Sense and Proof

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1 Frege, Carnap and the contrast between cognitive semantic sense

In a paper given at the 1997 SILFS Conference, [Pen98] claims that a strong tension between a semantic and a cognitive notion of sense was already present in Frege's writings.

Many authors have discussed recently this tension in Frege, but few of them remark that Carnap was probably the first to attempt to compose the tension.

Carnap ([Car47]) soon realized that his conception of intension, although suitable to treat a semantic notion of sense as truth condition on the lines developed by Wittgenstein, was not enough to solve the problem posed by Frege on belief contexts. He devised therefore the notion of intensional structure and intensional isomorphism, as a more fine-grained notion that the semantic notion o intension and intensional equivalence. For many reasons however his notion did not appear successful. In this short paper we try to develop a new perspective which tries to compose the tension between a semantic and a cognitive notion of sense, suggesting the use of proof theory as a mean to interpret the cognitive notion of sense.

In the Fregean view, sense is something which is there to be grasped from a speaker. The concept of grasping a sense or understanding is co-original with the notion of sense. However it is not clear in Frege what is the object of understanding. Frege himself made interesting remarks on the limitations of understanding complex mathematical formula with all their connections with other parts of mathematics. In his famous argument on the "intuitive difference of thought" (as it has been named by Gareth Evans) Frege claims that if it is possible to understand two sentences, and coherently believe one and disbelieve the other, then those sentences express different senses or thoughts. The Fregean example is the belief that Hesperus is a planet and Phosphorus is not a planet, given by the ignorance of the identity Hesperus=Phosporus. A lack of knowledge permits a person who hold erroneous beliefs to be considered rational, because the erroneous beliefs express two different thoughts, whose truth is not evident to the believer.

The criterion links sense identity to what a subject believes, to her limited accessibility to the information, to what she grasps, with a limited understanding. Here the limitation is given by empirical information, but there are many passages where Frege accepts the idea of limited computational capacity as a way to explain why two expressions expresses different senses.

The semantic notion of sense is linked to the criterion of logical equivalence: in an example proposed by Frege (letter to Husserl 1906, see [Fre76].) two sentences which express a logical equivalence like $A \to B$ and $\neg(A \wedge \neg B)$ express the same thought. Frege probably connects this idea with the criterion of immediate recognizability, given that most of his examples -given in his last writings on logic - deal with elementary logical equivalences between connectives. The idea of sameness of sense given by logical equivalence is coherent with the concept of sense as truth conditions made by Wittgenstein in his Tractatus.

The second, semantic notion clashes with the first if we try to apply also to the semantic notion the criterion of the intuitive difference of thoughts; if we may believe two sentences as having different truth value because of lack of information, why shouldn't we accept the same uncertain attitude towards two sentences with the same truth conditions, but such that we accept one of them while remaining uncertain on the truth condition of the other? Actually we may lack computational capacity, while understanding the basic notions used, the meaning of the connectives, the working of the symbolism. We may imagine a person who is so slow that she cannot realize that two logical equivalent sentences produce the same truth tables. Therefore she may believe that $A \to B$ is true while $\neg(A \wedge \neg B)$ is false; therefore the two sentences would express - contrary to what Frege says - two different (cognitive) senses. This possibility is enforced by what Frege says in a later paper on negation, when he assert that A and not not A express different thoughts.

We claim that the ambiguity in Frege's concept of sense depends on the lack of logical instruments that has been developed later with proof theory. And we think that proof theory may help to define different levels of understanding; in this way we may distinguish the semantic and the cognitive aspect of sense, composing the original tension, in a way which is different from the original suggestions given by Dummett with the choice of a verificationistic theory of meaning.

2 Understanding sense as truth conditions

Let us begin with an analysis of what it means "to understand sense as truth conditions". When we consider complex sentences, the understanding of truth conditions amounts to knowing the corresponding truth tables.

The sense of a complex sentence A is then known, when we know for which values of the atomic propositions occurring in A, the sentence is true.

Consider the example considered above, we have two logically equivalent sentences (1) $A \to B$ (2) $\neg(A \wedge \neg B)$.

If a subject understands $A \to B$ then, by definition of material implication, she knows that $A \to B$ is true when either A is false or B is true.

If the subject understands $\neg(A \wedge \neg B)$, then she knows that the sentence is true when it is false that A and B is true; which entails, if the meaning of the conjunction and the negation is known, that he knows that $\neg(A \wedge \neg B)$ is true when either A is false or B is true.

So the truth conditions of the sentences involved are the same. If we do not consider how subjects grasp those truth conditions, then we face the following problems.

