STRATEGIC DECISIONS UNDER UNCERTAINTY: SUPPLIER QUALITY IMPROVEMENT AND EXIT IN DUOPOLY

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Business Administration
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2018

Urbana, Illinois

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Abstract

This dissertation consists of three interrelated essays on firm-level decision problems when the exterior environment (e.g. product quality or market prospect) is uncertain and there are strategic interactions with other firms (e.g. competitors).

The first essay (Chapter 2) studies a buyers decision to improve its suppliers quality when the focal supplier is shared by another buyer who competes in the same market. Each buyers investment is a way to outperform the other buyer. However, the investment opportunity comes with spillover risk via the shared supplier. Given this risk-benefit tradeoff, we characterize the conditions under which the optimal timing of the first investment in shared suppliers is earlier (or later) than in sole suppliers. Also, we find that learning moderates the impact of competition and spillover on investment decisions, which suggests that the interplay between learning, spillover, and competition should be carefully examined to build sound investment strategies.

The second essay (Chapter 3) also examines buyers investment decisions in a buyer-supplier-buyer triad. However, we consider the case when market competition is not an integral part of the problem so that a buyer strives to free-ride on the other buyers investment in the shared supplier. Moreover, because the improved quality deteriorates over time by organizational forgetting, buyers should make such an investment decision repeatedly. This problem is thus a repeated free-rider problem. The main finding of this essay is that each buyer delays its investment in the hope of free-riding on the other only if the game is repeated and there is a unique equilibrium entailing inefficient delays. Due to this uniqueness of the equilibrium, we are able to construct the well-defined measure for the inefficiency from free-riding incentives and estimate this inefficiency by using primary data from a field study of an automotive manufacturer. The results from this estimation indicate that the inefficiency

can be substantial although it greatly varies depending on the supplier sectors.

The third essay (Chapter 4) investigates firms exit decision problems under uncertainty by employing the similar mathematical framework used in the second essay: The first firm to exit the market concedes the monopolists profit to the remaining firm. The extant literature in economics has predicted that the firms stay in the market longer than necessary. We revisit this problem with two realistic perturbations firms are asymmetric in their exit barriers and the market evolves stochastically. In contrast to the findings of the previous literature, we find that this perturbed model does not admit an MPE (Markov perfect equilibrium) resulting in inefficient (i.e. longer than necessary) stays. Therefore, this asserts the instability of an equilibrium with inefficient stays, which provides a novel rationale for selecting an equilibrium over the others.

To my kitty

Acknowledgments

From the beginning of my doctoral study until the completion of my dissertation, I have been deeply indebted to my advisor, Dharma Kwon, for his continuous supervision and support. He taught me how to recognize good research topics, turn an idea into a well-defined research problem, and make progress on research despite challenges. He is also a surpassing mentor who has enlightened me about building a sustainable career in academia as well as contributing to scholarly communities. I would like to express my gratitude to Richard Engelbrecht-Wiggans for his invaluable insights. He has asked many intriguing questions that led me towards thinking outside the box, which has encouraged me to dig deeper and figure out the gist of my work. I am thankful to Xin Chen for his detailed suggestions on my dissertation. As a top-notch expert in mathematical models, he made various comments that enabled me to write my dissertation with a much higher rigor. I am grateful to Yuqian Xu for her guidance. She particularly helped me effectively navigate the job market with my dissertation essays.

I wish to greatly thank Anupam Agrawal, George Georgiadis, and Suresh Muthulingam for the insightful discussions, comments, and contributions to my dissertation. Heartfelt thanks to the current and former faculty members of the UIUC Gies College of Business for their support during various stages of my doctoral study: Gopesh Anand, Dilip Chhajed, Michael Lim, Ujjal Mukherjee, Nicholas Petruzzi, Fataneh Taghaboni-Dutta, and Han Ye. Each of them helped me develop a wide range of academic skills.

I sincerely appreciate the UIUC Gies College of Business for assistantships since I entered the program and Irwin Family Foundation for generous fellowship. Owing to their extensive financial support, I could be much more productive as a researcher. I also would like to thank Diana Gonzales for her administrative support and encouraging words during my Ph.D. study.

In addition, I am thankful to the former Illinois OM doctoral students, Karthik Murali, Tianqin Shi, and Wenxin Xu, for their kind support, which has been extended after beginning their career at other academic institutions. I am also appreciative of the support from my colleagues in the Ph.D. program, especially Minjae Lee, Seunghoon Lee, and Jae Yoo. So many fun times we have had together nourished my soul both emotionally and intellectually.

Last but not least, I gratefully acknowledge the wholehearted devotion from my family. My mom, dad, and younger sister have always believed in me, and everything that I have accomplished, I owe to them. I thank Thor, the sweetest cat in the world, for cheering me up all the time. My deepest thanks go to my wife Yeonwoo Seo, who has provided me unconditional love and support. She has been standing by me along this long journey, and I could not have made it without her. I dedicate this dissertation to her.

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Chapter 1

Introduction

1.1 Supplier Quality Improvement: Investment in Shared Suppliers

Investment acquisition, capital investment, capacity expansion, market entry, or product development is one of the major corporate decisions to develop or sustain a firm's competitive advantage in the market. As the firms keep expanding the boundary of their supply chain network in the contemporary business environment, the success of investment in new products or quality improvements increasingly relies on other firms in the supply chain network.

Both scholars and practitioners thus have paid growing attentions to the examination of investment decisions in the context of supply chains. One stream of work has studied how a buyer firm can work with its supplier(s) to increase (or maximize) the values generated by the investment in product development, quality improvement, or market entry. This line of literature thus focuses on the vertical relationship in the supply chain. In another stream of research, on the other hand, scholars have examined how the interactions between the buyer firms in the same tier of the supply chain can influence their investment decisions. Hence, the second line of literature concerns about the horizontal relationships in the supply chain. In the two interrelated essays of this dissertation, we study the latter decision problem in a buyer-supplier-buyer triad, in order to investigate the effect of strategic interactions between the buyer firms on their supplier development strategies.

In Chapter 2, we investigate firms' optimal strategies to invest in their suppliers when (i) the benefits of such investments spillovers to other competing firms who source from the same suppliers and (ii) the firms are uncertain about the improvement potential of the supplier's quality. We formulate this problem as the investment game of two Bayesian firms who consider investment in the quality of their shared supplier; the firms have incomplete information on the quality improvement potential of the supplier, and each firm updates their beliefs based on the suppliers quality performance. Assuming that the firms' strategies depending only on their posterior belief, we first obtain pure strategy Markov perfect equilibria characterized by the investment thresholds of both firms. In particular, the equilibrium investment strategies are characterized by a region of preemption and a region of war of attrition. We then identify the conditions under which the optimal timing of the first investment in shared suppliers is earlier (or later) than in sole suppliers. Moreover, we examine how the interplay between spillover, competition, and learning rate affect the first investment time.

In Chapter 3, we consider the case when market competition between the buyer firms is not an integral part of this decision problem. More importantly, each buyer firm is facing a continued deterioration in the suppliers quality due to organizational forgetting, which requires each firm to decide when to invest in the suppliers quality repeatedly. The resulting game is thus a repeated stochastic war of attrition. After establishing that a pure strategy equilibrium always exists, we find that the repetitive nature of the investment opportunities induces a unique mixed strategy equilibrium leading to inefficient delays in investment. We also argue that pure strategy equilibrium is unstable relative to this unique mixed strategy equilibrium in the presence of the repetitive nature of the game. We then compare the inefficient equilibrium to the first-best solution and illustrate the resulting efficiency loss by using primary data collected from a field study. We conclude that coordination among the firms or divisions can potentially save substantial amount of money.

1.2 Exit in Duopoly: Selection of Equilibrium

The second part of this dissertation focuses on firms' strategic exit problems in a saturated and contracting market such as chemical, automobile, or magazine industry. Although these shrinking industries may have been given not as much attention as fast growing markets, it is indeed a primary concern to policy makers because they are major contributors to GDP of the country.

Management and economics scholars have studied firms' strategic exit decisions in declining markets by formulating it as a duopoly (sometimes oligopoly) game of exits. In a conventional duopoly game theoretic model of exit, the first firm to exit the market concedes the monopolist's profit to his opponent firm. In fact, this is one of the common applications of war of attrition models. Wars of attrition have been of primary interest in the management and economics literature since its introduction to the field by Maynard Smith (1974). Its prominent presence in this literature is its pervasiveness in managerial and economic problems. The fierce competition between Borders and Barnes & Noble after the appearance of Amazon is one of the examples.

In Chapter 4, we revisit this problem with two realistic perturbations asymmetric exit barriers and stochastic market evolution. While it is well known that war of attrition under complete information admits both pure strategy and mixed strategy equilibria (Tirole, 1988, Fudenberg and Tirole, 1986, Levin, 2004), raising the issue of equilibrium selection, we show that if the players payoffs are stochastic and the players exit payoffs are heterogeneous, then the game admits Markov perfect equilibria in pure strategies only. In other words, we find that the mixed strategy equilibria, which much of the extant literature has focused on, is unstable to realistic perturbations of the model. This thus indicates that when using such models to draw economic or managerial insights, it may be more prudent to focus on the pure strategy equilibria, which are efficient in the sense that no resources are wasted fighting over the winners payoff.

Chapter 2

Investment in Shared Suppliers: Effect of Learning, Spillover, and Competition

2.1 Introduction

Many manufacturing firms invest significant time, effort, expertise, and capital to improve their suppliers' cost, technical, and quality capabilities. Two key challenges often govern such investment decisions. The first challenge is around the uncertainty regarding the ability of suppliers to develop their capabilities. The second challenge is whether other firms, that source products from these suppliers, would benefit from the firm's investments because of spillover. A priori it is not clear how these challenges can impact the optimal investment strategies of a firm. This essay aims to shed light on these issues by investigating two research questions: First, when is it optimal for a firm to invest in quality improvement at a shared supplier if the gains from such investments can spill over to benefit other firms who also source from the same supplier? Second, how does spillover impact the timing of the first (leader's) investment?

When a firm invests to improve quality at shared suppliers, the improved quality performance can spill over to benefit other firms because the shared suppliers may serve as informal channels for knowledge transfer (Alcácer and Chung, 2007). For example, Spekman and Gibbons (2008) highlight that Pratt and Whitney was wary that their competitor Rolls Royce would also benefit when they worked with their shared supplier Dynamic Gunver Technologies. In addition, the uncertainty in the quality improvement potential of a shared supplier gets compounded by the fact that the quality performance observed by the buyers is noisy because of random variability (Oakland, 2007), which makes it difficult to ascertain the true quality of the supplier. Indeed, there are many instances where firms have failed to develop their supplier's capabilities despite significant investments. For instance, Boeing partnered with

Vought to develop the fuselage of the 787 Dreamliner aircraft. However, despite significant investments from Boeing, Vought was unable to develop the fuselage (Tang and Zimmerman, 2009) and Boeing had to takeover Vought's manufacturing facilities (Sanders, 2009).

These uncertainties lead to the natural question: Should a firm expedite or delay investment in shared suppliers? To examine the optimal investment strategy, we construct a game theoretical model that incorporates the spillover of quality, the uncertainty in the quality improvement potential of the shared supplier, and the noise in the quality performance observed by the buyers. In our model, two firms consider investing in their shared supplier. The firms do not know the true quality improvement potential of the supplier initially, but they share a common belief regarding the true type of the supplier. The performance of the supplier is noisy, so the firms cannot immediately detect the quality improvement potential of the supplier, but they can update their posterior beliefs based on the observed quality performance of the supplier. Each firm can invest in the supplier once at any point in time, and the improvement in the supplier quality benefits both firms at the same time due to spillover effects. Because the investment strategies of the firms affect each other's profits, we formulate the problem as a game.

This essay contributes two main results to the literature. First, we obtain a complete characterization of the Markov perfect equilibria (MPE) of the problem. In particular, we obtain two distinct types of equilibria. Second, we also investigate the impact of the presence of a competing firm on the thresholds and the timing of the investment in a shared supplier, and identify the conditions under which the leader's time to investment is hastened (or delayed) by competition.

2.2 Related Literature

This work draws on and contributes to several streams of the literature including that on buyer-supplier relations, on spillover in supply chains, and on optimal investment strategies in supply chains. In this section, we discuss the literature most relevant to our work, and highlight the key differences in our work.

There is a large body of work in the supply chain literature that examines buyer supplier relations using game theoretic models. Several papers have examined how contracts impact buyer supplier relations (e.g., Lim, 2001; Baiman et al., 2000; Corbett et al., 2005; Balachandran and Radhakrishnan, 2005; Tunca and Zenios, 2006). Scholars have also used game theoretic models to examine how the structure of buyer supplier relations affects innovation, capacity creation and risk sharing within supply chains (e.g., Plambeck and Taylor 2005 and lk et al. 2005, 2007). Our study adds to this literature as we examine buyers' strategy to improve the value within supply chains by investing in the supplier capabilities. Furthermore, our work differs from this broad body of work as we consider both competition and spillover in our game theoretic model.

The supply chain literature has also examined how competition moderates investment in suppliers and how spillover affects supply chains. For instance in the context of competition, scholars have shown that when buyers compete with each other, their preferences for contracting with the suppliers may differ (Feng and Lu, 2012) and that their investment strategies may be different for shared suppliers as compared to that for exclusive suppliers (Feng and Lu, 2013). Along similar lines, in the context of spillovers, scholars have shown that firms can benefit from other firms' investment in cost efficiency (e.g., Knott et al. 2009), inventory management (e.g., Yao et al. 2012), and establishing robust processes (Andritsos and Tang 2014). Within supply chains, spillovers have been studied in the context of a supplier investing in downstream buyers' product improvements (Harhoff 1996), and in the context of buyers investing in suppliers to manage capacity (Qi et al. 2015), increase reliability (Wang et al. 2014) or generate knowledge and reputational spillovers (Kang et al. 2009). In this essay, we integrate these streams and simultaneously explore the effect of competition and spillover on buyer investments at shared suppliers, with a focus on quality improvement. Hence, our study is also aligned with the work of scholars who have explored mechanisms to elicit improved quality performance from suppliers (e.g., Babich and Tang, 2012).

This essay also extends and complements the results in the body of work that focuses on duopoly investment games. Nielsen (2002) studies a case wherein two firms can make an entry in a market at any point in time, and the return to investment increases in the number of firms in the same market (due to some positive externalities). Thijssen et al. (2006) also study a duopoly model, but there is no dynamics of the posterior after the first investment because the follower learns the true profitability immediately.

In a closely related model, Kwon et al. (2016) study a three-stage game of entry into an uncertain market with positive or negative externalities. Their paper focuses on the impact of the follower's learning; in the first stage of their model, there is no dynamic signal, because of which the leader has no chance to learn and hence the first-stage equilibrium is simply static. In contrast, this essay studies the impact of the leader's learning on the subgame perfect (dynamic) equilibrium in the first stage because our model also incorporates a dynamic signal in the first stage. As a result, we obtain two distinct classes of equilibria with dynamic strategy profiles in the first stage. In Kwon et al. (2016), the first-stage equilibrium is simply characterized by a static equilibrium; consequently the two distinct classes of equilibria do not arise. Hence, the characterization of the dynamic equilibrium of the first stage and the leader's dynamic strategy of investment cannot be reproduced by the model of Kwon et al. (2016). Lastly, even though Kwon et al. (2016) also study the time to the first investment in the first stage, it is only in the context of a mixed strategy equilibrium of a static war of attrition game; in the pure strategy equilibria, the time to the first investment is zero. In our paper, the time to the first (leader's) investment is studied in the context of the dynamic learning of the leader in a pure strategy equilibrium. Thus, the meaning of the *time to the first investment* in the two papers is qualitatively different.

2.3 The Model

We consider a model of two identical manufacturing firms dealing with a shared (or common) supplier. The quality improvement potential of the supplier is measured by the return on investment in improving the quality of sourced goods, and is unknown to the two firms. Initially, the two firms place a moderate amount of investment in the supplier which may be considered as initial efforts of the firms to evaluate testing lots or developmental orders from the supplier. The return on investment is in the form of the improved profit earned on account of the improved quality of the sourced goods. For instance, the quality of the sourced goods may be measured by the defect level of the sourced components.

We assume that the quality improvement potential of the supplier can be one of two states: high or low. We call a supplier with the high improvement potential an H-type, and one with the low improvement potential an L-type. Neither firm knows the true type of the supplier, but they both share a common prior belief, i.e., the initial probability that the supplier is of H-type. The firms update their posterior beliefs based on the return to initial investment. Each firm can observe the other firm's profit, so they also share their posterior beliefs.[1]

Beyond the initial exploratory investment, both firms consider making followup investments to improve the quality of supplied goods. Without loss of generality, assume that firm 1 is the leader (L) that makes the first investment at time T_1, and firm 2 is the follower (F) that makes the second investment at time T_2. We model the profit process $X_{i,t}$ for firm $i \in \{1, 2\}$ as a Brownian motion that satisfies $dX_{i,t} = \mu(j)dt + \sigma dB_{i,t}$. Here $\sigma > 0$ is the constant volatility of the cumulative profit, and the drift $\mu(j) \in \{h(j), \ell(j)\}$ is the average profit per unit time for an index $j \in \{0, L, F, 2\}$ where we let j denote the number of investments made if $j = 0$ or 2, or the receiver of the return to investment after the first investment if $j = L$ or F. If no one has invested yet, then both firms' average profit per unit time is $\mu(0)$. If one firm has invested, then the leader receives $\mu(L)$ while the follower receives $\mu(F)$. If both firms have invested, then both receive $\mu(2)$. Note that $\mu(j) = h(j)$ for all j if the supplier is of H-type, and $\mu(j) = \ell(j)$ if the supplier is of L-type. The process $B_{i,t}$ is a white noise (Wiener process) in the profit stream for firm $i \in \{1, 2\}$. The true value of $\mu(j)$ is unknown, but it is publicly known to be either $h(j)$, if the supplier has a high improvement potential (supplier is of H type), or $\ell(j)$, if the supplier has a low improvement potential (supplier is of

[1] This is a reasonable assumption in many manufacturing environments because suppliers often use visual control systems in their facilities, and when buyers from the firm visit the suppliers' facilities, they can easily infer the quality of components supplied to other firms. For instance, in our multiple visits to TMV (an automotive firm that manufactures cars and commercial vehicles) and its suppliers, we found that buyer engineers engaged in quality improvement at suppliers. These suppliers not only supplied to other divisions of TMV, but also supplied to other automotive firms. During our visits to these suppliers, such buyer-driven improvements were visible, and suppliers openly discussed improvements done by firms other than TMV with us and TMV engineers. Therefore it is possible to make informed judgments about the investments by other buyers and the benefit in quality supplied to other buyers. Further, third party surveys such as JD Power help firms to gauge the quality performance of other firms and also help them to infer the performance of suppliers.

L type). We assume that both firms share the same prior belief about $\mu(j)$. At any point in time, either firm can invest in the supplier to improve the drift. Each investment, irrespective of when it is made, costs k, and each firm can make exactly one investment to improve the quality of the sourced goods. We assume that both firms are risk-neutral and have a common discount rate $r > 0$.

We can formulate the model in three stages. In the first stage (when $t < T_1$), no firm has invested yet. In the second stage (when $t \in [T_1, T_2)$), the leader earns $\mu(L)$ per unit time on average while the follower earns $\mu(F)$ per unit time. In the third stage when $t \geq T_2$, both firms have invested, and each firm earns $\mu(2)$ per unit time on average. For each stage, $\mu(j)$ is either $h(j)$ for an H-type or $\ell(j)$ for an L-type supplier ($h(j) > \ell(j)$). Then each firm i's profit processes $X_{i,t}$, $i = 1, 2$, are given as follows:

$$
\begin{aligned}
X_{i,t} &= \textstyle\int_0^t [\mu(0)ds + \sigma dB_{i,s}] \quad , && \text{for } t < T_1 \ , \\
X_{1,t} &= X_{1,T_1} + \textstyle\int_{T_1}^t [\mu(L)ds + \sigma dB_{1,s}] \, , && \text{for } t \in [T_1, T_2) \ , \\
X_{2,t} &= X_{2,T_1} + \textstyle\int_{T_1}^t [\mu(F)ds + \sigma dB_{2,s}] \, , && \text{for } t \in [T_1, T_2) \ , \\
X_{i,t} &= X_{i,T_2} + \textstyle\int_{T_2}^t [\mu(2)ds + \sigma dB_{i,s}] \, , && \text{for } t \geq T_2 \ .
\end{aligned}
$$

We assume that the processes $B_{1,t}$ and $B_{2,t}$ are mutually independent. Note that due to the unobservable noise terms $\sigma B_{i,t}$, neither firm can determine $\mu(j)$. Finally, we assume that the type (quality improvement potential) of the supplier does not change even after investments.

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be the probability space with which $X_{i,t}$, $\mu(j)$, and $B_{i,t}$ are measurable. We let $\{\mathcal{F}_t : t \geq 0\}$ denote the natural filtration with respect to the observable cumulative profit processes $\{X_{i,t} : t \geq 0\}$. The two firms have a common prior $p \equiv \mathbb{P}(\{\mu(0) = h(0)\})$, the probability that the supplier is of H-type.

Next, we construct the posterior updating process. Consider the first stage when both firms earn $\mu(0)$ per unit time on average. Both firms observe $X_{i,t}$, so they update their posterior beliefs by incorporating X_1 and X_2 simultaneously. To construct posterior beliefs, we define the following: the reduced volatility $\tilde{\sigma} \equiv \sigma/\sqrt{2}$; a new one-dimensional standard Brownian motion $W_t \equiv (B_{1,t} + B_{2,t})/\sqrt{2}$; and a new process $\tilde{X}_t \equiv (X_{1,t} + X_{2,t})/2$. The new Brownian motion W is unobservable, but the process $\tilde{X}$ is observable because it is constructed entirely from X_1 and X_2.

The updated posterior beliefs can be constructed from $\tilde{X}$ and t alone. Let $P_t = \mathbb{P}(\{\mu(0) = h(0)\}|\mathcal{F}_t)$ denote the posterior probability at time t. From Bayes rule (Peskir and Shiryaev 2006, pp. 288-289), we derive the following expression for P_t in terms of the observable process $\tilde{X}_t$:

$$P_t = \left(1 + \frac{1-P_0}{P_0} \exp\left\{-\frac{h(0)-\ell(0)}{\tilde{\sigma}^2} \cdot \left[\tilde{X}_t - \frac{h(0)+\ell(0)}{2} t\right]\right\}\right)^{-1} \quad \text{for } t < T_1 . \tag{2.1}$$

Next, we consider the second stage. In this stage, firm 1 earns $\mu(L)$ while firm 2 earns $\mu(F)$ per unit time on average. For notational convenience, let $\Lambda = \sqrt{2}\sqrt{[h(L)-\ell(L)]^2 + [h(F)-\ell(F)]^2}$ and define the following for the second stage:

$$\begin{aligned} \hat{X}_t &\equiv \frac{[h(L)-\ell(L)]X_{1,t} + [h(F)-\ell(F)]X_{2,t}}{\Lambda} , \\ \hat{h} &\equiv \frac{[h(L)-\ell(L)]h(L) + [h(F)-\ell(F)]h(F)}{\Lambda} , \\ \hat{\ell} &\equiv \frac{[h(L)-\ell(L)]\ell(L) + [h(F)-\ell(F)]\ell(F)}{\Lambda} . \end{aligned}$$

Finally, we remark that the posterior process for $t \in (T_1, T_2)$ (in the second stage) is given by

$$P_t = \left(1 + \frac{1-P_{T_1}}{P_{T_1}} \exp\left\{-\frac{\hat{h}-\hat{\ell}}{\tilde{\sigma}^2} \cdot \left[(\hat{X}_t - \hat{X}_{T_1}) - \frac{\hat{h}+\hat{\ell}}{2}(t-T_1)\right]\right\}\right)^{-1} . \tag{2.2}$$

It is straightforward to derive (2.2) from the strong Markov property of P_t and the Bayes rule as in (2.1).

Before we proceed, we list the assumptions for our analysis.

Assumption 2.1 *$\mu(0) < \mu(2) < \mu(L)$ and $\mu(F) < \mu(2)$.*

After the leader invests at time T_1, the leader earns an improved profit stream because the supplier's quality improves, so $\mu(L) > \mu(0)$. The product supplied to the other firm (follower) does not improve as much since the spillover is not perfect, so $\mu(F) < \mu(L)$. The follower's profit stream would additionally improve if he invests as well, so we assume that in stage 2,

$\mu(F) < \mu(2)$. Finally, if both firms improved their products due to investment in the shared supplier in stage 3, then the leader's profit stream would be somewhat reduced due to competition, therefore we specify $\mu(2) < \mu(L)$.

The next two assumptions regard $h_F \equiv h(2) - h(F)$, $\ell_F \equiv \ell(2) - \ell(F)$, $h_{L1} \equiv h(L) - h(0)$, $\ell_{L1} \equiv \ell(L) - \ell(0)$, and $\ell(L) - \ell(F)$:

Assumption 2.2 *$h_F/r > k > \ell_F/r$ and $h_{L1}/r > k > \ell_{L1}/r$.*

Assumption 2.3 *$(\ell(L) - \ell(F))/r - k < 0$.*

Assumption 2.3 implies that the difference between the leader's profit stream and the follower's profit stream is not extraordinarily large, and this allows us to focus on the parameter regimes that are analytically tractable.

Assumption 2.4 *$\hat{h} - \hat{\ell} > h(0) - \ell(0)$.*

Assumption 2.4 is automatically satisfied if a slightly stronger pair of assumptions $h(L) - \ell(L) > h(0) - \ell(0)$ and $h(F) - \ell(F) > h(0) - \ell(0)$ are simultaneously satisfied. These assumptions imply that the difference in quality increases after the first investment and that the signal-to-noise ratio (the rate of learning) increases with the investment, i.e., from $[h(0) - \ell(0)]/\tilde{\sigma}$ to $(\hat{h} - \hat{\ell})/\tilde{\sigma}$. Intuitively, this is justified because with additional investment in a given project the signal about the true quality gets strengthened and the investor learns faster about the project.

2.4 Game of Investment

In this section, we analyze the game-theoretic model proposed in Sec. 2.3. In Secs. 2.4.1 and 2.4.2, we first assume an asymmetric MPE in which one of the firms invests first and the other follows suit. Later in Sec. 2.4.3, we obtain MPEs and find that there are two qualitatively distinct types of pure strategy equilibria.

2.4.1 The Follower's Investment Policy

Following backward induction, consider the second stage, i.e., suppose that an investment has already been made by firm 1. Then firm 2's cumulative profit process satisfies $dX_{2,t} = \mu(F)dt + \sigma dB_{2,t}$ before investment, and $dX_{2,t} = \mu(2)dt + \sigma dB_{2,t}$ after investment. It follows that the second firm's objective is to maximize:

$$\begin{aligned} V_{F,\tau}(p) &= \mathbb{E}^p[\int_0^\tau e^{-rt}dX_{2,t} + \int_\tau^\infty e^{-rt}dX_{2,t} - e^{-r\tau}k] \\ &= \tfrac{1}{r}[ph(F) + (1-p)\ell(F)] + \mathbb{E}^p[g_F(P_\tau)e^{-r\tau}]\,, \end{aligned}$$

where $g_F(x) = \frac{1}{r}[xh_F + (1-x)\ell_F] - k$ and $h_F \equiv h(2) - h(F)$, $\ell_F \equiv \ell(2) - \ell(F)$. The function $g_F(x)$ is the payoff from immediate investment net of the payoff from investment at $t = \infty$. As per Assumption 2.2, we assume that $h_F/r > k > \ell_F/r$, i.e., investment is profitable for the follower only if the supplier is of H-type.

The follower's investment policy is determined by the optimal stopping time τ that maximizes $V_{F,\tau}(p)$. In order to develop the representation for τ , we lay some preliminaries. The posterior process P_t in the second stage evolves as in Eq. (2.2). Define a function

$$\psi_F(x) \equiv x^{(1+\gamma_F)/2}(1-x)^{(1-\gamma_F)/2}\,, \quad \text{where} \quad \gamma_F \equiv \sqrt{1 + \frac{8r\tilde{\sigma}^2}{(\hat{h}-\hat{\ell})^2}}\,. \qquad (2.3)$$

Here $\psi_F(x)$ is an increasing fundamental solution to the differential equation $\mathcal{A}_F\psi_F(p) = 0$ where

$$\mathcal{A}_F \equiv -r + \frac{1}{2}\left(\frac{\hat{h}-\hat{\ell}}{\tilde{\sigma}}\right)^2 p^2(1-p)^2\partial_p^2 \qquad (2.4)$$

is the characteristic differential operator (Oksendal, 2003, Chapter 7) for the process P_t of the second stage given by (2.2). The function $\psi_F(x)$ is introduced because the optimal solution $V_F^*(p) \equiv \sup_\tau V_{F,\tau}(p)$ satisfies the differential equation $\mathcal{A}_\mathcal{F}V_F^*(p) = -\mathbb{E}^p[\mu(F)]$ as per Oksendal (2003, Chapter 10).

Proposition 2.1 *The follower's optimal policy is to invest at the stopping time* $\tau_F = \inf\{t \geq 0 : P_t \geq \theta_F\}$ *where*

$$\theta_F = \frac{(\gamma_F+1)(k-\ell_F/r)}{(\gamma_F+1)(k-\ell_F/r) + (\gamma_F-1)(h_F/r-k)}\,, \qquad (2.5)$$

and the optimal payoff $V_F^*(p) \equiv \sup_\tau V_{F,\tau}(p)$ *is given by*

$$V_F^*(p) = \begin{cases} \frac{1}{r}[ph(F) + (1-p)\ell(F)] + \frac{g_F(\theta_F)}{\psi_F(\theta_F)}\psi_F(p) & \textit{if } p < \theta_F\,, \\ \frac{1}{r}[ph(F) + (1-p)\ell(F)] + g_F(p) & \textit{otherwise}\,. \end{cases} \quad (2.6)$$

Here $V_F^*(p)$ represents the optimal value function for the follower. For $p < \theta_F$, the value function gives the payoff for waiting until P_t hits the threshold θ_F, while for $p \geq \theta_F$, it gives the payoff for immediate investment. Proposition 2.1 asserts that the follower's optimal policy is to invest as soon as P_t hits the optimal upper threshold θ_F.

2.4.2 The Leader's Investment

We now consider the leader's investment problem in the first stage. Before the investment at time T_1, the firm 1's (leader) cumulative profit process satisfies $dX_{1,t} = \mu(0)dt + \sigma dB_{1,t}$, and the posterior process P_t is given by Eq. (2.1). After the investment, the firms 1's cumulative profit process satisfies $dX_{1,t} = \mu(L)dt + \sigma dB_{1,t}$. In this section, we only consider instances when the leader and the follower do not invest simultaneously. We explore the case of simultaneous investment in Sec. 2.4.3. Hence, we only consider values of posterior beliefs within the interval $(0, \theta_F)$. Let us define

$$\begin{aligned} h_{L1} &= h(L) - h(0) > 0, \quad \ell_{L1} = \ell(L) - \ell(0) > 0\,, \\ h_{L2} &= h(2) - h(L) < 0, \quad \ell_{L2} = \ell(2) - \ell(L) < 0\,, \end{aligned} \quad (2.7)$$

and establish the form of the firm 1's (leader) value function as follows.

Lemma 2.1 *Under the constraint that the follower invests at* $\tau_F = \inf\{t \geq 0 : P_t \geq \theta_F\}$ *and the leader invests before* τ_F*, the leader's return from investment at time* τ *is given by*

$$V_{L,\tau}(p) = \frac{1}{r}[ph(0) + (1-p)\ell(0)] + \mathbb{E}^p[g_L(P_\tau)e^{-r\tau}]\,, \quad (2.8)$$

where $g_L(x)$ *is a function defined in the interval* $[0, \theta_F]$ *given by*

$$g_L(x) \equiv \frac{1}{r}[xh_{L1} + (1-x)\ell_{L1}] - k + \frac{1}{r}[\theta_F h_{L2} + (1-\theta_F)\ell_{L2}]\frac{\psi_F(x)}{\psi_F(\theta_F)}\,. \quad (2.9)$$

Here $g_L(x)$ is the payoff from immediate investment net of the payoff from investment at $t = \infty$. As per Assumption 2.2, we assume that $h_{L1}/r > k > \ell_{L1}/r$. We also define

$$\psi_L(x) \equiv x^{(1+\gamma_L)/2}(1-x)^{(1-\gamma_L)/2} , \quad \text{and} \quad \gamma_L \equiv \sqrt{1 + \frac{8r\tilde{\sigma}^2}{(h(0)-\ell(0))^2}} ,$$

where $\psi_L(x)$ is an increasing fundamental solution to the differential equation $\mathcal{A}_L\psi_L(p) = 0$, and

$$\mathcal{A}_L \equiv -r + \frac{1}{2}\left(\frac{h(0)-\ell(0)}{\tilde{\sigma}}\right)^2 p^2(1-p)^2\partial_p^2 \tag{2.10}$$

is the characteristic differential operator (Oksendal, 2003, Chapter 7) for the process P_t of the first stage given by (2.1).

Now we can obtain the leader's optimal policy conditional on the follower's policy.

Proposition 2.2 *Assume $\mu(F) > \mu(0)$ and that the follower invests at time τ_F. Then the leader's optimal policy is to invest at the stopping time $\tau_L = \inf\{t \geq 0 : P_t \geq \theta_L\}$ for some $\theta_L \in (0, \theta_F]$, and the leader's optimal value function is given by*

$$V_L^*(p) \equiv \sup_\tau V_{1,\tau}(p) = \begin{cases} \frac{1}{r}[ph(0) + (1-p)\ell(0)] + \frac{g_L(\theta_L)}{\psi_L(\theta_L)}\psi_L(p) & \text{for } p < \theta_L , \\ \frac{1}{r}[ph(0) + (1-p)\ell(0)] + g_L(p) & \text{for } p \in [\theta_L, \theta_F] . \end{cases} \tag{2.11}$$

As per Oksendal (2003, Chapter 10), the optimal solution $V_L^*(p)$ satisfies $\mathcal{A}_L V_L^*(p) = -\mathbb{E}^p[\mu(0)]$, which is the reason we needed to define the fundamental solution $\psi_L(p)$ that satisfies $\mathcal{A}_L\psi_L(p) = 0$.

Here $V_L^*(p)$ represents the optimal value function for the leader. For $p < \theta_L$, the value function gives the payoff for waiting until P_t hits the threshold θ_L, while for $p \geq \theta_L$, it gives the payoff for immediate investment. This optimal policy can be understood as the intuitive notion that a leader invests immediately when the profit prospect is sufficiently high, i.e., higher than the threshold value θ_L. If P_0 is below θ_L, the leader waits until the posterior P_t hits θ_L and invests after then.

In addition, note that we assume $\mu(F) > \mu(0)$ in Proposition 2.2. When $\mu(F) < \mu(0)$, then there could exist a $\theta'_L > \theta_L$ such that the optimal continuation for the leader is $(0, \theta_L) \cup (\theta'_L, \theta_F)$, i.e., it is optimal for the leader to wait whenever P_t is within either $(0, \theta_L)$ or (θ'_L, θ_F) and to invest otherwise.[2] Note that at θ_L, the smooth-pasting (continuity of the first derivative) condition $g'_L(\theta_L)/g_L(\theta_L) = \psi'_L(\theta_L)/\psi_L(\theta_L)$ still holds as in the case of $\mu(F) > \mu(0)$. However, in all numerical examples of the case $\mu(F) < \mu(0)$ that we studied, the interval (θ_L, θ_F) is subsumed by the preemption region defined in Proposition 2.3, in which both firms try to preemptively invest before the other, so the existence the two disconnected continuation region for the leader is inconsequential.

Intuitively, the condition $\mu(F) < \mu(0)$ implies that, despite the spillover effect, competition diminishes the follower's profit when the leader's product quality improves. When the leader invests, then the quality of the shared supplier improves, and because of spillover at the shared supplier, quality of components supplied to the follower also improves. However, the leader's investment improves its own product quality more than it improves the follower's product quality, since the spillover is not perfect. The leader's superior product quality attracts more customers while the follower loses its own customers due to competition. Thus, even though both firms source and sell better products after the leader's investment, yet there is a possibility that the profit of the follower is lower than what it would have been had the leader not invested.

2.4.3 Markov Perfect Equilibria

In this subsection, we obtain the pure strategy MPE. As a preliminary step, we compare $V_{L,0}(\cdot)$ and $V_F^*(\cdot)$. The function $V_{L,0}(p)$ is the leader's expected return from immediate investment when the current posterior is p while $V_F^*(p)$ is the follower's optimal expected return when the leader has already invested. If $V_{L,0}(p) > V_F^*(p)$, then both firms would have an incentive to invest before the other firm does when the current posterior is p since the leader's return is greater. On the other hand, if $V_{L,0}(p) < V_F^*(p)$, then both firms would be discouraged from being the leader. Thus, the relative magnitudes of $V_{L,0}(p)$ and $V_F^*(p)$ determine whether a given region of p is of preemption type or a

[2] The proof of this statement can be made available upon request.

war of attrition type in equilibrium.

Proposition 2.3 *There exists a unique* $\theta_c \in (0, \theta_F)$ *such that* $V_{L,0}(p) < V_F^*(p)$ *for* $p \in (0, \theta_c)$ *and* $V_{L,0}(p) > V_F^*(p)$ *for* $p \in (\theta_c, \theta_F)$.

This proposition asserts that a critical θ_c always exists such that when $p \in (0, \theta_c)$ then the value of the leader from immediate investment is less than the value that can be obtained by the follower, and as a result neither firm would be willing to invest before the other. In contrast, when $p \in (\theta_c, \theta_F)$, then the value of the leader from immediate investment is higher than that can be obtained by the follower, and as a result each firm has the incentive to preemptively invest before the other. Thus, we characterize (θ_c, θ_F) as the *preemption* region. If both firms concurrently play a *preemption policy,* under which a firm invests immediately unless the other has invested, then there is ambiguity regarding which firm actually invests first and whether simultaneous investment occurs. For convenience, we assume that a simultaneous preemption strategy profile does not lead to simultaneous investment, which is a Pareto-dominated outcome. Instead, we assume that even if both take the preemption policies, they avoid simultaneous investment, and one of the firms ends up successfully preempting the other with a 50% chance, while the other one, upon being preempted, immediately switches to a follower's strategy of investing at time τ_F. This modeling assumption permits us to study pure strategy equilibria.

Theorem 2.1 *Assume* $\mu(F) > \mu(0)$.

(i) Suppose that $\theta_L \geq \theta_c$. *Then there is a pure strategy MPE in which* $(0, \theta_c)$ *is the continuation region for both firms,* $[\theta_c, \theta_F)$ *is the region in which both firms take the preemption policy, and for* $P_t \in [\theta_F, 1)$, *both firms invest immediately.*

(ii) Suppose that $\theta_L < \theta_c$. *Then there exist two pure strategy MPEs with the following characteristics: for* $P_t \in (0, \theta_c)$, *one firm (the leader) invests at* $\tau_L = \inf\{t \geq 0 : P_t \geq \theta_L\}$ *and the other (follower) invests at* $\tau_F = \inf\{t \geq 0 : P_t \geq \theta_F\}$*; for* $P_t \in [\theta_c, \theta_F)$, *both firms take the preemption policy; for* $P_t \in [\theta_F, 1)$, *both firms invest immediately.*

Note that the equilibria have qualitatively different characteristics for $\theta_L \geq \theta_c$ and $\theta_L < \theta_c$. If $\theta_L \geq \theta_c$, *type 1* equilibria occur: For any initial value of

$p < \theta_c$, both firms wait until P_t reaches θ_c, at which time both firms execute the preemption policy. If $\theta_L < \theta_c$, *type 2* equilibria occur: For any initial value of $p < \theta_L$, no firm invests until P_t reaches θ_L, and when $P_t = \theta_L$ for the first time, the leader invests. Note that in type 2 equilibria, the interval (θ_c, θ_F) is still the preemption region, but it is not reached in case the initial probability p is less than θ_c because the leader invests earlier than the time when $P_t = \theta_c$. Finally, the interval (θ_L, θ_c) is characterized as a *war of attrition region* in the sense that the follower is better off than the leader although a leader's optimal policy is to invest immediately. Figure 2.1 illustrates these differences, where we show $V_F^*(p)$, $V_{L,0}(p)$, and $V_L^*(p)$ for Type 2 equilibria in Figure 2.1(a) and Type 1 equilibria in Figure 2.1(b). The parameter values used are $r = 0.9$, $k = 2.5$, $\sigma = 1$, $h(L) = 6$, $l(L) = 3$, $h(2) = 5$, $\ell(2) = 2.4$ for both (a) and (b). For the case of Type 2 equilibria, we used $h(F) = 2.5$, $\ell(F) = 1.5$, $h(0) = 1.6$, $\ell(0) = 0.8$, for the case of Type 1 equilibria, we used $h(F) = 1.5$, $\ell(F) = 0.8$, $h(0) = 2$, $\ell(0) = 1.8$.

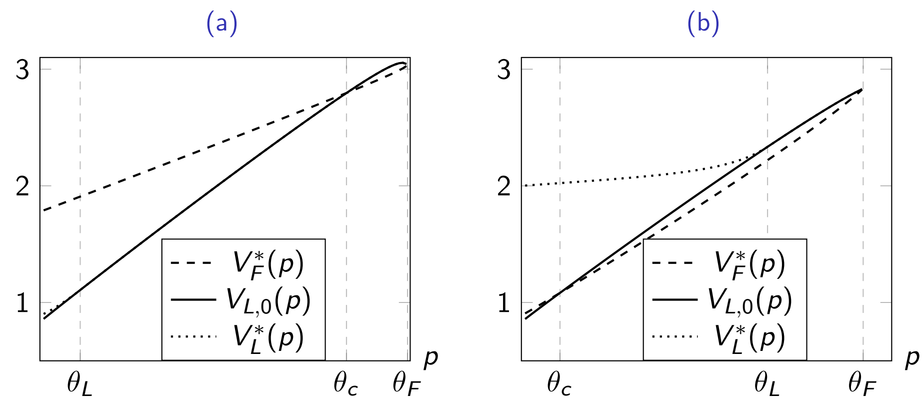
Although our paper limits its scope to pure strategy equilibria, mixed strategy equilibria are also possible when $V_{L,0}(p) < V_F^*(p)$ as shown by Thijssen et al. (2006). However, in realistic scenarios, one firm is often known to be more aggressive at investment than the rival firm, in which case the outcome should be a pure strategy equilibrium. In this essay, we focus on the characterization of pure strategy equilibria in the war of attrition regime.

Note that Theorem 2.1 applies only to the case of $\mu(F) > \mu(0)$. When, $\mu(F) < \mu(0)$, the characterization of equilibria is more complex because the leader's optimal value function $V_L^*(p)$ may have two disconnected continuation regions $(0, \theta_L)$ and (θ_L', θ_F) with $\theta_L' > \theta_L$. However, our extensive numerical experiments suggest that when $\mu(F) < \mu(0)$, we only obtain $\theta_L > \theta_c$, which gives us the equilibria of type 1. On the other hand, when $\mu(F) > \mu(0)$, our numerical experiments suggest that the equilibria are only of type 2. Therefore, we conjecture that the equilibria is characterized based on the relative magnitudes of $\mu(F)$ and $\mu(0)$.

Due to the complexity of the model, we could not obtain general conditions for type 1 or type 2 equilibria. However, we can obtain useful analytical insights in special cases of large values of $\tilde{\sigma}$.[3] In the limit $\tilde{\sigma} \to \infty$, we obtain

[3] It is difficult to obtain meaningful analytical insights in the limit $\tilde{\sigma} \to 0$ because $\lim_{\tilde{\sigma} \to 0} \theta_c = \lim_{\tilde{\sigma} \to 0} \theta_L = \lim_{\tilde{\sigma} \to 0} \theta_F = 1$.

Figure 2.1: $V_F^*(p)$, $V_{L,0}(p)$, and $V_L^*(p)$. (a) is for Type 2 equilibria and (b) is for Type 1 equilibria



the following from the definitions of θ_F, θ_c, and θ_L:

$$\lim_{\tilde{\sigma}\to\infty} \theta_F = \frac{kr - \ell(2) + \ell(F)}{h(2) - h(F) - \ell(2) + \ell(F)}, \tag{2.12}$$

$$\lim_{\tilde{\sigma}\to\infty} \theta_c = \frac{kr - \ell(L) + \ell(F)}{h(L) - h(F) - \ell(L) + \ell(F)}, \tag{2.13}$$

$$\lim_{\tilde{\sigma}\to\infty} \theta_L = \min\left\{\frac{kr - \ell(L) + \ell(0)}{h(L) - h(0) - \ell(L) + \ell(0)}, \lim_{\tilde{\sigma}\to\infty} \theta_F\right\}. \tag{2.14}$$

From these expressions, the following statements follow (after some algebra): *If $\mu(F) > \mu(0)$, then $\theta_c > \theta_L$ (type 2 equilibria occur) for sufficiently large values of $\tilde{\sigma}$. Likewise, if $\mu(F) < \mu(0)$, then $\theta_c < \theta_L$ (type 1 equilibria occur) for sufficiently large values of $\tilde{\sigma}$.*

The intuition for the statements above is as follows. If $\mu(F)$ is larger than $\mu(0)$, then the benefit for the follower is large, so θ_c is relatively higher. On the other hand, if $\mu(F)$ is smaller than $\mu(0)$, then there is greater disadvantage to becoming the follower, so there is higher incentive for preemption, which pushes θ_c to a level lower than θ_L. This intuition is explored in Figure 2.2, which illustrates θ_c and θ_L for $\mu(F) > \mu(0)$ and $\mu(F) < \mu(0)$. We observe that $\theta_c \geq \theta_L$ for $\mu(F) > \mu(0)$, and $\theta_c \leq \theta_L$ for $\mu(F) < \mu(0)$ even for small values of $\tilde{\sigma}$. (We used the same parameter values as in Figure 2.1 except for the values of σ.)

Lastly, in Figure 2.3, we illustrate a sample evolution of the posterior belief process P_t under a type 1 equilibrium for the same set of parameter values as

Figure 2.2: θ_c and θ_L as a function of $\tilde{\sigma}$. (a) is for $\mu(F) > \mu(0)$ and (b) is for $\mu(F) < \mu(0)$.

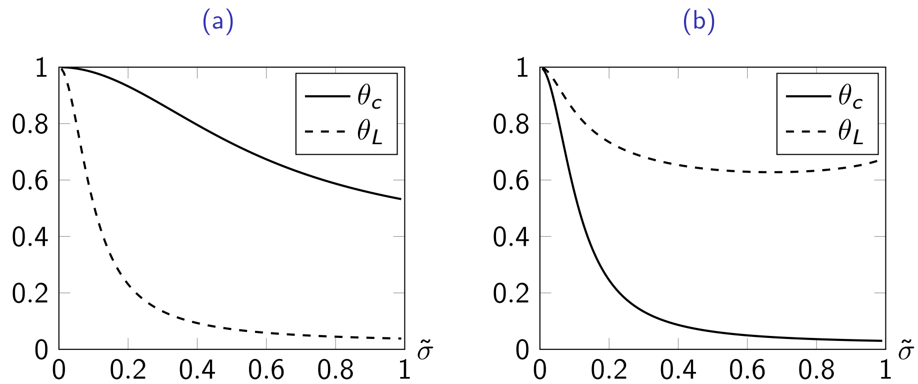
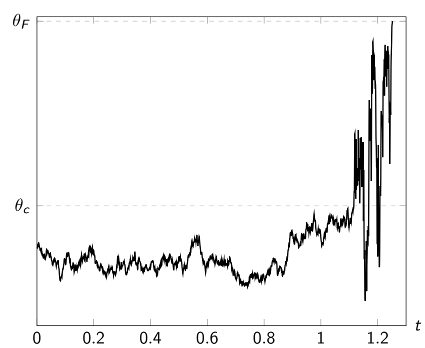


Figure 2.3: A sample path of P_t.



in Figure 2.1(b) except $\sigma = 0.2$. In this example, we set the initial prior P_0 at 0.25. In the time interval $(0, 1.116)$, none of the firms invests in the supplier, but they observe the supplier's performance and update the posterior dynamically. The posterior P_t hits θ_c at time $t = 1.116$, at which time the leader invests. In the time interval $(1.116, 1.252)$, the posterior process fluctuates more widely because of the higher rate of learning (Assumption 2.4), and quickly reaches θ_F at $t = 1.252$, at which time the follower also invests.

