$$L_0{}^{-1}f(z)=\sum_{n\neq\pm1}\frac{\widehat{f}(n)}{c_{\text{ww}}(kn)-c_{\text{ww}}(k)+\frac{5}{12}k^2\eta_0(1-n^2)}e^{inz}.$$

Hence we may rewrite (3.2.16) as

$$\eta_r=L_0^{-1}(I-\Pi)g(\eta_r;a,a^*,c_r)\quad\text{and}\quad 0=\Pi g(\eta_r;a,a^*,c_r). \tag{3.2.17}$$

Note that $L_0^{-1}:(1-\Pi)H^k(\mathbb{T})\to H^k(\mathbb{T})$ is bounded. We claim that

$$L_0^{-1}:(1-\Pi)H^k(\mathbb{T})\to H^{k+2}(\mathbb{T})$$

is bounded. As a matter of fact,

$$\left|\frac{n^2\widehat{f}(n)}{c_{\rm ww}(kn)-c_{\rm ww}(k)+\frac{5}{12}k^2\eta_0(1-n^2)}\right|\leqslant C|\widehat{f}(n)|$$

for some constant $C>0$ for $n\in\mathbb{Z}$ and $|n|$ sufficiently large. Therefore, for any $a,a^*\in\mathbb{C}$ and $c_r\in\mathbb{R}$,

$$L_0^{-1}(1-\Pi)g : H^{k+2}(\mathbb{T})\to H^{k+2}(\mathbb{T})$$

is bounded. Note that it depends analytically on its argument. Since $g(0;0,0,c_r)=0$ for any $c_r\in\mathbb{R}$, it follows from the implicit function theorem that a unique solution

$$\eta_2=\eta_r(a,a^*,c_r)$$

exists to the former equation of (3.2.17) near $\eta_r=0$ for $a\in\mathbb{C}$ and $|a|$ sufficiently small for any $c_r\in\mathbb{R}$. Note that η_2 depends analytically on its arguments and it satisfies (3.2.14) for $|a|$ sufficiently small for any $c_r\in\mathbb{R}$. The uniqueness implies

$$\eta_2(0,0,c_r)=0\qquad\text{for any } c_r\in\mathbb{R}. \tag{3.2.18}$$

Moreover, since (3.2.6)-(3.2.7) and, hence, (3.2.17) are invariant under (3.2.5) for any $z_0\in\mathbb{R}$, it follows that

$$\eta_2(a,a^*,c_r)(z+z_0)=\eta_2(ae^{iz_0},a^*e^{-iz_0},c_r)\quad\text{and}\quad\eta_2(a,a^*,c_r)(-z)=\eta_2(a,a^*,c_r)(z) \tag{3.2.19}$$

for any $z_0\in\mathbb{R}$ for any $a\in\mathbb{C}$ and $|a|$ sufficiently small, and $c_r\in\mathbb{R}$.

To proceed, we rewrite the latter equation in (3.2.17) as

$$\Pi g(\eta_2(a,a^*,c_r);a,a^*,c_r)=0$$

for $a\in\mathbb{C}$ and $|a|$ sufficiently small for $c_r\in\mathbb{R}$. This is solvable, provided that

$$\pi_\pm(a,a^*,c_r):=\left\langle g(\eta_2(a,a^*,c_r);a,a^*,c_r),ae^{iz}\pm a^*e^{-iz}\right\rangle_{L^2(\mathbb{T})}=0. \tag{3.2.20}$$

We use (3.2.19), where $z_0=-2\arg(a)$, and (3.2.20) to show that

$$\pi_-(a^*,a,c_r)=\pi_-(a,a^*,c_r)=-\pi_-(a^*,a,c_r).$$

Hence $\pi_-(a, a^*, c_r) = 0$ holds for any $a \in \mathbb{C}$ and $|a|$ sufficiently small for any $c_r \in \mathbb{R}$. Moreover, we use (3.2.19), where $z_0 = -\arg(a)$, and (3.2.20) to show that

$$\pi_+(a, a^*, c_r) = \pi_+(|a|, |a|, c_r).$$

Hence it suffices to solve $\pi_+(a, a, c_r) = 0$ for any $a, c_r \in \mathbb{R}$ and $|a|$ sufficiently small.

Substituting (3.2.15) into (3.2.20), where $\eta_r = \eta_2(a, a, c_r)$, we make an explicit calculation to arrive at

$$\pi_+(a, a, c_r) = a^2(\pi c_r + \pi_r(a, c_r)),$$

where

$$\begin{aligned}\pi_r(a, c_r) =& -2a^{-1}\langle(1 + \eta_0 + a\cos z + \eta_2(a, a, c_r)(z))^{3/2}, \cos z\rangle \\ &- \frac{5}{12}k^2(\langle\eta_2''(a, a, c_r)(z) - \eta_2(a, a, c_r)(z), \cos^2 z\rangle - a^{-1}\langle\eta_2\eta_2''(a, a, c_r)(z), \cos z\rangle) \\ &- \frac{13}{48}k^2a^{-1}(\langle\eta_2'(a, a, c_r)(z)^2, \cos z\rangle - \langle\eta_2'(a, a, c_r)(z), \sin 2z\rangle),\end{aligned}$$

and $\langle\cdot\,,\cdot\rangle$ means the $L^2(\mathbb{T})$ inner product. We merely pause to remark that π_r is well defined. As a matter of fact, $a^{-1}\eta_2$ is not singular for $a \in \mathbb{R}$ and $|a|$ sufficiently small by (3.2.18). Clearly, π_r and, hence, $\pi_\pm$ depend analytically on its arguments. Since $\pi_r(0, 0) = \partial\pi_r/\partial c_r(0, 0) = 0$ by (3.2.18), it follows from the implicit function theorem that a unique solution

$$c_r = c_1(a)$$

exists to $\pi_+(a, a, c_r) = 0$ and, hence, the latter equation of (3.2.17) near $c_r = 0$ for $a \in \mathbb{R}$ and $|a|$ sufficiently small. Clearly, c_1 depends analytically on a.

To recapitulate,

$$\eta_r = \eta_2(a, a, c_1(a)) \quad \text{and} \quad c_r = c_1(a)$$

uniquely solve (3.2.17) for $a \in \mathbb{R}$ and $|a|$ sufficiently small, and by virtue of (3.2.13),

$$\eta(a)(z) = \eta_0 + a\cos z + \eta_2(a, a, c_1(a))(z) \quad \text{and} \quad c(a) = c_0 + c_1(a) \tag{3.2.21}$$

uniquely solve (3.2.6)-(3.2.7) and, hence, (3.2.2) for $a \in \mathbb{R}$ and $|a|$ sufficiently small. Note that η is 2π periodic and even in z. Moreover, $\eta \in H^\infty(\mathbb{T})$.

For $a, b \in \mathbb{R}$ and $|a|, |b|$ sufficiently small, we write that

$$\eta(a; b, k)(z) := \eta_0(b, k) + a\cos z + a^2\eta_2(z) + a^3\eta_3(z) + \cdots \tag{3.2.22}$$

and

$$c(a;b,k) :=c_0(b,k) + ac_1 + a^2c_2 + \cdots , \tag{3.2.23}$$

where $\eta_2, \eta_3, \ldots$ are 2π periodic, even, and smooth functions of z, and $c_1, c_2, \cdots \in \mathbb{R}$.

We claim that $c_1 = 0$. As a matter of fact, note that (3.2.2) and, hence, (3.2.6)-(3.2.7) remain invariant under $z \mapsto z + \pi$ by (3.2.5). Since $\partial\eta/\partial a(0)(z) = \cos z$, however, $\eta(z) \neq \eta(z+\pi)$ must hold. Thus $\partial c/\partial a(0) = 0$. This proves the claim. If $\langle \eta_{j-1}, \eta_j \rangle_{L^2(\mathbb{T})} = 0$ for any integer $j \geqslant 1$, in addition, then $c_{2j-1} = 0$ for any integer $j \geqslant 1$. Hence c is even in a.

Substituting (3.2.22) and (3.2.23) into (3.2.2), we may calculate the small amplitude expansion. The proof is very similar to that in Chapter 2. Hence we omit the details.

Below we summarize the conclusion.

Lemma 3.2.1 (Existence of sufficiently small and periodic traveling waves)**.** *For any $k > 0$, $b \in \mathbb{R}$ and $|b|$ sufficiently small, a one parameter family of solutions of* (3.2.2) *exists, denoted $\eta(a;b,k)$ and $c(a;b,k)$, for $a \in \mathbb{R}$ and $|a|$ sufficiently small; $\eta \in H^\infty(\mathbb{T})$ and it is even in z; η and c depend analytically on a, and b, k. Moreover,*

$$\eta(a;b,k)(z) =b(1 - c_{\text{ww}}(k)) + a\cos z + a^2(h_0 + h_2\cos 2z) + O(a(a+b)^2), \tag{3.2.24}$$

$$c(a;b,k) =c_{\text{ww}}(k) + b\Big(\frac{3}{2} - \frac{5}{12}k^2\Big)(1 - c_{\text{ww}}(k)) + a^2c_2 + O(a(a+b)^2) \tag{3.2.25}$$

as $a, b \to 0$, where

$$h_0 = \Big(\frac{3}{8} - \frac{7}{96}k^2\Big)\frac{1}{c_{\text{ww}}(k) - 1}, \qquad h_2 = \Big(\frac{3}{8} - \frac{11}{32}k^2\Big)\frac{1}{c_{\text{ww}}(k) - c_{\text{ww}}(2k)}, \tag{3.2.26}$$

and

$$c_2 = \Big(\frac{3}{2} - \frac{5}{12}k^2\Big)h_0 + \Big(\frac{3}{4} - \frac{1}{2}k^2\Big)h_2 - \frac{3}{32}. \tag{3.2.27}$$

3.3 The spectral problem

For $k > 0$, $a, b \in \mathbb{R}$ and $|a|, |b|$ sufficiently small, let $\eta = \eta(a;b,k)$ and $c = c(a;b,k)$, denote a sufficiently small and $2\pi/k$ periodic traveling wave of (3.2.1) and (3.1.2), whose existence follows from the previous section. We address its modulational stability and instability.

Linearizing (3.2.1) about η in the coordinate frame moving at the speed c, we arrive at

$$\zeta_t + k\partial_z\Big(c_{\rm ww}(k|\partial_z|) - c - 3 + 3(1+\eta)^{1/2} + k^2\Big(\frac{5}{12}(\eta\partial_z^2 + \eta_{zz}) + \frac{13}{24}\eta_z\partial_z\Big)\Big)\zeta = 0,$$

where $c_{\rm ww}(k|\partial_z|)$ is in (3.1.2). Seeking a solution of the form $\zeta(z,t) = e^{\lambda kt}\zeta(z)$, $\lambda \in \mathbb{C}$, we arrive at

$$\begin{aligned}\lambda\zeta = \partial_z\Big(&- c_{\rm ww}(k|\partial_z|) + c + 3 - 3(1+\eta)^{1/2} - k^2\Big(\frac{5}{12}(\eta\partial_z^2 + \eta_{zz}) + \frac{13}{24}\eta_z\partial_z\Big)\Big)\zeta \\ =:&\ \mathcal{L}(a;b,k)\zeta.\end{aligned} \tag{3.3.1}$$

Next, we employ Floquet theory to study the $L^2(\mathbb{R})$-spectrum of $\mathcal{L}$. The corresponding Bloch operators are given by

$$\lambda\phi = e^{-i\xi z}\mathcal{L}(a;b,k)e^{i\xi z}\phi =: \mathcal{L}(\xi)(a;b,k)\phi \tag{3.3.2}$$

for some $\xi \in (-1/2, 1/2]$ and $\phi \in L^2(\mathbb{T})$. Note that

$$\mathrm{spec}_{L^2(\mathbb{T})}(\mathcal{L}(\xi)) = \overline{(\mathrm{spec}_{L^2(\mathbb{T})}(\mathcal{L}(-\xi)))}.$$

Hence it suffices to take $\xi \in [0, 1/2]$.

For an arbitrary ξ, one must in general study (3.3.2) by means of numerical computation. But, for $\xi > 0$ small and for λ in the vicinity of the origin in $\mathbb{C}$, we may take a spectral perturbation approach in [HJ15a, HJ15b, HP16b], for instance, to address it analytically.

We assume that $b = 0$. For nonzero b, one may explore in like manner. But the calculation becomes lengthy and tedious. Hence we do not discuss the details. We use the notation

$$\mathcal{L}(\xi, a) = \mathcal{L}(\xi)(a; 0, k). \tag{3.3.3}$$

For $a = 0$ — namely, the rest state — a straightforward calculation reveals that

$$\mathcal{L}(\xi, 0)e^{inz} = i\omega(n+\xi)e^{inz} \quad \text{for } n \in \mathbb{Z} \text{ and } \xi \in [0, 1/2], \tag{3.3.4}$$

where

$$\omega(n+\xi) = (\xi + n)(c_{\rm ww}(k) - c_{\rm ww}(k(n+\xi))). \tag{3.3.5}$$

For $\xi = 0$,

$$\omega(1) = \omega(-1) = \omega(0) = 0,$$

and $\omega(n) \neq 0$ otherwise. Hence, zero is an $L^2(\mathbb{T})$-eigenvalue of $\mathcal{L}(0,0)$ with multiplicity three. Moreover,

$$\cos z, \qquad \sin z, \quad \text{and} \quad 1 \tag{3.3.6}$$

are the associated eigenfunctions, real valued and orthogonal to each other. For $\xi > 0$ sufficiently small,

$$i\omega(\pm 1 + \xi) \quad \text{and} \quad i\omega(\xi)$$

are the $L^2(\mathbb{T})$-eigenvalues of $\mathcal{L}(\xi, 0)$ in the vicinity of the origin in $\mathbb{C}$, and (3.3.6) are the associated eigenfunctions.

For $a \in \mathbb{R}$ and $|a|$ sufficiently small and for $\xi = 0$, zero is an $L^2(\mathbb{T})$-eigenvalue of $\mathcal{L}(0,a)$ with algebraic multiplicity three and geometric multiplicity two, and

$$\begin{aligned}
\phi_1(z) &:= \Big(\eta_a - \frac{c_a}{c_b}\eta_b\Big)(a;0,k)(z) = \cos z + ap_1 + 2ah_2\cos 2z + O(a^2),\\
\phi_2(z) &:= -\frac{1}{a}\eta_z(a;0,k)(z) = \sin z + 2ah_2\sin 2z + O(a^2),\\
\phi_3(z) &:= \frac{1}{1 - c_{\text{ww}}(k)}\eta_b(k,a,0)(z) = 1 + O(a^2)
\end{aligned} \tag{3.3.7}$$

are the associated eigenfunctions, where

$$p_1 = 2h_0 - \frac{24c_2}{18 - 5k^2} = \frac{1}{18 - 5k^2}\Big(\frac{9}{4} - \frac{3}{16}\frac{(3 - 2k^2)(12 - 11k^2)}{c_{\text{ww}}(k) - c_{\text{ww}}(2k)}\Big) \tag{3.3.8}$$

and h_2 is defined in (3.2.26). The proof is nearly identical to that in Section 2.3, for instance. Hence we omit the details. For $a = 0$, note that (3.3.7) becomes (3.3.6).

3.4 The perturbation analysis

Recall that for $\xi > 0$ sufficiently small and for $a = 0$, the $L^2(\mathbb{T})$-spectrum of $\mathcal{L}(\xi, 0)$ contains three purely imaginary eigenvalues $i\omega(\pm 1 + \xi)$ and $i\omega(\xi)$ in the vicinity of the origin in $\mathbb{C}$, and (3.3.6) spans the associated eigenspace, which does not depend on ξ. For $\xi = 0$ for $a \in \mathbb{R}$ and $|a|$ sufficiently small, the spectrum of $\mathcal{L}(0,a)$ contains three eigenvalues at the origin, and (3.3.7) spans the associated eigenspace, which depends analytically on a.

For $\xi > 0$, $a \in \mathbb{R}$ and $\xi, |a|$ sufficiently small, it follows from perturbation theory (see [Kat76], for instance, for details) that the $L^2(\mathbb{T})$-spectrum of $\mathcal{L}(\xi, a)$ contains three eigenvalues in the vicinity of the origin in $\mathbb{C}$, and (3.3.7) spans the associated eigenspace. Let

$$\mathbf{L}(\xi,a)=\left(\frac{\langle\mathcal{L}(\xi,a)\phi_j,\phi_k\rangle}{\langle\phi_j,\phi_j\rangle}\right)_{j,k=1,2,3}\quad\text{and}\quad\mathbf{I}(a)=\left(\frac{\langle\phi_j,\phi_k\rangle}{\langle\phi_j,\phi_j\rangle}\right)_{j,k=1,2,3},\tag{3.4.1}$$

where ϕ_1,ϕ_2,ϕ_3 are in (3.3.7). Throughout the subsection, $\langle\cdot\,,\cdot\rangle$ means the $L^2(\mathbb{T})$ inner product. Note that $\mathbf{L}$ represents the action of $\mathcal{L}$ on the eigenspace, spanned by ϕ_1,ϕ_2,ϕ_3, and $\mathbf{I}$ is the projection of the identity onto the eigenspace. It follows from perturbation theory (see [Kat76], for instance for details) that for $\xi>0$, $a\in\mathbb{R}$ and $\xi,|a|$ sufficiently small, the eigenvalues of $\mathcal{L}(\xi,a)$ agree in location and multiplicity with the roots of $\det(\mathbf{L}-\lambda\mathbf{I})$ up to terms of order a.

For $a\in\mathbb{R}$ and $|a|$ sufficiently small, a Baker-Campbell-Hausdorff expansion reveals that

$$\mathcal{L}(\xi,a)=\mathcal{L}(0,a)+i\xi[\mathcal{L}(0,a),z]-\frac{1}{2}\xi^2[[\mathcal{L}(0,a),z],z]+O(\xi^3)$$

as $\xi\to0$, where $[\cdot\,,\cdot]$ means the commutator. We merely pause to remark that $[\mathcal{L},z]$ and $[[\mathcal{L},z],z]$ are well defined in the periodic setting even though z is not. We use (3.3.1), (3.3.2) and (3.2.24), (3.2.25) to write

$$\begin{aligned}\mathcal{L}(\xi,a)=&\mathcal{M}-a\partial_z\Big(\frac{3}{2}\cos z+k^2\Big(\frac{5}{12}\cos z(\partial_z^2-1)-\frac{13}{24}\sin z\partial_z\Big)\Big)\\&-i\xi a\Big(\frac{3}{2}\cos z+k^2\Big(\frac{5}{12}(2\partial_z\cos z\partial_z+\cos z(\partial_z^2-1))-\frac{13}{24}(\sin z\partial_z+\partial_z\sin z)\Big)\Big)\\&+O(\xi^3+\xi^2a+a^2)\end{aligned}\tag{3.4.2}$$

as $\xi,a\to0$, where

$$\mathcal{M}=\mathcal{L}(0,0)+i\xi[\mathcal{L}(0,0),z]-\frac{1}{2}\xi^2[[\mathcal{L}(0,0),z],z]$$

agrees with $\mathcal{L}(\xi,0)$ up to terms of order ξ^2 as $\xi\to0$. We may then resort to (3.3.4), (3.3.5), and we make an explicit calculation to find that

$$\begin{aligned}\mathcal{L}(\xi,0)e^{inz}=&in(c_{\rm ww}(k)-c_{\rm ww}(nk))e^{inz}\\&+i\xi(c_{\rm ww}(k)-c_{\rm ww}(nk)-kc'_{\rm ww}(nk))e^{inz}\\&-\frac{1}{2}\xi^2(2kc'_{\rm ww}(nk)+k^2c''_{\rm ww}(nk))e^{inz}+O(\xi^3)\end{aligned}$$

as $\xi \to 0$. Therefore, $\mathcal{M}1 = i\xi(c_{\rm ww}(k) - 1)$,

$$\mathcal{M}\begin{Bmatrix}\cos z\\ \sin z\end{Bmatrix} = -\,i\xi kc'_{\rm ww}(k)\begin{Bmatrix}\cos z\\ \sin z\end{Bmatrix} \pm \frac{1}{2}\xi^2(2kc'_{\rm ww}(k) + k^2c''_{\rm ww}(k))\begin{Bmatrix}\sin z\\ \cos z\end{Bmatrix},$$

and

$$\begin{aligned}\mathcal{M}\begin{Bmatrix}\cos 2z\\ \sin 2z\end{Bmatrix} = &\mp 2(c_{\rm ww}(k) - c_{\rm ww}(2k))\begin{Bmatrix}\sin 2z\\ \cos 2z\end{Bmatrix}\\ &+ i\xi(c_{\rm ww}(k) - c_{\rm ww}(2k) - 2kc'_{\rm ww}(2k))\begin{Bmatrix}\cos 2z\\ \sin 2z\end{Bmatrix}\\ &\pm \frac{1}{2}\xi^2(2kc'_{\rm ww}(2k) + k^2c''_{\rm ww}(2k))\begin{Bmatrix}\sin 2z\\ \cos 2z\end{Bmatrix}.\end{aligned}$$

We use (3.4.2), (3.3.7) and the above formula for $\mathcal{M}$, and we make a lengthy but explicit calculation to find that

$$\begin{aligned}\mathcal{L}\phi_1 =& -i\xi kc'_{\rm ww}(k)\cos z\\ &- i\xi a\Big(\frac{3}{4} - \frac{7}{48}k^2 - p_1(c_{\rm ww}(k) - 1)\Big)\\ &+ i\xi a\Big(-\frac{3}{4} + \frac{33}{16}k^2 + 2h_2(c_{\rm ww}(k) - c_{\rm ww}(2k) - 2kc'_{\rm ww}(2k))\Big)\cos 2z\\ &+ \frac{1}{2}\xi^2(2kc'_{\rm ww}(k) + k^2c''_{\rm ww}(k))\sin z + O(\xi^3 + \xi^2 a + a^2),\\ \mathcal{L}\phi_2 =& -i\xi kc'_{\rm ww}(k)\sin z\\ &+ i\xi a\Big(-\frac{3}{4} + \frac{33}{16}k^2 + 2h_2(c_{\rm ww}(k) - c_{\rm ww}(2k) - 2kc'_{\rm ww}(2k))\Big)\sin 2z\\ &- \frac{1}{2}\xi^2(2kc'_{\rm ww}(k) + k^2c''_{\rm ww}(k))\cos z + O(\xi^3 + \xi^2 a + a^2),\\ \mathcal{L}\phi_3 =& a\Big(3 - \frac{5}{6}k^2\Big)\sin z + i\xi(c_{\rm ww}(k) - 1) - i\xi a\Big(\frac{3}{2} - \frac{23}{24}k^2\Big)\cos z + O(\xi^3 + \xi^2 a + a^2)\end{aligned}$$

as $\xi, a \to 0$, where p_1 is in (3.3.8) and h_2 is in (3.2.26).

