Dynamic Incentive Regulation of Diffuse Pollution

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Abstract

Diffuse pollution from agriculture and extractive industries reduces air and water quality and contributes to climate change. We consider a setting in which a regulator must incentivize unobserved abatement given that firms have limited liability, and when they can enter and exit. We demonstrate that a simple dynamic incentive scheme can solve this difficult regulatory problem: firms pay a constant tax and receive rebates following periods of low pollution. We apply the model to water pollution from a fracking operation and simulate the contract to explore the volatility of the firm's payments and the costs of limited liability.

1 Introduction

Diffuse pollution results from emissions that cannot be traced to their source. One of the most significant impacts of these emissions is poor water quality. Runoff and leaching from agriculture and mining can contaminate surface and ground water bodies with excess nutrients (such as phosphorus and nitrogen) and toxic substances (such as arsenic). This contamination increases infant mortality and digestive cancer (Brainerd and Menon, 2014; Ebenstein, 2012) and is a major threat to natural assets such as Chesapeake Bay, the Gulf of Mexico and the Great Barrier Reef (GBRA, 2014; EPA, 2009). The regulation of diffuse emissions is complicated by a number of factors. First, the emissions of any individual source are not observed. Second, ambient, or measurable, pollution fluctuates with both random weather events and natural emissions and is therefore stochastic. Finally, in practice regulators are required to take into account "ability to pay" and are frequently constrained by statutory limits on penalties that they can levy. The US EPA for example is constrained by maximum limits under the Clean Water Act, the Clean Air Act and the Comprehensive Environmental Response, Compensation, and Liability Act among others.

We characterize the primary regulatory problem as one of moral hazard. Moral hazard arises because emissions are unobservable and because ambient pollution, which is observed, is stochastic. The regulator (principal) must design the optimal set of incentives to induce abatement from multiple polluters (firms). The regulator and firms interact over an infinite horizon subject to the firms' entry and exit decisions. These firms are risk neutral but at any point in time they are limited by credit constraints that are reflected in statutory limits on fines. The regulator is therefore constrained in that transfers from each firm have a strict lower bound.

We demonstrate that the regulation of diffuse emissions would be feasible via the combination of two simple incentive mechanisms: a constant tax flow and rebates that depend on the history of pollution (*ambient rebates*). The constant tax flow respects the lower bound constraint on the transfer flow, while the long run incentives provided by the rebates ensure that firms have sufficient incentive to abate. By leveraging dynamic incentives, the regulator is able to ensure that firms abate even if the fines that can be mandated are severely constrained in the short term.

We employ the continuous-time methods pioneered by Sannikov (2008) to solve the moral hazard problem. We model the accumulation of pollution as a Brownian motion whose drift is affected by the firms' emissions. These firms can be thought of as drilling sites with runoff that affects water quality. The nature of runoff and leakage means that contaminants enter air or water at many unknown points, and neither the volume nor the content contributed by an individual source can be measured. Firms can undertake costly abatement actions, for example exert costly effort to maintain the integrity of wells and containment ponds. It is either too costly to verify these actions or they are unverifiable.

We assume restoration activities are not feasible and consider the special case where it is optimal for the regulator to keep the stock of pollution constant, so that all active firms must abate their emissions.[1] More generally, one could imagine a setting in which a social planner must determine the optimal abatement effort and the allocation of that effort across firms. As long as the damage from emissions is sufficiently high, the contract we outline will be the optimal solution to this social planner's problem.

We compare the optimal contract and the firms' expected payoffs to two benchmarks: observable emissions and no transfer flow constraint. The setting with observable emissions can be taken to represent the regulation of sources of emissions that can be monitored; these sources are the focus of the vast bulk of current regulation. In this benchmark, the regulator levies a tax on any firm that fails to abate equal to the cost of their abatement. When emissions are not observed, but there is no constraint on transfers to and from the firm, we show that a firm's current payoff (i.e., not their future payoff) is tied to the flow of pollution. If there is an arbitrarily large negative shock to the pollution stock, a firm may then have to pay an arbitrarily large penalty. We view this feature of the contract without a transfer flow constraint as unrealistic.

We derive the comparative statics of the effect of changes in the parameters determin-

[1] Keohane et al. (2007) study a model in which the regulator is able to restore the stock of pollution beside controlling its flow by inducing abatement; they show that periodic restoration of the stock complements abatement of the flow of damages.

ing the firms' payoffs on the optimal contract and we apply the model to the problem of water pollution from a fracking operation, demonstrating the impact of limited liability on firm entry, exit and the cost of regulation across multiple scenarios.

Our paper intersects two independent literatures. The first is devoted to the regulation of diffuse emissions (see Fisher-Vanden and Olmstead, 2013, for a recent survey). Within this literature, the closest papers propose incentives that depend on the difference between the desired and realized aggregate ambient pollution stock. Segerson (1988) first proposed a combination of per-unit and lump sum taxes and rebates in a static environment. Subsequent extensions to dynamic and stochastic environments were then undertaken by Xepapadeas (1992) and Athanassoglou (2010).

As convincingly argued by Karp (2005), there are two important barriers to the implementation of this type of aggregate ambient taxation. First, when the evolution of pollution is stochastic, the burden of transfers from individual firms may be large and reasonably viewed as exceeding political or financial constraints. Second, the aggregate burden of taxes may exceed the social cost of damage and lead to inefficient levels of entry and exit. Unlike the optimal control models of Xepapadeas (1992) and Athanassoglou (2010), we find the contract that solves the regulator's principal-agent problem incorporating an exogenously given constraint on transfers from firms as well as firm entry and exit. Despite the differences in our approach, the mechanism that solves this problem shares some features of the original static scheme proposed by Segerson (1988). Unlike Segerson's proposal, however, in our solution taxation is always lump sum and rebates are a function of dynamic rather than static performance.

The second literature is concerned with the management of environmental damage in the presence of firms with limited liability. In the classic set-up a firm with limited liability undertakes an environmentally risky activity and exerts costly but unobserved safety care to reduce the probability of some severe environmental harm. A particular concern of this literature is to what extent the judgment-proof problem can be ameliorated by extending liability to financiers and other stakeholders (Pitchford, 1995; Boyer and Laffont, 1997; Hiriart and Martimort, 2006; Hiriart et al., 2011).[2] Our setting differs

[2] Empirically, Boomhower (2016) demonstrates that bankruptcy provisions, a form of limited liability,

in that we consider multiple firms whose unobserved action is abatement rather than safety precaution and where environmental damages are not limited to severe discrete events. In this setting, we show that making use of dynamic incentives can alleviate limited liability constraints in the regulation of environmentally damaging activities.

While others have adopted a principal-agent framework to study the regulation of diffuse emissions in static environments (Chambers and Quiggin, 1996), we are the first to do so in a dynamic, continuous time setting with a stochastic pollution flow. The dynamic incentive regulation we propose is related to, but different from, the incentive regulation of public utilities (for a survey, see Laffont and Tirole, 1993). The focus of the literature on incentive regulation is to elicit cost information from a public utility; the workhorse is a static, adverse selection model in which the public utility is a monopolist with private information about its cost. We, on the other hand, study a model of dynamic moral hazard.

Finally, our paper is related to the dynamic principal-agent literature. Dynamic contracts can differ significantly to those in static settings, yet dynamic models that use the standard optimal control approach can be intractable once one introduces realistic features like limited liability and entry and exit. To overcome these difficulties, we formulate our model as a dynamic continuous time agency problem using the tractable machinery introduced by Sannikov (2008). The main technical advantage of his methods is that the optimal contract can be found by solving an ordinary differential equation, allowing for a detailed description of the optimal contract and for simple comparative statics analysis.

The paper is structured as follows. Section 2 outlines the model and Section 3 derives the optimal contract. Section 4 contains the comparative statics analysis of the optimal contract. Section 5 simulates an application of the model to a fracking operation. Finally, Section 6 concludes.

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have implications for firm size and environmental outcomes in the oil and gas industry.

2 The Model

Consider an environmental regulator and a group of firms . The regulator must design a contract to manage pollution within a statutory-imposed limit on the size of penalties or levies and taking into account that firms make entry and exit decisions. Our focus is on the form of this contract. In general, a social planner ought to choose the path of abatement and a set of transfers that maximize social welfare by balancing the damage of pollution with the costs of implementing any level of abatement by any firm at any point in time. Solving such a problem quickly becomes intractable. Here we study an important special case when a) each active firm chooses only whether to pollute or abate all their emissions and b) it is optimal for pollution to remain constant and hence for the planner to induce all active firms to abate at every point in time.[3] Our approach allows us to focus on the structure of the dynamic incentives and to study entry and exit; the cost of assuming a binary abatement choice is that we must abstract from a detailed investigation of the optimal path of abatement.

In this setting, potential firms decide whether to enter the industry and if they have entered they decide whether to exit. At a random time t_0^i, firm i gets a chance to enter; it enters if the total expected value of entry is greater than its outside option R. Firm i exits at time T^i if its expected continuation value of staying in the regulated industry is below R.[4]

Let $\mathcal{N}$ be the set of potential firms and $\mathcal{N}_t \subseteq \mathcal{N}$ be the set of firms that have entered by time t; N_t is the cardinality of $\mathcal{N}_t$, or the number of firms that have entered by t. All firms are risk neutral. Firm i generates a profit flow π^i; that is, in a time period of length dt expected profit is $\pi^i dt$. In the absence of costly abatement, the operations of the firms affect pollution flow.

Polluting firms could be mining operations, drilling sites, or farms with runoff that

[3] These restrictions ensure that each firm's optimal abatement action is independent of the other firms' actions and of the pollution stock. One can show that abatement is optimal if it is assumed that the social cost of pollution is sufficiently high. Henceforth, we will maintain such an assumption and view abatement by all firms that are active as the socially optimal outcome.

[4] The assumption that firms do not enter if they expect just to make their outside option R and that once they have entered they do not exit when they just make R can be easily justified with the presence of small entry and exit costs.

affects water quality or with leaks that emit methane and other gases. To fix ideas, consider an unconventional drilling operation such as hydraulic fracturing of fracking.[5] Fracking can contaminate ground water and local water bodies if gases and fluids migrate from wells and containment ponds (Kuwayama et al., 2015). In a deterministic world abatement such as well and pond maintenance could eliminate pollution, but due to random weather events and natural concentrations, actual pollution is stochastic. Firms' hidden abatement actions only control the drift or expected volume of pollution, they do not have the ability to control the realized flow of pollution.

Active firms choose whether to pollute or abate. Hence we model firm i at time t as choosing an abatement action $a_t^i \in \{0, 1\}$. When firm i abates in a time period of length dt, it incurs an abatement cost $c^i dt$. We allow the abatement flow costs c^i to differ across firms and assume that $\pi^i > c^i$ so that it is still profitable for every firm to produce while abating. The change in the pollution stock is a function of each firm's abatement choice.

We specify the flow of pollution, or change in pollution stock, dY_t in the time interval $[t - dt, t]$ as:

$$dY_t = \mu_t dt + \sigma dZ_t \tag{1}$$

The actions of the firms determine the expected flow of pollution μ_t. Abatement in the time interval $[t - dt, t]$ determines the expected change in the pollution stock $\mu_t dt$ in $[t - dt, t]$. The change in pollution stock also depends on an exogenous shock to pollution arising from random weather events such as precipitation, that affect the flow of pollution measured by the regulator; this is captured by the standard Brownian motion term dZ_t. Viewing Brownian motion as the limit of a symmetric random walk and taking $\Delta = \sqrt{dt}$, we can think that, due to exogenous random factors, the stock of pollution in the time interval $[t - dt, t]$ increases by $\sigma\Delta$ with probability 0.5 and decreases by the same amount with probability 0.5. Thus, σ measures the volatility of the exogenous shock to pollution.

[5] Hydraulic fracturing, or fracking, involves fracturing underground rock and shale formations using a chemical/water mixture to release oil and gas resources that would be inaccessible using traditional extraction techniques.

Define $\mathcal{M}_t \subseteq \mathcal{N}_t$ as the set and M_t as the number of firms that have entered but are not polluting at time t. Firms who have entered include those who are still active and those who have exited. Firms not polluting at time t are firms who have exited, and firms who are abating. Thus, the set of polluting firms at time t is $\mathcal{P}_t = \mathcal{N}_t \backslash \mathcal{M}_t$ and P_t is the number of them. The expected flow of pollution μ_t is given by:

$$\mu_t = \alpha\left(\mathcal{P}_t\right) \tag{2}$$

where $\alpha(\cdot)$ is an increasing function, i.e, $\mathcal{P}'_t \subset \mathcal{P}_t$ implies $\alpha\left(\mathcal{P}_t\right) \geqslant \alpha\left(\mathcal{P}'_t\right)$, satisfying $\alpha(\varnothing) = 0$, so that if no firms are polluting in time interval $[t-dt, t]$, the drift of pollution is $\mu_t dt = 0$. If only firm i pollutes then the drift of pollution is $\mu_t = \alpha(\{i\}) = A^i$; that is, the expected change in of pollution in time interval $[t - dt, t]$ is $A^i dt$. While we do not explicitly model pollution decay, our model is isomorphic to a model in which pollution decays at rate ρdt and firms abate an amount $(A^i - \rho)dt$.[6] The firm's flow cost of abatement is a function of the level of pollution it abates, given by $c^i = \lambda A^i$.

The regulator observes the change in the pollution stock dY_t, but the abatement choices of firms are hidden by the noise created by the Brownian motion. The higher the level of volatility σ, the less information the regulator can deduce from the observed flow dY_t. Note that this flow is independent of the stock of pollution Y_t; it only depends on the firms' abatement actions and random shocks.

The regulator may motivate each firm to abate with transfers and the threat of shutdown. In each time interval $[t - dt, t]$, the transfer dI^i_t can either be from the regulator to a firm ($dI^i_t > 0$) or vice versa from a firm to the regulator ($dI^i_t < 0$). The next section focuses on determining the form of dI^i_t. Given this transfer, the payoff to the firm from their abatement choice a^i_t evolves according to:

$$d\Pi^i_t(a^i_t) = (\pi^i - c^i a^i_t)dt + dI^i_t \tag{3}$$

If firm i chooses to abate in time interval $[t - dt, t]$ then the flow of its payoff in the time

[6] Pollution decay is an important feature of many ambient pollution problems (e.g., methane and water pollution).

interval is $(\pi^i - c^i)dt + dI_t^i$. The regulator is constrained by statute and must ensure that transfers from the firm do not exceed a lower bound $dI_t^i \geqslant -\tau dt$. This constraint acts as a limited liability constraint by putting an upper bound on the amount that the regulator can force a firm to pay at any instant.

