Behaviour of L_q norms of the sinc_p function

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Abstract

An integral inequality due to Ball involves the L_q norm of the sinc_p function; the dependence of this norm on q as $q \to \infty$ is now understood. By use of recent inequalities involving $p-$trigonometric functions $(1 < p < \infty)$ we obtain asymptotic information about the analogue of Ball's integral when sin is replaced by $\sin_p$.

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1 Introduction

In [1], K. Ball proved that every section of the unit cube in $\mathbb{R}^n$ by an $(n-1)-$dimensional subspace has $(n-1)-$volume at most $\sqrt{2}$, which is attained if and only if this section contains an $(n-2)-$dimensional face of the cube. To show this, Ball made essential use of the inequality

$$\sqrt{q}\int_{-\infty}^{\infty}\left|\frac{\sin x}{x}\right|^q \mathrm{d}x \le \sqrt{2}\,\pi, \qquad q \ge 2,$$

in which equality holds if and only if $q = 2$.

It is now known (see [3]) that

$$\lim_{q\to\infty}\sqrt{q}\int_{-\infty}^{\infty}\left|\frac{\sin x}{x}\right|^q \mathrm{d}x = \sqrt{\frac{3\pi}{2}}. \tag{1}$$

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Moreover, the asymptotic properties of the q-norm of the sinc function were studied. In fact, more precise results of an asymptotic nature of the integral in Ball's integral inequality are now known (see [8] for more details).

Stimulated by applications to such differential operators as the p-Laplacian, there is now a large amount of recent work concerning generalisations of the sine function (and other trigonometric functions): see, for example, [5]. This encourages us to look for an extension of (1) to a more general setting. With this in mind, for $q \in (1,\infty)$, define

$$I_p(q) := q^{1/p} \int_0^\infty \left| \frac{\sin_p x}{x} \right|^q \mathrm{d}x$$

where $p \in (1,\infty)$ and $\sin_p$ is the generalised sine function. This function is defined to be the inverse of the function $F_p : [0,1] \longrightarrow [0, \frac{\pi_p}{2}]$ given by

$$F_p(y) := \int_0^y (1-t^p)^{-\frac{1}{p}} \mathrm{d}t, \tag{2}$$

where $\pi_p := 2F_p(1) = \frac{2\pi}{p \sin(\frac{\pi}{p})}$; it is increasing on $[0, \frac{\pi_p}{2}]$ and is extended to the whole of $\mathbb{R}$ to be a $2\pi_p$−periodic function (still denoted by $\sin_p$) by means of the rules

$$\sin_p(-x) = -\sin_p(x) \qquad \text{and} \qquad \sin_p\left(\frac{\pi_p}{2} - x\right) = \sin_p\left(\frac{\pi_p}{2} + x\right). \tag{3}$$

The choice $p = 2$ corresponds to the standard trigonometric setting: $\sin_2 \equiv \sin$, $\pi_2 = \pi$. Moreover, π_p is a decreasing function in $p \in (1,\infty)$ such that

$$\begin{cases} \pi_p \to \infty & \text{when} \quad p \to 1^+ \\ \pi_p \to 2 & \text{when} \quad p \to \infty. \end{cases} \tag{4}$$

The main purpose of this paper is to show that for each $p \in (1,\infty)$ there is an analogue of (1), namely

$$\lim_{q\to\infty} q^{1/p} \int_0^\infty \left| \frac{\sin_p x}{x} \right|^q dx = p^{-1}(p(p+1))^{1/p}\,\Gamma(1/p).$$

This is achieved by appropriate use of certain recently-obtained inequalities concerning $\sin_p$. Moreover, information is obtained about the asymptotic behaviour of the above integral as $q \to \infty$; this complements that known when $p = 2$.

2 Properties of the sinc_p function

Given $p \in (1,\infty)$, the function sinc_p is defined by

$$\mathrm{sinc}_p x = \begin{cases} \frac{\sin_p x}{x}, & x \in \mathbb{R}\backslash\{0\}, \\ 1, & x = 0. \end{cases}$$

It is even and its roots are the points $n\pi_p$ with $n \in \mathbb{Z}\backslash\{0\}$. Since $|\sin_p x| \le 1$ for all $x \in \mathbb{R}$, $\lim_{|x|\to\infty} \mathrm{sinc}_p x = 0$.

Lemma 2.1. *(a)* $|\operatorname{sinc}_p x| \le 1$ *for all* $x \in \mathbb{R}$.

