Forthcoming in G. Holmström-Hintikka, Lindström, S., and Sliwinski, R., (eds.) *Collected Papers of Stig Kanger with Essays on his Life And Work*, Volume II, Synthese Library Kluwer

AN EXPOSITION ANS DEVELOPMENT OF KANGER'S EARLY SEMANTICS FOR MODAL LOGIC*

Sten Lindström

1. *Introduction*

Stig Kanger — born of Swedish parents in China in 1924 — was professor of Theoretical Philosophy at Uppsala University from 1968 until his death in 1988. He received his Ph. D. from Stockholm University in 1957 under the supervision of Anders Wedberg. Kanger's dissertation, *Provability in Logic*, was remarkably short, only 47 pages, but also very rich in new ideas and results. By combining Gentzen-style techniques with a model theory à la Tarski, Kanger obtained new and simplified proofs of central metalogical results of classical predicate logic: Gödel's completeness theorem, Löwenheim-Skolem's theorem and Gentzen's Hauptsatz. The part that had the greatest impact, however, was the 15 pages devoted to modal logic. There Kanger developed a new semantic interpretation for quantified modal logic which had a close family resemblance to semantic theories that were developed around the same time by Jaakko Hintikka, Richard Montague and Saul Kripke (independently of each other and independently of Kanger).

Although his work did not receive much attention at the time, it is generally accepted nowadays that Kanger played a crucial role in the development of model-theoretic semantics for modal logic. The precise nature of his contribution is, however, not so widely appreciated. It is sometimes said that Kanger was the true originator of the so-called possible worlds or Kripke semantics for modal logic. Thus, Dagfinn Føllesdal, in his address to the Stig

* The present paper is a very slightly revised version of Lindström (1998). A shorter version was published as Lindström (1996). Various versions have been presented at seminars in Umeå, Uppsala, Boston, Pittsburgh and Los Angeles. I am especially indebted to Joseph Almog, Lennart Åqvist, Nuel Belnap, Thorild Dahlquist, Kit Fine, Bengt Hansson, Risto Hilpinen, Jaakko Hintikka, David Kaplan, Paul Needham, Peter Pagin, Wlodek Rabinowicz, Krister Segerberg and Rysiek Sliwinski for their very helpful comments and advice. A fellowship at the Swedish Collegium for Advanced Studies in the Social Sciences (SCASSS) during the Autumn of 1996 provided an excellent research environment for working on the paper.

Kanger Memorial Symposium in Uppsala August 1991, proposed the label "Kanger-Kripke-semantics" for the possible worlds approach.[1]

Such an assimilation of Kanger-type semantics to standard possible worlds semantics could, however, be misleading. Kanger's formal semantics for modal logic does not, at least explicitly, utilize the notion of a possible world. Nor is there in his early works from 1957 any mention or discussion of possibilia (possible worlds, counterfactual states of affairs, possible individuals). I do not think this is an accident. Kanger's goal was to generalize and extend the standard Tarski-style definition of truth in a model for first-order predicate languages to languages of quantified modal logic. He wanted to do so without introducing any new primitives in the metalanguage. In (1957b), he explicitly mentions it as an advantage of his approach that it does not presuppose any "intensional" entities like Fregean senses, meanings or intensions. Although he does not discuss the matter, one gets the decided impression that Kanger's ontology is no more hospitable toward possibilia than it is toward intensional entities.

There are, I shall argue, important differences, both of a conceptual and a technical nature, between Kanger's approach to modal semantics and the possible worlds approach. In this connection, one should distinguish between: (i) *possible worlds semantics proper:* a particular type of model-theoretic semantics for modal logic; and (ii) the *possible worlds interpretation* of modal concepts. Accordingly, we may pose the two questions: "Is Kanger's semantics a kind of possible worlds semantics?" and "Is Kanger's interpretation of modal notions a 'possible worlds interpretation'?". In this paper I shall argue that both questions should be answered in the negative. Kanger semantics differs from standard possible worlds semantics in many ways. As we shall see, the notions of logical truth and logical consequence are defined differently for the two kinds of semantics. Moreover, the underlying intuitions about modality are different.

2. *Semantics for quantified modal logic: from Carnap to Kripke*

The proof theoretic study of quantified modal logic was pioneered by Ruth Barcan Marcus (1946a, 1946b, 1947) and Rudolf Carnap (1946, 1947) who were the first to formulate axiomatic systems that combined quantification theory with (S5- and S4-type) modal logic. The attempts to interpret quantified modal logic by means of formal semantic methods also began with Carnap (1946, 1947), where he presented a semantics for logical necessity based on Leibniz's old idea that a proposition is necessarily true if and only if it is true in all possible worlds. In his formal semantics, Carnap used syntactic entities — state-descriptions — as representatives of possible worlds. Suppose that we are considering a first-order predicate

---

[1] At the 9th International Congress of Logic, Methodology and Philosophy of Science. See Føllesdal (1994).

language L with a countably infinite set of individual constants, an infinite sequence of n-ary predicate letters for each $n \geq 1$, but without function symbols or the identity symbol. In addition to Boolean connectives and quantifiers, the language L contains the modal operator $\Box$ for logical necessity. A *state-description* for L is simply a set of (closed) atomic sentences of L.[2] Carnap (1947), p. 9, writes "...the state-descriptions represent Leibniz's possible worlds or Wittgenstein's possible states of affairs". Relative to a state-description S, the notion of *truth* for sentences φ of L is defined as follows (we write $S \vDash \varphi$ for φ being true in S):

(1) $S \vDash P(a_1,..., a_n)$ iff $P(a_1,..., a_n) \in S$.

(2) $S \vDash \neg\varphi$ iff $S \nvDash \varphi$.

(3) $S \vDash (\varphi \rightarrow \psi)$ iff $S \nvDash \varphi$ or $S \vDash \psi$.

(4) $S \vDash \forall x\varphi$ iff, for every individual constant c, $S \vDash \varphi(c/x)$.

Here, $\varphi(c/x)$ is the result of substituting c for every free occurrence of x in φ.

Finally, Carnap gives the following truth clause for the operator $\Box$:

(5) $S \vDash \Box\varphi$ iff, for every state-description S', $S' \vDash \varphi$.

That is, the modal formula "it is (logically) necessary that φ" is true in a state-description S if and only if φ is true in every state-description S'.

Carnap defines *logical truth* as truth in all state-descriptions (we write $\vDash \varphi$ for φ being logically true). Hence,

(6) $S \vDash \Box\varphi$ iff $\vDash \varphi$.

(7) $S \vDash \neg\Box\varphi$ iff $\nvDash \varphi$.

(8) $\vDash \Box\varphi$, if $\vDash \varphi$; and $\vDash \neg\Box\varphi$, otherwise.

It is easy to verify that $\Box$ satisfies the usual laws of the system S5, together with the so-called Barcan formula and its converse, and the rule of necessitation,

(K) $\vDash \Box(\varphi \rightarrow \psi) \rightarrow \Box(\varphi \rightarrow \psi)$.

(T) $\vDash \Box\varphi \rightarrow \varphi$.

(S5) $\vDash \neg\Box\varphi \rightarrow \Box\neg\Box\varphi$.

(Ba) $\vDash \forall x\Box\varphi(x) \rightarrow \Box\forall x\varphi(x)$.

(CBa) $\vDash \Box\forall x\varphi(x) \rightarrow \forall x\Box\varphi(x)$.

(Nec) If $\vDash \varphi$, then $\vDash \Box\varphi$.

Let QC(L) be the set of all logically true sentences, according to the above semantics, in the language L.

---

[2] Actually Carnap's state descriptions are sets of literals (i.e., either atomic sentences or negated atomic sentences) that contain for each atomic sentence either it or its negation. However, for our purposes we may identify a state description with the set of atomic sentences that it contains. Also, in order to make things simple, I am not discussing here Carnap's treatment of identity statements.

Theorem. The set QC(L) is not recursively enumerable, so there is no formal system with this set as its theorems.

Proof: Suppose the set QC(L) is recursively enumerable. Let φ be any sentence of the non-modal fragment L_0 of L. Then, we have:

(i) Either $\vDash \Box\varphi$ or $\vDash \neg\Box\varphi$.
(ii) If $\vDash \Box\varphi$, then $\vDash \varphi$.
(iii) If $\vDash \neg\Box\varphi$, then $\nvDash \varphi$.

Since, by assumption, there is an effective enumeration of the logically true sentences of L, we can effectively decide which of $\vDash \Box\varphi$ or $\vDash \neg\Box\varphi$ holds. But, this means in virtue of (ii) and (iii) that we can effectively decide which of $\vDash \varphi$ or $\nvDash \varphi$ that holds. By Gödel's completeness theorem for the predicate calculus, for any sentence φ of L_0, $\vDash \varphi$ holds if and only if φ is a theorem of the first-order predicate calculus. But this is contrary to Church's theorem according to which the pure predicate calculus is undecidable (cf. Kleene 1967, § 45).

Q. E. D.

The next step in the development of a viable semantics for quantified modal logic was taken by Stig Kanger in his dissertation *Provability in Logic* (1957a). Kanger's ambition was to provide a language of quantified modal logic with a model-theoretic semantics à la Tarski. For that purpose Kanger introduced the notions of a *primary valuation* and a *system*. A primary valuation for a language L of quantified modal logic is a function v which for every non-empty domain D assigns an appropriate extension in D to every individual constant, individual variable, and predicate constant in L. A system is an ordered pair S = <D, v>, where D is a (non-empty) domain and v is a primary valuation. Notice that v does not only assign extensions to symbols relative to the designated domain D, but relative to *all* domains simultaneously.

In Kanger's dissertation there is to be found, for the first time in print, a semantics for modal operators in terms of so-called *accessibility relations:* each modal operator □ is associated with an accessibility relation $R_\Box$ between systems in terms of which the semantic evaluation clause for □ is spelled out:

$$S \vDash \Box\varphi \text{ iff for every system } S' \text{ such that } SR_\Box S',\ S' \vDash \varphi.$$

That is, □φ is true in the system S if and only if φ is true in every system S′ that is $R_\Box$-accessible from S. One particular modal operator that Kanger introduces is one he calls *logical necessity* and which he provides with the following semantic clause:

$$S \vDash \boxed{L}\,\varphi \text{ iff for every system } S',\ S' \vDash \varphi.$$

Thus, $R_{\boxed{L}}$ is the universal relation between systems.

Kanger points out that by imposing certain formal requirements on the accessibility relation, like reflexivity, symmetry, transitivity, etc., one can make the operator satisfy corresponding well-known axioms of modal logic. In this way, the introduction of accessibility relations made it possible to apply semantic and model-theoretic methods to the study of a variety of modal notions other than logical necessity. Although Kanger was the first to publish, other researchers, among them Hintikka and Montague, also came up with the idea of utilizing accessibility relations in the semantics of modal notions.