If firm i enters at time t_0^i then it must be the case that the total expected payoff of the contract to the firm is greater than its outside option. Let W_t^i be firm i's continuation value, or the firm's total expected discounted payoff from time t, conditional on not having exited (i.e., not having exercised its outside option) prior to t. This firm's total expected payoff at time t is the discounted expected value of all future instantaneous payoffs $d\Pi_s^i$ and depends on its abatement actions over time a_s^i. Taking the continuous time limit of dt converging to zero, we can write firm i's total expected payoff at t as:

$$W_t^i = \mathbb{E}_t \left[\exp^{\gamma t} \int_t^{T^i} \exp^{-\gamma s} d\Pi_s^i(a_s^i) + \exp^{-\gamma(T^i - t)} R \right] \tag{4}$$

where the expectation $\mathbb{E}_t$ is taken at time t and firms discount at rate γ. This firm exits at endogenously chosen time T^i when they exercise their outside option R (for example transferring business to another state).

The regulator's mandate is to keep the expected stock of pollution constant at the initial level Y_0. The regulator uses taxes and rebates in order to maximize a weighted social welfare function. The regulator places a weight of 1 on the cost of providing firms with incentives to abate at each point in time, and a weight of $(1-\phi) \in [0,1]$ on the initial discounted payoff, or promised utility level, $W_{t_0^i}^i$, of each firm i. The smaller weight attached to the firms' payoff reflects either the greater importance attached to consumer surplus, or the social cost of raising funds.[7] Thus, the regulator's problem is to maximize:

$$b_0 = \left[\sum_{i \in \mathcal{N}} \mathbb{E}_0 \left[\int_{t_0^i}^{T^i} -\exp^{-rs} dI_s^i + (1-\phi) W_{t_0^i} \right] \right] \tag{5}$$

by choosing dI_s^i and $W_{t_0^i}$ subject to the transfer flow constraints $dI_t^i \geqslant -\tau dt$ and to the

[7] Under the latter interpretation, one can decompose the weight attached to the transfers into a component having the same weight $1-\phi$ as the firms' payoff and the flow cost of raising funds ϕdI_t.

incentive constraints that it is optimal for each firm i to select $a_t^i = 1$ for all $t < T^i$.

The regulator discounts at rate $r < \gamma$ and is thus more patient than the firms, reflecting the fact that the regulator cares more about future generations than does any firm.

3 The Optimal Contract

We begin by discussing how the regulator chooses payments dI_t^i. We then go on to discuss $W_{t_0^i}$ - the value offered to a firm when they consider entering. When the regulator does not observe the actions of the firms that have entered, it must incentivize each firm i to abate by linking dI_t^i to what is observed, the flow of pollution dY_t. Given the repeated setting, the optimal contract could tie each firm's payoff to the entire history of pollution at time t. This makes the regulator's problem potentially very complex. However, as first shown by Spear and Srivastava (1987) in a discrete time principal-agent setting, the principal's problem can be reduced to a recursive problem with the continuation value, or expected future payoff, of the agent as a state variable.

As a state variable, the expected future payoff W_t^i summarizes all relevant information about the entire history of firm i's actions up to time t – it serves as a dynamic rating of the firm's past performance as well as their expected future payoff. If W_t^i is a dynamic rating of past performance, changes in it must be tied to the flow of pollution. Let dW_t^i be the change in promised utility in time period $[t - dt, t]$. We can then think that in time period $[t - dt, t]$ firm i obtains its instantaneous payoff $d\Pi_t^i = (\pi^i - c^i a^i)dt + dI_t^i$ and its promised payoff change dW_t^i. From the point of view of the regulator dW_t^i and the instantaneous transfer dI_t^i can be viewed as substitute incentive tools to provide firm i with incentives to abate. The availability of two tools in a dynamic setting proves extremely important when the regulator is faced with constraints on the instantaneous transfer.

More precisely, we may view of W_t^i as a performance rating that determines the future payments to the firm and dW_t^i as a change in the expected, contracted payments the firm is promised to receive in the future, contingent on the pollution state. As we shall see, in the optimal contract we can think that the firm holds a pollution derivative with index W_t^i that pays out once its value reaches a pre-determined threshold. Under abatement, the value of the index W_t^i depends purely on changes in the pollution stock which are driven by random natural fluctuations. In this sense, we can think of the contract as similar to a weather derivative whose index is a function of ambient temperature, and that pays out to the holder of the derivative if that index surpasses an agreed threshold.[8]

3.1 The Tax Rate and the Firms' Outside Option

Before we go on to specify how a firm's performance rating evolves over time, we need to consider the implications of firm exit for the regulator's problem. When a firm's abatement action is unobservable and the regulator is constrained by a transfer constraint, the regulator punishes pollution by lowering the firm's current performance rating and simultaneously lowering their promised future transfers dI_t^i. As we shall see, this implies that firms' promised payoffs evolve with shocks to pollution, and there is a risk that they exit if their future promised value falls below their outside option.

The exit of any active firm is privately and socially inefficient as the regulator will ensure that every firm abates all their emissions (i.e., causes no social damage) and the firm's profit net of its cost of abatement is positive. The regulator values the firm, however the threat of forcing the firm to exit is important to provide incentives to abate in the vicinity of the outside option. Rather than exiting, the firm could instead decide to stop abating forever. By doing so firm i could guarantee itself a payoff flow at least equal to $\pi^i - \tau$ and a discounted continuation payoff of $\frac{\pi^i-\tau}{\gamma}$. If this discounted payoff is higher than firm i's outside option payoff R, then the threat of forcing the firm to exit is not effective and the regulator cannot prevent the firm from polluting when the promised utility in the contract falls below $\frac{\pi^i-\tau}{\gamma}$. To put it differently, to control pollution the regulator must be able to force a firm to exit or abate. But the regulator cannot push the promised payoff W^i below $\frac{\pi^i-\tau}{\gamma}$. Thus, avoiding pollution requires that the outside

[8] Weather derivatives have been traded in the Chicago Mercantile Exchange since 1999.

option satisfies the constraint

$$R > \frac{\pi^i - \tau}{\gamma} \tag{6}$$

At the same time, it must be the case that it is socially beneficial for the firm to enter and abate forever rather than to take their outside option. Hence it must be the case that

$$R \leqslant \frac{\pi^i - c^i}{\gamma} \tag{7}$$

We combine these two inequalities in the following lemma.

Lemma 1. *A necessary condition for firm i either abating or exiting, but never polluting is:*

$$\tau > \pi^i - \gamma R \geqslant c^i \tag{8}$$

Lemma 1 provides a lower bound on the maximum levy flow set by the legislature. Specifically, the maximum tax flow τ must be greater than the difference between firm i's profit flow π^i and the flow value of their outside option γR. For each firm, the maximum levy flow must also be weakly greater than its cost of abatement. Henceforth, we will assume that condition (8) holds for each firm i.

3.2 The Evolution of the Firms' Values

To guarantee that firm i abates, the regulator must ensure that the total benefit that they receive from abating is greater than the total benefit they receive from polluting. But the regulator cannot observe firms' chosen actions, it observes only the noisy signal dY_t. Each firm's total benefit includes its current payoff and its future payoff. For incentive compatibility, one or both of these payoffs must be linked to the realization of pollution. Define the *instantaneous reward* of firm i in period $[t - dt, t]$ as $(\pi^i - c^i)dt + dI_t^i + dW_t^i$, the sum of the instantaneous payoff when abating and the change in their promised utility.

Thinking of the promised utility as a perpetuity leads to the intuitive conclusion that, in the absence of changes in pollution, the instantaneous reward, viewed as a periodic payment, ought to be equal to $\gamma W_t^i dt$, the discount rate times the present value of the perpetuity. The next lemma shows that in the optimal contract with random shocks to pollution the regulator provides incentives to each firm i by tying the instantaneous reward to the flow of pollution; increasing it above the periodic perpetuity payment $\gamma W_t^i dt$ if pollution decreases and decreasing it if pollution increases.[9] In addition, as a long series of pollution shocks that decrease the firm i's promised utility is possible, their promised utility could become as low as their outside option. When this occurs firm i exits.

Lemma 2. *At time $t < T^i$ there is a sensitivity β_t^i of each firm i's instantaneous reward to the flow of pollution such that:*

$$(\pi^i - c^i)dt + dI_t^i + dW_t^i = \gamma W_t^i dt + \beta_t^i dY_t \tag{9}$$

where

$$\beta_t^i \leqslant -\lambda \tag{10}$$

The instantaneous reward follows this process as long as $t < T^i$ where T^i is the first time at which $W_t^i < R$.

The proof of (9) is provided in the appendix. Here we prove why (10) must hold.

To ensure that each firm abates, the instantaneous reward must be linked to the flow of pollution dY_t and to the past performance measure given by the promised utility. The parameter β_t^i governs how sensitive firm i's instantaneous reward is to the signal dY_t. In other words, it governs how firm i's present and future payments change with the observed flow of pollution. To demonstrate that $\beta_t^i \leqslant -\lambda$ we focus on a firm's incentives to abate. We look for a Nash equilibrium among the firms, that is, we consider a firm's incentives to abate conditional on all other active firms choosing to abate. For incentive compatibility it is sufficient to derive conditions under which firm i chooses $a_t^i = 1$ at

[9] The conclusion does not depend on whether the transfer constraint is due to limited liability, or tax policy.

any arbitrary moment in time.

At time t a firm i must balance its payoff from deviating today with the reduction in its payoff tomorrow. The firm wishes to maximize the expected change in W_t^i plus their current payoff. By (9), under the terms of the contract, and assuming all other firms abate, if firm i chooses abatement, then $\mu_t = 0$ and their continuation value will evolve according to $dW_t^i + dI_t^i = \gamma W_t^i dt - (\pi^i - c^i)dt + \beta_t^i \sigma dZ_t$. If they instead choose not to abate they gain the avoided cost of abatement $c^i dt$ but the regulator will lower their promised utility. The firm's failure to abate affects the expected flow of pollution, which becomes $\mu_t = A^i$ and their expected future payoff changes by $d\left(W_t^i + dI_t^i\right)^D = \gamma W_t^i dt - (\pi^i - c^i)dt + \beta_t^i(A^i dt + \sigma dZ_t)$.[10] The regulator must ensure that the change in the firm's payoff from deviating (not abating at time t) is negative; that is $d\left(W_t^i + dI_t^i\right)^D + c^i dt \leqslant dW_t^i + dI_t^i$, or:

$$\beta_t^i A^i dt + c^i dt \leqslant 0.$$

The first term, $\beta_t^i A^i dt$, reflects the change to the firm's promised payoff. The second term is the immediate gain to firm i from not abating, the avoided cost of abatement $c^i dt$. Using $c^i = \lambda A^i$ and re-arranging:

$$\beta_t^i \leqslant -\lambda$$

Hence, for a positive pollution flow dY_t firm i's promised payoff must fall by at least λdY_t to ensure that firm i abates.

3.3 The Incentive Scheme

Lemma 9 provides a constraint on the two incentive tools that the regulator can use at any point in time, the transfer dI_t^i and the change in the promised utility dW_t^i. As incentive instruments they are perfect substitutes from the point of view of the firm, as they enter linearly in the instantaneous reward of the firm. We shall now show that

[10] The presence of the term $(\pi^i - c^i)dt$ in the expression of the evolution of the promised value reflects the fact that the regulator cannot observe the deviation and hence adjusts the promised value for the cost of abatement even if the firm shirks.

they are not perfect substitutes from the point of view of the regulator.

Define $\mathbf{W}_t$ as the vector of continuation payoffs for all the firms at time t. In order to determine the preferences of the regulator over the two incentive tools at her disposal, we need to characterize the regulator's expected value from the promised payoffs $\mathbf{W}_t$. Denote by $b(\mathbf{W}_t)$ the regulator's value function; that is, the highest payoff that the regulator can obtain from a set of contracts that induce all firms to abate and that provide the set of firms with a profile of promised utilities $\mathbf{W}_t$. Consider also the case of treating firms separately, dealing with each firm i as if all other firms abate. Denote by $f^i(W_t^i)$ the regulator's value function from firm i; that is, the highest payoff that the regulator can obtain from a contract that induces firm i to abate and that provides firm i with promised utility W_t^i, under the assumption that all other firms abate.

Lemma 3.

(a) *The regulator's value function $b(\mathbf{W})$ is separable in the firms' promised utilities W^i; i.e., $b(\mathbf{W}) = \sum_i f^i(W^i)$;*

(b) *The regulator's value function $f^i(W^i)$ from firm i is strictly concave in the interval $[R, \widehat{W}^i]$, where the threshold utility $\widehat{W}^i$ satisfies:*

$$\pi^i - c^i = \gamma \widehat{W}^i + r \left(f^i(\widehat{W}^i) - (1-\phi)\widehat{W}^i \right) \tag{11}$$

The proof of Lemma 3 is provided in the appendix.

Lemma 3 (a) shows that the regulator's value function is separable and can be written as the sum of the individual value functions $f^i(W^i)$ for each firm i. Intuitively, this is because linking the incentive schemes of different firms provides no benefit to the regulator in our setting. Lemma 3 (b) shows that each value function f^i for firm i is strictly concave for values of the promised utility between the outside option value R and the threshold $\widehat{W}^i$. The threshold $\widehat{W}^i$ represents the point at which the flow value of firm i's production (profit minus abatement cost) is equal to the flow value the firm derives from the contract plus the flow value of the regulator net of the value the regulator obtains from the firm's profit.

Lemma 3 allows us to specify the optimal contract and incentive payments. Proposition 1 outlines their form.

Proposition 1.

(*a*) *When firm i enters at time $t = t_0^i$, the regulator chooses a starting promised utility $W^i_{t_0^i}$, with $R < W^i_{t_0^i} \leqslant \widehat{W}^i$, where the threshold utility $\widehat{W}^i$ satisfies (11).*

(*b*) *The regulator chooses $\beta_t^i = -\lambda$ as the sensitivity of firm i's instantaneous reward to the flow of pollution.*

(*c*) *If at time $t > t_0^i$ it is $W_t^i < \widehat{W}^i$, then the regulator provides incentive payments:*

$$dI_t^i = -\tau dt \tag{12}$$

$$dW_t^i = \tau dt + \gamma W_t^i dt - (\pi^i - c^i)dt - \lambda dY_t \tag{13}$$

(*d*) *If at time $t > t_0^i$ it is $W_t^i = \widehat{W}^i$, then the regulator provides incentive payments:*

$$dI_t^i = \max\left\{-\tau dt, \gamma \widehat{W}^i dt - (\pi^i - c^i)dt - \lambda dY_t\right\} \tag{14}$$

$$dW_t^i = \min\left\{0, \gamma \widehat{W}^i dt - (\pi^i - c^i - \tau)dt - \lambda dY_t\right\} \tag{15}$$

The proof of Proposition 1 is in the appendix.

