SIMULTANEOUS DENSE AND NONDENSE ORBITS AND THE SPACE OF LATTICES

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ABSTRACT. We show that the set of points nondense under the $\times n$-map on the circle and dense for the geodesic flow after we identify the circle with a periodic horospherical orbit of the modular surface has full Haudorff dimension. We also show the analogous result for toral automorphisms on the 2-torus and a diagonal flow. Our results can be interpreted in number-theoretic terms: the set of well approximable numbers that are nondense under the $\times n$-map has full Hausdorff dimension. Similarly, the set of well approximable 2-vectors that are nondense under a hyperbolic toral automorphism has full Hausdorff dimension. Our result for numbers is the counterpart to a classical result of Kaufmann.

1. INTRODUCTION

Let $f : Y \to Y$ be a dynamical system on a set Y with a topology and let $\mathcal{S}$ be a finite subset. Let $ND(f)$ denote the set of points with nondense orbit under f and $ND(f, \mathcal{S})$ be the set of points whose orbit closures miss the subset $\mathcal{S}$. Let $D(f)$ denote the set of points with dense orbits. Given another map $\tilde{f} : Y \to Y$, one could ask for the size of the set $ND(f) \cap D(\tilde{f})$ or, more specifically, $ND(f, \mathcal{S}) \cap D(\tilde{f})$.[1] Such a question was first asked by V. Bergelson, M. Einsiedler, and the second-named author for commuting toral automorphisms and endomorphisms and commuting Cartan actions on certain cocompact homogeneous spaces in [1]. In this paper, we study this same question for certain noncompact phase spaces. One of our results (Corollary 1.2) is even an example of a noncommuting pair of maps. Consequently, the two main restrictions to the proof technique in [1] can be overcome, at least in these special cases. (Another example for the noncommuting case is the main result in [19], in which a different proof technique is used. See also [14].) For further details of the history of this question, see [1]. Note that our two main results have consequences for number theory (Corollary 1.6), which allows us to complement a classical result of Kaufman [12] (see Remark 1.7).

1.1. **Statement of results.** Let $X := X_d := \mathrm{SL}_d(\mathbb{R}) / \mathrm{SL}_d(\mathbb{Z})$, the space of unimodular lattices in $\mathbb{R}^d$. This space is noncompact and has finite Haar meaure, which we normalize to be a probability measure and denote by μ_X. Let

$$(1.1) \qquad h_d : \mathbb{R}^{d-1} \to X_d \quad \text{be defined by} \quad h_d(s) = \begin{pmatrix} 1_{d-1} & s \\ 0 & 1 \end{pmatrix} \quad \mathrm{mod} \ \mathrm{SL}_d(\mathbb{Z}).$$

[1] The proofs in [1] work for $ND(f, \mathcal{S}) \cap D(\tilde{f})$, not just $ND(f) \cap D(\tilde{f})$.

For $g \in \mathrm{SL}_d(\mathbb{R})$, the left translation by g on X_d is also denoted by g. Let $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ denote the d-dimensional torus, $\mathbb{T}$ denote the circle, and $[\cdot] : \mathbb{R}^d \to \mathbb{T}^d$ denote the natural projection map. Let $\mathcal{O}_f^{\mathbb{N}}(y)$ (where $\mathbb{N} = \{0, 1, 2, \ldots\}$ in this paper) denote the forward orbit of the point y under f and $\mathcal{O}_f^{\mathbb{Z}}(y)$ denote the forward and backwards orbit. Our two main results are the following two theorems.

Theorem 1.1. *Let $n \in \mathbb{Z}_{\geq 2}$, $T := T_n$ be the $\times n$-map on $\mathbb{T}$ and let $\mathcal{S} = \{s_1, \cdots, s_m\}$ be a finite set of points in $\mathbb{R}$. Let $t_0 > 0$ and $g = g_{t_0} := \mathrm{diag}(e^{t_0}, e^{-t_0}) \in \mathrm{SL}_2(\mathbb{R})$. Then the set*

$$(1.2) \qquad \{s \in \mathbb{R} : \overline{\mathcal{O}_T^{\mathbb{N}}([s])} \cap \mathcal{S} = \emptyset \quad and \quad \overline{\mathcal{O}_g^{\mathbb{N}}(h_2(s))} = X_2\}$$

has full Hausdorff dimension.