2.5 Impact of Competition on the Leader's Investment

In this section, we examine the impact of competition on the time to the first (leader's) investment at a shared supplier. In Section 2.5.1, we consider a benchmark model in which only one firm has the option of investment. In Section 2.5.2, we compare the benchmark result to the game-theoretic model regarding the conditional expected time to the first investment.

2.5.1 Benchmark Model

In the model considered in Section 2.4, suppose that only one firm is capable of investing in the supplier. Then the investing firm's value function from investing at time τ is given by

$$V_{B,\tau}(p) = \frac{1}{r}[ph(0) + (1-p)\ell(0)] + \mathbb{E}^p[g_B(P_\tau)e^{-r\tau}] \ ,$$

where $g_B(x) \equiv \frac{1}{r}[xh_{L1} + (1-x)\ell_{L1}] - k$. The only difference from the leader's value function in Section 2.4.2 is that there is no follow-up investment by another firm. Thus, the posterior process for $t < \tau$ is exactly given by (2.1). Since this is a single decision-maker problem, the solution is provided by a slight modification of Proposition 2.1. The optimal stopping time of investment is given by $\tau_B = \inf\{t > 0 : P_t \geq \theta_B\}$ where

$$\theta_B = \frac{(\gamma_L + 1)(k - \ell_{L1}/r)}{(\gamma_L + 1)(k - \ell_{L1}/r) + (\gamma_L - 1)(h_{L1}/r - k)} \tag{2.15}$$

is the optimal threshold of investment. Finally, the optimal value function is given by

$$\begin{aligned} V_B^*(p) &= \frac{1}{r}[ph(0) + (1-p)\ell(0)] + \frac{g_B(\theta_B)}{\psi_L(\theta_B)}\psi_L(p) \quad \text{for } p < \theta_B \ , \\ &= \frac{1}{r}[ph(0) + (1-p)\ell(0)] + g_B(p) \qquad \text{otherwise.} \end{aligned}$$

2.5.2 Impact of Competition

Our primary interest is on the time to the first investment in the first stage. We first establish the following:

Proposition 2.4 *Let $\tau_\theta = \inf\{t > 0 : P_t \geq \theta\}$ for some $\theta \in (0,1)$. Then the probability of $\tau_\theta < \infty$ is given by*

$$\mathbb{P}^p[\tau_\theta < \infty] = \begin{cases} p/\theta & \text{if } p < \theta\,, \\ 1 & \text{otherwise} \end{cases} ,$$

and the expectation of τ_θ conditional on $\tau_\theta < \infty$ for $p < \theta$ is given by

$$\mathbb{E}^p[\tau_\theta | \tau_\theta < \infty] = \frac{2\tilde{\sigma}^2}{[h(0) - \ell(0)]^2} \ln\left[\frac{\theta(1-p)}{(1-\theta)p}\right] . \qquad (2.16)$$

Note that $\mathbb{E}^p[\tau_\theta] = \infty$ for $p < \theta$ since $\mathbb{P}^p[\tau_\theta = \infty] > 0$ for any $p < \theta$.

In the equilibria that we studied in Section 2.4, the first investment takes place at τ_{θ_m} where $\theta_m \equiv \min\{\theta_c, \theta_L\}$ is the threshold for the first investment. Hence, we are interested in comparing $\mathbb{E}^p[\tau_{\theta_B} | \tau_{\theta_B} < \infty]$ with $\mathbb{E}^p[\tau_{\theta_m} | \tau_{\theta_m} < \infty]$. From (2.16), we have $\mathbb{E}^p[\tau_{\theta_B} | \tau_{\theta_B} < \infty] \geq \mathbb{E}^p[\tau_{\theta_m} | \tau_{\theta_m} < \infty]$ whenever $\theta_B \geq \theta_m$ and vice versa. Therefore we only need to compare θ_B to θ_m.

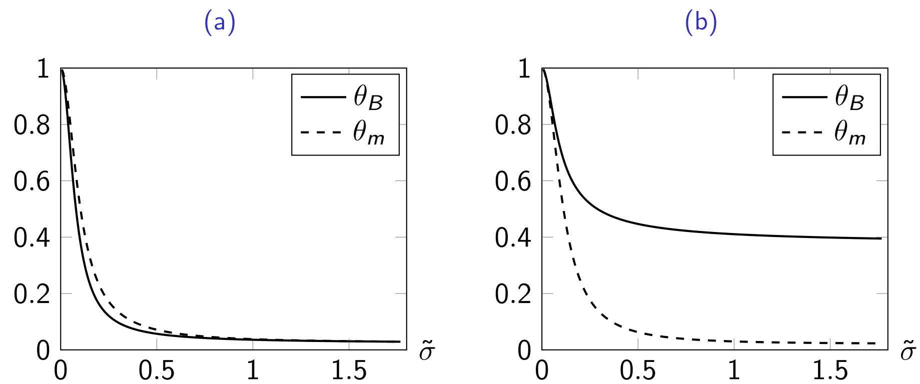
In order to gain meaningful analytical insights, we first examine the case of large values of $\tilde{\sigma}$. From (2.15), the large-$\tilde{\sigma}$ limit is given by

$$\lim_{\tilde{\sigma}\to\infty} \theta_B = \frac{kr - \ell_{L1}}{h_{L1} - \ell_{L1}} .$$

Note that $\lim_{\tilde{\sigma}\to\infty} \theta_B = \lim_{\tilde{\sigma}\to\infty} \theta_L$ if $\lim_{\tilde{\sigma}\to\infty} \theta_L < \lim_{\tilde{\sigma}\to\infty} \theta_F$.

We first examine the case of $\theta_L < \theta_c$ so that $\theta_m = \theta_L$. In this case, the leader can invest in the supplier at his own pace without being concerned about preemption from the competitor because the leader's optimal threshold is lower than that in the preemption region. Furthermore, we can intuitively argue that $\theta_B < \theta_L$ as follows: The existence of a competitor decreases the profit for the leader as compared to the benchmark case, and consequently, the leader's incentive to invest is lower, which delays the investment. In fact, we can have the following result.

Figure 2.4: θ_B and θ_m as a function of $\tilde{\sigma}$. (a) is for $\mu(F) > \mu(0)$ and (b) is for $\mu(F) < \mu(0)$.



Proposition 2.5 *Suppose $\mu(F) > \mu(0)$. Then, $\mu(L) > \mu(2)$ if and only if $\theta_B < \theta_L$.*

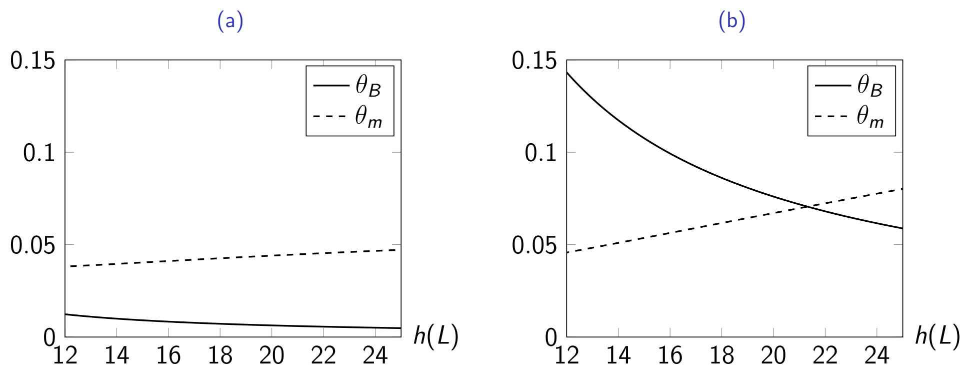
Figure 2.4(a) also confirms this result for general values of $\tilde{\sigma}$. (We used the same parameter values as in Figure 2.1 except for the values of σ.)

Next, from the results of Section 2.4.3, if $\mu(F) < \mu(0)$, then $\theta_m = \theta_c < \theta_B$ for large values of $\tilde{\sigma}$. In other words, if the follower's profit is sufficiently low, then the boundary θ_c of the preemption region is low because each firm would have a strong incentive to preempt the other. In this case, therefore, the preemptive threat is stronger than the loss in profit from the competitor, and we can conclude that the existence of a competitor hastens the first investment. Figure 2.4(b) illustrates this intuition.

In summary, our analytical results show that, in the large-$\tilde{\sigma}$ limit, the first investment is delayed by competition for sufficiently high profit (a high degree of spillover) for the follower, but it is hastened by competition for sufficiently low profit (a low degree of spillover) for the follower. These results are driven by the threat of preemption and the diminished profit of the leader due to the investment of the follower. Thus, we find that the impact of the interplay of competition and spillover on firm's investment strategy in shared supplier is non-trivial.

Lastly, we examine the impact of $h(L)$. At first glance, an increase of $h(L)$ would decrease both θ_B and θ_L because it would increase both $V_{B,\tau}(\cdot)$ and $V_{L,\tau}(\cdot)$. It might also hasten the preemption (decrease θ_c) because the leader's value function increases with $h(L)$ while the follower's value function

Figure 2.5: θ_B and θ_m as a function of $h(L)$. (a) is for $\mu(F) > \mu(0)$ and (b) is for $\mu(F) < \mu(0)$.



would not be affected by $h(L)$. Indeed, from (2.15), it is easy to verify that $\lim_{h(L)\to\infty} \theta_B = 0$. However, for sufficiently large values of $h(L)$, we find that $\theta_m = \min\{\theta_L, \theta_c\}$ does not converge to 0. This is because a large value of $h(L)$ allows the follower to learn the true type of the supplier very quickly due to a large value of the signal-to-noise ratio $\frac{\hat{h}-\hat{\ell}}{\tilde{\sigma}}$ (which represents the rate at which the follower learns about the true type of the supplier). If $\frac{\hat{h}-\hat{\ell}}{\tilde{\sigma}}$ is very large, then the follower can quickly invest in the supplier in case the supplier is of H-type, and consequently, the leader's advantage (stage 2) lasts for a very short time. Therefore, the leader's threshold θ_L or the preemption threshold θ_c are not as low as θ_B for large values of $h(L)$. We conclude that competition delays the first investment for sufficiently high values of $h(L)$. For small values of $h(L)$, however, $\theta_B > \theta_m$ can happen, depending on whether $\mu(F) > \mu(0)$ or $\mu(F) < \mu(0)$. Figure 2.5 illustrates numerical examples of θ_B and θ_m as a function of $h(L)$. (We used the same parameter values as in Figure 2.1 except that $\sigma = 0.5$.)

Recall that $h(L)$ is the leader's return from investing in a supplier when the supplier is of the H-type. Our analysis and numerical experiments show that when this return is very high, then competition always delays the first investment. At lower values of $h(L)$, the interplay of competition and quality spillover comes into play. When $\mu(F) > \mu(0)$, then, the follower's benefit is high due to spillover effect, so the firms' incentive to preempt is weak; therefore the first investment is delayed compared to the benchmark case. However, when $\mu(F) < \mu(0)$, then the follower's benefit is low due to severe competition effect, so the firms' incentive to preempt is strong. Hence, first investment is hastened compared to the benchmark case.

2.6 Discussions

In this section, we discuss the robustness of our assumptions and the relationship between our paper with the idea of strategic substitutes/complements of investments.[4]

2.6.1 Asymmetric Costs of Investment

If the investment costs (k) are asymmetric, then the equilibrium will be asymmetric, and each firm would have different values of θ_c and θ_L in equilibrium. In this subsection, we discuss the equilibrium resulting from asymmetric costs.

Suppose firm i's cost of investment is k_i. Let $V_F^{*(i)}$ be firm i's payoff as a follower, and let $V_{L,0}^{(i)}$ be firm i's payoff as a leader. For each $i = 1, 2$, let $\theta_c^{(i)}$ be the value of p at which $V_F^{*(i)}(p)$ and $V_{L,0}^{(i)}(p)$ cross each other. Then we have $\theta_c^{(1)} \neq \theta_c^{(2)}$ in general. In particular, for sufficiently small differences between k_1 and k_2, one can prove that $\theta_c^{(1)} > \theta_c^{(2)}$ if $k_1 > k_2$. The asymmetric thresholds $\theta_c^{(1)} > \theta_c^{(2)}$ can result in a qualitatively different equilibrium behavior for type 1 equilibria.

We consider a case of type 1 equilibrium in which $k_1 - k_2$ is a very small positive number. Due to $\theta_c^{(1)} > \theta_c^{(2)}$, we have $V_{L,0}^{(1)}(p) < V_F^{*(1)}(p)$ and $V_{L,0}^{(2)}(p) > V_F^{*(2)}(p)$ for $p \in (\theta_c^{(2)}, \theta_c^{(1)})$. In this case, the region $p > \theta_c^{(1)}$ remains as the mutual preemption regime. For the initial prior $p < \theta_c^{(1)}$, however, the equilibrium behavior is different.

If $p \in (\theta_c^{(2)}, \theta_c^{(1)})$, then firm 2 would prefer to be the leader than the follower. However, firm 2's payoff is maximized if it invests at a point infinitesimally close to $\theta_c^{(1)}$. (If $p = \theta_c^{(1)}$, then firm 1 would also have incentive to preempt firm 2, so firm 2's payoff is greater if it preempts firm 1 at a value slightly less than $\theta_c^{(1)}$.) On the other hand, firm 1 would not want to preempt firm 2 until p reaches $\theta_c^{(1)}$. Therefore, we conclude that firm 2 becomes the leader whenever the initial prior satisfies $p < \theta_c^{(1)}$, and the equilibrium strategy for firm 2 is to invest at $\theta_c^{(1)-}$.

[4]The detailed analyses of this section are available upon request.

In the case of a type 2 equilibrium, the asymmetry in k would only shift the values of $\theta_L^{(1)}$ for firm 1 and $\theta_L^{(2)}$ for firm 2, but the behavior of the firms remains qualitatively the same, i.e., the leader (firm i) will wait until P_t hits $\theta_L^{(i)}$ before investment, and the follower (firm j) will wait until $\theta_F^{(j)}$.

2.6.2 Absence of Uncertainty

Note that the impact of uncertainty in the supplier's improvement potential is core to the buyer firms' investment decisions in our model. If there is no uncertainty in the quality improvement potential of the supplier, then the buyer firms invest immediately if the (discounted) quality increase from the investment is larger than the associated cost, and they never invest otherwise. In other words, the problem reduces to a static model if the quality improvement potential is fully known to the investing firms.

2.6.3 Large Spillover

When the quality improvement by spillover from the other firm's investment is much higher than the competition effect (e.g. market stealing effect) between the two firms, we can have $\mu(2) > \mu(L)$. In this case, we have $h_{L2}, \ell_{L2} > 0$ as opposed to (2.7), and based on our analysis, we find that $V_F^*(p) > V_{L,0}(p)$ for all $p \leq \theta_F$, which indicates that the preemption regime does not exist. It follows that type 1 equilibria are not possible, and all equilibria are of type 2, where (θ_L, θ_F) is the war of attrition regime for some $\theta_L \in (0, \theta_F]$ and $(\theta_F, 1)$ is the symmetric investment regime. The intuition underlying this result is that if $\mu(2) > \mu(L)$, then the firms do not have an incentive to preempt the rival firm because the leader's profit stream $\mu(L)$ is less than the profit stream after the follower has invested. In addition, we also find from Proposition 2.5 as well as numerical analyses that $\theta_B > \theta_m$ for both $\mu(F) > \mu(0)$ and $\mu(F) < \mu(0)$, which means that the presence of the competitor hastens the investment.

2.6.4 Investments as Strategic Substitute or Complement

Although the concept of strategic substitutes and complements was originally defined for static investments, we extend it by considering the expected time to investment. Specifically, we consider the conditional expected time $\mathbb{E}^p[T_1|T_1 < \infty]$ to investment of firm 1 under two scenarios: (1) firm 2 has not invested, and firm 1 takes on the leader's role, and (2) firm 2 has already invested and firm 1 is the follower.

Based on our analysis, we find that $\mathbb{E}^p[T_1|T_1 < \infty]$ is shorter for Scenario 1 for sufficiently high p, by contrast $\mathbb{E}^p[T_1|T_1 < \infty]$ is shorter for Scenario 2 for sufficiently low p. This implies that firm 1 invests earlier when the other firm has not invested if p is sufficiently high, and invests earlier when the other firm has invested if p is sufficiently low.

Lastly, we discuss the intuitive explanations for these results. If p is sufficiently small, then the profit prospect is not very high, so there is no first mover advantage, and hence it is crucial for firm 1 to learn about the profitability of investment. Due to the inequality $\hat{h} - \hat{\ell} > h(0) - \ell(0)$, firm 1 can learn faster under Scenario 2 than under Scenario 1, which means it can quickly learn that the profitability of the investment is high under Scenario 2, so the expected time to investment is shorter as compared to that under Scenario 1. If p is sufficiently large, then firm 1 expects higher first mover advantage, so it has incentive to invest quickly under Scenario 1. On the other hand, under Scenario 2, the rival firm has already taken the leading role, so firm 1's reward from investment is smaller, and it does not have incentive to invest quickly.

2.7 Conclusion

Suppliers often cater to the requirements of multiple firms, and some of these firms may compete with one another. When firms consider investments to improve the quality performance of such shared suppliers they should account for two issues. First, there is uncertainty in the returns from the investment in quality improvement at the supplier. Second, improvements from the investments can spillover and benefit other firms that source from the shared suppliers. In this essay, we investigate how spillover, uncertainty, and competition affect the investment strategies of two Bayesian firms that can invest in quality improvement at the shared supplier. Our analyses reveal that the interplay of competition and spillover has differing impact on the investment strategies. We find two distinct types of equilibria depending on the relative effects of competition and spillover. Additionally, we find that a firm's expected time to investment in a shared supplier is delayed or hastened by competition depending on the leader's return from investing in a high quality supplier, i.e. $h(L)$. When $h(L)$ is high, then the presence of a competing investor *always* prompts the leader to delay its investment. In contrast when $h(L)$ is low then the interplay of competition and spillover has a more nuanced impact on the leader's investment decision: When spillover is high then the presence of a competing investor delays the leader's investment whereas when spillover is low, it hastens the leader's investment. Therefore, our results indicate that the leader's optimal investment strategy for the shared suppliers may vary depending on the interplay of (i) spillover (ii) competition and (iii) supplier capabilities.

This essay contributes by providing insights that can help firms craft appropriate strategies for improving quality at shared suppliers and by exploring the nature of spillover from investments at these suppliers. Our results suggest that the interplay of spillover and competition can be important for a significant range of parameters, and that manufacturing firms should take these into account to determine appropriate investment strategies at shared suppliers.

Chapter 3

Strategic Investment in Shared Suppliers with Quality Deterioration

3.1 Introduction

Firms increasingly consider outsourcing as a key business enabler rather than a simple cost-cutting measure (Plotkin, 2016). As the emphasis on outsourcing is increasing, firms are relying more on their suppliers for their product requirements, and their performance is affected by the quality of their sourcing. Many a time, firms work with their suppliers to address quality issues or develop new product development capabilities by investing in improvement initiatives that aim to address specific quality or capability issues. Such investments directly benefit the buyer firms due to the suppliers' improved performance. Supply chain practitioners call such investments *supplier development*, which is defined as "any activity initiated by a buying organization to improve the performance of its suppliers" (Krause et al., 2007). However, suppliers are often not exclusive. They may also cater to the needs of other buyers, some of whom may be competitors, while others may be non-competing.

When buyers share suppliers, the quality and capability related knowledge resident with suppliers due to a specific buyer's investments can spill over to other buyers. For example, Aune et al. (2013) analyze a triadic relationship between an electronics subcontractor Electra and its two important buyers, Ramo and Sensoil. The authors note that each of the two buyers recognized substantial benefits from improving safety-related qualifications of Electra's products. Moreover, because these firms belong to the same industrial cluster (but they do not directly compete in the market), both Ramo and Sensoil were conscious of the other's interests in such developments. However, while Ramo actively helped Electra develop these capabilities, Sensoil did not make any efforts to do the same. Indeed, Sensoil attempted to gain from Ramo's efforts by explicitly expressing to Electra its strong interest in such capabilities. This case demonstrates how a manufacturer can strategically aim to capitalize on capabilities that are developed elsewhere without investing in this development. Therefore, a buyer's investment decision on supplier development may need to take into account the impact on other manufacturers or even the investment strategies of other manufacturers. More specifically, strategic yet potentially inefficient delays in investment, as illustrated in Sensoil's tactics, may arise from the free-rider effect due to spillover of investment.

In a similar vein, Agrawal and Muthulingam (2015) examine an auto manufacturer with two independently controlled divisions that share many of their suppliers. Concerned about suppliers' quality, one of the divisions deployed quality improvement initiatives at its suppliers while the other division did not. Interestingly, one of the main findings of this study is that improvements in the shared suppliers' quality deteriorate over time due to depreciation of organizational learning, which is likely to yield another compounding factor in supplier development. To counter this effect, manufacturers need to make recurrent investments in their suppliers, and the investment game between manufacturers should be considered a repeated game.

In this paper, we examine supplier development investment strategies in the presence of spillover and quality deterioration as well as in the face of other strategic manufacturers. To this end, we formulate a continuous-time game of investment timing between two firms with a shared supplier. At each point in time, each firm decides whether to make a costly investment in the supplier for quality improvements *or* to wait for the other firm to do so. Firms are asymmetric in their investment costs. We take the supplier's quality as the state variable common to both firms, and the profit flow for both firms is a function of the state variable. To incorporate the quality deterioration over time as well as the inherent stochasticity of the quality, we model the state variable as a diffusion process with a negative drift. If an investment is made by either firm, the state variable is restored to an exogenously determined high value. However, the quality deteriorates over time again, and the same investment cycle begins. Thus, the game is infinitely repeated.

We detail equilibria both in pure and mixed strategies. First, we find that a pure strategy Markov perfect equilibrium (MPE) always exists. Next, as a preliminary step to characterizing mixed strategy equilibria, we examine the case where the stochasticity of the quality is ignored. In this special case, we find that a mixed strategy MPE exists under a moderate degree of asymmetry between the firms, and more importantly, we identify the characteristics of mixed strategy MPEs. Motivated by these characteristics, we propose a non-Markovian but subgame perfect extension of the equilibrium concept – named two-phase subgame perfect equilibria (SPE). We find that a two-phase mixed strategy SPE exists *only if* the game is repeated; there is no two-phase mixed strategy SPE if there is only one investment opportunity. This implies that the repetitive nature of the game induces a mixed strategy SPE, which results in inefficient delays in investment. We compare this inefficient equilibrium to the first-best solution in a field study on supplier development of an automobile manufacturer, and find that the efficiency loss from investment delays can be substantial.

Our paper makes two contributions to the literature. First, to the best of our knowledge, our work is the first to characterize the equilibria of a repeated stochastic war of attrition and establish that the repetitive nature of the game drives a mixed strategy equilibrium, which results in an uncoordinated outcome. Note that the equilibrium outcomes of our game-theoretic model are classified into the coordinated and uncoordinated categories: In a *coordinated* outcome, one of the firms invests as soon as the supplier's quality falls below its own optimal threshold, thereby eliminating the inefficiency from incentives to free-ride. In an *uncoordinated* outcome, which results only from a mixed strategy equilibrium, each firm delays the investment in the hope that the other firm will invest soon, thus yielding inefficiency.

Second, we find that a mixed strategy equilibrium is uniquely determined within a class of equilibria named two-phase SPE, in which the first phase is characterized by a stationary strategy and the second phase is characterized as an MPE. This finding is a novel result because attrition games generally admit a continuum of mixed strategy equilibria as shown by Hendricks et al. (1988). Moreover, the uniqueness of the mixed strategy equilibrium in an attrition game implies the uniqueness of the uncoordinated outcome, from which we are able to define the *unique* efficiency loss from the free-rider effect relative to the first-best solution. The unique estimate of the inefficiency can be highly practical and informative to supply chain managers whose priority is cost savings. For instance, a manufacturer considering supplier development can unambiguously determine the cost savings that can be achieved by coordinating its investment with other manufacturers sharing the same supplier.

3.2 Related Domains of Literature

In the context of supply chain operations, there is a stream of work on investment in shared suppliers. Muthlingam and Agrawal (2016) is the first documented field study that sheds light on the conditions under which spillover through shared suppliers happen between two manufacturing divisions of a firm. Wang et al. (2014) examine a two-stage model of two manufacturers investing in their shared supplier's reliability while competing with each other. They focus on characterizing the market competition in the presence of knowledge spillover. Qi et al. (2015) examine capacity investments in suppliers shared with competitors when the realized capacity is stochastic. The spillover effect that they consider is the spillover of production capacity rather than spillover of knowledge. They focus on how capacity contract structures mitigate the problem of competition. Wang et al. (2014) and Qi et al. (2015) focus on models of two manufacturers who directly compete with each other whereas our paper focuses on the issue of coordinated investment between two manufacturers who do not necessarily compete in the same market. Furthermore, our work incorporates the dynamics of the improvement and the timing of investment as essential constituents so that we can address the impact of quality deterioration and the time value of money. In a similar vein, Agrawal et al. (2016) examine investment timing strategies in the presence of spillover, but their focus is on the uncertainty of the supplier's quality improvement potential.

Attrition games are first examined by Maynard Smith (1974) in the context of biological evolution, but have since been applied to many economic problems of concession under uncertainty. For example, they have been applied to a continuous-time stochastic game of duopoly exit by Murto (2004) who examines the impact of uncertainty on the multiplicity of pure strategy MPEs. Decamps and Mariotti (2004) study an investment game under uncertain profitability, in which a second mover advantage arises due to the benefits of learning from the first mover. Obtaining a unique symmetric perfect Bayesian equilibrium, they investigate how an incentive to delay investment is affected by the cost and information structures. Thijssen et al. (2006) also examine a duopoly investment game under uncertainty of the profitability, albeit in the presence of both first and second mover advantages. Kwon et al. (2016) examine a related game of investment but with negative or positive externalities between the players and show that the mixed strategy equilibria exhibit nuanced comparative statics with respect to externalities.

A specific arena of interest in economics literature is the characterization of mixed strategy equilibria of attrition games. Hendricks et al. (1988) examine a continuous-time deterministic game of concession and completely characterize the equilibria. In particular, they show that there is a continuum of mixed strategy equilibria. In a stochastic timing game, Touzi and Vieille (2002) establish the formulation of mixed strategy of stopping times. Extending this stream of literature, Steg (2015) formulates the concept of subgame perfection in a stochastic game and obtains a symmetric mixed strategy SPE of a symmetric war of attrition. Kim et al. (2017) study two-player attrition games with a stochastic state variable and find that there is no mixed strategy MPE in the presence of stochasticity and asymmetry between the two players. Overall, while most of the extant literature focuses on one-shot symmetric games of attrition, our work contributes by examining repeated stochastic games of attrition in the presence of asymmetry between two players.

Lastly, our work addresses a problem of horizontal coordination in supply chains. Lower acquisition cost, enhanced bargaining power, and exclusive purchasing rights are often perceived as key motivations for horizontal coordination between the buyers. For instance, Snyder (1998) and Inderst and Wey (2007) argue that buyer alliances can enhance the buyers' bargaining power. Chipty and Snyder (1999) examine, in addition, the effects of buyer merger on market efficiency. Dana (2012) finds that buyer groups can benefit from the fiercer price competition among suppliers by committing to purchasing from a single supplier. Chen and Roma (2011) examine the impact of competition on the effectiveness of group purchasing. Other papers on buyer bargaining and alliances include Horn and Wolinsky (1988), Chipty (1995), and Stole and Zwiebel (1996). Agrawal (2014) models a raw material supply game between two competing buyers and shows that competing buyers can earn more profits if they cooperate than if they were operating as a hands-off monopoly; this effect increases as raw material becomes more dominant in sourcing. While these prior studies examine horizontal coordination under various contexts, our paper addresses the horizontal coordination problem between manufacturers in the context of supplier development and demonstrates the value of such a coordination by applying a game-theoretic model to a field study.

3.3 Model

In this section, we formulate the model and define the feasible strategies as well as the payoffs for the game. Suppose that two manufacturing firms, labeled by an index $i = 1, 2$ procure components from a shared supplier. The quality (e.g., how low the defect rate is) of the shared supplier is modeled as a diffusion process $X = \{X_t; t \geq 0\}$, which is defined on an interval $\mathscr{I} \subset \mathbb{R}$ and is a solution to the following stochastic differential equation:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \,. \tag{3.1}$$

Here $W = \{W_t; t \geq 0\}$ is the Wiener process, and the initial condition of X is denoted by $X_0 = x$. The drift and volatility functions $\mu : \mathscr{I} \to \mathbb{R}$ and $\sigma : \mathscr{I} \to \mathbb{R}$ are continuous on $\mathscr{I}$. Because there always exists a random variability in the quality performance (Oakland, 2007)[1], we assume $\sigma(\cdot) > 0$ throughout the paper unless otherwise specified. We also assume $\mu(\cdot) < 0$ on $\mathscr{I}$ to capture deterioration in the supplier's quality because of employee turnover, product changes, or equipment wear (Agrawal and Muthulingam, 2015). Whenever necessary, we express the quality of the shared supplier as X_t^x to indicate its initial value x. In the remainder of the paper, we set $\mathscr{I} = \mathbb{R}$ for simplicity although our main results hold for any interval $\mathscr{I}$.

The firms' payoffs are given by the expected value of the cumulative discounted cash flows. The time rate of profit flow is given by a continuous increasing function $\pi : \mathscr{I} \to \mathbb{R}$ of the state variable X, which satisfies the absolute integrability condition (Alvarez, 2001)

$$\mathbb{E}^x\left[\int_0^\infty |\pi(X_t)|e^{-rt}dt\right] < \infty \,, \tag{3.2}$$

where $\mathbb{E}^x[\cdot]$ is the expectation conditional on the initial value $X_0 = x$. Here $r > 0$ is the discount rate common to both firms.

[1] See also the literature in the statistical process control (SPC) for more information.

Firm i can invest (e.g., send its engineers to supplier plant to identify quality improvement opportunities or offer training sessions to workers) in the shared supplier and instantaneously[2] reset the quality X to a given level $\zeta \in \mathscr{I}$ (e.g., six sigma level) at the lump-sum cost of $k_i > 0$. Without loss of generality, we assume that firm 2 is strictly more cost-efficient than firm 1 by setting $k_1 > k_2$. Each firm has an opportunity to make an indefinite number of investments at their discretionary times. As a convention, we say that the game is *in stage* n if $n-1$ investments have been already made. Note that because the quality X is restored to ζ immediately after any investment, the value of X is ζ at the beginning of each stage.

3.3.1 Strategy

In this subsection, we specify the strategy space of the game. We first pay particular attention to Markov strategies (which we formally define below) under which a player's action at time t only depends on the current value X_t of the state variable. The concept of Markov strategies is essential not only for characterizing MPEs in Section 3.4, but also for examining SPEs in Section 3.5.

A firm's strategy is the timing of investment, which can be formally represented as a stopping time measurable with respect to the natural filtration $\mathcal{F}_X$ generated by the process X. Because we are interested in *mixed* strategy equilibria in which a strategy is a *randomized* stopping time (Touzi and Vieille, 2002), we represent a strategy as a cumulative distribution function (CDF) of investment timing. Here a CDF is an $\mathcal{F}_X$-adpated, right-continuous, and non-decreasing process that ranges in the interval $[0, 1]$. Then firm i's strategy for *each stage* n is defined as a collection $G_i^{(n)} := (G_i^{(n,x)})_{x \in \mathscr{I}}$ of CDFs. In fact, a CDF $G_i^{(n,x)}(t)$ is the probability that firm i will invest by time t in the nth stage given the initial condition $X_0 = x$. Note that $G_i^{(n,x)}$ must conform to Bayes' rule (Steg, 2015): for any $t \geq s \geq 0$, we have $G_i^{(n,x)}(t) = G_i^{(n,x)}(s^-) + [1 - G_i^{(n,x)}(s^-)]G_i^{(n,X_s^x)}(t-s)$. The complete specification of firm i's strategy is an infinite sequence of stage-wise strategies $\mathbf{G}_i = \{G_i^{(n)}\}_{n=1}^{\infty}$. Similar notations and conventions for CDFs are also used by Steg (2015).

[2]We relax this assumption in Section 3.6.1.

Following the canonical definition of a Markov strategy (Maskin and Tirole, 2001), which stipulates that the actions of the firms depend solely on the current value of the state variable, we require that the probability of a firm's future investment depends only on the current value of X. This stipulation further characterizes the evolution of the CDFs as follows. First, a discontinuity of a firm's CDF takes place at a hitting time $\tau_A = \inf\{t \geq 0 : X_t \in A\}$ for some subset A of $\mathbb{R}$, and the probability of a firm's investment at time τ_A depends only on the current value X_{τ_A}. Second, whenever the CDFs continuously evolve, the hazard rate of investment of a firm depends only on the current value of X. We take these two conditions as the formal definition of Markov strategies.

A technical challenge in the game of infinite investment opportunities is that the strategy space is vast. Hence, we elect to limit our scope of strategies to the set $\mathcal{S}^\infty$ of all strategies $\mathbf{G}_i = \{G_i^{(n)}\}_{n=1}^\infty$ such that $G_i^{(n)}$'s are identical for all $n \geq 1$. Note that the set $\mathcal{S}^\infty$ has the following convenient property:

Lemma 3.1 *For any $\mathbf{G}_j \in \mathcal{S}^\infty$, firm i's best response $\mathbf{G}_i \in \mathcal{S}^\infty$ exists.*

Lemma 3.1 establishes the following intuition: if firm j employs the same policy for every stage, then there exists a best response of firm i having identical policies for all stages. Lemma 3.1 justifies our focus on equilibria with strategy profiles of the form $\mathcal{S}^\infty \times \mathcal{S}^\infty$. This restriction not only simplifies our search for equilibria, but it also provides the simplest and most practical prescription of investment strategies for a firm to execute. Under this simplification, *firm i's strategy $\mathbf{G}_i$ can be simply represented by a single-stage strategy G_i* without the stage index n. For the remainder of the paper, we use this simplified notation unless otherwise specified.

A notable special case of a strategy G_i is one with a hitting time τ_i of some set $A \subset \mathbb{R}$ at which the CDF G_i^x jumps from 0 to 1 for all x. We call a strategy of this form a *pure* Markov strategy and denote it by $H(\tau_i)$ where $G_i^x(t) = H^x(t; \tau_i) := \mathbf{1}_{\{t \geq \tau_i\}}(t)$. In contrast, if a Markov strategy G_i cannot be represented as $H(\tau_i)$ for any hitting time τ_i, then G_i is called a *mixed* Markov strategy.

Lastly, we introduce the notion of *support* of a mixed Markov strategy G_i as the subset of the state space $\mathscr{I}$ in which firm i invests with a positive probability, i.e., $G_i(\cdot)$ strictly increases in time either continuously or discontinuously.

Formally, we define the support of G_i as follows:

$$\text{supp}(G_i) := \Big\{ x \in \mathscr{I} : \left. \frac{dG_i^y(t)}{dt} \right|_{t=\tau} > 0 \text{ or } \Delta G_i^y(\tau) \in (0,1) \\ \text{for any } y \in \mathscr{I} \text{ whenever } X_\tau^y = x \Big\} \tag{3.3}$$

where $\Delta G_i^y(\tau) := G_i^y(\tau) - G_i^y(\tau^-)$ denotes a jump at time τ.

3.3.2 Payoff

Following the terminology used by Hendricks et al. (1988), if firm i invests earlier than firm j in stage n, then we call firm i a *leader* and firm j a *follower* for stage n. We express firm i's payoff given a strategy profile $\mathcal{G} = (G_i, G_j) \in \mathcal{S}^\infty \times \mathcal{S}^\infty$ as the following recursive equation:

$$V_i(x;\mathcal{G}) = \mathbb{E}^x \Bigg[\iint_0^\infty \Big\{ \int_0^{s\wedge u} \pi(X_t) e^{-rt} dt \\ + e^{-r(s\wedge u)} [\mathbf{1}_{\{s<u\}} l_i^{\mathcal{G}} + \mathbf{1}_{\{s>u\}} f_i^{\mathcal{G}} + \mathbf{1}_{\{s=u\}} m_i^{\mathcal{G}}] \Big\} dG_i(s) dG_j(u) \Bigg] , \tag{3.4}$$

where

$$l_i^{\mathcal{G}} := V_i(\zeta;\mathcal{G}) - k_i \quad \text{and} \quad f_i^{\mathcal{G}} := V_i(\zeta;\mathcal{G})$$

are the rewards to the leader and the follower respectively. If both firms decide to invest at the same time, the reward to each firm is $m_i^{\mathcal{G}} := (l_i^{\mathcal{G}} + f_i^{\mathcal{G}})/2$ by following the convention from the literature (e.g. Dutta et al., 1995) that each firm has an equal chance to be the leader and the follower.

Note that the rewards from investment ($l_i^{\mathcal{G}}$, $f_i^{\mathcal{G}}$, and $m_i^{\mathcal{G}}$) depend on the subsequent payoff $V_i(\zeta;\mathcal{G})$ which also depends on $\mathcal{G}$. If there exists only one opportunity to invest in the shared supplier, however, the rewards to the leader and the follower do not depend on $\mathcal{G}$ and are given by

$$l_i := (R_r\pi)(\zeta) - k_i \quad \text{and} \quad f_i := (R_r\pi)(\zeta) , \tag{3.5}$$

respectively, where $(R_r\pi)(x) := \mathbb{E}^x[\int_0^\infty \pi(X_t)e^{-rt}dt]$ is the payoff from a perpetual stream of $\pi(X_t)$. The function $(R_r\pi)(\cdot)$ is well-defined due to the absolute integrability condition (3.2). In sum, the key feature in games of infinite investment opportunities is that the rewards from investment are recursively determined by the strategy profile $\mathcal{G}$. This feature leads to the emergence of a mixed strategy equilibrium for the game of infinite investment opportunities in Section 3.5, a result different from that of the game of a single investment opportunity.

3.4 Characterization of MPE

The primary goal of this section is to characterize pure and mixed strategy MPEs. A strategy profile (G_i^*, G_j^*) is said to be an equilibrium if $V_i(x; G_i^*, G_j^*) \geq V_i(x; G_i, G_j^*)$ for any $x \in \mathscr{I}$ and G_i. We first obtain pure strategy MPE in Section 3.4.1, and then we examine the mixed strategy MPE of deterministic games (i.e. $\sigma(\cdot) = 0$ in (3.1)) in Section 3.4.2. Mixed strategy equilibria of stochastic games ($\sigma(\cdot) > 0$) are discussed in Section 3.5 because they require a slightly extended solution concept of MPE.

3.4.1 Pure Strategy Equilibria

In this subsection, we construct pure strategy MPEs in which one firm never invests. We first obtain firm i's best response when firm j never invests, i.e., $G_j = H(\infty)$. Let $V_i^*(\cdot)$ denote the optimal value function for firm i given that firm j's strategy is $H(\infty)$. Then $V_i^*(\cdot)$ satisfies the following optimality equation:

$$V_i^*(x) = \sup_{\tau_i \geq 0} \mathbb{E}^x \left[\int_0^{\tau_i} \pi(X_t) e^{-rt} dt + [V_i^*(\zeta) - k_i] e^{-r\tau_i} \right]. \qquad (3.6)$$

The equation can be solved as an optimal stopping problem, and hence, we introduce a function $\phi : \mathscr{I} \to \mathbb{R}$ that denotes a decreasing fundamental solution to the differential equation $\mathcal{A}\phi(x) = 0$ where $\mathcal{A} := \frac{1}{2}\sigma^2(x)\partial_{xx} + \mu(x)\partial_x - r$ is the r-excessive characteristic operator (Oksendal, 2003) for the process X. In addition, we make the following assumption:

Assumption 3.1 *The function*

$$\beta_i(x) := \frac{l_i - (R_r\pi)(x)}{\phi(x) - \phi(\zeta)} \tag{3.7}$$

has a unique maximizer $\theta_i < \zeta$, where l_i is defined in (3.5). Furthermore, $\beta_i'(x) > 0$ for $x < \theta_i$ and $\beta_i'(x) < 0$ for $\theta_i < x < \zeta$.

This assumption[3] ensures that there exists a unique optimal solution to the optimal stopping problem (3.6). By the well-established theory of optimal stopping (Oksendal, 2003; Alvarez 2001, p.322), the value function for firm i is given by $\beta_i(\theta_i)\phi(x) + (R_r\pi)(x)$ for all $x \geq \theta_i$ if firm i's policy is to invest at the stopping time $\tau_i^* := \inf\{t \geq 0 : X_t \leq \theta_i\}$ with threshold θ_i. Note that $\beta_i(\theta_i)$ is the coefficient of $\phi(\cdot)$ in firm i's value function associated with the policy of investing at the threshold θ_i. Therefore, Assumption 3.1 ensures the unique optimal threshold θ_i, which we formally state in the following proposition:

Proposition 3.1 *The optimal stopping time τ_i^* that solves (3.6) is given by*

$$\tau_i^* := \inf\{t \geq 0 : X_t \leq \theta_i\}\,. \tag{3.8}$$

Proposition 3.1 establishes the optimal stopping time to invest as the first moment that the state X_t hits the lower threshold θ_i. Intuitively, a costly investment is worth making only if the quality X falls sufficiently low.

We now establish a pure strategy MPE of the form $(H(\infty), H(\tau_2^*))$. By virtue of Proposition 3.1, it suffices to show that $H(\infty)$ is a best response to $H(\tau_2^*)$. Towards this end, we first establish the following lemma:

Lemma 3.2 *$k_1 > k_2$ implies that $\theta_1 < \theta_2$.*

Intuitively, a firm with a lower investment cost has a higher incentive to invest. Thus, one may anticipate that firm 2, with lower investment cost, is the natural one to be the leader. The following proposition establishes that this is indeed always an MPE.

Proposition 3.2 *The strategy profile $(H(\infty), H(\tau_2^*))$ is an MPE.*

[3] Alvarez (2001, p.325) made an assumption similar to Assumption 3.1 to characterize the optimal stopping times.

On the other hand, if firm 2 *can* expect to be the follower in the *not too distant future*, it is willing to wait beyond τ_2^* until firm 1 invests at $\tau_1^* \geq \tau_2^*$. The following proposition establishes that $(H(\tau_1^*), H(\infty))$ is also a pure strategy MPE as long as the asymmetry $k_1 - k_2$ is not too large.

Proposition 3.3 *There exists $\kappa_p(k_2) > 0$ such that $(H(\tau_1^*), H(\infty))$ is an MPE if $k_1 - k_2 < \kappa_p(k_2)$.*

Notice that in either of the above pure strategy equilibria, it is always the case in every stage that only one of the firms invests and the other one free-rides. In this sense, each firm's role is coordinated in equilibrium as an investor or a free-rider so that the investor firm invests *as if there were no other firms*, thus eliminating costly delays in investment arising from incentives to free-ride.

3.4.2 Mixed Strategy Equilibria: Deterministic Game

We next provide the complete characterization of mixed strategy *SPEs* of the deterministic game. As shown by Kim et al. (2017), a mixed strategy MPE does not exist in an asymmetric stochastic attrition game; this raises the question of whether there exists a mixed strategy equilibrium that is not an MPE. As a preliminary step to addressing this question, we consider SPE yet limit our attention to the deterministic game in this subsection. In fact, because the state variable X of the deterministic game strictly decreases in time, the state variable is equivalent to the calendar time itself within each stage. In this special case, therefore, *the concept of SPEs is completely equivalent to that of MPE*s.

We first provide the complete form of the mixed strategy SPEs in the deterministic game; we later establish that this is indeed the only form of mixed strategy SPEs in Proposition 3.5. In a mixed strategy SPE, each stage begins with the initial value $X_0 = \zeta$ in the *peace phase* (phase 1) during which none of the firms invest until X hits a threshold $\theta \in \{\theta_1, \theta_2\}$. Once X hits θ, the *war phase* (phase 2) begins, during which each firm invests with non-zero probability. In particular, one of the firms (firm i) strategically assigns a non-zero probability (q_i) that it will invest as soon as X hits $\theta = \theta_i$ at time $\tau_i^* = \inf\{t \geq 0 : X_t \in (-\infty, \theta)\}$. If no firm invests at τ_θ, then for all $t > \tau_\theta$, firm i invests with an arrival rate

$$\lambda_i(X_t) = \frac{r(V_j(\zeta;\mathcal{G}) - k_j) - \pi(X_s)}{k_j} \,. \tag{3.9}$$

Here, by the conventional definition of an arrival rate, the probability that firm i invests within an infinitesimal time dt is $\lambda dt + o(dt)$. Therefore, based on the definition of the support in (3.3), $\mathrm{supp}(G_1) = \mathrm{supp}(G_2) = (-\infty, \theta)$ in phase 2.

The form of $\lambda_i(\cdot)$ can be intuitively explained by a heuristic argument. If both firms employ a mixed strategy in an equilibrium, each firm must be indifferent between immediate investment and investment in time dt. This indifference condition can be satisfied if the opponent firm invests with the right level of the arrival rate. To see this, note that the indifference condition at time t for firm j can be written as

$$l_j^{\mathcal{G}} = \pi(X_t)dt + f_j^{\mathcal{G}}\lambda_i(X_t)dt + [1 - \lambda_i(X_t)dt]e^{-rdt}l_j^{\mathcal{G}} + o(dt) \,. \tag{3.10}$$

Here the left-hand-side is the reward $l_j^{\mathcal{G}} = V_j(\zeta;\mathcal{G}) - k_j$ from immediate investment. The right-hand-side is the reward from investing in an infinitesimal time dt, which consists of the profit flow $\pi(X_t)dt$, the reward $f_j^{\mathcal{G}} = V_j(\zeta;\mathcal{G})$ from firm i's investment with probability $\lambda_i(X_t)dt$, and the discounted reward $e^{-rdt}l_j^{\mathcal{G}}$ from investment in time dt with probability $1 - \lambda_i(X_t)dt$. Then it is straightforward to show that (3.10) leads to (3.9).

From the description of the strategy profile above, we construct the corresponding CDFs as follows:

$$G_i^x(t) = \mathbf{1}_{\{X_t^x \le \theta_i\}}(t)\left\{1 - (1 - q_i)\exp\left[-\int_{\tau_i^*}^{t} \lambda_i(X_s^x)ds\right]\right\} , \tag{3.11}$$

$$G_j^x(t) = \mathbf{1}_{\{X_t^x \le \theta_i\}}(t)\left\{1 - \exp\left[-\int_{\tau_i^*}^{t} \lambda_j(X_s^x)ds\right]\right\} , \tag{3.12}$$

where $q_i \in (0,1)$. Here, $V_i(\zeta;\mathcal{G}) = (R_r\pi)(\zeta) + \beta_i(\theta_i)\phi(\zeta)$ and $V_j(\zeta;\mathcal{G}) = (R_r\pi)(\zeta) + \beta_j(\theta_i;q_i)\phi(\zeta)$ where $\beta_i(\cdot)$ is defined by (3.7), and $\beta_j(x;q) := [V_j(\zeta;\mathcal{G}) - (1-q)k_j - (R_r\pi)(x)]/\phi(x)$.

The following proposition establishes the sufficient conditions under which these CDFs constitute a mixed strategy SPE.

Proposition 3.4 *Suppose that $\mathcal{G} = (G_1, G_2)$ is a mixed strategy profile given by (3.11) and (3.12).*

(a) There exists $\kappa(k_2) > 0$ such that if $k_1 - k_2 < \kappa(k_2)$, then $\mathcal{G} = (G_1, G_2)$ is an equilibrium with supp$(G_i) = (-\infty, \theta_2)$, $i \in \{1, 2\}$ for sufficiently high values of q_2.

(b) There exists $\kappa_p(k_2) > 0$ such that if $k_1 - k_2 < \kappa_p(k_2)$, then $\mathcal{G} = (G_1, G_2)$ is an equilibrium with supp$(G_i) = (-\infty, \theta_1)$, $i \in \{1, 2\}$ for sufficiently high values of q_1.