Proposition 1 (a) places intuitive bounds on the promised utility $W^i_{t_0^i}$ that the firm is offered when it enters the market. For now we take the starting value $W^i_{t_0^i}$ of each firm i as given; it will be pinned down in the next subsection.

Proposition 1 (b) reflects the fact that an increase in the sensitivity of firm i's instantaneous reward to the flow of pollution increases the volatility of the incentive payments from the regulator to the firm. As the regulator's value function is concave in promised utility, the regulator dislikes variability and hence selects the smallest size of β_t^i (i.e., $\beta_t^i = -\lambda$) that incentivizes the firm to abate.

The concavity of the value function also implies that for low values of the promised utility lowering the transfer payment of the firm as much as possible is optimal (Proposition 1 (c)), while for high values of the promised utility it is optimal for the regulator to lower the change in promised utility as much as possible (Proposition 1 (d)). The threshold $\widehat{W}^i$ represents the cutoff promised utility level at which the regulator switches from using long-run incentives to using short-run incentives. In other words, at promised utility $\widehat{W}^i$, the regulator switches from (a) keeping dI_t^i as low as possible and linking dW_t^i to changes in the pollution stock, to (b) keeping dW_t^i as low as possible and linking dI_t^i to changes in the pollution stock.

Figures 1 and 2 depict a stylized example of a regulator's value function and the contract linking the firm's payoff to pollution flow. Consider a decrease in pollution. The regulator wants to reward firms. They can either reward them instantaneously with a transfer ($dI_t^i > 0$) or they can increase the promised future reward by raising the pollution derivative that pays out when reaching a given threshold, ($dW_t^i > 0$). The cost of rewarding the firm instantaneously is always $-\phi$, the cost of raising the transfer dI_t^i. The cost of rewarding the firm in the future is $f^i_{W_t^i}$ which is the shadow price, or marginal regulator's benefit, of an increase in the promised utility dW_t^i. The regulator is more patient than the firm, so paying firms sooner rather than later is beneficial. However, when the stock of future rewards (i.e., W_t^i) is low, the risk of a firm exiting is relatively high. The regulator can lower the risk that the firm exits by promising future rewards. Hence when $f^i_{W_t^i} > -\phi$ in Figure 1, dW_t^i is linked to changes in pollution stock in Figure 2. However, as the stock of future rewards increases, the probability of exit falls, and the cost of delaying the reward dominates. The threshold $\widehat{W}$ is exactly the point at which the cost of using current and future rewards is equal and the regulator switches from promising future rewards to a current rebate. Hence in Figure 1, when $f^i_{W_t^i} = -\phi$, dI_t^i is linked to changes in the pollution stock in Figure 2. Now consider an increase in pollution. The regulator wants to punish firms in this instance, but they face the limited liability constraint. However, the regulator can still punish the firm by reducing the stock of future rewards. If necessary the regulator will force the firm to exit because the stock of rewards is not as attractive as the firm's outside option ($W_t^i = R$).

Because of the lower bound on the transfer flow that the regulator can charge the firm, when $W_t^i < \widehat{W}^i$ the regulator sets $dI_t^i = -\tau$, which can be interpreted as a constant tax flow. Hence in any time period $[t-dt,t]$ the firm pays the regulator a constant fee or flat tax τdt. On the other hand, the regulator never lets the promised utility go above the threshold $\widehat{W}^i_t$. When the promised utility reaches the threshold level, the contract provides flow rebates following negative pollution shocks. If there is a negative pollution shock in period $[t-dt,t]$ when the promised utility is at the threshold, the firm obtains a rebate $\tau dt - dI^i_t > 0$ in addition to paying the flat tax $-\tau dt$. We call this an *ambient rebate* to reflect the fact that firms' receipt of rebates depend on the history of the flow of pollution.

To summarize, when $W^i_t < \widehat{W}^i$ it is cheaper for the regulator to motivate firm i by allowing the promised utility W^i_t to increase. When firm i's promised utility reaches $\widehat{W}^i$ it is cheaper for the regulator to reward the firm with a rebate in the present instead of promising a larger rebate in the future.

3.4 The Firms' Starting Value and Entry

We now go on to determine the firm's starting value and their entry decision. At time t^i_0 the regulator offers potential entrant i a contract with promised payoff $W^i_{t^i_0}$. Since $f^i(W^i)$ is concave there is a unique value W^i_* that maximizes $f^i(W^i)$. The regulator therefore chooses $W^i_{t^i_0} = W^i_*$ and the firm enters if and only if $W^i_* > R$. Thus, in effect, if the regulator's value function for firm i is decreasing in $[R, \widehat{W}^i]$, then the regulator blocks the firm's entry by offering a contract that yields the outside option payoff as the firm's starting value, but does not cover any, however small, entry cost.

3.5 Benchmarks and Extensions

To highlight the implication of hidden actions in the design of the contracts, we next outline the incentive scheme the regulator would offer if the firms' actions were observable. We then consider the optimal contract if there were no constraints on the instantaneous transfers that the firms can be asked to pay and the optimal contract in a static setting.

3.5.1 Observable Actions

The case of observable actions is a useful analytic benchmark, and also represents the regulation of point sources, those who are the target of the majority of current environmental regulation. The optimal contract offered by the regulator to firm i specifies a transfer dI_t^i and recommends the abatement actions $a_t^i = 1$ for all t and i. When the regulator observes whether or not firm i abates in "instant" dt, then it can condition the payments on the observed action.

If the firm does not abate, then the regulator can fine the firm up to the maximum amount; that is, it can set $(dI_t^i | a_t^i = 0) \geqslant -\tau dt$. If $\phi > 0$, then the regulator loses from any transfer to the firm and will choose the lowest possible transfer, $(dI_t^i | a_t^i = 0) = -\tau dt$. For incentive compatibility, that is to guarantee that the firm wants to abate, it must be that the transfer when the firm does abate minus the cost of abating are at least as large as the transfer when not abating. Formally, it must be $(dI_t^i | a_t^i = 1) - c^i dt \geqslant (dI_t^i | a_t^i = 0) = -\tau dt$. Again, if $\phi > 0$, the regulator prefers transfers as low as possible, so they will pick $(dI_t^i | a_t^i = 1) = (c^i - \tau)dt$.

Thus the optimal contract will specify $(dI_t^i | a_t^i = 0) = -\tau dt$ and $(dI_t^i | a_t^i = 1) = (c^i - \tau)dt$. With this contract, firm i always abates.[11] Without lump sum transfers, then firm i's promised utility would simply be the discounted value of the flow of profit minus the cost of abatement and the transfer flow or: $W_{t_0^i}^i = \frac{\pi^i - \tau}{\gamma}$. An extra dollar paid by the firm to the regulator costs the regulator $1 - \phi$ in reduced payoff to the firm. The regulator thus benefits from a transfer from the firm; net of any small entry cost to induce entry, the regulator charges the firm a lump sum transfer that puts its promised utility at R.

3.5.2 No Transfer Constraint

We now outline the features of the contract in the alternative benchmark case when there is no constraint on the transfer flow from the firm. A key difference in the contract is that when there is no transfer constraint, the regulator can make sure that the firm

[11] If $\phi = 0$, then any contract with $(dI_t^i | a_t^i = 1) - c^i dt \geqslant (dI_t^i | a_t^i = 0) \geqslant -\tau dt$ will induce the firm to abate and yields the same payoff to the regulator.

never exits. As outlined in Proposition 2, the unconstrained regulator uses the current transfer flow to reward abatement and punish pollution. Thus, each firm will have to pay arbitrarily large fee flows after an arbitrarily large positive shock that increases pollution. Since such shocks are bound to occur, the optimal contract of the model with no transfer constraints does not seem appealing or practical.[12]

Proposition 2. *If there are no constraints on the instantaneous transfer flow between the regulator and firm i, then:*

(a) *at t_0^i the regulator offers the firm the starting value $W^i_{t_0^i} = R$ and pays the firm any entry cost, so that the firm enters;*[13]

(b) *for all $t > t_0^i$ firm i's promised value is constant at R;*

(c) *for all $t > t_0^i$ the regulator pays the firm according to the incentive scheme $dI_t^i = (\gamma R - (\pi^i - c^i))dt + \beta^i dY_t$ with $\beta^i \leqslant -\lambda$;*

(d) *the regulator's value is $rf^i(R) = \pi^i - c^i + (1-\phi)rR - \gamma R$.*

The proof of the proposition is in the appendix.

3.5.3 A Static Setting

Had we adopted a static, or more precisely single-period, version of our model, it would not have been possible to study entry and exit by firms. The regulator would not have been constrained by the threat of exit of a socially efficient firm and the promise of future payments would not have been available as a tool to incentivize a firm to abate. The only way in which the regulator could have incentivized a firm to abate is by linking their reward to the pollution generated in the period. This could be accomplished by taxing the firm the maximum amount allowed by the firm's limited liability if the pollution stock at the end of the period is above a threshold and paying a bonus if it is below it.

[12] In the environmental, optimal control, literature, it is standard to assume that there are no transfer constraints, see Karp (2005) for a discussion.

[13] The regulator benefits from the firm entering, but it also benefits from reducing the firm's starting value. If we do not allow the regulator to induce entry by paying the firm's entry cost and we maintain the assumption that the firm does not enter if offered a starting value of R, then the problem has no solution, as the regulator wants to induce entry and to choose $W^i_{t_0^i}$ as close as possible to R.

One of the main insights of our paper is that in a dynamic setting the optimal contract also has the simple feature of charging the firm the maximum flow rate and of rewarding with rebates, but the rebates are tied to the promised utility, a measure of the industry's past abatement performance, not the pollution stock.

3.6 Parameter Uncertainty

The optimal contract of each firm depends on parameters of the firm's payoff that could be unknown or private information of the firm. It is beyond the scope of this paper to add the problem of screening for hidden information to the hidden action problem (pollution abatement) that is the focus of our analysis. It is however useful to point out the robustness of the optimal contract. First, if there is uncertainty about the true value of the unit abatement cost λ, then the regulator could guarantee that abatement still takes place by choosing the sensitivity $-\beta^i$ of the contract to the pollution flow to be equal to the highest value in the range of λ. As indicated by (10) in Lemma 2, as long as $-\beta^i \geqslant \lambda$ the firm will abate.

Second, uncertainty about the cost of abatement c^i (via either uncertainty in λ or A^i) and the profit flow π^i affects the rebates paid to the firm when the threshold $\widehat{W}^i$ is reached. Overestimating the profit flow or underestimating the abatement cost means that the promised utility specified in the contract is higher than the true expected future payoff of the firm; as a result the firm would tend to exit earlier. To avoid inefficient exit, it is again advantageous for the regulator to err on the cautious side and choose rebates for a profit flow on the low side of its possible range and an abatement cost on the high side of its range.

4 Comparative Statics

In this section we study the impact of changes in the parameters determining a firm's payoff on the optimal contract offered to that firm by the regulator.

We look at the effects on the regulator's value function $f^i(W^i)$ (i.e., the expected future accumulation of net tax receipts from firm i) for all possible values of the promised utility W^i, the effects on the firm's maximum promised utility $\widehat{W}^i$ (i.e., the threshold promised utility at which the firm is paid a bonus in case of a negative shock to pollution) and the effects on the firm's starting promised utility W^i_*.

Proposition 3. *For each firm i the following effects hold.*

(a) *At all levels of the firm's current promised utility W^i, the regulator's value is: (i) increasing in the transfer constraint τ, the outside option payoff R and the profit flow π^i; (ii) decreasing in the volatility of pollution σ, the discount rate γ, the level of abatement A^i and the unit cost of abatement λ.*

(b) *The threshold promised utility $\widehat{W}^i$ is: (i) increasing in the volatility of pollution σ and the profit flow π^i; (ii) decreasing in the transfer constraint τ, the outside option payoff R, the discount rate γ and the level of abatement A^i.*

(c) *The starting promised utility W^i_* is: (i) increasing in the outside option payoff R and the profit flow π^i; (ii) decreasing in the transfer constraint τ, the discount rate γ and the level of abatement A^i.*

The proof of the proposition is in the appendix.

An increase in the firm's profit flow π^i has a similar effect to a reduction in the abatement level A^i. They are beneficial for both firm and regulator, as they reduce the cost of providing long-run incentives. An increase in the firm's profit flow or a reduction in the needed abatement level allow the regulator to reduce the bonus payment to the firm when the promised utility reaches the threshold $\widehat{W}^i$ and thus make it convenient to rely for a longer time on long-run incentives by increasing the threshold $\widehat{W}^i$.

It is not surprising that an increase in the maximum instantaneous transfer τ that the firm can pay increases the regulator's value $f^i(W^i)$ at all levels of the state variable W^i. What is perhaps more surprising is that the increase in τ harms the firm, by lowering the starting value of the promised utility W^i_*. Intuitively, an increase in how much the firm may be taxed in each period reduces the cost of providing short-run incentives. This is also reflected in a reduction of the maximum level of long-run incentives, the threshold $\widehat{W}^i$, leading to the firm being paid bonuses earlier. It would seem that firm and regulator could both benefit when providing short-run incentives is easier, but this is not so, because the regulator gains from taxing the firm, irrespective of the incentive motive.

An increase in the unit cost of abatement λ has two effects; it increases the bonus that the firm must be paid at $\widehat{W}^i$ and also increases the magnitude of the sensitivity $\beta_t^i = -\lambda$ of firm i's instantaneous reward to changes in the pollution stock. Both effects have a negative impact on the regulator's value, which is thus decreasing in the unit cost of abatement. The first effect pushes towards a decrease in the threshold promised utility $\widehat{W}^i$, while the second pushes towards its increase. The combined effect depends on the curvature of the regulator's value function and is ambiguous. For similar reasons, the impact of an increase in the unit cost of abatement on the firm's starting value W_*^i is also ambiguous.

The regulator's value is decreasing in the volatility of pollution σ (consistent with the concavity of the regulator's value function), reflecting the fact that this volatility makes the signal dY_t a poorer indication of the firms' actions, making it harder to provide the firm with incentives to abate. As pollution becomes more volatile, the firm's promised utility also does and thus it is beneficial, in order to delay exit, to raise the threshold promised utility $\widehat{W}^i$ which triggers bonus payments to the firm. The effect on the starting value of the firm is however ambiguous, as it depends on changes in the curvature of the regulator's value function.

An increase in the firms' outside option payoff R is directly beneficial to the regulator, as it increases the regulator's payoff when the firm exits. It is thus not surprising that it raises the regulator's value at all levels of the promised utility to the firm and that it also benefits the firm by raising the starting value W_*^i. The reason why it reduces the threshold promised utility $\widehat{W}^i$ is also intuitive. As the firm's exit is a smaller loss, the regulator is willing to risk an earlier exit by reducing its reliance on long-run incentives.

