Discrete Integrable Systems and Geometric Numerical Integration

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The candidate confirms that the work submitted is his own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

- Parts of Chapters 2 and 3 and Chapter 4 are based on the work in [7], Alsallami SAM, Niesen J, and Nijhoff FW. Closed-form modified Hamiltonians for integrable numerical integration schemes. *Nonlinearity*, **31**(11): 5110, 2018.

The contribution of the candidate to the work was to carry out the majority of the computations and provide the main proofs. The ideas were developed in discussion with the co-authors, J Niesen and FW Nijhoff. The candidate wrote the draft of the paper which, after all the co-authors amendments, was brought to a publishable form.

To my parents, wife and lovely daughter

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Abstract

This thesis deals with discrete integrable systems theory and modified Hamiltonian equations in the field of geometric numerical integration. Modified Hamiltonians are used to show that symplectic schemes for Hamiltonian systems are accurate over long times. However, for nonlinear systems the series defining the modified Hamiltonian equation usually diverges.

The first part of the thesis demonstrates that there are nonlinear systems where the modified Hamiltonian has a closed-form expression and hence converges. These systems arise from the theory of discrete integrable systems. Specifically, they arise as reductions of a lattice version of the Korteweg-de Vries (KdV) partial differential equation. We present cases of one and two degrees of freedom symplectic mappings, for which the modified Hamiltonian equations can be computed as a closed form expression using techniques of action-angle variables, separation of variables and finite-gap integration. These modified Hamiltonians are also given as power series in the time step by Yoshida's method based on the Baker-Campbell-Hausdorff series. Another example is a system of an implicit dependence on the time step, which is obtained by dimensional reduction of a lattice version of the modified KdV equation.

The second part of the thesis contains a different class of discrete-time system, namely the Boussinesq type, which can be considered as a higher-order counterpart of the KdV type. The development and analysis of this class by means of the Bäcklund transformation, staircase reductions and Dubrovin equations forms one of the major parts of the thesis. First, we present a new derivation of the main equation, which is a nine-point lattice Boussinesq equation, from the Bäcklund transformation for the continuous Boussinesq equation. Second, we focus on periodic reductions of the lattice equation and derive all necessary ingredients of the corresponding finite-dimensional models. Using the corresponding monodromy matrix and applying techniques from Lax pair and r-matrix structure analysis to the Boussinesq mappings, we study the dynamics in terms of the so-called Dubrovin equations for the separated variables.

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Chapter 1

Introduction

This thesis brings together ideas from discrete-time integrable systems and geometric numerical integration. The term “discrete-time integrable systems” combines two aspects: discreteness and integrability. On the one hand, the term “discrete-time” means that the evolution is given by the iteration of a mapping of dynamical systems into the phase space under consideration. On the other hand, “integrable” means that there is a sufficient number of independent invariants, i.e. first integrals under the discrete dynamics, which are preserved under the mapping and are in involution, cf. [21]. This definition will be made more precise later in the chapter. The term “geometric numerical integration” refers to numerical methods which preserve geometric properties of the flow of a differential equation, as expanded upon in section 1.4. There are many aspects to integrable systems and geometric integration, some of which we shall meet as this thesis progresses. To be more specific, this thesis is concerned with integrable partial difference equations (lattice equations), e.g. in [24,56,87,89,95,115] and modified Hamiltonian equations, e.g. in [52, 102, 105, 106, 109].

Given a numerical scheme, the modified Hamiltonian is a Hamiltonian whose flow interpolates the iterations of the scheme, i.e. its flow coincides with the computed points [74]. The modified Hamiltonian is generally a divergent asymptotic series in the step size of the numerical scheme [52, 102]. The necessary truncation of the series induces an error which can be made exponentially small in the step size. However, no error needs to be induced when the expansion for the modified Hamiltonian is convergent. Examples of the modified Hamiltonian for the harmonic oscillator and discussions of this situation can be found in [105, 106, 109]. However, for nonlinear systems the convergence of the modified

Hamiltonian can only be achieved in exceptional circumstances, e.g. when the numerical scheme is integrable. The principal interest here is that the numerical scheme is looked at as a discrete-time system in its own right. Thus, the research will be conducted on the interface between the areas of discrete integrable systems and geometric integration in numerical analysis.

In the one-degree-of-freedom case, which by the number of degrees of freedom we mean the number of pairs in Hamilton's equations, an autonomous Hamiltonian system always has a conserved quantity, since the Hamiltonian itself is a first integral. For discrete-time systems this is no longer true, and it is exceptional to have an invariant, which only happens if the system is integrable [72, 99, 100, 121]. The discrete Hamiltonian system case will be encountered in a classical mechanics context in section 1.2.2 below. On the one hand, a non-integrable map, however, is not expected to possess a globally defined invariant function on its phase space and since the modified Hamiltonian is by construction an invariant of the numerical scheme, such a map the modified Hamiltonian cannot exist as a proper function (i.e. it cannot have a convergent modified Hamiltonian). On the other hand, if the numerical scheme is integrable, there must be a link between the invariant and the modified Hamiltonian, possibly through a transcendental relation, cf. [41, 42].

In the case of the multiple-degrees-of-freedom, the situation is more subtle; for complete integrability the system needs to possess as many independent invariants as there are degrees of freedom. This situation is explored throughout this thesis with regard to the two-degrees-of-freedom case. However, to have a convergent modified Hamiltonian, it might not be necessary to have more than one first integral, and the system may need to be only partially integrable (such a situation corresponds to so-called quasi-integrable systems). In this thesis we will not explore the latter possibility, but consider the numerical scheme viewed as a dynamical map which is symplectic and completely integrable; i.e. it possesses a full set of invariants which are independent and in involution with respect to the Poisson bracket, cf. [121] (these terms will be defined later in the chapter). The integrable numerical schemes that we consider in this thesis arise as reductions of both linear and nonlinear integrable lattice equations, which are integrable partial difference equations on a quadrilateral lattice, cf. [56, 83, 87, 90, 115]. These equations also arise as numerical algorithms, e.g. Padé approximant and convergence acceleration algorithms [51], and they are also important for the study of numerically induced chaos [112]. The prime examples of such a system are the lattice version of the *Korteweg-de Vries*

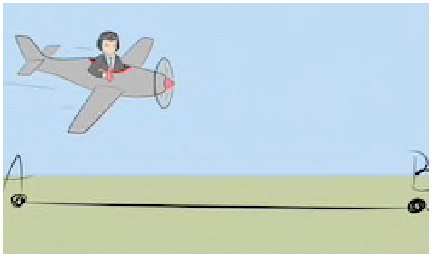
(KdV) [24, 56, 58, 83, 90, 95] and *Boussinesq* (BSQ) [79, 87, 97, 115, 123, 130] *partial differential equations*, which will be studied in detail in this thesis.

In this introductory chapter we collect a number of basic topics within the area which are most relevant to the remainder of the text. Our aim is to give a general overview of the concepts and examples involved, as well as to review some helpful groundwork. In the next section we introduce the context of the thesis "in lay terms", explaining what the thesis is about. Section 1.2 introduces some important ideas of classical mechanics needed for the main topic of the thesis: canonical transformation, Hamiltonian evolution, generating functions, and integrability and action-angle variables. In section 1.3 we discuss the connection of discrete to continuous equations and vice versa; in particular, we go over the steps how to construct the discrete version of KdV equation starting from the continuous case as well as the other way around. We then give a short overview of quadrilateral lattice equations and the fundamental property of multi-dimensional consistency of integrable lattice models. We also give a brief explanation of the common procedure in the literature for constructing integrable mappings as reductions of lattice equations; in particular, we show how the lattice KdV equation gives rise to the generalized McMillan maps, followed by a discussion of the discrete integrable systems as considered in this thesis. In section 1.4 we introduce the term geometric numerical integration and give a brief overview of its history. We then review Yoshida's method, which gives the modified Hamiltonians as power series in the time step, followed by a short overview of the convergence of modified Hamiltonians. Section 1.5 contains an outline of the thesis.

1.1 Thesis overview in lay terms

This section is dedicated to my family and friends who are not scientific.

Let us here try to explain the context of the thesis in lay terms. Many phenomena in the real world involve a movement of objects along trajectories, such as a planet moving in the orbit, a plane moving from one city to another or a horse jumping over a fence.



In mathematics, trajectories are described using mathematical expressions called *differential equations*, which are formulas for quantities that describe the coordinates of motion. Predicting the behaviour of such a trajectory, i.e. predicting where the trajectory goes, is very important. However, this is in fact a difficult mathematical problem that requires solving complicated formulas. Thus, whether or not we are able to predict the outcome of this trajectory depends on whether the mathematical equations describe irregular or regular motion. It also depends on whether the mathematical nature of the system of equations is chaotic or integrable.

In the integrable case there is predictability, as the motion is regular in the mathematical sense, whereas in the chaotic case there is not predictability, because the motion sensitively depends on where we start the motion (the initial conditions). Many situations in nature are of the latter class; however, there are also cases that are regular motion of curves. In either case the exact trajectory is difficult to compute, and we need an approximation technique that takes advantage of computers in order to solve the problem. Instead of considering the system as continuous, the computer requires a discretization that treats the system as a series of jumps or through a stroboscope. An everyday example is the motion frames of a horse, as shown in figure 1.1.

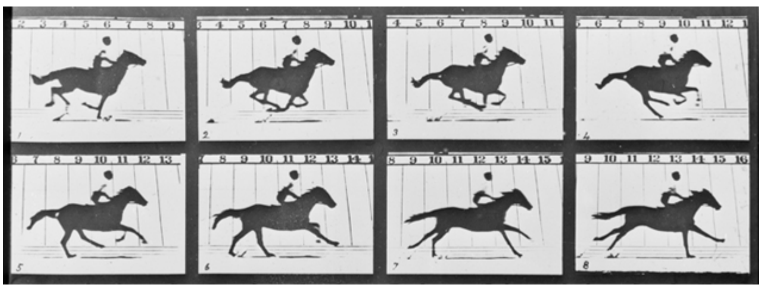


Figure 1.1: Horse in motion frames.

The mathematical analogues of these motion frames are approximations that have different mathematical properties. The optimal way to make these approximations is done via algorithms that preserve several geometric quantities. It turns out that those algorithms that respect a certain conversation law are actually more powerful than other algorithms that do not. Hence, we would need to find some exact stroboscopes that interpolate precisely.

In some cases the algorithms will eventually become uncontrolled if we would continue for a long enough time. Basically, in the non-integrable case the algorithms may look good in the initial steps, but they will become bad after a while. Therefore, an asymptotic series may offer a good approximation asymptotically in the first few terms, but it will diverge after a while. By contrast, this will never happen in the integrable case in which we are interested. There is an energy, called "shadow energy" in some contexts, that is conserved in the integrable case, but there are also cases where it is not conserved.

How to calculate this energy is what we consider in this thesis. We use technical results from the theory of discrete motion in the integrable systems in order to find an exact formula for this energy. Specifically, we study the energy that arises from discrete motion in the theory of water waves, see figure 1.2 for illustration of water waves.

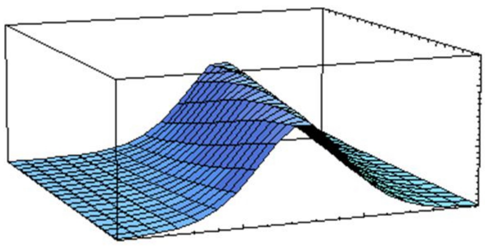


Figure 1.2: Water wave.

1.2 Classical mechanics

The dynamical mappings considered in this thesis are all canonical or symplectic. The maps are the discrete-time counterparts of finite-dimensional Hamiltonian systems. This class of mappings is introduced in more detail next.

1.2.1 Canonical transformation

The only phase space considered in this thesis is the oriented Euclidean space $\mathbb{R}^{2N}$.

Recall that the *Hamiltonian* H is a function of $2N$ canonical coordinates $\boldsymbol{q} = (q_1, \ldots, q_N)$ and $\boldsymbol{p} = (p_1, \ldots, p_N)$, that satisfies the $2N$ relations

$$\dot{p}_i := \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad \text{and} \quad \dot{q}_i := \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \,, \tag{1.1}$$

where $i = 1, \ldots, N$, which are known as Hamilton's equations. A coordinate transformation is canonical [69,75,114] if it preserves the form of Hamiltonians equations. Formally, let $\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{p})^T$ and

$$J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

be a $2N \times 2N$ block matrix, where I_N is the $N \times N$ identity matrix. Hamilton's equations can then be written as

$$\dot{\boldsymbol{x}} = J\nabla H, \quad \text{where} \quad \nabla H = (\partial/\partial\boldsymbol{q}, \partial/\partial\boldsymbol{p})^T H.$$

Let $\boldsymbol{x} \to \boldsymbol{y}(\boldsymbol{x})$, where $\boldsymbol{y} = (\widetilde{\boldsymbol{q}}, \widetilde{\boldsymbol{p}})^T$, be a transformation of coordinates, i.e.

$$\boldsymbol{q} \to \widetilde{\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{p}) \qquad \text{and} \qquad \boldsymbol{p} \to \widetilde{\boldsymbol{p}}(\boldsymbol{q}, \boldsymbol{p}) \,,$$

we have $\dot{\boldsymbol{y}} = (\mathfrak{J} J \mathfrak{J}^T)\nabla H$ where $\mathfrak{J}$ is the Jacobian matrix given by

$$\mathfrak{J}_{ij} = \frac{\partial y_i}{\partial x_j} = \begin{pmatrix} \partial\widetilde{q}_i/\partial q_j & \partial\widetilde{q}_i/\partial p_j \\ \partial\widetilde{p}_i/\partial q_j & \partial\widetilde{p}_i/\partial p_j \end{pmatrix} . \tag{1.2}$$

The form of Hamilton's equations is preserved if

$$\mathfrak{J} J \mathfrak{J}^T = J \,. \tag{1.3}$$

A transformation is canonical if its Jacobians satisfies (1.3).

Definition 1.2.1 ([52, 69]). Consider the canonical coordinates $\boldsymbol{q} = (q_1, \ldots, q_N)$ and momenta $\boldsymbol{p} = (p_1, \ldots, p_N)$.

(i) The Poisson bracket is defined as

$$\{F, G\} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) ,$$

where F and G are two arbitrary smooth functions defined on the phase space $(\boldsymbol{q}, \boldsymbol{p})$. Clearly, the Poisson brackets of the canonical coordinates and momenta are

$$\{q_i, q_j\} = 0 \,, \quad \{p_i, p_j\} = 0 \,, \quad \{q_i, p_j\} = \delta_{ij} \,, \tag{1.4}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. In addition, we have a symplectic structure given by the 2-form [8]

$$\omega(\boldsymbol{q}, \boldsymbol{p}) = \sum_{i=1}^{N} dq_i \wedge dp_i \,.$$

(ii) The Poisson bracket can be shown to satisfy the following properties

- bilinearity: $\{F, \alpha G + \beta H\} = \alpha\{F, G\} + \beta\{F, H\}$, where α and β do not depend on the coordinates of the phase space,
- skew-symmetry: $\{F, G\} = -\{G, F\}$,
- Leibniz rule: $\{F, GH\} = \{F, G\}H + G\{F, H\}$,
- Jacobi identity: $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$.

Note that any transformation sends $(\boldsymbol{q}, \boldsymbol{p})$ to $(\widetilde{\boldsymbol{q}}, \widetilde{\boldsymbol{p}})$ such that

$$\{\widetilde{q}_i, \widetilde{q}_j\} = 0 \,, \qquad \{\widetilde{p}_i, \widetilde{p}_j\} = 0 \,, \qquad \{\widetilde{q}_i, \widetilde{p}_j\} = \delta_{ij} \,,$$

is a canonical transformation [114].

1.2.2 Hamiltonian evolution

In the Hamiltonian formalism the time evolution is guaranteed to be canonical. Define the differential operator D_G by

$$D_G F := \{F, G\} \,.$$

This allows us to write the continuous-time Hamilton's equations of motion in the form [106],

$$\dot{\boldsymbol{p}} = D_H \boldsymbol{p} = \{\boldsymbol{p}, H\} = -\frac{\partial H}{\partial \boldsymbol{q}} \,, \qquad \dot{\boldsymbol{q}} = D_H \boldsymbol{q} = \{\boldsymbol{q}, H\} = \frac{\partial H}{\partial \boldsymbol{p}} \,, \tag{1.5}$$

where $H(\boldsymbol{q}, \boldsymbol{p}, t)$ is the Hamiltonian function. Introducing the notation

$$\boldsymbol{z} := (q_1, \ldots, q_N; p_1, \ldots, p_N)$$

of the canonical coordinates $\boldsymbol{q}$ and momenta $\boldsymbol{p}$ that is not explicitly a function of time t, we have

$$\dot{\boldsymbol{z}} = D_H \, \boldsymbol{z} = \{\boldsymbol{z}, H\}. \tag{1.6}$$

Thus, the integrated t-flow of the equations of motion, or the exact time evolution of $\boldsymbol{z}$, from $t = 0$ to $t = \tau$ can formally be written as [128],

$$\boldsymbol{z}(\boldsymbol{q}(\tau), \boldsymbol{p}(\tau)) = e^{\tau D_H} \boldsymbol{z}(\boldsymbol{q}(0), \boldsymbol{p}(0)) = \sum_{k=0}^{\infty} \frac{1}{k!} \, (\tau D_H)^k \, \boldsymbol{z}(\boldsymbol{q}, \boldsymbol{p}) \, .$$

Consequently, the solutions of Hamilton's equations are characterized by their symplecticity. Furthermore, if the Hamiltonian is not explicitly a function of t, then it is by the skew-symmetry property a constant of the evolution, i.e. $\dot{H} = \{H, H\} = 0$.

The field of symplectic integration is concerned with the preservation of the symplectic nature of the solutions of a Hamiltonian evolution under discretization for the purpose of numerical integration. The iteration of a canonical transformation is considered to be a discrete-time evolution when considering discrete-time systems. Relevant to this thesis is the discrete-time evolution introduced in the following proposition.

Proposition 1.2.1. *Consider the smooth functions $T(\boldsymbol{p})$ and $V(\boldsymbol{q})$, and the following discrete-time evolution*

$$\widetilde{\boldsymbol{p}} := e^{D_V} e^{D_T} \boldsymbol{p} \, , \qquad \widetilde{\boldsymbol{q}} := e^{D_V} e^{D_T} \boldsymbol{q} \, . \tag{1.7}$$

Then, the mapping $(\boldsymbol{q}, \boldsymbol{p}) \to (\widetilde{\boldsymbol{q}}, \widetilde{\boldsymbol{p}})$ is canonical (symplectic) and given by

$$\widetilde{\boldsymbol{p}} - \boldsymbol{p} = -\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}} = -\frac{\partial H(\boldsymbol{q}, \widetilde{\boldsymbol{p}})}{\partial \boldsymbol{q}} \, , \qquad \widetilde{\boldsymbol{q}} - \boldsymbol{q} = \frac{\partial T(\widetilde{\boldsymbol{p}})}{\partial \widetilde{\boldsymbol{p}}} = \frac{\partial H(\boldsymbol{q}, \widetilde{\boldsymbol{p}})}{\partial \widetilde{\boldsymbol{p}}} \, , \tag{1.8}$$

where

$$H(\boldsymbol{q}, \widetilde{\boldsymbol{p}}) = T(\widetilde{\boldsymbol{p}}) + V(\boldsymbol{q}) \, . \tag{1.9}$$

Proof

Consider the smooth function $\boldsymbol{z}(\boldsymbol{q},\boldsymbol{p})$. Then, we have

$$\begin{aligned}
e^{D_V}e^{D_T}\boldsymbol{z}(\boldsymbol{q},\boldsymbol{p}) &= e^{\{\cdot,V\}}e^{\{\cdot,T\}}\boldsymbol{z}(\boldsymbol{q},\boldsymbol{p}) = e^{-\frac{\partial V}{\partial \boldsymbol{q}}\frac{\partial}{\partial \boldsymbol{p}}}\, e^{\frac{\partial T}{\partial \boldsymbol{p}}\frac{\partial}{\partial \boldsymbol{q}}}\, \boldsymbol{z}(\boldsymbol{q},\boldsymbol{p}) \\
&= e^{-\frac{\partial V}{\partial \boldsymbol{q}}\frac{\partial}{\partial \boldsymbol{p}}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\partial T}{\partial \boldsymbol{p}}\frac{\partial}{\partial \boldsymbol{q}}\right)^k \boldsymbol{z}(\boldsymbol{q},\boldsymbol{p}) \\
&= e^{-\frac{\partial V}{\partial \boldsymbol{q}}\frac{\partial}{\partial \boldsymbol{p}}}\left(\boldsymbol{z}(\boldsymbol{q},\boldsymbol{p}) + \frac{\partial T}{\partial \boldsymbol{p}}\frac{\partial \boldsymbol{z}(\boldsymbol{q},\boldsymbol{p})}{\partial \boldsymbol{q}} + \frac{1}{2!}\left(\frac{\partial T}{\partial \boldsymbol{p}}\right)^2 \frac{\partial^2 \boldsymbol{z}(\boldsymbol{q},\boldsymbol{p})}{\partial \boldsymbol{q}^2} + \cdots\right) \\
&= e^{-\frac{\partial V}{\partial \boldsymbol{q}}\frac{\partial}{\partial \boldsymbol{p}}}\, \boldsymbol{z}\left(\boldsymbol{q} + \frac{\partial T}{\partial \boldsymbol{p}}, \boldsymbol{p}\right) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\partial V}{\partial \boldsymbol{q}}\frac{\partial}{\partial \boldsymbol{p}}\right)^k \boldsymbol{z}\left(\boldsymbol{q} + \frac{\partial T}{\partial \boldsymbol{p}}, \boldsymbol{p}\right) \\
&= \boldsymbol{z}\left(\boldsymbol{q} + \frac{\partial T}{\partial \boldsymbol{p}}, \boldsymbol{p}\right) - \frac{\partial V}{\partial \boldsymbol{q}}\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{p}} + \frac{1}{2!}\left(\frac{\partial V}{\partial \boldsymbol{q}}\right)^2 \frac{\partial^2 \boldsymbol{z}}{\partial \boldsymbol{p}^2} - \cdots \\
&= \boldsymbol{z}\left(\boldsymbol{q} + \frac{\partial T}{\partial \widetilde{\boldsymbol{p}}}, \boldsymbol{p} - \frac{\partial V}{\partial \boldsymbol{q}}\right) = \boldsymbol{z}\left(\widetilde{\boldsymbol{q}}, \widetilde{\boldsymbol{p}}\right).
\end{aligned}$$

Thus, the mapping (1.8) is generated by the evolution (1.7). The symplecticity property can easily be proved by the following steps. First, we have

$$\begin{aligned}
d\widetilde{\boldsymbol{q}} \wedge d\widetilde{\boldsymbol{p}} = d\left(\frac{\partial H}{\partial \widetilde{\boldsymbol{p}}} + \boldsymbol{q}\right) \wedge d\widetilde{\boldsymbol{p}} &= \left(\left(1 + \frac{\partial^2 H}{\partial \widetilde{\boldsymbol{p}}\,\partial \boldsymbol{q}}\right) d\boldsymbol{q} + \frac{\partial^2 H}{\partial \widetilde{\boldsymbol{p}}^2} d\widetilde{\boldsymbol{p}}\right) \wedge d\widetilde{\boldsymbol{p}} \\
&= \left(1 + \frac{\partial^2 H}{\partial \widetilde{\boldsymbol{p}}\,\partial \boldsymbol{q}}\right) d\boldsymbol{q} \wedge d\widetilde{\boldsymbol{p}}\,.
\end{aligned}$$

Second, we have

$$\begin{aligned}
d\boldsymbol{q} \wedge d\boldsymbol{p} = d\boldsymbol{q} \wedge d\left(\widetilde{\boldsymbol{p}} + \frac{\partial H}{\partial \boldsymbol{q}}\right) &= d\boldsymbol{q} \wedge \left(\left(1 + \frac{\partial^2 H}{\partial \widetilde{\boldsymbol{p}}\,\partial \boldsymbol{q}}\right) d\widetilde{\boldsymbol{p}} + \frac{\partial^2 H}{\partial \boldsymbol{q}^2} d\boldsymbol{q}\right) \\
&= \left(1 + \frac{\partial^2 H}{\partial \widetilde{\boldsymbol{p}}\,\partial \boldsymbol{q}}\right) d\boldsymbol{q} \wedge d\widetilde{\boldsymbol{p}}\,.
\end{aligned}$$

One obtains $d\widetilde{\boldsymbol{q}} \wedge d\widetilde{\boldsymbol{p}} = d\boldsymbol{q} \wedge d\boldsymbol{p}$. □

It must be stressed that the discrete Hamiltonian given in (1.9) is not a Hamiltonian function as such, but a generating function for the canonical transformation formation (1.8), which is a different function than the Hamiltonian given above in the continuous-time setting in equations (1.5) and (1.6) [8,106]. Equations (1.8) are called discrete-time Hamilton equations. Note that the characteristic property of the conservation of the Hamiltonian, not depending explicitly on time, does not hold in this discrete-time case.

1.2.3 Generating functions

In some contexts [47], canonical transformations are usually considered to arise from a generating function. In fact, any canonical transformation implies the existence of a corresponding generating function. Generating functions act as a bridge between two sets of canonical coordinates. There is a simple method to construct canonical transformations between coordinates $(\boldsymbol{q}, \boldsymbol{p})$ and $(\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{P}})$, cf. [114].

Consider a function $F(\boldsymbol{q}, \boldsymbol{\mathcal{Q}})$ of the original $\boldsymbol{q}$ and the final $\boldsymbol{\mathcal{Q}}$. Let

$$\boldsymbol{p} = \frac{\partial F}{\partial \boldsymbol{q}} \,. \tag{1.10}$$

After inverting, this equation can be thought of as defining the new coordinates $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{Q}}(\boldsymbol{q}, \boldsymbol{p})$, where the new canonical momenta $\boldsymbol{\mathcal{P}}$ are given by

$$\boldsymbol{\mathcal{P}} = -\frac{\partial F}{\partial \boldsymbol{\mathcal{Q}}} \,. \tag{1.11}$$

We can see how it works with just a single-degree-of-freedom (it generalizes trivially to the case of several degrees of freedom). We can look at the Poisson bracket

$$\{\mathcal{Q}, \mathcal{P}\} = \frac{\partial \mathcal{Q}(q,p)}{\partial q}\frac{\partial \mathcal{P}(q,p)}{\partial p} - \frac{\partial \mathcal{Q}(q,p)}{\partial p}\frac{\partial \mathcal{P}(q,p)}{\partial q} \,.$$

At this point we need to play with partial derivatives. Equation (1.11) defines $\mathcal{P} = \mathcal{P}(q, \mathcal{Q})$, so we have

$$\frac{\partial \mathcal{P}(q,p)}{\partial p} = \frac{\partial \mathcal{Q}(q,p)}{\partial p}\frac{\partial \mathcal{P}(q,\mathcal{Q})}{\partial \mathcal{Q}} \quad \text{and} \quad \frac{\partial \mathcal{P}(q,p)}{\partial q} = \frac{\partial \mathcal{P}(q,\mathcal{Q})}{\partial q} + \frac{\partial \mathcal{Q}(q,p)}{\partial q}\frac{\partial \mathcal{P}(q,\mathcal{Q})}{\partial \mathcal{Q}} \,.$$

Inserting this into the Poisson bracket gives

$$\{\mathcal{Q}, \mathcal{P}\} = -\frac{\partial \mathcal{Q}(q,p)}{\partial p}\frac{\partial \mathcal{P}(q,\mathcal{Q})}{\partial q} = \frac{\partial \mathcal{Q}(q,p)}{\partial p}\frac{\partial^2 F}{\partial q \partial \mathcal{Q}} = \frac{\partial \mathcal{Q}(q,p)}{\partial p}\frac{\partial p(q,\mathcal{Q})}{\partial \mathcal{Q}} = 1 \,.$$

The function $F(q, \mathcal{Q})$ is known as a generating function of the first kind and is denoted by F_1 [114].

There are actually three further types of generating functions related to the first by Legendre transforms such that each is a function of one of the original coordinates and one of the new coordinates. Furthermore, we can have the following generating functions and can check that all define canonical transformations [8, 47, 114]:

$$F_2(q,\mathcal{P}): \quad p = \frac{\partial F_2}{\partial q} \quad \text{and} \quad \mathcal{Q} = \frac{\partial F_2}{\partial \mathcal{P}}\,, \tag{1.12a}$$

$$F_3(p,\mathcal{Q}): \quad q = -\frac{\partial F_3}{\partial p} \quad \text{and} \quad \mathcal{P} = -\frac{\partial F_3}{\partial \mathcal{Q}}\,, \tag{1.12b}$$

$$F_4(p,\mathcal{P}): \quad q = -\frac{\partial F_4}{\partial p} \quad \text{and} \quad \mathcal{Q} = \frac{\partial F_4}{\partial \mathcal{P}}\,. \tag{1.12c}$$

So, the generating function can be written as a function of independent variables in one of only these four forms. In this thesis, two types of the generating functions above are considered. The generating function F_1 is denoted by F, and the generating function F_2 is denoted by S.

1.2.4 Integrability and action-angle variables

In classical mechanics the question of integrability for a dynamical system with finite degrees of freedom is well understood. The key concept in the language of integrability is that of an integral of motion. A first integral of motion for a Hamiltonian system is a smooth function, say I, defined on a phase space such that $\dot{I} = 0$ under the flow implied by Hamilton's equations, in other words, I is a constant of the motion. Equivalently, with Hamiltonian H the first integral of motion I satisfies $\{I, H\} = 0$. Note that the Hamiltonian itself is a first integral. For a dynamical system with N degrees of freedom defined in terms of coordinates $\boldsymbol{q}$, conjugate momenta $\boldsymbol{p}$ and a Hamiltonian $H(\boldsymbol{q},\boldsymbol{p})$, if there exist N integrals of motion $I_i(\boldsymbol{q},\boldsymbol{p})$ such that the I_i

(i) are independent functions and do not depend explicitly on time ,

(ii) are in involution with respect to the Poisson structure, i.e. $\{I_i, I_j\} = 0$,

then the system is said to be *completely integrable (Liouville integrable)* [125] and can be integrated by quadratures (i.e. given in terms of integrals).

In classical mechanics, Hamilton-Jacobi theory provides a mechanism by which problems may be solved using coordinate transformations [8, 47]. One transformation which has historically proven useful is the transformation from a set of coordinates and momenta, $\boldsymbol{q}$

and $\boldsymbol{p}$ in H, to the action-angle variables, $\mathfrak{Q}$ and $\mathfrak{P}$. The action-angle variables constitute a system of coordinates and momenta, $\mathfrak{Q}$ and $\mathfrak{P}$, in which the new Hamiltonian K is a function only of the momenta $\mathfrak{P}$; it is independent of $\mathfrak{Q}$. Since the transformation is canonical, Hamilton's equations of motion are preserved.

If the action-angle variables exist, a canonical transformation may be carried out using a F_2 generating function, cf. (1.12a), which can be used to generate the action-angle variables. F_2 is a function of the old coordinates $\boldsymbol{q}$ and the new momenta $\mathfrak{P}$, which gives the old momenta $\boldsymbol{p}$ and new coordinates $\mathfrak{Q}$ as:

$$\boldsymbol{p} = \frac{\partial F_2(\boldsymbol{q}, \mathfrak{P})}{\partial \boldsymbol{q}} , \qquad \mathfrak{Q} = \frac{\partial F_2(\boldsymbol{q}, \mathfrak{P})}{\partial \mathfrak{P}} \tag{1.13}$$

with

$$K = H + \frac{\partial F_2}{\partial t} \tag{1.14}$$

the transformed Hamiltonian [47]. The new momenta $\mathfrak{P}$ are called *actions* and the new coordinates $\mathfrak{Q}$ are called *angles*. From equation (1.13) it is possible to find one set of coordinates in terms of the other [93]:

$$\boldsymbol{q} = \phi(\mathfrak{Q}, \mathfrak{P}) , \qquad \boldsymbol{p} = \varphi(\mathfrak{Q}, \mathfrak{P}) , \tag{1.15a}$$

$$\mathfrak{Q} = \chi(\boldsymbol{q}, \boldsymbol{p}) , \qquad \mathfrak{P} = \psi(\boldsymbol{q}, \boldsymbol{p}) . \tag{1.15b}$$

Equations (1.13) or (1.15) are the equations of the canonical transformation.

One valuable feature of action-angle variables lies in the fact that the transformed Hamiltonian $K(\mathfrak{P})$ depends on $\mathfrak{P}$ only:

$$E = H(\boldsymbol{q}, \boldsymbol{p}) \quad \Rightarrow \quad E = K(\mathfrak{P}) ,$$

where E is the energy. Thus, since the transformation $(\boldsymbol{q}, \boldsymbol{p}) \to (\mathfrak{Q}, \mathfrak{P})$ carried out using a generating function F_2 is canonical, Hamilton's equations of motion hold, and we have [47, 113]:

$$\dot{\mathfrak{P}} = -\frac{\partial K}{\partial \mathfrak{Q}} = 0 , \qquad \dot{\mathfrak{Q}} = \frac{\partial K}{\partial \mathfrak{P}} = \boldsymbol{\nu}(\mathfrak{P}) , \tag{1.16}$$

where $\boldsymbol{\nu}$ are the frequencies of oscillation of the systems. Observe that in (1.16) the first equation tells us that $\mathfrak{P}$ are the integrals of motion whereas the second equation tells us that K is obtained by integrating $\boldsymbol{\nu}$ with respect to $\mathfrak{P}$ up to an arbitrary function of the invariant. Hence, in the sense of the Liouville-integrability introduced above, the integrable systems are solvable by quadratures.

Example:

As this technique is used throughout this thesis, let us now clarify the idea of action-angle variables by considering the example of the nonlinear simple pendulum [8, 47, 69],

$$H = \frac{1}{2}p^2 + V(q) \,, \qquad \text{where} \qquad V(q) = \nu_0^2 \cos q \,. \tag{1.17}$$

The red line, as shown in figure 1.3, divides the phase space into three regions. If the phase space is bounded, as in region 2 for the pendulum, generalized coordinate can be taken as angle and the conjugate momentum is called action. The idea of action-angle variables in bounded regions is to find new canonical coordinates $(\mathcal{Q}, \mathcal{P})$ for which each phase curve is labeled uniquely by $\mathcal{P}$, which is constant along a phase curve, and each point on a phase curve is labeled by a single-valued function $\mathcal{Q}$.

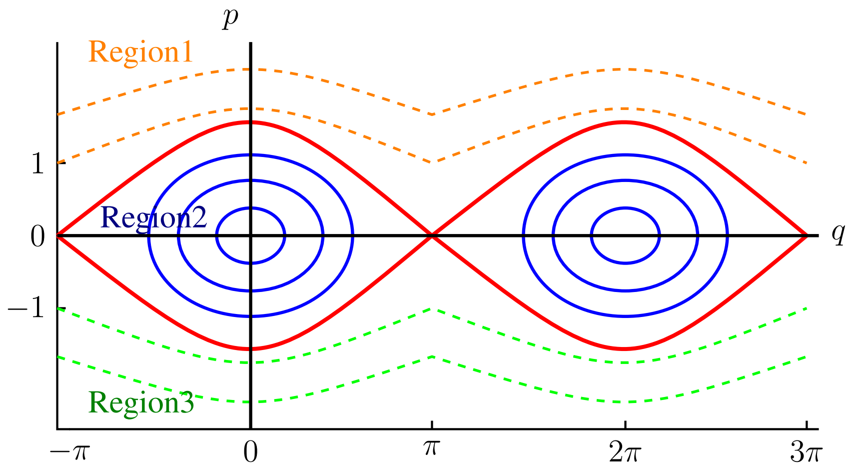


Figure 1.3: Sketch of level sets of $H(q, p)$ for the simple pendulum. Orbits follow these level sets.

Since the action-angle variables exist, the canonical transformation $(q, p) \to (\mathcal{Q}, \mathcal{P})$ can be given by means of a generating function F_2, so that it is a function of the old position q and the new momentum $\mathcal{P}$, it is usually denoted as $S(q, \mathcal{P})$. This function gives the old momentum p and new coordinate $\mathcal{Q}$ as

$$p = \frac{\partial S(q, \mathcal{P})}{\partial q} \,, \qquad \mathcal{Q} = \frac{\partial S(q, \mathcal{P})}{\partial \mathcal{P}} \,, \tag{1.18}$$

with the transformed Hamiltonian $K = H + \partial S/\partial t$. Thus, we require new canonical variables $(\mathcal{Q}, \mathcal{P})$ such that

$$\dot{\mathcal{P}} = -\frac{\partial K}{\partial \mathcal{Q}} = 0 \qquad \text{and} \qquad \dot{\mathcal{Q}} = \frac{\partial K}{\partial \mathcal{P}} = \nu(\mathcal{P}) = \text{constant} \equiv C \ ,$$

which implies that K only depends on $\mathcal{P}$. This means that $\mathcal{Q}(t) = C\,t + \mathcal{Q}(0)$, and the function K is obtained by integrating ν with respect to $\mathcal{P}$ up to an arbitrary function of the invariant.

Let us find the action $\mathcal{P}$, which is the constant of the motion. To find $\mathcal{P}$ we use the fact that canonical transformation preserves phase space area, i.e. the Jacobian of the transformation is one, and also the reflection symmetry of the Hamiltonian. Thus, we have

$$2\pi\,\mathcal{P} = 2\int_{q_1}^{q_2} p\,dq = 2\int_{q_1}^{q_2} \sqrt{2\,(H - V(q))}\,dq \ , \tag{1.19}$$

where $q_1 = -\arccos(H/\nu_0^2)$ and $q_2 = \arccos(H/\nu_0^2)$ are the intercept of the orbit with q-axis and we have used the reflection symmetry. Hence, we obtain

$$\mathcal{P} = \frac{1}{\pi}\int_{q_1}^{q_2} \sqrt{2\,(H - V(q))}\,dq \ , \tag{1.20}$$

which defines $\mathcal{P}(H)$. Now, by integrating the first system of equations (1.18), we get the generating function S up to an arbitrary function of the invariant

$$S(q, \mathcal{P}) = \int_{q^0}^{q} \sqrt{2\,(H(\mathcal{P}) - V(q'))}\,dq' \ . \tag{1.21}$$

Consequently, the new angle variable is obtained from

$$\mathcal{Q} = \frac{\partial S}{\partial \mathcal{P}} = \int_{q^0}^{q} \frac{1}{\sqrt{2\,(H(\mathcal{P}) - V(q'))}}\,dq' \ . \tag{1.22}$$

1.3 Discrete integrable systems

Discrete integrable systems, namely those with their independent variables taking discrete values, have many applications in several fields in mathematics and physics, such as special function theory, algebraic geometry, classical mechanics, quantum field theory and geometric numerical integration in numerical analysis (for the purpose of this thesis, we shall focus more closely on the applications of the numerical analysis field especially on modified Hamiltonians). Since discrete systems are essential in numerical analysis, they are also a rapidly growing field of computer science [48].

Discrete systems and their integrability received a lot of attention in late 1970s. Examples of discrete systems were studied by Hirota in a series of papers [58–61]. In particular, he derived discrete analogues of many already famous partial differential equations (PDEs), such as the KdV, modified KdV and sine-Gordon (sG) equations. The KdV equation was the first PDE shown to possess what is now known as a Lax pair. This class of equation is introduced in more detail in the next section.

In the early 1980s, Capel, Nijhoff, Quispel et al. provided some of the first systematic tools for direct construction of integrable lattice equations [85, 90, 98]. This was a starting point for new systems of discrete equations to appear in the literature. In the early 1990s, Grammaticos, Papageorgiou and Ramani proposed the first test for integrability in the discrete case, known as *singularity confinement* [50], as a proper candidate for a discrete analogue of the Painlevé property. However, as mentioned in [57], singularity confinement is only a necessary but not sufficient criterion for predicting integrability. Another important integrability test involves 3-*dimensional consistency* (or *consistency-around-the-cube* (CAC)) and, by extension, *multi-dimensional consistency* [56]. This was proposed as a feature of integrable partial difference equations (P∆Es) independently by Nijhoff [91] and Bobenko and Suris [14]. As a consequence of this property, one can construct the Lax pairs of the discrete system.

1.3.1 Discrete equations from Bäcklund transformations

This section is devoted to show how to derive equations on the discrete-time (difference equations) from the corresponding equations on the continuous-time using the Bäcklund transformations. Specifically, we discuss the lattice KdV equation, which is the space-time discretization of the partial differential equation

$$u_t = u_{xxx} + 6\,u\,u_x\,; \qquad u(x,0) = u_0(x)\,, \tag{1.23}$$

where $u = u(x,t)$ is an appropriate field variable, $x \in \mathbb{R}$ is a space coordinate in the direction of propagation and $t \in \mathbb{R}^+$ is time. This equation has multiple applications of physical significance, but was first derived in 1895 by Korteweg and de Vries [66] in a study of shallow water waves. The KdV equation (1.23) was a key milestone in a big contraversy on the nature of waves, not least following the famous "real life" observation of a solitary wave by John Scott Russell in 1834, see e.g. [3, 77]. The numerical solutions of the KdV equation are obtained by using centred finite-differences in the spatial coordinate x and a fourth-order Runge-Kutta method for the temporal coordinate t, see e.g. [71]. This method was chosen over straight finite-difference methods, such as the finite-difference scheme of Zabusky & Kruskal [129], because of its stability.

The KdV equation is particularly notable as the prototypical example of an exactly solvable model, i.e. a nonlinear partial differential equation whose solutions can be exactly and precisely specified. It can be solved by means of a method known as the *inverse scattering transform* [3, 9]. In particular, inverse scattering allows the solution of the initial value problem for the KdV equation when $u_0(x)$ is suitably bounded as $x \to \pm\infty$, see e.g. [70]. In [44] from 1967, Gardner, Greene, Kruskal and Miura, by the method of inverse scattering transform, showed that the initial value problem for the KdV equation (1.23) can be solved by considering an associated linear problem

$$\psi_{xx} + u\,\psi = \lambda\,\psi\,, \tag{1.24a}$$

$$\psi_t = 4\,\psi_{xxx} + 6\,u\,\psi_x + 3\,u_x\,\psi\,, \tag{1.24b}$$

where $\psi = \psi(x,t;\lambda)$ is a scalar function of x and t. The consistency condition of the system (1.24) leads to the KdV equation [56]. The first equation (1.24a) has the form of a linear spectral problem

$$\mathcal{L}\,\psi = \lambda\,\psi\,, \quad \text{where} \quad \mathcal{L} := \partial_x^2 + u \tag{1.25a}$$

is called the Schrödinger differential operator. The parameter λ is an eigenvalue of the operator $\mathcal{L}$, which depends on t if u depends on it; however, it is independent of x. The second equation (1.24b) (deformation equation) can be written as

$$\psi_t = \mathcal{A}\,\psi\,; \qquad \mathcal{A} := 4\,\partial_x^3 + 6\,u\,\partial_x + 3\,u_x\,, \tag{1.25b}$$

which describes the time evolution of the eigenfunction $\psi(x,t)$. By computing the compatibility condition $(\psi_{xx})_t = (\psi_t)_{xx}$, where we assume ψ is not identically zero, the linear system (1.24) is self-consistent, i.e.

$$(\psi_{xx})_t = (\psi_t)_{xx} \quad \text{if and only if} \quad \lambda_t - u_t + u_{xxx} + 6\,u\,u_x = 0\,.$$

Hence, under the condition $\lambda_t = 0$, the system (1.24) is compatible, provided u satisfies the KdV equation. Such a system of equations (1.25) is called a *Lax pair*, after Peter Lax who provided a method for describing such linear problems [68].

Miura and Bäcklund transformations

The KdV equation (1.23) possesses an important transformation called the *Miura transformation* [73]. This transformation gives rise to many insights about the solutions of KdV equation. To find the Miura transformation for the KdV, consider equation (1.24a) written as

$$u = \lambda - \frac{\psi_{xx}}{\psi}, \quad (\psi \neq 0)\ . \tag{1.26}$$

Inserting this into (1.24b), one obtains

$$\psi_t = \psi_{xxx} - \frac{3\,\psi_x\,\psi_{xx}}{\psi} + 6\,\lambda\,\psi_x\ . \tag{1.27}$$

By now introducing the variable $v := \partial_x \log \psi$, one obtains from (1.26) the equation

$$u = \lambda - v_x - v^2\ , \tag{1.28}$$

which is the Miura transformation. As a consequence of equation (1.27), we can derive a PDE in the variable v,

$$v_t = v_{xxx} - 6\,v^2\,v_x + 6\,\lambda\,v_x\ , \tag{1.29}$$

which, for $\lambda = 0$, is known as the *modified KdV* (mKdV) equation.

Another important type of transformation, which derived from the Miura transformation, is a *Bäcklund transformation (BT)* [56, 84]. To derive the BT, consider the Miura transformation for both possible signs $\pm v$, namely one sign in transforming from u to v and another from v to $\widetilde{u}$, i.e.

$$\widetilde{u} = \lambda + v_x - v^2 \quad \text{and} \quad u = \lambda - v_x - v^2\ . \tag{1.30}$$

Adding and subtracting the two relations above, one obtains

$$\widetilde{u} + u = 2\,(\lambda - v^2)\ , \tag{1.31a}$$

$$\widetilde{u} - u = 2\,v_x\ . \tag{1.31b}$$

If we introduce the variable w by taking $u = w_x$, the KdV equation then leads to

$$w_t = w_{xxx} + 3\,w_x^2 \tag{1.32}$$

after one integration in x. This equation is called the *potential KdV* (pKdV) equation. In terms of the new dependent variable w, the equation (1.31b) can be integrated to $\widetilde{w} - w = 2\,v$. By inserting this into equation (1.31a), one obtains

$$\left(\widetilde{w} + w\right)_x = 2\,\lambda - \frac{1}{2}\left(\widetilde{w} - w\right)^2 . \tag{1.33a}$$

This equation provides us with the x-dependent part of BT, and to find the t-dependent part we use the pKdV equation (1.32). Adding equation (1.32) for w and $\widetilde{w}$ and using equation (1.33a) to reduce $w_{xxx} + \widetilde{w}_{xxx}$, one obtains

$$\left(\widetilde{w} + w\right)_t = \left(\widetilde{w} - w\right)\left(w_{xx} - \widetilde{w}_{xx}\right) + 2\left(w_x^2 + w_x\,\widetilde{w}_x + \widetilde{w}_x^2\right) . \tag{1.33b}$$

The system of equations (1.33) together constitute the Bäcklund transformation for the KdV equation. Further, we have the following proposition:

Proposition 1.3.1. *The system of equations (1.33) defines a transformation from a given solution $w(x,t)$ of the pKdV equation (1.32) to a new solution $\widetilde{w}(x,t)$ of the same equation.*

Proof

Consider the consistency condition $\partial_t\left(\widetilde{w} + w\right)_x = \partial_x\left(\widetilde{w} + w\right)_t$. From the left hand side we have

$$\partial_t\left(\widetilde{w} + w\right)_x = \partial_t\left[2\,\lambda - \frac{1}{2}\left(\widetilde{w} - w\right)^2\right] = -(\widetilde{w} - w)(\widetilde{w} - w)_t\,. \tag{1.34}$$

From the right hand side we have

$$\begin{aligned}\partial_x\left(\widetilde{w} + w\right)_t &= \left(\widetilde{w}_x - w_x\right)\left(w_{xx} - \widetilde{w}_{xx}\right) + \left(\widetilde{w} - w\right)\left(w_{xxx} - \widetilde{w}_{xxx}\right)\\ &\quad + 2\left(2\,w_x\,w_{xx} + w_{xx}\,\widetilde{w}_x + w_x\,\widetilde{w}_{xx} + 2\,\widetilde{w}_x\,\widetilde{w}_{xx}\right)\\ &= \left(\widetilde{w} - w\right)\left(w_{xxx} - \widetilde{w}_{xxx}\right) + 3\left(\widetilde{w}_x + w_x\right)\left(w_{xx} + \widetilde{w}_{xx}\right) . \end{aligned}\tag{1.35}$$

By differentiating the x-dependent part (1.33a) with respect to x, i.e.

$$\left(\widetilde{w} + w\right)_{xx} = -\left(\widetilde{w} - w\right)\left(\widetilde{w} - w\right)_x \ ,$$

and subsequently inserting the latter equation into equation (1.35) we obtain

$$\begin{aligned}\partial_x\left(\widetilde{w}+w\right)_t &= \left(\widetilde{w}-w\right)\left(w_{xxx}-\widetilde{w}_{xxx}\right)-3\left(\widetilde{w}_x+w_x\right)\left(\widetilde{w}-w\right)\left(\widetilde{w}_x-w_x\right)\\ &= \left(\widetilde{w}-w\right)\left[\left(w_{xxx}+3\,w_x^2\right)-\left(\widetilde{w}_{xxx}+3\,\widetilde{w}_x^2\right)\right]. \end{aligned} \tag{1.36}$$

Thus, from the consistency condition we obtain

$$\begin{aligned} & -\left(\widetilde{w}-w\right)\left(\widetilde{w}-w\right)_t = \left(\widetilde{w}-w\right)\left[\left(w_{xxx}+3\,w_x^2\right)-\left(\widetilde{w}_{xxx}+3\,\widetilde{w}_x^2\right)\right].\\ \Rightarrow\quad & \left(\widetilde{w}-w\right)\left(\widetilde{w}_t-w_t+w_{xxx}+3\,w_x^2-\widetilde{w}_{xxx}-3\,\widetilde{w}_x^2\right)=0\\ \Rightarrow\quad & \widetilde{w}=w \quad \text{or} \quad w_t=w_{xxx}+3\,w_x^2 \quad\Rightarrow\quad \widetilde{w}_t=\widetilde{w}_{xxx}+3\,\widetilde{w}_x^2\,. \end{aligned}$$

Hence, if w solves the pKdV equation, then either $\widetilde{w}=w$ or $\widetilde{w}$ obeys the pKdV equation. □

Given a seed solution w of the pKdV equation, then inserting this into (1.33a) we obtain a first-order nonlinear ordinary differential equation for $\widetilde{w}$. This equation is of the form:

$$\widetilde{w}_x = -\frac{1}{2}\,\widetilde{w}^2 + a(x)\,\widetilde{w} + b(x)\,,$$

where the right-hand side is a quadratic in $\widetilde{w}$ and in which $a(x)$ and $b(x)$ can be arbitrary functions of x. This is a well-known type of differential equation called a *Riccati equation*, cf. [103], which is in general solvable through a linearization procedure. After solving this equation we have some integration constants that may depend on t. These constants can be determined from equation (1.33b).

By considering the $BT: v \mapsto \widetilde{v}$ given by the set of equations

$$\left(\widetilde{v}\,v\right)_x = 2\sqrt{\lambda}\left(\widetilde{v}^2-v^2\right), \tag{1.37a}$$

$$\left(\widetilde{v}\,v\right)_t = 4\sqrt{\lambda}\left(\widetilde{v}\,\widetilde{v}_{xx}-v\,v_{xx}-2\,\widetilde{v}_x^2+2\,v_x^2\right), \tag{1.37b}$$

we can also show, in exactly the same manner as that of proposition 1.3.1, that if v is a solution of the potential mKdV equation

$$v_t = v_{xxx} - 3\,\frac{v_x\,v_{xx}}{v}\,, \tag{1.38}$$

then $\widetilde{v}$ is also a solution of the same equation [56].

Permutability property of BTs

A very important property of BTs is the *permutability property*, which obtained by the iteration of BTs. Suppose BT_λ and BT_μ are two different transformations with parameters λ and μ respectively

$$BT_\lambda : w \overset{\lambda}{\longmapsto} \widetilde{w} \qquad (\widetilde{w} + w)_x = 2\,\lambda - \frac{1}{2}\,(\widetilde{w} - w)^2 \,, \tag{1.39a}$$

$$BT_\mu : w \overset{\mu}{\longmapsto} \widehat{w} \qquad (\widehat{w} + w)_x = 2\,\mu - \frac{1}{2}\,(\widehat{w} - w)^2 \,. \tag{1.39b}$$

There is in fact a choice of composing these BTs: either start with BT_λ and subsequently apply BT_μ, or the other way around. In this way we get iterated solutions, respectively,

$$\widehat{\widetilde{w}} = BT_\mu \circ BT_\lambda\, w \quad \text{and} \quad \widetilde{\widehat{w}} = BT_\lambda \circ BT_\mu\, w \,.$$

Both ways of composing the BTs lead to the same result, i.e. $\widehat{\widetilde{w}} = \widetilde{\widehat{w}}$. This leads to the famous permutability property of BTs derived in 1973 by Wahlquist and Estabrook [122]:

Theorem 1.3.1 ([56, p. 58]). *The BTs given by (1.39) for different parameters λ and μ commute, and hence we have the following commutation diagram of BTs:*

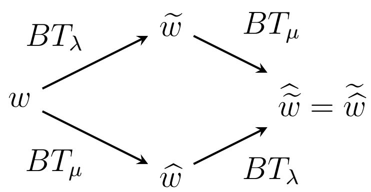


Figure 1.4: Permutability property of BTs.

In fact, performing the iterated BTs, we are led, in addition to (1.39), to the relations:

$$BT_\mu : \widetilde{w} \overset{\mu}{\longmapsto} \widehat{\widetilde{w}} \qquad (\widehat{\widetilde{w}} + \widetilde{w})_x = 2\,\mu - \frac{1}{2}\,(\widehat{\widetilde{w}} - \widetilde{w})^2 \,, \tag{1.40a}$$

$$BT_\lambda : \widehat{w} \overset{\lambda}{\longmapsto} \widetilde{\widehat{w}} \qquad (\widetilde{\widehat{w}} + \widehat{w})_x = 2\,\lambda - \frac{1}{2}\,(\widetilde{\widehat{w}} - \widehat{w})^2 \,. \tag{1.40b}$$

Assuming $\widehat{\widetilde{w}} = \widetilde{\widehat{w}}$ and eliminating all x-derivatives from equations (1.39) and (1.40), we obtain

$$(\widetilde{w} - \widehat{w})(\widehat{\widetilde{w}} - w) = 4\,(\lambda - \mu) \,. \tag{1.41}$$

Note that once we know w, $\widetilde{w}$ and $\widehat{w}$, we can solve $\widehat{\widetilde{w}}$ from equation (1.41).

In a similar way as before, letting $v \mapsto \widehat{v}$ denotes a similar BT with parameter μ instead of λ, for which we have

$$(\widehat{v}\,v)_x = 2\sqrt{\mu}\left(\widehat{v}^2 - v^2\right), \tag{1.42a}$$

$$(\widehat{v}\,v)_t = 4\sqrt{\mu}\left(\widehat{v}\,\widehat{v}_{xx} - v\,v_{xx} - 2\,\widehat{v}_x^2 + 2\,v_x^2\right), \tag{1.42b}$$

we can eliminating all x-derivatives from equations (1.37) and (1.42) using the permutability property of BTs. Hence, we obtain

$$\sqrt{\lambda}\,v\,\widehat{v} + \sqrt{\mu}\,\widehat{v}\,\widehat{\widetilde{v}} = \sqrt{\mu}\,v\,\widetilde{v} + \sqrt{\lambda}\,\widetilde{v}\,\widehat{\widetilde{v}}\,. \tag{1.43}$$

Transition to lattice equations

It is clear that by iterating the BTs with two different parameters we create from one seed solution w a lattice of solutions, as shown in figure 1.5.

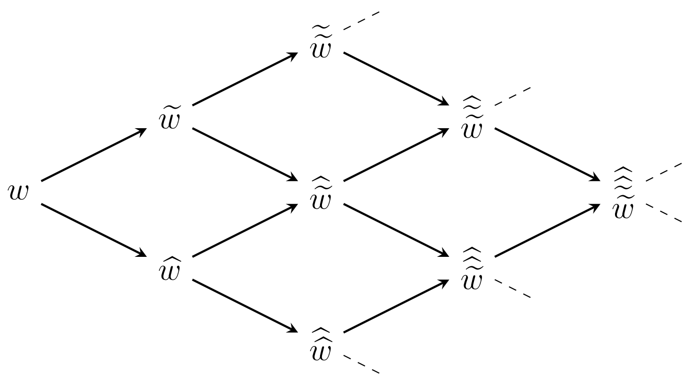


Figure 1.5: Lattice of BTs.

We can now introduce an enumeration of the solutions $w_{n,m} = BT_\lambda^n \circ BT_\mu^m\, w$ where BT_λ and BT_μ commute. Thus, we can rewrite the permutability condition (1.41) as

$$(w_{n+1,m} - w_{n,m+1})(w_{n+1,m+1} - w_{n,m}) = a^2 - b^2\,, \tag{1.44}$$

in which, for later convenience, we have identified the lattice parameters $\lambda := \frac{1}{4}\,a^2$ and $\mu := \frac{1}{4}\,b^2$. The shifts along the lattice $w_{n,m} \mapsto w_{n+1,m}$ and $w_{n,m} \mapsto w_{n,m+1}$ corresponding to the application of the BTs BT_λ and BT_μ. Similarly, we can rewrite the permutability condition (1.43) as a difference equation of the form

$$a\, v_{n,m}\, v_{n,m+1} + b\, v_{n,m+1}\, v_{n+1,m+1} = b\, v_{n,m}\, v_{n+1,m} + a\, v_{n+1,m}\, v_{n+1,m+1}\ . \qquad (1.45)$$

Equations (1.44) and (1.45) are, respectively, discrete versions of the potential KdV (1.32) and mKdV (1.38) equations, noting that initial value problems for PΔEs are discussed in section 1.3.2. We note that we will be concerned with equations (1.44) and (1.45) in chapter 2.

It is not hard to derive the discrete analogue of the KdV equation, in contrast to the discrete pKdV equation (1.44), cf. [56]. To do this, recall that the difference between the continuous KdV equation (1.23) and its potential version (1.32) was simply that its solutions were connected through taking a derivative $u = w_x$. However, on the lattice there are many ways in which can do the analogue of "taking the x-derivative", for instance we can replace it by taking a difference, such as $\Delta_n w_{n,m} \equiv w_{n+1,m} - w_{n,m}$, but this is not the only choice. Alternatively, we can take a difference between vertices farther away or across diagonals, such as:

$$\Delta w_{n,m} = w_{n+1,m+1} - w_{n,m} \quad \text{or} \quad \Delta w_{n,m} = w_{n,m+1} - w_{n+1,m}\ ,$$

or a host of other choices. In the case of the lattice pKdV equation (1.44) the natural choice of the discrete analogue of the derivative seems to be either one of the choices given above, namely a difference across the diagonal. This suggest that as lattice KdV variables we would take either one of the two choices:

$$Q = w_{n,m} - w_{n+1,m+1} \quad \text{or} \quad R = w_{n,m+1} - w_{n+1,m}\ .$$

It is then straightforward from (1.44) to have $Q\,R = a^2 - b^2$. Obviously, we also have

$$Q_{n,m+1} - Q_{n+1,m} = R_{n,m} - R_{n+1,m+1}\ . \qquad (1.46)$$

Hence, we obtain the following equation for Q (we can also write the equation for R),

$$Q_{n,m+1} - Q_{n+1,m} = (a^2 - b^2)\left(\frac{1}{Q_{n,m}} - \frac{1}{Q_{n+1,m+1}}\right), \qquad (1.47)$$

which is the discrete version of the KdV equation.

Continuum limits of lattice equations

By *continuum limits* of lattice equations we mean the limiting equations that we retrieve from the discrete equations by shrinking the lattice grid to a continuous set of values corresponding to spatial and temporal coordinates. We obtain the transition from difference to differential equation by carrying out Taylor expansions, namely by using expansions of the form

$$f(x+h) = f(x) + \frac{h}{1!}\frac{df}{dx} + \frac{h^2}{2!}\frac{d^2 f}{dx^2} + \cdots ,$$

where h is the step-size parameter, in the discrete equation and then by expanding power-by-power in the lattice parameter h.