To proceed, we take the $L^2(\mathbb{T})$-inner product of the above and (3.3.7), and we make a lengthy but explicit calculation to find that

$$\begin{aligned}
\langle \mathcal{L}\phi_1, \phi_1\rangle = \langle \mathcal{L}\phi_2, \phi_2\rangle &= -\frac{1}{2} i\xi k c_{\rm ww}'(k) + O(\xi^3+\xi^2 a + a^2),\\
\langle \mathcal{L}\phi_1, \phi_2\rangle = -\langle \mathcal{L}\phi_2, \phi_1\rangle &= \frac{1}{4}\xi^2(2kc_{\rm ww}'(k) + k^2 c_{\rm ww}''(k)) + O(\xi^3+\xi^2 a + a^2),\\
\langle \mathcal{L}\phi_1, \phi_3\rangle = i\xi a\Big(-\frac{3}{4} + \frac{36}{48}k^2 + p_1(c_{\rm ww}(k)-1)\Big) &+ O(\xi^3+\xi^2 a + a^2),
\end{aligned}$$

and

$$\begin{aligned}
\langle \mathcal{L}\phi_2, \phi_3\rangle &= 0 + O(\xi^3+\xi^2 a + a^2),\\
\langle \mathcal{L}\phi_3, \phi_1\rangle &= i\xi a\Big(-\frac{3}{4} + \frac{23}{48}k^2 + p_1(c_{\rm ww}(k)-1)\Big) + O(\xi^3+\xi^2 a + a^2),\\
\langle \mathcal{L}\phi_3, \phi_2\rangle &= a\Big(\frac{3}{4} - \frac{5}{24}k^2\Big) + O(\xi^3+\xi^2 a + a^2),\\
\langle \mathcal{L}\phi_3, \phi_3\rangle &= i\xi(c_{\rm ww}(k)-1) + O(\xi^3+\xi^2 a + a^2)
\end{aligned}$$

as $\xi, a \to 0$, where p_1 is in (3.3.8). Moreover, we take the $L^2(\mathbb{T})$-inner products of (3.3.7), and we make an explicit calculation to find that

$$\begin{aligned}
&\langle \phi_1, \phi_1\rangle = \langle \phi_2, \phi_2\rangle = \frac{1}{2} + O(\xi^3+\xi^2 a + a^2), \quad \langle \phi_1, \phi_2\rangle, \langle \phi_2, \phi_3\rangle = 0 + O(\xi^3+\xi^2 a + a^2),\\
&\langle \phi_1, \phi_3\rangle = ap_1 + O(\xi^3+\xi^2 a + a^2), \quad \langle \phi_3, \phi_3\rangle = 1 + O(\xi^3+\xi^2 a + a^2)
\end{aligned}$$

as $\xi, a \to 0$, where p_1 is in (3.3.8). Together, (3.4.1) becomes

$$\begin{aligned}
\mathbf{L}(\xi, a) = {}& a\Big(\frac{3}{4} - \frac{5}{24}k^2\Big)\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix} \qquad (3.4.3)\\
&+ i\xi\begin{pmatrix} -kc_{\rm ww}'(k) & 0 & 0\\ 0 & -kc_{\rm ww}'(k) & 0\\ 0 & 0 & c_{\rm ww}(k)-1\end{pmatrix}\\
&+ i\xi a\begin{pmatrix} 0 & 0 & -\frac{3}{4}+\frac{7}{24}k^2 + 2p_1(c_{\rm ww}(k)-1)\\ 0 & 0 & 0\\ -\frac{3}{4}+\frac{23}{48}k^2 + p_1(c_{\rm ww}(k)-1) & 0 & 0\end{pmatrix}\\
&+ \xi^2(kc_{\rm ww}'(k) + \tfrac{1}{2}k^2 c_{\rm ww}''(k))\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + O(\xi^3+\xi^2 a + a^2),
\end{aligned}$$

and

$$\mathbf{I}(a) = \mathbf{I} + ap_1 \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + O(a^2) \tag{3.4.4}$$

as $\xi, a \to 0$, where p_1 is in (3.3.8) and $\mathbf{I}$ is the 3×3 identity matrix.

3.5 The modulational instability index

For $\xi > 0$, $a \in \mathbb{R}$ and $\xi, |a|$ sufficiently small, we turn the attention to the roots of

$$\det(\mathbf{L} - \lambda \mathbf{I})(\xi, a; k) =: p_3(\xi, a; k)\lambda^3 + ip_2(\xi, a; k)\lambda^2 + p_1(\xi, a; k)\lambda + ip_0(\xi, a; k),$$

where $\mathbf{L}$ and $\mathbf{I}$ are in (3.4.3) and (3.4.4). Details are similar to Section 2.5. Hence we merely hit the main points.

Let

$$q(-i\xi\lambda)(\xi, a; k) = (i\xi^3(q_3\lambda^3 - q_2\lambda^2 - q_1\lambda + q_0)(\xi, a; k),$$

where $p_j = \xi^{3-j}q_j$, $j = 0, 1, 2, 3$. Note that $q_0, q_1, \dots, q_3$ are real valued and depend analytically on ξ, a and k for any $\xi > 0$ and $|a|$ sufficiently small. Moreover, they are odd in ξ and even in a. For $a \in \mathbb{R}$ and $|a|$ sufficiently small, a periodic traveling wave $\eta(a; 0, k)$ and $c(a; 0, k)$ of (3.2.1) and (3.1.2) is modulationally unstable, provided that q possesses a pair of complex roots or, equivalently,

$$\mathrm{disc}(k; \xi, a) := (18q_3q_2q_1q_0 + q_2^2q_1^2 + 4q_2^3q_0 + 4q_3q_1^3 - 27q_3^2q_0^2)(k; \xi, a) < 0$$

for $\xi > 0$ and small, and it is modulationally stable if $\mathrm{disc}(k; \xi, a) > 0$. Note that $\mathrm{disc}(k; \xi, a)$ is even in ξ and a. Hence we write that

$$\mathrm{disc}(k; \xi, a) =: \mathrm{disc}(k; \xi, 0) + a^2\Delta(k) + O(a^2(\xi^2 + a^2))$$

as $a \to 0$ for $\xi > 0$ and small. We then use (3.4.3) and (3.4.4), and we make a Mathematica calculation to show that

$$\mathrm{disc}(k; \xi, 0) = k^2 i_1^2(k)\Big(\xi i_2^2 + \frac{1}{4}\xi^3 k^2 i_1^2\Big)^2(k) > 0$$

as $\xi \to 0$. Therefore, if $\Delta(k) < 0$ then $\mathrm{disc}(k; \xi, a) < 0$ for $\xi > 0$ and sufficiently small, depending on $a \in \mathbb{R}$ and $|a|$ sufficiently small, implying modulational instability, whereas if $\Delta(k) > 0$ then $\text{disc}(k;\xi,a) > 0$ for $\xi > 0$, $a \in \mathbb{R}$ and $\xi, |a|$ sufficiently small, implying modulational stability. We use (3.4.3) and (3.4.4), and we make a Mathematica calculation to find Δ explicitly.

Below we summarize the conclusion.

Theorem 3.5.1 (Modulational instability index)**.** *A sufficiently small and $2\pi/k$-periodic traveling wave of* (3.2.1) *and* (3.1.2) *is modulationally unstable, provided that*

$$\Delta(k) := \frac{i_1(k)i_2(k)}{i_3(k)} i_4(k) < 0, \tag{3.5.1}$$

where

$$\begin{aligned} i_1(k) =&(kc_{\text{ww}}(k))'', \\ i_2(k) =&(kc_{\text{ww}}(k))' - 1, \\ i_3(k) =&c_{\text{ww}}(k) - c_{\text{ww}}(2k), \\ i_4(k) =&\Big(3i_2 - i_2 i_3 + 6i_3 - \frac{1}{12}k^2(57i_2 + 34i_3) + \frac{1}{108}k^4(198i_2 + 35i_3)\Big)(k), \end{aligned} \tag{3.5.2a}$$

and $c_{\text{ww}}(k)$ is in (3.1.2)*. It is modulationally stable if $\Delta(k) > 0$.*

3.6 Results

Since $(kc_{\text{ww}}(k))' < 1$ for any $k > 0$ and decreases monotonically over the interval $(0,\infty)$ by brutal force, $i_1(k) < 0$ and $i_2(k) < 0$ for any $k > 0$. Since $c_{\text{ww}}(k) > 0$ for any $k > 0$ and decreases monotonically over the interval $(0,\infty)$, $i_3(k) > 0$ for any $k > 0$.

We use (3.5.2a) and make an explicit calculation to show that

$$\lim_{k\to 0+} \frac{i_4(k)}{k^2} = \frac{9}{2} \quad \text{and} \quad \lim_{k\to\infty} i_4(k) = -\infty.$$

Hence $\Delta(k) > 0$ for $k > 0$ sufficiently small, implying the modulational stability, and it is negative for $k > 0$ sufficiently large, implying the modulational instability. The intermediate value theorem asserts a root of i_4. A numerical evaluation of (3.5.2a) reveals a unique root k_c, say, of i_4 over the interval $(0,\infty)$ such that $i_4(k) > 0$ if $0 < k < k_c$ and it is negative if $k_c < k < \infty$. Upon close inspection (see Figure 3.1), moreover, $k_c = 1.420\ldots$.

We have the following result.

Corollary 3.6.1. *A sufficiently small and $2\pi/k$ periodic traveling wave of* (3.2.1) *and* (3.1.2)

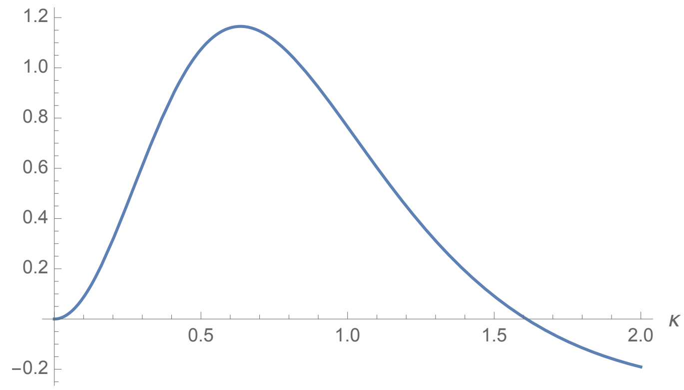


Figure 3.1: The graph of $i_4(k)$ for $k \in (0, 1.5)$.

is modulationally unstable if $k > k_c$*, where* $k_c = 1.420\ldots$ *is a unique root of* i_4 *in* (3.5.2a) *over the interval* $(0, \infty)$*. It is modulationally stable if* $0 < k < k_c$*.*

3.7 The Camassa-Holm equation

We may write (3.1.6), in the case of $a = 1/12$, after normalization of parameters, as

$$\eta_t + c_{\text{CH}}(|\partial_x|)\Big(2(1+\eta)^{3/2} - 2\eta + \frac{7}{24}\eta\eta_{xx} + \frac{7}{48}\eta_x^2\Big)_x = 0, \tag{3.7.1}$$

where

$$\widehat{c_{\text{CH}}(|\partial_x|)f}(k) = \frac{12-k^2}{12+k^2}\widehat{f}(k). \tag{3.7.2}$$

Note that $c_{\text{CH}}(k)$ approximates (3.1.2).

For any $k > 0$, we may repeat the argument in Section 3.2 to determine sufficiently small and $2\pi/k$-periodic traveling waves. Specifically, a two parameter family of periodic traveling waves of (3.7.1) and (3.7.2) exists, denoted

$$\eta(a, b; k)(z) \qquad \text{where } z = k(x - c(a, b; k)t),$$

for $a, b \in \mathbb{R}$ and $|a|, |b|$ sufficiently small; η is 2π periodic, even, and smooth in z. Moreover,

$$\begin{aligned} \eta(a, b; k)(z) =& b(1 - c_{\text{CH}}(k)) + a\cos z + a^2(h_0 + h_2\cos 2z) + O(a(a^2+b^2)),\\ c(a, b; k) =& c_{\text{CH}}(k) + b\Big(\frac{3}{2} - \frac{7}{24}k^2\Big)c_{\text{CH}}(k)(1 - c_{\text{CH}}(k)) + a^2c_2 + O(a(a^2+b^2)) \end{aligned}$$

as $a, b \to 0$, where

$$h_0 = \frac{36 - 7k^2}{96(c_{\rm CH}(k) - 1)}, \qquad h_2 = \frac{(12 - 7k^2)c_{\rm CH}(2k)}{32(c_{\rm CH}(k) - c_{\rm CH}(2k))},$$
$$c_2 = c_{\rm CH}(k)\Big(\frac{36 - 7k^2}{24}h_0 + \frac{12 - 7k^2}{16}h_2 - \frac{3}{32}\Big).$$

We then proceed as in Section 3.5, to determine a modulational instability index

$$\Delta_{\rm CH}(k) := \frac{i_1(k)i_2(k)}{i_3(k)}i_4(k),$$

where

$$\begin{aligned} i_1(k) =&(kc_{\rm CH}(k))'', \\ i_2(k) =&(kc_{\rm CH}(k))' - 1, \\ i_3(k) =&c_{\rm CH}(k) - c_{\rm CH}(2k), \\ i_4(k) =&1296(c_{\rm CH}(2k)i_2(k) + 2i_3(k)) - 432i_2(k)i_3(k) \\ &- 1512k^2c_{\rm CH}(2k)i_2(k) + 49k^4(9c_{\rm CH}(2k)i_2(k) - 2i_3(k)). \end{aligned}$$

We omit the details.

A straightforward calculation shows that $\frac{i_2 i_4}{i_3}(k) < 0$ for any $k > 0$ while $i_1(k)$ changes its sign from negative to positive across $k = 6$. Therefore, a sufficiently small and $2\pi/k$ periodic traveling wave of (3.7.1) and (3.7.2) is modulationally unstable if $k > 6$. For other values of a in (3.1.6), the result is qualitatively the same. Thus, the Camassa-Holm equation seems to predict the Benjamin-Feir instability of a Stokes wave. But the mechanism of instability is same as the BBM equation, whereas it does not take place in the water wave problem.

Chapter 4

The full-dispersion shallow water equations

The Euler's equations which describe surface gravity waves are bi-directional. The scalar or uni-directional shallow water models lack two-wave interactions which is a property of water waves. In this chapter, we introduce two bi-directional shallow water equations which generalizes nonlinear shallow water equations to include full-dispersion of water waves. We prove the existence of periodic traveling waves of both these equations and calculate their small amplitude asymptotics. We establish that their sufficiently small, periodic wave train is spectrally unstable to long wavelength perturbations, provided that the wave number is greater than a critical value, like the Benjamin-Feir instability of a Stokes wave.

4.1 The equation

As Whitham [Whi74] emphasized, "the breaking phenomenon is one of the most intriguing long-standing problems of water wave theory." The *nonlinear shallow water equations*,

$$\begin{aligned} &\eta_t + u_x + (u\eta)_x = 0, \\ &u_t + \eta_x + uu_x = 0, \end{aligned} \tag{4.1.1}$$

approximate the physical problem when the characteristic wavelength is of a larger order than the undisturbed fluid depth, and they explain wave breaking. That is, the solution remains bounded, whereas its slope becomes unbounded in finite time. Here $t \in \mathbb{R}$ is proportional to elapsed time, and $x \in \mathbb{R}$ is the spatial variable in the primary direction of wave propagation; $\eta = \eta(x,t)$ represents the surface displacement, and $u = u(x,t)$ is the fluid particle velocity at the rigid flat bottom. Note that the phase speed for the linear part of (4.1.1) is 1 for any wave number, whereas the speed of a $2\pi/k$-periodic wave near the rest state of water (see [Whi74], for instance) is

$$c_{\rm ww}(k) = \sqrt{\frac{\tanh(k)}{k}}. \tag{4.1.2}$$

This motivates us to propose the *full-dispersion shallow water equations*,

$$\begin{aligned} &\eta_t + u_x + (u\eta)_x = 0, \\ &u_t + c_{\text{ww}}^2(|\partial_x|)\eta_x + uu_x = 0, \end{aligned} \tag{4.1.3}$$

where c_{ww} is in (4.1.2). They combine the dispersion relation of water waves and the nonlinear shallow water equations, and they extend the Whitham equation to permit bidirectional propagation. Moreover, proposed in [MKD15] are

$$\begin{aligned} &\eta_t + c_{\text{ww}}^2(|\partial_x|)u_x + (u\eta)_x = 0, \\ &u_t + \eta_x + uu_x = 0, \end{aligned} \tag{4.1.4}$$

as a Boussinesq-Whitham model. We call (4.1.3), full-dispersion shallow water equation - I (FDSW-I) and (4.1.4), full-dispersion shallow water equation - II (FDSW-II). We provide the complete analysis for FDSW-I and the analysis for FDSW-II follows along the same lines, therefore, we only hit the main points in Section 4.7.

4.2 Periodic traveling waves

By a traveling wave of (4.1.3)-(4.1.2), we mean a solution which propagates at a constant velocity without change of form. That is, η and u are functions of $x - ct$ for some $c > 0$, the wave speed. Under the assumption, we will go to a moving coordinate frame, changing $x - ct$ to x, whereby t will disappear. The result becomes, by quadrature,

$$\begin{aligned} &-c\eta + u + u\eta = (1 - c^2)b_1, \\ &-cu + c_{\text{ww}}^2(|\partial_x|)\eta + \frac{1}{2}u^2 = (1 - c^2)b_2 \end{aligned}$$

for some b_1, $b_2 \in \mathbb{R}$; $1 - c^2$ is for convenience. We seek a periodic traveling wave of (4.1.3)-(4.1.2). That is, η and u are 2π periodic functions of

$$z := kx \qquad \text{for some } k > 0, \quad \text{the wave number,}$$

and they solve

$$\begin{aligned} &-c\eta + u + u\eta = (1 - c^2)b_1, \\ &-cu + c_{\text{ww}}^2(k|\partial_z|)\eta + \frac{1}{2}u^2 = (1 - c^2)b_2. \end{aligned} \tag{4.2.1}$$

Note that

$$c_{\text{ww}}^2(k|\partial_z|) : H^s(\mathbb{T}) \to H^{s+1}(\mathbb{T}) \qquad \text{for any } k > 0 \quad \text{for any integer } s \geqslant 0. \tag{4.2.2}$$

Note that

$$c_{\text{ww}}^2(k|\partial_z|)e^{inz} = c_{\text{ww}}^2(nk)e^{inz} \qquad \text{for } n \in \mathbb{Z}, \tag{4.2.3}$$

or, equivalently, $c_{\text{ww}}^2(k|\partial_z|)(1) = 1$,

$$c_{\text{ww}}^2(k|\partial_z|)(\cos nz) = c_{\text{ww}}^2(nk)\cos nz \quad \text{and} \quad c_{\text{ww}}^2(k|\partial_z|)(\sin nz) = c_{\text{ww}}^2(nk)\sin nz.$$

Note that (4.2.1) remains invariant under

$$z \mapsto z + z_0 \quad \text{and} \quad z \mapsto -z \tag{4.2.4}$$

for any $z_0 \in \mathbb{R}$. Hence, in particular, we may assume that η and u are even.

We state an existence result for periodic traveling waves of (4.1.3)-(4.1.2) and their small amplitude expansion.

Theorem 4.2.1 (Existence of sufficiently small, periodic wave trains)**.** *For any $k > 0$, b_1, $b_2 \in \mathbb{R}$ and $|b_1|$, $|b_2|$ sufficiently small, a one parameter family of solutions of* (4.2.1) *exists, denoted $\eta(a; k, b_1, b_2)(z)$, $u(a; k, b_1, b_2)(z)$, and $c(a; k, b_1, b_2)$, for $a \in \mathbb{R}$ and $|a|$ sufficiently small; η and u are 2π periodic, even, and smooth in z, and c is even in a; η, u, and c depend analytically on a, and k, b_1, b_2. Moreover,*

$$\begin{aligned}\eta(a; k, b_1, b_2)(z) =& \eta_0(k, b_1, b_2) + a\cos z + a(b_1 c_{\text{ww}}(k) + b_2)\cos z \\ &+ a^2(h_0 + h_2\cos 2z) + O(a(a + b_1 + b_2)^2),\end{aligned} \tag{4.2.5a}$$

$$\begin{aligned}u(a; k, b_1, b_2)(z) =& u_0(k, b_1, b_2) + ac_{\text{ww}}(k)\cos z + \frac{1}{2}ac_{\text{ww}}(k)(b_1 c_{\text{ww}}(k) + b_2)\cos z \\ &+ a^2 c_{\text{ww}}(k)\Big(h_0 - \frac{1}{2} + \Big(h_2 - \frac{1}{2}\Big)\cos 2z\Big) + O(a(a + b_1 + b_2)^2),\end{aligned} \tag{4.2.5b}$$

and

$$c(a; k, b_1, b_2) = c_0(k, b_1, b_2) + \frac{3}{4}a^2 c_{\text{ww}}(k)(2h_0 + h_2 - 1) + O(a(a + b_1 + b_2)^2) \tag{4.2.5c}$$

as $a,\ b_1,\ b_2 \to 0$*;*

$$\eta_0(k,b_1,b_2) = b_1 c_{\rm ww}(k) + b_2 + O((b_1+b_2)^2), \tag{4.2.6a}$$
$$u_0(k,b_1,b_2) = b_1 + b_2 c_{\rm ww}(k) + O((b_1+b_2)^2), \tag{4.2.6b}$$

and

$$c_0(k,b_1,b_2) = c_{\rm ww}(k) + b_1\Big(\frac{1}{2}c_{\rm ww}^2(k)+1\Big) + \frac{3}{2}b_2 c_{\rm ww}(k) + O((b_1+b_2)^2) \tag{4.2.6c}$$

as $b_1,\ b_2 \to 0,$ *where*

$$h_0 = \frac{3}{4}\frac{c_{\rm ww}^2(k)}{c_{\rm ww}^2(k)-1} \quad \textit{and} \quad h_2 = \frac{3}{4}\frac{c_{\rm ww}^2(k)}{c_{\rm ww}^2(k)-c_{\rm ww}^2(2k)}. \tag{4.2.7}$$

As a preliminary, we establish the smoothness of solutions of (4.2.1).

