SCHMIDT GAMES AND MARKOV PARTITIONS

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Abstract. Let T be a C^2-expanding self-map of a compact, connected, C^∞, Riemannian manifold M. We correct a minor gap in the proof of a theorem from the literature: the set of points whose forward orbits are nondense has full Hausdorff dimension. Our correction allows us to strengthen the theorem. Combining the correction with Schmidt games, we generalize the theorem in dimension one: given a point $x_0 \in M$, the set of points whose forward orbit closures miss x_0 is a winning set. Finally, our key lemma, the No Matching lemma, may be of independent interest in the theory of symbolic dynamics or the theory of Markov partitions.

1. Introduction

Let $T : M \to M$ be a C^2-expanding self-map of a compact[1], connected, C^∞, Riemannian manifold M with volume measure σ. In this note, we study the set of points whose forward orbits are nondense. Since there is an ergodic T-invariant probability measure equivalent to σ, this set has zero volume [7]. It is, however, large in terms of Hausdorff dimension, as M. Urbański has shown [11]:

Theorem 1.1. *Let T be as above. If V is a nonempty open subset of M, then the Hausdorff dimension of the set of all points contained in V whose forward orbits under T are nondense in M equals* $\dim M$.

Besides this theorem, there are a number of other related results in [11] such as the analog theorem for certain Anosov diffeomorphisms. The proofs of these various results are elegant, but they all contain a (essentially the same) minor gap. There are two corrections of this gap for Theorem 1.1. The first one, by the current author, will be discussed in detail and proved in Section 4 below. Its proof will also yield a stronger theorem (Theorem 1.3 below). The second one, by Mariusz Urbański, the original author of the theorem, will be briefly mentioned in the same section. Corrections for the other results should be very similar to these two.

The other chief concern of this note is a result of S. G. Dani showing that the sets of points with nondense forward orbits are winning for certain systems. Being winning is stronger than having full Hausdorff dimension. We will define and discuss winning sets in Subsection 5.1. Dani's theorem is, explicitly, the following [3] (see Subsection 5.1 for the definition of 1/2-winning):

Theorem 1.2. *Let f be a semisimple, surjective linear endomorphism of the torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ where $n \geq 1$. The set of points whose forward orbit closures miss the identity element 0 in $\mathbb{T}^n$ is $1/2$-winning.*

Our Theorem 1.4 below will generalize (in part) Dani's result in dimension one.

[1] One can also study non-compact manifolds; see [2], [6], and [5].

Finally, we note that there are some interesting results where one considers points whose orbits (eventually) avoid certain uncountable sets [4].

1.1. **Statement of Results.** We give a correction for the proof of Theorem 1.1 and note that our proof allows us to show the following stronger theorem:

THEOREM 1.3. *Let T be as above. Given $x_1, \cdots, x_p \in M$, the set of points whose forward orbit closures miss $x_1, \cdots, x_p$ has full Hausdorff dimension (i.e. $= \dim M$).*

This theorem will be proved in Section 4.

Remark. A similar result, proved using a line of reasoning different from that in [11] or this note, can be found in [1]. The proof in [1] uses higher dimensional nets and Kolmogorov's consistency theorem from probability theory, while the proofs in [11] and this note are based on elementary properties of Markov partitions. See Section 6 for a discussion of Theorem 1.3 and the result in [1].

The key lemma used to correct the proof of Theorem 1.1 and to prove Theorem 1.3 is Lemma 3.1 (the No Matching lemma) below. It is also the key lemma used to prove our other main result (being α-winning, which is stronger than having full Hausdorff dimension, is defined in Subsection 5.1):

THEOREM 1.4. *Let M be the circle $S^1 := \mathbb{R}/\mathbb{Z}$ and T be as above. Given a point $x_0 \in M$, the set of points whose forward orbit closures miss x_0 is α-winning for some $0 < \alpha \leq 1/2$.*

This theorem will be proved in Subsection 5.2 . Using the properties of Schmidt games (see Subsection 5.1 for details on these games), we obtain

COROLLARY 1.5. *Let $\mathfrak{T}$ be any finite set of C^2-expanding self-maps of S^1 and $A \subset S^1$ be any countable set. Then the set of points whose forward orbit closures under any map in $\mathfrak{T}$ that miss A is α-winning for some $0 < \alpha \leq 1/2$.*

Remark. Hence, we have generalized in dimension one Theorems 1.1 and 1.3 (and also the aforementioned result in [1]) and (in part) Theorem 1.2. See Theorem 5.3 and Corollary 5.7 below for more precise statements of Theorem 1.4 and Corollary 1.5 respectively.

2. BACKGROUND ON MARKOV PARTITIONS AND SYMBOLIC DYNAMICS

In this section, we summarize the basic facts about and setup the relevant notation for Markov partitions and symbolic dynamics. These facts and this notation will help us prove our two main results.

2.1. **The Basics of Markov Partitions.** Part of this subsection will follow the development in [11]. Let M and σ be as above. If $A \subset M$, let us denote its topological closure in M by $\overline{A}$. Recall that a C^1-map $f : M \to M$ is **expanding** if (perhaps after a smooth change of Riemannian metric) there exists a real number $\lambda > 1$ such that

$$\|D_x f(v)\| \geq \lambda \|v\|$$

for all $x \in M$ and for all v in the tangent space of M at x [7]. A **Markov partition for T** is a finite collection $\mathcal{R} := \{R_1, \cdots, R_s\}$ of nonempty subsets of M such that

$$M = R_1 \cup \cdots \cup R_s \tag{2.1}$$

$$R_j = \overline{\text{Int}R_j} \text{ for every } j = 1, \cdots, s \tag{2.2}$$

$$\text{Int}R_i \cap \text{Int}R_j = \emptyset \text{ for all } 1 \le i \ne j \le s \tag{2.3}$$

$$\sigma(R_j \backslash \text{Int}R_j) = 0 \text{ for all } j = 1, \cdots, s \tag{2.4}$$

$$\text{For every } j \in \{1, \cdots, s\}, T(R_j) \text{ is a union of elements of } \mathcal{R}. \tag{2.5}$$

The **diameter** of a Markov partition is the maximum diameter over all its elements. Because T is expanding, T is injective on any set $B \subset M$ if $\text{diam}(B)$ is smaller than a constant $\delta_T > 0$. A Markov partition **with small diameter** is a Markov partition whose diameter $< \delta_T$. All Markov partitions in this note have small diameters (such Markov partitions always exist; see [7] and [9]).

Let A be the transition matrix associated to $\mathcal{R}$ (see [11] for a reference). The Markov partition $\mathcal{R}$ encodes the dynamics of T in the usual way, namely via a semi-conjugacy from the subshift of finite type given by A. Since we intricately manipulate elements of the subshift (and even parts of elements, as we shall see), we need more notation. We refer to the set $\{1, \cdots, s\}$ as an **alphabet** (or, in particular, the alphabet for $\mathcal{R}$) and its elements as **letters**.[2] A **string** is a bi-infinite, infinite, or finite sequence of the letters of the given alphabet. Thus, every element of a string has at most one predecessor and at most one successor. Given a string with an element i that has no successor, a **concatenation or appending (on the right)** is a new string identical to the given string except that a successor is chosen from the given alphabet for this element i. (Note that, depending on context, a repeated concatenation may be referred to simply as a concatenation.) Given any string α, define the **length of α,** $l(\alpha)$, to be the number of elements in α. A string is **finite** if it is a finite sequence. A finite string is **reducible** if it is of the form $a \cdots a$ where $a = \alpha_0 \cdots \alpha_r$ is a string of length $r+1$. A finite string is **irreducible** if it is not reducible.

Let $h \le t$ be integers. A **(h,t)-string** α is a string $\alpha_h \alpha_{h+1} \cdots \alpha_t$ with the given indices, and a **substring** of α is a string $\alpha_i \cdots \alpha_j$ where $h \le i \le j \le t$. Also, given $h \le i \le j \le t$, the **(i,j)-substring** of α is the string $\alpha_i \cdots \alpha_j$. An (i,j)-string γ is a **(extrinsic) substring** of α if there exists a $(k, k+j-i)$-substring of α such that $\alpha_k = \gamma_i, \alpha_{k+1} = \gamma_{i+1}, \cdots, \alpha_{k+j-i} = \gamma_j$. For convenience, $(0,t)$-strings will also be called **t-strings**. Given $n \le N$, a n-string β is **equivalent** to a N-string α (or a N-string α is **equivalent** to a n-string β) if $\alpha_0 = \beta_0, \cdots, \alpha_n = \beta_n$.

