ON THE ARITHMETIC OF ABELIAN VARIETIES

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Abstract. We prove some new results on the arithmetic of abelian varieties over function fields of one variable over finitely generated (infinite) fields. Among other things, we introduce certain new natural objects 'discrete Selmer groups' and 'discrete Shafarevich-Tate groups', and prove that they are finitely generated $\mathbb{Z}$-modules. Further, we prove that in the isotrivial case, the discrete Shafarevich-Tate group vanishes and the discrete Selmer group coincides with the Mordell-Weil group. One of the key ingredients to prove these results is a new specialisation theorem for first Galois cohomology groups, which generalises Néron's specialisation theorem for rational points of abelian varieties.

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§0. Introduction. Let k be a field of characteristic 0, and $C \to \operatorname{Spec} k$ a smooth, separated and geometrically connected (not necessarily proper) *algebraic curve* over k. Write $K = k(C)$ for the function field of C, C^{cl} for the set of closed points of C, and $k(c)$ for the residue field of C at $c \in C^{\mathrm{cl}}$. Let $\mathcal{A} \to C$ be an *abelian scheme* over C. Write $A \overset{\mathrm{def}}{=} \mathcal{A}_K \overset{\mathrm{def}}{=} \mathcal{A} \times_C \operatorname{Spec} K$ for the *generic fibre* of $\mathcal{A}$. For each $c \in C^{\mathrm{cl}}$, write $\mathcal{A}_c \overset{\mathrm{def}}{=} \mathcal{A} \times_C \operatorname{Spec} k(c)$ for the fibre of $\mathcal{A}$ at c, K_c for the completion of K at c, and $A_c \overset{\mathrm{def}}{=} A \times_K K_c$. Thus, A (resp. $\mathcal{A}_c$, resp. A_c) is an abelian variety over K (resp. $k(c)$, resp. K_c). Consider the Kummer exact sequence

$$0 \to A(K)^\wedge \to H^1(G_K, TA) \to TH^1(G_K, A) \to 0,$$

where TA is the (full) Tate module of A, $A(K)^\wedge$ is the completion $\varprojlim_{N>0} A(K)/NA(K)$ of the group $A(K)$ of K-rational points of A (which coincides with the profinite completion of $A(K)$, if $A(K)$ is finitely generated), and $TH^1(G_K, A)$ is the (full) Tate module of the Galois cohomology group $H^1(G_K, A)$ classifying K-principal homogeneous spaces under A. Similarly, for each closed point $c \in C^{\mathrm{cl}}$, we have the Kummer exact sequences

$$0 \to A_c(K_c)^\wedge \to H^1(G_{K_c}, TA_c) \to TH^1(G_{K_c}, A_c) \to 0$$

and

$$0 \to \mathcal{A}_c(k(c))^\wedge \to H^1(G_{k(c)}, T\mathcal{A}_c) \to TH^1(G_{k(c)}, \mathcal{A}_c) \to 0.$$

We have a natural commutative diagram

(0.1)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(K)^\wedge & \longrightarrow & H^1(G_K, TA) & \longrightarrow & TH^1(G_K, A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_c A_c(K_c)^\wedge & \longrightarrow & \prod_c H^1(G_{K_c}, TA_c) & \longrightarrow & \prod_c TH^1(G_{K_c}, A_c) & \longrightarrow & 0 \end{array}$$

where the horizontal sequences are the above Kummer exact sequences, the vertical maps are natural restriction maps, and the product is taken over all closed points $c \in C^{\mathrm{cl}}$. In fact, when k is finitely generated over $\mathbb{Q}$, diagram (0.1) can be identified with the following natural commutative diagram (cf. Proposition 2.1 (ii)):

(0.2)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}(C)^\wedge & \longrightarrow & H^1(\pi_1(C), TA) & \longrightarrow & TH^1_{\text{ét}}(C, \mathcal{A}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_c \mathcal{A}_c(k(c))^\wedge & \longrightarrow & \prod_c H^1(G_{k(c)}, T\mathcal{A}_c) & \longrightarrow & \prod_c TH^1(G_{k(c)}, \mathcal{A}_c) & \longrightarrow & 0 \end{array}$$

where the upper horizontal sequence is a Kummer exact sequence for the étale site of C, the lower horizontal sequence is as above, the vertical maps are natural restriction maps, and the product is taken over all closed points $c \in C^{\mathrm{cl}}$.

Just as in the case where K is a number field, define the *profinite Selmer group*

$$\mathrm{Sel}(A) \stackrel{\mathrm{def}}{=} \mathrm{Sel}(A, C) \stackrel{\mathrm{def}}{=} \mathrm{Ker}\big(H^1(G_K, TA) \to \prod_c TH^1(G_{K_c}, A_c)\big),$$

and the *Shafarevich-Tate group*

$$Ш(A) \stackrel{\mathrm{def}}{=} Ш(A, C) \stackrel{\mathrm{def}}{=} \mathrm{Ker}\big(H^1(G_K, A) \to \prod_c H^1(G_{K_c}, A_c)\big).$$

Thus, we have a natural exact sequence

$$0 \to A(K)^\wedge \to \mathrm{Sel}(A) \to TШ(A) \to 0,$$

where $TШ(A)$ is the Tate module of $Ш(A)$. For an integer $N > 0$, define the *N-Selmer group* by

$$\mathrm{Sel}_N(A) \stackrel{\mathrm{def}}{=} \mathrm{Sel}(A, C)_N \stackrel{\mathrm{def}}{=} \mathrm{Ker}\big(H^1(G_K, A[N]) \to \prod_c H^1(G_{K_c}, A_c)\big),$$

so that $\mathrm{Sel}(A) = \varprojlim_{N>0} \mathrm{Sel}_N(A)$.

One of our main results is the following, which improves a result of [Lang-Tate] (cf. Proposition 2.10, Proposition 3.9 (i) and Remark 3.13).

Proposition A. *Assume that k is* **finitely generated** *over $\mathbb{Q}$. Then for each integer $N > 0$, the N-Selmer group $\mathrm{Sel}_N(A)$, as well as the subgroup $Ш(A)[N]$ of N-torsion points of $Ш(A)$, is* **finite***.*

The proof of Proposition A follows from the following specialisation result (cf. Proposition 1.8).

Proposition B. *Assume that k is* **Hilbertian** *(cf [Serre2], 9.5). Then for each integer $N > 0$, there exists a* **finite** *subset $S \subset C^{\mathrm{cl}}$ (of cardinality ≤ 2), depending on N, such that the natural restriction map*

$$H^1(\pi_1(C), A[N]) \to \prod_{c \in S} H^1(G_{k(c)}, \mathcal{A}_c[N]),$$

is **injective***.*

We also prove the following analogous specialisation result for the Galois cohomology of the l-adic Tate module of A. (cf. Proposition 1.4).

Proposition C. *Assume that k is* **Hilbertian***. Let l be a prime number. Then there exists a* **finite** *subset $S \subset C^{\mathrm{cl}}$ of cardinality ≤ 2, depending on l, such that the natural restriction map*

$$H^1(\pi_1(C), T_l A) \to \prod_{c \in S} H^1(G_{k(c)}, T_l \mathcal{A}_c)$$

is **injective***.*

In the case where either k is **finitely generated** over $\mathbb{Q}$ or the $\overline{k}$-trace of $A_{K\overline{k}} \overset{\mathrm{def}}{=} A \times_K K\overline{k}$ is trivial, one can prove that there exists a finite subset $S \subset C^{\mathrm{cl}}$ as in Proposition C **of cardinality** 1 (cf. Proposition 1.2 and Proposition 1.4). We do not know (even in the finitely generated case) if an analogue of Proposition C holds for the Galois cohomology of the full Tate module TA.

As a consequence of Proposition C, one deduces the following (cf. Proposition 2.2).

Proposition D. *Assume that k is* **Hilbertian***. Then the middle and left vertical maps in diagrams (0.1) and (0.2) are* **injective***.*

For the rest of this introduction we will assume that k is **finitely generated** over $\mathbb{Q}$. We will identify $A(K)^\wedge$, $H^1(G_K, TA)$, and $\prod_c \mathcal{A}_c(k(c))^\wedge$ with their images in $\prod_c H^1(G_{k(c)}, T\mathcal{A}_c)$. For each closed point $c \in C^{\mathrm{cl}}$ the group $\mathcal{A}_c(k(c))$ of $k(c)$-rational points of $\mathcal{A}_c$ is finitely generated as $k(c)$ is finitely generated over $\mathbb{Q}$ (Mordell-Weil Theorem, cf. [Lang-Néron]), hence injects into its profinite completion $\mathcal{A}_c(k(c))^\wedge$. We identify $\mathcal{A}_c(k(c))$ with its image in $\mathcal{A}_c(k(c))^\wedge$. We define the **discrete** Selmer group by

$$\mathfrak{Sel}(A) \overset{\mathrm{def}}{=} \mathfrak{Sel}(A, C) \overset{\mathrm{def}}{=} \mathrm{Sel}(A) \bigcap \prod_c \mathcal{A}_c(k(c)) \subset \prod_c H^1(G_{k(c)}, T\mathcal{A}_c).$$

Note that $A(K) \subset \mathfrak{Sel}(A)$. We define the **discrete** Shafarevich-Tate group by

$$\mathfrak{Sha}(A) \overset{\mathrm{def}}{=} \mathfrak{Sha}(A, C) \overset{\mathrm{def}}{=} \mathfrak{Sel}(A)/A(K).$$

We conjecture the following (cf. Conjecture 3.8).

Conjecture E. The equality $\mathfrak{Sel}(A) = A(K)$ (or, equivalently, $\mathfrak{Sha}(A) = 0$) holds.

Concerning Conjecture E, we prove the following (cf. Proposition 2.5, Proposition 3.3 and Proposition 3.7).

Proposition F. *The discrete Selmer group* $\mathfrak{Sel}(A)$ *is a* **finitely generated** $\mathbb{Z}$-*module. The discrete Shafarevich-Tate group* $\mathfrak{Sha}(A)$ *is a* **finitely generated free** $\mathbb{Z}$*-module.*

Proposition G. *Assume that there exists a prime number* l *such that the* l*-primary part* $\text{Ш}(A)[l^\infty]$ *of the torsion group* $\text{Ш}(A)$ *is finite. Then the assertion of Conjecture E holds.*

Our results are most complete in the case where the abelian variety A is *isotrivial*. In this case we prove the following (cf. Theorem 4.1).

Theorem H. *Assume that the abelian variety* A *is* **isotrivial**, *i.e.,* $A_{\overline{K}}$ *descends to an abelian variety over* $\overline{k}$*. Then the Shafarevich-Tate group* $\text{Ш}(A)$ *is* **finite**. *In particular, the assertion of Conjecture E holds in this case.*

Although we assumed above that $\text{char}(k) = 0$, we prove similar results in arbitrary characteristics.

Some of the results in this paper have applications in anabelian geometry. More precisely, Conjecture E and Theorem H have applications to Grothendieck's anabelian section conjecture (cf. [Saïdi1], §0, for a precise statement of this conjecture). One can prove that the validity of Conjecture E above implies that the section conjecture (for π_1 of proper hyperbolic curves) over finitely generated fields reduces to the case of number fields. Using among others Theorem H, one can also prove that if the section conjecture holds for all proper hyperbolic curves over all number fields then it holds for all proper hyperbolic curves over all finitely generated fields which are defined over a number field (cf. [Saïdi2], §5).

Finally, we explain the content of each section briefly. In §1, we prove Propositions B and C. In §2 and §3, we prove Propositions A, D, F and G. In §4, we prove Theorem H.

Notations. Next, we fix notations that will be used throughout this paper.

Given a (profinite) group G and a (continuous) G-module C, we write $C^G \stackrel{\text{def}}{=} H^0(G, C)$.

Let H be an abelian group. For an integer $N > 0$, we write $H/N \stackrel{\text{def}}{=} H/NH$ and $H[N] \stackrel{\text{def}}{=} \{h \in H \mid Nh = 0\}$. We write $H^\wedge \stackrel{\text{def}}{=} \varprojlim_{N>0} H/N$, and $H^{\text{prof}} \stackrel{\text{def}}{=} \varprojlim_{H' \subset H,\ (H:H')<\infty} H/H'$ for the profinite completion of H. Thus, we have natural homomorphisms $H \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \to H^\wedge \to H^{\text{prof}}$, which are isomorphisms when H is finitely generated. We write $H^{\text{tor}} \stackrel{\text{def}}{=} \bigcup_{N>0} H[N]$ for the torsion subgroup of H, and set $H/\{\text{tor}\} \stackrel{\text{def}}{=} H/H^{\text{tor}}$.

For a prime number l, we write $H^{\wedge,l} \stackrel{\text{def}}{=} \varprojlim_{n\geq 0} H/l^n$ for the l-adic completion of H, and $H^{\text{pro-}l} \stackrel{\text{def}}{=} \varprojlim_{H' \subset H,\ (H:H'):l\text{-power}} H/H'$ for the pro-l completion of H. Thus, we have natural homomorphisms $H \otimes_{\mathbb{Z}} \mathbb{Z}_l \to H^\wedge \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_l = H^{\wedge,l} \to H^{\text{pro-}l}$, which are isomorphisms when H is finitely generated. We write $H^{\text{tor},l}$ and $H^{\text{tor},l'}$ for the l-primary part and the prime-to-l part, respectively, of the torsion abelian group H^{tor}, and set $H/\{l\text{-tor}\} \stackrel{\text{def}}{=} H/H^{\text{tor},l}$ and $H/\{l'\text{-tor}\} \stackrel{\text{def}}{=} H/H^{\text{tor},l'}$.

Let $\mathfrak{Primes}$ be the set of all prime numbers. For a nonempty subset $\Sigma \subset \mathfrak{Primes}$, we say that N is a Σ-integer if N is divisible only by primes in Σ. We set $\widehat{\mathbb{Z}}^{\Sigma} \stackrel{\text{def}}{=} \varprojlim_{N:\Sigma\text{-integer}>0} \mathbb{Z}/N = \prod_{l\in\Sigma} \mathbb{Z}_l$, write $H^{\wedge,\Sigma} \stackrel{\text{def}}{=} \varprojlim_{N:\Sigma\text{-integer}>0} H/N = \prod_{l\in\Sigma} H^{\wedge,l}$ (thus, $H^{\wedge,\{l\}} = H^{\wedge,l}$) for the Σ-adic completion of H, and write $T^{\Sigma}H \stackrel{\text{def}}{=} \varprojlim_{N:\Sigma\text{-integer}>0} H[N] = \prod_{l\in\Sigma} T_l H$ (where $T_l H \stackrel{\text{def}}{=} T^{\{l\}}H$) for the Σ-adic Tate module of H. (Note that $T^{\Sigma}H$ is always torsion-free.) We write $H^{\text{tor},\Sigma} \stackrel{\text{def}}{=} \bigcup_{N:\Sigma\text{-integer}>0} H[N] = \oplus_{l\in\Sigma} H^{\text{tor},l}$ for the Σ-primary torsion subgroup of H, and set $H/\{\Sigma\text{-tor}\} \stackrel{\text{def}}{=} H/H^{\text{tor},\Sigma}$. Set $\Sigma' \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \Sigma$.

Let B be an abelian variety over a field κ of characteristic $p \geq 0$ with algebraic closure $\overline{\kappa}$ and separable closure $\kappa^{\text{sep}} \subset \overline{\kappa}$. We write $B[N]$, B^{tor}, $T^{\Sigma}B$, $T_l B$ instead of $B(\kappa^{\text{sep}})[N]$, $B(\kappa^{\text{sep}})^{\text{tor}}$, $T^{\Sigma}B(\kappa^{\text{sep}})$, $T_l B(\kappa^{\text{sep}})$, respectively. Write $\mathfrak{Primes}^{\dagger} \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \{p\}$. For a nonempty subset $\Sigma \subset \mathfrak{Primes}^{\dagger}$, recall the Kummer exact sequence

$$0 \to B(\kappa)^{\wedge,\Sigma} \to H^1(G_\kappa, T^{\Sigma}B) \to T^{\Sigma}H^1(G_\kappa, B) \to 0, \tag{0.3}$$

where $G_\kappa \stackrel{\text{def}}{=} \text{Gal}(\kappa^{\text{sep}}/\kappa)$ is the absolute Galois group of κ and $H^1(G_\kappa, B) \stackrel{\text{def}}{=} H^1(G_\kappa, B(\kappa^{\text{sep}}))$ is the first Galois cohomology group, which arises from the Kummer exact sequence of G_κ-modules

$$0 \to B[N] \to B(\kappa^{\text{sep}}) \xrightarrow{N} B(\kappa^{\text{sep}}) \to 0,$$

where N denotes the map of multiplication by a Σ-integer $N > 0$. Note that the above sequence (0.3) induces a natural isomorphism $(B(\kappa)^{\wedge,\Sigma})^{\text{tor}} \xrightarrow{\sim} H^1(G_\kappa, T^{\Sigma}B)^{\text{tor}}$, as $T^{\Sigma}H^1(G_\kappa, B)$ is torsion-free.