Lemma 4.2.2 (Regularity)**.** *If* $\eta,\ u \in H^1(\mathbb{T})$ *solve* (4.2.1) *for some* $c>0$*, and* $k>0,\ b_1,\ b_2 \in \mathbb{R}$ *and if* $c - \|u\|_{L^\infty(\mathbb{T})} \geqslant \epsilon > 0$ *for some* ϵ *then* $\eta,\ u \in H^\infty(\mathbb{T})$*.*

Proof. We differentiate (4.2.1) to arrive at

$$-c\eta' + u' + u\eta' + u'\eta = 0 \quad \text{and} \quad -cu' + c_{\rm ww}^2(k|\partial_z|)\eta' + uu' = 0,$$

whence

$$\eta' = \frac{1+\eta}{c-u}u' \quad \text{and} \quad u' = \frac{1}{c-u}c_{\rm ww}^2(k|\partial_z|)\eta'. \tag{4.2.8}$$

Here and elsewhere, the prime means ordinary differentiation.

Note that $\frac{1}{c-u} : H^1(\mathbb{T}) \to H^1(\mathbb{T})$ by hypothesis. Since $\eta' \in L^2(\mathbb{T})$ by hypothesis, it follows from the latter equation of (4.2.8) and (4.2.2) that $u' \in H^1(\mathbb{T})$. It then follows from the former equation of (4.2.8) and a Sobolev inequality that $\eta' \in H^1(\mathbb{T})$. In other words, η, $u \in H^2(\mathbb{T})$. A bootstrap argument completes the proof. □

Throughout, we use

$$\mathbf{u} = \begin{pmatrix}\eta\\ u\end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix}\zeta\\ v\end{pmatrix} \tag{4.2.9}$$

whenever it is convenient to do so.

Let $\mathbf{f} : H^1(\mathbb{T}) \times H^1(\mathbb{T}) \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to H^1(\mathbb{T}) \times H^1(\mathbb{T})$ such that

$$\mathbf{f}(\mathbf{u}, c; k, b_1, b_2) = \begin{pmatrix} -c\eta + u + u\eta - (1-c^2)b_1 \\ -cu + c_{\text{ww}}^2(k|\partial_z|)\eta + \frac{1}{2}u^2 - (1-c^2)b_2 \end{pmatrix}. \tag{4.2.10}$$

It is well defined by (4.2.2) and a Sobolev inequality. We seek a solution $\mathbf{u} \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$, $c > 0$, and $k > 0$, b_1, $b_2 \in \mathbb{R}$ of

$$\mathbf{f}(\mathbf{u}, c; k, b_1, b_2) = \mathbf{0} \tag{4.2.11}$$

satisfying $c - \|u\|_{L^\infty(\mathbb{T})} \geqslant \epsilon > 0$ for some ϵ and, by virtue of Lemma 4.2.2, a solution $\mathbf{u} \in H^\infty(\mathbb{T}) \times H^\infty(\mathbb{T})$ of (4.2.1). Note that $\mathbf{f}$ is invariant under (4.2.4) for any $z_0 \in \mathbb{R}$. Hence we may assume that $\mathbf{u}$ is even.

For any $c > 0$, and $k > 0$, b_1, $b_2 \in \mathbb{R}$, note that

$$\partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}, c; k, b_1, b_2) = \begin{pmatrix} u - c & 1 + \eta \\ c_{\text{ww}}^2(k|\partial_z|) & u - c \end{pmatrix} : H^1(\mathbb{T}) \times H^1(\mathbb{T}) \to H^1(\mathbb{T}) \times H^1(\mathbb{T})$$

is continuous by (4.2.2) and a Sobolev inequality. For any $\mathbf{u} \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$, and $k > 0$, b_1, $b_2 \in \mathbb{R}$, moreover,

$$\partial_c\mathbf{f}(\mathbf{u}, c; k, b_1, b_2) = \begin{pmatrix} -\eta + 2cb_1 \\ -u + 2cb_2 \end{pmatrix} : \mathbb{R} \to H^1(\mathbb{T}) \times H^1(\mathbb{T})$$

is continuous. Here (by abuse of notation) ∂ means Fréchet differentiation. Since

$$\partial_k\mathbf{f}(\mathbf{u}, c; k, b_1, b_2) = \begin{pmatrix} 0 \\ \frac{1}{k}(\operatorname{sech}^2(k|\partial_z|) - c_{\text{ww}}^2(k|\partial_z|)) \end{pmatrix},$$

and

$$\partial_{b_1}\mathbf{f}(\mathbf{u}, c; k, b_1, b_2) = \begin{pmatrix} c^2 - 1 \\ 0 \end{pmatrix}, \qquad \partial_{b_2}\mathbf{f}(\mathbf{u}, c; k, b_1, b_2) = \begin{pmatrix} 0 \\ c^2 - 1 \end{pmatrix}$$

are continuous, likewise, $\mathbf{f}$ depends continuously differentiably on its arguments. Furthermore, since the Fréchet derivatives of $\mathbf{f}$ with respect to $\mathbf{u}$, and c, b_1, b_2 of all orders $\geqslant 3$ are zero everywhere by brutal force, and since c_{ww}^2 is a real analytic function, $\mathbf{f}$ is a real analytic operator.

For any $c > 0$ for any $k > 0$, b_1, $b_2 \in \mathbb{R}$ and $|b_1|$, $|b_2|$ sufficiently small, note that

$$\begin{aligned} \eta_0(c; k, b_1, b_2) &= b_1c + b_2 + O((b_1 + b_2)^2), \\ u_0(c; k, b_1, b_2) &= b_1 + b_2c + O((b_1 + b_2)^2) \end{aligned} \tag{4.2.12}$$

make a constant solution of (4.2.10)-(4.2.11) and, hence, (4.2.1). Let $\mathbf{u}_0 = \begin{pmatrix} \eta_0 \\ u_0 \end{pmatrix} (c; k, b_1, b_2)$. It follows from the implicit function theorem that if non-constant solutions of (4.2.10)-(4.2.11) and, hence, (4.2.1) bifurcate from $\mathbf{u} = \mathbf{u}_0$ for some $c = c_0$ then, necessarily,

$$\mathbf{L}_0 := \partial_{\mathbf{u}} \mathbf{f}(\mathbf{u}_0, c_0; k, b_1, b_2) : H^1(\mathbb{T}) \times H^1(\mathbb{T}) \to H^1(\mathbb{T}) \times H^1(\mathbb{T})$$

is not an isomorphism. Here $\mathbf{u}_0$ depends on c_0. But we suppress it for simplicity of notation. Note that

$$\mathbf{L}_0 \mathbf{u}_1 e^{\pm inz} = \begin{pmatrix} u_0 - c_0 & 1 + \eta_0 \\ c_{\text{ww}}^2(k|\partial_z|) & u_0 - c_0 \end{pmatrix} \mathbf{u}_1 e^{\pm inz} = \mathbf{0} \qquad \text{for } n \in \mathbb{Z} \tag{4.2.13}$$

for some nonzero $\mathbf{u}_1$ if and only if

$$(c_0 - u_0)^2 = c_{\text{ww}}^2(nk)(1 + \eta_0). \tag{4.2.14}$$

For $b_1 = b_2 = 0$ and, hence, $\eta_0 = u_0 = 0$ by (4.2.12), it simplifies to $c_0 = \pm c_{\text{ww}}(nk)$ — the phase velocity of a $2\pi/nk$ periodic wave in the linear theory; $\pm$ indicate right and left propagating waves, respectively. Without loss of generality, here we restrict the attention to $n = 1$ and we assume the $+$ sign. For $|b_1|$ and $|b_2|$ sufficiently small, (4.2.14) becomes

$$c_0 = c_{\text{ww}}(k) + b_1 \Big(\frac{1}{2} c_{\text{ww}}^2(k) + 1 \Big) + \frac{3}{2} b_2 c_{\text{ww}}(k) + O((b_1 + b_2)^2).$$

Substituting it into (4.2.12), we find

$$\begin{aligned} \eta_0(k, b_1, b_2) &= b_1 c_{\text{ww}}(k) + b_2 + O((b_1 + b_2)^2), \\ u_0(k, b_1, b_2) &= b_1 + b_2 c_{\text{ww}}(k) + O((b_1 + b_2)^2). \end{aligned}$$

They agree with (4.2.6). In the sequel, $\mathbf{u}_0 = \begin{pmatrix} \eta_0 \\ u_0 \end{pmatrix} (k, b_1, b_2)$ and $c_0 = c_0(k, b_1, b_2)$.

Since $c_{\text{ww}}(k) > c_{\text{ww}}(nk)$ for $n = 2, 3, \dots$ pointwise in $\mathbb{R}$ (see Figure 4.1), a straightforward calculation reveals that for any $k > 0$, b_1, $b_2 \in \mathbb{R}$ and $|b_1|$, $|b_2|$ sufficiently small, the $H^1(\mathbb{T}) \times H^1(\mathbb{T})$ kernel of $\mathbf{L}_0 = \partial_{\mathbf{u}} \mathbf{f}(\mathbf{u}_0, c_0; k, b_1, b_2)$ is two dimensional and spanned by $\mathbf{u}_1 e^{\pm iz}$ for some nonzero $\mathbf{u}_1$ satisfying (4.2.13). Note from (4.2.13) and (4.2.6) that

$$\mathbf{u}_1 = \begin{pmatrix} 1 + \eta_0 \\ c_0 - u_0 \end{pmatrix} = \begin{pmatrix} 1 + b_1 c_{\text{ww}}(k) + b_2 \\ c_{\text{ww}}(k) + \frac{1}{2} b_1 c_{\text{ww}}^2(k) + \frac{1}{2} b_2 c_{\text{ww}}(k) \end{pmatrix} + O((b_1 + b_2)^2) \tag{4.2.15}$$

as b_1, $b_2 \to 0$ up to the multiplication by a constant. This agrees with (4.2.5a) and (4.2.5b)

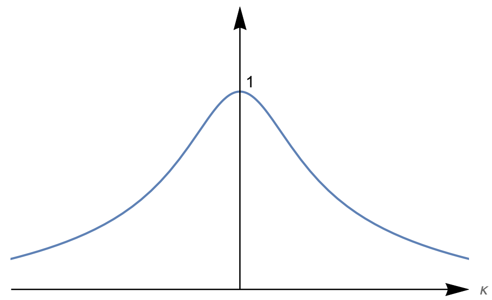


Figure 4.1: Schematic plot of $c_{\rm ww}$.

at the order of a.

Moreover, a straightforward calculation reveals that for any $k > 0$, b_1, $b_2 \in \mathbb{R}$ and $|b_1|$, $|b_2|$ sufficiently small, the $H^1(\mathbb{T}) \times H^1(\mathbb{T})$ co-kernel of $\mathbf{L}_0$ is two dimensional and spanned by $\mathbf{u}_1^{\perp} e^{\pm iz}$ for some $\mathbf{u}_1^{\perp}$ orthogonal to $\mathbf{u}_1$. In particular, $\mathbf{L}_0$ is a Fredholm operator of index zero.

For any $k > 0$, b_1, $b_2 \in \mathbb{R}$ and $|b_1|$, $|b_2|$ sufficiently small, we turn the attention to non-constant solutions of (4.2.10)-(4.2.11) and, hence, (4.2.1) bifurcating from $\mathbf{u} = \mathbf{u}_0$ and $c = c_0$, where η_0, u_0, and c_0 are in (4.2.6). A Lyapunov-Schmidt procedure is instrumental for the purpose. Here the proof follows along the same line as in Chapter 2 and 3, but with suitable modifications to accommodate product spaces. Throughout the subsection, k, and b_1, b_2 are fixed and suppressed for simplicity of notation.

Recall $\mathbf{f}(\mathbf{u}_0, c_0) = \mathbf{0}$, where $\mathbf{f}$ is in (4.2.10). Recall $\mathbf{L}_0 \mathbf{u}_1 e^{\pm iz} = 0$, where $\mathbf{L}_0$ is in (4.2.13) and $\mathbf{u}_1$ is in (4.2.15). We write that

$$\mathbf{u}(z) = \mathbf{u}_0 + \frac{1}{2}\mathbf{u}_1(ae^{iz} + a^* e^{-iz}) + \mathbf{u}_r(z) \quad \text{and} \quad c = c_0 + c_r, \tag{4.2.16}$$

and we require that $a \in \mathbb{C}$, $\mathbf{u}_r = \begin{pmatrix} \eta_r \\ u_r \end{pmatrix} \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$ be even and

$$\begin{aligned} \langle \mathbf{u}_r, \mathbf{u}_1(ae^{iz} + a^* e^{-iz}) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} &((1+\eta_0)\eta_r(z)(ae^{iz} + a^* e^{-iz}) \\ &+ (c_0 - u_0) u_r(z)(ae^{iz} + a^* e^{-iz}))\, dz = 0, \end{aligned} \tag{4.2.17}$$

and $c_r \in \mathbb{R}$. Here and elsewhere, the asterisk means complex conjugation; $\langle \cdot\,, \cdot \rangle$ is the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-inner product.

Substituting (4.2.16) into (4.2.10)-(4.2.11), we use $\mathbf{f}(\mathbf{u}_0; c_0) = \mathbf{0}$, and (4.2.13), (4.2.15), and we make an explicit calculation to arrive at

$$\begin{aligned}\mathbf{L}_0\mathbf{u}_r =& -\begin{pmatrix} \left(\frac{1}{2}(c_0-u_0)(ae^{iz}+a^*e^{-iz})+u_r\right)\left(\frac{1}{2}(1+\eta_0)(ae^{iz}+a^*e^{-iz})+\eta_r\right) \\ \frac{1}{2}\left(\frac{1}{2}(c_0-u_0)(ae^{iz}+a^*e^{-iz})+u_r\right)^2 \end{pmatrix} \\ &+ c_r\begin{pmatrix} \frac{1}{2}(1+\eta_0)(ae^{iz}+a^*e^{-iz})+\eta_r \\ \frac{1}{2}(c_0-u_0)(ae^{iz}+a^*e^{-iz})+u_r \end{pmatrix} \\ =:& \mathbf{g}(\mathbf{u}_r;a,a^*,c_r)\end{aligned} \tag{4.2.18}$$

up to terms of order c_r as $c_r \to 0$. Recall that $\mathbf{f}$ is a real analytic operator. Hence $\mathbf{g}$ depends analytically on its arguments. Clearly, $\mathbf{g}(\mathbf{0};0,0,c_r)=0$ for any $c_r \in \mathbb{R}$.

Recall that $\mathbf{L}_0$ is a Fredholm operator of index zero,

$$\ker \mathbf{L}_0 = \operatorname{span}\{\mathbf{u}_1 e^{\pm iz}\} \quad \text{and} \quad \text{co-ker}\, \mathbf{L}_0 = \operatorname{span}\{\mathbf{u}_1^\perp e^{\pm iz}\},$$

where $\mathbf{u}_1$ and $\mathbf{u}_1^\perp$ are orthogonal to each other. Let Π denote the spectral projection of $L^2(\mathbb{T})\times L^2(\mathbb{T})$ onto the kernel of $\mathbf{L}_0$. Specifically, if $\mathbf{v} = \sum_{n\in\mathbb{Z}} \begin{pmatrix} \widehat{\zeta}(n) \\ \widehat{v}(n) \end{pmatrix} e^{inz}$ in the Fourier series then

$$\begin{aligned}\Pi\mathbf{v} =& \langle \mathbf{v}, \mathbf{u}_1 e^{iz}\rangle \mathbf{u}_1 e^{iz} + \langle \mathbf{v}, \mathbf{u}_1 e^{-iz}\rangle \mathbf{u}_1 e^{-iz} \\ =& ((1+\eta_0)(\widehat{\zeta}(1)e^{iz}+\widehat{\zeta}(-1)e^{-iz}) + (c_0-u_0)(\widehat{v}(1)e^{iz}+\widehat{v}(-1)e^{-iz}))\mathbf{u}_1.\end{aligned}$$

Since $\Pi\mathbf{u}_r = 0$ by (4.2.17), we may recast (4.2.18) as

$$\mathbf{L}_0\mathbf{u}_r = (1-\Pi)\mathbf{g}(\mathbf{u}_r;a,a^*,c_r) \quad \text{and} \quad \mathbf{0} = \Pi\mathbf{g}(\mathbf{u}_r;a,a^*,c_r). \tag{4.2.19}$$

Moreover, $\mathbf{L}_0 : (1-\Pi)(H^1(\mathbb{T})\times H^1(\mathbb{T})) \to \operatorname{range}\mathbf{L}_0$ is invertible. Specifically, if

$$\mathbf{v} = \begin{pmatrix} 1+\eta_0 \\ u_0-c_0 \end{pmatrix}(v_{+1}e^{iz}+v_{-1}e^{-iz}) + \sum_{n\neq\pm1}\begin{pmatrix} \widehat{\zeta}(n) \\ \widehat{v}(n) \end{pmatrix} e^{inz},$$

for some constants $v_{\pm1}$, belongs to the range of $\mathbf{L}_0$ by (4.2.13) then

$$\begin{aligned}\mathbf{L}_0^{-1}\mathbf{v}(z) =& \begin{pmatrix} 0 \\ 1 \end{pmatrix}(v_{+1}e^{iz}+v_{-1}e^{-iz}) \\ &+ \sum_{n\neq\pm1}\frac{1}{(u_0-c_0)^2 - c_{\text{ww}}^2(nk)(1+\eta_0)}\begin{pmatrix} u_0-c_0 & -1-\eta_0 \\ -c_{\text{ww}}^2(kn) & u_0-c_0 \end{pmatrix}\begin{pmatrix} \widehat{\zeta}(n) \\ \widehat{v}(n) \end{pmatrix} e^{inz}.\end{aligned}$$

It is well defined since (4.2.14) holds true if and only if $n = \pm 1$. Hence we may recast (4.2.19) as

$$\mathbf{u}_r = \mathbf{L}_0^{-1}(1-\Pi)\mathbf{g}(\mathbf{u}_r; a, a^*, c_r) \quad \text{and} \quad \mathbf{0} = \Pi\mathbf{g}(\mathbf{u}_r; a, a^*, c_r). \tag{4.2.20}$$

Clearly, $\mathbf{L}_0^{-1}(1-\Pi)\mathbf{g}$ depends analytically on its arguments. Since $\mathbf{g}(\mathbf{0}; 0, 0, c_r) = 0$ for any $c_r \in \mathbb{R}$, it follows from the implicit function theorem that a unique solution $\mathbf{u}_r = \mathbf{u}_2(a, a^*, c_r)$ exists to the former equation of (4.2.20) near $\mathbf{u}_r = \mathbf{0}$ for $a \in \mathbb{C}$ and $|a|$ sufficiently small for any $c_r \in \mathbb{R}$. Note that $\mathbf{u}_2$ depends analytically on its arguments and it satisfies (4.2.17) for $|a|$ sufficiently small for any $c_r \in \mathbb{R}$. The uniqueness implies

$$\mathbf{u}_2(0, 0, c_r) = \mathbf{0} \qquad \text{for any } c_r \in \mathbb{R}. \tag{4.2.21}$$

Moreover, since (4.2.10)-(4.2.11) and, hence, (4.2.20) are invariant under (4.2.4) for any $z_0 \in \mathbb{R}$, it follows that

$$\mathbf{u}_2(a, a^*, c_r)(z + z_0) = \mathbf{u}_2(ae^{iz_0}, a^*e^{-iz_0}, c_r)(z) \quad \text{and} \quad \mathbf{u}_2(a, a^*, c_r)(-z) = \mathbf{u}_2(a, a^*, c_r)(z) \tag{4.2.22}$$

for any $z_0 \in \mathbb{R}$ for any $a \in \mathbb{C}$, $|a|$ sufficiently small, and $c_r \in \mathbb{R}$.

To proceed, we write the latter equation of (4.2.20) as

$$\Pi\mathbf{g}(\mathbf{u}_2(a, a^*, c_r); a, a^*, c_r) = \mathbf{0}$$

for $a \in \mathbb{C}$ and $|a|$ sufficiently small for any $c_r \in \mathbb{R}$. This is solvable, provided that

$$\pi_\pm(a, a^*, c_r) := \langle \mathbf{g}(\mathbf{u}_2(a, a^*, c_r); a, a^*, c_r), \mathbf{u}_1(ae^{iz} \pm a^*e^{-iz})\rangle = 0; \tag{4.2.23}$$

$\langle \cdot, \cdot \rangle$ is the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ inner product. We use (4.2.22), where $z_0 = -2\arg(a)$, and (4.2.23) to show that

$$\pi_-(a^*, a, c_r) = \pi_-(a, a^*, c_r) = -\pi_-(a^*, a, c_r).$$

Hence $\pi_-(a, a^*, c_r) = 0$ holds true for any $a \in \mathbb{C}$ and $|a|$ sufficiently small for any $c_r \in \mathbb{R}$. Moreover, we use (4.2.22), where $z_0 = -\arg(a)$, and (4.2.23) to show that

$$\pi_+(a, a^*, c_r) = \pi_+(|a|, |a|, c_r).$$

Hence it suffices to solve $\pi_+(a, a, c_r) = 0$ for a, $c_r \in \mathbb{R}$ and $|a|$ sufficiently small.