Finally, a **valid** string is a string given by the transition matrix A as follows: for every element i of the string with a successor j, $A_{i,j} = 1$. For $n \in \mathbb{N} \cup \{0\}$, let $\Sigma(n)$ denote the set of valid n-strings. For $\alpha \in \Sigma(n)$, define

$$R_\alpha := R_{\alpha_0} \cap T^{-1}(R_{\alpha_1}) \cap \cdots \cap T^{-n}(R_{\alpha_n}).$$

[2] In this note, we assume that all alphabets have at least two letters.

Thus, for all $n \in \mathbb{N} \cup \{0\}$, $R_\alpha \neq \emptyset$ and has the following properties (see [11] and [7]):

$$\cup_{\alpha \in \Sigma(n)} R_\alpha = M \tag{2.6}$$

$$R_\alpha = \overline{\mathrm{Int} R_\alpha} \tag{2.7}$$

$$\mathrm{Int} R_\alpha \cap \mathrm{Int} R_\beta = \emptyset \text{ for every distinct pair } \alpha, \beta \in \Sigma(n) \tag{2.8}$$

$$T(R_\alpha) = R_{\alpha_1 \cdots \alpha_n} \tag{2.9}$$

$$T^{-1}(R_\alpha) = \cup_{\{i | A_{i,\alpha_0}=1\}} R_{i\alpha} \tag{2.10}$$

$$R_\alpha = \cup_{\{i | A_{\alpha_n,i}=1\}} R_{\alpha i} \tag{2.11}$$

$$\sigma(R_\alpha \backslash \mathrm{Int} R_\alpha) = 0 \tag{2.12}$$

$$\mathrm{diam}(R_\alpha) < \delta_T \lambda^{-n}. \tag{2.13}$$

In the next subsection, we will discuss further properties at length.

2.1.1. *The Bounded Distortion Property and Boundary Points.* Let $J(T)(x) = |\det D_x T|$ denote the Jacobian of T at the point x. An important property of Markov partitions is the bounded distortion property ([11] and see [7] for a proof):

Theorem 2.1. *There exists a constant $C \geq 1$ such that*

$$\frac{J(T^n)(y)}{J(T^n)(x)} \leq C$$

for all $n \geq 1$, $\alpha \in \Sigma(n)$, and $x, y \in R_\alpha$.

Bounded distortion has two further refinements that are important to us. A little more notation is needed before we can state these. Define

$$G(n) := \{R_\alpha \mid \alpha \in \Sigma(n)\};$$

call $G(n)$ the n^{th} **generation of** $\mathcal{R}$. Hence, $G(0) = \mathcal{R}$. If γ is a valid string, let G_γ denote the **generation that** R_γ **belongs to**. Also, denote the **set of boundary points of all elements of all generations of** $\mathcal{R}$ by $\partial(\mathcal{R})$ (or ∂, if the context implies the Markov partition). And, let us refer to the points in the full volume set $M \backslash \partial$ as **interior points**.

For later use, we will need to further distinguish subsets of ∂. Let ∂_n denote **the set of all boundary points of all elements of** $G(n)$. Clearly, a chain of inclusions $\partial_0 \subset \partial_1 \subset \cdots$ exists. A point in ∂_0 has **weight** 0. For $n \geq 1$, a point in $\partial_n \backslash \partial_{n-1}$ has **weight** n.

Moreover, given $\gamma \in \Sigma(n)$, let us define, for later use, the following sets of valid concatenations of γ:

$$\Sigma_\gamma(q) := \{\delta \in \Sigma(n+q) \mid \delta \text{ is equivalent to } \gamma\}.$$

Returning to the notation for the refinements of bounded distortion, let us define a **lower constant of bounded distortion**

$$\varepsilon(q) := \min_{\delta \in \Sigma(q)} \frac{\sigma(R_\delta)}{\sigma(R_{\delta_0})} > 0$$

and an **upper constant of bounded distortion**

$$1 \geq \mathcal{E}(q) := \max_{\delta \in \Sigma(q)} \frac{\sigma(R_\delta)}{\sigma(R_{\delta_0})} > 0.$$

It is clear that both $\mathcal{E}(q)$ and $\varepsilon(q)$ are weakly monotonically decreasing functions of q, both of which tend to 0 as q tends to ∞. Finally, let $R_{\min} \in \mathcal{R}$ be an element with smallest σ, let $R_{\max} \in \mathcal{R}$ be an element with largest σ, and define

$$r = \frac{\sigma(R_{\min})}{\sigma(R_{\max})}.$$

Our first refinement of bounded distortion (Theorem 2.1) is the following (note that C is from the theorem):

LEMMA 2.2. *For every element $R_\alpha \in G(N)$ and every element $R_{\alpha\beta} \in G(N+n)$,*

$$\frac{\varepsilon(n)}{C} \leq \frac{\sigma(R_{\alpha\beta})}{\sigma(R_\alpha)} \leq C\mathcal{E}(n).$$

Our second refinement of bounded distortion is the following (again, C is from the theorem):

LEMMA 2.3. *Let $N \in \mathbb{N}$ and η be a valid finite string of length at least 2. Let $R_{\eta\alpha}$ be an element of $G(N)$ (contained in R_η) of largest σ; let $R_{\eta\beta}$ be an element of $G(N)$ of smallest σ. Then*

$$\frac{\sigma(R_{\eta\beta})}{\sigma(R_{\eta\alpha})} \geq \frac{r}{C}.$$

The proofs of these two refinements follow, essentially, a similar calculation in [11] and are omitted.

Another useful property (again, easy to see and whose proof is omitted) is

LEMMA 2.4. *The following hold (for Markov partitions with small diameter):*

$$T(\partial) \subset \partial \text{ and } T^{-1}(\partial) \subset \partial.$$

2.1.2. *Representations.* Let $\Sigma(\infty)$ denote the set of valid infinite strings $\alpha_0\alpha_1\cdots$ indexed by $\mathbb{N} \cup \{0\}$. If $\alpha \in \Sigma(\infty)$, then R_α is a unique point in M. Conversely, if $x \in M$, then there exists an $\alpha \in \Sigma(\infty)$ such that $x = R_\alpha$. A **representation** of $x \in M$ is an element $\alpha \in \Sigma(\infty)$ for which $x = R_\alpha$. A representation may not be unique.

Let us, henceforth, denote the $(0, Q)$-substring of a $\gamma \in \Sigma(\infty)$ by $\gamma(Q)$. The following two facts about representations are easy to see and their proofs are omitted:

LEMMA 2.5. *A point $x \in M$ has non-unique representations $\iff$ $x \in \partial$. The set of points with non-unique representations is σ-null.*

LEMMA 2.6. *Let $x \in M$ be a point with representations $\gamma^1, \cdots, \gamma^r$. Then, for every $Q \in \mathbb{N} \cup \{0\}$, there exists an open neighborhood U of x such that $U \subset \cup_{t=1}^{r} \mathit{Int} R_{\gamma^t(Q)} \cup \partial$.*

2.1.3. *Two Facts about Preimages.* Finally, we have two basic facts about missing preimages. These are easy to see and their proofs are omitted.

LEMMA 2.7. *Let E be a set of points whose forward orbits miss an open set U. Then E is also a set of points whose forward orbits miss the open set $T^{-n}(U)$ for any $n \in \mathbb{N}$.*

LEMMA 2.8. *Let E be a set of points whose forward orbit closures miss a point y. Then E is also a set of points whose forward orbit closures miss $T^{-n}(y)$ for any $n \in \mathbb{N}$.*

2.1.4. *A Lower Bound for Hausdorff Dimension.* In this subsection, we follow a simplified version of the development in [11]. Let $K \subset M$ be compact. For $k \in \mathbb{N}$, let E_k denote a finite collection of compact subsets of K with positive volume. (Recall that volume measure is denoted by σ.) We require the following to hold:

$$\text{The union of the elements of } E_1 \text{ is } K. \tag{2.14}$$

$$\text{For distinct } F, G \in E_k, \sigma(F \cap G) = 0. \tag{2.15}$$

$$\text{Every element } F \in E_{k+1} \text{ is contained in an element } G \in E_k. \tag{2.16}$$

Let us define the following notation:

- Let $\cup E_k$ denote the union of all elements of E_k.
- Let $E := \cap_{k=1}^{\infty} \cup E_k$.
- Define, for every $F \in E_k$,

$$\text{density}(E_{k+1}, F) := \frac{\sigma(\cup E_{k+1} \cap F)}{\sigma(F)}.$$

- Let $\Delta_k := \inf\{\text{density}(E_{k+1}, F) \mid F \in E_k\}$.
- Let $d_k := \sup\{\text{diam}(F) \mid F \in E_k\}$.