One source of inspiration for Kanger's use of accessibility relations in modal logic was no doubt the work of Jónsson and Tarski (1951) on representation theorems for Boolean algebras with operators.[3] Jónsson and Tarski define operators $\Diamond$ on arbitrary subsets X of a set U in terms of binary relations $R \subseteq U \times U$ in the following way:

$$\Diamond X = \{x \in U: \exists y \in X(yRx)\},$$

that is $\Diamond X$ is the image of X under R. They also point to correspondences between properties of $\Diamond$ and properties of R. Among other things, they prove a representation theorem for so-called closure algebras that, via the Tarski-Lindenbaum construction, yields the completeness theorem for propositional S4 with respect to Kripke models with a reflexive and transitive accessibility relation. However, Jónsson and Tarski do not say anything about the relevance of their work to modal logic. Perhaps they considered the connection too obvious or of too little importance to mention it.

A semantic approach to first order modal predicate logic that has a close resemblance to Kanger's was developed by Montague (1960).[4] Like Kanger, Montague starts out from the standard model-theoretic semantics for non-modal first-order languages and extends it to languages with modal operators. He defines an *interpretation* for an ordinary first-order predicate language L to be a triple S = <D, I, g>, where (i) D is a non-empty set (the *domain*); (ii) I is a function that assigns appropriate denotations in D to the non-logical constants (predicate symbols and individual constants) of L;[5] and (iii) a function g that assigns values in D to the individual variables of L. For each non-logical constant or variable X, let S(X) be the *semantic value* (i.e., *denotation* for non-logical constants and *value* for variables) of X in the interpretation S. Then the notion of truth relative S is defined as follows:

---

[3] On p. 39 in (1957a) Kanger makes an explicit reference to Jónsson and Tarski (1951).

[4] Montague (1960) writes: "The present paper was delivered before the Annual Spring Conference in Philosophy at the University of California, Los Angeles, in May, 1955. It contains no results of any great technical interest; I therefore did not initially plan to publish it. But some closely analogous, though not identical, ideas have recently been announced by Kanger [(1957b)], [(1957c)] and by Kripke in [(1959)]. In view of this fact, together with the possibility of stimulating further research, it now seems not wholly inappropriate to publish my early contribution."

[5] We are not going to consider languages that contain function symbols.

(1) $S \vDash P(t_1,..., t_n)$ iff $\langle S(t_1),..., S(t_n)\rangle \in S(P)$.

(2) $S \vDash (t_1 = t_2)$ iff $S(t_1) = S(t_2)$.

(3) $S \vDash \neg\varphi$ iff $S \nvDash \varphi$.

(4) $S \vDash (\varphi \to \psi)$ iff $S \nvDash \varphi$ or $S \vDash \psi$.

(5) $S \vDash \forall x\varphi$ iff for every object $a \in D$, $S(a/x) \vDash \varphi$.

Here, S(a/x) is the interpretation which is exactly like S, except for assigning the object a to the variable x as its value.

Montague now asks the same question as Kanger: How can this definition of the truth-relation $\vDash$ be generalized to first-order languages with modal operators? As we recall, Kanger solved the problem by modifying the notion of an interpretation: a Kanger-type interpretation (what he called 'a system') assigns denotations to the non-logical constants and values to the variables not only for one single domain (the 'actual' one) but for all domains. Montague's approach is simpler than Kanger's: he keeps the notion of an interpretation S of first-order logic intact, and just adds semantic evaluation clauses for the modal operators. As in the Kanger semantics, each modal operator $\Box$ is associated with an accessibility relation $R_\Box$. Now, however accessibility relations are relations between interpretations $S = \langle D, I, g\rangle$ of the underlying non-modal first-order language. The semantic clause corresponding to the operator $\Box$, with associated accessibility relation $R_\Box$, is:

(6) $S \vDash \Box\varphi$ iff for every interpretation S' such that $SR_\Box S'$, $S' \vDash \varphi$.

Montague observes that for each variable x the universal quantifier $\forall x$, that binds the variable x, can be viewed as a modal operator with the accessibility relation $R_{\forall x}$ defined by the condition:

$\langle D, I, g\rangle R_{\forall x}\langle D', I', g'\rangle$ iff $D = D'$, $I = I'$ and $g(y) = g'(y)$ for every variable y different from x.

Given this definition, (5) becomes a special case of (6).

Montague associates with the operator $\boxed{\mathrm{L}}$ of *logical necessity* the accessibility relation $R_{\boxed{\mathrm{L}}}$ defined by:

$\langle D, I, g\rangle R_{\boxed{\mathrm{L}}} \langle D', I', g'\rangle$ iff $D = D'$ and $g = g'$.

Thus, his semantic clause for $\boxed{\mathrm{L}}$ becomes:

(1) $\langle D, I, g\rangle \vDash \boxed{\mathrm{L}}\, \varphi$ iff for every I' defined over D, $\langle D, I', g\rangle \vDash \varphi$.

This semantic clause should be compared with Kanger's stricter condition:

(2) $S \vDash \boxed{\mathrm{L}}\, \varphi$ iff for every system S', $S' \vDash \varphi$.

The difference between (1) and (2) corresponds to a difference between two different conceptions of logical truth: Tarski's (1936) conception and the modern model-theoretic one. According to Tarski (1936), truth and logical truth are properties that primarily apply to *inter-*

preted formal languages. An *interpreted* first-order language L comes with a *domain of discourse* D and an interpretation function I that gives the denotations in D of the non-logical constants of L. Hence, the (absolute) notion of *truth* is well-defined for such a language. A formula φ of L is *true relative to an assignment* g of values in L to the variables of L if $\langle D, I, g\rangle \models \varphi$. φ is *true*, simpliciter, if it is true relative to every assignment, that is if it is true in the intended model $\langle D, I\rangle$.[6] Now, according to Tarski (1936), a sentence (closed formula) φ of L is *logically true* if it is true and its truth is invariant with respect to all reinterpretations of its non-logical constants relative to the given domain of discourse D. That is, φ is logically true if and only if, for every interpretation $\langle D, I', g'\rangle$ with the given domain D, $\langle D, I', g'\rangle \models \varphi$. If φ is a sentence (closed formula), then its truth-value is independent of the assignment of values to the variables. Hence, Tarski's (1936) definition yields for sentences φ of L:

$$\varphi \text{ is logically true iff for every } I' \text{ defined over } D,\ \langle D, I', g\rangle \models \varphi,$$

where D is the domain of discourse of L and g is any assignment of values in D to the variables. Hence, for closed formulas of the underlying first-order language, Montague's truth-clause (1) above coincides with Tarski's (1936) definition of logical truth. It also accords well with Carnap's definition of logical truth as truth in all state descriptions, provided that the language contains names for all the objects in a fixed domain of discourse.

Tarski's (1936) definition yields the result that all true sentences of an interpreted language that do not contain any non-logical symbols are logically true. So, if the domain of discourse is infinite (and provided that = is taken as a logical constant), then all the sentences of the form:

(n) $\exists x_1 \ldots \exists x_n (x_1 \neq x_2 \wedge \ldots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \ldots \wedge x_2 \neq x_n \wedge \ldots \wedge x_{n-1} \neq x_n)$

$n \geq 2$, are logically true. So according to Montague (1960), the sentences:

($\boxed{L}$ n) $\boxed{L}\, \exists x_1 \ldots \exists x_n (x_1 \neq x_2 \wedge \ldots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \ldots \wedge x_2 \neq x_n \wedge \ldots \wedge x_{n-1} \neq x_n)$

---

[6] Here we ignore the possibility of D not being a set but a *proper class* and I not assigning sets but proper classes as extensions to the predicate symbols of L. If this were the case, then the intended interpretation of L would not be a *model* in the formal sense of model theory. Of course, there are interpreted first-order languages whose intended interpretations are not models in the formal sense, e. g., the first-order language of set theory, with the proper class V of all sets as its domain and the proper class $\{\langle x, y\rangle : x \text{ is a member of } y\}$ as the interpretation of '$\in$'. This opens up the possibility for a sentence φ of an interpreted formal language of being true although it may be false in all models in the sense of model theory. This possibility is precluded for first-order languages, by the Löwenheim-Skolem theorem: the truth of φ implies its consistency (by the intuitive soundness of first-order logic), which in turn, by the Löwenheim-Skolem theorem, implies φ having a (countable) model. But for formal languages that are able to express such notions as ' there is a proper class of x's such that φ' the implication:

(1) if φ is true, then φ is true in some model (which is built up from *sets*).

fails. Let, for example, φ be the sentence ''there is a proper class of x's such that x = x'. This example is due to McGee (1992).

are *true* for an interpreted language with an infinite domain. Notice, however, that according to Montague (1960), it is not the case that:

$$\vDash \boxed{L}\, \exists x_1 \ldots \exists x_n(x_1 \neq x_2 \wedge \ldots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \ldots \wedge x_2 \neq x_n \wedge \ldots \wedge x_{n-1} \neq x_n)$$

that is the sentences ($\boxed{L}$ n) are not *logically true*. This is so, since the definition of logical truth that Montague adopts in the metalanguage is the standard model-theoretic one of truth in all interpretations, not just truth in all interpretations with a given domain. In conclusion, Montague's and Kanger's respective truth clauses for $\boxed{L}$ yield, for modality-free sentences φ,

$\boxed{L}\, \varphi$ is true iff φ is logically true in the sense of Tarski (1936). *(Montague)*
$\boxed{L}\, \varphi$ is true iff φ is logically true in the model-theoretic sense. *(Kanger)*

There are other possibilities for interpreting logical necessity within Montague's framework, one interesting alternative being:

(3) $\langle D, I, g\rangle \vDash \boxed{L}\, \varphi$ iff for every model $\langle D', I'\rangle$ such that $D \subseteq D'$, $\langle D', I', g\rangle \vDash \varphi$.

Notice that on Montague's interpretation (1) of $\boxed{L}$, both the *Barcan formula*:

(Ba) $\forall x \boxed{L}\, \varphi(x) \rightarrow \boxed{L}\, \forall x \varphi(x)$

and its converse:

(CBa) $\boxed{L}\, \forall x \varphi(x) \rightarrow \forall x \boxed{L}\, \varphi(x)$

are logically true. However, on the interpretation (3), (CBa) still holds, but the Barcan formula fails. Kanger's interpretation (2), finally, makes both (Ba) and (CBa) fail. As a matter of fact, on the interpretation (2) the following is a logical truth:

$$\boxed{L}\, \varphi(x_1,\ldots, x_n) \leftrightarrow \boxed{L}\, \forall x_1 \ldots \forall x_n \varphi(x_1,\ldots, x_n).$$

Hence, if we write, for example,

$$\exists x_1 \boxed{L}\, \varphi(x_1,\ldots, x_n),$$

the initial quantifier does not really "bind" the free occurrences of the variable x_1 in the formula φ.