Proposition 3.4 asserts that the deterministic game can admit a continuum of mixed strategy SPEs, in which each stage is decomposed precisely into two distinct phases with the common support between the two firms. Indeed, we can prove that there is no other form of SPEs in the deterministic game, thus establishing that the equilibria depicted in (3.11) and (3.12) are the only possible forms of mixed strategy SPEs in this case.

Proposition 3.5 *Any SPE of the deterministic game belongs to the class of equilibria given in (3.11) and (3.12).*

Motivated by this complete characterization of mixed strategy SPEs, we examine whether a stochastic game can also admit a mixed strategy SPE having this characteristics in the next section.

3.5 Mixed Strategy SPE: Stochastic Game

In this section, we investigate mixed strategy SPEs of stochastic games. In Section 3.5.1, we first introduce a concept of *two-phase mixed strategy SPE*, which is slightly extended beyond MPE. We then construct a mixed strategy equilibrium, which is *unique* within this class of SPEs. In fact, this class of SPEs are the stochastic analog of mixed strategy SPEs obtained in Section 3.4.2. In Section 3.5.2, we discuss the stability of pure strategy equilibria. Particularly, we demonstrate that pure strategy MPEs are not stable in a repeated investment model, thus yielding that a mixed strategy equilibrium is the most likely outcome of the game in the long run.

3.5.1 Two-phase Mixed Strategy SPE

We begin this section by introducing the notion of two-phase mixed strategy SPEs. To this end, it is instructive to recall the salient characteristics of mixed strategy SPEs of the deterministic game given in Section 3.4.2. In mixed strategy SPEs of the deterministic game, there is the common support $(-\infty, \theta)$ within which the CDFs evolve continuously at an arrival rate $\lambda_i(\cdot)$ dependent on the state variable X. Moreover, at the hitting time of the common support, one of the firms immediately invests with a non-zero probability q. This is also a typical form of a mixed strategy equilibrium in a continuous-time deterministic game (Hendricks et al., 1988).

We can generalize this form of SPEs to stochastic games. As in the deterministic game, a mixed strategy equilibrium of the stochastic game can be also parameterized by a probability $q \in (0,1)$ with which one of the firms invests at the hitting time τ_Γ of the common support $\Gamma = (-\infty, \theta)$ for some threshold θ; for time $t > \tau_\Gamma$, the CDFs are continuous in time until the end of the current stage. Thus, each stage of the game comprises of two phases: phase 1, the period before the hitting time τ_Γ of Γ, and phase 2, the period after τ_Γ.

In such a mixed strategy equilibrium of stochastic games, however, we show below that the firms' payoffs depend not only on the state variable but also on the phase of the game, so it is *not* an MPE. Hence, it is imperative to extend the space of mixed strategy equilibria to the set of equilibria represented by $\mathcal{G} = (G_i, G_j)$ that satisfy the following three conditions: (1) $\mathcal{G}$ is *subgame perfect.* (2) G_i and G_j share the same support Γ in phase 2. (3) One of the firms invests at time τ_Γ with a probability $q \in (0,1)$; the CDFs in the second phase are continuous in time. We call this set of equilibria *two-phase mixed strategy SPE* and denote it by $\mathcal{E}$.

These three conditions constitute an argument for the claim that two-phase mixed strategy SPE is a simple yet natural extension of MPE: Condition (1) is a usual refinement of an equilibrium in dynamic games. Condition (2), which stipulates a common support Γ of mixed strategies, is also an easily justifiable condition. This is because mixed strategies should be employed by both firms *contemporaneously* in an equilibrium. Intuitively, in a mixed strategy equilibrium, each firm randomizes a set of investment timings with a probability distribution chosen in such a way that the opponent is indifferent among its own set of investment timings, thus incentivized to randomly mix the set of timings. Condition (3) is equivalent to the following statement: (3)' *One of the firms invests at time* τ_Γ *with a probability* $q \in (0,1)$*; the subsequent strategy profile in phase 2 is MPE.* The equivalence between conditions (3) and (3)' holds because all SPEs with continuous CDFs are MPE[4]. Moreover, because phase 1 is terminated at a hitting time τ_Γ of X, each firm's strategy is Markov in phase 1 as well. It thus follows that both firms' strategies are Markov in each phase[5].

Now, we provide the form of the two-phase mixed strategy SPE denoted by $\mathcal{G}^*$ and establish later that it is indeed a unique equilibrium of the class $\mathcal{E}$ in Theorem 3.1. The strategy profile $\mathcal{G}^*$ is characterized by a common support $\Gamma = (-\infty, \theta_2)$ where θ_2 is defined in Assumption 3.1. In the first phase, both firms simply wait until $\tau_\Gamma = \inf\{t \geq 0 : X_t \in \Gamma\}$ without investing. At time τ_Γ, firm 2 invests in the shared supplier with probability q_2^* given in (3.17). Hence, firm 2's CDF has a discontinuity at time τ_Γ while firm 1's CDF does not. If firm 2 does invest at time τ_Γ, then the current stage of the game ends, and the game moves forward to the next stage. On the other hand, if firm 2 does not invest at time τ_Γ, which happens with probability $1 - q_2^*$, then the second phase commences at τ_Γ, and both firms' CDFs are continuous in time. More specifically, at each point in time t, firm i takes a strategy to invest with probability $\mathbf{1}_{\{X_t \in \Gamma\}} \frac{r l_j^{\mathcal{G}^*} - \pi(X_t)}{k_j} dt$ for the next time time interval $(t, t+dt)$. Because the probability of investment depends only on X under this strategy, the second phase can be characterized as an MPE until the current stage ends. The form of the arrival rate $\mathbf{1}_{\{X_t \in \Gamma\}} \frac{r l_j^{\mathcal{G}^*} - \pi(X_t)}{k_j}$ can be intuitively derived from the same argument as that for the arrival rate (3.9) in the deterministic game.

There are two notable characteristics of this equilibrium: (i) The threshold of the common support is given by θ_2, and (ii) only firm 2 has a non-zero probability q_2^* of investment at the time of entry into Γ. Because firm 2 has a stronger incentive to invest, it is understandable that it has some non-zero probability q of investment at time τ_Γ. Generally, firm 2's investment probability q increases the likelihood that firm 1 will be the follower in subsequent

[4]This can be easily established by the arguments used in Kim et al. (2017).

[5]It can be proved that the mixed strategy equilibrium in Theorem 3.1 is unique within a class of SPE in which the first phase ends at some hitting time and the firms' strategies are Markov within each of the two phases. However, we do not provide the proof here as it is beyond the scope of this paper.

stages, thus boosting firm 1's reward to be the leader in the current stage. For sufficiently high value of q, therefore, firm 1's reward to be the leader coincides with that of firm 2. In fact, $q = q_2^*$ is exactly the value that aligns the two firms' rewards from investment, thereby rendering a common support $\Gamma = (-\infty, \theta_2)$.

Formally, the mixed strategy profile $\mathcal{G}^*$ can be described as below. Because each stage of the game consists of two phases, we let $V_i^{\mathrm{I}}(\cdot; \mathcal{G}^*)$ and $V_i^{\mathrm{II}}(\cdot; \mathcal{G}^*)$ denote the payoff functions in phases 1 and 2 respectively. Similarly, we let G_i^{I} and G_i^{II} denote firm i's CDFs in phases 1 and 2 respectively. Then we can first express the CDF of firm 2 as follows:

$$G_2^{\mathrm{I},x}(t) = \mathbf{1}_{\{t \geq \tau_\Gamma\}}(t)\Big[q_2^* + (1 - q_2^*)G_2^{\mathrm{II},x}(t - \tau_\Gamma)\Big] \text{ for } x \geq \theta_2 \,, \tag{3.13}$$

$$G_2^{\mathrm{II},x}(t) = 1 - \exp\left[-\int_0^t \frac{\mathbf{1}_{\{X_s^x \in \Gamma\}}(s)[r l_1^{\mathcal{G}^*} - \pi(X_s^x)]}{k_1} ds\right] \,. \tag{3.14}$$

Here $l_1^{\mathcal{G}^*} = V_1^{\mathrm{I}}(\zeta; \mathcal{G}^*) - k_1$ is firm 1's expected reward from investment and $G_2^{\mathrm{I},x}(t) = G_2^{\mathrm{II},x}(t)$ for $x < \theta_2$ because $x < \theta_2$ means that the game is in phase 2. Note that firm 2's CDF is discontinuous at $t = \tau_\Gamma$ between the two phases. In contrast, the CDF of firm 1 is always continuous in time, and thus, it can be written without a phase index as follows:

$$G_1^x(t) = 1 - \exp\left[-\int_0^t \frac{\mathbf{1}_{\{X_s^x \in \Gamma\}}(s)[r l_2^{\mathcal{G}^*} - \pi(X_s^x)]}{k_2} ds\right] \,, \tag{3.15}$$

where $l_2^{\mathcal{G}^*} = V_2^{\mathrm{I}}(\zeta; \mathcal{G}^*) - k_2$. By the well-established theory of optimal stopping (Oksendal, 2003; Alvarez 2001, p.322), the corresponding payoff functions are given as

$$V_2^{\mathrm{I,II}}(x; \mathcal{G}^*) = \begin{cases} (R_r\pi)(x) + \beta_2(\theta_2)\phi(x) & \text{for } x > \theta_2 \\ l_2^{\mathcal{G}^*} & \text{otherwise} \,. \end{cases}$$

$$V_1^{\mathrm{I}}(x; \mathcal{G}^*) = (R_r\pi)(x) + \beta_1^{\mathrm{I}}(\theta_2; q_2^*)\phi(x) \,,$$

$$V_1^{\mathrm{II}}(x; \mathcal{G}^*) = \begin{cases} (R_r\pi)(x) + \beta_1(\theta_2)\phi(x) & \text{for } x > \theta_2 \\ l_1^{\mathcal{G}^*} & \text{otherwise} \,. \end{cases}$$

where $\beta_i(\theta_2)$ is defined by (3.7), and $\beta_1^{\mathrm{I}}(x; q) := [V_1^{\mathrm{I}}(\zeta; \mathcal{G}^*) - (1 - q)k_1 -$

$(R_r\pi)(x)]/\phi(x)$. Note that firm 1's payoff depends on the phase. In phase 1, if firm 1 anticipates that firm 2 will invest at time τ_Γ with probability q_2^*, then firm 1's payoff depends on q_2^*, which appears in the expression for the coefficient $\beta_1^{\mathrm{I}}(\theta_2; q_2^*)$.

We now establish the necessary and sufficient conditions for this equilibrium:

Theorem 3.1 *The strategy profile $\mathcal{G}^*$ is an equilibrium which exists if and only if*

$$k_1 - k_2 < \kappa(k_2) := \frac{k_2\phi(\zeta)}{\phi(\theta_2) - \phi(\zeta)}\ , \tag{3.16}$$

and

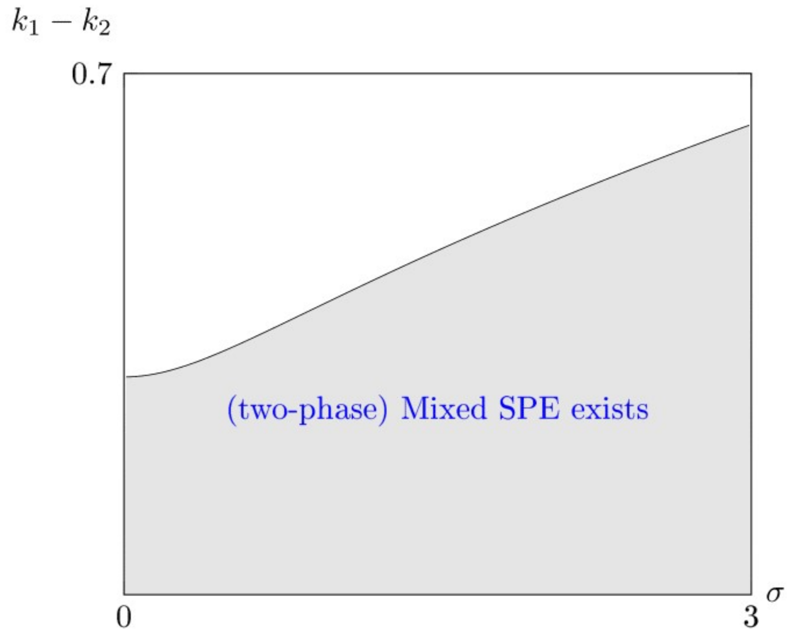
$$q_2^* = \frac{(k_1 - k_2)\phi(\theta_2)}{k_1\phi(\zeta)}\ . \tag{3.17}$$

Furthermore, $\mathcal{G}^*$ is the only equilibrium in $\mathcal{E}$.

We remark that q_2^* is uniquely given by (3.17), and hence, $\mathcal{G}^*$ is the unique mixed strategy equilibrium that belongs to $\mathcal{E}$. Our extensive numerical experiments indicate that $\mathcal{G}^*$ exists for a wide range of model parameter values. For illustration, see Figure 3.1 where the profit flow is of the form $\pi(x) = x$ with $\mu(x) = \mu < 0$ and $\sigma(x) = \sigma > 0$. The two-phase mixed strategy SPE exists only if the cost differential $k_1 - k_2$ of the two firms is below $\kappa(k_2)$. This is because if the cost asymmetry is sufficiently large, then the more efficient firm is strongly compelled to take the leading role in investment so that the outcome is easily coordinated, which leads to a pure strategy equilibrium. In contrast, if the cost asymmetry is moderate, then the two firms are more or less similar to each other so that neither firm is strongly compelled to take the investor's role, which leads to an uncoordinated outcome; either firm can be an investor or a free-rider depending on the realization of the mixed strategy profile. This failure of coordination can yield substantial inefficiency as demonstrated by the field study in Section 3.6.

We next discuss the impact of the repetitive nature of the game for the existence of a two-phase mixed strategy SPE. In stark contrast to Theorem 3.1, it turns out that the game of a single investment opportunity does not admit a two-phase mixed strategy SPE, thus identifying the repetitive nature as a key determinant of the equilibrium characterization of our game.

Figure 3.1: Existence of mixed strategy SPE with respect to $k_1 - k_2$ and $\sigma(x) = \sigma$. We set $\pi(x) = x$ with $r = 0.5$ $\mu(x) = \mu = -0.5$, $k_2 = 0.15$, and $\zeta = 2.5$.



To understand these contrasting results, we first recall that the CDFs of a two-phase mixed strategy SPE must be continuous in time in phase 2. Under such a strategy profile, it can be readily verified that each firm's payoff function in phase 2 is the same as the optimal payoff function given the other firm's strategy of never investing; this is because the payoff function under a mixed strategy equilibrium is equal to the reward from immediate investment when the state variable X is within the support of the equilibrium strategies. Next, note that in the single-investment model, the reward from investment $(R_r\pi)(\zeta) - k_i$ depends *only* on k_i. Thus, $k_1 > k_2$ must *always* lead to $\theta_1 < \theta_2$, i.e., the asymmetry in optimal thresholds of becoming the leader. Because of this asymmetric rewards to be the leader, the incentives for investment are asymmetric between the two firms so that the two firms cannot have a common support for a mixed strategy. In contrast, in the infinite-investment model, even if $k_1 > k_2$, the rewards to be the leader can be equalized, i.e., $V_1^{\mathrm{I}}(\zeta; \mathcal{G}) - k_1 = V_2^{\mathrm{I}}(\zeta; \mathcal{G}) - k_2$ because firm 2 can allocate sufficiently high probability q_2^* of investment at τ_Γ in subsequent stages to increase firm 1's payoff $V_1^{\mathrm{I}}(\zeta; \mathcal{G})$. Thus, a common support of a mixed strategy is possible for an infinite-investment model even if $k_1 > k_2$.

Lastly, a question may arise as to whether there are n-phase SPEs with $n > 2$. Let us consider an n-phase strategy of a firm that stipulates a non-zero probability $q_m > 0$ of investment at the m-th entry of X into Γ where $1 \leq m \leq n$ for some $n > 2$. Because X is a diffusion process, it crosses the boundary of Γ many times within an infinitesimal amount of time (Morters and Peres, 2010, Remark 8.2). It implies that the n entries into Γ will take place almost at the same time. Hence, the total probability that the firm will invest at τ_Γ is

$$q_T = q_1 + (1 - q_1)q_2 + (1 - q_1)(1 - q_2)q_3 + ... + q_n \prod_{m=1}^{n-1} (1 - q_m) .$$

Thus, any *n-phase* strategy effectively reduces to a two-phase strategy with probability q_T of investment at τ_Γ.

3.5.2 Instability of Pure Strategy Equilibria

Although we have found that a two–phase mixed strategy SPE uniquely exists in a game of infinite investment opportunities, the outcome of the game appears to be ambiguous because pure strategy equilibria also exist as established in Section 3.4.1. In this section, we address this issue and examine the stability of equilibria that we have obtained. In particular, we argue that pure strategy equilibria are unstable relative to the unique mixed strategy SPE $\mathcal{G}^*$, especially because our game is infinitely repeated.

To see this, recall first that in either of the pure strategy equilibria, one of the firms is designated as the investor while the other is the free-rider. This outcome is possible if there is only one investment opportunity. Because our game is infinitely repeated, however, the firm designated as the investor can view it as an extremely unfair outcome. Therefore, the designated firm i might rather want to employ a mixed strategy G_i in $\mathcal{G}^*$, *especially if it brings in a higher payoff.* Once firm i begins employing a mixed strategy for a few stages, then firm j will realize that the pure strategy profile is not being played anymore, and it will also shift to a mixed strategy G_j.

In fact, we can verify that the designated investor of a pure strategy equilibrium is not worse off by initiating the shift to the mixed strategy equilibrium. Consider first $\mathcal{H}_1 = (H(\tau_1^*), H(\infty))$ in which firm 1 is the designated investor. Regarding $\beta_i(\cdot)$ in Assumption 3.1 as the objective function $\beta_i(x; k)$ where x is a decision variable and k is a parameter, we can use Envelope Theorem to $\beta_i(x; k)$ to prove that $k_1 > k_2$ implies $\beta_2(\theta_2) > \beta_1(\theta_1)$. It thus follows that $V_1^{\mathrm{I}}(\zeta; \mathcal{G}^*) = V_2^{\mathrm{I}}(\zeta; \mathcal{G}^*) + (k_1 - k_2) > V_2^{\mathrm{I}}(\zeta; \mathcal{G}^*) > V_1^{\mathrm{I}}(\zeta; \mathcal{H}_1)$ where the last inequality follows from Lemma B.6. On the other hand, if firm 2 is the designated investor in the pure strategy MPE $\mathcal{H}_2 = (H(\infty), H(\tau_1^*))$, it is straightforward to verify from Lemma B.6 that $V_2^{\mathrm{I}}(\zeta; \mathcal{G}^*) = V_2^{\mathrm{I}}(\zeta; \mathcal{H}_2)$, which implies that firm 2 is indifferent between the two equilibria. However, even in the MPE $\mathcal{H}_2$, if firm 2 forgoes investment at time $\tau_2 = \inf\{t \geq 0 : X_t < \theta_2\}$ by mistake, then firm 1 may misconstrue this as a signal that firm 2 has begun to play a mixed strategy. Due to the possibility of such mistakes or errors combined with the existence of the mixed strategy equilibrium $\mathcal{G}^*$, the pure strategy equilibrium is not stable in the long run. In sum, we argue that if a mixed strategy equilibrium exists, then pure strategy equilibria are unstable in the infinitely repeated game and the unique mixed strategy equilibrium is the more likely outcome.

3.6 Field Study

In this section, we apply our model to a field study of a large automotive manufacturer, which we call AMC, to demonstrate the efficiency loss from the mixed strategy equilibrium relative to the first-best solution. The field data contains longitudinal quality performance of suppliers shared by two distinct business units: "the car business" (hereafter Car) and "the commercial vehicles business" (hereafter Commercial). These two business units are managed by separate unit leaders with distinct organizational hierarchies and distant manufacturing factories. Moreover, these two business units do not compete with each other in the end product markets. Car and Commercial procure many components from the same suppliers, and assess the quality performance of their suppliers using a variety of evaluation methods. In 2006, Car initiated a supplier quality improvement program by assigning a team of engineers to this program. The engineers in this team were responsible for working with the suppliers of Car; however, 65 percent of Car suppliers use the same manufacturing facilities to supply similar components to Commercial. For three years since 2006, approximately 2,000 quality improvement initiatives were implemented at about 200 suppliers. During this period, only Car engineers were involved in the program and Commercial did not take any efforts towards improving its suppliers' quality. The shared suppliers can be classified into 12 different categories based on the industry sector they belong to. In order to impute our model parameters for each industry sector, we select a representative supplier on which we have collected the most accurate information based on multiple personal on-site visits. Our quality data comprises the time series data on these suppliers' quality (defect rate) around this period.

One notable observation of this supplier development initiative is that the supplier quality does not improve instantaneously, but instead it improves gradually over time. To incorporate this feature into our model, we now assume that each investment takes place over a certain time period during which the supplier's quality gradually albeit stochastically improves. As our numerical study will show, this modification does not alter the main insights discussed in Section 3.5 obtained from the simplified model in Section 3.3.

3.6.1 Model with Gradual Improvement

In this subsection, we first present the model with gradual quality improvement. Because this model is a close variant of the one presented in Section 3.3, we only highlight the salient differences between the two models.

When no investment is being made, the quality X of the shared supplier follows (3.1) with a negative drift $\mu(\cdot) = \mu < 0$ and positive volatility $\sigma(\cdot) = \sigma > 0$. Unlike the model in Section 3.3, however, the investment in the shared supplier by either firm takes time. During the period of investment by either firm, the quality of the supplier increases on average. Hence, we model the quality Y during the investment period as a stochastic process satisfying the following stochastic differential equation:

$$dY_t = \bar{\mu}dt + \bar{\sigma}dW_t \,, \tag{3.18}$$

where $\bar{\mu} = \mu + \bar{u} > 0$ indicates that (i) the quality is expected to improve gradually over time, and (ii) the increment in average rate of quality change is the unit-time effort level $\bar{u}$. The duration of investment ends when Y_t reaches the satisfactory level ζ, at which time the next stage of the game commences. Therefore, each stage consists of a non-investment period which ends at the time τ_I of investment and an investment period which is terminated when Y_t reaches ζ. Each firm's strategy is characterized by the timing τ_I of investment. Each investment by firm i entails an upfront cost $k_i > 0$ and a running cost $c > 0$ per unit-time effort level. For simplicity of the model, we assume that contemporaneous investment by both firms does not happen. This assumption is justified especially in case it is inefficient and suboptimal for at least one of the firms whenever both firms invest contemporaneously.

Given a strategy profile $\mathcal{G}$, the rewards to the leader and the follower now depend on the value of the state variable X_{τ_I} at which the improvement project is initiated. More specifically, if the investment is initiated when $X_{\tau_I} = y$, then the rewards $l_i^{\mathcal{G}}(y)$ and $f_i^{\mathcal{G}}(y)$ are given by

$$l_i^{\mathcal{G}}(y) := f_i^{\mathcal{G}}(y) - \mathbb{E}^y\left[\int_0^{\tau_\zeta} c\bar{u}e^{-rt}dt\right] - k_i$$
$$f_i^{\mathcal{G}}(y) := \mathbb{E}^y\left[\int_0^{\tau_\zeta} \pi(Y_t)e^{-rt}dt + e^{-r\tau_\zeta}V_i(\zeta;\mathcal{G})\right],$$

where $\tau_\zeta := \inf\{t \geq 0 : Y_t^y \geq \zeta\}$ is the first hitting time of $[\zeta, \infty)$. We can then express firm i's expected payoff $V_i(x;\mathcal{G})$ under a strategy profile $\mathcal{G} = (G_i, G_j)$ as

$$\mathbb{E}^x\Bigg[\iint_0^\infty \Big\{\int_0^{s\wedge u} \pi(X_t)e^{-rt}dt$$
$$+ e^{-r(s\wedge u)}[\mathbf{1}_{\{s<u\}}l_i^{\mathcal{G}}(X_s) + \mathbf{1}_{\{s>u\}}f_i^{\mathcal{G}}(X_u) + \mathbf{1}_{\{s=u\}}m_i^{\mathcal{G}}(X_s)]\Big\}dG_i(s)dG_j(u)\Bigg].$$

Utilizing the standard theory of optimal stopping (Oksendal, 2003, Chapter 10), we can obtain the optimal stopping time $\tau_i^* = \inf\{t \geq 0 : X_t^x \leq \theta_i\}$ to $V_i^*(x) = \sup_{\tau_i \geq 0} \mathbb{E}^x[\int_0^{\tau_i} \pi(X_t)e^{-rt}dt + l_i^{(H(\tau_i),H(\infty))}(X_{\tau_i})e^{-r\tau_i}]$ for an optimally chosen threshold θ_i. Then it can be verified that all the results we have obtained up to Section 3.5 remain true with the newly defined threshold θ_i and the reward functions $l_i^{\mathcal{G}}(y)$ and $f_i^{\mathcal{G}}(y)$.

Next, we describe the methods that we use to impute the model parameters from our field data.

Quality X, Y: The quality performance data P_t is given as parts per million (ppm) defects for each month t. Note that the higher the P_t, the poorer is the quality. In order to make this data fit to our model, therefore, we set X_t and Y_t as $-P_t$ so that X_t (or Y_t) increases with the quality. Also, from the time series data during the non-investment and investment periods, we estimate the drifts μ, $\bar{\mu}$ and the volatilities $\sigma, \bar{\sigma}$ of X and Y as the slope of the trend line of this time series data and the standard deviation of the monthly changes of the time series data, respectively.

Investment costs k_i and c: The investment cost to the manufacturers consists of the one-time upfront cost k_i and the operating cost c per each month for the duration of an improvement project. These costs are obtained from internal data of AMC, and we find some variations of these costs across the industry sectors of the target supplier. The investment costs arise due to a variety of factors - some quality issues need investment in process improvements (such as poka-yoke or new machines), while others may need changing the engineering designs or may need improvement in quality assurance routines. To accommodate the asymmetry in investment costs between the manufacturers, we use the calculated cost to estimate the more efficient manufacturer's investment cost (k_2), and assume that the less efficient manufacturer's investment cost (k_1) is 20% higher than k_2.

Satisfactory level of quality ζ: The supplier development project aims to improve the quality to a sufficiently high reset level ζ. According to the data obtained from AMC, ζ is 10 ppm for most supplier sectors. In three supplier sectors out of twelve, ζ is set as six sigma (i.e., 3.4 ppm).

Profit Rate $\pi(\cdot)$: It is not easy to estimate how a manufacturer's profit rate is affected by a particular supplier's quality level because (1) there can be various channels through which defects have an effect on the profit rate and (2) some of defects (especially as getting closer to the market on the value chain) happen so rarely that it is difficult to measure their impact on the profit rate. Hence, we use the cost incurred from processing one defective product of a supplier as the proxy for the profit rate as a function of the quality so that we set $\pi(x) = xd$ where d is the cost incurred from each instance of defective unit in the inspection or product assembly stage (i.e., before the product is sold in the market). Note that this is a conservative estimate for the cost incurred from quality issues because it does not include the quality-related costs that can be incurred in other stages of the value chain (for instance, warranty expenses can be incurred to manufacturers after their products are sold in the market). We find from the data obtained from AMC that these costs d of defects substantially vary across the supplier industry sectors.

Discount Rate r: We use the weighted average cost of capital (WACC) for the automotive industry in the country where our field data was collected. Using the public financial data, we calculate WACC for this focal industry as 1.11% per month in US dollars (i.e., $r = 0.011$).

3.6.2 Comparison to First-Best Solution

Unless the two firms are able to collaborate on the joint investment, they are likely to end up in a mixed strategy equilibrium because the pure strategy equilibria are likely to be unstable due to the repetitive nature of our game as argued in Section 3.5.2. Recall that in a mixed strategy equilibrium, both firms wait for the other firm to invest until their own randomly chosen time. The resulting delays in investment typically yield low system efficiency. Note that the first-best solution can be achieved by a collaborative investment to maximize the joint payoff and share the surplus, and such cooperative development efforts towards the shared supplier are occasionally observed in practice (Aune et al., 2013, p.101). Moreover, such a collaboration is especially feasible in the context of our field study because the two firms are in fact two divisions of a single firm. Hence, exploiting the uniqueness of the mixed strategy equilibrium we have established, it is a legitimate empirical question to examine the loss of efficiency resulting from incentives to free-ride compared to the first-best solution.

In the first-best solution, the two firms cooperate to maximize the joint payoff. We let $V_B^*(\cdot)$ denote the optimal joint payoff from the collaborative investment that satisfies

$$\begin{aligned} V_B^*(x) &= \sup_{\tau>0} \mathbb{E}^x \left[\int_0^{\tau} 2\pi(X_t) e^{-rt} dt + l_B(X_\tau) e^{-r\tau} \right] \\ &= \mathbb{E}^x \left[\int_0^{\tau^*} 2\pi(X_t) e^{-rt} dt + l_B(X_{\tau^*}) e^{-r\tau^*} \right], \end{aligned}$$

where

$$l_B(y) := \mathbb{E}^y \left[\int_0^{\tau_\zeta} [\pi(Y_t) - c\bar{u}] e^{-rt} dt + e^{-r\tau_\zeta} V_B^*(\zeta) \right] - k_2,$$

and τ^* is the optimal time of joint investment. It can be straightforwardly verified (Oksendal, 2003; Alvarez, 2001) that there exists some optimal threshold θ_B such that τ^* is of the form $\tau^* = \inf\{t \geq 0 : X_t \leq \theta_B\}$. Here we assume that the efficiency of the joint investment is as high as that of the more efficient firm, and thus, the cost of investment is k_2, which is the lesser cost of investment between the two firms.

In the numerical study below, we evaluate the efficiency gaps between the first-best solution and the mixed strategy equilibrium obtained in Section 3.5. Towards that end, we compute the absolute and relative gaps

$$e_{abs} := V_B^*(\zeta) - [V_1(\zeta; \mathcal{G}^*) + V_2(\zeta; \mathcal{G}^*)] \quad \text{and} \quad e_{rel} := e_{abs}/[-V_B^*(\zeta)]$$

between $V_B^*(\zeta)$ and $V_1(\zeta; \mathcal{G}^*) + V_2(\zeta; \mathcal{G}^*)$, by which we can measure *cost savings* from the collaborative investment. Note that the value functions $V_B^*(\cdot)$ and $V_i(\cdot; \mathcal{G}^*)$ are negative numbers because they only include costs. Table 3.1 summarizes the imputed parameter values as well as the efficiency results from our numerical analysis for each supplier sector.

It is immediately clear from Table 3.1 that the imputed parameter values, especially μ and d, greatly vary across the industry sector of the target suppliers. For instance, the average rate of quality deterioration ($|\mu|$) in Precision Plastic Molding sector is nearly 30 times greater than that in Forgings sector. Similarly, the average rate of quality improvements ($\bar{\mu}$) in Plastics industry is nearly 20 times as high as that in Glass industry. Moreover, Glass sector has the highest cost of defects (d), which is \$374 per each instance of defect, while it is only \$12 in Plastics sector. There also exist some variations in the upfront cost (k_2) and the satisfactory quality level (ζ).

Interestingly, these differences in parameter values generate substantial variations in the inefficiency of the mixed strategy equilibrium across the supplier sector. For example, the relative efficiency gap (e_{rel}) for Glass sector is estimated as less than 1% whereas the one for Rubber Tubes sector is nearly 25%. These figures indicate not only that the efficiency loss can be indeed quite severe, but also that the characteristics of the target supplier industry can make significant differences in these efficiency losses. In particular, all the three supplier sectors related to Precision (Machining, Plastic Molding, and Rubber Molding) exhibit high efficiency loss, which may suggest that the

Table 3.1: Imputed parameter values and numerical analysis results; $r = 1.11\%$, $k_1 = 1.2k_2$, and $c = \$2.5K$.

Supplier Sector	μ (ppm / mo.)	σ (ppm / mo.)	$\bar{\mu}$ (ppm / mo.)	$\bar{\sigma}$ (ppm / mo.)	k_2 (\$ K)	ζ (ppm)	d (\$)	e_{abs} (\$ K)	e_{rel} (%)	ep_{rel} (%)
Castings	-0.449	0.169	0.115	0.047	3	10	61	140	7.48	0.31
Electrical	-0.145	0.047	0.042	0.016	3	10	28	48	5.77	0.29
Electronics	-0.238	0.103	0.035	0.017	3	10	35	71	6.83	0.25
Forgings	-0.019	0.011	0.077	0.037	6	10	58	21	1.30	0.20
Glass	-0.035	0.014	0.017	0.009	3	10	374	42	0.41	0.05
Machining	-0.248	0.065	0.094	0.016	3	10	53	86	5.47	0.29
Plastics	-0.289	0.145	0.315	0.127	3	10	12	82	19.24	0.89
Precision Machining	-0.348	0.128	0.110	0.052	7	3.4	123	158	11.18	0.83
Precision Plastic Molding	-0.577	0.107	0.096	0.034	6	3.4	58	181	23.03	0.95
Precision Rubber Molding	-0.458	0.112	0.171	0.081	6	10	18	136	19.82	0.84
Rubber Tubes	-0.165	0.067	0.091	0.021	3	3.4	15	50	24.26	1.23
Sheet-Metal	-0.171	0.058	0.154	0.028	3	10	19	56	9.30	0.54

inherent characteristics of the supplier sectors can be a useful indicator for the efficiency loss. Moreover, although the absolute efficiency gap (e_{abs}) is marginal in some supplier sectors, the total amount of monetary values which can be saved from collaboration can be still substantial, ranging in millions of dollars, if we add other quality-related costs such as warranty claim expenses[6].

Towards a deeper understanding of where this loss of efficiency comes from, we further examine the relative efficiency gap (ep_{rel}) between the first-best solution and the pure strategy equilibrium $(H(\infty), H(\tau_2^*))$. Note that the system efficiency of the pure strategy equilibrium $(H(\infty), H(\tau_2^*))$ is higher than that of the mixed strategy equilibrium because the firms *coordinate* the investment. (But it is still lower than that of the first-best solution because each of them is self-interested.) We find that the efficiency loss in a pure strategy equilibrium is at most 1% across all the supplier sectors, which is arguably negligible. This finding has two notable implications. First, the lack of coordination in the mixed strategy equilibrium is the main driving force behind its significant efficiency loss. Hence, if no mixed strategy equilibrium exists, then the firms have no choice but to coordinate the investment, in which case substantial efficiency loss does not take place. Second, because the mixed strategy equilibrium is induced by the repetitive nature of our game, the repetition of investment opportunities is the key factor inducing inefficiency, which underscores the needs to consider the repetitive nature of investment opportunities.

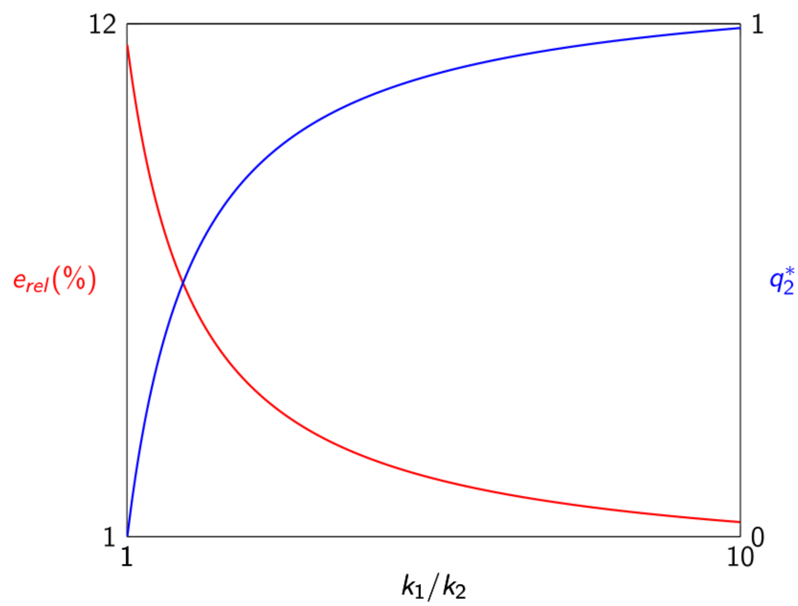
In addition, we investigate the impact of imputed model parameter values on the efficiency loss. Generally speaking, the absolute efficiency gap (e_{abs}) increases in the cost d per defect, which is quite intuitive. Comparing the efficiency gaps of Precision Plastic Molding and Sheet-Metal sectors, which have almost the same values of d, we observe that the efficiency losses (both e_{abs} and e_{rel}) increase in $|\mu|$. This can be explained by the fact that a greater value of $|\mu|$ translates into a higher rate of the quality deterioration, which implies a higher inefficiency of the mixed strategy equilibrium. Examining Plastics and Precision Rubber Molding sectors side by side, however, they exhibit almost the same efficiency gaps although their quality deterioration rates are quite different. This indicates that there exist several moving parts

[6]Each year, U.S. auto makers spend a half billion dollars of warranty costs on average (WarrantyWeek, 2016).

driving these results, and thus, our analysis can serve as a starting point for further explorations on the determinants of the cost savings from collaboration.

Last but not least, we examine the impact of asymmetry k_1/k_2 on the efficiency loss from a mixed strategy equilibrium. In Figure 3.2, note that a mixed strategy equilibrium exists even when k_1 is 10 times bigger than k_2. This confirms our finding in Section 3.5 that a mixed strategy equilibrium exists in a wide range of parameter values. Furthermore, as we increase k_1 while holding k_2 constant, the efficiency loss from a mixed strategy equilibrium decreases. Intuitively, as the asymmetry increases, the coordination between the firms becomes easier, and the efficiency loss from the mixed strategy equilibrium becomes smaller. Formally speaking, greater asymmetry forces firm 2 to place a higher value of q_2^* in equilibrium, which results in a higher equilibrium payoff to firm 1 because of a higher chance q_2^* of becoming the free-rider.

Figure 3.2: Impact of asymmetry on efficiency loss e_{rel} and firm 2's jump q_2^* in Precision Machining Sector



3.7 Extensions and Discussions

3.7.1 Game of $N > 2$ Firms

In this subsection, we extend our model to the case when there are more than two manufacturing firms who source from a shared supplier and consider making an investment in the supplier. Although the key results in this subsection can be extended to the model with any $N > 2$, we will describe our model in the case of $N = 3$ below for a simpler exposition and discuss how this can be generalized to the game of any $N > 2$ firms wherever relevant.

Suppose now that *three* manufacturing firms, labeled by an index $i = 1, 2, 3$ procure components from a shared supplier. Other features of the model – the quality of the shared supplier as a diffusion process (3.1), the profit rate satisfying (3.2), and firm i spending the lump-sum cost $k_i > 0$ for an investment in the supplier – remain the same.

Note that, for each stage, firm i becomes a leader if it invests earlier than *any other* firms $-i$ whereas it becomes a follower otherwise[7]. Therefore, what matters from the point of view of firm i is just the *minimum* of investment timings of all the other firms $-i$. It is thus useful to consider a CDF $G_{-i}(t)$, which is the probability that *at least* one firm $j \neq i$ other than firm i will invest by time t. In fact, G_{-i} can be written as

$$1 - G_{-i}(t) = \prod_{j \neq i} (1 - G_j(t)) \,. \tag{3.19}$$

We can then write each firm i's payoff given a strategy profile $\mathcal{G} = (G_1, G_2, G_3)$ as the following recursive equation:

$$V_i(x; \mathcal{G}) = \mathbb{E}^x \Bigg[\iint_0^\infty \Big\{ \int_0^{s \wedge t} \pi(X_t) e^{-rt} dt + e^{-r(s \wedge t)} [\mathbf{1}_{\{s<t\}} l_i^{\mathcal{G}} + \mathbf{1}_{\{s>t\}} f_i^{\mathcal{G}} + \mathbf{1}_{\{s=t\}} m_i^{\mathcal{G}}] \Big\} dG_i(s) dG_{-i}(t) \Bigg] \,, \tag{3.20}$$

where $l_i^{\mathcal{G}} = V_i(\zeta; \mathcal{G}) - k_i$ and $f_i^{\mathcal{G}} = V_i(\zeta; \mathcal{G})$ are the rewards to the leader and the follower respectively. If $\mathcal{G}$ specifies that any of the three firms invest at

[7] Following an usual convention in game theory, we use the notation $-i$ to denote any other players than player i.

the same time, the reward to each (investing) firm is either $m_i^{\mathcal{G}} = (l_i^{\mathcal{G}} + f_i^{\mathcal{G}})/2$ or $m_i^{\mathcal{G}} = (l_i^{\mathcal{G}} + f_i^{\mathcal{G}})/3$ depending on the number of simultaneous investors.

Recall from Section 3.5 that the CDF G of the equilibrium strategy profile is of the form $G^x(t) = 1 - \exp[-\int_0^t \lambda(X_s^x)ds]$ where $\lambda(X_s)ds$ is the investment probability for the next time interval $(s, s+ds)$. Now, if the CDFs of each firm $j \neq i$ is of this form with the corresponding investment rate $\lambda_j(\cdot)$, then we can obtain from (3.19) that

$$
\begin{aligned}
1 - G_{-i}(t) &= \prod_{j\neq i}(1 - G_j(t)) \\
&= \prod_{j\neq i} \exp\left[-\int_0^t \lambda_j(X_s^x)ds\right] = \exp\left[-\int_0^t \sum_{j\neq i} \lambda_j(X_s^x)ds\right] .
\end{aligned}
$$

It thus follows that if each firm $j \neq i$ invests with the rate of $\lambda_j(\cdot)$, then it is as if all the other firms $-i$ *jointly* invests with the rate of $\sum_{j\neq i}\lambda_j(\cdot)$. This argument will be frequently used throughout this subsection when we establish the equilibrium strategy profiles.

We now examine the characterization of mixed strategy equilibria by using the concept of two-phase mixed strategy SPE, which was introduced in Section 3.5, with natural modification of its definition to accommodate the feasible equilibrium structures in the game of $N > 2$ firms. More specifically, we define the set of two-phase mixed strategy SPEs is the set $\mathcal{E}$ of equilibrium strategy profile $\mathcal{G} = (G_1, G_2, G_3)$ satisfying the three conditions: (1) $\mathcal{G}$ is subgame perfect. (2) *At least two* G_i's share the same support in phase 2. (3) One of the firms invests at time τ_Γ with a probability $q \in (0,1)$; the CDFs in phase 2 are continuous in time.

Case 1: Symmetric Firms ($k_1 = k_2 = k_3$)

As an initial step to examining a game of $N > 2$ firms, we first consider the case when all the three firms are equally cost-efficient, i.e., $k := k_1 = k_2 = k_3$. In this symmetric case, we can prove the unique existence of a two-phase mixed strategy SPE $\mathcal{G}_N^0 = (G_1, G_2, G_3)$. Towards an intuitive illustration of this analytical result as in Section 3.5.1, we first provide the form of this equilibrium strategy profile $\mathcal{G}_N^0$, and formally establish later that it is an equilibrium in Proposition 3.6. The strategy profile $\mathcal{G}_N^0$ is characterized by a common support $\Gamma_N = (-\infty, \theta_i)$ where θ_i is given in Assumption 3.1, which is exactly the same among the three firms (i.e. $\theta := \theta_1 = \theta_2 = \theta_3$) because of the symmetry in their costs. In addition, there are two phases under the strategy profile $\mathcal{G}_N^0$. In phase 1, all the firms wait until $\tau_{\Gamma_N} = \inf\{t \geq 0 : X_t \in \Gamma_N\}$. In phase 2, which begins at the hitting time τ_{Γ_N}, each firm i invests with some probability $\lambda_i(X_t)dt$ for the time interval $(t, t+dt)$. More precisely, the investment rate $\lambda_i(\cdot)$ for each firm i can be expressed as

$$\lambda_1(x) = \mathbf{1}_{\{x\in\Gamma_N\}}\frac{1}{2}[\nu_2(x) + \nu_3(x) - \nu_1(x)] \tag{3.21}$$

$$\lambda_2(x) = \mathbf{1}_{\{x\in\Gamma_N\}}\frac{1}{2}[\nu_1(x) + \nu_3(x) - \nu_2(x)] \tag{3.22}$$

$$\lambda_3(x) = \mathbf{1}_{\{x\in\Gamma_N\}}\frac{1}{2}[\nu_1(x) + \nu_2(x) - \nu_3(x)] \tag{3.23}$$

where $\nu_i(x) = [rl_i^{\mathcal{G}_N^0} - \pi(x)]/k_i$ for $i = 1, 2, 3$. Note that $\sum_{j\neq i} \lambda_j(x) = \nu_i(x)$ whenever $x \in \Gamma_N$ so as to make firm i indifferent between immediate investment and investment in time dt whenever $X_t \in \Gamma_N$. Indeed, because the strategy profile $\mathcal{G}_N^0$ stipulates $l^{\mathcal{G}_N^0} := l_1^{\mathcal{G}_N^0} = l_2^{\mathcal{G}_N^0} = l_3^{\mathcal{G}_N^0}$, the symmetric cost-efficiency implies that $\nu_i(\cdot)$'s are all the same, and thus, so are $\lambda_i(\cdot)$'s for $i = 1, 2, 3$. In other words, each firm i's strategy G_i of the equilibrium strategy profile $\mathcal{G}_N^0 = (G_1, G_2, G_3)$ is identical to each other and can be expressed as follows:

$$G_i^x(t) = 1 - \exp\left[-\int_0^t \lambda_i(X_s^x)ds\right] ,$$

where $\lambda_i(x) = \mathbf{1}_{\{x\in\Gamma_N\}}[rl^{\mathcal{G}_N^0} - \pi(x)]/2k$ for $i = 1, 2, 3$. Moreover, the corresponding payoff functions are written as

$$V_i(x; \mathcal{G}_N^0) = \begin{cases} (R_r\pi)(x) + \beta(\theta)\phi(x) & \text{for } x > \theta \\ l^{\mathcal{G}_N^0} & \text{otherwise} \end{cases} ,$$

where $\beta(\cdot)$ is defined by (3.7), which is identical among the three firms.

We now show that the strategy profile $\mathcal{G}_N^0$ is always a unique equilibrium in $\mathcal{E}$ when the firms are equally cost-efficient.

Proposition 3.6 *Suppose that $k_1 = k_2 = k_3$. Then the strategy profile $\mathcal{G}_N^0$ is a unique equilibrium in $\mathcal{E}$.*

Intuitively, a game of symmetric players should admit a symmetric equilibrium.

What may not be straightforward is, however, its uniqueness in the equilibrium class $\mathcal{E}$. In fact, because the strategy profile $\mathcal{G}_N^0$ is an MPE and the equilibrium class $\mathcal{E}$ includes all the MPE, $\mathcal{G}_N^0$ is indeed a unique mixed strategy MPE of this game. This result is driven by the fact that if any of the firms invests with some probability $q > 0$ at the hitting time of Γ_N, then the other firms would be strictly better off from investing outside Γ_N, which thus leads to failing to satisfy the indifference condition – one of the necessary condition for a mixed strategy equilibrium.

It is straightforward to see that the results in the game of three manufacturing firms above can be carrying over into the game of any number of firms. In general, if there are N firms who are equally cost-efficient at investing in the shared supplier, then the strategy profile $\mathcal{G}_N^0$ described above, with $\lambda_i(x) = \mathbf{1}_{\{x\in\Gamma_N\}}[rl^{\mathcal{G}_N^0} - \pi(x)]/(N-1)k$ for any firm i, constitutes a unique equilibrium in $\mathcal{E}$. Note that we have $\sum_{j\neq i}\lambda_j(x) = \nu_i(x)$ by this choice of $\lambda_{-i}(\cdot)$'s so that firm i is indifferent between immediate investment and investment in time dt whenever $X_t \in \Gamma_N$.