We further require the following to hold:

$$\Delta_k > 0 \tag{2.17}$$

$$d_k < 1 \tag{2.18}$$

$$\lim_{k \to \infty} d_k = 0. \tag{2.19}$$

Following [5], let us call $\{E_k\}_{k \in \mathbb{N}}$ a **strongly tree-like** collection. Let $HD(\cdot)$ denote Hausdorff dimension. The following lemma for this strongly tree-like collection is proved in [11] by adapting a proof from [8] (both proofs are based on Frostman's lemma):

Lemma 2.9. *It holds that*

$$HD(E) \geq \dim M - \limsup_{k \to \infty} \frac{\sum_{j=1}^{k} \log \Delta_j}{\log d_k}.$$

Remark. The upper index of summation is k (not $k-1$ as in [11]). See [5], but note that what is referred to as the "j-th stage density" must be > 0. For a more general version of this lemma, see [5] or [11]. For a version involving higher dimensional nets, see [1].

2.2. **The Basics of Strings.** Let us continue our discussion of the basic facts of strings from Subsection 2.1.

2.2.1. *Partial String Matches.* Let $n \leq N$, and let γ be a n-string and α, a N-string. A **match of** γ **with** α is an $(i, i+n)$-substring of α given by $\alpha_i = \gamma_0, \alpha_{i+1} = \gamma_1, \cdots, \alpha_{i+n} = \gamma_n$. Whenever γ is a substring of α, there is at least one such match. A **partial match of** γ **with** α is an (i, N)-substring of α given by $\alpha_i = \gamma_0, \alpha_{i+1} = \gamma_1, \cdots, \alpha_N = \gamma_m$ where $m < n$. Consequently, $i > N - n$. Call i the **head (of the partial match)**.

Note that if two partial matches of γ with α have heads $i < j$, then a "right shift and crop" of the one with the smaller head will produce the one with the larger head. This is just pattern matching.

2.2.2. *Valid Strings and Matching.* Let us now specialize to valid strings (defined in Subsection 2.1) for a Markov partition with small diameter $\mathcal{R} := \{R_1, \cdots, R_s\}$.

By (2.5) and (2.11), there exists a letter for which concatenation on the right of any valid finite string produces a valid finite string. But, there exist Markov partitions such that, for some letter i, only one letter j produces a valid 1-string when concatenated on the right; such i is called a **degenerate letter**. A letter that is not degenerate is **nondegenerate**. A **block** of a string is a substring composed of exactly one nondegenerate letter, which is found at the largest index. Note that given the initial letter in a block, the only valid concatenation on the right of the initial letter is the one that produces the rest of the block. By (2.13), there exists an integer B, called the **maximal block length**, such that for every B-string α, $\sigma(R_{\alpha_0}) > \sigma(R_\alpha)$. A **general block** of a string is a substring composed of exactly one nondegenerate letter. A **reverse block** of a string is a substring composed of exactly one nondegenerate letter, which is found at the smallest index. A **double general block** of a string is a substring composed of a block followed by a reverse block. (Hence, a double general block has exactly two nondegenerate letters; they are adjacent.)

The following lemma and corollary are easy to see and their proofs are omitted:

Lemma 2.10. *In a string composed of only degenerate letters, each letter is distinct.*

Corollary 2.11. *The maximal block length of the Markov partition $\{R_1, \cdots, R_s\}$ is at most s. The maximal length of any general block is at most $2s - 1$; any double general block, at most $2s$.*

3. The No Matching Lemma

We are now ready to prove the key lemma used in our two main results. This lemma may be of independent interest in the theory of symbolic dynamics or the theory of Markov partitions.

Lemma 3.1 (No Matching). *Let $N \geq n \geq 8s - 4$. Let γ be any n-string such that γ_{n-1} is nondegenerate* ***except*** *those of the following kind:*

$$\gamma = a^0 \cdots a^m$$

where

$$a^0 = \cdots = a^{m-1}$$

are general blocks and either

$$a^m \text{ is a general block not equivalent to } a^0 a^0 \tag{3.1}$$

or

$$a^m \text{ is a double general block not equivalent to } a^0 a^0. \tag{3.2}$$

And let α be a N-string such that no match of γ with α exists. Then there exists a choice of substrings b^0 and b^1 of length at most s such that for any letters $\beta_0, \beta_1, \cdots \beta_k$, no match of γ with the $N + n$-string $\alpha b^0 b^1 \beta_0 \cdots \beta_k$ exists.

Remark. It is possible for both (3.1) and (3.2) to hold for the same string γ.

Remark. If no match of γ with $\alpha b^0 b^1 \beta_0 \cdots \beta_k$ exists, then no match of γ with $\alpha b^0 b^1 \beta_0 \cdots \beta_{k'}$ for any $0 \leq k' \leq k$ exists.

Proof. By Corollary 2.11, all n-strings contain at least four general blocks. Note that for the exceptional n-strings, we obtain $m \geq 3$ by Corollary 2.11.

There are three cases:

Case 1: No partial matches of γ with α exist.

Choose any letters for $b^0 b^1 \beta_0 \cdots \beta_k$ that make $\alpha b^0 b^1 \beta_0 \cdots \beta_k$ a valid string.

Case 2: There exists exactly one partial match of γ with α.

Hence, there exists exactly one choice of letter for the initial letter of b^0 (namely a choice for b_0^0) which would produce a match of γ with $\alpha b^0 b^1 \beta_0 \cdots \beta_k$.

If α_N is nondegenerate, let b_0^0 not be this letter. Since no other partial matches of γ with α exists, we are free to take any letters for the remainder as long as they form a valid string.

If α_N is degenerate, then $\alpha_N b^0$ must contain the block starting with α_N. Also, because γ_{n-1} is nondegenerate, γ must contain this block (not just partially contain it). By the definition of block, we have a choice of letter to concatenate on the right of the block. Choose the letter that is different from what is in γ. Hence, we have made a choice for b^0, and again we are free to take any letters for the remainder as long as they form a valid string.

Case 3: There are at least two partial matches of γ with α.

Let i be the smallest head and j be the second smallest head. Let γ^i correspond to the partial match with i as head; γ^j, with j as head.

Now γ^i is the concatenation of the same substring of length $j - i \geq 1$. Denote the substring by $c = \gamma_0 \cdots \gamma_{j-i-1}$. Then,

$$\gamma^i = c \cdots c \gamma_0 \cdots \gamma_r$$

where $0 \leq r \leq j - i - 1$. (Note that γ^i may contain only one substring c.) For an nonnegative integer t, let $0 \leq \bar{t} < j - i$ denote the representative of t mod $j - i$. By Lemma 2.10, c contains at least one nondegenerate letter. (Moreover, i and j imply that c is irreducible.) There are two cases:

Case 3A: The substring c is a general block.

If γ_r is the one nondegenerate letter in c, then choose b^0 to be a 0-string and $b_0^0 \neq \gamma_{\overline{r+1}}$. Now, if b_0^0 is nondegenerate, we have that $\gamma^i b^0 = c \cdots c\tilde{c}$ where $\tilde{c}$ is a double general block. Otherwise, if b_0^0 is degenerate, we have that $\gamma^i b^0 = c \cdots c\tilde{c}$ where $\tilde{c}$ is a general block.

