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SINGLE AND DOUBLE INTEGRALS FOR AREA IN ADVANCED PLACEMENT CALCULUS

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One of the most fascinating and useful applications of elementary integral calculus is its application to area. It is at the time of applying integration to area that calculus students begin to see the power of integration. It is the area application that first destroys the myth that integration is simply antidifferentiation or reversing the derivative process. Indeed, the geometric interpretation of the integral is light years from the geometric interpretation of the derivative. This fact should be strongly emphasized in the calculus classroom.

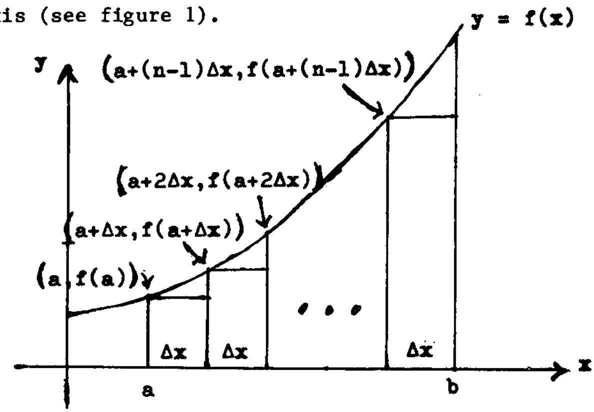
A firm understanding of just how the area between a curve f(x) and an axis is computed can best be demonstrated using the Riemann integration technique of inscribing 'n' rectangles having equal length bases, finding the sum of their areas, and then letting the number of rectangles approach infinity as a limit. In this way exact areas can be computed and confidence in the fundamental theorem of integral calculus,

$$\lim_{\Delta x \to 0} \sum_{a}^{b} f(x) \ \Delta x = \int_{a}^{b} f(x) \ dx$$

is developed. This intuitive approach is especially needed in Advanced Placement High School calculus before a more rigorous proof of the fundamental theorem is attempted. The student needs to have a feeling for the theorem before formal proof. In this way, a better understanding of the topic is developed.

It is easy to build a sound foundation in the student's understanding of this concept. In general, the approach is to take a given function, $y = f(x)$, and inscribe n-rectangles between it and the x-axis (see figure 1).

FIGURE 1



Then Δx is $\frac{b-a}{n}$ where n is the total number of rectangles. Note that Δx remains constant and is the base of all n of the rectangles. Further, emphasis is needed that $\Delta x = \frac{b-a}{n}$ is a function of n and that as n approaches infinity, Δx approaches zero. Constants a and b are fixed and not equal.

The approximate area, A_a, between $f(x)$ and the x-axis is given by the sum of the areas of the n-rectangles drawn under the curve.

$$A_a = f(a)\,\Delta x + f(a+\Delta x)\,\Delta x + f(a+2\,\Delta x)\,\Delta x + \ldots + f(a+(n-1)\,\Delta x)\,\Delta x$$

$$= \sum_{k=0}^{n-1} f(a+k\cdot\Delta x)\,\Delta x = \sum_{k=1}^{n} f(a+(k-1)\cdot\Delta x)\,\Delta x.$$

Of course, as the number of rectangles is increased between a and b, the value of Δx gets smaller, and the value of A_a becomes a better approximation to the exact area A_{exact}. In other words,

$$A_{exact} = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(a+k\cdot\Delta x)\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} f(a+(k-1)\,\Delta x)\,\Delta x.$$

However, it is by working specific examples that student confidence in the fundamental theorem can be established. Specific examples will require a knowledge of sigma notation properties (see table 1). The properties are usually proven using mathematical induction in a precalculus course and can be repeated for emphasis in a lecture on this material.

TABLE 1
PROPERTIES OF THE SIGMA NOTATION

$$\sum_{k=1}^{n} c = cn$$

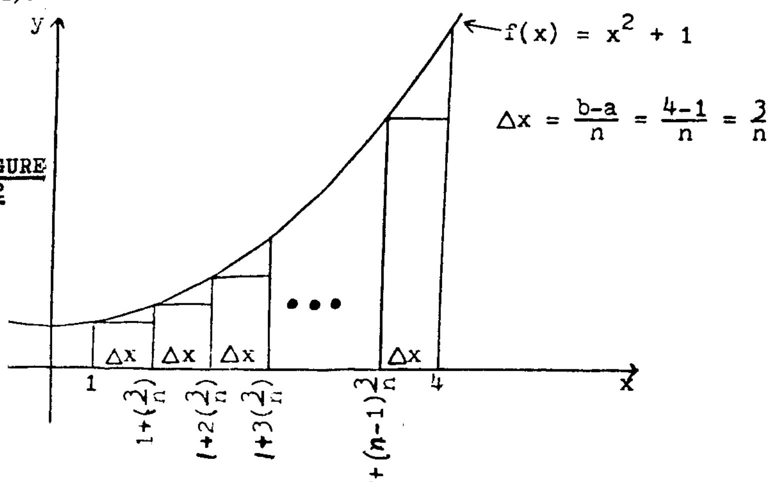
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

As a specific example, consider the area between the graph of $f(x) = x^2 + 1$ and the x-axis on the interval $1 \leq x \leq 4$ (see figure 2).