Finally, the regulator's value is decreasing in the firm's discount rate γ. If the firm is more impatient, then the regulator's use of the short-run incentive of a tax in the present is more costly for the firm and the long-run incentive of future rebates is less effective. The firm is also worse off, as the starting promised utlity of a less patient firm is lower. In addition, since long-run incentives are less effective, relying on them is less valuable to the regulator and the threshold promised utility $\widehat{W}^i$ also decreases.

5 Numerical Application to Fracking

In this section we use a numerical application of the model to demonstrate how the limited liability constraint generates inefficiency. There are several ways in which the transfer constraint is costly. First, firms may not enter. Second, once they have entered, firms may exit after a series of positive pollution shocks. Finally, because they are less patient than the regulator, rewarding firms in the future is relatively more costly, yet the regulator uses future promises to reward current abatement and to manage the possibility of exit. We provide an example of each of these channels using an application to water pollution caused by a fracking operation.

The unconventional gas industry, and in particular the practice of fracking, has been associated with contamination of natural waterways and potable water supplies (Kuwayama et al., 2015; Hill et al., 2012). We assume that a fracking firm operates in the vicinity of a waterway and that leakage of flowback water containing bromide from the firm's operations can pollute this waterway if there is inadequate monitoring and maintenance of containment ponds and other facilities. The volatility of bromide is driven by precipitation and naturally occurring deposits. The regulator can only observe bromide concentrations at a downstream water quality monitor.

5.1 Scenario Parameters

In Table 1 we outline parameters for four scenarios. In each scenario we consider a single firm and rescale time such that $t_0^i = 0$. Henceforth we drop the superscript i. Across scenarios we assume that the expected impact of flowback water on levels of bromide ranges from $A = 0.005$ to $A = 0.05$ mg/L. For reference, Wilson and Van Briesen (2013) find bromide concentrations of between 0.05 and 0.09 mg/L in the Monongahela River, a river potentially affected by fracking in the Marcellus shale. The lower bound of the range we assume is lower than the 0.01 mg/L detection limit in that study. The upper bound of the range is the EPA's maximum containment load for bromide. We assume that the volatility of pollution ranges from $\sigma = 0.025$ to $\sigma = 0.03$ mg/L. The standard deviation of bromide concentrations in the Monongahela River was found to be high: ranging from 0.04 to 0.09 across testing sites.

The firm's profit is parameterized to approximate daily net revenue of fracking operations cited in the media and does not vary across scenarios.[14] Similarly, the unit cost of abatement (λ) is held fixed across scenarios leading to total abatment costs (c) of \$195 to \$1950. To parameterize the unit cost of abatement (monitoring and maintenance effort) we assume that it takes one full time employee to reduce pollution by 0.01 mg/L. We take the average wage for the gas industry in Pennsylvania from the U.S. Bureau of Labor Statistics to then calculate an approximate unit abatement cost at the daily level (see Cruz et al., 2014).

5.2 Results

For each scenario we numerically solve for the optimal contract to keep levels of bromide constant. This provides us with the rebate threshold $\widehat{W}$ and the regulator's value at the threshold $f(\widehat{W})$, both reported in Table 2. The solution to the regulator's problem also gives the starting values $W_0 = W_*$ for the firm and $f(W_0) = f(W_*)$ for the regulator. We then apply Proposition 2 to calculate expected firm and regulator outcomes in the case with no transfer constraint.

We start by highlighting the two cases where limited liability inefficiently deters entry. Table 2 shows that in both Scenarios 2 and 3, the regulator's value is maximized at the outside option of the firm and thus the regulator offers the firm $W_0 = R$.[15] Thus with any small positive entry cost, the firm decides not to enter. The only difference between Scenarios 2 and 4 is the outside option of the firm. Across these scenarios at least, the outside option of the firm is therefore critical to firm entry.[16] This is not

[14] See the Appendix for further detail on these sources and the calculations made.
[15] Appendix Figure 6 shows the computed value functions for each scenario.
[16] Scenarios 1 and 3 differ in the outside option of the firm and the maximum tax constraint. However imposing $\tau = 721.25$ from Scenario 1 in Scenario 3 does not change the entry outcome, hence entry is

surprising, as a high outside option lowers the cost to the regulator of detering firm entry. Contrasting these scenarios to the benchmark case of no transfer constraint shows the loss in total payoffs associated with the existence of limited liability. The total starting value in Scenario 2 (the sum of the regulator's starting value and the firm's starting value) with limited liability is approximatly 11% lower (9,409 vs 10,588). In Scenario 3, the loss is approximately 26% (32,846 vs 44,371). As the firm's expected values are the same across scenarios, this loss in welfare is experienced solely by the regulator.

When firms do enter, limited liability also results in a substantial net loss, and this loss is entirely borne by the regulator. Table 2 shows that, in Scenarios 1 and 4, the firm does better with constraints on the transfers it pays to the regulator. The total difference in payoffs due to limited liability is approximately 50% in Scenario 1 and over 60% in Scenario 4.

Table 2 reports expected firm and regulator outcomes at t_0. The probability of exit and the volatility of transfers for $t > 0$ are also important quantities. To explore them, we simulate 10,000 Brownian motion paths dZ_t^j, $j = 1, .., 10000$. For each path, or simulation j we calculate the evolution of the firm's promised utility W_t^j and the transfer flow dI_t^j. For every t, we then calculate the mean and standard deviation of promised utility and transfer flows and the probability of exit by $t = 60$. Figures 3 and 4 and Table 3 summarize the results.[17] In each Figure, the left-hand panel plots promised utility and the right-hand panels plots the transfer flow. Across scenarios we plot the path of promised utility and transfers from the same simulated pollution path. The mean and standard deviation of transfers over time[18] are reported in Table 3 along with the probability of exit.

Scenario 1 provides an example of where limited liability can lead to firm exit. In Scenario 1 the probability of exit by $t = 60$ is 10%, leading to the decline in the mean promised utility in the left panel of Figure 3. The mean transfer from the firm to the regulator as well as the volatility of the transfer are significantly higher without limited

not determined by the transfer constraint in this case either.

[17] In the Appendix we also plot the mean regulator's payoff across simulations and the single path of the regulator's payoff.

[18] For each path j, we calculate the standard deviation of transfers, we then average this standard deviation to generate a measure of the average volatility of transfers over time.

liability. This difference in volatility is evidenced by the transfers plotted for a single path of pollution for the two cases in the left hand panel of Figure 5. For this single path, the unconstrained transfer from the firm to the regulator exceeds the limited liability constraint 25% of the time.

In contrast, Scenario 4 provides an example where we observe no exit and the volatility of the transfers is approximately the same with or without limited liability. Figure 5 shows that the transfers are indeed very similar with or without a constraint on the transfer. With high pollution volatility, a low outside option and a relatively high maximum tax constraint, the regulator maintains the firm's promised value at the threshold to minimize the chance of receiving a low payoff if the firm exits. Relative to the no constraint case this is costly to the regulator and beneficial to the firm.

6 Conclusion

Firms who produce emissions that can be measured are the target of most environmental regulation. Firms whose emissions cannot be measured also cause significant damage but they remain relatively unregulated. We study a setting in which a regulator is tasked with enforcing abatement when emissions from firms cause stochastic pollution.

If the regulator could observe emissions, then it would tax emitting firms. When individual emissions are not observable but the regulator is unconstrained in its use of incentives, the regulator charges firms when pollution increases, and provides rebates when pollution decreases. This implies unboundedly large transfer flows from the firm to the regulator after arbitrarily large positive shocks to pollution. In practice, regulators cannot charge unboundedly large taxes but are constrained in the transfers they can mandate from firms. These constraints realistically arise either from the political process (e.g., as a result of industry lobbying) or as the result of credit constraints.

Our main contribution is to show that if the regulator faces a constraint on the size of the tax it can levy, then it can exploit dynamic incentives to induce firms to abate. To do so the regulator taxes a constant amount in the present and promises future ambient rebates: expected rebates rise if ambient pollution decreases, and fall if ambient pollution increases.

We then use a numerical application to fracking to demonstrate the costs of these constraints. We show that under constraints on the maximum tax, firm entry is lowered, the probability of firm exit is raised and the delay of incentive payments, which is necessary to motivate abatement, comes at significant cost to the regulator.

The mechanism we propose relies on polluting firms being taxed or rebated as a function of ambient pollution levels, and being incentivized by the use of future rebates that are contingent on current abatement actions. A reasonable question is whether such a scheme is practical or politically feasible. The Everglades Forever Act, designed to reduce phosphorous load into the Everglades Protection Area in Florida, suggests that it is. Like our mechanism, the Everglades Forever Act imposes a minimum tax on agricultural producers contributing to runoff into the Everglades, and provides reductions in this tax if aggregate phosphorous reduction targets are met. The scheme also involves a dynamic component, allowing current reductions to offset future tax payments.

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Tables and Figures

Table 1: Parameters for Fracking Application

Firm's discount factor (γ)	0.1			
Profit flow (\$) (π^i)	2,500			
Unit abatement cost (\$) (λ)	39,000			
Regulator's discount factor (r)	0.01			
Weight on firm's utility (ϕ)	0.5			
Scenarios	**(1)**	**(2)**	**(3)**	**(4)**
Abatement level (mg/L) (A)	0.005	0.05	0.005	0.05
Volatility (mg/L) (σ)	0.025	0.03	0.025	0.03
Abatement cost (\$) (c)	195	1,950	195	1,950
Outside option (\$) (R)	17,288	5,225	21,898	4,125
Maximum transfer (\$) (τ)	781.25	5,000	320	5,000

Notes: Table 1 reports the parameters for the application of the model to water pollution caused by fracking. The top panel shows parameters that do not vary across scenario. The bottom panels shows parameters that do vary across scenarios.

Table 2: Numerical Solution to Optimal Contract

Scenarios	**(1)**	**(2)**	**(3)**	**(4)**
Transfer constraint				
Rebate threshold ($\widehat{W}$)	22,050	5,531	23,120	5,491
Regulator's value at threshold ($f(\widehat{W})$)	21,026	2,461	10,415	2,835
Firm's starting value ($W_0 = W_*$)	20,467	5,225	21,898	4,546
Regulator's starting value ($f(W_0) = f(W_*)$)	21,595	2,612	10,949	3,246
Total starting value ($f(W_0) + W_0$)	42,062	9,409	32,846	7,792
No transfer constraint				
Regulator's starting value ($f(R)$)	66,269	5,362	22,474	15,812
Firm's starting value ($W_0 = R$)	17,288	5,225	21,898	4,125
Total starting value ($f(W_0) + W_0$)	83,556	10,588	44,371	19,938

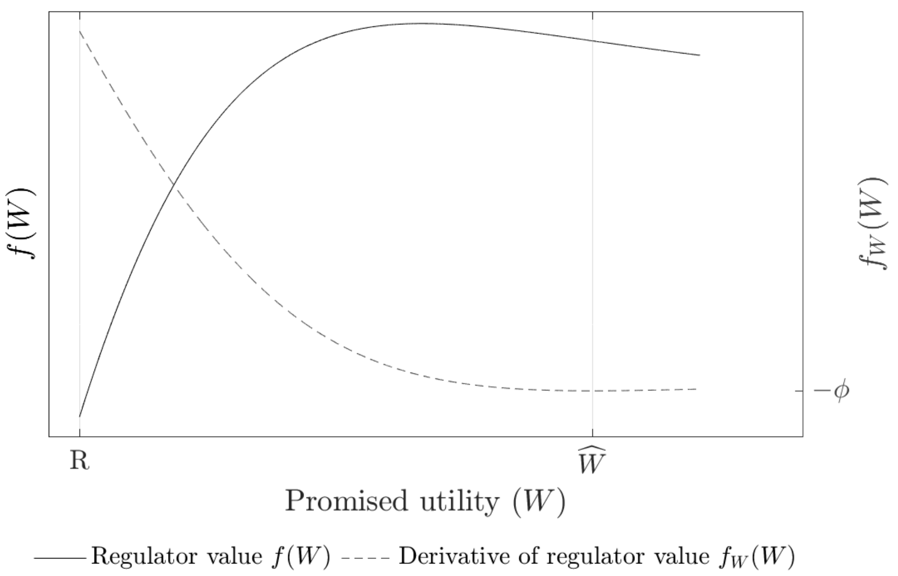
Notes: The first panel of Table 2 reports computed values for the rebate threshold, the regulator's value at the rebate threshold, and both the firm and regulator's starting values under a maximum transfer or limited liability constraint. The second panel reports starting values when there is no such transfer constraint. See Table 1 for scenario parameters.

Table 3: Simulation Results

Scenarios	**(1)**	**(4)**
Transfer constraint		
Mean transfer to firm (dI)	-105	-4
SD transfer	636	1170
Probability of exit	0.1	0
No transfer constraint		
Mean transfer to firm (dI)	-576	-138
SD transfer	974	1169
Probability of exit	0	0

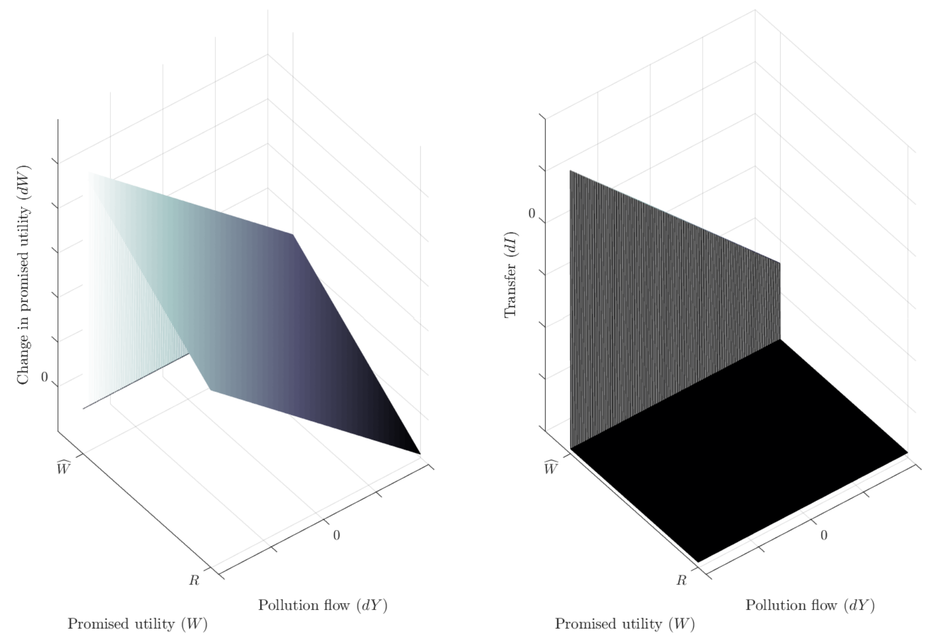
Notes: Table 3 reports the mean, standard deviation (SD) and probability of exit by $t = 60$ from 10,000 simulations of the transfer payments and firm continuation values for Scenarios 1 and 4. See Table 1 for scenario parameters. The standard deviation measures the average volatility of transfers over time, it is the mean of the standard deviation of transfer payments in each of the 10,000 simulations.