(b) The function sinc_p *is strictly decreasing on the interval* $\left(0, \frac{\pi_p}{2}\right)$.

Proof. *(a)* Observe that for $x \in \left(0, \frac{\pi_p}{2}\right]$ we have the p-analogue of the classical Jordan inequality [4, Proposition 2.3],

$$\frac{2}{\pi_p} \le \operatorname{sinc}_p x < 1 \qquad \forall x \in \left(0, \frac{\pi_p}{2}\right]. \tag{5}$$

On the other hand, for $x \in \left(\frac{\pi_p}{2}, \infty\right)$ and since $\pi_p \in (2,\infty)$ for all $p \in (1,\infty)$ we conclude that

$$|\operatorname{sinc}_p x| = \left|\frac{\sin_p x}{x}\right| \le \frac{1}{|x|} < 1.$$

Since sinc_p is an even function, *(a)* is complete.
(b) Observe that

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{sinc}_p x = \frac{\cos_p x}{x^2}(x - \tan_p x)$$

and $\cos_p x > 0$ for any $x \in (0, \frac{\pi_p}{2})$. Let $g(x) = x - \tan_p x$. Then, $g'(x) = -\tan_p^p x < 0$ for all $x \in (0, \frac{\pi_p}{2})$ and we conclude that $g(x)$ is strictly decreasing in this interval. Then $g(x) < g(0) = 0$ for $x \in (0, \frac{\pi_p}{2})$. The result follows. ☐

3 The p-version of Ball's integral inequality

For $p, q \in (1,\infty)$ define

$$I_p(q) := q^{1/p} \int_0^\infty \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x.$$

Here we establish the existence of $\lim_{q\to\infty} I_p(q)$. We use Laplace's method for integrals which suggests approximating the integrand in a neighbourhood by simpler functions for which the integral can be evaluated after proving that the integral on the complementing interval is very small when q is large enough.

Lemma 3.1. *Let* $p, q \in (1,\infty)$. *For any real* $\alpha > 0$, *we have*

$$\int_\alpha^\infty \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x \le \begin{cases} \frac{1}{q-1}\alpha^{1-q} & \text{for} \quad \alpha \ge 1, \\ \left(\frac{\sin_p \alpha}{\alpha}\right)^q (1-\alpha) + \frac{1}{q-1} & \text{for} \quad \alpha < 1. \end{cases}$$

Moreover,

$$\lim_{q\to\infty} q^{1/p} \int_\alpha^\infty \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x = 0, \qquad \forall \alpha > 0.$$

Proof. We first discuss the case $\alpha \in [1,\infty)$.

$$\begin{aligned}\int_\alpha^\infty \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x &= \lim_{\beta\to\infty}\int_\alpha^\beta \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x \\ &\le \lim_{\beta\to\infty}\int_\alpha^\beta x^{-q}\mathrm{d}x \\ &= \lim_{\beta\to\infty}\frac{1}{1-q}\left[\frac{1}{\beta^{q-1}}-\frac{1}{\alpha^{q-1}}\right] = \frac{1}{q-1}\alpha^{1-q}.\end{aligned}$$

Then,

$$\lim_{q\to\infty} q^{1/p}\int_\alpha^\infty \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x \le \lim_{q\to\infty}\frac{q^{1/p}}{(q-1)\alpha^{q-1}} = 0.$$

For $\alpha \in (0,1)$, we have

$$\int_\alpha^\infty \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x = \int_\alpha^1 \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x + \int_1^\infty \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x.$$

From the previous case we have,

$$q^{1/p}\int_1^\infty \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x \le \frac{q^{1/p}}{q-1},$$

which approaches zero as $q\to\infty$.
For the remaining integral, the corresponding interval $(\alpha,1)$ is a subset of $(0,\frac{\pi_p}{2})$ since $\pi_p \in (2,\infty)$. According to Lemma 2.1,

$$0 < \frac{\sin_p x}{x} < \frac{\sin_p \alpha}{\alpha} < 1, \qquad \forall x \in (\alpha,1).$$

Then,

$$\int_\alpha^\infty \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x < \left(\frac{\sin_p \alpha}{\alpha}\right)^q (1-\alpha) + \frac{1}{q-1}.$$

Using Lemma 2.1 *(a)*, we conclude that

$$\lim_{q\to\infty} q^{1/p}\int_\alpha^\infty \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x \le \lim_{k\to\infty} q^{1/p}\left[\left(\frac{\sin_p \alpha}{\alpha}\right)^q (1-\alpha) + \frac{1}{q-1}\right] = 0.$$