COROLLARY 1.2. *Let the assumptions be as in Theorem 1.1 and let $\mathcal{G} : \mathbb{T} \to \mathbb{T}$ be the Gauss map. Then the set*

$$\{s \in \mathbb{R} : \overline{\mathcal{O}_T^{\mathbb{N}}([s])} \cap \mathcal{S} = \emptyset \quad and \quad \overline{\mathcal{O}_{\mathcal{G}}^{\mathbb{N}}([s])} = \mathbb{T}\}$$

has full Hausdorff dimension.

Proof. Suppose that $s \in \mathbb{R}$ and

$$(1.3) \qquad \overline{\mathcal{O}_g^{\mathbb{N}}(h(s))} = X_2.$$

It follows from [4, Theorem 1.7] that the continued fraction expansion of s contains all patterns. Therefore $\overline{\mathcal{O}_{\mathcal{G}}^{\mathbb{N}}([s])} = \mathbb{T}$. Thus Corollary 1.2 follows from Theorem 1.1. $\square$

We say two natural numbers $n, m \geq 2$ are *multiplicatively dependent* if there exists integers $p, q \geq 1$ such that $n^p = m^q$. Otherwise, n and m are *multiplicatively independent*. For T_m, define

$$\widetilde{D}(T_m) := \{s \in \mathbb{R} : \overline{\mathcal{O}_{T_m}^{\mathbb{N}}([s])} = \mathbb{T}\}.$$

Here, T_m is the $\times m$-map, or, equivalently, the multiplication map on $\mathbb{T}$ by m. Also for $t > 0$, let

$$\widetilde{D}(g_t, X_d) := \{s \in \mathbb{R}^{d-1} : \overline{\mathcal{O}_{g_t}^{\mathbb{N}}(h_d(s))} = X_d\}.$$

Our method of the proof of Theorem 1.1 can also be used to obtain

Theorem 1.3. *Let the assumptions be as in Theorem 1.1. Let $\{T_m\}$ be all multiplication maps on $\mathbb{T}$ by natural numbers m which are multiplicatively independent from n. Let $\{t_i\}$ be a sequence of positive real numbers. Then the set*

$$\{s \in \mathbb{R} : \overline{\mathcal{O}_{T_n}^{\mathbb{N}}([s])} \cap \mathcal{S} = \emptyset\} \cap \left(\cap_m \widetilde{D}(T_m) \right) \cap \left(\cap_i \widetilde{D}(g_{t_i}, X_2) \right)$$

has full Hausdorff dimension.

Theorem 1.4. *Let $T = T_\alpha$ be an automorphism of $\mathbb{T}^2$ induced by the left multiplication of a hyperbolic matrix $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ and let $\mathcal{S} = \{s_1, \cdots, s_m\}$ be a finite set of points in $\mathbb{R}^2$. Let $t_0 > 0$ and $g = g_{t_0} := \mathrm{diag}(e^{t_0}, e^{t_0}, e^{-2t_0}) \in \mathrm{SL}_3(\mathbb{R})$. Then the set*

$$(1.4) \qquad \{s \in \mathbb{R}^2 : \overline{\mathcal{O}_T^{\mathbb{Z}}([s])} \cap \mathcal{S} = \emptyset \quad and \quad \overline{\mathcal{O}_g^{\mathbb{N}}(h_3(s))} = X_3\}$$

has full Hausdorff dimension.