Figure 1: Regulator's Value



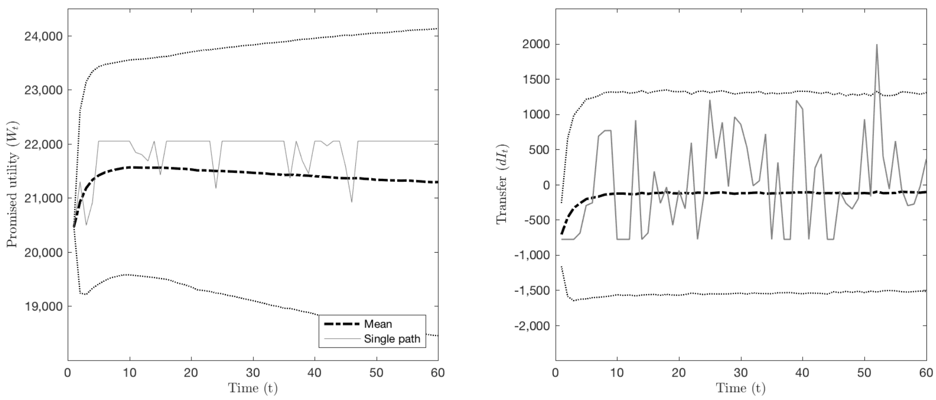
Notes: Figure 1 depicts a stylized regulator's value function.

Figure 2: Contract Values



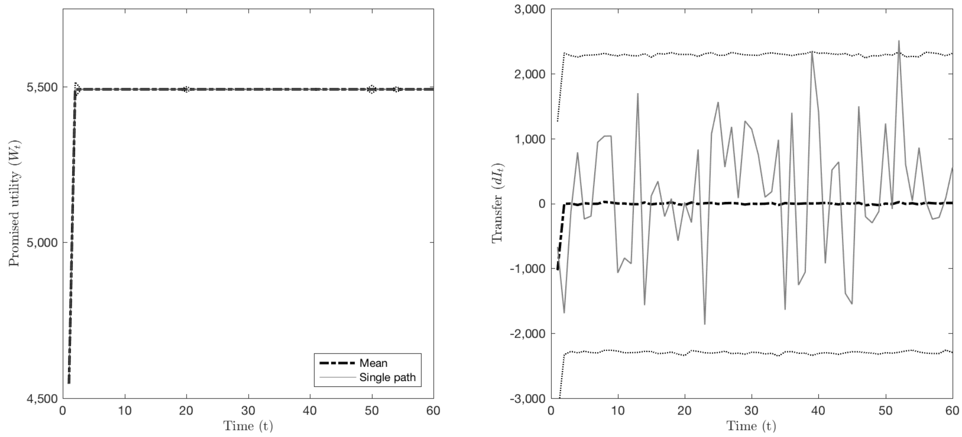
Notes: Figure 2 depicts the contract that links the firm's payoffs to the pollution flow. The left panel shows changes in the firm's promised value given their current promised value and the pollution flow. The right panel shows transfer flows to the firm given their current promised value and the pollution flow.

Figure 3: Scenario 1 Firm's Expected Payoff and Transfers



Notes: Figure 3 depicts results for simulations of Scenario 1. See Table 3 for the parameter values. The left panel shows the firm's mean expected payoff (dark dashed line) across 10,000 simulations of pollution and a single pollution path (grey solid line). The right panel shows average transfers across simulations (dark dashed line) and a single pollution path (grey solid line). Single pollution path is the same across plots. In both plots, dotted lines are the mean of the variable +/- 1.96*standard deviation.

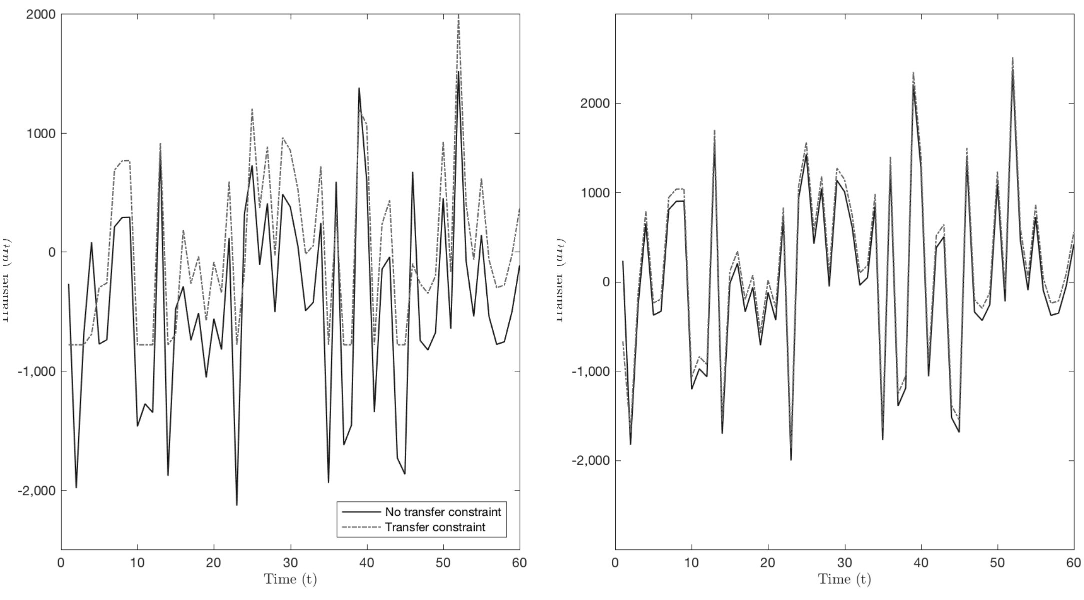
Figure 4: Scenario 4 Firm's Expected Payoff and Transfers



Notes: Figure 4 depicts results for Scenario 4. See Table 3 for the parameter values. The left panel shows the firm's mean expected payoff (dark dashed line) across 10,000 simulations of pollution and a single pollution path (grey solid line). The right panel shows average transfers across simulations (dark dashed line) and a single pollution path (grey solid line). Single pollution path is the same across plots. In both plots, dotted lines are the mean of the variable +/- 1.96*standard deviation.

Figure 5: Transfers in Unconstrained Benchmark

(a) Scenario 1 (b) Scenario 4



Notes: Figure 5 shows transfers in the benchmark case without a transfer constraint for a single path of pollution (solid line) and reproduces the transfer from the same pollution path for the model with a transfer constraint (dashed line). Left hand panel shows transfers for Scenario 1, right hand panel shows transfers for Scenario 4. Scenario parameters outlined in Table 3.

Appendix

In all proofs in this appendix, to simplify notation we rescale time when firm i enters so that with the new time scale $t_0^i = 0$.

Proof of Lemma 2. Recall from (4) that i's promised utility, the total expected payoff firm i receives from choosing abatement from time t after some history of reports and choices a_s^i, $0 \leqslant s \leqslant t$, given that all other firms follow the recommended actions from time t, is

$$W_t^i = \mathbb{E}_t \left[\int_t^{T^i} \exp^{-\gamma(s-t)} d\Pi_s^i(a_s^i) + \exp^{-\gamma(T^i-t)} R \right]$$

Consider the case where for $0 \leqslant s \leqslant t$ the firm follows the recommended actions and abates (i.e., $a_s^i = 1 \ \forall \ 0 \leqslant s \leqslant t$). It is sufficient to show that equation (9) holds for this case (DeMarzo and Sannikov, 2006). Define

$$V_t^i = \left[\int_0^t \exp^{-\gamma s} d\Pi_s^i(a_s^i = 1) + \exp^{-\gamma t} W_t^i \right]$$

Then V_t^i is the total past and future expected payoff of the firm from action choice $a_s^i = 1 \ \forall \ s \in [0, t]$. Indeed $V_t^i = \mathbb{E}_t[V_{t+s}^i]$; the total past and future expected payoff from choosing the abatement actions, where the expectation is taken over information available at time t, is constant over time. In other words, V_t^i is a martingale. By the martingale representation theorem there is a process $\tilde{\beta}_t^i$ such that $dV_t^i = \exp^{-\gamma t} \tilde{\beta}_t^i dZ_t$. Differentiating V_t^i with respect to t, and using the martingale property:

$$\begin{aligned} dV_t^i &= \exp^{-\gamma t} d\Pi_t^i(a_t^i = 1) - \gamma \exp^{-\gamma t} W_t^i dt + \exp^{-\gamma t} dW_t^i \\ \exp^{-\gamma t} \tilde{\beta}_t^i dZ_t &= \exp^{-\gamma t} d\Pi_t^i(a_t^i = 1) - \gamma \exp^{-\gamma t} W_t^i dt + \exp^{-\gamma t} dW_t^i \\ dW_t^i &= \gamma W_t^i dt - d\Pi_t^i(a_t^i = 1) + \tilde{\beta}_t^i dZ_t \end{aligned}$$

Now, letting $\tilde{\beta}_t^i = \beta_t^i \sigma$, substituting for dY_t from (1) with $\mu_t = 0$, and $d\Pi_t^i$ from (3) with

$a_t^i = 1$:

$$dW_t^i = \gamma W_t^i dt - (\pi^i - c^i)dt - dI_t^i + \beta_t^i dY_t$$

The proof that $\beta_t^i \leqslant -\lambda$ is provided in the text. $\square$

Proof of Lemma 3. We first show that the regulator's value function is separable. Consider two firms i and j. By Lemma 2, when all firms abate the evolution of the continuation value dW_t^i is correlated with dW_t^j through the pollution shock σdZ_t but W_t^i does not depend directly on W_t^j. The regulator therefore cannot gain by linking the payoffs of the firms and will choose to adjust dI^i and dW^i independently of dI^j and dW^j; that is, the regulator's value function is separable in W_t^i and W_t^j.[19]

With separability we can focus on the regulator's instantaneous value from firm i, $rf^i(W^i)dt$. Rescale time so that firm i's chance of entering occurs at $t_0^i = 0$ and define G_t^i as the total regulator's payoff from firm i up to and from time t, conditional on firm i entering and not exiting before t:

$$G_t^i = \int_0^t - \exp^{-rs} dI_s^i + \exp^{-rt} f^i(W_t^i) + (1-\phi)(W_0^i - \exp^{-rt} W_t^i) \tag{16}$$

Differentiating with respect to time and applying Ito's Lemma:

$$\begin{aligned} \exp^{rt} dG_t^i + rf^i(W_t^i)dt = & - dI_t^i + f_{W^i}^i(W_t^i)dW_t^i + \frac{1}{2}(\sigma\beta_t^i)^2 f_{W^iW^i}^i(W_t^i)dt \\ & + (1-\phi)rW_t^i dt - (1-\phi)dW_t^i \end{aligned} \tag{17}$$

Under the optimal policy dG_t^i is a martingale and hence its expected value is zero. Thus

[19] The separability of the value function relies on the assumption that the recommended actions of active firms (abatement) are independent of the total number of active or exited firms. Without this assumption, the expected stopping time $\mathbb{E}(T^j)$ and the incentives dI_t^j of firm j may affect firm i's recommended actions or incentive compatibility constraint and separability may fail.

we have:

$$rf^i(W_t^i)dt = -\left(dI_t^i + dW_t^i\right) + \left(\phi + f_{W^i}^i(W_t^i)\right)dW_t^i + \frac{1}{2}(\sigma\beta_t^i)^2 f_{W^iW^i}^i(W_t^i)dt + (1-\phi)rW_t^i dt \tag{18}$$

Let $\widehat{W}^i$ be the lowest value of $W^i \geqslant R$ such that $f_{W^i}^i(\widehat{W}^i) = -\phi$ (the value matching condition). Optimality of the threshold $\widehat{W}^i$ requires that the slope of the value function f^i be constant at $\widehat{W}^i$; that is, $f_{W^iW^i}^i(\widehat{W}^i) = 0$ (the super contact condition; e.g., see Dumas, 1991). Replacing the value matching and the super contact conditions into (18) and using equation (9) yields the condition

$$rf^i(\widehat{W}^i)dt = (\pi^i - c^i)dt + \widehat{W}^i\left[(1-\phi)r - \gamma\right]dt \tag{19}$$

To prove the concavity of $f^i(W^i)$ in $\left[R, \widehat{W}^i\right]$, it is sufficient to show that $f_{W^iW^i}^i(W^i) < 0$ for all W^i such that $R \leqslant W^i < \widehat{W}^i$. We first show that for $R \leqslant W^i < \widehat{W}^i$:

$$rf^i(W^i) + W^i(\gamma - (1-\phi)r) < \pi^i - c^i \tag{20}$$

Condition (20) states that the flow payoff of the regulator plus the expected flow utility of firm i net of the flow payoff of the regulator due to firm i must be less than the flow value of the firm's production net of abatement cost. To verify this intuitive condition we use the fact that $f_{W^i}^i(W^i) > -\phi$ for $W^i < \widehat{W}^i$ and equation (19).

We wish to evaluate $rf^i(W^i) + W^i(\gamma - (1-\phi)r)$ for $W^i < \widehat{W}^i$. Taking a first order Taylor approximation, for small $\Delta > 0$ we have that $f(\widehat{W}^i - \Delta) \approx f^i(\widehat{W}^i) - \Delta f_{W^i}^i(\widehat{W}^i - \Delta)$. As $f_{W^i}^i(\widehat{W}^i - \Delta) > -\phi$, it is $f^i(\widehat{W}^i - \Delta) < f^i(\widehat{W}^i) + \Delta\phi$. Then, letting $W^i = \widehat{W}^i - \Delta$, (20) follows, as it must be:

$$\begin{aligned} rf^i(\widehat{W}^i - \Delta) + (\widehat{W}^i - \Delta)(\gamma - (1-\phi)r) &< r(f^i(\widehat{W}^i) + \Delta\phi) + (\widehat{W}^i - \Delta)(\gamma - (1-\phi)r) \\ &= rf^i(\widehat{W}^i) + \widehat{W}^i(\gamma - (1-\phi)r) - \Delta(\gamma - r) \\ &< rf^i(\widehat{W}^i) + \widehat{W}^i(\gamma - (1-\phi)r) \\ &= \pi^i - c^i, \end{aligned}$$

where the second inequality comes from $\gamma > r$ and the last equality uses (19).