In PΔEs of the form we have seen such as

$$(a - b + u_{n,m+1} - u_{n+1,m})\,(a + b + u_{n,m} - u_{n+1,m+1}) = a^2 - b^2 , \tag{1.48}$$

which is in fact the explicit form of equation (1.44) by the change of dependent variable $w_{n,m} = u_{n,m} - n\,a - m\,b$, cf. [56], we have independent variables $u_{n,m}$ and two shifts $u_{n+1,m}$ and $u_{n,m+1}$. By interpreting these variables in the discrete equation (1.48) as

$$\begin{aligned}
u_{n,m} &= u\left(x_0 + \frac{n}{a} + \frac{m}{b}, t_0 + \frac{n}{3\,a^3} + \frac{m}{3\,b^3}\right) = u(x,t) ,\\
u_{n+1,m} &= u\left(x_0 + \frac{n}{a} + \frac{m}{b} + \frac{1}{a}, t_0 + \frac{n}{3\,a^3} + \frac{m}{3\,b^3} + \frac{1}{3\,a^3}\right) = u\left(x + \frac{1}{a}, t + \frac{1}{3\,a^3}\right),\\
u_{n,m+1} &= u\left(x_0 + \frac{n}{a} + \frac{m}{b} + \frac{1}{b}, t_0 + \frac{n}{3\,a^3} + \frac{m}{3\,b^3} + \frac{1}{3\,b^3}\right) = u\left(x + \frac{1}{b}, t + \frac{1}{3\,b^3}\right),
\end{aligned}$$

we can do Taylor expansions with respect to

$$\frac{1}{a}\,,\quad \frac{1}{b}\,,\quad \frac{1}{3\,a^3}\quad \text{and}\quad \frac{1}{3\,b^3}\,.$$

Let us now investigate the effect of this limit on the equation (1.48). For small parameter $1/b$, consider Taylor series expansions of the form:

$$\begin{aligned}
u_{n,m+1} = u &+ \frac{1}{b}\,\partial_x u + \frac{1}{2\,b^2}\,\partial_x^2 u + \frac{1}{3\,b^3}\left(\partial_t u + \frac{1}{2}\,\partial_x^3 u\right) + \frac{1}{3\,b^4}\,\partial_x\partial_t u\\
&+ \frac{1}{6\,b^5}\,\partial_x^2\partial_t u + \frac{1}{18\,b^6}\,\partial_x^3\partial_t u + O\left(\frac{1}{b^7}\right).
\end{aligned} \tag{1.49}$$

Inserting this into the equation (1.48) we obtain

$$a^2 - b^2 = \left[a - b + \left(u + \frac{1}{b}\,\partial_x u + \frac{1}{2\,b^2}\,\partial_x^2 u + \cdots\right) - u_{n+1,m}\right] \times \left[a + b + u - \left(u_{n+1,m} + \frac{1}{b}\,\partial_x u_{n+1,m} + \frac{1}{2\,b^2}\,\partial_x^2 u_{n+1,m} + \cdots\right)\right].$$

Expanding in powers of $1/b$, we obtain as coefficient of the leading term of order $O(1)$ the *differential-difference equation* (DΔE):

$$\partial_x\left(u_{n+1,m} + u\right) = 2\,a\left(u_{n+1,m} - u\right) - \left(u_{n+1,m} - u\right)^2 . \tag{1.50}$$

Now, for the small parameter $1/a$, we can consider Taylor series expansions of the form:

$$u_{n+1,m} = u + \frac{1}{a}\,\partial_x u + \frac{1}{2\,a^2}\,\partial_x^2 u + \frac{1}{3\,a^3}\left(\partial_t u + \frac{1}{2}\,\partial_x^3 u\right) + \frac{1}{3\,a^4}\,\partial_x\partial_t u + \frac{1}{6\,a^5}\,\partial_x^2\partial_t u + \frac{1}{18\,a^6}\,\partial_x^3\partial_t u + O\left(\frac{1}{a^7}\right). \tag{1.51}$$

Inserting this into the equation (1.50) and expanding in powers of $1/a$, we obtain as coefficient of the leading term of order $O\left(1/a^2\right)$ the following PDE:

$$u_t = \frac{1}{4}\,u_{xxx} + \frac{3}{2}\,u_x^2 , \tag{1.52}$$

which is the pKdV equation (coinciding with (1.32)) in terms of the variable w up to a change of independent variables, i.e. after $x \mapsto x/2$ and $t \mapsto t/2$ we recover the equation for w.

We see that we have come now full circle: we started out with the continuous KdV equation, derived its BTs, which using the permutability property led to the construction of a lattice of solutions. The relations between these solutions are reinterpreted as a PΔE on the two-dimensional lattice. As a dynamical equation the PΔE was seen as a discretization of some continuum equations. The full continuum limit now turns out to be the equation we started out from, namely the KdV itself.

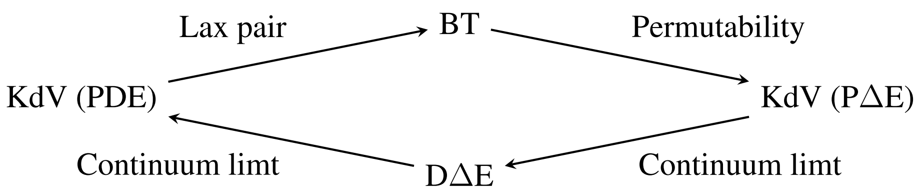


Figure 1.6: A diagram illustrating the relationship between the continuous and discrete KdV equations.

A very nice property of lattice equations is the Lax pair, which its existence is guaranteed by the multi-dimensional consistency property. We will next, relevant to chapters 2 and 3 of the thesis, briefly explain the multi-dimensional consistency property.

1.3.2 Quadrilateral lattice equations and 3D-consistency

Quadrilateral P∆Es

Let us consider partial difference equations of the following form, which we call quadrilateral P∆Es:

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) = 0 \,, \tag{1.53}$$

where the fields $u := u(n, m)$ are the dependent dynamical variables with $n, m \in \mathbb{Z}$. The variables n and m play the role of independent discrete variables. To explain the notations in equation (1.53), we note that $\widetilde{u} := u(n + 1, m)$, $\widehat{u} := u(n, m + 1)$ and $\widehat{\widetilde{u}} := u(n + 1, m + 1)$ are the shifted variables defining the different values of u at the vertices around an elementary plaquette on the lattice, as shown in figure 1.7. The fields $u = u(n, m)$ and its shifts are assigned to the vertices of the square lattice.

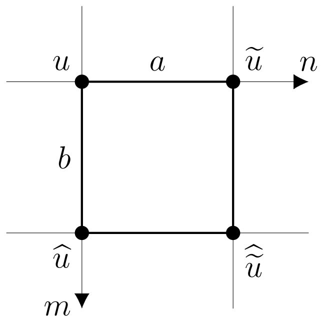


Figure 1.7: An elementary plaquette on the lattice.

The notation is inspired by the one for BTs, which as we have seen in the previous section, give rise to purely algebraic equation as a consequence of their permutability property. However, here we will forget, at first instance, about this connection and consider lattice equations of the form (1.53) in their own right as a P∆E on a two-dimensional lattice.

A classification of P∆Es in the same way as of PDEs does not yet exist. Related to the question of classification, there is the issue of whether solutions of a given P∆E arise from boundary value problems or from initial value problems [56]. We may consider a P∆E of the form (1.53) to have features reminiscent of hyperbolic PDEs. If the equation $Q = 0$ can be solved uniquely for each elementary quadrilateral, then one may pose *initial value problems* (IVPs) in ways very similar to hyperbolic type of equations such as the KdV equation itself (i.e. as a discrete nonlinear evolution equation).

The simple approach is to consider IVP where we assign values of the dependent variable u along horizontal array of vertices in the lattice, i.e. values for $u_{n,0}$, for all n, considering the variable m to be the temporal discrete variable. It is easy to see, however, that such an IVP leads to a nonlocal problem if we use (1.53) as an iteration scheme. To calculate any value $u_{n,1}$, say for given n, we would need to involve all initial values $u_{n',0}$ with $n' < n$, and furthermore have to assume limiting behaviour as $n \to -\infty$.

However, there is nothing that tells us that we should identify the n- and m-axes as the spatial and temporal axes respectively. The lattice picture allows us to play other and more natural games. Thus, changing the perspective slightly, we may tilt the lattice and rather consider initial value data to be imposed on configurations like a "sawtooth" or a "ladder", such as the one shown in figure 1.8.

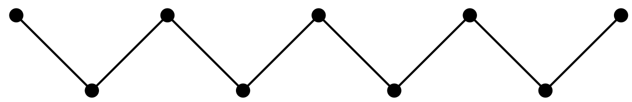


Figure 1.8: An evolution from initial values given on the staircase.

Hyperbolic PDEs have another feature that is shared with their discrete analogues. The domain on dependence of the solution at any point (x, t) encapsulates "memory" of the initial values that contributed to the evaluation of the solution u at that point [56]. In fact, any given point has a backward shadow (the analogue of the so-called "light-cone") of points the values of u on which determine the value at that point, as indicated by figure 1.9. The point at the bottom of the cone being fully determined by the initial values at the top of the cone and only and exclusively by those values. A more discussion of configurations and IVPs for lattice equations can be found in refs. [6, 116–118].

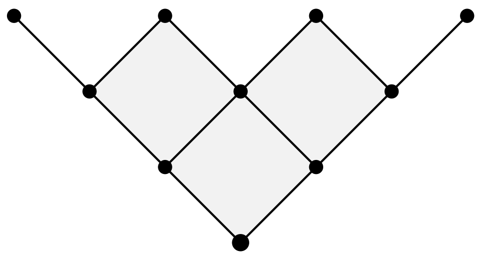


Figure 1.9: An initial value for an equation of the type (1.53) forms a light-cone.

3D-consistency

We now consider a class of quadrilateral PΔEs (1.53) in which, apart from the independent discrete variables n and m on which the variable $u_{n,m}$ depends, there are parameters a and b which associate with these independent variables. We can think of these parameters as being the parameters which measure the lattice spacing in the directions associated with n and m, and we refer to them as *lattice parameters*. These parameters will play a crucial role as we shall meet as this thesis progresses. Thus, the equations under question will take the form:

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; a, b) = 0 \ , \tag{1.54}$$

where the spectral parameters a and b are complex-valued lattice parameters associated with the n and m directions.

Suppose now $u := u(n, m, h)$ are 3-dimensional fields and hence we mention that $\widetilde{u} := u(n + 1, m, h)$, $\widehat{u} := u(n, m + 1, h)$ and $\overline{u} := u(n, m, h + 1)$. We assign initial values u, $\widetilde{u}$, $\widehat{u}$ and $\overline{u}$ to the vertices of the cube, as shown in figure 1.10, where $^{-}$ denotes a shift in the third independent variable h, which is associated with the lattice parameter c[1].

[1]It should be noted that the shift in the third independent variable h, i.e. the "bar shift" $^{-}$, will be replaced by a "breve shift" $\breve{}$ in the next chapter.

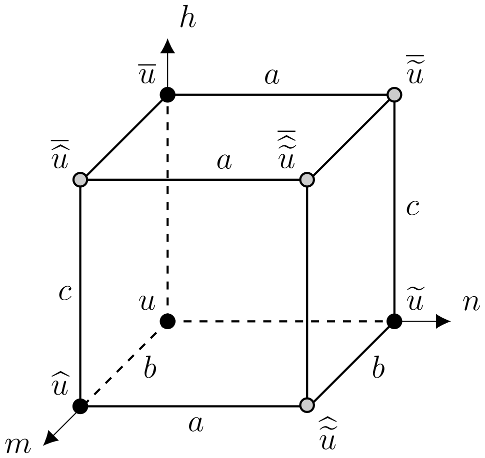


Figure 1.10: Consistency around the cube.

Applying the equation $Q = 0$ on three elementary plaquettes of the cube in figure 1.10 yields the system of equations

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; a, b) = 0 \,, \tag{1.55a}$$

$$Q(u, \widetilde{u}, \overline{u}, \overline{\widetilde{u}}; a, c) = 0 \,, \tag{1.55b}$$

$$Q(u, \widehat{u}, \overline{u}, \overline{\widehat{u}}; c, b) = 0 \,, \tag{1.55c}$$

obtained from the bottom, back and left faces of the cube, respectively. Since equations (1.55) are uniquely solvable, i.e. assuming Q such that when three points are given we can solve for the fourth point, we can uniquely determine the values $\widehat{\widetilde{u}}$, $\overline{\widetilde{u}}$ and $\overline{\widehat{u}}$. Now, the 3D-consistency property indicates that for any choice of initial values u, $\widetilde{u}$, $\widehat{u}$ and $\overline{u}$, equation $Q = 0$ produces the same value $\overline{\widehat{\widetilde{u}}}$ when solved using the top, front or right faces of the cube.

An illustrative example: The lattice pKdV

As an illustrative example we shall take the discrete version of the pKdV equation which is given by

$$Q(w, \widetilde{w}, \widehat{w}, \widehat{\widetilde{w}}; a, b) = (\widehat{w} - \widetilde{w})(w - \widehat{\widetilde{w}}) + b^2 - a^2 = 0 \,. \tag{1.56}$$

Solving equation (1.56) for $\hat{\tilde{w}}$, we obtain

$$\hat{\tilde{w}} = w + \frac{a^2 - b^2}{\tilde{w} - \hat{w}} \, . \tag{1.57}$$

Using the back and left faces of the cube 1.10, we can solve equations

$$Q(w, \tilde{w}, \overline{w}, \overline{\tilde{w}}; a, c) = 0 \, , \tag{1.58a}$$

$$Q(w, \hat{w}, \overline{w}, \overline{\hat{w}}; c, b) = 0 \, , \tag{1.58b}$$

to obtain solutions for $\overline{\tilde{w}}$ and $\overline{\hat{w}}$, respectively, namely

$$\overline{\tilde{w}} = w + \frac{a^2 - c^2}{\tilde{w} - \overline{w}} \, , \tag{1.59a}$$

$$\overline{\hat{w}} = w + \frac{c^2 - b^2}{\overline{w} - \hat{w}} \, . \tag{1.59b}$$

Now, if we shift (1.57) in the h-direction and then substitute $\overline{\tilde{w}}$ and $\overline{\hat{w}}$ by (1.59), we deduce

$$\overline{\hat{\tilde{w}}} = \frac{(a^2 - b^2)\tilde{w}\,\hat{w} + (b^2 - c^2)\hat{w}\,\overline{w} + (c^2 - a^2)\overline{w}\,\tilde{w}}{(c^2 - b^2)\tilde{w} + (b^2 - a^2)\overline{w} + (a^2 - c^2)\hat{w}} \, . \tag{1.60}$$

It is obvious that we could have obtained exactly the same result for $\overline{\hat{\tilde{w}}}$ if we had alternatively shifted (1.59a) in the m-direction and substituted $\hat{\tilde{w}}$ and $\overline{\hat{w}}$ by their values, or shifted (1.59b) in the n-direction and substituted $\hat{\tilde{w}}$ and $\overline{\tilde{w}}$. Thus, the discrete potential KdV equation (1.56) is 3D-consistent (i.e. it obeys the CAC property). We note that this property will be applied to the linearized lattice KdV equation in chapter 2.

Lax pair for lattice pKdV

As discussed in the previous section, the multi-dimensional consistency property explained above gives rise to the existence of a Lax pair. The idea is to consider the shift in third lattice direction $\overline{w}$ associated with lattice parameter c as a new dependent variable $\mathrm{w} := \overline{w}$ associated with parameter $k := c$. Proceeding in this way, the lattice KdV equations (1.59) are then given by

$$(\tilde{\mathrm{w}} - w)\,(\mathrm{w} - \tilde{w}) = k^2 - a^2 \quad \Rightarrow \quad \tilde{\mathrm{w}} = \frac{w\,\mathrm{w} + (k^2 - a^2 - w\,\tilde{w})}{\mathrm{w} - \tilde{w}} \, , \tag{1.61a}$$

$$(\hat{\mathrm{w}} - w)\,(\mathrm{w} - \hat{w}) = k^2 - b^2 \quad \Rightarrow \quad \hat{\mathrm{w}} = \frac{w\,\mathrm{w} + (k^2 - b^2 - w\,\hat{w})}{\mathrm{w} - \hat{w}} \, , \tag{1.61b}$$

which give equations for w in terms of w. Writing the fractional form:

$$\mathrm{w}_{n,m} =: \frac{\psi_{n,m}}{\phi_{n,m}} - n\,a - m\,b\,,$$

then the equations (1.61) can be written as

$$\widetilde{\psi} = \frac{\widetilde{\phi}\,[w\,\psi + (k^2 - a^2 - w\,\widetilde{w})\,\phi]}{\psi - \widetilde{w}\,\phi} + a\,\widetilde{\phi}\,, \tag{1.62a}$$

$$\widehat{\psi} = \frac{\widehat{\phi}\,[w\,\psi + (k^2 - b^2 - w\,\widehat{w})\,\phi]}{\psi - \widehat{w}\,\phi} + b\,\widehat{\phi}\,. \tag{1.62b}$$

Since at least one of the two functions ψ or ϕ can be chosen freely, we make a choice for $\widetilde{\phi}$ so that this reduces equation (1.62a) to the linearized equations

$$(a-k)\,\widetilde{\phi} = -\widetilde{w}\,\phi + \psi\,, \tag{1.63a}$$

$$(a-k)\,\widetilde{\psi} = \left[k^2 - a^2 - \widetilde{w}\,(a+w)\right]\phi + (a+w)\,\psi\,. \tag{1.63b}$$

We introduce the two-component vector $\Phi := (\phi, \psi)^T$ to write equations (1.63) as a system of 2×2 matrix form

$$(a-k)\,\widetilde{\Phi} = \begin{pmatrix} -\widetilde{w} & 1 \\ k^2 - a^2 - \widetilde{w}\,(a+w) & a+w \end{pmatrix} \Phi\,. \tag{1.64}$$

An entirely similar construction can be performed for the hat shift (1.62b), which together with the equation (1.64) produces the Lax pair for the lattice pKdV equation (1.56),

$$(a-k)\widetilde{\Phi}(k) = \mathcal{L}(k)\Phi(k)\,, \qquad (b-k)\widehat{\Phi}(k) = \mathcal{M}(k)\Phi(k)\,, \tag{1.65}$$

in which $\mathcal{L}(k)$ and $\mathcal{M}(k)$ are given by

$$\mathcal{L}(k) = \begin{pmatrix} -\widetilde{w} & 1 \\ k^2 - a^2 - \widetilde{w}\,(a+w) & a+w \end{pmatrix}, \tag{1.66a}$$

$$\mathcal{M}(k) = \begin{pmatrix} -\widehat{w} & 1 \\ k^2 - b^2 - \widehat{w}\,(b+w) & b+w \end{pmatrix}. \tag{1.66b}$$

The consistency condition of the spectral problem (1.65), i.e. $\widehat{\widetilde{\Phi}} = \widetilde{\widehat{\Phi}}$, leads to the condition (sometimes called discrete zero-curvature condition),

$$\widehat{\mathcal{L}}\,\mathcal{M} = \widetilde{\mathcal{M}}\,\mathcal{L}\,. \tag{1.67}$$

Requiring this condition to hold produces the lattice pKdV equation (1.56) as a compatibility condition.

1.3.3 Periodic reductions: The generalized McMillan maps

As discussed above, if the values of the field variables at three of the four sites around an elementary plaquette are known, the lattice pKdV equation

$$(a - b + \widehat{u} - \widetilde{u})\,(a + b - \widehat{\widetilde{u}} + u) = a^2 - b^2 \tag{1.68}$$

gives the value of the field variable at the fourth site. This allows us to consider initial value problems on the lattice. Periodic initial value problems were first considered in [24, 95], where the lattice pKdV (1.68) was one of the partial difference equations considered. This led to multi-dimensional families of mappings that turned out to be integrable. The *mappings of KdV-type* were one of the first families of multi-dimensional generalizations of the McMillan mapping [72, 100] to be derived. A periodic initial value problem is considered here. This term is explained in this section.

Reductions of lattice equations to integrable symplectic mappings have been considered since the early nineties [24, 95, 96]. Reduction is obtained by imposing a periodic initial value problem, with the evolution of the data progressing through the lattice according to a dynamical map which is constructed by implementing the quadrilateral lattice KdV equation (1.68).

Let us now consider initial value problems for (1.68) on the lattice. We assign initial data on a staircase on the lattice. By staircase we mean a sequence of neighbouring lattice sites with n and m nondecreasing, as illustrated for example in figure 1.11. From the fact that equation (1.68) at each site involves only the four variables situated on the four lattice sites around a simple plaquette, it follows that the information on these staircases evolves diagonally through the lattice along parallel staircases. Furthermore, because of the convexity of the staircase configuration, the initial-value problem is well posed. Although staircases of variable length and height stairsteps can be considered, for the sake of clarity we use a standard staircase with an even-periodic configuration of initial data. It would also be possible to consider an odd-periodic configuration of initial data [56, 95]; however, this term will not be considered in this thesis. In fact, one can show that several families of staircases give rise to equivalent mappings.

Suppose we take the standard staircase through the origin $(n, m) = (0, 0)$, and assign initial data on the lattice along this staircase, as in figure 1.11, namely

$$u(j, j) =: \mathrm{u}_{2j}\ , \qquad u(j+1, j) =: \mathrm{u}_{2j+1} \qquad (j \in \mathbb{Z})\ .$$

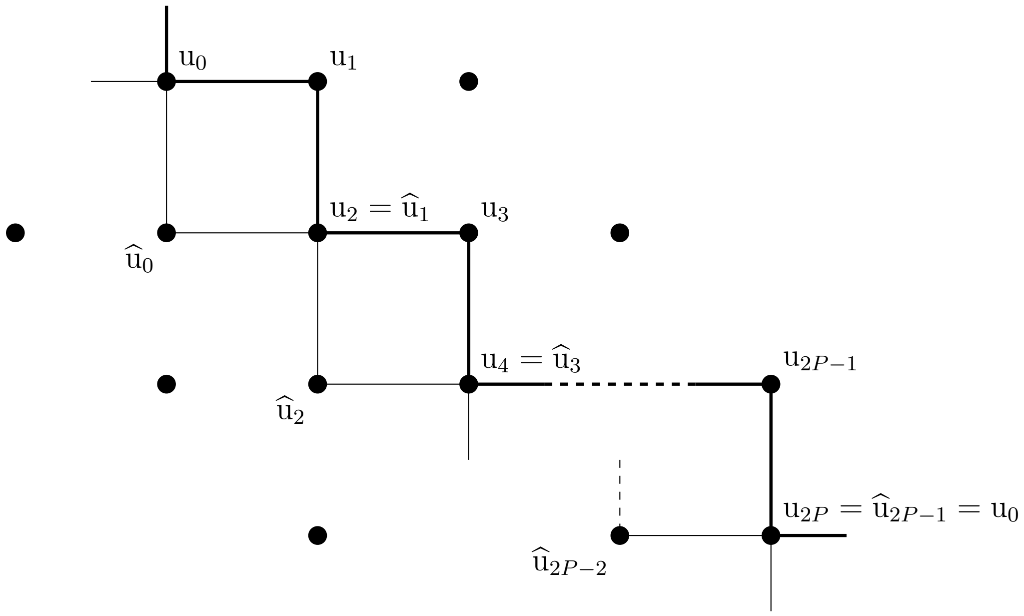


Figure 1.11: Standard staircase of periodic initial data on lattice, with even period $2P$.

The data on the next staircase one vertical step down can be calculated by applying the lattice KdV equation (1.68). Thus, performing iterations by updating the lattice variables u along a vertical shift in the m-direction, we define

$$u(j, j+1) =: \widehat{\mathrm{u}}_{2j} \quad \text{and} \quad u(j+1, j+1) =: \widehat{\mathrm{u}}_{2j+1} \ .$$

We therefore have from (1.68) evolution equations for the u_j,

$$\widehat{\mathrm{u}}_{2j+1} = \mathrm{u}_{2j+2} \ , \qquad \widehat{\mathrm{u}}_{2j} = \mathrm{u}_{2j+1} - \delta + \frac{\epsilon\,\delta}{\epsilon - \mathrm{u}_{2j+2} + \mathrm{u}_{2j}} \ , \tag{1.69}$$

in which $\delta := a - b$ and $\epsilon := a + b$. Introducing the variables for the differences on odd and even sites of the staircase, i.e.

$$X_j := \mathrm{u}_{2j+1} - \mathrm{u}_{2j-1} \ , \qquad Y_j := \mathrm{u}_{2j+2} - \mathrm{u}_{2j} \qquad (j \in \mathbb{Z}) \ ,$$

mapping (1.69) is reduced to the measure-preserving rational mapping

$$\widehat{X}_j = Y_j \ , \qquad \widehat{Y}_j = X_{j+1} + \frac{\epsilon\,\delta}{\epsilon - Y_{j+1}} - \frac{\epsilon\,\delta}{\epsilon - Y_j} \ . \tag{1.70}$$

The mapping (1.70) is a multidimensional generalization of the *McMillan mapping* [95]. We can now impose the even periodicity condition

$$\mathrm{u}_{2(j+P)} = \mathrm{u}_{2j} \ , \qquad \mathrm{u}_{2(j+P)+1} = \mathrm{u}_{2j+1} \qquad (j \in \mathbb{Z}).$$

Note that $P\,(P = 2, 3, \dots)$ can be interpreted as the period along the two diagonals of the lattice corresponding to the staircase. It is easy to see that these periodic conditions are compatible with the lattice equation and hence will be preserved after iteration of the mapping. This means that we have to supply (1.70) with the periodicity constraints

$$\sum_{j=1}^{P} X_j = 0 \,, \qquad \sum_{j=1}^{P} Y_{j-1} = 0 \,. \tag{1.71}$$

Equation (1.70) for $j = 1, 2, \dots, P$ together with constraint (1.71) is a $2(P-1)$-dimensional integrable mapping. The simplest case $P = 2$ corresponds to the well-known McMillan mapping [72]. The McMillan map was also obtained as a special case of an eighteen-parameter family of two-dimensional integrable mappings by Quispel et al [100].

1.3.4 Discrete-time integrability

As integrability of map technology is used throughout this thesis, we offer a short review of integrability in the discrete sense in this section. It is entirely analogous to that of Liouville integrability in the continuous-time case given in section 1.2.4. Veselov adapted the Liouville-Arnold theorem [8] to discrete maps in [121]. In the discrete case, the analogue of Hamiltonian evolution is taken by a canonical (symplectic) map, i.e.

$$p_i(n+1) = f_i\left(\boldsymbol{q}(n), \boldsymbol{p}(n)\right), \qquad q_i(n+1) = g_i\left(\boldsymbol{q}(n), \boldsymbol{p}(n)\right),$$

which preserves the Poisson bracket structure

$$\{q_i(n), q_j(n)\} = \{p_i(n), p_j(n)\} = 0 \,, \qquad \{q_i(n), p_j(n)\} = \delta_{ij},$$

for $i, j = 1, 2, ..., N$. Here the variable n serves as the discrete time.

Definition 1.3.1 ([56, p. 161]). For a canonical map on the phase space $\mathbb{R}^{2N}$ defined in terms of coordinates $\boldsymbol{q}$ and conjugate momenta $\boldsymbol{p}$, if there exists N smooth functions $I_i(\boldsymbol{q}, \boldsymbol{p})$ such that the I_i

(i) are conserved quantities (invariants of the map), i.e.

$$I_i\left(\boldsymbol{q}(n+1), \boldsymbol{p}(n+1)\right) = I_i\left(\boldsymbol{q}(n), \boldsymbol{p}(n)\right),$$

(ii) are in involution with respect to the Poisson bracket, i.e. $\{I_i, I_j\} = 0$,

(iii) are independent functions throughout the phase space,

then the mapping is said to be *integrable*.

Example: As an illustrative example of a discrete integrable map, we consider the example of *McMillan map* [72] (see also [94]). In the early seventies, McMillan found a four-parameter family of rational mappings of the plane, together with their invariants. The form for the transformation chosen was the area-preserving mappings

$$p(n+1) = q(n) \, , \qquad q(n+1) = -p(n) + f(q(n)) \, , \tag{1.72}$$

where n denotes the discrete time and in which $f(q(n))$ is a nonlinear function given by

$$f(q(n)) = -\frac{B\, q(n)^2 + D\, q(n)}{A\, q(n)^2 + B\, q(n) + C} \, .$$

By an area-preserving map such as (1.72), we mean a mapping satisfies the following property

$$\frac{\partial(q(n+1), p(n+1))}{\partial(q(n), p(n))} = \det \mathfrak{J} = 1 \, ,$$

where $\mathfrak{J}$ is the Jacobian that we encountered in section 1.2.1. It is clear that the dynamical mapping (1.72) is a rational mapping of a specific form. McMillan has shown that the most general form of an invariant is biquadratic [72]:

$$\begin{aligned} I(q(n), p(n)) = {} & A\, p(n)^2 q(n)^2 + B \left[p(n)^2 q(n) + p(n)\, q(n)^2\right] \\ & + C \left[p(n)^2 + q(n)^2\right] + D\, p(n)\, q(n) \, . \end{aligned} \tag{1.73}$$

For example, if $A = 1$, $B = 0$, $C = 1$ and $D = -2\,\alpha$, $(\alpha > 1)$, then (1.72) yields the area-preserving map:

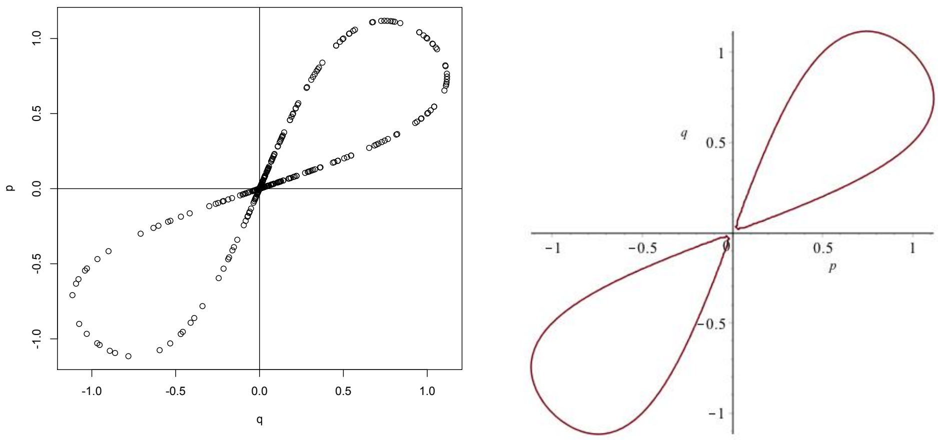
$$p(n+1) = q(n) \, , \qquad q(n+1) = \frac{2\,\alpha\, q(n)}{1 + q(n)^2} - p(n) \, , \tag{1.74}$$

whose invariant curve is given by

$$I(q,p) = p^2 q^2 + p^2 + q^2 - 2\,\alpha\, p\, q \, . \tag{1.75}$$

In the sense of definition 1.3.1, the mapping (1.74) is integrable.

Figure 1.12(a) displays the behaviour of McMillan map (1.74) for the (arbitrary) parameter value $\alpha = 1.5$ and initial conditions $(p_0, q_0) = (1, 0.5)$, for iterates (time-one iterate) $n = 0 \ldots 1000$. The invariant of the mapping is shown in figure 1.12(b) for the same parameter value.



(a) Iterates of mapping (1.74). (b) Invariant (1.75) (lemniscate).

Figure 1.12: McMillan map along with invariant for the parameter $\alpha = 1.5$.

As noted above, part of this thesis is concerned with the study of modified Hamiltonian equations in geometric integration field and their connections to the theory of integrable systems. In the next section, we will address a number of aspects of the geometric integration theory that come together in this subject.

1.4 Geometric numerical integration

Geometric numerical integrators are numerical methods that preserve geometric properties of the flow of a differential equation, such as symplectic integrators for Hamiltonian systems, symmetric integrators for reversible systems, methods preserving first integrals and numerical methods on manifolds and integrators for constrained mechanical systems [18, 52, 69]. An important aspect of geometric numerical integration is the explanation of the relationship between preservation of the geometric properties of the numerical scheme and the favorable error propagation in long-time integration. This is done using the idea of *backward error analysis*, where the numerical one-step map is interpreted as the flow of a modified differential equation which is constructed as an asymptotic series [52,69]. In this way, geometric properties of the numerical integrator translate into structure preservation on the level of the modified equations.

It is a well-known and essential fact that for Hamiltonian systems the modified system is Hamiltonian if the numerical scheme is symplectic. One then speaks of a *modified Hamiltonian* or an interpolating Hamiltonian (it is also called "shadow Hamiltonian" in some contexts) [69, 105, 106, 109]. The existence of a modified Hamiltonian is an indicator of the validity of statistical estimates calculated from long-time integration of chaotic Hamiltonian systems [102]. Energy drifts caused by numerical instability are better detected by evaluating the modified Hamiltonian than the original Hamiltonian [109]. Computing the modified Hamiltonian is also useful to illustrate and guide backward error analysis.

In the field of geometric numerical integration, modified Hamiltonians are used to show that symplectic schemes for Hamiltonian systems are accurate over long times. In 1990 Yoshida provided a new method [128] of constructing a modified Hamiltonian based on the Baker-Campbell-Hausdorff (BCH) series [92, 120], which plays a key role in geometric numerical integration theory. This class of method is introduced next.

1.4.1 Yoshida construction of modified Hamiltonian

Symplectic integrators are numerical integration schemes for N-degrees-of-freedom Hamiltonian systems $(\boldsymbol{q}, \boldsymbol{p}, H)$ which conserve the symplectic two-form $\sum_i dq_i \wedge dp_i$, where $i = 1, \ldots, N$. In $2N$-dimensional phase space, recall the definition of the Poisson bracket from 1.2.1, that is

$$\{F, G\} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) , \tag{1.76}$$

where q_i, p_i are coordinates of the phase space, $i = 1, \ldots, N$. As discussed in section 1.2.2, defining the differential operator D_G by

$$D_G F := \{F, G\} , \tag{1.77}$$

we then have $\dot{\boldsymbol{z}} = D_H\, \boldsymbol{z} = \{\boldsymbol{z}, H\}$ where $\boldsymbol{z} = (q_1, \ldots, q_N; p_1, \ldots, p_N)$. Hence, we have

$$e^{tD_H}(\boldsymbol{z}(\boldsymbol{q}, \boldsymbol{p})) \, .$$

To apply Yoshida's method [128] (see also [69]), consider the symplectic Euler method

$$\begin{aligned} p_i(n+1) - p_i(n) &= -\tau \, \frac{\partial H(q_i(n), p_i(n+1))}{\partial q_i} \, , \\ q_i(n+1) - q_i(n) &= \tau \, \frac{\partial H(q_i(n), p_i(n+1))}{\partial p_i} \, . \end{aligned} \tag{1.78}$$

Hence, we apply to a Newtonian type Hamiltonian, i.e.

$$H(q_1, \ldots, q_N; p_1, \ldots, p_N) = T(p_1, \ldots, p_N) + V(q_1, \ldots, q_N) \, .$$

Equations (1.78) actually form a discrete analogue of the usual Hamilton's equations, so we will refer to them as discrete Hamilton equations and call the corresponding H the discrete Hamiltonian. Thus, the discrete Hamiltonian is the generating function for the canonical transformation formation (1.78). The symplectic Euler method is represented by

$$e^{\tau D_V} e^{\tau D_T} \, .$$

The BCH formula [26, 92, 120] tells us that the product of two exponentials can be expressed as a single exponential, i.e.

$$e^X e^Y = e^Z \, .$$

Thus, Z can be given by the series

$$\begin{aligned} Z = \log\left(e^X \, e^Y\right) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\sum_{p,q=0}^{\infty} \frac{1}{p! \, q!} \, X^p \, Y^q - 1 \right)^k \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum \frac{X^{p_1} Y^{q_1} \cdots X^{p_k} Y^{q_k}}{p_1! \, q_1! \cdots p_k! \, q_k!} \, . \end{aligned}$$

Recall that the commutator $[X, Y]$ is defined as $X\,Y - Y\,X$, we then have

$$Z = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{p_i, q_i} \frac{[X^{p_1} Y^{q_1} \cdots X^{p_k} Y^{q_k}]}{\left(\sum_i^k (p_i + q_i)\right) p_1! \, q_1! \cdots p_k! \, q_k!} \, , \tag{1.79}$$

where the inner summation is taken over all non-negative integers $p_1, q_1, \ldots, p_k, q_k$ such that $p_1 + q_1 > 0, \ldots, p_k + q_k > 0$ and $[X^{p_1} Y^{q_1} \cdots X^{p_k} Y^{q_k}]$ denotes the right nested commutator, i.e.

$$[X^{p_1} Y^{q_1} \cdots X^{p_k} Y^{q_k}] = [\underbrace{X, [X, \cdots, [X,}_{p_1} \underbrace{[Y, [Y, \cdots [Y,}_{q_1} \cdots \underbrace{[X, [X, \cdots [X,}_{p_k} \underbrace{[Y, [Y, \cdots Y]]}_{q_k} \cdots]] \, .$$

Hence, Z is given by an infinite series of nested commutators, which reads from (1.79) for the first few terms as

$$\begin{aligned} Z = X + Y &+ \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) - \frac{1}{24}[Y,[X,[X,Y]]] \\ &- \frac{1}{720}([X,[X,[X,[X,Y]]]] + [Y,[Y,[Y,[Y,X]]]]) \\ &+ \frac{1}{360}([Y,[X,[X,[X,Y]]]] + [X,[Y,[Y,[Y,X]]]]) \\ &+ \frac{1}{120}([Y,[X,[Y,[X,Y]]]] + [X,[Y,[X,[Y,X]]]]) + \cdots . \end{aligned}$$

Proposition 1.4.1. *Recall the differential operator D_G defined in equation (1.77). Then, the following relation*

$$[D_F, D_G] = D_{\{G,F\}}$$

holds.

Proof

Let F, G and H be any smooth functions defined on the phase space $(\boldsymbol{q}, \boldsymbol{p})$. Using the Jacobi identity, we have

$$\begin{aligned} [D_F, D_G]\, H = (D_F D_G - D_G D_F)\, H &= D_F D_G H - D_G D_F H \\ &= D_F\{H,G\} - D_G\{H,F\} = \{\{H,G\},F\} - \{\{H,F\},G\} \\ &= -\{F,\{H,G\}\} - \{G,\{F,H\}\} = \{H,\{G,F\}\} \\ &= D_{\{G,F\}} H \,. \end{aligned}$$

□

Now, the product $e^{\tau D_V} e^{\tau D_T}$ can be written as $e^{\tau D_{H^*}}$, where using

$$[D_F, D_G] = D_{\{G,F\}} \,,$$

we have

$$H^* = T + V + \frac{\tau}{2}\{T,V\} + \frac{\tau^2}{12}(\{T,\{T,V\}\} + \{V,\{V,T\}\}) + \cdots . \tag{1.80}$$

Thus, the symplectic Euler method follows the exact τ evolution of H^*. In other words, H^* is the modified Hamiltonian. For the canonical Poisson bracket (1.76), this result can be written as

$$H^* = H + \tau H_1 + \tau^2 H_2 + \cdots$$

where

$$H_1 = -\frac{1}{2}\sum_{i=1}^{N} \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i}\,, \quad H_2 = \frac{1}{12}\sum_{i,j=1}^{N}\left(\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial q_j \partial q_i} + \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial q_j}\frac{\partial^2 H}{\partial p_j \partial p_i}\right), \cdots .$$

It should be noted that the BCH formula can be applied to a general form without restriction of $H = T + V$, see e.g. [128].

An illustrative example:

As Yoshida's method is used throughout this thesis, let us apply this method to the example of harmonic oscillator [47]:

$$H(q,p) = \frac{1}{2}\left(p^2 + q^2\right), \tag{1.81}$$

noting that a detailed description of how to construct discrete harmonic oscillators as reductions of lattice equations is exposed in section 2.7, as well as the relevant modified Hamiltonians are discussed in sections 3.2 and 3.3.

The time evolution of the system (1.81) is given by the Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial q} = -q\,, \qquad \dot{q} = \frac{\partial H}{\partial p} = p\,. \tag{1.82}$$

The symplectic Euler method, when applied to Hamiltonian (1.81), is expressed as

$$p_{n+1} = p_n - \tau\, q_n\,, \qquad q_{n+1} = q_n + \tau\, p_{n+1}\,, \tag{1.83}$$

which conserves the quantity

$$I(q_n, p_n) = p_n^2 + q_n^2 - \tau\, p_n\, q_n\,. \tag{1.84}$$

Now, letting $T = p^2/2$ and $V = q^2/2$, and using the BCH formula (1.80) the expansion for the modified Hamiltonian when applied to (1.81) is written as:

$$\begin{aligned} H^*(q,p) = {} & \frac{1}{2}p^2 + \frac{1}{2}q^2 - \frac{\tau}{2}\,p\,q + \frac{\tau^2}{12}p^2 + \frac{\tau^2}{12}q^2 - \frac{\tau^3}{12}\,p\,q \\ & + \frac{\tau^4}{60}p^2 + \frac{\tau^4}{60}q^2 - \frac{\tau^5}{60}\,p\,q + O(\tau^6)\,. \end{aligned} \tag{1.85}$$

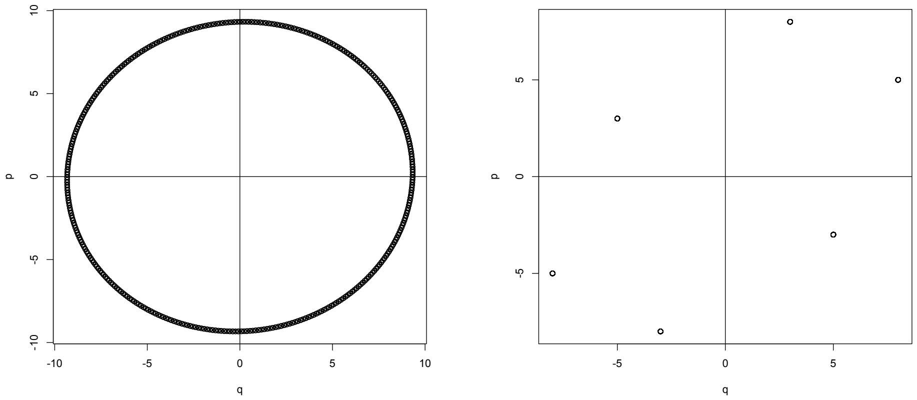
This equation can also be written as

$$H^*(q,p) = f(\tau)\, I(q,p) \; ,$$

in which

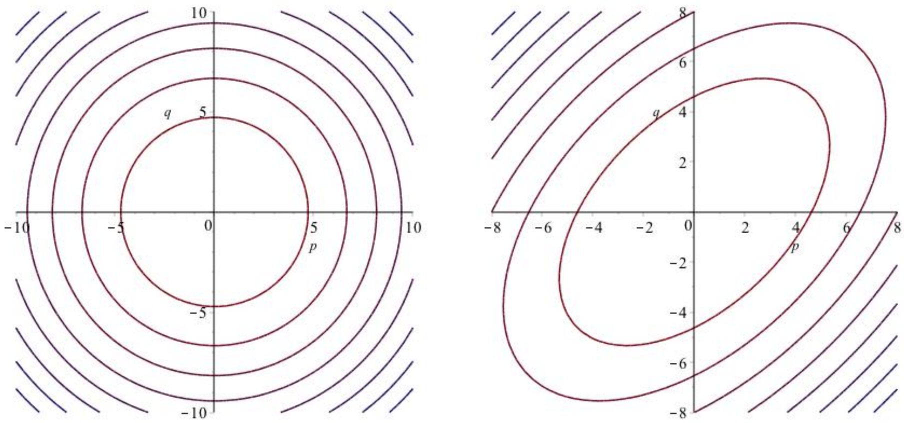
$$f(\tau) = \frac{1}{2} + \frac{\tau^2}{12} + \frac{\tau^4}{60} + O(\tau^6) \; .$$

Figure 1.13 displays the behaviour of symplectic mapping (1.83) for the initial conditions $(p_0, q_0) = (5, 8)$, for various values of τ and iterates $n = 0 \ldots 1000$. The level sets of equations (1.81) and (1.85) are shown in figure 1.14.



(a) For $\tau = 0.05$. (b) For $\tau = 1$.

Figure 1.13: Iterates of mapping (1.83) for various values of τ.



(a) Harmonic oscillator system. (b) Modified Hamiltonian for $\tau = 1$.

Figure 1.14: Level sets of Hamiltonians (1.81) and (1.85).

1.4.2 Convergence of modified Hamiltonian

It is well known that for linear systems (quadratic Hamiltonians), the expansion of the modified Hamiltonian is convergent. It used to be believed that a discretization arising from a Hamiltonian of the form

$$H(q,p) = \frac{1}{2}\,p^2 + V(q)\;, \tag{1.86}$$

where $\partial V/\partial q$ is nonlinear, the BCH expansion for the modified Hamiltonian does not converge [52,106,109]. If the series does not converge, the modified Hamiltonian may not exist as a proper function and as such provides only a formal invariant, i.e. the modified Hamiltonian formally exist as a series using the BCH expansion. However, one paper [41] gives examples of systems coming from the theory of discrete integrable systems, for which there exists a closed-form expression for the modified Hamiltonian, indicating that the BCH expansion should converge. Generally, if the BCH expansion converges, the modified Hamiltonian defines an invariant for the mapping. In fact, a sufficient condition for this convergence is the existence of an underlying invariant curve in the relevant phase space which globally defined in the relevant integrable system.

Example of non-integrable map [121]:
Applying the symplectic Euler method (1.78) to the Hamiltonian (1.86) where

$$V(q) = \frac{1}{2}\,(\alpha\, q^2 + \frac{1}{2}\beta\, q^4)\;; \qquad \alpha,\beta \in \mathbb{R}^+,$$

we obtain

$$p_{n+1} - p_n = -\tau\,\frac{\partial H(q_n,p_{n+1})}{\partial q_n}\;, \qquad q_{n+1} - q_n = \tau\,\frac{\partial H(q_n,p_{n+1})}{\partial p_{n+1}}\;.$$

This implies

$$q_{n+1} - 2\,q_n + q_{n-1} = -\tau^2\,(\alpha\, q_n + \beta\, q_n^3)\;, \tag{1.87}$$

which is a natural discretization of the equation of motion that arises from (1.86) via Hamilton's equations. This symplectic mapping is not integrable since it does not have an invariant [41, 121]. Thus, the BCH expansion for the modified Hamiltonian

$$\begin{aligned} H^*(q,p) = {} & \frac{1}{2}\,p^2 + \frac{1}{4}\,q^2\,(2\,\alpha + \beta\, q^2) - \frac{\tau}{2}\,p\,q\,(\alpha + \beta\, q^2) + \frac{\tau^2}{12}\,p^2\,(\alpha + 3\,\beta\, q^2) \\ & + \frac{\tau^2}{12}\,q^2\,(\alpha + \beta\, q^2)^2 - \frac{\tau^3}{12}\,p\,q\,(\alpha + \beta\, q^2)(\alpha + 3\,\beta\, q^2) + O(\tau^4) \end{aligned} \tag{1.88}$$

does not converge.

Thus, the above discussion provides a link to the theory of discrete integrable systems. Throughout this thesis we give further examples of systems coming from the theory of discrete integrable systems which, when viewed as an application of the symplectic Euler method, have a closed-form expression for the modified Hamiltonian. The corresponding Hamiltonian systems are associated with the interpolating flow of these integrable mappings. In particular, we give examples of one- and two-degrees-of-freedom systems which arise from linear and nonlinear integrable lattice equations. The construction of such examples arise from periodic initial value problems of the lattice version of KdV type are explicated in the next chapter.

This concludes a brief introduction of some of the main aspects of integrable systems and geometric integration, especially those of relevance throughout the rest of the thesis.

1.5 Organization of the thesis

The character of chapter 2 is introductory. The main results of the thesis are presented in chapters 3–6. Specifically, this thesis is organized as follows.

Chapter 2 is concerned with lattice equations and integrable mappings. In particular, we show how the nonlinear integrable lattice KdV and mKdV equations give rise to finite-dimensional integrable mappings which, are close to the identity mappings, do indeed carry a spectral interpretation. Hence, the invariants can be calculated systematically from the monodromy matrix constructed from the Lax pair. Close-to-the-identity linear mappings are also derived from the linearized lattice KdV model, and we exploit the multi-dimensional consistency to derive commuting maps. In all the aforementioned cases, the Lax pair description of the mappings are given. Such mappings, which are close-to-the-identity mappings, are interesting in their own right since they are looked at as numerical schemes and used to demonstrate the connection between the invariants and the relevant modified Hamiltonians.

Chapter 3 considers the discrete-time integrable KdV models from chapter 2. The first part of the chapter considers the linear system; the modified Hamiltonians are studied for the linearized mappings of the KdV type. It is found that the modified Hamiltonians of the map and its commuting map are in involution with respect to the variables of the original maps. Furthermore, the latter ideas are extended with regard to the two-degrees-of-freedom system in order to learn more and complete the picture. The second part of the chapter considers one-dimensional discrete-time systems of nonlinear KdV models; the modified Hamiltonian of the symplectic mapping, as given by Yoshida's approach, is given and written in a closed-form expression in terms of an elliptic integral by using an action-angle variables technique. Moreover, we show how, using the mKdV example, one can obtain a closed-form expression for the modified Hamiltonian of an implicit scheme, although this is predictably more complex in the implicit case.

Chapter 4 deals with the methods of separation of variables and finite-gap integration for the mappings of the KdV type. More concretely, chapter 4 aims to study the modified Hamiltonian of a two-degrees-of-freedom KdV system. The transition to multiple degrees of freedom brings important new features such as the finite-gap integration technique [10]. The multicomponent case requires the technique of separation of variables [110, 119], which we discuss for the transformation to action-angle variables and involves some approach of the theory of genus-two abelian features, as developed in section 4.7. In addition, we derive the discrete dynamics in terms of the separated variables and establish an interpolating flow for the map by considering the invariants, followed by a diagram which presents commuting canonical transformations that are associated with generating functions.

Chapter 5 is devoted to the exploration of an integrable lattice BSQ model that was presented in [87] and proposed as three-component systems by Tongas and Nijhoff [115]. Specifically, beginning with the BSQ equation on the continuous-time, parallel to the KdV theory that has been discussed in section 1.3.1, we show how to derive the discrete version of the BSQ equation from the Bäcklund transformation. By considering the lattice BSQ equation, we derive multi-dimensional families of integrable dynamical mappings using a periodic staircase initial value problem, with vertical and diagonal shift evolutions of the data progressing through the lattice. Lax pairs of the mappings are derived for discrete-time evolutions along both directions. The big Lax matrix structure suggests a possible avenue for the identification of the reduction of variables, so we present derivations of the big Lax pair which lead to the mappings in terms of the reduced variables as conjugation matrices. Moreover, we examine the cases of periods one and two reduction for both evolutions as examples, in order to illustrate the general theory. Finally, we review the classical r-matrix and Yang-Baxter structures for the mappings of lattice hierarchy of equations.

Chapter 6 considers the BSQ model that we set up in chapter 5. A general form of the monodromy matrix which has a natural grading in terms of the spectral parameter is deduced, leading to three different cases which are associated with the length of chain on the staircase in the lattice. With the use of the Weierstrass gap sequence technique [39], a novel formulation of the spectral curve for generalized families of mappings is also uncovered. The method of separation of variables is applied to the BSQ mappings, where the separated coordinates appear as the poles of the properly normalized eigenvector of the corresponding monodromy matrix. Moreover, by taking a slightly different approach, we also present a new machinery of defining the symmetric function of the separation variables. Finally, we use a different approach from the one expounded in [33] to obtain a novel expression for the continuous Dubrovin equations in more general formula, where we use the characteristic behaviour of the monodromy matrix and the techniques of separation of variables and r-matrix structure, followed by a discussion of the discrete Dubrovin equations.

Chapter 7 concludes the thesis with a summary of the results and contains a discussion and some ideas for future work.

The following diagrams 1.15 and 1.16 represent the main objects we respectively study in chapters 2–4 and in chapters 5 and 6.

Chapter 2	Chapter 3	Chapter 6
Chapter 4	Chapter 5	

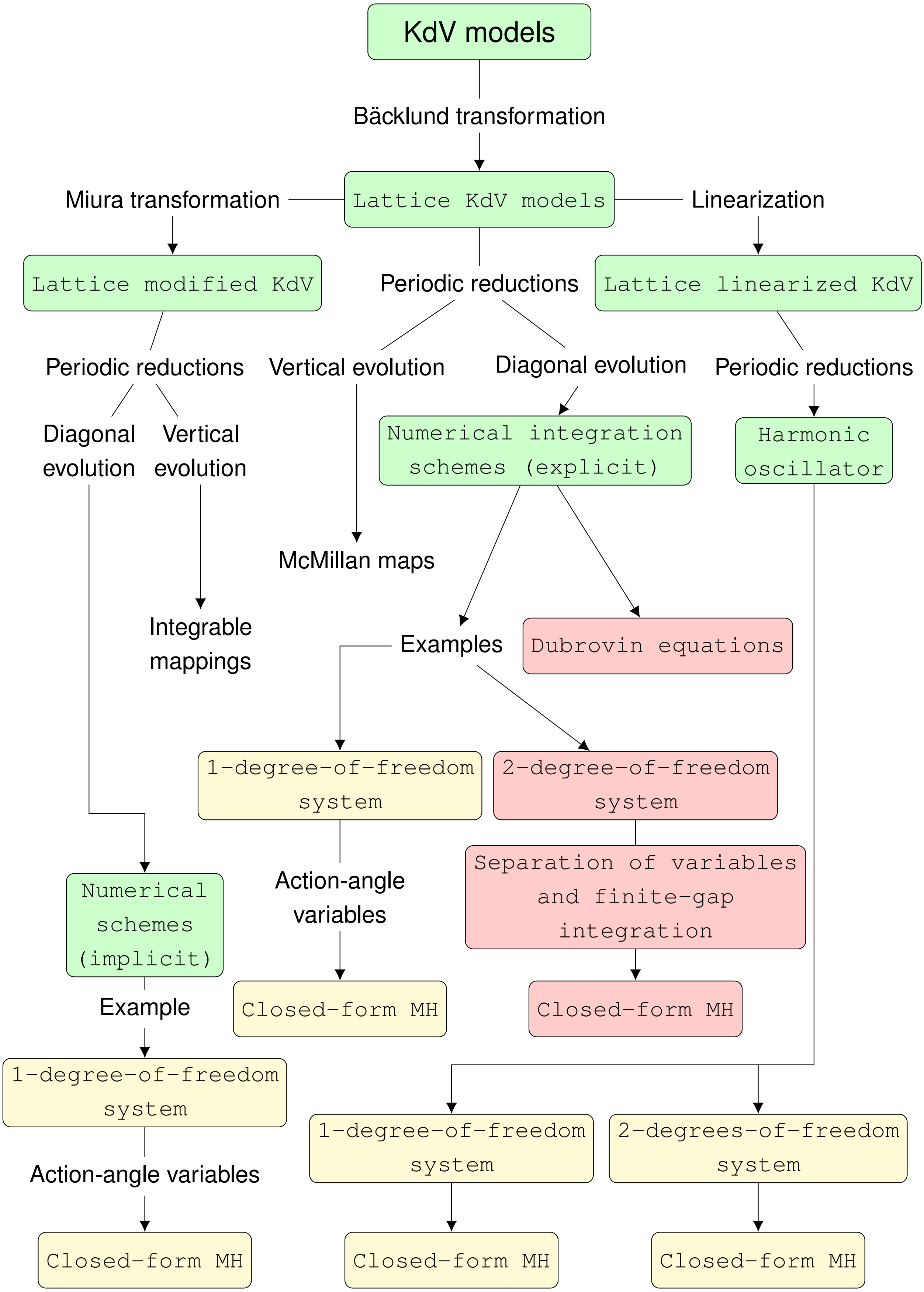


Figure 1.15: A simplified diagram of the main objects in Chapters 2, 3 and 4.

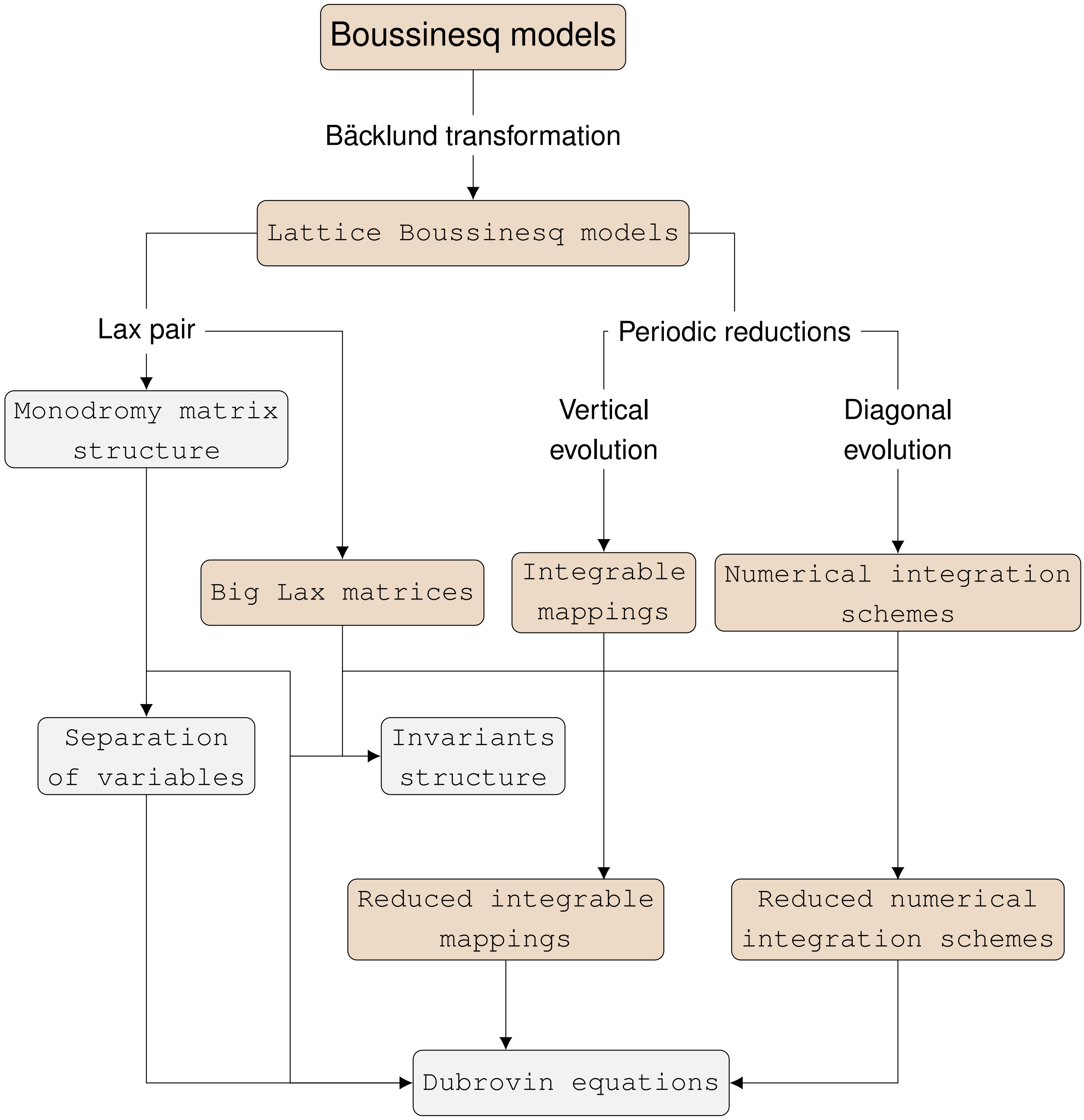


Figure 1.16: A simplified diagram of the main objects in Chapters 5 and 6.

Chapter 2

Lattice KdV models and dynamical mappings

2.1 Overview

The various manifestations of the lattice Korteweg-de Vries (KdV) equation are considered and highlighted in this chapter. The lattice KdV equation (as well as other lattice analogues of other integrable PDEs) were obtained in [90] using a discrete version of the direct linearization method introduced in [43]. In [124] a full hierarchy of KdV and higher-order KdV equations were derived by performing a particular continuum limit. A periodic initial value problem was imposed on the lattice KdV equation [24, 95]. This led to finite-dimensional reductions of the lattice (i.e. multi-dimensional families of rational mappings). These mappings are integrable in the sense of the definition given in section 1.3.4 of the introduction and also possess the usual paraphernalia of integrable systems, such as Lax pairs. Of course, with as much justification, one could call these mappings "KdV-type mappings"[2].

The modified Hamiltonian plays a prominent role in the context of what follows in this thesis. The majority of this chapter is dedicated to a construction of certain integrable mappings of KdV models. The KdV mappings for discrete-time evolution along the di-

[2]This term is already used in the literature for mappings from a periodic initial value problem, so we use that term.

agonal direction is the object of our interest in this chapter. The principal interest here is that the discrete-time systems are considered as numerical integrators in their own right.

One interesting characteristic of the resulting discrete systems is the multi-dimensional consistency discussed in section 1.3.2 of the introduction. This means that these systems can be extended into a 3-D lattice in a consistent way. For some integrable models commuting maps have been found analogously to the multi-dimensional consistency of the corresponding lattice models, e.g. in [64, 65, 126, 127]. The construction of commuting mappings for discrete-time evolution along the diagonal direction is also an object of our interest in this chapter, but we restrict ourselves to the linear case of KdV mappings.