In other words, Kanger's interpretation does not allow us to "quantify in" past the modal operator $\boxed{L}$. The interpretations (1) and (3), on the other hand, give coherent sense to "quantifying in". For example, given interpretation (3), the following sentence is clearly meaningful. As a matter of fact, it is easily seen to be false in any domain with two elements:

$$\exists x \exists y(x \neq y \wedge \forall z \boxed{L}(z = x \vee z = y)) \rightarrow \exists x \exists y(x \neq y \wedge \boxed{L}\, \forall z(z = x \vee z = y)).$$

The semantical frameworks of Carnap (1947), Kanger (1957a, (1957b) and Montague (1960) — although important developments in themselves, were still not what we nowadays call possible worlds semantics. Something was still missing: an ingredient that was provided by Jaakko Hintikka and Saul Kripke.[7]

In order to see what was missing, let us just make a seemingly small change in Carnap’s semantics. Instead of considering the set of *all* state-descriptions, we shall only consider those state-descriptions that belong to some arbitrary non-empty set **C**. Let us call such a set **C** of state-descriptions a *Carnap frame*. The notion of truth in a state-description now becomes relativized to a Carnap frame. Thus for any $S \in \mathbf{C}$,

(1) $S \vDash_{\mathbf{C}} P(a_1,..., a_n)$ iff $P(a_1,..., a_n) \in S$.

(2) $S \vDash_{\mathbf{C}} \neg\varphi$ iff $S \nvDash_{\mathbf{C}} \varphi$.

(3) $S \vDash_{\mathbf{C}} (\varphi \rightarrow \psi)$ iff $S \nvDash_{\mathbf{C}} \varphi$ or $S \vDash_{\mathbf{C}} \psi$.

(4) $S \vDash_{\mathbf{C}} \forall x\varphi$ iff, for every individual constant c, $S \vDash_{\mathbf{C}} \varphi(c/x)$.

(5) $S \vDash_{\mathbf{C}} \Box\varphi$ iff, for every state-description S' in **C**, $S' \vDash_{\mathbf{C}} \varphi$.

Observe how the relativization of the definition to **C** changes the meaning of $\Box$. Instead of $\Box\varphi$ meaning that φ is logically true, it now means that φ is *universally true* relative to the given set of state descriptions. The metatheoretic definition of logical truth must be changed accordingly: φ is logically true ($\vDash \varphi$) if and only if, for every **C** and every $S \in \mathbf{C}$, $S \vDash_{\mathbf{C}} \varphi$. It is not difficult to axiomatize the quantified modal logic that corresponds to this semantics. As a matter of fact, Hintikka’s and Kripke’s innovation is essential for proving completeness results.

The set **C** represents the space of possible worlds: not every state-description may correspond to a genuine possibility. The last step to possible worlds semantics, taken by Kripke, is to let the possible worlds be explicitly represented in the semantics by an index set W. Hence, by replacing the non-empty set of state-descriptions by an indexed family $\mathbf{C} = \{S_w\}_{w \in W}$ of state-descriptions and utilizing accessibility relations between indices (possible worlds), we get something rather similar to possible worlds semantics. Sentences are now evaluated at indices rather than at state-descriptions:

(1) $w \vDash_{\mathbf{C}} P(a_1,..., a_n)$ iff $P(a_1,..., a_n) \in S_w$.

(2) $w \vDash_{\mathbf{C}} \neg\varphi$ iff $w \nvDash_{\mathbf{C}} \varphi$.

(3) $w \vDash_{\mathbf{C}} (\varphi \rightarrow \psi)$ iff $w \nvDash_{\mathbf{C}} \varphi$ or $w \vDash_{\mathbf{C}} \psi$.

(4) $w \vDash_{\mathbf{C}} \forall x\varphi$ iff, for every individual constant c, $w \vDash_{\mathbf{C}} \varphi(c/x)$.

(5) $w \vDash_{\mathbf{C}} \Box\varphi$ iff, for every $u \in W$ such that wRu, $u \vDash_{\mathbf{C}} \varphi$.

Finally, we get a full-fledged possible worlds semantics if we replace the use of state-descriptions by an indexed family of domains $\{D_w\}_{w \in W}$ together with an interpretation function I that assigns, for each possible world $w \in W$, an appropriate semantic value to the non-logical symbols of L.

---

[7] Cf. Kripke (1959, 1963) and Hintikka (1957a, 1957b, 1961).

In Carnap’s, Kanger’s and Montague’s early theories, the space of possibilities is represented by one comprehensive collection containing *all* state descriptions, systems or models, respectively. Hence, every state description, system, or model is thought of as representing a genuine possibility. Hintikka, Kripke and modern possible worlds semantics are instead working with semantic interpretations in which the space of possibilities is represented by an arbitrary non-empty set **K** of model sets (in the case of Hintikka) or possible worlds (Kripke). Following Hintikka’s (1980, 1989) terminology, one may say that the early theories of Carnap, Kanger, and Montague were considering *standard interpretations* only, where one quantifies over what is, in some formal sense, *all* the possibilities. In the possible worlds approach, one also considers *non-standard* interpretations, where arbitrary non-empty sets of possibilities are considered.[8] The consideration of interpretations (model structures) that are non-standard in this sense — in combination with the use of accessibility relations between worlds in each interpretation — made it possible for Kripke (1959, 1963b and 1965) to prove completeness theorems for various systems of propositional and quantified modal logic (T, B, S4, etc.).

Kripke’s major innovation was perhaps his use — within each model structure — of a set of abstract points (indices, “possible worlds”) to represent the space of possibilities. This innovation made it possible for Montague — building on ideas from Carnap — to represent intensional entities (senses, intensions) by set-theoretic functions from points (representing possible worlds) to extensions.[9] These Carnap-Montague intensions could then be used to give model-theoretic interpretations of comprehensive fragments of natural language. Montague grammar combined categorial grammar with the idea of representing meanings of lexical items with functions from points (indices, “possible worlds”) to extensions.

We end this section by presenting a version of Kripke’s (1963a) semantics for modal predicate logic with identity, where the notion of a *possible world* is an explicit ingredient of the semantic theory.

A (Kripke) *model structure* for a language L of first-order modal predicate logic is a quintuple $S = \langle W, D, R, E, w_0\rangle$ where, (i) W is a non-empty set (of *possible worlds*); (ii) D is a non-empty set (of *possible objects*); (iii) $R \subseteq W \times W$ is the *accessibility relation* of the structure; (iv) E is a function which assigns to each $w \in W$ a subset E_w of D (intuitively E_w is the set of objects that *exist* in w); and (v) w_0 is a designated element of W (the *actual world*). It is required that $D = \bigcup_{w\in W}E_w$, i. e., that every possible object exists in at least one world.

A *Kripke model* is an ordered pair $\mathfrak{M} = \langle S, I\rangle$, where $S = \langle W, D, R, E, w_0\rangle$ is a model structure and I is a function satisfying: (vi) for each $w \in W$, I assigns to each n-ary predicate constant P of L a subset I(w, P) of D^n; and (vii) I assigns to each individual constant c of L an

---

[8] For the standard-non-standard distinction, see also Cocchiarella (1975).

[9] Cf. Montague (1974) and the papers reprinted therein.

element $I(c) \in D$. A model $\mathfrak{M}$ of the form $\langle S, I \rangle$ is said to be *based on* the model structure S.

Observe that $I(w, P)$ is not necessarily a subset of $(E_w)^n$, i. e., the extension of P in w may involve objects that do not exist in w. Moreover, individual constants are treated as *rigid designators*, i. e., they are assigned denotations in a world-independent way. An *assignment* in $\mathfrak{M}$ is a function g which assigns to each variable x an element g(x) in D. For any term t in L, we define $\mathfrak{M}(t, g)$ to be g(t) if t is a variable; and I(t) if t is an individual constant.

With these notions in place, we can define what it means for a formula φ to be *true in a world* w *with respect to the model* $\mathfrak{M}$ *and the assignment* g (in $\mathfrak{M}$) (in symbols, $w \vDash_{\mathfrak{M}} \varphi[g]$):

(1) $w \vDash_{\mathfrak{M}} P(t_1,..., t_n)[g]$ iff $\langle \mathfrak{M}(t_1, g),..., \mathfrak{M}(t_n, g)\rangle \in I(w, P)$.

(2) $w \vDash_{\mathfrak{M}} (t_1 = t_n)[g]$ iff $\mathfrak{M}(t_1, g) = \mathfrak{M}(t_2, g)$.

(3) $w \vDash_{\mathfrak{M}} \neg\varphi[g]$ iff $w \nvDash_{\mathfrak{M}} \varphi[g]$.

(4) $w \vDash_{\mathfrak{M}} (\varphi \to \psi)[g]$ iff $w \nvDash_{\mathfrak{M}} \varphi[g]$ or $w \vDash_{\mathfrak{M}} \psi[g]$.

(5) $w \vDash_{\mathfrak{M}} \forall x\varphi[g]$ iff, for every $a \in E_w$, $w \vDash_{\mathfrak{M}} \varphi[g(a/x)]$.

(6) $w \vDash_{\mathfrak{M}} \Box\varphi[g]$ iff, for every $u \in W$ such that wRu, $u \vDash_{\mathfrak{M}} \varphi[g]$.

We say that φ is *true with respect to the model* $\mathfrak{M}$ *and the assignment* g (in symbols $\vDash_{\mathfrak{M}} \varphi[g]$), iff φ is true at the actual world w_0 with respect to $\mathfrak{M}$ and g. φ is *true in the model* $\mathfrak{M}$ (in symbols, $\vDash_{\mathfrak{M}} \varphi$), if for every assignment g, $\vDash_{\mathfrak{M}} \varphi[g]$. φ is *true in a model structure* S ($\vDash_S \varphi$)if it is true in every model based on S. Let **K** be a class of model structures. We say that φ is ***K**-valid* if φ is true in every $S \in$ **K**.

We are especially interested in model structures where R is the *universal relation* in W, i. e., in which:

(6) $w \vDash_{\mathfrak{M}} \Box\varphi$ iff, for every $u \in W$, $u \vDash_{\mathfrak{M}} \varphi$.

Let **QS5=** be the class of such structures. As Kripke (1963a) showed, neither the Barcan formula nor its converse is (**QS5=**)-valid. It should also be pointed out that Kripke's semantics validates the *Law of Identity*,

(L=) $\forall x(x = x)$,

as well as the principle of *Indiscernibility of Identicals*,

(I=) $\forall x\forall y[x = y \to (\varphi(x/z) \to \varphi(y/z))]$,

applicable without restrictions also to modal contexts φ(z).

From these principles, together with the rule of Necessitation it is easy to infer:

(□=) $\forall x\forall y(x = y \to \Box(x = y))$ *(Necessity of Identity)*

(□≠) $\forall x\forall y(x \neq y \to \Box(x \neq y))$. *(Necessity of Distinctness)*

The latter principles are controversial. Hintikka has repeatedly stressed that they are unacceptable for *propositional attitude constructions* like “John knows that ...”, “George the IV believes that...", etc.[10] On the other hand, Kripke's formal semantics fits well with his conception of *metaphysical necessity* — as it is expounded in Kripke (1980) — and the possible worlds metaphysics that goes with it. In section 6.1, I shall argue for the thesis that (**QS5=**) is the (first-order) quantified modal logic corresponding to Kripke's conception of metaphysical necessity. And I shall write $\boxed{\mathrm{M}}$ rather than $\Box$ when the necessity operator is interpreted as metaphysical necessity. The dual operator, metaphysical possibility, is written as $\Diamond_{\mathrm{M}}$, that is, $\Diamond_{\mathrm{M}}\varphi =_{\mathrm{df.}} \neg\boxed{\mathrm{M}}\neg\varphi$.