Case 2: A Single Firm with the Lowest Cost ($k_1 = k_2 > k_3$)

Next, we depart from a game of symmetric players and investigate more realistic cases when some of the firms are more cost-efficient than the others. We begin with the case of $k_1 = k_2 > k_3$, i.e., when there is a single cost leader – firm 3 in this case – and the other firms are equally cost-efficient. Let us first define $\bar{k} := k_1 = k_2$ for notational simplicity. As will be shown below, we can obtain a unique two-phase mixed strategy SPE $\mathcal{G}_N^* = (G_1, G_2, G_3)$, which has a similar structure to the strategy profile $\mathcal{G}^*$ obtained in Theorem 3.1. To be more specific, the strategy profile $\mathcal{G}_N^*$ is characterized by a common support $\Gamma_N = (-\infty, \theta_3)$ where θ_3 is the optimal threshold for the most cost-efficient firm (firm 3), given in Assumption 3.1. Moreover, in the first phase of each stage of the game, all the three firms just wait until $\tau_{\Gamma_N} = \inf\{t \geq 0 : X_t \in \Gamma_N\}$, at which time firm 3 invests with probability $q_3^* = (\bar{k} - k_3)/\bar{k}\cdot\phi(\theta_3)/\phi(\zeta)$. In case firm 3 does not invest at time τ_{Γ_N}, which occurs with probability $1 - q_3^*$, the second phase begins immediately and all the firms' CDFs are continuous in time. In particular, at each point in time t, each firm i invests with probability $\lambda_i(X_t)dt$ for the time interval $(t, t+dt)$ where $\lambda_i(\cdot)$'s are given in (3.21) - (3.23) and $\nu_i(x) = [rl_i^{\mathcal{G}_N^*} - \pi(x)]/k_i$ for $i = 1, 2, 3$.

Note that, similarly to the case of symmetric firms, $\mathcal{G}_N^*$ stipulates that

$l_i^{\mathcal{G}_N^*}$'s are the same for all i; however, because only firm 1 and firm 2 have the same investment cost (higher than firm 3), we have $\nu_1(x) = \nu_2(x) < \nu_3(x)$ so that $\lambda_1(x) = \lambda_2(x) = \mathbf{1}_{\{x\in\Gamma_N\}}\nu_3(x)/2$ and $\lambda_3(x) = \mathbf{1}_{\{x\in\Gamma_N\}}[2\nu_1(x) - \nu_3(x)]/2$. Therefore, each firm i's strategy G_i of the equilibrium strategy profile $\mathcal{G}_N^* = (G_1, G_2, G_3)$ can be written as follows:

$$
\begin{aligned}
G_i^{(\mathrm{I,II}),x}(t) &= 1 - \exp\left[-\int_0^t \mathbf{1}_{\{X_s^x\in\Gamma_N\}}\frac{\nu_3(X_s^x)}{2}ds\right] \ , \ i = 1, 2 \\
G_3^{\mathrm{I},x}(t) &= \begin{cases} \mathbf{1}_{\{t\geq\tau_{\Gamma_N}\}}(t)\Big[q_3^* + (1-q_3^*)G_2^{\mathrm{II},x}(t-\tau_{\Gamma_N})\Big] & \text{for } \ x \geq \theta_3 \ , \\ G_3^{\mathrm{II},x}(t) & \text{otherwise} \ . \end{cases} \\
G_3^{\mathrm{II},x}(t) &= 1 - \exp\left[-\int_0^t \mathbf{1}_{\{X_s^x\in\Gamma_N\}}\frac{[2\nu_1(X_s^x) - \nu_3(X_s^x)]}{2}ds\right] \ ,
\end{aligned}
$$

which are of the same form as (3.13) - (3.15) with the adjusted investment rates $\lambda_i(\cdot)$ for each firm i. Similarly, the corresponding payoff functions $V_i(\cdot;\mathcal{G}_N^*)$ are of the same form as the ones for the game of two firms in Section 3.5.1.

We now establish the conditions under which the strategy profile $\mathcal{G}_N^*$ is an equilibrium in $\mathcal{E}$.

Proposition 3.7 *Suppose that $\bar{k} = k_1 = k_2 > k_3$. Then the strategy profile $\mathcal{G}_N^*$ is an equilibrium, which exists if and only if*

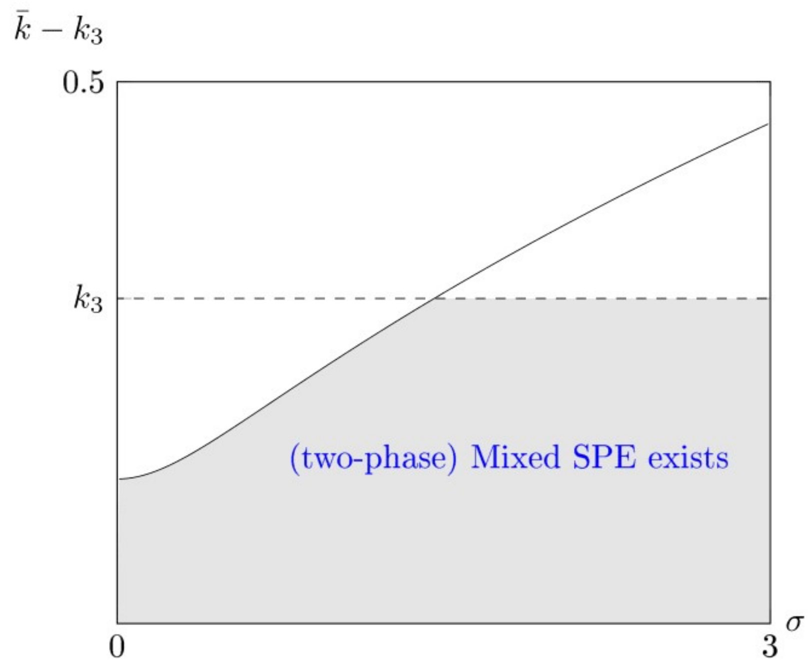
$$
\bar{k} - k_3 < \min\{\kappa(k_3) := \frac{k_3\phi(\zeta)}{\phi(\theta_3) - \phi(\zeta)}, k_3\} \ ,
$$

where $q_3^* = (\bar{k} - k_3)/\bar{k}\cdot\phi(\theta_3)/\phi(\zeta)$. Also, $\mathcal{G}_N^*$ is the only equilibrium in $\mathcal{E}$.

Similarly as in the game of two asymmetric firms discussed in Section 3.5.1, the two-phase mixed strategy SPE exists only if the cost differential $\bar{k} - k_3$ between the three firms is moderate, i.e., below a certain threshold $\kappa(k_3)$; the higher the cost asymmetry is, the easier the firms' roles can be coordinated as explained in Theorem 3.1.

In the game of *three* firms, however, there is an additional condition for the existence of the two-phase mixed strategy SPE, which is $\bar{k} < 2k_3$. This constraint comes from the fact that firm 3's investment rate $\lambda_3(\cdot)$ must be non-negative because G_3 is not a non-decreasing process (thus not a CDF) otherwise. In addition, our numerical experiments suggest that this additional condition ($\bar{k} < 2k_3$) can sometimes further limit the existence of a mixed strategy equilibrium. See Figure 3.3. In other words, there are some parameter regimes where we have $\kappa(k_3) > k_3$ so that a two-phase mixed strategy SPE, which could have existed if there were only one firm other than firm 3, does not exist because there are two equally cost-efficient firms other than the cost leader (firm 3).

Figure 3.3: Existence of mixed strategy SPE with respect to $\bar{k} - k_3$ and $\sigma(x) = \sigma$ when there are *three* firms with $\bar{k} = k_1 = k_2 > k_3$. We set $\pi(x) = x$ with $r = 0.9$ $\mu(x) = \mu = -0.5$, $k_2 = 0.3$, and $\zeta = 2.5$.



Indeed, it is not difficult to obtain this additional condition in the game of any $N > 2$ firms. Suppose that there are $N > 2$ manufacturing firms who consider an investment in their shared supplier and that firm N is the cost leader, i.e., $\bar{k} > k_N$ where $\bar{k}$ is the investment cost of all the other firms (firm i, $i = 1, ..., N-1$). In order for a mixed strategy profile of the form $\mathcal{G}_N^*$ to be an equilibrium, the indifference condition for each firm i within the common support Γ_N requires that the investment rate $\lambda_i(\cdot)$ of firm i must be given as

$$\lambda_i(x) = \mathbf{1}_{\{x \in \Gamma_N\}} \frac{1}{N-1} \left[\sum_{j \neq i} \nu_j(x) - (N-2)\nu_i(x) \right],$$

where $\nu_i(x) = [rl_i^{\mathcal{G}_N^*} - \pi(x)]/k_i$ for $1 \le i \le N$. Then because $l_i^{\mathcal{G}_N^*}$'s and k_i's are all the same for $1 \le i \le N-1$, we can obtain

$$\lambda_i(x) = \mathbf{1}_{\{x \in \Gamma_N\}} \frac{1}{N-1} \nu_N(x) \, , i = 1, ..., N-1$$
$$\lambda_N(x) = \mathbf{1}_{\{x \in \Gamma_N\}} \frac{1}{N-1} \Big[(N-1)\nu_1(x) - (N-2)\nu_N(x) \Big] \, .$$

Using the fact that $l_i^{\mathcal{G}_N^*}$'s are all the same for all i, therefore, the requirement $\lambda_N(\cdot) \ge 0$ implies that the additional condition for the existence of mixed strategy equilibrium in the game of $N > 2$ firms is given by $\bar{k} < (N-1)/(N-2) \cdot k_N$, or equivalently,

$$\bar{k} - k_N < \frac{1}{N-2} k_N \, .$$

Because the right hand side of this inequality condition decreases in N, it follows that, *ceteris paribus*, the two-phase mixed strategy SPE of the form $\mathcal{G}_N^*$ is less likely to exist as the number of equally cost-inferior firms (firm i, $i = 1, ..., N-1$) increases.

Case 3: Multiple Firms with the Lowest Cost ($k_1 > k_2 = k_3$)

Now, we examine the case of $k_1 > k_2 = k_3$, i.e., when there are multiple cost leaders – firm 2 and firm 3 in this case. We define $\underline{k} := k_2 = k_3$ for notational simplicity. Note that, so far in the two-phase mixed strategy SPE of the games of asymmetric firms (in Section 3.5.1 or Case 2 above), the most cost-efficient firm invests with a non-zero probability $q > 0$ at the boundary of the common support of all the firms' CDFs. In this case when there are more than two firms who are the most cost-efficient, however, the strategy profile of this form cannot constitute an equilibrium. This is because any two non-zero probability masses cannot be put at the same point in the state space *in equilibrium*. In addition, any mixed strategy profile, in which only one of the cost leaders invests with a non-zero probability mass, cannot be an equilibrium either because this would make the other cost leader be better off from investing before the state X hits the supposedly common support.

Hence, the only possible form of a two-phase mixed strategy SPE must be atom-free, i.e., all the firms' CDFs are continuous in time everywhere. The characterization of mixed strategy equilibrium in this case, therefore, boils down to determining each firm's investment rate $\lambda_i(\cdot)$. Thus, we will now consider a candidate $\mathcal{G}_N^* = (G_1, G_2, G_3)$ for the equilibrium strategy profile and discuss the conditions under which the given strategy profile can be indeed an equilibrium.

First of all, it is straightforward to see that G_2 and G_3 must share the support $\Gamma_N = (-\infty, \theta)$ where $\theta = \theta_2 = \theta_3$ is the common threshold given in Assumption 3.1; all the firms' CDFs are continuous in time and these two firms (firm 2 and firm 3) have the identical investment cost. The support of firm 1's CDF G_1, however, should be strictly included in Γ_N; if the support of G_1 is Γ_N and it is a part of an equilibrium, then it must be optimal for firm 1 to invest whenever $X_t \leq \theta$ as a best response to G_{-1}. But this leads to a contradiction because the support of G_{-1} is $\Gamma_N = (-\infty, \theta)$ and firm 1's single-decision-maker optimal threshold is $\theta_1 < \theta$. Hence, the support of G_1 should be of the form $\underline{\Gamma}_N := (-\infty, \eta_1)$ where $\theta_1 < \eta_1 < \theta_3$ so that only firm 2 and firm 3 invest at the continuous rates for $X_t \in \overline{\Gamma}_N := (\eta_1, \theta)$ while all the three firms randomize an immediate investment and investment in time dt for $X_t \in \underline{\Gamma}_N$. Hence, $\tau_1 := \inf\{t \geq 0 : X_t \leq \eta_1\}$ is a solution to the optimal stopping problem $\sup_{\tau \geq 0} \mathbb{E}^x[\int_0^\tau \pi(X_t)e^{-rt}dt + l_1^{\mathcal{G}_N^*}e^{-r\tau}]$. In other words, the investment rate $\lambda_i(\cdot)$ for each firm i is expressed as

$$\lambda_1(x) = \mathbf{1}_{\{x \in \underline{\Gamma}_N\}} \frac{1}{2}[2\nu_3(x) - \nu_1(x)]$$
$$\lambda_2(x) = \mathbf{1}_{\{x \in \underline{\Gamma}_N\}} \frac{1}{2}\nu_1(x) + \mathbf{1}_{\{x \in \overline{\Gamma}_N\}}\nu_3(x)$$
$$\lambda_3(x) = \mathbf{1}_{\{x \in \underline{\Gamma}_N\}} \frac{1}{2}\nu_1(x) + \mathbf{1}_{\{x \in \overline{\Gamma}_N\}}\nu_2(x)$$

where $\nu_i(x) = [rl_i^{\mathcal{G}_N^*} - \pi(x)]/k_i$ for $i = 1, 2, 3$. Here we use the fact that $l_2^{\mathcal{G}_N^*} = l_3^{\mathcal{G}_N^*}$ and $k_2 = k_3$. Hence, each firm i's strategy G_i of the equilibrium strategy profile $\mathcal{G}_N^* = (G_1, G_2, G_3)$ can be expressed as $G_i^x(t) = 1 - \exp[-\int_0^t \lambda_i(X_s^x)ds]$. Note that because $l_1^{\mathcal{G}_N^*} < l_2^{\mathcal{G}_N^*} = l_3^{\mathcal{G}_N^*}$ and $k_1 > k_2 = k_3$, we always have $\nu_3(x) > \nu_1(x)$ so that the existence of mixed strategy equilibrium is not bounded by the non-negativity constraint of $\lambda_1(\cdot)$ as in the case of $k_1 = k_2 > k_3$. We now state the conditions under which the strategy profile $\mathcal{G}_N^*$ is a two-phase mixed strategy SPE.

Proposition 3.8 *Suppose that* $k_1 > k_2 = k_3$ *and* $\tau_1 := \inf\{t \geq 0 : X_t \leq \eta_1\}$

with $\theta_1 < \eta_1 < \theta$ *is a solution to the problem* $\sup_{\tau\geq 0}\mathbb{E}^x[\int_0^\tau \pi(X_t)e^{-rt}dt + l_1^{\mathcal{G}_N^*}e^{-r\tau}]$. *Then the strategy profile* $\mathcal{G}_N^*$ *is a two-phase mixed strategy SPE.*

The explicit form of the threshold η_1 cannot be obtained due to its technical difficulties. However, it can be intuitively understood that a solution to the given optimal stopping problem in the statement of Proposition 3.8 must be characterized by a single threshold η_1 with $\theta_1 < \eta_1 < \theta$. As firm 2 and firm 3 expands its boundary of the region where they invest at the rate of $\nu_3(x)$ and $\nu_2(x)$ respectively, firm 1's value function (particularly $l_1^{\mathcal{G}_N^*}$) should increase. And there can be a certain "matching point" η_1 with which τ_1 becomes the solution to the given optimal stopping problem associated with $l_1^{\mathcal{G}_N^*}$.

Case 4: Asymmetric Firms ($k_1 > k_2 > k_3$)

Lastly, we discuss the case of $k_1 > k_2 > k_3$, i.e, when all the three firms are asymmetric. In fact, we can regard this case as a combination of case 2 and case 3 above.

First of all, the candidate for a two-phase mixed strategy SPE $\mathcal{G}_N^* = (G_1, G_2, G_3)$ must be characterized by a common support $\Gamma_N = (-\infty, \theta_3)$ where θ_3 is the optimal threshold for the most cost-efficient firm (firm 3), given in Assumption 3.1. Moreover, in the first phase of each stage of the game, all of the firms wait until $\tau_{\Gamma_N} = \inf\{t \geq 0 : X_t \in \Gamma_N\}$, at which time firm 3 invests with probability $q_3^* = (k_2 - k_3)/k_2 \cdot \phi(\theta_3)/\phi(\zeta)$. The structure of the strategy profile $\mathcal{G}_N^*$ is thus of the same form as the one obtained in case 2 ($k_1 = k_2 > k_3$).

In addition, if firm 3 does not invest at time τ_{Γ_N}, which occurs with probability $1 - q_3^*$, the second phase begins immediately and all the firms' CDFs are continuous in time. In particular, both firm 2 and firm 3 invest with some positive probability during $X_t < \theta_3$ (i.e., the supports of G_2 and G_3 are both Γ_N) whereas the support of firm 1's CDF G_1 is of the form $\underline{\Gamma}_N = (-\infty, \eta_1)$ where $\theta_1 < \eta_1 < \theta_3$, which is of similar kind as the one obtained in case 3 ($k_1 > k_2 = k_3$). Here $\tau_1 = \inf\{t \geq 0 : X_t \leq \eta_1\}$ is a solution to the optimal stopping problem $\sup_{\tau\geq 0}\mathbb{E}^x[\int_0^\tau \pi(X_t)e^{-rt}dt + l_1^{\mathcal{G}_N^*}e^{-r\tau}]$. Therefore, the investment rate $\lambda_i(\cdot)$ for each firm i is expressed as

$$\lambda_1(x) = \mathbf{1}_{\{x\in\underline{\Gamma}_N\}}\frac{1}{2}[\nu_2(x)+\nu_3(x)-\nu_1(x)]$$
$$\lambda_2(x) = \mathbf{1}_{\{x\in\underline{\Gamma}_N\}}\frac{1}{2}[\nu_1(x)+\nu_3(x)-\nu_2(x)] + \mathbf{1}_{\{x\in\overline{\Gamma}_N\}}\nu_3(x)$$
$$\lambda_3(x) = \mathbf{1}_{\{x\in\underline{\Gamma}_N\}}\frac{1}{2}[\nu_1(x)+\nu_2(x)-\nu_3(x)] + \mathbf{1}_{\{x\in\overline{\Gamma}_N\}}\nu_2(x)$$

where $\nu_i(x) = [rl_i^{\mathcal{G}_N^*} - \pi(x)]/k_i$ for $i = 1,2,3$ and $\overline{\Gamma}_N := (\eta_1, \theta_3)$. In other words, each firm's strategy G_i can be expressed as follows:

$$G_i^{(\mathrm{I},\mathrm{II}),x}(t) = 1 - \exp\left[-\int_0^t \lambda_i(X_s^x)ds\right] ,\ i = 1,2,$$
$$G_3^{\mathrm{I},x}(t) = \begin{cases} \mathbf{1}_{\{t\geq\tau_{\Gamma_N}\}}(t)\Big[q_3^* + (1-q_3^*)G_2^{\mathrm{II},x}(t-\tau_{\Gamma_N})\Big] & \text{for } x \geq \theta_3 , \\ G_3^{\mathrm{II},x}(t) & \text{otherwise} . \end{cases}$$
$$G_3^{\mathrm{II},x}(t) = 1 - \exp\left[-\int_0^t \lambda_3(X_s^x)ds\right] ,$$

Note that we will obtain a similar condition as case 2 for the existence of a two-phase mixed strategy SPE of this form. This is because $k_1 > k_2 > k_3$ and $l_1^{\mathcal{G}_N^*} < l_2^{\mathcal{G}_N^*} = l_3^{\mathcal{G}_N^*}$, which makes $\nu_1(x) < \nu_2(x) < \nu_3(x)$. Hence, $\lambda_3(x) > 0$ for $x \in \underline{\Gamma}_N$ if and only if k_1 and k_2 are close enough to k_3, similarly as discussed in case 2 above. Although the precise condition analogous to case 2 cannot be obtained explicitly because of its analytical intractability, it is easy to see from $k_1 > k_2$ that there is a certain threshold $\kappa_2(k_1, k_3) < k_3$ such that $\lambda_3(\cdot)$ can be non-negative whenever needed if $k_2 - k_3 < \kappa_2(k_1, k_3)$.

We now state our discussions in the following proposition.

Proposition 3.9 *Suppose that $k_1 > k_2 > k_3$ and $\tau_1 := \inf\{t \geq 0 : X_t \leq \eta_1\}$ with $\theta_1 < \eta_1 < \theta$ is a solution to the problem $\sup_{\tau\geq 0}\mathbb{E}^x[\int_0^\tau \pi(X_t)e^{-rt}dt + l_1^{\mathcal{G}_N^*}e^{-r\tau}]$. Then the strategy profile $\mathcal{G}_N^*$ is an equilibrium, which exists if and only if*

$$k_2 - k_3 < \min\{\kappa_1(k_3) := \frac{k_3\phi(\zeta)}{\phi(\theta_3)-\phi(\zeta)}, \kappa_2(k_1,k_3)\} ,$$

where $q_3^* = (k_2 - k_3)/k_2 \cdot \phi(\theta_3)/\phi(\zeta)$.

To summarize the findings of this subsection where we discuss the game of $N > 2$ firms, we first find that the two-phase mixed strategy SPE exists in the investment game of more than two firms. However, it turns out that there is additional condition for the existence of a mixed strategy equilibrium if more than two firms consider an investment in the shared suppliers, and the parameter regime under which a two-phase mixed strategy SPE exists can be cut out because of this additional condition as shown in Figure 3.3. Because this additional condition gets stricter as the number of the firms increases (as argued in our discussion in case 2), we can conclude that the two-phase mixed strategy SPE is less likely to exist as there are more number of firms considering an investment in the shared suppliers.

3.7.2 Game of Finite ($M > 1$) Investment Opportunities

In this subsection, we examine the model where two manufacturing firms are allowed to make an investment only up to a finite number $M > 1$ of times. We will first describe our model in the case of $M = 2$ below for a simpler exposition and discuss how this can be generalized to the game of any $M > 2$ investment opportunities later.

Special Case: Game of $M = 2$ Investment Opportunities

Suppose that two manufacturing firms are given *two* opportunities to invest in their shared supplier. Unlike the game of infinite investment opportunities, we do not have to (and it is natural not to) restrict our attention to the set $\mathcal{S}^\infty$ of all strategies such that each stage-wise strategy must be identical. We thus write firm i's strategy as $\mathbf{G}_i = (G_i^{(1)}, G_i^{(2)})$ to accommodate the possibilities that $G_i^{(1)}$ and $G_i^{(2)}$, CDFs for stage 1 and stage 2 respectively, are distinct. We also express a strategy profile for each stage as $\mathcal{G}^{(1)} = (G_1^{(1)}, G_2^{(1)})$ and $\mathcal{G}^{(2)} = (G_1^{(2)}, G_2^{(2)})$. Given stage-wise strategy profiles $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$, therefore, we can define firm i's payoffs $V_i^{(1)}(\cdot)$ and $V_i^{(2)}(\cdot)$ for each stage as follows:

$$
\begin{aligned}
V_i^{(1)}(x;\mathcal{G}^{(1)},\mathcal{G}^{(2)}) = \mathbb{E}^x\Bigg[\iint_0^\infty \Big\{\int_0^{s\wedge u} &\pi(X_t)e^{-rt}dt \\
&+ e^{-r(s\wedge u)}[\mathbf{1}_{\{s<u\}}(V_i^{(2)}(\zeta;\mathcal{G}^{(2)})-k_i) \\
&+ \mathbf{1}_{\{s>u\}}V_i^{(2)}(\zeta;\mathcal{G}^{(2)})]\Big\}dG_i^{(1)}(s)dG_j^{(1)}(u)\Bigg], \qquad (3.24)\\
V_i^{(2)}(x;\mathcal{G}^{(2)}) = \mathbb{E}^x\Bigg[\iint_0^\infty \Big\{\int_0^{s\wedge u} &\pi(X_t)e^{-rt}dt + e^{-r(s\wedge u)}[\mathbf{1}_{\{s<u\}}((R_r\pi)(\zeta)-k_i) \\
&+ \mathbf{1}_{\{s>u\}}(R_r\pi)(\zeta)]\Big\}dG_i^{(2)}(s)dG_j^{(2)}(u)\Bigg], \qquad (3.25)
\end{aligned}
$$

where we ignore the case of simultaneous investments because it is an innocuous simplification (in the sense that simultaneous investments can never happen in equilibrium).

Assuming $k_1 > k_2$ as in the game of infinite investment opportunities, we now discuss the characterization of mixed strategy equilibria in the game of two investment opportunities by using the concept of two-phase mixed strategy SPE. Note that if the game enters stage 2, firms have only one investment opportunity left, which implies that it is as if they are in the game of single investment opportunity. Because it was already showed in Section 3.5.1 that only pure strategy MPEs exist in the game of single investment opportunity with the cost asymmetry ($k_1 > k_2$), the possible equilibrium strategy profile $\mathcal{G}^{(2)}$ in stage 2 must be in pure strategies, i.e., $\mathcal{G}^{(2)} = (H(\infty), H(\tau_2^{(2)}))$ or $\mathcal{G}^{(2)} = (H(\tau_1^{(2)}), H(\infty))$ where $\tau_i^{(2)} := \inf\{t \geq 0 : X_t \leq \theta_i^{(2)}\}$ is a solution to the optimal stopping time problem

$$
\sup_\tau \mathbb{E}^x\left[\int_0^\tau \pi(X_t)e^{-rt}dt + e^{-r\tau}[(R_r\pi)(\zeta)-k_i]\right].
$$

Therefore, it only remains to examine if there can exist a two-phase mixed strategy SPE in stage 1.

Recall that the CDFs of a two-phase mixed strategy SPE is continuous in time in phase 2, and under such a strategy profile, each firm's payoff function in phase 2 is identical to the optimal payoff function given the other firm never investing. It thus follows that a two-phase mixed strategy SPE exists in stage 1 if and only if the rewards from investment $V_i^{(2)}(\zeta;\mathcal{G}^{(2)}) - k_i$ ($i = 1, 2$) in stage 1 are identical between the two firms. Because we assume $k_1 > k_2$, however, these rewards from investment in stage 1 can be identical if and only if $V_1^{(2)}(\zeta; \mathcal{G}^{(2)}) > V_2^{(2)}(\zeta; \mathcal{G}^{(2)})$, which requires $\mathcal{G}^{(2)} = (H(\infty), H(\tau_2^{(2)}))$ [8].

Given the choice $\mathcal{G}^{(2)} = (H(\infty), H(\tau_2^{(2)}))$ of the equilibrium strategy profile in stage 2, we can now prove that there exists a two-phase mixed strategy SPE in the game of two investment opportunities if and only if k_1 and k_2 (with $k_1 > k_2$ as before) satisfy the specific equation. In particular, a strategy profile $\mathcal{G}^{(1)} = (G_1^{(1)}, G_2^{(1)})$ with a common support $\Gamma^{(1)} = (-\infty, \theta_2^{(1)})$ and firm 2's investment probability $q_2 \in [0, 1)$ at $\tau_2^{(2)}$ is a mixed strategy equilibrium in stage 1 if and only if

$$k_1 = \left(1 + \frac{\phi(\zeta)}{\phi(\theta_2^{(2)})}\right) k_2 \,. \tag{3.26}$$

Here $\theta_2^{(1)}$ is the threshold for the stopping time $\tau_2^{(1)} := \inf\{t \geq 0 : X_t \leq \theta_2^{(1)}\}$, which is a solution to the optimal stopping time problem

$$\sup_{\tau} \mathbb{E}^x \left[\int_0^{\tau} \pi(X_t) e^{-rt} dt + e^{-r\tau} [V_2^{(2)}(\zeta; \mathcal{G}^{(2)}) - k_2] \right] .$$

The condition (3.26) can be obtained by deriving the following relation from a simple algebra with (3.25) and $\mathcal{G}^{(2)} = (H(\infty), H(\tau_2^{(2)}))$:

$$[V_2^{(2)}(\zeta; \mathcal{G}^{(2)}) - k_2] - [V_1^{(2)}(\zeta; \mathcal{G}^{(2)}) - k_1] = k_1 - k_2 - \frac{\phi(\zeta)}{\phi(\theta_2^{(2)})} k_2 \,.$$

In other words, the choice of k_1 in (3.26) is made so as to make the rewards from investment in stage 1 are equal between the two firms. Note also that firm 2's investment probability mass q_2 can take on any values smaller than 1, which implies that a two-phase mixed strategy SPE in stage 1 is not uniquely determined in contrast to the game of infinite investment opportunities.

General Analysis: Game of $M \geq 2$ Investment Opportunities

Next, we will construct a generalized (sufficient) condition for the existence

[8] In other words, we cannot obtain a two-phase mixed strategy SPE in stage 1 if the two firms are in the equilibrium $\mathcal{G}^{(2)} = (H(\tau_1^{(2)}), H(\infty))$ in stage 2; as will be shown later, however, if the number M of investment opportunities is more than two, we can obtain a mixed strategy equilibrium in stage 1 with the equilibrium strategy profile being $(H(\tau_1^{(M)}), H(\infty))$ in the last stage M.

of a two-phase mixed strategy SPE in the game of any $M \geq 2$ investment opportunities. Suppose that there are $M \geq 2$ opportunities to invest in their shared supplier. Similarly as above, we can express a strategy profile for each stage $m \leq M$ as $\mathcal{G}^{(m)} = (G_1^{(m)}, G_2^{(m)})$ and write firm i's payoffs $V_i^{(m)}(\cdot)$ for each stage $m \leq M$ as follows:

$$\begin{aligned} V_i^{(m)}(x; \mathcal{G}^{(m)+}) = \mathbb{E}^x \Bigg[\iint_0^\infty \Bigg\{ \int_0^{s\wedge u} \pi(X_t) e^{-rt} dt \\ + e^{-r(s\wedge u)} [\mathbf{1}_{\{s<u\}} (V_i^{(m+1)}(\zeta; \mathcal{G}^{(m+1)+}) - k_i) \\ + \mathbf{1}_{\{s>u\}} V_i^{(m+1)}(\zeta; \mathcal{G}^{(m+1)+})] \Bigg\} dG_i^{(1)}(s) dG_j^{(1)}(u) \Bigg], \end{aligned} \tag{3.27}$$

where $\mathcal{G}^{(m)+} := (\mathcal{G}^{(m)}, \mathcal{G}^{(m+1)}, ..., \mathcal{G}^{(M)})$, $m \leq M$, is a simplifying notation that indicates the collection of strategy profiles for stages n ranging from m to M. We also let $\mathcal{G}^{(M+1)} = \mathcal{G}^{(M+1)+} = \phi$ and $V_i^{(M+1)}(\zeta; \mathcal{G}^{(M+1)+}) = (R_r \pi)(\zeta)$ for notational consistency and completeness. By using these mathematical notions, we can then (backward recursively) define $\tau_i^{(m)} := \inf\{t \geq 0 : X_t \leq \theta_i^{(m)}\}$, $m \leq M$, as a solution to the following optimal stopping time problem

$$\sup_\tau \mathbb{E}^x \left[\int_0^\tau \pi(X_t) e^{-rt} dt + e^{-r\tau} [V_i^{(m+1)}(\zeta; \mathcal{G}^{(m+1)+}) - k_i] \right], \tag{3.28}$$

where $\mathcal{G}^{(n)} = (H(\tau_i^{(n)}), H(\infty))$, $n \geq m+1$. Note that $\tau_i^{(m)}$ is firm i's optimal time to invest in stage m given the other firm never investing and these stopping times will be a building block for constructing mixed strategy equilibria as in the game of infinite investment opportunities.

One of the main goals of this subsection is to investigate the impact of an increase in M (the number of investment opportunities) on the equilibrium characterization. Hence, we particularly focus on the conditions under which there is a mixed strategy equilibrium in the *first* stage (stage 1) because the first stage shows up in the game of *any* $M \geq 2$ investment opportunities. More specifically, we will prove the following two statements as a generalization of what we established in the game of two investment opportunities: (i) In the game of any $M \geq 2$ investment opportunities, if $k_1 = [1 + \phi(\zeta)/\phi(\theta_2^{(M)})]k_2$, then there is a mixed strategy equilibrium in stage 1. (ii) The game of $M \geq 2$ investment opportunities admits *at least* $M - 1$ different values of k_1, holding

k_2 constant, with which we can obtain a mixed strategy equilibrium in stage 1. Towards this end, we illustrate an intuitive reasoning behind this generalized result in the case of $M = 3$, followed by the formal statement and discussion of the generalized result in the game of any $M > 1$ investment opportunities later.

Given $M = 3$, our first claim is that we can obtain a mixed strategy equilibrium in stage 1 if $k_1 = [1 + \phi(\zeta)/\phi(\theta_2^{(3)})]k_2$. To see this, set $k_1 = [1+\phi(\zeta)/\phi(\theta_2^{(3)})]k_2$ and observe from (3.27) that we can have $V_1^{(3)}(\zeta; \mathcal{G}^{(3)}) - k_1 = V_2^{(3)}(\zeta; \mathcal{G}^{(3)}) - k_2$ with this choice of k_1 and $\mathcal{G}^{(3)} = (H(\infty), H(\tau_2^{(3)}))$. It thus follows from the arguments in the game of two investment opportunities above that we can construct a two-phase mixed strategy SPE $\mathcal{G}^{(2)} = (G_1^{(2)}, G_2^{(2)})$ in stage 2 ($= M - 1$). Recall also that $\mathcal{G}^{(2)} = (G_1^{(2)}, G_2^{(2)})$ is characterized by a common support $\Gamma^{(2)} = (-\infty, \theta_2^{(2)})$ and firm 2's investment probability mass q_2 where q_2 can be any non-negative values less than 1 (i.e., $q_2 \in [0, 1)$). By putting this strategy profile $\mathcal{G}^{(2)} = (G_1^{(2)}, G_2^{(2)})$ with the choice of $q_2 = (k_1 - k_2)/k_1 \cdot \phi(\theta_2^{(2)})/\phi(\zeta)$ into (3.27), therefore, we can obtain

$$V_1^{(2)}(\zeta; \mathcal{G}^{(2)+}) - k_1 = V_2^{(2)}(\zeta; \mathcal{G}^{(2)+}) - k_2 \ ,$$

which is a necessary and sufficient condition for the existence of a two-phase mixed strategy SPE $\mathcal{G}^{(1)} = (G_1^{(1)}, G_2^{(1)})$ with a common support $\Gamma^{(1)} = (-\infty, \theta_2^{(1)})$ in the first stage. We lastly note that $k_1 = [1 + \phi(\zeta)/\phi(\theta_2^{(3)})]k_2$ implies

$$k_1 - k_2 = \frac{\phi(\zeta)}{\phi(\theta_2^{(3)})}k_2 < \frac{\phi(\zeta)}{\phi(\theta_2^{(2)})}k_2 < \frac{\phi(\zeta)}{\phi(\theta_2^{(2)}) - \phi(\zeta)}k_2 \ ,$$

which makes sure that $q_2 = (k_1 - k_2)/k_1 \cdot \phi(\theta_2^{(2)})/\phi(\zeta) < 1$. Here we use the fact $\theta_2^{(3)} < \theta_2^{(2)}$ and $\phi(\cdot)$ is a strictly decreasing function.

The second claim, given $M = 3$, is that there is another value of k_1 with which we can obtain a mixed strategy equilibrium in the first stage. This additional choice of k_1 is given as

$$k_1 = \left(1 + \frac{\phi(\zeta)}{\phi(\theta_2^{(2)})} + \frac{\phi^2(\zeta)}{\phi(\theta_2^{(2)})\phi(\theta_2^{(3)})}\right)k_2 \ . \qquad (3.29)$$

The condition (3.29) can be obtained by deriving the following relation from a simple algebra with (3.27), $\mathcal{G}^{(2)} = (H(\infty), H(\tau_2^{(2)}))$, and $\mathcal{G}^{(3)} = (H(\infty), H(\tau_2^{(3)}))$:

$$\begin{aligned}&[V_2^{(2)}(\zeta;\mathcal{G}^{(2)+}) - k_2] - [V_1^{(2)}(\zeta;\mathcal{G}^{(2)+}) - k_1] \\ &= k_1 - k_2 - \left(\frac{\phi(\zeta)}{\phi(\theta_2^{(2)})} + \frac{\phi^2(\zeta)}{\phi(\theta_2^{(2)})\phi(\theta_2^{(3)})}\right)k_2 \,.\end{aligned}$$

In other words, the choice of k_1 in (3.29) is made so as to make the rewards from investment in stage 1 are equal between the two firms, given that firm 2 invests with probability 1 in stages 2 and 3. In addition, because $V_2^{(2)}(\zeta;\mathcal{G}^{(2)+}) - k_2$ is the same between two cases – when $\mathcal{G}^{(2)} = (G_1^{(2)}, G_2^{(2)})$ obtained in our first claim above and when $\mathcal{G}^{(2)} = (H(\infty), H(\tau_2^{(2)}))$ as given here, we can conclude that we can obtain a two-phase mixed strategy SPE $\mathcal{G}^{(1)} = (G_1^{(1)}, G_2^{(1)})$ with a common support $\Gamma^{(1)} = (-\infty, \theta_2^{(1)})$ in stage 1, which is of the same form as that obtained in our first claim above.

We now formally state the generalized version of these two claims in the game of any $M \geq 2$ investment opportunities:

Proposition 3.10 *Suppose that two firms with investment costs $k_1 > k_2$ are given $M \geq 2$ investment opportunities and define*

$$k_1^{(m)} := \left[1 + \sum_{k=m}^{M}\left(\prod_{l=m}^{k}\frac{\phi(\zeta)}{\phi(\theta_2^{(l)})}\right)\right]k_2 \,,\; 2 \leq m \leq M \,. \tag{3.30}$$

Then if $k_1 = k_1^{(m)}$, it follows that

(a) We can have a two-phase mixed strategy SPE $\mathcal{G}^{(1)} = (G_1^{(1)}, G_2^{(1)})$ with the support $\Gamma^{(1)} = (-\infty, \theta_2^{(1)})$ and $q_2^{(1)} \in [0,1)$ in the first stage of the game.

(b) The equilibrium strategy profiles $\mathcal{G}^{(n)} = (G_1^{(n)}, G_2^{(n)})$, $n \geq 2$, in all the subsequent stages $n \geq 2$ are uniquely determined as

$$(G_1^{(n)}, G_2^{(n)}) = \begin{cases} (G_1^{(n)}(\theta_2^{(n)}), G_2^{(n)}(\theta_2^{(n)}; q_2^{(n)})) & \text{for } n < m \\ (H(\infty), H(\tau_2^{(n)})) & \text{otherwise}\,, \end{cases}$$

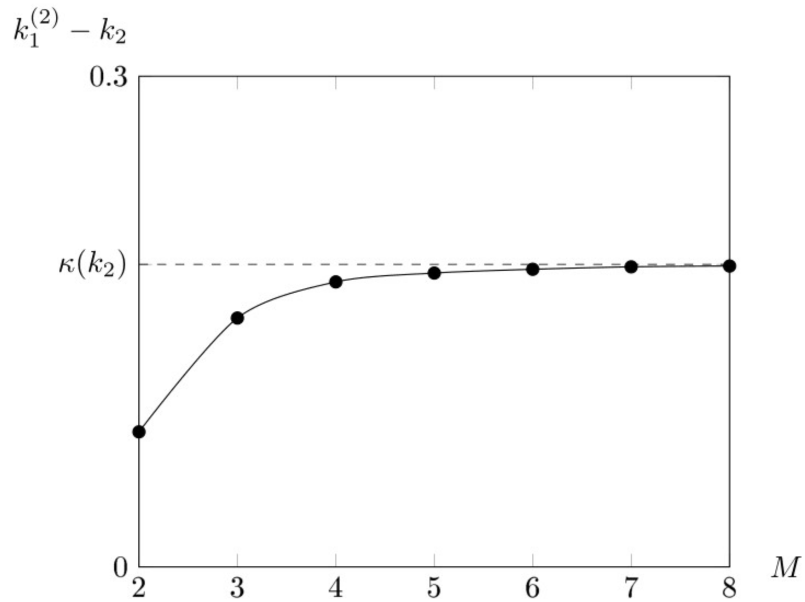
where $q_2^{(n)} = (k_1 - k_2)/k_1 \cdot \phi(\theta_2^{(n)})/\phi(\zeta) < 1$.

Moreover, we have $k_1^{(2)} - k_2 \uparrow \kappa(k_2)$ and $q_2^{(2)} \uparrow q_2^*$ as $M \to \infty$ where $\kappa(k_2)$ and q_2^* are given in Theorem 3.1.

Proposition 3.10 has a couple of implications worth discussing. First of all, it asserts that we can obtain a two-phase mixed strategy SPE for a larger number of k_1's (holding k_2 constant) as the number M of investment opportunities increases. In other words, a two-phase mixed strategy SPE is more likely to exist as there are more number of opportunities to invest in the shared suppliers. This thus corroborates the impact of the repetitive nature on the equilibrium characterization – repeated investment opportunities inducing a mixed strategy equilibrium – which is one of our major findings in Section 3.5.1.

Second, $k_1^{(2)}$ – the newly added value of k_1 as a result of one additional investment opportunity – is a strictly increasing function of the number M of investment opportunities, which implies that the higher repetition of investments induces a mixed strategy equilibrium in the larger level of cost asymmetry between the two firms. See Figure 3.4. Hence, this point complements our finding in Section 3.5.1 on the inducement of a mixed strategy equilibrium by the repetitive nature in the sense that it illustrates how the repeated investment opportunities can increase the likelihood of a mixed strategy equilibrium.

Figure 3.4: $k_1^{(2)} - k_2$ as a function of the number M of investment opportunities. We set $\pi(x) = 1 - e^{-px}$ with $r = 0.5$, $\mu(x) = \mu = -0.5$, $\sigma(x) = \sigma = 1$, $k_2 = 1$, $p = 0.5$, and $\zeta = 2.5$.



Lastly, the game of $M \geq 2$ investment opportunities admits *at least* $M-1$ different values of k_1, holding k_2 constant, with which we can obtain a two-phase mixed strategy SPE. In fact, it is not difficult to see that the number of k_1's, which allows a mixed strategy equilibrium, is *strictly larger than* $M-1$ in the game of $M > 2$ investment opportunities. In the game of $M = 3$ investment opportunities, for instance, if k_1 and k_2 satisfy

$$k_1 = \left(\frac{1 + \phi(\zeta)/\phi(\theta_2^{(2)})}{1 + \phi^2(\zeta)/(\phi(\theta_2^{(2)})\phi(\theta_1^{(3)}))} \right) k_2 \, , \tag{3.31}$$

then we can obtain a mixed strategy equilibrium $\mathcal{G}^{(1)} = (G_1^{(1)}, G_2^{(1)})$ in stage 1 with $\mathcal{G}^{(2)} = (H(\infty), H(\tau_2^{(2)}))$ and $\mathcal{G}^{(3)} = (H(\tau_1^{(3)}), H(\infty))$. Note here that $k_1 > k_2$ because $\phi(\zeta)/\phi(\theta_1^{(3)}) < 1$. Similarly as before, this condition can be obtained by deriving the following relation from a simple algebra with (3.27), $\mathcal{G}^{(2)} = (H(\infty), H(\tau_2^{(2)}))$, and $\mathcal{G}^{(3)} = (H(\tau_1^{(3)}), H(\infty))$:

$$\begin{aligned} &[V_2^{(2)}(\zeta; \mathcal{G}^{(2)+}) - k_2] - [V_1^{(2)}(\zeta; \mathcal{G}^{(2)+}) - k_1] \\ &= k_1 - k_2 - \frac{\phi(\zeta)}{\phi(\theta_2^{(2)})} k_2 + \frac{\phi^2(\zeta)}{\phi(\theta_2^{(2)})\phi(\theta_1^{(3)})} k_1 \, , \end{aligned}$$

where $\theta_2^{(2)}$ is the threshold for a solution to the optimal stopping problem (3.28) with $\mathcal{G}^{(3)} = (H(\tau_1^{(3)}), H(\infty))$. Because it can be clearly seen that the choice of k_1 given in (3.31) is different from any of $k_1^{(m)}$ in (3.30), we can conclude that the number of k_1's, with which we can obtain a mixed strategy equilibrium, is strictly bigger than $M-1$ in the game of $M > 2$ investment opportunities. Moreover, as the number M of investment opportunities grows, the number of possible combinations of pure strategy MPEs in stages $n \geq 2$ increases. This thus suggests that the number of k_1's allowing a mixed strategy equilibrium increases at the faster rate as the number M of investment opportunities grows.

Overall, we can summarize our discussions on Proposition 3.10 that our findings on the game of infinite investment opportunities (Theorem 3.1) can be seen as an approximated result for the game of finite ($M \geq 2$) investment opportunities when M is large enough.

We close this subsection by discussing the characterization of pure strategy MPEs in the game of finite investment opportunities. In the game of finite investment opportunities, stage-dependent equilibrium strategies are sometimes more natural. Note that a lower investment cost does not always mean a higher incentive to invest in any stage $n < M$ of the game of M investment opportunities. This is because the rewards from investment are determined not only by the cost efficiency (k) but also by the value from investment ($V(\zeta;\cdot)$), which is a function of strategy profiles in the subsequent stages ($\mathcal{G}^{(m)}$, $n < m \leq M$). In the game of two investment opportunities, for instance, although $k_1 > k_2$ makes firm 2 have a higher incentive to invest than firm 1 in stage 2, it could be the other way around in stage 1 (i.e. firm 1 could be more incentivized to invest than firm 2 in stage 1) if $[V_1^{(2)}(\zeta;\mathcal{G}^{(2)}) - k_1] - [V_2^{(2)}(\zeta;\mathcal{G}^{(2)}) - k_2] > 0$, which can happen when k_1 and k_2 are close enough; more precisely, $k_1 - k_2 < \phi(\zeta)/\phi(\theta_2) \cdot k_2$ in the game of two investment opportunities.

In fact, this observation is related to our result in Section 3.5.1 about the existence condition for the two-phase mixed strategy SPE in the game of infinite investment opportunities. Note that, in the game of $M = 2$ investment opportunities, firm 2 keeps a higher incentive to invest than firm 1 in both stage 1 and stage 2 if and only if $k_1 - k_2 > \frac{\phi(\zeta)}{\phi(\theta_2)} k_2$. Similarly, in the game of $M = 3$ investment opportunities, it can be seen from calculating $V_1^{(2)}(\zeta;\mathcal{G}^{(2)},\mathcal{G}^{(3)})$ and $V_2^{(2)}(\zeta;\mathcal{G}^{(2)},\mathcal{G}^{(3)})$ by using $V_1^{(2)}(\zeta;\mathcal{G}^{(3)})$ and $V_2^{(2)}(\zeta;\mathcal{G}^{(3)})$ where $\mathcal{G}^{(2)} = \mathcal{G}^{(3)} = (H(\infty), H(\tau_2^*))$ that firm 2 is more incentivized to invest than firm 1 in all the stages $m \leq M = 3$ if and only if $k_1 - k_2 > \left(\frac{\phi(\zeta)}{\phi(\theta_2)} + \frac{\phi^2(\zeta)}{\phi^2(\theta_2)}\right) k_2$. In the game of $M \geq 2$ investment opportunities, therefore, it can be easily seen that firm 2 has a higher incentive to invest than firm 1 in all the stages $m \leq M$ if and only if

$$k_1 - k_2 > \sum_{m=1}^{M-1} \left(\frac{\phi(\zeta)}{\phi(\theta_2)}\right)^m k_2 \to \kappa(k_2) \text{ as } M \to \infty$$

where $\kappa(k_2) = \frac{\phi(\zeta)}{\phi(\theta_2) - \phi(\zeta)} k_2$ is given in (3.16). Therefore, the condition under which firm 2 has a higher incentive to invest than firm 1 in all the stages $m \leq M$ converges to the condition for the non-existence of the two-phase mixed strategy SPE in the game of infinite investment opportunities as M goes to infinity. Moreover, this implies that if $k_1 - k_2 < \kappa(k_2)$, then a pure strategy MPE $(H(\infty), H(\tau_2^*))$, in which firm 2 is a designated investor for all stages, is not a natural choice of equilibrium in the game of M investment opportunities as the number M of opportunities increases; this is because firm 1's reward from investment in stage 1 increases as firm 2 becomes an investor more number of times in the subsequent stages, and eventually, the reward from investment for firm 1 exceeds that for firm 2 if the number of investment opportunities is large enough.