Otherwise, γ_r is degenerate, and thus the choices of b^0 are fixed until after we reach the next nondegenerate letter, b_q^0. Because γ_{n-1} is nondegenerate, $\gamma^i b_0^0 \cdots b_q^0$ is a substring of γ. Moreover, $\gamma_r b_0^0 \cdots b_q^0$ must appear together in γ^i because of the repeating substring. Hence, it is a substring of cc. If $q + 1 = l(c)$, then $\gamma_r = b_q^0$, a contradiction. Hence, $q + 1 < l(c)$, and we can choose b_{q+1}^0 to be different from the letter that follows the substring $\gamma_r b_0^0 \cdots b_q^0$ in cc. Now, if b_{q+1}^0 is nondegenerate, we have that $\gamma^i b^0 = c \cdots c\tilde{c}$ where $\tilde{c}$ is a double general block. Otherwise, if b_{q+1}^0 is degenerate, we have that $\gamma^i b^0 = c \cdots c\tilde{c}$ where $\tilde{c}$ is a general block.

Hence, only γ^i may possibly be completed to a match of γ with $\alpha b^0 b^1 \beta_0 \cdots \beta_k$. If $\gamma^i b^0$ is not equivalent to γ, we are done.

Thus, let $\gamma^i b^0$ be equivalent to γ. There are two cases. If $\gamma^i b^0 = c\tilde{c}$, then there is another nondegenerate letter after $\gamma^i b^0$ in γ because γ contains at least four general blocks. Otherwise, $\gamma^i b^0 = c \cdots c\tilde{c}$ where $\tilde{c}$ is either a general or double general block not equivalent to cc (as constructed above). Thus, there is at least another nondegenerate letter after the substring $\tilde{c}$ in γ because we exclude strings of the form (3.1) and (3.2).

Let $\tilde{c}_t$ be the last letter of $\tilde{c}$ (i.e. $\tilde{c}_t = b^0_{q+1}$). If $\tilde{c}_t$ is nondegenerate, choose b^1 to be a 0-string where b^1_0 is a different letter than what follows $\gamma^i b^0$ in γ. If $\tilde{c}_t$ is degenerate, then the substring $\tilde{c}_t b^1_0 \cdots b^1_p$ up to the next nondegenerate letter (i.e. b^1_p) in γ is determined. (Since γ_{n-1} is nondegenerate, b^1_p comes before γ_n.) Since this is a block, we have a choice of letters for b^1_{p+1}; pick it so that it is different from that in γ. Hence, γ^i cannot produce a match either. We may now pick any letters for the remainder as long as they produce a valid string.

Case 3B: The substring c is not a general block.

Hence, c contains at least two nondegenerate letters (not necessarily distinct). Also, $j-i \geq 2$.

If γ_r is a nondegenerate letter in c, then choose b^0 to be a 0-string and $b^0_0 \neq \gamma_{l(\gamma^i)}$. Otherwise, the choices of b^0 are fixed until after we reach the next nondegenerate letter, b^0_q. Because γ_{n-1} is nondegenerate, $\gamma^i b^0_0 \cdots b^0_q$ is a substring of γ. Pick b^0_{q+1} to be different from the letter that follows $\gamma^i b^0_0 \cdots b^0_q$ in γ. Hence, $\gamma^i b^0$ cannot complete to a match.

Because of the repeating substrings, we know the $j-i$ adjacent letters, namely a substring of cc, that are needed for γ^j or any partial match with larger head to produce a match of γ with $\alpha b^0 b^1 \beta_0 \cdots \beta_k$. (Note that every letter of c appears in such a substring.)

Let k be any head greater than or equal to j. Assume $\gamma^k b^0$ can be completed to γ.

Now if γ_r is nondegenerate, then $b^0 = \gamma_{\overline{r+1}}$, a 0-string. And one of the letters $\gamma_{\overline{r+1}}, \cdots, \gamma_{\overline{r+j-i-1}}$ is also nondegenerate. If b^0 is nondegenerate, choose b^1_0 to be different from $\gamma_{\overline{r+2}}$. Otherwise, b^0 is degenerate, and thus the block beginning with b^0 must be at most $l(c)-1$ in length. There are at least two ways to concatenate a letter to the end of this block. Pick, for b^1, one that is different from the one in γ.

Otherwise, γ_r is degenerate. Assume that b^0_{q+1} is nondegenerate. Hence, the $q+1$-string b^0 has length strictly less than $l(c)$ since otherwise $b^0_{q+1} = \gamma_r$, a contradiction. Thus, for $\gamma^k b^0$ to complete to γ, we know, by the length, exactly the letter that is required to be concatenated on the right. Let b^1 be a 0-string such that b^1_0 is not this letter. Otherwise, b^0_{q+1} is degenerate, and b^0 has exactly one nondegenerate letter, namely b^0_q. Choose the beginning of b^1 to be the rest of the block starting with b^0_{q+1}. Let b^1_h be the nondegenerate letter at the end of this block. If $q+2+h+1 = l(c)$, then $b^0 b^1_0 \cdots b^1_h$ are all the letters in c, and therefore $b^1_h = \gamma_r$, a contradiction. Thus, for $\gamma^k b^0 b^1_0 \cdots b^1_h$ to complete to γ, we know, by the length, exactly the letter that is required to be concatenated on the right. Let b^1_{h+1} not be this letter. We may now pick any letters for the remainder as long as they produce a valid string.

The proof of the lemma is complete. □

4. The Proof of Theorem 1.3

In this section, we prove Theorem 1.3. This proof, which follows the proof scheme in [11], is also the author's correction of the proof of Theorem 1.1. Choose a Markov partition with small diameter $\mathcal{R} := \{R_1, \cdots, R_s\}$. It is easy to see that the number of representations of every point is less than or equal to some natural number P_0. For a point $x \in M$, define the **adjacency set of x in generation N**:

$$\Phi_N(x) = \{R \in G(N) \mid R \ni x\}.$$

Now, recall that $\overline{A}$ denotes the topological closure of a subset $A \subset M$. Also, for $x \in M$, $\mathcal{O}_T^+(x)$ denotes the forward orbit of x under the self-map T, and, for $\alpha \in \Sigma(\infty)$, $\alpha(n)$ denotes $\alpha_0 \cdots \alpha_n$. Finally, we have the following caveat:

Remark. In this section, if we let T act on a representation, we are implicitly using the aforementioned semi-conjugacy, as this action denotes left shift.

Recall the statement of Theorem 1.3. By Lemma 2.8, if any two of the $x_1, \cdots, x_p$ have forward orbits that intersect, we may replace both of these points with a point in the intersection of their forward orbits and still prove the theorem. Repeat. Hence, without loss of generality, we may assume that $x_1, \cdots, x_p$ have pairwise disjoint forward orbits.

Let $\bar{\gamma}^1, \cdots, \bar{\gamma}^P$ be all possible representations of $x_1, \cdots, x_p$ (all representations of the same point are included in this list). Hence, $P \leq pP_0$. Also, there exists a least generation $\tilde{n}$ such that $|G(\tilde{n})| > P$ and $\delta_T \lambda^{-\tilde{n}} < 1$.

Let us collect these representations thus:

$$\{\bar{\gamma}^1\}, \cup_{t=0}^{3s}\{T^t(\bar{\gamma}^2)\}, \cdots, \cup_{t=0}^{3s}\{T^t(\bar{\gamma}^P)\}.$$

From each collection, pick exactly one element; call this element $\tilde{\gamma}^j$. Because of the pairwise disjoint orbits, the chosen elements are distinct representations. Hence, there exists $\tilde{N} \in \mathbb{N}$ such that for all $n \geq \tilde{N}$, $\tilde{\gamma}^1(n-2s), \cdots, \tilde{\gamma}^P(n-2s)$ are distinct. Repeat over all such possible combinations, and take the largest $\tilde{N}$.

If $T^t(\bar{\gamma}^j) = a \cdots$ for some general block a, set $Q_{j,t} = 8s - 4$. Otherwise, after the first general block a, there exists a general block b of least last index $J \geq 1$ such that $a \neq b$ (i.e. a is not equivalent to b), and set $Q_{j,t} = \max(J + 2s, 8s - 4)$. Set $Q = \max\{Q_{j,t} \mid j = 1, \cdots, P \text{ and } t = 0, \cdots, 3s\}$.