3. *Metalinguistic versus object-level interpretation of modalities*

According to the dominant intuition underlying Kanger's work, modal operators of the object language, "It is logically necessary that", "It is set-theoretically necessary that", etc., are obtained "from above" as it were, by "reflecting" corresponding predicates of the metalanguage: "truth in all interpretations", "truth in every interpretation that is normal with respect to the set-theoretic vocabulary", etc. According to this intuition, it is (analytically) necessary that all bachelors are unmarried, just because the sentence "All bachelors are unmarried" is an analytic truth. Writing $\boxed{\mathrm{N}}\varphi$ for "it is analytically necessary that φ", where φ is a sentence of the object language, we have:

(1) $\boxed{\mathrm{N}}\varphi$ is true iff φ is an analytic truth.

By relativizing this schema to a semantic interpretation S, we get:

(2) $\boxed{\mathrm{N}}\varphi$ is true w. r. t. S iff φ is an analytic truth w. r. t. S.

The basic idea can be expressed in the form of a *reflection principle*:

(R) $S \models \mathbf{O}\varphi$ iff $P(\varphi, S)$,

where **O** is an operator of the object language and P is a metalinguistic predicate that may be satisfied by a formula φ and an interpretation S.[11] By itself, the schema (R) only gives "truth-conditions" for formulas containing **O**. But if we think of (R) as explaining the meaning of the operator **O** in terms of the metalinguistic predicate P, we have what may be called a *metalinguistic interpretation* of **O**.

Kanger's interpretation of modal operators is usually metalinguistic in this sense. Consider, for example, his introduction of the operator $\boxed{\mathrm{L}}$ of *logical necessity*. The underlying intuition is expressed by the schema:

(3) $\boxed{\mathrm{L}}\varphi$ is true iff the sentence φ is logically true.

---

[10] See, for example, Hintikka (1969).

[11] We leave it open, for the time being, exactly what is meant by a (semantic) interpretation.

In virtue of the standard model-theoretic definition of "logical truth" as "truth in every interpretation of the object language", we get:

(4) $\boxed{L}\,\varphi$ is true iff φ is true in every interpretation.

(4) corresponds to the following clause of Kanger semantics:

$\boxed{L}\,\varphi$ is true in the domain D relative to the (primary) valuation v of the object language L iff φ is true in every domain D′ relative to *every valuation* of L.

Analytic truth is explicated by Kanger in the following way:

(5) φ is an analytic truth in the domain D relative to the (primary) valuation v iff φ is true in any domain D′ relative v.

This corresponds to the following semantic clause for the operator $\boxed{N}$:

(6) $\boxed{N}\,\varphi$ is true in D relative v iff for every D′, φ is true in D′ relative v.

The possible worlds approach, by contrast, is based on the intuition that there is a multitude of ways in which the world could have been different. On this conception, necessity is not primarily a metalinguistic notion: the necessary truths are rather the truths that would have remained true had any of these possibilities been realized. It is necessarily true that 2 + 2 = 4 just because, for any of the ways ("possible worlds") in which the world could have been different, had the world been different in that way, the proposition that 2 + 2 = 4 would still have been true. These two intuitions about the source of necessity are clearly distinct and indeed, as we shall see, give rise to very different formal semantic theories.

To further clarify the relationship between Kanger's semantics for modal logic and possible worlds semantics, we may make use of a terminology due to Etchemendy (1990) and distinguish between interpretational and representational semantics. Intuitively, the truth-value of a sentence is determined by two factors: (i) the *interpretation* of the symbols in L; and (ii) the *facts* (the way the world is). We may keep the world fixed but let the interpretation of the symbols vary. This is the basic idea behind the standard definitions of *logical truth* and *logical consequence* in the model-theoretic tradition after Tarski (1936). Etchemendy calls it *interpretational semantics*. On this approach, set-theoretic models represent ways of reinterpreting the language. Or, we may keep the interpretation of the linguistic units fixed, but consider what would have happened to the truth-value of the sentence had the facts been different. A semantics based on the latter idea Etchemendy calls *representational*. On this approach set-theoretic models represent ways in which the world could have been different (possible worlds). The classical example of representational semantics is, of course, Carnap's semantics in terms of state descriptions.

We can use either of the two ideas, the representational or the interpretational, to interpret modal operators. For example, (i) and (ii) below are alternative ways of interpreting a necessity operator:

(i) $\boxed{L}$ φ is true (in the actual domain relative to the intended interpretation) iff φ is true in every domain relative to *every interpretation* of the (non-logical) symbols of φ.

(ii) $\boxed{M}$ φ is true (in the actual world relative to the intended interpretation) iff φ is true and would have remained true (relative to the intended interpretation of the language) even if the world had been different in any of the ways in which it (metaphysically) could have been different.

The definition of $\boxed{L}$ is an example of the interpretational approach: it corresponds to Kanger's explication of *logical necessity*. The definition of $\boxed{M}$, corresponding to Kripke's notion of *metaphysical necessity*, is an example of the representational approach.

Both ideas play a role in Kanger's semantics, but the first one is clearly the one that dominates his thinking. The second idea turns up, in a relatively mild form, in his semantics for *analytic necessity*: it is analytically true that φ (in the actual domain and on the intended interpretation of object language) iff φ is true in every domain. In order for this definition to make intuitive sense, we must allow domains that contain not only actually existing objects but also merely possible objects, i.e., objects that do not exist but might have existed — or at least representatives of such objects. Otherwise, all true universal generalizations of the form $\forall x(Fx \rightarrow Gx)$ would on the proposed definition be analytically true. We could not distinguish between the analytically true sentence "All bachelors are unmarried" and the presumably synthetic truth "All bachelors are less than 150 years old". Kanger expresses the idea behind his definition of analyticity: "Our definition of analyticity may be regarded as an explication ... of the idea that an analytic proposition is a proposition that is true in every possible universe."[12] Actual truth is defined with respect to the domain that Kanger describes as "the class of all 'real' individuals".[13] We see that alternative domains of individuals play somewhat the same role in Kanger semantics as possible worlds in standard possible worlds semantics. Kripke's notion of metaphysical necessity — truth in every metaphysically possible world — is , however, foreign to Kanger's way of thinking. Modal operators are viewed as projections of metalinguistic concepts rather than as expressing genuine properties of propositions.

---

[12] Kanger (1970), p. 49.

[13] Kanger (1957b), p. 4. Cf. also Kanger (1970), p. 50.

4. *Kanger's semantics for first-order quantified modal logic*

We shall now take a closer look at Kanger's semantics for quantified modal logic. We consider a first-order predicate language L with identity and a family of unary modal operators $\{\Box_i : i \in I\}$.

A *domain* is a non-empty set D. A (primary)*valuation* (for L) is a function v, which, given any domain D, assigns: (i) an element of D to each term t of L; (ii) a set of ordered n-tuples of elements in D to each n-place predicate constant P in L.

We may, whenever convenient, think of a valuation as consisting of two components: (i) an *interpretation* I that assigns denotations I(D, σ) to the non-logical constants of L relative to domains D; and (ii) an *assignment* g that assigns, for each domain D and each variable x, a *value* g(D, x) to x in D. Of course, $v = I \cup g$. For any I and D, we define I_D to be the unary function defined by letting $I_D(\sigma) = I(D, \sigma)$. v_D and g_D are defined analogously.

A *system* is an ordered pair S = <D, v>, where D is a domain and v is a valuation. When we want to emphasize the distinction between the interpretation I and assignment g, we also write a system as: S = <D, I, g>. Kanger defines the notion of a formula φ being *true in a system* S = <D, v> (in symbols, $S \vDash \varphi$) in the following way:[14]

(1) $S \vDash (t_1 = t_2)$ iff $v(D, t_1) = v(D, t_2)$.

(2) $S \vDash P(t_1,..., t_n)$ iff $\langle v(D, t_1),...,v(D, t_n)\rangle \in v(D, P)$.

(3) $S \nvDash \bot$.

(4) $S \vDash (\varphi \to \psi)$ iff $S \nvDash \varphi$ or $S \vDash \psi$.

(5) $\langle D, I, g\rangle \vDash \forall x\varphi$ iff $\langle D, I, g'\rangle \vDash \varphi$, for each g' such that $g' =_x g$.

(6) $\langle D, I, g\rangle \vDash \exists x\varphi$ iff $\langle D, I, g'\rangle \vDash \varphi$ for some g' such that $g' =_x g$.

(7) for every operator $\Box$, $S \vDash \Box\varphi$ iff $\forall S'$, if $SR_\Box S'$, then $S' \vDash \varphi$.

Explanation: g' is like g except possibly at x (also written, $g' =_x g$) if and only if, for each domain D and each variable y other than x, $g'(D, y) = g(D, y)$. In the above definition, $R_\Box$ is a binary relation between systems that is associated with the modal operator $\Box$. $R_\Box$ is the *accessibility relation* associated with the operator $\Box$.

Among the modal operators in L, there are two designated ones $\boxed{N}$ ("analytic necessity") and $\boxed{L}$ ("logical necessity") with the semantic clauses:

$\langle D, I, g\rangle \vDash \boxed{N}\,\varphi$ iff for every domain D′, $\langle D', I, g\rangle \vDash \varphi$.

$\langle D, I, g\rangle \vDash \boxed{L}\,\varphi$ iff for every system S, $S \vDash \varphi$.

We define a *Kanger model* for L to be an ordered pair $\mathfrak{M}$ = <D, I> of a domain and an interpretation. Notice that for any D and any I, the structure $\langle D, I_D\rangle$ is a model for the non-

[14] Kanger uses the notation T(D, v, φ) = 1 instead of our $\langle D, I\rangle \vDash \varphi[g]$ and he speaks of the operation T which, for every domain D, every primary valuation v and every sentence φ, assigns one of the truth-values 0 or 1 to φ as the *secondary valuation* for L.

modal fragment L_0 of L. Hence, an interpretation I can also be represented as a family $\{<D, I_D>\}_{D \in \mathbf{D}}$ of first-order models indexed by the collection D of all domains. Notice also that this family contains exactly one model for each domain D. A Kanger model can then be written as an ordered pair of the form $\mathfrak{M} = <\mathbf{D}, \{<D, I_D>\}_{D \in \mathbf{D}}>$. A system, finally, can be represented as an ordered pair $<\mathfrak{M}, g>$ of a Kanger model $\mathfrak{M}$ and an assignment g.

We say that formula φ is *true* in a Kanger model $\mathfrak{M}$ (in symbols, $\mathfrak{M} \models \varphi$), if it is true in every system S of the form $<\mathfrak{M}, g>$. If g and g′ are assignments that agree for each domain on all the variables that occur free in φ, then $<\mathfrak{M}, g> \models \varphi$ iff $<\mathfrak{M}, g'> \models \varphi$. Hence, a sentence (closed formula) is *true* in a Kanger model $\mathfrak{M} = \{<D, I_D>\}_{D \in \mathbf{D}}$ iff it is true in some system $<\mathfrak{M}, g>$.