3.8 Summary and Conclusions

In this paper, we examine a two-player stochastic game of investment. Our study is motivated by real industry scenarios, in which two manufacturers consider investment in their shared suppliers. Our model captures three salient features of this game of investment: spillover of investments via the shared supplier, stochasticity of the supplier's quality, and the need for continued investment in the supplier's quality. In the model that we examine, an investment by either firm spills over to the other, the quality of the supplier is modeled as a (stochastic) diffusion process, and each firm has an indefinite number of opportunities to invest in the quality of the supplier. Formulating the problem as a repeated stochastic war of attrition, we characterize the equilibria in both pure and mixed strategies.

Our first key finding is that there always exists a pure strategy equilibrium where each firm's role is coordinated, either as an investor or a free-rider, so that any costly delays in investment are curtailed. In characterizing mixed strategy equilibria, we also find that the repetitive nature of the investment opportunities induces a mixed strategy equilibrium, which results in inefficient delays in investment because of the coordination failure. More specifically, the stochastic game admits a *unique* two-phase mixed strategy SPE in a wide-range of model parameters *if* there are *multiple* investment opportunities. This result is in stark contrast with (i) the single investment game in which there is *no* two-phase mixed strategy SPE and (ii) the deterministic game where there is a *continuum* of two-phase mixed strategy SPE.

Based on the uniqueness of the mixed strategy equilibrium, we are able to estimate the inefficiency resulting from the free-riding incentives. Using primary data collected from a field study, we demonstrate that the efficiency gap between the mixed strategy equilibrium and its associated first-best solution can be substantial although this gap varies across industry sectors to some degree. Moreover, we find that the efficiency loss in the mixed strategy equilibrium is mostly driven by the coordination failure: If the game consists only of a single stage or the cost asymmetry is sufficiently high, then the mixed strategy equilibrium disappears and only a pure strategy equilibrium is possible, in which case the efficiency loss will be negligible, as illustrated in the efficiency gap between the pure strategy equilibrium and the first-best solution.

We also examine the extended model in which more than two manufacturing firms consider an investment in their shared suppliers. We find that the mixed strategy equilibrium exists in a wide range of parameter values as in our base model although we additionally discover that the parameter regime where the mixed strategy equilibrium exists is shrinking as the number of the firms increases.

Overall, our work shows that when independent firms consider investing in shared suppliers, it is important to recognize that the loss of efficiency arising from a mixed strategy equilibrium can be very high, in which case the financial gains from the coordinated investment will be worth the effort. Therefore, a firm considering supplier development should attempt to coordinate or collaborate on its investments with other firms sharing the same supplier: If the firms are divisions of one company, then they should make a collaborative investment; if the firms are not within the boundary of one company, then they might be able to converge on a more efficient pure strategy equilibrium through prior communication or exchange of resources.

Chapter 4

Equilibrium Selection in the War of Attrition under Complete Information

4.1 Introduction

In the classic war of attrition, the first player to quit concedes a prize to his opponent. Each player trades off the cost associated with fighting against the value of the prize. These features are common in many managerial and economic problems. Oligopolists in a declining industry may bear losses in anticipation of profitability following a competitor's exit (Ghemawat and Nalebuff, 1985). For example, the rise of Amazon in the mid-1990s made the business model of Barnes & Noble and Borders obsolete, turning traditional bookselling into a declining market. As the demand shrank sharply, these two major players at the time had to cut down slack in their capacities, but each would prefer its competitors to carry the painful burden of closing stores or exiting the market altogether (Newman, 2011). Similarly, the presently low price of crude oil is often attributed to a war of attrition among Saudi Arabia, its Persian Gulf OPEC allies, and non-OPEC rivals such as Russia and the many shale-oil producers in the United States (Reed, 2016). Other examples of wars of attrition include the provision of public goods (Bliss and Nalebuff, 1984), lobbying (Becker, 1983), labor disputes (Greenhouse, 1999), court of law battles (McAfee, 2009), races to dominate a market (Ghemawat, 1997), technology standard races (Bulow and Klemperer, 1999), all-pay auctions (Krishna and Morgan, 1997), and bargaining games (Abreu and Gul, 2000).

It is well known that the canonical model of war of attrition admits equilibria in both pure and mixed strategies (Tirole (1988), Fudenberg and Tirole (1996), Levin (2004), and others). Naturally then, this raises the issue of equilibrium selection. To select an equilibrium, the extant literature has focused on three perturbations of the model: First, by assuming that with a *small* probability, each player behaves irrationally by never quitting. Second, by considering hybrid all-pay auctions, in which the winner's costs are a convex combination of his own planned exit time and the loser's exit time. Third, by imposing a time limit, after which the prize is awarded at random. For details, see Myatt (2005) and the references therein. In this paper, we show that in a canonical war of attrition under complete information, if the players' flow payoffs whilst fighting for the prize vary stochastically and their exit payoffs are heterogeneous, then the game admits only pure-strategy Markov Perfect equilibria (hereafter MPE). If the players are sufficiently heterogeneous, then the game admits a unique pure-strategy MPE. This result shows that the mixed-strategy equilibria are unstable to a natural perturbation of the model, and thus it provides a rationale for focusing on the pure-strategy equilibria in wars of attrition.

In our model, two competing oligopolists contemplate exiting a market. While both firms remain in the market, each receives a flow payoff that depends on the stochastically fluctuating market conditions (*e.g.*, the price of a relevant commodity); hereafter the *state*. At every moment, each firm can exit the market, in which case, it collects its outside option. Its rival then obtains a (larger) *winner's* payoff, which depends on the state at the time of exit; *e.g.*, the net present discounted monopoly profit in that market. We assume that all payoff-relevant parameters are common knowledge, so this is a game with complete information. As the state follows a Markov process, we restrict attention to Markov Perfect strategies, wherein at every moment, each firm conditions its probability of exit on the current state.

We begin by characterizing the best response of a firm who anticipates that its rival will never exit the market. We show that a firm will optimally exit at the first moment that the state drifts below a threshold. Moreover, this *myopically optimal* threshold increases in the firm's outside option. This is intuitive: the better is a firm's outside option, the less it is willing to endure poor market conditions before exiting the market. Proposition 1 shows that there exists a pure-strategy MPE in which the firm with the larger outside option exits the market at its myopically optimal threshold and its rival never exits. If the heterogeneity in outside options is not too large, then there exists another pure-strategy MPE in which the firm with the lower outside option exits the market at its own myopically optimal threshold and its rival never exits.

Towards our main result, we establish two lemmas. The first shows that in any mixed-strategy MPE, even if the state evolves deterministically, (a) both firms must be randomizing between remaining in the market and exiting on a common set of states (*i.e.*, their strategies must have common support), and (b) strategies must be continuous in the interior of their support (*i.e.*, during any interval $(t, t + dt)$, the probability that a firm exits the market must be of order dt). The second lemma shows that if the state evolves stochastically, then each firm's strategy must be continuous everywhere, including at the boundary of its support, and its support must be equal to the half-line below the myopically optimal exit threshold.

The main result follows immediately: If the market conditions are stochastic and the firms have heterogeneous outside options, in which case their myopically optimal exit thresholds differ, then the game admits no mixed-strategy MPE. These ingredients are necessary for the game to admit only pure-strategy MPE: if the firms have identical outside options or the market conditions are deterministic, then there exists a mixed-strategy MPE which we characterize.

4.2 Related Literature

First and foremost, this paper contributes to the literature on wars of attrition, which has received widespread attention since the seminal work of Maynard Smith (1974). Our model is closest related to Hendricks et al. (1988) and Murto (2004). The former provides a complete characterization of the equilibria (in both pure and mixed strategies) in a war of attrition under complete information, in which the players' payoffs vary deterministically over time. The latter considers stochastic payoffs, but restricts attention to pure-strategy Markov Perfect equilibria. In contrast, we allow payoffs to vary stochastically, and we show that if players are heterogeneous, then the game admits MPE in pure strategies *only*.

Our paper also contributes to a strand of literature that contemplates equilibrium selection in the war of attrition. Fudenberg and Tirole (1996) consider a game of exit in a duopoly, in which players are uncertain about their rivals' cost of remaining in the market. In the unique equilibrium, each firm exits at a deterministic time that decreases in its cost. In Kornhauser et al. (1989), with a small probability, each player irrational and never quits. They show that this approach selects a unique equilibrium in which the weaker player quits immediately. Kambe (1999), and Abreu and Gul (2000) analyze a bargaining game in which two players seek to divide some surplus, and each player is behavioral with some probability. They show that this gives rise to a unique equilibrium that entails delay. This is in contrast to our model, where players are rational and possess complete information about the parameters of the game.

Touzi and Vieille (2002) introduces the concept of mixed strategies in continuous-time Dynkin games (a class of stopping games), and proves that the game admits minimax solutions in mixed strategies. Steg (2015) characterizes equilibria in both pure and mixed strategies in a family of continuous-time stochastic timing games. Whereas these papers consider games with identical players, we focus on games with non-homogeneous players and show that the set of equilibria differ drastically from the case with homogenous players. Riedel and Steg (2017) examines mixed-strategy equilibria in continuous-time stopping games with heterogeneous players, but they restrict attention to preemption games, whereas our model is one of a war of attrition.

Finally, our paper is also related to the literature in real option games in the context of timing decisions with externalities under uncertainty. Dixit and Pindyck (1994) establishes the fundamental framework for analyzing real options and real option games. Grenadier (2002), Decamps and Mariotti (2004), and Mason and Weeds (2010) examine the interplay between the option value of waiting and externalities due to competition, learning, and network effects. However, these papers focus on the role of a preemptive threat in real option games while our work is focused on a free-riding incentive.

4.3 Model

We consider a war of attrition with complete information between two oligopolistic firms. Time is continuous, and both firms discount time at rate $r > 0$. At every moment, each firm decides whether to remain or exit the market.

While both firms remain in the market, each firm earns a flow profit $\pi(X_t)$, where $\pi : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, while X_t is a scalar that captures the *market conditions* that the firms operate in (*e.g.*, the size of the market, or the price of raw materials).[1] The market conditions fluctuate over time according to

$$dX_t^x = \mu(X_t^x)dt + \sigma(X_t^x)dB_t\,, \tag{4.1}$$

where X_t^x is defined on $\mathscr{I} := (a,b) \subset \mathbb{R}$, $X_0^x = x$, the functions $\mu : \mathscr{I} \to \mathbb{R}$ and $\sigma : \mathscr{I} \to \mathbb{R}_+$ are continuous, and B_t is a Wiener process.[2,3,4]

If firm i chooses to exit at time t, then it receives its outside option l_i, and firm $j \neq i$ receives $w(X_t^x)$, where $w : \mathscr{I} \to \mathbb{R}$ is the expected discounted payoff of a monopoly flow profit $\pi^M(\cdot) > \pi(\cdot)$. We say that firm j is the *winner*, and firm i is the *loser*. We set the convention that $l_1 \leq l_2$; *i.e.*, firm 2 has a larger outside option than firm 1. We assume that $w(x) > l_2$ for all $x \in \mathscr{I}$ so that the winner's reward is always larger than that of the loser. The game ends as soon as a firm exits the market. If both firms exit at the same moment, then each firm obtains the outside option l_i or $w(X_t^x)$ with probability $1/2$ each.

Finally, we make the following technical assumptions (see also Alvarez, 2001).

1. $\pi(\cdot)$ satisfies the absolute integrability condition $\mathbb{E}^x \left[\int_0^\infty |e^{-rt}\pi(X_t^x)|\, dt\right] < \infty$.

2. For each i, there exists some $x_{ci} \in \mathscr{I}$ such that $\pi(x_{ci}) = rl_i$.

The first assumption ensures that each firm's payoff is well-defined, whereas the second guarantees the existence of an internal optimal exit threshold.

4.3.1 Markov Strategies

At every moment t, each firm chooses the probability with which to remain in the market to maximize its expected discounted payoff. We assume that both

[1] For simplicity, we assume that the firms earn identical flow profits. Our results are generalizable to allow heterogeneous flow profits.

[2] We use the superscript in X_t^x to denote its dependence on the initial value x at time 0.

[3] Special cases in which $\sigma(\cdot) = 0$ have been analyzed extensively (Ghemawat and Nalebuff (1985), Hendricks et al. (1988), and others). Therefore, we restrict attention to $\sigma(\cdot) > 0$ in the main body of this paper, and for completeness, we revisit the case in which $\sigma(\cdot) = 0$ in Appendix A.

[4] The boundary points a and b are assumed to be *natural* (Borodin and Salminen, 1996, p.18-20); *i.e.*, neither a, nor b can be reached by X_t^x in finite time. For example, if X_t is a standard diffusion process, then $\mathscr{I} = \mathbb{R}$. If X_t is a geometric diffusion process, then $\mathscr{I} = (0, \infty)$.

firms employ Markov strategies, so for any x, their decision at time t depends only on the current state X_t^x. We make the definition of a Markov strategy mathematically precise below. Each firm i's strategy can be defined as a family of cumulative distribution functions (hereafter CDF) $G_i := (G_i^x)_{x \in \mathscr{I}}$ of stopping times with respect to $\mathcal{F}^X$ for each $x \in \mathscr{I}$.[5] We say that a pair (G_1, G_2) is a *strategy profile.* For each i, G_i must be time-consistent, or equivalently, conform to Bayes' rule; *i.e.*, for any $t \geq s \geq 0$, $G_i^x(t) = G_i^x(s^-) + [1 - G_i^x(s^-)]G_i^{X_s^x}(t - s)$.

For example, suppose that $X_t = x$ and neither firm has yet exited by time t. Then firm i employs the strategy G_i^x, and if neither firm exits, then at $t + dt$, the state evolves to $x + dX_t$, at which moment firm i employs the strategy $G_i^{x+dX_t}$. This definition extends the concept of randomized stopping times (Touzi and Vieille, 2002) and subgames (Steg, 2015) to a continuous-time stochastic game.

Note that firm i exits at time t with positive probability when G_i^x either has an (upward) jump at t, or it is continuously (strictly) increasing at t. The Markov property requires that any jump in G_i^x occurs at a hitting time $\tau_E = \inf\{t \geq 0 : X_t \in E\}$ for some set $E \subset (a, b)$, and that the probability of exit at τ_E depends only on $X_{\tau_E}^x$. If G_i^x is continuously (strictly) increasing at t, then the hazard rate of exit is a function of X_t^x alone.

A special case of a strategy G_i is one in which there exists a stopping time (a hitting time of a set E) τ_i at which G_i^x jumps from 0 to 1. We call a strategy of this form a *pure* Markov strategy and denote it by $H(\tau_i)$, where $G_i^x(t) = H^x(t; \tau_i) := \mathbf{1}_{\{t \geq \tau_i\}}(t)$. In contrast, if a Markov strategy G_i cannot be represented by $H(\tau_i)$ for any stopping time τ_i, then we refer to it as a *mixed* Markov strategy.

Lastly, we define the *support* of a mixed-strategy as a subset of the state space in which firm i randomizes between remaining in the market and exiting;

[5] By a CDF of stopping times, we refer to an $\mathcal{F}^X$-adapted, right-continuous, and non-decreasing process that ranges in the interval $[0, 1]$, where $\mathcal{F}^X := \{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by X.

i.e.,

$$\text{supp}(G_i) := \Bigg\{ x \in \mathscr{I} : \frac{dG_i^y(t)}{dt}\bigg|_{t=\tau} > 0 \text{ or } \Delta G_i^y(\tau) \in (0,1)$$
$$\text{for any } y \in \mathscr{I} \text{ whenever } X_\tau^y = x \Bigg\},$$

where $\Delta G_i^y(\tau) = G_i^y(\tau) - G_i^y(\tau^-)$ denotes a jump at time τ.

4.4 Preliminaries

In this section, we introduce notation and establish a Lemma that will be helpful for the subsequent analysis. In particular, in Section 4.4.1, we characterize each firm's expected discounted payoff given an arbitrary strategy profile. Then in Section 4.4.2, we characterize the best response of a firm which anticipates that its rival will never exit the market.

4.4.1 Payoffs

We begin by defining the conditional expected payoff of firm i, given the history $\mathcal{F}_t$ of X starting at $X_0^x = x$ and its rival exiting the market at t, at which time it becomes the *winner*:

$$W_i^x(t) = \int_0^t \pi(X_s^x)e^{-rs}ds + w(X_t^x)e^{-rt}\,. \tag{4.2}$$

Firm i receives the flow payoff $\pi(X_s^x)$ during $[0,t)$, whereas at time t, its rival, firm $-i$ exits and firm i receives the winner's payoff $w(X_t^x)$. Similarly, we define the conditional expected payoff of firm i, given the history $\mathcal{F}_t$ of X starting at $X_0^x = x$ and it exiting the market at t, at which time it becomes the *loser*:

$$L_i^x(t) = \int_0^t \pi(X_s^x)e^{-rs}ds + l_i e^{-rt}\,, \tag{4.3}$$

If both firms exit at t, then we assume that either firm becomes the winner with equal probability, so each firm obtains conditional expected payoff $M_i^x(t) =$

$(L_i^x(t) + W_i^x(t))/2$. We define

$$S_i^x(t; G_{-i}) = \int_0^{t^-} W_i^x(s) dG_{-i}^x(s) + M_i^x(t) \Delta G_{-i}^x(t) + L_i^x(t)[1 - G_{-i}^x(t)] \,. \quad (4.4)$$

Note that $S_i^x(t; G_{-i})$ denotes the expected payoff of firm i conditional on the history $\mathcal{F}_t$ of X starting at $X_0^x = x$, exiting at t, and its rival employing strategy G_{-i}. The first term captures the payoff associated with becoming the winner at any time before t. The second term captures the payoff associated with both firms exiting simultaneously at t, and the last term captures the payoff associated with becoming the loser at t.

Finally, define firm i's expected payoff under an arbitrary strategy profile (G_i, G_{-i}) starting at $X_0^x = x$ by

$$V_i(x; G_i, G_{-i}) = \mathbb{E}^x \left[\int_0^\infty S_i(t; G_{-i}) dG_i^x(t) \right] . \quad (4.5)$$

We say that a strategy profile (G_1^*, G_2^*) is an MPE if for each i, $V_i(x; G_i^*, G_{-i}^*) \geq V_i(x; G_i, G_{-i}^*)$ for all $x \in \mathscr{I}$ and any G_i.

4.4.2 Best Response to $H(\infty)$

As a building block towards characterizing the MPE of the game, we begin by characterizing firm i's best response to $H(\infty)$; *i.e.*, the best response of firm i when its opponent's strategy is to never exit the market. In this case, $G_{-i}^x(t) = 0$ for any $x \in \mathscr{I}$ and $t < \infty$, so firm i's best response can be determined by solving the following optimal stopping problem:

$$\sup_{\tau_i} V_i(x; H(\tau_i), H(\infty)) = \sup_{\tau_i} \mathbb{E}^x[L_i(\tau_i)] = \sup_{\tau_i} \mathbb{E}^x \left[\int_0^{\tau_i} \pi(X_t) e^{-rt} dt + l_i e^{-r\tau_i} \right] . \quad (4.6)$$

We use Proposition 2 in Alvarez (2001, p.334) to establish the following lemma.

Lemma 4.1 *For each $i \in \{1, 2\}$, there exists a unique threshold θ_i^* such that firm i optimally exits at*

$$\tau_i^* = \inf \{ t \geq 0 : \; X_t^x \leq \theta_i^* \} \; ; \quad (4.7)$$

i.e., at the first time such that $X_t^x \leq \theta_i^*$*. If* $l_1 < l_2$*, then* $\theta_1^* < \theta_2^*$*.*

This is intuitive: the firm's value of remaining in the market decreases as the market conditions deteriorate, and once they become sufficiently poor, the firm is better off exiting and collecting its outside option. As the firms earn identical flow payoffs while they remain in the market, the firm with the higher outside option optimally exits at a higher threshold.

4.5 Markov Perfect Equilibria

In this section, we characterize the MPE of this game. We begin by characterizing the pure-strategy MPE in Section 4.5.1. In Section 4.5.2, we establish our main result: if the market conditions fluctuate stochastically (*i.e.*, $\sigma(\cdot) > 0$) and the firms are heterogeneous (*i.e.*, $l_1 < l_2$), then this game has no mixed-strategy MPE.

4.5.1 Pure-strategy MPE

The following result shows that the strategy profile $(H(\infty), H(\tau_2^*))$ constitutes an MPE, and under certain conditions, $(H(\tau_1^*), H(\infty))$ also constitutes an MPE, where τ_1^*, τ_2^* are defined in Lemma 4.1.

Proposition 4.1 *The strategy profile* $(G_1, G_2) = (H(\infty), H(\tau_2^*))$ *is a pure-strategy MPE. Moreover, there exists a threshold* $\kappa > 0$ *that is independent of* l_1 *such that* $(G_1, G_2) = (H(\tau_1^*), H(\infty))$ *is also a pure-strategy MPE if* $l_2 < l_1 + \kappa$*.*

If firm i expects its rival to never exit the market, then by Lemma 4.1, it will optimally exit at the first time such that $X_t^x \leq \theta_i^*$. Therefore, it suffices to show that if firm i employs the strategy $H(\tau_i^*)$, then its opponent's best response is to never exit.

Suppose that firm 1 expects its rival to exit at the first moment that $X_t^x \leq \theta_2^*$. Recall from Lemma 4.1 that $\theta_1^* \leq \theta_2^*$, and so firm 1 has no incentive to exit until at least $X_t^x \leq \theta_1^*$. Therefore, firm 1 expects that the game will end before the state drifts below θ_1^*, and hence the strategy $G_1 = H(\infty)$ is incentive compatible. If instead firm 2 anticipates that its rival employs the strategy $H(\tau_1^*)$, then it can sustain the strategy $H(\infty)$ as long as it does not need to wait too long in the time interval (τ_2^*, τ_1^*) until firm 1 exits, and so $H(\infty)$ is a best response for firm 2 as long as $l_2 - l_1$ is not too large.

Note that we restrict attention to single-threshold strategies, so $(H(\tau_1^*), H(\infty))$ and $(H(\infty), H(\tau_2^*))$ are the sole candidates for pure-strategy MPE. As shown in Murto (2004), there may also exist pure-strategy equilibria with multiple exit thresholds.[6] However, these pure-strategy MPE with multiple thresholds do not affect our characterization of mixed-strategy MPE, and so we do not consider them in this paper.

4.5.2 Mixed-strategy MPE

We begin by establishing two Lemmas, which outline a set of necessary conditions that any mixed-strategy MPE must satisfy. Below we let Γ^o denote the interior of Γ, and $\overline{\Gamma^o}$ denote the closure of Γ^o.[7]

Lemma 4.2 *Suppose that (G_1, G_2) constitutes a mixed-strategy MPE. Then:*

(a) The supports of G_1 and G_2 have common interior Γ^o.

(b) If $x \in \overline{\Gamma^o}$, then both $G_1^x(t)$ and $G_2^x(t)$ are continuous at any $t = \tau$ such that $\Pr(X_\tau^x \in \overline{\Gamma^o}) > 0$.

It is helpful to convey the intuition with a heuristic derivation. (The formal arguments are relegated to Appendix B.) In the interior of the support of G_i, firm i must be indifferent between exiting immediately and remaining in the market; *i.e.*,

$$l_i = \frac{dG_{-i}^x(t)}{1 - G_{-i}^x(t)} w(X_t^x) + \left(1 - \frac{dG_{-i}^x(t)}{1 - G_{-i}^x(t)}\right) [\pi(X_t^x)dt + (1 - rdt)l_i] \,. \quad (4.8)$$

[6] In particular, Murto (2004) shows that there may exist an equilibrium in which each firm i exits at the first moment such that $X_t^x \in (-\infty, a_i] \cup [b_i, \theta_i^*]$ for some $a_i < b_i$; *i.e.*, firm i does not exit within some interval (a_i, b_i) below the threshold θ_i^*. Intuitively, if $x \in (a_1, b_1)$ and $b_1 - x$ is sufficiently small, then firm 2 can be better off waiting until X_t^x hits a_1 or b_1 and becoming the winner rather than exiting immediately. Finally, note that if the initial state $x \geq \max\{\theta_1^*, \theta_2^*\}$, then the outcome of this equilibrium coincides with the outcome of the equilibrium characterized in Proposition 4.1.

[7] Clearly, Γ^o is always a subset of Γ. Note also that it can be a *proper* subset; *i.e.*, $\Gamma^o \subsetneq \Gamma$, if and only if there are some *point* components of Γ that are not in in Γ^o. A point component of a set A is defined as a singleton point set $\{a\}$ such that $a \in A$ but disconnected from $A \backslash \{a\}$.

where ${dG^x_{-i}(t)}/{[1-G^x_{-i}(t)]}$ represents the probability that firm $-i$ will exit during $(t, t+dt)$, conditional on not having exited until t. The left-hand-side of (4.8) represents firm i's payoff in case it exits at t. If it remains in the market, then with probability ${dG^x_{-i}(t)}/{[1-G^x_{-i}(t)]}$ it receives the winner's payoff $w(X^x_t)$, whereas with the complementary probability, it earns the flow payoff $\pi(X^x_t)$ during $(t, t+dt)$, and its (discounted) continuation profit l_i at $t+dt$.[8] It follows from (4.8) that firm $-i$'s probability of exit during $(t, t+dt)$, where $dt \simeq 0$ must equal

$$\frac{dG^x_{-i}(t)}{1-G^x_{-i}(t)} = \frac{rl_i - \pi(X^x_t)}{w(X^x_t) - l_i} dt \,. \tag{4.9}$$

Notice that if $\pi(X^x_t) > rl_i$, then the right-hand-side of (4.8) is strictly larger than l_i, and so firm i strictly prefers to remain in the market regardless of its rival's strategy.

Suppose that there exists a non-empty interval that is in the interior of the support of G_i but not of G_{-i}. Then for at least some x in that interval, we must have $\pi(x) < rl_i$ and $dG^x_{-i}(0) = 0$. This implies that the right-hand-side of (4.8) is strictly smaller than l_i, so firm i strictly prefers to exit. However, this contradicts that x is in the interior of the support of G_i, so we conclude that the supports of G_1 and G_2 share the same interior.

Second, observe from (4.9) that in the interior of the common support of G_1 and G_2, ${dG^x_i(t)}/{dt}$ is finite for each i, which implies that strategies are continuous. This is intuitive: if a firm's strategy were discontinuous at some state in the interior of its support, then its rival would strictly prefer to remain in the market when that state is reached in order to increase the probability of obtaining the winner's payoff.

Lemma 4.2 holds irrespective of whether the market conditions fluctuate stochastically (*i.e.*, $\sigma(\cdot) > 0$), or deterministically. The following lemma establishes two additional necessary conditions that any mixed-strategy MPE must satisfy when $\sigma(\cdot) > 0$.

Lemma 4.3 *Suppose that $\sigma(\cdot) > 0$, and (G_i, G_j) constitutes a mixed-strategy MPE. Then:*

(a) $G^x_1(t)$ and $G^x_2(t)$ are continuous in t for all $x \in \mathscr{I}$, i.e., they have no mass points (discontinuities of the CDFs).

[8] We ignore the event that both firms exit the market simultaneously. As the proof shows, this is an innocuous simplification.

(b) The support $\Gamma = (a, \theta_1^*) = (a, \theta_2^*)$*, where* θ_i^* *is given in Lemma 4.1.*

Lemma 4.3(a) establishes that if $\sigma(\cdot) > 0$, then CDFs of an MPE must be continuous even if the initial value x is not in $\overline{\Gamma^o}$. To see why, we first recall that a discontinuity of a CDF must be a hitting time $\tau_E = \inf\{t \geq 0 : X_t^x \in E\}$ for some set $E \subset \mathbb{R}$ and for all x such that $\Pr(\tau_E < \infty) > 0$. Then because X is irreducible if $\sigma(\cdot) > 0$, Lemma 4.2(b) implies that if $\sigma(\cdot) > 0$, then a discontinuity of a CDF cannot take place while X is within $\overline{\Gamma^o}$ for $\forall x \in \mathscr{I}$, *i.e.*, irrespective of the initial value x. Hence, a mass point can exist only outside $\overline{\Gamma^o}$, in which case Γ must have a point component. However, we can further show that Γ cannot have a point component. Suppose to the contrary that $\{y\}$ is a point component of Γ. Then both firms assign a non-zero probability of exit when X hits y. However, in that case, one firm may decide never to exit when $X = y$ thereby increasing the probability of being the winner. Thus, an equilibrium does not allow a point component of Γ, and so $\overline{\Gamma} = \overline{\Gamma^o}$. Thus, we obtain the result that the CDFs of an MPE are continuous in time.

Next, recall that even if firm i anticipates that its rival will never exit the market, it is not willing to exit before X goes below θ_i^*. Hence, if firm i expects its rival to exit in finite time with positive probability, then this would, *ceteris paribus*, only decrease firm i's incentive to exit. Consequently, firm i *always* strictly prefers to remain in the market whenever $X > \theta_i^*$, which implies $\Gamma \subseteq (a, \theta_i^*)$. To see that this inclusion is indeed an equality, suppose that, for some $\theta < \theta_i^*$, firm $-i$ (and hence firm i) exits with positive probability on (a, θ). Then because firm $-i$ does not exit at any $X > \theta$ and its strategy has no mass points, starting at θ_i^*, firm i's expected payoff from exiting at the first time that $X_t^{\theta_i^*} \leq \theta$ is strictly less than l_i by Lemma 4.1. Therefore, firm i strictly prefers to exit at θ_i^*, which contradicts the premise that $\Gamma = (a, \theta)$ where $\theta < \theta_i^*$.

Recall from Lemma 4.1 that if $l_1 < l_2$, then $\theta_1^* < \theta_2^*$. Therefore, we have the following immediate implication.

Theorem 4.1 *Suppose that* $\sigma(\cdot) > 0$ *and* $l_1 < l_2$*. Then this game admits no mixed-strategy MPE.*

While the assumptions that payoffs are deterministic and firms are symmetric may be a good approximation of a particular setting, in reality, payoffs are not set in stone and no firms are exactly alike. This theorem, together with Proposition 4.1, shows that in this case, the game admits at most two MPE, both in pure strategies.

Because the result holds irrespective of the degree of uncertainty (and heterogeneity), it shows that mixed-strategy MPE are unstable to a natural perturbation of the canonical model, and provides an equilibrium selection argument for wars of attrition under complete information.

Finally, we point out that both ingredients are necessary to eliminate mixed-strategy MPE. To highlight this point, in the following section and in Appendix A, we characterize a mixed-strategy MPE for the case in which firms are homogeneous and payoffs are deterministic, respectively.

Special Case: Homogeneous Firms ($l_1 = l_2$)

In this section, we consider the case in which the firms are homogeneous (*i.e.,* $l_1 = l_2$), and we characterize the unique mixed-strategy MPE. It follows from Lemmas 4.2 and 4.3 that if (G_1, G_2) constitutes a mixed-strategy MPE, then each G_i must satisfy (4.8) on $\Gamma = (a, \theta_1^*)$, where $\theta_1^* = \theta_2^*$ is given in Lemma 4.1. Solving (4.8) subject to the boundary condition $G_i^x(0) = 0$ for every i and $x \in \Gamma$ yields

$$G_i^x(t) = 1 - \exp\left[-\int_0^t \frac{\mathbf{1}_{\{X_s^x \in \Gamma\}}(s)[rl_j - \pi(X_s^x)]}{w(X_s^x) - l_j} ds \right] . \tag{4.10}$$

Observe that $G_i^x(t)$ is a CDF of stopping times because it is right-continuous with left limits and non-decreasing in t. Moreover, its hazard rate depends only on the state X_t^x, confirming that it is a Markov strategy. The following Proposition shows that the strategy profile $(G_1, G_2) = (G_1^x, G_2^x)_{x \in \mathscr{I}}$ indeed constitutes the unique mixed-strategy MPE.

Proposition 4.2 *Suppose that* $\sigma(\cdot) > 0$ *and* $l_1 = l_2$. *Then* $(G_1, G_2) = (G_1^x, G_2^x)_{x \in \mathscr{I}}$*, where* $\Gamma = (a, \theta^*)$ *and* $\theta^* = \theta_1^* = \theta_2^*$ *constitutes a mixed-strategy MPE.*

4.6 Concluding Remarks

It is well known that canonical war of attrition games under complete information admit equilibria in pure strategies, as well as in mixed strategies (*e.g.*, Tirole (1988), Levin (2004), and others). We study such a two-player model and we show that if (i) the players' payoffs during the "war phase" are stochastic, *and* (ii) their exit payoffs are heterogeneous, then the game admits only pure-strategy MPE. That is, any degree of Brownian uncertainty in the players' in-war payoffs, and any amount of heterogeneity in their exit payoffs is sufficient to destabilize the mixed-strategy MPE. The main implication of this result is that in contrast to much of the extant literature, it may be more prudent to focus on the pure-strategy MPE of the game.

This paper opens several avenues for future research. First, the result on non-existence of mixed-strategy MPE in conjunction with the fact that the pure-strategy MPE are asymmetric, raises the question of equilibrium selection. Second, the recent literature on wars of attrition has focused on games with private information, such as Abreu and Gul (2000) who show that in such a setting, there exists a unique Bayesian Nash equilibrium, which entails costly fighting between the players. It is of interest to explore the stability of that equilibrium.

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Appendix A

Appendix of Chapter 2

A.1 Mathematical Proofs

Proof of Proposition 2.1: We employ Theorem 3(A) of Alvarez (2001) to prove this proposition. For convenience, we follow the convention of Alvarez (2001) and define a function $\Pi(p) = g_F(p)/\psi_F(p)$. Our goal is to prove that $\tau_F = \inf\{t > 0 : P_t \geq \theta_F\}$ is the optimal stopping time. In order to prove it, by virtue of Theorem 3(A) of Alvarez (2001), we only need to prove that $g_F(\cdot)$ is non-decreasing, that $\Pi(\cdot)$ attains a unique global interior maximum at θ_F, and that $\Pi(\cdot)$ is non-increasing for $p > \theta_F$.

First, $g_F(\cdot)$ is obviously an increasing function. Second, we can prove that $\Pi(\cdot)$ attains a unique global interior maximum at θ_F. The first-order necessary condition is given by

$$\frac{d\Pi(p)}{dp} = \frac{\psi_F(p)g_F'(p) - \psi_F'(p)g_F(p)}{\psi_F^2(p)} = 0\,,$$

which has a unique solution at $p = \theta_F$, which can be shown after some algebra. Furthermore,

$$\frac{d}{dp}[\psi_F(p)g_F'(p) - \psi_F'(p)g_F(p)] = \psi_F(p)g_F''(p) - \psi_F''(p)g_F(p) = -\psi_F''(p)g_F(p)\,.$$

Note that $\psi_F''(p) > 0$ for all p and that $g_F(\theta_F) = \psi_F(\theta_F)g_F'(\theta_F)/\psi_F'(\theta_F) > 0$ from the first-order condition. Thus, $-\psi_F''(\theta_F)g_F(\theta_F) < 0$, which implies that $\Pi(\cdot)$ attains a unique global interior maximum at θ_F. Lastly, we infer that $\Pi(\cdot)$ is non-increasing for $p > \theta_F$ because $-\psi_F''(p)g_F(p) < 0$ for all $p > \theta_F$ which implies that $d\Pi(p)/dp < 0$ for all $p > \theta_F$. ■

Proof of Lemma 2.1: The leader's value function is

$$V_{L,\tau}(p) = E^p[\int_0^\tau \mu(0)e^{-rt}dt + \int_0^\tau \sigma e^{-rt}dB_t^1 + (R_1(P_\tau) - k)e^{-r\tau}] \,,$$

with the constraint $\tau \leq \tau_F$. Here $R_1(\cdot)$ denotes the expected return immediately after the first investment:

$$\begin{aligned} R_1(p) &= E^p[\int_0^{\tau_F} \mu(L)e^{-rt}dt + \int_0^{\tau_F} \sigma e^{-rt}dB_t^1 + \int_{\tau_F}^{\infty} \mu(2)e^{-rt}dt + \int_{\tau_F}^{\infty} \sigma e^{-rt}dB_t^1] \\ &= E^p[\mu(L) + e^{-r\tau_F}\mu_{L2}]/r \,. \end{aligned}$$

Here τ_F is the stopping time at which the follower invests. From the fact that $P_{\tau_F} = \theta_F$,

$$E^p[e^{-r\tau_F}\mu_{L2}] = E^p[E^p[e^{-r\tau_F}\mu_{L2}|\mathcal{F}_{\tau_F}]] = [\theta_F h_{L2} + (1-\theta_F)\ell_{L2}]E^p[e^{-r\tau_F}] \,.$$

Let $f(p) \equiv E^p[e^{-r\tau_F}]$. Then $f(p)$ must satisfy $\mathcal{A}_F f(p) = 0$ for $p \leq \theta_F$ and the boundary condition $f(\theta_F) = 1$ (Oksendal (2003)). It is easy to verify that $f(p) = \psi_F(p)/\psi_F(\theta_F)$ for $p \leq \theta_F$. ■

Proof of Proposition 2.2: We first note that $g_L(\theta_F) > g_F(\theta_F) > 0$ due to $\mu(F) > \mu(0)$, $\lim_{p\to 0} g_L(p) < 0$ by Assumption 2.2, and $g_L(\cdot)$ is strictly concave on $(0, \theta_F)$ by Assumption 2.1. Thus, there exists a unique $p_L \in (0, \theta_F)$ such that $g_L(p) > 0$ if and only if $p > p_L$. Thus, we have

$$\begin{aligned} \mathcal{A}_L g_L(p) &= -r g_L(p) \\ &\quad + \frac{1}{2r\psi_F(\theta_F)}[\theta_F h_{L2} + (1-\theta_F)\ell_{L2}] \left(\frac{h(0)-\ell(0)}{\tilde{\sigma}}\right)^2 p^2(1-p)^2\partial_p^2\psi_F(p) < 0 \end{aligned}$$

for $p \in (p_L, \theta_F)$ by the convexity of $\psi_F(\cdot)$ and Assumption 2.1. Moreover, by using $\mathcal{A}_F\psi_F(p) = 0$, we can obtain

$$\begin{aligned} \mathcal{A}_L g_L(p) &= -[p h_{L1} + (1-p)\ell_{L1} - kr] \\ &\quad - [\theta_F h_{L2} + (1-\theta_F)\ell_{L2}]\frac{\psi_F(p)}{\psi_F(\theta_F)}\left[1 - \left(\frac{h(0)-\ell(0)}{\hat{h}-\hat{\ell}}\right)^2\right] \,, \end{aligned}$$

from which we can see that $\mathcal{A}_L g_L(\cdot)$ is convex on $(0, \theta_F)$ by Assumption 2.4. Combining this with the fact $\lim_{p\to 0} \mathcal{A}_L g_L(p) > 0$, there must exist a unique $p_c \in (0, p_L)$ such that $\mathcal{A}_L g_L(p) > 0$ if and only if $p < p_c$. By the argument given in Chapter 10, Oksendal (2003) and Theorem 2 of Alvarez (2001), the continuation region for this optimal stopping problem must be of the form $(0, \theta_L)$ for some $\theta_L \in (p_c, \theta_F]$. Similarly in the proof of Proposition **2.1**, we define $\Pi(p) = g_L(p)/\psi_L(p)$ and $\Pi_1(p) = \psi_L(p)g_L'(p) - \psi_L'(p)g_L(p)$ with $\Pi'(p) = \Pi_1(p)/\psi_L^2(p)$, and apply Theorem 3(A) of Alvarez (2001) in the following three possibilities:

(i) Suppose $\lim_{p\to\theta_F} g_L'(p) \geq 0$ and $\Pi_1(\theta_F) < 0$. Then, the strict concavity of $g_L(\cdot)$ implies that $g_L'(p) > 0$ for all $p \in (0, \theta_F)$ from which the first condition of Theorem 3(A) of Alvarez (2001) is satisfied. In addition, $g_L(p_L) = 0$ implies that $\Pi_1(p_L) = \psi_L(p_L)g_L'(p_L) > 0$ and $\Pi_1'(p) = \psi_L(p)g_L''(p) - \psi_L''(p)g_L(p) < 0$ for all $p \in (p_L, \theta_F)$ because $\psi_L(\cdot)$ is convex on $(0, \theta_F)$ and $g_L(p) > 0$ if and only if $p > p_L$. Therefore, from $\Pi_1(\theta_F) < 0$, we can see that there exists a unique $\theta_L \in (p_L, \theta_F)$ at which $\Pi(\cdot)$ attains its global maximum with $\Pi'(\theta_L) = \Pi_1(\theta_L) = 0$ and the second condition of Theorem 3(A) holds. Furthermore, $\Pi_1'(p) < 0$ for all $p \in (p_L, \theta_F)$ implies that $\Pi_1'(p) < 0$ for all $p \in (\theta_L, \theta_F)$, which completes the sufficiency of the optimality of θ_L by Alvarez (2001).

(ii) Suppose $\lim_{p\to\theta_F} g_L'(p) \geq 0$ and $\Pi_1(\theta_F) \geq 0$. The only difference is $\Pi_1(p) > 0$ for all $p \in (p_L, \theta_F)$ because of $\Pi_1(p_L) > 0$ and $\Pi_1'(p) < 0$ for all $p \in (p_L, \theta_F)$. It follows that $\Pi(\cdot)$ is strictly increasing within the interval (p_L, θ_F), and its maximum is attained at the boundary θ_F. This implies that the value function $V_{L,\tau_\theta}(p; \theta)$ is maximized with $\theta = \theta_F$, and thus, we can conclude that the optimal policy is to invest at $\tau_L = \inf\{t > 0 : P_t \geq \theta_F\}$.

(iii) Suppose $\lim_{x\uparrow\theta_F} g_L'(x) < 0$. Then, from the strict concavity of $g_L(\cdot)$, there exists a unique $p_b < \theta_F$ at which $g_L(\cdot)$ is maximized with $g_L'(p_b) = 0$, and hence, $g_L(\cdot)$ is increasing if and only if $p < p_b$. This implies that $\Pi(p) = g_L(p)/\psi_L(p)$ is decreasing for all $p > p_b$, and thus, the maximum of $\Pi(\cdot)$ on $(0, p_b)$ is its global maximum on $(0, \theta_F)$. Moreover, $\Pi_1(p_b) = \psi_L(p_b)g_L'(p_b) - \psi_L'(p_b)g_L(p_b) = -\psi_L'(p_b)g_L(p_b) < 0$ because $g_L(p_b) \geq g_L(\theta_F) > 0$. Therefore, we can apply the arguments used in the case (i) to the sub-interval $(0, p_b)$ instead of $(0, \theta_F)$, and we can show the existence of the optimal threshold $\theta_L < p_b < \theta_F$ with $\Pi_1(\theta_L) = 0$, which completes the proof. ■

Proof of Proposition 2.3: We first define $D(p) \equiv V_{L,0}(p) - V_F^*(p)$. From (2.6), (2.8), and (2.9), we find that $D(p)$ is either convex or concave within the interval $(0, \theta_F)$. Furthermore, we find $D(0) = (\ell(L) - \ell(F))/r - k < 0$ by

Assumption 2.3, and $D(\theta_F) = 0$. We also obtain

$$D'(p) = \frac{1}{r}[h(L) - h(F) - \ell(L) + \ell(F)] \\ + \frac{\psi_F'(p)}{\psi_F(\theta_F)}\{k - \frac{1}{r}[\theta_F(h(L) - h(F)) + (1-\theta_F)(\ell(L) - \ell(F))]\}\,. \quad \text{(A.1)}$$

Under Assumptions 2.1 and 2.2, we can prove that $\lim_{p\uparrow\theta_F} D'(p) < 0$ always holds. From (A.1) and (2.3),

$$\lim_{p\uparrow\theta_F} D'(p) = -\frac{\gamma_F - 1}{2(1-\theta_F)}\left[\frac{h(L) - h(F)}{r} - k\right] + \frac{\gamma_F + 1}{2\theta_F}\left[k - \frac{\ell(L) - \ell(F)}{r}\right], \quad \text{(A.2)}$$

can be derived. In addition, from (2.5), we find that (A.2) is negative if and only if the following value is negative:

$$-\frac{h(L) - h(F) - kr}{h(2) - h(F) - kr} + \frac{kr - \ell(L) + \ell(F)}{kr - \ell(2) + \ell(F)} \quad \text{(A.3)}$$

By Assumptions 2.1 and 2.2, we have $h(L) - h(F) - kr > h(2) - h(F) - kr > 0$, and thus, the first term of (A.3) is always negative, and its absolute value is larger than one. By Assumption 2.2, on the other hand, we have $kr - \ell(2) + \ell(F) > 0$. Hence, if $kr - \ell(L) + \ell(F) < 0$, then the second term in (A.3) is negative in which case (A.3) is negative. If $kr - \ell(L) + \ell(F) \geq 0$, then by $\mu(F) < \mu(2) < \mu(L)$ from Assumption 2.1, we find $kr - \ell(2) + \ell(F) > kr - \ell(L) + \ell(F) \geq 0$,which implies that the second term in (A.3) is less than one. Since the first term of (A.3) is less than -1, we can have $\lim_{p\uparrow\theta_F} D'(p) < 0$.

Next, we prove that, if $\lim_{p\uparrow\theta_F} D'(p) < 0$, then there exists $\theta_c \in (0, \theta_F)$ such that $D(p) < 0$ for $p < \theta_c$ and $D(p) > 0$ for $p \in (\theta_c, \theta_F)$. Suppose first that $k - \frac{1}{r}[\theta_F(h(L) - h(F)) + (1-\theta_F)(\ell(L) - \ell(F))] \geq 0$. Then $D(p)$ is convex on $(0, \theta_F)$. The convexity of $D(\cdot)$ with $D(0) < 0$ and $D(\theta_F) = 0$ implies that $\lim_{p\uparrow\theta_F} D'(p) > 0$ must be satisfied, which contradicts our finding above. Therefore, we should have $k - \frac{1}{r}[\theta_F(h(L) - h(F)) + (1-\theta_F)(\ell(L) - \ell(F))] < 0$ so that $D(p)$ is strictly concave on $(0, \theta_F)$. Then, from the concavity of $D(\cdot)$ with $D(0) < 0$ and $D(\theta_F) = 0$, it follows that $D(p)$ changes its sign exactly once at some $\theta_c \in (0, \theta_F)$. ■

Proof of Theorem 2.1: First, note that $[\theta_F, 1)$ is the region of simultaneous investment in the sense that the best response is to invest immediately whenever $P_t \in [\theta_F, 1)$ if the opponent's strategy is to invest immediately.

Second, we consider the region $[\theta_c, \theta_F)$. Suppose firm 2's strategy is to preemptively invest whenever $P_t \in [\theta_c, \theta_F)$, which means firm 2 invests immediately if firm 1 has not invested yet. If the current posterior is $p \in [\theta_c, \theta_F)$, firm 1 can either wait or take a preemption policy. If firm 1 waits, firm 2 will preemptively invest, in which case firm 1 receives $V_F(p)$. If firm 1 takes a preemption policy, then there is 50% chance that it will be the leader or the follower, so its value function is $[V_{L,0}(p) + V_F(p)]/2 > V_F(p)$. Thus, firm 1's best response is to take a preemption policy in $[\theta_c, \theta_F)$. We conclude that there are pure strategy MPEs in which $[\theta_c, \theta_F)$ is the preemption region and $[\theta_F, 1)$ is the simultaneous investment region.

(i) Suppose that $\theta_L \geq \theta_c$. We fix firm 2's strategy as prescribed in (i) of the Proposition. From the proof of Proposition 2.2, note that $\lim_{p\to 0} \mathcal{A}_L g_L(p) > 0$ and that there exists a p_c such that $\mathcal{A}_L g_L(p) > 0$ for $p \in (0, p_c)$ and $\mathcal{A}_L g_L(p) < 0$ for $p \in (p_c, \theta_F)$. Thus, by Theorem 2 of Alvarez (2001), we find that firm 1's best response is to have the continuation region of the form $(0, \theta)$ for some $\theta \leq \theta_c$ and invest immediately for $[\theta, \theta_c]$. Let $V_\theta(p)$ be firm 1's value function associated with the continuation region $(0, \theta)$. From the fact that $\theta_L \geq \theta_c$, it is straightforward to show that $\partial V_\theta(p)/\partial\theta > 0$ for all $\theta \leq \theta_c$, so the optimal choice of θ is θ_c.