2.2 Lattice KdV system

Recall that the lattice version of the KdV partial differential equation relates the values of a field variable u around an elementary plaquette as follows [58, 65, 83, 90]:

$$(a-b+\widehat{u}-\widetilde{u})\,(a+b-\widehat{\widetilde{u}}+u)=a^2-b^2\ , \tag{2.1}$$

which is equation (1.68). As in section 1.3.2, $u:=u(n,m)$ is the dynamical variable at the lattice site (n,m) with $n,m\in\mathbb{Z}$, and $\widetilde{\ }$ and $\widehat{\ }$ are shorthand notations for translations on the lattice: i.e. $\widetilde{u}:=u(n+1,m)$ and $\widehat{u}:=u(n,m+1)$, as depicted in figure 2.1. Furthermore, a and b are complex-valued lattice parameters. As noted in section 1.3.2, equation (2.1) arises as the compatibility condition of a pair of linear problems (a Lax pair) defining the shifts of two component vector functions $\Phi(k)$ in the n and m directions

$$(a-k)\widetilde{\Phi}(k)=\mathcal{L}(k)\Phi(k)\ ,\qquad (b-k)\widehat{\Phi}(k)=\mathcal{M}(k)\Phi(k)\ , \tag{2.2}$$

where $\mathcal{L}(k)$ is given by

$$\mathcal{L}(k)=\begin{pmatrix} a-\widetilde{u} & 1\\ k^2-a^2+(a-\widetilde{u})(a+u) & a+u\end{pmatrix}\ , \tag{2.3}$$

and where $\mathcal{M}(k)$ is given by a similar matrix obtained from (2.3) by making the replacements $a\to b$ and $\widetilde{\ }\to\widehat{\ }$. The parameter k is the spectral parameter.

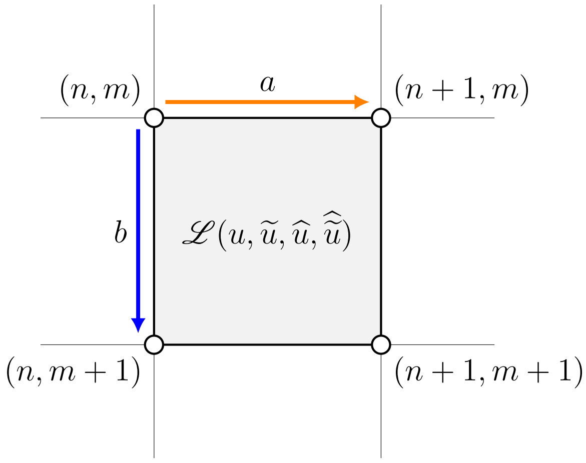


Figure 2.1: Elementary plaquette.

An important feature of equation (2.1) is that it arises from discrete Euler-Lagrange equations on the four-point Lagrangian

$$\mathscr{L}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = u\,(\widetilde{u}-\widehat{u}) + (p^2-q^2)\log\left(a+b+u-\widehat{\widetilde{u}}\right)\;; \tag{2.4}$$

$$\widehat{\widetilde{\left(\frac{\partial\mathscr{L}}{\partial u}\right)}} + \widehat{\left(\frac{\partial\mathscr{L}}{\partial \widetilde{u}}\right)} + \widetilde{\left(\frac{\partial\mathscr{L}}{\partial \widehat{u}}\right)} + \left(\frac{\partial\mathscr{L}}{\partial \widehat{\widetilde{u}}}\right) = 0\;.$$

The action for the KdV lattice equation (2.1) reads

$$\mathcal{S} = \sum_{n,m\in\mathbb{Z}} \mathscr{L}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1})\;. \tag{2.5}$$

As discussed in the previous chapter, we know that if the values of the field variables at three of the four sites around an elementary plaquette are known, the lattice KdV equation (2.1) gives the value of the field variable at the fourth site. Hence, we are allowed to consider initial value problems on the lattice. In the next section, a periodic initial value problem is considered.

2.3 Periodic reductions

Reduction is obtained by imposing a periodic initial value problem, with the evolution of the data progressing through the lattice according to a dynamical map which is constructed by implementing the quadrilateral lattice KdV equation (2.1).

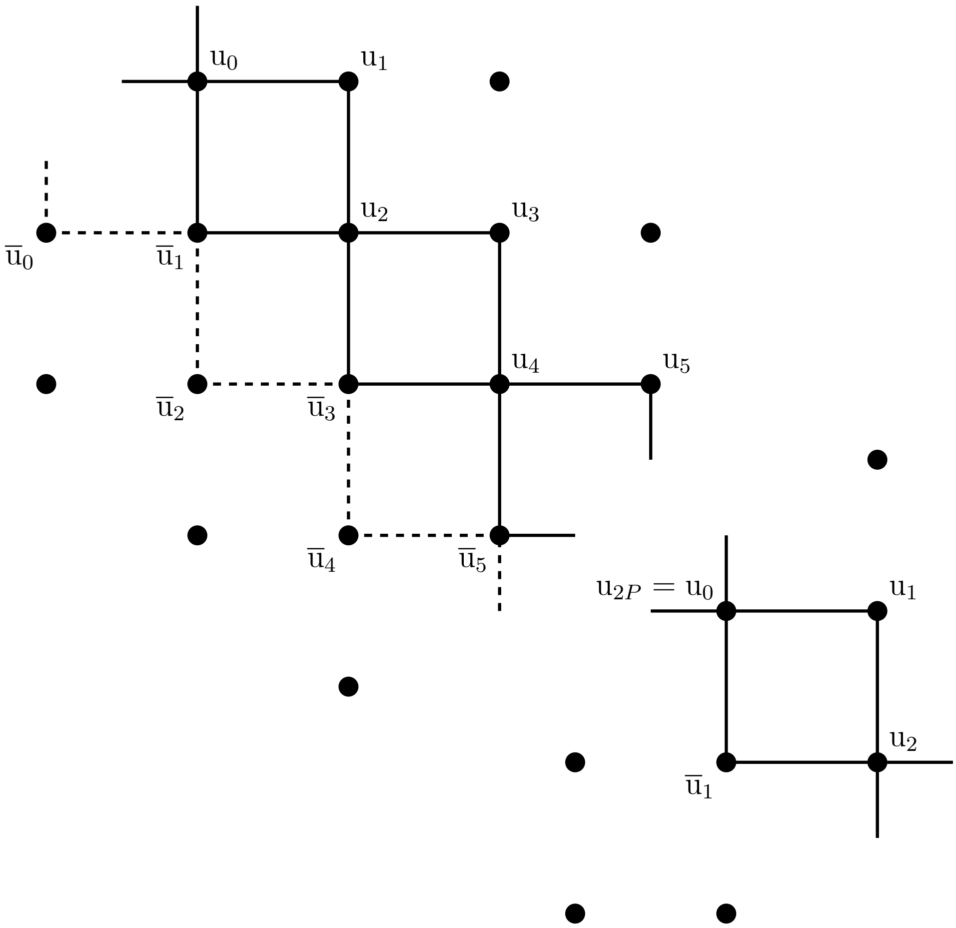


Figure 2.2: Periodic staircase on the lattice, with even period $2P$.

We choose initial data on the standard staircase through the origin $(n, m) = (0, 0)$ and assign initial data on the lattice along this staircase as depicted in figure 2.2, namely

$$u(j, j) =: \mathrm{u}_{2j} , \qquad u(j+1, j) =: \mathrm{u}_{2j+1} \qquad (j \in \mathbb{Z}) .$$

In this thesis, we take a different point of view from the one expounded in [95] which was introduced in section 1.3.3 of the introduction. What is new here is that we perform iterations by updating the lattice variables u along a diagonal shift rather than vertical shift (m direction); we define

$$u(j-1, j+1) =: \overline{\mathrm{u}}_{2j} , \qquad u(j, j+1) =: \overline{\mathrm{u}}_{2j+1} ,$$

using the lattice KdV (2.1). In this way we obtain the mapping

$$\overline{\mathrm{u}}_{2j+1} = \mathrm{u}_{2j+1} - \delta + \frac{\epsilon\,\delta}{\epsilon - \mathrm{u}_{2j+2} + \mathrm{u}_{2j}} , \quad \overline{\mathrm{u}}_{2j} = \mathrm{u}_{2j} - \delta + \frac{\epsilon\,\delta}{\epsilon - \overline{\mathrm{u}}_{2j+1} + \overline{\mathrm{u}}_{2j-1}} , \qquad (2.6)$$

in which $\delta = a - b$ and $\epsilon = a + b$. We choose a diagonal shift so that the mapping is close to the identity mapping if δ is small. Therefore, the mapping is viewed as an application of the symplectic Euler method. By introducing the differences

$$X_j := \mathrm{u}_{2j+1} - \mathrm{u}_{2j-1} \,, \qquad Y_j := \mathrm{u}_{2j+2} - \mathrm{u}_{2j} \qquad (j \in \mathbb{Z}) \,,$$

equation (2.6) can be reduced to the rational mapping

$$\overline{X}_j = X_j + \frac{\epsilon\,\delta}{\epsilon - Y_j} - \frac{\epsilon\,\delta}{\epsilon - Y_{j-1}} \,, \quad \overline{Y}_j = Y_j - \frac{\epsilon\,\delta}{\epsilon - \overline{X}_j} + \frac{\epsilon\,\delta}{\epsilon - \overline{X}_{j+1}} \,. \tag{2.7}$$

We now impose the even periodicity condition

$$\mathrm{u}_{2(j+P)} = \mathrm{u}_{2j} \,, \qquad \mathrm{u}_{2(j+P)+1} = \mathrm{u}_{2j+1} \qquad (j \in \mathbb{Z}).$$

Equation (2.7) for $j = 1, 2, \ldots, P$ together with constraint

$$\sum_{j=1}^{P} X_j = 0 \,, \qquad \sum_{j=1}^{P} Y_{j-1} = 0 \,, \tag{2.8}$$

is a $2(P-1)$-dimensional integrable mapping.

The trivial case is $N = 2P = 2$, when we have only two different initial values $\mathrm{u}_0, \mathrm{u}_1$ on the staircase, as illustrated in figure 2.3(a). We take the discrete-time evolution along the diagonal direction and indicate it by the bar shift. Then, from the lattice equation (2.1) applied on the two quadrilaterals in figure 2.3(a), we obtain the linear set $\overline{\mathrm{u}}_0 = \mathrm{u}_0$ and $\overline{\mathrm{u}}_1 = \mathrm{u}_1$.

The $N = 4$ (simplest) case is the next case of interest, as here we find a system of one-degree-of-freedom with one invariant.

2.3.1 Period 2 reduction

Consider the $P = 2$ reduction illustrated in figure 2.3(b). We begin with initial data u_0, u_1, u_2, u_3 and u_4, and let $\mathrm{u}_4 = \mathrm{u}_0$, according to figure 2.3(b). This unit is assumed to repeat periodically through the lattice to form a connected periodic staircase. Applying

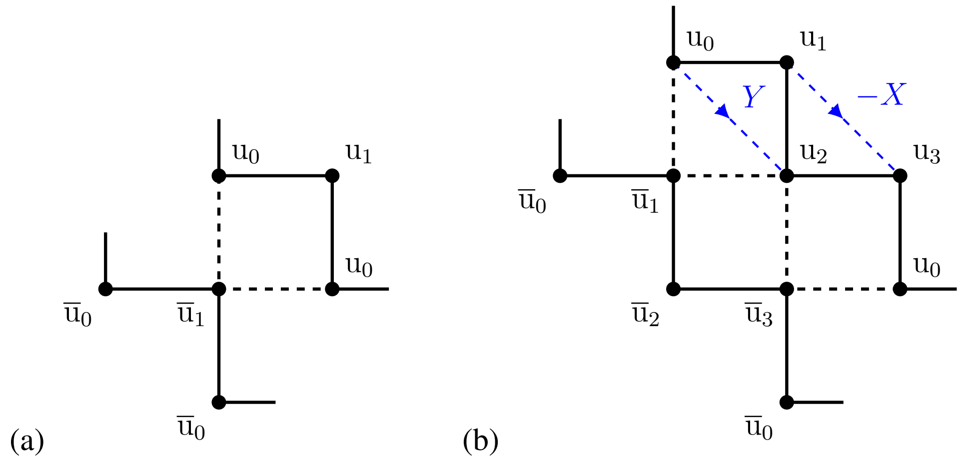


Figure 2.3: Configuration of points for periods 1 and 2 reduction.

the lattice equation (2.1) to the four plaquettes in figure 2.3(b), we obtain the dynamical mapping $(\mathrm{u}_0, \mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3) \to (\overline{\mathrm{u}}_0, \overline{\mathrm{u}}_1, \overline{\mathrm{u}}_2, \overline{\mathrm{u}}_3)$ associated with this periodic reduction:

$$\overline{\mathrm{u}}_1 = \mathrm{u}_1 - \delta + \frac{\epsilon\,\delta}{\epsilon - \mathrm{u}_2 + \mathrm{u}_0}\,, \qquad \overline{\mathrm{u}}_0 = \mathrm{u}_0 - \delta + \frac{\epsilon\,\delta}{\epsilon - \overline{\mathrm{u}}_1 + \overline{\mathrm{u}}_3}\,, \tag{2.9a}$$

$$\overline{\mathrm{u}}_3 = \mathrm{u}_3 - \delta + \frac{\epsilon\,\delta}{\epsilon - \mathrm{u}_0 + \mathrm{u}_2}\,, \qquad \overline{\mathrm{u}}_2 = \mathrm{u}_2 - \delta + \frac{\epsilon\,\delta}{\epsilon - \overline{\mathrm{u}}_3 + \overline{\mathrm{u}}_1}\,. \tag{2.9b}$$

By introducing the reduced variables $X := \mathrm{u}_1 - \mathrm{u}_3$ and $Y := \mathrm{u}_2 - \mathrm{u}_0$ as illustrated in figure 2.3(b), the four-component map (2.9) can be reduced to a two-component map

$$\overline{X} = X + \frac{2\,\epsilon\,\delta\,Y}{\epsilon^2 - Y^2}\,, \qquad \overline{Y} = Y - \frac{2\,\epsilon\,\delta\,\overline{X}}{\epsilon^2 - \overline{X}^2}\,. \tag{2.10}$$

It is obvious that this map is close to the identity mapping if δ is small. Furthermore, according to the definition of integrability in section 1.3.4 of the introduction the map (2.10) is integrable and possesses an exact invariant

$$\mathcal{I}(X, Y) = X^2\,Y^2 - \epsilon^2\,X^2 - \epsilon^2\,Y^2 - 2\,\epsilon\,\delta\,X\,Y\,, \tag{2.11}$$

noting that the construction of this quantity will be detailed in section 2.5.

2.3.2 Period 3 reduction

As the next case of interest, let us consider the $P = 3$ reduction depicted in figure 2.4. This case gives rise to a system of two-degrees-of-freedom (i.e. 4-dimensional mapping)

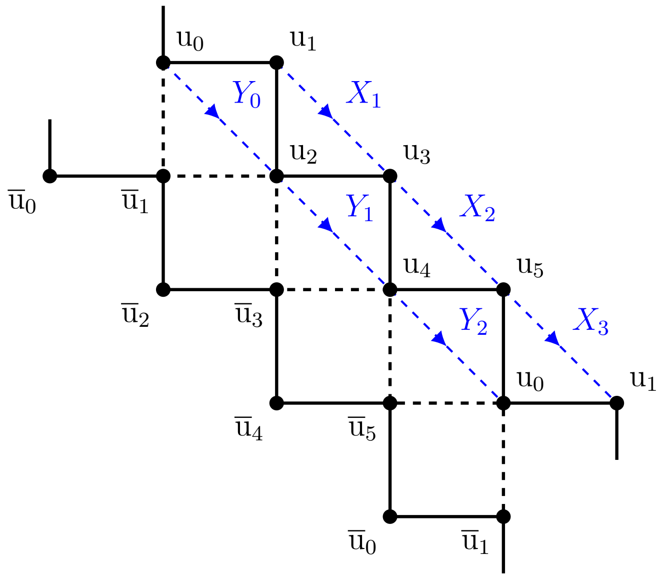


Figure 2.4: Configuration of points for period 3 reduction.

with two commuting invariants. We take u_0, u_1, u_2, u_3, u_4 and u_5 as initial data supplied for the configuration in the figure. This configuration is repeated periodically across an infinite staircase in the lattice. Again, from the lattice equation (2.1) applied to the six quadrilaterals in figure 2.4 we get the dynamical mapping $(\mathrm{u}_0,\mathrm{u}_1,\mathrm{u}_2,\mathrm{u}_3,\mathrm{u}_4,\mathrm{u}_5) \to (\overline{\mathrm{u}}_0,\overline{\mathrm{u}}_1,\overline{\mathrm{u}}_2,\overline{\mathrm{u}}_3,\overline{\mathrm{u}}_4,\overline{\mathrm{u}}_5)$:

$$\overline{\mathrm{u}}_1 = \mathrm{u}_1 - \delta + \frac{\epsilon\,\delta}{\epsilon - \mathrm{u}_2 + \mathrm{u}_0}\,, \qquad \overline{\mathrm{u}}_0 = \mathrm{u}_0 - \delta + \frac{\epsilon\,\delta}{\epsilon - \overline{\mathrm{u}}_1 + \overline{\mathrm{u}}_5}\,, \tag{2.12a}$$

$$\overline{\mathrm{u}}_3 = \mathrm{u}_3 - \delta + \frac{\epsilon\,\delta}{\epsilon - \mathrm{u}_4 + \mathrm{u}_2}\,, \qquad \overline{\mathrm{u}}_2 = \mathrm{u}_2 - \delta + \frac{\epsilon\,\delta}{\epsilon - \overline{\mathrm{u}}_3 + \overline{\mathrm{u}}_1}\,, \tag{2.12b}$$

$$\overline{\mathrm{u}}_5 = \mathrm{u}_5 - \delta + \frac{\epsilon\,\delta}{\epsilon - \mathrm{u}_0 + \mathrm{u}_4}\,, \qquad \overline{\mathrm{u}}_4 = \mathrm{u}_4 - \delta + \frac{\epsilon\,\delta}{\epsilon - \overline{\mathrm{u}}_5 + \overline{\mathrm{u}}_3}\,. \tag{2.12c}$$

We introduce the reduced variables

$$X_1 := \mathrm{u}_3 - \mathrm{u}_1\,, \quad X_2 := \mathrm{u}_5 - \mathrm{u}_3\,, \quad X_3 := \mathrm{u}_1 - \mathrm{u}_5\,,$$

$$Y_0 := \mathrm{u}_2 - \mathrm{u}_0\,, \quad Y_1 := \mathrm{u}_4 - \mathrm{u}_2\,, \quad Y_2 := \mathrm{u}_0 - \mathrm{u}_4\,,$$

as in figure 2.4. Using the periodicity constraints

$$X_1 + X_2 + X_3 = 0\,, \qquad Y_0 + Y_1 + Y_2 = 0\,,$$

one can reduce the map (2.12) to a dynamical map in terms of four variables X_1, X_2, Y_1, Y_2

$$\overline{X}_1 = X_1 + \frac{\epsilon\,\delta}{\epsilon - Y_1} - \frac{\epsilon\,\delta}{\epsilon + Y_1 + Y_2}\,, \qquad \overline{Y}_1 = Y_1 + \frac{\epsilon\,\delta}{\epsilon - \overline{X}_2} - \frac{\epsilon\,\delta}{\epsilon - \overline{X}_1}\,, \tag{2.13a}$$

$$\overline{X}_2 = X_2 + \frac{\epsilon\,\delta}{\epsilon - Y_2} - \frac{\epsilon\,\delta}{\epsilon - Y_1}\,, \qquad \overline{Y}_2 = Y_2 + \frac{\epsilon\,\delta}{\epsilon + \overline{X}_1 + \overline{X}_2} - \frac{\epsilon\,\delta}{\epsilon - \overline{X}_2}\,. \tag{2.13b}$$

Once again we get a rational map which is close to the identity mapping if δ is small, noting that a Hamiltonian structure for the mapping (2.13) will be established in section 4. As discussed in section 2.5, invariants of the map (2.13) are given by

$$\begin{aligned} \mathcal{I}_1 &= x_1\,x_2\,x_3\,y_0\,y_1\,y_2 - \epsilon\,\delta\,(x_1\,x_2\,y_1\,y_2 + x_1\,x_3\,y_0\,y_1 + x_2\,x_3\,y_0\,y_2) \\ &\quad + \epsilon^2\,\delta^2\,(x_1\,y_1 + x_2\,y_2 + x_3\,y_0)\,, \end{aligned} \tag{2.14a}$$

$$\begin{aligned} \mathcal{I}_2 &= x_1\,x_2\,x_3\,y_0\,y_1\,y_2 + \epsilon\,\delta\,(x_1\,x_2\,y_0\,y_1 + x_1\,x_3\,y_0\,y_2 + x_2\,x_3\,y_1\,y_2) \\ &\quad + \epsilon^2\,\delta^2\,(x_1\,y_0 + x_2\,y_1 + x_3\,y_2)\,, \end{aligned} \tag{2.14b}$$

in which we have used the shorthand notation

$$x_j := \epsilon - X_j\,,\ \ y_j := \epsilon - Y_j\ \ (j = 1, 2)\,,\ \ y_0 := \epsilon + Y_1 + Y_2\,,\ \ x_3 := \epsilon + X_1 + X_2\,.$$

In the sense of the definition given in section 1.3.4, the map (2.13) is integrable, and hence, it has a Lax pair as discussed in the next section.

2.4 Lax matrices

In this section, Lax matrices are derived for the previously given mappings of KdV type. The Lax matrices are derived using the Lax matrices for the lattice KdV equation.

The Lax pair for the lattice KdV consists of an $\mathcal{L}$-part, which effects a horizontal translation $u_{n,m} \to u_{n+1,m}$, and an $\mathcal{M}$-part, which effects a vertical translation $u_{n,m} \to u_{n,m+1}$, as shown in figure 2.1: that is

$$\mathcal{L} := U\,\mathrm{A}\,\widetilde{U}^{-1}\,, \qquad \mathcal{M} := U\,\mathrm{B}\,\widehat{U}^{-1}\,,$$

in which

$$U = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \quad \mathrm{A} = \begin{pmatrix} a & 1 \\ k^2 & a \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} b & 1 \\ k^2 & b \end{pmatrix}.$$

As discussed in section 2.2, the associated linear problem is

$$\mathcal{L}\,\Phi = (a-k)\,\widetilde{\Phi}\,, \qquad \mathcal{M}\,\Phi = (b-k)\,\widehat{\Phi}.$$

Like the initial value configuration that we have considered, the Lax matrices $\mathcal{L}$, $\mathcal{M}$ can be written as

$$\mathcal{L} = \mathrm{U}_{2j}\,\mathrm{A}\,\mathrm{U}_{2j+1}^{-1}\,, \qquad \mathcal{M} = \mathrm{U}_{2j+1}\,\mathrm{B}\,\mathrm{U}_{2j+2}^{-1}\,.$$

Let us now consider the two paths shown in figure 2.5, where the clockwise path (red path) is effected by

$$\overline{\mathrm{U}}_{2j+3}\,\mathrm{A}^{-1}\,\overline{\mathrm{U}}_{2j+2}^{-1}\,\mathrm{U}_{2j+2}\,\mathrm{B}\,\overline{\mathrm{U}}_{2j+3}^{-1}\,\mathrm{U}_{2j+1}\,\mathrm{B}\,\mathrm{U}_{2j+2}^{-1}\,\mathrm{U}_{2j}\,\mathrm{A}\,\mathrm{U}_{2j+1}^{-1}$$

and the anticlockwise path (green path) is effected by

$$\overline{\mathrm{U}}_{2j+1}\,\mathrm{B}\,\overline{\mathrm{U}}_{2j+2}^{-1}\,\overline{\mathrm{U}}_{2j}\,\mathrm{A}\,\overline{\mathrm{U}}_{2j+1}^{-1}\,\overline{\mathrm{U}}_{2j+1}\,\mathrm{A}^{-1}\,\overline{\mathrm{U}}_{2j}^{-1}\,\mathrm{U}_{2j}\,\mathrm{B}\,\overline{\mathrm{U}}_{2j+1}^{-1}\,.$$

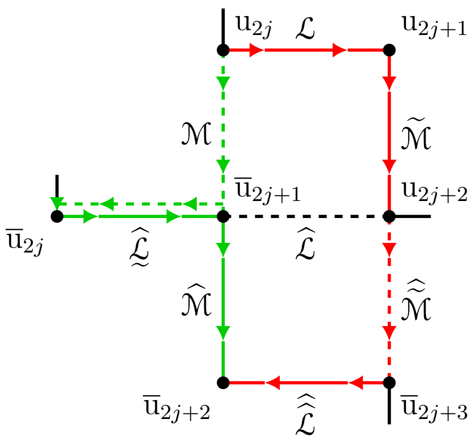


Figure 2.5: The Lax matrices for the mappings are derived by considering the compatibility of the two paths shown.

These two paths yield the equality

$$\widehat{\widehat{\mathcal{L}}}^{-1}\,\widehat{\widetilde{\mathcal{M}}}\,\widetilde{\mathcal{M}}\,\mathcal{L} = \widehat{\mathcal{M}}\,\underset{\sim}{\widehat{\mathcal{L}}}\,\underset{\sim}{\widehat{\mathcal{L}}}^{-1}\,\mathcal{M} \quad\Leftrightarrow\quad \widehat{\widetilde{\mathcal{M}}}\,\widetilde{\mathcal{M}}\,\mathcal{L} = \widehat{\widehat{\mathcal{L}}}\,\widehat{\mathcal{M}}\,\mathcal{M}\,, \tag{2.15}$$

which follows as a straightforward consequence from the zero-curvature condition (1.67). By equating the two paths, we have

$$\begin{aligned}
&\overline{\mathrm{U}}_{2j+3}\,\mathrm{A}^{-1}\,\overline{\mathrm{U}}_{2j+2}^{-1}\,\mathrm{U}_{2j+2}\,\mathrm{B}\,\overline{\mathrm{U}}_{2j+3}^{-1}\,\mathrm{U}_{2j+1}\,\mathrm{B}\,\mathrm{U}_{2j+2}^{-1}\,\mathrm{U}_{2j}\,\mathrm{A}\,\mathrm{U}_{2j+1}^{-1}\\
&= \overline{\mathrm{U}}_{2j+1}\,\mathrm{B}\,\overline{\mathrm{U}}_{2j+2}^{-1}\,\overline{\mathrm{U}}_{2j}\,\mathrm{A}\,\overline{\mathrm{U}}_{2j+1}^{-1}\,\overline{\mathrm{U}}_{2j+1}\,\mathrm{A}^{-1}\,\overline{\mathrm{U}}_{2j}^{-1}\,\mathrm{U}_{2j}\,\mathrm{B}\,\overline{\mathrm{U}}_{2j+1}^{-1}\,,\\
\Longrightarrow\ &\left(\mathrm{B}\,\overline{\mathrm{U}}_{2j+2}^{-1}\,\overline{\mathrm{U}}_{2j}\,\mathrm{A}\,\overline{\mathrm{U}}_{2j+1}^{-1}\,\overline{\mathrm{U}}_{2j-1}\right)\left(\overline{\mathrm{U}}_{2j-1}^{-1}\,\overline{\mathrm{U}}_{2j+1}\,\mathrm{A}^{-1}\,\overline{\mathrm{U}}_{2j}^{-1}\,\mathrm{U}_{2j}\,\mathrm{B}\,\overline{\mathrm{U}}_{2j+1}^{-1}\,\mathrm{U}_{2j-1}\right)\\
&= \left(\overline{\mathrm{U}}_{2j+1}^{-1}\,\overline{\mathrm{U}}_{2j+3}\,\mathrm{A}^{-1}\,\overline{\mathrm{U}}_{2j+2}^{-1}\,\mathrm{U}_{2j+2}\,\mathrm{B}\,\overline{\mathrm{U}}_{2j+3}^{-1}\,\mathrm{U}_{2j+1}\right)\left(\mathrm{B}\,\mathrm{U}_{2j+2}^{-1}\,\mathrm{U}_{2j}\,\mathrm{A}\,\mathrm{U}_{2j+1}^{-1}\,\mathrm{U}_{2j-1}\right).
\end{aligned}$$

By renaming the matrix products in the brackets, we can write the latter equation in the form of the discrete-time system

$$\overline{\mathbf{L}}_j\,\mathbf{M}_j = \mathbf{M}_{j+1}\,\mathbf{L}_j\ , \tag{2.16}$$

in which

$$\mathbf{L}_j = \mathrm{B}\,\mathrm{U}_{2j+2}^{-1}\,\mathrm{U}_{2j}\,\mathrm{A}\,\mathrm{U}_{2j+1}^{-1}\,\mathrm{U}_{2j-1}\ ,\quad \mathbf{M}_j = \overline{\mathrm{U}}_{2j-1}^{-1}\,\overline{\mathrm{U}}_{2j+1}\,\mathrm{A}^{-1}\,\overline{\mathrm{U}}_{2j}^{-1}\,\mathrm{U}_{2j}\,\mathrm{B}\,\overline{\mathrm{U}}_{2j+1}^{-1}\,\mathrm{U}_{2j-1}\ ,$$

which can be used to construct the integrals of mappings.

By using a special property of the Lax matrices $\mathbf{L}_j$ and $\mathbf{M}_j$, it turns out that we can perform at each site of the staircase a gauge transformation[3] such that we obtain expressions for Lax matrices in terms of the reduced variables X_j and Y_j only. In this way we can factorize the Lax matrices $\mathbf{L}_j$ and $\mathbf{M}_j$:

$$\mathbf{L}_j = \mathcal{B}\,L_j\,\mathcal{B}^{-1}\ ,\qquad \mathbf{M}_j = \mathcal{B}\,M_j\,\mathcal{B}^{-1}\ ,\qquad \mathcal{B} = \begin{pmatrix} 1 & 0\\ b & 1\end{pmatrix},$$

in which

$$L_j(\lambda) = V_{2j}\,V_{2j-1}\ ;\qquad V_j = \begin{pmatrix} v_j & 1\\ \lambda_j & 0\end{pmatrix}, \tag{2.17a}$$

$$M_j(\lambda) = \begin{pmatrix} -\epsilon\,\delta/\overline{v}_{2j-1} & 1\\ \lambda & -\overline{v}_{2j-1}\end{pmatrix}\begin{pmatrix} v_{2j-1}-\epsilon\,\delta/v_{2j} & 1\\ \lambda & 0\end{pmatrix}, \tag{2.17b}$$

[3] A discussion of gauge theories can be found in [2].

where $v_{2j} := \epsilon - Y_j$, $v_{2j-1} := \epsilon - X_j$, $\lambda_{2j} := k^2 - b^2 = \lambda$ and $\lambda_{2j-1} := \lambda - \epsilon\,\delta$, noting that we have used the factorization

$$\mathrm{A} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ k^2 - a^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ k^2 - b^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix},$$

in order to achieve the Lax matrices (2.17). Thus, the condition (2.16) becomes

$$\overline{L}_j\, M_j = M_{j+1}\, L_j\ , \tag{2.18}$$

which is known as *Zakharov-Shabat* condition. By a direct calculation, it can be verified that the mapping of KdV type (2.7) is generated by the Zakharov-Shabat equation (2.18) as a compatibility condition.

Having obtained the linear system in Zakharov-Shabat form (2.18), we can construct a matrix $T(\lambda)$, called the *monodromy matrix*, by gluing the elementary translation matrices $L_j(\lambda)$ along the staircase over one period P, leading to

$$T(\lambda) := \prod_{j=0}^{\overset{\curvearrowleft}{P-1}} L_j(\lambda), \tag{2.19}$$

where the $\curvearrowleft$ indicates that the factors in the product are arranged from right to left. By construction, the monodromy matrix (2.19) has a particularly simple behaviour under application of the mapping, namely

$$\overline{T}(\lambda) = M_{P+1}(\lambda)\, T(\lambda)\, M_1(\lambda)^{-1}, \tag{2.20}$$

where use has been made of the compatibility condition of the Lax representation and the periodicity condition $M_{P+1} = M_1 \equiv M$. Equation (2.20) implies that $\mathrm{tr}\,(\overline{T}) = \mathrm{tr}\,(T)$ where $\mathrm{tr}\,(T) := \mathrm{trace}\,(T)$. Expanding the trace in powers of the spectral parameter λ leading to a sufficient number of invariants.

2.5 Invariants

The derivations of the invariants for the discrete-time evolution given by the mappings presented in section 2.3 are given in this section. In fact, all mappings obtained in section 2.3 exhibit $P-1$ nontrivial integrals which can be found in a way by exploiting either the Lax matrices (2.2) of the original lattice KdV equation or the Lax matrices (2.17).

In order to do this using the Lax (2.2), we need to define the monodromy matrix $\mathcal{T}(k)$ as

$$\mathcal{T}(k) = \prod_{v=0}^{\overset{\curvearrowleft}{N-1}} \mathcal{L}(\mathrm{u}_{v+1}, \mathrm{u}_v) \;, \tag{2.21}$$

in which the translation matrices $\mathcal{L}(\mathrm{u}_{v+1}, \mathrm{u}_v)$ represent either the Lax matrix in the n-direction (v even), i.e. $\mathcal{L}(k)$, or the Lax matrix in the m-direction (v odd), i.e. $\mathcal{M}(k)$. They are of the form

$$\mathcal{L}(\mathrm{u}_{v+1}, \mathrm{u}_v) = \begin{pmatrix} a_{v+1} - \mathrm{u}_{v+1} & 1 \\ k^2 - a_{v+1}(\mathrm{u}_{v+1} - \mathrm{u}_v) - \mathrm{u}_{v+1}\mathrm{u}_v & a_{v+1} + \mathrm{u}_v \end{pmatrix} , \tag{2.22}$$

where $a_{v+1} = a$ if v is even and $a_{v+1} = b$ if v is odd. This leads to (cf. [95])

$$\begin{aligned} \mathrm{tr}\,\mathcal{T}(k) = & \prod_{v=1}^{N} (a_v + a_{v+1} + \mathrm{u}_{v-1} - \mathrm{u}_{v+1}) \\ & + \sum_{j=1}^{N} (k^2 - a_j^2) \prod_{\substack{v=1 \\ v \neq j-1, j}}^{N} (a_v + a_{v+1} + \mathrm{u}_{v-1} - \mathrm{u}_{v+1}) \\ & + \sum_{i<j=1}^{N} (k^2 - a_i^2)(k^2 - a_j^2) \prod_{\substack{v=1 \\ v \neq i-1, i, j-1, j}}^{N} (a_v + a_{v+1} + \mathrm{u}_{v-1} - \mathrm{u}_{v+1}) \\ & + \cdots . \end{aligned} \tag{2.23}$$

The coefficients of the powers of k are the integrals of the mappings in the sense of definition 1.3.1. It should be noted that the monodromy matrix $T(\lambda)$ from (2.19) is the same as $\mathcal{T}(k)$ from (2.21), apart from a similarity transformation corresponding to a gauge transformation at the beginning and end points of the chain from 0 to $2P$ (which by periodicity is performed by the same multiplying matrix).

Alternatively, as a consequence of the discrete-time evolution (2.20), the $\mathrm{tr}\, T(\lambda)$ of the monodromy matrix (2.19) is invariant under the mapping (2.7), so we can generate $P-1$ invariants by developing the trace in powers of the spectral parameter λ.

Let us now look at the two cases considered in section 2.3, namely the periods $P = 2$ and $P = 3$ reductions (observe that we have omitted the trivial case $P = 1$ since it has no invariant). In the case of $N = 2P = 4$, the trace of the monodromy matrix $\mathcal{T}(k)$ reads as follows:

$$\begin{aligned}
\operatorname{tr} \mathcal{T}(k) &= \operatorname{tr}\{\mathcal{L}(\mathrm{u}_0, \mathrm{u}_3)\,\mathcal{L}(\mathrm{u}_3, \mathrm{u}_2)\,\mathcal{L}(\mathrm{u}_2, \mathrm{u}_1)\,\mathcal{L}(\mathrm{u}_1, \mathrm{u}_0)\} \\
&= \operatorname{tr}\left\{\mathrm{U}_3\,\mathrm{B}\,\mathrm{U}_0^{-1}\,\mathrm{U}_2\,\mathrm{A}\,\mathrm{U}_3^{-1}\,\mathrm{U}_1\,\mathrm{B}\,\mathrm{U}_2^{-1}\,\mathrm{U}_0\,\mathrm{A}\,\mathrm{U}_1^{-1}\right\} \\
&= \operatorname{tr}\left\{\mathrm{B}\,\mathrm{U}_0^{-1}\,\mathrm{U}_2\,\mathrm{A}\,\mathrm{U}_3^{-1}\,\mathrm{U}_1\,\mathrm{B}\,\mathrm{U}_2^{-1}\,\mathrm{U}_0\,\mathrm{A}\,\mathrm{U}_1^{-1}\,\mathrm{U}_3\right\} \\
&= \operatorname{tr}\left\{\begin{pmatrix} \epsilon + Y & 1 \\ k^2 - b^2 & 0 \end{pmatrix}\begin{pmatrix} \epsilon + X & 1 \\ k^2 - a^2 & 0 \end{pmatrix}\begin{pmatrix} \epsilon - Y & 1 \\ k^2 - b^2 & 0 \end{pmatrix}\begin{pmatrix} \epsilon - X & 1 \\ k^2 - a^2 & 0 \end{pmatrix}\right\} \\
&= 2\,k^4 + (3\,\epsilon^2 - \delta^2)\,k^2 + X^2\,Y^2 - \epsilon^2\,(X^2 + Y^2) - 2\,\epsilon\,\delta\,X\,Y \\
&\quad + \frac{1}{8}\,(\epsilon^4 + \delta^4) - \frac{1}{4}\,\epsilon^2\,\delta^2,
\end{aligned} \tag{2.24}$$

where $X = \mathrm{u}_1 - \mathrm{u}_3$, $Y = \mathrm{u}_2 - \mathrm{u}_0$, $\delta = a - b$ and $\epsilon = a + b$. Since the trace is invariant and the spectral parameter k^2 is arbitrary, equation (2.24) yields the integral of the mapping (2.10)

$$\mathcal{I} = X^2\,Y^2 - \epsilon^2\,X^2 - \epsilon^2\,Y^2 - 2\,\epsilon\,\delta\,X\,Y\,, \tag{2.25}$$

cf. (2.11), which is an elliptic curve that can be parametrized in terms of Jacobi elliptic functions, thus leading to explicit solutions to the corresponding mapping.

An alternative way of computing the invariant (2.25) is obtained by expanding the trace of the monodromy matrix $T(\lambda)$ (transfer matrix) in powers of the spectral parameter λ which leads to the same result. This can be seen in

$$\begin{aligned}
\operatorname{tr} T(\lambda) &= \operatorname{tr}\{L_1(\lambda)\,L_0(\lambda)\} \\
&= \operatorname{tr}\left\{\begin{pmatrix} \epsilon - Y_1 & 1 \\ \lambda & 0 \end{pmatrix}\begin{pmatrix} \epsilon - X_1 & 1 \\ \lambda - \epsilon\,\delta & 0 \end{pmatrix}\begin{pmatrix} \epsilon - Y_0 & 1 \\ \lambda & 0 \end{pmatrix}\begin{pmatrix} \epsilon - X_2 & 1 \\ \lambda - \epsilon\,\delta & 0 \end{pmatrix}\right\} \\
&= 2\,\lambda^2 + 2\,\epsilon\,(2\,\epsilon - \delta)\,\lambda + X^2\,Y^2 - \epsilon^2\,(X^2 + Y^2) - 2\,\epsilon\,\delta\,X\,Y \\
&\quad + (\epsilon\,\delta - \epsilon^2)^2,
\end{aligned} \tag{2.26}$$

in which $X_2 := X, X_1 := -X, Y_0 := Y$ and $Y_1 := -Y$. Again, since the trace is invariant and the spectral parameter λ is arbitrary, equation (2.26) yields the integral (2.25).

For $N = 2P = 6$, we get for the trace of the monodromy matrix $T(\lambda)$ in powers of the spectral parameter λ:

$$\begin{aligned}
\operatorname{tr} T(\lambda) &= \operatorname{tr}\{L_2(\lambda)\, L_1(\lambda)\, L_0(\lambda)\} \\
&= \operatorname{tr}\left\{\begin{array}{l} \begin{pmatrix} \epsilon - Y_2 & 1 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} \epsilon - X_2 & 1 \\ \lambda - \epsilon\,\delta & 0 \end{pmatrix} \begin{pmatrix} \epsilon - Y_1 & 1 \\ \lambda & 0 \end{pmatrix} \\ \cdot \begin{pmatrix} \epsilon - X_1 & 1 \\ \lambda - \epsilon\,\delta & 0 \end{pmatrix} \begin{pmatrix} \epsilon - Y_0 & 1 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} \epsilon - X_3 & 1 \\ \lambda - \epsilon\,\delta & 0 \end{pmatrix} \end{array}\right\} \\
&= 2\,\lambda^3 + 3\,\epsilon\,(3\,\epsilon - \delta)\,\lambda^2 + \left[(1/\epsilon\,\delta)\,(\mathcal{I}_2 - \mathcal{I}_1) - 3\,\epsilon^2\,\delta\,(3\,\epsilon - \delta)\right]\lambda \\
&+ \mathcal{I}_1 - \epsilon^3\,\delta^3,
\end{aligned} \tag{2.27}$$

where

$$X_1 = \mathrm{u}_3 - \mathrm{u}_1, \quad X_2 = \mathrm{u}_5 - \mathrm{u}_3, \quad X_3 = \mathrm{u}_1 - \mathrm{u}_5\,,$$
$$Y_0 = \mathrm{u}_2 - \mathrm{u}_0, \quad Y_1 = \mathrm{u}_4 - \mathrm{u}_2, \quad Y_2 = \mathrm{u}_0 - \mathrm{u}_4\,.$$

Equation (2.27) yields the invariants $\mathcal{I}_1$ and $\mathcal{I}_2$ which are conserved by the mapping (2.13). These invariants are calculated as

$$\begin{aligned}
\mathcal{I}_1 &= x_1\,x_2\,x_3\,y_0\,y_1\,y_2 - \epsilon\,\delta\,(x_1\,x_2\,y_1\,y_2 + x_1\,x_3\,y_0\,y_1 + x_2\,x_3\,y_0\,y_2) \\
&\quad + \epsilon^2\,\delta^2\,(x_1\,y_1 + x_2\,y_2 + x_3\,y_0)\,,
\end{aligned} \tag{2.28a}$$

$$\begin{aligned}
\mathcal{I}_2 &= x_1\,x_2\,x_3\,y_0\,y_1\,y_2 + \epsilon\,\delta\,(x_1\,x_2\,y_0\,y_1 + x_1\,x_3\,y_0\,y_2 + x_2\,x_3\,y_1\,y_2) \\
&\quad + \epsilon^2\,\delta^2\,(x_1\,y_0 + x_2\,y_1 + x_3\,y_2)\,,
\end{aligned} \tag{2.28b}$$

cf. (2.14), in which we have used the abbreviations

$$x_j = \epsilon - X_j\,, \quad y_j = \epsilon - Y_j \quad (j = 1, 2)\,, \quad y_0 = \epsilon + Y_1 + Y_2\,, \quad x_3 = \epsilon + X_1 + X_2\,.$$

The canonical structure will allow us to show the critical integrability property that the two invariants are in involution with each other, with respect to the canonical Poisson bracket. This involutivity with respect to a properly chosen symplectic structure is discussed in section 4.7 of chapter 4.

2.6 Lattice modified KdV system

By considering a staircase with diagonal shift evolution, the lattice KdV equation (2.1) already gives rise to finite-dimensional integrable mappings which are viewed as an application of the symplectic Euler method. Such mappings where the time step is an "implicit" dependence is the object of our interest in this section. These mappings arise from periodic initial value problems of the lattice modified KdV (mKdV) equation [83, 95]

$$a\,v\,\widehat{v} + b\,\widehat{v}\,\widehat{\widetilde{v}} = b\,v\,\widetilde{v} + a\,\widetilde{v}\,\widehat{\widetilde{v}}\,, \tag{2.29}$$

which is essentially equation (1.45). The parameters a and b denote as before the lattice parameters, and the notations for the translations in the lattice direction are as before in the lattice KdV case. A Lax representation for equation (2.29) is constructed [56] in the same way as in section 1.3.2 and given by

$$(a-k)\widetilde{\Psi}(k) = \mathfrak{L}(k)\Psi(k)\,, \quad (b-k)\widehat{\Psi}(k) = \mathfrak{M}(k)\Psi(k)\,, \tag{2.30}$$

where $\mathfrak{L}(k)$ and $\mathfrak{M}(k)$ are given by

$$\mathfrak{L}(k) = \begin{pmatrix} a & \widetilde{v} \\ k^2/v & a\,\widetilde{v}/v \end{pmatrix}, \quad \mathfrak{M}(k) = \begin{pmatrix} b & \widehat{v} \\ k^2/v & b\,\widehat{v}/v \end{pmatrix}.$$

The lattice equation (2.29) now arises from the compatibility of the matrix spectral problem (2.30), which is the zero-curvature condition $\widehat{\mathfrak{L}}\,\mathfrak{M} = \widetilde{\mathfrak{M}}\,\mathfrak{L}$.

Equation (2.29) is related to the lattice KdV equation via a Miura transformation as follows:

$$a - b + \widehat{u} - \widetilde{u} = \frac{a\,\widetilde{v} - b\,\widehat{v}}{v}\,. \tag{2.31}$$

On the level of the linear system, this reflects a gauge transformation of the form

$$k^2\,\Psi(k) = \mathfrak{U}\,\Phi(k)\,, \quad \mathfrak{U} = \begin{pmatrix} s & v \\ k^2 & 0 \end{pmatrix},$$

in which $s = (a-\widetilde{u})\,v - a\,\widetilde{v}$ and $\widetilde{s} = a\,v - (a+u)\,\widetilde{v}$, leading to

$$\mathfrak{L}(k) = \widetilde{\mathfrak{U}}\,\mathcal{L}\,\mathfrak{U}^{-1}\,, \quad \widetilde{\mathfrak{U}}\,F\,\mathfrak{U}^{-1} = \frac{\widetilde{v}}{k^2}\,E\,, \quad \text{where} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It should be noted that a similar relation with the same s holds for $\mathfrak{M}(k)$, just by replacing $a \to b$ and $\widetilde{\ } \to \widehat{\ }$, i.e.

$$\mathfrak{M}(k) = \widehat{\mathcal{U}}\, \mathcal{M}\, \mathcal{U}^{-1}.$$

Again, there is an action principle, now not directly in terms of the variable $v_{n,m}$, but in terms of its logarithm. Thus, by writing $v_{n,m} := e^{w_{n,m}}$, the equation (2.29) can also be derived via discrete Euler-Lagrange equations on the four-point Lagrangian

$$\begin{gathered}\mathscr{L}(w,\widetilde{w},\widehat{w},\widehat{\widetilde{w}}) = w\,(\widetilde{w}-\widehat{w}) + F(w-\widehat{\widetilde{w}}+\sigma) - F(w-\widehat{\widetilde{w}}+\rho)\;;\\ \widehat{\widetilde{\left(\frac{\partial\mathscr{L}}{\partial w}\right)}} + \widehat{\left(\frac{\partial\mathscr{L}}{\partial \widetilde{w}}\right)} + \widetilde{\left(\frac{\partial\mathscr{L}}{\partial \widehat{w}}\right)} + \left(\frac{\partial\mathscr{L}}{\partial \widehat{\widetilde{w}}}\right) = 0\,,\end{gathered} \tag{2.32}$$

where, for the action, we sum across every plaquette in the lattice

$$\mathcal{S} = \sum_{n,m\in\mathbb{Z}} \mathscr{L}(w_{n,m}, w_{n+1,m}, w_{n,m+1}, w_{n+1,m+1})\;. \tag{2.33}$$

In equation (2.32), the function F is given by

$$F(x) = \int_{-\infty}^{x} \log\left(1+e^{w}\right) dw\;,$$

which, in fact, is directly related to the Euler dilogarithm function; the σ and ρ are parameters related to the lattice parameters a and b, namely by

$$e^{\sigma} = b/a\;, \qquad e^{\rho} = a/b\;.$$

We note that everything that holds true for the lattice KdV equation holds true also for the lattice mKdV equation, such as periodic reductions, Lax pairs and invariants.

2.6.1 Reduced mapping and Lax system

Let us now consider initial value problems for (2.29) on the lattice in precisely the same way as before with initial data on staircases like the one depicted in figure 2.2, namely

$$v(j,j) =: \mathrm{v}_{2j}\;, \qquad v(j+1,j) =: \mathrm{v}_{2j+1} \qquad (j\in\mathbb{Z})\;.$$

We perform iterations by updating the lattice variables v along a diagonal shift, i.e.

$$v(j-1,j+1) =: \overline{\mathrm{v}}_{2j}\;, \qquad v(j,j+1) =: \overline{\mathrm{v}}_{2j+1}\;,$$

using the lattice mKdV (2.29). One obtains the mapping

$$\overline{\mathrm{v}}_{2j+1} = \mathrm{v}_{2j+1} \frac{\mathrm{v}_{2j} + r\,\mathrm{v}_{2j+2}}{\mathrm{v}_{2j+2} + r\,\mathrm{v}_{2j}} \;, \qquad \overline{\mathrm{v}}_{2j} = \mathrm{v}_{2j} \frac{\mathrm{v}_{2j-1} + r\,\mathrm{v}_{2j+1}}{\mathrm{v}_{2j+1} + r\,\mathrm{v}_{2j-1}} \;, \tag{2.34}$$

where $r = a/b$. Again, we use a diagonal shift so that the mapping is close to the identity mapping if r is close to 1. Similarly as in section 2.3, we can reduce the system (2.34) in terms of the logarithmic variables

$$X_j := \log \frac{\mathrm{v}_{2j+1}}{\mathrm{v}_{2j-1}} \;, \qquad Y_j := \log \frac{\mathrm{v}_{2j+2}}{\mathrm{v}_{2j}} \qquad (j \in \mathbb{Z}) \;.$$

The reduced mapping turns out to be

$$\overline{X}_j = X_j + \log \frac{(r\; + e^{Y_{j-1}})(1 + r\, e^{Y_j})}{(r\; + e^{Y_j})(1 + r\, e^{Y_{j-1}})} \;, \quad \overline{Y}_j = Y_j + \log \frac{(r\; + e^{\overline{X}_j})(1 + r\, e^{\overline{X}_{j+1}})}{(r\; + e^{\overline{X}_{j+1}})(1 + r\, e^{\overline{X}_j})} \;. \tag{2.35}$$

The Lax system for the mapping (2.35) can be constructed by using the relevant Lax representation given in factorized form as follows:

$$\mathfrak{L} = V^{-1}\,\mathrm{A}\,\widetilde{V} \;, \qquad \mathfrak{M} = V^{-1}\,\mathrm{B}\,\widehat{V} \;,$$

in which

$$V = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} , \quad \mathrm{A} = \begin{pmatrix} a & 1 \\ k^2 & a \end{pmatrix} , \quad \mathrm{B} = \begin{pmatrix} b & 1 \\ k^2 & b \end{pmatrix} .$$

In a similar way as in section 2.4, we can find from the Zakharov-Shabat system for the lattice equation (2.29) a linear system for the mapping in terms of the reduced variables X_j, Y_j,

$$\Psi_{j+1} = L_j \Psi_j \;, \quad \overline{\Psi}_j = M_j \Psi_j \;,$$

where L_j and M_j are given by

$$L_j = \begin{pmatrix} b & y_j \\ k^2 & b\,y_j \end{pmatrix} \begin{pmatrix} a & x_j \\ k^2 & a\,x_j \end{pmatrix} ,$$

$$M_j = \begin{pmatrix} -a & (1 + r\,x_j)/(r + x_j) \\ k^2\,\overline{x}_j^{-1} & -a\,\overline{x}_j^{-1}(1 + r\,x_j)/(r + x_j) \end{pmatrix} \begin{pmatrix} b & x_j(1 + r\,y_j)/(r + y_j) \\ k^2 & b\,x_j(1 + r\,y_j)/(r + y_j) \end{pmatrix} ,$$

in which we define the shorthand $x_j \equiv e^{X_j}$ and $y_j \equiv e^{Y_j}$. The Zakharov-Shabat system

$$\overline{L}_j\, M_j = M_{j+1}\, L_j \;,$$

leads to the mapping (2.35) as compatibility condition. We again impose the periodicity condition

$$\sum_{j=1}^{P} X_j = 0 \,, \qquad \sum_{j=1}^{P} Y_{j-1} = 0 \,. \tag{2.36}$$

The mapping (2.35) together with the condition (2.36) is a $2(P-1)$-dimensional mapping that exhibits $P-1$ invariants, which can be constructed in the same way as in the case of the lattice KdV by using the monodromy matrix. The invariants can also be constructed from the invariants of the KdV by using the Miura transformation (2.31).

2.6.2 Periods 2 and 3 reduction

As noted above, the mapping (2.35) together with the conditions (2.36) is a $2(P-1)$-dimensional mapping that exhibits $P-1$ invariants. In the case of $P = 2$, the corresponding mapping is given by the area-preserving mapping

$$\overline{X} = X + 2\log\frac{1 + r\,e^{Y}}{r + e^{Y}} \,, \qquad \overline{Y} = Y + 2\log\frac{r + e^{\overline{X}}}{1 + r\,e^{\overline{X}}} \,, \tag{2.37}$$

in which $X \equiv \log(\mathrm{v}_3/\mathrm{v}_1)$ and $Y \equiv \log(\mathrm{v}_0/\mathrm{v}_2)$. We get for the trace of the monodromy matrix $T(k)$

$$\mathrm{tr}\, T(k) = 2\,k^4 + b^2\,k^2\,\mathcal{I} + 2\,a^2\,b^2 \,. \tag{2.38}$$

We find the invariant

$$\mathcal{I} = e^{X-Y} + e^{Y-X} + 2\,r\,(e^{X} + e^{Y} + e^{-X} + e^{-Y}) + r^2\,(e^{X+Y} + e^{-(X+Y)}) \,, \tag{2.39}$$

which is an elliptic curve, cf. [37, proposition 2.4.3, p. 58], since it defines a smooth biquadratic curve, is conserved by the mapping (2.37).

In the case of $P = 3$, equation (2.35) together with the periodicity condition (2.36) yields a four-dimensional mapping which reads in terms of variables X_1, X_2, Y_1, Y_2

$$\overline{X}_1 = X_1 + \log\frac{(1 + r\,e^{Y_1})(1 + r\,e^{Y_1+Y_2})}{(r + e^{Y_1})(r + e^{Y_1+Y_2})} \,, \quad \overline{Y}_1 = Y_1 + \log\frac{(r + e^{\overline{X}_1})(1 + r\,e^{\overline{X}_2})}{(r + e^{\overline{X}_2})(1 + r\,e^{\overline{X}_1})} \,,$$
$$\overline{X}_2 = X_2 + \log\frac{(r + e^{Y_1})(1 + r\,e^{Y_2})}{(r + e^{Y_2})(1 + r\,e^{Y_1})} \,, \qquad \overline{Y}_2 = Y_2 + \log\frac{(r + e^{\overline{X}_2})(r + e^{\overline{X}_1+\overline{X}_2})}{(1 + r\,e^{\overline{X}_2})(1 + r\,e^{\overline{X}_1+\overline{X}_2})} \,,$$

in which $X_1 \equiv \log(\mathrm{v}_3/\mathrm{v}_1)$, $X_2 \equiv \log(\mathrm{v}_5/\mathrm{v}_3)$, $Y_1 \equiv \log(\mathrm{v}_4/\mathrm{v}_2)$ and $Y_2 \equiv \log(\mathrm{v}_0/\mathrm{v}_4)$. The trace of the monodromy matrix $T(k)$ is given by

$$\mathrm{tr}\, T(k) = 2\,k^6 + b^2\,k^4\,(\mathcal{I}_2 + 2\,r) + a\,b^3\,k^2\,(\mathcal{I}_1 + 2\,r) + 2\,a^3\,b^3\,. \tag{2.40}$$

The quantities $\mathcal{I}_1$ and $\mathcal{I}_2$ are conserved by the mapping; they are calculated from (2.40) as

$$\begin{aligned}
\mathcal{I}_1 &= x_1\,x_2\,y_0\,y_1 + x_1\,x_3\,y_0\,y_2 + x_2\,x_3\,y_1\,y_2 + x_1\,y_0 + x_2\,y_1 + x_3\,y_2 \\
&\quad + r\,(x_1\,x_2\,y_1 + x_1\,x_3\,y_0 + x_2\,x_3\,y_2 + x_1\,y_0\,y_1 + x_2\,y_1\,y_2 + x_3\,y_0\,y_2) \\
&\quad + r\,(x_1\,x_2 + x_1\,x_3 + x_2\,x_3 + y_0\,y_1 + y_0\,y_2 + y_1\,y_2) \\
&\quad + r^2\,(x_1\,x_2\,y_1\,y_2 + x_1\,x_3\,y_0\,y_1 + x_2\,x_3\,y_0\,y_2 + x_1\,y_1 + x_2\,y_2 + x_3\,y_0)\,, && \text{(2.41a)}\\
\mathcal{I}_2 &= x_1\,x_2\,y_0 + x_1\,x_3\,y_2 + x_2\,x_3\,y_1 + x_1\,y_0\,y_2 + x_2\,y_0\,y_1 + x_3\,y_1\,y_2 \\
&\quad + r\,(x_1\,x_2 + x_1\,x_3 + x_2\,x_3 + y_0\,y_1 + y_0\,y_2 + y_1\,y_2 + x_1\,y_2 + x_2\,y_0 + x_3\,y_1) \\
&\quad + r\,(x_1\,x_2\,y_0\,y_2 + x_1\,x_3\,y_1\,y_2 + x_2\,x_3\,y_0\,y_1) \\
&\quad + r^2\,(x_1\,x_2\,y_2 + x_1\,x_3\,y_1 + x_2\,x_3\,y_0 + x_1\,y_1\,y_2 + x_2\,y_0\,y_2 + x_3\,y_0\,y_1)\,, && \text{(2.41b)}
\end{aligned}$$

in which

$$x_j \equiv e^{X_j}\,, \quad y_j \equiv e^{Y_j} \quad (j = 1, 2)\,, \quad y_0 \equiv e^{-(Y_1+Y_2)}\,, \quad x_3 \equiv e^{-(X_1+X_2)}\,.$$

It should be pointed out that the latter case, that is $P = 3$, will not be considered in this thesis, but are addressed for the convenience of the reader. Next, the construction of commuting discrete maps arising from the linearized lattice KdV equation will be considered.

2.7 Linearized lattice KdV system

The linearized lattice KdV equation, with which this section is concerned, gives rise to the discrete harmonic oscillator system by applying a periodic initial value problem. Recall the lattice KdV equation of section 2.2

$$(a - b + u_{n,m+1} - u_{n+1,m})\,(a + b + u_{n,m} - u_{n+1,m+1}) = a^2 - b^2, \tag{2.42}$$

which is (2.1) in explicit form. Here we are interested in the *linearization* of the lattice KdV equation (2.42). By a linearization of a nonlinear lattice equation such as (2.42), we mean the linear equation obtained by expanding the dependent variable around a specific known solution of the nonlinear equation and taking the dominant term in the expansion [56]. The simplest linearizations are obtained by taking a trivial solution such as the zero solution if it exists. It is easy to see that (2.42) admits the solution $u_{n,m} = 0$: i.e. u vanishes for all n, m.

By setting $u_{n,m} = \eta\, \omega_{n,m}$ and expanding up to linear terms in the small parameter η, we obtain the following linear equation for ω,

$$(a+b)(\omega_{n+1,m} - \omega_{n,m+1}) = (a-b)(\omega_{n,m} - \omega_{n+1,m+1}) \ . \tag{2.43}$$

This equation is, in fact, the space-time discretization of the partial differential equation

$$w_t = w_{xxx} \ , \tag{2.44}$$

which is obtained as linearization of equation (1.32). It is easily verified that equation (2.43) obeys the consistency-around-the-cube (CAC) property of section 1.3.2 in the same way as the full nonlinear equation. The linear lattice equation (2.43) arises naturally from the Adler, Bobenko and Suris (ABS) list [5].

Let us now consider the equation of interest in this section, which is in the linear quadrilateral equation in terms of the dynamical variable u:

$$(a+b)(\widetilde{u} - \widehat{u}) = (a-b)(u - \widehat{\widetilde{u}}) \ . \tag{2.45}$$

This equation arises as the compatibility condition of the linear system,

$$(a-k)\widetilde{\Phi}(k) = \mathsf{L}(k)\Phi(k) \ , \qquad (b-k)\widehat{\Phi}(k) = \mathsf{M}(k)\Phi(k) \ , \tag{2.46}$$

where $\mathsf{L}(k)$ and $\mathsf{M}(k)$ are given by

$$\mathsf{L} := U\,\mathsf{A}\,\widetilde{U}^{-1} \ , \qquad \mathsf{M} := U\,\mathsf{B}\,\widehat{U}^{-1} \ , \tag{2.47}$$

in which

$$U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} , \qquad \mathsf{A} = \begin{pmatrix} a+k & 0 \\ 0 & a-k \end{pmatrix} , \qquad \mathsf{B} = \begin{pmatrix} b+k & 0 \\ 0 & b-k \end{pmatrix} .$$

We note that in a similar way to the lattice KdV equation, the Bäcklund transform also gives rise to the Lax representation (2.46) for the lattice equation (2.45), cf. [65]. An alternative way of deriving the Lax pair (2.46) is obtained by linearizing the Lax pair of the nonlinear case, i.e. the Lax pair (2.2).