A formula φ is said to be *valid (logically true)* if it is true in every system $<D, v>$. A formula φ is a *logical consequence* of a set Γ of formulas (in symbols, $\Gamma \models \varphi$) if φ is true in every system in which all the formulas in Γ are true.

Kanger defines a *proposition* as an ordered pair $<\varphi, I>$ of a sentence (closed formula) φ and an interpretation I.[15] An n-ary *predicate* is defined as an ordered pair $<P, I>$, where P is an n-ary predicate constant and I is an interpretation. A *name* is a pair $<c, I>$, of a constant c and an interpretation I. The proposition $<\varphi, I>$ is said to be *true* in the domain D if $<D, I> \models \varphi$. $<\varphi, I>$ is called *analytic* if it is true in every domain D. Notice that for any domain D,

(1) $<\boxed{N}\,\varphi, I>$ is true in D iff $<\varphi, I>$ is analytic.

(2) $<\boxed{L}\,\varphi, I>$ is true in D iff φ is logically true.

The notion of (*non-relativized) truth* is explained as follows:[16] Let D_0 be the set of all "real" individuals. Let I_0 be the interpretation which, for every domain D, assigns to the non-logical symbols the denotations that they have according to the intended interpretation of L.[17] A proposition $<\varphi, I>$ is *true* if it is true in the domain D_0. A sentence φ is *true* if the proposition $<\varphi, I_0>$ is true. φ is said to be *analytic* if $<\varphi, I_0>$ is analytic. Notice that $\boxed{N}\,\varphi$ is true iff φ is analytic. Similarly, $\boxed{L}\,\varphi$ is true iff φ is logically true.

Let us say that a modal operator $\Box$ is *ontological* if there exists a relation R such that for every domain D, every valuation v and every formula φ:[18]

$$<D, v> \models \Box\varphi \text{ iff for every } D', \text{ if } DRD', \text{ then } <D', v> \models \varphi.$$

---

[15] Actually he uses valuations here, but since the assignment of values to the variables is immaterial in this context it is more natural to work with interpretations.

[16] Cf. Kanger (1957b), p. 4.

[17] Kanger (1957b) uses the terminology "standard usage of L" instead of "intended interpretation of L".

[18] Here our terminology differs slightly from Kanger's. Our ontological operators correspond to what Kanger calls purely ontological operators. Cf. Kanger (1957a) p. 34.

In (1970) Kanger presents a version of his semantics where he considers ontological operators only. That is, every operator $\Box$ is associated with a relation $R_{\Box}$ between domains rather than between systems. In (1972) the ontological approach has become his official semantics for modal logic.

An important difference between Kanger's semantics for modal operators and that of Kripke is that Kanger assigns to each modal operator a *fixed* accessibility relation, *once and for all*, while Kripke ("standard possible worlds semantics") lets the accessibility relations vary from one model structure to another. One could say that Kanger treats the modal operators as *logical symbols* – having fixed interpretations – while Kripke treats them as *non-logical symbols* – i.e., allowing their interpretations to vary.

5. The treatment of "quantifying in" in Kanger semantics

In *Provability in Logic* (1957a) and in "The morning star paradox"(1957b) Kanger gives the following evaluation clause for the universal quantifier:

(i) $<D, I, g> \models \forall x\varphi$ iff $<D, I, g'> \models \varphi$ for every assignment g′ such that $g' =_x g$.

According to this clause, the formula $\forall x\varphi$ is true in the system <D, I, g> if and only if φ is true in every system <D, I, g′> which is exactly like <D, I, g> except, possibly, for the values it assigns to the variable x in the various domains.

In (1957c) and (1957d) he considers the following alternative clause:

(ii) $<D, I, g> \models \forall x\varphi$ iff $<D, I, g'> \models \varphi$ for every g′ such that (i) $g' =_x g$; and (ii) $g'(D', x) = g(D', x)$ for all domains D′ that are distinct from D.

That is, the formula $\forall x\varphi$ is true in the system <D, I, g> if and only if φ is true in every system <D, I, g′> which is exactly like <D, I, g> except possibly, for the value it assigns to the variable x in the domain D.

Kanger (1957c) suggests the following informal readings of these two alternatives:

(x)φ is true iff φ is true for every interpretation of x
(Ux)φ is true iff φ is true whatever x may denote,

writing (x) and (Ux) for the quantifiers with the first and the second evaluation clause, respectively. If φ is a non-modal formula, then, of course,

$$(x)\varphi \leftrightarrow (Ux)\varphi$$

is valid. However, (Ux) is really an odd quantifier, which does not even validate:

$$(Ux) \Diamond_{N} Fx \rightarrow (Uy) \Diamond_{N} Fy.$$

Consider, namely, a system S = <D, I, g> such that Fx is true in some domain D′ different from D, and Fy is false in every domain. Suppose also that the extension of F in D is the empty set. Then, $\langle N\rangle Fx$ and $\neg\langle N\rangle Fy$ are both true in S. By the semantic clause for (Ux), $(Ux)\langle N\rangle Fx$ will also be true in S. But, $\neg\langle N\rangle Fy$ is true for every assignment g' satisfying the conditions (i) $g' =_y g$; and (ii) for all domains D' that are distinct from D, $g'(D', y) = g(D', y)$. Hence, $(Uy)\langle N\rangle Fy$ is false in S.

That is, substitutivity of alphabetic variants fails for the operator (Ux). For this reason, I shall not consider it further. So, when I speak of Kanger's interpretation of the quantifiers, I shall understand the treatment in (1957a) and (1957b), according to which the universal quantifier $\forall$ has the semantic clause (i) and the existential quantifier has the dual clause:

(iii) $\langle D, I, g\rangle \models \exists x\varphi$ iff $\langle D, I, g'\rangle \models \varphi$ for some g' such that $g' =_x g$.

In order to get a clearer understanding of Kanger's treatment of quantification, I shall speak of selection functions that pick out from each domain an element of that domain as *individual concepts.* To be more precise, an individual concept, in this sense, is a function f, with the collection of all domains as its range, such that for every domain D, $f(D) \in D$. We can think of a system $S = \langle D, I, g\rangle$ as assigning to each individual constant c the individual concept $\{\langle D, I(D, c)\rangle: D \text{ is a domain}\}$ and to each variable x the individual concept $\{\langle D, g(D, x)\rangle: D \text{ is a domain}\}$. The formula $P(t_1,..., t_n)$ is true in $S = \langle D, I, g\rangle$ if and only if the individual concepts designated by $t_1,..., t_n$ pick out objects in the domain D that stand in the relation I(D, P) to each other. The identity symbol designates the relation of *coincidence* between individual concepts (at the "actual" domain D). That is. $t_1 = t_2$ is true in a system $S = \langle D, I, g\rangle$ if and only if the individual concepts designated by t_1 and t_2, respectively, pick out one and the same object in the domain D of S.

Kanger's quantifier $\forall x$, with the semantic clause (i), can now be thought of as an objectual quantifier that ranges not over the "individuals" in the "actual" domain D, but over the (constant) domain of all individual concepts. That is, $\forall x\varphi$ is true in a system $\langle D, I, g\rangle$ if and only if φ is true in every system which is exactly like $\langle D, I, g\rangle$ except, possibly, for the individual concept that it assigns to the variable x. Note, that interpreted in this way, the range of the quantifiers $\forall x$ and $\exists x$ is independent not only of the domain D but also of the system S: the range of the quantifiers $\forall x$ and $\exists x$ is fixed, once and for all, to be the collection of absolutely all individual concepts. While formulas of the form $t_1 = t_2$ express coincidence, identity between individual concepts is expressed by formulas of the form $\boxed{N}(t_1 = t_2)$.

Writing $x \equiv y$ for $\boxed{N}(x = y)$, the following principles are valid:

(L≡) $\forall x(x \equiv x)$

(I ≡) $\forall x\forall y(x \equiv y \to (\varphi(x/z) \to \varphi(y/z)))$,

that is, ≡ satisfies the formal laws of the identity relation.

None of the sentences:

(n) $\exists x_1...\exists x_n(x_1 \neq x_2 \wedge ... \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge ...\wedge x_2 \neq x_n \wedge ...$
$\wedge\, x_{n-1} \neq x_n)$,

saying that the "actual" domain has at least n elements (for n ≥ 2), is logically true according to Kanger's semantics. In contrast all sentences of the form:

$$\exists x_1 \ldots \exists x_n(\neg(x_1 \equiv x_2) \wedge \ldots \wedge \neg(x_2 \equiv x_3) \wedge \ldots \wedge \neg(x_2 \equiv x_n) \wedge \ldots \wedge \neg(x_{n-1} \equiv x_n)),$$

are logically true. Intuitively, these sentences say that there are, for each n, at least n individual concepts.

Consider now Kanger's (1957b) discussion of the so-called Morning Star paradox. The paradox arises from the following premises:

(1) $\boxed{N}$(Hesperus = Hesperus)

(2) Phosphorus = Hesperus

(3) $\neg\boxed{N}$(Phosphorus = Hesperus) ,

where "Phosphorus" and "Hesperus" are two proper names (individual constants) and $\boxed{N}$ is to be read "it is analytically necessary that". We assume that "Phosphorus" is used by the language community as a name for a certain bright heavenly object visible in the morning and that "Hesperus" is used for some bright heavenly object visible in the evening. Unbeknown to the community, however, these objects are one and the same, namely, the planet Venus.

"Hesperus = Hesperus" being an instance of the Law of Identity is clearly an analytic truth. It follows that the premise (1) is true. (2) is true, as a matter of fact. "Phosphorus = Hesperus" is obviously not an analytic truth, "Phosphorus" and "Hesperus" being two different names with quite distinct uses. So, (3) is true. However, using sentential logic together with the following laws of predicate logic:

(UI) $\forall x\varphi \rightarrow \varphi(t/x)$, *(Universal Instantiation)*

(I=) $\forall x\forall y(x = y \rightarrow (\varphi(x/z) \rightarrow \varphi(y/z)))$,

(Indiscernibilityof Identicals)

we can infer from (2) and (3):

(4) $\neg\boxed{N}$(Hesperus = Hesperus) .

But (1) and (4) contradict each other, so something must have gone wrong in this argument.

According to Kanger's diagnosis of the Morning Star paradox it is (I=) that is at fault. Given Kanger's semantic clause (i) for the universal quantifier and his semantic treatment of individual terms, (UI) is valid but (I=) is not. Only the following restricted version of (I=) is valid:

(I=′) $\forall x\forall y(x = y \rightarrow (\varphi(x/z) \rightarrow \varphi(y/z)))$, provided that no free occurrence of z in φ is within the scope of a modal operator.

With this change in the underlying logic, (4) can no longer be inferred from (2) and (3).