(ii) Suppose that $\theta_L < \theta_c$. We fix firm 2's strategy as the follower's strategy prescribed in (ii) of the Proposition. By Proposition 2.2, firm 1's best response is to take $(0, \theta_L)$ as the continuation region, and $[\theta_L, \theta_c]$ as the region of immediate investment. ■

Proof of Proposition 2.4: If $p \geq \theta$, then the proposition trivially follows. Suppose $p < \theta$. We note that

$$\mathbb{P}^p[\tau_\theta < \infty] = \lim_{r\to 0} E^p[e^{-r\tau_\theta}] = \lim_{r\to 0} \frac{\psi_L(p)}{\psi_L(\theta)} = \frac{p}{\theta} ,$$

from the dependence of $\psi_L(\cdot)$ on r through γ_L.

$$E^p[\tau_\theta \mathbf{1}_{\{\tau_\theta < \infty\}}] = \lim_{r\to 0} \frac{\partial}{\partial r} E^p[e^{-r\tau_\theta}] = \frac{2\tilde{\sigma}^2 p}{[h(0) - \ell(0)]^2 \theta} \ln\left[\frac{\theta(1-p)}{(1-\theta)p}\right] .$$

Finally, $E^p[\tau_\theta | \tau_\theta < \infty] = E^p[\tau_\theta \mathbf{1}_{\{\tau_\theta < \infty\}}]/\mathbb{P}^p[\tau_\theta < \infty]$. ■

Proof of Proposition 2.5: Recall from the proof of Proposition 2.2 that

$$\begin{aligned}\Pi_1(p) &= \psi_L(p)g_L'(p) - \psi_L'(p)g_L(p) \\ &= -\frac{1}{2r}\frac{\psi_L(p)}{p(1-p)}\Big[(\gamma_L - 1)(h_{L1} - rk)p + (\gamma_L + 1)(\ell_{L1} - rk)(1-p) \\ &+ (\gamma_L - \gamma_F)(\theta_F h_{L2} + (1-\theta_F)\ell_{L2})\frac{\psi_F(p)}{\psi_F(\theta_F)}\Big] .\end{aligned}$$

We can also obtain a similar algebraic expression for $\Pi_1'(\cdot)$ and find a function $B(p)$ with $B(0) < 0$ such that $\Pi_1'(p) < 0$ if and only if $B(p) > 0$. In addition, if $\mu(L) > \mu(2)$, $B(\cdot)$ is concave with $B(\theta_F) > 0$, and if $\mu(L) < \mu(2)$, then $B(\cdot)$ is increasing in $(0, \theta_F)$. Thus, there must exist a unique $p_s \in (0, \theta_F)$ such that $\Pi_1'(p) > 0$ if and only if $p < p_s$ for either $\mu(L) > \mu(2)$ or $\mu(L) < \mu(2)$.

Define $Q(p) \equiv (\gamma_L - 1)(h_{L1} - rk)p + (\gamma_L + 1)(\ell_{L1} - rk)(1-p) + (\gamma_L - \gamma_F)(\theta_F h_{L2} + (1-\theta_F)\ell_{L2})\psi_F(p)/\psi_F(\theta_F)$. Then, $\Pi_1(p) < 0$ if and only if $Q(p) > 0$. However,

$$\begin{aligned}Q(\theta_B) &= (\gamma_L - 1)(h_{L1} - rk)\theta_B + (\gamma_L + 1)(\ell_{L1} - rk)(1-\theta_B) \\ &+ (\gamma_L - \gamma_F)(\theta_F h_{L2} + (1-\theta_F)\ell_{L2})\frac{\psi_F(\theta_B)}{\psi_F(\theta_F)} \\ &= (\gamma_L - \gamma_F)(\theta_F h_{L2} + (1-\theta_F)\ell_{L2})\frac{\psi_F(\theta_B)}{\psi_F(\theta_F)} ,\end{aligned}$$

which means $Q(\theta_B) > 0$ if and only if $\mu(2) > \mu(L)$. Thus, $Q(\theta_L) = \Pi_1(\theta_L) = 0$ completes the proof. ∎

Appendix B

Appendix of Chapter 3

B.1 Mathematical Preliminaries

In this Appendix, we develop some preliminary facts necessary for analyzing the game of infinite investment opportunities. Consider a pure strategy profile $(\mathbf{T}_i, \mathbf{T}_j)$ where the strategy for stage n is represented by $H(\tau_i^{(n)})$, i.e., $G_i^{(n)}$ jumps at $\tau_i^{(n)}$ from 0 to 1[1]. We then express firm i's expected payoff under $(\mathbf{T}_i, \mathbf{T}_j)$ as

$$U_i(x; \mathbf{T}_i, \mathbf{T}_j) = \Phi_i(x; H(\tau_i^{(1)}), H(\tau_j^{(1)})) + \sum_{n=2}^{\infty} \mathbb{E}^x \left[\Big(\prod_{m=1}^{n-1} e^{-r(\tau_i^{(m)} \wedge \tau_j^{(m)})} \Big) \Phi_i(\zeta; H(\tau_i^{(n)}), H(\tau_j^{(n)})) \right], \quad \text{(B.1)}$$

where, for any $z \in \mathscr{I}$,

$$\Phi_i(z; H(\tau_i^{(n)}), H(\tau_j^{(n)})) := \mathbb{E}^z \left[\int_0^{\tau_i^{(n)} \wedge \tau_j^{(n)}} \pi(X_t) e^{-rt} dt - (\mathbf{1}_{\{\tau_i^{(n)} < \tau_j^{(n)}\}} k_i + \mathbf{1}_{\{\tau_i^{(n)} = \tau_j^{(n)}\}} \frac{1}{2} k_i) e^{-r(\tau_i^{(n)} \wedge \tau_j^{(n)})} \right]. \quad \text{(B.2)}$$

Note that $U_i(x; \mathbf{T}_i, \mathbf{T}_j)$ is well-defined due to the absolute integrability condition (3.2). The function $\Phi_i(\cdot; \cdot, \cdot)$ is firm i's expected cumulative discounted profit accrued during the given stage.

We first establish Lemmas B.1 and B.2, which directly lead to Lemma 3.1. For some $\mathbf{T}_j \in \mathcal{S}^\infty$, we define $U_i^*(x; \mathbf{T}_j) := \sup_{\mathbf{T}_i} U_i(x; \mathbf{T}_i, \mathbf{T}_j)$.

[1] Although we develop Lemmas B.1 – B.3 only with pure strategy profiles for the sake of expositions, the same results straightforwardly generalize to mixed strategy profiles as well. See Lemma B.5 for details.

Lemma B.1 $U_i^*(x;\mathbf{T}_j)$ *satisfies*

$$U_i^*(x;\mathbf{T}_j) = \sup_{\tau}\Big\{\Phi_i(x;H(\tau),H(\tau_j^{(1)})) + \mathbb{E}^x[e^{-r(\tau\wedge\tau_j^{(1)})}U_i^*(\zeta;\mathbf{T}_j)]\Big\}\,. \qquad \text{(B.3)}$$

Equation (B.3) is called the *optimality equation* (Ross, 1995).

Proof of Lemma B.1: First, note that the opponent's strategy $\mathbf{T}_j$ can be seen as another state variable on which firm i's strategy depends. Because $\mathbf{T}_j \in \mathcal{S}^\infty$, each $H(\tau_j^{(n)})$ has an identical distribution for each stage n. Therefore, given $\mathbf{T}_j \in \mathcal{S}^\infty$, the problem of finding firm i's best response reduces to one of finding the optimal policy for a single decision maker. Hence, equation (B.3) holds by the arguments provided by Ross (1995, p.31). ■

Lemma B.2 *For* $\mathbf{T}_j \in \mathcal{S}^\infty$, *we define* $\mathbf{T}_i^* = \{H(\tau_i^{*(n)})\}_{n=1}^\infty \in \mathcal{S}^\infty$ *where* $\tau_i^{*(n)}$ *solves the right-hand-side of (B.3). Then*

$$U_i(x;\mathbf{T}_i^*,\mathbf{T}_j) = \sup_{\mathbf{T}_i} U_i(x;\mathbf{T}_i,\mathbf{T}_j) = U_i^*(x;\mathbf{T}_j)\,.$$

Proof of Lemma B.2: For notational convenience, we denote $\hat{\tau}^{(n)} := \tau_i^{*(n)}$ for all $n \geq 1$. Because $(\mathbf{T}_i^*,\mathbf{T}_j) \in \mathcal{S}^\infty \times \mathcal{S}^\infty$, we have

$$U_i(x;\mathbf{T}_i^*,\mathbf{T}_j) = \Phi_i(x;H(\hat{\tau}^{(1)}),H(\tau_j^{(1)})) + \mathbb{E}^x\left[e^{-r(\hat{\tau}^{(1)}\wedge\tau_j^{(1)})}U_i(\zeta;\mathbf{T}_i^*,\mathbf{T}_j)\right]. \qquad \text{(B.4)}$$

Moreover, we know from Lemma B.1 that

$$U_i^*(x;\mathbf{T}_j) = \Phi_i(x;H(\hat{\tau}^{(1)}),H(\tau_j^{(1)})) + \mathbb{E}^x\left[e^{-r(\hat{\tau}^{(1)}\wedge\tau_j^{(1)})}U_i^*(\zeta;\mathbf{T}_j)\right]. \qquad \text{(B.5)}$$

Subtracting (B.5) from (B.4), we obtain

$$U_i(x;\mathbf{T}_i^*,\mathbf{T}_j) - U_i^*(x;\mathbf{T}_j) = \mathbb{E}^x[e^{-r(\hat{\tau}^{(1)}\wedge\tau_j^{(1)})}][U_i(\zeta;\mathbf{T}_i^*,\mathbf{T}_j) - U_i^*(\zeta;\mathbf{T}_j)]\,, \qquad \text{(B.6)}$$

for all $x \in \mathscr{I}$. Particularly, this equation holds for $x = \zeta$, and hence we have

$$U_i(\zeta;\mathbf{T}_i^*,\mathbf{T}_j) - U_i^*(\zeta;\mathbf{T}_j) = \mathbb{E}^\zeta[e^{-r(\hat{\tau}^{(1)}\wedge\tau_j^{(1)})}][U_i(\zeta;\mathbf{T}_i^*,\mathbf{T}_j) - U_i^*(\zeta;\mathbf{T}_j)]\,. \qquad \text{(B.7)}$$

Since $\hat{\tau}^{(1)} \wedge \tau_j^{(1)} > 0$ (a.s.) if the current state value is ζ, we have $\mathbb{E}^{\zeta}[e^{-r(\hat{\tau}^{(1)} \wedge \tau_j^{(1)})}] < 1$ in (B.7), which implies $U_i(\zeta; \mathbf{T}_i^*, \mathbf{T}_j) - U_i^*(\zeta; \mathbf{T}_j) = 0$. Inserting this result in (B.6) completes the proof. ■

Lemmas B.1 and B.2 assert that there is a best response in $\mathcal{S}^\infty$ to the opponent's strategy in $\mathcal{S}^\infty$. The following two lemmas establish the method of obtaining the best response payoff functions.

Lemma B.3 $U_i^*(x; \mathbf{T}_j)$ *is the unique solution to the optimality equation (B.3).*

Proof of Lemma B.3: Suppose that $\tilde{U}_i(\cdot)$ is a function on $\mathscr{I}$ that satisfies the optimality equation (B.3). We let $\tilde{\tau}$ denote the stopping time with which the corresponding supremum is attained. Because $U_i^*(x; \mathbf{T}_j)$ also satisfies (B.3), by using similar arguments to Ross (1995, p.34), we obtain

$$\tilde{U}_i(x) - U_i^*(x; \mathbf{T}_j) = \mathbb{E}^x[e^{-r(\tilde{\tau} \wedge \tau_j^{(1)})}][\tilde{U}_i(\zeta) - U_i^*(\zeta; \mathbf{T}_j)] \,, \tag{B.8}$$

for all $x \in \mathscr{I}$. Setting $x = \zeta$ in (B.8) and reversing the roles of $\tilde{U}_i(\cdot)$ and $U_i^*(\cdot; \mathbf{T}_j)$, the desired result follows from the same arguments as in the proof of Lemma B.2 and Ross (1995, p.34). ■

Lemma B.4 *For any* $\mathbf{T}_j \in \mathcal{S}^\infty$*, a strategy* $\mathbf{T}_i \in \mathcal{S}^\infty$ *is a best response to* $\mathbf{T}_j$ *if*

$$\begin{aligned} &\Phi_i(x; H(\tau_i^{(1)}), H(\tau_j^{(1)})) + \mathbb{E}^x[e^{-r(\tau_i^{(1)} \wedge \tau_j^{(1)})} U_i(\zeta; \mathbf{T}_i, \mathbf{T}_j)] \\ = \sup_\tau &\left\{ \Phi_i(x; H(\tau), H(\tau_j^{(1)})) + \mathbb{E}^x[e^{-r(\tau \wedge \tau_j^{(1)})} U_i(\zeta; \mathbf{T}_i, \mathbf{T}_j)] \right\} . \end{aligned}$$

Proof of Lemma B.4: For $(\mathbf{T}_i, \mathbf{T}_j) \in \mathcal{S}^\infty \times \mathcal{S}^\infty$, we have $U_i(x; \mathbf{T}_i, \mathbf{T}_j) = \Phi_i(x; H(\tau_i^{(1)}), H(\tau_j^{(1)})) + \mathbb{E}^x[e^{-r(\tau_i^{(1)} \wedge \tau_j^{(1)})} U_i(\zeta; \mathbf{T}_i, \mathbf{T}_j)]$. Thus, this lemma is a direct consequence of Lemma B.3. ■

The following lemma establishes that the statements of Lemmas B.1 – B.3 carry over to any mixed strategy profile $(\mathbf{G}_i, \mathbf{G}_j)$.

Lemma B.5 *Firm i's expected payoff associated* $U_i(x; \mathbf{G}_i, \mathbf{G}_j)$ *with* $(\mathbf{G}_i, \mathbf{G}_j)$

is given by

$$
\begin{aligned}
&\widetilde{\Phi}_i(x; G_i^{(1)}, G_j^{(1)}) \\
&+ \sum_{n=2}^{\infty} \mathbb{E}^x \Big[\int \!\cdots\! \int_0^{\infty} \Big\{ \big(\prod_{m=1}^{n-1} e^{-r(s_i^{(m)} \wedge s_j^{(m)})} \big) \widetilde{\Phi}_i(\zeta; G_i^{(n)}, G_j^{(n)}) \Big\} \\
&\prod_{m=1}^{n-1} \big(dG_i^{(m)}(s_i^{(m)}) dG_j^{(m)}(s_j^{(m)}) \big) \Big] ,
\end{aligned}
$$

where, for any $z \in \mathcal{I}$ *and* $k \geq 1$, $\widetilde{\Phi}_i(z; G_i^{(n)}, G_j^{(n)})$ *is defined as*

$$
\begin{aligned}
&\mathbb{E}^z \Big[\iint_0^{\infty} \Big\{ \int_0^{s_i^{(n)} \wedge s_j^{(n)}} \pi(X_t) e^{-rt} dt \\
&- (\mathbf{1}_{\{s_i^{(n)} < s_j^{(n)}\}} k_i + \mathbf{1}_{\{s_i^{(n)} = s_j^{(n)}\}} \frac{1}{2} k_i) e^{-r(s_i^{(n)} \wedge s_j^{(n)})} \Big\} dG_i^{(n)}(s_i^{(n)}) dG_j^{(n)}(s_j^{(n)}) \Big] .
\end{aligned}
$$

Note that $U_i(x; \mathbf{G}_i, \mathbf{G}_j)$ and $\widetilde{\Phi}_i(z; G_i^{(n)}, G_j^{(n)})$ are mixed strategy profile analogs of (B.1) and (B.2).

Proof of Lemma B.5: It directly follows from the definition of $U_i(x; \mathbf{G}_i, \mathbf{G}_j)$ and a series of algebra. ■

For any strategy profile $(\mathbf{G}_i, \mathbf{G}_j) \in \mathcal{S}^{\infty} \times \mathcal{S}^{\infty}$, we have $V_i(x; G_i, G_j) = U_i(x; \mathbf{G}_i, \mathbf{G}_j)$ where $V_i(\cdot; \cdot, \cdot)$ is given in (3.4). For the remainder of the Appendices, therefore, we write the arguments with $V_i(\cdot; \cdot, \cdot)$ and (G_i, G_j) unless otherwise specified. The following lemma can be used to obtain the optimal stopping time τ_i^* in Proposition 3.1.

Lemma B.6 *Consider a pure strategy* $H(\tau^{\theta})$ *where* $\tau^{\theta} := \inf\{t \geq 0 : X_t \leq \theta\}$ *for some* $\theta < \zeta$ *and* $\forall n \geq 1$. *Then firm i's expected payoff under the strategy profile* $(G_i, G_j) = (H(\tau^{\theta}), H(\infty))$ *is given by*

$$
\begin{aligned}
V_i(x; H(\tau^{\theta}), H(\infty)) &= \mathbb{E}^x \Big[\int_0^{\tau^{\theta}} \pi(X_t) e^{-rt} dt + [V_i(\zeta; H(\tau^{\theta}), H(\infty)) - k_i] e^{-r\tau^{\theta}} \Big] \\
&= \begin{cases} (R_r \pi)(x) + \beta_i(\theta) \phi(x) & \text{for } x \geq \theta , \\ V_i(\zeta; H(\tau^{\theta}), H(\infty)) - k_i & \text{for } x < \theta , \end{cases}
\end{aligned}
$$

where $\beta_i(\cdot)$ *is defined in (3.7).*

Proof of Lemma B.6: First, we obtain the following from the definition in

(B.1):

$$
\begin{aligned}
&V_i(x; H(\tau^\theta), H(\infty)) \\
&= \Phi_i(x; H(\tau^\theta), H(\infty)) + \sum_{n=2}^{\infty} \mathbb{E}^x \left[\Big(\prod_{m=1}^{n-1} e^{-r\tau^\theta} \Big) \Phi_i(\zeta; H(\tau^\theta), H(\infty)) \right] \\
&= \mathbb{E}^x \left[\int_0^{\tau^\theta} \pi(X_t) e^{-rt} dt + [V_i(\zeta; H(\tau^\theta), H(\infty)) - k_i] e^{-r\tau^\theta} \right] , \qquad \text{(B.9)}
\end{aligned}
$$

for all $x \in \mathscr{I}$. Using $\mathbb{E}^x[e^{-r\tau^\theta}] = \phi(x)/\phi(\theta)$ for $x > \theta$, we can solve (B.9) for $x = \zeta$ to obtain the relation $V_i(\zeta; H(\tau^\theta), H(\infty)) - k_i = [\phi(\theta) l_i - \phi(\zeta)(R_r\pi)(\theta)]/[\phi(\theta) - \phi(\zeta)]$. Substituting this back in (B.9) completes the proof. ■

When examining a mixed strategy profile $\mathcal{G} = (G_i, G_j)$, it is useful to define the following stochastic process:

$$
S_i^{\mathcal{G}}(t; G_j) := \int_0^{t-} F_i^{\mathcal{G}}(s) dG_j(s) + M_i^{\mathcal{G}}(t) \Delta G_j(t) + L_i^{\mathcal{G}}(t)[1 - G_j(t)] \,, \qquad \text{(B.10)}
$$

where

$$
L_i^{\mathcal{G}}(t) := \int_0^t \pi(X_s) e^{-rs} ds + l_i^{\mathcal{G}} e^{-rt}, \;\; F_i^{\mathcal{G}}(t) := \int_0^t \pi(X_s) e^{-rs} ds + f_i^{\mathcal{G}} e^{-rt}, \qquad \text{(B.11)}
$$

$$
\text{and } M_i^{\mathcal{G}}(t) := \frac{1}{2}[L_i^{\mathcal{G}}(t) + F_i^{\mathcal{G}}(t)].
$$

Here, $\mathbb{E}^x[L_i^{\mathcal{G}}(t)]$ and $\mathbb{E}^x[F_i^{\mathcal{G}}(t)]$ are respectively firm i's expected payoffs associated with becoming the leader and the follower at time t for the *current* stage, given that $\mathcal{G}$ is employed for every *subsequent* stage. Thus, $\mathbb{E}^x[S_i^{\mathcal{G}}(t; G_j)]$ is firm i's expected payoff associated with investing at time t given G_j for the current stage and given that $\mathcal{G}$ is the strategy profile employed by the firms for every subsequent stage. Observe that $V_i(\cdot; \mathcal{G}) = V_i(\cdot; G_i, G_j)$, given in (3.4), can be alternatively written as

$$
V_i(x; G_i, G_j) = \mathbb{E}^x \left[\int_0^\infty S_i^{\mathcal{G}}(t; G_j) dG_i(t) \right] . \qquad \text{(B.12)}
$$

The following lemma will be used to characterize best responses in mixed strategies. Here we use the convention that $\overline{A}$ denotes the closure of a set A.

Lemma B.7 *Given a strategy profile $\mathcal{G} = (G_i, G_j)$, G_i is a best response to G_j if and only if*

$$\mathbb{E}^x[S_i^{\mathcal{G}}(\bar{\tau}; G_j)] = \sup_{\tau} \mathbb{E}^x[S_i^{\mathcal{G}}(\tau; G_j)] \,, \tag{B.13}$$

for all $x \in \mathscr{I}$, whenever $X_{\bar{\tau}}^x \in \overline{supp(G_i)}$.

Proof of Lemma B.7: This lemma is a direct consequence of the inequality $V_i(x; G_i, G_j) \leq \sup_\tau \mathbb{E}^x[S_i^{\mathcal{G}}(\tau; G_j)]$ obtained from Lemma 3.1. in Steg (2015). ■

B.2 Mathematical Proofs

Proof of Lemma 3.1: It directly follows from Lemmas B.1 and B.2. ■

Proof of Proposition 3.1: It directly follows from Theorem 3 of Alvarez (2001) and Lemma B.6. ■

Proof of Lemma 3.2: Note first that, for each $i \in \{1, 2\}$, θ_i is defined as a unique maximizer of $\beta_i(\cdot)$ given in (3.7) with (i) $\beta_i^{'}(\theta_i) = 0$, and (ii) $\beta_i^{'}(x) > 0$ for $x < \theta_i$ and $\beta_i^{'}(x) < 0$ for $x > \theta_i$. Recalling $l_i = (R_r\pi)(\zeta) - k_i$ and differentiating (3.7) yields $\beta_1^{'}(x) - \beta_2^{'}(x) = \phi^{'}(x)[k_1 - k_2]/[\phi(x) - \phi(\zeta)]^2 < 0$ where the inequality follows from $\phi^{'}(\cdot) < 0$ and $k_1 > k_2$. Thus, we have $\beta_1^{'}(\theta_2) < \beta_2^{'}(\theta_2) = 0$, which implies that $\theta_2 > \theta_1$ from the property (ii) of $\beta_1(\cdot)$ described above. ■

Proof of Proposition 3.2: By the definition of τ_2^*, it is straightforward to verify that $H(\tau_2^*)$ is firm 2's best response to $H(\infty)$. Thus, it only remains to prove that $H(\infty)$ is firm 1's best response to $H(\tau_2^*)$. From Lemma B.4, we only need to show the following:

$$\begin{aligned} &\Phi_1(x; H(\infty), H(\tau_2^*)) + \mathbb{E}^x[e^{-r\tau_2^*} V_1(\zeta; H(\infty), H(\tau_2^*))] \\ = &\sup_\tau \left\{ \Phi_1(x; H(\tau), H(\tau_2^*)) + \mathbb{E}^x[e^{-r(\tau \wedge \tau_2^*)} V_1(\zeta; H(\infty), H(\tau_2^*))] \right\} . \end{aligned}$$

where τ_i^* is given in (3.8). Then the goal is to find τ that solves the maximization problem

$$\sup_\tau \left\{ \Phi_1(x; H(\tau), H(\tau_2^*)) + \mathbb{E}^x[e^{-r(\tau \wedge \tau_2^*)} V_1(\zeta; H(\infty), H(\tau_2^*))] \right\} . \tag{B.14}$$

First, consider $\tau_\theta := \inf\{t \geq 0 : X_t \leq \theta\}$ for some $\theta > \theta_2$. Then we have

$$\begin{aligned} V_1(\zeta; H(\infty), H(\tau_2^*)) &> V_1(\zeta; H(\tau_2^*), H(\infty)) \\ &> V_1(\zeta; H(\tau_\theta), H(\infty)) = V_1(\zeta; H(\tau_\theta), H(\tau_2^*)) \,. \end{aligned} \tag{B.15}$$

The first inequality holds because the value of becoming the follower is always greater than that of becoming the leader. The second inequality comes from Lemma B.6, Assumption 3.1, and $\theta > \theta_2 > \theta_1$. The equality holds because of $\theta > \theta_2$. We now claim that τ_θ cannot be a solution to (B.14). Towards a contradiction, suppose that τ_θ is a solution. Then it implies that employing $H(\tau_\theta)$ for all the stages n must be at least not worse than employing $H(\infty)$ for all stages, which means that $V_1(\zeta; H(\tau_\theta), H(\tau_2^*)) \geq V_1(\zeta; H(\infty), H(\tau_2^*))$. However, this contradicts the inequality (B.15).

Next, we show that any τ_θ with $\theta \leq \theta_2$ cannot be a solution to (B.14), either. For $x \leq \theta_2$, it is always better to wait for an instant to invest than to invest immediately because $f_i^{\mathcal{G}} > m_i^{\mathcal{G}}$. Therefore, ∞ is the solution to (B.14) as desired. ■

Proof of Proposition 3.3: First, we define $\kappa_p \equiv \kappa_p(k_2) > 0$ as the solution to

$$\beta_{F2}(\theta_2 - \kappa_p) = \beta_2(\theta_2) \,, \tag{B.16}$$

where $\beta_2(\cdot)$ is given in (3.7) and $\beta_{F2}(\theta) := [f_2 - (R_r\pi)(\theta)]/[\phi(\theta) - \phi(\zeta)]$.

We claim that κ_p is unique if it exists for two reasons: (1) $\beta_{F2}(\theta) > \beta_2(\theta)$ for $\forall \theta \in \mathscr{I}$ because $f_2 > l_2$. (2) $\beta'_{F2}(\theta) > \beta'_2(\theta)$ for $\forall \theta < \theta_2$ because

$$\begin{aligned} \beta'_{F2}(\theta) &= \Big\{-(R_r\pi)'(\theta)[\phi(\theta) - \phi(\zeta)] - \phi'(\theta)[f_2 - (R_r\pi)(\theta)]\Big\}/[\phi(\theta) - \phi(\zeta)]^2 \\ &> \Big\{-(R_r\pi)'(\theta)[\phi(\theta) - \phi(\zeta)] - \phi'(\theta)[l_2 - (R_r\pi)(\theta)]\Big\}/[\phi(\theta) - \phi(\zeta)]^2 \\ &= \beta'_2(\theta) > 0 \,. \end{aligned}$$

Here the first inequality follows because $f_2 > l_2$, and the last inequality follows from Assumption 3.1. If κ_p that satisfies (B.16) does not exist, we set $\kappa_p = \infty$ as a convention.

We now show that if $\theta_2 - \theta_1 < \kappa_p$, then $V_2(\zeta; H(\tau_1^*), H(\infty)) > V_2(\zeta; H(\infty), H(\tau_2^*))$.

Note that $\theta_1 > \theta_2 - \kappa_p$ implies that $\beta_{F2}(\theta_1) > \beta_{F2}(\theta_2 - \kappa_p) = \beta_2(\theta_2)$ because $\beta'_{F2}(\theta) > 0$ for $\forall\theta < \theta_2$ and (B.16). Thus, we obtain

$$\begin{aligned} V_2(\zeta; H(\tau_1^*), H(\infty)) &= (R_r\pi)(\zeta) + \phi(\zeta)\beta_{F2}(\theta_1) \\ &> (R_r\pi)(\zeta) + \phi(\zeta)\beta_2(\theta_2) = V_2(\zeta; H(\infty), H(\tau_2^*)) \,, \end{aligned}$$

where the equalities follow from Lemma B.6.

Lastly, we only need to prove that $H(\infty)$ is firm 2's best response to $H(\tau_1^*)$ if $V_2(\zeta; H(\tau_1^*), H(\infty)) > V_2(\zeta; H(\infty), H(\tau_2^*))$. The rest of the proof proceeds just as the proof of Proposition 3.2, and it will be omitted. ■

Proof of Proposition 3.4(a): First, we prove that $\mathcal{G} = (G_1, G_2)$ with $\text{supp}(G_i) = (-\infty, \theta_2]$ is an equilibrium with $q_2 = q_2^*$. Note that if $k_1 - k_2 < \kappa(k_2) := k_2\phi(\zeta)/[\phi(\theta_2) - \phi(\zeta)]$, then we have $q_2^* < 1$ so that $\mathcal{G}$ is a valid mixed strategy profile. Also, solving the equation $V_1(\zeta; \mathcal{G}) = (R_r\pi)(\zeta) + \beta_1(\theta_2; q_2)\phi(\zeta)$ for $V_1(\zeta; \mathcal{G}) - k_1$ yields

$$\begin{aligned} V_1(\zeta; \mathcal{G}) - k_1 &= \frac{\phi(\theta_2)[(R_r\pi)(\zeta) - k_1] - \phi(\zeta)[(R_r\pi)(\theta_2) - k_1 q_2^*]}{\phi(\theta_2) - \phi(\zeta)} \\ &= \frac{\phi(\theta_2)[(R_r\pi)(\zeta) - k_2] - \phi(\zeta)(R_r\pi)(\theta_2)}{\phi(\theta_2) - \phi(\zeta)} = V_2(\zeta; \mathcal{G}) - k_2 \,. \end{aligned}$$

Then because $x_2 = \theta_2$ where x_i is given by $\pi(x_i) = r[V_i(\zeta; \mathcal{G}) - k_i]$, $i \in \{1, 2\}$, it follows from the strict monotonicity of $\pi(\cdot)$ that $x_1 = \theta_2$. Moreover, for any time t with $X_t < \theta_2$, $G_j(\cdot)$ is given as

$$\frac{dG_j(t)}{[1 - G_j(t)]} = \frac{[rl_i^{\mathcal{G}} - \pi(X_t)]dt}{k_i} = \frac{-dL_i^{\mathcal{G}}(t)}{[F_i^{\mathcal{G}}(t) - L_i^{\mathcal{G}}(t)]} \,,$$

where we use the definitions of $L_i^{\mathcal{G}}(t)$ and $F_i^{\mathcal{G}}(t)$ in (B.11) for the last equality. Then because $G_j(t)$ and $L_i^{\mathcal{G}}(t)$ are both monotone and continuous for any $X_t < \theta_2$, we can use the arguments in the proof of Lemma 3 in Hendricks et al. (1988) to obtain $S_i^{\mathcal{G}}(u; G_j) - S_i^{\mathcal{G}}(v; G_j) = 0$ whenever $X_u < X_v < \theta_2$. Now, because $x_i = \theta_2$ implies that $\arg\sup\{\tau : L_i^{\mathcal{G}}(\tau)\} = \tau_2^*$ for $i \in \{1, 2\}$, we obtain $S_i^{\mathcal{G}}(t; G_j) = L_i^{\mathcal{G}}(t) < L_i^{\mathcal{G}}(\tau_2^*) < q_2^* F_i^{\mathcal{G}}(\tau_2^*) + (1 - q_2^*) L_i^{\mathcal{G}}(\tau_2^*)$ for any $X_t > \theta_2$ where we use the facts $F_i^{\mathcal{G}}(\cdot) > L_i^{\mathcal{G}}(\cdot)$ and $\lim_{t \uparrow \tau_2^*} G_j(t) = 0$. Hence, using the facts that $S_1^{\mathcal{G}}(\tau_2^{*+}; G_2) = q_2^* F_1^{\mathcal{G}}(\tau_2^*) + (1 - q_2^*) L_1^{\mathcal{G}}(\tau_2^*)$ and $S_2^{\mathcal{G}}(\tau_2^{*+}; G_1) = L_2^{\mathcal{G}}(\tau_2^*)$, we obtain $S_i^{\mathcal{G}}(t; G_j) = \sup_\tau S_i^{\mathcal{G}}(\tau; G_j)$, $i \in \{1, 2\}$ if and only if $X_t < \theta_2$, which proves that the given strategy profile is an equilibrium.

Next, we show that $\mathcal{G} = (G_1, G_2)$ with $\text{supp}(G_i) = (-\infty, \theta_2)$ is an equilibrium with $q_2 (\geq q_2^*)$ close to 1. Note once again that if $k_1 - k_2 < \kappa(k_2)$, then we have $q_2^* < 1$ so that high enough q_2 implies $x_1 > \theta_2$. Hence, it is sufficient to prove that $S_1^{\mathcal{G}}(t_1; G_2) < S_1^{\mathcal{G}}(\tau_2^{*+}; G_2)$ where $t_1 := \inf\{t \geq 0 : X_t \leq x_1\}$. Towards a contradiction, set $X_0 = \zeta$ and suppose that $S_1^{\mathcal{G}}(t_1; G_2) \geq S_1^{\mathcal{G}}(\tau_2^{*+}; G_2)$. Then because $V_1(\zeta; \mathcal{G}) = S_1^{\mathcal{G}}(\tau_2^{*+}; G_2)$, we can obtain $V_1(\zeta; H(\infty), H(\tau_2^*)) - V_1(\zeta; \mathcal{G}) = (1 - q_2) k_1 e^{-r\tau_2^*}/(1 - e^{-r\tau_2^*})$. Thus, for any given $\epsilon > 0$, we can find $\delta > 0$ such that $1 - q_2 < \delta$ implies $V_1(\zeta; H(\infty), H(\tau_2^*)) - V_1(\zeta; \mathcal{G}) < \epsilon$. Also, because $x_1 > \theta_2$, we should have $S_1^{\mathcal{G}}(t_1; G_2) = L_1^{\mathcal{G}}(t_1)$. Thus, our assumption $S_1^{\mathcal{G}}(t_1; G_2) \geq S_1^{\mathcal{G}}(\tau_2^{*+}; G_2)$ implies that firm 1 would be no worse by employing $H(t_1)$ in every stage than playing G_1. Combining all, we obtain

$$V_1(\zeta; H(t_1), H(\infty)) = V_1(\zeta; H(t_1), G_2) \geq V_1(\zeta; \mathcal{G}) = V_1(\zeta; H(\infty), H(\tau_2^*)) - \epsilon \,.$$

Letting $\epsilon \to 0$ yields a contradiction to (B.15). This completes the proof. ■

Proof of Proposition 3.4(b): Set $\kappa_p(k_2)$ as the solution to (B.16) in the proof of Proposition 3.3 and assume that $k_1 - k_2 < \kappa_p(k_2)$. First, define $t_2 := \inf\{t \geq 0 : X_t \leq x_2\}$ where x_2 satisfies $\pi(x_2) = r[V_2(\zeta; \mathcal{G}) - k_2]$, and observe that

$$\begin{aligned} V_2(\zeta; H(\infty), H(t_2)) &= (R_r\pi)(\zeta) + \beta_2(x_2)\phi(\zeta) \\ &< (R_r\pi)(\zeta) + \beta_2(\theta_2)\phi(\zeta) \\ &< (R_r\pi)(\zeta) + \beta_{F2}(\theta_1)\phi(\zeta) = V_2(\zeta; H(\tau_1^*), H(\infty)) \,. \end{aligned} \tag{B.17}$$

Here the first inequality follows from Assumption 3.1, and the second inequality holds because $k_1 - k_2 < \kappa_p(k_2)$. Now we show that $S_2^{\mathcal{G}}(t_2; G_1) < S_2^{\mathcal{G}}(\tau_1^{*+}; G_1)$. Towards a contradiction, set $X_0 = \zeta$ and suppose that $S_2^{\mathcal{G}}(t_2; G_1) \geq S_2^{\mathcal{G}}(\tau_1^{*+}; G_1)$. Then because $V_2(\zeta; \mathcal{G}) = S_2^{\mathcal{G}}(\tau_1^{*+}; G_2)$, we can obtain

$$V_2(\zeta; H(\tau_1^*), H(\infty)) - V_2(\zeta; \mathcal{G}) = (1 - q_1) k_2 e^{-r\tau_1^*}/(1 - e^{-r\tau_1^*}) \,.$$

Thus, for any given $\epsilon > 0$, we can find $\delta > 0$ such that $1 - q_1 < \delta$ implies $V_2(\zeta; H(\tau_1^*), H(\infty)) - V_2(\zeta; \mathcal{G}) < \epsilon$. Also, because $x_2 (\geq \theta_2)$ increases in q_1 and

$\theta_2 > \theta_1$, we have $x_2 > \theta_1$, i.e., $t_2 < \tau_1^*$, and thus it follows that $S_2^{\mathcal{G}}(t_2; G_1) = L_2^{\mathcal{G}}(t_2)$. Then because our assumption $S_2^{\mathcal{G}}(t_2; G_1) \geq S_2^{\mathcal{G}}(\tau_1^{*+}; G_1)$ implies that firm 2 would be no worse by employing $H(t_2)$ in every stage than playing G_2, we must obtain

$$V_2(\zeta; H(\infty), H(t_2)) = V_2(\zeta; G_1, H(t_2)) \geq V_2(\zeta; \mathcal{G}) = V_2(\zeta; H(\tau_1^*), H(\infty)) - \epsilon\,,$$

where letting $\epsilon \to 0$ results in a contradiction to (B.17). This completes the proof. ■

Proof of Proposition 3.5: Assume that $\mathcal{G}$ is a properly mixed strategy equilibrium. We define $\tilde{\theta}_i$ as the unique solution to the equation

$$\pi(\tilde{\theta}_i) = r l_i^{\mathcal{G}}\,,$$

and define $\theta = \min\{\tilde{\theta}_1, \tilde{\theta}_2\}$. We also let τ_θ denote the hitting time of $(-\infty, \theta)$. Without loss of generality, we assume that $\theta = \tilde{\theta}_1 \leq \tilde{\theta}_2$. (The case of $\theta = \tilde{\theta}_2 \leq \tilde{\theta}_1$ can be proved in exactly the same way.) We prove the proposition by proving the following three claims: (i) Firm i never invests when $X_t > \tilde{\theta}_i$ in equilibrium. (ii) For all $t > \tau_\theta$ such that $X_t < \theta$, each firm i invests with an arrival rate of $\lambda_i(X_t)$. (iii) When $X_t \in (\tilde{\theta}_1, \tilde{\theta}_2]$, firm 2 never invests. In particular, $q_2 = 0$.

(i) At some time t such that $X_t > \tilde{\theta}_i$, suppose that the set of firm i's best responses (one of the constituents of the mixed strategy) includes immediate investment so that its equilibrium payoff is $V_i(X_t; \mathcal{G}) = l_i^{\mathcal{G}}$. Below we prove that firm i can improve its payoff by investing at time $t + dt$, i.e., an infinitesimal time dt later.

First, suppose that firm j's strategy is to invest in the time interval $(t, t+dt)$ with probability $p_j \in (0, 1]$ such as when firm j's CDF is discontinuous within the time interval $(t, t + dt)$. Then firm i's payoff from investing at $t + dt$ is

$$p_j f_i^{\mathcal{G}} + (1 - p_j) l_i^{\mathcal{G}} + O(dt) > l_i^{\mathcal{G}}\,.$$

Thus, it contradicts the assumption that immediate investment is a constituent of the mixed strategy for firm i.

Second, suppose that firm j's strategy is to invest in the time interval $(t, t + dt)$ with probability $\lambda_j dt + O(dt^2)$ for some positive constant λ_j such as when firm j's CDF is continuously increasing within the time interval $(t, t+dt)$. Then firm i's payoff from investing at $t + dt$ is

$$\pi(X_t)dt + f_i^{\mathcal{G}}\lambda_j dt + (1 - \lambda_j dt)(1 - r dt)V_i(X_t; \mathcal{G}) + O(dt^2)$$
$$= l_i^{\mathcal{G}} + (\pi(X_t) - r l_i^{\mathcal{G}})dt + (f_i^{\mathcal{G}} - l_i^{\mathcal{G}})\lambda_j dt + O(dt^2) .$$

Because $\pi(X_t) > r l_i^{\mathcal{G}}$ and $f_i^{\mathcal{G}} > l_i^{\mathcal{G}}$, the expression above is larger than $l_i^{\mathcal{G}}$. This contradicts the assumption that immediate investment is a constituent of the mixed strategy for firm i. We conclude that firm i's best response is to never invest whenever $X_t > \tilde{\theta}_i$ (or equivalently, when $t < \tau_\theta$.)

(ii) Recall that we assume that the game begins with the initial condition $X_0 = \zeta$. Suppose $t_0 > \tau_\theta$ so that $X_0 < \theta$. At time t_0, we define for $t > t_0$

$$L_i(t) = e^{r t_0} L_i^{\mathcal{G}}(t) \ , \ F_i(t) = e^{r t_0} F_i^{\mathcal{G}}(t) \ ,$$

where $L_i^{\mathcal{G}}(\cdot)$ and $F_i^{\mathcal{G}}(\cdot)$ are defined in (B.11). Note that $L_i(t)$ and $F_i(t)$ are the payoffs to the leader and the follower at time t_0 when investment is anticipated to be made at time t. Note also that $F_i(t) > L_i(t)$. Furthermore, $L_i(t)$ is decreasing in t for all $t > t_0$ because $\pi(X_t) < l_i^{\mathcal{G}}$. Hence, we can apply Theorem 3 of Hendricks et al. (1988) and conclude that the only possible mixed strategy equilibria are the ones in which each firm i invests with an arrival rate of $\lambda_i(X_t)$ given by (3.9).

(iii) From claim (i), we already know that firm 1 does not invest at all when $X_t \in (\tilde{\theta}_1, \tilde{\theta}_2]$. Firm 2's payoff $V_2(x; \mathcal{G})$ obviously always satisfies $V_2(x; \mathcal{G}) \geq l_2^{\mathcal{G}}$; furthermore,

$$\mathcal{A}V_2(x; \mathcal{G}) = \mu(x)\partial_x V_2(x; \mathcal{G}) - rV_2(x; \mathcal{G}) + \pi(x) = 0$$

whenever $V_2(x; \mathcal{G}) > l_2^{\mathcal{G}}$ and $x \in (\tilde{\theta}_1, \tilde{\theta}_2]$ because it is suboptimal to invest when $V_2(X_t; \mathcal{G}) > l_2^{\mathcal{G}}$. We note that $rV_2(x; \mathcal{G}) - \pi(x) \geq r l_2^{\mathcal{G}} - \pi(x) > 0$ for all $x < \tilde{\theta}_2$, so $\partial_x V_2(x; \mathcal{G}) < 0$ whenever $V_2(X_t; \mathcal{G}) > l_2^{\mathcal{G}}$.

Suppose that one of firm 2's best responses (one of the constituents of the mixed strategy) under the equilibrium $\mathcal{G}$ is to wait until the hitting time of some point $y < \tilde{\theta}_2$ and invest when $X_t = y$. By the argument above, we have $V_2(x; \mathcal{G}) \leq l_2^{\mathcal{G}}$ for all $x < y$; since $V_2(x; \mathcal{G}) < l_2^{\mathcal{G}}$ contradicts the assumption that $\mathcal{G}$ is an equilibrium, we conclude that $V_2(x; \mathcal{G}) = l_2^{\mathcal{G}}$ for all $x < y$. It also means that $V_2(x;\mathcal{G})$ can be achieved by investing at the stopping time $\tau_y = \inf\{t \geq 0 : X_t \leq y\}$. However, because $\pi(x) < rl_2^{\mathcal{G}}$ for all $x < \tilde{\theta}_2$, we have $L_i^{\mathcal{G}}(\tau_{\tilde{\theta}_2}) > L_i^{\mathcal{G}}(\tau_y)$ where $\tau_{\tilde{\theta}_2} = \inf\{t \geq 0 : X_t \leq \tilde{\theta}_2\}$ is the hitting time of $(-\infty, \tilde{\theta}_2]$; this can be verified because $dL_i^{\mathcal{G}}(t) = [\pi(X_t) - rl_i^{\mathcal{G}}]e^{-rt}dt$. Therefore, we conclude that firm 2 never invests when $X_t \in (\tilde{\theta}_1, \tilde{\theta}_2]$.

From (i), (ii), and (iii), we conclude that the only possible forms of mixed strategy SPE are the ones given in Proposition 3.4. ■

Proof of Theorem 3.1: We prove this theorem in two steps: (i) Prove that $\mathcal{G}^*$ is an equilibrium, and (ii) prove that $\mathcal{G}^*$ is the only equilibrium in $\mathcal{E}$.

(i) For notational simplicity, we write $\mathcal{G}^* = \mathcal{G} = (G_1, G_2)$ and drop the asterisk for part (i) of the proof. We first prove that G_1 is a best response to G_2 if the condition (3.16) is satisfied. Note that, if condition (3.16) holds, then we have $q_2^* = (1 - k_2/k_1)\phi(\theta_2)/\phi(\zeta) < 1$. We let $\mathbb{E}^{\{\mathrm{I},\mathrm{II}\},x}[\cdot]$ denote the expectation conditional on phase (I or II) and $X_0 = x \in \mathscr{I}$. Then because $\mathrm{supp}(G_1)$ is chosen as $(-\infty, \theta_2)$, it suffices to establish the following relations by the virtue of Lemma B.7:

$$\mathbb{E}^{\{\mathrm{I},\mathrm{II}\},x}[S_1^{\mathcal{G}}(u;G_2)] = \mathbb{E}^{\{\mathrm{I},\mathrm{II}\},x}[S_1^{\mathcal{G}}(v;G_2)] \text{ for any } X_u^x, X_v^x < \theta_2\,, \tag{B.18}$$

$$\mathbb{E}^{\mathrm{I},x}[S_1^{\mathcal{G}}(t;G_2)] < \mathbb{E}^{\mathrm{I},x}[S_1^{\mathcal{G}}(u;G_2)] \text{ for any } X_t^x \geq \theta_2 > X_u^x\,, \tag{B.19}$$

$$\mathbb{E}^{\mathrm{II},x}[S_1^{\mathcal{G}}(t;G_2)] < \mathbb{E}^{\mathrm{II},x}[S_1^{\mathcal{G}}(u;G_2)] \text{ for any } X_t^x > \theta_2 \geq X_u^x\,, \tag{B.20}$$

where $\tau_2 := \inf\{t \geq 0 : X_t < \theta_2\}$.