The equation (2.45) also arises from a discrete action principle, namely

$$\mathcal{S} = \sum_{n,m\in\mathbb{Z}} \left[u_{n,m} \left(u_{n+1,m} - u_{n,m+1} \right) + s\, u_{n,m} \left(u_{n+1,m+1} - u_{n,m} \right) \right] , \tag{2.48}$$

where $s := (a-b)/(a+b)$. The Euler-Lagrange equations for (2.48), which are obtained by variation of $\mathcal{S}$ with respect to the variables $u_{n,m}$, i.e.

$$\frac{\delta \mathcal{S}}{\delta u_{n,m}} = 0 \ ,$$

lead to the equation

$$u_{n+1,m} - u_{n,m+1} + u_{n-1,m} - u_{n,m-1} + s \left(u_{n+1,m+1} - 2\, u_{n,m} + u_{n-1,m-1} \right) = 0 \ ,$$

which is a consequence of equation (2.45).

We now look for a reduction of the linearized lattice KdV equation (2.45) to an integrable symplectic mapping. As discussed in section 2.3, we define $2P$ initial conditions $\mathrm{u}_0, \mathrm{u}_1, \ldots, \mathrm{u}_{2P-1}$ along a staricase as shown in figure 2.2. The linearized KdV equation (2.45) defines a dynamical map $(\mathrm{u}_0, \mathrm{u}_1, \ldots, \mathrm{u}_{2P-1}) \to (\overline{\mathrm{u}}_0, \overline{\mathrm{u}}_1, \ldots, \overline{\mathrm{u}}_{2P-1})$ as follows:

$$\overline{\mathrm{u}}_{2j+1} = \mathrm{u}_{2j+1} + s \left(\mathrm{u}_{2j+2} - \mathrm{u}_{2j} \right) , \qquad \overline{\mathrm{u}}_{2j} = \mathrm{u}_{2j} + s \left(\overline{\mathrm{u}}_{2j+1} - \overline{\mathrm{u}}_{2j-1} \right) , \tag{2.49}$$

which by imposing periodic initial conditions on the staircase, i.e.

$$\mathrm{u}_{2j+P} = \mathrm{u}_{2j} \ , \qquad \mathrm{u}_{2j+P+1} = \mathrm{u}_{2j+1} \ ,$$

reduces to a finite-dimensional mapping of dimension $2P$.

As before in section 2.3, by introducing the reduced variables for the differences on odd and even sites of the staircase, i.e.

$$x_j := \mathrm{u}_{2j+1} - \mathrm{u}_{2j-1} \ , \qquad y_j := \mathrm{u}_{2j+2} - \mathrm{u}_{2j} \ ,$$

the mapping can be even further reduced to a $(2P-2)$-dimensional one which reads

$$\overline{x}_j = x_j + s\,(y_j - y_{j-1})\;, \quad \overline{y}_j = y_j + s\,(\overline{x}_{j+1} - \overline{x}_j)\;, \qquad (j = 1, \ldots, P)\;, \tag{2.50}$$

where we impose the periodicity condition

$$\sum_{j=1}^{P} x_j = 0\;, \qquad \sum_{j=1}^{P} y_{j-1} = 0\;. \tag{2.51}$$

Using the same approach as in section 2.4, the Lax pair for the mapping (2.50) can be constructed by using the relevant Lax representation (2.46). They are given in terms of reduced variables x_j and y_j:

$$L_j = V_{2j}\,V_{2j-1}\;; \qquad V_j = \begin{pmatrix} a_j + k & -(a_j + k)\,v_j \\ 0 & a_j - k \end{pmatrix}, \tag{2.52a}$$

$$M_j = \begin{pmatrix} k - a & -(k+a)\,\overline{x}_j \\ 0 & -(k+a) \end{pmatrix} \begin{pmatrix} b + k & (b+k)(x_j - s\,y_j) - (b-k)\,s\,\overline{x}_j \\ 0 & b - k \end{pmatrix}, \tag{2.52b}$$

where $v_{2j} := y_j$, $v_{2j-1} := x_j$, $a_{2j} := b$ and $a_{2j-1} := a$. The Zakharov-Shabat equations

$$\overline{L}_j\,M_j = M_{j+1}\,L_j\;, \tag{2.53}$$

lead to mapping (2.50) as compatibility condition.

As discussed in section 2.5, in general for an integrable system, it is then possible to extract invariants from such a Lax pair by the construction of a monodromy matrix $T(k)$. The Zakharov-Shabat condition (2.53) guarantees the preservation of the spectral data of $T(k)$ under the map. However, for the linear reduction mapping, these spectral data are trivial, which means no invariants are encoded. It remains an open question what is the Lax pair for the mapping that generates invariants. Nonetheless, the invariants do exist, and can be obtained by linearizing the invariants of the nonlinear case. The invariants are *quadratic*, they appear at the quadratic order in η as the limits of the invariants of the nonlinear case.

2.7.1 Commuting discrete map

As mentioned above, the linear lattice KdV equation (2.45) obeys the 3-D consistency (CAC) property. In fact, it has multi-dimensional consistency, see for example [91]. So,

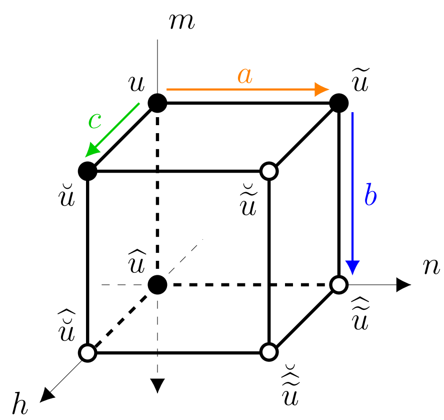


Figure 2.6: 3-dimensional consistency.

we can introduce a third direction to the reduction with parameter c and the shifted variables $\breve{\mathrm{u}}_i$, as shown in figure 2.6. This leads us to a system of the same equations in all three lattice directions:

$$(a+b)(\widetilde{u}-\widehat{u}) = (a-b)(u-\widehat{\widetilde{u}}) \,, \tag{2.54a}$$

$$(b+c)(\widehat{u}-\breve{u}) = (b-c)(u-\widehat{\breve{u}}) \,, \tag{2.54b}$$

$$(c+a)(\breve{u}-\widetilde{u}) = (c-a)(u-\widetilde{\breve{u}}) \,. \tag{2.54c}$$

As before, assigning initial data on the lattice along a staircase as depicted in figure 2.7, equations in (2.54) give a set of equations

$$\overline{\mathrm{u}}_{2j-1} = \mathrm{u}_{2j-1} + s\,(\mathrm{u}_{2j} - \mathrm{u}_{2j-2}) \,, \qquad \breve{\mathrm{u}}_{2j+1} = \mathrm{u}_{2j+2} + t'\,(\grave{\mathrm{u}}_{2j+3} - \mathrm{u}_{2j+1}) \,, \tag{2.55a}$$

$$\grave{\mathrm{u}}_{2j} = \mathrm{u}_{2j} + t\,(\overline{\mathrm{u}}_{2j-1} - \grave{\mathrm{u}}_{2j+1}) \,, \qquad \grave{\mathrm{u}}_{2j+1} = \mathrm{u}_{2j+1} + t\,(\mathrm{u}_{2j} - \breve{\mathrm{u}}_{2j+1}) \,, \tag{2.55b}$$

where $t := (c-a)/(c+a)$ and $t' := (b-c)/(b+c)$. These equations lead to a system of equations for the dynamical mapping $(\mathrm{u}_0, \ldots, \mathrm{u}_{2P-1}) \to (\grave{\mathrm{u}}_0, \ldots, \grave{\mathrm{u}}_{2P-1})$:

$$\grave{\mathrm{u}}_{2j+1} = \mathrm{u}_{2j+1} + t\,(\mathrm{u}_{2j} - \mathrm{u}_{2j+2}) - t\,t'\,(\grave{\mathrm{u}}_{2j+3} - \mathrm{u}_{2j+1}) \,, \tag{2.56a}$$

$$\grave{\mathrm{u}}_{2j} = \mathrm{u}_{2j} - t\,(\grave{\mathrm{u}}_{2j+1} - \mathrm{u}_{2j-1}) + s\,t\,(\mathrm{u}_{2j} - \mathrm{u}_{2j-2}) \,. \tag{2.56b}$$

In figure 2.7 the blue path denotes the bar evolution $^{-}$ and the orange path denotes the grave evolution $\grave{}$. It is clear that this mapping is close to the identity mapping if t is

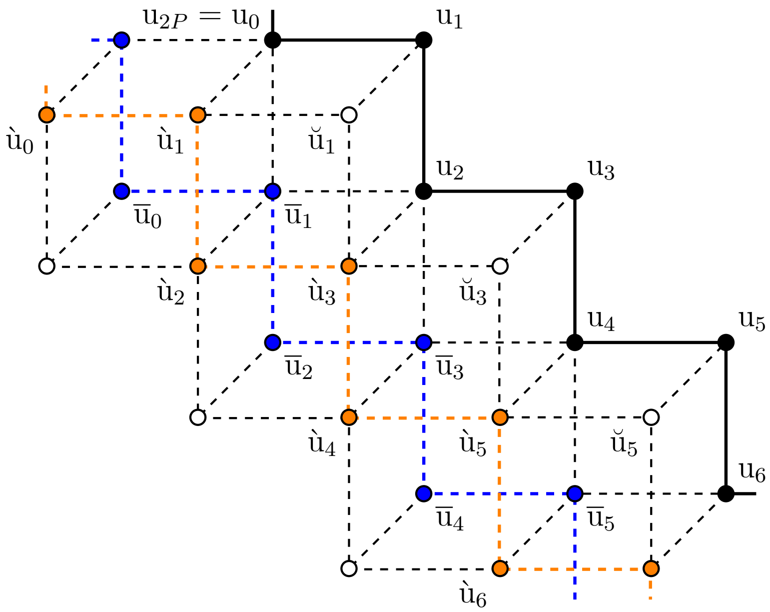


Figure 2.7: Commuting mapping: the system (2.45) extended to 3-D.

small. Again, we use reduction variables x_j and y_j, which yield the reduced mapping $(x_1, \ldots, x_P; y_0, \ldots, y_{P-1}) \to (\grave{x}_1, \ldots, \grave{x}_P; \grave{y}_0, \ldots, \grave{y}_{P-1})$:

$$\grave{x}_j = x_j - t\,(y_j - y_{j-1}) - t\,t'\,(\grave{x}_{j+1} - x_j)\,, \tag{2.57a}$$

$$\grave{y}_j = y_j - t\,(\grave{x}_{j+1} - x_j) + s\,t\,(y_j - y_{j-1})\,, \tag{2.57b}$$

in which we impose the periodicity condition (2.51). Considering the periodicity condition (2.51) and writing the vector

$$\mathbf{X} := (x_1, \ldots, x_{P-1}; y_1, \ldots, y_{P-1})^T,$$

we can pose the map (2.57) in a matrix form:

$$\grave{\mathbf{X}} = \mathbf{A}^{-1}\,\mathbf{B}\,\mathbf{X}\,, \tag{2.58}$$

in which $\mathbf{A}$ and $\mathbf{B}$ are $2(P-1) \times 2(P-1)$ matrices that are independent of x_j and y_j.

Let us now look for the Lax pair of mappings (2.57). The Lax pair consists of an L-part that effects an n direction $u_{n,m,h} \to u_{n+1,m,h}$, an M-part that effects an m direction

$u_{n,m,h} \to u_{n,m+1,h}$, and an N-part that effects an h direction $u_{n,m} \to u_{n,m,h+1}$, as shown in figure 2.6,

$$\mathsf{L} := U\,\mathsf{A}\,\widetilde{U}^{-1}\,, \qquad \mathsf{M} := U\,\mathsf{B}\,\widehat{U}^{-1}\,, \qquad \mathsf{N} := U\,\mathsf{C}\,\breve{U}^{-1}\,, \tag{2.59}$$

in which

$$U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \ \mathsf{A} = \begin{pmatrix} a+k & 0 \\ 0 & a-k \end{pmatrix}, \ \mathsf{B} = \begin{pmatrix} b+k & 0 \\ 0 & b-k \end{pmatrix}, \ \mathsf{C} = \begin{pmatrix} c+k & 0 \\ 0 & c-k \end{pmatrix}.$$

Consider the two paths shown in figure 2.8. Like the initial value configuration that we

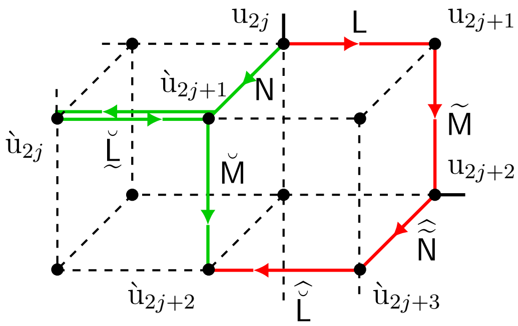


Figure 2.8: Compatibility condition leading to the ZS system for the mapping (2.57).

have considered, by equating the two paths we have

$$\begin{aligned}
&\grave{\mathrm{U}}_{2j+3}\,\mathsf{A}^{-1}\,\grave{\mathrm{U}}^{-1}_{2j+2}\,\mathrm{U}_{2j+2}\,\mathsf{C}\,\grave{\mathrm{U}}^{-1}_{2j+3}\,\mathrm{U}_{2j+1}\,\mathsf{B}\,\mathrm{U}^{-1}_{2j+2}\,\mathrm{U}_{2j}\,\mathsf{A}\,\mathrm{U}^{-1}_{2j+1} \\
&= \grave{\mathrm{U}}_{2j+1}\,\mathsf{B}\,\grave{\mathrm{U}}^{-1}_{2j+2}\,\grave{\mathrm{U}}_{2j}\,\mathsf{A}\,\grave{\mathrm{U}}^{-1}_{2j+1}\,\grave{\mathrm{U}}_{2j+1}\,\mathsf{A}^{-1}\,\grave{\mathrm{U}}^{-1}_{2j}\,\mathrm{U}_{2j}\,\mathsf{C}\,\grave{\mathrm{U}}^{-1}_{2j+1}\,, \\
\Longrightarrow\ &\left(\mathsf{B}\,\grave{\mathrm{U}}^{-1}_{2j+2}\,\grave{\mathrm{U}}_{2j}\,\mathsf{A}\,\grave{\mathrm{U}}^{-1}_{2j+1}\,\grave{\mathrm{U}}_{2j-1}\right)\left(\grave{\mathrm{U}}^{-1}_{2j-1}\,\grave{\mathrm{U}}_{2j+1}\,\mathsf{A}^{-1}\,\grave{\mathrm{U}}^{-1}_{2j}\,\mathrm{U}_{2j}\,\mathsf{C}\,\grave{\mathrm{U}}^{-1}_{2j+1}\,\mathrm{U}_{2j-1}\right) \\
&= \left(\grave{\mathrm{U}}^{-1}_{2j+1}\,\grave{\mathrm{U}}_{2j+3}\,\mathsf{A}^{-1}\,\grave{\mathrm{U}}^{-1}_{2j+2}\,\mathrm{U}_{2j+2}\,\mathsf{C}\,\grave{\mathrm{U}}^{-1}_{2j+3}\,\mathrm{U}_{2j+1}\right)\left(\mathsf{B}\,\mathrm{U}^{-1}_{2j+2}\,\mathrm{U}_{2j}\,\mathsf{A}\,\mathrm{U}^{-1}_{2j+1}\,\mathrm{U}_{2j-1}\right).
\end{aligned}$$

Hence, we can write the latter equation in the form of the discrete-time Zakharov-Shabat system

$$\grave{L}_j\,N_j = N_{j+1}\,L_j\,, \tag{2.60}$$

in which

$$L_j = V_{2j}\, V_{2j-1}\,; \qquad V_j = \begin{pmatrix} a_j + k & -(a_j+k)\, v_j \\ 0 & a_j - k \end{pmatrix}, \tag{2.61a}$$

$$N_j = \begin{pmatrix} k-a & -(k+a)\,\grave{x}_j \\ 0 & -(k+a) \end{pmatrix} \begin{pmatrix} c+k & (c+k)(\mathrm{u}_{2j-1} - \grave{\mathrm{u}}_{2j+1}) - (c-k)(\grave{\mathrm{u}}_{2j} - \mathrm{u}_{2j}) \\ 0 & c-k \end{pmatrix}, \tag{2.61b}$$

where $v_{2j} := y_j$, $v_{2j-1} := x_j$, $a_{2j} := b$ and $a_{2j-1} := a$. Once again, it is easily verified that mapping (2.57) follows from the Zakharov-Shabat equation (2.60) as a compatibility condition. It should be noted that the Lax matrices L_j are the same for both mappings (2.50) and (2.57), but the corresponding Lax matrices, M_j and N_j respectively, are different.

2.7.2 Periods 2 and 3 reduction

Let us consider the $P = 2$ reduction[4]. Using the linear lattice equation (2.45), one can arrive at the area-preserving reduced mapping

$$\overline{x} = x + 2\,s\,y\,, \qquad \overline{y} = y - 2\,s\,\overline{x}\,, \tag{2.62}$$

in which $x := \mathrm{u}_1 - \mathrm{u}_3$ and $y := \mathrm{u}_2 - \mathrm{u}_0$. Writing the vector $\mathbf{x} := (x, y)^T$, we pose the map $(x, y) \to (\overline{x}, \overline{y})$ in a matrix form

$$\overline{\mathbf{x}} = \mathsf{S}\,\mathbf{x}\,, \qquad \text{with matrix} \quad \mathsf{S} = \begin{pmatrix} 1 & 2\,s \\ -2\,s & 1 - 4\,s^2 \end{pmatrix}. \tag{2.63}$$

As the map is linear, we seek a quadratic invariant. This has the form

$$\mathcal{J} = \mathbf{x}^T\, J\, \mathbf{x}\,,$$

for some matrix J. The condition for invariance is then

$$\overline{\mathcal{J}} = \mathcal{J} \quad \Rightarrow \quad \overline{\mathbf{x}}^T J\, \overline{\mathbf{x}} = \mathcal{J} \quad \Rightarrow \quad \mathsf{S}^T J\, \mathsf{S} = J\,. \tag{2.64}$$

[4] An odd-periodic configuration of initial data, which we omit this in this thesis, was considered for one dimensional reduction, i.e. $P = 2$, in [64, 65], but for a different discrete map.

Using the condition (2.64) we are able to find a matrix

$$J = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} , \tag{2.65}$$

satisfying the condition and hence an invariant of the mapping (2.62) has the form

$$\mathcal{I}(x, y) = x^2 + y^2 + 2\, s\, x\, y \; . \tag{2.66}$$

Alternatively, this invariant can be obtained by linearizing the invariant (2.11), it appears at the quadratic order in the small parameter.

Using the system of equations (2.54) the commuting mapping of the map (2.62) is given by the area-preserving mapping

$$\grave{x} = x - 2\, t\, y + t\, t' \,(\grave{x} + x) \; , \qquad \grave{y} = y + 2\, s\, t\, y + t\,(\grave{x} + x) \; . \tag{2.67}$$

Again, we can write the map $(x, y) \to (\grave{x}, \grave{y})$ in a matrix form

$$\grave{\mathbf{x}} = \mathtt{T}\, \mathbf{x}; \qquad \mathbf{x} = (x, y)^T \; , \tag{2.68}$$

with matrix

$$\mathtt{T} = \frac{1}{1 - t\, t'} \begin{pmatrix} 1 + t\, t' & -2\, t \\ 2\, t & (1 + 2\, s\, t)(1 - t\, t') - 2\, t^2 \end{pmatrix} .$$

Additionally, considering the matrix form for the invariant in (2.64), we can describe the evolution of invariant (2.66) under the commuting flow by

$$\grave{\mathcal{I}} = \grave{\mathbf{x}}^T\, J\, \grave{\mathbf{x}} = \mathbf{x}^T\, \mathtt{T}^T\, J\, \mathtt{T}\, \mathbf{x} \; . \tag{2.69}$$

However, the calculation shows that for the same matrix J (2.65), we have $\mathtt{T}^T\, J\, \mathtt{T} = J$. Hence the invariant is also preserved under the commuting map $\grave{\mathcal{I}} = \mathcal{I}$. Since the invariant of the bar evolution $^-$ is also invariant under the grave evolution $\grave{}$, we again have an invariant and integrability for this second evolution.

Recalling now the matrix forms of the two mappings (equations (2.63) and (2.68)), it is then clear that the two maps (2.62) and (2.67) commute:

$$(\grave{\bar{x}}, \grave{\bar{y}}) = (\bar{\grave{x}}, \bar{\grave{y}}) \; , \tag{2.70}$$

since we have $[\mathtt{S},\mathtt{T}]=0$. This last relation relies on the parameter identity

$$-s\,t\,t' = s + t + t' \,, \tag{2.71}$$

which is a reformulation of the critical partial fraction identity

$$\frac{1}{s\,t'} + \frac{1}{t\,t'} + \frac{1}{s\,t} + 1 = 0 \,. \tag{2.72}$$

For the case of $P = 3$ reduction, the corresponding mapping reads in terms of four variables x_1, x_2, y_1, y_2:

$$\overline{x}_1 = x_1 + s\,(2\,y_1 + y_2) \,, \qquad \overline{y}_1 = y_1 - s\,(\overline{x}_1 - \overline{x}_2) \,, \tag{2.73a}$$

$$\overline{x}_2 = x_2 - s\,(y_1 - y_2) \,, \qquad \overline{y}_2 = y_2 - s\,(\overline{x}_1 + 2\,\overline{x}_2) \,, \tag{2.73b}$$

whereas the commuting mapping of the map (2.73) reads as

$$\grave{x}_1 = x_1 - t\,(2\,y_1 + y_2) - t\,t'\,(\grave{x}_2 - x_1) \,, \tag{2.74a}$$

$$\grave{y}_1 = y_1 - t\,(\grave{x}_2 - x_1) + s\,t\,(2\,y_1 + y_2) \,, \tag{2.74b}$$

$$\grave{x}_2 = x_2 - t\,(y_2 - y_1) + t\,t'\,(\grave{x}_1 + \grave{x}_2 + x_2) \,, \tag{2.74c}$$

$$\grave{y}_2 = y_2 + t\,(\grave{x}_1 + \grave{x}_2 + x_2) + s\,t\,(y_2 - y_1) \,. \tag{2.74d}$$

As before, by writing the vector $\mathbf{X} := (x_1, x_2, y_1, y_2)^T$, we can write both maps (2.73) and (2.74) in matrix form

$$\overline{\mathbf{X}} = \mathbf{S}\,\mathbf{X} \,, \qquad \grave{\mathbf{X}} = \mathbf{T}\,\mathbf{X} \,, \tag{2.75}$$

with matrices

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 2\,s & s \\ 0 & 1 & -s & s \\ -s & s & -3\,s^2+1 & 0 \\ -s & -2\,s & 0 & -3\,s^2+1 \end{pmatrix} , \qquad \mathbf{T} = \mathbf{A}^{-1}\,\mathbf{B} \,, \tag{2.76}$$

where the matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} 1 & t\,t' & 0 & 0 \\ -t\,t' & 1-t\,t' & 0 & 0 \\ 0 & t & 1 & 0 \\ -t & -t & 0 & 1 \end{pmatrix} , \qquad \mathbf{B} = \begin{pmatrix} 1+t\,t' & 0 & -2\,t & -t \\ 0 & 1+t\,t' & t & -t \\ t & 0 & 1+2\,s\,t & s\,t \\ 0 & t & -s\,t & 1+s\,t \end{pmatrix} .$$

Taking into account the parameter identity (2.71), it is clear that the commutator $[\mathtt{S}, \mathtt{T}]$ is zero, and hence the two maps (2.73) and (2.74) commute:

$$(\grave{\bar{x}}_1, \grave{\bar{x}}_2, \grave{\bar{y}}_1, \grave{\bar{y}}_2) = (\bar{\grave{x}}_1, \bar{\grave{x}}_2, \bar{\grave{y}}_1, \bar{\grave{y}}_2) \ . \tag{2.77}$$

In the same way as in the case of $P = 2$, we can derive two quadratic invariants of the mapping $\mathcal{I}_1$ and $\mathcal{I}_2$, which are invariant under both maps (2.73) and (2.74). After some simplification, these have the form

$$\mathcal{I}_1 = \frac{3}{2}\, s\, (x_1\, y_1 + x_1\, y_2 + x_2\, y_2) + x_1^2 + x_1\, x_2 + x_2^2 + y_1^2 + y_1\, y_2 + y_2^2 \ , \tag{2.78a}$$

$$\mathcal{I}_2 = x_1\, y_1 - x_1\, y_2 + 2\, x_2\, y_1 + x_2\, y_2 \ . \tag{2.78b}$$

With respect to a properly chosen symplectic structure the two invariants are in involution, i.e. $\{\mathcal{I}_1, \mathcal{I}_2\} = 0$. A detailed description of the symplectic structure of the mappings and the connection between the invariants and the relevant modified Hamiltonian of discrete harmonic oscillator is explored in chapter 3.

2.8 Summary

Periodic initial value problems were applied to the lattice version of KdV type. A new way of performing iterations was considered in this chapter. This was done by updating the lattice variables along a diagonal shift, which resulted in new families of integrable mappings that are close to the identity mapping if the step size is small. Further, we have also extended the idea to 3-D lattice, which produced families of commuting mappings, where we have restricted ourselves to the case of linearized lattice KdV equation. In both the KdV and mKdV cases, we have found similar structures of mappings and invariants. In fact, both cases are examples of discrete dynamical systems that are covered by the discrete version of the Arnold-Liouville theorem as formulated by Veselov [121].

It was established in [24] that the invariants of the mappings are in involution. It is a hallmark of integrability that these invariants themselves generate commuting flows using the Poisson bracket structures which are compatible with the discrete mappings. The connection between the invariants and the relevant modified Hamiltonians is displayed in chapters 3 and 4.

Chapter 3

Modified Hamiltonians for numerical integrations of KdV type

3.1 Overview

As discussed in chapter 1, integrable mappings have been systematically constructed and studied, e.g. in [49, 99, 100, 111, 121]. In the previous chapter, a specific set of examples were constructed from periodic initial value problems on the lattice partial difference equations. These are examples of discrete integrable systems which have a closed-form expression for the modified Hamiltonian flow when viewed as an application of the symplectic Euler method as will be shown in this chapter.

The outline of this chapter is as follows. In section 3.2, we present a first example arising from the simplest reduction of the linearized lattice Korteweg-de Vries (KdV) equation. The modified Hamiltonian of the symplectic mapping, as given in Yoshida's approach, is then deduced and expressed in closed form. The exact solution for the modified Hamiltonian flow is also given. Having obtained the commuting mapping, one then speaks of a commuting modified Hamiltonian. So, we derive the commuting modified Hamiltonian where it can be written down in closed form. Due to the closed form expressions, we show that these modified Hamiltonians are in involution. In section 3.3, we extend the ideas in section 3.2 through a transition to multiple-degrees-of-freedom, but we restrict ourselves to a two-degrees-of-freedom system. We present a second example arising from the period-three reduction of the linearized lattice KdV equation. We again show the derivation of the discrete Hamiltonian which acts as the generating function for the mapping. We also do the same for the modified Hamiltonian and the commuting modified Hamiltonian as in the first example.

By considering periodic reduction of the nonlinear case of the KdV models, we present examples of integrable numerical integration schemes from which the modified Hamiltonians can be written in closed-form expressions. However, we focus on the one-degree-of-freedom case obtained from the simplest reduction, leaving the systems with two-degrees-of-freedom to chapter 4. So we present a third example arising from the nonlinear lattice KdV equation in section 3.4. We show the derivation of the modified Hamiltonian using Yoshida's method, expressed in closed form using action-angle variables. We then present a fourth example coming from the nonlinear modified KdV (mKdV) case in section 3.5, and do the same for the modified Hamiltonian as in the third example.

3.2 A discrete-time harmonic oscillator

From the start it should be made clear that some of the results of this section are not new; these results, in different forms, can be found in the literature. Expressions for the modified (or interpolating) Hamiltonian can be found in [11] and [12].

3.2.1 The discrete map

As discussed in section 2.7 of the previous chapter, the linearized lattice KdV equation (2.45) leads to integrable mapping (2.62) when considering a staircase with the simplest case ($P = 2$), namely the one-dimensional reduction. Let us recall the system of equations (2.62), i.e.

$$\overline{p} = p - \tau\, q\,, \qquad \overline{q} = q + \tau\, \overline{p}\,, \tag{3.1}$$

in which we identify $x := -p$, $y := q$ and $\tau := 2\, s$. This equation is in fact a discrete harmonic oscillator system, which preserves the standard symplectic structure $d\overline{q} \wedge d\overline{p} = dq \wedge dp$. This leads to standard Poisson brackets

$$\{q, q\} = \{p, p\} = 0\,, \quad \{q, p\} = 1\,.$$

Thus, the mapping (3.1) is indeed a canonical transformation with the generating function

$$H_\tau(q,\overline{p}) = \frac{\tau}{2}\left(\overline{p}^2 + q^2\right), \tag{3.2}$$

through the equations

$$\overline{p} - p = -\frac{\partial H_\tau}{\partial q}, \qquad \overline{q} - q = \frac{\partial H_\tau}{\partial \overline{p}}. \tag{3.3}$$

We can regard equations (3.3) as discrete analogue of the Hamilton equations, where (3.2) can be regarded as the discrete Hamiltonian. In contrast to the usual approach in geometric integration where the Hamiltonian of a differential equation is elevated to become a discrete Hamiltonian for the mapping, here we take the generating function of an integrable dynamical map as a discrete Hamiltonian without ab initio having a differential equation. In fact, there is an underlying continuous Hamiltonian flow whose Hamiltonian is given by an invariant of the map which is different from the discrete Hamiltonian. More specifically, the mapping (3.1) conserves the quantity

$$\mathcal{I}_\tau(q,p) = p^2 + q^2 - \tau\, p\, q, \tag{3.4}$$

and, therefore, by definition 1.3.1 is an integrable map.

Let us now apply Yoshida's method to derive the modified Hamiltonian of the symplectic Euler discretization from (3.3) which, of course, it can be written in a closed-form expression. Recall that the well-known Baker-Campbell-Hausdorff (BCH) formula (1.80) when applied to a Hamiltonian of the form

$$H(q,p) = T(p) + V(q)$$

gives the expansion for the modified Hamiltonian

$$H^* = T + V + \frac{\tau}{2}\{T,V\} + \frac{\tau^2}{12}\left(\{T,\{T,V\}\} + \{V,\{V,T\}\}\right) + \cdots .$$

In our case we have

$$T \equiv \frac{1}{2}p^2, \quad V \equiv \frac{1}{2}q^2 \quad \text{and} \quad H \equiv \frac{1}{\tau}H_\tau .$$

Thus, when applied to (3.2), the BCH formula (1.80) gives

$$\frac{1}{\tau}H_\tau^* = T + V + \frac{\tau}{2}\{T,V\} + \frac{\tau^2}{12}\left(\{T,\{T,V\}\} + \{V,\{V,T\}\}\right) + \cdots ,$$

leading to the following expansion for the modified Hamiltonian:

$$\begin{aligned} H^*_\tau = & \frac{\tau}{2}\,p^2 + \frac{\tau}{2}\,q^2 - \frac{\tau^2}{2}\,p\,q + \frac{\tau^3}{12}\,p^2 + \frac{\tau^3}{12}\,q^2 - \frac{\tau^4}{12}\,p\,q \\ & + \frac{\tau^5}{60}\,p^2 + \frac{\tau^5}{60}\,q^2 - \frac{\tau^6}{60}\,p\,q + O(\tau^7)\;. \end{aligned} \tag{3.5}$$

It is clear that this modified Hamiltonian can also be expressed in terms of the invariant (3.4),

$$H^*_\tau(\mathcal{I}_\tau) = \frac{\tau}{2}\,\mathcal{I}_\tau + \frac{\tau^3}{12}\,\mathcal{I}_\tau + \frac{\tau^5}{60}\,\mathcal{I}_\tau + O(\tau^7)\;. \tag{3.6}$$

In the sense of section 1.4.2 of the introduction, this modified Hamiltonian can be written in closed form as well,

$$H^*_\tau(q,p) = F(\tau)\,\mathcal{I}_\tau(q,p)\;, \tag{3.7}$$

where F is a function expressed in terms of the discrete-time evolution τ. To find the function F, we must look for an exact solution to the true Hamiltonian flow arising from (3.7), and the general solution to the equation of motion arising from (3.3).

The Hamilton equations of the modified system (3.7) are

$$\dot{q} = \frac{\partial H^*_\tau}{\partial p} = F(\tau)(2\,p - \tau\,q)\;, \tag{3.8a}$$

$$\dot{p} = -\frac{\partial H^*_\tau}{\partial q} = -F(\tau)(2\,q - \tau\,p)\;, \tag{3.8b}$$

where the dot $(\cdot)$ denotes differentiation with respect to the continuous-time flow variable t. By converting equation (3.8a) to the second order, after making some substitutions, one obtains the second-order differential equation

$$\ddot{q} + F^2\left(4 - \tau^2\right)q = 0\;. \tag{3.9}$$

It is obvious that the exact solution to the differential equation (3.9) is given by

$$q(t) = c_1\cos\left(t\,F\sqrt{4-\tau^2}\right) + c_2\sin\left(t\,F\sqrt{4-\tau^2}\right)\,, \tag{3.10}$$

where t is the continuous-time flow variable and c_1, c_2 are constants.

Recalling the symplectic mapping in (3.1), by eliminating p from this map we write a second-order difference equation in q,

$$\overline{q} + (\tau^2 - 2)\,q + \underline{q} = 0\,, \tag{3.11}$$

where the underlined $\underline{q}$ indicates a backwards step. The most general solution to the homogeneous difference equation (3.11) is given by

$$q(n) = \alpha_1 \cos(\mu\, n) + \alpha_2 \sin(\mu\, n)\,, \qquad \mu := \cos^{-1}\left(1 - \frac{\tau^2}{2}\right), \tag{3.12}$$

where n is the discrete variable and α_1, α_2 are constants [56, 65]. In fact, this solution is nothing but the numerical solution of the continuous-time harmonic oscillator.

We note that the solution (3.12) for the difference equation (3.11) has a clear relation to the solution (3.10) for the differential equation (3.9). Letting $t = n$, the exact solution of the modified system, i.e. (3.10), and the numerical solution of the original system, i.e. (3.12), are the same, cf. [52]. Hence, we have

$$\begin{aligned} &\alpha_1 \cos\left[n \cos^{-1}\left(1 - \frac{\tau^2}{2}\right)\right] + \alpha_2 \sin\left[n \cos^{-1}\left(1 - \frac{\tau^2}{2}\right)\right] \\ &= c_1 \cos\left[n\, F\sqrt{4-\tau^2}\right] + c_2 \sin\left[n\, F\sqrt{4-\tau^2}\right], \end{aligned} \tag{3.13}$$

leading to

$$F\sqrt{4-\tau^2} = \cos^{-1}\left(1 - \frac{\tau^2}{2}\right) \quad \Rightarrow \quad F(\tau) = \frac{1}{\sqrt{4-\tau^2}} \cos^{-1}\left(1 - \frac{\tau^2}{2}\right).$$

Consequently, we obtain

$$H^*_\tau = \frac{\mathcal{J}_\tau}{\sqrt{4-\tau^2}} \cos^{-1}\left(1 - \frac{\tau^2}{2}\right). \tag{3.14}$$

This is a closed-form expression for the modified Hamiltonian of the symplectic mapping from (3.3).

3.2.2 The commuting map

Recall the commuting mapping of the map (2.62), i.e. the mapping (2.67),

$$\acute{x} = x - 2\,t\,y + t\,t'\,(\acute{x} + x)\,, \qquad \acute{y} = y + 2\,s\,t\,y + t\,(\acute{x} + x)\,. \tag{3.15}$$

We eliminate x from this map to produce a second-order difference equation in y,

$$\acute{y} + \mathsf{c}\,y + \underset{\grave{}}{y} = 0\,, \qquad \mathsf{c} := \frac{2\,(2\,s\,t + t\,t' + 1)}{t\,t' - 1}\,, \tag{3.16}$$

in which we have used the parameter identity (2.71). This equation has the same form as (3.11), that of a discrete harmonic oscillator, along with invariant

$$\mathscr{I}(y, \grave{y}) = y^2 + \grave{y}^2 + \mathsf{c}\, y\, \grave{y}\ .$$

Again, it easy to see that the equation of motion (3.16) has a solution which is given by

$$y(m) = \beta_1 \cos(\nu\, m) + \beta_2 \sin(\nu\, m)\ , \qquad \nu := \cos^{-1}(-\mathsf{c}/2)\,, \tag{3.17}$$

where m is the discrete variable and β_1, β_2 are constants. For convenience, we note that we allow n to label the bar evolution, and m to label the grave evolution, i.e. $y = y_{n,m}$, $\overline{y} = y_{n+1,m}$ and $\grave{y} = y_{n,m+1}$.

Equation (3.16) can be expressed by a Lagrangian-type generating function, with the equation arising from discrete Euler-Lagrange equations as

$$\mathscr{L}(y, \grave{y}) = \frac{1}{2}\,(\grave{y} - y)^2 - \frac{1}{2}\,\mathsf{c}\, y^2 - y^2\ ; \qquad \left(\frac{\partial \grave{\mathscr{L}}}{\partial y}\right) + \left(\frac{\partial \mathscr{L}}{\partial \grave{y}}\right) = 0\ . \tag{3.18}$$

We can define a momentum conjugate to the variable y by [24],

$$\grave{Y} = \frac{\partial \mathscr{L}}{\partial \grave{y}} = \grave{y} - y\ .$$

A Legendre transformation then leads us to introduce the discrete-time Hamiltonian

$$\mathscr{H}(y, \grave{Y}) = \grave{Y}\,(\grave{y} - y) - \mathscr{L} = \frac{1}{2}\,\grave{Y}^2 + \frac{1}{2}\,(\mathsf{c} + 2)\, y^2\ , \tag{3.19}$$

which is the generating function of the mapping viewed as a canonical transformation. One has the discrete-time Hamilton equations

$$\grave{Y} - Y = -\frac{\partial \mathscr{H}}{\partial y}\ , \qquad \grave{y} - y = \frac{\partial \mathscr{H}}{\partial \grave{Y}}\ , \tag{3.20}$$

leading to the equation for the mapping under consideration. In terms of the conjugate variables y, Y, we have a standard symplectic structure $d\grave{y} \wedge d\grave{Y} = dy \wedge dY$.

In order to write the symplectic mapping (3.20) in the form which is viewed as the symplectic Euler method (1.78), we introduce the canonical transformation $(y, Y) \to (q_{\mathsf{c}}, p_{\mathsf{c}})$,

$$q_{\mathsf{c}} := y\,(\mathsf{c} + 2)^{1/4}\ , \qquad p_{\mathsf{c}} := Y\,(\mathsf{c} + 2)^{-1/4}\ .$$

The discrete Hamiltonian (3.19) is then written in terms of the conjugate variables $q_{\mathsf{c}}, p_{\mathsf{c}}$,

$$H_{\mathsf{h}}(q_{\mathsf{c}}, \grave{p}_{\mathsf{c}}) = \frac{\mathsf{h}}{2}\left(\grave{p}_{\mathsf{c}}^2 + q_{\mathsf{c}}^2\right); \qquad \mathsf{h} := \sqrt{\mathsf{c}+2}\,, \tag{3.21}$$

with a discrete Hamilton equations

$$\grave{p}_{\mathsf{c}} - p_{\mathsf{c}} = -\frac{\partial H_{\mathsf{h}}}{\partial q_{\mathsf{c}}}\,, \qquad \grave{q}_{\mathsf{c}} - q_{\mathsf{c}} = \frac{\partial H_{\mathsf{h}}}{\partial \grave{p}_{\mathsf{c}}}\,. \tag{3.22}$$

We also have the Poisson bracket structure

$$\{q_{\mathsf{c}}, q_{\mathsf{c}}\} = \{p_{\mathsf{c}}, p_{\mathsf{c}}\} = 0\,, \quad \{q_{\mathsf{c}}, p_{\mathsf{c}}\} = 1\,,$$

which are preserved by the map in (3.22). The mapping (3.22) conserves the function

$$\mathcal{I}_{\mathsf{h}}(q_{\mathsf{c}}, p_{\mathsf{c}}) = p_{\mathsf{c}}^2 + q_{\mathsf{c}}^2 - \mathsf{h}\, p_{\mathsf{c}}\, q_{\mathsf{c}}\,. \tag{3.23}$$

Now, to derive the modified Hamiltonian for the system (3.21), we follow the same process as has been previously studied for the system (3.2). We apply the BCH formula (1.80) to the Hamiltonian in (3.21), where we consider h to be the step size of the mapping (3.22), and obtain the expansion for the modified Hamiltonian:

$$\begin{aligned} H_{\mathsf{h}}^* = &\frac{\mathsf{h}}{2}\, p_{\mathsf{c}}^2 + \frac{\mathsf{h}}{2}\, q_{\mathsf{c}}^2 - \frac{\mathsf{h}^2}{2}\, p_{\mathsf{c}}\, q_{\mathsf{c}} + \frac{\mathsf{h}^3}{12}\, p_{\mathsf{c}}^2 + \frac{\mathsf{h}^3}{12}\, q_{\mathsf{c}}^2 - \frac{\mathsf{h}^4}{12}\, p_{\mathsf{c}}\, q_{\mathsf{c}} \\ &+ \frac{\mathsf{h}^5}{60}\, p_{\mathsf{c}}^2 + \frac{\mathsf{h}^5}{60}\, q_{\mathsf{c}}^2 - \frac{\mathsf{h}^6}{60}\, p_{\mathsf{c}}\, q_{\mathsf{c}} + O(\mathsf{h}^7)\,. \end{aligned} \tag{3.24}$$

In a similar way as before, we can derive a closed-form expression for the modified Hamiltonian (3.24). By comparing the solution to the equation of motion arising from (3.22), with the exact solution to the differential equation

$$\ddot{q}_{\mathsf{c}} + F_{\mathsf{c}}(\mathsf{h})^2\left(4 - \mathsf{h}^2\right) q_{\mathsf{c}} = 0\,, \tag{3.25}$$

which derives from the true Hamiltonian flow arising from (3.24), we arrive at the following closed form for the modified Hamiltonian of the mapping from (3.22),

$$H_{\mathsf{h}}^* = \frac{\mathcal{I}_{\mathsf{h}}}{\sqrt{4 - \mathsf{h}^2}} \cos^{-1}\left(1 - \frac{\mathsf{h}^2}{2}\right). \tag{3.26}$$

Importantly, this shows that the expansion in (3.24) converges.

Having obtained the closed-form expressions for the modified Hamiltonians (3.14) and (3.26), one can show that these modified Hamiltonians are in involution with respect to the following Poisson bracket:

$$\{H^*_\tau, H^*_{\mathsf{h}}\} = \frac{\partial H^*_\tau}{\partial y}\frac{\partial H^*_{\mathsf{h}}}{\partial x} - \frac{\partial H^*_\tau}{\partial x}\frac{\partial H^*_{\mathsf{h}}}{\partial y} = 0 \,. \tag{3.27}$$

In fact, on the basis of the integrals (3.4) and (3.23), using the relations

$$q = y \,, \qquad p = -x \,,$$
$$q_{\mathsf{c}} = y\sqrt{\mathsf{h}} \,, \quad p_{\mathsf{c}} = \frac{y}{\sqrt{\mathsf{h}}}\left(\mathsf{h}^2 + 2\,s\,t - \frac{2\,t^2}{1-t\,t'}\right) + \frac{t\,x}{\sqrt{\mathsf{h}}}\left(1 + \frac{1+t\,t'}{1-t\,t'}\right) ,$$

and the fact (2.71), one can show by direct calculation that $\det\left[\partial(\mathcal{I}_\tau, \mathcal{I}_{\mathsf{h}})/\partial(x,y)\right] = 0$, which proves the relation (3.27).

A natural extension to the ideas in section 3.2 is to apply them to the two-dimensional evolution of the higher period staircase reduction, described in section 2.7.2. We will do this in the next section and there are again two commuting discrete maps and correspondingly two commuting invariants.

3.3 A coupled system of discrete-harmonic oscillators

In this section, we consider the discrete integrable systems that arise as $P = 3$ reduction of the linearized lattice KdV equation. This gives us an advantage to learn more and obtain good insights into the essential structures behind the modified Hamiltonians. In [65] the discrete-harmonic oscillator arising from period $P = 3$ reduction of the linearized lattice KdV equation was considered for the vertical time evolution, where it was found that the paired equations for the discrete flow are entangled. As we shall see next, however, for the diagonal time evolution we find a separated pair of discrete harmonic oscillators.

3.3.1 The discrete map

Let us remind ourselves of the integrable mapping considered in section 2.7.2,

$$\overline{x}_1 = x_1 + s\,(2\,y_1 + y_2) \,, \qquad \overline{y}_1 = y_1 - s\,(\overline{x}_1 - \overline{x}_2) \,, \tag{3.28a}$$

$$\overline{x}_2 = x_2 - s\,(y_1 - y_2) \,, \qquad \overline{y}_2 = y_2 - s\,(\overline{x}_1 + 2\,\overline{x}_2) \,. \tag{3.28b}$$

By eliminating x_1 and x_2, we derive a coupled system of equations for a discrete flow in variables y_1 and y_2,

$$\overline{y}_1 + (3\,s^2 - 2)\,y_1 + \underline{y}_1 = 0\,, \qquad \overline{y}_2 + (3\,s^2 - 2)\,y_2 + \underline{y}_2\,. \tag{3.29}$$

These are a separated pair of discrete harmonic oscillators in y_1 and y_2, with two quadratic invariants:

$$\mathscr{I}_1(y_1, \overline{y}_1) = y_1^2 + \overline{y}_1{}^2 + (3\,s^2 - 2)\,y_1\,\overline{y}_1\,, \tag{3.30a}$$

$$\mathscr{I}_2(y_2, \overline{y}_2) = y_2^2 + \overline{y}_2{}^2 + (3\,s^2 - 2)\,y_2\,\overline{y}_2\,. \tag{3.30b}$$

Solving the equations in (3.29) individually one obtains the general solutions

$$y_1(n) = \alpha_1 \cos(\mu\,n) + \beta_1 \sin(\mu\,n)\,, \qquad y_2(n) = \alpha_2 \cos(\mu\,n) + \beta_2 \sin(\mu\,n)\,, \tag{3.31}$$

where $\mu := \cos^{-1}\left(1 - \frac{3}{2}\,s^2\right)$, and n is a discrete variable. The equations in (3.29) are generated as discrete Euler-Lagrange equations

$$\overline{\frac{\partial \mathscr{L}}{\partial y_1}} + \frac{\partial \mathscr{L}}{\partial \overline{y}_1} = 0\,, \qquad \overline{\frac{\partial \mathscr{L}}{\partial y_2}} + \frac{\partial \mathscr{L}}{\partial \overline{y}_2} = 0\,,$$

from the Lagrangian

$$\mathscr{L}(y_1, y_2, \overline{y}_1, \overline{y}_2) = \frac{1}{2}\left[(\overline{y}_1 - y_1)^2 + (\overline{y}_2 - y_2)^2\right] - \frac{3}{2}\,s^2\,(y_1^2 + y_2^2)\,. \tag{3.32}$$

The Lagrangian above defines the momenta conjugate to y_1, y_2 by the usual formula $\overline{Y}_i = \partial\mathscr{L}/\partial\overline{y}_i$, so that

$$\overline{Y}_1 = \overline{y}_1 - y_1\,, \qquad \overline{Y}_2 = \overline{y}_2 - y_2\,, \tag{3.33}$$

with respect to which we have the invariant Poisson structure

$$\{y_i, Y_j\} = \delta_{ij}\,, \qquad i, j = 1, 2\,,$$

preserved under the mapping. Defining the discrete-time Hamiltonian by

$$\begin{aligned}\mathscr{H}(y_1, y_2, \overline{Y}_1, \overline{Y}_2) &= \overline{Y}_1\,(\overline{y}_1 - y_1) + \overline{Y}_2\,(\overline{y}_2 - y_2) - \mathscr{L} \\ &= \frac{1}{2}\left(\overline{Y}_1^2 + \overline{Y}_2^2\right) + \frac{3}{2}\,s^2\,(y_1^2 + y_2^2)\,,\end{aligned} \tag{3.34}$$

we get the discrete-time Hamilton equations

$$\overline{Y}_1 - Y_1 = -\frac{\partial \mathscr{H}}{\partial y_1}\,, \qquad \overline{y}_1 - y_1 = \frac{\partial \mathscr{H}}{\partial \overline{Y}_1}\,, \tag{3.35a}$$

$$\overline{Y}_2 - Y_2 = -\frac{\partial \mathscr{H}}{\partial y_2}\,, \qquad \overline{y}_2 - y_2 = \frac{\partial \mathscr{H}}{\partial \overline{Y}_2}\,, \tag{3.35b}$$

which preserve the symplectic structure $\sum_i d\overline{y}_i \wedge d\overline{Y}_i = \sum_i dy_i \wedge dY_i$.

In a similar way as before, we define a canonical transformation

$$q_i = \left(3\,s^2\right)^{1/4} y_i\,, \qquad p_i = \left(3\,s^2\right)^{-1/4} Y_i\,,$$

and hence obtain a generating function H_s in terms of the variables q_1, q_2, p_1 and p_2,

$$H_\mathsf{s}(q_1, q_2, \overline{p}_1, \overline{p}_2) = \frac{\mathsf{s}}{2}\left(\overline{p}_1^2 + \overline{p}_2^2\right) + \frac{\mathsf{s}}{2}\left(q_1^2 + q_2^2\right)\,, \qquad \mathsf{s} := s\sqrt{3}\,. \tag{3.36}$$

The Hamilton equations are

$$\overline{p}_1 - p_1 = -\frac{\partial H_\mathsf{s}}{\partial q_1}\,, \qquad \overline{q}_1 - q_1 = \frac{\partial H_\mathsf{s}}{\partial \overline{p}_1}\,, \tag{3.37a}$$

$$\overline{p}_2 - p_2 = -\frac{\partial H_\mathsf{s}}{\partial q_2}\,, \qquad \overline{q}_2 - q_2 = \frac{\partial H_\mathsf{s}}{\partial \overline{p}_2}\,, \tag{3.37b}$$

which conserves two independent invariants,

$$\mathcal{I}_{\mathsf{s}1} = p_1^2 + q_1^2 - \mathsf{s}\, p_1\, q_1 \quad \text{and} \quad \mathcal{I}_{\mathsf{s}2} = p_2^2 + q_2^2 - \mathsf{s}\, p_2\, q_2\,. \tag{3.38}$$

Note that we have the Poisson bracket structure

$$\{q_i, q_j\} = \{p_i, p_j\} = 0\,, \quad \{q_i, p_j\} = \delta_{ij}\,, \qquad i, j = 1, 2\,,$$

and so with respect to these, the two invariants in (3.38) are in involution: $\{\mathcal{I}_{\mathsf{s}1}, \mathcal{I}_{\mathsf{s}2}\} = 0$.

Now, by applying the BCH formula to the Hamiltonian from (3.36), one obtains the expansion for the modified Hamiltonian:

$$\begin{aligned} H_\mathsf{s}^* =& \frac{\mathsf{s}}{2}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) - \frac{\mathsf{s}^2}{2}\left(p_1\, q_1 + p_2\, q_2\right) \\ &+ \frac{\mathsf{s}^3}{12}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) - \frac{\mathsf{s}^4}{12}\left(p_1\, q_1 + p_2\, q_2\right) \\ &+ \frac{\mathsf{s}^5}{60}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) - \frac{\mathsf{s}^6}{60}\left(p_1\, q_1 + p_2\, q_2\right) + O(\mathsf{s}^7)\,. \end{aligned} \tag{3.39}$$

We can also express the latter equation in terms of the invariants, i.e.

$$H^*_{\mathsf{s}} = \frac{\mathsf{s}}{2}\,(\mathcal{J}_{\mathsf{s}1} + \mathcal{J}_{\mathsf{s}2}) + \frac{\mathsf{s}^3}{12}\,(\mathcal{J}_{\mathsf{s}1} + \mathcal{J}_{\mathsf{s}2}) + \frac{\mathsf{s}^5}{60}\,(\mathcal{J}_{\mathsf{s}1} + \mathcal{J}_{\mathsf{s}2}) + O(\mathsf{s}^7)\,. \tag{3.40}$$

Writing the modified Hamiltonian (3.40) in the form,

$$H^*_{\mathsf{s}} = G(\mathsf{s})\,(\mathcal{J}_{\mathsf{s}1} + \mathcal{J}_{\mathsf{s}2})\,, \tag{3.41}$$

and following the same prescription in section 3.2, we can find the exact solutions to the true Hamiltonian flow arising from (3.41),

$$q_1(t) = c_1 \cos\left(t\,G\sqrt{4-\mathsf{s}^2}\right) + c_2 \sin\left(t\,G\sqrt{4-\mathsf{s}^2}\right), \tag{3.42a}$$

$$q_2(t) = c_3 \cos\left(t\,G\sqrt{4-\mathsf{s}^2}\right) + c_4 \sin\left(t\,G\sqrt{4-\mathsf{s}^2}\right), \tag{3.42b}$$

where c_1, c_2, c_3 and c_4 are constants. Comparing these with the classical solutions to the equations of discrete harmonic oscillators arising from (3.37), it is then not hard to see that an appropriate choice of G is given by

$$G(\mathsf{s}) = \frac{1}{\sqrt{4-\mathsf{s}^2}} \cos^{-1}\left(1 - \frac{\mathsf{s}^2}{2}\right).$$

Hence, the closed form for the modified Hamiltonian of the mapping in (3.37) reads as

$$H^*_{\mathsf{s}} = \frac{(\mathcal{J}_{\mathsf{s}1} + \mathcal{J}_{\mathsf{s}2})}{\sqrt{4-\mathsf{s}^2}} \cos^{-1}\left(1 - \frac{\mathsf{s}^2}{2}\right). \tag{3.43}$$

3.3.2 The commuting map

Recalling now the commuting mapping of the map from (3.28), i.e. the mapping (2.74),

$$\grave{x}_1 = x_1 - t\,(2\,y_1 + y_2) - t\,t'\,(\grave{x}_2 - x_1)\,, \tag{3.44a}$$

$$\grave{y}_1 = y_1 - t\,(\grave{x}_2 - x_1) + s\,t\,(2\,y_1 + y_2)\,, \tag{3.44b}$$

$$\grave{x}_2 = x_2 - t\,(y_2 - y_1) + t\,t'\,(\grave{x}_1 + \grave{x}_2 + x_2)\,, \tag{3.44c}$$

$$\grave{y}_2 = y_2 + t\,(\grave{x}_1 + \grave{x}_2 + x_2) + s\,t\,(y_2 - y_1)\,, \tag{3.44d}$$

in a similar manner as (3.29) we can derive a separated pair of equations for a discrete flow in variables y_1 and y_2,

$$\grave{y}_1 + \mathsf{a}\,y_1 + \underset{\grave{}}{y}_1 = 0\,, \qquad \grave{y}_2 + \mathsf{a}\,y_2 + \underset{\grave{}}{y}_2\,; \qquad \mathsf{a} := \frac{2\,t\,(3\,s + 2\,t') + 4}{t\,t' - 2}\,. \tag{3.45}$$

The calculations involved to produce equations (3.45) are large, but have been verified by MAPLE (Math software).

Similar to the previous example, using the separated pair of discrete harmonic oscillators in (3.45) we arrive at the following discrete-time Hamiltonian in terms of the variables $q_{\mathsf{a}1}, q_{\mathsf{a}2}, p_{\mathsf{a}1}$ and $p_{\mathsf{a}2}$,

$$H_{\mathsf{b}}(q_{\mathsf{a}1}, q_{\mathsf{a}2}, \grave{p}_{\mathsf{a}1}, \grave{p}_{\mathsf{a}2}) = \frac{\mathsf{b}}{2}\,(\grave{p}_{\mathsf{a}1}^2 + \grave{p}_{\mathsf{a}2}^2) + \frac{\mathsf{b}}{2}\,(q_{\mathsf{a}1}^2 + q_{\mathsf{a}2}^2)\,, \qquad \mathsf{b} := \sqrt{\mathsf{a}+2}\,, \tag{3.46}$$

where we have

$$\grave{p}_{\mathsf{a}1} - p_{\mathsf{a}1} = -\frac{\partial H_{\mathsf{b}}}{\partial q_{\mathsf{a}1}}\,, \qquad \grave{q}_{\mathsf{a}1} - q_{\mathsf{a}1} = \frac{\partial H_{\mathsf{b}}}{\partial \grave{p}_{\mathsf{a}1}}\,, \tag{3.47a}$$

$$\grave{p}_{\mathsf{a}2} - p_{\mathsf{a}2} = -\frac{\partial H_{\mathsf{b}}}{\partial q_{\mathsf{a}2}}\,, \qquad \grave{q}_{\mathsf{a}2} - q_{\mathsf{a}2} = \frac{\partial H_{\mathsf{b}}}{\partial \grave{p}_{\mathsf{a}2}}\,. \tag{3.47b}$$

This mapping conserves two independent invariants,

$$\mathcal{I}_{\mathsf{b}1} = p_{\mathsf{a}1}^2 + q_{\mathsf{a}1}^2 - \mathsf{b}\,p_{\mathsf{a}1}\,q_{\mathsf{a}1} \quad \text{and} \quad \mathcal{I}_{\mathsf{b}2} = p_{\mathsf{a}2}^2 + q_{\mathsf{a}2}^2 - \mathsf{b}\,p_{\mathsf{a}2}\,q_{\mathsf{a}2}\,. \tag{3.48}$$

Again, with respect to the Poisson bracket structure

$$\{q_{\mathsf{a}i}, q_{\mathsf{a}j}\} = \{p_{\mathsf{a}i}, p_{\mathsf{a}j}\} = 0\,, \quad \{q_{\mathsf{a}i}, p_{\mathsf{a}j}\} = \delta_{ij}\,, \qquad i,j = 1,2\,,$$

the two invariants (3.48) are also in involution: $\{\mathcal{I}_{\mathsf{b}1}, \mathcal{I}_{\mathsf{b}2}\} = 0$.

The expansion for the modified Hamiltonian of the system in (3.46) is given by an expression similar to that obtained from (3.39); we simply replace p_i with $p_{\mathsf{a}i}$, q_i with $q_{\mathsf{a}i}$ and s with b. Hence, the closed form for the modified Hamiltonian of the mapping (3.47) is

$$H_{\mathsf{b}}^* = \frac{(\mathcal{I}_{\mathsf{b}1} + \mathcal{I}_{\mathsf{b}2})}{\sqrt{4-\mathsf{b}^2}}\cos^{-1}\left(1 - \frac{\mathsf{b}^2}{2}\right). \tag{3.49}$$

Once again, it is not difficult to see that the two modified Hamiltonians in (3.43) and (3.49) are in involution, i.e.

$$\{H_{\mathsf{s}}^*, H_{\mathsf{b}}^*\} = \sum_{i=1}^{2}\left(\frac{\partial H_{\mathsf{s}}^*}{\partial y_i}\frac{\partial H_{\mathsf{b}}^*}{\partial x_i} - \frac{\partial H_{\mathsf{s}}^*}{\partial x_i}\frac{\partial H_{\mathsf{b}}^*}{\partial y_i}\right) = 0\,. \tag{3.50}$$

This must be true since we have

$$\sum_{i=1}^{2}\det\frac{\partial(\mathcal{I}_{\mathsf{s}1}, \mathcal{I}_{\mathsf{b}1})}{\partial(x_i, y_i)} = \sum_{i=1}^{2}\det\frac{\partial(\mathcal{I}_{\mathsf{s}2}, \mathcal{I}_{\mathsf{b}2})}{\partial(x_i, y_i)} = \sum_{i=1}^{2}\det\frac{\partial(\mathcal{I}_{\mathsf{s}1}, \mathcal{I}_{\mathsf{s}2})}{\partial(x_i, y_i)} = \sum_{i=1}^{2}\det\frac{\partial(\mathcal{I}_{\mathsf{b}1}, \mathcal{I}_{\mathsf{b}2})}{\partial(x_i, y_i)} = 0\,,$$

which implies

$$\sum_{i=1}^{2} \det \frac{\partial(\mathcal{I}_{s1}, \mathcal{I}_{b2})}{\partial(x_i, y_i)} = \sum_{i=1}^{2} \det \frac{\partial(\mathcal{I}_{s2}, \mathcal{I}_{b1})}{\partial(x_i, y_i)} = 0 \, .$$

As we have seen, the discrete harmonic oscillator examples, which arise by exploiting the multi-dimensional consistency of the linearized lattice KdV equation, display some useful techniques. From a discrete-time perspective, there are a number of points of interest, especially for integrable systems. Specifically, the discrete-time systems are viewed as numerical integration schemes. As we shall observe next, for some important nonlinear integrable systems, which are not in the Newtonian form $H = p^2/2 + V(q)$ where $\partial V/\partial q$ is nonlinear, the expansions for the modified Hamiltonians of the relevant symplectic mappings do converge.