Let

$$q_0 = \max(\tilde{N}, 2sP + 1, Q, \tilde{n}).$$

By Lemma 2.10, there exists a sequence of integers $\{q_i\}$ greater than or equal to q_0 such that $\bar{\gamma}^1_{q_i - 1}$ is nondegenerate. Let us now fix a $q \in \{q_i\}$ and set

$$\gamma^1 := \bar{\gamma}^1_0 \cdots \bar{\gamma}^1_q.$$

Thus, Lemma 3.1 applies to every such γ^1.

For each $2 \leq j \leq P$, there exists, by Lemma 2.10, a least $K_j \in \{0, \cdots, s-1\}$ such that $(T^{K_j}(\bar{\gamma}^j))_{q-1}$ is nondegenerate. Set

$$\gamma^j := (T^{K_j}(\bar{\gamma}^j))_0 \cdots (T^{K_j}(\bar{\gamma}^j))_q.$$

Note that Lemma 3.1 can individually apply to each γ^j.

Define

$$E_k := E_k(q) := \{R_\alpha \mid \alpha \in \Sigma(kq) \text{ and } T^n(R_\alpha) \cap \cup_{j=1}^P \mathrm{Int} R_{\gamma^j} = \emptyset$$
$$\text{for every } n = 0, 1, \cdots, (k-1)q\}.$$

Hence, $R_\alpha \in E_k$ if and only if all of the $\gamma^1, \cdots, \gamma^P$ are not substrings of α.

As in [11], we wish to show that $\{E_k\}$ is strongly tree-like so that we can apply Lemma 2.9. Let

$$K := \cup E_1$$

and

$$E(q) := \cap_{k=1}^{\infty} \cup E_k.$$

We are now ready to show the following proposition:

Proposition 4.1.

$$HD(E(q)) \geq \dim M + \frac{\log \frac{\varepsilon(2spP_0)}{C}}{q \log \lambda}$$

for all $q \in \{q_i\}$.

Proof. As in [11], we note that

$$d_k < \delta_T \lambda^{-qk}$$

by (2.13). Verifying (2.15), (2.16), (2.18), and (2.19) is routine and can be found in [11].

We, however, must correct the estimate of Δ_k.[3] (With this estimate, we will also verify (2.17).) Let $R_\alpha \in E_k$. Thus, no match of any of the γ^j's exists with α. If, for a γ^j, there are no partial matches, then any valid concatenation (of the correct length) of α will produce an element of E_{k+1}. For the remaining γ^j's, there are partial matches, and we will apply Lemma 3.1 serially. Each of these remaining γ^j's has a partial match with smallest head. Pick one of these γ^j's (needs not be unique) with the least smallest head h; call it γ. Let γ' be one of the γ^j's except for γ, and denote the smallest head of γ' by h'. Thus $h \leq h'$.

Let b^0 and b^1 be chosen as in Lemma 3.1 applied to γ. Then there is no match of γ with $\alpha b^0 b^1 \beta_0 \cdots \beta_{k'}$ where $\beta_0, \cdots, \beta_{k'}$ are any letters that make $\alpha b^0 b^1 \beta_0 \cdots \beta_{k'} \in \Sigma((k+1)q)$.

Sublemma 4.2. *There is no match of* γ' *with* $\alpha b^0 b^1$.

Proof. Assume not. Let us denote $\alpha = \alpha_0 \cdots \alpha_N$. Since $l(b^0 b^1) \leq 2s$, all but at most the last $2s$ letters of γ' are in the partial match with head h'. Consequently, all but at most the last $2s$ letters of γ are, likewise, in the partial match with head h. It is easy to see that

$$N + 2s - h' \geq q,$$

---

[3] The minor gap from [11] lies at this step. It is easy to see that the original proof will not work (for example) for the multiplication by 2 map on the circle $S^1 := \mathbb{R}/\mathbb{Z}$ with Markov partition given by the dyadic partition

$$\mathcal{D} = \{R_1, \cdots, R_{2^s}\} \text{ where } R_i = [\frac{i-1}{2^s}, \frac{i}{2^s}]$$

and with transition matrix

$$A := \begin{pmatrix} B \\ B \end{pmatrix} \text{ where } B := \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & & & & & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ & & & & & \vdots & & & & & \\ 0 & 0 & \cdots & & & & & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Also, note that Urbański (the author of Theorem 1.1) has given a second correction using his original proof scheme and the Baire category theorem on certain orbits under powers of the map under consideration (via personal communication).

and therefore

$$h' - h \leq 2s$$

since $h \geq N - q$. Now, by construction, $T^{h'-h}(\gamma)$ and γ' disagree on at least the last $2s+1$ letters, a contradiction as both are partial matches with α. □

Remove γ from consideration. Now pick, among the remaining, one with the least smallest head (again, needs not be unique), and repeat applying Lemma 3.1 with α replaced by $\alpha b^0 b^1$ until no more γ^j's remain (possible since $q > 2sP$). Therefore, after serially applying Lemma 3.1, we obtain

$$R_{\alpha b^0 b^1 \cdots b^{2P-1} b^{2P} \beta_0 \cdots \beta_{k'}} \in E_{k+1}$$

where $\beta_0, \cdots \beta_{k'}$ are any allowed letters. Thus, $R_{\alpha b^0 b^1 \cdots b^{2P-1} b^{2P}}$ is a union of elements of E_{k+1}.

By Lemma 2.2 and the monotonicity of $\varepsilon(\cdot)$ (recall the definition of $\varepsilon(\cdot)$ from Subsection 2.1.1),

$$\varepsilon(2sP) \leq C \frac{\sigma(R_{\alpha b^0 b^1 \cdots b^{2P-1} b^{2P}})}{\sigma(R_\alpha)}.$$

Consequently, $\sigma(\cup E_{k+1} \cap R_\alpha) \geq \frac{\varepsilon(2sP)}{C} \sigma(R_\alpha)$. Thus,

$$\text{density}(E_{k+1}, R_\alpha) \geq \frac{\varepsilon(2sP)}{C}.$$

Hence,

$$\frac{\sum_{j=1}^k \log \Delta_j}{\log d_k} \leq \frac{k \log \frac{\varepsilon(2sP)}{C}}{\log \delta_T - qk \log \lambda}.$$

Applying Lemma 2.9, we obtain

$$HD(E(q)) \geq \dim M + \frac{\log \frac{\varepsilon(2sP)}{C}}{q \log \lambda} \geq \dim M + \frac{\log \frac{\varepsilon(2spP_0)}{C}}{q \log \lambda}$$

for all $q \in \{q_i\}$. □

Lemma 4.3. *The set $E(q)$ is also a set of points whose forward orbits miss neighborhoods of $x_1, \cdots, x_p$.*

Proof. For interior points in $\{x_1, \cdots, x_p\}$, apply Lemma 2.7.

Let $x \in \{x_1, \cdots, x_p\}$ be a boundary point. It is easy to see that there exists an open set $U \ni x$ such that $U \subset \cup \Phi_{q+s}(x)$. We claim that all the points in $E(q)$ have forward orbits which miss the open set U. Assume not. Then there exist a $y \in E(q)$, which corresponds to an $\alpha \in \Sigma(\infty)$, and n such that $T^n(y) \in U$. Let $k \in \mathbb{N}$ be chosen so that $kq \geq n + q + s$. Hence, $T^n(R_{\alpha(kq)}) \cap U \neq \emptyset$. Thus, $T^n(R_{\alpha(kq)}) \subset R_\beta$ for some $\beta \in \Phi_{q+s}(x)$. Now β is equivalent to one of the representations of x; say it is $\bar{\gamma}^j$. Thus, $T^{n+K_j}(R_{\alpha(kq)}) \subset T^{K_j}(R_\beta) \subset T^{K_j}(R_{\bar{\gamma}^j(q+K_j)}) = R_{\gamma^j}$, a contradiction. □

We can now prove the theorem:

Proof of Theorem 1.3. Applying Proposition 4.1 and Lemma 4.3 and setting $q_i \to \infty$, we have shown our desired result: for any points $x_1, \cdots, x_p \in M$,

$$F_T(x_1, \cdots, x_p) := \{x \in M \mid \{x_1, \cdots, x_p\} \cap \overline{\mathcal{O}_T^+(x)} = \emptyset\}$$

has full Hausdorff dimension (i.e. $= \dim M$). □

Remark. If one simply wishes to correct the proof of Theorem 1.1, one can significantly simplify the above proof by considering missing only one point, an interior point.

In the next section, we will see how the author's correction leads to a generalization in dimension one.