Substituting (9) into (18), taking expectations and re-arranging, we can write the regulator's value function from firm i for $W_t^i < \widehat{W}$ as:

$$\begin{aligned} rf^i(W_t^i) = &-\phi dI_t^i + f_{W^i}^i(W_t^i)(\gamma W_t^i - (\pi^i - c^i) - dI_t^i) \\ &+ (1-\phi)((r-\gamma)W_t^i + \pi^i - c^i) + \frac{1}{2}(\beta_t^i\sigma)^2 f_{W^iW^i}^i(W_t^i) \end{aligned} \tag{21}$$

Hence, for $R \leqslant W^i < \widehat{W}^i$ we have:

$$\begin{aligned} \frac{(\sigma\beta^i)^2}{2} f_{W^iW^i}^i(W^i) &= rf^i(W^i) + \phi dI_t^i - f_{W^i}^i(W^i)(\gamma W^i - (\pi^i - c^i) - dI_t^i) \\ &\qquad - (1-\phi)\left[(r-\gamma)W^i + (\pi^i - c^i)\right] \\ &\leqslant rf^i(W^i) + \phi dI_t^i + \phi(\gamma^i W^i - (\pi^i - c^i) - dI_t^i) \\ &\qquad - (1-\phi)\left[(r-\gamma)W^i + (\pi^i - c^i)\right] \\ &= rf^i(W^i) + W^i(\gamma - (1-\phi)r) - (\pi^i - c^i) \\ &< 0 \end{aligned}$$

The first inequality follows from $f_{W^i}^i(W^i) \geqslant -\phi$ and $\gamma W^i - (\pi^i - c^i) - dI_t^i) > 0$ and the second inequality follows from (20). Hence for $R \leqslant W^i \leqslant \widehat{W}^i$ the function f^i is strictly concave. □

Proof of Proposition 1. The regulator chooses dI_t^i and dW_t^i to maximize the rhs of (18), or (21), subject to $dI_t^i \geqslant -\tau dt$ and equation (9) in Lemma 2, which can be written as $dW_t^i + dI_t^i = \gamma W_t^i dt - (\pi^i - c^i)dt + \beta_t^i dY_t$.

By Lemma 3, f^i is concave for $W_t^i < \widehat{W}^i$. Since, by definition, $f_{W^i}^i(\widehat{W}^i) = -\phi$ (the value matching condition), when $W_t^i < \widehat{W}^i$ it must be $f_{W^i}^i(W_t^i)dt > -\phi$; thus, the rhs of (18) is increasing in dW_t^i and by (9) it is decreasing in dI_t^i. It follows that when $W_t^i < \widehat{W}^i$ it is optimal to set $dI_t^i = -\tau dt$ and hence, by (9), $dW_t^i = \tau + \gamma W_t^i dt - (\pi^i - c^i)dt + \beta_t^i dY_t$.

On the contrary, the concavity of f^i implies that when $W_t^i \geqslant \widehat{W}^i$ the rhs of (18) is (weakly) decreasing in W_t^i, since $f_{W^i}^i(W_t^i) = -\phi$. Thus, the regulator is better off by not letting firm i's promised utility exceed $\widehat{W}^i$. This has two implications.

First, it cannot be that $W^i_{t^i_0} > \widehat{W}^i$ when firm i first enters. Since it cannot be that $W^i_{t^i_0} < R$ either (firm i cannot receive less than its outside option), it must be that $R \leqslant W^i_{t^i_0} \leqslant \widehat{W}^i$. This proves part (a) of the proposition.

Second, if at time $t > t^i_0$, $W^i_t = \widehat{W}^i$ the regulator must make sure that $dW^i_t \leqslant 0$. By (9) in order to ensure that $dW^i_t \leqslant 0$ at $W^i_t = \widehat{W}^i$ it must be:

$$\begin{aligned} dI^i_t &= \max\left\{-\tau dt, \gamma\widehat{W}^i dt - (\pi^i - c^i)dt + \beta^i_t dY_t\right\} \\ dW^i_t &= \min\left\{0, \gamma\widehat{W}^i dt - (\pi^i - c^i - \tau)dt + \beta^i_t dY_t\right\} \end{aligned}$$

To conclude the proof of parts (c) and (d) and prove part (b) of the proposition, we need to show that $\beta^i_t = -\lambda$.

Since f^i is strictly concave in W^i_t, $f^i_{W^iW^i}(W^i_t) < 0$ for $W^i_t < \widehat{W}^i$, the regulator will seek to minimize the magnitude of β^i_t subject to the incentive compatibility constraint $\beta^i_t \leqslant -\lambda$. Hence $\beta^i_t = -\lambda$ and for $W^i_t < \widehat{W}^i$ (21) becomes:

$$\begin{aligned} rf^i(W^i_t) =&\phi\tau + f^i_{W^i}(W^i_t)(\gamma W^i_t - (\pi^i - c^i) + \tau) \\ &+ (1-\phi)((r-\gamma)W^i_t + (\pi^i - c^i) + \frac{1}{2}(\lambda\sigma)^2 f^i_{W^iW^i}(W^i_t) \end{aligned} \tag{22}$$

Lemma 4 shows that there is a unique solution to the boundary value problem defined by (22) with boundary conditions $f^i(R) = (1-\phi)R$ and $f^i(\widehat{W}^i) = \frac{\pi^i - c^i - (\gamma - (1-\phi)r)\widehat{W}^i}{r}$.

$\square$

Lemma 4. *There is a unique solution to the boundary value problem defined by (22) with boundary conditions* $f^i(R) = (1-\phi)R$ *and* $f^i(\widehat{W}^i) = \frac{\pi^i - c^i - (\gamma - (1-\phi)r)\widehat{W}^i}{r}$.

Proof. Consider the family of initial value problems defined by (22) with initial conditions $f^i(R) = (1-\phi)R$ and $f^i_{W^i}(R) = k$ with $k \geqslant -1$. Since it has bounded derivatives, the value function $f^i(W^i)$ on the left hand side of (22) is Lipschitz continuous. Then, by a standard result in the theory of ordinary differential equations (e.g., see Hirsh and Smale, 1974, Theorem 1, page 162) the initial value problem has a unique global solution, which is continuous in ϕ, R and k.

Any solution of the boundary value problem must be the solution of an initial value problem with $f^i_{W^i}(R) = k$ for some $k \geqslant -\phi$. We will argue that there exists a unique $k = k^*$ for which the solution of the boundary value problem coincides with the solution of the initial value problem. First we will prove a few preliminary claims. The first claim says that the slope of the solution of the initial value problem with a greater slope at R has a greater slope at all $W^i \leqslant \widehat{W}^i$.

Claim 1. *Let $f^i(\cdot; k)$ be the solution of the initial value problem defined by (22) with initial conditions $f^i(R) = (1-\phi)R$ and $f^i_{W^i}(R) = k$. If $k > k'$, then $f^i_{W^i}(W^i; k) > f^i_{W^i}(W^i; k')$ for all $W^i \in [R, \widehat{W}^i)$.*

Proof of the Claim. The proof is by contradiction. Let $k > k'$; by definition $f^i_{W^i}(R; k) > f^i_{W^i}(R; k')$. Suppose that, contrary to the claim, there is a $\widetilde{W}^i$ such that: (i) $f^i_{W^i}(W^i; k) > f^i_{W^i}(W^i; k')$ for $W^i < \widetilde{W}^i$, (ii) $f^i_{W^i}(\widetilde{W}^i; k) = f^i_{W^i}(\widetilde{W}^i; k')$, and (iii) $f^i_{W^i}(W^i; k) < f^i_{W^i}(W^i; k')$ for $\widetilde{W}^i < W^i < \widetilde{W}^i + \delta$, for some $\delta > 0$. First note that (i) and (ii) imply: (iv) $f^i(\widetilde{W}^i; k) > f^i(\widetilde{W}^i; k')$. Second, observe that (ii), (iv) and (22) imply $f^i_{W^iW^i}(\widetilde{W}^i; k) > f^i_{W^iW^i}(\widetilde{W}^i; k')$, or equivalently, $\frac{f^i_{W^i}(\widetilde{W}^i+\varepsilon;k)-f^i_{W^i}(\widetilde{W}^i;k)}{\varepsilon} > \frac{f^i_{W^i}(\widetilde{W}^i+\varepsilon;k')-f^i_{W^i}(\widetilde{W}^i;k')}{\varepsilon}$ for a sufficiently small ε. The latter inequality together with (ii) implies $f^i_{W^i}(\widetilde{W}^i + \varepsilon; k) > f^i_{W^i}(\widetilde{W}^i + \varepsilon; k')$, contradicting (iii) and concluding the proof of the claim. □

Define the function:

$$g^i(W^i) = \frac{\pi^i - c^i - (\gamma - (1-\phi)r)W^i}{r} \tag{23}$$

Next, observe that, by (19), for the solution $f^i(W^i; k)$ of an initial value problem also to be a solution of the boundary value problem it is necessary that there exists a $\widehat{W}^i$ such that $f^i(\widehat{W}^i; k) = g^i(\widehat{W}^i)$ and $f^i_{W^i}(\widehat{W}^i; k) = -\phi$; that is $f^i(\cdot; k)$ crosses $g^i(\cdot)$ at slope $-\phi$. The next claim shows that such a condition is also sufficient, as the solution of the initial value problem that satisfies it is strictly concave to the left of $\widehat{W}^i$ and hence $\widehat{W}^i$ is the lowest value of W^i such that $f^i_{W^i}(W^i) = -\phi$.

Claim 2. *Let $f^i(\cdot; k)$ be the solution of the initial value problem defined by (22) with initial conditions $f^i(R) = (1-\phi)R$ and $f^i_{W^i}(R) = k$. If for some $\widehat{W}^i$ it is $f^i_{W^i}(\widehat{W}^i; k) \geqslant$*

$-\phi$, *$f^i(\widehat{W}^i;k) = g^i(\widehat{W}^i)$ and $f^i(W^i;k) < g^i(W^i)$ for all $W^i \in [R, \widehat{W}^i)$, then $f^i(W^i;k)$ is strictly concave for all $W^i \in [R, \widehat{W}^i)$.*

Proof of the Claim. Observe that (22) can be written as

$$r\left[f^i(W^i;k) - g^i(W^i)\right] = \left[\phi + f^i_{W^i}(W^i;k)\right]\left[\tau - rg^i(W^i) + (1-\phi)rW^i\right] + \frac{(\lambda\sigma)^2}{2} f^i_{W^iW^i}(W^i) \tag{24}$$

Note that $\left[\tau - rg^i(W^i) + (1-\phi)rW^i\right] > 0$ by Lemma 1. Hence, $f^i_{W^i}(W^i;k) \geqslant -\phi$ and $f^i(W^i;k) < g^i(W^i)$ imply $f^i_{W^iW^i}(W^i;k) < 0$. It only remains to show that $f^i_{W^i}(\widehat{W}^i;k) \geqslant -\phi$ and (24) imply $f^i_{W^i}(W^i;k) \geqslant -\phi$ for all $W^i \in [R, \widehat{W}^i)$. Using a first order Taylor's approximation, we have: $f^i_{W^i}(W^i - \varepsilon;k) = f^i_{W^i}(W^i;k) - \varepsilon f^i_{W^iW^i}(W^i;k) + o(\varepsilon)$. Consider $\widehat{W}^i - \varepsilon$ for an arbitrarily small ε. As the left hand side of (24) is negative and $f^i_{W^i}(\widehat{W}^i - \varepsilon;k) = -\phi + o(\varepsilon)$, it is immediate that $f^i_{W^iW^i}(\widehat{W}^i - \varepsilon;k) < 0$. An analogous argument shows, more generally, that $f^i_{W^iW^i}(W^i;k) < 0$ implies that $f^i_{W^iW^i}(W^i - \varepsilon;k) < 0$ for all $W^i \in (R, \widehat{W}^i]$, concluding the proof of the claim. □

We now show that for k sufficiently large the solution $f^i(\cdot;k)$ of the initial value problem has positive slope when it crosses $g^i(\cdot)$ and hence cannot be a solution of the boundary value problem.

Claim 3. *Let $f^i(\cdot;k)$ be the solution of the initial value problem defined by (22) with initial conditions $f^i(R) = (1-\phi)R$ and $f^i_{W^i}(R) = k$. There exists $\bar{k}$ such that for some $\widehat{W}^i$ it is $f^i_{W^i}(\widehat{W}^i;\bar{k}) > 0$, $f^i(\widehat{W}^i;\bar{k}) = g^i(\widehat{W}^i)$ and $f^i(W^i;\bar{k}) < g^i(W^i)$ for all $W^i \in [R, \widehat{W}^i)$.*

Proof of the Claim. Using a first order Taylor's approximation, we have: $f^i(R+\varepsilon;k) = f^i(R;k) + \varepsilon f^i_{W^i}(R;k) + o(\varepsilon) = (1-\phi)R + \varepsilon k + o(\varepsilon)$. By choosing $k = \bar{k} = \frac{g^i(R)-(1-\phi)R}{\varepsilon}$, we can make sure that $f^i(R+\varepsilon;\bar{k}) = g^i(R) + o(\varepsilon) > g^i(R+\varepsilon)$. Since $f^i(R;\bar{k}) < g^i(R)$, by continuity of $f^i(\cdot;\bar{k})$ there must exists a $\widehat{W}^i$ such that $f^i(\widehat{W}^i;\bar{k}) = g^i(\widehat{W}^i)$ and $f^i(W^i;\bar{k}) < g^i(W^i)$ for all $W^i \in [R, \widehat{W}^i)$. To conclude the proof of the claim, we only need to show that $f^i_{W^i}(\widehat{W}^i;\bar{k}) > 0$, or, taking a first order Taylor approximation, $f^i_{W^i}(\widehat{W}^i;\bar{k}) = f^i_{W^i}(R;\bar{k}) + (\widehat{W}^i - R)f^i_{W^iW^i}(\widehat{W}^i;\bar{k}) + o(\widehat{W}^i - R) > 0$. This follows from recalling that $f^i_{W^i}(R;\bar{k}) = \bar{k}$, $\widehat{W}^i - R < \varepsilon$, and that by (24) $f^i_{W^iW^i}(\widehat{W}^i;\bar{k}) = \frac{2}{(\lambda\sigma)^2}\left[rg^i(\widehat{W}^i) - (1-\phi)r\widehat{W}^i - \tau\right]\left[\phi + f^i_{W^i}(\widehat{W}^i;\bar{k})\right]$. $\square$

We now consider $k = -\phi$ and show that $f^i(\cdot;-\phi)$ cannot first cross $g^i(\cdot)$ at a slope greater than or equal to $-\phi$ (and hence cannot be a solution of the boundary value problem). Suppose, to the contrary, that $f^i_{W^i}(\widehat{W}^i;-\phi) \geqslant -\phi$, $f^i(\widehat{W}^i;-\phi) = g^i(\widehat{W}^i)$ and $f^i(W^i;-\phi) < g^i(W^i)$ for all $W^i \in [R, \widehat{W}^i)$. Then by Claim 2 $f^i(\cdot;-\phi)$ must be linear with slope $-\phi$, as it must be concave and it has slope $-\phi$ at R by definition. However the left hand side of (24) is negative for $W^i < \widehat{W}^i$, which implies that $f^i(\cdot;-\phi)$ must be strictly concave, a contradiction.