As Kanger himself points out, there are still some difficulties left. From (UI) and the equivalence $\exists x\varphi \leftrightarrow \neg\forall x\neg\varphi$, we get:

(EG) $\varphi(t/x) \rightarrow \exists x\varphi$. *(Existential Generalization)*

But, in view of (1) - (3) and the Law of Identity, the following sentences are true:

(5) $\text{Phosphorus} = \text{Hesperus} \wedge \neg\boxed{N}(\text{Phosphorus} = \text{Hesperus})$.

(6) $\text{Hesperus} = \text{Hesperus} \wedge \boxed{N}(\text{Hesperus} = \text{Hesperus})$.

So, it follows that:

(7) $\exists x(x = \text{Hesperus} \wedge \neg\boxed{N}(x = \text{Hesperus}))$.

(8) $\exists x(x = \text{Hesperus} \wedge \boxed{N}(x = \text{Hesperus}))$.

Although unintuitive, this result is perfectly compatible with the interpretation of the quantifiers as ranging over individual concepts and of the identity symbol as designating coincidence between individual concepts. According to this interpretation, (7) and (8) mean:

(7′) There is an individual concept x which actually coincides with the individual concept Hesperus but does not do so by analytical necessity.

(8′) There is an individual concept x which not only happens to coincide with the individual concept Hesperus but does so by analytic necessity.

As Quine (1947) was the first to point out, however, (7) and (8) are incompatible with interpreting $\forall x$ and $\exists x$ as objectual quantifiers meaning “for all objects x (in the domain D)” and “for at least one object x (in D)” and letting the identity sign stand for genuine identity between objects (in D). Because, under this interpretation, (7) and (8) have the readings:

(7″) There is an object x (in the actual domain D) which is identical with Hesperus and which is not necessarily identical with Hesperus.

(8″) There is an object x (in the actual domain D) which is identical with Hesperus and which is necessarily identical with Hesperus.

meaning that one and the same object, Hesperus, both is and is not necessarily identical with Hesperus, which is absurd. So Kanger’s semantics for quantified modal logic is incompatible with interpreting the quantifiers as ranging over actually existing individuals (as opposed to individual concepts) and at the same time interpreting = as identity between individuals.

In Kanger’s semantics there are no means of identifying individuals from one domain to another. In particular, the truth-values of formulas will not be affected if we make all the domains disjoint, by systematically replacing every domain D by the set:

$$\{\langle D, a\rangle : a \in D\}.$$

In other words, set-theoretic relations between domains like inclusion, overlap and disjointness, have no semantic significance.

Suppose we make the claim:

(9) Something is such that it is the number of planets but might not have been so.

It seems reasonable to formalize this claim in quantified modal logic as:

(10) $\exists x(Px \wedge \neg\Box Px)$.

We cannot use any of the Kanger's quantifiers for this purpose, however. Suppose, namely, that:

$$g(D, x) \in I(D, P), DR_{\Box}D', D \neq D', g(D', x) \notin I(D', P).$$

Intuitively this means that one thing is the number of planets in the domain D and *one thing or another* is not the number of planets in the modal alternative D′ to D. From this, we should not be able to conclude (10). But on any of Kanger's interpretations of the universal quantifier, (10) follows. So his approach does not allow us to express the claim that *one and the same* object has a given property in one domain and lacks that property in another domain.

Now, we might ask how we could repair Kanger's semantics in order to allow for genuine quantification over individuals. There are many possibilities. One that is particularly straightforward technically is to adapt Kripke's (1963a) treatment of quantification to Kanger's approach. This means that we modify the notion of an assignment g in such a way that an individual variable x is assigned an object g(x) in a domain-independent way. That is, we make two changes with respect to Kanger's notion of an assignment: (i) the value g(D, x) of an individual variable x in a domain D is no longer required to be a member of D; (ii) for all domains D and D′, we require that g(D, x) = g(D′, x). After these changes are made, an *assignment* simply becomes a function g that assigns to each variable x an object g(x). We then adopt the following evaluation clauses for the universal and existential quantifiers:

$\langle D, I, g\rangle \models \forall x\varphi$ iff $\langle D, I, g'\rangle \models \varphi$, for every g′ such that (i) g′ is like g except possibly at x; and (ii) $g'(x) \in D$.
$\langle D, I, g\rangle \models \exists x\varphi$ iff $\langle D, I, g'\rangle \models \varphi$, for some g′ such that (i) g′ is like g except possibly at x; and (ii) $g'(x) \in D$.

With these clauses our previous objection to Kanger's approach that it could not make correct sense of sentences like

$$\exists x(Px \wedge \neg\Box Px),$$

seems to have been met. For this sentence to be true at $\langle D, I, g\rangle$, there needs to be an element $a \in D$ such that:

$$a \in I(D, P) \text{ and for some } D', DR_{\Box}D' \text{ and } a \notin I(D', P).$$

We have been able to express the claim that one and the same individual has the property P in domain D and lacks that property in some domain D′ that is possible relative to D.

Let us now see how the modified semantics might handle the Morning Star paradox. In this semantics, = is interpreted as genuine identity between objects. Accordingly the logical principles for = are the expected ones:

(L=) $\forall x(x = x)$. *(Law of Identity)*

(I=) $\forall x \forall y(x = y \rightarrow (\varphi(x/z) \rightarrow \varphi(y/z)))$. *(Indiscernibility of Identicals)*

However, instead of (UI) we have:

(UI′) $\forall x\varphi \wedge \exists x(x = y) \rightarrow \varphi(y/x)$.

Now, how should we handle individual constants within the modified Kanger semantics? An intuitively appealing approach is to assign denotations to constants in a domain-dependent way as before, but not require the denotation I(D, c) of a constant c relative to a domain D to be a member of D. With this treatment of individual constants, we cannot infer from (I=) to:

(11) Phosphorus = Hesperus $\rightarrow$
$(\boxed{N}(\text{Phosphorus} = \text{Phosphorus}) \rightarrow \boxed{N}(\text{Phosphorus} = \text{Hesperus}))$,

unless the following requirements are met:

$\exists x\boxed{N}(x = \text{Phosphorus})$, $\exists x\boxed{N}(x = \text{Hesperus})$.

But these conditions hold, only if:

$\exists x\boxed{N}(x = \text{Phosphorus})$, $\boxed{N}(\text{Phosphorus} = \text{Hesperus})$.

The last of these conditions contradicts (3), so it cannot be assumed. It would, presumably, hold only if “Phosphorus” and “Hesperus” were synonymous. Hence, we cannot infer (11) from (I=). We can also verify, directly, that the modified semantics does not allow the inference from (2) and (3) to (4). So the Morning Star paradox, in the form that Kanger presented it, is resolved.

Let us say that a modal operator $\Box$ is a *constant assignment* operator, if there exists a binary relation $R_{\Box}$ between Kanger models $<D, I>$, such that for every assignment g,

$<D, I, g> \models \Box\varphi$ if and only if for every Kanger model $<D', I'>$ such that $<D, I>R_{\Box}<D', I'>$, $<D', I', g> \models \varphi$.

Intuitively, constant assignment operators do not affect the values that are assigned to free variables within their scopes. In contrast, Kanger’s operator of logical necessity, which is not a constant assignment operator, binds all free variables within its scope. Hence, quantifying in, past this quantifier, does not make sense. As Kaplan (1986) has shown, however, we can interpret logical necessity in a way that admits of quantifying in. Within the present framework, Kaplan’s treatment of logical necessity amounts to the following semantic clause:

$<D, I, g> \models \boxed{L}\,\varphi$ if and only if for every Kanger model $<D', I'>$,
$<D', I', g> \models \varphi$.

Although, we do have for this operator, Barcan Marcus’s (1947) Necessity of Identity principle:

($\Box$=) $\forall x \forall y(x = y \rightarrow \boxed{L}(x = y))$, we, of course, do not have:

(12) Phosphorus = Hesperus → $\boxed{\text{L}}$(Phosphorus = Hesperus) ,

and this, we think, is as it should be.

It is, of course, very doubtful whether Kanger would have approved of these changes to his semantics.

6. *The set-domain versus the class-domain semantics*

Kanger presents his semantic theory within an informal set-theoretic framework, where all sets are treated on a par, as genuine objects that can be members of other sets.[19] This theory — what we might call *naive Kanger semantics* — is, however, threatened by paradoxes: given normal assumptions about sets the theory is inconsistent. For example, primary valuations are themselves non-empty sets, that is, domains. It follows that for any valuation v, v itself belongs to the first argument domain of v. In consequence, for an individual constant c, $\langle\langle v, c\rangle, v(v, c)\rangle \in v$. This is contrary to the ordinary assumption of set theory that sets are well-founded. This consequence may not seem especially serious from Kanger's point of view, since in section 5.6 of (1957a) he briefly discusses the introduction of non-well-founded sets in set theory. (Cf. K. B. Hansen's contribution to the present volume).

However, there are more serious problems. Consider any valuation v. Its first argument domain is the collection U of all domains. But in order for v to be a set, the collection U must also be a set. Let $\wp^+(U)$ be the set of all non-empty subsets of U. Then, by Cantor's theorem, card($\wp^+(U)$) > card(U). But this contradicts the fact that $\wp^+(U) \subseteq U$. Thus, given normal set-theoretic assumptions the collections U and v cannot be sets. But this is in apparent contradiction to Kanger's free use of the valuations v in various set-theoretic constructions, for instance, when he defines the notions of a system, a proposition, truth in a system, logical truth, etc., or when he introduces accessibility relations between systems.

We are going to consider two alternative ways of modifying Kanger's approach in such a way that the threat of paradox is avoided: *the set-domain approach* and the *class-domain approach*.

6.1. Kanger set-domain semantics

A natural way of modifying naive Kanger semantics is to assume, as before, that the *domains* are arbitrary non-empty *sets* but that the valuation functions are proper classes. The elements of a domain are, of course, either individuals *(urelements)* or sets. A (primary) valuation v is

[19] In his definitions Kanger speaks of "classes" rather than "sets", but this terminological difference is inessential, since he does not make any distinctions within the category of all classes but rather treats all classes that he speaks of as genuine objects that can be members of other classes.

now defined in the same way as before, except that it is taken to be a functional (proper) *class* rather than a functional *set*. The same holds for the notions of an interpretation I and an assignment g. The models $<D, I_D>$ for the non-modal base language L_0 are *sets*. An *interpretation* is a *proper class* $\{<D, I_D>\}_{D\in \mathbf{D}}$ of such models (relational structures) indexed by the class D of all non-empty sets.

A *Kanger (set-domain) model* consists of a domain D and an interpretation I = $\{<D, I_D>\}_{D\in \mathbf{D}}$. Such a model (we use the notation (D, I) for it) cannot be a set nor, in fact, a class, since it is intuitively an ordered pair, one of the components of which is a proper class. Although, we cannot define it in the usual systems of set-theory (Zermelo-Fraenkel, Gödel-Bernays-von Neumann, Kelley-Morse), it makes intuitive sense to speak of the ordered pair (D, I).

An *assignment* g is now a functional class which for any domain D and any variable x assigns a value g(D, x) in D to x.