First, for example, we cannot concede the possibility that a subject might believe (1) true and (2) false. Suppose for example that a subject, by mistake or logical confusion, believes that $A \to B$ is true and believes that $\neg(A \wedge \neg B)$ is false. Then, she believes that it is the case that A is false or B is true, since she believes (1) true. But if she believes that $\neg(A \wedge \neg B)$ is false, then she believes that $(A \wedge \neg B)$ is true; so she believes that A is true and B is false. Since the information the subject should manage is not coherent (A is false or B is true, A is true and B is false), the only way in which we may claim that she believes (1) true and (2) false is claiming that she is accepting a contradiction. We have no means to say that she doesn't realize that she is contradicting. We may however ask whether it is possible that a subject believes that (1) is true while having no opinion about the truth value of (2).

Even in this case, if we keep the hypotheses at issue, we cannot represent such situation. If a subject believes that $A \to B$ is true, then she believes that A is false or B is true, and since these are precisely the truth-conditions of $\neg(A \wedge \neg B)$ and the subject knows that if these condition are satisfied then $\neg(A \wedge \neg B)$ is true (since she understands both sentences), then we are forced to say that the subject believes that (2) is also true. But why does she believe it? Simply because of our definitions, since we cannot deny it without contradicting our definitions.

From this argument, it follows that if a subject believes a sentence A, then he must believe all the sentences that are logically equivalent to A, no matter how complex they are. Moreover, the subjects *immediately* believes all the logically equivalent sentences, since we just proved it as a fact simply entailed by our hypotheses. With our notion of understanding sense as truth conditions we are compelled to make our speaker logically omniscient.

3 A weaker notion of understanding

The logical omniscience of a subject described by the assumptions we made concerning the notion of understanding a sentence is completely useless from a cognitive, or computational, point of view.

It seems that if we assume the definition of understanding a sentence as mere grasping truth conditions, even if we try to employ a cognitive notion of sense (e. g. the one presented by the immediate recognizability criterion) we are not allowed to define a cognitive difference between logically equivalent sentences.

If however we look closer at the same argument we mentioned to find out the truth conditions of the sentences involved, we note that the formula $\neg(A \wedge \neg B)$ requires more calculation than $A \rightarrow B$.

The notion of sense of a sentences in mere terms of truth conditions fails to capture all the information concerning the complexity of the process of grasping the truth condition, which seem to be the relevant aspect in a cognitive notion of sense.

In the following sections, we will advance a proposal for defining a cognitive notion of sense which is coherent and compatible with an objective one.

A proper notion of cognitive sense should be grounded not on a strong notion of understanding requiring full grasping of truth conditions, but on a weaker notion of understanding based on the idea of limited knowledge (or even on bounded rationality) [1].

We will therefore need to enrich the notion of sense considering another aspect of sense that seems more suitable to deal with cognitive aspects, namely the notion of sense as computing procedure hinted at by Frege (see [fre93].).

4 Limited knowledge and procedures

Assuming that understanding a formula is not necessarily understanding directly its truth conditions, but understanding its mode of composition and the meaning of the connectives, we can state the problem at issue with the following question:

> (Q) Assuming that a subject understands (1) $A \rightarrow B$ and (2) $\neg(A \wedge \neg B)$, and moreover accepts (1) $A \rightarrow B$, what does she need in order to accept also (2) $\neg(A \wedge \neg B)$?

The point is that if we aim to describe a cognitive notion of sense, we cannot consider the process of understanding of the two sentence as imme-

[1] See [Pen03b]

diate [2]. We need to consider the process of understanding, say (2) given (1), as a process mediated by a procedure, or a computation. Otherwise we would lose the information concerning the complexity of the process of understanding sentences which is essential for a cognitive notion of sense. Therefore we reformulate the truth conditional approach considering the procedure of grasping truth conditions as a constitutive feature of understanding sentences.

Our basic definition is then that a subject understands a sentence when she can perform a *procedure* of grasping truth conditions of the sentence.

A good way to represent procedures, as we shall see in more detail in the next section, is the notion of proof which may be defined within some suitable logical calculus. The notion of proof, or more generally the notion of justification, has been applied for example by Michael Dummett [3] to define his justification semantics.

However, our proposal is to keep the truth conditional approach – since it allows to state clearly the relationship between a cognitive notion of sense and an objective, or semantic, notion of sense – enriching it by means of a notion of procedure, rather than proposing an alternative semantic theory based on different key concepts.[4] It is useful to remark that the approach we are suggesting is not to be intended as a representation of the explicit knowledge speakers have. We are not claiming that it is always the case that someone who accepts a sentence is able to justify it showing a proof. We are rather suggesting that the notion of proof is a useful tool for representing implicit knowledge[5] speakers show when they understand sentences.