§1. A Specialisation Theorem for H^1. Let S be a locally noetherian, regular, integral scheme. Write K for the function field of S, $k(t)$ for the residue field of S at $t \in S$, and $p_t (\geq 0)$ for the characteristic of $k(t)$. Write $\text{char}(S) \stackrel{\text{def}}{=} \{p_t \mid t \in S\} \subset \mathfrak{Primes} \cup \{0\}$. Let $\mathcal{A} \to S$ be an *abelian scheme* over S. We write $A \stackrel{\text{def}}{=} \mathcal{A}_K \stackrel{\text{def}}{=} \mathcal{A} \times_S \operatorname{Spec} K$ for the *generic fibre* of $\mathcal{A}$, and, for each $t \in S$, we write $\mathcal{A}_t \stackrel{\text{def}}{=} \mathcal{A} \times_S \operatorname{Spec} k(t)$ for the fibre of $\mathcal{A}$ at t. Thus, A (resp. $\mathcal{A}_t$) is an abelian variety over K (resp. over $k(t)$).

Let η be a geometric point of S with values in the generic point of S. Then η determines an algebraic closure $\overline{K}$ and a separable closure K^{sep} of K, Write $G_K = \text{Gal}(K^{\text{sep}}/K)$ for the absolute Galois group of K, and $\pi_1(S) = \pi_1(S, \eta)$ for the étale fundamental group of S. Thus, we have a natural exact sequence of profinite groups

$$1 \to I_S \to G_K \to \pi_1(S) \to 1$$

where I_S is defined so that the sequence is exact.

Write S^1 for the set of points of codimension 1 of S. For each $t \in S^1$, the local ring $\mathcal{O}_{S,t}$ is a discrete valuation ring, and let $(G_K \supset) D_t \supset I_t$ be a *decomposition group* and an *inertia group* associated to t. Thus, D_t and I_t are only defined up to conjugation in G_K. We have a natural exact sequence

$$1 \to I_t \to D_t \to G_{k(t)} \to 1$$

where $G_{k(t)} \stackrel{\text{def}}{=} \mathrm{Gal}(k(t)^{\text{sep}}/k(t))$. Then, by purity, the group I_S is (topologically) normally generated by the subgroups I_t, where t runs over all points in S^1. We have a natural exact sequence

$$1 \to I_t^{\text{w}} \to I_t \to I_t^{\text{t}} \to 1,$$

where the wild inertia group I_t^{w} is defined to be the unique Sylow-p_t subgroup of I_t (resp. the trivial subgroup $\{1\} \subset I_t$) for $p_t > 0$ (resp. $p_t = 0$), and the tame inertia group I_t^{t} is defined by $I_t^{\text{t}} \stackrel{\text{def}}{=} I_t/I_t^{\text{w}}$. Note that I_t^{t} is naturally isomorphic to $\widehat{\mathbb{Z}}^{p_t'}(1)$, where $\widehat{\mathbb{Z}}^{p_t'} \stackrel{\text{def}}{=} \widehat{\mathbb{Z}}^{\mathfrak{Primes}\setminus\{p_t\}}$, and the "(1)" denotes a Tate twist.

Lemma 1.1. *Let $\Sigma \subset \mathfrak{Primes} \setminus \mathrm{char}(S)$ be a nonempty subset. Then:*
(i) *The G_K-module $T^\Sigma A$ (hence, in particular, $\prod_{l\in\Sigma} A[l]$) has a natural structure of $\pi_1(S)$-module.*
(ii) *For each $l \in \Sigma$, write $\pi_1(S)[A,l]$ for the kernel of the natural map $\pi_1(S) \to \mathrm{Aut}(A[l])$ (cf. (i)), $\pi_1(S)[A,l]^l$ for the maximal pro-l quotient of $\pi_1(S)[A,l]$, and $\pi_1(S)(A,l)$ for the kernel of the natural surjection $\pi_1(S)[A,l] \twoheadrightarrow \pi_1(S)[A,l]^l$. Define $\pi_1(S)[A,\Sigma] \stackrel{\text{def}}{=} \bigcap_{l\in\Sigma} \pi_1(S)[A,l]$, and $\pi_1(S)(A,\Sigma) \stackrel{\text{def}}{=} \bigcap_{l\in\Sigma} \pi_1(S)(A,l)$, where the intersection is over all prime integers $l \in \Sigma$. Further, let $\Pi_S^{A,\Sigma} \stackrel{\text{def}}{=} \pi_1(S)/\pi_1(S)(A,\Sigma)$. (Note that $\pi_1(S)(A,\Sigma)$ is a normal subgroup of $\pi_1(S)$ since $\pi_1(S)(A,l)$ is a characteristic subgroup of $\pi_1(S)[A,l]$.) Thus, we have a natural exact sequence*

$$1 \to \pi_1(S)[A,\Sigma]/\pi_1(S)(A,\Sigma) \to \Pi_S^{A,\Sigma} \to \pi_1(S)/\pi_1(S)[A,\Sigma] \to 1,$$

where

$$\pi_1(S)/\pi_1(S)[A,\Sigma] \hookrightarrow \prod_{l\in\Sigma}(\pi_1(S)/\pi_1(S)[A,l]) \hookrightarrow \prod_{l\in\Sigma}\mathrm{Aut}(A[l])$$

and

$$\pi_1(S)[A,\Sigma]/\pi_1(S)(A,\Sigma) = \prod_{l\in\Sigma}\mathrm{Im}(\pi_1(S)[A,\Sigma] \to \pi_1(S)[A,l]^l).$$

Then the $\pi_1(S)$-module $T^\Sigma A$ has a natural structure of $\Pi_S^{A,\Sigma}$-module.
(iii) *The natural inflation map $H^1(\Pi_S^{A,\Sigma}, T^\Sigma A) \to H^1(\pi_1(S), T^\Sigma A)$ is an isomorphism. The natural inflation map $H^1(\pi_1(S), T^\Sigma A) \to H^1(G_K, T^\Sigma A)$ is an injection in general, and an isomorphism if $k(t)$ is finitely generated over the prime field for each $t \in S^1$.*
(iv) *Let N be a Σ-integer > 0. Then the natural inflation map $H^1(\Pi_S^{A,\Sigma}, A[N]) \to H^1(\pi_1(S), A[N])$ is an isomorphism and the natural inflation map $H^1(\pi_1(S), A[N]) \to H^1(G_K, A[N])$ is an injection.*

Proof. (i) For each $t \in S^1$, any inertia group I_t associated to t acts trivially on $T^\Sigma A$, as follows from the well-known criterion of good reduction for abelian varieties (cf. [Serre-Tate], Theorem 1). Thus, I_S acts trivially on $T^\Sigma A$, and $T^\Sigma A$ has a natural structure of $\pi_1(S)$-module.
(ii) This follows from (i) and the fact that, for each $l \in \Sigma$, $\mathrm{Ker}(\mathrm{Aut}(T_l A) \to \mathrm{Aut}(A[l]))$ ($\simeq \mathrm{Ker}(\mathrm{GL}_{2d}(\mathbb{Z}_l) \to \mathrm{GL}_{2d}(\mathbb{F}_l))$, where $d \stackrel{\text{def}}{=} \dim A$) is pro-$l$.
(iii) First, note that the inflation maps for various H^1 are always injective, and that various H^1 with coefficients in $T^\Sigma A$ decompose into the direct product of H^1 with coefficients in $T_l A$ for $l \in \Sigma$. Thus, it suffices to prove, for each $l \in \Sigma$, that

the inflation map $H^1(\Pi_S^{A,l}, T_lA) \to H^1(\pi_1(S), T_lA)$ is an isomorphism in general (where $\Pi_S^{A,l} \stackrel{\text{def}}{=} \Pi_S^{A,\{l\}}$), and that the inflation map $H^1(\pi_1(S), T_lA) \to H^1(G_K, T_lA)$ is an isomorphism if $k(t)$ is finitely generated over the prime field for each $t \in S^1$.

For the first assertion, consider the inflation-restriction exact sequence

$$0 \to H^1(\Pi_S^{A,l}, T_lA) \xrightarrow{\text{inf}} H^1(\pi_1(S), T_lA) \xrightarrow{\text{res}} H^1(\pi_1(S)(A,l), T_lA)^{\Pi_S^{A,l}}.$$

We claim that $H^1(\pi_1(S)(A,l), T_lA) = 0$. Indeed, this follows from the fact that $H^1(\pi_1(S)(A,l), T_lA) = \text{Hom}(\pi_1(S)(A,l), T_lA)$, and $\pi_1(S)(A,l) = \text{Ker}(\pi_1(S)[A,l] \twoheadrightarrow \pi_1(S)[A,l]^l)$.

For the second assertion, consider the inflation-restriction exact sequence

$$0 \to H^1(\pi_1(S), T_lA) \xrightarrow{\text{inf}} H^1(G_K, T_lA) \xrightarrow{\text{res}} H^1(I_S, T_lA)^{\pi_1(S)}.$$

We claim that $H^1(I_S, T_lA)^{\pi_1(S)} = 0$. Indeed, this follows from the fact that $H^1(I_S, T_lA) = \text{Hom}(I_S, T_lA)$ and that, for each $t \in S^1$, one has $\text{Hom}(I_t, T_lA)^{G_{k(t)}} = \text{Hom}(I_t^{\text{t},l}, T_lA)^{G_{k(t)}} = 0$, where $I_t^{\text{t},l}$ is the maximal pro-l quotient of I_t^{t}. This last statement follows from the fact that the action of $G_{k(t)}$ on $I_t^{\text{t},l} \xrightarrow{\sim} \mathbb{Z}_l(1)$ and T_lA has different weights. More precisely, $T_l\mathcal{A}_t \simeq T_lA$ as $G_{k(t)}$-modules, and the $G_{k(t)}$-representation $I_t^{\text{t},l} \otimes \mathbb{Q}_l$ (resp. $T_l\mathcal{A}_t \otimes \mathbb{Q}_l$) is pure of weight -2 (resp. pure of weight -1) since $k(t)$ is finitey generated (cf. [Jannsen], 2). Thus, $\text{Hom}(I_t^{\text{t},l}, T_l\mathcal{A}_t)^{G_{k(t)}} = 0$ follows (cf. loc. cit., Fact 2).

(iv) As the inflation maps for various H^1 are always injective and various H^1 with coefficients in $A[N]$ decompose into the direct product of H^1 with coefficients in $A[l^r]$ for $l \in \Sigma$ and some $r \geq 0$, it suffices to prove, for each $l \in \Sigma$, that the inflation map $H^1(\Pi_S^{A,l}, A[l^r]) \to H^1(\pi_1(S), A[l^r])$ is an isomorphism. For this, consider the inflation-restriction exact sequence

$$0 \to H^1(\Pi_S^{A,l}, A[l^r]) \xrightarrow{\text{inf}} H^1(\pi_1(S), A[l^r]) \xrightarrow{\text{res}} H^1(\pi_1(S)(A,l), A[l^r])^{\Pi_S^{A,l}}.$$

We claim that $H^1(\pi_1(S)(A,l), A[l^r]) = 0$. Indeed, this follows from the fact that $H^1(\pi_1(S)(A,l), A[l^r]) = \text{Hom}(\pi_1(S)(A,l), A[l^r])$, and $\pi_1(S)(A,l) = \text{Ker}(\pi_1(S)[A,l] \twoheadrightarrow \pi_1(S)[A,l]^l)$. $\square$

Next, let k be a field of characteristic $p \geq 0$, and $C \to \text{Spec}\, k$ a smooth, separated and geometrically connected (not necessarily proper) *algebraic curve* over k. Write $K = k(C)$ for the function field of C, C^{cl} for the set of closed points of C (which coincides with the set C^1 of codimension 1 of C), and $k(c)$ for the residue field of C at $c \in C^{\text{cl}}$. Let $\mathcal{A} \to C$ be an *abelian scheme* over C. We write $A \stackrel{\text{def}}{=} \mathcal{A}_K \stackrel{\text{def}}{=} \mathcal{A} \times_C \text{Spec}\, K$ for the *generic fibre* of $\mathcal{A}$, and, for each $c \in C^{\text{cl}}$, we write $\mathcal{A}_c \stackrel{\text{def}}{=} \mathcal{A} \times_C \text{Spec}\, k(c)$ for the fibre of $\mathcal{A}$ at c. Thus, A (resp. $\mathcal{A}_c$) is an abelian variety over K (resp. over $k(c)$).

Let η be a geometric point of C with values in the generic point of C. Then η determines algebraic closures $\overline{K}$ and $\overline{k}$ and separable closures K^{sep} and k^{sep} of K and k, respectively, and a geometric point $\overline{\eta}$ of $C_{\overline{k}} \stackrel{\text{def}}{=} C \times_k \overline{k}$. Write $G_k = \text{Gal}(k^{\text{sep}}/k)$, $G_K = \text{Gal}(K^{\text{sep}}/K)$ and $G_{Kk^{\text{sep}}} = \text{Gal}(K^{\text{sep}}/Kk^{\text{sep}})$ for the absolute Galois groups of k, K and Kk^{sep}, respectively. Write $\pi_1(C) = \pi_1(C, \eta)$ and $\pi_1(C_{\overline{k}}) = \pi_1(C_{\overline{k}}, \overline{\eta})$

for the étale fundamental groups of C and $C_{\overline{k}}$, respectively. Thus, we have natural exact sequences of profinite groups

$$1 \to \pi_1(C_{\overline{k}}) \to \pi_1(C) \to G_k \to 1, \tag{1.1}$$

$$1 \to G_{Kk^{\mathrm{sep}}} \to G_K \to G_k \to 1, \tag{1.2}$$

and

$$1 \to I_C \to G_K \to \pi_1(C) \to 1, \tag{1.3}$$

where I_C is defined so that sequence (1.3) is exact. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & I_C & = & I_C & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & G_{Kk^{\mathrm{sep}}} & \longrightarrow & G_K & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \pi_1(C_{\overline{k}}) & \longrightarrow & \pi_1(C) & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

For each $c \in C^{\mathrm{cl}}$, write K_c (resp. $\widehat{\mathcal{O}}_{C,c}$) for the completion of K (resp. $\mathcal{O}_{C,c}$) at c, and $A_c \overset{\mathrm{def}}{=} A \times_K K_c$. Thus, K_c (resp. $\widehat{\mathcal{O}}_{C,c}$) is a complete discrete valuation field (resp. ring) of equal characteristic $p \geq 0$ with residue field $k(c)$, and A_c is an abelian variety over K_c. Let $(G_K \supset)\ D_c \supset I_c$ be a *decomposition group* and an *inertia group* associated to c. Thus, D_c and I_c are only defined up to conjugation in G_K. We have a natural exact sequence

$$1 \to I_c \to D_c \to G_{k(c)} \to 1, \tag{1.4}$$

where $G_{k(c)} \overset{\mathrm{def}}{=} \mathrm{Gal}(k(c)^{\mathrm{sep}}/k(c))$ is identified with the image of D_c in G_k (cf. sequence (1.2)). Then the group I_C (cf. sequence (1.3)) is (topologically) normally generated by the subgroups I_c, where c runs over all points in C^{cl}. We have a natural exact sequence

$$1 \to I_c^{\mathrm{w}} \to I_c \to I_c^{\mathrm{t}} \to 1,$$

where the wild inertia group I_c^{w} is defined to be the unique Sylow-p subgroup of I_c (resp. the trivial subgroup $\{1\} \subset I_c$) for $p > 0$ (resp. $p = 0$), and the tame inertia group I_c^{t} is defined by $I_c^{\mathrm{t}} \overset{\mathrm{def}}{=} I_c/I_c^{\mathrm{w}}$. Note that I_c^{t} is naturally isomorphic to $\widehat{\mathbb{Z}}^{\dagger}(1)$, where $\mathfrak{Primes}^{\dagger} \overset{\mathrm{def}}{=} \mathfrak{Primes} \setminus \{p\}$, $\widehat{\mathbb{Z}}^{\dagger} \overset{\mathrm{def}}{=} \widehat{\mathbb{Z}}^{\mathfrak{Primes}^{\dagger}}$, and the "(1)" denotes a Tate twist.