To prove (B.18), we first use $\lim_{t\uparrow\tau_2} G_2^{\mathrm{I},x}(t) = 0$ and $G_2^{\mathrm{I},x}(\tau_2) = q_2^*$ to obtain

$$\begin{aligned} V_1^{\mathrm{I}}(\zeta;\mathcal{G}) &= \mathbb{E}^{\mathrm{I},\zeta}[S_1^{\mathcal{G}}(\tau_2^+;G_2)] = \mathbb{E}^{\mathrm{I},\zeta}[q_2^* F_2^{\mathcal{G}}(\tau_2^+) + (1-q_2^*)L_2^{\mathcal{G}}(\tau_2^+)] \\ &= (R_r\pi)(\zeta) + \frac{[V_1^{\mathrm{I}}(\zeta;\mathcal{G}) - (1-q_2^*)k_1] - (R_r\pi)(\theta_2)}{\phi(\theta_2)}\phi(\zeta)\,, \end{aligned}$$

which we can solve for $V_1^{\mathrm{I}}(\zeta;\mathcal{G}) - k_1$, after which we have

$$\begin{aligned} V_1^{\mathrm{I}}(\zeta;\mathcal{G}) - k_1 &= \frac{\phi(\theta_2)[(R_r\pi)(\zeta) - k_1] - \phi(\zeta)[(R_r\pi)(\theta_2) - k_1 q_2^*]}{\phi(\theta_2) - \phi(\zeta)} \\ &= \frac{\phi(\theta_2)[(R_r\pi)(\zeta) - k_2] - \phi(\zeta)(R_r\pi)(\theta_2)}{\phi(\theta_2) - \phi(\zeta)} = V_2^{\mathrm{I}}(\zeta;\mathcal{G}) - k_2\,, \end{aligned}$$

where we use the expression of q_2^* in (3.17). Now, observe that if $x < \theta_2$, then $\mathbb{E}^{\mathrm{I},x}[\cdot] = \mathbb{E}^{\mathrm{II},x}[\cdot]$; if $x \geq \theta_2$, then

$$\mathbb{E}^{\mathrm{I},x}[S_1^{\mathcal{G}}(u;G_2)] = \mathbb{E}^{\mathrm{I},x}\Bigg[\int_0^{\tau_2} \pi(X_t)e^{-rt}dt \\ + e^{-r\tau_2}[q_2^* V_1^{\mathrm{I}}(\zeta;\mathcal{G}) + (1-q_2^*)\mathbb{E}^{\mathrm{II},\theta}[S_1^{\mathcal{G}}(u-\tau_2;G_2)]\Bigg],$$

for any $X_u^x < \theta_2$. Hence, it is sufficient to establish (B.18) for phase II only. Then because $G_2(\cdot)$ has no discontinuity in phase II and $V_1^{\mathrm{I}}(\zeta;\mathcal{G}) - k_1 = V_2^{\mathrm{I}}(\zeta;\mathcal{G}) - k_2$, it follows that $\sup_\tau \mathbb{E}^{\mathrm{II},x}[L_1^{\mathcal{G}}(\tau)] = \mathbb{E}^{\mathrm{II},x}[L_1^{\mathcal{G}}(u)]$ for any $X_u^x < \theta_2$. From this result, we establish $\mathbb{E}^{\mathrm{II},x}[S_1^{\mathcal{G}}(u;G_2)] = \mathbb{E}^{\mathrm{II},x}[S_1^{\mathcal{G}}(v;G_2)]$ for any $X_u^x, X_v^x < \theta_2$ by using the arguments in Theorem 5.1 of Steg (2015).

To prove (B.19) and (B.20), observe that, for any $X_t^x > \theta_2$, we have

$$\mathbb{E}^{\{\mathrm{I},\mathrm{II}\},x}[S_1^{\mathcal{G}}(t;G_2)] = \mathbb{E}^{\{\mathrm{I},\mathrm{II}\},x}[L_1^{\mathcal{G}}(t)] \leq \sup_\tau \mathbb{E}^{\{\mathrm{I},\mathrm{II}\},x}[L_1^{\mathcal{G}}(\tau)] = \mathbb{E}^{\{\mathrm{I},\mathrm{II}\},x}[L_1^{\mathcal{G}}(\tau_2)]\,.$$

Here the first equality follows from $\lim_{t\uparrow\tau_2} G_2^{\mathrm{I},x}(t) = 0$, and the last equality follows from $V_1^{\mathrm{I}}(\zeta;\mathcal{G}) - k_1 = V_2^{\mathrm{I}}(\zeta;\mathcal{G}) - k_2$. To prove (B.19), therefore, it is sufficient to use (1) $\mathbb{E}^{\mathrm{I},x}[S_1^{\mathcal{G}}(\tau_2^+;G_2)] = \mathbb{E}^{\mathrm{I},x}[q_2^* F_1^{\mathcal{G}}(\tau_2) + (1-q_2^*)L_1^{\mathcal{G}}(\tau_2)]$ and (B.18), (2) $\mathbb{E}^{\mathrm{I},x}[S_1^{\mathcal{G}}(\tau_2;G_2)] = \mathbb{E}^{\mathrm{I},x}[q_2^* M_1^{\mathcal{G}}(\tau_2) + (1-q_2^*)L_1^{\mathcal{G}}(\tau_2)]$, and (3) $F_1^{\mathcal{G}}(\cdot) > M_1^{\mathcal{G}}(\cdot) > L_1^{\mathcal{G}}(\cdot)$. To prove (B.20), use $\mathbb{E}^{\mathrm{II},x}[S_1^{\mathcal{G}}(\tau_2;G_2)] = \mathbb{E}^{\mathrm{II},x}[L_1^{\mathcal{G}}(\tau_2)]$ and (B.18).

Conversely, it can be proven in a similar fashion that G_2 is also a best response to G_1; the only difference in this case is that G_1 has no discontinuity in either phase I or II, which only makes the arguments simpler. This completes the proof of the if-and-only-if statement of the theorem.

(ii) In this part, we establish that the common support must be given by $\Gamma = (-\infty, \theta_2)$ in phase 2 of SPE $\mathcal{G} \in \mathcal{E}$. As a first step, we prove that the support for firm i must be of the form $(-\infty, \theta_i^*)$ where θ_i^* is the threshold of investment timing $\tau_i = \inf\{t \geq 0 : X_t < \theta_i^*\}$ that solves the stopping problem

$$\mathbb{E}^{\mathrm{II},x}\left[\int_0^{\tau_i} \pi(X_t)e^{-rt}dt + [V_i^I(\zeta;\mathcal{G}) - k_i]e^{-r\tau_i}\right] \\ = \sup_{\tau\geq 0} \mathbb{E}^{\mathrm{II},x}\left[\int_0^{\tau} \pi(X_t)e^{-rt}dt + [V_i^I(\zeta;\mathcal{G}) - k_i]e^{-r\tau}\right] = V_i^*(x)\,.$$

(By Alvarez (2001), the optimal stopping time τ_i is characterized by a single threshold θ_i^*.) This also implies that $\theta_1^* = \theta_2^*$ must hold if the two firms must share the same support.

As a preliminary step, we list necessary conditions satisfied by $\mathcal{G}$ and Γ. We let Γ^C denote the complement of Γ; because Γ is closed, Γ^C is an open set. Note that $V_i^{\text{II}}(x;\mathcal{G}) \geq l_i^{\mathcal{G}}$ for all x and that $V_i^{\text{II}}(x;\mathcal{G}) = l_i^{\mathcal{G}}$ for all $x \in \Gamma$ because the firms employ a mixed strategy whenever $X_t \in \Gamma$. Furthermore, because the CDFs are continuous in time in phase 2, the following Hamilton-Jacobi-Bellman (HJB) equations are satisfied:

$$\begin{aligned} \mathcal{A}V_i^{\text{II}}(x;\mathcal{G}) + \pi(x) &= 0 \quad \text{for } x \in \Gamma^C \\ \mathcal{A}V_i^{\text{II}}(x;\mathcal{G}) + \pi(x) + \lambda_j(x)[f_i^{\mathcal{G}} - V_i^{\text{II}}(x;\mathcal{G})] &= 0 \quad \text{for } x \in \Gamma\,, \end{aligned} \tag{B.21}$$

where $\lambda_j(x)$ is the rate of arrival of firm j's investment when $x \in \Gamma$. Equation (B.21) is derived from the fact that for $X_t \in \Gamma$ and within the time interval $(t, t+dt)$ there is probability of $\lambda_j(X_t)dt$ that firm j would invest, leading to firm i's payoff of $f_i^{\mathcal{G}}$, and probability of $1 - \lambda_j(X_t)dt$ that firm i's payoff is $V_i^{\text{II}}(X_{t+dt};\mathcal{G})$ at the end of the time interval. Note also that $V_i^{\text{II}}(x;\mathcal{G}) = l_i^{\mathcal{G}}$ whenever $x \in \Gamma$ by the property of a mixed strategy equilibrium. Hence, (B.21) can be re-written as $-rl_i^{\mathcal{G}} + \pi(x) + \lambda_j(x)(f_i^{\mathcal{G}} - l_i^{\mathcal{G}})$, which leads to $\lambda_j(x) = k_i^{-1}(rl_i^{\mathcal{G}} - \pi(x))$. (From the constraint that $\lambda_j(x) \geq 0$, we can also derive the fact that $rl_i^{\mathcal{G}} - \pi(x) \geq 0$ must be satisfied for all $x \in \Gamma$.)Lastly, we note that $V_i^{\text{II}}(x;\mathcal{G})$ must be continuously differentiable for all $x \in \Gamma \cup \Gamma^C$ by the property of diffusive processes and non-singularity of $\lambda_j(\cdot)$. In particular, $V_i^{\text{II}}(x;\mathcal{G})$ is continuously differentiable at the boundary of Γ.

In sum, $V_i^{\text{II}}(x;\mathcal{G})$ satisfies the HJB equation $\mathcal{A}V_i^{\text{II}}(x;\mathcal{G}) + \pi(x) = 0$ for an open set Γ^C, $\mathcal{A}V_i^{\text{II}}(x;\mathcal{G}) + \pi(x) \leq 0$ for all $x \in \Gamma$ because $\lambda_j(x)[f_i^{\mathcal{G}} - V_i^{\text{II}}(x;\mathcal{G})] \geq 0$, the inequality $V_i^{\text{II}}(x;\mathcal{G}) \geq l_i^{\mathcal{G}}$ for $x \in \Gamma^C$, and the equality $V_i^{\text{II}}(x;\mathcal{G}) = l_i^{\mathcal{G}}$ for all $x \in \Gamma$. In addition, $V_i^{II}(x;\mathcal{G})$ is continuously differentiable at the boundary of Γ. By the verification theorem for optimal stopping problems (Oksendal, 2003), these are exactly the sufficient conditions under which $\tau_\Gamma = \inf\{t \geq 0 : X_t \in \Gamma\}$ (the hitting time of Γ) is the optimal stopping time that maximizes the following:

$$\sup_{\tau \geq 0} \mathbb{E}^{\text{II},x}\left[\int_0^\tau \pi(X_t)e^{-rt}dt + [V_i^I(\zeta;\mathcal{G}) - k_i]e^{-r\tau}\right] = V_i^*(x)\,.$$

Since $V_i^*(x)$ is a unique function of x, we conclude that $V_i^{\mathrm{II}}(x;\mathcal{G}) = V_i^*(x)$ and furthermore that $\Gamma = (-\infty, \theta_i^*]$ by the functional form of $V_i^*(x)$. We can apply the same logic to firm j, so that $\theta_i^* = \theta_j^*$ must be satisfied.

In phase 1, if firm i's strategy is to invest with probability q_i at time τ_Γ, then firm i's payoff function is given by $V_i^{\mathrm{I}}(x;\Gamma) = V_i^*(x)$ so that $\theta_i^* = \theta_i$, which implies that $\theta_1^* = \theta_2^* = \theta_2$ if $q_2 > 0$ but $\theta_1^* = \theta_2^* = \theta_1$ if $q_1 > 0$. We note that q_2^* is uniquely given by ((3.17)) from the condition that $\theta_1^* = \theta_2^*$. (See part (i) of this proof). Hence, no other values of q_2^* is possible in a mixed strategy SPE in $\mathcal{E}$. Furthermore, it is straight forward to verify that $q_i q_j = 0$, i.e., only one firm has non-zero probability of investment at the end of phase 1; if $q_1 > 0$ and $q_2 > 0$, then each firm can improve its payoff by not investing at at time τ_Γ. In order to complete the proof, therefore, it is sufficient to prove that q_1 cannot be non-zero.

Set $X_0 = x > \theta_2$ and $\tau_1 := \inf\{t \geq 0 : X_t \leq \theta_1\}$, and suppose that $\mathcal{G}^q := (G_1, G_2)$ is an equilibrium where firm 1 places probability $q_1 \in [0,1)$ of investment at the end of phase 1. Then by Lemma B.7, we must have $\mathbb{E}^{\mathrm{II},x}[S_2^{\mathcal{G}^q}(\tau_1; G_1)] = \sup_\tau \mathbb{E}^{\mathrm{II},x}[S_2^{\mathcal{G}^q}(\tau; G_1)]$ in Phase II of the game. By using $\lim_{t\uparrow\tau_1} G_1^{\mathrm{I},x}(t) = 0$ and $G_1^{\mathrm{I},x}(\tau_1) = q$, we can have

$$
\begin{aligned}
V_2^{\mathrm{I}}(\zeta;\mathcal{G}^q) &= \mathbb{E}^{\mathrm{I},x}[S_2^{\mathcal{G}^q}(\tau_1^+; G_1)] = \mathbb{E}^{\mathrm{I},x}[qF_2^{\mathcal{G}^q}(\tau_1) + (1-q)L_2^{\mathcal{G}^q}(\tau_1)] \\
&= (R_r\pi)(\zeta) + \Big[\{V_2^{\mathrm{I}}(\zeta;\mathcal{G}^q) - (1-q)k_2\} - (R_r\pi)(\theta_1)\Big]\phi(\zeta)/\phi(\theta_1)\,,
\end{aligned}
$$

from which it can be seen that $L_2^{\mathcal{G}^q} = V_2^{\mathrm{I}}(\zeta;\mathcal{G}^q) - k_2$ increases in q. Because $\sup_\tau \mathbb{E}^{\mathrm{II},x}[L_2^{\mathcal{G}^q}(\tau)] = \mathbb{E}^{\mathrm{II},x}[L_2^{\mathcal{G}^q}(\tau_2)]$ when $q = 0$, we must have $\sup_\tau \mathbb{E}^{\mathrm{II},x}[L_2^{\mathcal{G}^q}(\tau)] = \mathbb{E}^{\mathrm{II},x}[L_2^{\mathcal{G}^q}(\tau_2^q)]$ with some $\tau_2^q < \tau_2$ for any $q > 0$. However, $G_1^{\mathrm{II},x}(\tau_1) = 0$ and $\tau_2 < \tau_1$ imply that

$$
\mathbb{E}^{\mathrm{II},x}[S_2^{\mathcal{G}^q}(\tau_1; G_1)] = \mathbb{E}^{\mathrm{II},x}[L_2^{\mathcal{G}^q}(\tau_1^*)] < \mathbb{E}^{\mathrm{II},x}[L_2^{\mathcal{G}^q}(\tau_2^q)] = \mathbb{E}^{\mathrm{II},x}[S_2^{\mathcal{G}^q}(\tau_2^q; G_1)]\,,
$$

which is a contradiction. Therefore, $\mathcal{G}^q$ cannot be an equilibrium for any $q \in [0,1)$. ■

Proof of Proposition 3.6: We prove this theorem in two steps: (i) Prove that $\mathcal{G}_N^0$ is an equilibrium, and (ii) prove that $\mathcal{G}_N^0$ is the only equilibrium in $\mathcal{E}$.

(i) For notational simplicity, we write $\mathcal{G}_N^0 = \mathcal{G} = (G_1, G_2, G_3)$. By the symmetry between the firms, it is then enough to prove that G_1 is a best response to G_{-1} where G_{-i} is defined in (3.19). Because supp(G_1) is chosen as $(-\infty, \theta)$, where $\theta = \theta_1 = \theta_2 = \theta_3$ is the common threshold given in Assumption 3.1, it suffices to establish the following relations by the virtue of Lemma B.7:

$$\mathbb{E}^x[S_1^{\mathcal{G}_N^0}(u; G_{-1})] = \mathbb{E}^x[S_1^{\mathcal{G}_N^0}(v; G_{-1})] \text{ for any } X_u^x, X_v^x < \theta\,,$$
$$\mathbb{E}^x[S_1^{\mathcal{G}_N^0}(t; G_{-1})] < \mathbb{E}^x[S_1^{\mathcal{G}_N^0}(u; G_{-1})] \text{ for any } X_t^x \geq \theta > X_u^x\,.$$

Then the rest of the proof proceeds as in the proof of Proposition 4.2 where we are using the arguments in Theorem 5.1 of Steg (2015).

(ii) This part of the proof can be done by using the same arguments in the part (ii) of the proof of Theorem 3.1. ■

Proof of Proposition 3.7: The proof of this proposition is similar to Theorem 3.1. To see why, we first note that both firm 1 and firm 2 have the same investment cost $\bar{k} = k_1 = k_2$, which is higher than that (k_3) of firm 3. Therefore, the equilibrium strategies and payoffs of firm 1 and firm 2 are of the same structure as that of firm 1 in Theorem 3.1 while firm 3's equilibrium strategy and payoff is of the same form as that of firm 2 in Theorem 3.1. Then by proceeding similarly in the proof of Theorem 3.1, we can first show that the choice of $q_3^* = (1 - k_3/\bar{k})\phi(\theta_3)/\phi(\zeta) < 1$ will make $V_i^{\mathrm{I}}(\zeta; \mathcal{G}_N^*) - k_i = V_3^{\mathrm{I}}(\zeta; \mathcal{G}_N^*) - k_3$ for $i = 1, 2$, which establishes

$$\mathbb{E}^{\mathrm{I},x}[S_i^{\mathcal{G}_N^*}(u; G_{-i})] = \mathbb{E}^{\mathrm{I},x}[S_i^{\mathcal{G}_N^*}(v; G_{-i})] \text{ for any } X_u^x, X_v^x < \theta\,,$$

where $i = 1, 2$. In the second phase, because $G_{-i}(\cdot)$ has no discontinuity and $V_i^{\mathrm{I}}(\zeta; \mathcal{G}_N^*) - k_i = V_3^{\mathrm{I}}(\zeta; \mathcal{G}_N^*) - k_3$, it must follow that $\sup_\tau \mathbb{E}^{\mathrm{II},x}[L_i^{\mathcal{G}}(\tau)] = \mathbb{E}^{\mathrm{II},x}[L_i^{\mathcal{G}}(u)]$ for any $X_u^x < \theta_3$. Combining this result with our choices of $\lambda_i(\cdot)$ to be $\sum_{j\neq i} \lambda_j(x) = v_i(x)$ whenever $x < \theta_3$, we can establish $\mathbb{E}^{\mathrm{II},x}[S_i^{\mathcal{G}_N^*}(u; G_{-i})] = \mathbb{E}^{\mathrm{II},x}[S_i^{\mathcal{G}_N^*}(v; G_{-i})]$ for any $X_u^x, X_v^x < \theta_3$ by using the arguments in Theorem 5.1 of Steg (2015). Note that the condition $\bar{k} - k_3 < k_3$ is satisfied if and only if $\lambda_3(x) > 0$ whenever $x < \theta_3$.

The rest part of the proof can be done similarly as in the proof of Theorem 3.1. ■

Proof of Proposition 3.8: Note that under the suggested equilibrium strategy profile $\mathcal{G}_N^* = (G_1, G_2, G_3)$, all the CDFs G_i's are continuous in time in both phases. Let us put $\theta = \theta_2 = \theta_3$ where these θ_i's are defined in Assumption 3.1. Then because $\sum_{j\neq i} \lambda_j(x) = v_i(x)$ for $i = 2, 3$ whenever $x < \theta$, we can establish from the arguments of Theorem 5.1 of Steg (2015) that

$$\mathbb{E}^x[S_i^{\mathcal{G}_N^*}(u; G_{-i})] = \mathbb{E}^x[S_i^{\mathcal{G}_N^*}(v; G_{-i})] \text{ for any } X_u^x, X_v^x < \theta\,, \tag{B.22}$$

$$\mathbb{E}^x[S_i^{\mathcal{G}_N^*}(t; G_{-i})] < \mathbb{E}^x[S_i^{\mathcal{G}_N^*}(u; G_{-i})] \text{ for any } X_t^x \geq \theta > X_u^x\,, \tag{B.23}$$

where $i = 2, 3$.

Also, we can understand the condition that $\tau_1 = \inf\{t \geq 0 : X_t \leq \eta_1\}$ with $\theta_1 < \eta_1 < \theta$ is a solution to the problem $\sup_{\tau\geq 0} \mathbb{E}^x[\int_0^\tau \pi(X_t)e^{-rt}dt + l_1^{\mathcal{G}_N^*}e^{-r\tau}]$ as follows: Suppose for a moment that the support of G_2 and G_3 is $(-\infty, \theta_1)$, i.e., both firm 2 and firm 3 invest if and only if $X_t \in (-\infty, \theta_1)$. Then the solution to the optimal stopping problem $\sup_{\tau\geq 0} \mathbb{E}^x[\int_0^\tau \pi(X_t)e^{-rt}dt + l_1^{\mathcal{G}_N^*}e^{-r\tau}]$ is simply given as $\tau_1^* = \inf\{t \geq 0 : X_t \leq \theta_1\}$ and firm 1 invests with some probability if and only if $X_t \in (-\infty, \theta_1)$. Suppose now that firm 2 and firm 3 employs strategies where they invest with the rate of $v_3(x)$ and $v_2(x)$ respectively above θ_1, specifically, whenever $X_t \in (\bar{\eta}, \theta)$ with some $\bar{\eta} > \theta_1$. Then this will increase $l_1^{\mathcal{G}_N^*}$ because firm 1 expects a higher probability to become a free-rider in all the subsequent stages, which implies that the solution to the given optimal stopping problem $\sup_{\tau\geq 0} \mathbb{E}^x[\int_0^\tau \pi(X_t)e^{-rt}dt + l_1^{\mathcal{G}_N^*}e^{-r\tau}]$ can be given as $\underline{\tau}_1 = \inf\{t \geq 0 : X_t \leq \underline{\eta}\}$ with $\theta_1 < \underline{\eta} < \bar{\eta} < \theta$. Therefore, as firm 2 and firm 3 expands its boundary of the region where they invest at the rate of $\nu_3(x)$ and $\nu_2(x)$ respectively, i.e., as $\bar{\eta}$ goes down, there can exist a fixed (or matching) point η_1 with $\theta_1 < \eta_1 < \theta$ such that $\tau_1 = \inf\{t \geq 0 : X_t \leq \eta_1\}$ becomes the solution to the optimal stopping problem $\sup_{\tau\geq 0} \mathbb{E}^x[\int_0^\tau \pi(X_t)e^{-rt}dt + l_1^{\mathcal{G}_N^*}e^{-r\tau}]$.

Hence, from firm 1's point of view, $G_{-1}(\cdot)$ is continuous in time in both phases and it is assumed that $\tau_1 = \inf\{t \geq 0 : X_t \leq \eta_1\}$ is a solution to the optimal stopping problem $\sup_{\tau\geq 0} \mathbb{E}^x[\int_0^\tau \pi(X_t)e^{-rt}dt + l_1^{\mathcal{G}_N^*}e^{-r\tau}]$ where $\theta_1 < \eta_1 < \theta$. Note that the existence of Then because $\sum_{j\neq 1} \lambda_j(x) = v_1(x)$ whenever $x < \eta_1$, we can establish from the arguments of Theorem 5.1 of Steg (2015) that (B.22) and (B.23) for $i = 1$. This completes the proof. ■

Proof of Proposition 3.9: As explained in the body of the paper, the suggested strategy profile $\mathcal{G}_N^* = (G_1, G_2, G_3)$ is a combination of the equilibrium strategy profiles obtained in case 2 ($k_1 = k_2 > k_3$) and case 3 ($k_1 > k_2 = k_3$). Therefore, the proof of this proposition can be obtained by applying the arguments used in the proofs of Proposition 3.7 and Proposition 3.8. More specifically, the easiest step is to establish that G_2 is a best response to G_{-2}. First, by using the arguments used in the proof of Proposition 3.7, it is easy to see that the choice of $q_3^* = (1 - k_3/k_2)\phi(\theta_3)/\phi(\zeta) < 1$ will make $V_2^{\mathrm{I}}(\zeta; \mathcal{G}_N^*) - k_2 = V_3^{\mathrm{I}}(\zeta; \mathcal{G}_N^*) - k_3$ for $i = 1, 2$, which establishes

$$\mathbb{E}^{\mathrm{I},x}[S_2^{\mathcal{G}_N^*}(u; G_{-2})] = \mathbb{E}^{\mathrm{I},x}[S_2^{\mathcal{G}_N^*}(v; G_{-2})] \text{ for any } X_u^x, X_v^x < \theta_3 .$$

In the second phase, because $G_{-2}(\cdot)$ is continuous and $V_2^{\mathrm{I}}(\zeta; \mathcal{G}_N^*) - k_2 = V_3^{\mathrm{I}}(\zeta; \mathcal{G}_N^*) - k_3$, we have $\sup_\tau \mathbb{E}^{\mathrm{II},x}[L_2^{\mathcal{G}_N^*}(\tau)] = \mathbb{E}^{\mathrm{II},x}[L_2^{\mathcal{G}_N^*}(u)]$ for any $X_u^x < \theta_3$. Combining this result with $\sum_{j \neq 2} \lambda_j(x) = v_2(x)$ whenever $x < \theta_3$, we can establish $\mathbb{E}^{\mathrm{II},x}[S_2^{\mathcal{G}_N^*}(u; G_{-2})] = \mathbb{E}^{\mathrm{II},x}[S_2^{\mathcal{G}_N^*}(v; G_{-2})]$ for any $X_u^x, X_v^x < \theta_3$ by using the arguments in Theorem 5.1 of Steg (2015).

In addition, it is straightforward to show that G_3 is a best response to G_{-3} because $G_{-3}(\cdot)$ is always continuous in time and $\mathrm{supp}(G_3) = \mathrm{supp}(G_{-3})$, which means that we can here use the arguments in Theorem 5.1 of Steg (2015) once again.

Lastly, the existence of η_1 with $\tau_1 = \inf\{t \geq 0 : X_t \leq \eta_1\}$ becoming the solution to the optimal stopping problem $\sup_{\tau \geq 0} \mathbb{E}^x[\int_0^\tau \pi(X_t)e^{-rt}dt + l_1^{\mathcal{G}_N^*} e^{-r\tau}]$ can be understood in the same way as that in the proof of Proposition 3.8. Then the rest of the proof for showing that G_1 is a best response to G_{-1} can be proceeded in a similar way as that in the proof of Proposition 3.8. ■

Proof of Proposition 3.10: Fix $M \geq 2$ and choose any m with $2 \leq m \leq M$. Let $k_1 = k_1^{(m)}$ where $k_1^{(m)}$ is given in (3.30). It is then enough to show that the strategy profile $(\mathcal{G}^{(n)})_{n \geq 1}$ described in the statement of the proposition constitutes an equilibrium.

First, we can easily see that if $k_1 = k_1^{(m)}$ and $\mathcal{G}^{(n)} = (H(\infty), H(\tau_2^{(n)}))$, $n \geq m$, then we have $V_1^{(m)}(\zeta; \mathcal{G}^{(m)+}) - k_1 = V_2^{(m)}(\zeta; \mathcal{G}^{(m)+}) - k_2$ by going through a serious of algebra with the definition (3.27) and $\mathcal{G}^{(n)}$'s, $n \geq m$. It thus follows that we can construct a two-phase mixed strategy SPE $\mathcal{G}^{(m-1)} = (G_1^{(m-1)}, G_2^{(m-1)})$ with a common support $\Gamma^{(m-1)} = (-\infty, \theta_2^{(m-1)})$ and any choice of $q_2^{(m-1)} \in [0, 1)$.

In addition, by putting this mixed strategy equilibrium $\mathcal{G}^{(m-1)}$ with the described choice of $q_2 = q_2^{(m-1)} = (k_1 - k_2)/k_1 \cdot \phi(\theta_2^{(m-1)})/\phi(\zeta)$ into (3.27), we can also verify that $V_1^{(m-1)}(\zeta; \mathcal{G}^{(m-1)+}) - k_1 = V_2^{(m-1)}(\zeta; \mathcal{G}^{(m-1)+}) - k_2$, which implies again that we can obtain a two-phase mixed strategy SPE $\mathcal{G}^{(m-2)} = (G_1^{(m-2)}, G_2^{(m-2)})$ with a common support $\Gamma^{(m-2)} = (-\infty, \theta_2^{(m-2)})$ and any choice of $q_2^{(m-2)} \in [0, 1)$.

Therefore, for any n with $2 \leq n < m$, we can construct a two-phase mixed strategy SPE $\mathcal{G}^{(n)} = (G_1^{(n)}, G_2^{(n)})$ with the support $\Gamma^{(n)} = (-\infty, \theta_2^{(m-1)})$ and $q_2 = q_2^{(n)}$, from which we can eventually obtain $V_1^{(2)}(\zeta; \mathcal{G}^{(2)+}) - k_1 = V_2^{(2)}(\zeta; \mathcal{G}^{(2)+}) - k_2$. Because this is a necessary and sufficient condition for the existence of a mixed strategy equilibrium in the first stage, it follows that the specified strategy profile is indeed an equilibrium involving mixed strategies as desired.

Also, it is easy to see that we have $q_2^{(n)} < 1$ where $n < m$ if and only if

$$k_1 - k_2 < \frac{\phi(\zeta)}{\phi(\theta_2^{(n)}) - \phi(\zeta)} k_2 \,.$$

Note that if $k_1 = k_1^{(m)}$, then we obtain

$$\begin{aligned} k_1 - k_2 &= \sum_{k=m}^{M} \left(\prod_{l=m}^{k} \frac{\phi(\zeta)}{\phi(\theta_2^{(l)})} \right) k_2 < \sum_{k=m}^{M} \left(\frac{\phi(\zeta)}{\phi(\theta_2^{(m)})} \right)^{k-m+1} k_2 \\ &< \sum_{k=1}^{\infty} \left(\frac{\phi(\zeta)}{\phi(\theta_2^{(m)})} \right)^{k} k_2 = \frac{\phi(\zeta)}{\phi(\theta_2^{(m)}) - \phi(\zeta)} k_2 < \frac{\phi(\zeta)}{\phi(\theta_2^{(n)}) - \phi(\zeta)} k_2 \,, \end{aligned}$$

where we use the fact that $\theta_2^{(k)}$ is an increasing sequence and $\phi(\cdot)$ is a strictly decreasing function. This simply implies that $q_2^{(n)}$ is always well defined by any choices of $k_1 = k_1^{(m)}$ with $m > n$, as desired.

Lastly, it can be seen from (3.30) that $(k_1^{(2)} - k_2)/k_2$ converges to a geometric series with a constant ratio $\phi(\zeta)/\phi(\theta_2)$ as $M \to \infty$ because $\theta_2^{(2)} \to \theta_2$ as $M \to \infty$. This thus completes the proof. ∎

Appendix C

Appendix of Chapter 4

C.1 Mixed-strategy MPE when $\sigma(\cdot) = 0$

In this section, we consider the case in which $\sigma(\cdot) = 0$. This case was previously analyzed by Hendricks et al. (1988) using a similar model. We present it here for completeness, and to highlight that the non-existence of mixed-strategy MPE requires two ingredients: first, that the firms' flow payoffs while they remain in the market are stochastic, and second, that the firms have heterogeneous outside options. To facilitate the analysis and following Hendricks et al. (1988), we will assume in this section that $\mu(\cdot) \leq 0$; *i.e.*, the market conditions deteriorate over time.

We shall construct a mixed-strategy MPE in which both firms remain in the market whenever $X_t^x > \theta_1^*$, and they randomize on the set $\Gamma = (\alpha, \theta_1^*)$. Lemma 2 holds, and because $\mu(\cdot) \leq 0$, the set Γ is absorbing; *i.e.*, if $X_t^x \in \Gamma$, then $X_s^x \in \Gamma$ for all $s > t$ with probability 1. Therefore, for every $x \in \Gamma$, each firm i's strategy $G_i^x(t)$ must satisfy (4.10). If $x \notin \Gamma$, then each firm i's strategy may have a discontinuity of size $1 - p_i$ at $\tau_\Gamma = \inf\{t \geq 0 : X_t^x \in \Gamma\}$, and solving (4.8) subject to the boundary condition $G_i^x(\tau_\Gamma) = 1 - p_i$ where $p_i \in [0, 1]$ yields that each firm i's strategy must satisfy

$$G_i^x(t) = \mathbf{1}_{\{X_t^x \in \Gamma\}}(t) \left\{ 1 - p_i \exp\left[-\int_0^t \frac{\mathbf{1}_{\{X_s^x \in \Gamma\}}(s)[rl_j - \pi(X_s^x)]}{w(X_s^x) - l_j} ds \right] \right\}. \quad \text{(C.1)}$$

Observe that the strategy profile $(G_1^x, G_2^x)_{x \in \mathscr{I}}|_{p_1, p_2}$ is Markov, and the following proposition shows that for an appropriate choice of p_1 and p_2, it constitutes a mixed-strategy MPE.

Proposition C.1 *Suppose that $\sigma(\cdot) = 0$. Then there exists $\kappa(l_2) > 0$ such that $(G_1^x, G_2^x)_{x \in \mathscr{I}}|_{p_1, p_2 = 1}$ is a mixed-strategy MPE with $0 < p_1 < \bar{p}(l_1, l_2)$ as* *long as* $l_2 - l_1 < \kappa(l_2)$.

Recall from Proposition 4.1 that whenever $X_t^x \in (\theta_1^*, \theta_2^*]$, firm 1 strictly prefers to remain in the market, whereas firm 2 strictly prefers to exit immediately. In order for firm 2 to wait until X_t^x enters the randomization set Γ, in equilibrium, firm 1 must exit at the moment such that $X_t^x = \theta_1^*$ with sufficiently high probability. When $X_t^x \in \Gamma^o$, it follows from Lemma (4.2) that both firms must exit at the rate given by ((4.9)).

C.2 Mathematical Preliminaries

We first define the following functions that will be used in Appendices C.2 and C.3.

$$R(x) := \mathbb{E}^x\left[\int_0^\infty \pi(X_t)e^{-rt}dt\right], \tag{C.2}$$

$$\beta_i(x) := \frac{l_i - R(x)}{\phi(x)}, \tag{C.3}$$

where $\phi : \mathscr{I} \to \mathbb{R}$ satisfies the differential equation[1] $\frac{1}{2}\sigma^2(x)\phi^{''}(x) + \mu(x)\phi^{'}(x) - r\phi(x) = 0$ with the properties of $\phi(\cdot) > 0$ and $\phi^{'}(\cdot) < 0$. The function $R(\cdot)$ is well-defined because we assume that $\pi(\cdot)$ satisfies the absolute integrability condition in Section 3.3.

Lemma C.1 *The function* $\beta_i(x)$ *has a unique interior maximum at* $\theta_i^* \leq x_{ci}$ *where* $\pi(x_{ci}) = rl_i$. *Furthermore,* $\beta_i^{'}(x) > 0$ *for* $x < \theta_i^*$ *and* $\beta_i^{'}(x) < 0$ *for* $x > \theta_i^*$.

Proof of Lemma C.1: To prove this lemma, it is enough to examine the behavior of the first derivative of $\beta_i(x) = [l_i - R(x)]/\phi(x)$.

According to the theory of diffusive processes (Alvarez, 2001, p.319), the

[1] This second-order linear ordinary differential equation (ODE) always has two linearly independent fundamental solutions, one of which is monotonically decreasing (see Alvarez, 2001, p.319). Note that if $f(\cdot)$ solves this equation, then so does $cf(\cdot)$ for any constant $c \in \mathbb{R}$ because it is a homogeneous equation. Hence, we can always find the one which is always positive.

function $R(\cdot)$, given in (C.2), can be expressed as

$$R(x) = \frac{\phi(x)}{B}\int_a^x \psi(y)\pi(y)m^{'}(y)dy + \frac{\psi(x)}{B}\int_x^b \phi(y)\pi(y)m^{'}(y)dy\,. \qquad \text{(C.4)}$$

Here, a and b are the two boundaries of the state space $\mathscr{I}$, $\psi(\cdot)$ and $\phi(\cdot)$ are the increasing and decreasing fundamental solutions to the differential equation $\frac{1}{2}\sigma^2(x)f^{''}(x)+\mu(x)f^{'}(x)-rf(x)=0$, $B=[\psi^{'}(x)\phi(x)-\psi(x)\phi^{'}(x)]/S^{'}(x)$ is the *constant* Wronskian determinant of $\psi(\cdot)$ and $\phi(\cdot)$, $S^{'}(x)=\exp(-\int 2\mu(x)/\sigma^2(x)dx)$ is the density of the scale function of X, and $m^{'}(y)=2/[\sigma^2(y)S^{'}(y)]$ is the density of the speed measure of X.

By using (C.4), differentiation of $R(x)$ with respect to x leads to

$$R^{'}(x)\phi(x) - R(x)\phi^{'}(x) = S^{'}(x)\int_x^b \phi(y)\pi(y)m^{'}(y)dy\,. \qquad \text{(C.5)}$$

Moreover, because $l_i = \mathbb{E}^x[\int_0^\infty rl_ie^{-rt}dt]$, we can write

$$R(x) - l_i = \mathbb{E}^x\left[\int_0^\infty [\pi(X_t)-rl_i]e^{-rt}dt\right], \qquad \text{(C.6)}$$

which implies that we can treat the functional $R(x)-l_i$ as the expected cumulative present value of a flow payoff $\pi(\cdot)-rl_i$. Combining (C.5) and (C.6), therefore, we obtain

$$\beta_i^{'}(x) = -\frac{R^{'}(x)\phi(x)-[R(x)-l_i]\phi^{'}(x)}{\phi^2(x)} = -\frac{S^{'}(x)}{\phi^2(x)}\int_x^b \phi(y)[\pi(y)-rl_i]m^{'}(y)dy\,. \qquad \text{(C.7)}$$

Now, because $\pi(\cdot)$ is strictly increasing and $\pi(x_{ci}) = rl_i$, it must be the case that $\pi(x) < rl_i$ for $x < x_{ci}$ and $\pi(x) > rl_i$ for $x > x_{ci}$. Thus, $\beta_i^{'}(x) < 0$ for all $x > x_{ci}$. Note also that if $x < K < x_{ci}$, then

$$\begin{aligned}
&\int_x^b \phi(y)[\pi(y)-rl_i]m^{'}(y)dy \\
&= \int_x^K \phi(y)[\pi(y)-rl_i]m^{'}(y)dy + \int_K^b \phi(y)[\pi(y)-rl_i]m^{'}(y)dy \\
&\le \frac{[\pi(K)-rl_i]}{r}\left(\frac{\phi^{'}(K)}{S^{'}(K)} - \frac{\phi^{'}(x)}{S^{'}(x)}\right) + \int_K^b \phi(y)[\pi(y)-rl_i]m^{'}(y)dy \to -\infty\,,
\end{aligned}$$

as $x \downarrow a$ because a is a natural boundary, which implies that $\lim_{x \downarrow a} \beta_i^{'}(x) = \infty$. Here we use $\phi^{'}(x) < 0$ and $\pi(x) < \pi(K) < rl_i$ for $x < K$. It thus follows that $\beta_i^{'}(\theta_i^*) = 0$ for some $\theta_i^* \leq x_{ci}$, which implies that $\int_{\theta_i^*}^b \phi(y)[\pi(y) - rl_i]m^{'}(y)dy = 0$ because $S^{'}(x) > 0$ and $\phi(x) > 0$ in (C.7). Moreover, note that $\int_x^b \phi(y)[\pi(y) - rl_i]m^{'}(y)dy$ is increasing in $x < x_{ci}$ because $\pi(y) < rl_i$ for $\forall y < x_{ci}$, thus yielding $\int_x^b \phi(y)[\pi(y) - rl_i]m^{'}(y)dy < 0$ if $x < \theta_i^* \leq x_{ci}$ and $\int_x^b \phi(y)[\pi(y) - rl_i]m^{'}(y)dy > 0$ if $\theta_i^* < x \leq x_{ci}$. Combining this with (C.7), we obtain the unique existence of θ_i^* such that $\beta_i^{'}(x) > 0$ for $\forall x < \theta_i^*$ and $\beta_i^{'}(x) < 0$ for $\forall x > \theta_i^*$, which completes the proof. ■

Lemma C.2 *A mixed-strategy G_i is a best response to a mixed-strategy G_j if and only if, for each $x \in \mathscr{I}$,*

$$\mathbb{E}^x[S_i(\hat{\tau}; G_j)] = \sup_{\tau} \mathbb{E}^x[S_i(\tau; G_j)] \,, \tag{C.8}$$

whenever $X_{\hat{\tau}}^x \in \overline{supp(G_i)}$ almost surely.

Lemma C.2 implies that each pure-strategy, which is involved in a mixed-strategy best response, must itself be a best response.

Proof of Lemma C.2: This lemma follows from Lemma 3.1. in Steg (2015). Define the right-continuous inverse of G_i^x as

$$\tau_i^{G,x}(z) := \inf\{s \geq 0 : G_i^x(s) > z\} \,, \forall z \in [0,1] \,, \tag{C.9}$$

which satisfies $\tau_i^{G,x}(z) \leq t$ if and only if $G_i^x(t) \geq z$. Then we can obtain the change-of-variable formula between G_i^x and $\tau_i^{G,x}(z)$ as the following:

$$\mathbb{E}^x\left[\int_0^\infty S_i(t; G_j) dG_i^x(t)\right] = \mathbb{E}^x\left[\int_0^1 S_i(\tau_i^{G,x}(z); G_j) dz\right] .$$

By using this change-of-variable, we have

$$\begin{aligned}
V_i(x;G_i,G_j) &= \mathbb{E}^x\left[\int_0^\infty S_i(t;G_j)dG_i^x(t)\right] \\
&= \mathbb{E}^x\left[\int_0^1 S_i(\tau_i^{G,x}(z);G_j)dz\right] \\
&= \int_0^1 \mathbb{E}^x[S_i(\tau_i^{G,x}(z);G_j)]dz \\
&\le \int_0^1 \sup_\tau \mathbb{E}^x[S_i(\tau;G_j)]dz = \sup_\tau \mathbb{E}^x[S_i(\tau;G_j)]\,, \qquad \text{(C.10)}
\end{aligned}$$

where the first equality follows from (3.4) and the first inequality follows because $\tau_i^{G,x}(z)$ is a stopping time with respect to the state X for each z. Note that the relation holds for any pair of mixed strategies (G_i, G_j).

Now, suppose that $\mathbb{E}^x[S_i^x(\hat{\tau};G_j)] = \sup_\tau \mathbb{E}^x[S_i^x(\tau;G_j)]$ whenever $X_{\hat{\tau}}^x \in \overline{\text{supp}(G_i)}$ almost surely. Observe that $\tau_i^{G,x}(z)$ is in the support of G_i^x for any z; this is because $t > \tau_i^{G,x}(z)$ implies that $G_i^x(t) > z = G_i^x(\tau_i^{G,x}(z))$ if $G_i^x(\cdot)$ has no jump at $\tau_i^{G,x}(z)$, or $\Delta G_i^x(\tau_i^{G,x}(z)) > 0$ if $G_i^x(\cdot)$ has a jump at $\tau_i^{G,x}(z)$. Hence, $\mathbb{E}^x[S_i^x(\tau_i^{G,x}(z);G_j)] = \sup_\tau \mathbb{E}^x[S_i^x(\tau;G_j)]$ for any $z \in [0,1]$ by our assumption, which implies that, for any mixed-strategy $\tilde{G}_i$,

$$\begin{aligned}
V_i(x;G_i,G_j) &= \mathbb{E}^x\left[\int_0^\infty S_i(t;G_j)dG_i^x(t)\right] \\
&= \mathbb{E}^x\left[\int_0^1 S_i(\tau_i^{G,x}(z);G_j)dz\right] \\
&= \int_0^1 \mathbb{E}^x[S_i(\tau_i^{G,x}(z);G_j)]dz \\
&= \int_0^1 \sup_\tau \mathbb{E}^x[S_i(\tau;G_j)]dz = \sup_\tau \mathbb{E}^x[S_i(\tau;G_j)] \ge V_i(x;\tilde{G}_i,G_j)\,,
\end{aligned}$$

where the last inequality follows from (C.10). Thus, G_i is a best response to G_j. Conversely, if G_i is a best response to G_j and $X_{\hat{\tau}}^x \in \overline{\text{supp}(G_i)}$, then we must have $\mathbb{E}^x[S_i(\hat{\tau};G_j)] = \sup_\tau \mathbb{E}^x[S_i(\tau;G_j)]$; otherwise firm i could earn higher payoff by taking $X_{\hat{\tau}}^x$ out of the support of G_i. This completes the proof.

■

Lemma C.3 *Define the process*

$$J_i(t) := \sup_{\tau \geq t} \mathbb{E}^x[L_i(\tau)|\mathcal{F}_t] = \int_0^t \pi(X_s^x)e^{-rs}ds + V_i^*(X_t^x)e^{-rt}\,, \qquad \text{(C.11)}$$

where $V_i^*(\cdot)$ *is defined in (4.6). Then we have the following results:*

(a) $J_i(t) \geq L_i(t)$ *for all* $t \geq 0$*, and the equality holds if* $t \geq \tau_i^*$ *where* τ_i^* *is given in (4.7). Moreover,* $J_i(\cdot)$ *can be expressed as* $J_i(t) = N_i(t) - D_i^{\theta_i^*}(t)$ *where* $N_i(\cdot)$ *is a uniformly integrable martingale, and* $D_i^y(\cdot)$ *is a non-decreasing and predictable process given by*

$$dD_i^y(t) := \mathbf{1}_{\{X_t^x \leq y\}}[rl_i - \pi(X_t^x)]e^{-rt}dt \text{ with } D_2^y(0) = 0\,. \qquad \text{(C.12)}$$

(b) For any stopping times τ_a, τ_b *with* $\tau_a \leq \tau_b$ *and a mixed-strategy* G_j*,*

$$\mathbb{E}^x\left[\int_{\tau_a}^{\tau_b} N_i(t)dG_j^x(t)|\mathcal{F}_{\tau_a}\right] = -\mathbb{E}^x\left[N_i(\tau_b)[1-G_j^x(\tau_b)]|\mathcal{F}_{\tau_a}\right] + N_i(\tau_a)[1-G_j^x(\tau_a)]\,, \qquad \text{(C.13)}$$

where $\mathcal{F}_t$ *is the natural filtration generated by the state* X *(Oksendal, 2003).*

Proof of Lemma C.3(a): Comparing (4.3) and (C.11), it can be clearly seen that $J_i(t) \geq L_i(t)$ for all $t \geq 0$. Also, $J_i(t) = L_i(t)$ if $t \geq \tau_i^*$ by the definition of $V_i^*(\cdot)$. According to the theory of optimal stopping, it is well-known that the process $J_i(\cdot)$ is the Snell envelope of the process $L_i(\cdot)$ and it is of Class D (Steg, 2015). Hence, we can apply Doob-Meyer decomposition theorem to $J_i(\cdot)$, which implies that $J_i(\cdot)$ can be decomposed into a uniformly integrable martingale and a unique, non decreasing, predictable process. Because $J_i(t)$ (more precisely, $V_i^*(X_t^x)$) is a twice differentiable function of X, the exact form of $N_i(\cdot)$ and $D_i^{\theta_i^*}(\cdot)$ can be obtained by applying Ito formula to $J_i(\cdot)$. ■

Proof of Lemma C.3(b): For a mixed-strategy G_j, consider the right-continuous inverse $\tau_j^{G,x}(z)$ given in (C.9). Observe that

$$
\begin{aligned}
\mathbb{E}^x\left[\int_{\tau_a}^{\tau_b} N_i(t)dG_j^x(t)|\mathcal{F}_{\tau_a}\right] &= \mathbb{E}^x\left[\int_0^1 N_i(\tau_j^{G,x}(z))\mathbf{1}_{[\tau_a,\tau_b]}(\tau_j^{G,x}(z))dz|\mathcal{F}_{\tau_a}\right] \\
&= \int_0^1 \mathbb{E}^x\left[N_i(\tau_j^{G,x}(z))\mathbf{1}_{[\tau_a,\tau_b]}(\tau_j^{G,x}(z))|\mathcal{F}_{\tau_a}\right]dz \\
&= \int_0^1 \mathbb{E}^x\left[\mathbb{E}^x\left[N_i(\tau_b)|\mathcal{F}_{\tau_j^{G,x}(z)}\right]\mathbf{1}_{[\tau_a,\tau_b]}(\tau_j^{G,x}(z))|\mathcal{F}_{\tau_a}\right]dz \\
&= \int_0^1 \mathbb{E}^x\left[\mathbb{E}^x\left[N_i(\tau_b)\mathbf{1}_{[\tau_a,\tau_b]}(\tau_j^{G,x}(z))|\mathcal{F}_{\tau_j^{G,x}(z)}\right]|\mathcal{F}_{\tau_a}\right]dz \\
&= \int_0^1 \mathbb{E}^x\left[N_i(\tau_b)\mathbf{1}_{[\tau_a,\tau_b]}(\tau_j^{G,x}(z))|\mathcal{F}_{\tau_a}\right]dz \\
&= \mathbb{E}^x\left[N_i(\tau_b)\int_0^1 \mathbf{1}_{[\tau_a,\tau_b]}(\tau_j^{G,x}(z))dz|\mathcal{F}_{\tau_a}\right] \\
&= \mathbb{E}^x\left[N_i(\tau_b)[G_j(\tau_b)-G_j(\tau_a)]|\mathcal{F}_{\tau_a}\right] \\
&= \mathbb{E}^x\left[-N_i(\tau_b)[1-G_j(\tau_b)]+N_i(\tau_b)[1-G_j(\tau_a)]|\mathcal{F}_{\tau_a}\right] \\
&= -\mathbb{E}^x\left[N_i(\tau_b)[1-G_j(\tau_b)]|\mathcal{F}_{\tau_a}\right]+N_i(\tau_a)[1-G_j(\tau_a)]\,,
\end{aligned}
$$

where the first equality holds from the change-of-variable from $G_j(t)$ to $\tau_j^{G,x}(z)$, the third equality follows from the fact $\mathbb{E}^x\left[N_i(\tau_b)|\mathcal{F}_{\tau_j^{G,x}(z)}\right] = N_i(\tau_j^{G,x}(z))$ because $N_i(\cdot)$ is a martingale, the fourth equality holds because $\mathbf{1}_{[\tau_a,\tau_b]}(\tau_j^{G,x}(z))$ is measurable with respect to the filtration $\mathcal{F}_{\tau_j^{G,x}(z)}$, the fifth equality follows from the smoothing law of the conditional expectation because $\mathcal{F}_{\tau_a} \subseteq \mathcal{F}_{\tau_j^{G,x}(z)}$, the seventh equality follows from $\int_0^1 \mathbf{1}_{[\tau_a,\tau_b]}(\tau_j^{G,x}(z))dz = G_j(\tau_b)-G_j(\tau_a)$ by the definition of $\tau_j^{G,x}(z)$, and the last equality follows because $\mathbb{E}^x\left[N_i(\tau_b)|\mathcal{F}_{\tau_a}\right] = N_i(\tau_a)$ ($N_i(\cdot)$ is a martingale) and $G_j(\tau_a)$ is measurable with respect to $\mathcal{F}_{\tau_a}$. Thus, the desired relation is established. ■

C.3 Mathematical Proofs

Proof of Lemma 4.1: The proof of this lemma is available in Alvarez (2001), but here, we provide an alternative argument for an intuitive understanding of the results of this lemma. To that end, we will use the optimality conditions, which are known as "value matching" and "smooth pasting" conditions (Samuelson, 1965; McKean, 1965; Merton, 1973).