Let T_α and T_β be two commuting hyperbolic automorphisms of $\mathbb{T}^2$ and $\mathcal{A} := (T_\alpha, T_\beta)$ be the $\mathbb{Z}^2$-action on $\mathbb{T}^2$ that is generated by T_α and T_β. A $\mathbb{Z}^2$-action $\mathcal{A}'$ on a torus $\mathbb{T}^{d'}$ is an *algebraic factor* of $\mathcal{A}$ if there is a surjective toral homomorphism $\varphi : \mathbb{T}^2 \to \mathbb{T}^{d'}$ such that $\mathcal{A}' \circ \varphi = \varphi \circ \mathcal{A}$ and is a *rank-one factor* if, in addition, $\mathcal{A}'(\mathbb{Z}^2)$ has a finite-index subgroup consisting of the powers of a single map. For T_β, let

$$\widetilde{D}(T_\beta) := \{s \in \mathbb{R}^2 : \overline{\mathcal{O}^{\mathbb{N}}_{T_\beta}([s])} = \mathbb{T}^2\}.$$

Our method of the proof of Theorem 1.4 can also be used to obtain

Theorem 1.5. *Let the assumptions be as in Theorem 1.4. Let $\{T_\beta\}$ be all hyperbolic toral automorphisms of the 2-torus that commute with T_α and such that the algebraic $\mathbb{Z}^2$-actions (T_α, T_β) are all without rank-one factors. Let $\{t_i\}$ be a sequence of positive real numbers. Then the set*

$$\{s \in \mathbb{R}^2 : \overline{\mathcal{O}^{\mathbb{Z}}_{T_\alpha}([s])} \cap \mathcal{S} = \emptyset\} \cap \big(\cap_\beta \widetilde{D}(T_\beta) \big) \cap \big(\cap_i \widetilde{D}(g_{t_i}, X_3) \big)$$

has full Hausdorff dimension.

Finally, we note that our two main theorems have consequences for number theory. Recall that a classical object in the theory of Diophantine approximation is the set of well approximable d-vectors $WA(d)$, which is defined to be the complement in $\mathbb{R}^d$ of the set of badly approximable vectors $BA(d)$.

COROLLARY 1.6. *Let the assumptions be as in Theorems 1.3 and 1.5. The sets*

$$\{s \in \mathbb{R} : \overline{\mathcal{O}^{\mathbb{N}}_{T_n}([s])} \cap \mathcal{S} = \emptyset\} \cap \big(\cap_m \widetilde{D}(T_m) \big) \cap WA(1)$$

$$\{s \in \mathbb{R}^2 : \overline{\mathcal{O}^{\mathbb{Z}}_{T_\alpha}([s])} \cap \mathcal{S} = \emptyset\} \cap \big(\cap_\beta \widetilde{D}(T_\beta) \big) \cap WA(2)$$

have full Hausdorff dimension.

Proof. By Dani correspondence, $WA(d)$ is the set of d-vectors whose corresponding trajectories are not bounded and, thus, a superset of the d-vectors whose corresponding trajectories are dense under the diagonal element g. The result is now immediate from our two theorems. $\square$

Remark 1.7. It is a classical result of Kaufman [12] from 1980 (see also Queffélec-Ramaré [16] for extensions of [12]) that the set $BA(1) \cap D(T_n)$ has full Hausdorff dimension; also see [9, Section 1.3.3] and [10]. The first assertion in Corollary 1.6 immediately implies the other mixed case, namely, $ND(T_n) \cap WA(1)$ has full Hausdorff dimension. It is known (see e.g. Pollington [15]) that the intersection of $WA(1)$ with the set of numbers which are non-normal in every base has full Hausdorff dimension.

1.2. **Ideas in the proof.** The purpose of this paper is to study simultaneous dense and nondense orbits, first studied in [1], in the context of a noncompact phase space and, in the case of Corollary 1.2, for noncommuting maps. The basic idea here follows that in [1]; namely, form a large closed fractal of nondense orbits, apply measure rigidity to obtain Haar measure (up to a constant), and then use properties of entropy and measures to conclude our full Hausdorff dimension results. The main difficulty in the cases that we consider in this paper is the possibility of the escape of all the mass, namely that the weak-* limit of measures goes to 0. Using results in [5, 11, 17], we show that not all the mass can escape when the Hausdorff dimension is large and, thus, we still obtain dense orbits.