Substituting (4.2.18) into (4.2.23), where $\mathbf{u}_r = \mathbf{u}_2(a, a, c_r) =: \begin{pmatrix} \eta_2 \\ u_2 \end{pmatrix}(a, c_r)$, we make an explicit calculation to arrive at

$$\pi_+(a,a,c_r) =: a^2(c_r((1+\eta_0)^2 + (c_0-u_0)^2) + \pi_r(a,c_r))$$

for a, $c_r \in \mathbb{R}$ and $|a|$ sufficiently small, where

$$\begin{aligned}\pi_r(a,c_r) = &- a^2(1+\eta_0)((c_0-u_0)\langle \cos z\, \eta_2(a,c_r), \cos z\rangle \\ &\qquad + (1+\eta_0)\langle \cos z\, u_2(a,c_r), \cos z\rangle + a^{-1}\langle (\eta_2 u_2)(a,c_r), \cos z\rangle) \\ &- \tfrac{1}{2}a^2(c_0-u_0)(2(c_0-u_0)\langle \cos z\, u_2(a,c_r), \cos z\rangle + a^{-1}\langle u_2^2(a,c_r), \cos z\rangle);\end{aligned}$$

$\langle \cdot\,,\cdot\rangle$ means the $L^2(\mathbb{T})$ inner product. We merely pause to remark that π_r is well defined. Indeed, $a^{-1}\eta_2$ and $a^{-1}u_2$ are not singular for $a \in \mathbb{R}$ and $|a|$ sufficiently small by (4.2.21). Clearly, π_r and, hence, π_+ depend analytically on their arguments. Since $\pi_r(0,0) = 0$ and $(\partial_{c_r}\pi_r)(0,0) = 0$ by (4.2.21), it follows from the implicit function theorem that a unique solution $c_r = c_2(a)$ exists to $\pi_+(a,a,c_r) = 0$ and, hence, the latter equation of (4.2.20) near $c_r = 0$ for $a \in \mathbb{R}$ and $|a|$ sufficiently small. Clearly, c_2 depends analytically on a.

To summarize,

$$\mathbf{u}_r = \mathbf{u}_2(a,a,c_2(a)) \quad \text{and} \quad c_r = c_2(a)$$

uniquely solve (4.2.18) for $a \in \mathbb{R}$ and $|a|$ sufficiently small, and by virtue of (4.2.16),

$$\mathbf{u}(a)(z) = \mathbf{u}_0 + a\mathbf{u}_1 \cos z + \mathbf{u}_2(a,a,c_2(a))(z) \quad \text{and} \quad c(a) = c_0 + c_2(a) \tag{4.2.24}$$

uniquely solve (4.2.10)-(4.2.11) and, hence, (4.2.1) for $a \in \mathbb{R}$ and $|a|$ sufficiently small. Note that $\mathbf{u}_2$ and, hence, $\mathbf{u}$ are 2π periodic and even in z. Since $\mathbf{u}_2$ and c_2 are near $\mathbf{0}$ and 0, Lemma 4.2.2 implies that $\mathbf{u}$ is smooth in z. We claim that c is even in a. Indeed, note that (4.2.1) and, hence, (4.2.10)-(4.2.11) remain invariant under $z \mapsto z+\pi$ by (4.2.4). Since $(\partial_a \mathbf{u})(0)(z) = \mathbf{u}_1 \cos z$, however, $\mathbf{u}(z) \neq \mathbf{u}(z+\pi)$ must hold true. Thus $(\partial_a c)(0) = 0$. This proves the claim. Clearly, $\mathbf{u}$ and c depend analytically on $a \in \mathbb{R}$ and $|a|$ sufficiently small. This completes the existence proof.

It remains to verify (4.2.5). Throughout the subsection, $k > 0$ is fixed and suppressed for simplicity of notation; b_1, $b_2 \in \mathbb{R}$ and $|b_1|$, $|b_2|$ sufficiently small are fixed.

Recall from the existence proof that (4.2.24) depends analytically on a, b_1, $b_2 \in \mathbb{R}$ and $|a|$,

$|b_1|$, $|b_2|$ sufficiently small. We write that

$$\begin{aligned}\eta(a;b_1,b_2)(z) =&\eta_0(b_1,b_2)+a(1+\eta_0(b_1,b_2))\cos z\\&+a^2\eta_2(z)+a^3\eta_3(z)+O(a^4+a^2(b_1+b_2)+a(b_1+b_2)^2),\\u(a;b_1,b_2)(z) =&u_0(b_1,b_2)+a(c_0-u_0)(b_1,b_2)\cos z\\&+a^2u_2(z)+a^3u_3(z)+O(a^4+a^2(b_1+b_2)+a(b_1+b_2)^2),\end{aligned}$$

and

$$c(a;b_1,b_2) =c_0(b_1,b_2)+a^2c_2+O(a^4+a^2(b_1+b_2)+a(b_1+b_2)^2)$$

as $a,b_1,b_2\to 0$, where η_0, u_0, and c_0 are in (4.2.6), and we require that η_2, u_2, and η_3, u_3 be 2π periodic, even, and smooth functions of z, and $c_2\in\mathbb{R}$. We merely pause to remark that η_2, u_2, η_3, u_3, and c_2 do *not* depend on b_1 and b_2, whereas η_0, u_0, and c_0 do. In the following sections, we restrict the attention to periodic traveling waves of (4.1.3)-(4.1.2) for $a\in\mathbb{R}$ and $|a|$ sufficiently small for $b_1=b_2=0$, and we calculate the spectrum of the associated linearized operator up to the order of a. (The index formulae would become unwieldy when terms of order a^2 were to be added.) For the purpose, it suffices to calculate solutions explicitly up to terms of orders a^2, and ab_1, ab_2.

Substituting the above into (4.2.1), we recall that η_0, u_0, and c_0 solve (4.2.1), and we make an explicit calculation to arrive, at the order of a, at

$$\begin{aligned}&-c_0(1+\eta_0)\cos z+(c_0-u_0)\cos z=0,\\&-c_0(c_0-u_0)\cos z+c_{\text{ww}}^2(k|\partial_z|)(1+\eta_0)\cos z=0.\end{aligned}$$

This holds true up to terms of orders b_1 and b_2 by (4.2.3), (4.2.6c), and (4.2.13), (4.2.15).

To proceed, we assume $b_1=b_2=0$ and, hence, $\eta_0=u_0=0$ and $c_0=c_{\text{ww}}(k)$ by (4.2.6). At the order of a^2, we gather

$$\begin{aligned}&-c_0\eta_2+u_2+c_0\cos^2 z=0,\\&-c_0u_2+c_{\text{ww}}^2(k|\partial_z|)\eta_2+\tfrac12 c_0^2\cos^2 z=0.\end{aligned}$$

We then use (4.2.3), (4.2.6c) and we make an explicit calculation to find

$$\eta_2(z)=h_0+h_2\cos 2z\quad\text{and}\quad u_2(z)=h_0-\frac12+\Big(h_2-\frac12\Big)\cos 2z, \tag{4.2.25}$$

where h_0 and h_2 are in (4.2.7). Continuing, at the order of a^3, we gather

$$\begin{aligned}
&-c_0\eta_3 - c_2\cos z + u_3 + u_2\cos z + c_0\eta_2\cos z = 0,\\
&-c_0u_3 - c_2c_0\cos z + c_{\rm ww}^2(k|\partial_z|)\eta_3 + c_0u_2\cos z = 0.
\end{aligned}$$

Taking $L^2(\mathbb{T})$-inner products, we use (4.2.3) and (4.2.25), so that

$$\begin{aligned}
&-c_0\langle\eta_3,\cos z\rangle - c_2 + \langle u_3,\cos z\rangle + h_0 - \frac{1}{2} + \frac{1}{2}\Big(h_2 - \frac{1}{2}\Big) + c_0\Big(h_0 + \frac{1}{2}h_2\Big) = 0,\\
&-c_0\langle u_3,\cos z\rangle - c_2c_0 + c_{\rm ww}^2(k)\langle\eta_3,\cos z\rangle + c_0\Big(h_0 - \frac{1}{2} + \frac{1}{2}\Big(h_2 - \frac{1}{2}\Big)\Big) = 0.
\end{aligned}$$

We then use (4.2.6c) and we make an explicit calculation to find

$$c_2 = \frac{3}{4}c_{\rm ww}(k)(2h_0 + h_2 - 1).$$

This completes the proof.

4.3 The spectral problem

Let $\eta = \eta(a;k,b_1,b_2)$, $u = u(a;k,b_1,b_2)$, and $c = c(a;k,b_1,b_2)$, for some $a \in \mathbb{R}$ and $|a|$ sufficiently small, $k > 0$, b_1, $b_2 \in \mathbb{R}$ and $|b_1|$, $|b_2|$ sufficiently small, denote a $2\pi/k$-periodic wave train of (4.1.3)-(4.1.2), whose existence follows from Theorem 4.2.1. We address its stability and instability to "slow modulations". Throughout the section, we employ the notation of (4.2.9) whenever it is convenient to do so.

We linearize (4.1.3)-(4.1.2) about $\mathbf{u}$ in the coordinate frame moving at the speed c. Recall that $\mathbf{u}$ and c solve (4.2.1) and $z = kx$. The result becomes

$$\partial_t\mathbf{v} = k\partial_z\begin{pmatrix} c-u & -1-\eta \\ -c_{\rm ww}^2(k|\partial_z|) & c-u\end{pmatrix}\mathbf{v}.$$

We seek a solution of the form $\mathbf{v}(z,t) = e^{\lambda kt}\mathbf{v}(z)$, $\lambda \in \mathbb{C}$, to arrive at

$$\lambda\mathbf{v} = \partial_z\begin{pmatrix} c-u & -1-\eta \\ -c_{\rm ww}^2(k|\partial_z|) & c-u\end{pmatrix}\mathbf{v} =: \mathcal{L}(a;k,b_1,b_2)\mathbf{v}, \tag{4.3.1}$$

where

$$\mathcal{L} : H^1(\mathbb{R})\times H^1(\mathbb{R}) \subset L^2(\mathbb{R})\times L^2(\mathbb{R}) \to L^2(\mathbb{R})\times L^2(\mathbb{R}).$$

We take a Floquet theory approach to characterize the $L^2(\mathbb{R}) \times L^2(\mathbb{R})$-spectrum of $\mathcal{L}$ in a convenient form. The Bloch operators are given by,

$$\mathcal{L}(\xi) := e^{-i\xi z}\partial_z \begin{pmatrix} c-u & -1-\eta \\ -c_{\text{ww}}^2(k|\partial_z|) & c-u \end{pmatrix} e^{i\xi z} \tag{4.3.2}$$

for $\xi \in (-1/2, 1/2]$. Note that (4.3.2), when $\pm\xi$ are taken in pair, remains invariant under

$$\lambda \mapsto \lambda^* \quad \text{and} \quad \boldsymbol{\phi} \mapsto \boldsymbol{\phi}^*,$$

and under

$$\lambda \mapsto -\lambda \quad \text{and} \quad z \mapsto -z.$$

Hence we may assume $\xi \in [0, 1/2]$.

For an arbitrary ξ, one must study (4.3.2) numerically except for few cases — for instance, completely integrable systems (see [BHJ16], for instance, for references). But, for $\xi > 0$ small and for λ in the vicinity of the origin in $\mathbb{C}$, we may take a spectral perturbation approach in [HJ15a,HP16b], for instance, to address it analytically. This is the subject of investigation here.

In the remaining of the section, $k > 0$ is suppressed for simplicity of notation, unless specified otherwise. We assume $b_1 = b_2 = 0$. For nonzero b_1 and b_2, one may explore in like manner. But the calculation becomes lengthy and tedious. Hence we do not discuss the details. We use

$$\mathcal{L}(\xi, a) = \mathcal{L}(\xi)(a; k, 0, 0) \tag{4.3.3}$$

for simplicity of notation.

We begin by discussing the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-spectra of $\mathcal{L}(\xi, 0)$ for $\xi \in [0, 1/2]$. This is the linearization of (4.1.3)-(4.1.2) about $\eta = u = 0$ and $c = c_{\text{ww}}(k)$ — namely, the rest state — in the moving coordinate frame.

Note from (4.3.2) and (4.2.6) that

$$\mathcal{L}(\xi, 0) = e^{-i\xi z}\partial_z \begin{pmatrix} c_{\text{ww}}(k) & -1 \\ -c_{\text{ww}}^2(k|\partial_z|) & c_{\text{ww}}(k) \end{pmatrix} e^{i\xi z}.$$

We use (4.2.3) and make an explicit calculation to show that

$$\mathcal{L}(\xi,0)\mathbf{e}(n+\xi,\pm) = i\omega(n+\xi,\pm)\mathbf{e}(n+\xi,\pm) \qquad \text{for } n\in\mathbb{Z} \text{ and } \xi\in[0,1/2], \tag{4.3.4}$$

where

$$\omega(n+\xi,\pm) = (n+\xi)(c_{\text{ww}}(k)\pm c_{\text{ww}}(k(n+\xi))) \quad \text{and} \quad \mathbf{e}(n+\xi,\pm)(z) = \begin{pmatrix} 1 \\ \mp c_{\text{ww}}(k(n+\xi)) \end{pmatrix} e^{inz}. \tag{4.3.5}$$

Hence for any $\xi\in[0,1/2]$, the spectrum of $\mathcal{L}(\xi,0)$ consists of two families of infinitely many and purely imaginary eigenvalues, each with finite multiplicity. In particular, the rest state of (4.1.3)-(4.1.2) is spectrally stable to square integrable perturbations.

The spectrum of the linear operator associated with the water wave problem consists of $i\omega(n+\xi,\pm)$ for $n\in\mathbb{Z}$ and $\xi\in[-1/2,1/2)$; see [Whi74], for instance, for details. To compare, the spectrum of the linear operator for the Whitham equation consists of $i\omega(n+\xi,-)$ for $n\in\mathbb{Z}$ and $\xi\in[-1/2,1/2)$; see [HJ15a], for instance, for details. Perhaps, this is because the Whitham equation merely includes unidirectional propagation. In the following section, we discuss the effects of bidirectional propagation in (4.1.3)-(4.1.2).

As $|a|$ increases, the eigenvalues in (4.3.4) move around and they may leave the imaginary axis to lose the spectral stability. Recall that the spectrum of $\mathcal{L}(\pm\xi,a)$ is symmetric with respect to the reflections in the real and imaginary axes for any $\xi\in[0,1/2]$ for any $a\in\mathbb{R}$ and admissible. Hence a necessary condition of the spectral instability is that a pair of eigenvalues on the imaginary axis collide.

Note that the eigenfunctions in (4.3.4) vary, analytically, with $\xi\in[0,1/2]$. To compare, the eigenfunctions of the linear operator for the Whitham equation do not depend on ξ; see [HJ15a], for instance, for details.

To proceed, for $\xi=0$, note from (4.3.5) that

$$\omega(0,+)=\omega(0,-)=\omega(1,-)=\omega(-1,-)=0.$$

Since

$$\cdots<\omega(-3,-)<\omega(-2,-)<0<\omega(2,-)<\omega(3,-)<\cdots$$

and

$$\cdots<\omega(-2,+)<\omega(-1,+)<0<\omega(1,+)<\omega(2,+)<\dots$$

by brutal force, zero is an $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-eigenvalue of $\mathcal{L}(0,0)$ with multiplicity four. Note that

$$
\begin{aligned}
\boldsymbol{\phi}_1(z) &:= \frac{1}{2}(\mathbf{e}(1,-)+\mathbf{e}(-1,-))(z) = \begin{pmatrix} 1 \\ c_{\mathrm{ww}}(k) \end{pmatrix} \cos z, \\
\boldsymbol{\phi}_2(z) &:= \frac{1}{2i}(\mathbf{e}(1,-)-\mathbf{e}(-1,-))(z) = \begin{pmatrix} 1 \\ c_{\mathrm{ww}}(k) \end{pmatrix} \sin z, \\
\boldsymbol{\phi}_3(z) &:= \frac{1}{2}((c_{\mathrm{ww}}(k)+2)\mathbf{e}(0,+)-(c_{\mathrm{ww}}(k)-2)\mathbf{e}(0,-))(z) = \begin{pmatrix} 2 \\ -c_{\mathrm{ww}}(k) \end{pmatrix}, \\
\boldsymbol{\phi}_4(z) &:= \frac{1}{2}((c_{\mathrm{ww}}(k)-2)\mathbf{e}(0,+)+(c_{\mathrm{ww}}(k)+2)\mathbf{e}(0,-))(z) = \begin{pmatrix} c_{\mathrm{ww}}(k) \\ 2 \end{pmatrix}
\end{aligned}
\tag{4.3.6}
$$

are the associated eigenfunctions, real valued and orthogonal to each other.

For $\xi \neq 0$, since $\omega(n+\xi,+)$ increases in $n+\xi$ for any $n \in \mathbb{Z}$ and $\xi \in (0,1/2]$, and since $\omega(n+\xi,-)$ decreases in $n+\xi$ if $-1/2 < n+\xi < 1/2$ and increases if $n+\xi < -1$ or $n+\xi > 1$ by brutal force, it follows that

$$\omega(1/2,-) \leqslant \omega(0+\xi,\pm), \omega(\pm 1+\xi,-) \leqslant \omega(1/2,+),$$

and

$$
\begin{aligned}
&\cdots < \omega(-2+\xi,-) < \omega(\xi,-) < 0 < \omega(-1+\xi,-) < \omega(1+\xi,-) < \omega(2+\xi,-) < \cdots, \\
&\cdots < \omega(-2+\xi,+) < \omega(-1+\xi,+) < 0 < \omega(\xi,+) < \omega(1+\xi,+) < \omega(2+\xi,+) < \cdots.
\end{aligned}
$$

Hence $\omega(n+\xi,\pm) \neq 0$ for any $n \in \mathbb{Z}$ and $\xi \in (0,1/2]$. But in [HP16a], we observe infinitely many collisions of purely imaginary eigenvalues of $\mathcal{L}(\xi,0)$ away from the origin. To compare, no eigenvalues of the linear operator for the Whitham equation (see (1)) collide other than at the origin; see [HJ15a], for instance.

Continuing, for $\xi > 0$ and sufficiently small, $i\omega(\xi,\pm)$ and $i\omega(\pm 1+\xi,-)$ are $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-

eigenvalues of $\mathcal{L}(\xi,0)$ in the vicinity of the origin in $\mathbb{C}$. Moreover, (by abuse of notation)

$$\begin{aligned}
\boldsymbol{\phi}_1(z) :=& \frac{1}{2}\sqrt{c_{\rm ww}^2(k)+1}\left(\frac{\mathbf{e}(1+\xi,-)}{\|\mathbf{e}(1+\xi,-)\|}+\frac{\mathbf{e}(-1+\xi,-)}{\|\mathbf{e}(-1+\xi,-)\|}\right)(z) \\
=& \begin{pmatrix} 1 \\ c_{\rm ww}(k)\end{pmatrix}\cos z + i\xi\frac{kc'_{\rm ww}(k)}{c_{\rm ww}^2(k)+1}\begin{pmatrix} -c_{\rm ww}(k) \\ 1\end{pmatrix}\sin z + \xi^2\mathbf{p}_2\cos z + O(\xi^3), \\
\boldsymbol{\phi}_2(z) :=& \frac{1}{2i}\sqrt{c_{\rm ww}^2(k)+1}\left(\frac{\mathbf{e}(1+\xi,-)}{\|\mathbf{e}(1+\xi,-)\|}-\frac{\mathbf{e}(-1+\xi,-)}{\|\mathbf{e}(-1+\xi,-)\|}\right)(z) \\
=& \begin{pmatrix} 1 \\ c_{\rm ww}(k)\end{pmatrix}\sin z - i\xi\frac{kc'_{\rm ww}(k)}{c_{\rm ww}^2(k)+1}\begin{pmatrix} -c_{\rm ww}(k) \\ 1\end{pmatrix}\cos z + \xi^2\mathbf{p}_2\sin z + O(\xi^3), \\
\boldsymbol{\phi}_3(z) :=& \frac{1}{2}((c_{\rm ww}(k)+2)\mathbf{e}(\xi,+)-(c_{\rm ww}(k)-2)\mathbf{e}(\xi,-))(z) \\
=& \begin{pmatrix} 2 \\ -c_{\rm ww}(k)\end{pmatrix} + \frac{1}{6}\xi^2k^2c_{\rm ww}(k)\begin{pmatrix} 0 \\ 1\end{pmatrix} + O(\xi^3), \\
\boldsymbol{\phi}_4(z) :=& \frac{1}{2}((c_{\rm ww}(k)-2)\mathbf{e}(\xi,+)+(c_{\rm ww}(k)+2)\mathbf{e}(\xi,-))(z) \\
=& \begin{pmatrix} c_{\rm ww}(k) \\ 2\end{pmatrix} - \frac{1}{3}\xi^2k^2\begin{pmatrix} 0 \\ 1\end{pmatrix} + O(\xi^3)
\end{aligned} \tag{4.3.7}$$

span the associated eigenspace, orthogonal to each other, where $\|\cdot\| = \sqrt{\langle\cdot\,,\cdot\rangle_{L^2(\mathbb{T})\times L^2(\mathbb{T})}}$ and

$$\mathbf{p}_2 = \frac{1}{2}\frac{k^2}{c_{\rm ww}^2(k)+1}\begin{pmatrix} c'_{\rm ww}(k)^2\dfrac{2c_{\rm ww}^2(k)-1}{c_{\rm ww}^2(k)+1} - (c_{\rm ww}c''_{\rm ww})(k) \\ -3\dfrac{(c_{\rm ww}(c'_{\rm ww})^2)(k)}{c_{\rm ww}^2(k)+1} + c''_{\rm ww}(k)\end{pmatrix}. \tag{4.3.8}$$

For $\xi=0$, note that $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_3$, $\boldsymbol{\phi}_4$ become (4.3.6). Recall that $c_{\rm ww}$ is a real analytic function. Hence they depend analytically on $\xi \in [0,1/2]$.

Note that $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ vary with $\xi > 0$ and sufficiently small to the linear order. In the following subsection, we take this into account and construct an eigenspace for ξ, $a \neq 0$ and sufficiently small, which varies analytically with ξ and a; see (4.3.10) for details. Consequently, the spectral perturbation calculation in Section 4.4 becomes lengthy and complicated. To compare, the eigenfunctions of the linear operator for the Whitham equation do not depend on ξ for any $a \in \mathbb{R}$ and admissible; see [HJ15a], for instance, for details.

Note that $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are complex valued. For real valued functions, one must take $\pm\xi$ in pair and deal with six functions. But the spectral perturbation calculation in Section 4.4 involves complex valued operators anyway. Hence this is not worth the effort.

We turn the attention to the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-spectra of $\mathcal{L}(\xi, a)$ in the vicinity of the origin in $\mathbb{C}$, for $\xi \in [0, 1/2]$ for $a \in \mathbb{R}$ and $|a|$ sufficiently small.