3.4 The KdV map example

Our aim in this section is to investigate the modified Hamiltonian for a mapping of "non-Newtonian" form, which arising from period $P = 2$ reduction of "nonlinear" integrable lattice KdV equation. This case gives rise to a one-degree-of-freedom Hamiltonian system, which is associated with discrete integrable dynamics.

3.4.1 Hamiltonian system and Yoshida's method

Consider the integrable mapping

$$\overline{p} = p + \frac{2\,\epsilon\,\delta\,q}{\epsilon^2 - q^2} \, , \qquad \overline{q} = q - \frac{2\,\epsilon\,\delta\,\overline{p}}{\epsilon^2 - \overline{p}^2} \, , \tag{3.51}$$

which is the mapping (2.10) in which we identify $X := p$ and $Y := q$. We have the standard Poisson brackets

$$\{q, q\} = \{p, p\} = 0 \, , \quad \{q, p\} = 1 \, ,$$

which are preserved by the map (3.51). Indeed, the mapping (3.51) is a canonical transformation with the generating function

$$H(q, \overline{p}) = \epsilon\,\delta \log\left(\epsilon^2 - \overline{p}^2\right) + \epsilon\,\delta \log\left(\epsilon^2 - q^2\right) \, , \tag{3.52}$$

through the discrete Hamilton equations

$$\overline{p} - p = -\frac{\partial H}{\partial q}\,, \qquad \overline{q} - q = \frac{\partial H}{\partial \overline{p}}\,. \tag{3.53}$$

The mapping in (3.51) possesses an exact invariant

$$\mathcal{I} = p^2\,q^2 - \epsilon^2\,(p^2 + q^2) - 2\,\epsilon\,\delta\,p\,q\,, \tag{3.54}$$

and, therefore, it is an integrable map (i.e. it is a symplectic mapping with an invariant).

By considering δ in the mapping (3.51) to be the step size (i.e. δ plays the rule of τ in the expansion (1.80)) and applying the BCH series (1.80) to the Hamiltonian (3.52), we obtain the following expansion for the modified Hamiltonian:

$$\begin{aligned} H^* &= \delta\,\epsilon\log(\epsilon^2 - p^2) + \delta\,\epsilon\log(\epsilon^2 - q^2) - \frac{2\,\delta^2\,\epsilon^2\,p\,q}{(\epsilon^2 - p^2)(\epsilon^2 - q^2)} \\ &\quad - \frac{2\,\delta^3\,\epsilon^3\,(\epsilon^2\,p^2 + \epsilon^2\,q^2 + 2\,p^2\,q^2)}{3\,(\epsilon^2 - p^2)^2(\epsilon^2 - q^2)^2} - \frac{4\,\delta^4\,\epsilon^4\,p\,q\,(\epsilon^2 + p^2)(\epsilon^2 + q^2)}{3\,(\epsilon^2 - p^2)^3(\epsilon^2 - q^2)^3} \\ &\quad + O(\delta^5)\,. \end{aligned} \tag{3.55}$$

At this stage, it is not obvious that this expansion converges, however, we will show that we actually have a closed-form expression for the modified Hamiltonian. This will provide a connection between the modified Hamiltonian and the invariant, taking into account that if the modified Hamiltonian exists, then it must be expressed in terms of the invariant. As the mapping is integrable, we can employ a transformation to action-angle variables to derive an interpolating Hamiltonian where the canonical momentum is an invariant of the map.

3.4.2 Action-angle variables derivation of the interpolating Hamiltonian

A change to action-angle variables is a canonical (symplectic) transformation to a new set of phase space coordinates, such that the new momenta are invariants of the system and the coordinates evolve in a linear fashion. This can be viewed as an application of the Hamilton-Jacobi method, cf. [47]. A discussion of action-angle variables for integrable mappings can be found in refs. [8, 16].

In the one-degree-of-freedom system under consideration, the relevant canonical transformation $(q,p) \longrightarrow (\mathcal{Q},\mathcal{P})$ is given by means of a generating function, $S(q,\mathcal{P})$ as

$$p = \frac{\partial S\,(q,\mathcal{P})}{\partial q}\,, \qquad \mathcal{Q} = \frac{\partial S\,(q,\mathcal{P})}{\partial \mathcal{P}}\,, \tag{3.56}$$

with

$$K(\mathcal{Q},\mathcal{P}) = H + \frac{\partial S}{\partial t}\,, \tag{3.57}$$

being the transformed Hamiltonian, noting that in the case of a transformation to action-angle variables the K only depends on $\mathcal{P}$. In fact, the modified Hamiltonian H^* and the discrete Hamiltonian (3.52) respectively coincide with the canonical transformed Hamiltonian K and the Hamiltonian H in (3.57). Note that in our case $\partial S/\partial t = 0$ since S is not explicitly a function of t.

For the system (3.53) the new momentum is defined to be $\mathcal{P} = \mathcal{I}$ where $\mathcal{I}$ is the invariant of the map given in (3.54). Basically, the inversion that we have $\mathcal{P}$ is a function of q and p, and so that we can invert that to be p as a function of q and $\mathcal{P}$. Thus, integrating the first system of equations (3.56), we get S up to an arbitrary function of the invariant

$$S(q,\mathcal{P}) = \int_{q^0}^{q} \frac{\epsilon\,\delta\,q' + \sqrt{\epsilon^2\,\delta^2\,q'^2 - (\epsilon^2 - q'^2)(\epsilon^2\,q'^2 + \mathcal{P})}}{q'^2 - \epsilon^2}\,dq'\,, \tag{3.58}$$

and consequently we obtain

$$\mathcal{Q}(q,\mathcal{P}) = \int_{q^0}^{q} \frac{1}{2\sqrt{\delta^2\,\epsilon^2\,q'^2 - (\epsilon^2 - q'^2)(\epsilon^2\,q'^2 + \mathcal{P})}}\,dq'\,. \tag{3.59}$$

In the context of the mapping (3.51), the relevant continuous Hamiltonian flow is the one whose Hamiltonian is given by the invariant of the map for which we have the Hamilton's equations

$$\dot{p} = -\frac{\partial \mathcal{P}}{\partial q}\,, \qquad \dot{q} = \frac{\partial \mathcal{P}}{\partial p}\,. \tag{3.60}$$

This system actually defines an interpolating flow where the trajectory of the system (3.60) and the orbit of the map share the level set of the invariant. Hence, we integrate (3.60) to obtain

$$t = \int_{0}^{\mathcal{E}(t|\epsilon,\delta,\mathcal{P})} \frac{1}{2\sqrt{\delta^2\,\epsilon^2\,q^2 - (\epsilon^2 - q^2)(\epsilon^2\,q^2 + \mathcal{P})}}\,dq\,, \tag{3.61}$$

which defines the relevant elliptic function $\mathcal{E}(t|\epsilon,\delta,\mathcal{P})$ in terms of an elliptic integral of the first kind [54]. The modified Hamiltonian H^* coincides with the canonical transformed

Hamiltonian K obtained by applying the canonical transformation (3.56), viewed as a function of $\mathcal{Q}, \mathcal{P}$. Hence, Hamilton's equations in the new variables imply

$$\dot{\mathcal{P}} = -\frac{\partial H^*}{\partial \mathcal{Q}} = 0 \,, \qquad \dot{\mathcal{Q}} = \frac{\partial H^*}{\partial \mathcal{P}} = \nu \,, \tag{3.62}$$

which tell us on the one hand that H^* is a function of $\mathcal{P}$ alone, and on the other hand that H^* is obtained by integrating ν with respect to $\mathcal{P}$ up to an arbitrary function of the invariant.

In order to apply the canonical transformation to the map (3.51), we introduce the "frequency" ν as the discrete time-one step

$$\nu = \int_q^{\overline{q}} \frac{\partial p}{\partial \mathcal{P}} \, dq' \,, \quad \text{so that} \quad \overline{\mathcal{Q}} - \mathcal{Q} = \nu \,, \tag{3.63}$$

which crucially depends on $\mathcal{P}$ only. Since the time t flow interpolates the map (3.51), the iteration of the map is a time-one step stroboscope of the continuous time flow, so the integral (3.61) can be subdivided into uniform time-one iterates. Thus, we can choose an initial point at q, and use the system (3.51) to compute $\overline{q}$, starting from $q = 0$ we then obtain

$$\overline{q} = \frac{2\,\epsilon^2\,\delta\,\sqrt{-\mathcal{P}}}{\epsilon^4 + \mathcal{P}} \,.$$

Thus, the frequency (3.63) is given as

$$\nu = \int_0^{\frac{2\,\epsilon^2\,\delta\,\sqrt{-\mathcal{P}}}{\epsilon^4+\mathcal{P}}} \frac{1}{2\,\sqrt{\delta^2\,\epsilon^2\,q^2 - (\epsilon^2 - q^2)(\epsilon^2\,q^2 + \mathcal{P})}} \, dq \,, \tag{3.64}$$

and hence

$$H^*(\mathcal{P}) = \int^{\mathcal{P}} \int_0^{\frac{2\,\epsilon^2\,\delta\,\sqrt{-\mathcal{P}'}}{\epsilon^4+\mathcal{P}'}} \frac{1}{2\,\sqrt{\delta^2\,\epsilon^2\,q^2 - (\epsilon^2 - q^2)(\epsilon^2\,q^2 + \mathcal{P}')}} \, dq \, d\mathcal{P}' \,. \tag{3.65}$$

The inside integral is a definite integral and the outside integral is an indefinite integral which is determined up to an integration constant. Equation (3.65) is a closed-form expression for the modified Hamiltonian of the map (3.51).

Writing equation (3.64) as a series in δ and fixing the integration constant such that for $\delta = 0$, the integration over $\mathcal{P}'$ vanishes, we obtain

$$H^*(\mathcal{P}) = \delta\,\epsilon \log\left(\mathcal{P} + \epsilon^4\right) + \frac{2\,\delta^3\,\epsilon^3\,\mathcal{P}}{3\,(\mathcal{P} + \epsilon^4)^2} - \frac{4\,\delta^5\,\epsilon^5\,\mathcal{P}\,(\mathcal{P} - 2\,\epsilon^4)}{15\,(\mathcal{P} + \epsilon^4)^4} + O(\delta^7) \,. \tag{3.66}$$

In the light of our present investigation we expect that expansions (3.55) and (3.66) are the same. In order to prove a full match, we would need to be able to compute all terms in the BCH series which in view of the complexity seems an impossible task. In fact, we envisage using the correspondence between the closed-form modified Hamiltonian and BCH series as a way of understanding the structure of the latter in the integrable case. Therefore, we will prove the match up to order δ^4 which can be seen in two steps. Firstly, we insert the invariant $\mathcal{P} = \mathcal{I}$ from (3.54) into (3.66) and subsequently expand each term in orders of δ, one obtains for the first few

$$\delta\,\epsilon \log\,(\mathcal{P}+\epsilon^4) = \delta\,\epsilon \log\,(\epsilon^2-p^2)(\epsilon^2-q^2) - \frac{2\,\delta^2\,\epsilon^2\,p\,q}{(\epsilon^2-p^2)(\epsilon^2-q^2)} - \frac{2\,\delta^3\,\epsilon^3\,p^2\,q^2}{(\epsilon^2-p^2)^2(\epsilon^2-q^2)^2} - \frac{8\,\delta^4\,\epsilon^4\,p^3\,q^3}{3\,(\epsilon^2-p^2)^3(\epsilon^2-q^2)^3} - O(\delta^5)\,,$$

$$\frac{2\,\delta^3\,\epsilon^3\,\mathcal{P}}{3\,(\mathcal{P}+\epsilon^4)^2} = \frac{2\,\delta^3\,\epsilon^3\,(p^2\,q^2-\epsilon^2\,p^2-\epsilon^2\,q^2)}{3\,(\epsilon^2-p^2)^2(\epsilon^2-q^2)^2} + \frac{4\,\delta^4\,\epsilon^4\,p\,q\,(p^2\,q^2-\epsilon^2\,p^2-\epsilon^2\,q^2-\epsilon^4)}{3\,(\epsilon^2-p^2)^3(\epsilon^2-q^2)^3} + O(\delta^5)\,,$$

and so forth. Secondly, we rearrange the series arising from the previous step by combining all terms with the same order of δ, one obtains

$$\begin{aligned} H^* &= \delta\,\epsilon \log\,(\mathcal{P}+\epsilon^4) + \frac{2\,\delta^3\,\epsilon^3\,\mathcal{P}}{3\,(\mathcal{P}+\epsilon^4)^2} + O(\delta^5) \\ &= \delta\,\epsilon \log\,(\epsilon^2-p^2)(\epsilon^2-q^2) - \frac{2\,\delta^2\,\epsilon^2\,p\,q}{(\epsilon^2-p^2)(\epsilon^2-q^2)} - \frac{2\,\delta^3\,\epsilon^3\,(\epsilon^2\,p^2+\epsilon^2\,q^2+2\,p^2\,q^2)}{3\,(\epsilon^2-p^2)^2(\epsilon^2-q^2)^2} \\ &\quad - \frac{4\,\delta^4\,\epsilon^4\,p\,q\,(\epsilon^2+p^2)(\epsilon^2+q^2)}{3\,(\epsilon^2-p^2)^3(\epsilon^2-q^2)^3} + O(\delta^5)\,, \end{aligned}$$

which is the same expansion for the modified Hamiltonian obtained by applying the BCH series, i.e. expansion (3.55).

3.5 The mKdV map example

Next, consider the integrable mapping (2.37), which arises from the simplest case ($P = 2$) reduction of integrable lattice mKdV equation. For convenience, this mapping is given by

$$\overline{p} = p + 2\log\frac{1+r\,e^q}{r+e^q}\,, \qquad \overline{q} = q + 2\log\frac{r+e^{\overline{p}}}{1+r\,e^{\overline{p}}}\,, \tag{3.67}$$

where we identify $X := p$ and $Y := q$. We have the standard invariant symplectic structure

$$d\overline{q} \wedge d\overline{p} = dq \wedge dp \ ,$$

which is preserved by the map (3.67). The mapping (3.67) is in fact a canonical transformation, and the generating function H (Hamiltonian) of the mapping is in this case found to be

$$H(q,\overline{p}) = 2\int_0^{\overline{p}} \log\frac{r+e^{\xi}}{1+r\,e^{\xi}}\,d\xi + 2\int_0^{q} \log\frac{r+e^{\xi}}{1+r\,e^{\xi}}\,d\xi \ . \tag{3.68}$$

Note that this Hamiltonian can be written in terms of dilogarithm functions using the well-known integral representation. The discrete-time Hamilton equations are written as

$$\overline{p} - p = -\frac{\partial H}{\partial q} \ , \qquad \overline{q} - q = \frac{\partial H}{\partial \overline{p}} \ ,$$

and the mapping (3.67) conserves the function

$$\mathcal{I} = e^{p-q} + e^{q-p} + 2\,r\left(e^{p} + e^{q} + e^{-p} + e^{-q}\right) + r^2\left(e^{p+q} + e^{-(p+q)}\right) . \tag{3.69}$$

In this example we actually have a different type of situation from what we had in the previous example since the step size in the mapping (3.67) is much more hidden (implicit). By applying the BCH formula (1.80) to the Hamiltonian (3.68), the expansion for the modified Hamiltonian is given as

$$\begin{aligned} H^* = {} & 2\int_0^{p} \log\frac{r+e^{\xi}}{1+r\,e^{\xi}}\,d\xi + 2\int_0^{q} \log\frac{r+e^{\xi}}{1+r\,e^{\xi}}\,d\xi - 2\log\frac{1+r\,e^{p}}{r+e^{p}}\log\frac{1+r\,e^{q}}{r+e^{q}} \\ & - \frac{2\,e^{q}(r^2-1)}{3\,(1+r\,e^{q})(r+e^{q})}\left(\log\frac{1+r\,e^{p}}{r+e^{p}}\right)^2 - \frac{2\,e^{p}(r^2-1)}{3\,(1+r\,e^{p})(r+e^{p})}\left(\log\frac{1+r\,e^{q}}{r+e^{q}}\right)^2 \\ & - \frac{4\,e^{p+q}(r^2-1)^2}{3\,(1+r\,e^{q})(r+e^{q})(1+r\,e^{p})(r+e^{p})}\log\frac{1+r\,e^{p}}{r+e^{p}}\log\frac{1+r\,e^{q}}{r+e^{q}} \\ & + \cdots \ . \end{aligned} \tag{3.70}$$

Setting $\tau := r-1$, we write the expansion for the modified Hamiltonian in (3.70) as a series in orders of τ,

$$\begin{aligned} H^* = {} & 2\,\tau\left[p+q-2\,\log\,(1+\mathrm{e}^{p})(1+\mathrm{e}^{q})\right] \\ & - \tau^2\left[p+q-2\,\log\,(1+\mathrm{e}^{p})(1+\mathrm{e}^{q}) + \frac{4\,(\mathrm{e}^{p+q}+1)}{(1+\mathrm{e}^{p})(1+\mathrm{e}^{q})}\right] \\ & + \frac{2\,\tau^3}{3}\left[\frac{3\,(\mathrm{e}^{p+2q}+\mathrm{e}^{2p+q}+\mathrm{e}^{p}+\mathrm{e}^{q})}{(1+\mathrm{e}^{p})^2(1+\mathrm{e}^{q})^2} + \frac{2\,(3\,\mathrm{e}^{2(p+q)}+8\,\mathrm{e}^{p+q}+3)}{(1+\mathrm{e}^{p})^2(1+\mathrm{e}^{q})^2}\right] \\ & + \frac{2\,\tau^3}{3}\left[p+q-2\,\log\,(1+\mathrm{e}^{p})(1+\mathrm{e}^{q})\right] + O(\tau^4) \ . \end{aligned} \tag{3.71}$$

Once again, it is not obvious that this expansion converges, however, finding the connection between the modified Hamiltonian and the invariant will essentially assert that there is a closed-form expression for the modified Hamiltonian.

Again, since the mapping (3.67) is integrable we can follow the action-angle prescription of section 3.4.2 to derive the modified Hamiltonian. Indeed, we can define the invariant (3.69) as the new momentum $\mathcal{P}$ and its canonically conjugated variable as the new coordinate $\mathcal{Q}$. Similarly as in example 3.4, the corresponding frequency ν is given by

$$\nu = \int_{q}^{\overline{q}} \frac{dq'}{\sqrt{(2\,r^2+\mathcal{P}+2)\,(\mathcal{P}-2\,r^2-4\,r\,e^{q'}-4\,r\,e^{-q'}-2)}} \,. \tag{3.72}$$

Choosing an initial point at q starting from $q=0$, one then obtains

$$\overline{q} = 2\log\left(\frac{r+\Delta}{1+r\,\Delta}\right),$$

in which we define the shorthand

$$\Delta \equiv \frac{\mathcal{P}-4\,r-\sqrt{(2\,r^2+\mathcal{P}+2)\,(\mathcal{P}-2\,r^2-8\,r-2)}}{2\,(1+r)^2}.$$

The integral (3.72) is also an elliptic integral of the first kind.

By using the Hamilton's equations with the action-angle coordinates, and denoting the new Hamiltonian by H^*, we obtain

$$H^*(\mathcal{P}) = \int^{\mathcal{P}} \int_0^{2\log\left(\frac{r+\Delta}{1+r\,\Delta}\right)} \frac{dq\,d\mathcal{P}'}{\sqrt{(2\,r^2+\mathcal{P}'+2)\,(\mathcal{P}'-2\,r^2-4\,r\,e^{q}-4\,r\,e^{-q}-2)}} \,, \tag{3.73}$$

up to an integration constant. Equation (3.73) is a closed-form expression for the modified Hamiltonian of the map (3.67).

Again, setting $\tau = r-1$ and fixing the constant of integration over $\mathcal{P}$ such that for $\tau = 0$, the integral vanishes, we write the expression in (3.73) as a series in τ,

$$\begin{aligned} H^*(\mathcal{P}) = &-2\,\tau\log(\mathcal{P}+4) + \tau^2\left(\log(\mathcal{P}+4) - \frac{8}{(\mathcal{P}+4)}\right) \\ &- \frac{2\,\tau^3}{3}\left(\log(\mathcal{P}+4) - \frac{4}{(\mathcal{P}+4)} - \frac{24}{(\mathcal{P}+4)^2}\right) + O(\tau^4)\,. \end{aligned} \tag{3.74}$$

Equation (3.74) can once again be expanded in orders of τ and, after following the same procedure of section 3.4.2, the matching with series (3.71) will be seen.

3.6 Summary

By considering discrete integrable mappings, which arise by reduction from integrable lattice equations, we have presented examples of integrable numerical integration schemes which have closed-form modified Hamiltonians.

The discrete map and its commuting map led to a novel perspective on the discrete harmonic oscillators that arise from the equations of motion for two different evolutions. Applying a BCH formula led to modified Hamiltonians which were found to have closed-form expressions. With respect to the variables of original maps, we have found that those modified Hamiltonians are in involution.

The ideas can also be extended to higher dimensional cases, by considering the discrete and commuting discrete maps for higher periodic reductions, which definitely gives a transparent picture of the connection between the invariants and the relevant modified Hamiltonians. We examined the next case (in two-degrees-of-freedom system) as examples, deriving the modified Hamiltonians. We have also found that these modified Hamiltonians are in involution with respect to the original variables.

It is generally understood that common numerical methods applied to Hamiltonians of the Newtonian form $H = p^2/2 + V(q)$, where $\partial V(q)/\partial q$ is nonlinear, have an expansion for the modified Hamiltonian which does not converge. However, in the light of this thesis, there exist special symplectic integrators that do allow convergent expansion for the modified Hamiltonians for some non-Newtonian Hamiltonian systems, which are associated with discrete integrable dynamics. The corresponding Hamiltonian systems are associated with the interpolating flow of these integrable mappings. We have presented an example of the Hamiltonian system for the simplest case (i.e. one-degree-of-freedom) which arises from the reduction of the nonlinear lattice KdV equation. In this case, we obtained a closed-form expression in terms of an elliptic integral and compare it with the expansion that we get from Yoshida's approach.

Furthermore, we broadened the perspective by looking at the extension to an implicit scheme where the mKdV case is a key example. In particular, the example of the mKdV map exhibits an implicit dependence on the time step which could be of relevance to certain implicit schemes in numerical analysis.

Chapter 4

The KdV model: Separation of variables and finite-gap integration

4.1 Overview

The examples presented in the previous chapter already display some useful techniques from Hamiltonian mechanics. As we shall observe next, in the nonlinear case the transition to higher-degrees-of-freedom systems requires some further technology from the theory of discrete integrable systems. We will restrict ourselves to two-degrees-of-freedom systems to keep the discussion transparent. The higher-degrees-of-freedom case has to be treated by the method of separation of variables in order to arrive at the action-angle variables, and we restrict ourselves to the case of KdV mappings.

In the two-degrees-of-freedom case the situation becomes more complicated, which requires some mathematical techniques, such as separation of variables [110, 119] and finite-gap integration [10, 46][5]. Instead of transforming $(\boldsymbol{q}, \boldsymbol{p})$ directly to the action-angle variables $(\mathcal{Q}, \mathcal{P})$, here we look for a canonical transformation to separated variables $(\boldsymbol{q}, \boldsymbol{p}) \longrightarrow (\boldsymbol{\mu}, \boldsymbol{\eta})$ and a further transformation $(\boldsymbol{\mu}, \boldsymbol{\eta}) \longrightarrow (\mathcal{Q}, \mathcal{P})$, combined with a description of the dynamics of the separated variables $(\boldsymbol{\mu}, \boldsymbol{\eta}) \longrightarrow (\overline{\boldsymbol{\mu}}, \overline{\boldsymbol{\eta}})$. We employ the key technique of separation of variables to the mappings of KdV type in section 4.2.

[5] We note that in this section we are going to apply these techniques for higher-degrees-of-freedom of genus g.

Then, we impose the Poisson bracket for the monodromy matrix, followed by the Poisson brackets between its entries, to establish the separation of variables transform as a canonical transformation, in section 4.3. We derive the discrete dynamics in terms of the separated variables (discrete Dubrovin equations) for this case in section 4.4. We establish the continuous-time evolution for the auxiliary spectrum generated by the invariants in section 4.5. We introduce a canonical transformation to action-angle variables on the basis of structure in terms of the separation variables in section 4.6. Finally in section 4.7, considering the two-degrees-of-freedom case, we exploit the schemes that we set up to derive a closed-form expression for the modified Hamiltonian.

4.2 Separation of variables

The method of separation of variables (SoV) plays an important role in studying Liouville integrable systems. The SoV originated from the development of Hamiltonian mechanics as a method to separate an N-degrees-of-freedom system into one-degree-of-freedom system through the Hamilton-Jacobi equation for particular Hamiltonians. The SoV approach has been applied to many families of finite-dimensional integrable systems, cf. refs. [8, 67, 110].

Taking the monodromy matrix $T(\lambda)$ in the form

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} , \tag{4.1}$$

it is well known that in the periodic problems for equations of KdV type, the roots $\{\mu_j\}$, $j = 1, \ldots, g$, of the polynomial $B(\lambda)$ define the so-called *auxiliary spectrum* [113] and they play the role of separation variables [119]. These roots correspond to the poles of the Baker-Akhiezer[6] (BA) function by which we mean an eigenfunction of the Lax representation normalized in such way that its analytic behaviour as a function of the spectral parameter λ is characterized through the singularity structure. The entry $B(\lambda)$ has the following factorization

$$B(\lambda) = B_g \prod_{j=1}^{g} (\lambda - \mu_j) , \tag{4.2}$$

[6]This terminology appears in the theory of discrete integrable systems, in particular, in finite-gap integration.

where B_g is a Casimir. This polynomial leads to the expressions for $B_0/B_g, \ldots, B_{g-1}/B_g$ as elementary symmetric functions of the zeroes $\mu_1, \ldots, \mu_g$. Similarly, the coefficients of $C(\lambda)$ are symmetric functions of its zeroes $\gamma_1, \ldots, \gamma_g$.

The linear problem for the BA function is the eigenvector

$$T(\lambda)\,\phi(\lambda) = \eta(\lambda)\,\phi(\lambda)\ , \tag{4.3}$$

of the monodromy matrix $T(\lambda)$ corresponding to the eigenvalue $\eta(\lambda)$ of the spectral curve

$$\det\left(T(\lambda) - \eta\mathbb{I}\right) = \eta^2 - \operatorname{tr} T(\lambda)\,\eta + \det T(\lambda) = 0\ . \tag{4.4}$$

This curve (4.4) defines a hyperelliptic curve of genus $g = P-1$, where P denotes the period reduction defined in chapter 2. The linear problem (4.3) provides that a normalization of the eigenvectors ϕ is fixed

$$\vec{\alpha}\,\phi = 1\ , \tag{4.5}$$

where $\vec{\alpha}$ is a row vector suitably chosen. The pair (λ, η) can be thought of as a point of the spectral curve (4.4). The BA function ϕ is then a meromorphic function on the spectral curve. In our case the following normalization works,

$$\vec{\alpha} = (1, 0)\ . \tag{4.6}$$

From the linear equation (4.3) and normalization (4.5) we derive that

$$\vec{\alpha}\,T\,\phi = \eta\ ,$$

and hence,

$$\phi = \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\,T(\lambda) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \eta \end{pmatrix}. \tag{4.7}$$

The separation variables μ_j are defined to be the values of λ. Hence, the following determinant has to vanish on the separation variables μ_j,

$$B(\lambda) = \det \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\,T(\lambda) \end{pmatrix} = 0\ . \tag{4.8}$$

It turns out that with this choice of $\vec{\alpha}$, the variables (μ_j, η_j), where $\eta_j = \eta(\mu_j)$, are canonical, as we will show later. No general recipe is known how to guess the proper (that is producing canonical variables) normalization for the BA function. It is easy to see that the pairs (μ_j, η_j) thus defined satisfy the separation equations (4.4), which express the fact that the (μ_j, η_j) are lying on the spectral curve.

We note that the functions $A(\lambda)$ and $D(\lambda)$ of the entries of the monodromy matrix $T(\lambda)$ satisfy the following relations

$$A(\mu_j) = \mathfrak{y}^+(\mu_j) = \eta_j \quad \text{and} \quad D(\mu_j) = \mathfrak{y}^-(\mu_j) = \eta'_j \,. \tag{4.9}$$

What remains is to establish the SoV transform as a canonical transformation, i.e. to verify that the Poisson brackets between the separation variables have the canonical structure. To do this we use the information about the Poisson brackets between the entries of monodromy matrix $T(\lambda)$ provided by the classical r, s-matrix structure.

4.3 Poisson brackets and r-matrix structures

Recall the reduced rational mapping of KdV type, for discrete-time evolution along the diagonal shift, considered in chapter 2,

$$\overline{x}_j = x_j - \frac{\omega}{y_j} + \frac{\omega}{y_{j-1}} \,, \qquad \overline{y}_j = y_j + \frac{\omega}{\overline{x}_j} - \frac{\omega}{\overline{x}_{j+1}} \,, \tag{4.10}$$

in which we have defined the shorthand $x_j := \epsilon - X_j$, $y_j := \epsilon - Y_j$ and $\omega := \epsilon\,\delta$. This mapping is a $2\,(P-1)$-dimensional where we impose the periodicity condition

$$\sum_{j=1}^{P} x_j = \sum_{j=1}^{P} y_{j-1} = P\epsilon := C \,, \quad \text{where } C \text{ is a Casimir.}$$

Let us now remind the reader of the Lax matrices of the mapping of KdV type (4.10). These matrices depends on a discrete variable labelling the sites along a chain of length P, and are given by

$$L_j(\lambda) = V_{2j}\, V_{2j-1} \,; \qquad V_j = \begin{pmatrix} v_j & 1 \\ \lambda_j & 0 \end{pmatrix} , \tag{4.11a}$$

$$M_j(\lambda) = \begin{pmatrix} \lambda_{2j} + (\omega/\overline{v}_{2j-1})\,(\omega/v_{2j} - v_{2j-1}) & -\omega/\overline{v}_{2j-1} \\ \lambda_{2j}\,(v_{2j-1} - \overline{v}_{2j-1} - \omega/v_{2j}) & \lambda_{2j} \end{pmatrix} , \tag{4.11b}$$

where $v_{2j} := y_j, v_{2j-1} := x_j, \lambda_{2j} := \lambda$ and $\lambda_{2j-1} := \lambda - \omega$, with λ the spectral parameter. The mapping equations (4.10) are produced by the Zakharov-Shabat condition,

$$\overline{L}_j \, M_j = M_{j+1} \, L_j \;, \qquad j = 1, \ldots, N \;, \tag{4.12}$$

for matrices (4.11).

The key object here is the monodromy matrix $T(\lambda)$ obtained by gluing the elementary translation matrices L_n along a line connecting the sites 1 and $P+1$ over one period P, namely,

$$T(\lambda) := \prod_{n=1}^{\overset{\curvearrowleft}{P}} L_n(\lambda) \;, \tag{4.13}$$

which is essentially the monodromy matrix (2.19). The mappings of KdV type (4.10) have sufficiently many independent invariants for integrability, guaranteed by the existence of the monodromy matrix $T(\lambda)$ (4.13). However, for the mappings to be integrable, these invariants must be in involution with respect to the Poisson structure

$$\{x_n, y_m\} = \delta_{n,m} - \delta_{n,m+1} \;, \tag{4.14}$$

which can be shown by constructing an r, s-matrix structure for the Lax pair [88]. The classical r, s-matrix structure for the mapping of KdV type is given by [82]:

$$r_{12}^{-} = \frac{\mathscr{P}_{12}}{\lambda_1 - \lambda_2} \;, \quad s_{12}^{-} = \frac{1}{\lambda_1} E \otimes F \;, \quad s_{12}^{+} = \frac{1}{\lambda_2} F \otimes E \;, \quad r_{12}^{+} = r_{12}^{-} + s_{12}^{+} - s_{12}^{-} \;, \tag{4.15}$$

in which $\lambda_\alpha = k_\alpha^2 - b^2, \alpha = 1, 2$, and the permutation matrix $\mathscr{P}_{12}$ and the matrices E, F are given by

$$\mathscr{P}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

With the use of these matrices, the Poisson bracket structure (4.14) can be encoded on the Lax matrices (4.11) as

$$\begin{aligned} \{\overset{1}{L}_n, \overset{2}{L}_m\} = -\delta_{n,m+1} \, \overset{1}{L}_n \, s_{12}^{+} \, \overset{2}{L}_m + \delta_{n+1,m} \, \overset{2}{L}_m \, s_{12}^{-} \, \overset{1}{L}_n \\ + \delta_{n,m} \, (r_{12}^{+} \, \overset{1}{L}_n \, \overset{2}{L}_m - \overset{1}{L}_n \, \overset{2}{L}_m \, r_{12}^{-}) \;. \end{aligned} \tag{4.16}$$

This is known as the discrete version of the non-ultralocal Poisson bracket structure [82, 88]. In equation (4.16) the superscripts 1 and 2 for the operator matrices L_n denote the corresponding factor on which this L_n acts (acting trivially on the other factors), i.e.

$$\overset{1}{L}_n := L_n(\lambda_1) \otimes \mathbb{I} \quad \text{and} \quad \overset{2}{L}_n := \mathbb{I} \otimes L_n(\lambda_2) ,$$

where λ_1 and λ_2 are distinct spectral parameters.

To ensure that the skew-symmetry and Jacobi identities hold for the Poisson bracket (4.16), the following conditions must be hold, respectively,

$$r_{12}^{\pm}(\lambda_1, \lambda_2) = -r_{21}^{\pm}(\lambda_2, \lambda_1) , \qquad s_{12}^{-}(\lambda_1, \lambda_2) = s_{21}^{+}(\lambda_2, \lambda_1) , \tag{4.17}$$

and

$$\left[r_{12}^{\pm}, r_{13}^{\pm}\right] + \left[r_{12}^{\pm}, r_{23}^{\pm}\right] + \left[r_{13}^{\pm}, r_{23}^{\pm}\right] \;=\; 0 , \tag{4.18a}$$

$$\left[s_{12}^{\pm}, s_{13}^{\pm}\right] + \left[s_{12}^{\pm}, r_{23}^{\pm}\right] + \left[s_{13}^{\pm}, r_{23}^{\pm}\right] \;=\; 0 . \tag{4.18b}$$

The relation (4.18a) is nothing but the usual classical limit of the famous Yang-Baxter equation [63, 107]. We note that a more general proof of equations (4.18) is given in chapter 5.

Proposition 4.3.1. *The Poisson bracket for the monodromy matrix $T(\lambda)$ in terms of the r, s-matrix structure follows from the discrete version of the non-ultralocal Poisson bracket structure (4.16) and reads*

$$\{\overset{1}{T}, \overset{2}{T}\} = r_{12}^{+} \overset{1}{T} \overset{2}{T} - \overset{2}{T} \overset{1}{T} r_{12}^{-} - \overset{1}{T} s_{12}^{+} \overset{2}{T} + \overset{2}{T} s_{12}^{-} \overset{1}{T} . \tag{4.19}$$

Proof

Consider the equations (4.13) and (4.16). Using these equations we can establish

$$
\begin{aligned}
\{\overset{1}{T},\overset{2}{L}_n\} &= \{\overset{1}{L}_P\,\overset{1}{L}_{P-1}\cdots\overset{1}{L}_1,\ \overset{2}{L}_n\} \\
&= \{\overset{1}{L}_P,\ \overset{2}{L}_n\}\,\overset{1}{L}_{P-1}\cdots\overset{1}{L}_1 + \sum_{j=2}^{P-1}\overset{1}{L}_P\cdots\overset{1}{L}_{j+1}\,\{\overset{1}{L}_j,\ \overset{2}{L}_n\}\,\overset{1}{L}_{j-1}\cdots\overset{1}{L}_1 \\
&\quad + \overset{1}{L}_P\cdots\overset{1}{L}_2\,\{\overset{1}{L}_1,\ \overset{2}{L}_n\} \\
&= \{\overset{1}{L}_P,\ \overset{2}{L}_n\}\,\overset{1}{L}_{P-1}\cdots\overset{1}{L}_1 + \overset{1}{L}_P\cdots\overset{1}{L}_{n+2}\left(-\overset{1}{L}_{n+1}\,s^{+}_{12}\,\overset{2}{L}_n\right)\overset{1}{L}_n\cdots\overset{1}{L}_1 \\
&\quad + \overset{1}{L}_P\cdots\overset{1}{L}_{n+1}\left(r^{+}_{12}\,\overset{1}{L}_n\,\overset{2}{L}_n - \overset{2}{L}_n\,\overset{1}{L}_n\,r^{-}_{12}\right)\overset{1}{L}_{n-1}\cdots\overset{1}{L}_1 \\
&\quad + \overset{1}{L}_P\cdots\overset{1}{L}_n\left(\overset{2}{L}_n\,s^{-}_{12}\,\overset{1}{L}_{n-1}\right)\overset{1}{L}_{n-2}\cdots\overset{1}{L}_1 + \overset{1}{L}_P\cdots\overset{1}{L}_2\,\{\overset{1}{L}_1,\ \overset{2}{L}_n\}\ .
\end{aligned}
\tag{4.20}
$$

At this point it is useful to establish the commutation relations for the monodromy matrix T using the fundamental commutation relations of the matrices L_n. Introduce the following decomposition of the monodromy matrix (4.13)

$$
T = T^{+}_n \cdot T^{-}_{n-1}\ ,
$$

in which

$$
T^{+}_n(\lambda) = \prod_{j=n}^{P} L_j(\lambda) \quad \text{and} \quad T^{-}_{n-1}(\lambda) = \prod_{j=1}^{n-1} L_j(\lambda)\ .
$$

Thus, we can rewrite the equation (4.20) as

$$
\begin{aligned}
\{\overset{1}{T},\overset{2}{L}_n\} &= \{\overset{1}{L}_P,\overset{2}{L}_n\}\,\overset{1}{T}{}^{-}_{P-1} + \overset{2}{L}_n\,\overset{1}{T}{}^{+}_n\left(s^{-}_{12} - r^{-}_{12}\right)\overset{1}{T}{}^{-}_{n-1} \\
&\quad - \overset{1}{T}{}^{+}_{n+1}\left(s^{+}_{12} - r^{+}_{12}\right)\overset{1}{T}{}^{-}_n\,\overset{2}{L}_n + \overset{2}{T}{}^{+}_2\,\{\overset{1}{L}_1,\ \overset{2}{L}_n\}\ .
\end{aligned}
\tag{4.21}
$$

Now, using equation (4.21) we have that

$$
\begin{aligned}
\{\overset{1}{T},\overset{2}{T}\} &= \sum_{n=1}^{P} \overset{2}{L}_P \cdots \overset{2}{L}_{n+1}\,\{\overset{1}{T},\,\overset{2}{L}_n\}\,\overset{2}{L}_{n-1}\cdots \overset{2}{L}_1 \\
&= \{\overset{1}{T},\overset{2}{L}_P\}\,\overset{2}{T}{}^{-}_{P-1} + \sum_{n=2}^{P-1} \overset{2}{T}{}^{+}_{n+1}\,\{\overset{1}{T},\,\overset{2}{L}_n\}\,\overset{2}{T}{}^{-}_{n-1} + \overset{2}{T}{}^{+}_{2}\,\{\overset{1}{T},\overset{2}{L}_1\} \\
&= \{\overset{1}{T},\overset{2}{L}_P\}\,\overset{2}{T}{}^{-}_{P-1} + \sum_{n=2}^{P-1} \overset{2}{T}{}^{+}_{n+1}\,\{\overset{1}{L}_P,\,\overset{2}{L}_n\}\,\overset{1}{T}{}^{-}_{P-1}\,\overset{2}{T}{}^{-}_{n-1} \\
&\quad + \sum_{n=2}^{P-1}\Big(\overset{2}{T}{}^{+}_{n}\,\overset{1}{T}{}^{+}_{n}\,\big(s^{-}_{12}-r^{-}_{12}\big)\,\overset{1}{T}{}^{-}_{n-1}\,\overset{2}{T}{}^{-}_{n-1} - \overset{2}{T}{}^{+}_{n+1}\,\overset{1}{T}{}^{+}_{n+1}\,\big(s^{+}_{12}-r^{+}_{12}\big)\,\overset{1}{T}{}^{-}_{n}\,\overset{2}{T}{}^{-}_{n}\Big) \\
&\quad + \sum_{n=2}^{P-1} \overset{2}{T}{}^{+}_{n+1}\,\overset{1}{T}{}^{+}_{2}\{\overset{1}{L}_1,\,\overset{2}{L}_n\}\,\overset{2}{T}{}^{-}_{n-1} + \overset{2}{T}{}^{+}_{2}\,\{\overset{1}{T},\overset{2}{L}_1\} \\
&= \{\overset{1}{L}_P,\overset{2}{L}_P\}\,\overset{1}{T}{}^{-}_{P-1}\,\overset{2}{T}{}^{-}_{P-1} + \overset{2}{T}{}^{+}_{P}\,\overset{1}{T}{}^{+}_{n}\,\big(s^{-}_{12}-r^{-}_{12}\big)\,\overset{1}{T}{}^{-}_{n-1}\,\overset{2}{T}{}^{-}_{P-1} \\
&\quad - \overset{1}{T}{}^{+}_{n+1}\,\big(s^{+}_{12}-r^{+}_{12}\big)\,\overset{1}{T}{}^{-}_{n}\,\overset{2}{T}{}^{-}_{P} + \overset{1}{T}{}^{+}_{2}\,\{\overset{1}{L}_1,\overset{2}{L}_P\}\,\overset{2}{T}{}^{-}_{P-1} \\
&\quad + \sum_{n=2}^{P-1} \overset{2}{T}{}^{+}_{n+1}\,\{\overset{1}{L}_P,\,\overset{2}{L}_n\}\,\overset{1}{T}{}^{-}_{P-1}\,\overset{2}{T}{}^{-}_{n-1} + \sum_{n=2}^{P-1} \overset{2}{T}{}^{+}_{n+1}\,\overset{1}{T}{}^{+}_{2}\{\overset{1}{L}_1,\,\overset{2}{L}_n\}\,\overset{2}{T}{}^{-}_{n-1} \\
&\quad + \sum_{n=2}^{P-1}\Big(\overset{2}{T}{}^{+}_{n}\,\overset{1}{T}{}^{+}_{n}\,\big(s^{-}_{12}-r^{-}_{12}\big)\,\overset{1}{T}{}^{-}_{n-1}\,\overset{2}{T}{}^{-}_{n-1} - \overset{2}{T}{}^{+}_{n+1}\,\overset{1}{T}{}^{+}_{n+1}\,\big(s^{+}_{12}-r^{+}_{12}\big)\,\overset{1}{T}{}^{-}_{n}\,\overset{2}{T}{}^{-}_{n}\Big) \\
&\quad + \overset{2}{T}{}^{+}_{2}\,\{\overset{1}{L}_P,\overset{2}{L}_1\}\,\overset{1}{T}{}^{-}_{P-1} + \overset{2}{T}{}^{+}_{1}\,\overset{1}{T}{}^{+}_{n}\,\big(s^{-}_{12}-r^{-}_{12}\big)\,\overset{1}{T}{}^{-}_{n-1} \\
&\quad - \overset{2}{T}{}^{+}_{2}\,\overset{1}{T}{}^{+}_{n+1}\,\big(s^{+}_{12}-r^{+}_{12}\big)\,\overset{1}{T}{}^{-}_{n}\,\overset{2}{T}{}^{-}_{1} + \overset{2}{T}{}^{+}_{2}\,\overset{1}{T}{}^{+}_{2}\,\{\overset{1}{L}_1,\overset{2}{L}_1\}\,.
\end{aligned}
$$

Finally, using equation (4.16) we have

$$
\begin{aligned}
\{\overset{1}{T},\overset{2}{T}\} &= r^{+}_{12}\,\overset{1}{T}\,\overset{2}{T} - \overset{2}{T}\,\overset{1}{T}\,r^{-}_{12} - \overset{1}{T}{}^{+}_{2}\,\overset{1}{T}{}^{-}_{1}\,s^{+}_{12}\,\overset{2}{T}{}^{+}_{P}\,\overset{2}{T}{}^{-}_{P-1} + \overset{2}{T}{}^{+}_{P}\,\overset{1}{T}{}^{+}_{n}\,\big(s^{-}_{12}-r^{-}_{12}\big)\,\overset{1}{T}{}^{-}_{n-1}\,\overset{2}{T}{}^{-}_{P-1} \\
&\quad - \overset{1}{T}{}^{+}_{n+1}\,\big(s^{+}_{12}-r^{+}_{12}\big)\,\overset{1}{T}{}^{-}_{n}\,\overset{2}{T}{}^{-}_{P} - \overset{2}{T}{}^{+}_{P}\,\overset{1}{T}{}^{+}_{P}\,s^{+}_{12}\,\overset{1}{T}{}^{-}_{P-1}\,\overset{2}{T}{}^{-}_{P-1} \\
&\quad + \overset{2}{T}{}^{+}_{2}\,\overset{1}{T}{}^{+}_{2}\,\big(s^{-}_{12}-r^{-}_{12}\big)\,\overset{1}{T}{}^{-}_{1}\,\overset{2}{T}{}^{-}_{1} - \overset{2}{T}{}^{+}_{P}\,\overset{1}{T}{}^{+}_{P}\,\big(s^{+}_{12}-r^{+}_{12}\big)\,\overset{1}{T}{}^{-}_{P-1}\,\overset{2}{T}{}^{-}_{P-1} \\
&\quad + \overset{2}{T}{}^{+}_{2}\,\overset{2}{T}{}^{-}_{1}\,s^{-}_{12}\,\overset{1}{T}{}^{+}_{P}\,\overset{1}{T}{}^{-}_{P-1} + r^{+}_{12}\,\overset{1}{T}\,\overset{2}{T} - \overset{2}{T}\,\overset{1}{T}\,r^{-}_{12} + \overset{2}{T}{}^{+}_{1}\,\overset{1}{T}{}^{+}_{n}\,\big(s^{-}_{12}-r^{-}_{12}\big)\,\overset{1}{T}{}^{-}_{n-1} \\
&\quad - \overset{2}{T}{}^{+}_{2}\,\overset{1}{T}{}^{+}_{n+1}\,\big(s^{+}_{12}-r^{+}_{12}\big)\,\overset{1}{T}{}^{-}_{n}\,\overset{2}{T}{}^{-}_{1} + \overset{2}{T}{}^{+}_{2}\,\overset{1}{T}{}^{+}_{2}\,s^{-}_{12}\,\overset{1}{T}{}^{-}_{1}\,\overset{2}{T}{}^{-}_{1} \\
&= r^{+}_{12}\,\overset{1}{T}\,\overset{2}{T} - \overset{2}{T}\,\overset{1}{T}\,r^{-}_{12} - \overset{1}{T}\,s^{+}_{12}\,\overset{2}{T} + \overset{2}{T}\,s^{-}_{12}\,\overset{1}{T}\,.
\end{aligned}
$$

Hence, the proposition follows. □

In the classical case the traces of powers of the monodromy matrix are invariant under the mapping as a consequence of the discrete-time evolution

$$\overline{T}(\lambda) = M_{P+1}(\lambda)\, T(\lambda)\, M_1(\lambda)^{-1}, \qquad M = \begin{pmatrix} \lambda + (\omega/\overline{x})\,(\omega/y - x) & -\omega/\overline{x} \\ \lambda\,(x - \overline{x} - \omega/y) & \lambda \end{pmatrix}, \quad (4.22)$$

where M is M_1, and the periodicity condition $M_{P+1} = M_1$. Thus, this leads to a sufficient number of invariants which are obtained by expanding the traces in powers of the spectral parameter λ. The dynamical map in terms of the monodromy matrix is preserved by the Poisson bracket as a consequence of the compatibility condition of a discrete-time Zakharov-Shabat system (4.12). The involution property of the classical invariants, which was proven in [24], follows also from the Poisson bracket

$$\{\mathrm{tr}\, T(\lambda),\, \mathrm{tr}\, T(\lambda')\} = 0\,, \quad (4.23)$$

which in turn follows from equation (4.19) as

$$\begin{aligned}
\{\mathrm{tr}\, T(\lambda),\, \mathrm{tr}\, T(\lambda')\} &= \mathrm{tr}_2\, \mathrm{tr}_1\, \{\overset{1}{T}, \overset{2}{T}\} \\
&= \mathrm{tr}_2\, \mathrm{tr}_1\, [r^+_{12}\, \overset{1}{T}\, \overset{2}{T} - \overset{2}{T}\, \overset{1}{T}\, r^-_{12} - \overset{1}{T}\, s^+_{12}\, \overset{2}{T} + \overset{2}{T}\, s^-_{12}\, \overset{1}{T}] \\
&= \mathrm{tr}_2\, [\mathrm{tr}_1\, r^+_{12}\, \overset{1}{T}\, \overset{2}{T} - \overset{2}{T}\, \mathrm{tr}_1\, \overset{1}{T}\, r^-_{12} - \mathrm{tr}_1\, \overset{1}{T}\, s^+_{12}\, \overset{2}{T} + \overset{2}{T}\, \mathrm{tr}_1\, s^-_{12}\, \overset{1}{T}] \\
&= \mathrm{tr}_2\, [\mathrm{tr}_1((r^+_{12} - s^+_{12})\overset{1}{T})\, \overset{2}{T} - \overset{2}{T}\, \mathrm{tr}_1((r^-_{12} - s^-_{12})\overset{1}{T})]\,,
\end{aligned}$$

in which $\mathrm{tr}_1 := \mathrm{tr} \otimes \mathbb{I}$ and $\mathrm{tr}_2 := \mathbb{I} \otimes \mathrm{tr}$. Since we have $r^+_{12} - s^+_{12} = r^-_{12} - s^-_{12}$ from equation (4.15), we obtain

$$\{\mathrm{tr}\, T(\lambda),\, \mathrm{tr}\, T(\lambda')\} = \mathrm{tr}_2\, \{\mathrm{tr}_1[(r^+_{12} - s^+_{12})\, \overset{1}{T}],\, \overset{2}{T}\} = 0\,.$$

Next, we would like to perform the Poisson bracket on the entries of monodromy matrix. To do this recall the normalization vector (4.6), i.e. $\alpha = (1, 0)$. By introducing the vector $\beta = (0, 1)$, we have

$$A(\lambda) = \alpha\, T\, \alpha^T, \quad B(\lambda) = \alpha\, T\, \beta^T, \quad C(\lambda) = \beta\, T\, \alpha^T, \quad D(\lambda) = \beta\, T\, \beta^T.$$

Thus, we can extract the Poisson brackets between the entries of the monodromy matrix (4.1) as the following

$$\{A(\lambda_1), A(\lambda_2)\} = (\alpha \otimes \alpha)\,\{\overset{1}{T}, \overset{2}{T}\}\,(\alpha \otimes \alpha)^T\,, \quad (4.24a)$$

$$\{A(\lambda_1), B(\lambda_2)\} = (\alpha \otimes \alpha)\,\{\overset{1}{T}, \overset{2}{T}\}\,(\alpha \otimes \beta)^T\,, \quad (4.24b)$$

$$\{A(\lambda_1), D(\lambda_2)\} = (\alpha \otimes \beta)\,\{\overset{1}{T}, \overset{2}{T}\}\,(\alpha \otimes \beta)^T\,, \quad (4.24c)$$

$$\{B(\lambda_1), B(\lambda_2)\} = (\alpha \otimes \alpha)\,\{\overset{1}{T}, \overset{2}{T}\}\,(\beta \otimes \beta)^T\,, \quad (4.24d)$$

$$\{D(\lambda_1), B(\lambda_2)\} = (\beta \otimes \alpha)\,\{\overset{1}{T}, \overset{2}{T}\}\,(\beta \otimes \beta)^T\,, \quad (4.24e)$$

$$\{D(\lambda_1), D(\lambda_2)\} = (\beta \otimes \beta)\,\{\overset{1}{T}, \overset{2}{T}\}\,(\beta \otimes \beta)^T\,. \quad (4.24f)$$

Using equations (4.15) and (4.19), the Poisson brackets between these entries are calculated from the right-hand side of equations (4.24) as

$$\{A(\lambda_1), A(\lambda_2)\} = \frac{1}{\lambda_2}B(\lambda_1)C(\lambda_2) - \frac{1}{\lambda_1}B(\lambda_2)C(\lambda_1)\,, \quad (4.25a)$$

$$\{A(\lambda_1), B(\lambda_2)\} = \frac{A(\lambda_2)B(\lambda_1) - B(\lambda_2)A(\lambda_1)}{\lambda_1 - \lambda_2} + \frac{1}{\lambda_2}B(\lambda_1)D(\lambda_2)\,, \quad (4.25b)$$

$$\{A(\lambda_1), D(\lambda_2)\} = \frac{\lambda_1\,B(\lambda_1)C(\lambda_2) - \lambda_2\,B(\lambda_2)C(\lambda_1)}{\lambda_1\,(\lambda_1 - \lambda_2)}\,, \quad (4.25c)$$

$$\{B(\lambda_1), B(\lambda_2)\} = 0\,, \quad (4.25d)$$

$$\{D(\lambda_1), B(\lambda_2)\} = \frac{\lambda_2\,D(\lambda_1)B(\lambda_2) - \lambda_1\,D(\lambda_2)B(\lambda_1)}{\lambda_2\,(\lambda_1 - \lambda_2)}\,, \quad (4.25e)$$

$$\{D(\lambda_1), D(\lambda_2)\} = 0\,. \quad (4.25f)$$

These results lead to the following proposition:

Proposition 4.3.2. *The separation variables (μ_j, η_j), where $j = 1, \ldots, g$, are canonical, i.e. they possess the following Poisson brackets:*

$$\{\mu_i, \mu_j\} = \{\eta_i, \eta_j\} = 0, \qquad \{\eta_i, \mu_j\} = \delta_{ij}\eta_i\,. \quad (4.26)$$

Proof

The proof consists of two parts. Firstly, one needs to show that $\{\mu_i, \mu_j\} = \{\eta_i, \eta_j\} = 0$. Secondly, one needs to show that $\{\eta_i, \mu_j\} = \delta_{ij}\eta_i$. Let us start by showing the first part, i.e. $\{\mu_i, \mu_j\} = \{\eta_i, \eta_j\} = 0$. Consider the equation (4.25d), i.e.

$$\{B(\lambda_1), B(\lambda_2)\} = 0\,,$$

where

$$B(\lambda_1) = B_g \prod_{i=1}^{g} (\lambda_1 - \mu_i) \quad \text{and} \quad B(\lambda_2) = B_g \prod_{j=1}^{g} (\lambda_2 - \mu_j) \; .$$

Then, we have

$$\begin{aligned}\{B(\lambda_1), B(\lambda_2)\} &= \{B_g \prod_{i=1}^{g}(\lambda_1 - \mu_i), B_g \prod_{j=1}^{g}(\lambda_2 - \mu_j)\} = \{\sum_{i=0}^{g-1} \lambda_1^i \, B_i, \sum_{j=0}^{g-1} \lambda_2^j \, B_j\} \\ &= \{B_g \sum_{i=0}^{g-1} \lambda_1^i (-1)^{g-i} S_{g-i}(\boldsymbol{\mu}), B_g \sum_{j=0}^{g-1} \lambda_2^j (-1)^{g-j} S_{g-j}(\boldsymbol{\mu})\} = 0 \; ,\end{aligned}$$

where $S_k(\boldsymbol{\mu})$ are the symmetric products defined by

$$S_k(\mu_1, \ldots, \mu_g) \equiv \sum_{i_1 < i_2 < \cdots < i_k}^{g} \mu_{i_1} \mu_{i_2} \ldots \mu_{i_k} \; , \qquad S_0(\mu_1, \ldots, \mu_g) = 1 \; . \tag{4.27}$$

Thus, we have

$$\begin{aligned} &\{B_g \sum_{i=0}^{g-1} \lambda_1^i (-1)^{g-i} S_{g-i}(\boldsymbol{\mu}), B_g \sum_{j=0}^{g-1} \lambda_2^j (-1)^{g-j} S_{g-j}(\boldsymbol{\mu})\} = 0 \\ \Rightarrow \quad &\sum_{j=1}^{g} \{\mu_j, \sum_{i=0}^{g-1} \lambda_1^i (-1)^{g-i} S_{g-i}(\boldsymbol{\mu})\} \prod_{\substack{l=1 \\ l \neq j}}^{g} (\lambda_2 - \mu_l) = 0 \\ \Rightarrow \quad &\sum_{i,j=1}^{g} \{\mu_i, \mu_j\} \prod_{\substack{k,l=1 \\ k \neq i, l \neq j}}^{g} (\lambda_1 - \mu_k)(\lambda_2 - \mu_l) = 0 \; .\end{aligned}$$

Now, for all λ_1, λ_2, this equation implies

$$\sum_{\substack{i,j=1 \\ i<j}}^{g} \{\mu_i, \mu_j\}(\mu_i - \mu_j) \prod_{\substack{k=1 \\ k \neq i,j}}^{g} (\lambda_1 - \mu_k)(\lambda_2 - \mu_k) = 0 \; .$$

As discussed in the next section, the discriminant $R(\lambda)$ of the curve (4.4) cannot be zero. Thus, we have $\mu_i - \mu_j \neq 0$ for all $i \neq j$, and consequently we must have $\{\mu_i, \mu_j\} = 0$. Taking $\lambda_1 = \mu_i$ and $\lambda_2 = \mu_j$ in either equation (4.25a), (4.25c) or (4.25f), we can immediately show that $\{\eta_i, \eta_j\} = 0$.

Now in order to show the second part, i.e. $\{\eta_i, \mu_j\} = \delta_{ij} \eta_i$, one needs to prove

$$\{\eta_i, \mu_i\} = \eta_i \quad \text{and} \quad \{\eta_i, \mu_j\} = 0 \quad \text{where} \quad i \neq j \; .$$

To prove these equations we start from the relation (4.25e). Taking $\lambda_1 = \mu_i$ in (4.25e), we have

$$\{D(\mu_i), B(\lambda_2)\} = \{\eta_i, B_g \prod_{j=1}^{g}(\lambda_2 - \mu_j)\} = \frac{\eta_i\, B_g \prod_{j=1}^{g}(\lambda_2 - \mu_j)}{\mu_i - \lambda_2}$$

$$\Rightarrow \quad \{\eta_i, B_g \sum_{j=0}^{g-1} \lambda_2^j (-1)^{g-j} S_{g-j}(\boldsymbol{\mu})\} = -\eta_i\, B_g \prod_{\substack{j=1\\ i\neq j}}^{g}(\lambda_2 - \mu_j)$$

$$\Rightarrow \quad \sum_{k=0}^{g-1} \lambda_2^k (-1)^{g-k} \sum_{j=1}^{g} \{\eta_i, \mu_j\} S_{g-1-k}(\boldsymbol{\mu})_{|\mu_j=0} = -\eta_i \prod_{\substack{j=1\\ i\neq j}}^{g}(\lambda_2 - \mu_j)\,.$$

Since λ_2 is arbitrary, then the latter equation can be posed in a matrix form,

$$\begin{pmatrix} \{\eta_i, \mu_1\} \\ \vdots \\ \vdots \\ \{\eta_i, \mu_g\} \end{pmatrix} = \begin{pmatrix} -S_{g-1}(\boldsymbol{\mu})_{|\mu_1=0} & \cdots & -S_{g-1}(\boldsymbol{\mu})_{|\mu_g=0} \\ \vdots & & \vdots \\ -S_1(\boldsymbol{\mu})_{|\mu_1=0} & \cdots & -S_1(\boldsymbol{\mu})_{|\mu_g=0} \\ -1 & \cdots & -1 \end{pmatrix}^{-1} \begin{pmatrix} -\eta_i\, S_{g-1}(\boldsymbol{\mu})_{|\mu_i=0} \\ \vdots \\ -\eta_i\, S_1(\boldsymbol{\mu})_{|\mu_i=0} \\ -\eta_i \end{pmatrix}.$$

Using the definition of symmetric products $S_k(\boldsymbol{\mu})$, $k = 1, \ldots, g-1$, it can then be seen that this system implies $\{\eta_i, \mu_j\} = \eta_i$ when $i = j$ and $\{\eta_i, \mu_j\} = 0$ otherwise. Hence, the second part of the proposition follows. □

Based on the observation from (4.9) we note that if we use $\eta(\mu_j) = A(\mu_j)$ instead of $\eta(\mu_j) = D(\mu_j)$ in the proposition 4.3.2, the separation variables (μ_j, η_j) possess the Poisson brackets $\{\mu_i, \mu_j\} = \{\eta_i, \eta_j\} = 0$ and $\{\mu_i, \eta_j\} = \delta_{ij}\eta_j$. To prove this, we need to use relation (4.25b).