2. Proof of Theorems 1.1 and 1.4

We will prove Theorems 1.1 and 1.4 together (deferring the proof of Propositions 2.3 and 2.6 to Sections 4 and 5, respectively). Let

$$X := X_2 \text{ or } X_3,$$
$$g_t := \text{diag}(e^t, e^{-t}) \text{ or } \text{diag}(e^t, e^t, e^{-2t}),$$
$$g := g_{t_0} \text{ where } t_0 \text{ is a fixed positive number,}$$
$$I := \mathbb{R}/\mathbb{Z} \text{ or } \mathbb{R}^2/\mathbb{Z}^2,$$
$$T := T_n \text{ or } T_\alpha,$$
$$\mathcal{O}_T(s) := \mathcal{O}_{T_n}^{\mathbb{N}}(s) \text{ or } \mathcal{O}_{T_\alpha}^{\mathbb{Z}}(s).$$

The map $h = h_2$ or h_3 defined in (1.1) induces a map $I \to X$ all of which we denote by h. Since T_α is a hyperbolic matrix, it has two real eigenvalues, one of which has absolute value > 1. Let λ_α denote this eigenvalue.

Fix a finite set $\mathcal{S} := \{s_1, \cdots, s_m\}$ of points in I. In I, pick a sequence of open balls U_q centered at 0 with radius $\to 0$ as $q \to \infty$. Define the following closed T-invariant set

$$E(q) := E_{T,\mathcal{S}}(q) := \{s \in I \mid \mathcal{O}_T(s) \cap \cup_{i=1}^m (U_q + s_i) = \emptyset\}.$$

Proposition 2.1. *The set $\cup_q E(q)$ is winning and, therefore, has full Haudorff dimension.*

Proof. Apply [18, Corollary 1.5] and [3, Theorem 1.1]. $\square$

Consequently, $E(q)$ has Hausdorff dimension as close as we like to full Hausdorff dimension, provided we choose q large enough. Large Hausdorff dimension implies that the topological entropy of T restricted to such $E(q)$ is close to $h_{top}(T)$:

Proposition 2.2 (Proposition 2.4 of [1]). *Let $F \subset I$ be a closed T-invariant set. Then we have that*

$$h_{top}(T|_F) \geq h_{top}(T) - (d - \dim F) \log(\lambda_1)$$

where $\lambda_1 := n$ if $T = T_n$ or $\lambda_1 := |\lambda_\alpha|$ if $T = T_\alpha$.

The variational principle now gives the existence of a T-invariant Borel probability measure $\nu := \nu(q)$, whose support lies in $E(q)$, such that $h_\nu(T|_{E(q)})$ is as close to $h_{top}(T)$ as we like, provided we choose q large enough. By ergodic decomposition, we may restrict to an ergodic component and thus assume that ν is ergodic.

Proposition 2.3. *Let ν be a T-invariant, ergodic probability measure on I with positive entropy. Then any weak-$*$ limit μ of*

$$(2.1) \qquad \frac{1}{N} \sum_{i=0}^{N-1} (g^i)_*(h_*\nu) \quad as \quad N \to \infty$$

is equal to $c\mu_X$ where μ_X is the probability Haar measure on X and $0 \le c \le 1$.

We prove Proposition 2.3 in Section 4 using [17, Theorems 3.1, 5.1 and 6.1] of the first-named author. To guarantee that $c \ne 0$, we also need the the following proposition, ensuring that not all of the mass can escape. The proposition is essentially a corollary of [5, Theorem 1.6] and [11, Theorem 1.3].

Proposition 2.4. *Let ν be a T-invariant, ergodic probability measure on I with entropy $h_\nu(T) = \delta h_{top}(T)$ where $0 \le \delta \le 1$. Then any weak-$*$ limit μ of (2.1) will have total mass $\mu(X) \ge (1 + \dim(I))\delta - \dim(I)$.*

Proof. Since

$$(2.2) \qquad \frac{1}{t_0 N} \int_0^{t_0 N} (g_t)_*(h_*\nu)\, dt = \frac{1}{t_0} \int_0^{t_0} (g_t)_* \frac{1}{N} \sum_{k=0}^{N-1} (g^k)_*(h_*\nu)\, dt,$$

it suffices to show that any weak-$*$ limit μ_1 of

$$(2.3) \qquad \frac{1}{\tau} \int_0^\tau (g_t)_*(h_*\nu)\, dt \quad as \quad \tau \to \infty$$

has total mass $\mu_1(X) \ge (1 + \dim(I))\delta - \dim(I)$.