Note from (4.3.2) and (4.2.5) that

$$\begin{aligned}\mathcal{L}(\xi, a) =& e^{-i\xi z}\partial_z \begin{pmatrix} c-u & -1-\eta \\ -c_{\rm ww}^2(k|\partial_z|) & c-u \end{pmatrix} e^{i\xi z} \\ =& e^{-i\xi z}\partial_z \left(\begin{pmatrix} c_{\rm ww}(k) & -1 \\ -c_{\rm ww}^2(k|\partial_z|) & c_{\rm ww}(k) \end{pmatrix} + a \begin{pmatrix} -c_{\rm ww}(k) & -1 \\ 0 & -c_{\rm ww}(k) \end{pmatrix} \cos z \right) e^{i\xi z} + O(a^2)\end{aligned}$$

as $a \to 0$, whence

$$\|\mathcal{L}(\xi, a) - \mathcal{L}(\xi, 0)\|_{L^2(\mathbb{T})\times L^2(\mathbb{T}) \to L^2(\mathbb{T})\times L^2(\mathbb{T})} = O(a)$$

as $a \to 0$ uniformly for $\xi \in [0, 1/2]$. Recall that the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-spectrum of $\mathcal{L}(\xi, 0)$ contains four purely imaginary eigenvalues $i\omega(\xi, \pm)$, $i\omega(\pm 1 + \xi, -)$ in the vicinity of the origin in $\mathbb{C}$ for $\xi > 0$ and sufficiently small. Since $\mathcal{L}(\xi, a)$ depends analytically on $\xi \in [0, 1/2]$ and $a \in \mathbb{R}$ admissible, it follows from perturbation theory (see [Kat76, Section 4.3.1], for instance, for details) that the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-spectrum of $\mathcal{L}(\xi, a)$ contains four eigenvalues, denoted

$$\lambda_1(\xi, a), \lambda_2(\xi, a), \lambda_3(\xi, a), \lambda_4(\xi, a),$$

near the origin for $\xi > 0$, $a \in \mathbb{R}$ and ξ, $|a|$ sufficiently small.

Moreover, a straightforward calculation reveals that

$$|\lambda_j(\xi, 0) - \lambda_\ell(\xi, 0)| \geqslant \omega_0 > 0 \qquad \text{for } j, \ell = 1, 2, 3, 4 \text{ and } j \neq \ell$$

for any $\xi \geqslant \xi_0 > 0$ for any ξ_0 for some ω_0. Hence it follows from perturbation theory that λ_1, λ_2, λ_3, λ_4 remain purely imaginary for any $\xi \geqslant \xi_0 > 0$ for any ξ_0, for any $a \in \mathbb{R}$ and $|a|$ sufficiently small. In particular, a sufficiently small, periodic wave train of (4.1.3)-(4.1.2) is spectrally stable to "short wavelength perturbations" in the vicinity of the origin in $\mathbb{C}$. For $\xi = 0$, on the other hand, we demonstrate that four eigenvalues collide at the origin.

Lemma 4.3.1 (Spectrum of $\mathcal{L}(0, a)$)**.** *For $a \in \mathbb{R}$ and $|a|$ sufficiently small, zero is an $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-eigenvalue of $\mathcal{L}(0, a)$ with algebraic multiplicity four and geometric multiplicity* *three. Moreover, (by abuse of notation)*

$$\begin{aligned}
\boldsymbol{\phi}_1(z) :=& \frac{2}{c_{\mathrm{ww}}^2(k)+2}((\partial_{b_1}c)(\partial_a\mathbf{u}) - (\partial_a c)(\partial_{b_1}\mathbf{u}))(z) \qquad (4.3.9a)\\
=& \begin{pmatrix} 1 \\ c_{\mathrm{ww}}(k) \end{pmatrix}\cos z + a\begin{pmatrix} -3h_2\dfrac{c_{\mathrm{ww}}^2(k)}{c_{\mathrm{ww}}^2(k)+2} \\ c_{\mathrm{ww}}(k)\Big(\frac{1}{2} - 3h_2\dfrac{1}{c_{\mathrm{ww}}^2(k)+2}\Big) \end{pmatrix} \\
&+ 2a\begin{pmatrix} h_2 \\ c_{\mathrm{ww}}(k)\left(h_2 - \frac{1}{2}\right) \end{pmatrix}\cos 2z + O(a^2),\\
\boldsymbol{\phi}_2(z) :=& -\frac{1}{a}(\partial_z\mathbf{u})(z) \qquad (4.3.9b)\\
=& \begin{pmatrix} 1 \\ c_{\mathrm{ww}}(k) \end{pmatrix}\sin z + 2a\begin{pmatrix} h_2 \\ c_{\mathrm{ww}}(k)\left(h_2 - \frac{1}{2}\right) \end{pmatrix}\sin 2z + O(a^2),
\end{aligned}$$

where h_2 *is in* (4.2.7), *and*

$$\begin{aligned}
\boldsymbol{\phi}_3(z) :=& \frac{2}{c_{\mathrm{ww}}^2(k)-1}((\partial_{b_1}c)(\partial_{b_2}\mathbf{u}) - (\partial_{b_2}c)(\partial_{b_1}\mathbf{u}))(z) \qquad (4.3.9c)\\
=& \begin{pmatrix} 2 \\ -c_{\mathrm{ww}}(k) \end{pmatrix} + a\begin{pmatrix} 2 \\ c_{\mathrm{ww}}(k) \end{pmatrix}\cos z + O(a^2),\\
\boldsymbol{\phi}_4(z) :=& \frac{1}{3}\frac{c_{\mathrm{ww}}^2(k)+4}{c_{\mathrm{ww}}(k)}\Big((\partial_{b_2}\mathbf{u}) + \frac{c_{\mathrm{ww}}^2(k)-2}{c_{\mathrm{ww}}^2(k)+4}\boldsymbol{\phi}_{3,a}\Big)(z) \qquad (4.3.9d)\\
=& \begin{pmatrix} c_{\mathrm{ww}}(k) \\ 2 \end{pmatrix} + a\begin{pmatrix} c_{\mathrm{ww}}(k) \\ \frac{1}{2}c_{\mathrm{ww}}^2(k) \end{pmatrix}\cos z + O(a^2),
\end{aligned}$$

are the associated eigenfunctions. Specifically,

$$\mathcal{L}(0,a)\boldsymbol{\phi}_k = 0 \quad \textit{for } k=1,2,3, \quad \textit{and} \quad \mathcal{L}(0,a)\boldsymbol{\phi}_4 = \frac{1}{4}a(c_{\mathrm{ww}}^2(k)+1)\boldsymbol{\phi}_2.$$

For $a=0$, note that $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_3$, $\boldsymbol{\phi}_4$ becomes (4.3.6). Theorem 4.2.1 implies that they depend analytically on $a \in \mathbb{R}$ and $|a|$ sufficiently small.

Proof. Exploiting variations of (4.2.1) in the z, and a, b_1, b_2 variables, the proof is similar to that of [HJ15a, Lemma 3.1], for instance.

Differentiating (4.2.1) with respect to z and evaluating the result at $b_1 = b_2 = 0$, we infer from (4.3.1) that

$$\mathcal{L}(0,a)(\partial_z\mathbf{u}) = 0.$$

Hence zero is an eigenvalue of $\mathcal{L}(0,a)$ and $\partial_z\mathbf{u}$ is an associated eigenfunction. We then use

(4.2.5a) and (4.2.5b) to find (4.7). By the way, this is reminiscent of that (4.2.1) remains invariant under spatial translations.

Differentiating (4.2.1) with respect to a, and b_1, b_2, and evaluating at $b_1 = b_2 = 0$, we infer from (4.3.1) that

$$\mathcal{L}(0,a)(\partial_a \mathbf{u}) = -(\partial_a c)(\partial_z \mathbf{u}),$$

and

$$\mathcal{L}(0,a)(\partial_{b_1} \mathbf{u}) = -(\partial_{b_1} c)(\partial_z \mathbf{u}), \qquad \mathcal{L}(0,a)(\partial_{b_2} \mathbf{u}) = -(\partial_{b_2} c)(\partial_z \mathbf{u}).$$

Hence

$$\mathcal{L}(0,a)((\partial_{b_1} c)(\partial_a \mathbf{u}) - (\partial_a c)(\partial_{b_1} \mathbf{u})) = 0 \quad \text{and} \quad \mathcal{L}(0,a)((\partial_{b_1} c)(\partial_{b_2} \mathbf{u}) - (\partial_{b_2} c)(\partial_{b_1} \mathbf{u})) = 0.$$

We then use (4.2.5) and (4.2.6) to find (4.7.5) and (4.3.9c). Note that $\partial_{b_2} \mathbf{u}$ is a generalized eigenfunction. We use (4.2.5) and (4.2.6) to find (4.3.9d). This completes the proof. □

To recapitulate, for $\xi > 0$ and sufficiently small for $a = 0$, the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-spectrum of $\mathcal{L}(\xi, 0)$ contains four purely imaginary eigenvalues $i\omega(\xi, \pm)$, $i\omega(\pm 1 + \xi, -)$ in the vicinity of the origin in $\mathbb{C}$, and (4.3.7) spans the associated eigenspace, which depends analytically on ξ. For $\xi = 0$ for $a \in \mathbb{R}$ and $|a|$ sufficiently small, the spectrum of $\mathcal{L}(0, a)$ contains four eigenvalues at the origin, and (4.3.9) makes the associated eigenfunctions, which depends analytically on a.

For $\xi > 0$, $a \in \mathbb{R}$ and ξ, $|a|$ sufficiently small, the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-spectrum of $\mathcal{L}(\xi, a)$ contains four eigenvalues $\lambda_1(\xi, a)$, $\lambda_2(\xi, a)$, $\lambda_3(\xi, a)$, $\lambda_4(\xi, a)$ near the origin, and the associated eigenfunctions vary analytically from (4.3.7) and (4.3.9). Let (by abuse of notation)

$$
\begin{aligned}
\boldsymbol{\phi}_1(\xi,a)(z) ={}& \begin{pmatrix} 1 \\ c_{\rm ww}(k) \end{pmatrix} \cos z + i\xi \frac{kc'_{\rm ww}(k)}{c^2_{\rm ww}(k)+1} \begin{pmatrix} -c_{\rm ww}(k) \\ 1 \end{pmatrix} \sin z \\
&+ a \begin{pmatrix} -3h_2 \dfrac{c^2_{\rm ww}(k)}{c^2_{\rm ww}(k)+2} \\ c_{\rm ww}(k)\left(\frac{1}{2} - 3h_2 \dfrac{1}{c^2_{\rm ww}(k)+2}\right) \end{pmatrix} + 2a \begin{pmatrix} h_2 \\ c_{\rm ww}(k)\left(h_2 - \frac{1}{2}\right) \end{pmatrix} \cos 2z \\
&+ \xi^2 \mathbf{p}_2 \cos z + O(\xi^3 + \xi^2 a + a^2), \\
\boldsymbol{\phi}_2(\xi,a)(z) ={}& \begin{pmatrix} 1 \\ c_{\rm ww}(k) \end{pmatrix} \sin z - i\xi \frac{kc'_{\rm ww}(k)}{c^2_{\rm ww}(k)+1} \begin{pmatrix} -c_{\rm ww}(k) \\ 1 \end{pmatrix} \cos z \\
&+ 2a \begin{pmatrix} h_2 \\ c_{\rm ww}(k)\left(h_2 - \frac{1}{2}\right) \end{pmatrix} \sin 2z + \xi^2 \mathbf{p}_2 \sin z + O(\xi^3 + \xi^2 a + a^2), \\
\boldsymbol{\phi}_3(\xi,a)(z) ={}& \begin{pmatrix} 2 \\ -c_{\rm ww}(k) \end{pmatrix} + a \begin{pmatrix} 2 \\ c_{\rm ww}(k) \end{pmatrix} \cos z + \frac{1}{6}\xi^2 k^2 c_{\rm ww}(k) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(\xi^3 + \xi^2 a + a^2), \\
\boldsymbol{\phi}_4(\xi,a)(z) ={}& \begin{pmatrix} c_{\rm ww}(k) \\ 2 \end{pmatrix} + a \begin{pmatrix} c_{\rm ww}(k) \\ \frac{1}{2}c^2_{\rm ww}(k) \end{pmatrix} \cos z - \frac{1}{3}\xi^2 k^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(\xi^3 + \xi^2 a + a^2)
\end{aligned}
\tag{4.3.10}
$$

as ξ, $a \to 0$, where h_2 is in (4.2.7) and $\mathbf{p}_2$ is in (4.3.8). For $a = 0$, note that $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_3$, $\boldsymbol{\phi}_4$ become (4.3.7). For $\xi = 0$, they become (4.3.9). Hence $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_3$, $\boldsymbol{\phi}_4$ span the eigenspace associated with λ_1, λ_2, λ_3, λ_4 up to terms of orders ξ^2 and a as ξ, $a \to 0$.

It seems impossible to uniquely determine terms of orders ξa and higher in the eigenfunction expansion without ad hoc orthogonality conditions. Fortuitously, it turns out that they do not contribute to the modulational instability. Hence we may neglect them in (4.3.10). To compare, the eigenfunctions of the linear operator for the Whitham equation, which do not depend on ξ, extend to $a \neq 0$; see [HJ15a], for instance, for details. We are able to calculate terms of orders a^2 and higher in the eigenfunction expansion. But the index formulae become unwieldy. Hence we do not use them in the calculation in the following subsection.

4.4 The perturbation analysis

Recall that for $\xi > 0$, $a \in \mathbb{R}$ and ξ, $|a|$ sufficiently small, the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-spectrum of $\mathcal{L}(\xi,a)$ contains four eigenvalues $\lambda_1(\xi,a)$, $\lambda_2(\xi,a)$, $\lambda_3(\xi,a)$, $\lambda_4(\xi,a)$ in the vicinity of the origin in $\mathbb{C}$, and (4.3.10) spans the associated eigenspace up to terms of orders ξ^2 and a. Let

$$\mathbf{L}(\xi,a)=\left(\frac{\langle\mathcal{L}(\xi,a)\boldsymbol{\phi}_j(\xi,a),\boldsymbol{\phi}_\ell(\xi,a)\rangle}{\langle\boldsymbol{\phi}_j(\xi,a),\boldsymbol{\phi}_j(\xi,a)\rangle}\right)_{j,\ell=1,2,3,4} \tag{4.4.1}$$

and

$$\mathbf{I}(\xi,a)=\left(\frac{\langle\boldsymbol{\phi}_j(\xi,a),\boldsymbol{\phi}_\ell(\xi,a)\rangle}{\langle\boldsymbol{\phi}_j(\xi,a),\boldsymbol{\phi}_j(\xi,a)\rangle}\right)_{j,\ell=1,2,3,4}, \tag{4.4.2}$$

where $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_3$, $\boldsymbol{\phi}_4$ are in (4.3.10). Throughout the subsection, $\langle\,,\rangle$ means the $L^2(\mathbb{T})\times L^2(\mathbb{T})$-inner product. Note that $\mathbf{L}$ represents the action of $\mathcal{L}(\xi,a)$ on the eigenspace associated with λ_1, λ_2, λ_3, λ_4, up to the orders of ξ^2 and a as ξ, $a\to0$, after normalization, and $\mathbf{I}$ is the projection of the identity onto the eigenspace. It follows from perturbation theory (see [Kat76, Section 4.3.5], for instance, for details) that for $\xi>0$, $a\in\mathbb{R}$ and ξ, $|a|$ sufficiently small, the roots of $\det(\mathbf{L}-\lambda\mathbf{I})$ coincide with the eigenvalues of $\mathcal{L}(\xi,a)$ up to terms of orders ξ^2 and a.

For any $a\in\mathbb{R}$ and $|a|$ sufficiently small, we make a Baker-Campbell-Hausdorff expansion to write

$$\mathcal{L}(\xi,a)=\mathcal{L}_0+i\xi\mathcal{L}_1-\frac{1}{2}\xi^2\mathcal{L}_2+O(\xi^3)$$

as $\xi\to0$, where

$$\begin{aligned}
\mathcal{L}_0=&\mathcal{L}(0,a)=\partial_z\begin{pmatrix}c_{\text{ww}}(k) & -1\\ -c_{\text{ww}}^2(k|\partial_z|) & c_{\text{ww}}(k)\end{pmatrix}-a\partial_z\begin{pmatrix}c_{\text{ww}}(k) & 1\\ 0 & c_{\text{ww}}(k)\end{pmatrix}\cos z+O(a^2),\\
\mathcal{L}_1=&[\mathcal{L}_0,z]=\begin{pmatrix}c_{\text{ww}}(k) & -1\\ -[\partial_z c_{\text{ww}}^2(k|\partial_z|),z] & c_{\text{ww}}(k)\end{pmatrix}-a\begin{pmatrix}c_{\text{ww}}(k) & 1\\ 0 & c_{\text{ww}}(k)\end{pmatrix}\cos z+O(a^2),\\
\mathcal{L}_2=&[\mathcal{L}_1,z]=\begin{pmatrix}0 & 0\\ -[[\partial_z c_{\text{ww}}^2(k|\partial_z|),z],z] & 0\end{pmatrix}+O(a^2)
\end{aligned}$$

as $a\to0$, and $[\cdot,\,\cdot]$ means the commutator. The latter equalities follow from (4.3.1), (4.3.2), (4.2.5) and that $\mathcal{L}(\xi,a)$ depends analytically on ξ near $\xi=0$. We merely pause to remark that $\mathcal{L}_1$ and $\mathcal{L}_2$ are well defined in the periodic setting even though z is not. Indeed, $[\partial_z c_{\text{ww}}^2(k|\partial_z|),z]=c_{\text{ww}}^2(k|\partial_z|)+[c_{\text{ww}}^2(k|\partial_z|),z]\partial_z$ and

$$[c_{\text{ww}}^2(k|\partial_z|),z]e^{inz}=ie^{inz}\sum_{m\neq0}\frac{(-1)^{|m|+1}}{m}(-c_{\text{ww}}^2(kn)-c_{\text{ww}}^2(k(n+m)))e^{imz}\quad\text{for } n\in\mathbb{Z}$$

by brutal force. One may likewise represent $[[\partial_z c_{\text{ww}}^2(k|\partial_z|),z],z]$ in the Fourier series. Unfortunately, this is not convenient for an explicit calculation. We instead rearrange the above as

$$
\begin{aligned}
\mathcal{L}(\xi,a) =&\mathcal{L}(0,0)+i\xi[\mathcal{L}(0,0),z]-\frac{\xi^2}{2}[[\mathcal{L}(0,0),z],z]\\
&-a\partial_z\begin{pmatrix} c_{\rm ww}(k) & 1\\ 0 & c_{\rm ww}(k)\end{pmatrix}\cos z-i\xi a\begin{pmatrix} c_{\rm ww}(k) & 1\\ 0 & c_{\rm ww}(k)\end{pmatrix}\cos z+O(\xi^3+\xi^2a+a^2)\\
=:&\,\mathcal{M}-a\partial_z\begin{pmatrix} c(k) & 1\\ 0 & c(k)\end{pmatrix}\cos z-i\xi a\begin{pmatrix} c_{\rm ww}(k) & 1\\ 0 & c_{\rm ww}(k)\end{pmatrix}\cos z+O(\xi^3+\xi^2a+a^2)
\end{aligned}
\tag{4.4.3}
$$

as ξ, $a\to 0$, and note that $\mathcal{M}$ agrees with $\mathcal{L}(\xi,0)$ up to terms of order ξ^2 for $\xi>0$ and sufficiently small. We then resort to (4.2.3) and make an explicit calculation to find

$$
\begin{aligned}
\mathcal{L}(\xi,0)\begin{pmatrix}\zeta\\ v\end{pmatrix}e^{inz} =&in\begin{pmatrix} c_{\rm ww}(k)\zeta-v\\ -c_{\rm ww}^2(kn)\zeta+c_{\rm ww}(k)v\end{pmatrix}e^{inz}\\
&+i\xi\begin{pmatrix} c_{\rm ww}(k)\zeta-v\\ -\left(c_{\rm ww}^2(kn)+kn(c_{\rm ww}^2)'(kn)\right)\zeta+c_{\rm ww}(k)v\end{pmatrix}e^{inz}\\
&-i\xi^2\,kn\Big((c_{\rm ww}^2)'(kn)+\frac{1}{2}kn(c_{\rm ww}^2)''(kn)\Big)\zeta\begin{pmatrix}0\\1\end{pmatrix}e^{inz}+O(\xi^3)\\
=&\mathcal{M}\begin{pmatrix}\zeta\\ v\end{pmatrix}e^{inz}+O(\xi^3)
\end{aligned}
$$

as $\xi\to 0$, for any constants ζ, v and $n\in\mathbb{Z}$. For instance, since $c_{\rm ww}^2(0)=1$ and $(c_{\rm ww}^2)'(0)=0$, it follows that

$$
\mathcal{M}\begin{pmatrix}\zeta\\ v\end{pmatrix}=i\xi\begin{pmatrix} c_{\rm ww}(k)\zeta-v\\ c_{\rm ww}(k)v-\zeta\end{pmatrix}.
$$

One may likewise calculate $\mathcal{M}\begin{pmatrix}\zeta\\ v\end{pmatrix}\begin{Bmatrix}\cos nz\\ \sin nz\end{Bmatrix}$ explicitly up to the order of ξ^2. We omit the details.