□

Our main result is the following theorem:

Theorem 3.1. *Let* $p,q \in (1,\infty)$. *Then*

$$\lim_{q\to\infty} I_p(q) = \frac{1}{p}\Gamma\left(\frac{1}{p}\right)(p(p+1))^{1/p}.$$

Proof. From [6, 3.251, p. 324], for $\mu > 0$, $\nu > 0$, $\lambda > 0$ we have

$$\int_0^1 x^{\mu-1}(1-x^\lambda)^{\nu-1}\mathrm{d}x = \frac{1}{\lambda}B\left(\frac{\mu}{\lambda},\nu\right). \tag{6}$$

From [2, Theorem 1.1 (1)], for $0 < x < \left(1-\left(\frac{2}{\pi_p}\right)^{p(p+1)}\right)^{1/p}$ we have

$$\frac{x}{\sin_p^{-1} x} > (1-x^p)^{\frac{1}{p(p+1)}}. \tag{7}$$

Note that for all $p \in (1,\infty)$ we have

$$0 < \left(1-\left(\frac{2}{\pi_p}\right)^{p(p+1)}\right)^{1/p} < 1.$$

Let

$$\alpha_1 := \sin_p^{-1}\left(1-\left(\frac{2}{\pi_p}\right)^{p(p+1)}\right)^{1/p} \in \left(0,\frac{\pi_p}{2}\right).$$

Changing the variable to $y = \sin_p x$ and using the inequality in (7) we see that

$$\begin{aligned}
q^{1/p}\int_0^\infty \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x &> q^{1/p}\int_0^{\alpha_1}\left(\frac{\sin_p x}{x}\right)^q \mathrm{d}x \\
&= q^{1/p}\int_0^{\sin_p \alpha_1}\left(\frac{y}{\sin_p^{-1} y}\right)^q (1-y^p)^{-1/p}\mathrm{d}y \\
&> q^{1/p}\int_0^{\sin_p \alpha_1}(1-y^p)^{\frac{q}{p(p+1)}}(1-y^p)^{-1/p}\mathrm{d}y \\
&= q^{1/p}\left(\int_0^1 - \int_{\sin_p \alpha_1}^1 (1-y^p)^{\frac{q}{p(p+1)}}(1-y^p)^{-1/p}\mathrm{d}y\right) \\
&=: J_1(p,q) - J_2(p,q).
\end{aligned}$$

From (6) with $\mu = 1$, $\lambda = p$ and $\nu = \frac{q}{p(p+1)} - \frac{1}{p} + 1$ we get

$$\begin{aligned}
\lim_{q\to\infty} J_1(p,q) &= \lim_{q\to\infty} q^{1/p}\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\frac{\Gamma\left(\frac{q}{p(p+1)}-\frac{1}{p}+1\right)}{\Gamma\left(\frac{q}{p(p+1)}+1\right)} \\
&= \frac{1}{p}\Gamma\left(\frac{1}{p}\right)(p(p+1))^{1/p}.
\end{aligned} \tag{8}$$

The last equality is due to the fact that

$$\frac{\Gamma(q+a)}{\Gamma(q+b)} \sim q^{a-b} \quad \text{as} \quad q\to\infty, \tag{9}$$

which follows from Stirling's formula: see also [7, Problem 2, p.45]. Moreover,

$$\begin{aligned}\lim_{q\to\infty} J_2(p,q) &= \lim_{q\to\infty} q^{1/p}\int_{\sin_p \alpha_1}^{1} (1-y^p)^{\frac{q}{p(p+1)}}(1-y^p)^{-1/p}\mathrm{d}y\\ &\le \lim_{q\to\infty} q^{1/p}(1-\sin_p^p \alpha_1)^{\frac{q}{p(p+1)}}\int_{\sin_p \alpha_1}^{1} (1-y^p)^{-1/p}\mathrm{d}y\\ &\le \lim_{q\to\infty} q^{1/p}(1-\sin_p^p \alpha_1)^{\frac{q}{p(p+1)}}\int_{0}^{1} (1-y^p)^{-1/p}\mathrm{d}y\\ &= \frac{\pi_p}{2}\lim_{q\to\infty} q^{1/p}(1-\sin_p^p \alpha_1)^{\frac{q}{p(p+1)}} = 0. \qquad (10)\end{aligned}$$