We claim that for ν almost every $s \in I$ one has

$$(2.4) \qquad \lim_{r \searrow 0} \frac{\log \nu(B(s,r))}{\log r} = \delta\dim(I).$$

For $T = T_n$, the claim follows from the main theorem of [2]. For $T = T_\alpha$ the claim follows from [20, Lemma 3.2]. Therefore for any $\epsilon > 0$ (sufficiently small) there exists $r_0 > 0$ and a subset $I_\epsilon \subset I$ such that for $0 < r \le r_0$ we have

$$\nu(I_\epsilon) > 1 - \epsilon$$

and

$$(2.5) \qquad \nu(B(s,r)) \le r^{(\delta-\epsilon)\dim(I)} \quad \text{for } s \in I_\epsilon.$$

Therefore $\frac{1}{1-\epsilon}h_*(\nu|_{I_\epsilon})$ has dimension at least $(\delta - \epsilon)\dim(I)$ in the unstable horospherical direction of g_1 according to [5, Definition 1.5]. Therefore [5, Theorem 1.6] and [11, Theorem 1.3] imply that

$$\mu_1(X) \ge (1 - \epsilon)(1 + \dim(I))(\delta - \epsilon) - \dim(I).$$

The conclusion follows by letting ϵ decrease to 0. $\qquad\square$

We wish to understand the following dense set of forward orbits

$$D(g, X) := \{s \in I : \overline{\mathcal{O}_g^{\mathbb{N}}(h(s))} = X\}$$

in terms of the measure ν supported on $E(q)$.

Lemma 2.5. *The set $D(g, X)$ is T-invariant.*

Proof. In the setting of Theorem 1.1, it is proved in [4, Lemma 4.3] that the set

$$D'(g_t; X) := \{s \in I : \overline{\{g_t h(s) : t \geq 0\}} = X\}$$

is T-invariant. The same argument there replacing $D'(g_t; X)$ by $D(g, X)$ gives the lemma.

In the setting of Theorem 1.4, we let

$$(2.6) \qquad\qquad g_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $g_\alpha \in SL_3(\mathbb{Z})$ a simple calculation shows that

$$g_t(h(\alpha(s))) = g_\alpha g_t(h(s))$$

from which the lemma follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proposition 2.6. *Let ν be a T-invariant and ergodic probability measure on I. Let μ be a weak-$*$ limit of (2.1). Suppose that $\mu = c\mu_X$ for some $0 < c \leq 1$. Then*

$$\nu(D(g, X)) = 1.$$

We prove Proposition 2.6 in Section 5.

Remark 2.7. If we were to only prove Theorems 1.1 and 1.4, then we would only need for the conclusion of Proposition 2.6 to be $\nu(D(g, X)) > 0$, in which case Lemma 2.5 is not used. For Theorems 1.3 and 1.5, we need the full conclusion of Proposition 2.6 and thus also Lemma 2.5. The analogous remark can be made for [1, Theorem 3.2] and [1, Lemma 3.1].

2.1. **Finishing the proof of Theorems 1.1 and 1.4.** Apply Propositions 2.3, 2.4, and 2.6 to the measures $\nu := \nu(q)$ supported on $E(q)$, to obtain $\nu(D(g, X)) = 1$. Next apply the mass distribution principle to (2.5) with $\delta := \delta(q) < 1$ (where $\delta(q) \nearrow 1$ as $q \to \infty$) to obtain

$$\dim(D(g, X) \cap E(q)) \geq (\delta(q) - \epsilon)\dim(E(q)).$$

(We note that, since (2.4) holds with $\dim(I)$ replaced by $\dim(E(q))$, so does (2.5).) Letting $\epsilon \to 0$ and taking a union over q, we have that

$$\dim\big(D(g, X) \cap ND(T)\big) = \dim(I),$$

as desired.