First, the state space $\mathscr{I}$ must be the union of $C := \{x \in \mathscr{I} : V_i^*(x) > l_i\}$ and $\Gamma := \{x \in \mathscr{I} : V_i^*(x) = l_i\}$, which are mutually exclusive: This is because (1) X is a stationary process and the time horizon is infinite, and (2) the value function $V_i^*(\cdot)$ from an optimal stopping policy must be always no less than the reward l_i from stopping immediately. Hence, the problem to find an optimal stopping policy can be reduced to identify C or Γ.

Next, we find the differential equation that $V_i^*(x)$ must satisfy *if* $x \in C$. Note that the optimal value function $V_i^*(x)$ is the maximum of the reward from waiting an instant and the reward from stopping immediately. For any $x \in C$, therefore, the optimal stopping policy is to wait an instant dt, and hence, the optimal value function must satisfy the following equation:

$$V_i^*(x) = \pi(x)dt + (1 - rdt)\mathbb{E}^x[V_i^*(x) + dV_i^*(X_t)] . \tag{C.14}$$

Then applying Ito formula to $V_i^*(X_t)$ and using $\mathbb{E}^x[dB_t] = 0$ yields

$$\mathbb{E}^x[dV_i^*(X_t)] = [\mu(x)V_i^{*'}(x) + \frac{1}{2}\sigma^2(x)V_i^{*''}(x)]dt . \tag{C.15}$$

By plugging (C.15) into (C.14) and ignoring the term smaller than dt, we have

$$V_i^*(x) = \pi(x)dt + V_i^*(x) + [-rV_i^*(x) + \mu(x)V_i^{*'}(x) + \frac{1}{2}\sigma^2(x)V_i^{*''}(x)]dt ,$$

from which we obtain the following second-order linear differential equation:

$$\frac{1}{2}\sigma^2(x)V_i^{*''}(x) + \mu(x)V_i^{*'}(x) - rV_i^*(x) = -\pi(x) . \tag{C.16}$$

Thus, $V_i^*(\cdot)$ can be obtained by solving the differential equation (C.16). In fact, it can be seen from a series of algebra with the relation (C.4) that the function $R(\cdot) + A\phi(\cdot)$ with some constant $A \in \mathbb{R}$ is a solution to (C.16), and hence, we can guess $V_i^*(x) = R(x) + A\phi(x)$ with some constant A.

Intuitively, firm i must find it optimal to exit and receive his outside option l_i as soon as the state X hits some lower threshold θ_i. Hence, assume at the moment that the optimal stopping policy is given as $\tau^* := \inf\{t \geq 0 : X_t^x \leq$

$\theta_i\}$, which implies that θ_i is the boundary point of the region C. Now, we state the value matching condition and the smooth pasting condition, which results in two boundary conditions to the boundary value problem (C.16) with the free boundary θ_i:

$$V_i^*(\theta_i) = R(\theta_i) + A\phi(\theta_i) = l_i \tag{C.17}$$

$$V_i^{*'}(\theta_i) = R'(\theta_i) + A\phi'(\theta_i) = 0\,. \tag{C.18}$$

The value matching condition (C.17) and the smooth pasting condition (C.18) are the conditions that $V_i^*(\cdot)$ must satisfy at the boundary θ_i of C. We can first obtain $A = [l_i - R(\theta_i)]/\phi(\theta_i) = \beta_i(\theta_i)$ from (C.17). Then the condition (C.18) is equivalent to

$$\begin{aligned} 0 &= R^{'}(\theta_i) + \frac{l_i - R(\theta_i)}{\phi(\theta_i)}\phi^{'}(\theta_i) \\ &= \frac{R^{'}(\theta_i)\phi(\theta_i) + [l_i - R(\theta_i)]\phi^{'}(\theta_i)}{\phi(\theta_i)} = -\phi(\theta_i)\beta_i^{'}(\theta_i)\,. \end{aligned}$$

Because $\phi(\cdot) > 0$, it can be seen from Lemma C.1 that this condition is satisfied if and only if $\theta_i = \theta_i^*$, which implies that $A = \beta_i(\theta_i^*)$.

Lastly, it can be easily checked that $R(x) + \beta_i(\theta_i^*)\phi(x) \geq l_i$ for $\forall x \geq \theta_i^*$ and $\pi(x) < rl_i$ for $\forall x \leq \theta_i^* < x_{ci}$. By using the verification theorem (Oksendal, 2003, Theorem 10.4.1), therefore, the proposed value function $R(\cdot) + \beta_i(\theta_i^*)\phi(\cdot)$ is, in fact, the optimal value function $V_i^*(\cdot)$, as desired. ■

Proof of Proposition 4.1(a): Because it is shown in Lemma 4.1 that $G_2 = H(\tau_2^*)$, where $\tau_2^* = \inf\{t \geq 0 : X_t^x \leq \theta_2^*\}$ is given in (4.7), is firm 2's best response to $G_1 = H(\infty)$, it only remains to prove that $G_1 = H(\infty)$ is also firm 1's best response to $G_2 = H(\tau_2^*)$.

Let $H(\tau_1)$ be firm 1's best response to $H(\tau_2^*)$. Also, define $V_{W1}^*(x) := \sup_\tau V_1(x; H(\tau), H(\tau_2^*)) = V_1(x; H(\tau_1), H(\tau_2^*))$ be the corresponding payoff to firm 1. We denote the continuation region associated with τ_1 by C_1, i.e., $\tau_1 = \inf\{t \geq 0 : X_t^x \notin C_1\}$, and its complement by $\Gamma_1 = \mathscr{I} \backslash C_1$.

First, we show that $\Gamma_1 \cap (\theta_2^*, \infty) = \emptyset$. Toward a contradiction, suppose this is not the case. Then pick some $x \in \Gamma_1 \cap (\theta_2^*, \infty)$ and observe that $V_{W1}^*(x) = l_1$

due to $x \in \Gamma_1$. However,

$$\begin{aligned}
V_{W1}^*(x) \geq V_1(x; H(\infty), H(\tau_2^*)) &= \mathbb{E}^x\left[\int_0^{\tau_2^*} \pi(X_t)e^{-rt}dt + w(X_{\tau_2^*}^x)e^{-r\tau_2^*}\right] \\
&= R(x) + \left[\frac{w(\theta_2^*) - R(\theta_2^*)}{\phi(\theta_2^*)}\right]\phi(x) \\
&> R(x) + \left[\frac{l_1 - R(\theta_2^*)}{\phi(\theta_2^*)}\right]\phi(x) \\
&= R(x) + \beta_1(\theta_2^*)\phi(x) \\
&> R(x) + \beta_1(x)\phi(x) = l_1 \ ,
\end{aligned}$$

where the first inequality follows because $w(X_{\tau_2^*}^x) = w(\theta_2^*) > l_1$ and $\mathbb{E}^x[e^{-r\tau_2^*}] = \phi(x)/\phi(\theta_2^*)$ for $x > \theta_2^*$, and the second inequality holds because $x > \theta_2^* > \theta_1^*$ and $\beta_1^{'}(x) < 0$ for $x > \theta_1^*$ by Lemma C.1. This establishes the contradiction.

Second, we also prove that $\Gamma_1 \cap (-\infty, \theta_2^*] = \emptyset$. Towards a contradiction, suppose this is not the case. Then we can pick some $x \in \Gamma_1 \cap (-\infty, \theta_2^*]$ such that $V_{W1}^*(x) = m_1(x)$ because $\tau_2^* = \inf\{t \geq 0 : X_t^x \leq \theta_2^*\}$. However,

$$\begin{aligned}
&V_{W1}^*(x) \geq V_1(x; H(\infty), H(\tau_2^*)) \\
&= \mathbb{E}^x\left[\int_0^{\tau_2^*} \pi(X_t)e^{-rt}dt + w(X_{\tau_2^*}^x)e^{-r\tau_2^*}\right] = w(x) > m_1(x) \ ,
\end{aligned}$$

where the second equality uses that $\tau_2^* = 0$ when $X_0 = x \leq \theta_2^*$. This establishes the contradiction. Hence, we can conclude that $\Gamma_1 = \emptyset$ and $C_1 = \mathscr{I}$, which implies that $\tau_1 = \infty$. ■

Proof of Proposition 4.1(b): Consider the following condition:

$$V_2(x; H(\tau_1^*), H(\infty)) = \mathbb{E}^x\left[\int_0^{\tau_1^*} \pi(X_t)e^{-rt}dt + w(X_{\tau_1^*}^x)e^{-r\tau_1^*}\right] > l_2 \quad \forall x \in (\theta_1^*, \theta_2^*] \ . \tag{C.19}$$

First, we prove that (C.19) is sufficient for $(G_1, G_2) = (H(\tau_1^*), H(\infty))$ to be an MPE. Let $H(\tau_2)$ be firm 2's best response to $H(\tau_1^*)$, i.e., $V_{W2}^*(x) := \sup_\tau V_2(x; H(\tau_1^*), H(\tau)) = V_2(x; H(\tau_1^*), H(\tau_2))$ be the corresponding payoff. We denote the continuation region associated with τ_2 by C_2, i.e., $\tau_2 = \inf\{t \geq 0 : X_t^x \notin C_2\}$, and its complement by $\Gamma_2 = \mathscr{I} \backslash C_2$.

We now claim that $\Gamma_2 \cap (\theta_2^*, \infty) = \emptyset$. Towards a contradiction, suppose not. Then we can pick some $x \in \Gamma_2 \cap (\theta_2^*, \infty)$, which implies that $V_{W2}^*(x) = l_2$. However, because $\tau_1^* > \tau_2^*$ when $X_0 = x$, Lemma 4.1 implies that firm 2 could obtain a *strictly* higher payoff by exiting at $\tau_2^* > 0$ instead, i.e.,

$$V_{W2}^*(x) \geq V_2(x; H(\tau_1^*), H(\tau_2^*)) = \mathbb{E}^x \left[\int_0^{\tau_2^*} \pi(X_t) e^{-rt} dt + l_2 e^{-r\tau_2^*} \right] > l_2 \ ,$$

which is a contradiction. We next claim that $\Gamma_2 \cap (\theta_1^*, \theta_2^*] = \emptyset$. Towards a contradiction, suppose not. Then we can pick $x \in \Gamma_2 \cap (\theta_1^*, \theta_2^*]$, which implies that $V_{W2}^*(x) = l_2$. However, we have

$$V_{W2}^*(x) \geq V_2(x; H(\tau_1^*), H(\infty)) = \mathbb{E}^x \left[\int_0^{\tau_1^*} \pi(X_t) e^{-rt} dt + w(X_{\tau_1^*}^x) e^{-r\tau_1^*} \right] > l_2 \ ,$$

where the last inequality follows from (C.19). This establishes the contradiction. We further claim that $\Gamma_2 \cap (-\infty, \theta_1^*] = \emptyset$. If not, then there exists $x \in \Gamma_2 \cap (-\infty, \theta_1^*]$, which implies that both firms exit simultaneously when $X_t^y = x$, and hence, $V_{W2}^*(x) = m_2(x)$. Because $\tau_1^* = 0$ when $X_0 = x \leq \theta_1^*$, we have

$$\begin{aligned} V_{W2}^*(x) \geq V_2(x; H(\tau_1^*), H(\infty)) &= \mathbb{E}^x \left[\int_0^{\tau_1^*} \pi(X_t) e^{-rt} dt + w(X_{\tau_1^*}^x) e^{-r\tau_1^*} \right] \\ &= w(x) > m_2(x) \ , \end{aligned}$$

which is a contradiction. Combining the three claims above, therefore, we conclude that $\Gamma_2 = \emptyset$, which implies that $C_2 = \mathscr{I}$, and hence, $\tau_2 = \infty$.

Second, define $\underline{w} := \inf\{w(x) : x \in \mathscr{I}\}$ and $\beta_W(\theta) := [\underline{w} - R(\theta)]/\phi(\theta)$. Note that $\beta_W(\theta) > \beta_2(\theta)$ for $\forall \theta \in \mathscr{I}$ because $\underline{w} > l_2$. Also, observe that for $\forall \theta < \theta_2^*$, we have

$$\begin{aligned} \beta_W^{'}(\theta) &= \left\{ -R^{'}(\theta)\phi(\theta) - \phi^{'}(\theta)[\underline{w} - R(\theta)] \right\} / \phi^2(\theta) \\ &> \left\{ -R^{'}(\theta)\phi(\theta) - \phi^{'}(\theta)[l_2 - R(\theta)] \right\} / \phi^2(\theta) = \beta_2^{'}(\theta) > 0 \end{aligned}$$

where the first inequality follows because $\phi^{'}(\theta) < 0$, and the last inequality holds because $\beta_2^{'}(\theta) > 0$ for $\theta < \theta_2^*$ from Lemma C.1. Next, pick $\kappa_\theta > 0$ such that

$$\beta_W(\theta_2^* - \kappa_\theta) = \beta_2(\theta_2^*) , \tag{C.20}$$

where $\beta_2(\cdot)$ is defined in (C.3). If such κ_θ exists, it must be unique because $\beta_W'(\theta) > 0$ for $\theta < \theta_2^*$. If there does not exist κ_θ which satisfies C.20, then we let $\kappa_\theta = \infty$.

Finally, we show that (C.19) is satisfied if $\theta_2^* - \theta_1^* < \kappa_\theta$, which will complete the proof; this is because we can always find the unique $\kappa_l > 0$ for any given $\kappa_\theta > 0$ such that $\theta_2^* - \theta_1^* < \kappa_\theta$ if and only if $l_2 - l_1 < \kappa_l$ from the fact that θ_i^* given in (4.7) strictly increases in l_i. Suppose now that $\theta_2^* - \theta_1^* < \kappa_\theta$, i.e., $\theta_1^* > \theta_2^* - \kappa_\theta$. Note that $\beta_W'(\theta) > 0$ for $\forall \theta < \theta_2^*$, and recall that $\theta_1^* < \theta_2^*$. Therefore, $\beta_W(\theta_1^*) > \beta_W(\theta_2^* - \kappa_\theta) = \beta_2(\theta_2^*)$ by (C.20). Thus, for any $x \in (\theta_1^*, \theta_2^*]$,

$$\begin{aligned}
\mathbb{E}^x\left[\int_0^{\tau_1^*} \pi(X_t^x)e^{-rt}dt + w(\theta_1^*)e^{-r\tau_1^*}\right] &\geq \mathbb{E}^x\left[\int_0^{\tau_1^*} \pi(X_t^x)e^{-rt}dt + \underline{w}e^{-r\tau_1^*}\right] \\
&= R(x) + \phi(x)\beta_W(\theta_1^*) \\
&> R(x) + \phi(x)\beta_2(\theta_2^*) \\
&\geq R(x) + \phi(x)\beta_2(x) = l_2 ,
\end{aligned}$$

where the first inequality holds from the definition of $\underline{w}$, the first equality holds because $\mathbb{E}^x[e^{-r\tau_1^*}] = \phi(x)/\phi(\theta_1^*)$ for $x > \theta_1^*$, the second inequality follows because $\beta_W(\theta_1^*) > \beta_2(\theta_2^*)$, the last inequality holds because $\beta_2(\cdot)$ achieves its maximum at θ_2^* by Lemma C.1, and the last equality follows by the definition of $\beta_2(\cdot)$. Hence, (C.19) is satisfied, which establishes the desired result for $\kappa_\theta > 0$. ■

Proof of Lemma 4.2(a): Suppose that (G_1, G_2) is a mixed-strategy MPE. First, let us define $D_i := \{x \in \mathscr{I} : \pi(x) > rl_i\}$. We will show that D_i is a subset of the continuation region for firm i, i.e., $\overline{\text{supp}(G_i)} \cap D_i = \emptyset$. Towards a contradiction, suppose there exists some $x \in \overline{\text{supp}(G_i)} \cap D_i$. Because $\pi(\cdot)$ is continuous, for $\epsilon > 0$ sufficiently small, $\pi(y) > rl_i$ for all $y \in (x - \epsilon, x + \epsilon)$. Then using $\int_0^t -rl_i e^{-rs}ds = l_i(e^{-rt} - 1)$, we have

$$L_i(t) = l_i + \int_0^t [\pi(X_s^x) - rl_i]e^{-rs}ds > l_i , \ \forall t \in (0, \tau_\epsilon] , \tag{C.21}$$

where $\tau_\epsilon := \inf\{t \geq 0 : X_t^x \notin (x-\epsilon, x+\epsilon)\}$, and the inequality follows because $\pi_i(X_s^x) > rl_i$ for $\forall s < \tau_\epsilon$. Note that

$$\begin{aligned}\mathbb{E}^x[S_i(\tau_\epsilon; G_j)] &= \mathbb{E}^x\left[\int_0^{\tau_\epsilon} W_i(t)dG_j(t) + M_i(\tau_\epsilon)\Delta G_j(\tau_\epsilon) + L_i(\tau_\epsilon)[1 - G_j(\tau_\epsilon)]\right] \\ &> \mathbb{E}^x\left[\int_0^{\tau_\epsilon} L_i(t)dG_j(t) + L_i(\tau_\epsilon)\Delta G_j(\tau_\epsilon) + L_i(\tau_\epsilon)[1 - G_j(\tau_\epsilon)]\right] \\ &> \mathbb{E}^x\left[\int_0^{\tau_\epsilon} l_i dG_j(t) + l_i\Delta G_j(\tau_\epsilon) + l_i[1 - G_j(\tau_\epsilon)]\right] = l_i = \mathbb{E}^x[S_i(0; G_j)]\,,\end{aligned}$$

where the first inequality follows because $W_i(t) > M_i(t) > L_i(t)$, the second inequality follows from (C.21). This contradicts the supposition that $x \in \overline{\text{supp}(G_i)}$ because firm i can obtain a strictly greater expected payoff by adopting the strategy τ_ϵ. Therefore, it must be the case that $\overline{\text{supp}(G_i)} \cap D_i = \emptyset$.

Next, we prove that the *interiors* of supp(G_1) and supp(G_2) must coincide, which establishes the statement of this lemma. Towards a contradiction, suppose that there exists an open interval $E \subseteq \text{supp}(G_i)$ but $E \nsubseteq \text{supp}(G_j)$. Consider an exit strategy $\tau_E := \inf\{t > 0 : X_t^x \notin E\}$ for firm i, where $x \in E$. Then $\tau_E > 0$ a.s. because E is an open set. Fix some $\tau \in (0, \tau_E)$, and note that Lemma C.2 implies that $\mathbb{E}^x[S_i(\tau; G_j)] = \mathbb{E}^x[S_i(0; G_j)]$. Moreover, because $x \in E \nsubseteq \text{supp}(G_j)$, it must be the case that $G_j^x(\tau) = 0$.

Recall that $\overline{\text{supp}(G_i)} \cap D_i = \emptyset$, so $E \cap D_i = \emptyset$, which implies that $\pi(X_s^x) < rl_i$ for all $s \in [0, \tau)$ because E is an open set and $\pi(\cdot)$ strictly increases. Hence, we have

$$\mathbb{E}^x[S_i(\tau; G_j)] = \mathbb{E}^x[L_i(\tau)] = l_i + \mathbb{E}^x\left[\int_0^\tau [\pi(X_t) - rl_i]e^{-rt}dt\right] < l_i = \mathbb{E}^x[S_i(0; G_j)],$$

where the first equality follows because $G_j^x(\tau) = 0$, and hence, $G_j^x(t) = \Delta G_j^x(t) = 0$ for all $t \leq \tau$, the second equality follows from the definition of $L_i(\cdot)$, and the inequality follows from $\pi(X_s^x) < rl_i$ for all $s \in [0, \tau)$. Therefore, firm i can obtain a strictly greater payoff by exiting immediately, which contradicts the supposition that E is in the support of G_i. This completes the proof. ■

Proof of Lemma 4.2(b): Suppose that (G_1, G_2) is a mixed-strategy MPE, and let $\overline{\Gamma^o}$ be the closure of the common interior of supp(G_1) and supp(G_2). Note that Γ^o comprises of all the (open) component intervals of supp(G_1) and

$\mathrm{supp}(G_2)$; it thus excludes all the point components, if any, of either $\mathrm{supp}(G_1)$ or $\mathrm{supp}(G_2)$. (A point component of $\mathrm{supp}(G_i)$ is a singleton subset $\{c\}$ of $\mathrm{supp}(G_i)$ that is disconnected from the rest of $\mathrm{supp}(G_i)$.) $\overline{\Gamma^o}$ is simply augmented by the boundary points of all the component intervals of Γ^o.

Towards a contradiction, pick some $x \in \overline{\Gamma^o}$, some $i \in \{1,2\}$, $j \in \{1,2\}\backslash\{i\}$, and suppose that $G_j^x(\cdot)$ has a jump of size $q_\tau > 0$ at some τ such that $\mathbb{P}(X_\tau^x \in \overline{\Gamma^o}) > 0$. Defining $\tau' := \min\{\tau, \tau_E\}$ where $\tau_E := \inf\{t \geq 0 : X_t \notin \overline{\Gamma^o}\}$ is the exit time from $\overline{\Gamma^o}$, this supposition implies that $G_j^x(\cdot)$ has a jump of size $q_{\tau'} > 0$ at time τ' such that $\mathbb{P}(X_{\tau'}^x \in \overline{\Gamma^o}) = 1$. (Here we allow for the possibility that $G_j^x(\cdot)$ has an additional jump at time τ_E as well.) It must then follow from Lemma C.2 that $\mathbb{E}^x[S_i(\tau'; G_j)] \geq \mathbb{E}^x[S_i(\tau'^+; G_j)]$.

Now, we compare $\mathbb{E}^x[S_i(\tau'; G_j)]$ and $\mathbb{E}^x[S_i(\tau'^+; G_j)]$. First, observe that

$$\mathbb{E}^x[S_i(\tau'; G_j)] = \mathbb{E}^x\left[\int_0^{\tau'^-} W_i(t)dG_j^x(t) + M_i(\tau')\Delta G_j^x(\tau') + L_i(\tau')[1 - G_j^x(\tau')]\right],$$

where $\Delta G_j^x(\tau') = [1 - G_j^x(\tau'^-)]q_{\tau'}$ because we assume $G_j^x(\cdot)$ has a jump at τ'. On the other hand, we can similarly express $\mathbb{E}^x[S_i(\tau'^+; G_j)]$ as

$$\begin{aligned} &\mathbb{E}^x\left[\int_0^{\tau'} W_i(t)dG_j^x(t) + L_i(\tau')[1 - G_j^x(\tau')]\right] \\ = &\mathbb{E}^x\left[\int_0^{\tau'^-} W_i(t)dG_j^x(t) + W_i(\tau')\Delta G_j^x(\tau') + L_i(\tau')[1 - G_j^x(\tau')]\right], \end{aligned}$$

where the equality follows by breaking down the integral over $[0, \tau'^-]$ and $(\tau'^-, \tau']$. Then because $W_i(\cdot) > M_i(\cdot)$ and $\mathbb{E}^x[\Delta G_j^x(\tau')] = \mathbb{E}^x[[1-G_j^x(\tau'^-)]q_{\tau'}] > 0$, we conclude $\mathbb{E}^x[S_i(\tau'; G_j)] < \mathbb{E}^x[S_i(\tau'^+; G_j)]$, which is a contradiction. Hence, if $x \in \overline{\Gamma^o}$, then $G_j^x(t)$ must be continuous as long as $X_t \in \overline{\Gamma^o}$. ■

Proof of Lemma 4.3(a): Suppose that $\mathcal{G} := (G_i, G_j)$ is a mixed-strategy MPE. First, we claim that there is no point component $c \in \mathscr{I}$ of $\mathrm{supp}(G_i)$ such that $c \notin \mathrm{supp}(G_j)$. Note that this claim combined with Lemma 4.2(a) establishes that $\mathrm{supp}(G_i) = \mathrm{supp}(G_j)$, thus yielding $\overline{\mathrm{supp}(G_i)} = \overline{\mathrm{supp}(G_j)}$.

To prove this claim, observe first that there is no singleton set $\{c'\}$ that is a point component of both $\mathrm{supp}(G_i)$ and $\mathrm{supp}(G_j)$. This is because exit simultaneously with the opponent yields lower expected payoff compared to exit in an infinitesimal time. Now, consider a component interval (d, θ) of the common interior Γ^o for some $d \in [\alpha, \theta)$ and $\theta < \infty$, and suppose that there is a point component $c \in \mathscr{I}$ of $\text{supp}(G_i)$ such that $c \notin \text{supp}(G_j)$. We further suppose that the interval (θ, c) does not contain any other point component of $\text{supp}(G_i)$ or $\text{supp}(G_j)$.

It follows from the proof of Lemma 4.2(a) that $c \notin D_i$ where $D_i := \{x : \pi(x) > rl_i\}$, which implies that (θ, c) does not intersect with D_i because $\pi(\cdot)$ strictly increases. Also, because our assumption $c \notin \text{supp}(G_j)$ and Lemma 4.2(b) imply that $G_j^c(0) = G_j^\theta(0) = 0$, we obtain $V_i(c; \mathcal{G}) = V_i(\theta; \mathcal{G}) = l_i$; otherwise c and θ do not belong to $\overline{\text{supp}(G_i)}$ by definition of a mixed-strategy equilibrium. Then because $\pi(x) < rl_i$ for $\forall x \in (\theta, c)$, it is straightforward to verify that $V_i(x; \mathcal{G}) < l_i$ for $\forall x \in (\theta, c)$ (Oksendal, 2003). However, firm i can always achieve a higher payoff l_i from an immediate exit at any point $x \in (\theta, c)$, which contradicts the assumption that $\mathcal{G}$ is an equilibrium. Hence, such a point component c of $\text{supp}(G_i)$ cannot exist *above* a component interval (d, θ) of Γ^o. Lastly, because the exactly same procedure can be used to prove that there is no such point component c of $\text{supp}(G_i)$ *below* a component interval (θ, d) of Γ^o, we can conclude that there is no point component $c \in \mathscr{I}$ of $\text{supp}(G_i)$ such that $c \notin \text{supp}(G_j)$. This has proved that $\text{supp}(G_i) = \text{supp}(G_j) = \Gamma$ and $\overline{\Gamma} = \overline{\Gamma^o}$.

Finally, we prove the statement of this lemma. Suppose that G_i^x is discontinuous in time. Because we have proved above that no point component can exist outside $\overline{\Gamma}$ in equilibrium, the discontinuity cannot take place when $X_t \notin \overline{\Gamma}$, thus implying a discontinuity can only happen inside $\overline{\Gamma}$. Also, by definition of the Markov strategy, the discontinuity happens at $\tau_E = \inf\{t \geq 0 : X_t \in E\}$ for some $E \subset \overline{\Gamma}$ irrespective of the initial point x. Note that because X is an irreducible Markov chain if $\sigma(\cdot) > 0$, we have $\tau_E < \infty$ with positive probability irrespective of the initial point x. However, Lemma 4.2(b) stipulates that such a set E cannot intersect $\overline{\Gamma^o}$ if the initial point x is within $\overline{\Gamma^o}$. Because $\overline{\Gamma} = \overline{\Gamma^o}$, and because a Markov strategy does not depend on the initial value of the state variable, these two statements contradict each other unless $E = \emptyset$. Therefore, G_i^x must be continuous in time irrespective of x. ■

Proof of Lemma 4.3(b): Suppose that (G_1, G_2) is a mixed-strategy MPE. We have shown in Lemma 4.3(a) that if $\sigma(\cdot) > 0$, then $G_1^x(\cdot)$ and $G_2^x(\cdot)$ are continuous for all $x \in \mathscr{I}$. Let $\Gamma := \text{supp}(G_1)$=$\text{supp}(G_2)$ and define $C := \mathscr{I} \backslash \Gamma$. Recall from the proof of Lemma 4.2(a) that $\{x : \pi(x) > rl_i\} \subset C$ for each

$i \in \{1,2\}$.

First, we show that Γ is of the form (α, θ). Towards a contradiction, suppose that there exists an interval (a,b) such that $(a,b) \subseteq C$ and $a, b \in \overline{\Gamma}$. This implies that (a,b) is disconnected from $\{x : \pi(x) > rl_i\}$ for each i. Pick $x \in (a,b)$ and define $\tau_{(a,b)} := \inf\{t \geq 0 : X_t^x \notin (a,b)\}$.

It follows from Lemma C.2 that $\mathbb{E}^x[S_i(\tau_{(a,b)}; G_j)] = \sup_\tau \mathbb{E}^x[S_i(\tau; G_j)]$. Observe that

$$\begin{aligned}\mathbb{E}^x[S_i(\tau_{(a,b)}; G_j)] = \mathbb{E}^x[L_i(\tau_{(a,b)})] &= l_i + \mathbb{E}^x\left[\int_0^{\tau_{(a,b)}} [\pi(X_t) - rl_i] e^{-rt} dt\right] \\ &< l_i = \mathbb{E}^x[S_i(0; G_j)]\,,\end{aligned}$$

where the first equality follows from $G_j^x(\tau_{(a,b)}) = 0$, the second equality follows from the definition of $L_i(\cdot)$, and the inequality follows because $\pi(X_s^x) < rl_i$ for all $s \leq \tau_{(a,b)}$. This is a contradiction, which implies that $\Gamma = (-\infty, \theta)$ for some $\theta \leq x_{ci} \wedge x_{cj}$.

Second, pick some $x > \theta$, and define the strategy $\tau := \inf\{t \geq 0 : X_t^x \leq \theta\}$. We show that it must be the case $\theta = \theta_i^*$ for each i where θ_i^* is given in Lemma 4.1.

Towards a contradiction, suppose that $\theta < \theta_2^*$ and recall that $\theta_1^* \leq \theta_2^*$ because $l_1 \geq l_2$ by convention. Let $\tau_2^* = \inf\{t \geq 0 : X_t^x \leq \theta_2^*\}$ and note that $G_1(\tau_2^*) \leq G_1(\tau) = 0$ because $\theta < \theta_2^*$. Therefore, we have $\mathbb{E}^x[S_2(\tau; G_1)] = \mathbb{E}^x[L_2(\tau)] < \mathbb{E}^x[L_2(\tau_2^*)] = \mathbb{E}^x[S_2(\tau_2^*; G_1)]$ where the equalities follow from $G_1(\tau_2^*) = G_1(\tau) = 0$, and the inequality follows from Lemma 4.1. However, this contradicts that $\mathbb{E}^x[S_2(\tau; G_1)] = \sup_{\tilde{\tau}} \mathbb{E}^x[S_2(\tilde{\tau}; G_1)]$.

On the other hand, suppose that $\theta > \theta_2^*$, which implies that $\tau \leq \tau_2^*$. We first note that $W_2(\cdot)$ is a supermartingale because

$$\begin{aligned}\mathbb{E}^x\left[W_2(t) | \mathcal{F}_s\right] &= \mathbb{E}^x\left[\int_0^t \pi(X_v) e^{-rv} dv + e^{-rt} w(X_t) \mid \mathcal{F}_s\right] \\ &= \int_0^s \pi(X_v) e^{-rv} dv \\ &+ \mathbb{E}^x\left[\int_s^t \pi(X_v) e^{-rv} dv + \mathbb{E}^x\left[\int_t^\infty \pi^M(X_v) e^{-rv} dv \mid \mathcal{F}_t\right] \mid \mathcal{F}_s\right] \\ &< \int_0^s \pi(X_v) e^{-rv} dv + \mathbb{E}^x\left[\int_s^\infty \pi^M(X_v) e^{-rv} dv \mid \mathcal{F}_s\right] = W_2(s)\,.\end{aligned}$$

Here, we use the definition of $w(\cdot)$ and $\pi^M(\cdot) > \pi(\cdot)$. We next establish that

$$\begin{aligned}
\mathbb{E}^x\left[\int_\tau^{\tau_2^*} W_2(t)dG_1(t) \mid \mathcal{F}_\tau\right] &= \int_0^1 \mathbb{E}^x\left[W_2(\tau_1^{G,x}(z))\mathbf{1}_{[\tau,\tau_2^*]}(\tau_1^{G,x}(z))|\mathcal{F}_\tau\right]dz \\
&\geq \int_0^1 \mathbb{E}^x\left[\mathbb{E}^x\left[W_2(\tau_2^*)|\mathcal{F}_{\tau_1^{G,x}(z)}\right]\mathbf{1}_{[\tau,\tau_2^*]}(\tau_1^{G,x}(z))|\mathcal{F}_\tau\right]dz \\
&= \int_0^1 \mathbb{E}^x\left[\mathbb{E}^x\left[W_2(\tau_2^*)\mathbf{1}_{[\tau,\tau_2^*]}(\tau_1^{G,x}(z))|\mathcal{F}_{\tau_1^{G,x}(z)}\right]|\mathcal{F}_\tau\right]dz \\
&= \mathbb{E}^x\left[W_2(\tau_2^*)\int_0^1 \mathbf{1}_{[\tau,\tau_2^*]}(\tau_1^{G,x}(z))dz|\mathcal{F}_\tau\right] \\
&= \mathbb{E}^x\left[W_2(\tau_2^*)[G_1(\tau_2^*) - G_1(\tau)]|\mathcal{F}_\tau\right] \\
&> \mathbb{E}^x\left[L_2(\tau_2^*)G_1(\tau_2^*)|\mathcal{F}_\tau\right], \qquad \text{(C.22)}
\end{aligned}$$

where the first equality holds from the change-of-variable from $G_1(t)$ to $\tau_1^{G,x}(z)$ given in (C.9), the inequality holds because $W_2(\cdot)$ is a *supermartingale*, the second equality holds because $\mathbf{1}_{[\tau,\tau_2^*]}(\tau_1^{G,x}(z))$ is measurable with respect to the filtration $\mathcal{F}_{\tau_1^{G,x}(z)}$, the third equality follows from the tower rule of the conditional expectation because $\mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_1^{G,x}(z)}$, the fourth equality follows from $\int_0^1 \mathbf{1}_{[\tau,\tau_2^*]}(\tau_1^{G,x}(z))dz = G_1(\tau_2^*) - G_1(\tau)$ by the definition of $\tau_1^{G,x}(z)$, and the last inequality follows because $W_2(\cdot) > L_2(\cdot)$ and $G_1(\tau) = 0$ (recall that $\text{supp}(G_1) = (\alpha, \theta)$).

Using (C.22), therefore, we obtain

$$\begin{aligned}
&\mathbb{E}^x[S_2(\tau_2^*; G_1)] - \mathbb{E}^x[S_2(\tau; G_1)] \\
&= \mathbb{E}^x\left[\mathbb{E}^x\left[\int_\tau^{\tau_2^*} W_2(t)dG_1(t) + L_2(\tau_2^*)[1 - G_1(\tau_2^*)] - L_2(\tau) \mid \mathcal{F}_\tau\right]\right] \\
&> \mathbb{E}^x\left[\mathbb{E}^x\left[L_2(\tau_2^*)G_1(\tau_2^*) + L_2(\tau_2^*)[1 - G_1(\tau_2^*)] - L_2(\tau) \mid \mathcal{F}_\tau\right]\right] \\
&= \mathbb{E}^x\left[\mathbb{E}^x\left[L_2(\tau_2^*) - L_2(\tau) \mid \mathcal{F}_\tau\right]\right] = \mathbb{E}^x\left[L_2(\tau_2^*) - L_2(\tau)\right] > 0,
\end{aligned}$$

where the last equality follows from the tower rule of the conditional expectation and the last inequality follows from Lemma 4.1.

Therefore, it must be the case that $\theta = \theta_2^*$. By a symmetric argument, one can show that it must be the case that $\theta = \theta_1^*$. ∎

Proof of Theorem 4.1: By noting that $\theta_1^* = \theta_2^*$ if and only if $l_1 = l_2$, it is straightforward to see from Lemma 4.3(b) that if $l_1 \neq l_2$, then the game does not admit any mixed-strategy MPE, which completes the proof. ■

Proof of Proposition 4.2: Because $l_1 = l_2$, we must have $\theta^* := \theta_1^* = \theta_2^*$. Define $\tau^* := \inf\{t \geq 0 : X_t^x \leq \theta^*\}$ and it is enough to show that G_i is a best response to G_j by symmetry.

To that end, we will use Lemma C.2. More precisely, since it can be seen from (4.10) that the closure of the support of G_i^x is $(-\infty, \theta^*]$, we only need to prove the following two relations:

$$\mathbb{E}^x[S_i(u_1; G_j)] = \mathbb{E}^x[S_i(u_2; G_j)] \;\text{ for any }\; u, v \geq \tau^* \,, \tag{C.23}$$

$$\mathbb{E}^x[S_i(\tau; G_j)] < \mathbb{E}^x[S_i(\tau^*; G_j)] \;\text{ for any }\; \tau < \tau^* \,. \tag{C.24}$$

To show (C.23), choose any stopping times $u, v > \tau^*$ with $v > u$, and observe that

$$\begin{aligned}
&\mathbb{E}^x[S_i(v; G_j)] - \mathbb{E}^x[S_i(u; G_j)] \\
&= \mathbb{E}^x\left[\int_u^v F_i(s) dG_j^x(s) + L_i(v)[1 - G_j^x(v)] - L_i(u)[1 - G_j^x(u)]\right] .
\end{aligned} \tag{C.25}$$

Then it is enough to prove that the right side of (C.25) is equal to 0. By differentiating (4.10) with respect to time, we obtain

$$dG_j^x(s) = [1 - G_j^x(s)] \frac{dD_i^{\theta^*}(s)}{F_i(s) - L_i(s)} \,, \tag{C.26}$$

where $D_i^{\theta^*}(\cdot)$ is defined in (C.12). By applying integration by parts to (C.26), we have

$$\begin{aligned}
&\int_u^v [F_i(s) - L_i(s)] dG_j^x(s) = -\int_u^v [1 - G_j^x(s)] dD_i^{\theta^*}(s) \\
&= -\int_u^v D_i^{\theta^*}(s) dG_j^x(s) - D_i^{\theta^*}(v)[1 - G_j^x(v)] + D_i^{\theta^*}(u)[1 - G_j^x(u)] \,.
\end{aligned}$$

Then it follows from (C.25) that

$$S_i(v;G_j) - S_i(u;G_j) = \int\limits_u^v F_i(s)dG_j^x(s) + L_i(v)[1 - G_j^x(v)] - L_i(u)[1 - G_j^x(u)]\,.$$
$$= \int_u^v [D_i^{\theta^*}(s) + L_i(s)]dG_j^x(s) + [1 - G_j^x(v)][D_i^{\theta^*}(v) + L_i(v)]$$
$$- [1 - G_j^x(u)][D_i^{\theta^*}(u) + L_i(u)]\,.$$

By Lemma C.3(b), we now obtain

$$\mathbb{E}^x[S_i(v;G_j) - S_i(u;G_j)] = \mathbb{E}^x\Bigg[\int_u^v [D_i^{\theta^*}(s) + L_i(s) - N_i(s)]dG_j^x(s)$$
$$+ [1 - G_j^x(v)][D_i^{\theta^*}(v) + L_i(v) - N_i(v)]$$
$$- [1 - G_j^x(u)][D_i^{\theta^*}(u) + L_i(u) - N_i(u)]\Bigg]\,.$$

Lemma C.3(a) implies that $L_i(s) = J_i(s) = N_i(s) - D_i^{\theta^*}(s)$ for any $s \geq \tau^*$ and $u, v > \tau^*$, from which (C.23) follows.

To show (C.24), because $G_j^x(s) = 0$ for all $s \leq \tau^*$, we obtain, for any $\tau < \tau^*$,

$$\mathbb{E}^x[S_i(\tau;G_j)] = \mathbb{E}^x[L_i(\tau)] < \mathbb{E}^x[L_i(\tau^*)] = \mathbb{E}^x[S_i(\tau^*;G_j)]\,,$$

where the inequality follows from Lemma 4.1. This establishes (C.24), which completes the proof. ■

Proof of Proposition C.1: Note that we do not need the expectation notation throughout the proof of this proposition because there is no uncertainty when $\sigma(\cdot) = 0$ in (3.1). However, Lemma 4.1 and Lemma C.2 are still valid when $\sigma(\cdot) = 0$ so that we use those lemmas without expectation notation for notational simplicity.

First, we show that G_2 is also a best response to G_1 if $\theta_2^* - \theta_1^* < \kappa_\theta$ where κ_θ is given in (C.20). We will use Lemma C.2 and will prove the following relations:

$$S_2(u;G_1) = S_2(v;G_1) \text{ for any } u, v > \tau_1^*\,, \tag{C.27}$$
$$S_2(t;G_1) < S_2(u;G_1) \text{ for any } t \leq \tau_1^* < u\,. \tag{C.28}$$

To prove (C.27), choose any $u, v > \tau^*$ with $v > u$, and observe that

$$S_2(v; G_1) - S_2(u; G_1) = \int_u^v F_2(s) dG_1^x(s) + L_2(v)[1 - G_1^x(v)] - L_2(u)[1 - G_1^x(v)] \, . \tag{C.29}$$

Then it is enough to prove that the right side of (C.29) is equal to 0. For any $s > \tau_1^*$, observe from (4.10) and (C.1) that

$$G_1^x(s) = \begin{cases} (1 - p_1) + p_1 G_1^{\theta_1^*}(s - \tau_1^*) & \text{for } x > \theta_1^* \, , \\ G_1^x(s) & \text{for } x \leq \theta_1^* \, . \end{cases} \tag{C.30}$$

In addition, we can obtain from differentiating (4.10) that for any $s > \tau_1^*$,

$$\begin{aligned} dG_1^x(s) &= [1 - G_1^x(s)] \frac{[rl_2 - \pi(X_s^x)]}{w(X_s^x) - l_2} ds = [1 - G_1^x(s)] \frac{[rl_2 - \pi(X_s^x)] e^{-rs} ds}{W(s) - L_2(s)} \\ &= [1 - G_1^x(s)] \frac{-dL_2(s)}{W(s) - L_2(s)} \, , \end{aligned} \tag{C.31}$$

where $\mathbf{1}_{\{s \geq \tau_1^*\}}$ disappears because $\sigma(\cdot) = 0$. Hence, we can use integration by parts and obtain

$$\int_u^v [F_2(s) - L_2(s)] dG_1^x(s) = - \int_u^v [1 - G_1^x(s)] dL_2(s) \tag{C.32}$$

$$= - \int_u^v L_2(s) dG_1^x(s) - L_2(v)[1 - G_1^x(v)] + L_2(u)[1 - G_1^x(u)] \, . \tag{C.33}$$

After $\int_u^v L_2(s) dG_1^x(s)$ is cancelled out on the left side of (C.32) and the right side of (C.33), the resulting equation implies that the equation (C.29) is equal to 0, which establishes (C.27).

To show (C.28), we will use $\lim_{t \uparrow \tau_1^*} G_1^x(t) = 0$ and $G_1^x(\tau_1^*) = (1 - p_1)$. These imply that $S_2(t; G_1) = L_2(t)$ for all $t < \tau_1^*$, and that, for any $u > \tau_1^*$

$$S_2(\tau_1^*; G_1) = (1 - p_1) M_2(\tau_1^*) + p_1 L_2(\tau_1^*) < (1 - p_1) W_2(\tau_1^*) + p_1 L_2(\tau_1^*) = S_2(u; G_1) \, ,$$

because $p_1 < 1$ and (C.27). Thus, (C.28) holds if

$$\sup_{\tau} L_2(\tau) = L_2(\tau_2^*) < (1-p_1)W_2(\tau_1^*) + p_1 L_2(\tau_1^*)$$
$$= \int_0^{\tau_1^*} \pi(X_s)e^{-rs}ds + [(1-p_1)w(X_{\tau_1^*}^x) + p_1 l_2]e^{-r\tau_1^*} . \tag{C.34}$$

It can be then seen that (C.34) holds if, for any $x \in (\theta_1^*, \theta_2^*]$, we have

$$l_2 < \mathbb{E}^x\left[\int_0^{\tau_1^*} \pi(X_s^x)e^{-rs}ds + [(1-p_1)\underline{w} + p_1 l_2]e^{-r\tau_1^*}\right] = R(x) + \beta_2^p(\theta_1^*)\phi(x) , \tag{C.35}$$

where $\beta_2^p(\theta) := \{[(1-p_1)\underline{w} + p_1 l_2] - R(\theta)\}/\phi(\theta)$ and $\underline{w} = \inf\{W(x) : x \in \mathscr{I}\}$. However, because $\beta_2^p(\theta) < \beta_W(\theta)$ for all $p_1 < 1$ and $\theta \in \mathscr{I}$ where $\beta_W(\theta) = [\underline{w} - R(\theta)]/\phi(\theta)$ was used in the proof of Proposition 4.1(b), (C.35) holds if (C.19) does. Because we already proved in the proof of Proposition 4.1(b) that (C.19) is implied by the condition $\theta_2^* - \theta_1^* < \kappa_\theta$, we can conclude that the desired result follows.

Conversely, we show that G_1 is a best response to G_2. Since the closure of the support of G_1^x is $(\alpha, \theta_1^*]$, by the virtue of Lemma C.2, it is enough to establish the following relations:

$$S_1(u; G_2) = S_1(v; G_2) \text{ for any } u, v \geq \tau_1^* , \tag{C.36}$$
$$S_1(t; G_2) < S_1(u; G_2) \text{ for any } t < \tau_1^* \leq u . \tag{C.37}$$

To show (C.36), can be shown by using the arguments for (C.36) above with $p_2 = 0$.

To prove (C.37), because $G_2^x(\tau_1^*) = 0$, (C.36) implies that $S_1(u; G_2) = S_1(\tau_1^*; G_2) = L_1(\tau_1^*)$ for any $u > \tau_1^*$, and that $S_1(t; G_2) = L_1(t)$ for any $t < \tau_1^*$. Hence, for any $t < \tau_1^* \leq u$, we have $S_1(t; G_2) = L_1(t) < L_1(\tau_1^*) = S_1(u; G_2)$ where the inequality is due to Proposition 4.1. ■