A *system* is an ordered pair S = ($\mathfrak{M}$, g) consisting of a Kanger model $\mathfrak{M}$ and an assignment g.

The *accessibility relation* $R_\Box$ that is associated with a modal operator $\Box$ of L is a *collection* of ordered pairs (S, S′) of systems. Or, in the case of ontological operators: a collection of ordered pairs of domains.

The following notions are defined exactly as before: (i) $S \models \varphi$ and $\mathfrak{M} \models \varphi$; (ii) a formula φ being *valid* or *logically true* (written as $\models \varphi$); (iii) *logical consequence*, $\Gamma \models \varphi$, where Γ is a set of formulas and φ a formula.

The Kanger set-domain semantics presupposes a strong metatheory in which one can speak not only of the usual cumulative hierarchy of sets over a set of individuals, but also of *classes* that may contain sets and individuals, *collections* that may contain such classes, collections that may contain collections, and so on. Only the individuals and sets are regarded as genuine *objects*, while classes and collections are thought of as essentially predicative in nature. As a proper formal language for formalizing the metatheory, we think of a language of simple-type theory, where the individual variables (i.e., variables of type ι) range over the elements of the cumulative hierarchy and the predicate variables of type $(\alpha_1,..., \alpha_n)$ range over arbitrary n-ary relations whose i'th domain for $1 \leq i \leq n$ consist of the class of all entities of type α_i. So for instance, the variables of type (ι) range over arbitrary classes of objects (urelements and sets). Variables of type $((\iota), (\iota))$ range over arbitrary relations between classes of objects, etc. A metalanguage of this kind should be appropriate also to formalize the next version of Kanger semantics.

6.2. Kanger class-domain semantics

This semantics differs from the previous one in the following respects: A *domain* is now defined to be a non-empty *class* of elements of the cumulative set hierarchy, i.e., we also allow domains to be *proper classes.* A valuation is a function v, which given any domain D, assigns appropriate denotations over D to the non-logical constants (propositional constants, individual constants and predicate constants) and individual variables of L. We now allow the denotation of an n-place predicate constant P of L to be a *class* of n-tuples of elements in D. The notions of an interpretation and an assignment are adjusted accordingly. Hence, we allow models (D, I_D) for the base language L_0, where D is a class and I_D assigns appropriate classes to the non-logical predicate constants of L. Thus, Kanger models now have the form $\mathfrak{M}$ = (D, $\{(D, I_D)\}_{D \subseteq \mathbf{D}}$).

6.3. Comparing the two approaches

We might now ask how validity with respect to the class-domain semantics is related to validity with respect to the set-domain semantics. If the language L has sufficient expressive capacity, neither implies the other. Let, for example, L′ be the language which is obtained from our language L by adding the generalized quantifier ($\exists_{\text{abs inf}}$ x) as a new logical constant.[20] The intuitive reading of ($\exists_{\text{abs inf}}$ x)φ is

for absolutely infinitely many x, φ(x),

which means that the class of all objects a that satisfy φ(x) is a proper class. The conception of the *absolutely infinite* is due to Cantor. Intuitively, a class A is absolutely infinite if it does not have exactly κ elements for any cardinal number κ (compare, A being *infinite* if it does not have exactly n members for any natural number n).

Consider now the sentence ($\exists_{\text{abs inf}}$ x)(x = x). This sentence is true in a domain D iff the domain is a proper class. According to the class-domain semantics, there are domains that are proper classes. So, according to this semantics the following sentence:

(1) $\Diamond_N$ ($\exists_{\text{abs inf}}$ x)(x = x)

is true in every domain (and for every valuation v). Hence, it is *logically true* according to the Kanger class-domain semantics.

On the other hand, according to the Kanger set-domain semantics the sentence ($\exists_{\text{abs inf}}$ x)(x = x) is false in every domain. Hence, according to this semantics, it is instead

(2) ¬ $\Diamond_N$ ($\exists_{\text{abs inf}}$ x)(x = x)

---

[20] I have taken the quantifier ($\exists_{\text{abs inf}}$ x) from McGee (1992), where he uses it to show that there are interpreted formal languages for which the equivalence:

(M) φ is logically true iff φ is true in every model (in the standard model-theoretic sense of "model" according to which models are sets).

fails. He considers the language of set-theory with ($\exists_{\text{abs inf}}$ x) added to it. The sentence ($\exists_{\text{abs inf}}$ x)(x = x) is then an example of a true sentence which is not true in any model (whose domain is a set). So if (M) were correct then ¬($\exists_{\text{abs inf}}$ x)(x = x) would be an example of a false but logically true sentence. But there are of course no such sentences, so the equivalence (M) cannot hold in general.

that is a logical truth. Here we have a dramatic difference between the two semantic theories. Clearly, it is the class-domain semantics that yields the intuitively correct result in this case.

Instead of adding the "artificial" quantifier ($\exists_{abs\ inf}$ x) to L, we could instead have assumed that L contained a modal operator □ with the semantic clause:

$$(D, v) \models \Box\varphi \text{ iff } \forall D', \text{ if } D' \text{ is a proper class then } (D', v) \models \varphi.$$

Then, we would have for the dual operator ◇,

$$(D, v) \models \Diamond\varphi \text{ iff } \exists D', D' \text{ is a proper class and } (D', v) \models \varphi.$$

The class-domain semantics would then pronounce:

(3) $\Diamond\forall x(x = x)$

logically true, but according to the set-domain semantics it would instead be its negation:

(4) $\Box\exists x(x \neq x)$

that is logically true. Once again, the set-domain semantics gets the wrong result by arbitrarily excluding interpretations that are intuitively legitimate.

7. Logical versus metaphysical necessity

*7.1. On the adequacy of Kripke's logic (**QS5=**) as the logic of metaphysical necessity*

According to the metaphysical picture of modal reality inspired by Kripke's *Naming and Necessity* (1980), there is a space W of *possible worlds* in which the *actual world* w_0 is just one of the worlds. There is also a collection D of *possible objects* and there are *properties* that the possible objects can have and *relationships* that they can have to each other (we call these properties and relationships *attributes).* For each possible world w, n-place attribute A, and possible objects $a_1,..., a_n$, it is determinately either true or false that $A(a_1,..., a_n)$ holds in w. For each world w there is also the collection E_w of all the objects that exist in that world. Presumably, every possible object exists in at least one possible world. Also, certain objects like *pure sets* exist in all possible worlds. A reasonable existence principle for sets is that a set exists in a world w if and only if all its elements exist in that world. Given that sets are objects and that the pure sets exist in all possible worlds, it follows that the collection E_w of all individuals that exist in a world w is always a proper class rather than a set.

Consider now the first-order modal language L, with identity, and with $\boxed{M}$ as its only modal operator. In terms of the metaphysical picture described above, we can describe the *intended interpretation* of L and define the notion: "truth in a world w relative to an assignment g", for formulas of L. The intended interpretation consists of the following ingredients: (i) the class W of all metaphysically possible worlds; (ii) the class D of all possible objects; (iii) for each $w \in W$, a class $E_w \subseteq D$ of objects existing in the world w; (iv) the n-ary predicate symbols of L *designate* n-ary attributes and the individual constants of L designate possible objects; (v) the attributes have *extensions* relative to possible worlds (the extension of an n-ary attribute relative to a world w is a (possibly proper) class of n-tuples of possible objects). A sentence φ of the interpreted language L is *true* if it is true at the actual world (w. r. t. the intended interpretation) relative to every assignment.

Now, it may very well be that the collections of possible worlds, possible individuals, and individuals existing in particular worlds do not form sets. In that case, the intended interpretation of L does not correspond to a Kripke model structure. We cannot, then, directly conclude from a sentence φ being true to it being true in some model structure. As long as we are considering the language of first-order modal logic, however, we can still make this inference in an indirect way.[21] By the completeness theorem for Kripke's system of quantified S5, there exists a formal system **QS5=** which is (strongly) complete with respect to the set of all (**QS5=**)-structures.[22] Hence, for any sentence φ of L:

> φ is true ⇒ φ is (**QS5=**)-consistent (by the intuitive soundness of **QS5=** with respect to the intended interpretation) ⇒ φ is (**QS5=**)-satisfiable (by the completeness theorem for **QS5=**).

It follows that

> φ is (**QS5=**)-valid ⇒ ¬φ is not (**QS5=**)-satisfiable ⇒ ¬φ is not true ⇒ φ is true.

Now, consider the language L with the intended interpretation given above. We have a notion of truth for L, but what could it mean for a sentence of L to be *logically true*? The notion of the intended interpretation for L suggests a notion of an *interpretation* for L. An *interpretation* for L is just like a Kripke model $\mathfrak{M}$ = <S, I> based on a model structure S = <W, D, R, E, w_0> except that the collections W, D, R, E_w for w ∈ W, are not required to be sets but are allowed to be proper classes. (We are of course only considering interpretations

---

[21] If we are considering languages that are sufficiently strong in expressive power, then Kripke's model-theoretic semantics is not sufficient to capture the notions of metaphysical necessity and possibility. Consider, for example, the sentence:

(*) $\boxed{M}(\exists_{abs\ inf}\, x)(x = x)$.

This sentence is presumably true in the intended interpretation. However, there is no Kripke model structure where it is true.

[22] To be exact, we let **QS5=** be the system of *free* (i.e., 'free' of existential assumptions) modal predicate logic which is defined as follows. Axioms: (1) Any substitution instance of a theorem of propositional S5. (2) $\forall x\varphi \wedge \exists y(t = y) \rightarrow \varphi(t/x)$, provided that t is an individual constant or a variable that is free for x in φ. (3) $\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x\varphi \rightarrow \forall x\psi)$. (4) $\forall x\varphi \leftrightarrow \varphi$, provided x is not free in φ. (5) $\forall x\exists y(y = x)$. (6) t = t. (7) $t = t' \rightarrow (\varphi(t/x) \rightarrow \varphi(t'/x))$, provided that t is an individual constant or a variable that is free for x in φ. Deduction rules: (MP) If $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$. (Nec) If $\vdash \varphi$, then $\vdash \Box\varphi$. (UG) If $\vdash \varphi$, then $\vdash \forall x\varphi$. Cf. Garson (1984) and Hughes and Cresswell (1996), chap. 16-17, where the this and similar systems are formulated and proved to be complete with respect to Kripke's (1963a) semantics (these are the systems that Garson refer to as **Q1R**).

where R = D $\times$ D.) Hence, the intended interpretation becomes one of the interpretations. Let us call a sentence φ of L *supervalid* if it is true in every interpretation of L and *valid* if it is true in every (**QS5=**)-model (that is, if it is (**QS5=**)-valid).[23]
We now make the following *conjecture*:

> The logical truths of the language L of metaphysical necessity are precisely those sentences of L that are supervalid.

Adapting an argument due to Kreisel (1969), we can prove that supervalidity coincides with validity for the language L. The argument goes as follows:

Since Kripke (QS5=)-models are interpretations, we have:

(1) if φ is supervalid, then φ is valid.