3. Proof of Theorems 1.3 and 1.5

Let T_m be as in Theorem 1.3 and let ν be as in the proof of Theorem 1.1. We have $\nu(D(T_m)) = 1$ by [1, Theorem 3.2] and thus

$$\nu\bigg(\cap_i D(g_{t_i}, X_2) \cap \big(\cap_m D(T_m)\big)\bigg) = 1.$$

The rest of the proof of Theorem 1.3 is exactly the same as that for the Theorem 1.1. Replacing T_m with T_β from Theorem 1.5 gives the proof for Theorem 1.5.

4. Proof of Proposition 2.3

We claim that any weak-$*$ limit μ_1 of (2.3) is equal to $c\mu_X$. We first prove the proposition using this claim and then give its proof in the rest of this section.

Let μ be a weak-$*$ limit of (2.1). If $\mu = 0$ there is nothing to prove. Assume that $\mu(X) = c > 0$. It follows from (2.2) that

$$\mu_2 := \frac{1}{t_0} \int_0^{t_0} (g_t)_* \mu \, dt$$

is a weak-$*$ limit of (2.3). The claim above implies that $\mu_2 = c\mu_X$. On the other hand [8, Theorem 5.27] implies

$$(4.1) \qquad\qquad h_{c^{-1}\mu_2}(g) = \frac{1}{t_0} \int_0^{t_0} h_{c^{-1}(g_t)_*\mu}(g) \, dt.$$

It follows from [7, Corollary 7.10] that

$$h_{c^{-1}(g_t)_*\mu}(g) = h_{c^{-1}\mu}(g) \le h_{\mu_X}(g)$$

and that equality holds if and only if μ and hence $(g_t)_*\mu$ is equal to $c\mu_X$. Therefore (4.1) implies that $\mu = c\mu_X$ which completes the proof of the proposition.

Now we prove the claim in the setting of Theorem 1.1. The natural map

$$\pi : \mathrm{SL}_2(\mathbb{R}) \to PGL_2(\mathbb{R})$$

induces a diffeomorphism of homogeneous spaces

$$X_2 \to \widetilde{X}_2 := PGL_2(\mathbb{R})/PGL_2(\mathbb{Z}).$$

Let $n = p_1^{\sigma_1} \cdots p_k^{\sigma_k}$ be the prime decomposition of n. Let

$$L = \prod_{i=1}^k PGL_2(\mathbb{Q}_{p_i}), \quad \Gamma = PGL_2(\mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_k}]) \quad \text{and} \quad K = \prod_{i=1}^k PGL_2(\mathbb{Z}_{p_i}).$$

Then $Y = PGL_2(\mathbb{R}) \times L/\Gamma$ where Γ embeds diagonally is a finite volume homogeneous space. The natural map

$$\eta : Y \to X_2,$$

which maps $(\pi(g), h) \mod \Gamma$, where $g \in \mathrm{SL}_2(\mathbb{R})$ and $h \in K$, to $g \mod \mathrm{SL}_2(\mathbb{Z})$, has compact fibers.

Let

$$a = \left(\begin{array}{cc} n & 0 \\ 0 & 1 \end{array} \right)^{k+1} \in PGL_2(\mathbb{R}) \times L.$$

It follows from [17, Lemma 5.3] that there exists an a-invariant and ergodic probability measure $\tilde{\nu}$ on Y such that $\eta_* \tilde{\nu} = h_* \nu$ and $h_{\tilde{\nu}}(a) > 0$. Let μ_1 be a weak-$*$ limit of (2.3) for the sequence $\{\tau_i :\in \mathbb{N}\}$. By possibly passing to a subsequence we assume that

$$\tilde{\mu}_1 := \lim_{i \to \infty} \frac{1}{\tau_i} \int_0^{\tau_i} (\pi(g_t))_* ((\pi \circ h)_* \nu) \, dt$$

exists. If $\mu_1(X) = 0$ there is nothing to prove. So we assume that $\mu_1(X) = c > 0$. Since η has compact fibers we have $\tilde{\mu}_1(Y) = c > 0$. It follows from [17, Theorem 3.1] that all the ergodic

components of $\tilde{\mu}_1$ with respect to the group generated by a have positive entropy. Therefore [13, Theorem 1.1] implies that $\tilde{\mu}_1 = c\mu_Y$. Since $\eta\tilde{\mu}_1 = \mu_1$, we have

$$\mu_1 = c\mu_{X_2}$$

which completes the proof of the claim in the setting of Theorem 1.1.