4.4 Discrete Dubrovin equations

The Dubrovin equations [36] arise in the theory of finite-gap integration as the equations governing the dynamics of the auxiliary spectrum or equivalently of the poles of the Baker-Akhiezer function. In [86] the finite-gap integration of mapping reductions of the lattice KdV equation was considered, cf. also [23] for complementary results. As a byproduct difference analogues of the Dubrovin equations, cf. ref. [80], were derived which form the equations of the discrete motion of the auxiliary spectrum under the KdV mappings.

Let us first derive the equations of discrete motion for the diagonal evolution in terms of the auxiliary spectrum, which are different from the equations for the vertical evolution given in [80]. The monodromy matrix $T(\lambda)$ for genus g has a natural grading in terms of the spectral parameter λ:

$$T(\lambda) = \begin{pmatrix} \lambda^{g+1} + \lambda^g A_g + \cdots + A_0 & \lambda^g B_g + \lambda^{g-1} B_{g-1} + \cdots + B_0 \\ \lambda\,(\lambda^g C_g + \cdots + C_0) & \lambda\,(\lambda^g + \lambda^{g-1} D_{g-1} + \cdots + D_0) \end{pmatrix} . \tag{4.28}$$

We can write the trace of the monodromy matrix as

$$\operatorname{tr} T(\lambda) = I_0 + \sum_{j=1}^{g} I_j\,\lambda^j + 2\,\lambda^{g+1}\,, \quad I_j = A_j + D_{j-1}\,. \tag{4.29}$$

As noted in section 2.5, the coefficients I_j, $j = 0, \ldots, g-1$, are the invariants of the map, while the top coefficient $I_g = A_g + D_{g-1}$ is a Casimir with respect to the natural Poisson algebra associated with the dynamical map. The discriminant of the curve (4.4) takes on the form:

$$R(\lambda) = (\operatorname{tr} T)^2 - 4\,\det(T) = \left(2\,\lambda^{g+1} + \sum_{j=0}^{g} I_j\,\lambda^j\right)^2 - 4\,\lambda^{g+1}\,(\lambda-\omega)^{g+1}\,. \tag{4.30}$$

By considering the form of the monodromy matrix given in (4.1), from the discriminant of the curve (4.30) we have the fact

$$(A-D)(\lambda) = \kappa\sqrt{R(\lambda) - 4\,B(\lambda)\,C(\lambda)}\,, \tag{4.31}$$

where the κ denotes the sign $\kappa = \pm$ corresponding to the choice of sheet of the Riemann surface, subject to the condition $\overline{\kappa} = \kappa$. This condition follows from $\overline{\kappa} = \underset{\sim}{\widehat{\kappa}}$ since the bar shift $^-$ is the composition of two shifts $\kappa \to \underset{\sim}{\kappa} = -\kappa$ and $\kappa \to \widehat{\kappa} = -\kappa$, each of which provoke a change of sheet of the Riemann surface [80, 86].

From the fact (4.31) and with the use of equation (4.29) we obtain the following relation

$$2\,\lambda^{g+1} + \sum_{j=0}^{g} I_j\,\lambda^j - 2\,D(\lambda) = \kappa\sqrt{R(\lambda) - 4\,B(\lambda)\,C(\lambda)}\,, \tag{4.32}$$

taking into account that $\operatorname{tr} T(\lambda) = A(\lambda) + D(\lambda)$. Using the expression of $D(\lambda)$ from equation (4.28), we obtain

$$\sum_{j=1}^{g} \lambda^j \left(I_j - 2\, D_{j-1}\right) + I_0 = \kappa \sqrt{R(\lambda) - 4\, B(\lambda)\, C(\lambda)}\,. \tag{4.33}$$

By taking $\lambda = \mu_i$ in (4.33) we obtain

$$\sum_{j=1}^{g} \mu_i^j \left(I_j - 2\, D_{j-1}\right) + I_0 = \kappa_i \sqrt{R(\mu_i)}\,, \tag{4.34}$$

which can be written in a matrix form as follows

$$\begin{pmatrix} I_1 - 2D_0 \\ \vdots \\ I_g - 2D_{g-1} \end{pmatrix} = \begin{pmatrix} \mu_1 & \cdots & \mu_1^g \\ \vdots & & \vdots \\ \mu_g & \cdots & \mu_g^g \end{pmatrix}^{-1} \cdot \begin{pmatrix} \kappa_1 \sqrt{R(\mu_1)} - I_0 \\ \vdots \\ \kappa_g \sqrt{R(\mu_g)} - I_0 \end{pmatrix}.$$

Thus, we obtain

$$I_j - 2\, D_{j-1} = \left[\mathscr{M}^{-1} \left(\boldsymbol{\kappa} \sqrt{R(\boldsymbol{\mu})} - I_0\, \boldsymbol{e}\right)\right]_j\,, \qquad j = 1, \ldots, g\,, \tag{4.35}$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_g)^T$ denotes the vector with entries μ_j, I_0 denotes the invariant given in equation (4.29) and $\boldsymbol{e} = (1, 1, \ldots, 1)^T$, whereas $\mathscr{M}$ denotes the Vandermonde matrix

$$\mathscr{M} = \begin{pmatrix} \mu_1 & \cdots & \mu_1^g \\ \vdots & & \vdots \\ \mu_g & \cdots & \mu_g^g \end{pmatrix}. \tag{4.36}$$

From the Lax equation for the map (4.22) we have the following discrete relations for its entries

$$\left[\lambda - \frac{\omega}{\overline{x}}\left(x - \frac{\omega}{y}\right)\right] A - \frac{\omega}{\overline{x}}\, C = \overline{A}\left[\lambda - \frac{\omega}{\overline{x}}\left(x - \frac{\omega}{y}\right)\right] + \lambda\, \overline{B}\left(x - \overline{x} - \frac{\omega}{y}\right), \tag{4.37a}$$

$$\left[\lambda - \frac{\omega}{\overline{x}}\left(x - \frac{\omega}{y}\right)\right] B - \frac{\omega}{\overline{x}}\, D = \lambda\, \overline{B} - \frac{\omega}{\overline{x}}\, \overline{A}\,, \tag{4.37b}$$

$$\lambda\left(x - \overline{x} - \frac{\omega}{y}\right) A + \lambda\, C = \overline{C}\left[\lambda - \frac{\omega}{\overline{x}}\left(x - \frac{\omega}{y}\right)\right] + \lambda\, \overline{D}\left(x - \overline{x} - \frac{\omega}{y}\right), \tag{4.37c}$$

$$\lambda\left(x - \overline{x} - \frac{\omega}{y}\right) B + \lambda\, D = \lambda\, \overline{D} - \frac{\omega}{\overline{x}}\, \overline{C}\,. \tag{4.37d}$$

From (4.37) we establish that the top coefficients B_g and C_g can be taken to be equal and constant. Expanding equation (4.37a) in powers of λ we are lead to the following coupled equations

$$\overline{A}_0 = A_0 = I_0 \ , \qquad A_g - \frac{\omega}{\overline{x}} C_g = \overline{A}_g + \overline{B}_g \left(x - \overline{x} - \frac{\omega}{y} \right) , \tag{4.38}$$

while expanding equation (4.37b) we are lead to the following set of equations

$$\overline{A}_0 = B_0 \left(x - \frac{\omega}{y} \right) , \qquad \overline{B}_g = B_g \ , \tag{4.39a}$$

$$\frac{\omega}{\overline{x}} \left(\overline{A}_j - D_{j-1} \right) = \overline{B}_{j-1} - B_{j-1} + \frac{\omega}{\overline{x}} \left(x - \frac{\omega}{y} \right) B_j \ , \qquad j = 1, \dots, g \ . \tag{4.39b}$$

Using the equations (4.38) and (4.39a) one obtains

$$B_g \, \overline{x} - \frac{\omega}{\overline{x}} C_g = D_{g-1} - \overline{D}_{g-1} + \frac{B_g}{B_0} \overline{A}_0 \ , \tag{4.40}$$

taking into account that $\overline{A}_g + \overline{D}_{g-1} = A_g + D_{g-1}$. From the polynomial $B(\lambda)$ given in (4.2) we have the fact

$$B(\lambda) = B_g \prod_{j=1}^{g} (\lambda - \mu_j) = \sum_{j=0}^{g} \lambda^j B_j = B_g \sum_{j=0}^{g} \lambda^j (-1)^{g-j} S_{g-j}(\boldsymbol{\mu}) \ , \tag{4.41}$$

in which $S_{g-j}(\boldsymbol{\mu})$ are the functions of symmetric products defined in equation (4.27). Using equations (4.35), (4.39b), (4.40) and (4.41), we obtain a coupled system of set of first-order difference equations for the μ_j, namely

$$\begin{aligned} &\left[\mathscr{M}^{-1} \left(\boldsymbol{\kappa} \sqrt{R(\boldsymbol{\mu})} - I_0 \, \boldsymbol{e} \right) \right]_j + \left[\overline{\mathscr{M}}^{-1} \left(\overline{\boldsymbol{\kappa}} \sqrt{R(\overline{\boldsymbol{\mu}})} - I_0 \, \boldsymbol{e} \right) \right]_j \\ &= \frac{2 \, I_0 \, S_{g-j}(\boldsymbol{\mu})}{(-1)^j \prod_{i=1}^{g} \mu_i} + 2 \, B_g \, \frac{\overline{x}}{\omega} (-1)^{g-j+1} \left[S_{g-j+1}(\overline{\boldsymbol{\mu}}) - S_{g-j+1}(\boldsymbol{\mu}) \right] , \end{aligned} \tag{4.42a}$$

$$\begin{aligned} &\left[\mathscr{M}^{-1} \left(\boldsymbol{\kappa} \sqrt{R(\boldsymbol{\mu})} - I_0 \, \boldsymbol{e} \right) \right]_g - \left[\overline{\mathscr{M}}^{-1} \left(\overline{\boldsymbol{\kappa}} \sqrt{R(\overline{\boldsymbol{\mu}})} - I_0 \, \boldsymbol{e} \right) \right]_g \\ &= \frac{2 \, I_0}{(-1)^g \prod_{i=1}^{g} \mu_i} - 2 \, B_g \, \overline{x} + 2 \, C_g \, \frac{\omega}{\overline{x}} \ , \end{aligned} \tag{4.42b}$$

where $j = 1, \dots, g$. Equations (4.42) are the equations of the discrete dynamics for the diagonal evolution in terms of the separated variables, which are called *the discrete Dubrovin equations*. The actual discrete Dubrovin equations comprise two expressions

(4.42a), (4.42b), and coupled through the $\overline{x}$ which can be eliminated by combination of both.

In the case of $g = 1$ the discrete Dubrovin equations (4.42) reduce to set of two coupled equations, namely

$$\frac{1}{\mu}\sqrt{R(\mu)} + \frac{1}{\overline{\mu}}\sqrt{R(\overline{\mu})} = I_0\left(\frac{\mu - \overline{\mu}}{\mu\,\overline{\mu}}\right) + 2\,B_1\,\frac{\overline{x}}{\omega}\,(\mu - \overline{\mu})\,, \tag{4.43a}$$

$$\frac{1}{\overline{\mu}}\sqrt{R(\overline{\mu})} - \frac{1}{\mu}\sqrt{R(\mu)} = I_0\left(\frac{\mu + \overline{\mu}}{\overline{\mu}\,\mu}\right) + 2\,B_1\,\overline{x} - 2\,C_1\,\frac{\omega}{\overline{x}}\,, \tag{4.43b}$$

where we can use that $\overline{\kappa} = \kappa$ and fixed κ at an initial point. Solving equation (4.43a) for $\overline{x}$ and inserting into (4.43b) we obtain the actual discrete Dubrovin equation of second-degree, namely

$$\begin{aligned}(\mu - \overline{\mu} - \omega)\left(\frac{\sqrt{R(\overline{\mu})} - I_0}{\overline{\mu}}\right)^2 - (\mu - \overline{\mu} + \omega)\left(\frac{\sqrt{R(\mu)} + I_0}{\mu}\right)^2 \\ = \frac{2\,\omega}{\mu\,\overline{\mu}}\left(\sqrt{R(\overline{\mu})} - I_0\right)\left(\sqrt{R(\mu)} + I_0\right) - 4\,B_1\,C_1\,(\mu - \overline{\mu})^2\,,\end{aligned} \tag{4.44}$$

which is different from the first-degree Dubrovin equation given in [80] for a different discrete map. The Casimirs B_1 and C_1 are both equal to $2\,\epsilon$.

In the case of genus $g = 2$ the discrete Dubrovin equations are given by the following set of first-order difference equations

$$\begin{aligned}&\frac{(1/\mu_1)\sqrt{R(\mu_1)} - (1/\mu_2)\sqrt{R(\mu_2)}}{\mu_1 - \mu_2} - \frac{(1/\overline{\mu}_1)\sqrt{R(\overline{\mu}_1)} - (1/\overline{\mu}_2)\sqrt{R(\overline{\mu}_2)}}{\overline{\mu}_1 - \overline{\mu}_2} \\ &= I_0\left(\frac{1}{\mu_1\mu_2} + \frac{1}{\overline{\mu}_1\overline{\mu}_2}\right) - 2\,B_2\,\overline{x} + 2\,C_2\,\frac{\omega}{\overline{x}}\,,\end{aligned} \tag{4.45a}$$

$$\begin{aligned}&\frac{(1/\mu_1)\sqrt{R(\mu_1)} - (1/\mu_2)\sqrt{R(\mu_2)}}{\mu_1 - \mu_2} + \frac{(1/\overline{\mu}_1)\sqrt{R(\overline{\mu}_1)} - (1/\overline{\mu}_2)\sqrt{R(\overline{\mu}_2)}}{\overline{\mu}_1 - \overline{\mu}_2} \\ &= I_0\left(\frac{1}{\mu_1\mu_2} - \frac{1}{\overline{\mu}_1\overline{\mu}_2}\right) + 2\,B_2\,\frac{\overline{x}}{\omega}\,(\mu_1 - \overline{\mu}_1 + \mu_2 - \overline{\mu}_2)\,,\end{aligned} \tag{4.45b}$$

$$\begin{aligned}&\frac{(\mu_2/\mu_1)\sqrt{R(\mu_1)} - (\mu_1/\mu_2)\sqrt{R(\mu_2)}}{\mu_1 - \mu_2} + \frac{(\overline{\mu}_2/\overline{\mu}_1)\sqrt{R(\overline{\mu}_1)} - (\overline{\mu}_1/\overline{\mu}_2)\sqrt{R(\overline{\mu}_2)}}{\overline{\mu}_1 - \overline{\mu}_2} \\ &= I_0\left(\frac{\mu_1 + \mu_2}{\mu_1\mu_2} - \frac{\overline{\mu}_1 + \overline{\mu}_2}{\overline{\mu}_1\overline{\mu}_2}\right) + 2\,B_2\,\frac{\overline{x}}{\omega}\,(\mu_1\mu_2 - \overline{\mu}_1\overline{\mu}_2)\,,\end{aligned} \tag{4.45c}$$

where the Casimirs B_2 and C_2 are both equal to $3\,\epsilon$. As before eliminating $\overline{x}$ from (4.45), we get a coupled first-order difference equations of degree two for the variables μ_1 and

μ_2. In general, this system is difficult to solve directly. It is conceivable that the system can be solved using the techniques of Abel's paper [1], but we will not pursue this line of investigation here. Instead, we consider the integration by means of the continuous-time interpolating flow. Note that equations (4.44) and (4.45) describe the discrete evolution of the separation variables.

4.5 Interpolating flow

We will now establish the continuous evolution generated by the invariants. First, we need to find canonical separation variables which obey the standard Poisson bracket structure. Using (4.25b) and (4.25e) we obtain

$$\{A(\lambda_1)+D(\lambda_1), B(\lambda_2)\} = \frac{B(\lambda_1)\,[A(\lambda_2)-D(\lambda_2)] - B(\lambda_2)\,[A(\lambda_1)-D(\lambda_1)]}{\lambda_1-\lambda_2}\ . \tag{4.46}$$

Now, from the left-hand side of this equation we have

$$\begin{aligned}\{A(\lambda_1)+D(\lambda_1), B(\lambda_2)\} &= \{A(\lambda_1)+D(\lambda_1), B_g\prod_{j=1}^{g}(\lambda_2-\mu_j)\} \\ &= \{A(\lambda_1)+D(\lambda_1), B_g\sum_{j=0}^{g}\lambda_2^j(-1)^{g-j}S_{g-j}(\boldsymbol{\mu})\}\ .\end{aligned}$$

Since B_g is a Casimir and $S_0(\boldsymbol{\mu})=1$, we hence obtain

$$\{A(\lambda_1)+D(\lambda_1), B(\lambda_2)\} = \{A(\lambda_1)+D(\lambda_1), B_g\sum_{j=0}^{g-1}\lambda_2^j(-1)^{g-j}S_{g-j}(\boldsymbol{\mu})\} \tag{4.47}$$

$$= \sum_{j=1}^{g}\{\mu_j, A(\lambda_1)+D(\lambda_1)\}B_g\prod_{\substack{i=1\\ i\neq j}}^{g}(\lambda_2-\mu_i)\ . \tag{4.48}$$

Using this equation together with equation (4.46), we obtain

$$\begin{aligned}&\sum_{j=1}^{g}\{\mu_j, A(\lambda_1)+D(\lambda_1)\}B_g\prod_{\substack{i=1\\ i\neq j}}^{g}(\lambda_2-\mu_i) \\ &\quad= \frac{B(\lambda_1)\,[A(\lambda_2)-D(\lambda_2)] - B(\lambda_2)\,[A(\lambda_1)-D(\lambda_1)]}{\lambda_1-\lambda_2}\ .\end{aligned} \tag{4.49}$$

Taking $\lambda_1 = \lambda$ and $\lambda_2 = \mu_j$ in (4.49), we obtain

$$\{\mu_j, (A+D)(\lambda)\} = \frac{B(\lambda)(A-D)(\mu_j)}{(\lambda-\mu_j)\, B_g \prod_{i\neq j}(\mu_j-\mu_i)} \,, \tag{4.50}$$

with which we will be concerned later.

Equation (4.46) yields in the limit $\lambda_2 \longrightarrow \lambda_1$ the relation

$$\{A(\lambda)+D(\lambda), B(\lambda)\} = [A(\lambda)-D(\lambda)]\, B'(\lambda) - B(\lambda)\, [A(\lambda)-D(\lambda)]' \,, \tag{4.51}$$

where the prime denotes the differentiation with respect to λ. From the auxiliary spectrum (4.2), using $B(\mu_j) = 0$ and the relation

$$\{A+D, B\}(\mu_j) = (A-D)(\mu_j)\, B'(\mu_j) \,,$$

we obtain

$$\{(A+D)(\mu_j), \mu_j\} = (A-D)(\mu_j) \,, \tag{4.52}$$

whereas for $\mu_i \neq \mu_j$ we have

$$\{\mu_i, (A+D)(\mu_j)\} = 0 \,.$$

A set of canonical separation variables is introduced by taking the $\{\mu_j\}$ as the position variables, and $\{\nu_j\}$ as momenta variables defined by

$$2\cosh(\nu_j) \equiv \frac{\mathrm{tr}\, T(\mu_j)}{\sqrt{\det T(\mu_j)}} \quad \Rightarrow \quad \nu_j = \frac{1}{2}\log\left(\frac{A(\mu_j)}{D(\mu_j)}\right) , \tag{4.53}$$

leading to the Poisson brackets

$$\{\mu_i, \mu_j\} = \{\nu_i, \nu_j\} = 0 \,, \qquad \{\mu_i, \nu_j\} = \delta_{ij} \,, \qquad j = 1, \ldots, g \,. \tag{4.54}$$

In order to construct an interpolating flow for the map (4.22), we consider the continuous-time evolution for the auxiliary spectrum using $\mathrm{tr}\, T(\lambda)$ as the generating Hamiltonian of the flow[7] (a similar time evolution was considered in [23] by Cao & Xu), i.e.

$$\dot{\mu}_j = \frac{\partial \mu_j}{\partial t_\lambda} = \{\mu_j, \mathrm{tr}\, T(\lambda)\} = \frac{\sqrt{R(\mu_j)}\, B(\lambda)}{(\lambda-\mu_j)\, B_g \prod_{i\neq j}(\mu_j-\mu_i)} \,, \tag{4.55}$$

[7] In choosing a single parameter-flow generated by $\mathrm{tr}\, T(\lambda)$ we anticipate matching this flow to the map in the case of $g = 2$. For higher genus we would need a multiparameter combination of traces. We will not consider the latter extension in this thesis.

which follow from equations (4.50). Equations (4.55) are *the Dubrovin equations* for our case, and the value of λ will be fixed later. We note that an alternative approach for deriving the Dubrovin equations (4.55) is addressed in appendix A.

Resulting from the definition (4.53) and equation (4.55) we have the following immediate corollary:

Proposition 4.5.1. *The second companion equation for the conjugate variable ν_j is given by*

$$\dot{\nu}_j = \frac{\sqrt{\det T(\mu_j)}\, B(\lambda)}{(\lambda - \mu_j)\, B'(\mu_j)} \frac{\partial}{\partial \mu_j} \left(\frac{\operatorname{tr} T(\mu_j)}{\sqrt{\det T(\mu_j)}} \right). \tag{4.56}$$

Proof

Consider the definition (4.53), i.e.

$$2\cosh(\nu_j) = \frac{\operatorname{tr} T(\mu_j)}{\sqrt{\det T(\mu_j)}} . \tag{4.57a}$$

As consequence from this definition we have

$$2\sinh(\nu_j) = \frac{\sqrt{R(\mu_j)}}{\sqrt{\det T(\mu_j)}} . \tag{4.57b}$$

From equation (4.57a), we have

$$2\sinh(\nu_j)\, \dot{\nu}_j = \frac{\partial}{\partial t_\lambda} \left(\frac{\operatorname{tr} T(\mu_j)}{\sqrt{\det T(\mu_j)}} \right), \tag{4.58}$$

which, by using (4.57b), implies

$$\dot{\nu}_j \frac{\sqrt{R(\mu_j)}}{\sqrt{\det T(\mu_j)}} = \frac{\partial}{\partial \mu_j} \left(\frac{\operatorname{tr} T(\mu_j)}{\sqrt{\det T(\mu_j)}} \right) \left(\frac{B(\lambda) \sqrt{R(\mu_j)}}{(\lambda - \mu_j)\, B'(\mu_j)} \right).$$

Hence, the proposition follows. □

The coupled equations (4.55) and (4.56) derive from the Hamiltonian

$$H_\lambda(\mu_1, \ldots, \mu_g; \nu_1, \ldots, \nu_g) = \sum_{j=1}^{g} \frac{2\cosh(\nu_j) \sqrt{\det T(\mu_j)}\, B(\lambda) - B(\lambda) \operatorname{tr} T(\mu_j)}{(\lambda - \mu_j)\, B'(\mu_j)} , \tag{4.59}$$

which is different from Toda's equation, cf. [113], where he gave the potential term only. Using the Lagrange interpolation formula [113],

$$\frac{\lambda^k}{B(\lambda)} + \sum_{i=1}^{g} \frac{\mu_i^k}{B'(\mu_i)(\mu_i - \lambda)} = \begin{cases} 0 & , \quad 0 \leq k \leq g-1 \,, \\ 1/B_g & , \quad k = g \,, \end{cases} \tag{4.60}$$

we can integrate the Dubrovin equations (4.55), leading to the following Jacobi inversion problem

$$\lambda^k (t - t_0) = \sum_{j=1}^{g} \int_{\mu_j(t_0)}^{\mu_j(t)} \frac{\mu^k}{\sqrt{R(\mu)}} \, d\mu \,, \qquad k = 0, \ldots, g-1 \,, \tag{4.61}$$

on the hyperelliptic Riemann surface of genus $g = P - 1$ defined by

$$\eta^2 - \operatorname{tr} T(\lambda) \, \eta + \det T(\lambda) = 0 \,. \tag{4.62}$$

We note that the time variable $t = t(\lambda)$ depends on the parameter value λ. We also note that for different values of λ the compatibility flows commute as a consequence of equation (4.23).

4.6 Generating functions structures and action-angle variables

We can now introduce the canonical transformation to action-angle variables in terms of the spectral variables $\mu_1, \ldots, \mu_g, \nu_1, \ldots, \nu_g$. We look for a canonical transformation given by a generating function $G(\mu_1, \ldots, \mu_g; \mathcal{P}_1, \ldots, \mathcal{P}_g)$ in which the action variables $\mathcal{P}_1, \ldots, \mathcal{P}_g$ are the invariants appearing in (4.29); specifically $\mathcal{P}_j = I_{j-1}$ where $j = 1, \ldots, g$. This generating function is obtained from

$$G = \sum_{j=1}^{g} \int_{\mu_j^0}^{\mu_j} \nu_j(\mu; \mathcal{P}_1, \ldots, \mathcal{P}_g) \, d\mu = \sum_{j=1}^{g} \int_{\mu_j^0}^{\mu_j} \operatorname{arccosh} \left(\frac{1}{2} \frac{\operatorname{tr} T(\mu)}{\sqrt{\det T(\mu)}} \right) d\mu \,, \tag{4.63}$$

up to an arbitrary function of the invariants. Note that the integral (4.63) depends on μ's and $\mathcal{P}$'s only. The angle variables are obtained from

$$\mathcal{Q}_k = \frac{\partial G}{\partial \mathcal{P}_k} = \sum_{j=1}^{g} \int_{\mu_j(t_0)}^{\mu_j(t)} \frac{\mu^{k-1}}{\sqrt{R(\mu)}} \, d\mu \,, \qquad k = 1, \ldots, g \,. \tag{4.64}$$

As a consequence we have that the transformation to action-angle variables on the one hand and the mapping on the other hand form a ladder of commuting canonical transformations, as shown in figure 4.1. In fact, we can think of the transformation between the original variables and action-angle variables as the result of the composition of two generating functions which factorize the canonical transformation that leads to the modified Hamiltonian. To our knowledge, there is no existing general theory of composition of generating functions, and we basically give a description by the following commuting diagram for what we mean by that.

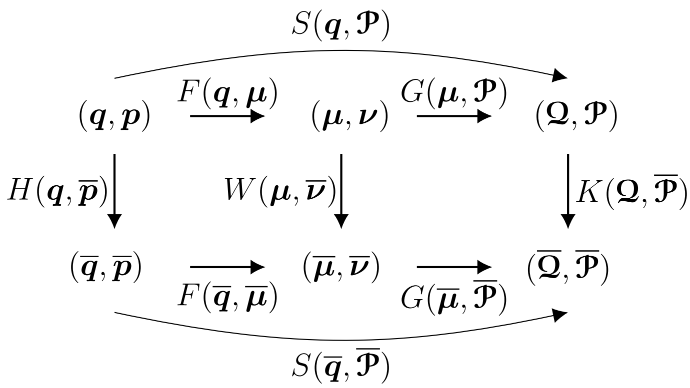


Figure 4.1: Commuting diagram of canonical transformations.

In this diagram, the S denotes the generating function of the canonical transformation from the original variables to the action-angle variables, the F denotes the generating function of the canonical transformation from the original variables to the separating variables, and the G denotes the generating function of the canonical transformation from the separating variables to the action-angle variables. The H is the action functional describing the canonical transformation which is the discrete mapping (Hamiltonian). The canonical transformation, with generating function W, realising the dynamical mapping in terms of the separation variables, is obviously, by construction, an integrable map itself. Unfortunately, however, it does not seem easy in general to obtain an explicit expression for the generating function W, since this requires the elimination of the invariants entering as coefficients of the spectral curve. The K is the action functional describing the canonical transformation in terms of the action-angle variables.

As an explicit example of the above diagram, let us consider the KdV map example 3.4. In this example the generating function S satisfying the defining equations (3.56) is given by (3.58). The generating functions F and G, which are respectively parametrized as

$$p = \frac{\partial F}{\partial q}\,, \qquad \nu = -\frac{\partial F}{\partial \mu}\,, \tag{4.65a}$$

$$\nu = \frac{\partial G}{\partial \mu}\,, \qquad \mathcal{Q} = \frac{\partial G}{\partial \mathcal{P}}\,, \tag{4.65b}$$

are given by

$$F(q,\mu) = (\omega - \mu)\log(\epsilon + q) + \mu \log(\epsilon - q) - \epsilon\, q\,, \tag{4.66}$$

$$G(\mu,\mathcal{P}) = \int_{\mu^0}^{\mu} \log\left(\frac{\sqrt{4\,\epsilon^2\mu' + \mathcal{P} + \alpha^2} + \sqrt{4\,\mu'\,(\mu'+\alpha) + \mathcal{P} + \alpha^2}}{\sqrt{4\,\epsilon^2\mu' + \mathcal{P} + \alpha^2} - \sqrt{4\,\mu'\,(\mu'+\alpha) + \mathcal{P} + \alpha^2}}\right) d\mu'\,, \tag{4.67}$$

up to an arbitrary function of the invariant, and in which $\alpha \equiv \epsilon^2 - \omega$. In equations (4.65) μ and ν are given by

$$\mu = \frac{1}{2\,\epsilon}\left(\epsilon\,\omega - \omega\, q - \epsilon^3 - \epsilon^2 p + \epsilon\, q^2 + p\, q^2\right), \tag{4.68}$$

$$\nu = \log\left(\frac{\epsilon + q}{\epsilon - q}\right), \tag{4.69}$$

where μ is the auxiliary spectrum and ν defined in (4.53). The generating function H satisfying the relations (3.53) is given by (3.52), whereas the generating function K satisfying the relations

$$\overline{\mathcal{P}} - \mathcal{P} = -\frac{\partial K}{\partial \mathcal{Q}}\,, \qquad \overline{\mathcal{Q}} - \mathcal{Q} = \frac{\partial K}{\partial \mathcal{P}}\,,$$

is given by (3.65), which is in fact the modified Hamiltonian of the map (3.51). We note that the generating function K is presented in terms of the variable $\mathcal{P}$ only since $\mathcal{P}$ is an invariant, i.e. $\overline{\mathcal{P}} = \mathcal{P}$.

Next, the schemes that we set up in the previous sections of this chapter will be illustrated for genus-two in the next section.

4.7 The modified Hamiltonian of two-degrees-of-freedom

As discussed in chapter 2, the lattice KdV equation (2.1) leads to the integrable mapping (2.13) when considering a staircase with period $P = 3$. For convenience, this mapping is introduced below:

$$\overline{X}_1 = X_1 + \frac{\epsilon\,\delta}{\epsilon - Y_1} - \frac{\epsilon\,\delta}{\epsilon + Y_1 + Y_2}\,, \qquad \overline{Y}_1 = Y_1 + \frac{\epsilon\,\delta}{\epsilon - \overline{X}_2} - \frac{\epsilon\,\delta}{\epsilon - \overline{X}_1}\,, \tag{4.70a}$$

$$\overline{X}_2 = X_2 + \frac{\epsilon\,\delta}{\epsilon - Y_2} - \frac{\epsilon\,\delta}{\epsilon - Y_1}\,, \qquad \overline{Y}_2 = Y_2 + \frac{\epsilon\,\delta}{\epsilon + \overline{X}_1 + \overline{X}_2} - \frac{\epsilon\,\delta}{\epsilon - \overline{X}_2}\,. \tag{4.70b}$$

This mapping is not a canonical transformation with respect to the standard symplectic structure, i.e.

$$d\overline{Y}_1 \wedge d\overline{X}_1 + d\overline{Y}_2 \wedge d\overline{X}_2 \neq dY_1 \wedge dX_1 + dY_2 \wedge dX_2\,.$$

So, let us now find the canonical variables of the integrable mapping (4.70). To do this we apply an action for the KdV lattice equation (2.1) for the case of $P = 3$.

4.7.1 Mapping action

The action for the KdV lattice equation (2.1) for the case of genus two follows from equation (2.5), and is given, according to figure 4.2, by

$$\begin{aligned}\mathcal{S}_{per} = \sum_t \big[&\mathrm{u}_0\,(\mathrm{u}_1 - \overline{\mathrm{u}}_1) + \epsilon\,\delta \log(\epsilon + \mathrm{u}_0 - \mathrm{u}_2) + \overline{\mathrm{u}}_1\,(\mathrm{u}_2 - \overline{\mathrm{u}}_2) + \epsilon\,\delta \log(\epsilon + \overline{\mathrm{u}}_1 - \overline{\mathrm{u}}_3) \\ &+ \mathrm{u}_2\,(\mathrm{u}_3 - \overline{\mathrm{u}}_3) + \epsilon\,\delta \log(\epsilon + \mathrm{u}_2 - \mathrm{u}_4) + \overline{\mathrm{u}}_3\,(\mathrm{u}_4 - \overline{\mathrm{u}}_4) + \epsilon\,\delta \log(\epsilon + \overline{\mathrm{u}}_3 - \overline{\mathrm{u}}_5) \\ &+ \mathrm{u}_4\,(\mathrm{u}_5 - \overline{\mathrm{u}}_5) + \epsilon\,\delta \log(\epsilon + \mathrm{u}_4 - \mathrm{u}_0) + \overline{\mathrm{u}}_5\,(\mathrm{u}_0 - \overline{\mathrm{u}}_0) + \epsilon\,\delta \log(\epsilon + \overline{\mathrm{u}}_5 - \overline{\mathrm{u}}_1)\big]\,,\end{aligned}$$

in which $\mathrm{u} = \mathrm{u}(t), \overline{\mathrm{u}} = \mathrm{u}(t+1)$ and $\mathcal{S}_{per}$ is $\mathcal{S}$ as given in (2.5) modulo periodicity. Using the reduced variables

$$X_1 := \mathrm{u}_3 - \mathrm{u}_1\,, \quad X_2 := \mathrm{u}_5 - \mathrm{u}_3\,, \quad X_3 := \mathrm{u}_1 - \mathrm{u}_5\,,$$
$$Y_0 := \mathrm{u}_2 - \mathrm{u}_0\,, \quad Y_1 := \mathrm{u}_4 - \mathrm{u}_2\,, \quad Y_2 := \mathrm{u}_0 - \mathrm{u}_4\,,$$

the latter equation can be reduced to the following form

$$\begin{aligned}\mathcal{S}_{per} = \sum_t \big[&Y_1\,\overline{X}_1 - Y_1\,X_1 - Y_2\,\overline{X}_3 + Y_2\,X_3 \\ &+ \epsilon\,\delta \log(\epsilon - Y_1)(\epsilon - Y_2)(\epsilon + Y_1 + Y_2) \\ &+ \epsilon\,\delta \log(\epsilon - \overline{X}_1)(\epsilon - \overline{X}_3)(\epsilon + \overline{X}_1 + \overline{X}_3)\big]\,,\end{aligned} \tag{4.71}$$

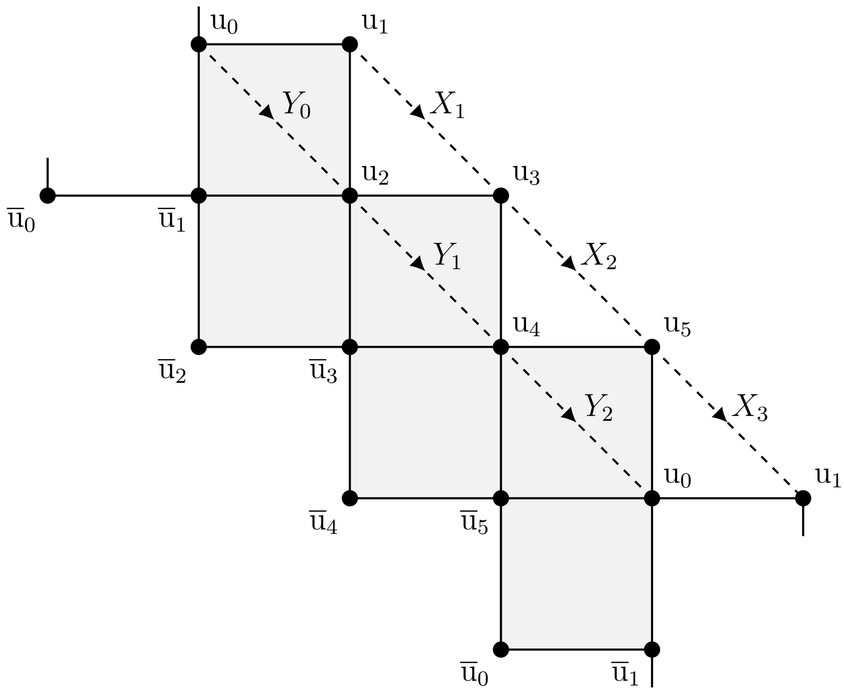


Figure 4.2: Action of mapping over three periods.

where $X_3 = -(X_1 + X_2)$ and $Y_0 = -(Y_1 + Y_2)$ using the periodicity constraints. The Euler-Lagrange equations for (4.71), which are obtained by variation of $\mathcal{S}_{per}$ with respect to the variables X_1, X_3, Y_1, Y_2, i.e.

$$\frac{\delta \mathcal{S}_{per}}{\delta X_1} = \underline{Y}_1 - Y_1 - \frac{\epsilon\,\delta}{\epsilon - X_1} + \frac{\epsilon\,\delta}{\epsilon + X_1 + X_3} = 0\,,$$
$$\frac{\delta \mathcal{S}_{per}}{\delta X_3} = Y_2 - \underline{Y}_2 - \frac{\epsilon\,\delta}{\epsilon - X_3} + \frac{\epsilon\,\delta}{\epsilon + X_1 + X_3} = 0\,,$$
$$\frac{\delta \mathcal{S}_{per}}{\delta Y_1} = \overline{X}_1 - X_1 - \frac{\epsilon\,\delta}{\epsilon - Y_1} + \frac{\epsilon\,\delta}{\epsilon + Y_1 + Y_2} = 0\,,$$
$$\frac{\delta \mathcal{S}_{per}}{\delta Y_2} = X_3 - \overline{X}_3 - \frac{\epsilon\,\delta}{\epsilon - Y_2} + \frac{\epsilon\,\delta}{\epsilon + Y_1 + Y_2} = 0\,,$$

lead to the equations

$$\overline{Y}_1 - Y_1 = \frac{\epsilon\,\delta}{\epsilon + \overline{X}_1 + \overline{X}_3} - \frac{\epsilon\,\delta}{\epsilon - \overline{X}_1}\,, \quad \overline{Y}_2 - Y_2 = \frac{\epsilon\,\delta}{\epsilon - \overline{X}_3} - \frac{\epsilon\,\delta}{\epsilon + \overline{X}_1 + \overline{X}_3}\,, \tag{4.72a}$$
$$\overline{X}_1 - X_1 = \frac{\epsilon\,\delta}{\epsilon - Y_1} - \frac{\epsilon\,\delta}{\epsilon + Y_1 + Y_2}\,, \quad \overline{X}_3 - X_3 = \frac{\epsilon\,\delta}{\epsilon + Y_1 + Y_2} - \frac{\epsilon\,\delta}{\epsilon - Y_2}\,, \tag{4.72b}$$

which is the integrable mapping (4.70) taking into account the periodicity constraints

$$X_1 + X_2 + X_3 = 0 \quad \text{and} \quad Y_0 + Y_1 + Y_2 = 0\,.$$

Now, our aim is to perform a Legendre transformation to establish a Hamiltonian structure for the mapping. A discussion of performing such a discrete-time Legendre transformation can be found in [16].

Consider a Lagrangian $\mathscr{L} = \mathscr{L}(X_1, \overline{X}_1, X_3, \overline{X}_3, Y_1, Y_2)$, such that the Euler-Lagrange equations

$$\overline{\frac{\partial \mathscr{L}}{\partial X_1}} + \frac{\partial \mathscr{L}}{\partial \overline{X}_1} = 0 \,, \qquad \overline{\frac{\partial \mathscr{L}}{\partial X_3}} + \frac{\partial \mathscr{L}}{\partial \overline{X}_3} = 0 \,,$$

correspond to the mapping under consideration. The discrete-time Hamiltonian is obtained through the following Legendre transformation

$$H(Y_1, Y_2, \overline{X}_1, \overline{X}_3) = Y_1\, X_1 - Y_1\, \overline{X}_1 + Y_2\, \overline{X}_3 - Y_2\, X_3 + \mathscr{L} \,.$$

Now, we have

$$\begin{aligned}
&\frac{\partial H}{\partial \overline{X}_1}\,\delta \overline{X}_1 + \frac{\partial H}{\partial \overline{X}_3}\,\delta \overline{X}_3 + \frac{\partial H}{\partial Y_1}\,\delta Y_1 + \frac{\partial H}{\partial Y_2}\,\delta Y_2 \\
&= Y_1\,\delta X_1 + X_1\,\delta Y_1 - Y_1\,\delta \overline{X}_1 - \overline{X}_1\,\delta Y_1 + \overline{X}_3\,\delta Y_2 + Y_2\,\delta \overline{X}_3 - Y_2\,\delta X_3 - X_3\,\delta Y_2 \\
&\quad + \frac{\partial \mathscr{L}}{\partial X_1}\,\delta X_1 + \frac{\partial \mathscr{L}}{\partial \overline{X}_1}\,\delta \overline{X}_1 + \frac{\partial \mathscr{L}}{\partial X_3}\,\delta X_3 + \frac{\partial \mathscr{L}}{\partial \overline{X}_3}\,\delta \overline{X}_3 + \frac{\partial \mathscr{L}}{\partial Y_1}\,\delta Y_1 + \frac{\partial \mathscr{L}}{\partial Y_2}\,\delta Y_2 \,.
\end{aligned}$$

Thus, one obtains a set of equations

$$\begin{aligned}
\frac{\partial H}{\partial \overline{X}_1} &= -Y_1 + \frac{\partial \mathscr{L}}{\partial \overline{X}_1} \,, & \frac{\partial H}{\partial \overline{X}_3} &= Y_2 + \frac{\partial \mathscr{L}}{\partial \overline{X}_3} \,, \\
\frac{\partial H}{\partial Y_1} &= X_1 - \overline{X}_1 + \frac{\partial \mathscr{L}}{\partial Y_1} \,, & \frac{\partial H}{\partial Y_2} &= \overline{X}_3 - X_3 + \frac{\partial \mathscr{L}}{\partial Y_2} \,, \\
0 &= Y_1 + \frac{\partial \mathscr{L}}{\partial X_1} \,, & 0 &= -Y_2 + \frac{\partial \mathscr{L}}{\partial X_3} \,,
\end{aligned}$$

and consequently we get

$$\begin{aligned}
\frac{\partial H}{\partial \overline{X}_1} &= \overline{Y}_1 - Y_1 \,, & \frac{\partial H}{\partial \overline{X}_3} &= Y_2 - \overline{Y}_2 \,, \\
\frac{\partial H}{\partial Y_1} &= X_1 - \overline{X}_1 \,, & \frac{\partial H}{\partial Y_2} &= \overline{X}_3 - X_3 \,.
\end{aligned}$$

By identifying

$$X_1 := p_1 \,, \quad X_3 := -p_2 \,, \quad Y_1 := q_1 \,, \quad Y_2 := q_2 \,,$$

one obtains the discrete-time Hamiltonian in terms of variables $q_1, q_2, \overline{p}_1, \overline{p}_2$

$$\begin{aligned} H(q_1, q_2, \overline{p}_1, \overline{p}_2) &= \epsilon\,\delta \log\,(\epsilon + \overline{p}_1 - \overline{p}_2)(\epsilon - \overline{p}_1)(\epsilon + \overline{p}_2) \\ &\quad + \epsilon\,\delta \log\,(\epsilon + q_1 + q_2)(\epsilon - q_1)(\epsilon - q_2)\,, \end{aligned} \tag{4.73}$$

which acts as the generating functional for the mapping, i.e. one has the discrete-time Hamilton equations

$$\overline{p}_1 - p_1 = -\frac{\partial H}{\partial q_1}\,, \qquad \overline{q}_1 - q_1 = \frac{\partial H}{\partial \overline{p}_1}\,, \tag{4.74a}$$

$$\overline{p}_2 - p_2 = -\frac{\partial H}{\partial q_2}\,, \qquad \overline{q}_2 - q_2 = \frac{\partial H}{\partial \overline{p}_2}\,, \tag{4.74b}$$

leading to the equations of the mapping

$$\overline{p}_1 = p_1 + \frac{\epsilon\,\delta}{\epsilon - q_1} - \frac{\epsilon\,\delta}{\epsilon + q_1 + q_2}\,, \qquad \overline{q}_1 = q_1 - \frac{\epsilon\,\delta}{\epsilon - \overline{p}_1} + \frac{\epsilon\,\delta}{\epsilon + \overline{p}_1 - \overline{p}_2}\,, \tag{4.75a}$$

$$\overline{p}_2 = p_2 + \frac{\epsilon\,\delta}{\epsilon - q_2} - \frac{\epsilon\,\delta}{\epsilon + q_1 + q_2}\,, \qquad \overline{q}_2 = q_2 + \frac{\epsilon\,\delta}{\epsilon + \overline{p}_2} - \frac{\epsilon\,\delta}{\epsilon + \overline{p}_1 - \overline{p}_2}\,. \tag{4.75b}$$

In terms of the conjugate variables q_1, q_2, p_1, p_2 we have the standard symplectic structure

$$\Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2\,,$$

leading to standard Poisson brackets

$$\{q_1, q_2\} = \{p_1, p_2\} = \{q_1, p_2\} = \{q_2, p_1\} = 0\,, \quad \{q_1, p_1\} = \{q_2, p_2\} = 1\,.$$

This structure is preserved by the map (4.75), and the mapping is in fact a canonical transformation, i.e. $\overline{\Omega} = \Omega$.

As noted around (2.14), the mapping (4.74) conserves the invariants

$$\begin{aligned} \mathcal{I}_1 &= x_1\,x_2\,x_3\,y_0\,y_1\,y_2 - \epsilon\,\delta\,(x_1\,x_2\,y_1\,y_2 + x_1\,x_3\,y_0\,y_1 + x_2\,x_3\,y_0\,y_2) \\ &\quad + \epsilon^2\,\delta^2\,(x_1\,y_1 + x_2\,y_2 + x_3\,y_0)\,, \end{aligned} \tag{4.76a}$$

$$\begin{aligned} \mathcal{I}_2 &= x_1\,x_2\,x_3\,y_0\,y_1\,y_2 + \epsilon\,\delta\,(x_1\,x_2\,y_0\,y_1 + x_1\,x_3\,y_0\,y_2 + x_2\,x_3\,y_1\,y_2) \\ &\quad + \epsilon^2\,\delta^2\,(x_1\,y_0 + x_2\,y_1 + x_3\,y_2)\,, \end{aligned} \tag{4.76b}$$

in which we have used the abbreviations

$$x_1 \equiv \epsilon - p_1\,,\;\; x_2 \equiv \epsilon + p_1 - p_2\,,\;\; x_3 \equiv \epsilon + p_2\,,\;\; y_0 \equiv \epsilon + q_1 + q_2\,,\;\; y_1 \equiv \epsilon - q_1\,,\;\; y_2 \equiv \epsilon - q_2\,.$$

The canonical structure allows us to show the integrability property that the two invariants are in involution with each other, with respect to the canonical Poisson bracket:

$$\{\mathcal{I}_1, \mathcal{I}_2\} = 0 \ .$$

The invariance and involutivity of these can be shown by direct calculations. The calculations involved are large, but have been verified by MAPLE. The invariants $\mathcal{I}_1$ and $\mathcal{I}_2$ will thus generate two commuting continuous flows to the mapping (4.74). So the mapping (4.74) satisfies the standard criteria for an integrable map: it has sufficient invariants in involution.

4.7.2 Closed-form modified Hamiltonian

Let us first derive the equation of modified Hamiltonian as given by Yoshida's method. By applying the BCH formula (1.80) to the Hamiltonian (4.73), where we consider δ as the step size of the mapping (4.75), one obtains the following expression for the modified Hamiltonian:

$$\begin{aligned} H^* = {} & \delta\,\epsilon \log\,(\epsilon + p_1 - p_2)(\epsilon - p_1)(\epsilon + p_2)(\epsilon + q_1 + q_2)(\epsilon - q_1)(\epsilon - q_2) \\ & - \frac{\delta^2\,\epsilon^2}{2}\left[\left(\frac{1}{\epsilon + p_1 - p_2} - \frac{1}{\epsilon - p_1}\right)\left(\frac{1}{\epsilon + q_1 + q_2} - \frac{1}{\epsilon - q_1}\right)\right. \\ & \left. - \left(\frac{1}{\epsilon + p_1 - p_2} - \frac{1}{\epsilon + p_2}\right)\left(\frac{1}{\epsilon + q_1 + q_2} - \frac{1}{\epsilon - q_2}\right)\right] + O(\delta^3) \ . \end{aligned} \tag{4.77}$$

We can also express the modified Hamiltonian in terms of the invariants:

$$H^* = \delta\,\epsilon \log\left(\frac{\mathcal{I}_1 + \mathcal{I}_2}{2}\right) + O(\delta^3) \ . \tag{4.78}$$

After inserting the invariants $\mathcal{I}_1$,$\mathcal{I}_2$ and expanding, equation (4.78) is seen to agree with equation (4.77) up to order δ^3. Observe that in equation (4.78), the term corresponding to δ^2 is zero. In fact, in the light of our present investigation we expect that for Hamiltonian systems of the KdV model, all terms of the interpolating Hamiltonian which correspond to δ^{2n} where $n \in \mathbb{N}$, are equal to zero.

Turning to the action-angle variables technique, accordance used Hamilton-Jacobi theory we need separation variables to obtain a closed-form expression for the modified Hamiltonian (4.77). As has been discussed earlier, a canonical transformation to action-angle variables in terms of the spectral variables μ_j, ν_j is introduced. Thus, in the two-degrees-of-freedom case we have a canonical transformation

$$(\mu_1, \mu_2, \nu_1, \nu_2) \longrightarrow (\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{P}_1, \mathcal{P}_2) \;, \tag{4.79}$$

which can be given by means of a generating function, $G(\mu_1, \mu_2, \mathcal{P}_1, \mathcal{P}_2)$ as

$$\nu_1 = \frac{\partial G}{\partial \mu_1} \;, \quad \nu_2 = \frac{\partial G}{\partial \mu_2} \;, \quad \mathcal{Q}_1 = \frac{\partial G}{\partial \mathcal{P}_1} \;, \quad \mathcal{Q}_2 = \frac{\partial G}{\partial \mathcal{P}_2} \;, \tag{4.80}$$

with

$$K_\lambda = H_\lambda + \frac{\partial G}{\partial t_\lambda} \;, \tag{4.81}$$

being the transformed Hamiltonian. The action variables (new momenta) $\mathcal{P}_1, \mathcal{P}_2$ are the invariants appearing in the trace of the monodromy matrix $T(\lambda)$,

$$\operatorname{tr} T(\lambda) = 2\,\lambda^3 + C\,\lambda^2 + \mathcal{P}_2\,\lambda + \mathcal{P}_1 \;, \tag{4.82}$$

where $\mathcal{P}_1 = \mathcal{I}_1 - \omega^3$, $\mathcal{P}_2 = (1/\omega)\,(\mathcal{I}_2 - \mathcal{I}_1 - C\,\omega^2)$ and $C := 3\,\epsilon\,(3\,\epsilon - \delta)$, and in which $\mathcal{I}_1, \mathcal{I}_2$ are the invariants given in equations (4.76).

The generating function G is obtained up to a function of the invariants from

$$G = \int_{\mu_1^0}^{\mu_1} \operatorname{arccosh}\left(\frac{1}{2}\frac{\operatorname{tr} T(\mu)}{\sqrt{\det T(\mu)}}\right) d\mu + \int_{\mu_2^0}^{\mu_2} \operatorname{arccosh}\left(\frac{1}{2}\frac{\operatorname{tr} T(\mu)}{\sqrt{\det T(\mu)}}\right) d\mu \;, \tag{4.83}$$

and consequently we have

$$\mathcal{Q}_1 = \int_{\mu_1^0}^{\mu_1} \frac{1}{\sqrt{R(\mu)}}\, d\mu + \int_{\mu_2^0}^{\mu_2} \frac{1}{\sqrt{R(\mu)}}\, d\mu \;, \tag{4.84a}$$

$$\mathcal{Q}_2 = \int_{\mu_1^0}^{\mu_1} \frac{\mu}{\sqrt{R(\mu)}}\, d\mu + \int_{\mu_2^0}^{\mu_2} \frac{\mu}{\sqrt{R(\mu)}}\, d\mu \;. \tag{4.84b}$$

Note that in the case of genus two, the discriminant of the curve (4.4) takes on the form:

$$\begin{aligned} R(\lambda) = {} & (4\,C + 12\,\omega)\,\lambda^5 + \left(C^2 - 12\,\omega^2 + 4\,\mathcal{P}_2\right)\lambda^4 \\ & + \left(4\,\omega^3 + 2\,C\,\mathcal{P}_2 + 4\,\mathcal{P}_1\right)\lambda^3 + \left(2\,C\,\mathcal{P}_1 + \mathcal{P}_2^2\right)\lambda^2 \\ & + 2\,\mathcal{P}_1\,\mathcal{P}_1\,\lambda + \mathcal{P}_1^2 \;. \end{aligned} \tag{4.85}$$

As discussed in section 4.5, a continuous-time evolution for the auxiliary spectrum can be introduced by considering $\operatorname{tr} T(\lambda)$ as the generating Hamiltonian of the flow. Using the trace (4.82) as the generating Hamiltonian of the flow, one can obtain the following set of equations

$$\dot{\nu}_1 = \frac{(\lambda-\mu_2)\sqrt{\det T(\mu_1)}}{\mu_1-\mu_2}\frac{\partial}{\partial\mu_1}\left(\frac{\operatorname{tr} T(\mu_1)}{\sqrt{\det T(\mu_1)}}\right), \qquad \dot{\mu}_1 = \frac{(\lambda-\mu_2)\sqrt{R(\mu_1)}}{\mu_1-\mu_2}, \tag{4.86a}$$

$$\dot{\nu}_2 = \frac{(\lambda-\mu_1)\sqrt{\det T(\mu_2)}}{\mu_2-\mu_1}\frac{\partial}{\partial\mu_2}\left(\frac{\operatorname{tr} T(\mu_2)}{\sqrt{\det T(\mu_2)}}\right), \qquad \dot{\mu}_2 = \frac{(\lambda-\mu_1)\sqrt{R(\mu_2)}}{\mu_2-\mu_1}, \tag{4.86b}$$

which derive from the Hamiltonian

$$H_\lambda(\mu_1,\mu_2,\nu_1,\nu_2) = \sum_{j=1,2}\prod_{i\neq j}(\lambda-\mu_i)\left(\frac{2\cosh(\nu_j)\sqrt{\det T(\mu_j)} - \operatorname{tr} T(\mu_j)}{\prod_{i\neq j}(\mu_j-\mu_i)}\right). \tag{4.87}$$

We note that the dot $(\cdot)$ in (4.86) denotes the differentiation with respect to the continuous-time flow variable t_λ. Using (4.60) we can integrate (4.86) to obtain

$$t-t_0 = \int_{\mu_1(t_0)}^{\mu_1(t)} \frac{1}{\sqrt{R(\mu)}}\,d\mu + \int_{\mu_2(t_0)}^{\mu_2(t)} \frac{1}{\sqrt{R(\mu)}}\,d\mu\,, \tag{4.88a}$$

$$\lambda(t-t_0) = \int_{\mu_1(t_0)}^{\mu_1(t)} \frac{\mu}{\sqrt{R(\mu)}}\,d\mu + \int_{\mu_2(t_0)}^{\mu_2(t)} \frac{\mu}{\sqrt{R(\mu)}}\,d\mu\,. \tag{4.88b}$$

In fact, the modified Hamiltonian H^* coincides with the canonical transformed Hamiltonian K_λ obtained by applying the canonical transformation (4.79), viewed as a function of $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{P}_1, \mathcal{P}_2$. Hamilton's equations in the new coordinates are given by

$$\dot{\mathcal{P}}_1 = -\frac{\partial H^*}{\partial \mathcal{Q}_1} = 0\,, \qquad \dot{\mathcal{Q}}_1 = \frac{\partial H^*}{\partial \mathcal{P}_1} = \nu_1\,, \tag{4.89a}$$

$$\dot{\mathcal{P}}_2 = -\frac{\partial H^*}{\partial \mathcal{Q}_2} = 0\,, \qquad \dot{\mathcal{Q}}_2 = \frac{\partial H^*}{\partial \mathcal{P}_2} = \nu_2\,, \tag{4.89b}$$

which tell us that H^* depends on $\mathcal{P}_1$ and $\mathcal{P}_2$ only.

As a consequence of Abel's theorem [17], for genus g we have the following relation

$$\sum_{j=1}^{g}\int_{(\mu_j,\eta_j)}^{(\overline{\mu}_j,\overline{\eta}_j)} \omega_k + \int_{\infty}^{(\omega,\eta(\omega))} \omega_k = 0 \qquad (\text{mod } \Lambda_g)\,, \tag{4.90}$$

which provides us with the solution of the discrete Dubrovin equations. In equation (4.90) ω_k is the normalized differential of the first kind (Abelian differential) and Λ_g is a period lattice of the associated Riemann surface. Using relation (4.90), we introduce the frequencies v_1 and v_2 (not to be confused with a canonical variables ν_1 and ν_2) as the discrete time-one step as the following:

$$\mathrm{v}_1 = \int_{\mu_1}^{\overline{\mu}_1} \frac{1}{\sqrt{R(\mu)}}\, d\mu + \int_{\mu_2}^{\overline{\mu}_2} \frac{1}{\sqrt{R(\mu)}}\, d\mu = -\int_{\infty}^{(\omega,\eta(\omega))} \frac{1}{\sqrt{R(\mu)}}\, d\mu\ , \tag{4.91a}$$

$$\mathrm{v}_2 = \int_{\mu_1}^{\overline{\mu}_1} \frac{\mu}{\sqrt{R(\mu)}}\, d\mu + \int_{\mu_2}^{\overline{\mu}_2} \frac{\mu}{\sqrt{R(\mu)}}\, d\mu = -\int_{\infty}^{(\omega,\eta(\omega))} \frac{\mu}{\sqrt{R(\mu)}}\, d\mu\ , \tag{4.91b}$$

so that

$$\overline{\mathcal{Q}}_1 - \mathcal{Q}_1 = \mathrm{v}_1 \quad \text{and} \quad \overline{\mathcal{Q}}_2 - \mathcal{Q}_2 = \mathrm{v}_2\ ,$$

and in which for the integral (4.88b) we have chosen a specific value of λ given by

$$\lambda = \left(\int_{\infty}^{(\omega,\eta(\omega))} \frac{\mu}{\sqrt{R(\mu)}}\, d\mu\right) \Big/ \left(\int_{\infty}^{(\omega,\eta(\omega))} \frac{1}{\sqrt{R(\mu)}}\, d\mu\right). \tag{4.92}$$

The integrals (4.91) provide us with the solution of the discrete Dubrovin equations (4.45).

As the system of equations (4.45) is given, we can then follow the KdV map example prescription of section 3.4.2 to compute the limits $\overline{\mu}_1$, $\overline{\mu}_2$ of the integrals (4.91) at some suitably chosen initial points μ_1, μ_2. However, the calculations involved are very large and cannot be reproduced here, instead we shall just give the steps how to compute these limits. Step one, consider the auxiliary spectrum μ_1, μ_2 (i.e. the roots of polynomial (4.2) for $g = 2$) which are expressed in terms of the variables q_1, p_1, q_2, p_2. Step two, let $q_1 = q_2 = 0$, and thus μ_1, μ_2, respectively, take the expressions

$$\mu_1 = \frac{1}{6}\left(3\,\omega + 2\,\epsilon\, p_2 - 4\,\epsilon^2 + \sqrt{\Delta}\right), \tag{4.93a}$$

$$\mu_2 = \frac{1}{6}\left(3\,\omega + 2\,\epsilon\, p_2 - 4\,\epsilon^2 - \sqrt{\Delta}\right), \tag{4.93b}$$

where we define the shorthand

$$\Delta := 40\,\epsilon^4 - 4\,\epsilon^3\, p_2 - 8\,\epsilon^2\, p_2^2 - 6\,(\mathcal{P}_2 + C\,\omega) - 3\,\omega^2\ .$$

Step three, use the coupled system of equations (4.76) at $q_1 = q_2 = 0$, i.e.

$$\mathcal{P}_1 = \epsilon^3\left(p_1\, p_2^2 - p_1^2\, p_2 + \epsilon^3\right) + \epsilon^2\left(\omega - \epsilon^2\right)\left(p_1^2 - p_1\, p_2 + p_2^2 + 3\,\omega\right) - \omega^3,$$

$$\mathcal{P}_2 = -2\,\epsilon^2\left(p_1^2 - p_1\, p_2 + p_2^2 - 3\,\epsilon^2\right) - C\,\omega,$$

to eliminate p_2 from the equations (4.93). Step four, rewrite the coupled system of equations (4.93) as a system for which μ_1, μ_2 depend on the variables $\mathcal{P}_1$, $\mathcal{P}_2$ only. Step five, use the system (4.45) and equation (4.85) together with the initial points μ_1, μ_2 obtained from the latter step to compute $\overline{\mu}_1$, $\overline{\mu}_2$.