Then from (8) and (10),

$$\liminf_{q\to\infty} q^{1/p}\int_0^\infty \left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x \ge \frac{1}{p}\Gamma\left(\frac{1}{p}\right)(p(p+1))^{1/p}. \qquad (11)$$

On the other hand, from [6, 3.251, p.325], for $0<\mu<p\nu$, $b>0$ and $p>0$ we have

$$\int_0^\infty x^{\mu-1}(1+bx^p)^{-\nu}\mathrm{d}x = \frac{1}{p}b^{-\mu/p}B\left(\frac{\mu}{p},\nu-\frac{\mu}{p}\right). \qquad (12)$$

From [2, Theorem 1.1 (1)], for $x\in(0,1)$ we have

$$\frac{x}{\sin_p^{-1} x} < \left(1+\frac{x^p}{p(p+1)}\right)^{-1}. \qquad (13)$$

Now let $\alpha_2 \in (0,\pi_p/2]$. Changing the variable to $y=\sin_p x$ and using the inequality in (13) we obtain

$$\begin{aligned}\int_0^{\alpha_2}\left(\frac{\sin_p x}{x}\right)^q \mathrm{d}x &= \int_0^{\sin_p \alpha_2}\left(\frac{y}{\sin_p^{-1} y}\right)^q (1-y^p)^{-1/p}\mathrm{d}y\\ &< \int_0^{\sin_p \alpha_2}\left(1+\frac{y^p}{p(p+1)}\right)^{-q}(1-y^p)^{-1/p}\mathrm{d}y\\ &< \left(1-\sin_p^p \alpha_2\right)^{-1/p}\int_0^{\sin_p \alpha_2}\left(1+\frac{y^p}{p(p+1)}\right)^{-q}\mathrm{d}y\\ &=: \left(1-\sin_p^p \alpha_2\right)^{-1/p} J_3(p,q). \qquad (14)\end{aligned}$$

From (12), with $\mu=1,\ b=(p(p+1))^{-1}$ and $\nu=q\in(1/p,\infty)$, we obtain

$$\begin{aligned}J_3(p,q) &< \frac{1}{p}(p(p+1))^{1/p}B\left(\frac{1}{p},q-\frac{1}{p}\right)\\ &= \frac{1}{p}(p(p+1))^{1/p}\Gamma\left(\frac{1}{p}\right)\frac{\Gamma\left(q-\frac{1}{p}\right)}{\Gamma(q)}. \qquad (15)\end{aligned}$$

With the help of (9) it follows that

$$\limsup_{q\to\infty} q^{1/p}\int_0^{\alpha_2}\left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x \le \left(1-\sin_p^p \alpha_2\right)^{-1/p}\frac{1}{p}(p(p+1))^{1/p}\,\Gamma\left(\frac{1}{p}\right).$$

Letting $\alpha_2 \to 0^+$ we conclude that,

$$\limsup_{q\to\infty} q^{1/p}\int_0^{\alpha_2}\left(\frac{\sin_p x}{x}\right)^q \mathrm{d}x \le \frac{1}{p}(p(p+1))^{1/p}\,\Gamma\left(\frac{1}{p}\right).$$

From Lemma 3.1 we conclude that

$$\begin{aligned}\limsup_{q\to\infty} q^{1/p}\int_0^{\infty}\left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x &= \limsup_{q\to\infty} q^{1/p}\left(\int_0^{\alpha_2}+\int_{\alpha_2}^{\infty}\left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x\right)\\ &\le \frac{1}{p}(p(p+1))^{1/p}\,\Gamma\left(\frac{1}{p}\right). \qquad (16)\end{aligned}$$

From (11) and (16) we deduce that

$$\lim_{q\to\infty} I_p(q) = \frac{1}{p}\Gamma\left(\frac{1}{p}\right)(p(p+1))^{1/p}.$$

□

4 Asymptotic Expansion

Here we investigate the asymptotic expansion of the $p-$Ball integral $I_p(q)$ by performing explicit calculations leading to a precise knowledge of the first two coefficients of the expansion. The study involved provides another proof of Theorem 3.1; the technique used is an adaptation of that developed in [8].