We use (4.4.3), (4.3.10), and the above formula for $\mathcal{M}$, and we make a lengthy and complicated, but explicit, calculation to show that

$$
\begin{aligned}
\boldsymbol{\mathcal{L}\phi}_1 =& -2i\xi\, k(c_{\text{ww}}c'_{\text{ww}})(k)\begin{pmatrix}0\\1\end{pmatrix}\cos z + i\xi\, kc'_{\text{ww}}(k)\begin{pmatrix}-1\\c_{\text{ww}}(k)\end{pmatrix}\cos z\\
&-\frac{1}{2}i\xi a\, c_{\text{ww}}(k)\begin{pmatrix}2\\c_{\text{ww}}(k)\end{pmatrix}(\cos 2z+1) + i\xi a\,\frac{kc'_{\text{ww}}(k)}{c^2_{\text{ww}}(k)+1}\begin{pmatrix}c^2_{\text{ww}}(k)-1\\-c_{\text{ww}}(k)\end{pmatrix}\cos 2z\\
&-\frac{1}{2}i\xi a\, c_{\text{ww}}(k)\begin{pmatrix}6h_2\dfrac{c^2_{\text{ww}}(k)-1}{c^2_{\text{ww}}(k)+2}+1\\-c_{\text{ww}}(k)\end{pmatrix}\\
&+\frac{1}{2}i\xi\, ac_{\text{ww}}(k)\begin{pmatrix}2\\c_{\text{ww}}(k)\Big(1-12k\dfrac{(c_{\text{ww}}c'_{\text{ww}})(2k)}{c^2_{\text{ww}}(k)-c^2_{\text{ww}}(2k)}\Big)\end{pmatrix}\cos 2z\\
&+\xi^2\, k(2(c_{\text{ww}}c'_{\text{ww}})(k)+k((c'_{\text{ww}})^2+c_{\text{ww}}c''_{\text{ww}})(k))\begin{pmatrix}0\\1\end{pmatrix}\sin z\\
&-\xi^2\, kc_{\text{ww}}(k)\begin{pmatrix}-1\\c_{\text{ww}}(k)\Big(1+2k\dfrac{(c_{\text{ww}}c'_{\text{ww}})(k)}{c^2_{\text{ww}}(k)+1}\Big)\end{pmatrix}\sin z\\
&+\frac{1}{2}\xi^2\, k^2\Big(2\frac{(c_{\text{ww}}(c'_{\text{ww}})^2)(k)}{c^2_{\text{ww}}(k)+1}-c''_{\text{ww}}(k)\Big)\begin{pmatrix}-1\\c_{\text{ww}}(k)\end{pmatrix}\sin z + O(\xi^3+\xi^2 a+a^2)
\end{aligned}
$$

as ξ, $a\to 0$, where h_2 is in (4.2.7). Moreover,

$$
\begin{aligned}
\boldsymbol{\mathcal{L}\phi}_2 =& -2i\xi\, k(c_{\text{ww}}c'_{\text{ww}})(k)\begin{pmatrix}0\\1\end{pmatrix}\sin z + i\xi\, kc'_{\text{ww}}(k)\begin{pmatrix}-1\\c_{\text{ww}}(k)\end{pmatrix}\sin z\\
&-\frac{1}{2}i\xi a\, c_{\text{ww}}(k)\begin{pmatrix}2\\c_{\text{ww}}(k)\end{pmatrix}\sin 2z + i\xi a\,\frac{kc'_{\text{ww}}(k)}{c^2_{\text{ww}}(k)+1}\begin{pmatrix}c^2_{\text{ww}}(k)-1\\-c_{\text{ww}}(k)\end{pmatrix}\sin 2z\\
&+\frac{1}{2}i\xi a\, c_{\text{ww}}(k)\begin{pmatrix}2\\c_{\text{ww}}(k)\Big(1-12k\dfrac{(c_{\text{ww}}c'_{\text{ww}})(2k)}{c^2_{\text{ww}}(k)-c^2_{\text{ww}}(2k)}\Big)\end{pmatrix}\sin 2z\\
&-\xi^2\, k(2c_{\text{ww}}c'_{\text{ww}}+k((c'_{\text{ww}})^2+c_{\text{ww}}c''_{\text{ww}}))(k)\begin{pmatrix}0\\1\end{pmatrix}\cos z\\
&+\xi^2\, kc_{\text{ww}}(k)\begin{pmatrix}-1\\c_{\text{ww}}(k)\Big(1+2k\dfrac{(c_{\text{ww}}c'_{\text{ww}})(k)}{c^2_{\text{ww}}(k)+1}\Big)\end{pmatrix}\cos z\\
&-\frac{1}{2}\xi^2\, k^2\Big(2\frac{(c_{\text{ww}}(c'_{\text{ww}})^2)(k)}{c^2_{\text{ww}}(k)+1}-c''_{\text{ww}}(k)\Big)\begin{pmatrix}-1\\c_{\text{ww}}(k)\end{pmatrix}\cos z + O(\xi^3+\xi^2 a+a^2),
\end{aligned}
$$

and

$$\begin{aligned}
\mathcal{L}\boldsymbol{\phi}_3 =& i\xi \begin{pmatrix} 3c_{\rm ww}(k) \\ -c_{\rm ww}^2(k)-2 \end{pmatrix} - 2i\xi a\, k(c_{\rm ww}c_{\rm ww}')(k) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos z + O(\xi^3+\xi^2 a+a^2), \\
\mathcal{L}\boldsymbol{\phi}_4 =& i\xi \begin{pmatrix} c_{\rm ww}^2(k)-2 \\ c_{\rm ww}(k) \end{pmatrix} + \frac{1}{2}a(c_{\rm ww}^2(k)+4) \begin{pmatrix} 1 \\ c_{\rm ww}(k) \end{pmatrix} \sin z \\
&- \frac{1}{2}i\xi a \begin{pmatrix} c_{\rm ww}^2(k)+4 \\ c_{\rm ww}(k)(c_{\rm ww}^2(k)+4+2k(c_{\rm ww}c_{\rm ww}')(k)) \end{pmatrix} \cos z + O(\xi^3+\xi^2 a+a^2)
\end{aligned}$$

as ξ, $a \to 0$.

To proceed, we take the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-inner products of the above and (4.3.10), and we make a lengthy and complicated, but explicit, calculation to show that

$$\begin{aligned}
\langle \mathcal{L}\boldsymbol{\phi}_1, \boldsymbol{\phi}_1 \rangle =& \langle \mathcal{L}\boldsymbol{\phi}_2, \boldsymbol{\phi}_2 \rangle \\
=& -\frac{1}{2}i\xi\, kc_{\rm ww}'(k)(c_{\rm ww}^2(k)+1) + O(\xi^3+\xi^2 a+a^2), \\
\langle \mathcal{L}\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \rangle =& -\langle \mathcal{L}\boldsymbol{\phi}_2, \boldsymbol{\phi}_1 \rangle \\
=& \frac{1}{2}\xi^2 \Big(kc_{\rm ww}'(k) + \frac{1}{2}k^2 c_{\rm ww}''(k) \Big) (c_{\rm ww}^2(k)+1) + O(\xi^3+\xi^2 a+a^2), \\
\langle \mathcal{L}\boldsymbol{\phi}_1, \boldsymbol{\phi}_3 \rangle =& \frac{2}{c_{\rm ww}(k)} \langle \mathcal{L}\boldsymbol{\phi}_1, \boldsymbol{\phi}_4 \rangle \\
=& -3i\xi a \Big(2h_2 c(k) \frac{c_{\rm ww}^2(k)-1}{c_{\rm ww}^2(k)+2} + c_{\rm ww}(k) + \frac{1}{6}kc_{\rm ww}'(k)(c_{\rm ww}^2(k)+2) \Big) \\
&+ O(\xi^3+\xi^2 a+a^2)
\end{aligned}$$

as ξ, $a \to 0$, where h_2 is in (4.2.7). Moreover,

$$\begin{aligned}
\langle \mathcal{L}\boldsymbol{\phi}_2, \boldsymbol{\phi}_3 \rangle =& \langle \mathcal{L}\boldsymbol{\phi}_2, \boldsymbol{\phi}_4 \rangle = 0 + O(\xi^3+\xi^2 a+a^2), \\
\langle \mathcal{L}\boldsymbol{\phi}_3, \boldsymbol{\phi}_1 \rangle =& -i\xi a\, c_{\rm ww}(k) \Big(6h_2 \frac{c_{\rm ww}^2(k)+1}{c_{\rm ww}^2(k)+2} + \frac{1}{2}(c_{\rm ww}^2(k)+2) + 2k(c_{\rm ww}c_{\rm ww}')(k) \Big) \\
&+ O(\xi^3+\xi^2 a+a^2), \\
\langle \mathcal{L}\boldsymbol{\phi}_3, \boldsymbol{\phi}_2 \rangle =& 0 + O(\xi^3+\xi^2 a+a^2), \\
\langle \mathcal{L}\boldsymbol{\phi}_3, \boldsymbol{\phi}_3 \rangle =& i\xi c_{\rm ww}(k)(c_{\rm ww}^2(k)+8) + O(\xi^3+\xi^2 a+a^2), \\
\langle \mathcal{L}\boldsymbol{\phi}_3, \boldsymbol{\phi}_4 \rangle =& \langle \mathcal{L}\boldsymbol{\phi}_4, \boldsymbol{\phi}_3 \rangle = i\xi(c_{\rm ww}^2(k)-4) + O(\xi^3+\xi^2 a+a^2),
\end{aligned}$$

and

$$
\begin{aligned}
\langle \mathcal{L}\boldsymbol{\phi}_4, \boldsymbol{\phi}_1\rangle =& -i\xi a\Big(\frac{1}{4}(c_{\mathrm{ww}}^4+3c_{\mathrm{ww}}^2)(k)+1+3h_2c_{\mathrm{ww}}^2(k)\frac{c_{\mathrm{ww}}^2(k)-1}{c_{\mathrm{ww}}^2(k)+2}+\frac{1}{2}k(c_{\mathrm{ww}}^3c_{\mathrm{ww}}')(k)\Big)\\
&+O(\xi^3+\xi^2a+a^2),\\
\langle \mathcal{L}\boldsymbol{\phi}_4, \boldsymbol{\phi}_2\rangle =& \frac{1}{4}a(c_{\mathrm{ww}}^2(k)+4)(c_{\mathrm{ww}}^2(k)+1)+O(\xi^3+\xi^2a+a^2),\\
\langle \mathcal{L}\boldsymbol{\phi}_4, \boldsymbol{\phi}_4\rangle =& i\xi c_{\mathrm{ww}}^3(k)+O(\xi^3+\xi^2a+a^2)
\end{aligned}
$$

as ξ, $a\to 0$, where h_2 is in (4.2.7).

Continuing, we take the $L^2(\mathbb{T})\times L^2(\mathbb{T})$-inner products of (4.3.10) and we make an explicit calculation to show that

$$
\begin{aligned}
\langle \boldsymbol{\phi}_1, \boldsymbol{\phi}_1\rangle =& \langle \boldsymbol{\phi}_2, \boldsymbol{\phi}_2\rangle = \frac{1}{2}(c_{\mathrm{ww}}^2(k)+1)-\frac{3}{4}\xi^2k^2\frac{c_{\mathrm{ww}}'(k)^2}{c_{\mathrm{ww}}^2(k)+1}+O(\xi^3+\xi^2a+a^2),\\
\langle \boldsymbol{\phi}_1, \boldsymbol{\phi}_2\rangle =& \langle \boldsymbol{\phi}_2, \boldsymbol{\phi}_1\rangle = 0+O(\xi^3+\xi^2a+a^2),\\
\langle \boldsymbol{\phi}_1, \boldsymbol{\phi}_3\rangle =& \langle \boldsymbol{\phi}_3, \boldsymbol{\phi}_1\rangle = a\Big(1-3h_2\frac{c_{\mathrm{ww}}^2(k)}{c_{\mathrm{ww}}^2(k)+2}\Big)+O(\xi^3+\xi^2a+a^2),\\
\langle \boldsymbol{\phi}_1, \boldsymbol{\phi}_4\rangle =& \langle \boldsymbol{\phi}_4, \boldsymbol{\phi}_1\rangle = \frac{1}{4}a\,c_{\mathrm{ww}}(k)(c_{\mathrm{ww}}^2(k)+6-12h_2)+O(\xi^3+\xi^2a+a^2)
\end{aligned}
$$

as ξ, $a\to 0$, where h_2 is in (4.2.7). Moreover,

$$
\begin{aligned}
\langle \boldsymbol{\phi}_2, \boldsymbol{\phi}_3\rangle =& \langle \boldsymbol{\phi}_3, \boldsymbol{\phi}_2\rangle = 2\langle \boldsymbol{\phi}_2, \boldsymbol{\phi}_4\rangle = 2\langle \boldsymbol{\phi}_4, \boldsymbol{\phi}_2\rangle = \frac{1}{2}i\xi a\,\frac{k(c_{\mathrm{ww}}c_{\mathrm{ww}}')(k)}{c_{\mathrm{ww}}^2(k)+1}+O(\xi^3+\xi^2a+a^2),\\
\langle \boldsymbol{\phi}_3, \boldsymbol{\phi}_3\rangle =& \langle \boldsymbol{\phi}_4, \boldsymbol{\phi}_4\rangle = c_{\mathrm{ww}}^2(k)+4+O(\xi^3+\xi^2a+a^2),\\
\langle \boldsymbol{\phi}_3, \boldsymbol{\phi}_4\rangle =& \langle \boldsymbol{\phi}_4, \boldsymbol{\phi}_3\rangle = 0+O(\xi^3+\xi^2a+a^2)
\end{aligned}
$$

as ξ, $a\to 0$.

Together, (4.4.1) becomes

$$
\begin{aligned}
\mathbf{L}(\xi, a) =& \frac{1}{4} a (c_{\mathrm{ww}}^2(k) + 1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
& + i\xi \begin{pmatrix} -kc_{\mathrm{ww}}'(k) & 0 & 0 & 0 \\ 0 & -kc_{\mathrm{ww}}'(k) & 0 & 0 \\ 0 & 0 & c_{\mathrm{ww}}(k) \dfrac{c_{\mathrm{ww}}^2(k) + 8}{c_{\mathrm{ww}}^2(k) + 4} & \dfrac{c_{\mathrm{ww}}^2(k) - 4}{c_{\mathrm{ww}}^2(k) + 4} \\ 0 & 0 & \dfrac{c_{\mathrm{ww}}^2(k) - 4}{c_{\mathrm{ww}}^2(k) + 4} & \dfrac{c_{\mathrm{ww}}^3(k)}{c_{\mathrm{ww}}^2(k) + 4} \end{pmatrix} \\
& + i\xi a\, L \begin{pmatrix} 0 & 0 & 2 & c_{\mathrm{ww}}(k) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - i\xi a \frac{1}{c_{\mathrm{ww}}^2(k) + 4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ L_{31} & 0 & 0 & 0 \\ L_{41} & 0 & 0 & 0 \end{pmatrix} \\
& + \frac{1}{2} \xi^2 k (2c_{\mathrm{ww}}'(k) + kc_{\mathrm{ww}}''(k)) \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + O(\xi^3 + \xi^2 a + a^2)
\end{aligned} \tag{4.4.4}
$$

as ξ, $a \to 0$, where

$$
\begin{aligned}
L =& -\frac{3c_{\mathrm{ww}}(k)}{c_{\mathrm{ww}}^2(k) + 1} \left(2h_2 \frac{c_{\mathrm{ww}}^2(k) - 1}{c_{\mathrm{ww}}^2(k) + 2} + 1 \right) - \frac{1}{2} kc_{\mathrm{ww}}'(k) \frac{c_{\mathrm{ww}}^2(k) + 2}{c_{\mathrm{ww}}^2(k) + 1}, \\
L_{31} =& c_{\mathrm{ww}}(k) \left(6h_2 \frac{c_{\mathrm{ww}}^2(k) + 1}{c_{\mathrm{ww}}^2(k) + 2} + \frac{1}{2} (c_{\mathrm{ww}}^2(k) + 2) + 2k (c_{\mathrm{ww}} c_{\mathrm{ww}}')(k) \right), \\
L_{41} =& \frac{1}{4} (c_{\mathrm{ww}}^4(k) + 3c_{\mathrm{ww}}^2(k) + 4) + 3h_2 c_{\mathrm{ww}}^2(k) \frac{c_{\mathrm{ww}}^2(k) - 1}{c_{\mathrm{ww}}^2(k) + 2} + \frac{1}{2} k (c_{\mathrm{ww}}^3 c_{\mathrm{ww}}')(k),
\end{aligned}
$$

and h_2 is in (4.2.7). Moreover, (4.4.2) becomes

$$\begin{aligned}
\mathbf{I}(\xi,a) =& \mathbf{I} + a\frac{2}{c_{\mathrm{ww}}^2(k)+1}\begin{pmatrix} 0 & 0 & 1-3h_2\dfrac{c_{\mathrm{ww}}^2(k)}{c_{\mathrm{ww}}^2(k)+2} & c_{\mathrm{ww}}(k)\left(\frac{1}{4}c_{\mathrm{ww}}^2(k)+\frac{3}{2}-3h_2\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&+ a\frac{1}{c_{\mathrm{ww}}^2(k)+4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1-3h_2\dfrac{c_{\mathrm{ww}}^2(k)}{c_{\mathrm{ww}}^2(k)+2} & 0 & 0 & 0 \\ c_{\mathrm{ww}}(k)\left(\frac{1}{4}c_{\mathrm{ww}}^2(k)+\frac{3}{2}-3h_2\right) & 0 & 0 & 0 \end{pmatrix} \\
&- \frac{1}{2}i\xi a\frac{k(c_{\mathrm{ww}}c'_{\mathrm{ww}})(k)}{(c_{\mathrm{ww}}^2(k)+1)^2(c_{\mathrm{ww}}^2(k)+4)}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2(c_{\mathrm{ww}}^2(k)+4) & c_{\mathrm{ww}}^2(k)+4 \\ 0 & c_{\mathrm{ww}}^2(k)+1 & 0 & 0 \\ 0 & c_{\mathrm{ww}}^2(k)+1 & 0 & 0 \end{pmatrix} \\
&+ O(\xi^3+\xi^2 a+a^2)
\end{aligned} \tag{4.4.5}$$

as ξ, $a \to 0$, where $\mathbf{I}$ means the 4×4 identity matrix. Note that the coefficient matrices are explicit functions of k.

For $a=0$, (4.4.4) and (4.4.5) become

$$\begin{aligned}
\mathbf{L}(\xi,0) =& i\xi\begin{pmatrix} -kc'_{\mathrm{ww}}(k) & 0 & 0 & 0 \\ 0 & -kc'_{\mathrm{ww}}(k) & 0 & 0 \\ 0 & 0 & c_{\mathrm{ww}}(k)\dfrac{c_{\mathrm{ww}}^2(k)+8}{c_{\mathrm{ww}}^2(k)+4} & \dfrac{c_{\mathrm{ww}}^2(k)-4}{c_{\mathrm{ww}}^2(k)+4} \\ 0 & 0 & \dfrac{c_{\mathrm{ww}}^2(k)-4}{c_{\mathrm{ww}}^2(k)+4} & c_{\mathrm{ww}}(k)\dfrac{c_{\mathrm{ww}}^2(k)}{c_{\mathrm{ww}}^2(k)+4} \end{pmatrix} \\
&+ \frac{1}{2}\xi^2 k(2c'_{\mathrm{ww}}(k)+kc''_{\mathrm{ww}}(k))\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + O(\xi^3)
\end{aligned}$$

and $\mathbf{I}(\xi,0)=\mathbf{I}$ as $\xi\to 0$. It is then easy to verify that the roots of $\det(\mathbf{L}-\lambda\mathbf{I})(\xi,0)$ coincide with the eigenvalues $i\omega(\pm 1+\xi,-)$ and $i\omega(\xi,\pm)$ of $\mathcal{L}(\xi,0)$ up to terms of order ξ^2 for $\xi>0$

and sufficiently small. For $\xi = 0$, (4.4.4) and (4.4.5) become

$$\mathbf{L}(0,a) = \frac{1}{4}a(c_{\text{ww}}^2(k)+1)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + O(a^2)$$

and $\mathbf{I}(0,a) = \mathbf{I} + O(a)$ for $a \to 0$. This is reminiscent of the Jordan block structure of $\mathcal{L}(0,a)$; see Lemma 4.3.1.

4.5 The modulational instability index

We turn the attention to the roots of

$$\begin{aligned} &\det(\mathbf{L}-\lambda\mathbf{I})(\xi)(a;k,0,0) \\ &\quad= p_4(\xi,a;k)\lambda^4 + ip_3(\xi,a;k)\lambda^3 + p_2(\xi,a;k)\lambda^2 + ip_1(\xi,a;k)\lambda + p_0(\xi,a;k) \\ &\quad=: p(\lambda)(\xi,a;k) \end{aligned} \tag{4.5.1}$$

for $\xi > 0$, $a \in \mathbb{R}$ and ξ, $|a|$ sufficiently small for $k > 0$, where $\mathbf{L}$ and $\mathbf{I}$ are in (4.4.4) and (4.4.5). Recall that they coincide with the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$-eigenvalues of $\mathcal{L}(\xi)(a;k,0,0)$ in the vicinity of the origin in $\mathbb{C}$ up to terms of orders ξ^2 and a as ξ, $a \to 0$.

Note that $p_0, p_1, \ldots, p_4$ depend analytically on ξ, a, and k for any $\xi > 0$ and $|a|$ sufficiently small for any $k > 0$. Recall that the spectrum of $\mathcal{L}(\xi,a)$ is symmetric with respect to the reflection in the imaginary axis for any $\xi \in [0,1/2]$ and $a \in \mathbb{R}$ admissible for any $k > 0$. Hence $p_0, p_1, \ldots, p_4$ are real valued. Recall that

$$\overline{\operatorname{spec}\mathcal{L}(\xi,a)} = \operatorname{spec}\mathcal{L}(-\xi,a).$$

Hence p_1 and p_3 are even in ξ, whereas p_0, p_2, p_4 are odd. Moreover, the spectrum of $\mathcal{L}(\xi,a)$ remains invariant under $a \mapsto -a$ by (4.2.4) for any $\xi \in [0,1/2]$ and $a \in \mathbb{R}$ admissible for any $k > 0$. Hence $p_0, p_1, \ldots, p_4$ are even in a.

For $\xi = 0$, Lemma 4.3.1 implies that $\lambda = 0$ is a root of $p(0,a;k)$ with multiplicity four for any $a \in \mathbb{R}$ and $|a|$ sufficiently small for any $k > 0$. Likewise, $\xi = 0$ is a root of $p(\cdot\,,a;k)(0)$ with multiplicity four. Thus we may define

$$q(-i\xi\lambda)(\xi,a;k) = \xi^4(q_4(\xi,a;k)\lambda^4 - q_3(\xi,a;k)\lambda^3 - q_2(\xi,a;k)\lambda^2 + q_1(\xi,a;k)\lambda + q_0(\xi,a;k)),$$

where

$$p_j(\xi, a; k) := \xi^{4-k} q_j(\xi, a; k) \qquad \text{for } j = 0, 1, \ldots, 4. \tag{4.5.2}$$

Note that $q_0, q_1, \ldots, q_4$ are real valued and depend analytically on ξ, a, and k for any $\xi > 0$, $|a|$ sufficiently small and for any $k > 0$. Moreover, they are odd in ξ and even in a. For $a \in \mathbb{R}$ and $|a|$ sufficiently small for $k > 0$, by virtue of Section 4.5, a sufficiently small, periodic wave train $\eta(a; k, 0, 0)$, $u(a; k, 0, 0)$ and $c(a; k, 0, 0)$ of (4.1.3)-(4.1.2) is modulationally unstable, provided that q possesses a pair of complex roots for $\xi > 0$ and small.