Now we prove the claim in the setting of Theorem 1.4. Let μ_1 be a weak-$*$ limit of (2.3). Without loss of generality we assume that $\mu_1(X_3) = c > 0$. It follows from [17, Theorem 3.1] that all the ergodic components of ν with respect to the group generated by g_α, where g_α is defined in (2.6), have positive entropy. Therefore [6, Corollary 1.4] implies that $\mu_1 = c\mu_{X_3}$ which completes the proof of the claim and hence the proposition.

5. Proof of Proposition 2.6

The proof is adapted from the proof of [1, Theorem 3.2]. Let d denote the distance function on X. Fix a countable dense subset $\{x_1, x_2, \cdots\} \in X$. Inductively define the following disjoint sets:

$$ND(1,1) := \left\{ s \in I : d\left(\overline{\{g^k h(s)\}_{k\in\mathbb{N}}}, x_1) \right) \geq 1 \right\}$$

$$ND(i,n) := \left\{ s \in I : d\left(\overline{\{g^k h(s)\}_{k\in\mathbb{N}}}, x_i) \right) \geq \frac{1}{n} \right\} \setminus \bigcup_{(j,m)<(i,n)} ND(j,m).$$

(Here we have fixed the total ordering $<$ of $\mathbb{N}\times\mathbb{N}$ given by $(i,n) < (i',n')$ if either $i+n < i'+n'$ or $i + n = i' + n'$ and $i < i'$.) It follows that $ND := I\backslash D(g, X)$ is the union of these sets.

Now assume that the conclusion is false; this, by Lemma 2.5, is equivalent to $\nu(ND) = 1$. Decompose $\nu = \sum_{i,n} \nu_{i,n}$ where $\nu_{i,n}$ is the restriction of ν to $ND(i,n)$. Using Banach–Alaoglu theorem, from the subsequence used to obtain μ, we may extract a further subsequence N_ℓ such that for all (i,n) the measure

$$\frac{1}{N_\ell} \sum_{k=0}^{N_\ell - 1} (g^k)_*(h_*\nu_{i,n})$$

converges in the weak-$*$ topology to $\mu_{i,n}$. We note that the $\mu_{i,n}$ are g-invariant measures on X and we have $\mu = \sum_{i,n} \mu_{i,n}$ by the disjointness of the sets $\{ND(i,n)\}$.

Let $B^M(y, r)$ denote an open ball in a metric space M around $y \in M$ of radius $r > 0$. By construction, we have that $\nu_{i,n}$-a.e. point s in I satisfies

$$h(s) \notin g^{-k}B^X(x_i, 1/n)$$

for all $k \geq 0$. Hence,

$$\nu_{i,n}(h^{-1}g^{-k}B^X(x_i, 1/n)) = (g^k)_*h_*\nu_{i,n}(B^X(x_i, 1/n)) = 0.$$

Since this holds for all $k \geq 0$, we have

$$\mu_{i,n}\left(B^X(x_i, 1/n)\right) = 0.$$

Finally, since $\mu_{i,n}$ is g-invariant, we have that

$$\bigcup_{k=0}^{\infty} g^{-k} B^X(x_i, 1/n)$$

is a $\mu_{i,n}$-null set, but it is also a full Haar measure set by the ergodicity of g. Consequently, $\mu_{i,n}$ is singular to Haar measure. Now, using the fact that a finite measure which is an infinite sum of finite measures, each of which is singular to μ_X, is singular to μ_X itself, we contradict the assumption that $\mu = c\mu_X$ for some $c > 0$. This completes the proof.

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