The problem (Q) therefore may be generally solved in our setting saying that a subject can understand (1) and (2) since she can manage a procedure to grasp their truth conditions, but - in case she believes one true and the other false - she may fail to realize she reaches a contradiction, since the subject may fail to manage the procedure of detecting the logical relationship between (1) and (2).

5 Proof theory for representing procedures

We sketch a formal setting for representing the definitions we proposed which allows us to keep both a classical truth conditional style semantics and

[2] see [Dum91].

[3] See [Dum79].

[4] The justification semantics proposed by Dummett uses the notion of proof as semantic value, and then it is committed with intuitionistc logic. Here we are presenting an approach which could be applied in different logical calculi, provided they have a proof theory and a truth values semantics, since it is not decided yet which formal calculus is adequate to represent semantic understanding.

[5] We refer to [Dum91], p. 139.

to account for the complexity of the procedure of grasping truth conditions.

We propose to use proof theory to represent the procedure of grasping truth conditions; in this way we can describe the failure of detecting logically equivalent sentence as a lack of logical competence due to the complexity of the sentences involved.

Here we will not refer precisely to a particular logical calculus, rather we will propose some general idea which may be applied in different logical frameworks[6] We consider two notions of sense: a *semantic* one and a *cognitive* one. Remark that in this way we may keep a notion of *co-tenability* of thoughts, which states some important intuition about the relationship between the meaning of a sentence and a subject who understands it.

(S) The semantic sense of a sentence A is the whole class of rules defining a proof of A, which lead to the truth conditions of A.

(C) The cognitive sense of a sentence A is a class of rules a subject manages, which lead to grasp (partial) truth conditions of A.

So the relationship between (C) and (S) can be stated in terms of an inclusion, namely the cognitive competence of a subject amounts to manage a subclass of the rules of inference which are defined in a logical calculus.

It is important to remark that the partial understanding we are defining depends on subjects just in the sense that subjects manage some of the rules required to build a proof of a given sentence. It doesn't mean that a subject has individual or private rules for building proofs[7]. Moreover, the partial comprehension can also be stated in terms of complexity bounds on the application of those rules. For example, if we assume that a subject is able to perform a *modus ponens*, we don't want to assume that he is able to draw the conclusion at any degree of complexity: we are not assuming she can

[6] An interesting choice would be to state our definitions within linear logic (see [Gir06]). Linear logic may be considered more general than intuitionistic or classical logic, in the sense that both can be embedded in linear logic, and it may also be considered as an analysis of the properties of classical and intuitionistic proofs. Briefly, linear logic allows to define where resources are actually needed to be bound and where we can assume an unbounded number of tokens. This aspect is crucial in order to go into the relationship between a cognitive and a semantic notion of sense. Moreover, linear logic has been applied to define formal grammars for natural languages working both for syntactical aspect of sentence understanding and for composition of meanings (see for example [Mor94] and [Car97]). However, it is not clear if we can consider the semantics of linear logic, based on the algebraic structure of phase space (see [Gir87]), as a truth value semantics: we would need in particular to investigate which notion of truth is formalized by that structure. We leave a deeper examination of this approach for further work.

[7] The fact that we don't allow individual or private strategies aims to keep some features of the Fregean anti-psychologism: the sense of a sentence doesn't depend on the representation nor depends on private aspects of the comprehension of meaning.

perform a proof consisting in an unbounded iterated or nested applications of *modus ponens*. Therefore, the class defining the cognitive notion will be a sub-class of the class involved in the objective notion of sense.[8]

We consider some example using natural deduction for classical logic. In order to get the truth conditions for (1) $A \to B$, a subject may be able to manage the following procedure, represented by the logical inference:

(π):

$$\frac{B}{A \to B}$$

So the subject knows that she is accepting $A \to B$, since she is accepting B.

In this way we may represent a partial comprehension of truth conditions, in the sense that we do not need to assume subjects are able to grasp all the possible case described by a truth table (in this example, that $A \to B$ may be true also if A is false). Moreover, in order to get truth conditions for (2), a subject may be able to perform the following procedure:

(π'):

$$\frac{\neg A}{\neg(A \wedge \neg B)}$$

Which doesn't entail a complete knowledge of truth conditions of (2). So we can assume that a subject can understand (1) and (2) and she can accept (1) but she can fail to accept (2) simply because she can fail to manage the procedure of detecting the connection between (1) and (2). Consider again Frege example. Suppose a subject understands (1) and (2), by means of the procedures we mentioned. Moreover, she considers (1) true. We can represent what the subject needs in order to accept also (2), for example by means of a proof like:

(π''):

$$\dfrac{\dfrac{\dfrac{\dfrac{[A \wedge \neg B]^1}{A}\,\wedge\,\mathrm{E} \qquad \begin{matrix}\pi \\ A \to B\end{matrix}}{B}\,\to\,\mathrm{E} \qquad \dfrac{[A \wedge \neg B]^1}{\neg B}\,\wedge\,\mathrm{E}}{\bot}\,\neg\,\mathrm{E}}{\neg(A \wedge \neg B)}\,\neg\,\mathrm{I},1$$

[8]Remark that it is difficult to speak of partial understanding using mere truth-conditional definition of sense since, as we saw, even if we try to define a partial understanding by means of the immediate recognizability criterion, we are led either to make the subject contradict himself or to make the subject logically omniscient.