Let

$$\begin{aligned}
\Delta_0 =& 256q_4^3q_0^3 - 192q_4^2q_3q_1q_0^2 - 128q_4^2q_2^2q_0^2 + 144q_4^2q_2q_1^2q_0 \\
&- 27q_4^2q_1^4 + 144q_4q_3^2q_2q_0^2 - 6q_4q_3^2q_1^2q_0 - 80q_4q_3q_2^2q_1q_0 \\
&+ 18q_4q_3q_2q_1^3 + 16q_4q_2^4q_0 - 4q_4q_2^3q_1^2 - 27q_3^4q_0^2 + 18q_3^3q_2q_1q_0 \\
&- 4q_3^3q_1^3 - 4q_3^2q_2^3q_0 + q_3^2q_2^2q_1^2,
\end{aligned}$$

and

$$\begin{aligned}
\Delta_1 =& - 8q_4q_2 - 3q_3^2, \\
\Delta_2 =& 64q_4^3q_0 - 16q_4^2q_2^2 - 16q_4q_3^2q_2 + 16q_4^2q_3q_1 - 3q_3^4.
\end{aligned}$$

They classify the nature of the roots of the quartic polynomial q. Specifically, if $\Delta_0 < 0$ then the roots of q are distinct, two real and two complex. If $\Delta_0 > 0$ and $\Delta_1 \geqslant 0$ then the roots are distinct and complex. If $\Delta_0 > 0$ and if $\Delta_1 < 0$, $\Delta_2 > 0$ then the roots of q are distinct and complex. If $\Delta_0 > 0$ and if $\Delta_1 < 0$, $\Delta_2 < 0$, on the other hand, then the roots are distinct and real. If $\Delta_0 = 0$ then at least two roots are equal; see [HP16b], for instance, for a complete proof. Note that Δ_0 is the discriminant of q.

Note that Δ_0, Δ_1, Δ_2 are even in ξ and a. We may write

$$\Delta_0(k; \xi, a) =: \Delta_0(k; \xi, 0) + a^2\Delta(k) + O(a^2(\xi^2 + a^2)),$$

and

$$\begin{aligned}
\Delta_1(k; \xi, a) =& \Delta_1(k; \xi, 0) + O(a^2), \\
\Delta_2(k; \xi, a) =& \Delta_2(k; \xi, 0) + O(a^2)
\end{aligned}$$

as $a \to 0$ for any $\xi > 0$ and sufficiently small for any $k > 0$. We then use (4.4.4), (4.4.5), (4.5.1), (4.5.2), and we make a Mathematica calculation to show that

$$\Delta_0(k;\xi,0) = 4\xi^2k^2(((kc_{\rm ww}(k))')^2-1)^4((kc_{\rm ww}(k))'')^2+O(\xi^4)>0,$$

and

$$\begin{aligned}\Delta_1(k;\xi,0) &= -4(2+(c_{\rm ww}(k)+kc'_{\rm ww}(k))^2)+O(\xi^2)<0,\\ \Delta_2(k;\xi,0) &= -16(1+2(c_{\rm ww}(k)+kc'_{\rm ww}(k))^2)+O(\xi^2)<0\end{aligned}$$

as $\xi\to 0$ for any $k>0$. Therefore, for $a\in\mathbb{R}$, $|a|$ sufficiently small and fixed, if $\Delta(k)<0$ for some $k>0$ then it is possible to find a sufficiently small $\xi_0>0$ such that $\Delta_0(k;\xi,a)<0$ and Δ_1, $\Delta_2<0$ for $\xi\in(0,\xi_0)$. Hence q possesses two real and two complex roots for $\xi\in(0,\xi_0)$, implying the modulational instability. We pause to remark that one must take ξ small enough so that $a^2\Delta(k)$ dominates $\Delta_0(k;\xi,0)=O(\xi^2)$. That means, the modulational instability is a nonlinear phenomenon. If $\Delta\geqslant 0$, on the other hand, then $\Delta_0>0$ and Δ_1, $\Delta_2<0$ for $\xi>0$ sufficiently small. Hence the roots of q are real for $\xi>0$ sufficiently small. Hence this implies the spectral stability in the vicinity of the origin in $\mathbb{C}$.

We use (4.4.4), (4.4.5), (4.5.1), (4.5.2), and we make a Mathematica calculation to find Δ explicitly, whereby we derive a modulational instability index for (4.1.3)-(4.1.2). We summarize the conclusion.

Theorem 4.5.1 (Modulational instability index)**.** *A sufficiently small, $2\pi/k$-periodic wave train of* (4.1.3)-(4.1.2) *is modulationally unstable, provided that*

$$\Delta(k) := \frac{i_1(k)i_2(k)}{i_3(k)}i_4(k)<0, \tag{4.5.3}$$

where

$$\begin{aligned}i_1(k) &= (kc_{\rm ww}(k))'', && (4.5.4a)\\ i_2(k) &= ((kc_{\rm ww}(k))')^2-1, && (4.5.4b)\\ i_3(k) &= c^2_{\rm ww}(k)-c^2_{\rm ww}(2k), && (4.5.4c)\end{aligned}$$

and

$$\begin{aligned}i_4(k) =& 3c^2_{\rm ww}(k)+5c^4_{\rm ww}(k)-2c^2_{\rm ww}(2k)(c^2_{\rm ww}(k)+2) \qquad (4.5.4d)\\ &+18kc^3_{\rm ww}(k)c'_{\rm ww}(k)+k^2(c'_{\rm ww})^2(k)(5c^2_{\rm ww}(k)+4c^2_{\rm ww}(2k)).\end{aligned}$$

It is spectrally stable to square integrable perturbations in the vicinity of the origin in $\mathbb{C}$ otherwise.

4.6 Results

Since $(kc_{\text{ww}}(k))' < 1$ for any $k > 0$ and decreases monotonically over the interval $(0, \infty)$ by brutal force, $i_1(k) < 0$ and $i_2(k) < 0$ for any $k > 0$. Since $c_{\text{ww}}(k) > 0$ for any $k > 0$ and decreases monotonically over the interval $(0, \infty)$ (see Figure 4.1), $i_3(k) > 0$ for any $k > 0$. Hence the sign of Δ coincides with that of i_4.

We use (4.7.11) and make an explicit calculation to show that

$$\lim_{k\to 0+} \frac{i_4(k)}{\sqrt{k}^5} = 9 \quad \text{and} \quad \lim_{k\to\infty} ki_4(k) = -3.$$

Hence $\Delta(k) > 0$ for $k > 0$ sufficiently small, implying the modulational stability, and it is negative for $k > 0$ sufficiently large, implying the spectral stability in the vicinity of the origin in $\mathbb{C}$. Moreover, the intermediate value theorem asserts a root of i_4, which changes the modulational stability and instability.

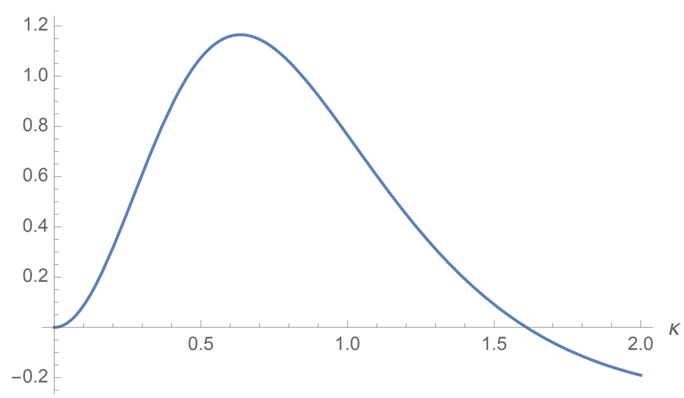


Figure 4.2: The graph of $i_4(k)$ for $k \in (0, 2)$.

It is difficult to analytically study the sign of i_4 further. On the other hand, a numerical evaluation of (4.7.11) reveals a unique root k_c, say, of i_4 over the interval $(0, \infty)$ (see Figure 4.2) such that $i_4(k) > 0$ if $0 < k < k_c$ and it is negative if $k_c < k < \infty$. Upon close inspection (see Figure 4.3), moreover, $k_c = 1.610\ldots$. We summarize the conclusion.

Corollary 4.6.1 (Critical wave number)**.** *A sufficiently small, $2\pi/k$-periodic wave train of* (4.1.3)-(4.1.2) *is modulationally unstable if $k > k_c$, where $k_c = 1.610\ldots$ is a unique root of i_4*

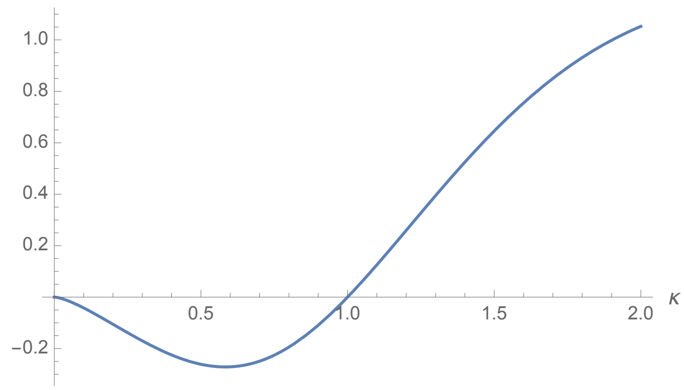


Figure 4.3: The graph of $i_4(1.61k^{-1})$ for $k \in (0, 2)$.

in (4.7.11) *over the interval* $(0, \infty)$. *It is spectrally stable to square integrable perturbations in the vicinity of the origin in* $\mathbb{C}$ *if* $0 < k < k_c$.

Corollary 4.6.1 qualitatively states the Benjamin-Feir instability of a Stokes' wave. Fortuitously, the critical wave number compares reasonably well with that in [BH67, Whi67] and [BM95]. The critical wave number for the Whitham equation is $1.146\dots$; see [HJ15a], for instance.

We point out that the critical wave number in [BH67, Whi67] and [BM95] was determined by an approximation of the numerical value of some explicit function of k, which seems difficult to calculate analytically. Therefore, it is not surprising that the proof of Corollary 4.6.1 ultimately relies on a numerical evaluation of the modulational instability index (4.5.3).

4.7 The full-dispersion shallow water equation-II

We derive a modulational instability index for (4.1.4)-(4.1.2). The details of the proof follow along Sections 4.2-4.6. We briefly present the main ideas and record relevant expressions.

By a traveling wave of (4.1.4)-(4.1.2), we mean a stationary solution of form $(\eta, u)(x, t) = (\eta, u)(x - ct)$ for some $c > 0$. Further, we take η and u to be 2π-periodic functions of $z = kx$. The result becomes, by quadrature,

$$\begin{aligned} &- c\eta + c_{\text{ww}}^2(k|\partial_z|)u + u\eta = (1 - c^2)b_1, \\ &- cu + \eta + \frac{1}{2}u^2 = (1 - c^2)b_2, \end{aligned} \tag{4.7.1}$$

for some $b_1, b_2 \in \mathbb{R}$.

The existence of a smooth η and u satisfying (4.7.1) follows from a Lyapunov-Schmidt procedure. The small amplitude expansion of solutions is given as

$$\begin{aligned}\eta(a;k,b_1,b_2)(z) =&\eta_0(k,b_1,b_2) + ac_{\rm ww}(k)\cos z + \frac{a}{2c_{\rm ww}(k)}(b_1c_{\rm ww}(k)+b_2)\cos z\\ &+a^2\Big(c_{\rm ww}(k)h_0 - \frac{1}{4} + \Big(c_{\rm ww}(k)h_2-\frac{1}{4}\Big)\cos 2z\Big) + O(a(a+b_1+b_2)^2),\\ u(a;k,b_1,b_2)(z) =&u_0(k,b_1,b_2) + a\cos z + a^2(h_0+h_2\cos 2z) + O(a(a+b_1+b_2)^2),\end{aligned}$$

and

$$c(a;k,b_1,b_2) =c_0(k,b_1,b_2) + \frac{3}{2}a^2\Big(h_0+\frac{1}{2}h_2 - \frac{1}{8c_{\rm ww}(k)}\Big) + O(a(a+b_1+b_2)^2)$$

as a, b_1, $b_2 \to 0$;

$$\begin{aligned}\eta_0(k,b_1,b_2) &= b_1c_{\rm ww}(k) + b_2 + O((b_1+b_2)^2),\\ u_0(k,b_1,b_2) &= b_1 + b_2c_{\rm ww}(k) + O((b_1+b_2)^2),\end{aligned}$$

and

$$c_0(k,b_1,b_2) = c_{\rm ww}(k) + \frac{3}{2}b_1 + \frac{1}{2c_{\rm ww}(k)}b_2(2c_{\rm ww}^2(k)+1) + O((b_1+b_2)^2)$$

as b_1, $b_2 \to 0$, where

$$h_0 = \frac{3}{4}\frac{c_{\rm ww}(k)}{c_{\rm ww}^2(k)-1} \quad \text{and} \quad h_2 = \frac{3}{4}\frac{c_{\rm ww}(k)}{c_{\rm ww}^2(k)-c_{\rm ww}^2(2k)}. \tag{4.7.4}$$

We linearize (4.1.4)-(4.1.2) about η and u in the coordinate frame moving at the speed $c>0$ and seek a solution of the form $\mathbf{v}(z,t) = e^{\lambda kt}\mathbf{v}(z)$, $\lambda \in \mathbb{C}$, to arrive at

$$\lambda\mathbf{v} = \partial_z\begin{pmatrix} c-u & -c_{\rm ww}^2(k|\partial_z|)-\eta \\ -1 & c-u\end{pmatrix}\mathbf{v} =: \mathcal{L}(a;k)\mathbf{v}.$$

Using Floquet theory, the $L^2(\mathbb{R})\times L^2(\mathbb{R})$-spectrum of $\mathcal{L}$ is decomposed into $L^2(\mathbb{T})\times L^2(\mathbb{T})$ spectra of $\mathcal{L}(\xi,a)$'s for $\xi \in (-1/2,1/2]$ defined by

$$\mathcal{L}(\xi,a)\mathbf{v}(\xi) := e^{-i\xi z}\mathcal{L}e^{i\xi z}\mathbf{v}(\xi).$$

For any $\xi \in (-1/2,1/2]$, the $L^2(\mathbb{T})\times L^2(\mathbb{T})$-spectrum of $\mathcal{L}(\xi,a)$ consists of eigenvalues with finite multiplicities. A straightforward calculation shows that zero is an eigenvalue of $\mathcal{L}(0,0)$ with multiplicity four. For $|a| \neq 0$, zero continues to be an eigenvalue of $\mathcal{L}(0,a)$ with a four-dimensional generalized eigenspace. For $|\xi|$ and $|a|$ small, we are interested in the eigenvalues of $\mathcal{L}(\xi,a)$ bifurcating from the zero eigenvalue of $\mathcal{L}(0,a)$. For this purpose, we extend the eigenspace for the zero eigenvalue of $\mathcal{L}(0,a)$ to construct a four-dimensional eigenspace for the bifurcating eigenvalues of $\mathcal{L}(\xi,a)$, for $|\xi|$ and $|a|$ small. This eigenspace is spanned by (see Lemma 4.3.1)

$$\begin{aligned}
\boldsymbol{\phi}_1(\xi,a)(z) = &\begin{pmatrix} c_{\rm ww}(k) \\ 1 \end{pmatrix} \cos z + i\xi \frac{kc'_{\rm ww}(k)}{c^2_{\rm ww}(k)+1} \begin{pmatrix} 1 \\ -c_{\rm ww}(k) \end{pmatrix} \sin z \\
&+ \frac{a}{4c_{\rm ww}(k)} \begin{pmatrix} -c_{\rm ww}(k)(1+4c_{\rm ww}(k)h_2) \\ 1-4c_{\rm ww}(k)h_2 \end{pmatrix} \\
&+ \frac{a}{2} \begin{pmatrix} 4c_{\rm ww}(k)h_2 - 1 \\ 4h_2 \end{pmatrix} \cos 2z + \xi^2 \mathbf{p}_2 \cos z + O(\xi^3 + \xi^2 a + a^2), \\
\boldsymbol{\phi}_2(\xi,a)(z) = &\begin{pmatrix} c_{\rm ww}(k) \\ 1 \end{pmatrix} \sin z - i\xi \frac{kc'_{\rm ww}(k)}{c^2_{\rm ww}(k)+1} \begin{pmatrix} 1 \\ -c_{\rm ww}(k) \end{pmatrix} \cos z \\
&+ \frac{a}{2} \begin{pmatrix} 4c_{\rm ww}(k)h_2 - 1 \\ 4h_2 \end{pmatrix} \sin 2z + \xi^2 \mathbf{p}_2 \sin z + O(\xi^3 + \xi^2 a + a^2), \\
\boldsymbol{\phi}_3(\xi,a)(z) = &\begin{pmatrix} 2c_{\rm ww}(k) \\ -1 \end{pmatrix} + a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos z - \frac{1}{6}\xi^2 k^2 c_{\rm ww}(k) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(\xi^3 + \xi^2 a + a^2), \\
\boldsymbol{\phi}_4(\xi,a)(z) = &\begin{pmatrix} 1 \\ 2c_{\rm ww}(k) \end{pmatrix} + \frac{a}{2c_{\rm ww}(k)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos z - \frac{1}{12}\xi^2 k^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(\xi^3 + \xi^2 a + a^2)
\end{aligned} \tag{4.7.5}$$

up to orders of ξ^2 and a as ξ, $a \to 0$, where h_2 is in (4.7.4) and

$$\mathbf{p}_2 = \frac{1}{2} \frac{k^2}{c^2_{\rm ww}(k)+1} \begin{pmatrix} -3\dfrac{(c_{\rm ww}(c'_{\rm ww})^2)(k)}{c^2_{\rm ww}(k)+1} + c''_{\rm ww}(k) \\ c'_{\rm ww}(k)^2 \dfrac{2c^2_{\rm ww}(k)-1}{c^2_{\rm ww}(k)+1} - (c_{\rm ww}c''_{\rm ww})(k) \end{pmatrix}.$$

For $|\xi|$ and $|a|$ small, the four eigenvalues of $\mathcal{L}(\xi,a)$ bifurcating from zero eigenvalue coincide with the roots of $\det(\mathbf{L} - \lambda \mathbf{I})$ up to orders of ξ^2 and a (see [Kat76, Section 4.3.5], for instance, for details), where

$$\mathbf{L}(\xi,a) = \left(\frac{\langle \mathcal{L}(\xi,a)\boldsymbol{\phi}_k(\xi,a), \boldsymbol{\phi}_\ell(\xi,a) \rangle}{\langle \boldsymbol{\phi}_k(\xi,a), \boldsymbol{\phi}_k(\xi,a) \rangle} \right)_{k,\ell=1,2,3,4} \tag{4.7.6}$$

and

$$\mathbf{I}(\xi,a) = \left(\frac{\langle \boldsymbol{\phi}_k(\xi,a), \boldsymbol{\phi}_\ell(\xi,a)\rangle}{\langle \boldsymbol{\phi}_k(\xi,a), \boldsymbol{\phi}_k(\xi,a)\rangle}\right)_{k,\ell=1,2,3,4}, \tag{4.7.7}$$

where $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_3$, $\boldsymbol{\phi}_4$ are in (4.7.5) and $\langle\,,\rangle$ means the $L^2(\mathbb{T})\times L^2(\mathbb{T})$-inner product. This amounts to the fact that restricted on a four-dimensional eigenspace, $\mathcal{L}(\xi,a)$ can be defined by the 4×4 matrix $\mathbf{L}(\xi,a)$ obtained by calculating its action on the basis $\{\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,\boldsymbol{\phi}_3,\boldsymbol{\phi}_4\}$. Therefore, the eigenvalues of the resulting matrix are given by the roots of its characteristic polynomial $\det(\mathbf{L}-\lambda\mathbf{I})$, where $\mathbf{I}$ is the projection of the identity onto the eigenspace.