For each $c \in C^{\mathrm{cl}}$, we have a natural commutative diagram (up to conjugation):

$$\begin{array}{ccccc} G_{K_c} & \overset{\sim}{\to} & D_c & \subset & G_K \\ & & \downarrow & & \downarrow \\ & & G_{k(c)} & \overset{s_c}{\hookrightarrow} & \pi_1(C) \end{array}$$

where the vertical arrows are natural surjections and the horizontal arrows are natural injections. (The map $s_c : G_{k(c)} = \pi_1(\operatorname{Spec} k(c)) \to \pi_1(C)$ is associated to the natural morphism $\operatorname{Spec} k(c) \to C$ with image c by functoriality of π_1.) Further, this diagram induces natural commutative diagrams

$$\begin{array}{ccc} H^1(\pi_1(C), A[N]) & \hookrightarrow & H^1(G_K, A[N]) \\ \downarrow & & \downarrow \\ H^1(G_{k(c)}, \mathcal{A}_c[N]) & \hookrightarrow & H^1(G_{K_c}, A_c[N]) \end{array}$$

for each $\mathfrak{Primes}^\dagger$-integer $N > 0$, and

$$\begin{array}{ccc} H^1(\pi_1(C), T^\Sigma A) & \hookrightarrow & H^1(G_K, T^\Sigma A) \\ \downarrow & & \downarrow \\ H^1(G_{k(c)}, T^\Sigma \mathcal{A}_c) & \hookrightarrow & H^1(G_{K_c}, T^\Sigma A_c) \end{array}$$

for each nonempty subset $\Sigma \subset \mathfrak{Primes}^\dagger$, where the horizontal arrows are inflation maps and the vertical arrows are natural restriction maps.

One of our main results in this section is the following.

Proposition 1.2. *Let* $\Sigma \subset \mathfrak{Primes}^\dagger$ *be a* **finite** *subset. Assume that* k *is* **finitely generated** *over the prime field and infinite. Then there exists a closed point* $c \overset{\mathrm{def}}{=} c(\Sigma) \in C^{\mathrm{cl}}$*, depending on (*$A$ *and)* Σ*, such that the natural restriction map* $H^1(\pi_1(C), T^\Sigma A) \to H^1(G_{k(c)}, T^\Sigma \mathcal{A}_c)$ *is* **injective***.*

Proof. For simplicity, write Q for the prime field of k and Z for the image of $\mathbb{Z}[\frac{1}{l}\ ;\ l \in \Sigma]$ in Q. Thus, we have $Q = \mathbb{Q}$ (resp. $Q = \mathbb{F}_p$) and $Z = \mathbb{Z}[\frac{1}{l}\ ;\ l \in \Sigma]$ (resp. $Z = \mathbb{F}_p$) when $p = 0$ (resp. $p > 0$). Then, as k is finitely generated over the perfect field Q, the system $\mathcal{A} \to C \to \operatorname{Spec} k \to \operatorname{Spec} Q$ admits a smooth model $\widetilde{\mathcal{A}} \to \mathcal{C} \to V \to U$. More precisely, $U = \operatorname{Spec} Z$; V is an integral scheme which is smooth over U and whose function field is isomorphic to (and is identified with) k; $\mathcal{C}$ is a smooth scheme over V whose generic fibre $\mathcal{C} \times_V k$ is k-isomorphic to (and is identified with) C; and $\widetilde{\mathcal{A}} \to \mathcal{C}$ is an abelian scheme such that $\widetilde{\mathcal{A}} \times_{\mathcal{C}} C$ is isomorphic to (and is identified with) $\mathcal{A} \to C$. Let $\pi_1(\mathcal{C})$ be the fundamental group of $\mathcal{C}$, with respect to the base point which is induced by the base point η in the beginning of §1. Thus, there exists a natural continuous surjective homomorphism $\pi_1(C) \twoheadrightarrow \pi_1(\mathcal{C}) \twoheadrightarrow \Pi_{\mathcal{C}}^{A,\Sigma}$. By Lemma 1.1 (ii), $T^\Sigma A$ has a natural structure of $\Pi_{\mathcal{C}}^{A,\Sigma}$-module, and, by Lemma 1.1 (iii), the inflation maps

$$H^1(\Pi_{\mathcal{C}}^{A,\Sigma}, T^\Sigma A) \to H^1(\pi_1(\mathcal{C}), T^\Sigma A) \to H^1(\pi_1(C), T^\Sigma A) \to H^1(G_K, T^\Sigma A)$$

are all isomorphisms. Recall that we have a natural exact sequence

$$1 \to \pi_1(\mathcal{C})[A,\Sigma]/\pi_1(\mathcal{C})(A,\Sigma) \to \Pi_{\mathcal{C}}^{A,\Sigma} \to \pi_1(\mathcal{C})/\pi_1(\mathcal{C})[A,\Sigma] \to 1,$$

where

$$\pi_1(\mathcal{C})/\pi_1(\mathcal{C})[A,\Sigma] \hookrightarrow \prod_{l\in\Sigma}(\pi_1(\mathcal{C})/\pi_1(\mathcal{C})[A,l]) \hookrightarrow \prod_{l\in\Sigma}\mathrm{Aut}(A[l])$$

and

$$\pi_1(\mathcal{C})[A,\Sigma]/\pi_1(\mathcal{C})(A,\Sigma) = \prod_{l\in\Sigma}\mathrm{Im}(\pi_1(\mathcal{C})[A,\Sigma] \to \pi_1(\mathcal{C})[A,l]^l).$$

In particular, as Σ is finite, $\pi_1(\mathcal{C})/\pi_1(\mathcal{C})[A,\Sigma]$ is finite.

Claim: The Frattini subgroup of $\Pi_{\mathcal{C}}^{A,\Sigma}$ is open.

Indeed, as $\pi_1(\mathcal{C})/\pi_1(\mathcal{C})[A,\Sigma]$ is finite and $\pi_1(\mathcal{C})[A,\Sigma]/\pi_1(\mathcal{C})(A,\Sigma)$ is a direct product of pro-l groups for $l \in \Sigma$, it suffices to prove that $\pi_1(\mathcal{C})[A,\Sigma]^l$ is a finitely generated pro-l group for each $l \in \Sigma$ (cf. [Serre2], 10.6, Proposition), or, equivalently, that $\pi_1(\mathcal{C})[A,\Sigma]^{\mathrm{ab}}/l$ is finite. Let $\mathcal{C}' \to \mathcal{C}$ be the finite étale cover corresponding to the open subgroup $\pi_1(\mathcal{C})[A,\Sigma] \subset \pi_1(\mathcal{C})$, so that $\pi_1(\mathcal{C})[A,\Sigma]$ is identified with $\pi_1(\mathcal{C}')$. Now, the desired finiteness follows from [Katz-Lang]. More precisely, let Q' be the algebraic closure of Q in k, which is finite over Q, and Z' the integral closure of Z in Q'. Then the morphism $\mathcal{C}' \to \mathrm{Spec}\, Z$ factors as $\mathcal{C}' \to \mathrm{Spec}\, Z' \to \mathrm{Spec}\, Z$, and the morphism $\mathcal{C}' \times_Z Q \to \mathrm{Spec}\, Q$ factors as $\mathcal{C}' \times_Z Q \to \mathrm{Spec}\, Q' \to \mathrm{Spec}\, Q$. The morphism $\mathcal{C}' \times_Z Q \to \mathrm{Spec}\, Q'$ is smooth and geometrically connected, hence there exist an open subscheme $\mathcal{C}'_1 \subset \mathcal{C}'$ (containing $\mathcal{C}' \times_Q Q'$) and an open subscheme $W \subset \mathrm{Spec}\, Z'$, such that $\mathcal{C}' \to \mathrm{Spec}\, Z'$ induces a smooth, surjective morphism $\mathcal{C}'_1 \to W$ of finite type with geometrically connected generic fibre. Now, by [Katz-Lang], Lemma 2 (2), we have an exact sequence

$$0 \to \mathrm{Ker}(\mathcal{C}'_1/W) \to \pi_1(\mathcal{C}'_1)^{\mathrm{ab}} \to \pi_1(W)^{\mathrm{ab}} \to 0,$$

where $\mathrm{Ker}(\mathcal{C}'_1/W) \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\pi_1(\mathcal{C}'_1)^{\mathrm{ab}} \to \pi_1(W)^{\mathrm{ab}})$, hence an exact sequence

$$\mathrm{Ker}(\mathcal{C}'_1/W)/l \to \pi_1(\mathcal{C}'_1)^{\mathrm{ab}}/l \to \pi_1(W)^{\mathrm{ab}}/l \to 0.$$

By [Katz-Lang], Theorem 1 (together with the fact that $l \in \mathfrak{Primes}^\dagger$), $\mathrm{Ker}(\mathcal{C}'_1/W)/l$ is finite, and $\pi_1(W)^{\mathrm{ab}}/l$ is finite by global class field theory (cf. [Katz-Lang], Proof of Theorem 4) (resp. as $\pi_1(W) \simeq \widehat{\mathbb{Z}}$) when $p = 0$ (resp. $p > 0$). Thus, $\pi_1(\mathcal{C}'_1)^{\mathrm{ab}}/l$ is finite, hence so is $\pi_1(\mathcal{C}')^{\mathrm{ab}}/l$ ($\twoheadleftarrow \pi_1(\mathcal{C}'_1)^{\mathrm{ab}}/l$). This finishes the proof of the claim.

By this claim and Hilbert's irreducibility theorem (cf. [Serre2], 10.6), there exists $c \in C^{\mathrm{cl}}$, such that the composite map $D_c \hookrightarrow \pi_1(C) \twoheadrightarrow \Pi_{\mathcal{C}}^{A,\Sigma}$, where $D_c \subset \pi_1(C)$ is a decomposition group at c (thus, D_c is only defined up to conjugation and $D_c \xrightarrow{\sim} G_{k(c)}$), is surjective. Hence, the natural map $H^1(\Pi_{\mathcal{C}}^{A,S}, T^\Sigma A) \xrightarrow{\sim} H^1(\pi_1(C), T^\Sigma A) \to H^1(D_c, T^\Sigma A)$ $(= H^1(G_{k(c)}, T^\Sigma \mathcal{A}_c))$ is injective, as desired. $\square$

Remark 1.3. A similar statement as in Proposition 1.2 holds when $T^{\Sigma}A$ is replaced by any finitely generated torsion-free $\widehat{\mathbb{Z}}^{\Sigma}$-module M on which $\pi_1(C)$ acts such that the action of $\pi_1(C)$ factors through $\pi_1(C) \twoheadrightarrow \pi_1(\mathcal{C})$ for some model $\mathcal{C}$ of C and that the weights of the Galois representation associated to M are distinct from the weight of the cyclotomic character.

More generally, we have the following result which generalises Proposition 1.2 to a wider class of base fields.

Proposition 1.4. *Let $\Sigma \subset \mathfrak{Primes}^{\dagger}$ be a* **finite** *subset. Assume that k is* **Hilbertian** *(cf [Serre2], 9.5). Then there exists a* **finite** *subset $S \stackrel{\text{def}}{=} S(\Sigma) \subset C^{\text{cl}}$ of cardinality ≤ 2, depending on (A and) Σ, such that the natural restriction map*

$$H^1(\pi_1(C), T^{\Sigma}A) \to \prod_{c \in S} H^1(G_{k(c)}, T^{\Sigma}\mathcal{A}_c)$$

is **injective**. *Moreover, in the case where the $\overline{k}$-trace $\text{Tr}_{K\overline{k}/\overline{k}}(A_{K\overline{k}})$ of $A_{K\overline{k}} \stackrel{\text{def}}{=} A \times_K K\overline{k}$ is trivial, there exists such a set S with $\sharp(S) = 1$.*

First, we prove the following.

Proposition 1.5. *Let $\Sigma \subset \mathfrak{Primes}^{\dagger}$ be a* **finite** *subset. Assume that k is* **Hilbertian**. *Let $M \subset H^1(\pi_1(C), T^{\Sigma}A)$ be a* **finitely generated** *$\widehat{\mathbb{Z}}^{\Sigma}$-submodule. Then there exists a closed point $c \in C^{\text{cl}}$, depending on Σ and M, such that the natural restriction map $M \to H^1(G_{k(c)}, T^{\Sigma}\mathcal{A}_c)$ is* **injective**.

Proof. Let L be the kernel of the natural map $\pi_1(C) \to \text{Aut}(T^{\Sigma}A) = \prod_{l \in \Sigma} \text{Aut}(T_l A)$. By definition, the action of $\pi_1(C)$ on $T^{\Sigma}A$ factors through $\pi_1(C) \twoheadrightarrow \pi_1(C)/L$, hence we have the following exact sequence:

$$\begin{array}{c} 0 \to H^1(\pi_1(C)/L, T^{\Sigma}A) \to H^1(\pi_1(C), T^{\Sigma}A) \to H^0(\pi_1(C)/L, H^1(L, T^{\Sigma}A)) \\ \| \\ \text{Hom}_{\pi_1(C)}(L, T^{\Sigma}A) \\ \cap \\ \text{Hom}(L, T^{\Sigma}A). \end{array}$$

Restricting to M, we get a homomorphism $M \to \text{Hom}(L, T^{\Sigma}A)$ or, equivalently, a homomorphism $L \to \text{Hom}(M, T^{\Sigma}A)$. Let L_M be the kernel of $L \to \text{Hom}(M, T^{\Sigma}A)$.

Claim: (i) $L_M \subset \pi_1(C)$ is a closed normal subgroup.
(ii) The Frattini subgroup of $\pi_1(C)/L_M$ is open.
(iii) $M \subset H^1(\pi_1(C), T^{\Sigma}A)$ is contained in the image of the inflation map

$$H^1(\pi_1(C)/L_M, T^{\Sigma}A) \hookrightarrow H^1(\pi_1(C), T^{\Sigma}A).$$

Indeed, as $\text{Hom}(L, T^{\Sigma}A)$ is the set of *continuous* homomorphisms from L to $T^{\Sigma}A$, the subgroup L_M is closed. As the image of $M \to \text{Hom}(L, T^{\Sigma}A)$ is contained in $\text{Hom}_{\pi_1(C)}(L, T^{\Sigma}A)$, the subgroup L_M is normal (not only in L but also) in $\pi_1(C)$. Thus, (i) follows. Next, we have the following exact sequence of profinite groups:

$$1 \to L/L_M \to \pi_1(C)/L_M \to \pi_1(C)/L \to 1.$$

As

$$\pi_1(C)/L \hookrightarrow \mathrm{Aut}(T^\Sigma A) = \prod_{l\in\Sigma} \mathrm{Aut}(T_l A) \simeq \prod_{l\in\Sigma} \mathrm{GL}_{2d}(\mathbb{Z}_l)$$

(where $d \stackrel{\mathrm{def}}{=} \dim(A)$) and

$$L/L_M \hookrightarrow \mathrm{Hom}(M, T^\Sigma A) = \prod_{l\in\Sigma} \mathrm{Hom}(M^l, T_l A) \simeq \prod_{l\in\Sigma} \mathbb{Z}_l^{2dr_l}$$

(where $M^l \stackrel{\mathrm{def}}{=} M \otimes_{\widehat{\mathbb{Z}}^\Sigma} \mathbb{Z}_l$ and $r_l \stackrel{\mathrm{def}}{=} \dim_{\mathbb{Q}_l}(M^l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$), (ii) follows from [Serre2], 10.6, Proposition. Finally, we have the inflation-restriction exact sequence

$$0 \to H^1(\pi_1(C)/L_M, T^\Sigma A) \to H^1(\pi_1(C), T^\Sigma A) \to H^1(L_M, T^\Sigma A)$$

arising from the exact sequence $1 \to L_M \to \pi_1(C) \to \pi_1(C)/L_M \to 1$. By the very definition of L_M, the image of $M \subset H^1(\pi_1(C), T^\Sigma A)$ in $H^1(L_M, T^\Sigma A)$ is trivial. Thus, the assertion of (iii) follows. This finishes the proof of the claim.

As in the proof of Proposition 1.2, (ii) of the above claim and the Hilbertian property of k imply (cf. [Serre2], 10.6) that there exists $c \in C^{\mathrm{cl}}$, such that the composite map D_c ($\stackrel{\sim}{\to} G_{k(c)}$) $\hookrightarrow \pi_1(C) \twoheadrightarrow \pi_1(C)/L_M$ is surjective. Hence, the composite map $H^1(\pi_1(C)/L_M, T^\Sigma A) \hookrightarrow H^1(\pi_1(C), T^\Sigma A) \to H^1(D_c, T^\Sigma A)$ ($= H^1(G_{k(c)}, T^\Sigma \mathcal{A}_c)$) is injective. This, together with (iii) of the above claim, finishes the proof of Proposition 1.5. □

Proof of Proposition 1.4. We have the inflation-restriction exact sequence

$$0 \to H^1(G_k, T) \to H^1(\pi_1(C), T^\Sigma A) \to H^1(\pi_1(C_{\overline{k}}), T^\Sigma A)$$

arising from the exact sequence $1 \to \pi_1(C_{\overline{k}}) \to \pi_1(C) \to G_k \to 1$, where $T \stackrel{\mathrm{def}}{=} (T^\Sigma A)^{\pi_1(C_{\overline{k}})}$.