5. A Generalization

In this section, we prove in dimension one a generalization of Theorems 1.1 and 1.3 (and also the aforementioned result in [1]) and, in part, a generalization of Theorem 1.2. In particular, we prove Theorem 1.4 (or, more precisely, Theorem 5.3 below). An immediate corollary is also obtained. We begin by introducing Schmidt games.

5.1. **The Basic Facts of Schmidt Games.** W. Schmidt introduced the games which now bear his name in [10]. Given a $0 < \kappa < 1$ and a ball U of M with radius r, let us denote, as in [10], U^κ to be the set of all balls $U' \subset U$ with radius equal to κr.

Let $0 < \alpha < 1$ and $0 < \beta < 1$. Let S be a subset of a complete metric space M. In an (α, β)-game, two players, Black and White, alternate choosing nested closed balls $B_1 \supset W_1 \supset B_2 \supset W_2 \cdots$ on M such that $W_n \in B_n^\alpha$ and $B_n \in W_{n-1}^\beta$. The second player, White, wins if the intersection of these balls lies in S. A set S is called (α, β)**-winning** if White can always win for the given α and β. A set S is called α**-winning** if White can always win for the given α and any β. A set S is called **winning** if it is α-winning for some α. Schmidt games have four important properties for us [10]:

Property (SG1). *The sets in $\mathbb{R}^n$ which are α-winning have full Hausdorff dimension.*

Property (SG2). *Countable intersections of α-winning sets are again α-winning.*

Property (SG3). *If a set is α-winning, then it is also α'-winning for all $0 < \alpha' \leq \alpha$.*

Property (SG4). *Let $0 < \alpha \leq 1/2$. If a set in a Banach space of positive dimension is α-winning, then the set with a countable number of points removed is also α-winning.*

5.2. **The Proof of the Generalization in Dimension One.** We will, in this subsection, specialize to the one-dimensional case: consider the system (S^1, σ, T) where $S^1 := \mathbb{R}/\mathbb{Z}$, σ is the probability Haar measure on S^1, and

$$T : S^1 \to S^1$$

is a C^2-expanding map.

It is clear from Krzyżewski and Szlenk's construction of a Markov partition with small diameter ([7], proof of Lemma 4) that

Lemma 5.1. *For any C^2-expanding map $T : S^1 \to S^1$, there exists a Markov partition with small diameter for which every element of every generation is path-connected.*

Endow S^1 with the usual metric, and let $d(A)$ denote the diameter of a set A. Using Lemma 5.1, we obtain a Markov partition with small diameter $\mathcal{R} := \{R_1, \cdots, R_s\}$, which we fix. Since the elements of each generation are intervals, we may use d and σ interchangeably on these elements.

Recall the definition of $\varepsilon(\cdot)$ from Subsection 2.1.1.

Lemma 5.2. *Let $\mathcal{R} := \{R_1, \cdots, R_s\}$ be a Markov partition with small diameter for which every element of every generation is path-connected. For any closed interval B such that*

$$d(B) < \min\{d(R_1), \cdots, d(R_s)\},$$

there exists $N \in \mathbb{N}$ for which an element $R_\eta \in G(N-1)$ can be chosen to satisfy

$$2d(R_\eta) \geq d(B) \geq \frac{\varepsilon(1)}{C} d(R_\eta). \tag{5.1}$$

Moreover, an element of $G(N)$ lies in both B and R_η and at least half of the interval B lies in R_η. Finally, if any element of any generation $R_\alpha \supset B$, then $R_\alpha \supset R_\eta$.

Remark. Although more than one value of N may make (5.1) true, we always agree to take the value of N as in the proof below. Hence, for each B there exists a unique N, namely $G(N)$ is the least generation in which an element of that generation lies completely in B.

Proof. **Case 1: $B \cap \partial_0 \neq \emptyset$.**

By length, B contains exactly one point y of weight 0. Thus, we have closed intervals B^+ and B^- such that

$$B = B^+ \cup B^-$$

where

$$\{y\} = B^+ \cap B^-.$$

Let $d(B^+) \geq d(B^-)$. (Note that B^- could possibly be just $\{y\}$.)

Now there exists a least $N \in \mathbb{N}$ such that $(\partial_N \backslash \{y\}) \cap B^+ \neq \emptyset$. Hence, there exists $R_\eta \in G(N-1)$ such that $B^+ \subset R_\eta$. Thus,

$$d(B) \leq 2d(R_\eta).$$

Let $z \in (\partial_N \backslash \{y\}) \cap B^+$ be closest to y. Then the interval between y and z in B^+ is an element of $G(N)$. Denote it by $R_{\eta i}$. Hence, by Lemma 2.2,

$$\frac{\varepsilon(1)}{C} d(R_\eta) \leq d(B).$$

Case 2: $B \cap \partial_0 = \emptyset$.

Thus, there exists a least $N \in \mathbb{N}$ such that $\partial_N \cap B \neq \emptyset$.

Case 2A: $|\partial_N \cap B| \geq 2$.

Thus, there exists $R_\eta \in G(N-1)$ such that $B \subset R_\eta$. Moreover, there exists an element $R_{\eta i}$ such that $R_{\eta i} \subset B$. As in Case 1, we obtain (5.1).

Case 2B: $|\partial_N \cap B| = 1$.

Let y be the point of weight N in B. Repeat the proof of Case 1 with this y. □

Also, recall that we denote the $(0, Q)$-substring of a $\gamma \in \Sigma(\infty)$ by $\gamma(Q)$. Finally, note that C is from Theorem 2.1. Our generalization is

THEOREM 5.3. *Let $x_0 \in S^1$. Then*

$$F_T(x_0) := \{x \in S^1 \mid x_0 \notin \overline{\mathcal{O}_T^+(x)}\}$$

is an $\frac{\varepsilon(7s+2)}{2C}$-winning set. (If $\mathcal{R}$ has no degenerate letters, we may replace $\varepsilon(7s+2)$ with $\varepsilon(5)$.)

Proof. Let $M := S^1$ and $F := F_T(x_0)$. Let $\gamma \in \Sigma(\infty)$ be a representation of x_0.

Let $n := \frac{\varepsilon(7s+2)}{2C}$ and $0 < m < 1$. We show that F is (n, m)-winning. Black starts, choosing B_1. Now there is a least $J \in \mathbb{N}$ such that for any choice of B_J,

$$d(B_J) < \min_{\xi \in \Sigma(1)} (d(R_\xi)).$$

(White chooses any allowed sets for $W_1, \cdots, W_{J-1}$. Black chooses B_J.)

By Lemma 5.2, there exist $N_0 \geq 1$ and an element $R_\eta \in G(N_0)$ that contains at least half of B_J. Since $n \leq 1/2$, choose $W_J \subset R_\eta$.

Let us now refine the notion of constants of bounded distortion:

$$\varepsilon_\eta(q) := \min_{\delta \in \Sigma_\eta(q)} \frac{\sigma(R_\delta)}{\sigma(R_\eta)} > 0$$

$$1 \geq \mathcal{E}_\eta(q) := \max_{\delta \in \Sigma_\eta(q)} \frac{\sigma(R_\delta)}{\sigma(R_\eta)} > 0.$$

For the given η, Lemma 2.3 implies that $\frac{\varepsilon_\eta(q)}{\mathcal{E}_\eta(q)} \geq r/C$.

SUBLEMMA 5.4. *$\mathcal{E}_\eta(q) \geq s^{-q}$.*

Proof. There are at most s^q elements of $G(l(\eta) - 1 + q)$ which are contained in R_η, i.e. $|\Sigma_\eta(q)| \leq s^q$, because there are only s possible letters to append (on the right) to any finite string.

Let $R_\alpha \in G(l(\eta) - 1 + q)$ be such that $\mathcal{E}_\eta(q) = \frac{\sigma(R_\alpha)}{\sigma(R_\eta)}$. Then R_α has the largest σ of any element of $G(l(\eta) - 1 + q)$ contained in R_η. Because all elements of the same generation have pairwise disjoint interiors and ∂ is σ-null, $s^q \sigma(R_\alpha) \geq \sum_{\beta \in \Sigma_\eta(q)} \sigma(R_\beta) = \sigma(R_\eta)$. □

Hence, $\varepsilon_\eta(q) \geq \frac{r}{Cs^q}$.