We now conclude the proof of the lemma by showing that there must be a unique solution of the initial value problem that crosses $g^i(\cdot)$ at slope $-\phi$. There are two cases. First, $f^i(\cdot;-\phi)$ crosses $g^i(\cdot)$ at a slope less than $-\phi$. Then by continuity in k of the solutions of the initial value problem and Claims 1 and 3, there exists a unique $k = k^*$ for which the solution of the boundary value problem coincides with the solution of the initial value problem. Second, $f^i(\cdot;-\phi)$ never crosses $g^i(\cdot)$. Then it must be the case that its slope, as W^i goes to infinity, is bounded above by the slope of $g^i(\cdot)$, $-\frac{\gamma}{r} + 1 - \phi$ which is less that $-\phi$. Appealing again to the continuity in k of the solutions of the initial value problem and Claims 1 and 3, we can conclude first that there there exists a $\tilde{k}$ for which $f^i(\cdot;\tilde{k})$ first crosses $g^i(\cdot)$ with slope approximately equal to $-\frac{\gamma}{r} + 1 - \phi < -\phi$ and then that there exists a unique $k = k^* > \tilde{k}$ for which the solution of the boundary value problem coincides with the solution of the initial value problem. $\square$

Proof of Proposition 2. Lemma 2 applies. In addition (18) holds, as in the proof of Proposition 1. Without a transfer constraint the regulator only faces the constraint that firm i exits when $W^i_t < R$. Using the same argument as in the proof of Proposition 1, the regulator's problem is to choose dI^i_t to maximize the right hand side of (22) without the constraint $dI^i_t \geqslant -\tau dt$. For the regulator's problem to have a solution, it must then be $f^i_{W^i}(W^i) = -\phi$ for all $W^i \geqslant R$. We can thus conclude that the upper threshold on the promised utility is $\widehat{W}^i = R$. Further, if $f^i_{W^i}(W^i) = -\phi$ then the starting promised utility is also R, $W^i_* = W^i_{t^i_0} = R$. The regulator keeps the promised utility of firm i constant at R; that is, $dW^i_t = 0$ for all $t > 0$. Imposing this and $W^i_t = R$ in (9), the expression for dW^i, yields:

$$dI^i_t = (\gamma R - (\pi^i - c^i))dt + \beta^i_t dY_t \tag{25}$$

and the regulator is indifferent to β^i subject to the incentive compatibility condition $\beta^i \leqslant -\lambda$. Finally, imposing $f^i_{W^i}(W^i) = -\phi$ and $f^i_{W^iW^i}(W^i) = 0$ and evaluating (22) at $W^i = R$ yields

$$rf^i(R) = \pi^i - c^i + (1-\phi)rR - \gamma R \tag{26}$$

□

Proof of Proposition 3 (a). As shown by DeMarzo and Sannikov (2006) (see their Lemmas 4 and 6), by using the envelope theorem and differentiating the HJB equation (22) and its boundary conditions, we can derive expressions for the change in the regulator's value as a result of a change in the parameters. We first determine the effect of a change in τ, σ and R, as the signs of these effects are immediate:

$$\frac{\partial f^i(W^i)}{\partial \tau} = \mathbb{E}\left[\int_0^{T^i} e^{-rt}(\phi + f^i_{W^i}(W^i_t))dt \;\middle|\; W^i_0 = W^i\right] \geqslant 0 \tag{27}$$

$$\frac{\partial f^i(W^i)}{\partial \sigma} = \mathbb{E}\left[\int_0^{T^i} e^{-rt}\sigma(\lambda)^2 f^i_{W^iW^i}(W^i_t))dt \;\middle|\; W^i_0 = W^i\right] \leqslant 0 \tag{28}$$

$$\frac{\partial f^i(W^i)}{\partial R} = \mathbb{E}\left[e^{-rT^i}(1-\phi) \;\middle|\; W^i_0 = W^i\right] > 0 \tag{29}$$

Note that the weak inequalities in (27) and (28) are strict if $\widehat{W}^i > R$.

To see that $\frac{\partial f^i(W^i)}{\partial \gamma} < 0$, $\frac{\partial f^i(W^i)}{\partial \pi^i} > 0$ and $\frac{\partial f^i(W^i)}{\partial A^i} < 0$, we use the argument of DeMarzo and Sannikov (2006). Suppose that firm i were either: (i) more patient, with true discount rate $\gamma' < \gamma$, (ii) had a higher profit flow $\pi^{i'} > \pi^i$, or (iii) had a lower abatement cost $A^{i'} < A^i$, but they were offered the contract designed for a firm with discount rate γ, flow profit π and abatement cost A^i. Then the firm would be better off while the regulator would have the same value. This can be seen by noting that the firm's transfer dI_t^i at $\widehat{W}^i$ in (14) increases in γ and $A^i = \frac{c^i}{\lambda}$ and decreases in π^i; hence the firm's transfer under the "wrong" parameters is higher than needed to maintain the promised utility flow and incentives for the true parameters. The contract for the "wrong" parameters is not optimal, as the regulator may reduce the value to the firm while preserving their participation and their incentive constraints. Hence it must be that the value to the regulator increases as the discount rate of the firm decreases, the profit flow increases or the abatement cost decreases.

To show that $\frac{\partial f^i(W^i)}{\partial \lambda} < 0$, we derive the expressions for $\frac{\partial f^i(W^i)}{\partial A^i}$ and $\frac{\partial f^i(W^i)}{\partial \lambda}$ and use the fact that the former is less than zero.

$$\frac{\partial f^i(W^i)}{\partial A^i} = \mathbb{E}\left[\int_0^{T^i} e^{-rt}\lambda(f^i_{W^i}(W^i_t) - (1-\phi))dt \;\middle|\; W^i_0 = W^i\right] \tag{30}$$

$$\begin{aligned}\frac{\partial f^i(W^i)}{\partial \lambda} &= \mathbb{E}\left[\int_0^{T^i} e^{-rt}\left[A^i(f^i_{W^i}(W^i_t) - (1-\phi)) + \sigma^2\lambda f^i_{W^iW^i}(W^i_t)\right]dt \;\middle|\; W^i_0 = W^i\right]\\ &= \frac{A^i}{\lambda}\frac{\partial f^i(W^i)}{\partial A^i} + \mathbb{E}\left[\int_0^{T^i} e^{-rt}\sigma^2\lambda f^i_{W^iW^i}(W^i_t)dt \;\middle|\; W^i_0 = W^i\right] < 0\end{aligned} \tag{31}$$

For future use, we also derive the following expressions:

$$\frac{\partial f^i(W^i)}{\partial \pi^i} = \mathbb{E}\left[\int_0^{T^i} e^{-rt}((1-\phi) - f^i_{W^i}(W^i_t))dt \;\middle|\; W^i_0 = W^i\right] \tag{32}$$

$$\frac{\partial f^i(W^i)}{\partial \gamma} = \mathbb{E}\left[\int_0^{T^i} e^{-rt}W^i_t(f^i_{W^i}(W^i_t) - (1-\phi))dt \;\middle|\; W^i_0 = W^i\right] \tag{33}$$

□

Proof of Proposition 3 (b).

We can derive expressions for the effect of parameters on the threshold $\widehat{W}$ by differentiating the boundary condition $rf(\widehat{W}^i) + \widehat{W}^i(\gamma - (1-\phi)r) = \pi^i - c^i$ with respect to each parameter. For any $\theta \neq r$ we have:

$$r\frac{\partial f^i(\widehat{W}^i)}{\partial \theta} + \left[rf^i_{W^i}(\widehat{W}^i) + \gamma - (1-\phi)r\right]\frac{\partial \widehat{W}^i}{\partial \theta} + \widehat{W}^i\frac{\partial(\gamma - (1-\phi)r)}{\partial \theta} = \frac{\partial(\pi^i - \lambda A^i)}{\partial \theta}$$

which yields, after using $f^i_{W^i}(\widehat{W}^i) = -\phi$ and rearranging:

$$(\gamma - r)\frac{\partial \widehat{W}^i}{\partial \theta} = -r\frac{\partial f^i(\widehat{W}^i)}{\partial \theta} - \widehat{W}^i\frac{\partial(\gamma - (1-\phi)r)}{\partial \theta} + \frac{\partial(\pi^i - \lambda A^i)}{\partial \theta}$$

By Proposition 3 (a), the signs of the effect of changes in τ, σ and R are immediate:

$$\frac{\partial \widehat{W}^i}{\partial \tau} = \frac{-r}{\gamma - r}\left[\frac{\partial f^i(\widehat{W}^i)}{\partial \tau}\right] \leqslant 0 \tag{34}$$

$$\frac{\partial \widehat{W}^i}{\partial \sigma} = \frac{-r}{\gamma - r}\left[\frac{\partial f^i(\widehat{W}^i)}{\partial \sigma}\right] \geqslant 0 \tag{35}$$

$$\frac{\partial \widehat{W}^i}{\partial R} = \frac{-r}{\gamma - r}\left[\frac{\partial f^i(\widehat{W}^i)}{\partial R}\right] < 0 \tag{36}$$

From (32), observe that $\frac{\partial f^i(\widehat{W}^i)}{\partial \pi^i} \leqslant \frac{1-e^{-rT^i}}{r}$ and hence

$$\frac{\partial \widehat{W}^i}{\partial \pi^i} = \frac{1}{\gamma - r}\left[1 - r\frac{\partial f^i(\widehat{W}^i)}{\partial \pi^i}\right] > 0 \tag{37}$$

From (33), observe that $\frac{\partial f^i(\widehat{W}^i)}{\partial \gamma} \geqslant -\frac{\widehat{W}^i(1-e^{-rT^i})}{r}$ and hence :

$$\frac{\partial \widehat{W}^i}{\partial \gamma} = \frac{-1}{\gamma - r}\left[\widehat{W}^i + r\frac{\partial f^i(\widehat{W}^i)}{\partial \gamma}\right] < 0 \tag{38}$$

From (30), observe that $\frac{\partial f^i(\widehat{W}^i)}{\partial A^i} \geqslant -\frac{\lambda(1-e^{-rT^i})}{r}$ and hence :

$$\frac{\partial \widehat{W}^i}{\partial A^i} = \frac{-1}{\gamma - r}\left[\lambda + r\frac{\partial f^i(\widehat{W}^i)}{\partial A^i}\right] < 0 \tag{39}$$

Finally, note from (31) that there is no simple lower bound on $\frac{\partial f^i(\widehat{W}^i)}{\partial \lambda}$, as it depends on the curvature of the value function $f^i_{W^iW^i}$ and hence the following expression is indeterminate:

$$\frac{\partial \widehat{W}^i}{\partial \lambda} = \frac{-1}{\gamma - r}\left[A^i + r\frac{\partial f^i(\widehat{W}^i)}{\partial \lambda}\right] \tag{40}$$

$\square$

Proof of Proposition 3 (c).

The optimality condition for the choice of W^i_* is $f^i_{W^i}(W^i_*) = 0$. Differentiating with respect to any parameter θ gives:

$$\frac{\partial W^i_*}{\partial \theta} = \frac{-1}{f^i_{W^iW^i}(W^i_*)} \cdot \frac{\partial}{\partial W^i}\left(\frac{\partial f^i(W^i_*)}{\partial \theta}\right)$$

Thus, to evaluate the effect of a change in the parameter θ we need to compare $\frac{\partial f^i(W^i_*)}{\partial \theta}$ with $\frac{\partial f^i(W^i_0)}{\partial \theta}$ for $W^i_0 \in [W^i_* - \epsilon, W^i_* + \epsilon]$.

First observe that the constant $1 - \phi$ in (29) is positive. Since an increase in W^i_0 from W^i_* to $W^i_* + \epsilon$ increases the stopping time T^i (the time at which W^i_t goes belows R and the firm exits); it immediately follows that $\frac{\partial W^i_*}{\partial R} > 0$.

From now on, define $\widehat{t}^i$ as the first time that the process W^i_t starting at W^i_0 reaches W^i_*.

Using (27) we may write:

$$\frac{\partial f^i(W^i)}{\partial \tau} = \mathbb{E}\left[\int_0^{\widehat{t}^i} e^{-rt}(\phi + f^i_{W^i}(W^i_t))dt + e^{-r\widehat{t}^i}\frac{\partial f^i(W^i_*)}{\partial \tau} \;\mid\; W^i_0 = W^i\right]$$

There are two cases. Suppose first that $\frac{\partial f^i(W^i_*)}{\partial \tau} > (\phi + f^i_{W^i}(W^i_*)) = \phi$. Take $W^i_0 = W^i_* + \epsilon$. Then, since $f^i_{W^i}(W^i) < 0$ for $W^i > W^i_*$, we have that $\frac{\partial f^i(W^i_*)}{\partial \tau} > (\phi + f^i_{W^i}(W^i_t))$ for all $t \in [0, \widehat{t}^i]$ and hence $\frac{\partial f^i(W^i_*)}{\partial \tau} > \frac{\partial f^i(W^i_* + \epsilon)}{\partial \tau}$. Second, suppose that $\frac{\partial f^i(W^i_*)}{\partial \tau} < \phi$. Take $W^i_0 = W^i_* - \epsilon$. Then, since $f^i_{W^i}(W^i) \geqslant 0$ for $W^i \leqslant W^i_*$, we have that $\frac{\partial f^i(W^i_*)}{\partial \tau} < (\phi + f^i_{W^i}(W^i_t))$ for all $t \in [0, \widehat{t}^i]$ and hence $\frac{\partial f^i(W^i_*)}{\partial \tau} < \frac{\partial f^i(W^i_* - \epsilon)}{\partial \tau}$. From the two cases we can thus conclude that $\frac{\partial}{\partial W^i}\left(\frac{\partial f^i(W^i_*)}{\partial \tau}\right) < 0$ and hence $\frac{\partial W^i_*}{\partial \tau} < 0$.