First, in the special case that $\mathrm{Tr}_{K\overline{k}/\overline{k}}(A_{K\overline{k}}) = 0$, $A(K\overline{k})$ is finitely generated by [Lang-Néron], hence, in particular, $A(K\overline{k})^{\mathrm{tor}}$ is finite and $T = (T^\Sigma A)^{\pi_1(C_{\overline{k}})} = T^\Sigma(A(K\overline{k})) = 0$. Thus, the restriction map $H^1(\pi_1(C), T^\Sigma A) \to H^1(\pi_1(C_{\overline{k}}), T^\Sigma A)$ is injective. The $\widehat{\mathbb{Z}}^\Sigma$-module $H^1(\pi_1(C_{\overline{k}}), T^\Sigma A) \simeq H^1(\Pi^{A,\Sigma}_{C_{\overline{k}}}, T^\Sigma A)$ (cf. Lemma 1.1 (iii)) is finitely generated by Lemma 1.6 (ii) below, since $\Pi^{A,\Sigma}_{C_{\overline{k}}}$ is finitely generated as a profinite group (cf. [Grothendieck], Exposé XIII, Corollaire 2.12. Note that $\Sigma \subset \mathfrak{Primes}^\dagger$). As $\widehat{\mathbb{Z}}^\Sigma$ is noetherian, $H^1(\pi_1(C), T^\Sigma A) \hookrightarrow H^1(\pi_1(C_{\overline{k}}), T^\Sigma A)$ is also finitely generated. Thus, the assertion follows from Proposition 1.5 in this case.

Lemma 1.6. (i) *Let Δ be a finitely generated group. Let M be a finitely generated $\mathbb{Z}$-module on which Δ acts. Then $H^1(\Delta, M)$ is a finitely generated $\mathbb{Z}$-module.*

If, moreover, either Δ or M is finite, then $H^1(\Delta, M)$ is finite.

(ii) *Let $\Sigma \subset \mathfrak{Primes}$ be any subset. Let Δ be a finitely generated profinite group. Let M be a finitely generated $\widehat{\mathbb{Z}}^\Sigma$-module on which Δ acts continuously. Then*

$$H^1(\Delta, M) \stackrel{\mathrm{def}}{=} \varprojlim_{N:\Sigma\text{-integer}>0} H^1(\Delta, M/N)$$

is a finitely generated $\widehat{\mathbb{Z}}^\Sigma$-module.

If, moreover, either Δ or M is finite, then $H^1(\Delta, M)$ is finite.

Proof. (i) Take a surjection $(\mathbb{Z})^{\oplus s} \twoheadrightarrow M$ of $\mathbb{Z}$-modules and a surjection $F_r \twoheadrightarrow \Delta$ of groups, where $F_r = \langle x_1, \ldots, x_r \rangle$ is a free group of finite rank r. Then we claim that $H^1(\Delta, M)$ is generated by (at most) rs elements as a $\mathbb{Z}$-module. Indeed, the inflation map $H^1(\Delta, M) \to H^1(F_r, M)$ is injective. By considering the standard resolution of the trivial F_r-module, we obtain $H^1(F_r, M) = M^{\oplus r}/(x_1 - 1, \ldots, x_r - 1)M$, which is generated by (at most) rs elements as a $\mathbb{Z}$-module. As $\mathbb{Z}$ is a PID, $H^1(\Delta, M) \hookrightarrow H^1(F_r, M)$ is also generated by (at most) rs elements as a $\mathbb{Z}$-module, as desired.

The second assertion follows from the first, together with the standard fact that $H^1(\Delta, M)$ is killed by $\sharp(\Delta)$ (resp. $\sharp(M)$) when $\sharp(\Delta) < \infty$ (resp. $\sharp(M) < \infty$).
(ii) Take a surjection $(\widehat{\mathbb{Z}}^\Sigma)^{\oplus s} \twoheadrightarrow M$ of $\widehat{\mathbb{Z}}^\Sigma$-modules and a surjection $\widehat{F}_r \twoheadrightarrow \Delta$ of profinite groups, where $\widehat{F}_r = \langle x_1, \ldots, x_r \rangle$ is a free profinite group of finite rank r. Then we claim that $H^1(\Delta, M)$ is generated by (at most) rs elements as a $\widehat{\mathbb{Z}}^\Sigma$-module. Indeed, write $M = \prod_{l \in \Sigma} M_l$ for the canonical decomposition corresponding to the decomposition $\widehat{\mathbb{Z}}^\Sigma = \prod_{l \in \Sigma} \mathbb{Z}_l$. Then we have $H^1(\Delta, M) = \prod_{l \in \Sigma} H^1(\Delta, M_l)$, hence it suffices to prove that $H^1(\Delta, M_l)$ is generated by (at most) rs elements as a $\mathbb{Z}_l$-module. Now, the inflation map $H^1(\Delta, M_l) \to H^1(\widehat{F}_r, M_l)$ is injective. By considering the standard resolution of the trivial $\widehat{F}_r$-module, we obtain $H^1(\widehat{F}_r, M_l) = M_l^{\oplus r}/(x_1 - 1, \ldots, x_r - 1)M_l$, which is generated by (at most) rs elements as a $\mathbb{Z}_l$-module. As $\mathbb{Z}_l$ is a PID, $H^1(\Delta, M_l) \hookrightarrow H^1(\widehat{F}_r, M_l)$ is also generated by (at most) rs elements as a $\mathbb{Z}_l$-module, as desired.

The second assertion follows from the first, together with the standard fact that $H^1(\Delta, M)$ is killed by $\sharp(\Delta)$ (resp. $\sharp(M)$) when $\sharp(\Delta) < \infty$ (resp. $\sharp(M) < \infty$). □

We shall return to the proof of Proposition 1.4. In general, fix any $c_0 \in C^{\mathrm{cl}}$. By Proposition 1.5, it suffices to prove that the kernel N of $H^1(\pi_1(C), T^\Sigma A) \to H^1(G_{k(c_0)}, T^\Sigma \mathcal{A}_{c_0})$ is finitely generated as a $\widehat{\mathbb{Z}}^\Sigma$-module. As we have already seen, the $\widehat{\mathbb{Z}}^\Sigma$-module $H^1(\pi_1(C_{\overline{k}}), T^\Sigma A)$ is finitely generated, hence, as $\widehat{\mathbb{Z}}^\Sigma$ is noetherian, the image of N in $H^1(\pi_1(C_{\overline{k}}), T^\Sigma A)$ is also finitely generated. Thus, it suffices to prove that the intersection of N and (the image of) $H^1(G_k, T)$ in $H^1(\pi_1(C), T^\Sigma A)$ is finitely generated. Here, we have

$$N \cap H^1(G_k, T) = \mathrm{Ker}(H^1(G_k, T) \to H^1(G_{k(c_0)}, T^\Sigma \mathcal{A}_{c_0})).$$

Since the map $H^1(G_k, T) \to H^1(G_{k(c_0)}, T^\Sigma \mathcal{A}_{c_0})$ factors as

$$H^1(G_k, T) \to H^1(G_{k(c_0)}, T) \to H^1(G_{k(c_0)}, T^\Sigma \mathcal{A}_{c_0}),$$

it suffices to prove that both $N_1 \stackrel{\mathrm{def}}{=} \mathrm{Ker}(H^1(G_k, T) \to H^1(G_{k(c_0)}, T))$ and $N_2 \stackrel{\mathrm{def}}{=} \mathrm{Ker}(H^1(G_{k(c_0)}, T) \to H^1(G_{k(c_0)}, T^\Sigma \mathcal{A}_{c_0}))$ are finitely generated (as $\widehat{\mathbb{Z}}^\Sigma$ is noetherian).

To prove that N_1 is finitely generated, let k_1/k be the normal closure of the finite extension $k(c_0)/k$. Then

$$N_1 \subset \mathrm{Ker}(H^1(G_k, T) \to H^1(G_{k_1}, T)) \simeq H^1(\mathrm{Aut}(k_1/k), T^{G_{k_1}}).$$

Thus, N_1 is finitely generated (in fact, finite) by Lemma 1.6 (ii).

To prove that N_2 is finitely generated, consider the long exact sequence associated to the exact sequence $0 \to T \to T^{\Sigma}\mathcal{A}_{c_0} \to (T^{\Sigma}\mathcal{A}_{c_0})/T \to 0$ of (continuous) $G_{k(c_0)}$-modules. Then we see that there exists a natural surjection

$$((T^{\Sigma}\mathcal{A}_{c_0})/T)^{G_{k(c_0)}} \twoheadrightarrow N_2,$$

from which N_2 is finitely generated. This finishes the proof of Proposition 1.4. $\square$

Question 1.7. Do the assertions of Proposition 1.2 and Proposition 1.4 also hold when $\sharp(\Sigma) = \infty$ (especially, when $\Sigma = \mathfrak{Primes}^{\dagger}$)?

For the cohomology with torsion coefficients, one has the following.

Proposition 1.8. *Assume that k is* **Hilbertian**. *Then, for each $\mathfrak{Primes}^{\dagger}$-integer $N > 0$, there exists $S \stackrel{\text{def}}{=} S(N) \subset C^{\text{cl}}$ of cardinality ≤ 2, depending on (A and) N, such that the natural restriction map*

$$H^1(\pi_1(C), A[N]) \to \prod_{c \in S} H^1(G_{k(c)}, \mathcal{A}_c[N])$$

is **injective**.

First, we prove the following.

Proposition 1.9. *Assume that k is* **Hilbertian** *and let N be a $\mathfrak{Primes}^{\dagger}$-integer > 0. Let $M \subset H^1(\pi_1(C), A[N])$ be a* **finite** *$\mathbb{Z}/N$-submodule. Then there exists a closed point $c \in C^{\text{cl}}$, depending on N and M, such that the natural restriction map $M \to H^1(G_{k(c)}, \mathcal{A}_c[N])$ is* **injective** *and that the natural restriction map $H^0(\pi_1(C), A[N]) \to H^0(G_{k(c)}, \mathcal{A}_c[N])$ is an isomorphism.*

Proof. The proof is similar to (and even simpler than) that of Proposition 1.5. As $A[N]$ is a finite discrete $\pi_1(C)$-module and $M \subset H^1(\pi_1(C), A[N])$ is finite, it follows from the definition of the profinite group cohomology that there exists an open normal subgroup $L_{0,M} \subset \pi_1(C)$ such that $L_{0,M}$ acts trivially on $A[N]$ and that $M \subset H^1(\pi_1(C), A[N])$ is contained in the image of the inflation map

$$H^1(\pi_1(C)/L_{0,M}, A[N]) \hookrightarrow H^1(\pi_1(C), A[N]).$$

Now, the Hilbertian property of k implies that there exists $c \in C^{\text{cl}}$, such that the composite map $D_c(\xrightarrow{\sim} G_{k(c)}) \hookrightarrow \pi_1(C) \twoheadrightarrow \pi_1(C)/L_{0,M}$ is surjective. Hence, the composite map $H^1(\pi_1(C)/L_{0,M}, A[N]) \hookrightarrow H^1(\pi_1(C), A[N]) \to H^1(D_c, A[N])$ $(= H^1(G_{k(c)}, \mathcal{A}_c[N]))$ (resp. $H^0(\pi_1(C)/L_{0,M}, A[N]) \xrightarrow{\sim} H^0(\pi_1(C), A[N]) \to H^0(D_c, A[N])$ $(= H^0(G_{k(c)}, \mathcal{A}_c[N]))$) is injective (resp. an isomorphism). This finishes the proof of Proposition 1.9. $\square$

Proof of Proposition 1.8. The proof is similar to (and even simpler than) that of Proposition 1.4. We have the inflation-restriction exact sequence

$$0 \to H^1(G_k, T_0) \to H^1(\pi_1(C), A[N]) \to H^1(\pi_1(C_{\overline{k}}), A[N])$$

arising from the exact sequence $1 \to \pi_1(C_{\overline{k}}) \to \pi_1(C) \to G_k \to 1$, where $T_0 \stackrel{\text{def}}{=} (A[N])^{\pi_1(C_{\overline{k}})}$. Fix any $c_0 \in C^{\text{cl}}$. By Proposition 1.9, it suffices to prove that the kernel N_0 of $H^1(\pi_1(C), A[N]) \to H^1(G_{k(c_0)}, \mathcal{A}_{c_0}[N])$ is finite. By Lemma 1.6 (ii), together with Lemma 1.1 (iv), $H^1(\pi_1(C_{\overline{k}}), A[N])$ is finite, hence the image of N_0 in $H^1(\pi_1(C_{\overline{k}}), A[N])$ is also finite. Thus, it suffices to prove that the intersection of N_0 and (the image of) $H^1(G_k, T_0)$ in $H^1(\pi_1(C), A[N])$ is finite. Here, we have

$$N_0 \cap H^1(G_k, T_0) = \mathrm{Ker}(H^1(G_k, T_0) \to H^1(G_{k(c_0)}, \mathcal{A}_{c_0}[N])).$$

Since the map $H^1(G_k, T_0) \to H^1(G_{k(c_0)}, \mathcal{A}_{c_0}[N])$ factors as

$$H^1(G_k, T_0) \to H^1(G_{k(c_0)}, T_0) \to H^1(G_{k(c_0)}, \mathcal{A}_{c_0}[N]),$$

it suffices to prove that both $N_{0,1} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(H^1(G_k, T_0) \to H^1(G_{k(c_0)}, T_0))$ and $N_{0,2} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(H^1(G_{k(c_0)}, T_0) \to H^1(G_{k(c_0)}, \mathcal{A}_{c_0}[N]))$ are finite.

To prove that $N_{0,1}$ is finite, let k_1/k be the normal closure of the finite extension $k(c_0)/k$. Then

$$N_{0,1} \subset \mathrm{Ker}(H^1(G_k, T_0) \to H^1(G_{k_1}, T_0)) \simeq H^1(\mathrm{Aut}(k_1/k), T_0^{G_{k_1}}).$$

Thus, $N_{0,1}$ is finite by Lemma 1.6 (ii).

To prove that $N_{0,2}$ is finite, consider the long exact sequence associated to the exact sequence $0 \to T_0 \to \mathcal{A}_{c_0}[N] \to \mathcal{A}_{c_0}[N]/T_0 \to 0$ of (discrete) $G_{k(c_0)}$-modules. Then we see that there exists a natural surjection

$$(\mathcal{A}_{c_0}[N]/T_0)^{G_{k(c_0)}} \twoheadrightarrow N_{0,2},$$

from which $N_{0,2}$ is finite. $\square$

The following form of Néron's specialisation theorem can be obtained as an application of the above injectivity results.

Proposition 1.10. *Let N be a $\mathfrak{Primes}^\dagger$-integer > 0.*
(i) *Assume that k is* **Hilbertian**. *Then there exists $S \stackrel{\mathrm{def}}{=} S(N) \subset C^{\mathrm{cl}}$ of cardinality ≤ 2, depending on (A and) N, such that the natural specialisation map $A(K)/N \to \prod_{c \in S} \mathcal{A}_c(k(c))/N$ is* **injective**.
(ii) *Assume that k is* **finitely generated** *over the prime field and infinite. Then there exists a closed point $c = c(N) \in C^{\mathrm{cl}}$, depending on (A and) N, such that the natural specialisation map $A(K)/N \to \mathcal{A}_c(k(c))/N$ is* **injective**, *that the natural specialisation map $A(K)[N] \to \mathcal{A}_c(k(c))[N]$ is an* **isomorphism**, *and that the natural specialisation map $A(K) \to \mathcal{A}_c(k(c))$ is* **injective** *and its cokernel admits no nontrivial N-torsion.*

Proof. For each $c \in C^{\mathrm{cl}}$, we have a natural commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{A}(C)/N & \longrightarrow & H^1(\pi_1(C), A[N]) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{A}_c(k(c))/N & \longrightarrow & H^1(G_{k(c)}, \mathcal{A}_c[N]) \end{array}$$

where the horizontal sequences arise from Kummer exact sequences over (the étale site of) C and over $k(c)$, and the vertical maps are natural specialisation/restriction maps. Note that $\mathcal{A}(C) \stackrel{\sim}{\to} A(K)$. Thus, (i) follows directly from Proposition 1.8.