Theorem 4.1. *There exist constants $\gamma_3, \gamma_4, \ldots$ such that for q large enough*

$$I_p(q) \sim (p(p+1))^{1/p}\left(\frac{1}{p}\Gamma\left(\frac{1}{p}\right)+\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\frac{(-p^2+p+1)(p+1)}{2p(2p+1)}\frac{1}{q}+\sum_{j=3}^{\infty}\Gamma(j+1/p)\frac{\gamma_j}{q^j}\right).$$

Proof. For $\alpha\in(0,1)$, let

$$\begin{aligned}J(q,\alpha) &:= q^{1/p}\int_0^{\alpha}\left(\frac{\sin_p x}{x}\right)^q \mathrm{d}x\\ &= q^{1/p}\int_0^{\alpha}\exp\left(\frac{-qx^p}{p(p+1)}\right)\left[\exp\left(\frac{x^p}{p(p+1)}\right)\frac{\sin_p x}{x}\right]^q \mathrm{d}x.\end{aligned}$$

By Lemma 3.1,

$$q^{1/p}\int_{\alpha}^{\infty}\left|\frac{\sin_p x}{x}\right|^q \mathrm{d}x \le q^{1/p}\left(\frac{\sin_p \alpha}{\alpha}\right)^q (1-\alpha)+\frac{q^{1/p}}{q-1}.$$

It is therefore enough to establish the existence of constants $\gamma_3, \gamma_4, \ldots$ such that

$$J(q,\alpha) \sim (p(p+1))^{1/p}\left(\frac{1}{p}\Gamma\left(\frac{1}{p}\right)+\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\frac{(-p^2+p+1)(p+1)}{2p(2p+1)}\frac{1}{q}+\sum_{j=3}^{\infty}\Gamma(j+1/p)\frac{\gamma_j}{q^j}\right).$$

Changing the variable to $u=\frac{x}{(p(p+1))^{1/p}}$ yields

$$J(q,\alpha)=q^{1/p}(p(p+1))^{1/p}\int_0^{\frac{\alpha}{(p(p+1))^{1/p}}}\exp\left(-qu^p\right)\left[\exp\left(u^p\right)\frac{\sin_p\left((p(p+1))^{1/p}u\right)}{(p(p+1))^{1/p}u}\right]^q \mathrm{d}u.$$

For the exponential term we have

$$\exp\left(u^p\right)=\sum_{j=0}^{\infty}\frac{u^{pj}}{j!}.$$

While for $\frac{\sin_p\left((p(p+1))^{1/p}u\right)}{(p(p+1))^{1/p}u}$, we have from [4, (2.17)] the power series expansion of $\sin_p^{-1}x$, and by the Lagrange reversion theorem this gives the existence of constants a_j such that

$$\frac{\sin_p\left((p(p+1))^{1/p}u\right)}{(p(p+1))^{1/p}u}=\sum_{j=0}^{\infty}a_j(p(p+1))^j u^{pj};$$

the series converges for sufficiently small u. The coefficients of the first three terms of this expansion involve $a_0=1$, $a_1=\frac{-1}{p(p+1)}$ and $a_2=\frac{-p^2+2p+1}{2p^2(p+1)(2p+1)}$. The Cauchy product formula then gives,

$$\exp\left(u^p\right)\frac{\sin_p\left((p(p+1))^{1/p}u\right)}{(p(p+1))^{1/p}u}=1+\sum_{j=2}^{\infty}b_ju^{pj},$$

where

$$b_j=\sum_{l=0}^{j}\frac{a_{j-l}(p(p+1))^{j-l}}{l!}$$

and the power series converges for sufficiently small u.

We know that for small values of u,

$$\left|\exp\left(u^p\right)\frac{\sin_p\left((p(p+1))^{1/p}u\right)}{(p(p+1))^{1/p}u}-1\right|=\left|\sum_{j=2}^{\infty}b_ju^{pj}\right|\le\sum_{j=2}^{\infty}\left|b_j\right|u^{pj}<1.$$

Note that the power series $\sum_{j=2}^{\infty}b_ju^{pj}$ is absolutely convergent for sufficiently small u.

Therefore by the Binomial expansion we get

$$\left[\exp\left(u^p\right)\frac{\sin_p\left((p(p+1))^{1/p}u\right)}{(p(p+1))^{1/p}u}\right]^q=1+q\left[\sum_{j=2}^{\infty}b_ju^{pj}\right]+\frac{q(q-1)}{2}\left[\sum_{j=2}^{\infty}b_ju^{pj}\right]^2 +\cdots+\frac{q(q-1)\ldots(q-m+1)}{m!}\left[\sum_{j=2}^{\infty}b_ju^{pj}\right]^m+\cdots$$