Define

$$H_k = H_k(Q) = \{R_\alpha \mid \alpha \in \Sigma(Q + k) \text{ and } T^n(R_\alpha) \cap \operatorname{Int} R_{\gamma(Q)} = \emptyset$$
$$\text{for every } n = 0, 1, \cdots, k\}.$$

There exists a least $P \in \mathbb{N}$ such that

(1) $P \geq 4s - 2$ and
(2) $\frac{4C^4 \delta_T \lambda^{-P}}{\varepsilon(1)\varepsilon(2s)\varepsilon(7s+2) r d(R_{\max})} < \frac{r\varepsilon(1)}{2C^2}$.

Also, there exists a least $L_0 \in \mathbb{N}$ such that $s^{-1/L_0} \geq \lambda^{-1/2}$.

SUBLEMMA 5.5. *For every $q \in \mathbb{N}$, there exists a least $p \in \mathbb{N}$ such that any allowed choice of B_{J+p} is a subset of R_δ for some $\delta \in \Sigma_\eta(q)$.*

Proof. Recall the definition of B^+ from the proof of Lemma 5.2.

Note that, by Lemma 5.2, $p \geq 1$. It suffices to show the sublemma for some p; that a least such p exists is then immediate. Let β be an element of $\Sigma_\eta(q)$ with smallest σ. Then $\frac{d(R_\beta)}{d(R_\eta)} \geq \frac{r}{Cs^q}$. Thus, there exists a large integer t such that $\frac{r}{Cs^q}d(R_\eta) > B_{J+t}$. Hence, $|B_{J+t} \cap \partial_{G_\beta}| \leq 1$ (i.e. there is at most one boundary point of the proper weight in B_{J+t}). Pick $W_{J+t} \subset B^+_{J+t}$. Hence, let $p = t+1$. □

By Sublemma 5.5, there exists a $L_1 \in \mathbb{N}$ such that B_{J+L_1} is contained in an element of $G(2P)$. Let $L := \max(L_0, L_1)$.

By Lemma 5.2, there exists a least $N \in \mathbb{N}$ for which we can choose an element $R_\delta \in G(N-1)$ such that

$$2d(R_\delta) \geq d(B_{J+L}) \geq \frac{\varepsilon(1)}{C}d(R_\delta). \tag{5.2}$$

Also, there exists $R_{\delta k} \subset B_{J+L}$ for some letter k. By construction, $R_{\delta k} \subset R_\eta$, and hence $R_\delta \subset R_\eta$ (because the generation that R_δ belongs to is later than or the same as that of R_η).

Also, since B_{J+L} is contained in an element of $G(2P)$, and B^+_{J+L} (see the proof of Lemma 5.2 for the meaning of the notation) is contained in an element of $G(N-1)$, $N-1 \geq 2P$.

Pick an integer $Q > N$ as follows. Choose integers $N_4 > N_3 > N_2 > N_1 \geq s+1$ as follows: γ_{N+N_1} is the next nondegenerate letter in γ following $\gamma(N+s)$, γ_{N+N_2} is the next nondegenerate letter in γ following $\gamma(N+N_1)$, γ_{N+N_3} is the next nondegenerate letter in γ following $\gamma(N+N_2)$, and γ_{N+N_4} is the next nondegenerate letter in γ following $\gamma(N+N_3)$. (By Corollary 2.11, $4+s \leq N_4 \leq 5s$.) If $\gamma(N+N_4+1)$ is of the form $a \cdots ab$ for a general block a and b is either a general block not equivalent to aa or a double general block not equivalent to aa, then $Q = N+N_1+1$; otherwise, choose $Q = N+N_4+1$. Hence, Lemma 3.1 applies to $\gamma(Q)$.

Now, by (5.2),

$$\begin{aligned}(mn)^L d(B_J) &\geq \frac{\varepsilon(1)}{C}d(R_\delta) \geq \frac{\varepsilon(1)}{C}\varepsilon_\eta(N-1-N_0)d(R_\eta)\\ &\geq \frac{\varepsilon(1)}{C}\varepsilon_\eta(N-1-N_0)d(B_J)/2 \geq \frac{\varepsilon(1)}{2C}\varepsilon_\eta(Q)d(B_J)\\ &\geq \frac{r\varepsilon(1)}{2C^2}\mathcal{E}_\eta(Q)d(B_J) \geq \frac{r\varepsilon(1)}{2C^2}s^{-Q}d(B_J).\end{aligned}$$

Thus,

$$m \geq \frac{r\varepsilon(1)}{2C^2}\lambda^{-Q/2}. \tag{5.3}$$

Since $Q - N \geq s+1$, $R_{\delta k}$ splits into at least two elements of $G(Q)$ by Corollary 2.11. One of these is not $R_{\gamma(Q)}$; call this element R_α. (Note that $Q \geq 8s-4$.) By Lemma 3.1, there exist strings b^0 and b^1, each of length at most s, such that for any valid choice of letters $\beta_0, \cdots, \beta_k$, where $l(b^0)+l(b^1)+k+1 \leq Q$, no match of $\gamma(Q)$ with $\alpha b^0 b^1 \beta_0 \cdots \beta_k$ exists. Thus,

$$d(R_\delta) \geq d(R_{\alpha b^0 b^1}) \geq \frac{\varepsilon(7s+2)}{C}d(R_\delta) \tag{5.4}$$

by Lemma 2.2. Consequently, by (5.2) and (5.4),

$$d(R_{\alpha b^0 b^1}) \geq n d(B_{J+L}) \geq \frac{\varepsilon(7s+2)\varepsilon(1)}{2C^2} d(R_{\alpha b^0 b^1}).$$

Since White must choose $W_{J+L} \in B^n_{J+L}$, White picks $W_{J+L} \subset R_{\alpha b^0 b^1}$. Black now chooses $B_{J+L+1} \in W^m_{J+L}$; hence,

$$m d(R_\alpha) \geq d(B_{J+L+1}) \geq \frac{m\varepsilon(7s+2)\varepsilon(1)\varepsilon(2s)}{2C^3} d(R_\alpha). \tag{5.5}$$

By Lemma 5.2 again, there exists $N' \in \mathbb{N}$ for which we can choose an element $R_{\eta'} \in G(N'-1)$ such that

$$2d(R_{\eta'}) \geq d(B_{J+L+1}) \geq \frac{\varepsilon(1)}{C} d(R_{\eta'}). \tag{5.6}$$

Also, there exists $R_{\eta' k'} \subset B_{J+L+1}$ for some letter k'. Now, by construction, $R_{\eta'} \subset R_{\alpha b^0 b^1}$. Hence, $N' - 1 \geq Q + l(b^0) + l(b^1)$. Define $q_{J+L+1} = N' - Q$.

Sublemma 5.6. *$l(b^0) + l(b^1) < q_{J+L+1} \leq Q$.*

Proof. Assume that $q_{J+L+1} \geq Q+1$. We have

$$d(R_{\eta'}) \leq \mathcal{E}(q_{J+L+1} - 1) C d(R_\alpha) \leq \mathcal{E}(Q) C d(R_\alpha).$$

Let $R_\beta \in G(Q)$ such that $\mathcal{E}(Q) = \frac{d(R_\beta)}{d(R_{\beta_0})}$. Since $d(R_{\beta_0}) \geq r d(R_{\max})$ and (2.13) holds, $d(R_{\eta'}) \leq \frac{\delta_T \lambda^{-Q}}{r d(R_{\max})} C d(R_\alpha)$.

Hence, by (5.5) and (5.6),

$$\begin{aligned} m &\leq \frac{4C^4 \delta_T \lambda^{-Q}}{\varepsilon(1)\varepsilon(2s)\varepsilon(7s+2) r d(R_{\max})} \\ &\leq \frac{4C^4 \delta_T \lambda^{-P}}{\varepsilon(1)\varepsilon(2s)\varepsilon(7s+2) r d(R_{\max})} \lambda^{-Q/2} < \frac{r\varepsilon(1)}{2C^2} \lambda^{-Q/2}, \end{aligned}$$

a contradiction of (5.3). □

Consequently, by Lemma 3.1, no match of $\gamma(Q)$ with any valid string beginning with $\alpha b^0 b^1$ in $\Sigma(Q + q_{J+L+1})$ exists.