Here π represents the procedure the subject can perform in order to understand and then accept (1). We used a natural deduction proof just to show an example, we could have chosen other proof theoretical calculi.[9]

So we can use the proof π'' to represent the process of accepting (2) given (1). In this way it is possible to argue about complexity bounds to put on the process itself. Of course our approach should take into account data describing subjects effective performances.

Consider now an example showing a kind of same level complexity and ask again the same question concerning what a subject needs in order to detect logically equivalent sentences. We consider commutative use of conjunction. May a subject accept $A \wedge B$ while not accepting $B \wedge A$? If she understands both sentences, then she grasps in a certain way a procedure represented by the rules for the conjunction.

If she accepts $A \wedge B$, then she should accept both A *and* B. But if this metalinguistic use of "and" is commutative, then she has to accept also $B \wedge A$. In this case, if we are in a commutative framework, we have two formulas which share a same level of understanding complexity, besides a common logical content; therefore we may say that a subject who accepts one of the two sentences is not rational if he doesn't accept the other, since she can manage both procedures. [10]

We conclude mentioning an example taken from the literature on the application of proof theoretical notions in formal semantics which show how proof theory can represent subjects' different performances also in case of quantified sentences[11]. Consider the problem of quantifier-scope ambiguity. The sentence "Someone loves everyone" allows two different readings, depending on the narrow ($\forall\exists$) or wide ($\exists\forall$) scope. A careful examination of

[9] We can read the proof in the following way. A subject can manage the procedure π that leads her to accept (1); in order to accept (2) a proof is required. Assume by contradiction, $A \wedge \neg B$. Eliminating conjunction, we obtain A and we obtain $\neg B$. From hypothesis (1) and from A, we obtain by eliminating conditional B. But B and $\neg B$ entail a contradiction ($\perp$), so we can apply the rule of introducing negation ($\neg I$) and discharge hypotheses marked by 1. We chose this example, and we used the intuitionistic rule for negation, just not to be committed a priori with classical logic. Actually the distinction between partial and full understanding stated in terms of proof can be reformulated for semantic theory that insist on other notion of semantic value.

[10] The aim of this proof theoretic approach to the complexity of understanding meaning would be a sort of normal form for the proof representing procedures of grasping truth conditions. Actually we cannot define a normal form procedure without considering empirical data concerning subjects effective performance. The notion of normalization of proof, which is a central issue in proof theory, may be a staring point in order to define classes of meanings sharing a same measure of such complexity, so to give a proof-theoretical account of the criterion of immediate recognizability.

[11] See [Mor00].

proofs representing the meanings of those sentences[12] it is possible to express the fact that the preferred reading ($\exists\forall$) has a lower complexity degree than the other. In this way it is possible to develop a quite precise notion of cognitive relationship between subject performances and meaning of a sentence.

Summing up, in case of two logically connected sentences, we can claim that it is rational, or better it is possible without contradiction, to accept a sentence while having no opinion concerning the other, when the complexity of the logically connected sentences is different.

So we provided the theoretical possibility for suspending judgement until, by reflection, calculation or other means, a subject can access a procedure for grasping the truth conditions of the sentence.

This approach points at a more sophisticated notion of rationality which can include the process of learning new procedures (for example, by means of interaction) rather than considering rationality as static set of features subjects have, or should have.

6 Conclusion

The proposal we presented allows to consider both a cognitive and a semantic notion of sense and, this is the most interesting point, we can see how the two notions interact: they are not distinct features, as it happens for example in many attempts to conciliate those two aspects of Fregean notion of sense. We stated the relationship between cognitive sense and semantic or objective sense in terms of a partial understanding speakers have of meaning.

The semantic notion of sense is given by the whole class of procedures, or proofs, of the given sentence that give the truth conditions of the sentence, while the cognitive notion of sense is defined as a partial access to semantic sense. Moreover we used a general notion of procedure, represented by the notion of proof in some suitable logical system. So it seems that the opposition between a truth-conditional semantics and a justification-semantics (Dummett) may be weakened considering the proof as a way to grasp truth conditions of a given sentence.

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[12] Actually, the representation at issue employs *proof nets*, which is the peculiar proof theory for linear logic.

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