FIGURE 2



$$A_a = f(1)\left(\frac{3}{n}\right) + f\left(1 + \frac{3}{n}\right)\left(\frac{3}{n}\right) + \ldots + f\left(1 + (n-1)\frac{3}{n}\right)\left(\frac{3}{n}\right)$$

$$= \frac{3}{n} \sum_{k=0}^{n-1} f\left(1 + k\left(\frac{3}{n}\right)\right)$$

Since $f(x) = x^2 + 1$,

then $f\left(1 + k\left(\frac{3}{n}\right)\right) = \left(1 + \frac{3k}{n}\right)^2 + 1 = 2 + \frac{6k}{n} + \frac{9k^2}{n^2}$

$$A_a = \frac{3}{n} \sum_{K=0}^{n-1} \left(2 + \frac{6k}{n} + \frac{9k^2}{n^2} \right) = \frac{3}{n} \left(\sum_{K=0}^{n-1} 2 + \frac{6}{n} \sum_{K=0}^{n-1} k + \frac{9}{n^2} \sum_{K=0}^{n-1} k^2 \right)$$

$$= \frac{3}{n} \left(2(n-1) + \frac{6(n-1)(n)}{2n} + \frac{9(n-1)(n)(2n-1)}{6n^2} \right)$$

$$= \frac{48n^2 - 57n + 9}{2n^2}$$

Now since A_a represents the approximate area under $x^2 + 1$ from 1 to 4 with n rectangles, taking the limit as n approaches infinity gives the exact area.

$$A_{exact} = \lim_{n \to \infty} \frac{48n^2 - 57n + 9}{2n^2} = \underline{24}$$

and $\displaystyle\int_1^4 (x^2 + 1)\,dx = \frac{x^3}{3} + x \Big|_1^4 = \underline{24}.$

Although this is just one example, it is strong evidence for the truth of the fundamental theorem. After several examples of this technique have been given with their integral counterpart, a formal proof of the fundamental theorem may be attempted.

Once the fundamental theorem has been established and the students are convinced of its truth, the teacher could take the opportunity to show the double-integral representation of area and its equivalence to the single integral and the fundamental theorem (see figure 3).

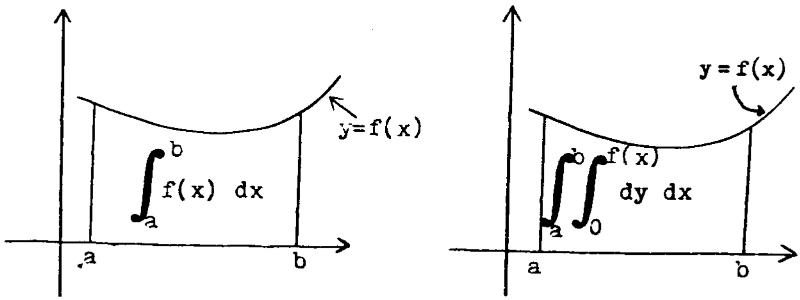


FIGURE 3

$$\int_a^b \int_0^{f(x)} dy\,dx = \int_a^b y \Big|_0^{f(x)} dx = \int_a^b f(x)\,dx.$$

The single integral approach can be thought of as the inscription of infinitely many rectangles under f(x) and above the x-axis on $a \leq x \leq b$. However, the double integral represents a summation of the area of infinitely many rectangles packed under f(x) and above the x-axis (see figure 4).

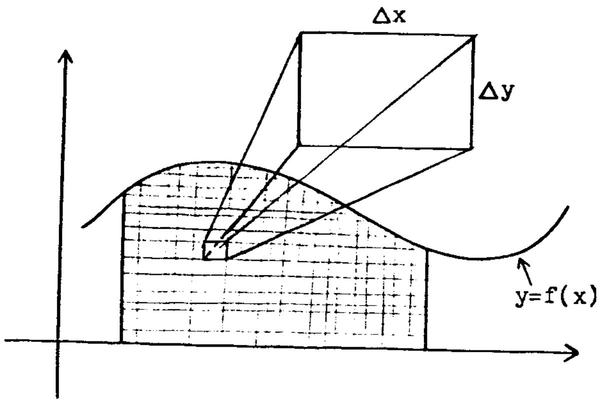


FIGURE 4

Each of the 'typical' rectangles has a length Δx and a width Δy. In fact, the logic is analogous to that used in the Riemann Integration approach. It is by summing the rectangle areas that we can approximate the exact area.

$$A_a = \sum_{\text{region}} \sum \Delta y \, \Delta x = \sum_{\text{region}} \sum \Delta x \, \Delta y$$

For convenience let $\Delta A = \Delta y \, \Delta x$. Now intuition leads the student to believe the true result that

$$\lim_{\Delta A \to 0} \sum \sum \Delta A = \int_a^b \int_0^{f(x)} dA = \iint dy \, dx.$$

The point here is that by developing the student's intuition one can build insight that can lead to understanding rather than memorization and frustration. It is after a concept is understood that formal proof should fit into advanced placement calculus. Rigor is important in mathematics and must not be totally omitted. However, a rigorous approach without a foundation in understanding is time wasted for most students.

WORD PROCESSING: A TEACHER'S AIDE

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With the growing presence of microcomputers in colleges and schools, used both administratively and for computer-assisted instruction(CAI), their use outside the classroom by teachers becomes more common. Whether the micro is at home or at school, it can be of considerable assistance, especially when the amount of work to be done exceeds the time in which to do it.

Many teachers, particularly in mathematics, are already acquainted with the value of computers for teaching mathematics and for student use in writing programs, entering formulas and solving problems. What is important, however, is to know that there are other microcomputer applications which make the teacher's work faster, more accurate, more uniform and much easier.