Now, by construction, B_{J+L+1} contains an element (i.e. $R_{\eta' k'}$ of $G(Q + q_{J+L+1})$) whose string begins with $\alpha b^0 b^1$. Let $\alpha' := \eta' k'$. Thus,

$$R_{\alpha'} \in H_{q_{J+L+1}}.$$

By Lemma 3.1, there exist strings b'^0 and b'^1, each of length at most s, such that for any valid choice of letters $\beta'_0, \cdots, \beta'_k$, where $l(b'^0) + l(b'^1) + k + 1 \leq Q$, no match of $\gamma(Q)$ with $\alpha' b'^0 b'^1 \beta'_0 \cdots \beta'_k$ exists. Thus,

$$d(R_{\eta'}) \geq d(R_{\alpha' b'^0 b'^1}) \geq \frac{\varepsilon(7s+2)}{C} d(R_{\eta'})$$

by Lemma 2.2.

As before, White chooses $W_{J+L+1} \subset R_{\alpha' b'^0 b'^1}$. Continue thus by induction.

Therefore, we obtain

$$\cap_{p=J+L+1}^{\infty} W_p \in \cap_{p=J+L+1}^{\infty} (\cup H_{\sum_{j=J+L+1}^{p} q_j}(Q)). \tag{5.7}$$

The latter set is a set of points whose forward orbits avoid $\mathrm{Int}R_{\gamma(Q)}$.

Denote

$$A_\gamma := \cup_{Q=2P+2}^{\infty} \cap_{p=J+L+1}^{\infty} (\cup H_{\sum_{j=J+L+1}^{p} q_j}(Q)).$$

By (5.7), A_γ is (n, m)-winning for all $0 < m < 1$.

If γ is the unique representation of x_0, then, by Lemma 2.5, $x_0 \in \mathrm{Int}R_{\gamma(Q)}$ for all $Q \in \mathbb{N} \cup \{0\}$. Hence, A_γ is the set of points whose forward orbits avoid a neighborhood of x_0. Thus, we are done for x_0 in this case.

If $\gamma^1, \cdots, \gamma^{r_0}$ are representations of x_0 for $r_0 > 1$, then $A := \cap_{t=1}^{r_0} A_{\gamma^t}$ is n-winning. The set of boundary points is the countable union of finite sets and hence countable (for $M = S^1$). Thus, $A \backslash \partial$ is n-winning.

Let $x \in A \backslash \partial$. Then there exist some $Q_1, \cdots, Q_{r_0}$ such that

$$\mathcal{O}_T^+(x) \cap \mathrm{Int}R_{\gamma^t(Q_t)} = \emptyset.$$

Let $Q := \max(Q^t)$. By Lemma 2.6, there exists an open neighborhood U of x_0 such that $U \subset \cup_{t=1}^{r_0} \mathrm{Int}R_{\gamma^t(Q)} \cup \partial$.

If there exists $q \geq 0$ such that $T^q(x) \in \partial$, then, by Lemma 2.4, $x \in \partial$, a contradiction. Thus, $\mathcal{O}_T^+(x) \cap \partial = \emptyset$. Hence, $\mathcal{O}_T^+(x) \cap U = \emptyset$. Thus, $A \backslash \partial$ is a set of points whose forward orbits avoid an open neighborhood of x_0. □

We have the following corollary. Let $\{T_n\}_{n=1}^N$ be any finite set of C^2-expanding self-maps of S^1. For each map, choose, via Lemma 5.1, a Markov partition with small diameter with only intervals as elements. Let s_n be the number of elements of the n^{th} Markov partition. Let ε_n be the lower constant of bounded distortion for the n^{th} Markov partition. Let C_n be the constant (from Theorem 2.1) for the n^{th} Markov partition. Let $\alpha = \min(\frac{\varepsilon_1(7s_1+2)}{2C_1}, \cdots, \frac{\varepsilon_N(7s_N+2)}{2C_N}) > 0$.

Corollary 5.7. *For each n, choose a (at most) countably infinite set $\{x_i^n\}_{i=1}^{\infty} \subset S^1$. Then*

$$\bigcap_{n=1}^{N} \bigcap_{i=1}^{\infty} F_{T_n}(x_i^n) \tag{5.8}$$

is α-winning.

Question 1. Is $F_T(x_0)$ α-winning for some α independent of the choice of Markov partition and of T itself (such as $\alpha = 1/2$ for example)?

6. Conclusion

In this note, we have presented a way of proving Theorem 1.3 using elementary methods of Markov partitions. As mentioned, A. G. Abercrombie and R. Nair have another method using higher dimensional nets and Kolmogorov's consistency theorem [1]. In addition to our result, their method also gives a lower bound for the Hausdorff dimension of the set of points whose forward orbits miss balls (of a radius which one can choose, subject to certain constraints) around the points $x_1, \cdots, x_p$. Instead of constructing good strings as we do, they construct a certain Borel measure on the set of points whose forward orbits miss the desired balls. This measure encapsulates the iterations of T and is zero on the strings which come too close to hitting the balls to be avoided. Thus, they are freed from considering matching.

Our method, on the other hand, is concerned with matching. In particular, the use of the No Matching lemma requires manipulation and coordination of elements of certain generations of the Markov partition, which the author only knows how to do when the points being missed are contained in these elements. If one would like to show a result concerning missing balls around points, then one must be able to manipulate and coordinate elements adjacent to the elements which contain the points being missed. This requirement is most clearly seen when one wishes to miss an interior point, as how close the point is to the boundary of the element (of the requisite generation of the Markov partition) determines how large a ball around this point our method allows us to miss. This sort of variation does not seem to allow us to give, without further modifications to our method, a lower bound like Abercrombie and Nair's.

However, our elementary method is very geometric since we handle elements of generations of the Markov partition directly. It is this geometric nature that allows us to generalize, in dimension one, Theorem 1.3 and Abercrombie and Nair's result to winning sets. Doing so has allowed us to obtain a considerable strengthening: the countable intersection property. With this property, we can generalize to finitely many maps and countably many points, as precisely stated in Corollary 5.7. (If we can answer Question 1 affirmatively, then we can generalize to countably many maps.) Can we also generalize to winning sets for higher dimensional manifolds, and can we prove a similar result for Anosov diffeomorphisms? Only starting with Subsection 5.2 did we specialize to dimension one. Much of the theory works for higher dimensions. How much will work and with what modifications?

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References

[1] A. G. Abercrombie and R. Nair, *An exceptional set in the ergodic theory of expanding maps on manifolds,* Monatsh. Math. **148** (2006), 1–17.

[2] S. G. Dani, *On badly approximable numbers, Schmidt games and bounded orbits of flows,* M. M. Dodson and J. A. G. Vickers (eds), *Number theory and dynamical systems,* London Mathematical Society Lecture Note Series **134**, Cambridge University Press, Cambridge, UK (1989).

[3] S. G. Dani, *On orbits of endomorphisms of tori and the Schmidt game,* Ergodic Theory Dynam. Systems **8** (1988), 523–529.

[4] D. Dolgopyat, *Bounded orbits of Anosov flows*, Duke Math. J. **87** (1997), 87–114.

[5] D. Kleinbock, *Nondense orbits of flows on homogeneous spaces*, Ergodic Theory Dynam. Systems **18** (1998), 373–396.

[6] D. Kleinbock and G. Margulis, *Bounded orbits of nonquasiunipotent flows on homogeneous spaces*, Amer. Math. Soc. Transl. **171** (1996), 141–172.

[7] K. Krzyżewski and W. Szlenk, *On invariant measures for expanding differentiable mappings*. Studia Math. **33** 1969, 83–92.

[8] C. McMullen, *Area and Hausdorff dimension of Julia sets of entire functions*, Trans. Am. Math. Soc. **300** (1987), 329–342.

[9] D. Ruelle, *Thermodynamic formalism*, Encyclopedia of Mathematics and its Applications **5**, Addison-Wesley, Reading, MA (1978).

[10] W. Schmidt, *Badly approximable numbers and certain games,* Trans. Amer. Math. Soc. **123** (1966), 178–199.

[11] M. Urbański, *The Hausdorff dimension of the set of points with nondense orbit under a hyperbolic dynamical system,* Nonlinearity **4** (1991), 385–397.

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