Denoting the new Hamiltonian by $H^*(\mathcal{P}_1, \mathcal{P}_2)$ one obtains

$$H^*(\mathcal{P}_1, \mathcal{P}_2) = \int^{(\mathcal{P}_1, \mathcal{P}_2)} [\nu_1 \, d\mathcal{P}_1 + \nu_2 \, d\mathcal{P}_2] \,, \tag{4.94}$$

over any curve in the $\mathcal{P}_1, \mathcal{P}_2$-plane up to an integration constant, and where the expression between brackets is a differential 1-form. The key reason is that on the basis of the integrals (4.91), using the dependence of the discriminant $R(\mu)$ on $\mathcal{P}_1$ and $\mathcal{P}_2$, one can prove that

$$\frac{\partial \nu_1}{\partial \mathcal{P}_2} = \frac{\partial \nu_2}{\partial \mathcal{P}_1} \,, \tag{4.95}$$

asserting that (ν_1, ν_2) is a conservative vector field. This can be seen as follows:

Recalling the discriminant of the curve (4.85), i.e.

$$\begin{aligned} R(\lambda) = {} & (4\,C + 12\,\omega)\,\lambda^5 + \left(C^2 - 12\,\omega^2 + 4\,\mathcal{P}_2\right)\lambda^4 \\ & + \left(4\,\omega^3 + 2\,C\,\mathcal{P}_2 + 4\,\mathcal{P}_1\right)\lambda^3 + \left(2\,C\,\mathcal{P}_1 + \mathcal{P}_2^2\right)\lambda^2 \\ & + 2\,\mathcal{P}_1\,\mathcal{P}_1\,\lambda + \mathcal{P}_1^2 \,, \end{aligned}$$

and the trace of the monodromy matrix $\operatorname{tr} T(\lambda)$, i.e.

$$\operatorname{tr} T(\lambda) = 2\,\lambda^3 + C\,\lambda^2 + \mathcal{P}_2\,\lambda + \mathcal{P}_1 \,,$$

from equations (4.91), we have

$$\frac{\partial \nu_1}{\partial \mathcal{P}_2} = -\int_{\infty}^{(\omega,\eta(\omega))} \frac{-1}{2\,[R(\mu)]^{3/2}}\, d\mu \left(\frac{\partial R(\mu)}{\partial \mathcal{P}_2}\right) = \int_{\infty}^{(\omega,\eta(\omega))} \frac{\mu \operatorname{tr} T(\mu)}{[R(\mu)]^{3/2}}\, d\mu \,, \tag{4.96a}$$

$$\frac{\partial \nu_2}{\partial \mathcal{P}_1} = -\int_{\infty}^{(\omega,\eta(\omega))} \frac{-\mu}{2\,[R(\mu)]^{3/2}}\, d\mu \left(\frac{\partial R(\mu)}{\partial \mathcal{P}_1}\right) = \int_{\infty}^{(\omega,\eta(\omega))} \frac{\mu \operatorname{tr} T(\mu)}{[R(\mu)]^{3/2}}\, d\mu \,. \tag{4.96b}$$

Hence, the integral (4.94) is independent of the integration path in $\mathcal{P}_1, \mathcal{P}_2$-plane chosen and thus leads to a well-defined function of $(\mathcal{P}_1, \mathcal{P}_2)$ obeying the relations (4.89). Equation (4.94) is a closed-form expression for the modified Hamiltonian of the mapping from (4.74).

4.8 Summary

In this chapter we broadened the perspective by looking at the extension to multiple-degrees-of-freedom. In this case, we are dealing with the more complicated situation where the underlying spectral representation of the underlying of integrable systems is associated with a higher-genus algebraic curve. This case forced us to deal with some new mathematical techniques, such as separation of variables and finite-gap integration.

One striking aspect was the interplay between the discrete- and continuous-time evolutions of the separation variables. We uncovered a novel formulation of the discrete Dubrovin equations for the diagonal evolution in terms of the auxiliary spectrum, which are different from the equations for the vertical evolution given by Nijhoff [80]. By considering the continuous-time evolution for the auxiliary spectrum, we used the trace of monodromy matrix as the generating Hamiltonian of the flow to construct an interpolating flow for the discrete map. This structure leads to interesting formulations of the first-order system of coupled ordinary differential equations, which are the Dubrovin equations for our case (potential term) and their companion equations for the conjugate variables (kinetic term).

Another point of interest is the diagram of commuting canonical transformations between the original, separating and action-angle variables, which follows as a consequence of the transformation to action-angle variables on the one hand and the dynamical mapping on the other hand. In order to arrive at the action-angle variables, we looked for the canonical transformations from the original variables to the separated variables and from the separated variables to the action-angle variables. Further, we looked for the dynamical transformation of the separated variables. Finally, we examined the schemes that we set up for the case of genus-two, and uncovered a novel expression for the modified Hamiltonian, from which it was written down in closed form.

Chapter 5

Lattice Boussinesq models and dynamical mappings

5.1 Overview

As we have seen in chapter 1, the Korteweg-de Vries (KdV) partial differential equation is discretized by a one-component quadrilateral equation. In the previous chapters of the thesis, the various manifestations of the discrete KdV equation are considered and highlighted. A natural extension to the research of the previous chapters is to consider the Boussinesq (BSQ) partial differential equation

$$u_{tt} + \frac{1}{3}\, u_{xxxx} + 4\, u_x\, u_{xx} = 0\;,$$

which in particular the second derivative in time [13]. (This equation will be encountered in section 5.2 below.) This means that it cannot be discretized by a one-component quadrilateral equation; rather, we need either a multicomponent quadrilateral system or an equation defined on a larger stencil.

The discrete BSQ models [79, 83, 87, 115, 130] are considered and highlighted in this chapter. The direct linearization of the BSQ system was first introduced in [97]. In [87] it was shown that in the more general case for an extra parameter ω, which is the $\mathcal{N}$-th root of unity, i.e. $\omega = \exp(2i\pi/\mathcal{N})$, the entire hierarchy of equations of what they called the “Gel’fand-Dikii (GD) hierarchy of equations” are linearized. The KdV and BSQ equations appeared as the lowest order members of the GD hierarchy of equations. For

$\mathcal{N} = 2$ the equations in this class are the KdV-type equations, whereas for $\mathcal{N} = 3$ the equations are of BSQ-type. In [87] the lattice GD hierarchy of equations were presented along with their relevant Lax pairs and a gauge transformation between the two Lax pairs. The lattice version of BSQ system was proposed by Tongas and Nijhoff in [115], where it was given as three-component systems which obey the property of multi-dimensional consistency. In [55] a one-parameter generalization was presented of the three-component BSQ system after a point transformation on the dependent variables. Subsequently, the direct linearization of the extended lattice BSQ system was introduced in [130]. The extended lattice BSQ system leads to the KdV class of lattice system. Added to the KdV model investigated in the previous chapters of the thesis, the BSQ model gives us an extra elevation to learn and see what is going on in general. Further, an important aspect of researching the BSQ system is the extrapolation to higher $\mathcal{N}$ values.

This chapter is organized as follows: In the next section we give the derivation of the discrete BSQ equation using the Bäcklund transformation. In the following section we consider the lattice BSQ system and explain how to present the lattice BSQ equation as three-component systems. In section 5.4 we present the general scheme to derive the mappings of BSQ type for two different discrete-time evolutions (vertical and diagonal evolutions) and construct the Lax pairs of the mappings. Moreover, we present the method to derive the big Lax matrix (dual Lax) which leads us to appropriate reduced variables in which the mappings and invariants are expressed, followed by illustrative examples of period one and two reductions for vertical and diagonal discrete-time evolutions. Finally, in section 5.5 we present the classical r-matrix structure for the mappings of lattice GD hierarchy of equations, where for $\mathcal{N} = 3$ we obtain the r-matrix structure for the mappings of of BSQ type. We also give the proof of the classical Yang-Baxter structure for the mappings of the lattice GD hierarchy of equations.

5.2 Discrete Boussinesq from Bäcklund transformation

The equation of interest in the context of what follows is the discrete version of the BSQ partial differential equation [87]. Therefore, in this section we will show how to derive the lattice version of the BSQ equation from the Bäcklund transformation (BT). Recall that the BSQ equation is a partial differential equation of the form:

$$u_{tt} + \frac{1}{3}\, u_{xxxx} + 4\, u_x\, u_{xx} = 0\,; \qquad u(x,0) = u_0(x)\,,\ \ u_t(x,0) = u_1(x)\,, \tag{5.1}$$

where $u = u(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is an appropriate field variable, x is a space coordinate in the direction of propagation and t is time. This equation arises in the theory of water waves and also in several physical applications [13]. The BSQ equation (5.1) is a soliton equation solvable by inverse scattering when the initial value problems $u_0(x)$ and $u_1(x)$ are suitably bounded as $x \to \pm\infty$, cf. [3, 22, 32].

In order to derive the discrete BSQ equation, we first establish the BT for the BSQ equation (5.1). The BT for the BSQ equation was given by Chen [30] in a form that does not contain any Bäcklund parameter. He also gave a superposition formula, but one which still contains derivatives with respect to the continuous variables. The issue of a purely algebraic superposition formula was discussed in a recent paper by Rasin and Schiff [101], where they derived the lattice BSQ equation in three-component form as was originally given in [115]. For the sake of parallel exposition with the KdV model presented in section 1.3.1, we here write down the proper BT for the BSQ equation with parameter, and derive from it the corresponding lattice BSQ equation in 9-point scalar form as originally given in [87].

Proposition 5.2.1. *The system of equations*

$$\partial_t\,(\widetilde{u} - u) = \partial_x^2\,(\widetilde{u} + u) - 2\,a\,\partial_x\,(\widetilde{u} - u) + \partial_x\,(\widetilde{u} - u)^2\ , \tag{5.2a}$$

$$\begin{aligned}\frac{3}{4}\,\partial_t\,(\widetilde{u} + u) = {} & \frac{3}{2}\,a\,\partial_x\,(\widetilde{u} + u) - \frac{3}{2}\,(\widetilde{u} - u)\,\partial_x\,(\widetilde{u} + u) - \frac{1}{4}\,\partial_x^2\,(\widetilde{u} - u) \\ & - 3\,a^2\,(\widetilde{u} - u) + 3\,a\,(\widetilde{u} - u)^2 - (\widetilde{u} - u)^3\,,\end{aligned} \tag{5.2b}$$

where the accent $\widetilde{\ }$ *defined in section 1.3.1 of the introduction, constitute the Bäcklund transformation for the BSQ equation (5.1).*

Proof

Consider the consistency condition $\partial_t\,\partial_x^2\,(\widetilde{u} + u) = \partial_x^2\,\partial_t\,(\widetilde{u} + u)$. From equation (5.2a) we have

$$\frac{3}{4}\,\partial_t\,\partial_x^2\,(\widetilde{u} + u) = \frac{3}{4}\,\partial_t\left[\partial_t\,(\widetilde{u} - u) + 2\,a\,\partial_x\,(\widetilde{u} - u) - \partial_x\,(\widetilde{u} - u)^2\right], \tag{5.3a}$$

whereas from equation (5.2b) we have

$$\frac{3}{4}\,\partial_x^2\,\partial_t\,(\widetilde{u}+u) = \frac{3}{2}\,a\,\partial_x^3\,(\widetilde{u}+u) - \frac{3}{2}\,\partial_x^2\,[(\widetilde{u}-u)\,\partial_x\,(\widetilde{u}+u)] - \frac{1}{4}\,\partial_x^4\,(\widetilde{u}-u) \\ -3\,a^2\,\partial_x^2\,(\widetilde{u}-u) + 3\,a\,\partial_x^2\,(\widetilde{u}-u)^2 - \partial_x^2\,(\widetilde{u}-u)^3\,. \tag{5.3b}$$

Thus, from the consistency condition and using equations (5.3) we obtain

$$\frac{1}{4}\,\partial_x^4\,(\widetilde{u}-u) + \frac{3}{4}\,\partial_t\left[\partial_t\,(\widetilde{u}-u) + 2\,a\,\partial_x\,(\widetilde{u}-u) - \partial_x\,(\widetilde{u}-u)^2\right] + \frac{3}{2}\,\partial_x\left[(\partial_x\widetilde{u})^2 - (\partial_x u)^2\right] \\ = \frac{3}{2}\,a\,\partial_x^3\,(\widetilde{u}+u) - 3\,a^2\,\partial_x^2\,(\widetilde{u}-u) + 3\,a\,\partial_x^2\,(\widetilde{u}-u)^2 - \partial_x^2\,(\widetilde{u}-u)^3 \\ -\frac{3}{2}\,\partial_x\left[(\widetilde{u}-u)\,\partial_x^2\,(\widetilde{u}+u)\right]. \tag{5.4}$$

Now, from equations (5.2) we can establish the following relations

$$\frac{3}{2}\,a\,\partial_x\,\partial_t\,(\widetilde{u}-u) = \frac{3}{2}\,a\,\partial_x^3\,(\widetilde{u}+u) - 3\,a^2\,\partial_x^2\,(\widetilde{u}-u) + \frac{3}{2}\,a\,\partial_x^2\,(\widetilde{u}-u)^2\;,$$
$$\frac{3}{4}\,\partial_x\,\partial_t\,(\widetilde{u}-u)^2 = \frac{3}{2}\,\partial_x\left[(\widetilde{u}-u)\,\partial_x^2\,(\widetilde{u}+u)\right] - \frac{3}{2}\,a\,\partial_x^2\,(\widetilde{u}-u)^2 + \partial_x^2\,(\widetilde{u}-u)^3\,.$$

Using these relations, the equation (5.4) can be written as

$$\frac{1}{4}\,\partial_x^4\,(\widetilde{u}-u) + \frac{3}{4}\,\partial_t\left[\partial_t\,(\widetilde{u}-u) + 2\,a\,\partial_x\,(\widetilde{u}-u) - \partial_x\,(\widetilde{u}-u)^2\right] + \frac{3}{2}\,\partial_x\left[(\partial_x\widetilde{u})^2 - (\partial_x u)^2\right] \\ = \frac{3}{2}\,a\,\partial_x\,\partial_t\,(\widetilde{u}-u) - \frac{3}{4}\,\partial_x\,\partial_t\,(\widetilde{u}-u)^2\,,$$

which implies

$$\left[\frac{1}{4}\,\partial_x^4\,\widetilde{u} + \frac{3}{4}\,\partial_t^2\,\widetilde{u} + \frac{3}{2}\,\partial_x\,(\partial_x\,\widetilde{u})^2\right] - \left[\frac{1}{4}\,\partial_x^4\,u + \frac{3}{4}\,\partial_t^2\,u + \frac{3}{2}\,\partial_x\,(\partial_x\,u)^2\right] = 0\,. \tag{5.5}$$

Hence, if u obeys the BSQ equation, then either $\widetilde{u}=u$ or $\widetilde{u}$ obeys the BSQ equation. □

As discussed in chapter 1, an important property of the BTs is the permutability property, which obtained by the iteration of the BTs. One valuable feature of the permutability property is that it is used to derive the discrete equation. This leads to the following theorem:

Theorem 5.2.1. *Suppose BT_a and BT_b are the Bäcklund transformation (5.2) for two different parameters a and b respectively*

$$BT_a : u \overset{a}{\longmapsto} \widetilde{u} \quad \textit{and} \quad BT_b : u \overset{b}{\longmapsto} \widehat{u}\,.$$

Then, the proper permutability condition is written as

$$\frac{a^3-b^3}{a-b+\widehat{\widetilde{u}}-\widetilde{\widetilde{u}}}-\frac{a^3-b^3}{a-b+\widehat{\widehat{u}}-\widehat{\widetilde{u}}} = (a-b+\widehat{\widehat{\widetilde{u}}}-\widehat{\widetilde{\widetilde{u}}})\,(2\,a+b+\widehat{u}-\widehat{\widehat{\widetilde{u}}}) \\ -(a-b+\widehat{u}-\widetilde{u})\,(2\,a+b+u-\widehat{\widetilde{\widetilde{u}}})\,, \tag{5.6}$$

which is known as the lattice version of the BSQ equation.

Proof

Consider the system of equations (5.2), i.e.

$$\partial_t\,(\widetilde{u}-u) = \partial_x^2\,(\widetilde{u}+u)-2\,a\,\partial_x\,(\widetilde{u}-u)+\partial_x\,(\widetilde{u}-u)^2\ , \tag{5.7a}$$

$$\frac{3}{4}\,\partial_t\,(\widetilde{u}+u) = \frac{3}{2}\,a\,\partial_x\,(\widetilde{u}+u)-\frac{3}{2}\,(\widetilde{u}-u)\,\partial_x\,(\widetilde{u}+u)-\frac{1}{4}\,\partial_x^2\,(\widetilde{u}-u) \\ -3\,a^2\,(\widetilde{u}-u)+3\,a\,(\widetilde{u}-u)^2-(\widetilde{u}-u)^3\,. \tag{5.7b}$$

To eliminate all t and x-derivatives from the system (5.7), we first need to write the system of equations for the hat shift

$$\partial_t\,(\widehat{u}-u) = \partial_x^2\,(\widehat{u}+u)-2\,b\,\partial_x\,(\widehat{u}-u)+\partial_x\,(\widehat{u}-u)^2\ , \tag{5.8a}$$

$$\frac{3}{4}\,\partial_t\,(\widehat{u}+u) = \frac{3}{2}\,b\,\partial_x\,(\widehat{u}+u)-\frac{3}{2}\,(\widehat{u}-u)\,\partial_x\,(\widehat{u}+u)-\frac{1}{4}\,\partial_x^2\,(\widehat{u}-u) \\ -3\,b^2\,(\widehat{u}-u)+3\,b\,(\widehat{u}-u)^2-(\widehat{u}-u)^3\,. \tag{5.8b}$$

Performing the iterated BTs

$$BT_b:\widetilde{u}\overset{b}{\longmapsto}\widehat{\widetilde{u}} \quad \text{and} \quad BT_a:\widehat{u}\overset{a}{\longmapsto}\widetilde{\widehat{u}}$$

for the systems of equations (5.7) and (5.8) respectively and assuming $\widehat{\widetilde{u}}=\widetilde{\widehat{u}}$, we are led to the relations

$$\partial_t\,(\widehat{\widetilde{u}}-\widehat{u}) = \partial_x^2\,(\widehat{\widetilde{u}}+\widehat{u})-2\,a\,\partial_x\,(\widehat{\widetilde{u}}-\widehat{u})+\partial_x\,(\widehat{\widetilde{u}}-\widehat{u})^2\,, \tag{5.9a}$$

$$\frac{3}{4}\,\partial_t\,(\widehat{\widetilde{u}}+\widehat{u}) = \frac{3}{2}\,a\,\partial_x\,(\widehat{\widetilde{u}}+\widehat{u})-\frac{3}{2}\,(\widehat{\widetilde{u}}-\widehat{u})\,\partial_x\,(\widehat{\widetilde{u}}+\widehat{u})-\frac{1}{4}\,\partial_x^2\,(\widehat{\widetilde{u}}-\widehat{u}) \\ -3\,a^2\,(\widehat{\widetilde{u}}-\widehat{u})+3\,a\,(\widehat{\widetilde{u}}-\widehat{u})^2-(\widehat{\widetilde{u}}-\widehat{u})^3, \tag{5.9b}$$

$$\partial_t\,(\widehat{\widetilde{u}}-\widetilde{u}) = \partial_x^2\,(\widehat{\widetilde{u}}+\widetilde{u})-2\,b\,\partial_x\,(\widehat{\widetilde{u}}-\widetilde{u})+\partial_x\,(\widehat{\widetilde{u}}-\widetilde{u})^2\,, \tag{5.9c}$$

$$\frac{3}{4}\,\partial_t\,(\widehat{\widetilde{u}}+\widetilde{u}) = \frac{3}{2}\,b\,\partial_x\,(\widehat{\widetilde{u}}+\widetilde{u})-\frac{3}{2}\,(\widehat{\widetilde{u}}-\widetilde{u})\,\partial_x\,(\widehat{\widetilde{u}}+\widetilde{u})-\frac{1}{4}\,\partial_x^2\,(\widehat{\widetilde{u}}-\widetilde{u}) \\ -3\,b^2\,(\widehat{\widetilde{u}}-\widetilde{u})+3\,b\,(\widehat{\widetilde{u}}-\widetilde{u})^2-(\widehat{\widetilde{u}}-\widetilde{u})^3. \tag{5.9d}$$

By combining (5.7a) with (5.9c) and (5.8a) with (5.9a), we respectively obtain

$$\partial_t\,(\widehat{\widetilde{u}} - u) = \partial_x^2\,(\widehat{\widetilde{u}} + 2\,\widetilde{u} + u) - 2\,\partial_x\,[\,b\,\widehat{\widetilde{u}} - a\,u + (a-b)\,\widetilde{u}\,] + \partial_x\,[(\widehat{\widetilde{u}} - \widetilde{u})^2 + (\widetilde{u} - u)^2]\,,$$

$$\partial_t\,(\widehat{\widetilde{u}} - u) = \partial_x^2\,(\widehat{\widetilde{u}} + 2\,\widehat{u} + u) - 2\,\partial_x\,[\,a\,\widehat{\widetilde{u}} - b\,u - (a-b)\,\widehat{u}\,] + \partial_x\,[(\widehat{\widetilde{u}} - \widehat{u})^2 + (\widehat{u} - u)^2]\,.$$

Eliminating t-derivatives from these equations we obtain

$$\partial_x\,(\widehat{u} - \widetilde{u}) = (a - b + \widehat{u} - \widetilde{u})\,(\widehat{\widetilde{u}} + u - \widehat{u} - \widetilde{u})\,. \tag{5.10}$$

As before, combining (5.7b) with (5.9b) and (5.8b) with (5.9d), we respectively obtain

$$\begin{aligned}
\frac{3}{4}\,\partial_t\,(\widehat{\widetilde{u}} + \widehat{u} + \widetilde{u} + u) = {} & \frac{3}{2}\,a\,\partial_x\,(\widehat{\widetilde{u}} + \widehat{u} + \widetilde{u} + u) - \frac{1}{4}\,\partial_x^2\,(\widehat{\widetilde{u}} - u + \widetilde{u} - \widehat{u}) \\
& - \frac{3}{2}\,[(\widetilde{u} - u)\,\partial_x\,(\widetilde{u} + u) + (\widehat{\widetilde{u}} - \widehat{u})\,\partial_x\,(\widehat{\widetilde{u}} + \widehat{u})] \\
& - 3\,a^2\,(\widehat{\widetilde{u}} - u + \widetilde{u} - \widehat{u}) + 3\,a\,[(\widetilde{u} - u)^2 + (\widehat{\widetilde{u}} - \widehat{u})^2] \\
& - (\widetilde{u} - u)^3 - (\widehat{\widetilde{u}} - \widehat{u})^3\,, \\
\frac{3}{4}\,\partial_t\,(\widehat{\widetilde{u}} + \widehat{u} + \widetilde{u} + u) = {} & \frac{3}{2}\,b\,\partial_x\,(\widehat{\widetilde{u}} + \widehat{u} + \widetilde{u} + u) - \frac{1}{4}\,\partial_x^2\,(\widehat{\widetilde{u}} - u + \widehat{u} - \widetilde{u}) \\
& - \frac{3}{2}\,[(\widehat{u} - u)\,\partial_x\,(\widehat{u} + u) + (\widehat{\widetilde{u}} - \widetilde{u})\,\partial_x\,(\widehat{\widetilde{u}} + \widetilde{u})] \\
& - 3\,b^2\,(\widehat{\widetilde{u}} - u + \widehat{u} - \widetilde{u}) + 3\,b\,[(\widehat{u} - u)^2 + (\widehat{\widetilde{u}} - \widetilde{u})^2] \\
& - (\widehat{u} - u)^3 - (\widehat{\widetilde{u}} - \widetilde{u})^3\,.
\end{aligned}$$

Eliminating t-derivatives from these equations we obtain

$$\begin{aligned}
\frac{1}{2}\,\partial_x^2\,(\widetilde{u} - \widehat{u}) = {} & 3\,a\,[(\widetilde{u} - u)^2 + (\widehat{\widetilde{u}} - \widehat{u})^2] - 3\,b\,[(\widehat{u} - u)^2 + (\widehat{\widetilde{u}} - \widetilde{u})^2] + (\widehat{u} - u)^3 \\
& - (\widehat{\widetilde{u}} - \widehat{u})^3 - (\widetilde{u} - u)^3 + (\widehat{\widetilde{u}} - \widetilde{u})^3 - 3\,(a^2 - b^2)\,(\widehat{\widetilde{u}} - u) \\
& - 3\,(a^2 + b^2)\,(\widetilde{u} - \widehat{u}) + \frac{3}{2}\,(a - b)\,\partial_x\,(\widehat{\widetilde{u}} + \widehat{u} + \widetilde{u} + u) \\
& - \frac{3}{2}\,[(\widetilde{u} - u)\,\partial_x\,(\widetilde{u} + u) + (\widehat{\widetilde{u}} - \widehat{u})\,\partial_x\,(\widehat{\widetilde{u}} + \widehat{u})] \\
& + \frac{3}{2}\,[(\widehat{u} - u)\,\partial_x\,(\widehat{u} + u) + (\widehat{\widetilde{u}} - \widetilde{u})\,\partial_x\,(\widehat{\widetilde{u}} + \widetilde{u})]\,.
\end{aligned} \tag{5.11}$$

Now, what remains is to eliminate all x-derivatives from equations (5.10) and (5.11). From equation (5.10) we can obtain

$$\partial_x^2\,(\widehat{u} - \widetilde{u}) = (a - b + \widehat{u} - \widetilde{u})\,[(\widehat{\widetilde{u}} + u - \widehat{u} - \widetilde{u})^2 + \partial_x\,(\widehat{\widetilde{u}} + u - \widehat{u} - \widetilde{u})]\,. \tag{5.12}$$

Using equations (5.11) and (5.12) we obtain

$$
\begin{aligned}
&(\widehat{u}-\widetilde{u})\,[\,3\,\widehat{\widetilde{u}}^{2}+3\,u^2-3\,(\widehat{u}+\widetilde{u})\,(\widehat{\widetilde{u}}+u)+2\,(\widehat{u}^2+\widetilde{u}\,\widehat{u}+\widetilde{u}^2\,]-3\,(a^2-b^2)\,(\widehat{\widetilde{u}}-u)\\
&+3\,(a^2+b^2)\,(\widehat{u}-\widetilde{u})+3\,a\,[(\widetilde{u}-u)^2+(\widehat{\widetilde{u}}-\widehat{u})^2]-3\,b\,[(\widehat{u}-u)^2+(\widehat{\widetilde{u}}-\widetilde{u})^2]\\
&+\frac{1}{2}\,(a-b+\widehat{u}-\widetilde{u})\,[(\widehat{\widetilde{u}}+u-\widehat{u}-\widetilde{u})^2+\partial_x\,(\widehat{\widetilde{u}}+u-\widehat{u}-\widetilde{u})]\\
&+\frac{3}{2}\,[(\widehat{u}-\widetilde{u})\,\partial_x\,(\widehat{\widetilde{u}}+\widehat{u}+\widetilde{u}+u)-(\widehat{\widetilde{u}}+u-\widehat{u}-\widetilde{u})\,\partial_x\,(\widehat{u}-\widetilde{u})]\\
&+\frac{3}{2}\,(a-b)\,\partial_x\,(\widehat{\widetilde{u}}+\widehat{u}+\widetilde{u}+u)=0\,.
\end{aligned}
$$

By simplifying this expression, we arrive at the following formula

$$
\begin{aligned}
&(a-b+\widehat{u}-\widetilde{u})\,[\,3\,\widehat{\widetilde{u}}^{2}+3\,u^2-3\,(\widehat{u}+\widetilde{u})\,(\widehat{\widetilde{u}}+u)+2\,(\widehat{u}^2+\widetilde{u}\,\widehat{u}+\widetilde{u}^2\,]\\
&-3\,(a^2-b^2)\,(\widehat{\widetilde{u}}-u)+3\,(a^2+b^2)\,(\widehat{u}-\widetilde{u})+(a-b)\,(\widehat{u}-\widetilde{u})^2\\
&-3\,(a+b)\,(\widehat{u}-\widetilde{u})\,(\widehat{\widetilde{u}}-u)-(a-b+\widehat{u}-\widetilde{u})\,(\widehat{\widetilde{u}}+u-\widehat{u}-\widetilde{u})^2\\
&+(a-b+\widehat{u}-\widetilde{u})\,\partial_x\,(2\,\widehat{\widetilde{u}}+2\,u+\widehat{u}+\widetilde{u})=0\,.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\frac{a^3-b^3}{a-b+\widehat{u}-\widetilde{u}}&=\partial_x\,(\widehat{\widetilde{u}}+u)+\frac{1}{2}\,\partial_x\,(\widehat{u}+\widetilde{u})+\frac{1}{2}\,(\widehat{u}^2+\widetilde{u}^2)-\frac{1}{2}\,(\widehat{u}+\widetilde{u})\,(\widehat{\widetilde{u}}+u)\\
&+\widehat{\widetilde{u}}^{2}-u\,\widehat{\widetilde{u}}+u^2-\frac{3}{2}\,(\widehat{\widetilde{u}}-u)\,(a+b)+\frac{1}{2}\,(\widehat{u}-\widetilde{u})\,(a-b)\\
&+a^2+a\,b+b^2\,.
\end{aligned}
\tag{5.13}
$$

By shifting equation (5.13) one-step for the hat $\widehat{\ }$ and one-step for the tilde $\widetilde{\ }$ separately and subtracting the resulting, we obtain

$$
\begin{aligned}
\frac{a^3-b^3}{a-b+\widehat{\widehat{u}}-\widehat{\widetilde{u}}}-\frac{a^3-b^3}{a-b+\widehat{\widetilde{u}}-\widetilde{\widetilde{u}}}&=\partial_x\,(\widehat{\widehat{\widetilde{u}}}-\widehat{\widetilde{\widetilde{u}}}+\widehat{u}-\widetilde{u})+\frac{1}{2}\,\partial_x\,(\widehat{\widehat{u}}-\widetilde{\widetilde{u}})\\
&+\frac{1}{2}\,(\widehat{\widehat{u}}^{2}-\widetilde{\widetilde{u}}^{2})+\widehat{\widehat{\widetilde{u}}}^{2}-\widehat{\widetilde{\widetilde{u}}}^{2}+\widetilde{u}\,\widehat{\widetilde{\widetilde{u}}}-\widehat{u}\,\widehat{\widehat{\widetilde{u}}}+\widehat{u}^2-\widetilde{u}^2\\
&-\frac{1}{2}\,(\widehat{\widehat{u}}+\widehat{\widetilde{u}})\,(\widehat{\widehat{\widetilde{u}}}+\widehat{u})+\frac{1}{2}\,(\widehat{\widetilde{u}}+\widetilde{\widetilde{u}})\,(\widehat{\widetilde{\widetilde{u}}}+\widetilde{u})\\
&-\frac{3}{2}\,(a+b)\,(\widehat{\widehat{\widetilde{u}}}-\widehat{\widetilde{\widetilde{u}}}+\widetilde{u}-\widehat{u})\\
&+\frac{1}{2}\,(a-b)\,(\widehat{\widehat{u}}-2\,\widehat{\widetilde{u}}+\widetilde{\widetilde{u}})\,.
\end{aligned}
\tag{5.14}
$$

Now, using equation (5.10) we can derive the following equations

$$\partial_x\,(\widehat{\widehat{\widetilde{u}}} - \widehat{\widetilde{\widetilde{u}}}) = (a - b + \widehat{\widehat{\widetilde{u}}} - \widehat{\widetilde{\widetilde{u}}})\,(\widehat{\widehat{\widetilde{\widetilde{u}}}} + \widehat{\widetilde{u}} - \widehat{\widehat{\widetilde{u}}} - \widehat{\widetilde{\widetilde{u}}})\,,$$
$$\partial_x\,(\widehat{\widehat{u}} - \widetilde{\widetilde{u}}) = (a - b + \widehat{\widehat{u}} - \widehat{\widetilde{u}})\,(\widehat{\widehat{\widetilde{u}}} + \widehat{u} - \widehat{\widehat{u}} - \widehat{\widetilde{u}}) + (a - b + \widehat{\widetilde{u}} - \widetilde{\widetilde{u}})\,(\widehat{\widetilde{\widetilde{u}}} + \widetilde{u} - \widehat{\widetilde{u}} - \widetilde{\widetilde{u}})\,.$$

Thus, using these equations together with equation (5.10), we can eliminate all x-derivatives from equation (5.14) and hence we obtain

$$\begin{aligned}\frac{a^3 - b^3}{a - b + \widehat{\widehat{u}} - \widehat{\widetilde{u}}} - \frac{a^3 - b^3}{a - b + \widehat{\widetilde{u}} - \widetilde{\widetilde{u}}} = {} & (a - b + \widehat{u} - \widetilde{u})\,(\widehat{\widetilde{u}} + u - \widehat{u} - \widetilde{u}) \\ & + (a - b + \widehat{\widehat{\widetilde{u}}} - \widehat{\widetilde{\widetilde{u}}})\,(\widehat{\widehat{\widetilde{\widetilde{u}}}} + \widehat{\widetilde{u}} - \widehat{\widehat{\widetilde{u}}} - \widehat{\widetilde{\widetilde{u}}}) \\ & + \frac{1}{2}\,(a - b)\,(\widehat{\widehat{\widetilde{u}}} + \widehat{\widetilde{\widetilde{u}}} + \widehat{u} + \widetilde{u} - 4\,\widehat{\widetilde{u}}) \\ & - \frac{3}{2}\,(a + b)\,(\widehat{\widehat{\widetilde{u}}} - \widehat{\widetilde{\widetilde{u}}} + \widetilde{u} - \widehat{u}) + \widehat{\widetilde{u}}\,(\widehat{\widetilde{\widetilde{u}}} - \widehat{\widehat{\widetilde{u}}} + \widetilde{u} - \widehat{u}) \\ & + \widehat{\widehat{\widetilde{u}}}\,(\widehat{\widehat{\widetilde{u}}} - \widehat{u}) + \widehat{\widetilde{\widetilde{u}}}\,(\widetilde{u} - \widehat{\widetilde{\widetilde{u}}}) + \widehat{u}^2 - \widetilde{u}^2\,. \end{aligned} \tag{5.15}$$

Computing this, we are led to the permutability condition (5.6). □

Next, we consider the lattice BSQ equation derived in this section and we apply periodic initial value problems to the lattice equation in order to construct integrable mappings.

5.3 Lattice Boussinesq system

Recall that the lattice version of the BSQ equation given in (5.6) holds on elementary plaquettes in the lattice on the variable u and reads in the form of the following 9-point equation on the two-dimensional lattice:

$$\begin{aligned}\frac{a^3 - b^3}{a - b + \widehat{\widetilde{u}} - \widetilde{\widetilde{u}}} - \frac{a^3 - b^3}{a - b + \widehat{\widehat{u}} - \widehat{\widetilde{u}}} = {} & (a - b + \widehat{\widehat{\widetilde{u}}} - \widehat{\widetilde{\widetilde{u}}})\,(2\,a + b + \widehat{u} - \widehat{\widehat{\widetilde{\widetilde{u}}}}) \\ & - (a - b + \widehat{u} - \widetilde{u})\,(2\,a + b + u - \widehat{\widetilde{\widetilde{u}}})\,. \end{aligned} \tag{5.16}$$

As in the previous chapters, the dependent variable u is defined on lattice points labelled by discrete variables (n, m) with $n, m \in \mathbb{Z}$, which are variables shifting by units, and with lattice parameters a and b, each associated with the n and m directions on the lattice respectively.

As noted above, the equation (5.16) appears in the class of $\mathcal{N} = 3$ of the GD hierarchy of equations and lives on a 9-point stencil as illustrated in figure 5.1.

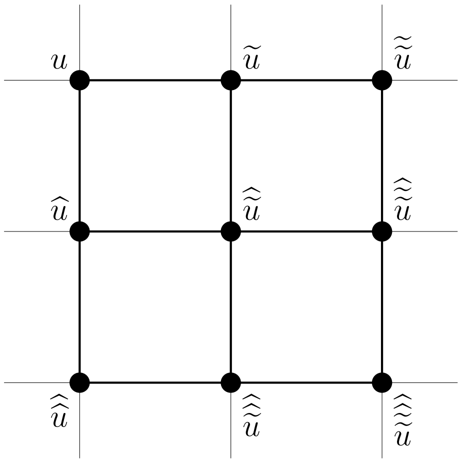


Figure 5.1: Nine-point stencil.

For the sake of clarity we prefer to use a notation with elementary lattice shifts denoted by accents $\widetilde{\ }$ and $\widehat{\ }$, i.e. for $u := u(n,m)$, we have $\widetilde{u} := u(n+1,m)$ and $\widehat{u} := u(n,m+1)$, for the convenience of the reader. Thus, as a consequence of this notation, we have

$$\widehat{\widetilde{u}} := u(n{+}1, m{+}1)\ , \quad \widehat{\widetilde{\widetilde{u}}} := u(n{+}2, m{+}1)\ , \quad \widehat{\widehat{\widetilde{u}}} := u(n{+}1, m{+}2)\ , \quad \widehat{\widehat{\widetilde{\widetilde{u}}}} := u(n{+}2, m{+}2)\ .$$

We now have the following proposition:

Proposition 5.3.1. *The equation (5.16) can be written as the following coupled system of* 3*-field equations for* u*,* v *and* w *around an elementary quadrilateral:*

$$v + \widetilde{w} = a\,(\widetilde{u} - u) + \widetilde{u}\,u\ , \tag{5.17a}$$

$$v + \widehat{w} = b\,(\widehat{u} - u) + \widehat{u}\,u\ , \tag{5.17b}$$

$$\widehat{\widetilde{v}} + w = a\,b - (a + b + u)(a + b - \widehat{\widetilde{u}}) + \frac{a^3 - b^3}{a - b + \widehat{u} - \widetilde{u}}\ . \tag{5.17c}$$

Proof

To show that the system (5.17) leads to the lattice BSQ equation given in (5.16), we need to eliminate the variables v and w from the three equations (5.17). From equations (5.17a) and (5.17b) we obtain the following relations

$$\widetilde{w} - \widehat{w} = a\,(\widetilde{u} - u) - b\,(\widehat{u} - u) + u\,(\widetilde{u} - \widehat{u}) \;, \tag{5.18a}$$

$$\widehat{v} - \widetilde{v} = a\,(\widehat{\widetilde{u}} - \widehat{u}) - b\,(\widehat{\widetilde{u}} - \widetilde{u}) + \widehat{\widetilde{u}}\,(\widehat{u} - \widetilde{u}) \;. \tag{5.18b}$$

Using these equations we can derive

$$\begin{aligned}(\widetilde{w} - \widehat{w}) - (\widehat{\widehat{\widetilde{v}}} - \widehat{\widetilde{\widetilde{v}}}) = {} & a\,(\widetilde{u} - u) - b\,(\widehat{u} - u) + u\,(\widetilde{u} - \widehat{u}) \\ & - a\,(\widehat{\widehat{\widetilde{\widetilde{u}}}} - \widehat{\widehat{\widetilde{u}}}) + b\,(\widehat{\widehat{\widetilde{\widetilde{u}}}} - \widehat{\widetilde{\widetilde{u}}}) - \widehat{\widehat{\widetilde{\widetilde{u}}}}\,(\widehat{\widehat{\widetilde{u}}} - \widehat{\widetilde{\widetilde{u}}}) \;,\end{aligned} \tag{5.19}$$

while using equation (5.17c) we derive

$$\begin{aligned}(\widetilde{w} + \widehat{\widetilde{\widetilde{v}}}) - (\widehat{w} + \widehat{\widehat{\widetilde{v}}}) = {} & \frac{a^3 - b^3}{a - b + \widehat{\widetilde{u}} - \widetilde{\widetilde{u}}} - \frac{a^3 - b^3}{a - b + \widehat{\widehat{u}} - \widehat{\widetilde{u}}} \\ & + (a + b + \widehat{u})(a + b - \widehat{\widehat{\widetilde{u}}}) - (a + b + \widetilde{u})(a + b - \widehat{\widetilde{\widetilde{u}}}) \;.\end{aligned} \tag{5.20}$$

From the two equations (5.19) and (5.20) the proposition follows. □

As discussed in chapter 1, the existence of Lax pairs that produce lattice equations is guaranteed by the multi-dimensional consistency property. The construction of the Lax pair that produces the lattice BSQ equation (5.16) can be performed in the same way as in section 1.3.2 of the introduction. The Lax system for the discrete equation (5.16) is given by a 3×3 matrix Lax system [87, 123],

$$\widetilde{\Phi}(k) = \mathcal{L}(k)\,\Phi(k) \;, \qquad \widehat{\Phi}(k) = \mathcal{M}(k)\,\Phi(k) \;, \tag{5.21}$$

in which

$$\mathcal{L}(k) = \begin{pmatrix} a - \widetilde{u} & 1 & 0 \\ -\widetilde{v} & a & 1 \\ k^3 + a^3 - (a^2 + a\,u + w)(a - \widetilde{u}) - \widetilde{v}\,(a + u) & -w & a + u \end{pmatrix} , \tag{5.22}$$

and $\mathcal{M}(k)$ is obtained from $\mathcal{L}(k)$ by replacing a with b and $\widetilde{\ }$ with $\widehat{\ }$. The compatibility condition of the Lax system

$$\widehat{\mathcal{L}}(k)\,\mathcal{M}(k) = \widetilde{\mathcal{M}}(k)\,\mathcal{L}(k) \;, \tag{5.23}$$

leads to the system of equations (5.17), which in turn lead to equation (5.16).

The lattice BSQ equation (5.16) can also be obtained from an action [87], namely

$$\mathcal{S} = \sum_{n,m\in\mathbb{Z}} \mathscr{L}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) \ , \tag{5.24}$$

in which

$$\mathscr{L} = \sigma\,\delta \log\left(\delta + \widehat{u} - \widetilde{u}\right) + u\left(b\,\widehat{u} - a\,\widetilde{u}\right) - (\epsilon + u)(\epsilon - \widehat{\widetilde{u}})(\delta + \widehat{u} - \widetilde{u}) \ , \tag{5.25}$$

where $\epsilon := a+b$, $\delta := a-b$ and $\sigma := a^2 + a\,b + b^2$. The discrete Euler-Lagrange equation

$$\widehat{\widetilde{\left(\frac{\partial \mathscr{L}}{\partial u}\right)}} + \widehat{\left(\frac{\partial \mathscr{L}}{\partial \widetilde{u}}\right)} + \widetilde{\left(\frac{\partial \mathscr{L}}{\partial \widehat{u}}\right)} + \left(\frac{\partial \mathscr{L}}{\partial \widehat{\widetilde{u}}}\right) = 0 \ , \tag{5.26}$$

yields the lattice BSQ equation (5.16).

As an application we now consider finite-dimensional mappings that are associated with the lattice BSQ equation (5.16) in the next section.

5.4 Staircase reductions

In chapter 2, we described how to construct integrable mappings from the lattice KdV equation when considering diagonal periodic initial value problems on the lattice. Considering initial value problems for the lattice BSQ equation that was presented in the previous section, we follow the same reduction procedure as has been studied for lattice KdV equation.

Let us now consider a staircase consisting of P alternating horizontal and vertical steps (see section 1.3.3). Taking such a standard staircase through the origin $(n, m) = (0, 0)$, we assign initial data on the lattice along a staircase of length N, as indicated in figure 5.2, as follows

$$\mathrm{a}(j, j) =: \mathbf{a}_{2j} \ , \quad \mathrm{a}(j+1, j) =: \mathbf{a}_{2j+1} \ ; \qquad \mathrm{a} := (u, v, w) \ , \tag{5.27}$$

with periodicity property $\mathbf{a}_{i+N} = \mathbf{a}_i$ where $\mathbf{a}_i = (\mathrm{u}_i, \mathrm{v}_i, \mathrm{w}_i)$. The lattice BSQ equation (5.17) describes on each elementary quadrilateral the evolution of the variables $\mathbf{a}_i$.

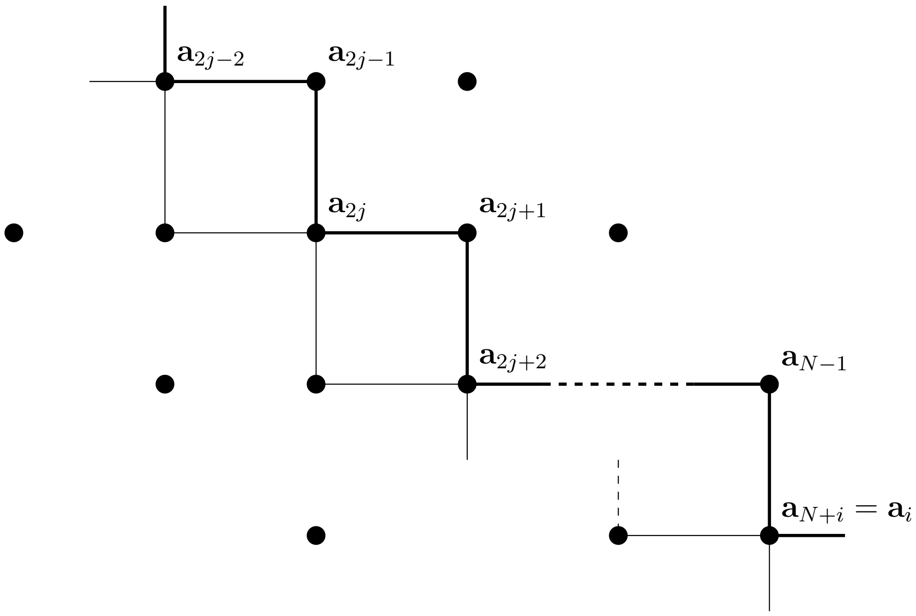


Figure 5.2: Periodic staircase on the lattice for period N.

Reductions of lattice KdV models to integrable mappings for discrete-time evolution in the vertical shift have been considered since the early nineties [24, 95]. As this thesis is concerned with discrete-time integrable systems and modified Hamiltonians, reductions of such models for discrete-time evolution in the diagonal shift was proposed in chapter 2. For lattice BSQ models, algebraic-geometric solutions have not been researched in depth, in particular mapping reductions associated with periodic initial value problems on the lattice have not been considered and mappings for diagonal evolution have never been studied to our knowledge. Thus, the construction of the mappings of BSQ type for discrete-time evolutions in the vertical shift, i.e. in the m-direction, and along the diagonal shift, i.e. close to the identity mapping, are both considered in this chapter.

5.4.1 Vertical evolution reduction

In this section, evolution in the m-direction (vertical shift) is used to construct the mapping. This leads to multi-dimensional families of mappings that turned out to be integrable. We define

$$\mathrm{a}(j, j+1) =: \widehat{\mathbf{a}}_{2j} \quad \text{and} \quad \mathrm{a}(j+1, j+1) =: \widehat{\mathbf{a}}_{2j+1}$$

like the one depicted in figure 1.11 , using the system of equations (5.17), we have evolution equations for the u_i, v_i, w_i,

$$\mathrm{v}_{2j+2} + \mathrm{w}_{2j} = \frac{\sigma\,\delta}{\delta + \widehat{\mathrm{u}}_{2j} - \mathrm{u}_{2j+1}} + \epsilon\,(\mathrm{u}_{2j+2} - \mathrm{u}_{2j}) + \mathrm{u}_{2j}\,\mathrm{u}_{2j+2} - \sigma \,, \tag{5.28a}$$

$$\widehat{\mathrm{v}}_{2j} - \mathrm{v}_{2j+1} = a\,(\mathrm{u}_{2j+2} - \widehat{\mathrm{u}}_{2j}) - b\,(\mathrm{u}_{2j+2} - \mathrm{u}_{2j+1}) + \mathrm{u}_{2j+2}\,(\widehat{\mathrm{u}}_{2j} - \mathrm{u}_{2j+1}) \,, \tag{5.28b}$$

$$\widehat{\mathrm{w}}_{2j} - \mathrm{w}_{2j+1} = b\,(\widehat{\mathrm{u}}_{2j} - \mathrm{u}_{2j}) - a\,(\mathrm{u}_{2j+1} - \mathrm{u}_{2j}) + \mathrm{u}_{2j}\,(\widehat{\mathrm{u}}_{2j} - \mathrm{u}_{2j+1}) \,, \tag{5.28c}$$

$$(\widehat{\mathrm{u}}, \widehat{\mathrm{v}}, \widehat{\mathrm{w}})_{2j+1} = (\mathrm{u}, \mathrm{v}, \mathrm{w})_{2j+2} \,. \tag{5.28d}$$

As discussed later in the next chapter, we find that all mappings of the BSQ models exhibit

$$\begin{cases} 3n-3 \quad \text{nontrivial integrals for period} \quad N = 3n-2 \,, \\ 3n-2 \quad \text{nontrivial integrals for period} \quad N = 3n-1 \;\text{ or }\; N = 3n \,, \end{cases}$$

where $n \in \mathbb{Z}^+$.

The Lax matrices of the lattice BSQ equation in (5.21) are factorized as

$$\mathcal{L}(k) := U(a)\,\Lambda(\lambda_a)\,\widetilde{U}'(a); \qquad \lambda_a \equiv k^3 + a^3 \,, \tag{5.29}$$

in which

$$U(a) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ -w & a+u & 1 \end{pmatrix} , \quad \Lambda_a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda_a & 0 & 0 \end{pmatrix} , \quad U'(a) = \begin{pmatrix} 1 & 0 & 0 \\ a-u & 1 & 0 \\ -a^2 + a\,u - v & 0 & 1 \end{pmatrix} ,$$

and a similar equation with $a \to b$ and $\widetilde{\ } \to \widehat{\ }$ for $\mathcal{M}(k)$. This factorization can be shown to lead to a reduction of variables, which will turn out to be the canonical variables of the system. In fact, considering the configuration of translations on the lattice as indicated in figure 5.3 , we can derive a discrete-time Zakharov-Shabat system from the Lax representation of the lattice. Using equation (5.29) we have

$$\begin{aligned} &\mathrm{U}_{2j+2}(b)\,\Lambda_b\,\widehat{\mathrm{U}}'_{2j+2}(b)\,\mathrm{U}_{2j+1}(b)\,\Lambda_b\,\mathrm{U}'_{2j+2}(b)\,\mathrm{U}_{2j}(a)\,\Lambda_a\,\mathrm{U}'_{2j+1}(a)\,\mathrm{U}_{2j-1}(b) \\ &= \widehat{\mathrm{U}}_{2j+1}(b)\,\Lambda_b\,\widehat{\mathrm{U}}'_{2j+2}(b)\,\widehat{\mathrm{U}}_{2j}(a)\,\Lambda_a\,\widehat{\mathrm{U}}'_{2j+1}(a)\,\mathrm{U}_{2j}(b)\,\Lambda_b\,\widehat{\mathrm{U}}'_{2j}(b)\,\mathrm{U}_{2j-1}(b) \,, \\ \Longrightarrow\; &\Lambda_b\,\widehat{\mathrm{U}}'_{2j+2}(b)\,\widehat{\mathrm{U}}_{2j}(a)\,\Lambda_a\,\widehat{\mathrm{U}}'_{2j+1}(a)\,\widehat{\mathrm{U}}_{2j-1}(b)\left(\Lambda_b\,\widehat{\mathrm{U}}'_{2j}(b)\,\mathrm{U}_{2j-1}(b)\right) \\ &= \left(\Lambda_b\,\widehat{\mathrm{U}}'_{2j+2}(b)\,\mathrm{U}_{2j+1}(b)\right)\Lambda_b\,\mathrm{U}'_{2j+2}(b)\,\mathrm{U}_{2j}(a)\,\Lambda_a\,\mathrm{U}'_{2j+1}(a)\,\mathrm{U}_{2j-1}(b) \,. \end{aligned}$$

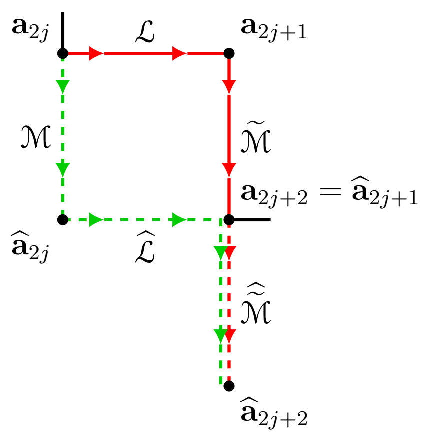


Figure 5.3: Two alternative routes through the lattice.

This relation leads to the Zakharov-Shabat system

$$\widehat{L}_j\, M_j = M_{j+1}\, L_j \ , \tag{5.30}$$

with the identifications

$$L_j = \Lambda_b\, \mathrm{U'}_{2j+2}(b)\, \mathrm{U}_{2j}(a)\, \Lambda_a\, \mathrm{U'}_{2j+1}(a)\, \mathrm{U}_{2j-1}(b) \ , \quad M_j = \Lambda_b\, \widehat{\mathrm{U}}'_{2j}(b)\, \mathrm{U}_{2j-1}(b) \ .$$

The Lax matrices L_j and M_j are given by

$$L_j(\lambda) = V_{2j}\, V_{2j-1} \ ; \quad V_j = \begin{pmatrix} a_j + a_{j+1} - \mathrm{u}_{j+2} & 1 & 0 \\ a_j\, \mathrm{u}_{j+2} - (a_j^2 + \mathrm{v}_{j+2} + \mathrm{w}_j) & a_{j+1} + \mathrm{u}_j & 1 \\ \lambda_j & 0 & 0 \end{pmatrix} , \tag{5.31a}$$

$$M_j(\lambda) = \begin{pmatrix} 2b - \widehat{\mathrm{u}}_{2j} & 1 & 0 \\ b\, \widehat{\mathrm{u}}_{2j} - (b^2 + \widehat{\mathrm{v}}_{2j} + \mathrm{w}_{2j-1}) & b + \mathrm{u}_{2j-1} & 1 \\ \lambda_{2j} & 0 & 0 \end{pmatrix} , \tag{5.31b}$$

in which $a_{2j} := b$, $a_{2j-1} := a$, $\lambda_{2j} := k^3 + b^3 = \lambda$ and $\lambda_{2j-1} := \lambda + \sigma\, \delta$. The Zakharov-Shabat condition (5.30) generates the equations of motion for the mapping (5.28) as a compatibility condition. We can further factorize the matrix V_j in (5.31a) as $V_n = \Lambda_n\, \mathrm{W}_n$ in which

$$\mathrm{W}_n = \mathrm{U'}_{n+2}(a_n)\, \mathrm{U}_n(a_{n+1}) =: \mathbb{I} + \sum_{i>j=1}^{3} v_{i,j}(n) E_{i,j} \ ,$$

where $E_{i,j}$ are the 3×3 elementary matrices defined by $(E_{i,j})_{k,l} = \delta_{i,k}\,\delta_{j,l}$. We can impose the following commutation relations between the fields $v_{i,j}$ [87],

$$\{v_{2,1}(n), v_{2,1}(m)\} = 0 \,, \qquad \{v_{3,1}(n), v_{2,1}(m)\} = \delta_{n,m+1} \,, \tag{5.32a}$$

$$\{v_{3,1}(n), v_{3,1}(m)\} = 0 \,, \qquad \{v_{3,2}(n), v_{2,1}(m)\} = 0 \,, \tag{5.32b}$$

$$\{v_{3,2}(n), v_{3,1}(m)\} = \delta_{n,m+1} \,, \qquad \{v_{3,2}(n), v_{3,2}(m)\} = 0 \,. \tag{5.32c}$$

The associated monodromy matrix, $T(\lambda)$, is obtained by gluing the elementary matrices V_n labelling the sites along a chain of length N, namely

$$T(\lambda) := \prod_{n=1}^{\curvearrowleft N} V_n(\lambda) \,. \tag{5.33}$$

The discrete-time evolution of the monodromy matrix is given by

$$\widehat{T}(\lambda) = M(\lambda)\, T(\lambda)\, M(\lambda)^{-1} \,, \tag{5.34}$$

where M is M_1, the discrete-time evolution operator at lattice site 1. As discussed in chapter 2, the trace of the monodromy matrix is invariant under the discrete mapping. Thus, equation (5.34) leads to a sufficient number of invariants which are obtained by expanding the trace in powers of the spectral parameter λ.

As we are also interested in the integrable mappings which are viewed as numerical integrators, we next consider the time-step evolution in the diagonal shift.

5.4.2 Diagonal evolution reduction

The class of discrete systems to be considered in this section is the mappings for discrete-time evolution along the diagonal direction. This leads to multi-dimensional families of integrable mappings that are close to the identity mapping. In fact, these mappings can be viewed as applications of the symplectic Euler method. By considering initial value problems for the lattice equation (5.17) on the lattice, we choose initial data on the standard staircase through the origin $(n, m) = (0, 0)$ and assign initial data on the lattice along this staircase as in (5.27). Defining

$$\mathrm{a}(j-1, j+1) =: \bar{\mathbf{a}}_{2j} \,, \qquad \mathrm{a}(j, j+1) =: \bar{\mathbf{a}}_{2j+1} \,,$$

like the one shown in figure 2.2, we therefore obtain from equation (5.17) the mapping

$$\begin{aligned}
\overline{\mathrm{v}}_{2j+1}+\overline{\mathrm{w}}_{2j-1} &= \frac{\sigma\,\delta}{\delta+\overline{\mathrm{u}}_{2j}-\mathrm{u}_{2j}}+\epsilon\,(\overline{\mathrm{u}}_{2j+1}-\overline{\mathrm{u}}_{2j-1})+\overline{\mathrm{u}}_{2j-1}\,\overline{\mathrm{u}}_{2j+1}-\sigma, && \text{(5.35a)}\\
\mathrm{v}_{2j+2}+\mathrm{w}_{2j} &= \frac{\sigma\,\delta}{\delta+\overline{\mathrm{u}}_{2j+1}-\mathrm{u}_{2j+1}}+\epsilon\,(\mathrm{u}_{2j+2}-\mathrm{u}_{2j})+\mathrm{u}_{2j}\,\mathrm{u}_{2j+2}-\sigma, && \text{(5.35b)}\\
\overline{\mathrm{v}}_{2j}-\mathrm{v}_{2j} &= a\,(\overline{\mathrm{u}}_{2j+1}-\overline{\mathrm{u}}_{2j})-b\,(\overline{\mathrm{u}}_{2j+1}-\mathrm{u}_{2j})+\overline{\mathrm{u}}_{2j+1}\,(\overline{\mathrm{u}}_{2j}-\mathrm{u}_{2j}), && \text{(5.35c)}\\
\overline{\mathrm{v}}_{2j+1}-\mathrm{v}_{2j+1} &= a\,(\mathrm{u}_{2j+2}-\overline{\mathrm{u}}_{2j+1})-b\,(\mathrm{u}_{2j+2}-\mathrm{u}_{2j+1})+\mathrm{u}_{2j+2}\,(\overline{\mathrm{u}}_{2j+1}-\mathrm{u}_{2j+1}), && \text{(5.35d)}\\
\overline{\mathrm{w}}_{2j}-\mathrm{w}_{2j} &= b\,(\overline{\mathrm{u}}_{2j}-\overline{\mathrm{u}}_{2j-1})-a\,(\mathrm{u}_{2j}-\overline{\mathrm{u}}_{2j-1})+\overline{\mathrm{u}}_{2j-1}\,(\overline{\mathrm{u}}_{2j}-\mathrm{u}_{2j}), && \text{(5.35e)}\\
\overline{\mathrm{w}}_{2j+1}-\mathrm{w}_{2j+1} &= b\,(\overline{\mathrm{u}}_{2j+1}-\mathrm{u}_{2j})-a\,(\mathrm{u}_{2j+1}-\mathrm{u}_{2j})+\mathrm{u}_{2j}\,(\overline{\mathrm{u}}_{2j+1}-\mathrm{u}_{2j+1}). && \text{(5.35f)}
\end{aligned}$$

This mapping is close to the identity mapping if δ is small (i.e. $a \to b$), however, it is not straightforward to see that since the mapping here is written as a coupled system of 3-field equations for u, v and w.