We omit all the details of the calculation as it is very similar to FDSW-I and report that (4.7.6) becomes

$$\begin{aligned}
\mathbf{L}(\xi,a) =&\frac{1}{4}a(c_{\text{ww}}^2(k)+1)\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\end{pmatrix}\\
&+i\xi\begin{pmatrix} -kc_{\text{ww}}'(k) & 0 & 0 & 0\\ 0 & -kc_{\text{ww}}'(k) & 0 & 0\\ 0 & 0 & c_{\text{ww}}(k)\frac{4c_{\text{ww}}^2(k)+5}{4c_{\text{ww}}^2(k)+1} & -\frac{4c_{\text{ww}}^2(k)-1}{4c_{\text{ww}}^2(k)+1}\\ 0 & 0 & -\frac{4c_{\text{ww}}^2(k)-1}{4c_{\text{ww}}^2(k)+1} & c_{\text{ww}}(k)\frac{4c_{\text{ww}}^2(k)-3}{4c_{\text{ww}}^2(k)+1}\end{pmatrix}\\
&+i\xi a\,L\begin{pmatrix} 0&0&2c_{\text{ww}}(k)&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}+i\xi a\frac{1}{2(4c_{\text{ww}}^2(k)+1)}\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ L_{31}&0&0&0\\ L_{41}&0&0&0\end{pmatrix}\\
&+\frac{1}{2}\xi^2k(2c_{\text{ww}}'(k)+kc_{\text{ww}}''(k))\begin{pmatrix} 0&1&0&0\\ -1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}+O(\xi^3+\xi^2a+a^2)
\end{aligned} \tag{4.7.8}$$

as ξ, $a\to 0$, where

$$\begin{aligned}
L =&\frac{1}{2c_{\text{ww}}(k)(c_{\text{ww}}^2(k)+1)}(4c_{\text{ww}}(k)(1-c_{\text{ww}}^2(k))h_2-kc_{\text{ww}}(k)c_{\text{ww}}'(k)-1-5c_{\text{ww}}^2(k)),\\
L_{31} =&4c_{\text{ww}}(k)(1-c_{\text{ww}}^2(k))h_2-2-c_{\text{ww}}^2(k),\\
L_{41} =&\frac{1}{2c_{\text{ww}}(k)}(4c_{\text{ww}}(k)(1-c_{\text{ww}}^2(k))h_2-2(c_{\text{ww}}^2(k)+1)-4c_{\text{ww}}^4(k)),
\end{aligned}$$

and h_2 is in (4.7.4). Moreover, (4.7.7) becomes

$$\begin{aligned}
\mathbf{I}(\xi, a) =&\mathbf{I} + a\frac{4c_{\text{ww}}(k)(1-2c_{\text{ww}}^2(k))h_2 - 1}{4c_{\text{ww}}(k)(c_{\text{ww}}^2(k)+1)(4c_{\text{ww}}^2(k)+1)}\begin{pmatrix} 0 & 0 & 2(4c_{\text{ww}}^2(k)+1) & 0 \\ 0 & 0 & 0 & 0 \\ c_{\text{ww}}^2(k)+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&+ a\frac{1-6c_{\text{ww}}(k)h_2}{2(c_{\text{ww}}^2(k)+1)(4c_{\text{ww}}^2(k)+1)}\begin{pmatrix} 0 & 0 & 0 & 2(4c_{\text{ww}}^2(k)+1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_{\text{ww}}^2(k)+1 & 0 & 0 & 0 \end{pmatrix} \\
&- i\xi a\frac{kc'_{\text{ww}}(k)}{4c_{\text{ww}}(k)(c_{\text{ww}}^2(k)+1)^2(4c_{\text{ww}}^2(k)+1)} \\
&\qquad\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4c_{\text{ww}}(k)(4c_{\text{ww}}^2(k)+1) & 2(4c_{\text{ww}}^2(k)+1) \\ 0 & 2c_{\text{ww}}(k)(c_{\text{ww}}^2(k)+1) & 0 & 0 \\ 0 & c_{\text{ww}}^2(k)+1 & 0 & 0 \end{pmatrix} \\
&+ O(\xi^3 + \xi^2 a + a^2)
\end{aligned} \tag{4.7.9}$$

as $\xi,\ a \to 0$, where $\mathbf{I}$ is the 4×4 identity matrix. The analysis of the roots of quartic polynomial, $\det(\mathbf{L} - \lambda\mathbf{I})$, in λ can be analyzed using discriminants (see Section 4.5, for details), and we derive a modulational instability index for (4.1.4) and (4.1.2) given by

$$\Delta(k) := \frac{i_1(k)i_2(k)}{i_3(k)}i_4(k), \tag{4.7.10}$$

where

$$\begin{aligned}
i_1(k) =&(kc_{\text{ww}}(k))'', \\
i_2(k) =&((kc_{\text{ww}}(k))')^2 - 1, \\
i_3(k) =&c_{\text{ww}}^2(k) - c_{\text{ww}}^2(2k),
\end{aligned}$$

and

$$i_4(k) =9c_{\text{ww}}^2(k)i_2(k) + i_3(k)(3 + 15c_{\text{ww}}^2(k) + 6kc_{\text{ww}}(k)c'_{ww}(k) - k^2(c'_{\text{ww}}(k))^2). \tag{4.7.11}$$

Again, a straightforward analysis shows that $i_1(k) < 0$ and $i_2(k) < 0$ for any $k > 0$

while $i_3(k) > 0$ for any $k > 0$. A numerical evaluation of (4.7.11) reveals a unique root $k_c = 1.008\ldots$ of i_4 over the interval $(0, \infty)$ such that $i_4(k) > 0$ if $0 < k < k_c$ and it is negative if $k_c < k < \infty$ (see Figures 4.4 and 4.5). Therefore, a sufficiently small $2\pi/k$-periodic traveling wave of (4.1.4) and (4.1.2) is modulationally unstable if $k > k_c$. It is modulationally stable if $0 < k < k_c$.

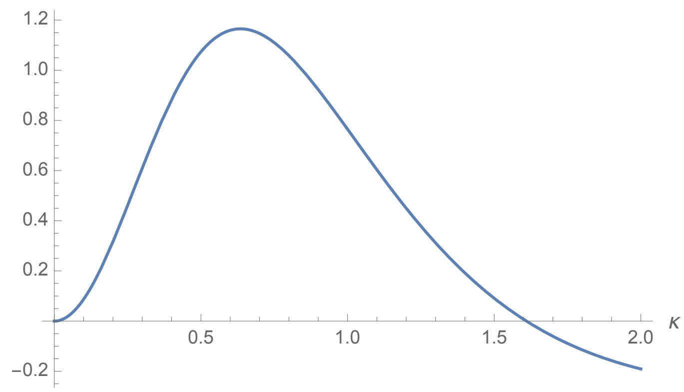


Figure 4.4: The graph of $i_4(k)$ for $k \in (0, 2)$.

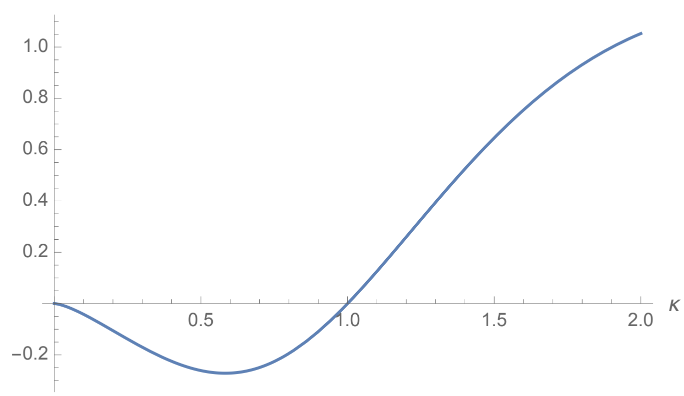


Figure 4.5: The graph of $i_4(1.008k^{-1})$ for $k \in (0, 2)$.

Chapter 5

The Effects of Surface Tension

In this chapter, we consider the effects of surface tension on modulational instability. The full-dispersion shallow water models can be easily modified to incorporate surface tension. The existence of periodic traveling waves can be established by Lyapunov-Schmidt procedure same as in zero surface tension case. The Floquet-Bloch theory and perturbation analysis can be applied as it is and we derive a modulational instability index. We obtain a stability diagram for each model in $k - k\sqrt{T}$ plane and compare the results with the physical problem obtained in [DR77] and [Kaw75].

5.1 The equation

The full-dispersion shallow water models can be modified to incorporate surface tension by replacing $c_{\mathrm{ww}}(k)$ by $c_{\mathrm{ww}}(k;T)$, where,

$$c_{\mathrm{ww}}(k;T) := \sqrt{(1+Tk^2)\frac{\tanh k}{k}}, \tag{5.1.1}$$

where T is the coefficient of surface tension. In this chapter, we compare the effects of surface tension on modulational instability in the Whitham equation (3.1.3), full-dispersion shallow water equations (4.1.3) and (4.1.4) and full-dispersion Camassa-Holm equation (3.2.1) with surface tension. The results of this chapter appeared in [Pan17].

Properties of $c_{\mathrm{ww}}(\cdot\,;T)$: For any $T > 0$, since

$$c^2_{\mathrm{ww}}(k;T) = (1+Tk^2)c^2_{\mathrm{ww}}(k),$$

note that $c^2_{\mathrm{ww}}(\cdot\,;T)$ is even and real analytic, and $c^2_{\mathrm{ww}}(0;T) = 1$. Moreover, $c^2_{\mathrm{ww}}(|\partial_x|;T)$ may be regarded equivalent to $1+|\partial_x|$ in the L^2-Sobolev space setting. In particular, $c^2_{\mathrm{ww}}(|\partial_x|;T) : H^{s+1}(\mathbb{R}) \to H^s(\mathbb{R})$ for any $s \in \mathbb{R}$.

When $T \geqslant 1/3$, note that $c_{\mathrm{ww}}(\cdot\,;T)$ increases monotonically and unboundedly away from the origin. When $0 < T < 1/3$, on the other hand, $c'_{\mathrm{ww}}(0;T) = 0$, $c''_{\mathrm{ww}}(0;T) < 0$ and $c_{\mathrm{ww}}(k;T) \to \infty$ as $k \to \infty$. Hence $c_{\mathrm{ww}}(\cdot\,;T)$ possesses a unique minimum over the interval $(0,\infty)$; see Figure 5.1.

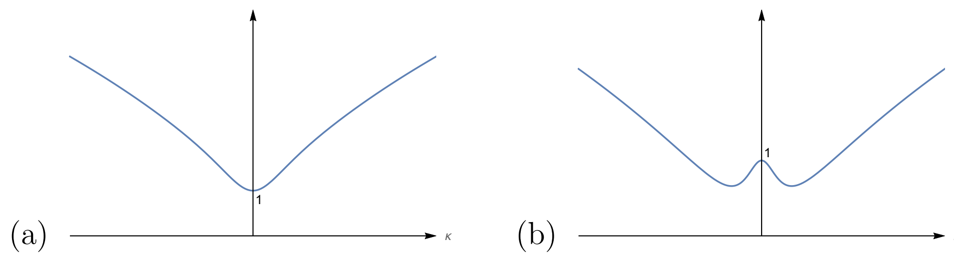


Figure 5.1: Schematic plots of $c_{\mathrm{ww}}(\cdot\,;T)$ when (a) $T \geqslant 1/3$ and (b) $0 < T < 1/3$.

5.2 Periodic traveling waves

Here the existence proof follows along the same line as that in Sections 2.2, 3.2 or 4.2. Interested reader may look at [HJ15b] for the Whitham equation, [HP16a] and [Pan17] for the full-dispersion shallow water equations (4.1.3) and (4.1.4) respectively and [HP17] for the full-dispersion Camassa-Holm equation.

The main difference from the zero surface tension case in the existence proof is the kernel of the linearized operator. When $T \geqslant 1/3$, since $c_{\mathrm{ww}}(k;T) < c_{\mathrm{ww}}(nk;T)$ for any $n = 2, 3, \dots$ pointwise in $\mathbb{R}$ (see Figure 5.1a), the kernel and co-kernel of the linearized operator is two dimensional. Hence, non-constant solutions bifurcate from the constant solution.

When $0 < T < 1/3$, on the other hand, for any integer $n \geqslant 2$, it is possible to find some k such that $c_{\mathrm{ww}}(k;T) = c_{\mathrm{ww}}(nk;T)$ (see Figure 5.1b). If $c_{\mathrm{ww}}(k;T) \neq c_{\mathrm{ww}}(nk;T)$ for any $n = 2, 3, \dots$ then the kernel and co-kernel of the linearized operator is likewise two dimensional. Hence, non-constant solutions bifurcate from the constant solution. But if $c_{\mathrm{ww}}(k;T) = c_{\mathrm{ww}}(nk;T)$ for some integer $n \geqslant 2$, resulting in the resonance of the fundamental mode and the n-th harmonic, then the kernel is four dimensional.

To proceed, for any $T > 0$, for any $k > 0$ satisfying

$$c_{\mathrm{ww}}(k;T) \neq c_{\mathrm{ww}}(nk;T), \qquad n = 2, 3, \dots, \tag{5.2.1}$$

we may repeat the Lyapunov-Schmidt procedure as in Sections 2.2, 3.2 and 4.2 to establish that a one parameter family of solutions exist. If $c_{\mathrm{ww}}(k;T) = c_{\mathrm{ww}}(nk;T)$ for some integer

$n \geqslant 2$ for some $k > 0$ then the proof breaks down.

5.3 The modulational stability index

The modulational instability analysis developed in earlier chapters can be applied as it is. The modulational instability index is given by

$$\Delta(k;T) := \frac{i_1(k;T)i_2(k;T)}{i_3(k;T)} i_4(k;T), \tag{5.3.1}$$

where

$$\begin{aligned}
i_1(k;T) =&(kc_{\text{ww}}(k;T))'', \\
i_2(k;T) =&((kc_{\text{ww}}(k;T))')^2 - 1 =: i_2^+ i_2^-(k;T), \\
i_3(k;T) =&c_{\text{ww}}^2(k;T) - c_{\text{ww}}^2(2k;T) =: i_3^+ i_3^-(k;T),
\end{aligned}$$

and

$$\begin{aligned}
i_4(k;T) =&(2i_3^- + i_2^-)(k;T), \text{ for Whitham,} \\
i_4(k;T) =&\Big(3i_2^- - i_2^- i_3^- + 6i_3^- - \tfrac{1}{12}k^2(57i_2^- + 34i_3^-) + \tfrac{1}{108}k^4(198i_2^- + 35i_3^-)\Big)(k;T), \text{ for FDCH,} \\
i_4(k;T) =&3c_{\text{ww}}^2(k;T) + 5c_{\text{ww}}^4(k;T) - 2c_{\text{ww}}^2(2k;T)(c_{\text{ww}}^2(k;T) + 2) + 18kc_{\text{ww}}^3(k;T)c'_{\text{ww}}(k;T) \\
&+ k^2(c'_{\text{ww}})^2(k;T)(5c_{\text{ww}}^2(k;T) + 4c_{\text{ww}}^2(2k;T)), \text{ for FDSW-I,} \\
i_4(k;T) =&9c_{\text{ww}}^2(k;T)i_2(k;T) + i_3(k;T)(3 + 15c_{\text{ww}}^2(k;T)) \\
&+ i_3(k;T)(6kc_{\text{ww}}(k;T)c'_{\text{ww}}(k;T) - k^2(c'_{\text{ww}}(k;T))^2), \text{ for FDSW-II.}
\end{aligned}$$

A sufficiently small, $2\pi/k$-periodic wave train is modulationally unstable, provided that $\Delta(k;T) < 0$. It is spectrally stable to square integrable perturbations in the vicinity of the origin in $\mathbb{C}$ otherwise. A change in sign of $\Delta(k;T)$ and thus, in stability occurs when one of the factors i_j's, $j = 1, 2, 3, 4$ vanishes. Notice that for a fixed T, all these factors explicitly depend on the wave number k, the phase velocity $c_{\text{ww}}(k;T)$, and the group velocity $(kc_{\text{ww}}(k;T))'$. Therefore, the vanishing of each of the factor is associated with some resonance in the wave (see [HP16a]). Specifically,

(R1) $i_1(k;T)$ is derivative of the group velocity and therefore, if $i_1(k;T) = 0$ at some k, the group velocity achieves an extremum at the wave number k;

(R2) $i_2(k;T)$ is the difference between the group velocity and the phase velocity in the long wave limit as $k \to 0$, that is, $\pm c_{\text{ww}}(0; T) = \pm 1$ and therefore, if $i_2(k; T) = 0$ at some k; it results in the "resonance of short and long waves;"

(R3) $i_3(k; T)$ is the difference between the phase velocities of the fundamental mode, $\pm c_{\text{ww}}(k)$ and second harmonic, $\pm c_{\text{ww}}(2k; T)$ and therefore, $i_3(k; T) = 0$ at some k implies "second harmonic resonance;"

(R4) $i_4(k; T)$ is the only factor which captures the nonlinearity, and we expect $i_4(k; T)$ to vanish when dispersion effects balance the nonlinear effects.

We describe the modulational instability through the diagram, Figure 5.2. In k-$k\sqrt{T}$ plane, four curves are corresponding to each mechanism split the plane into three regions of stability and three regions of instability. Any fixed $T > 0$ corresponds to a line passing through the origin of slope $\sqrt{T}$.

For $0 < T < 1/3$, the line crosses all the curves producing three intervals of stable wave numbers and three intervals of unstable wave numbers. Therefore, for $0 < T < 1/3$, all the four mechanisms (R1) to (R4) contribute towards modulational instability.

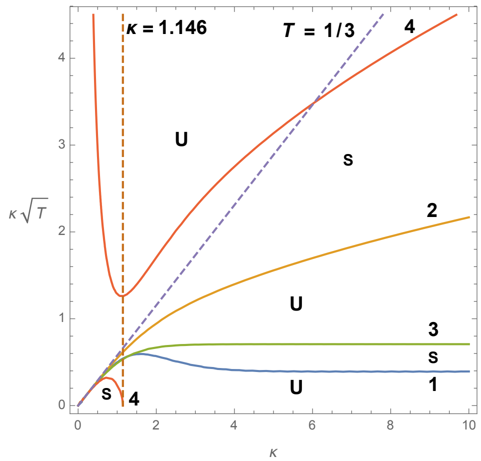
On the other hand, for $T > 1/3$, the line through the origin only crosses the Curve 4 corresponding to $i_4(k; T) = 0$, see Figure 5.2. In this case, the modulational instability is caused only by the mechanism (R4) similar to the case $T = 0$. For every $T > 1/3$, there is a critical wave number $k_c(T)$ such that a sufficiently small $2\pi/k$-periodic traveling wave is modulationally unstable if $k > k_c(T)$. The limit $\lim_{T\to\infty} k_c(T)$ is finite for the physical problem, FDSW-I and FDCH but infinity for the Whitham and FDSW-II equations. The result becomes inconclusive for $T = 1/3$.

The effects of surface tension on modulational instability in all these models along with the full water wave problem have been compared in Figure 5.2. The diagrams corresponding to model equations, Figure 5.2a,5.2b,5.2c,5.2d, contain four curves corresponding to each mechanism from (R1) to (R4). The diagram corresponding to the physical problem, Figure 5.2e, has five curves and by a direct comparison with the model equations, it can be deduced that Curves 2, 3 and 4 are coming from mechanisms (R1), (R2) and (R3) respectively since the full water wave problem shares dispersion with all these models. Moreover, Curves 1 and 5 of Figure 5.2e can be results of the interaction between dispersion and nonlinearity of the full water wave problem, like other models.

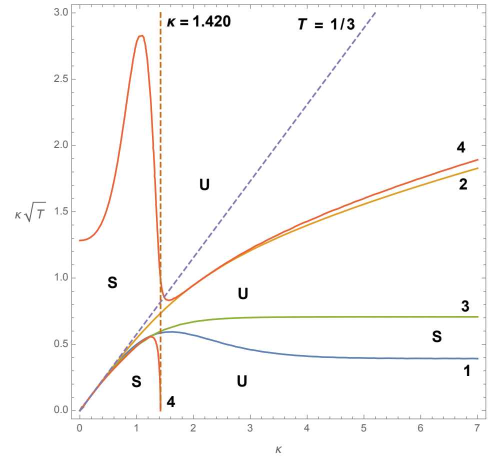
In Figure 5.2, a fixed $T > 0$ corresponds to a line passing through the origin. For small surface tensions, in the physical problem, the wave numbers are divided into three intervals of stability and three intervals of instability, see Figure 5.2e. All the models agree with the physical problem for small surface tensions, more precisely, for $0 < T < 1/3$.

The physical problem reveals that for sufficiently large surface tension, the stability changes to instability only once about a critical wave number much like Benjamin-Feir instability for $T = 0$, see Figure 5.2e. In all the models, for $T > 1/3$, there is a critical wave number $k_c(T)$ about which the stability changes to instability and therefore, all the models agree qualitatively with the physical problem. The difference in models arises when we look at $\lim_{T\to\infty} k_c(T)$. The physical problem suggests that this limit is finite and approximately equal to 1.121. As we can see from Figure 5.2, $\lim_{T\to\infty} k_c(T)$ diverges for the Whitham equation and FDSW-II model. In other words, all sufficiently small periodic traveling waves of the Whitham and FDSW-I model are modulationally stable in the large surface tension limit, which is unphysical as suggested by the physical problem. On the other hand, for FDSW-I model, $\lim_{T\to\infty} k_c(T) \approx 1.054$. Therefore, in the large surface tension limit, the FDSW-I model explains the effects of surface tension similar to the physical problem. For the FDCH equation, $\lim_{T\to\infty} k_c(T) \approx 1.283$ and it offers an improvement over the Whitham equation.

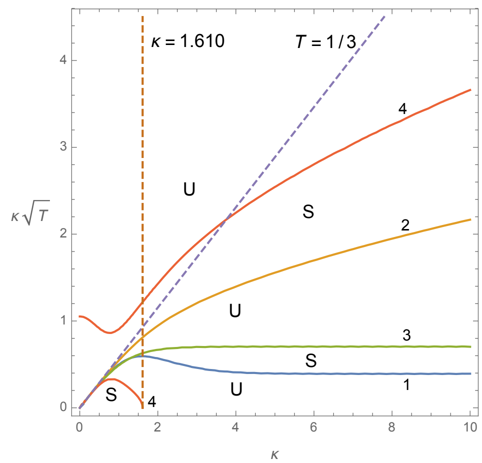
The comparative study suggests that although both FDSW-I and FDSW-II are bi-directional shallow water models extending nonlinear shallow water equations to include full-dispersion of water waves, the FDSW-I is a better model as far as modulational instability is concerned.



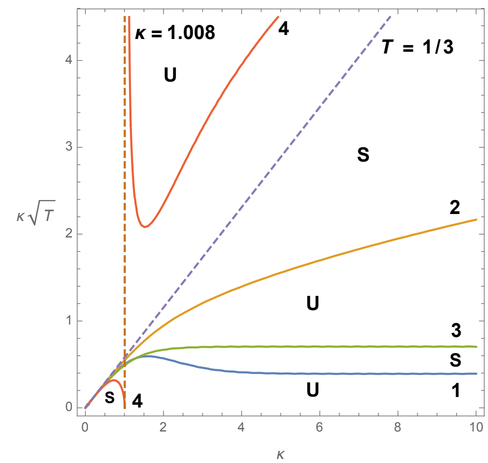
(a) Whitham



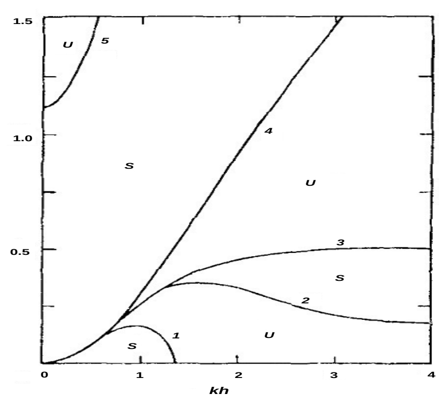
(b) FDCH



(c) FDSW-I



(d) FDSW-II



(e) Full water wave

Figure 5.2: Stability diagram for sufficiently small, periodic wave trains of models indicated. "S" and "U" denote stable and unstable regions. In Figures 5.2a-5.2d, solid curves represent roots of the modulational instability index and are labeled according to their mechanism. Figure 5.2a, 5.2b, 5.2c and 5.2d are adapted from [HJ15b], [HP17], [HP16a] and [Pan17] respectively. Figure 5.2e is taken from [DR77].

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