Next, if the characteristic p of k is > 0 (resp. 0), we may replace C by a nonempty open subset on which the p-rank of fibres of $\mathcal{A} \to C$ is constant (resp. by C itself). Then, for any $c \in C^{\mathrm{cl}}$, the specialisation map $A(K)^{\mathrm{tor}} \to \mathcal{A}_c(k(c))^{\mathrm{tor}}$ is injective.

Now, assume that k is finitely generated over the prime field, then K is also finitely generated over the prime field, hence $A(K)$ is a finitely generated abelian group (cf. [Lang-Néron]). Applying Proposition 1.9 to the finite $\mathbb{Z}/N$-submodule $M = A(K)/N \stackrel{\sim}{\leftarrow} \mathcal{A}(C)/N \hookrightarrow H^1(G_K, A[N])$, we see that there exists $c \in C^{\mathrm{cl}}$ such that $A(K)/N \to \mathcal{A}_c(k(c))/N$ is injective and that $A(K)[N] \to \mathcal{A}_c(k(c))[N]$ is an isomorphism. It follows from these facts, together with [Serre2], 11.1, Criterion, that $A(K) \to \mathcal{A}_c(k(c))$ is injective. (Strictly speaking, this argument is applicable only when $N \geq 2$. However, the assertion of (ii) for $N = 1$ is included in that for (any) $N > 1$.) Further, as $A(K)/N \to \mathcal{A}_c(k(c))/N$ is injective and $A(K)[N] \to \mathcal{A}_c(k(c))[N]$ is an isomorphism, the cokernel of $A(K) \hookrightarrow \mathcal{A}(k(c))$ admits no nontrivial N-torsion. (Use the Snake Lemma.) This finishes the proof of Proposition 1.10. □

§**2. Selmer Groups.** We follow the notations in §1. Moreover, let $\Sigma \subset \mathfrak{Primes}^{\dagger}$ be any **nonempty** subset.

We have a natural commutative diagram

(2.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(K)^{\wedge,\Sigma} & \longrightarrow & H^1(G_K, T^{\Sigma}A) & \longrightarrow & T^{\Sigma}H^1(G_K, A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_c \mathcal{A}_c(K_c)^{\wedge,\Sigma} & \longrightarrow & \prod_c H^1(G_{K_c}, T^{\Sigma}A_c) & \longrightarrow & \prod_c T^{\Sigma}H^1(G_{K_c}, A_c) & \longrightarrow & 0 \end{array}$$

where the horizontal sequences are Kummer exact sequences over K and K_c, the vertical maps are natural restriction maps, and the product is taken over all closed points $c \in C^{\mathrm{cl}}$. We have another natural commutative diagram

(2.2)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}(C)^{\wedge,\Sigma} & \longrightarrow & H^1(\pi_1(C), T^{\Sigma}A) & \longrightarrow & T^{\Sigma}H^1_{\text{ét}}(C, \mathcal{A}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_c \mathcal{A}_c(k(c))^{\wedge,\Sigma} & \longrightarrow & \prod_c H^1(G_{k(c)}, T^{\Sigma}\mathcal{A}_c) & \longrightarrow & \prod_c T^{\Sigma}H^1(G_{k(c)}, \mathcal{A}_c) & \longrightarrow & 0. \end{array}$$

where the horizontal sequences are Kummer exact sequences over (the étale site of) C and over $k(c)$, the vertical maps are natural restriction maps, and the product is taken over all closed points $c \in C^{\mathrm{cl}}$.

Proposition 2.1. (i) *There exists a natural injective map from diagram (2.2) to diagram (2.1). Further, the maps on the upper left and lower left terms are isomorphisms.*
(ii) *Assume that k is* **finitely generated** *over the prime field. Then the map from diagram (2.2) to diagram (2.1) in (i) is an isomorphism.*

Proof. (i) For the upper rows of diagrams (2.1) and (2.2), we have a natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}(C)^{\wedge,\Sigma} & \to & H^1(\pi_1(C), T^{\Sigma}A) & \to & T^{\Sigma}H^1_{\text{ét}}(C, \mathcal{A}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A(K)^{\wedge,\Sigma} & \to & H^1(G_K, T^{\Sigma}A) & \to & T^{\Sigma}H^1(G_K, A) & \to & 0 \end{array}$$

obtained by taking the pullback of étale cohomology groups via the natural morphism $\mathrm{Spec}(K) \to C$. Here the left vertical map coincides with the map induced by the natural map $\mathcal{A}(C) \to A(K)$, which is an isomorphism by (a variant of) the valuative criterion for properness. The middle vertical map is injective by Lemma 1.1 (iii). Thus, the right vertical map is also injective.

For the lower rows of diagrams (2.1) and (2.2), we have a natural commutative diagram for each $c \in C^{\mathrm{cl}}$

(2.3)

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}_c(k(c))^{\wedge,\Sigma} & \to & H^1(G_{k(c)}, T^\Sigma \mathcal{A}_c) & \to & T^\Sigma H^1(G_{k(c)}, \mathcal{A}_c) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \mathcal{A}_{R_c}(R_c)^{\wedge,\Sigma} & \to & H^1(\pi_1(R_c), T^\Sigma A_c) & \to & T^\Sigma H^1_{\text{ét}}(R_c, \mathcal{A}_{R_c}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_c(K_c)^{\wedge,\Sigma} & \to & H^1(G_{K_c}, T^\Sigma A_c) & \to & T^\Sigma H^1(G_{K_c}, A_c) & \to & 0 \end{array}$$

where $R_c \stackrel{\mathrm{def}}{=} \widehat{\mathcal{O}}_{C,c}$ (thus, K_c is the field of fractions of R_c) and $\mathcal{A}_{R_c} \stackrel{\mathrm{def}}{=} \mathcal{A} \times_C R_c$. Here, the upper vertical maps are obtained by taking the pullback of étale cohomology groups via the natural morphisms $\mathrm{Spec}(k(c)) \to \mathrm{Spec}(R_c)$ and the modulo c reduction on $\mathcal{A}_{R_c}$, and the lower vertical maps are obtained by taking the pullback of étale cohomology groups via the natural morphism $\mathrm{Spec}(K_c) \to \mathrm{Spec}(R_c)$. The upper left vertical map is an isomorphism, since the reduction map $\mathcal{A}_{R_c}(R_c) \to \mathcal{A}_c(k(c))$ is surjective with kernel being N-divisible for every $\mathfrak{Primes}^\dagger$-integer $N > 0$. The upper middle vertical map is an isomorphism, as $G_{k(c)} \xrightarrow{\sim} \pi_1(R_c)$ and $T^\Sigma A_c \xrightarrow{\sim} T^\Sigma \mathcal{A}_c$. Thus, the upper right vertical map is also an isomorphism. The lower left vertical map coincides with the map induced by the natural map $\mathcal{A}_{R_c}(R_c) \to A_c(K_c)$, which is an isomorphism by the valuative criterion for properness. The middle vertical map is injective by Lemma 1.1 (iii). Thus, the lower right vertical map is also injective.

Finally, it follows from various functoriality properties that the above maps form a map from diagram (2.2) to diagram (2.1) with the desired properties.

(ii) By Lemma 1.1 (iii), the maps $H^1(\pi_1(C), T^\Sigma A) \to H^1(G_K, T^\Sigma A)$ and

$$H^1(\pi_1(R_c), T^\Sigma A_c)(\xrightarrow{\sim} H^1(G_{k(c)}, T^\Sigma \mathcal{A}_c)) \to H^1(G_{K_c}, T^\Sigma A_c)$$

for $c \in C^{\mathrm{cl}}$ are isomorphisms. The assertion follows from this, together with (i). □

Proposition 2.2. *Assume that k is* **Hilbertian**. *Then the middle and left vertical maps in diagrams (2.1) and (2.2) are* **injective**. *(For the kernels of the right vertical maps, see §3.)*

Proof. The middle vertical map in diagram (2.2), say r_Σ, is identified with the product of $r_{\{l\}}$ ($l \in \Sigma$). Thus, the injectivity of r_Σ follows immediately from Proposition 1.4.

Next, as in the proof of Lemma 1.1, we have a natural commutative diagram

$$\begin{array}{ccccccc} 0 & \to & H^1(\pi_1(C), T^\Sigma A) & \to & H^1(G_K, T^\Sigma A) & \to & H^1(I_C, T^\Sigma A) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \prod_c H^1(G_{k(c)}, T^\Sigma \mathcal{A}_c) & \to & \prod_c H^1(G_{K_c}, T^\Sigma A_c) & \to & \prod_c H^1(I_c, T^\Sigma A_c) \end{array}$$

where the horizontal sequences are inflation-restriction exact sequences, the left and the middle vertical maps are the middle vertical maps in diagrams (2.2) and

(2.1), respectively, and the right vertical map is a natural restriction map. As shown above, the left vertical map is injective. The right vertical map is also injective, since $H^1(I_C, T^\Sigma A) = \mathrm{Hom}(I_C, T^\Sigma A)$, $H^1(I_c, T^\Sigma A_c) = \mathrm{Hom}(I_c, T^\Sigma A_c)$, $T^\Sigma A \overset{\sim}{\to} T^\Sigma A_c$, and I_C is (topologically) normally generated by I_c ($c \in C^{\mathrm{cl}}$). Thus, the middle vertical map (that is, the middle vertical map in diagram (2.1)) is injective.

Finally, the left vertical maps in diagrams (2.1) and (2.2) are injective, as the middle vertical maps therein are injective. □

For the rest of this paper, we will assume that k is **finitely generated** over the prime field and infinite. (Thus, in particular, k is Hilbertian.) We will identify $A(K)^{\wedge,\Sigma}$, $H^1(\pi_1(C), T^\Sigma A)$, and $\prod_c \mathcal{A}_c(k(c))^{\wedge,\Sigma}$ with their images in $\prod_c H^1(G_{k(c)}, T^\Sigma \mathcal{A}_c)$.

Definition 2.3. (i) For each $\mathfrak{Primes}^\dagger$-integer $N > 0$, we define the **N-Selmer group**

$$\mathrm{Sel}_N(A) \overset{\mathrm{def}}{=} \mathrm{Sel}_N(A, C) \overset{\mathrm{def}}{=} \mathrm{Ker}\big(H^1(G_K, A[N]) \to \prod_c H^1(G_{K_c}, A_c)\big).$$

(ii) We define the **Σ-adic Selmer group**

$$\begin{aligned}\mathrm{Sel}^\Sigma(A) \overset{\mathrm{def}}{=} \mathrm{Sel}^\Sigma(A, C) &\overset{\mathrm{def}}{=} \mathrm{Ker}\big(H^1(G_K, T^\Sigma A) \to \prod_c T^\Sigma H^1(G_{K_c}, A_c)\big)\\ &= \mathrm{Ker}\big(H^1(\pi_1(C), T^\Sigma A) \to \prod_c T^\Sigma H^1(G_{k(c)}, \mathcal{A}_c)\big),\end{aligned}$$

so that $\mathrm{Sel}^\Sigma(A) = \varprojlim_{N:\Sigma\text{-integer}>0} \mathrm{Sel}_N(A)$.

We have natural injective maps $A(K)/\{\Sigma'\text{-tor}\} \hookrightarrow A(K)^{\wedge,\Sigma}$ and $\mathcal{A}_c(k(c))/\{\Sigma'\text{-tor}\} \hookrightarrow \mathcal{A}_c(k(c))^{\wedge,\Sigma}$ ($c \in C^{\mathrm{cl}}$), as $A(K)$ and $\mathcal{A}_c(k(c))$ are finitely generated $\mathbb{Z}$-modules (cf. [Lang-Néron]). We will identify $\prod_c(\mathcal{A}_c(k(c))/\{\Sigma'\text{-tor}\})$ with its image in $\prod_c \mathcal{A}_c(k(c))^{\wedge,\Sigma}$.

Definition 2.4. We define the **Σ-discrete Selmer group**

$$\mathfrak{Sel}^\Sigma(A) \overset{\mathrm{def}}{=} \mathfrak{Sel}^\Sigma(A, C) \overset{\mathrm{def}}{=} H^1(\pi_1(C), T^\Sigma A) \bigcap \prod_c (\mathcal{A}_c(k(c))/\{\Sigma'\text{-tor}\}) \subset \prod_c H^1(G_{k(c)}, T^\Sigma A).$$

Note that $\mathfrak{Sel}^\Sigma(A) \subset \mathrm{Sel}^\Sigma(A)$ by definition. One of our main results in this section is the following.

Proposition 2.5. *The Σ-discrete Selmer group $\mathfrak{Sel}^\Sigma(A)$ is a* **finitely generated** *$\mathbb{Z}$-module.*

First, we prove the following.

Lemma 2.6. *The following holds*

$$\begin{aligned}&H^1(\pi_1(C), T^\Sigma A) \bigcap \prod_c (H^1(G_{k(c)}, T^\Sigma A)^{\mathrm{tor}})\\ &= \mathfrak{Sel}^\Sigma(A) \bigcap \prod_c (\mathcal{A}_c(k(c))^{\mathrm{tor}}/\{\Sigma'\text{-tor}\})\\ &= \mathfrak{Sel}^\Sigma(A)^{\mathrm{tor}} = A(K)^{\mathrm{tor}}/\{\Sigma'\text{-tor}\}.\end{aligned}$$

Proof. For each of the three desired equalities, the containment relation $\supset$ clearly holds. Thus, it suffices to prove that

$$H^1(\pi_1(C), T^\Sigma A) \bigcap \prod_c (H^1(G_{k(c)}, T^\Sigma A)^{\mathrm{tor}}) \subset A(K)^{\mathrm{tor}}/\{\Sigma'\text{-tor}\}.$$

So, take any element $\alpha = (\alpha_l)_{l\in\Sigma}$ of $H^1(\pi_1(C), T^\Sigma A) \bigcap \prod_c (H^1(G_{k(c)}, T^\Sigma A)^{\mathrm{tor}})$, where we write $\alpha_l \in H^1(\pi_1(C), T_l A)$ for the l-component of the cohomology class α. For each prime $l \in \Sigma$, there exists, by Proposition 1.2, a point $c \in C^{\mathrm{cl}}$ (depending on l) such that the natural restriction map $H^1(\pi_1(C), T_l A) \to H^1(G_{k(c)}, T_l \mathcal{A}_c)$ is injective. As the injective image of $\alpha_l \in H^1(\pi_1(C), T_l A)$ in $H^1(G_{k(c)}, T_l \mathcal{A}_c)$ lies in $H^1(G_{k(c)}, T_l \mathcal{A}_c)^{\mathrm{tor}}$, we have

$$\alpha_l \in H^1(\pi_1(C), T_l A)^{\mathrm{tor}} = (A(K)^{\wedge, l})^{\mathrm{tor}} = A(K)^{\mathrm{tor}}/\{l'\text{-tor}\} = A(K)^{\mathrm{tor}, l}.$$

Here, the first equality follows from the fact that $T_l H^1_{\text{ét}}(C, \mathcal{A})$ is torsion-free, the second equality follows from the fact that $A(K)$ is a finitely generated $\mathbb{Z}$-module, and the third equality follows as $A(K)^{\mathrm{tor}}$ is a torsion abelian group. In particular, $\alpha_l = 0$ for all but finitely many $l \in \Sigma$, as $A(K)^{\mathrm{tor}}$ is a finite abelian group. Now, we conclude that $\alpha = (\alpha_l)_{l\in\Sigma} \in (A(K)/\{\Sigma'\text{-tor}\})^{\mathrm{tor}} = A(K)^{\mathrm{tor}}/\{\Sigma'\text{-tor}\}$, as desired. □

Proposition 2.7. *Let $\Sigma_1 \subset \Sigma_2 \subset \mathfrak{Primes}^\dagger$ be nonempty subsets. Then there exists a natural exact sequence*

$$0 \to A(K)^{\mathrm{tor}, \Sigma_1'}/A(K)^{\mathrm{tor}, \Sigma_2'} \to \mathfrak{Sel}^{\Sigma_2}(A) \to \mathfrak{Sel}^{\Sigma_1}(A),$$

where the map $\mathfrak{Sel}^{\Sigma_2}(A) \to \mathfrak{Sel}^{\Sigma_1}(A)$ is induced by the projection $H^1(\pi_1(C), T^{\Sigma_2} A) \to H^1(\pi_1(C), T^{\Sigma_1} A)$.