Since the right hand side of the Binomial expansion is bounded from above by

$$1+q\left[\sum_{j=2}^{\infty}|b_j|u^{pj}\right]+\frac{q(q-1)}{2}\left[\sum_{j=2}^{\infty}|b_j|u^{pj}\right]^2$$
$$+\cdots+\frac{q(q-1)\ldots(q-m+1)}{m!}\left[\sum_{j=2}^{\infty}|b_j|u^{pj}\right]^m+\cdots=\left[1+\sum_{j=2}^{\infty}|b_j|u^{pj}\right]^q,$$

we may rearrange terms and, for small enough u, obtain (17) . Hence,

$$\left[\exp\left(u^p\right)\frac{\sin_p\left((p(p+1))^{1/p}u\right)}{(p(p+1))^{1/p}u}\right]^q=\sum_{j=0}^{\infty}c_ju^{pj}, \tag{17}$$

where $c_0=1$, $c_1=0$ and $c_2=qb_2=\frac{qp(-p^2+p+1)}{2(2p+1)}$. Observe that the other coefficients $c_j=c_j(q)$ $(j\geq 3)$ can be obtained by the following rearrangements:

$$qb_3u^{3p}=c_3u^{3p},\quad \left(qb_4+\frac{q(q-1)}{2}b_2^2\right)u^{4p}=c_4u^{4p},\ldots$$

Specialised to our case, [9, Theorem 8.1, p. 86] (with $x=q$, $p(t)=u^p$, $q(t)=\exp(u^p)\frac{\sin_p\left((p(p+1))^{1/p}u\right)}{(p(p+1))^{1/p}u}$, $s=pj$, $\lambda=1$ and $\mu=p$) establishes the existence of real constants $\gamma_0,\gamma_1,\ldots$ such that

$$\begin{aligned}J(q,\alpha)&\sim q^{1/p}(p(p+1))^{1/p}\sum_{j=0}^{\infty}\Gamma(j+1/p)\frac{\gamma_j}{q^{j+1/p}}\\&=(p(p+1))^{1/p}\sum_{j=0}^{\infty}\Gamma(j+1/p)\frac{\gamma_j}{q^j},\quad (q\to\infty)\end{aligned}$$

where

$$\gamma_0=\frac{1}{p},\quad \gamma_1=0\quad\text{and}\quad \gamma_2=\frac{q(-p^2+p+1)}{2(2p+1)}.$$

□

Remark 4.1. *The asymptotic expansion in Theorem 4.1 complements that of [8] when $p=2$; it involves the coefficients b_j of the expansion of* $\exp(u^p)\frac{\sin_p\left((p(p+1))^{1/p}u\right)}{(p(p+1))^{1/p}u}$ *which depend on the constants a_j of the power series of the function* sinc_p. *So far the first three terms in the expansion of* $\sin_p$ *are known and no regular pattern has been obtained for the other subsequent terms. It remains to see whether or not higher-order terms in the expansion of $I_p(q)$ can be determined.*

5 Concluding remarks

In this section we present some results obtained from Theorem 3.1. The proofs are natural adaptations of those given in [3] and are therefore omitted.

For $q \in (1,\infty)$ and $n \in \mathbb{N} \cup \{0\}$, define

$$\varphi_p(n,q) := \int_0^\infty \left(\ln \left| \frac{\sin_p x}{x} \right| \right)^n \left| \frac{\sin_p x}{x} \right|^q \, \mathrm{d}x.$$

Note that $\varphi_p(0,q) := \varphi_p(q) = I_p(q)$.

A more general result of $p-$Ball integral inequality can also be achieved by induction for any non-negative integer n.

Lemma 5.1. *For $n \in \mathbb{N} \cup \{0\}$ and $p \in (1,\infty)$. Then*

$$\lim_{q\to\infty} q^{n+\frac{1}{p}} \varphi_p(n,q) = (-1)^n \frac{1}{p} \Gamma\left(\frac{1}{p}\right) (p(p+1))^{1/p} \, \Gamma\left(n + \frac{1}{p}\right).$$

The following gives the analyticity of the function $\varphi_p(q)$ in a region containing $(1,\infty)$. The proof makes use of the L_q-Lebesgue integrability of the sinc_p functions when $p, q \in (1,\infty)$.

Corollary 5.1. *Let $q \in (1,\infty)$. For $1-q < z < q-1$,*

$$\varphi_p(q-z) = \sum_{n=0}^{\infty} (-1)^n \varphi_p(n,q) \frac{z^n}{n!},$$

where $\varphi_p^{(n)}(q) = \varphi_p(n,q)$.

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