The completeness theorem for the system (**QS5=**) yields:

(2) if φ is valid, then φ is (**QS5=**)-provable.

However, the system (**QS5=**) is intuitively sound with respect to supervalidity. That is the axioms are easily seen to be supervalid and the only rule of inference, modus ponens, preserves supervalidity. Hence:

(3) if φ is (**QS5=**)-provable, then φ is supervalid.

(2) together with (3) yield:

(4) if φ is valid, then φ is supervalid.

Hence, the notions of validity and supervalidity are coextensional for the language L. From this together with the conjecture, we conclude that (**QS5=**) is the first-order logic of metaphysical necessity.[24]

7.2. Logical necessity

An (interpreted) sentence φ is metaphysically necessary if it is true in every possible world. It is logically necessary if it is true for every domain and every interpretation of its non-logical symbols. Given a certain conception of modal reality, I have argued that Kripke's (1963a) semantics for quantified S5 adequately captures the logic of metaphysical necessity. This means that the *logic* of metaphysical necessity is relatively meager. Although there are, on the Kripkean metaphysical picture, a wealth of metaphysically necessary truths, only a few of them are also logically necessary. For example, if the axioms of Zermelo-Fraenkel set

---

[23] The term *"supervalidity"* is due to Boolos (1985). The concept itself goes back to Kreisel (1969).

[24] Here, we have, of course, presupposed Kripke's picture of metaphysical reality. Given another picture, for example that of Lewis (1985), we get a different logic of metaphysical necessity (but still a form of quantified S5).

theory are true, they are presumably true in all possible worlds, and hence metaphysically necessary. But they are not *truths of logic*, not even of the logic of metaphysical necessity.

Consider now the sentences saying that there are at least n ($n \geq 1$) individuals:

$$\text{(n)} \qquad \exists x_1 \ldots \exists x_n(x_1 \neq x_2 \wedge \ldots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \ldots \wedge x_2 \neq x_n \wedge \ldots \wedge x_{n-1} \neq x_n).$$

Each of these sentences is presumably metaphysically necessary. So for each positive n, the following is a truth of metaphysics:

$$(\boxed{\text{M}}\ \text{n}) \qquad \boxed{\text{M}}\ \exists x_1 \ldots \exists x_n(x_1 \neq x_2 \wedge \ldots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \ldots \wedge x_2 \neq x_n \wedge \ldots \wedge x_{n-1} \neq x_n).$$

It is, of course, not a logical truth. We do not have for any $n \geq 1$,[25]

$$\models_{\textbf{(QS5=)}} \boxed{\text{M}}\ \exists x_1 \ldots \exists x_n(x_1 \neq x_2 \wedge \ldots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \ldots \wedge x_2 \neq x_n \wedge \ldots \wedge x_{n-1} \neq x_n).$$

Nor do we have for any n,

$$\models_{\textbf{(QS5=)}} \Diamond_{\text{M}}\ \exists x_1 \ldots \exists x_n(x_1 \neq x_2 \wedge \ldots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \ldots \wedge x_2 \neq x_n \wedge \ldots \wedge x_{n-1} \neq x_n).$$

In sharp contrast to this, Kanger's semantics for logical necessity validates every instance of

$$\models \Diamond_{\text{L}}\ \exists x_1 \ldots \exists x_n(x_1 \neq x_2 \wedge \ldots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \ldots \wedge x_2 \neq x_n \wedge \ldots \wedge x_{n-1} \neq x_n).$$

This is as I think it should be. It is a logical truth that it is logically possible that there are at least n objects.

When comparing Kanger's semantics for modal logic with Kripke's we come to the conclusion that the former (at least in its class-domain version) is adequate for the notion of logical necessity, while the latter adequately captures a form of metaphysical necessity. Neither semantics can handle adequately the notion that is captured by the other. To devise a semantics that can treat both notions is a challenge that still remains to be met.

As we have seen, Kanger's model-theoretic semantics for quantified modal logic differs in many respects from modern possible worlds semantics. However, it raises sufficiently many questions both of a technical and of a philosophical kind to motivate an interest that is not merely historical.

References

Barcan (Marcus), R., 1946a, "A functional calculus of first order based on strict implication", *The Journal of Symbolic Logic* **11**, pp. 1-16.

Barcan (Marcus), R., 1946b, "The deduction theorem in a functional calculus of first order based on strict implication", *The Journal of Symbolic Logic* **11**, pp. 115-118.

[25] Kripke's (1963) semantics allows the domains of quantification to be empty.

Barcan (Marcus), R., 1947, "The identity of individuals in a strict functional calculus of second order", *The Journal of Symbolic Logic* **12**, pp. 12-15.

Boolos, G., 1985, "Nominalist platonism", *The Philosophical review,* **XCIV**, No. 3, pp. 327-344.

Carnap, R., 1946, "Modalities and quantification", *The Journal of Symbolic Logic* **11**, pp. 33-64.

Carnap, R., 1947, *Meaning and Necessity: A Study in Semantics and Modal Logic,* University of Chicago Press, Chicago. Second edition with supplements, 1956.

Cocchiarella, N., 1975, "On the primary and secondary semantics of logical necessity", *Journal of Philosophical Logic* **4**, pp. 13-27.

Etchemendy, J., 1990, *The Concept of Logical Consequence,* Harvard University Press, Cambridge, Massachusetts.

Føllesdal, D., 1994, "Stig Kanger in memoriam" in D. Prawitz and Westerståhl, D., (eds.) *Logic, Methodology and Philosophy of Science* **IX,** pp. 885-888. Elsevier, Amsterdam.

Garson, J. W., 1984, "Quantification in Modal Logic", in D. Gabbay and Guenthner, F., (eds.) *Handbook of Philosophical Logic*, Vol **II**, Ch. 5, 249-307, pp. 249-307, D. Reidel, Dordrecht.

Hintikka, J., 1957a, "Quantifiers in deontic logic", Societas Scientiarum Fennica, *Commentationes humanarium litterarum*, vol. **23**, no. 4.

Hintikka, J., 1957b, "Modality as referential multiplicity", *Ajatus* **20**, pp. 49-64.

Hintikka, J., 1961, "Modality and quantification", *Theoria* **27**, pp. 110-128.

Hintikka, J., 1969, "Semantics for propositional attitudes", in J. W. Davies *et al.* (ed.), *Philosophical Logic*, pp. 21-45, D. Reidel, Dordrecht. Reprinted in J. Hintikka, *Models for Modalities*, D. Reidel, 1969; and in L. Linsky (ed.), *Reference and Modality*, Oxford University Press, London, 1971.

Hintikka, J. 1980, "Standard vs. nonstandard logic: higher-order, modal, and first-order logics", in E. Agazzi (ed.) *Modern Logic — A Survey*, pp. 283-296, D. Reidel, Dordrecht.

Hintikka, J. 1989, "Is alethic modal logic possible?", in J. Hintikka & M. B. Hintikka: *The Logic of Epistemology and the Epistemology of Logic.* Kluwer, Dordrecht.

Hughes, G. E. and Cresswell, M. J., 1996, *A New Introduction to Modal Logic*, Routledge, London and New York.

Jónsson, B. and Tarski, A., 1951, "Boolean algebras with operators", *American Journal of Mathematics*, vol. **73** (1951), pp. 891-939, vol. 74 (1952), pp. 127-162.

Kanger, S., 1957a, *Provability in Logic,* Acta Universitatis Stockholmiensis, Stockholm Studies in Philosophy **1**, Almqvist & Wiksell, Stockholm.

Kanger, S., 1957b, "The morning star paradox", *Theoria* **23**, pp. 1-11.

Kanger, S., 1957c, "A note on quantification and modalities", *Theoria* **23**, pp. 133-134.

Kanger, S., 1957d, "On the characterization of modalities", *Theoria* **23**, pp. 152-155.

Kanger, S., 1970, *New Foundations for Ethical Theory*, in Hilpinen (ed.), *Deontic Logic: Introductory and Systematic Readings*, Reidel, pp. 36-58. Earlier mimeographed version: *New Foundations for Ethical Theory, Part* **1**, Stockholm 1957.

Kanger, S., 1972, "Law and logic", *Theoria* **38**, pp. 105-132.

Kaplan, D., 1986, "Opacity", in Hahn and Schilpp (eds.), *The Philosophy of W. V. Quine*, The Library of Living Philosophers, Volume XVIII, Open Court, La Salle, Illinois.

Kleene, S. C., 1967, *Mathematical Logic*, John Wiley & Sons, inc., New York.

Kreisel, G., 1969, "Informal rigour and completeness proofs", in I. Lakatos (ed.), *Problems in the philosophy of mathematics*, North-Holland, Amsterdam.

Kripke, S., 1959 "A completeness theorem in modal logic", *The Journal of Symbolic Logic*, **24**, pp. 1-14.

Kripke, S., 1963a, "Semantical considerations on modal logic", *Acta Philosophica Fennica*, fasc. **16**, Helsinki, pp. 83-94.

Kripke, S., 1963b, "Semantical analysis of modal logic I, normal propositional calculi", *Zeitschrift für mathematische Logic und Grundlagen der Mathematik* 9 (1963), 67-96.

Kripke, S., 1965, "Semantical analysis of modal logic II, non-normal propositional calculi", in *The Theory of Models*, edited by I. W. Addison, L. Henkin, A. Tarski, pp. 206-220, North-Holland, Amsterdam.

Kripke, S., 1980, *Naming and Necessity*, Basil Blackwell, Oxford.

Lewis, D., 1985 , *On the Plurality of Worlds*, Basil Blackwell, Oxford.

Lindström, S., 1996, "Modality without worlds: Kanger's early semantics for modal logic", in *Odds and Ends, Philosophical Essays Dedicated to Wlodek Rabinowicz on the Occasion of His Fiftieth Birthday*, Uppsala Philosophical Studies, **45**, Department of Philosophy, Uppsala University.

Lindström, S., 1998, "An Exposition and Development of Kanger's Early Semantics for Modal Logic", in P. W. Humphreys and J. H. Fetzer (eds.) *The New Theory of Reference – Kripke, Marcus, and Its Origins.* Kluwer, 1998.

McGee, V., 1992, "Two problems with Tarski's theory of consequence", *Proceedings of the Aristotelian Society*, new series, vol. **92**, pp. 273-292.

Montague, R., 1960, "Logical necessity, physical necessity, ethics and quantifiers", *Inquiry* **4**, pp. 259-269. Reprinted in Montague (1974).

Montague, R., 1974, *Formal Philosophy: Selected Papers of Richard Montague*. Edited and with an Introduction by Richmond H. Thomason. Yale University Press, New Haven.

Quine, W. V, 1947, "The problem of interpreting modal logic", *The Journal of Symbolic Logic* **12**, pp. 43-48.

Tarski, A., 1936, “Über den Begriff der logischen Folgerung”, *Actes du Congrés International de Philosophie Scientifique*, vol. **7**, Paris, pp. 1-11 (English translation: “On the concept of logical consequence”, pp. 409-420 in *Logic, Semantics, Metamathematics*, second edition, Hackett Indianapolis, 1983).