Proof. We have

$$\begin{aligned}
&\mathrm{Ker}(\mathfrak{Sel}^{\Sigma_2}(A) \to \mathfrak{Sel}^{\Sigma_1}(A)) \\
&= \mathfrak{Sel}^{\Sigma_2}(A) \cap \mathrm{Ker}\left(\prod_c (\mathcal{A}_c(k(c))/\{\Sigma_2'\text{-tor}\}) \to \prod_c (\mathcal{A}_c(k(c))/\{\Sigma_1'\text{-tor}\})\right) \\
&= \mathfrak{Sel}^{\Sigma_2}(A) \cap \prod_c (\mathcal{A}_c(k(c))^{\mathrm{tor}, \Sigma_1'}/\mathcal{A}_c(k(c))^{\mathrm{tor}, \Sigma_2'}) \\
&= (A(K)^{\mathrm{tor}}/A(K)^{\mathrm{tor}, \Sigma_2'}) \cap \prod_c (\mathcal{A}_c(k(c))^{\mathrm{tor}, \Sigma_1'}/\mathcal{A}_c(k(c))^{\mathrm{tor}, \Sigma_2'}) \\
&= (A(K)^{\mathrm{tor}, \Sigma_1'}/A(K)^{\mathrm{tor}, \Sigma_2'}),
\end{aligned}$$

where the third equality follows from Lemma 2.6. □

The proof of Proposition 2.5 will follow immediately from the following, since $\mathcal{A}_c(k(c))$ is a finitely generated $\mathbb{Z}$-module for every $c \in C^{\mathrm{cl}}$ (cf. [Lang-Néron]).

Proposition 2.8. *There exists a closed point $c \in C^{\mathrm{cl}}$ such that the natural restriction map*

$$\mathfrak{Sel}^\Sigma(A) \to \mathcal{A}_c(k(c))/\{\Sigma'\text{-tor}\}$$

is **injective**.

Proof. First, let $\Sigma_1 \subset \Sigma_2 \subset \mathfrak{Primes}^\dagger$ be nonempty subsets, and $c \in C^{\mathrm{cl}}$. Then the natural commutative diagram

$$\begin{array}{ccc} H^1(\pi_1(C), T^{\Sigma_2}A) & \to & H^1(G_{k(c)}, T^{\Sigma_2}\mathcal{A}_c) \\ \downarrow & & \downarrow \\ H^1(\pi_1(C), T^{\Sigma_1}A) & \to & H^1(G_{k(c)}, T^{\Sigma_1}\mathcal{A}_c), \end{array}$$

where the horizontal maps are natural restriction maps and the vertical maps are natural projections, restricts to a natural commutative diagram

$$\begin{array}{ccc} \mathfrak{Sel}^{\Sigma_2}(A) & \to & \mathcal{A}_c(k(c))/\{\Sigma_2'\text{-tor}\} \\ \downarrow & & \downarrow \\ \mathfrak{Sel}^{\Sigma_1}(A) & \to & \mathcal{A}_c(k(c))/\{\Sigma_1'\text{-tor}\}. \end{array}$$

Now, suppose that the assertion holds for Σ_1. Then there exists a closed point $c \in C^{\mathrm{cl}}$ such that the lower horizontal map of the latter commutative diagram is injective. Thus,

$$\begin{aligned} & \mathrm{Ker}(\mathfrak{Sel}^{\Sigma_2}(A) \to \mathcal{A}_c(k(c))/\{\Sigma_2'\text{-tor}\}) \\ = \ & \mathrm{Ker}(\mathfrak{Sel}^{\Sigma_2}(A) \to \mathcal{A}_c(k(c))/\{\Sigma_2'\text{-tor}\}) \cap \mathrm{Ker}(\mathfrak{Sel}^{\Sigma_2}(A) \to \mathfrak{Sel}^{\Sigma_1}(A)) \\ = \ & \mathrm{Ker}(\mathfrak{Sel}^{\Sigma_2}(A) \to \mathcal{A}_c(k(c))/\{\Sigma_2'\text{-tor}\}) \cap A(K)^{\mathrm{tor},\Sigma_1'}/A(K)^{\mathrm{tor},\Sigma_2'} = 0, \end{aligned}$$

where the second equality follows from Proposition 2.7 and the third equality follows from the fact that the reduction map $A(K)^{\mathrm{tor},\mathfrak{Primes}^\dagger} \to \mathcal{A}_c(k(c))^{\mathrm{tor},\mathfrak{Primes}^\dagger}$ is injective, as $\mathcal{A}[N]$ is finite étale over C for any $\mathfrak{Primes}^\dagger$-integer $N > 0$. Thus, the assertion also holds for Σ_2.

Now, to prove the assertion, we may assume that Σ is finite, by replacing Σ with any nonempty finite subset. (We may even assume $\sharp(\Sigma) = 1$.) Then, by Proposition 1.2, there exists a closed point $c \in C^{\mathrm{cl}}$, such that the natural restriction map

$$H^1(\pi_1(C), T^\Sigma A) \to H^1(G_{k(c)}, T^\Sigma \mathcal{A}_c)$$

is injective. In particular, the natural restriction map

$$\mathfrak{Sel}^\Sigma(A) \to \mathcal{A}_c(k(c))/\{\Sigma'\text{-tor}\}$$

is also injective, as desired. $\square$

Proposition 2.9. (i) *For each Σ-integer $N > 0$, the natural map $\mathfrak{Sel}^\Sigma(A) \to H^1(\pi_1(C), A[N])$ induces a natural* **injective** *map*

$$\mathfrak{Sel}^\Sigma(A)/N \hookrightarrow \mathrm{Sel}_N(A) \subset H^1(\pi_1(C), A[N]).$$

(ii) *The natural map $\mathfrak{Sel}^\Sigma(A) \to H^1(\pi_1(C), T^\Sigma A)$ induces a natural* **injective** *map*

$$\mathfrak{Sel}^\Sigma(A)^{\wedge,\Sigma} \hookrightarrow \mathrm{Sel}^\Sigma(A) \subset H^1(\pi_1(C), T^\Sigma A).$$

Proof. (i) As $H^1(\pi_1(C), A[N])$ is killed by N, the natural map $\mathfrak{Sel}^\Sigma(A) \to H^1(\pi_1(C), A[N])$ induces a natural map $\iota_N : \mathfrak{Sel}^\Sigma(A)/N \to H^1(\pi_1(C), A[N])$. As $\mathfrak{Sel}^\Sigma(A) \subset \mathrm{Sel}^\Sigma(A)$, the image of ι_N is contained in $\mathrm{Sel}_N(A)$. Thus, it suffices to prove that ι_N is injective. Consider the natural exact sequence

$$0 \to \mathfrak{Sel}^\Sigma(A) \xrightarrow{\iota} H^1(\pi_1(C), T^\Sigma A) \to \mathrm{Coker}(\iota) \to 0,$$

where ι is the natural injection. We claim that $\mathrm{Coker}(\iota)^{\mathrm{tor},\Sigma} = 0$ holds. Indeed, first we have the equality $\mathrm{Coker}(\iota)^{\mathrm{tor},\Sigma} = (\mathrm{Sel}^\Sigma(A)/\mathfrak{Sel}^\Sigma(A))^{\mathrm{tor},\Sigma}$, which follows from the fact that $H^1(\pi_1(C), T^\Sigma A)/\mathrm{Sel}^\Sigma(A)$ injects into $\prod_c T^\Sigma H^1(G_{k(c)}, \mathcal{A}_c)$, which is torsion-free. Moreover, we have a natural injective map $\mathrm{Sel}^\Sigma(A)/\mathfrak{Sel}^\Sigma(A) \to \prod_c \mathcal{A}_c(k(c))^{\wedge,\Sigma}/(\mathcal{A}_c(k(c))/\{\Sigma'\text{-tor}\})$ and the Σ-torsion of the latter group is trivial, as follows easily from the fact that the Σ-torsion of $\widehat{\mathbb{Z}}^\Sigma/\mathbb{Z}$ is trivial, and the groups $\mathcal{A}_c(k(c))$ are finitely generated.

Now, for each Σ-integer $N > 0$, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{Sel}(A) & \longrightarrow & H^1(\pi_1(C), T^\Sigma A) & \longrightarrow & \mathrm{Coker}(\iota) & \longrightarrow & 0 \\
 & & {\scriptstyle N}\downarrow & & {\scriptstyle N}\downarrow & & {\scriptstyle N}\downarrow & & \\
0 & \longrightarrow & \mathfrak{Sel}(A) & \longrightarrow & H^1(\pi_1(C), T^\Sigma A) & \longrightarrow & \mathrm{Coker}(\iota) & \longrightarrow & 0
\end{array}$$

where the vertical maps are the maps of multiplication by N. Thus, by the Snake Lemma, we have a natural exact sequence

$$0 = \mathrm{Coker}(\iota)[N] \to \mathfrak{Sel}^\Sigma(A)/N \to H^1(\pi_1(C), T^\Sigma A)/N.$$

Now the assertion follows, as $H^1(\pi_1(C), T^\Sigma A)/N \hookrightarrow H^1(\pi_1(C), A[N])$.
(ii) By (i), the natural map

$$\begin{array}{ccccc}
\mathfrak{Sel}^\Sigma(A)^{\wedge,\Sigma} & & \mathrm{Sel}^\Sigma(A) & & H^1(\pi_1(C), T^\Sigma A) \\
\| & & \| & & \| \\
\varprojlim_N \mathfrak{Sel}^\Sigma(A)/N & \to & \varprojlim_N \mathrm{Sel}_N(A) & \subset & \varprojlim_N H^1(\pi_1(C), A[N]),
\end{array}$$

where N runs over all Σ-integers > 0, is injective, as desired. □

Proposition 2.10. *For each* $\mathfrak{Primes}^\dagger$*-integer* $N > 0$*, the* N*-Selmer group* $\mathrm{Sel}_N(A)$ *is* **finite**.

Let N be a $\mathfrak{Primes}^\dagger$-integer > 0. As $\mathcal{A}[N]$ is finite étale over C, the G_K-module $A[N]$ is unramified, i.e., the group $I_C = \mathrm{Ker}(G_K \twoheadrightarrow \pi_1(C))$ acts trivially on $A[N]$. Thus, $A[N]$ has a natural structure of $\pi_1(C)$-module, and we have a natural inflation-restriction exact sequence

$$0 \to H^1(\pi_1(C), A[N]) \xrightarrow{\mathrm{inf}} H^1(G_K, A[N]) \xrightarrow{\mathrm{res}} \mathrm{Hom}(I_C, A[N])^{\pi_1(C)}.$$

Lemma 2.11. *The following holds:* $\mathrm{Sel}_N(A) \subset H^1(\pi_1(C), A[N])$.

Proof. As in diagram (2.3), we have a natural commutative diagram for each $c \in C^{\mathrm{cl}}$

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}_{R_c}(R_c)/N & \to & H^1(\pi_1(R_c), \mathcal{A}_{R_c}[N]) & \to & H^1_{\text{ét}}(R_c, \mathcal{A}_{R_c})[N] & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_c(K_c)/N & \to & H^1(G_{K_c}, A_c[N]) & \to & H^1(G_{K_c}, A_c)[N] & \to & 0, \end{array}$$

where the vertical maps are obtained by taking the pullback of étale cohomology groups via the natural morphism $\mathrm{Spec}(K_c) \to \mathrm{Spec}(R_c)$. The left vertical map coincides with the map induced by the natural map $\mathcal{A}_{R_c}(R_c) \to A_c(K_c)$, which is an isomorphism by the valuative criterion for properness. The middle vertical map is injective by Lemma 1.1 (iv). Thus, the right vertical map is also injective. By definition, the image of $\mathrm{Sel}_N(A)$ in $H^1(G_{K_c}, A[N])$ is contained in (the image of) $A_c(K_c)/N \xleftarrow{\sim} \mathcal{A}_{R_c}(R_c)/N$, hence, in particular, in (the image of) $H^1(\pi_1(R_c), \mathcal{A}_{R_c}[N])$. It follows from this that the image of $\mathrm{Sel}_N(A)$ in $H^1(I_c, A[N]) = \mathrm{Hom}(I_c, A[N])$ is trivial for all $c \in C^{\mathrm{cl}}$. Thus, the image of $\mathrm{Sel}_N(A)$ in $\mathrm{Hom}(I_C, A[N])$ is trivial, as desired, since I_C is (topologically) normally generated by the various I_c for $c \in C^{\mathrm{cl}}$. □

Proof of Proposition 2.10. By Proposition 1.8, there exists $S \subset C^{\mathrm{cl}}$ of cardinality ≤ 2, depending on (A and) N, such that the natural restriction map

$$H^1(\pi_1(C), A[N]) \to \prod_{c \in S} H^1(G_{k(c)}, \mathcal{A}_c[N])$$

is injective. In particular, $\mathrm{Sel}_N(A)$ injects into $\prod_{c \in S} \mathcal{A}_c(k(c))/N$, which is finite as S is finite and $\mathcal{A}_c(k(c))$ is finitely generated for each c. This finishes the proof of Proposition 2.10. □

§**3. The Shafarevich-Tate Group.** We use the same notations as in §1 and §2. In particular, k is a field which is **finitely generated** over the prime field of characteristic $p \geq 0$ and infinite; $C \to \mathrm{Spec}\, k$ is a smooth, separated and geometrically connected algebraic curve over k with function field $K = k(C)$; and $\mathcal{A} \to C$ is an abelian scheme over C with generic fibre $A = \mathcal{A}_K = \mathcal{A} \times_C \mathrm{Spec}\, K$.

Definition 3.1. We define the **Shafarevich-Tate group**

$$Ш(A) \stackrel{\mathrm{def}}{=} Ш(A, C) \stackrel{\mathrm{def}}{=} \mathrm{Ker}\big(H^1(G_K, A) \to \prod_c H^1(G_{K_c}, A_c)\big),$$

where the product is taken over all closed points $c \in C^{\mathrm{cl}}$. We set $Ш(A)^{(\dagger)} \stackrel{\mathrm{def}}{=} Ш(A)$ (resp. $Ш(A)^{(\dagger)} \stackrel{\mathrm{def}}{=} Ш(A)/\{p\text{-tor}\}$), when the characteristic p of k is 0 (resp. > 0).

Note that the abelian group $Ш(A)$ is a torsion group since the Galois cohomology group $H^1(G_K, A)$ is torsion. (In particular, $Ш(A)^{(\dagger)}$ is naturally identified with $Ш(A)^{\mathrm{tor}, p'}$ ($\subset Ш(A)$) when $p > 0$.) For each $\mathfrak{Primes}^\dagger$-integer $N > 0$, we have a natural exact sequence

$$0 \to A(K)/N \to \mathrm{Sel}_N(A) \to Ш(A)[N] \to 0,$$

and

$$0 \to A(K)^{\wedge, \Sigma} \to \mathrm{Sel}^\Sigma(A) \to T^\Sigma Ш(A) \to 0.$$

Definition 3.2. We define the **Σ-discrete Shafarevich-Tate group**

$$\mathfrak{Sha}^{\Sigma}(A) \stackrel{\text{def}}{=} \mathfrak{Sha}^{\Sigma}(A, C) \stackrel{\text{def}}{=} \mathfrak{Sel}^{\Sigma}(A)/(A(K)/\{\Sigma'\text{-tor}\}).$$

We set $\mathfrak{Sha}^{\Sigma}(A)^{(\dagger)} \stackrel{\text{def}}{=} \mathfrak{Sha}^{\Sigma}(A)$ (resp. $\mathfrak{Sha}^{\Sigma}(A)^{(\dagger)} \stackrel{\text{def}}{=} \mathfrak{Sha}^{\Sigma}(A)/\{p\text{-tor}\}$), when the characteristic p of k is 0 (resp. > 0).