Using (33) we have:

$$\frac{\partial f^i(W^i)}{\partial \gamma} = \mathbb{E}\left[\int_0^{\hat{t}^i} e^{-rt} W^i_t (f^i_{W^i}(W^i_t) - (1-\phi))dt + e^{-r\hat{t}^i}\frac{\partial f^i(W^i_*)}{\partial \gamma} \mid W^i_0 = W^i\right]$$

There are two cases. Suppose first that $\frac{\partial f^i(W^i_*)}{\partial \gamma} > W^i_*(f^i_{W^i}(W^i_*) - (1-\phi))$. Take $W^i_0 = W^i_* + \epsilon$. Then, since $W^i(f^i_{W^i}(W^i) - (1-\phi))$ is decreasing in W^i for $W^i \geqslant W^i_*$, we have that $\frac{\partial f^i(W^i_*)}{\partial \gamma} > W^i_t(f^i_{W^i}(W^i_t) - (1-\phi))$ for all $t \in [0, \hat{t}^i]$ and hence $\frac{\partial f^i(W^i_*)}{\partial \gamma} > \frac{\partial f^i(W^i_*+\epsilon)}{\partial \gamma}$. Second, suppose that $\frac{\partial f^i(W^i_*)}{\partial \gamma} < W^i_*(f^i_{W^i}(W^i_*) - (1-\phi)) = -(1-\phi)W^i_*$. Take $W^i_0 = W^i_* - \epsilon$. Then, since $W^i f^i_{W^i}(W^i) \geqslant 0$ for $W^i \leqslant W^i_*$, we have that $\frac{\partial f^i(W^i_*)}{\partial \gamma} < W^i_t(f^i_{W^i}(W^i_t) - (1-\phi))$ for all $t \in [0, \hat{t}^i]$ and hence $\frac{\partial f^i(W^i_*)}{\partial \gamma} < \frac{\partial f^i(W^i_*-\epsilon)}{\partial \gamma}$. From the two cases we can thus conclude that $\frac{\partial}{\partial W^i}\left(\frac{\partial f^i(W^i_*)}{\partial \gamma}\right) < 0$ and hence $\frac{\partial W^i_*}{\partial \gamma} < 0$.

Using (32) we have:

$$\frac{\partial f^i(W^i)}{\partial \pi^i} = \mathbb{E}\left[\int_0^{\hat{t}^i} e^{-rt}((1-\phi) - f^i_{W^i}(W^i_t))dt + e^{-r\hat{t}^i}\frac{\partial f^i(W^i_*)}{\partial \pi^i} \mid W^i_0 = W^i\right]$$

Here there are also two cases. Suppose first that $\frac{\partial f^i(W^i_*)}{\partial \pi^i} < (1-\phi) - f^i_{W^i}(W^i_*)$. Take $W^i_0 = W^i_* + \epsilon$. Then, since $(1-\phi) - f^i_{W^i}(W^i)$ is increasing in W^i, we have that $\frac{\partial f^i(W^i_*)}{\partial \pi^i} < (1-\phi) - f^i_{W^i}(W^i_t)$ for all $t \in [0, \hat{t}^i]$ and hence $\frac{\partial f^i(W^i_*)}{\partial \pi^i} < \frac{\partial f^i(W^i_*+\epsilon)}{\partial \pi^i}$. Second, suppose that $\frac{\partial f^i(W^i_*)}{\partial \pi^i} > (1-\phi) - f^i_{W^i}(W^i_*) = (1-\phi)$. Take $W^i_0 = W^i_* - \epsilon$. Then, since $f^i_{W^i}(W^i) \geqslant 0$ for $W^i \leqslant W^i_*$, we have that $\frac{\partial f^i(W^i_*)}{\partial \pi^i} > (1-\phi) - f^i_{W^i}(W^i_t)$ for all $t \in [0, \hat{t}^i]$ and hence $\frac{\partial f^i(W^i_*)}{\partial \pi^i} > \frac{\partial f^i(W^i_*-\epsilon)}{\partial \pi^i}$. From the two cases we can thus conclude that $\frac{\partial}{\partial W^i}\left(\frac{\partial f^i(W^i_*)}{\partial \pi^i}\right) > 0$ and hence $\frac{\partial W^i_*}{\partial \pi^i} > 0$.

Using (30) we have:

$$\frac{\partial f^i(W^i)}{\partial A^i} = \mathbb{E}\left[\int_0^{\hat{t}^i} e^{-rt}\lambda(f^i_{W^i}(W^i_t) - (1-\phi))dt + e^{-r\hat{t}^i}\frac{\partial f^i(W^i_*)}{\partial A^i} \mid W^i_0 = W^i\right]$$

Proceding as before, suppose first that $\frac{\partial f^i(W^i_*)}{\partial A^i} > \lambda(f^i_{W^i}(W^i_*) - (1-\phi))$. Take $W^i_0 =$

$W^i_* + \epsilon$. Then, since $\lambda(f^i_{W^i}(W^i) - (1-\phi))$ is decreasing in W^i for $W^i \geqslant W^i_*$, we have that $\frac{\partial f^i(W^i_*)}{\partial A^i} > \lambda(f^i_{W^i}(W^i_t) - (1-\phi))$ for all $t \in [0, \hat{t}^i]$ and hence $\frac{\partial f^i(W^i_*)}{\partial A^i} > \frac{\partial f^i(W^i_*+\epsilon)}{\partial A^i}$. Second, suppose that $\frac{\partial f^i(W^i_*)}{\partial A^i} < \lambda(f^i_{W^i}(W^i_*) - (1-\phi)) = -\lambda(1-\phi)$. Take $W^i_0 = W^i_* - \epsilon$. Then, since $\lambda f^i_{W^i}(W^i) \geqslant 0$ for $W^i \leqslant W^i_*$, we have that $\frac{\partial f^i(W^i_*)}{\partial A^i} < \lambda(f^i_{W^i}(W^i_t) - (1-\phi))$ for all $t \in [0, \hat{t}^i]$ and hence $\frac{\partial f^i(W^i_*)}{\partial A^i} < \frac{\partial f^i(W^i_*-\epsilon)}{\partial A^i}$. From the two cases we can thus conclude that $\frac{\partial}{\partial W^i}\left(\frac{\partial f^i(W^i_*)}{\partial A^i}\right) < 0$ and hence $\frac{\partial W^i_*}{\partial A^i} < 0$.

Finally note from (28) and (31) that $\frac{\partial f^i(W^i)}{\partial \sigma}$ and $\frac{\partial f^i(W^i)}{\partial \lambda}$ depend on the curvature of the value function $f^i_{W^i W^i}$ and thus the impact of changes in σ and λ on W^i_* are indeterminate. $\square$

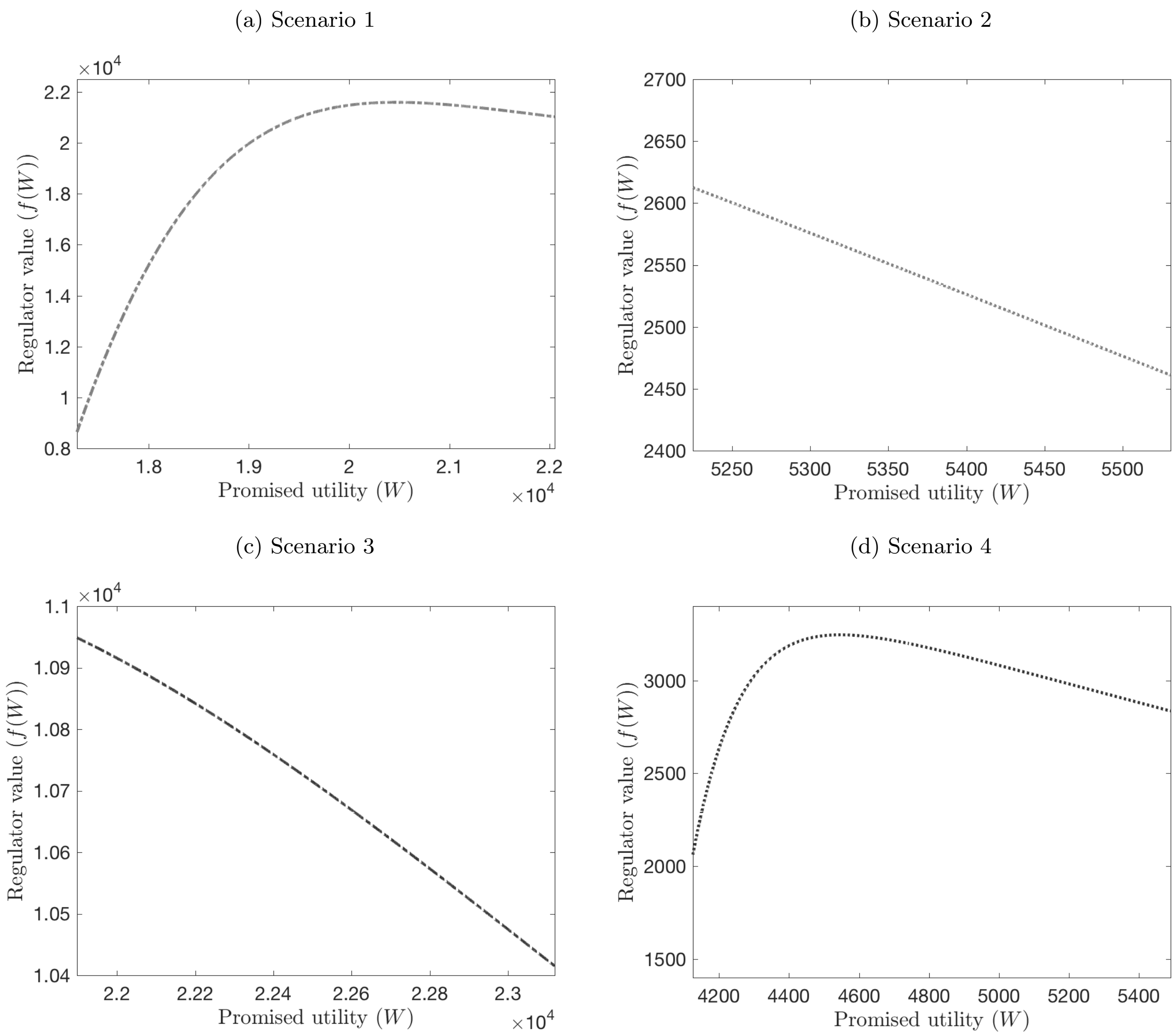
Details for the Simulation Scenarios in Section 5

To parameterize profits and abatment costs we use the following information:

- Annual industry revenue 3.5B $\approx$ 1.6M per fracking well. [20]
- Costs of well construction of 5M.[21]
- Average well lasts 7.5 years.[22]
- Daily revenue versus levelised daily costs $\approx$ 2.5K daily profit.
- Average pay in the PA oil and gas industry 142K $\approx$ \$390 per day.

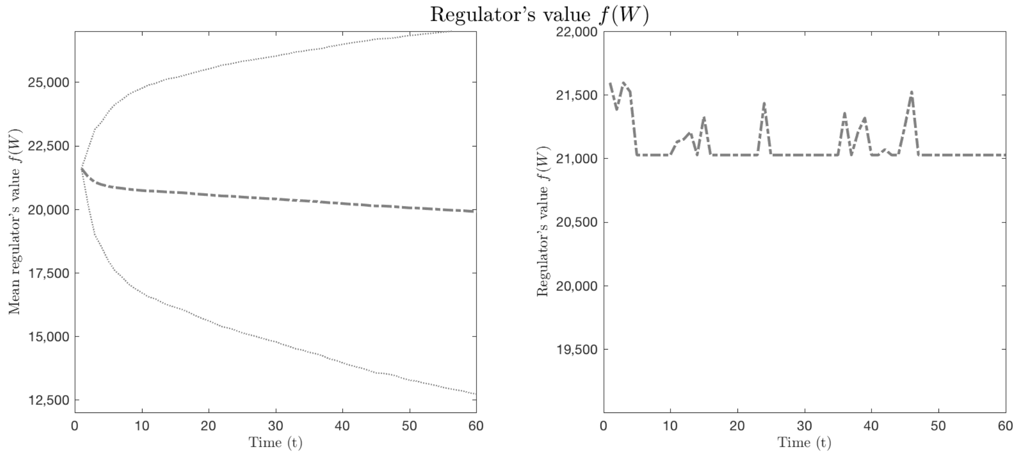
[20] http://thetimes-tribune.com/news/pa-gas-drilling-brought-3-5-billion-in-2011-1.1311378
[21] http://thetimes-tribune.com/news/pa-gas-drilling-brought-3-5-billion-in-2011-1.1311378
[22] http://www.marcellus-shale.us/marcellus-production.htm

Figure 6: Regulator's Value Function



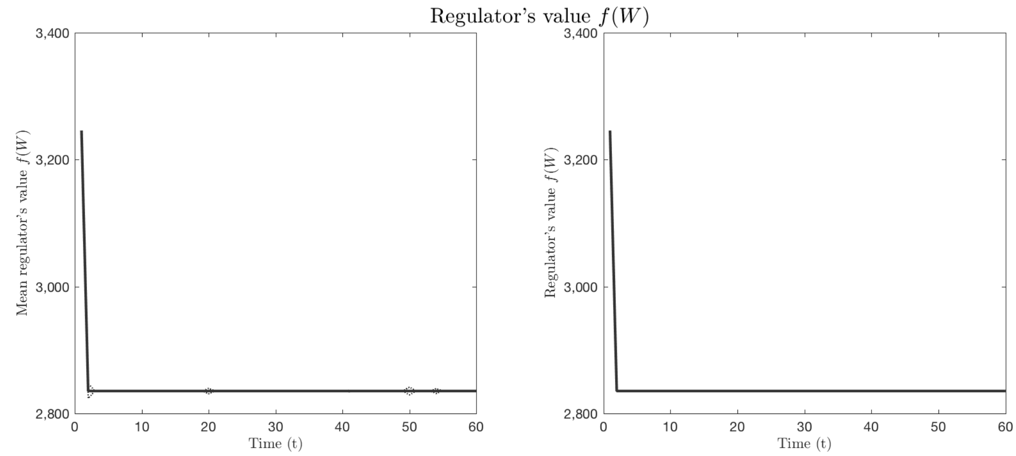
Notes: Figure 6 shows a computed regulator's value function for each the scenarios outlined in Table 3.

Figure 7: Regulator's Value in Simulations, Scenario 1



Notes: Figure 7 depicts the regulator's value for Scenario 1. The left hand panel shows the mean regulator's value. Dotted lines are the mean of the regulator's value +/- 1.96*standard deviation. The right hand panel shows the regulator's value for a single simulation. Parameters for the simulations can be found in Table 3.

Figure 8: Regulator's Value in Simulations, Scenario 4



Notes: Figure 8 depicts the regulator's value for Scenario 1. The left hand panel shows the mean regulator's value. Dotted lines are the mean of the regulator's value +/- 1.96*standard deviation. The right hand panel shows the regulator's value for a single simulation. Parameters for the simulations can be found in Table 3.