By definition, we have a natural exact sequence

$$0 \to A(K)/\{\Sigma'\text{-tor}\} \to \mathfrak{Sel}^{\Sigma}(A) \to \mathfrak{Sha}^{\Sigma}(A) \to 0.$$

Proposition 3.3. *The Σ-discrete Shafarevich-Tate group $\mathfrak{Sha}^{\Sigma}(A)$ is a* **finitely generated** *$\mathbb{Z}$-module. Further, $\mathfrak{Sha}^{\Sigma}(A)^{(\dagger)}$ is a* **finitely generated free** *$\mathbb{Z}$-module.*

Proof. The first assertion follows immediately from Proposition 2.5. To prove the second assertion, it suffices to show that for each prime number $l \in \mathfrak{Primes}^{\dagger}$, $\mathfrak{Sha}^{\Sigma}(A)$ admits no nontrivial l-torsion. It follows from Lemma 2.6, together with the Snake Lemma, that this last condition is equivalent to the injectivity of the natural map $(A(K)/\{\Sigma'\text{-tor}\})/l \to \mathfrak{Sel}^{\Sigma}(A)/l$. By definition, we have a natural map $\mathfrak{Sel}^{\Sigma}(A) \to \mathcal{A}_c(k(c))/\{\Sigma'\text{-tor}\}$ for each $c \in C^{\text{cl}}$ whose composite with the natural map $A(K)/\{\Sigma'\text{-tor}\} \to \mathfrak{Sel}^{\Sigma}(A)$ coincides with the specialisation map $A(K)/\{\Sigma'\text{-tor}\} = \mathcal{A}(C)/\{\Sigma'\text{-tor}\} \to \mathcal{A}_c(k(c))/\{\Sigma'\text{-tor}\}$. Thus, to prove the desired injectivity, it suffices to show that for each prime number $l \in \mathfrak{Primes}^{\dagger}$, there exists $c \in C^{\text{cl}}$ (which may depend on l), such that the specialisation map $(A(K)/\{\Sigma'\text{-tor}\})/l \to (\mathcal{A}_c(k(c))/(\Sigma'\text{-tor}))/l$ is injective. This last assertion follows from Proposition 1.10 (ii). Indeed, by Proposition 1.10 (ii), there exists a closed point $c \in C^{\text{cl}}$, such that the natural specialisation map $A(K)/l \to \mathcal{A}_c(k(c))/l$ is injective, and that the natural specialisation map $A(K) \to \mathcal{A}_c(k(c))$ is injective and the cokernel $\mathcal{A}_c(k(c))/A(K)$ admits no nontrivial l-torsion. Further, as $\mathcal{A}_c(k(c))^{\text{tor},\Sigma'}/A(K)^{\text{tor},\Sigma'} \hookrightarrow \mathcal{A}_c(k(c))/A(K)$, the cokernel $\mathcal{A}_c(k(c))^{\text{tor},\Sigma'}/A(K)^{\text{tor},\Sigma'}$ also admits no nontrivial l-torsion, or, equivalently, we have $A(K)^{\text{tor},\Sigma'}[l^{\infty}] \xrightarrow{\sim} \mathcal{A}_c(k(c))^{\text{tor},\Sigma'}[l^{\infty}]$. Consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(K)^{\text{tor},\Sigma'}/l & \longrightarrow & A(K)/l & \longrightarrow & (A(K)/\{\Sigma'\text{-tor}\})/l & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A}_c(k(c))^{\text{tor},\Sigma'}/l & \longrightarrow & \mathcal{A}_c(k(c))/l & \longrightarrow & (\mathcal{A}_c(k(c))/\{\Sigma'\text{-tor}\})/l & \longrightarrow & 0 \end{array}$$

where the vertical maps are natural specialisation maps and the injectivity of the horizontal map $A(K)^{\text{tor},\Sigma'}/l \to A(K)/l$ (resp. $\mathcal{A}_c(k(c))^{\text{tor},\Sigma'}/l \to \mathcal{A}_c(k(c))/l$) follows from the fact that $A(K)^{\text{tor},\Sigma'}/l = 0$ (resp. $\mathcal{A}_c(k(c))^{\text{tor},\Sigma'}/l = 0$) if $l \in \Sigma$ and $(A(K)/\{\Sigma'\text{-tor}\})[l] = 0$ (resp. $(\mathcal{A}_c(k(c))/\{\Sigma'\text{-tor}\})[l] = 0$) if $l \in \Sigma'$. Thus, by the Snake Lemma, to prove the injectivity of the map $(A(K)/\{\Sigma'\text{-tor}\})/l \to (\mathcal{A}_c(k(c))/\{\Sigma'\text{-tor}\})/l$, it suffices to prove that the map $A(K)^{\text{tor},\Sigma'}/l \to \mathcal{A}_c(k(c))^{\text{tor},\Sigma'}/l$ is an isomorphism, which follows from the above-mentioned isomorphism $A(K)^{\text{tor},\Sigma'}[l^{\infty}] \xrightarrow{\sim} \mathcal{A}_c(k(c))^{\text{tor},\Sigma'}[l^{\infty}]$. This finishes the proof of Proposition 3.3. $\square$

Proposition 3.4. *Let $\Sigma_1 \subset \Sigma_2 \subset \mathfrak{Primes}^\dagger$ be nonempty subsets. Then the natural projection $T^{\Sigma_2}Ш(A) \twoheadrightarrow T^{\Sigma_1}Ш(A)$ induces an* **injective** *map $\mathfrak{Sha}^{\Sigma_2}(A) \hookrightarrow \mathfrak{Sha}^{\Sigma_1}(A)$.*

Proof. Consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(K)/\{\Sigma_2'\text{-tor}\} & \longrightarrow & \mathfrak{Sel}^{\Sigma_2}(A) & \longrightarrow & \mathfrak{Sha}^{\Sigma_2}(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A(K)/\{\Sigma_1'\text{-tor}\} & \longrightarrow & \mathfrak{Sel}^{\Sigma_1}(A) & \longrightarrow & \mathfrak{Sha}^{\Sigma_1}(A) & \longrightarrow & 0 \end{array}$$

where the left vertical map is (resp. the middle and right vertical maps are) induced by the identity map $A(K) \to A(K)$ (resp. the natural projection $H^1(\pi_1(C), T^{\Sigma_2}A) \to H^1(\pi_1(C), T^{\Sigma_1}A)$). By the Snake Lemma, the desired injectivity follows from Proposition 2.7, together with the surjectivity of the left vertical map $A(K)/\{\Sigma_2'\text{-tor}\} \to A(K)/\{\Sigma_1'\text{-tor}\}$. $\square$

Proposition 3.5. (i) *For each Σ-integer $N > 0$, the natural map $\mathfrak{Sha}^{\Sigma}(A) \to Ш(A)[N]$ induces a natural* **injective** *map*

$$\mathfrak{Sha}^{\Sigma}(A)/N \hookrightarrow Ш(A)[N] \subset H^1_{\text{ét}}(C, \mathcal{A})[N].$$

(ii) *The natural map $\mathfrak{Sha}^{\Sigma}(A) \to T^{\Sigma}Ш(A)$ induces a natural* **injective** *map*

$$\mathfrak{Sha}^{\Sigma}(A)^{\wedge,\Sigma} \hookrightarrow T^{\Sigma}Ш(A) \subset T^{\Sigma}H^1_{\text{ét}}(C, \mathcal{A}).$$

Proof. (i) Consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} & & A(K)/N & \longrightarrow & \mathfrak{Sel}^{\Sigma}(A)/N & \longrightarrow & \mathfrak{Sha}^{\Sigma}(A)/N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A(K)/N & \longrightarrow & \mathrm{Sel}_N(A) & \longrightarrow & Ш(A)[N] & \longrightarrow & 0 \end{array}$$

where the left (resp. middle, resp. right) vertical map is the identity map (resp. induced by the natural map $\mathfrak{Sel}^{\Sigma}(A) \to H^1(\pi_1(C), A[N])$ as in Proposition 2.9 (i), resp. induced by the natural map $\mathfrak{Sha}^{\Sigma}(A) \to Ш(A)[N]$). It follows from this first that the left upper horizontal map $A(K)/N \to \mathfrak{Sel}^{\Sigma}(A)/N$ is injective. Now, applying the Snake Lemma to the above diagram, we see that the desired injectivity follows from Proposition 2.9 (i).
(ii) By (i), the natural map

$$\begin{array}{ccccc} \mathfrak{Sha}^{\Sigma}(A)^{\wedge,\Sigma} & & T^{\Sigma}Ш(A) & & T^{\Sigma}H^1(C, \mathcal{A}) \\ \| & & \| & & \| \\ \varprojlim_N \mathfrak{Sha}^{\Sigma}(A)/N & \to & \varprojlim_N Ш(A)[N] & \subset & \varprojlim_N H^1(C, \mathcal{A})[N], \end{array}$$

where N runs over all Σ-integers > 0, is injective, as desired. $\square$

Proposition 3.6. *The natural map* $\mathfrak{Sha}^{\Sigma}(A) \to T^{\Sigma}Ш(A)$ *induces a natural* **injective** *map*

$$\mathfrak{Sha}^{\Sigma}(A)^{(\dagger)} \hookrightarrow T^{\Sigma}Ш(A) \subset T^{\Sigma}H^1_{\text{ét}}(C, \mathcal{A}).$$

Proof. Consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{Sha}^{\Sigma}(A)^{\text{tor},\dagger'} & \longrightarrow & \mathfrak{Sha}^{\Sigma}(A) & \longrightarrow & \mathfrak{Sha}^{\Sigma}(A)^{(\dagger)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (\mathfrak{Sha}^{\Sigma}(A)^{\text{tor},\dagger})^{\wedge,\Sigma} & \longrightarrow & \mathfrak{Sha}^{\Sigma}(A)^{\wedge,\Sigma} & \longrightarrow & (\mathfrak{Sha}^{\Sigma}(A)^{(\dagger)})^{\wedge,\Sigma} & \longrightarrow & 0
\end{array}$$

where $\mathfrak{Sha}^{\Sigma}(A)^{\text{tor},\dagger'}$ stands for $\mathfrak{Sha}^{\Sigma}(A)^{\text{tor},(\mathfrak{Primes}^{\dagger})'}$ (i.e., $\mathfrak{Sha}^{\Sigma}(A)^{\text{tor},\dagger'} = \mathfrak{Sha}^{\Sigma}(A)^{\text{tor},p}$ for $p > 0$ and $\mathfrak{Sha}^{\Sigma}(A)^{\text{tor},\dagger'} = 0$ for $p = 0$) and the lower horizontal sequence is the Σ-adic completion of the upper horizontal sequence. As $\Sigma \subset \mathfrak{Primes}^{\dagger}$, we have $(\mathfrak{Sha}^{\Sigma}(A)^{\text{tor},\dagger'})^{\wedge,\Sigma} = 0$, hence $\mathfrak{Sha}^{\Sigma}(A)^{\wedge,\Sigma} \xrightarrow{\sim} (\mathfrak{Sha}^{\Sigma}(A)^{(\dagger)})^{\wedge,\Sigma}$. Thus, the desired injectivity is equivalent (cf. Proposition 3.5 (ii)) to the injectivity of the right vertical map $\mathfrak{Sha}^{\Sigma}(A)^{(\dagger)} \to (\mathfrak{Sha}^{\Sigma}(A)^{(\dagger)})^{\wedge,\Sigma}$, which follows from the fact (cf. Proposition 3.3) that $\mathfrak{Sha}^{\Sigma}(A)^{(\dagger)}$ is a finitely generated free $\mathbb{Z}$-module, as $\Sigma \neq \emptyset$. □

Proposition 3.7. *Assume that there exists* $l \in \Sigma$ *such that* $T_l Ш(A) = 0$. *Then* $\mathfrak{Sha}^{\Sigma}(A)^{(\dagger)} = 0$.

Proof. We have

$$\mathfrak{Sha}^{\Sigma}(A)^{(\dagger)} \hookrightarrow \mathfrak{Sha}^{\{l\}}(A)^{(\dagger)} \hookrightarrow T_l Ш(A) = 0,$$

by Proposition 3.4 and Proposition 3.6. Thus, the assertion follows. □

We conjecture the following.

Conjecture 3.8. We have $\mathfrak{Sha}^{\Sigma}(A) = 0$ or, equivalently, $\mathfrak{Sel}^{\Sigma}(A) = A(K)/\{\Sigma'\text{-tor}\}$ unconditionally.

Proposition 3.9. *Let* N *be a* $\mathfrak{Primes}^{\dagger}$*-integer* > 0. *Then:*
(i) $Ш(A)[N]$ *is* **finite**.
(ii) $Ш(A)/N$ *is* **finite**.

Proof. (i) This follows from Proposition 2.10, as we have an exact sequence

$$0 \to A(K)/N \to \text{Sel}_N(A) \to Ш(A)[N] \to 0.$$

(ii) This follows from (i), together with Lemma 3.10 below. □

Lemma 3.10. *Let* M *be a torsion abelian group and* $N > 0$ *an integer. Assume that* $M[N]$ *is finite. Then* M/N *is finite.*

Proof. There are several ways of proving this elementary fact. For example, consider the following decreasing sequence:

$$M[N] \supset NM[N^2] \supset \cdots \supset N^{n-1}M[N^n] \supset N^n M[N^{n+1}] \supset \cdots,$$

which stabilises as $M[N]$ is finite. Thus, there exists an integer $n_0 > 0$, such that $C \overset{\text{def}}{=} N^{n_0-1}M[N^{n_0}] = N^{n-1}M[N^n]$ ($\subset M[N]$) for all $n > n_0$. Now, for each $n > n_0$, consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M[N^{n-1}] & \longrightarrow & M[N^n] & \xrightarrow{N^{n-1}} & C & \longrightarrow & 0 \\ & & {\scriptstyle N}\downarrow & & {\scriptstyle N}\downarrow & & {\scriptstyle N=0}\downarrow & & \\ 0 & \longrightarrow & M[N^{n-1}] & \longrightarrow & M[N^n] & \xrightarrow{N^{n-1}} & C & \longrightarrow & 0. \end{array}$$

By the Snake Lemma, we have an exact sequence:

$$0 \to C \to M[N^{n-1}]/N \to M[N^n]/N \to C \to 0.$$

Thus, in particular, $\sharp(M[N^n]/N)$ stabilises and is bounded, hence

$$M/N = M[N^\infty]/N = (\varinjlim_{n\geq 0} M[N^n])/N = \varinjlim_{n\geq 0}(M[N^n]/N)$$

is finite, as desired. □

Proposition 3.11. *Let $A \to A'$ be an isogeny of abelian varieties over K. Then it induces a natural homomorphism $Ш(A) = Ш(A, C) \to Ш(A', C) = Ш(A')$, hence a natural homomorphism $Ш(A)^{(\dagger)} \to Ш(A')^{(\dagger)}$. Further, the kernel and the cokernel of the latter homomorphism are both finite.*

Proof. By definition, any homomorphism $A \to A'$ over K induces $Ш(A) \to Ш(A')$, hence $Ш(A)^{(\dagger)} \to Ш(A')^{(\dagger)}$, functorially. Now, if $f : A \to A'$ is an isogeny, then there exist an isogeny $g : A' \to A$ and an integer $N > 0$, such that $g \circ f = N \cdot \mathrm{id}_A$ and $f \circ g = N \cdot \mathrm{id}_{A'}$. We define $N^\dagger$ to be the maximal $\mathfrak{Primes}^\dagger$-integer dividing N. (Thus, $N/N^\dagger$ is 1 (resp. the maximal p-power dividing N) if $p = 0$ (resp. $p > 0$).) By functoriality, these equalities imply $\mathrm{Ker}(Ш(A) \to Ш(A')) \subset Ш(A)[N]$, hence $\mathrm{Ker}(Ш(A)^{(\dagger)} \to Ш(A')^{(\dagger)}) \hookrightarrow Ш(A)[N^\dagger]$, and $Ш(A')/N \twoheadrightarrow \mathrm{Coker}(Ш(A) \to Ш(A'))$, hence $Ш(A')/N^\dagger \twoheadrightarrow \mathrm{Coker}(Ш(A)^{(\dagger)} \to Ш(A')^{(\dagger)})$. Now, the desired finiteness follows from Proposition 3.9. □

Proposition 3.12. *Let k'/k be a finite extension of fields (finitely generated over the prime field), $C' \to \operatorname{Spec} k'$ a smooth, separated and geometrically connected algebraic curve over k', and $C' \to C$ a dominant k-morphism. Write $K' = k'(C')$ for the function field of C', and let K'/K be the (finite) extension of function fields induced by $C' \to C$. Then it induces a natural homomorphism $Ш(A) = Ш(A, C) \to Ш(A_{K'}, C') = Ш(A_{K'})$, hence a natural homomorphism $Ш(A)^{(\dagger)} \to Ш(A_{K'})^{(\dagger)}$. Further, the kernel of the former (resp. latter) homomorphism is finite, if K'/K is separable (resp. in general).*

Proof. By definition, $C' \to C$ induces $Ш(A) = Ш(A, C) \to Ш(A_{K'}, C') = Ш(A_{K'})$, hence $Ш(A)^{(\dagger)} \to Ш(A_{K'})^{(\dagger)}$, functorially. First, assume that K'/K is separable. Then, replacing K'/K by its Galois closure K''/K, k' by the algebraic closure k'' of k in K'' and C' by the smooth locus over k'' of the integral closure of C' in K'', if necessary, we may reduce the finiteness of $\mathrm{Ker}(Ш(A) \to Ш(A_{K'}))$ to the case where K'/K is Galois. In this case, we have

$$\mathrm{Ker}(Ш(A) \to Ш(A_{K'})) \subset \mathrm{Ker}(H^1(G_K, A) \to H^1(G_{K'}, A_{K'})) = H^1(\mathrm{Gal}(K'/K), A(K')),$$

which is finite by Lemma 1.6 (i) (together with [Lang-Néron]), as desired.

Thus, to prove the finiteness of $\mathrm{Ker}(Ш(A)^{(\dagger)} \to Ш(A_{K'})^{(\dagger)})$ in general, we may assume $p > 0$. Then, for any finite extension K'/K, there exist $n \geq 0$ and a finite separable extension $K''/K^{\frac{1}{p^n}}$ such that $K'' \supset K'$. The algebraic closures of k in K' and $K^{\frac{1}{p^n}}$ are k' and $k^{\frac{1}{p^n}}$, respectively, and let k'' be the algebraic closure of k in K'', which is separable over $k^{\frac{1}{p^n}}$. Let $\tilde{C}'$, $C^{\frac{1}{p^n}}$ and $\tilde{C}''$ be the integral closure of C in K', $K^{\frac{1}{p^n}}$ and K'', respectively. Then $\tilde{C}'$, $C^{\frac{1}{p^n}}$ and $\tilde{C}''$ are regular, separated, geometrically connected curves over k', $k^{\frac{1}{p^n}}$ and k'', respectively. Further, $\tilde{C}' \supset C'$ is generically smooth over k', $(C^{\frac{1}{p^n}} \to \mathrm{Spec}\, k^{\frac{1}{p^n}}) \simeq (C \to \mathrm{Spec}\, k)$ is smooth, and $\tilde{C}''$ is generically étale over $C^{\frac{1}{p^n}}$, hence generically smooth over $k^{\frac{1}{p^n}}$, and over k''. Let $C'' \subset \tilde{C}''$ be the intersection of the smooth locus of $\tilde{C}'' \to \mathrm{Spec}\, k''$ and the inverse image of C' under the (finite) morphism $\tilde{C}'' \to \tilde{C}'$. Then C'' is smooth, separated, geometrically connected curve over k''.

Now, the natural homomorphism $Ш(A, C) \to Ш(A_{K''}, C'')$ factors in two ways, as $Ш(A, C) \to Ш(A_{K'}, C') \to Ш(A_{K''}, C'')$ and as $Ш(A, C) = Ш(A_{K^{\frac{1}{p^0}}}, C^{\frac{1}{p^0}}) \to Ш(A_{K^{\frac{1}{p}}}, C^{\frac{1}{p}}) \to Ш(A_{K^{\frac{1}{p^2}}}, C^{\frac{1}{p^2}}) \to \cdots \to Ш(A_{K^{\frac{1}{p^n}}}, C^{\frac{1}{p^n}}) \to Ш(A_{K''}, C'')$. It follows from this that the finiteness of $\mathrm{Ker}(Ш(A)^{(\dagger)} \to Ш(A_{K'})^{(\dagger)})$ is reduced to that of $\mathrm{Ker}(Ш(A_{K^{\frac{1}{p^i}}}, C^{\frac{1}{p^i}})^{(\dagger)} \to Ш(A_{K^{\frac{1}{p^{i+1}}}}, C^{\frac{1}{p^{i+1}}})^{(\dagger)})$ $(i = 0, 1, \ldots, n-1)$ and that of $\mathrm{Ker}(Ш(A_{K^{\frac{1}{p^n}}}, C^{\frac{1}{p^n}})^{(\dagger)} \to Ш(A_{K''}, C'')^{(\dagger)})$. The latter finiteness follows from the above argument, as $K''/K^{\frac{1}{p^n}}$ is separable. For the former finiteness, it suffices to prove it for $i = 1$. In this case, the inclusion $K \subset K^{\frac{1}{p}}$ can be identified with the inclusion $\sigma : K \hookrightarrow K$, $x \mapsto x^p$. Under this identification, we have $A_{K^{\frac{1}{p}}} = A \times_{K,\sigma} K$ and the homomorphism $Ш(A, C) \to Ш(A_{K^{\frac{1}{p}}}, C^{\frac{1}{p}})$ is identified with the homomorphism $Ш(A, C) \to Ш(A \times_{K,\sigma} K, C)$ induced by the relative Frobenius K-morphism $A \to A \times_{K,\sigma} K$. Thus, the desired finiteness of $\mathrm{Ker}(Ш(A)^{(\dagger)} \to Ш(A_{K^{\frac{1}{p}}})^{(\dagger)})$ follows from Proposition 3.11. □

Remark 3.13. For simplicity, write Q for the prime field of k and Z for the image of $\mathbb{Z}$ in Q. Thus, we have $Q = \mathbb{Q}$ (resp. $Q = \mathbb{F}_p$) and $Z = \mathbb{Z}$ (resp. $Z = \mathbb{F}_p$) when $p = 0$ (resp. $p > 0$). Then, as k is finitely generated over the perfect field Q, the system $C \to \mathrm{Spec}\, k \to \mathrm{Spec}\, Q$ admits a smooth model $\mathcal{C} \to V \to U$. More precisely, $U = \mathrm{Spec}\, Z$; V is an integral scheme which is smooth over U and whose function field is isomorphic to (and is identified with) k; $\mathcal{C}$ is a smooth scheme over V whose generic fibre $\mathcal{C} \times_V k$ is k-isomorphic to (and is identified with) C. Let $\mathcal{C}^1$ denote the set of points of codimension 1 of $\mathcal{C}$, hence we have $C^{\mathrm{cl}} \subset \mathcal{C}^1$. For each $c \in \mathcal{C}^1$, let K_c be the completion of K at c, and $A_c \stackrel{\mathrm{def}}{=} A \times_K K_c$, just as in the case of $c \in C^{\mathrm{cl}}$. We define

$$\mathrm{Sel}_N(A, \mathcal{C}) \stackrel{\mathrm{def}}{=} \mathrm{Ker}\big(H^1(G_K, A[N]) \to \prod_{c \in \mathcal{C}^1} H^1(G_{K_c}, A_c)\big)$$

for each $\mathfrak{Primes}^\dagger$-integer $N > 0$, and

$$Ш(A, \mathcal{C}) \stackrel{\mathrm{def}}{=} \mathrm{Ker}\big(H^1(G_K, A) \to \prod_{c \in \mathcal{C}^1} H^1(G_{K_c}, A_c)\big).$$

Thus, we have

$$\mathrm{Sel}_N(A,\mathcal{C}) \subset \mathrm{Sel}_N(A) = \mathrm{Sel}_N(A,C)$$

and

$$Ш(A,\mathcal{C}) \subset Ш(A) = Ш(A,C).$$

In [Lang-Tate], Theorem 3 and Theorem 5, it is shown that $\mathrm{Sel}_N(A,\mathcal{C})$ and $Ш(A,\mathcal{C})[N]$ are finite. Thus, Proposition 2.10 and Proposition 3.9 (i) can be regarded as an improvement of these results, respectively.

§**4. The isotrivial case.** We use the same notations as in §1, §2 and §3. In particular, k is a field which is **finitely generated** over the prime field of characteristic $p \geq 0$ and infinite; $C \to \mathrm{Spec}\, k$ is a smooth, separated and geometrically connected algebraic curve over k with function field $K = k(C)$; and $\mathcal{A} \to C$ is an abelian scheme over C with generic fibre $A = \mathcal{A}_K = \mathcal{A} \times_C \mathrm{Spec}\, K$.

Theorem 4.1. *Assume that A is* **essentially isotrivial**, *i.e., $A_{\overline{K}}$ is isogenous to an abelian variety over $\overline{K}$ that descends to an abelian variety over $\overline{k}$. Then $Ш(A)^{(\dagger)}$ is* **finite**.

Proof. By Proposition 3.12 and Proposition 3.11, we may assume that $C(k) \neq \emptyset$, that C admits a (unique) smooth compactification C^{cpt}, and that A is constant, i.e., A descends to an abelian variety $\widetilde{A}$ over k ($\widetilde{A} \times_k K = A$). In order to prove that $Ш(A)^{(\dagger)}$ is finite, it suffices to show that $Ш(A)[l^\infty]$ is finite for all $l \in \mathfrak{Primes}^\dagger$ and that $Ш(A)[l^\infty] = 0$ for all but finitely many $l \in \mathfrak{Primes}^\dagger$. Further, as $Ш(A)[l^n]$ is finite for all $n \geq 0$ by Proposition 3.9, the condition that $Ш(A)[l^\infty]$ is finite is equivalent to: $T_l Ш(A) = 0$.

Let $l \in \mathfrak{Primes}^\dagger$. We view $(T_l\widetilde{A} \xrightarrow{\sim})T_lA$, which is fixed by $\pi_1(C_{\overline{k}})$, as a G_k-module, and we identify $H^1(G_k, T_lA)$ with $H^1(G_k, T_l\widetilde{A})$. We have a natural inflation-restriction exact sequence

$$0 \to H^1(G_k, T_l\widetilde{A}) \xrightarrow{\mathrm{inf}} H^1(\pi_1(C), T_lA) \xrightarrow{\mathrm{res}} \mathrm{Hom}(\pi_1(C_{\overline{k}}), T_lA)^{G_k}.$$

First, observe that (cf. Definition 2.3 (ii) for the definition of $\mathrm{Sel}^{\{l\}}(A)$)

$$\mathrm{Sel}^{\{l\}}(A) \cap H^1(G_k, T_l\widetilde{A}) = A(K)^{\wedge,l} \cap H^1(G_k, T_l\widetilde{A}) = \widetilde{A}(k)^{\wedge,l}$$

holds in $H^1(\pi_1(C), T_lA)$. Indeed, the inclusions

$$\mathrm{Sel}^{\{l\}}(A) \cap H^1(G_k, T_l\widetilde{A}) \supset A(K)^{\wedge,l} \cap H^1(G_k, T_l\widetilde{A}) \supset \widetilde{A}(k)^{\wedge,l}$$

are clear. To prove $\mathrm{Sel}^{\{l\}}(A) \cap H^1(G_k, T_l\widetilde{A}) \subset \widetilde{A}(k)^{\wedge,l}$, fix $c \in C(k) \neq \emptyset$. Then the composite of the inflation map $H^1(G_k, T_l\widetilde{A}) \to H^1(\pi_1(C), T_lA)$ and the restriction map $H^1(\pi_1(C), T_lA) \to H^1(G_{k(c)}, T_l\mathcal{A}_c) = H^1(G_k, T_l\widetilde{A})$ at c is the identity. As the image of $\mathrm{Sel}^{\{l\}}(A)$ under the restriction map $H^1(\pi_1(C), T_lA) \to H^1(G_{k(c)}, T_l\mathcal{A}_c) = H^1(G_k, T_l\widetilde{A})$ at c is included in $\widetilde{A}(k)^{\wedge,l}$ by definition, we obtain the desired inclusion.

Now, let $\varphi : A(K)^{\wedge,l} \to \mathrm{Hom}(\pi_1(C_{\overline{k}}), T_lA)^{G_K}$ be the composite of the natural maps $A(K)^{\wedge,l} \hookrightarrow H^1(\pi_1(C), T_lA) \to \mathrm{Hom}(\pi_1(C_{\overline{k}}), T_lA)^{G_K}$. Thus, we have a natural map

$$T_lШ(A) = \mathrm{Sel}^{\{l\}}(A)/A(K)^{\wedge,l} \to \mathrm{Hom}(\pi_1(C_{\overline{k}}), T_lA)^{G_k}/\varphi(A(K)^{\wedge,l}),$$

which is injective as

$$\mathrm{Sel}^{\{l\}}(A) \cap H^1(G_k, T_l\widetilde{A}) = \widetilde{A}(k)^{\wedge,l} \subset A(K)^{\wedge,l}.$$

To prove that $T_l Ш(A) = 0$, it suffices to show that $\mathrm{Hom}(\pi_1(C_{\overline{k}}), T_l A)^{G_k}/\varphi(A(K)^{\wedge,l}) = 0$. The Tate conjecture for abelian varieties holds over finitely generated fields by Tate, Zarhin, Mori in positive characteristic and by Faltings in characteristic 0 (cf. [Tate], [Zarhin1], [Moret-Bailly] and [Faltings1]). As a consequence, we have a natural isomorphism

$$\mathrm{Hom}_k(J, \widetilde{A}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \xrightarrow{\sim} \mathrm{Hom}(T_l J, T_l\widetilde{A})^{G_k} = \mathrm{Hom}(\pi_1(C^{\mathrm{cpt}}_{\overline{k}})^{\mathrm{ab},l}, T_l\widetilde{A})^{G_k},$$

where J denotes the jacobian variety of C^{cpt}. We also have natural isomorphisms

$$\mathrm{Hom}(\pi_1(C^{\mathrm{cpt}}_{\overline{k}})^{\mathrm{ab},l}, T_l\widetilde{A})^{G_k} \xrightarrow{\sim} \mathrm{Hom}(\pi_1(C_{\overline{k}})^{\mathrm{ab},l}, T_l\widetilde{A})^{G_k} \xrightarrow{\sim} \mathrm{Hom}(\pi_1(C_{\overline{k}}), T_l\widetilde{A})^{G_k},$$

where the first isomorphism follows from a standard weight argument. More precisely, we can assume without loss of generality (after possibly replacing k by a finite extension) that $C^{\mathrm{cpt}} \setminus C = \{c_0, c_1, \dots, c_n\} \subset C^{\mathrm{cpt}}(k)$. Thus, $I_C \stackrel{\mathrm{def}}{=} I_C^{(\mathrm{ab},l)} \stackrel{\mathrm{def}}{=} \mathrm{Ker}\left(\pi_1(C_{\overline{k}})^{\mathrm{ab},l} \twoheadrightarrow \pi_1(C^{\mathrm{cpt}}_{\overline{k}})^{\mathrm{ab},l}\right) \xrightarrow{\sim} \mathrm{Coker}\left(\mathbb{Z}_l(1) \xrightarrow{\mathrm{diag}} \oplus_{i=0}^n \mathbb{Z}_l(1)\right) \xrightarrow{\sim} \oplus_{i=1}^n \mathbb{Z}_l(1)$ as G_k-module. Now the G_k-representation $I_C \otimes \mathbb{Q}_l$ (resp. $T_l\widetilde{A} \otimes \mathbb{Q}_l$) is pure of weight -2 (resp. pure of weight -1) (cf. [Jannsen], 2), and $\mathrm{Hom}(I_C, T_l\widetilde{A})^{G_k} = 0$ follows (cf. loc. cit. Fact 2). Hence $\mathrm{Hom}(\pi_1(C^{\mathrm{cpt}}_{\overline{k}})^{\mathrm{ab},l}, T_l\widetilde{A})^{G_k} \xrightarrow{\sim} \mathrm{Hom}(\pi_1(C_{\overline{k}})^{\mathrm{ab},l}, T_l\widetilde{A})^{G_k}$. Further, as $C^{\mathrm{cpt}}(k) \neq \emptyset$, the natural map

$$A(K) = \mathrm{Mor}_k(C^{\mathrm{cpt}}, \widetilde{A}) \to \mathrm{Hom}_k(J, \widetilde{A})$$

induced by the Albanese property of J is surjective. Thus, the above map $\varphi : A(K)^{\wedge,l} \to \mathrm{Hom}(\pi_1(C_{\overline{k}}), T_l\widetilde{A})^{G_k}$ is surjective and $\mathrm{Hom}(\pi_1(C_{\overline{k}}), T_l\widetilde{A})^{G_k}/\varphi(A(K)^{\wedge,l})$ is trivial, as desired.

Next, we prove that $Ш(A)[l^\infty] = 0$, or, equivalently, $Ш(A)[l] = 0$, for all but finitely many $l \in \mathfrak{Primes}^\dagger$. Indeed, this follows from a similar argument as above using the following truncated version of the Tate conjecture

$$\mathrm{Hom}_k(J, \widetilde{A}) \otimes_{\mathbb{Z}} \mathbb{Z}/l\mathbb{Z} \xrightarrow{\sim} \mathrm{Hom}(J[l], \widetilde{A}[l])^{G_k},$$

which holds for all but finitely many $l \in \mathfrak{Primes}^\dagger$ (cf. [Zarhin2], [Zarhin3] in positive characteristic and [Faltings2], VI, §3 in characteristic 0). □

Theorem 4.2. *Assume that A is* **essentially isotrivial***. Then the assertion of Conjecture 3.8 holds (resp. holds up to p-torsion) if $p = 0$ (resp. $p > 0$). More precisely, we have $\mathfrak{Sha}^\Sigma(A)^{(\dagger)} = 0$.*

Proof. This follows immediately from Theorem 4.1 and Proposition 3.7. □

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