By consideration of the two paths like the one depicted in figure 2.5, we can find from the Zakharov-Shabat system for the lattice equation (5.17), a Lax system for the mapping (5.35). The Lax description of the mapping (5.35) is given by an L_j-matrix, which is the same L_j-matrix given in (5.31a), and an M'_j-matrix, which is given by

$$M'_j(\lambda) = \overline{\mathrm{U}}^{-1}_{2j-1}(b)\,\overline{\mathrm{U}}'^{-1}_{2j+1}(a)\,\Lambda_a\,\overline{\mathrm{U}}^{-1}_{2j}(a)\,\mathrm{U}_{2j}(b)\,\Lambda_b\,\overline{\mathrm{U}}'_{2j+1}(b)\,\mathrm{U}_{2j-1}(b)\ . \tag{5.36}$$

The Zakharov-Shabat equation, i.e.

$$\overline{L}_j\,M'_j = M'_{j+1}\,L_j\ , \tag{5.37}$$

leads to the mapping (5.35) as compatibility condition. The discrete-time evolution is given by

$$\overline{T}(\lambda) = M'(\lambda)\,T(\lambda)\,M'(\lambda)^{-1}\ . \tag{5.38}$$

The "local" Lax pair, which we have studied above, is not the unique Lax description for the BSQ model. In fact, for various members of the lattice GD hierarchy, it is possible to move to an alternative, *big* Lax matrix (dual Lax) formulation [87], i.e. involving matrices of the size of the number of steps over a one-period staircase. The big Lax leads to the proper reduced variables in which the mappings and their invariants are most conveniently expressed. In the following section, we consider the big Lax pair in more depth.

5.4.3 Big Lax matrix

As we shall observe in section 5.4.4, in order to obtain the final reduction of the mapping it is convenient to work with a representation in terms of a big Lax pair. To move to the big Lax matrix formulation, let us first recall the monodromy matrix $T(\lambda)$ from (5.33),

$$T_N(\lambda) = V_N(\lambda) \dots V_1(\lambda) \;, \tag{5.39}$$

where $N = 2P$ is an even period. Consider the eigenvectors θ_1 of the monodromy matrix $T(\lambda)$,

$$T(\lambda)\,\theta_1 = \eta\,\theta_1 \;. \tag{5.40}$$

The local matrices V_j from (5.39) define a sequence of vectors θ_j

$$\theta_{j+1} := V_j\,\theta_j \quad \Rightarrow \quad \theta_{N+1} = \eta\,\theta_1 \;,$$

such that for a vector $\theta_j := (\phi_j, \chi_j, \psi_j)^T$, we have

$$\begin{pmatrix} \phi_{j+1} \\ \chi_{j+1} \\ \psi_{j+1} \end{pmatrix} = \begin{pmatrix} (a_{j+1} + a_j - \mathrm{u}_{j+2})\,\phi_j + \chi_j \\ [\,a_j\,\mathrm{u}_{j+2} - (a_j^2 + \mathrm{v}_{j+2} + \mathrm{w}_j)]\,\phi_j + (a_{j+1} + \mathrm{u}_j)\,\chi_j + \psi_j \\ \lambda_j\,\phi_j \end{pmatrix} . \tag{5.41}$$

By eliminating χ_j and ψ_j from these equations we obtain set of equations for ϕ_j,

$$\begin{aligned} \phi_{j+1} = {} & (2\,a_j + a_{j+1} + \mathrm{u}_{j-1} - \mathrm{u}_{j+2})\,\phi_j + \mathrm{u}_{j+1}\,(a_{j-1} + a_j + \mathrm{u}_{j-1})\,\phi_{j-1} \\ & - (a_{j-1}^2 + a_j\,a_{j-1} + a_j^2 + a_{j-1}\,\mathrm{u}_{j-1} + a_j\,\mathrm{u}_{j-1} + \mathrm{v}_{j+1} + \mathrm{w}_{j-1})\,\phi_{j-1} \\ & + \lambda_{j-2}\,\phi_{j+2} \;, \end{aligned} \tag{5.42}$$

where $j = 3, \dots, N+2$. Defining the new vector $\Phi = (\phi_1, \dots, \phi_N)^T$, with the use of $\phi_{N+1} = \eta\,\phi_1$ we can cast equations (5.42) into the matrix form

$$\mathbf{L}(\eta)\,\Phi = \lambda\,\Phi \;. \tag{5.43}$$

Equation (5.43) is a global (big) spectral form corresponding to the local Lax matrix (5.31a), with an $N \times N$ Lax matrix given by

$$\mathbf{L}(\eta) = \Sigma_\eta^3 + \Sigma_\eta^2\,X + \Sigma_\eta\,Y + \Delta \;, \tag{5.44}$$

where the Σ_η is the $N \times N$ shift matrix defined by

$$\Sigma_\eta = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ \eta & & & & 0 \end{pmatrix} , \tag{5.45}$$

and the matrices X, Y and Δ are diagonal matrices with

$$X := \text{diag}\,(X_1, \ldots, X_N)\,,$$

$$Y := \text{diag}\,(Y_1, \ldots, Y_N)\,,$$

$$\Delta := \text{diag}\,(\underbrace{-\sigma\,\delta, 0, -\sigma\,\delta, 0, \ldots, -\sigma\,\delta, 0}_{N-\text{times}})\ .$$

By considering the $N \times N$ elementary matrices $E_{i,j}$, the big Lax matrix $\mathbf{L}(\eta)$ is more easily expressed by

$$\mathbf{L}(\eta) = \Sigma_\eta^3 + \Sigma_\eta^2 \left(\sum_{j=1}^{N} X_j\, E_{j,j} \right) + \Sigma_\eta \left(\sum_{j=1}^{N} Y_j\, E_{j,j} \right) - \sigma\,\delta \sum_{j=1}^{N/2} E_{2j-1,2j-1}\ . \tag{5.46}$$

In terms of $E_{i,j}$, the matrix Σ_η in (5.45) can be written as:

$$\Sigma_\eta := \sum_{j=1}^{N-1} E_{j,j+1} + \eta\, E_{N,1}\ ,$$

noting that it satisfies the following simple commutation relation with the diagonal matrices $E_{j,j}$ according to the simple rule

$$E_{j,j}\,\Sigma_\eta = \Sigma_\eta\, E_{j+1,j+1}\ .$$

The variables X_j, Y_j in (5.46) are chosen to be the reduced variables of the mapping reduction and given by

$$X_j := \mathrm{u}_{j+2} - \mathrm{u}_{j-1} - \epsilon - a_j\ , \tag{5.47a}$$

$$Y_{j-1} := \epsilon\,(\mathrm{u}_{j-1} - \mathrm{u}_{j+1}) - \mathrm{u}_{j-1}\,\mathrm{u}_{j+1} + \mathrm{v}_{j+1} + \mathrm{w}_{j-1} + \sigma\ , \tag{5.47b}$$

in which $j = 2, \ldots, N+1$, $a_{2j} \equiv b$ and $a_{2j-1} \equiv a$. Therefore, in terms of the reduced variables X_j and Y_j the evolution equations from (5.28) read

$$\widehat{X}_{2j-1} = X_{2j} - \frac{\sigma\delta}{Y_{2j-2}}, \qquad \widehat{Y}_{2j-1} = Y_{2j}\ , \tag{5.48a}$$

$$\widehat{X}_{2j} = X_{2j+1} + \frac{\sigma\delta}{Y_{2j+2}}, \qquad \widehat{Y}_{2j} = Y_{2j+1} + \sigma\delta\left(\frac{\widehat{X}_{2j+1}}{Y_{2j+2}} - \frac{X_{2j+1}}{Y_{2j}}\right), \tag{5.48b}$$

while the evolution equations from (5.35) read

$$\overline{X}_{2j-1} = X_{2j-1} + \frac{\sigma\delta}{Y_{2j}} - \frac{\sigma\delta}{\overline{Y}_{2j+1}}, \quad \overline{Y}_{2j-1} = Y_{2j-1} - \sigma\delta\left(\frac{X_{2j-1}}{Y_{2j+2}} - \frac{X_{2j}}{Y_{2j}} + \frac{\sigma\delta}{Y_{2j}Y_{2j+2}}\right), \tag{5.49a}$$

$$\overline{X}_{2j} = X_{2j} + \frac{\sigma\delta}{\overline{Y}_{2j+1}} - \frac{\sigma\delta}{Y_{2j+2}}, \qquad \overline{Y}_{2j} = Y_{2j} + \sigma\delta\left(\frac{\overline{X}_{2j+1}}{\overline{Y}_{2j+1}} - \frac{\overline{X}_{2j}}{\overline{Y}_{2j-1}} + \frac{\sigma\delta}{\overline{Y}_{2j-1}\overline{Y}_{2j+1}}\right). \tag{5.49b}$$

For the simplest case $P = 1$, the big Lax matrix (5.46) is actually smaller than the corresponding small Lax matrix (5.31a), and is given by the 2×2 matrix

$$\mathbf{L}(\eta) = \begin{pmatrix} \eta\, X_1 - \sigma\,\delta & \eta + Y_2 \\ \eta\,(Y_1 + \eta) & \eta\, X_2 \end{pmatrix}, \tag{5.50}$$

in which

$$X_1 := \mathrm{u}_1 - \mathrm{u}_0 - a - \epsilon\ , \qquad Y_1 := \mathrm{v}_1 + \mathrm{w}_1 - \mathrm{u}_1^2 + \sigma\ ,$$

$$X_2 := \mathrm{u}_0 - \mathrm{u}_1 - b - \epsilon\ , \qquad Y_2 := \mathrm{v}_0 + \mathrm{w}_0 - \mathrm{u}_0^2 + \sigma\ .$$

For $P = 2$, the big Lax matrix $\mathbf{L}(\eta)$ is given by the 4×4 matrix

$$\mathbf{L}(\eta) = \begin{pmatrix} -\sigma\,\delta & Y_2 & X_3 & 1 \\ \eta & 0 & Y_3 & X_4 \\ \eta\, X_1 & \eta & -\sigma\,\delta & Y_4 \\ \eta\, Y_1 & \eta\, X_2 & \eta & 0 \end{pmatrix}, \tag{5.51}$$

in which

$$
\begin{aligned}
X_1 &:= \mathrm{u}_3 - \mathrm{u}_0 - a - \epsilon\,, & Y_1 &:= \mathrm{v}_3 + \mathrm{w}_1 - \mathrm{u}_1\,\mathrm{u}_3 + \epsilon\,(\mathrm{u}_1 - \mathrm{u}_3) + \sigma\,,\\
X_2 &:= \mathrm{u}_0 - \mathrm{u}_1 - b - \epsilon\,, & Y_2 &:= \mathrm{v}_0 + \mathrm{w}_2 - \mathrm{u}_0\,\mathrm{u}_2 + \epsilon\,(\mathrm{u}_2 - \mathrm{u}_0) + \sigma\,,\\
X_3 &:= \mathrm{u}_1 - \mathrm{u}_2 - a - \epsilon\,, & Y_3 &:= \mathrm{v}_1 + \mathrm{w}_3 - \mathrm{u}_1\,\mathrm{u}_3 + \epsilon\,(\mathrm{u}_3 - \mathrm{u}_1) + \sigma\,,\\
X_4 &:= \mathrm{u}_2 - \mathrm{u}_3 - b - \epsilon\,, & Y_4 &:= \mathrm{v}_2 + \mathrm{w}_0 - \mathrm{u}_0\,\mathrm{u}_2 + \epsilon\,(\mathrm{u}_0 - \mathrm{u}_2) + \sigma\,.
\end{aligned}
$$

Having written the big Lax matrix as a cubic in Σ_η as shown in equation (5.46), it is not difficult to identify factorizations of $\mathbf{L}(\eta)$, of which there are three possible ways to factor off a linear term. Choosing

$$\mathbf{L}(\eta) = \mathbf{N}(\eta)\,\mathbf{M}(\eta)\,, \tag{5.52}$$

with

$$\mathbf{M}(\eta) = \Sigma_\eta - \sigma\,\delta \sum_{j=1}^{N/2} \frac{1}{Y_{2j}}\,E_{2j-1,2j-1}\,, \tag{5.53a}$$

$$\mathbf{N}(\eta) = \Sigma_\eta^2 + \Sigma_\eta \left(\sum_{j=1}^{N} X_{j+1}\,E_{j,j} \right) + \sum_{j=1}^{N} Y_{j+1}\,E_{j,j} + \sigma\,\delta \sum_{j=1}^{N/2} \left(\frac{1}{Y_{2j+2}}\,E_{2j-1,2j} + \frac{X_{2j+2}}{Y_{2j+2}}\,E_{2j,2j} \right), \tag{5.53b}$$

the mapping (5.48) then arises from a conjugation with the matrix $\mathbf{M}(\eta)$,

$$\widehat{\mathbf{L}}(\eta)\,\mathbf{M}(\eta) = \mathbf{M}(\eta)\,\mathbf{L}(\eta) \quad \Rightarrow \quad \widehat{\mathbf{L}}(\eta) = \mathbf{M}(\eta)\,\mathbf{N}(\eta)\,. \tag{5.54}$$

The time evolution of the Lax matrix (5.54) guarantees the invariance of the spectral curve

$$\det\left(\mathbf{L}(\eta) - \lambda \mathbb{I}\right) = 0\,. \tag{5.55}$$

We note that using the identity

$$\det{}_{N\times N}(\mathbf{L}(\eta) - \lambda\mathbb{I}) = \det{}_{3\times 3}\left(T(\lambda) - \eta\mathbb{I}\right)\,, \tag{5.56}$$

we can find invariants, from the characteristic polynomial of the big Lax matrix (5.46), which are the same as those yielded by the monodromy matrix $T(\lambda)$ from (5.33) [87].

Furthermore, these invariants are in involution with respect to the Poisson brackets [87]:

$$\{Y_i, Y_j\} = \delta_{j,i+1} X_j - \delta_{i,j+1} X_i \,, \tag{5.57a}$$

$$\{X_i, X_j\} = 0 \,, \tag{5.57b}$$

$$\{Y_i, X_j\} = \delta_{j,i+2} - \delta_{i,j+1} \,, \tag{5.57c}$$

in which $\delta_{i,j} = 1$ when $i = j$ (mod $2P$) and $\delta_{i,j} = 0$ otherwise.

5.4.4 Periods 1 and 2 reduction

To illustrate the general theory above we here consider the periods of one and two reduction, which respectively give rise to integrable systems of one and three degrees-of-freedom.

Period 1

Consider the $P = 1$ ($N = 2$) reduction for discrete-time evolution along the "vertical" direction. Equation (5.28) yields the evolution equations for $\mathrm{u}_0, \mathrm{v}_0, \mathrm{w}_0, \mathrm{u}_1, \mathrm{v}_1, \mathrm{w}_1$,

$$\mathrm{v}_0 + \mathrm{w}_0 = \frac{\sigma\,\delta}{\delta + \widehat{\mathrm{u}}_0 - \mathrm{u}_1} + \mathrm{u}_0^2 - \sigma \,, \qquad (\widehat{\mathrm{u}}, \widehat{\mathrm{v}}, \widehat{\mathrm{w}})_1 = (\mathrm{u}, \mathrm{v}, \mathrm{w})_0 \,, \tag{5.58a}$$

$$\widehat{\mathrm{v}}_0 - \mathrm{v}_1 = a\,(\mathrm{u}_0 - \widehat{\mathrm{u}}_0) - b\,(\mathrm{u}_0 - \mathrm{u}_1) + \mathrm{u}_0\,(\widehat{\mathrm{u}}_0 - \mathrm{u}_1) \,, \tag{5.58b}$$

$$\widehat{\mathrm{w}}_0 - \mathrm{w}_1 = b\,(\widehat{\mathrm{u}}_0 - \mathrm{u}_0) - a\,(\mathrm{u}_1 - \mathrm{u}_0) + \mathrm{u}_0\,(\widehat{\mathrm{u}}_0 - \mathrm{u}_1) \,. \tag{5.58c}$$

This mapping in terms of the reduced variables X_1, X_2, Y_1, Y_2 reads

$$\widehat{X}_1 = X_2 - \frac{\sigma\,\delta}{Y_2} \,, \qquad \widehat{Y}_1 = Y_2 \,, \tag{5.59a}$$

$$\widehat{X}_2 = X_1 + \frac{\sigma\,\delta}{Y_2} \,, \qquad \widehat{Y}_2 = Y_1 + \frac{\sigma\,\delta}{Y_2}\left(\widehat{X}_1 - X_1\right) . \tag{5.59b}$$

In fact, it is easily verified that this mapping also arises from the Zakharov-Shabat equation (5.54) as compatibility condition, where the corresponding matrices $\mathbf{L}$ and $\mathbf{M}$ are in this case given by

$$\mathbf{L}(\eta) = \begin{pmatrix} \eta\,X_1 - \sigma\,\delta & \eta + Y_2 \\ \eta\,(Y_1 + \eta) & \eta\,X_2 \end{pmatrix} , \qquad \mathbf{M}(\eta) = \begin{pmatrix} -\sigma\,\delta/Y_2 & 1 \\ \eta & 0 \end{pmatrix} .$$

The canonical nature of the mapping (5.59) is given by the following:

Proposition 5.4.1. *The mapping (5.59) is symplectic with respect to the Poisson bracket structure (5.57), i.e.*

$$\{Y_1, Y_2\} = X_2 - X_1 \,, \qquad \{Y_1, X_1\} = 1 \,, \qquad \{Y_1, X_2\} = -1 \,, \tag{5.60a}$$

$$\{Y_2, X_1\} = -1 \,, \qquad \{Y_2, X_2\} = 1 \,, \qquad \{X_1, X_2\} = 0 \,. \tag{5.60b}$$

In other words, the Poisson brackets (5.60) are preserved by the map (5.59).

Proof

To prove this statement, one needs to verify the following Poisson brackets

$$\{\widehat{Y}_1, \widehat{Y}_2\} = \widehat{X}_2 - \widehat{X}_1 \,, \qquad \{\widehat{Y}_1, \widehat{X}_1\} = 1 \,, \qquad \{\widehat{Y}_1, \widehat{X}_2\} = -1 \,,$$

$$\{\widehat{Y}_2, \widehat{X}_1\} = -1 \,, \qquad \{\widehat{Y}_2, \widehat{X}_2\} = 1 \,, \qquad \{\widehat{X}_1, \widehat{X}_2\} = 0 \,,$$

which are in fact can be shown by direct calculations. By considering the Poisson bracket structure (5.60), it is consequently not hard to derive the following relations

$$\{Y_2, \frac{1}{Y_2}\} = 0 \,, \qquad \{Y_1, \frac{1}{Y_2}\} = \frac{1}{Y_2^2}\,(X_1 - X_2) \,, \qquad \{\frac{1}{Y_2}, X_1\} = \frac{1}{Y_2^2} \,, \tag{5.61a}$$

$$\{X_2, \frac{1}{Y_2}\} = \frac{1}{Y_2^2} \,, \qquad \{X_1, \frac{1}{Y_1}\} = \frac{1}{Y_1^2} \,, \qquad \{\frac{1}{Y_1}, X_2\} = \frac{1}{Y_1^2} \,. \tag{5.61b}$$

Now, using equations (5.60) in addition to (5.61), we have

$$\begin{aligned}
\{\widehat{Y}_1, \widehat{Y}_2\} &= \{Y_2, Y_1 + \frac{\sigma\,\delta}{Y_2}\,(\widehat{X}_1 - X_1)\} = \{Y_2, Y_1\} + \sigma\,\delta\{Y_2, \frac{1}{Y_2}\widehat{X}_1\} - \sigma\,\delta\{Y_2, \frac{1}{Y_2}X_1\} \\
&= \{Y_2, Y_1\} + \{Y_2, X_2\}\frac{\sigma\,\delta}{Y_2} - \{Y_2, X_1\}\frac{\sigma\,\delta}{Y_2} + \sigma\,\delta\{Y_2, \frac{1}{Y_2}\}\left(X_2 - X_1 - \frac{2\,\sigma\,\delta}{Y_2}\right) \\
&= X_1 - X_2 + \frac{2\,\sigma\,\delta}{Y_2} = \widehat{X}_2 - \widehat{X}_1 \,, \\
\{\widehat{Y}_1, \widehat{X}_1\} &= \{Y_2, X_2 - \frac{\sigma\,\delta}{Y_2}\} = \{Y_2, X_2\} - \sigma\,\delta\{Y_2, \frac{1}{Y_2}\} = \{Y_2, X_2\} = 1 \,, \\
\{\widehat{Y}_1, \widehat{X}_2\} &= \{Y_2, X_1 + \frac{\sigma\,\delta}{Y_2}\} = \{Y_2, X_1\} + \sigma\,\delta\{Y_2, \frac{1}{Y_2}\} = \{Y_2, X_1\} = -1 \,,
\end{aligned}$$

$$\begin{aligned}
\{\widehat{Y}_2, \widehat{X}_1\} &= \{Y_1, X_2\} - \sigma\,\delta\{Y_1, \frac{1}{Y_2}\} - \sigma^2\,\delta^2\{\frac{1}{Y_2}\widehat{X}_1, \frac{1}{Y_2}\} + \sigma\,\delta\{\frac{1}{Y_2}\widehat{X}_1, X_2\} \\
&\quad - \sigma\,\delta\{\frac{1}{Y_2}X_1, X_2\} + \sigma^2\,\delta^2\{\frac{1}{Y_2}X_1, \frac{1}{Y_2}\} \\
&= \{Y_1, X_2\} - \sigma\,\delta\{Y_1, \frac{1}{Y_2}\} + \frac{\sigma\,\delta}{Y_2}\left(\{X_2, X_2\} - \{X_1, X_2\}\right) + \sigma^2\,\delta^2\{X_1, \frac{1}{Y_2}\} \\
&\quad - \sigma^2\,\delta^2\{\frac{1}{Y_2}, \frac{1}{Y_2}\}\left(X_2 - X_1 - \frac{2\,\sigma\,\delta}{Y_2}\right) - \sigma\,\delta\{X_2, \frac{1}{Y_2}\}\left(X_2 - X_1 - \frac{\sigma\,\delta}{Y_2}\right) \\
&= -1 - \frac{\sigma\,\delta}{Y_2^2}\,(X_1 - X_2) - \frac{\sigma^2\,\delta^2}{Y_2^3} - \frac{\sigma\,\delta}{Y_2^2}\left(X_2 - X_1 - \frac{\sigma\,\delta}{Y_2}\right) = -1\,, \\
\{\widehat{Y}_2, \widehat{X}_2\} &= \{Y_1, X_1\} + \sigma\,\delta\{Y_1, \frac{1}{Y_2}\} + \sigma^2\,\delta^2\{\frac{\widehat{X}_1}{Y_2}, \frac{1}{Y_2}\} + \sigma\,\delta\{\frac{\widehat{X}_1}{Y_2}, X_1\} \\
&\quad - \sigma\,\delta\{\frac{X_1}{Y_2}, X_1\} - \sigma^2\,\delta^2\{\frac{X_1}{Y_2}, \frac{1}{Y_2}\} \\
&= 1 + \frac{\sigma\,\delta}{Y_2^2}\,(X_1 - X_2) + \frac{\sigma^2\,\delta^2}{Y_2^3} + \frac{\sigma\,\delta}{Y_2^2}\left(X_2 - X_1 - \frac{\sigma\,\delta}{Y_2}\right) = 1\,.
\end{aligned}$$

Hence, the proposition follows. □

As discussed in chapter 6, the invariants for the BSQ mappings are obtained by expanding the traces of both matrices $T(\lambda)$ and $T^2(\lambda)$ in powers of the spectral parameter λ. In the case under consideration, the trace of the monodromy matrix $T(\lambda)$ is a Casimir with respect to the Poisson structure (5.60). Thus, the invariant for the map (5.59) is obtained by expanding the trace of the corresponding monodromy matrix $T^2(\lambda)$ in powers of the spectral parameter λ, or equivalently by $\det\left(\mathbf{L}_{2\times 2}(\eta) - \lambda\mathbb{I}\right)$. Therefore, the mapping (5.59) possesses an exact invariant

$$\mathcal{I} = Y_1\,Y_2 - \sigma\,\delta\,X_1\,. \tag{5.62}$$

We note that the quantities

$$X_1 + X_2 = -3\,\epsilon \quad \text{and} \quad Y_1 + Y_2 - X_1\,X_2 = C \tag{5.63}$$

are Casimirs with respect to the Poisson brackets (5.60). Thus, by eliminating X_2 and Y_2 from the map (5.59), using these Casimirs, we arrive at the following 2-dimensional rational mapping

$$\widehat{X} = \frac{\sigma\,\delta}{X\,(X + 3\,\epsilon) + Y - C} - X - 3\,\epsilon\,, \qquad \widehat{Y} = C - Y - X\,(X + 3\,\epsilon)\,, \tag{5.64}$$

carrying the following invariant

$$\mathcal{I}(X,Y) = Y^2 + X\,Y\,(X + 3\,\epsilon) + \sigma\,\delta\,X - C\,Y\,. \tag{5.65}$$

A continuous-time interpolating flow to the mapping (5.64) is generated by (5.65) together with the Poisson bracket $\{Y, X\} = 1$.

By deriving the "diagonal" evolution equations from equation (5.35) for $P = 1$ similarly to the standard treatment of the vertical evolution equations, we find a close to the identity integrable mapping in terms of X_1, X_2, Y_1, Y_2,

$$\overline{X}_1 = X_1 + \frac{\sigma\,\delta}{Y_2} - \frac{\sigma\,\delta}{\overline{Y}_1}\,, \qquad \overline{Y}_1 = Y_1 - \frac{\sigma\,\delta}{Y_2}\left(X_1 - X_2 + \frac{\sigma\,\delta}{Y_2}\right)\,, \tag{5.66a}$$

$$\overline{X}_2 = X_2 + \frac{\sigma\,\delta}{\overline{Y}_1} - \frac{\sigma\,\delta}{Y_2}\,, \qquad \overline{Y}_2 = Y_2 + \frac{\sigma\,\delta}{\overline{Y}_1}\left(\overline{X}_1 - X_2 + \frac{\sigma\,\delta}{Y_2}\right). \tag{5.66b}$$

This mapping also preserves the structure (5.60) and conserves the quantity (5.62). Again, using the Casimirs (5.63) we eliminate X_2 and Y_2 from this map to produce a 2-dimensional area-preserving rational mapping in terms of X and Y,

$$\overline{X} = X - \frac{\sigma\,\delta}{X\,(X + 3\,\epsilon) + Y - C} - \frac{\sigma\,\delta}{\overline{Y}}\,, \tag{5.67a}$$

$$\overline{Y} = Y + \frac{\sigma\,\delta}{X\,(X + 3\,\epsilon) + Y - C}\left(2\,X - \frac{\sigma\,\delta}{X\,(X + 3\,\epsilon) + Y - C} + 3\,\epsilon\right), \tag{5.67b}$$

which has the invariant (5.65).

It should be noted that in the KdV model the first non-trivial mapping occurs for period $P = 2$, leading to 2-dimensional rational mappings which actually correspond to elliptic curves. One may conjecture that under a certain transformation these mappings can be obtained from the mappings of BSQ type for period $P = 1$, which also correspond to elliptic curves. It is not currently clear how the KdV mappings for this particular case ($P = 2$) relate to the mappings under consideration, but this approach could prove a fruitful avenue of research in the future.

Period 2

Considering the period $P = 2$ ($N = 4$) reduction equation (5.28), for the "vertical" evolution, yields the evolution equations for $\mathrm{u}_0, \mathrm{v}_0, \mathrm{w}_0, \mathrm{u}_1, \mathrm{v}_1, \mathrm{w}_1, \mathrm{u}_2, \mathrm{v}_2, \mathrm{w}_2, \mathrm{u}_3, \mathrm{v}_3, \mathrm{w}_3$,

$$\mathrm{v}_2 + \mathrm{w}_0 = \frac{\sigma\,\delta}{\delta + \widehat{\mathrm{u}}_0 - \mathrm{u}_1} + \epsilon\,(\mathrm{u}_2 - \mathrm{u}_0) + \mathrm{u}_0\,\mathrm{u}_2 - \sigma\,, \tag{5.68a}$$

$$\mathrm{v}_0 + \mathrm{w}_2 = \frac{\sigma\,\delta}{\delta + \widehat{\mathrm{u}}_2 - \mathrm{u}_3} + \epsilon\,(\mathrm{u}_0 - \mathrm{u}_2) + \mathrm{u}_0\,\mathrm{u}_2 - \sigma\,, \tag{5.68b}$$

$$\widehat{\mathrm{v}}_0 - \mathrm{v}_1 = a\,(\mathrm{u}_2 - \widehat{\mathrm{u}}_0) - b\,(\mathrm{u}_2 - \mathrm{u}_1) + \mathrm{u}_2\,(\widehat{\mathrm{u}}_0 - \mathrm{u}_1)\,, \tag{5.68c}$$

$$\widehat{\mathrm{w}}_0 - \mathrm{w}_1 = b\,(\widehat{\mathrm{u}}_0 - \mathrm{u}_0) - a\,(\mathrm{u}_1 - \mathrm{u}_0) + \mathrm{u}_0\,(\widehat{\mathrm{u}}_0 - \mathrm{u}_1)\,, \tag{5.68d}$$

$$\widehat{\mathrm{v}}_2 - \mathrm{v}_3 = a\,(\mathrm{u}_0 - \widehat{\mathrm{u}}_2) - b\,(\mathrm{u}_0 - \mathrm{u}_3) + \mathrm{u}_0\,(\widehat{\mathrm{u}}_2 - \mathrm{u}_3)\,, \tag{5.68e}$$

$$\widehat{\mathrm{w}}_2 - \mathrm{w}_3 = b\,(\widehat{\mathrm{u}}_2 - \mathrm{u}_2) - a\,(\mathrm{u}_3 - \mathrm{u}_2) + \mathrm{u}_2\,(\widehat{\mathrm{u}}_2 - \mathrm{u}_3)\,, \tag{5.68f}$$

$$(\widehat{\mathrm{u}}, \widehat{\mathrm{v}}, \widehat{\mathrm{w}})_1 = (\mathrm{u}, \mathrm{v}, \mathrm{w})_2\,, \qquad (\widehat{\mathrm{u}}, \widehat{\mathrm{v}}, \widehat{\mathrm{w}})_3 = (\mathrm{u}, \mathrm{v}, \mathrm{w})_0\,, \tag{5.68g}$$

which read in terms of eight reduced variables $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4$, namely

$$\widehat{X}_1 = X_2 - \frac{\sigma\,\delta}{Y_4}\,, \qquad \widehat{Y}_1 = Y_2\,, \tag{5.69a}$$

$$\widehat{X}_2 = X_3 + \frac{\sigma\,\delta}{Y_4}\,, \qquad \widehat{Y}_2 = Y_3 + \sigma\,\delta\left(\frac{\widehat{X}_3}{Y_4} - \frac{X_3}{Y_2}\right), \tag{5.69b}$$

$$\widehat{X}_3 = X_4 - \frac{\sigma\,\delta}{Y_2}\,, \qquad \widehat{Y}_3 = Y_4\,, \tag{5.69c}$$

$$\widehat{X}_4 = X_1 + \frac{\sigma\,\delta}{Y_2}\,, \qquad \widehat{Y}_4 = Y_1 + \sigma\,\delta\left(\frac{\widehat{X}_1}{Y_2} - \frac{X_1}{Y_4}\right). \tag{5.69d}$$

Again, using the Lax matrices $\mathbf{L}_{4\times4}(\eta)$ and $\mathbf{M}_{4\times4}(\eta)$ relevant to the case $P = 2$, the Zakharov-Shabat equation (5.54) leads to the mapping (5.69) as compatibility condition.

Proposition 5.4.2. *The mapping (5.69) is symplectic with respect to the following Poisson bracket structure*

$$\{Y_i, Y_j\} = \delta_{j,i+1}X_j - \delta_{i,j+1}X_i\,, \tag{5.70a}$$

$$\{X_i, X_j\} = 0\,, \tag{5.70b}$$

$$\{Y_i, X_j\} = \delta_{j,i+2} - \delta_{i,j+1}\,, \tag{5.70c}$$

where $i, j = 1, \ldots, 4$.

The proof of this proposition is, once again, by direct computation in exactly the same manner as that of proposition 5.4.1. The integrals of mapping (5.69) are calculated from the $\operatorname{tr} T(\lambda)$ and $\operatorname{tr} T^2(\lambda)$ (or equivalently from $\det(\mathbf{L}_{4\times 4}(\eta) - \lambda \mathbb{I})$) as:

$$\mathcal{I}_1 = X_1\,X_2\,X_3\,X_4 - X_1\,X_2\,Y_3 - X_1\,X_4\,Y_2 - X_2\,X_3\,Y_4 - X_3\,X_4\,Y_1 + Y_1\,Y_3 + Y_2\,Y_4 - \sigma\,\delta\,(X_2 + X_4)\,, \tag{5.71a}$$

$$\mathcal{I}_2 = Y_1\,Y_2\,Y_3\,Y_4 + \sigma\,\delta\,(Y_1\,Y_2\,X_4 + Y_3\,Y_4\,X_2) + \sigma^2\,\delta^2\,X_2\,X_4\,, \tag{5.71b}$$

$$\mathcal{I}_3 = X_1\,Y_2\,Y_3 + X_2\,Y_3\,Y_4 + X_3\,Y_1\,Y_4 + X_4\,Y_1\,Y_2 - \sigma\,\delta\,(X_1\,X_3 - X_2\,X_4)\,. \tag{5.71c}$$

With respect to the Poisson structure (5.70) the three invariants in (5.71) are in involution,

$$\{\mathcal{I}_1, \mathcal{I}_2\} = \{\mathcal{I}_1, \mathcal{I}_3\} = \{\mathcal{I}_2, \mathcal{I}_3\} = 0\,.$$

So the map (5.69) satisfies the standard criteria for an integrable map: it has sufficient invariants in involution. Note that the following quantities

$$X_1 + X_2 + X_3 + X_4 = -6\,\epsilon \quad \text{and} \quad X_1\,X_3 + X_2\,X_4 + Y_1 + Y_2 + Y_3 + Y_4 = C \tag{5.72}$$

are Casimirs with respect to the Poisson brackets (5.70). As before, using these Casimirs we eliminate X_4 and Y_4 from the resulting mapping (5.69) to produce a 6-dimensional rational mapping in $X_1, X_2, X_3, Y_1, Y_2, Y_3$,

$$\widehat{X}_1 = X_2 - \frac{\sigma\,\delta}{Y_4}\,, \qquad \widehat{Y}_1 = Y_2\,, \tag{5.73a}$$

$$\widehat{X}_2 = X_3 + \frac{\sigma\,\delta}{Y_4}\,, \qquad \widehat{Y}_2 = Y_3 + \sigma\,\delta\left(\frac{X_4}{Y_4} - \frac{X_3}{Y_2} - \frac{\sigma\,\delta}{Y_2\,Y_4}\right), \tag{5.73b}$$

$$\widehat{X}_3 = X_4 - \frac{\sigma\,\delta}{Y_2}\,, \qquad \widehat{Y}_3 = Y_4\,, \tag{5.73c}$$

in which we identify

$$X_4 := -\,(X_1 + X_2 + X_3 + 6\,\epsilon)\,, \tag{5.74a}$$

$$Y_4 := (X_1 + X_2 + X_3 + 6\,\epsilon)\,X_2 - X_1\,X_3 - Y_1 - Y_2 - Y_3 + C\,. \tag{5.74b}$$

By considering the identifications (5.74), this mapping has the invariants (5.71).

In the case of "diagonal" evolution, the corresponding mapping is derived from the corresponding evolution equations as:

$$\overline{X}_1 = X_1 + \frac{\sigma\,\delta}{Y_2} - \frac{\sigma\,\delta}{\overline{Y}_3}\,, \qquad \overline{Y}_1 = Y_1 + \sigma\,\delta\left(\frac{X_2}{Y_2} - \frac{X_1}{Y_4} - \frac{\sigma\,\delta}{Y_2\,Y_4}\right), \tag{5.75a}$$

$$\overline{X}_2 = X_2 + \frac{\sigma\,\delta}{\overline{Y}_3} - \frac{\sigma\,\delta}{Y_4}\,, \qquad \overline{Y}_2 = Y_2 + \sigma\,\delta\left(\frac{\overline{X}_3}{\overline{Y}_3} - \frac{\overline{X}_2}{\overline{Y}_1} + \frac{\sigma\,\delta}{\overline{Y}_1\,\overline{Y}_3}\right), \tag{5.75b}$$

$$\overline{X}_3 = X_3 + \frac{\sigma\,\delta}{Y_4} - \frac{\sigma\,\delta}{\overline{Y}_1}\,, \qquad \overline{Y}_3 = Y_3 + \sigma\,\delta\left(\frac{X_4}{Y_4} - \frac{X_3}{Y_2} - \frac{\sigma\,\delta}{Y_2\,Y_4}\right), \tag{5.75c}$$

$$\overline{X}_4 = X_4 + \frac{\sigma\,\delta}{\overline{Y}_1} - \frac{\sigma\,\delta}{Y_2}\,, \qquad \overline{Y}_4 = Y_4 + \sigma\,\delta\left(\frac{\overline{X}_1}{\overline{Y}_1} - \frac{\overline{X}_4}{\overline{Y}_3} + \frac{\sigma\,\delta}{\overline{Y}_1\,\overline{Y}_3}\right). \tag{5.75d}$$

This mapping is also symplectic with respect to the Poisson bracket structure (5.70). The functions (5.71) are also conserved by the map (5.75). Once again, we use the Casimirs (5.72) to eliminate X_4 and Y_4 from the map (5.75) in order to arrive at 6-dimensional rational mapping, namely

$$\overline{X}_1 = X_1 + \frac{\sigma\,\delta}{Y_2} - \frac{\sigma\,\delta}{\overline{Y}_3}\,, \qquad \overline{Y}_1 = Y_1 + \sigma\,\delta\left(\frac{X_2}{Y_2} - \frac{X_1}{Y_4} - \frac{\sigma\,\delta}{Y_2\,Y_4}\right), \tag{5.76a}$$

$$\overline{X}_2 = X_2 + \frac{\sigma\,\delta}{\overline{Y}_3} - \frac{\sigma\,\delta}{Y_4}\,, \qquad \overline{Y}_2 = Y_2 + \sigma\,\delta\left(\frac{\overline{X}_3}{\overline{Y}_3} - \frac{\overline{X}_2}{\overline{Y}_1} + \frac{\sigma\,\delta}{\overline{Y}_1\,\overline{Y}_3}\right), \tag{5.76b}$$

$$\overline{X}_3 = X_3 + \frac{\sigma\,\delta}{Y_4} - \frac{\sigma\,\delta}{\overline{Y}_1}\,, \qquad \overline{Y}_3 = Y_3 + \sigma\,\delta\left(\frac{X_4}{Y_4} - \frac{X_3}{Y_2} - \frac{\sigma\,\delta}{Y_2\,Y_4}\right), \tag{5.76c}$$

taking into account the identified equations (5.74), which possesses the invariants (5.71).

We note that in this thesis we will not pursue the establishing of Hamiltonian structure for the mappings above as has been done for the KdV mappings, but we defer consideration of this point to a future study.

5.5 Classical r-matrix and Yang-Baxter structures

The integrable mappings (5.28) and (5.35) have sufficiently many independent invariants for integrability, guaranteed by the existence of the monodromy matrix (5.33). However, these invariants must be in involution in the sense of the definition given in section 1.3.4 of the introduction. As we have spoken of the involutivity property of the invariants, one can speak of an r, s-matrix structure for the Lax pair. Using equations (5.31) and (5.32), the fundamental Poisson structure for the Lax matrices L_n is given by [87]:

$$\begin{aligned}\{\overset{1}{L}_n(\lambda_1), \overset{2}{L}_m(\lambda_2)\} = \delta_{n,m+1}\, \overset{1}{L}_n(\lambda_1)\, s^+_{12}\, \overset{2}{L}_m(\lambda_2) - \delta_{n+1,m}\, \overset{2}{L}_m(\lambda_2)\, s^-_{12}\, \overset{1}{L}_n(\lambda_1) \\ - \delta_{n,m}\, [\, r^+_{12}\, \overset{1}{L}_n(\lambda_1)\, \overset{2}{L}_m(\lambda_2) - \overset{1}{L}_n(\lambda_1)\, \overset{2}{L}_m(\lambda_2)\, r^-_{12}\,]\,,\end{aligned} \tag{5.77}$$

in which

$$\overset{1}{L}_n := L_n(\lambda_1) \otimes \mathbb{I} \quad \text{and} \quad \overset{2}{L}_n := \mathbb{I} \otimes L_n(\lambda_2)\,,$$

and where λ_1 and λ_2 are distinct spectral parameters. This equation forms a so-called non-ultralocal Poisson structure [40]. The matrices $r^{\pm}_{12}$ and $s^{\pm}_{12}$ in (5.77) are given by

$$r^-_{12} = \frac{1}{\lambda_1 - \lambda_2} \sum_{i,j=1}^{3} E_{i,j} \otimes E_{j,i}\,, \qquad s^-_{12} = \frac{1}{\lambda_1} \sum_{i=1}^{2} E_{i,3} \otimes E_{3,i}\,, \tag{5.78a}$$

$$s^+_{12} = \frac{1}{\lambda_2} \sum_{i=1}^{2} E_{3,i} \otimes E_{i,3}\,, \qquad r^+_{12} = r^-_{12} + s^+_{12} - s^-_{12}\,. \tag{5.78b}$$

In fact, the r, s-matrix structure for mappings arising from a lattice version of the hierarchy of GD equations are deduced in [87]. They are given in the form

$$r^-_{12} = \frac{1}{\lambda_1 - \lambda_2} \sum_{i,j=1}^{\mathcal{N}} E_{i,j} \otimes E_{j,i}\,, \qquad s^-_{12} = \frac{1}{\lambda_1} \sum_{i=1}^{\mathcal{N}-1} E_{i,\mathcal{N}} \otimes E_{\mathcal{N},i}\,, \tag{5.79a}$$

$$s^+_{12} = \frac{1}{\lambda_2} \sum_{i=1}^{\mathcal{N}-1} E_{\mathcal{N},i} \otimes E_{i,\mathcal{N}}\,, \qquad r^+_{12} = r^-_{12} + s^+_{12} - s^-_{12}\,. \tag{5.79b}$$

We note that the r, s-matrix structures for the KdV mappings, i.e. (4.15), and for the BSQ mappings, i.e. (5.78), are simply special cases of this general result.

The requirements on the Poisson bracket (5.77) for the Jacobi identity and skew-symmetry yield conditions on $r^{\pm}_{12}$ and $s^{\pm}_{12}$ in (5.78) that must hold, most notably the usual classical Yang-Baxter equation

$$\left[r^{\pm}_{12}, r^{\pm}_{13}\right] + \left[r^{\pm}_{12}, r^{\pm}_{23}\right] + \left[r^{\pm}_{13}, r^{\pm}_{23}\right] = 0\,, \tag{5.80}$$

together with

$$\left[s^{\pm}_{12}, s^{\pm}_{13}\right] + \left[s^{\pm}_{12}, r^{\pm}_{23}\right] + \left[s^{\pm}_{13}, r^{\pm}_{23}\right] = 0\,. \tag{5.81}$$

In general, we have the following statement:

Proposition 5.5.1. *The classical r, s-matrix structure for the mappings of Gel'fand-Dikii hierarchy type, i.e. equations (5.79), form a new solution of the non-ultralocal version of the classical Yang-Baxter equation which reads as in equations (5.80) and (5.81).*

Proof

Let us start by introducing the permutation matrix $\mathscr{P}$ and a matrix $\mathscr{S}$, respectively

$$\mathscr{P} := \sum_{i,j=1}^{\mathcal{N}} E_{i,j} \otimes E_{j,i} \text{ and } \mathscr{S} := \sum_{i=1}^{\mathcal{N}-1} E_{i,\mathcal{N}} \otimes E_{\mathcal{N},i} \;\Rightarrow\; \mathscr{S}^T := \sum_{i=1}^{\mathcal{N}-1} E_{\mathcal{N},i} \otimes E_{i,\mathcal{N}} \,. \tag{5.82}$$

We then have

$$\mathscr{P}_{12}\,\mathscr{P}_{13} = \mathscr{P}_{23}\,\mathscr{P}_{12} = \mathscr{P}_{13}\,\mathscr{P}_{23} := \mathscr{P}_{123}\,, \tag{5.83a}$$

$$\mathscr{P}_{13}\,\mathscr{P}_{12} = \mathscr{P}_{12}\,\mathscr{P}_{23} = \mathscr{P}_{23}\,\mathscr{P}_{13} := \mathscr{P}_{321}\,, \tag{5.83b}$$

$$\mathscr{S}_{12}\,\mathscr{P}_{23} = \mathscr{P}_{23}\,\mathscr{S}_{13}\,, \quad \mathscr{S}_{13}\,\mathscr{P}_{23} = \mathscr{P}_{23}\,\mathscr{S}_{12}\,, \tag{5.83c}$$

$$\mathscr{S}^T_{12}\,\mathscr{P}_{23} = \mathscr{P}_{23}\,\mathscr{S}^T_{13}\,, \quad \mathscr{S}^T_{13}\,\mathscr{P}_{23} = \mathscr{P}_{23}\,\mathscr{S}^T_{12}\,. \tag{5.83d}$$

It follows as a direct consequence of equations (5.83c) and (5.83d) that

$$\mathscr{S}_{12}\,\mathscr{S}_{13} = \mathscr{S}_{13}\,\mathscr{S}_{12} \quad \text{and} \quad \mathscr{S}^T_{12}\,\mathscr{S}^T_{13} = \mathscr{S}^T_{13}\,\mathscr{S}^T_{12}\,. \tag{5.83e}$$

By setting $\lambda_1 - \lambda_2 := \lambda_{12}$, $\lambda_1 - \lambda_3 := \lambda_{13}$ and $\lambda_2 - \lambda_3 := \lambda_{23}$, from the left-hand side of equation (5.80) we have

$$\begin{aligned}
&[r^-_{12}, r^-_{13}] + [r^-_{12}, r^-_{23}] + [r^-_{13}, r^-_{23}] \\
&= \frac{1}{\lambda_{12}\,\lambda_{13}}[\mathscr{P}_{12}, \mathscr{P}_{13}] + \frac{1}{\lambda_{12}\,\lambda_{23}}[\mathscr{P}_{12}, \mathscr{P}_{23}] + \frac{1}{\lambda_{13}\,\lambda_{23}}[\mathscr{P}_{13}, \mathscr{P}_{23}] \\
&= \frac{1}{\lambda_{12}\,\lambda_{13}}(\mathscr{P}_{123} - \mathscr{P}_{321}) + \frac{1}{\lambda_{12}\,\lambda_{23}}(\mathscr{P}_{321} - \mathscr{P}_{123}) + \frac{1}{\lambda_{13}\,\lambda_{23}}(\mathscr{P}_{123} - \mathscr{P}_{321}) \\
&= \frac{1}{\lambda_{13}}\left(\frac{1}{\lambda_{12}} + \frac{1}{\lambda_{23}}\right)(\mathscr{P}_{123} - \mathscr{P}_{321}) + \frac{1}{\lambda_{12}\,\lambda_{23}}(\mathscr{P}_{321} - \mathscr{P}_{123}) \\
&= \frac{1}{\lambda_{12}\,\lambda_{23}}(\mathscr{P}_{123} - \mathscr{P}_{321}) + \frac{1}{\lambda_{12}\,\lambda_{23}}(\mathscr{P}_{321} - \mathscr{P}_{123}) \\
&= 0\,.
\end{aligned}$$

Now, from the left-hand side of equation (5.81) and using equations (5.83c) and (5.83e) we have

$$\begin{aligned}
&\left[s^{-}_{12}, s^{-}_{13}\right] + \left[s^{-}_{12}, r^{-}_{23}\right] + \left[s^{-}_{13}, r^{-}_{23}\right] \\
&= \frac{1}{\lambda_1^2}\left[\mathscr{S}_{12}, \mathscr{S}_{13}\right] + \frac{1}{\lambda_1\,\lambda_{23}}\left[\mathscr{S}_{12}, \mathscr{P}_{23}\right] + \frac{1}{\lambda_1\,\lambda_{23}}\left[\mathscr{S}_{13}, \mathscr{P}_{23}\right] \\
&= \frac{1}{\lambda_1^2}\left(\mathscr{S}_{12}\,\mathscr{S}_{13} - \mathscr{S}_{13}\,\mathscr{S}_{12}\right) + \frac{1}{\lambda_1\lambda_{23}}\left(\mathscr{S}_{12}\,\mathscr{P}_{23} - \mathscr{P}_{23}\,\mathscr{S}_{12} + \mathscr{S}_{13}\,\mathscr{P}_{23} - \mathscr{P}_{23}\,\mathscr{S}_{13}\right) \\
&= 0\,.
\end{aligned}$$

In a similar way to the latter proof, using equations (5.83d) and (5.83e) we can show that

$$\left[s^{+}_{12}, s^{+}_{13}\right] + \left[s^{+}_{12}, r^{+}_{23}\right] + \left[s^{+}_{13}, r^{+}_{23}\right] = 0\,.$$

By considering the relation $r^{+} = r^{-} + s^{+} - s^{-}$ and using the information provided above, we are able to show that equation (5.80) must hold for r^{+}. □

A consequence of the relation (5.77) is that the Poisson bracket between the monodromy matrices $T(\lambda)$ is given by

$$\{\overset{1}{T}, \overset{2}{T}\} = \overset{1}{T}\, s^{+}_{12}\, \overset{2}{T} - \overset{2}{T}\, s^{-}_{12}\, \overset{1}{T} - r^{+}_{12}\, \overset{1}{T}\,\overset{2}{T} + \overset{1}{T}\,\overset{2}{T}\, r^{-}_{12}\,. \tag{5.84}$$

We note that the proof of equation (5.84) proceeds in exactly the same way as the proof of equation (4.19). Having obtained equation (5.84), taking the trace over both components of the tensor product and using (5.78), the right-hand side of (5.84) vanishes identically, and we find

$$\{\mathrm{tr}\, T(\lambda),\ \mathrm{tr}\, T(\lambda')\} = 0\,, \tag{5.85}$$

implying that the integrals of the mapping are in involution. Similarly as in section 4.3, it is clearly sufficient to prove that the Poisson bracket between the trace of the monodromy matrix for different values of λ and λ' say, vanishes.

5.6 Summary

We have developed a novel Bäcklund transformation for the BSQ model that leads to the famous 9-point lattice BSQ equation. We gave a detailed description of deriving the discrete BSQ from the Bäcklund transformation. By considering the lattice BSQ equation we applied periodic initial value problems along a staircase in the lattice, which led to finite-dimensional integrable mappings. Evolutions in the vertical and diagonal shifts were taken to correspond to the mappings as discrete-time evolutions. This family of reductions have a local Lax pair inherited from the lattice equation.

Considering an alternative approach, we examined the big Lax matrix structure for the mappings of BSQ type in order to obtain the final reduction of the mappings and introduce proper reduced variables in which the mappings and invariants are conveniently expressed. Moreover, we examined the cases of periods one and two reduction for both evolutions we considered as examples, deriving the mappings and their invariants.

Finally, we discussed the r-matrix structure belonging to the local Lax pair of the lattice GD hierarchy, which is used to prove the involutivity of the invariants and the preservation of the Poisson bracket structure under the mapping, where it was shown that it obeyed the relation of the classical Yang-Baxter equation.

Chapter 6

The Boussinesq model: Monodromy matrix and Dubrovin equations

6.1 Overview

As we have seen in chapter 4, studying modified Hamiltonians for higher-degrees-of-freedom systems arising from the reduction of lattice KdV model requires some mathematical structures, such as separation of variables, finite-gap integration and Dubrovin equations. These have never been studied for lattice Boussinesq (BSQ) model, which have needed to be set up in order to study the modified Hamiltonians for higher-degrees-of-freedom. Thus, in this chapter we set up the finite-gap integration schemes and construct separation variables for the BSQ system that we set up in the previous chapter. The separated coordinates appear as the poles of the properly normalized eigenvector (Baker-Akhiezer function) of the corresponding monodromy matrix. Two different ways of defining the auxiliary spectrum are analyzed in this chapter. Continuous Dubrovin equations were presented for the BSQ hierarchy in [33, 34] and for the modified BSQ hierarchy by Geng in [45]. In this chapter, we use a different approach to obtain a more general form of the Dubrovin equations for mappings of the BSQ model, where we use the characteristic behaviour of the monodromy matrix and structures of separation of variables and r-matrix analysis.

The outline of this chapter is as follows. The analytic behaviour of the monodromy matrix is investigated in section 6.2. Based on the characteristic polynomial of the monodromy matrix and with the use of the Weierstrass gap sequence technique [39], we introduce a general form of the trigonal curve that depends on the length of chain on the staircase in the lattice. Using two different methods we employ the technique of separation of variables to the mappings of BSQ type in section 6.3. The canonicity of the separated variables is also verified with the use of the r-matrix technique. In section 6.4, by introducing an interpolating flow for the map, specifically by considering the invariants of the monodromy matrix, we derive a novel expression for the continuous Dubrovin equations. Finally, section 6.5 contains a discussion of the discrete Dubrovin equations.

6.2 Monodromy matrix and invariants structures

As the monodromy matrix plays an important role in the context of what follows in this chapter, we demonstrate the behaviour of the monodromy matrix as a polynomial function of the spectral parameter λ in next. Hence, we derive a general form of the monodromy matrix associated with genus of the spectral curve. We also propose general forms of the invariants for the discrete-time evolution given by the mappings, which are inspired by the novel monodromy matrix structure.

Let us first write the monodromy matrix (5.33) in the following form:

$$T(\lambda) = \prod_{n=1}^{\overset{\frown}{N}} V_n(\lambda) := \begin{pmatrix} A(\lambda) & B(\lambda) & C(\lambda) \\ D(\lambda) & E(\lambda) & F(\lambda) \\ G(\lambda) & H(\lambda) & J(\lambda) \end{pmatrix} . \tag{6.1}$$

Essentially, in the context of the mappings of BSQ-type, the discrete-time evolution implies the invariance under the map of a spectral curve

$$\det\left(T(\lambda) - \eta\mathbb{I}\right) = -\eta^3 + \mathrm{tr}\, T(\lambda)\, \eta^2 - S(\lambda)\, \eta + \det T(\lambda) = 0\,, \tag{6.2}$$

where $S(\lambda)$ is the sum of the principal minors of size two, i.e.

$$S(\lambda) = E\,J - F\,H + A\,J - C\,G + A\,E - B\,D\,.$$

Using the Weierstrass gap sequence technique [39], we find that the spectral curve (6.2) defines a trigonal curve [39] of genus

$$g = \begin{cases} 3n-3 & \text{if} \quad N = 3n-2\,, \\ 3n-2 & \text{if} \quad N = 3n-1 \ \text{ or } \ N = 3n\,, \end{cases} \tag{6.3}$$

in which N is the length of chain on the staircase, and where $n \in \mathbb{Z}^+$. For convenience, a detailed description of the Weierstrass gap technique is applied to specific cases of period reduction in appendix B.

Now, for genus g the matrix $T(\lambda)$ from (6.1) takes one of three distinct possible forms depending on whether the length N of the chain is $3n-2$, $3n-1$ or $3n$. Writing the monodromy matrix (6.1) as a power series in λ, namely

$$T = \begin{pmatrix} \lambda^{k_l} A_{k_l} + \cdots + A_0 & \lambda^{k_l-1} B_{k_l-1} + \cdots + B_0 & \lambda^{k_l-1} C_{k_l-1} + \cdots + C_0 \\ \lambda^{k_l} D_{k_l} + \cdots + D_0 & \lambda^{k_l} E_{k_l} + \cdots + E_0 & \lambda^{k_l-1} F_{k_l-1} + \cdots + F_0 \\ \lambda \left(\lambda^{k_l-1} G_{k_l-1} + \cdots + G_0\right) & \lambda \left(\lambda^{k_l-1} H_{k_l-1} + \cdots + H_0\right) & \lambda \left(\lambda^{k_l-1} J_{k_l-1} + \cdots + J_0\right) \end{pmatrix}, \tag{6.4}$$

the three cases are indicated by an integer l (taking values $l = 2$ or $l = 3$) such that the leading power of λ depends on l by

$$k_l := \frac{1}{3}(g + l) .$$

Thus, the three cases associated with genus g are characterized by the following values for the top coefficients:

$$\text{Case } N = 3n-2 : \quad \left\{ \begin{array}{c} A_{k_l} = E_{k_l} = J_{k_l-1} = 0 \\ C_{k_l-1} = D_{k_l} = H_{k_l-1} = 0 \\ B_{k_l-1} = F_{k_l-1} = G_{k_l-1} = 1 \\ l = 3 \end{array} \right\}, \tag{6.5a}$$

$$\text{Case } N = 3n-1 : \quad \left\{ \begin{array}{c} A_{k_l} = E_{k_l} = J_{k_l-1} = 0 \\ C_{k_l-1} = D_{k_l} = H_{k_l-1} = 1 \\ l = 2 \end{array} \right\}, \tag{6.5b}$$

$$\text{Case } N = 3n : \quad \left\{ \begin{array}{c} A_{k_l} = E_{k_l} = J_{k_l-1} = 1 \\ l = 2 \end{array} \right\}. \tag{6.5c}$$

The distinction of the three cases is related to the variance in the choice of periodic initial data in the two-dimensional lattice described by the lattice BSQ equation, see section 5.4, where we can impose periodicity on chains (staircases) with periods $3n-2$, $3n-1$ or $3n$. The period $3n-2$ corresponds to the curve of genus $g = 3n-3$ whereas the periods $3n-1$ and $3n$ correspond to the curve of genus $g = 3n-2$.

As a consequence of either discrete-time evolution (5.34) or (5.38), both traces $\mathrm{tr}\, T(\lambda)$ and $\mathrm{tr}\, T^2(\lambda)$ of the monodromy matrix (6.1) are invariant under the BSQ mappings. Hence, using both traces we can generate either $3n-3$, or $3n-2$ independent invariants, depending on whether the length N of the chain is, respectively, $3n-2$, or $3n-1$ or $3n$, by developing both traces in powers of the spectral parameter λ.

The trace of the monodromy matrix $T(\lambda)$ is written as

$$\mathrm{tr}\, T(\lambda) = \begin{cases} I_0 + \sum_{j=1}^{k_l-1} I_j\, \lambda^j & \text{if} \quad N = 3n-2\,, \\ I_0 + \sum_{j=1}^{k_l-1} I_j\, \lambda^j & \text{if} \quad N = 3n-1\,, \\ I_0 + \sum_{j=1}^{k_l-1} I_j\, \lambda^j + 3\,\lambda^{k_l} & \text{if} \quad N = 3n\,, \end{cases} \tag{6.6}$$

where $I_j = A_j + E_j + J_{j-1}$. The top coefficients I_j in (6.6) are given by

$$I_{k_l-1} = A_{k_l-1} + E_{k_l-1} + J_{k_l-2} \quad \text{for all } \; N\,, \tag{6.7}$$

while the lowest coefficients are given by

$$I_0 = A_0 + E_0\,. \tag{6.8}$$

The trace of the monodromy matrix $T^2(\lambda)$ is written as

$$\mathrm{tr}\, T^2(\lambda) = \begin{cases} I'_0 + \sum_{j=1}^{2k_l-2} I'_j\, \lambda^j & \text{if} \quad N = 3n-2\,, \\ I'_0 + \sum_{j=1}^{2k_l-1} I'_j\, \lambda^j & \text{if} \quad N = 3n-1\,, \\ I'_0 + \sum_{j=1}^{2k_l-1} I'_j\, \lambda^j + 3\,\lambda^{2k_l} & \text{if} \quad N = 3n\,. \end{cases} \tag{6.9}$$

where $I'_j = A_j^2 + E_j^2 + J_{j-1}^2 + 2\,(B_{j-1}\, D_j + C_{j-1}\, G_{j-1} + F_{j-1}\, H_{j-1})$. The top coefficients I'_j in (6.9) are given by

$$I'_{2k_l-2} = A_{k_l-1}^2 + E_{k_l-1}^2 + J_{k_l-2}^2 + 2\,(D_{k_l-1} + C_{k_l-2} + H_{k_l-2}) \text{ if } N = 3n-2\,, \tag{6.10a}$$

$$I'_{2k_l-1} = 2\,(B_{k_l-1} + G_{k_l-1} + F_{k_l-1}) \qquad \text{if } N = 3n-1\,, \tag{6.10b}$$

$$\begin{aligned} I'_{2k_l-1} &= 2\,(A_{k_l-1} + E_{k_l-1} + J_{k_l-2}) \\ &\quad + 2\,(B_{k_l-1} D_{k_l} + C_{k_l-1} G_{k_l-1} + F_{k_l-1} H_{k_l-1}) \qquad \text{if } N = 3n\,, \end{aligned} \tag{6.10c}$$

while the lowest coefficients are given by

$$I'_0 = A_0^2 + 2\, B_0 D_0 + E_0^2 \,. \tag{6.11}$$

The coefficients $\{I_0, \ldots, I_{k_l-2}; I'_0, \ldots, I'_{2k_l-3}\}$ of the traces (6.6) and (6.9) are the invariants of the map when $N = 3n-2$, whereas the coefficients $\{I_0, \ldots, I_{k_l-2}; I'_0, \ldots, I'_{2k_l-2}\}$ are the invariants of the map when $N = 3n-1$ or $N = 3n$. We note that the top coefficients in equations (6.7) and (6.10) are Casimirs with respect to the natural Poisson brackets associated with the dynamical maps. That these are indeed Casimirs can be verified by using the r-matrix structure, in particular by using equation (5.84).

The determinant of the monodromy matrix $T(\lambda)$ takes one of four distinct possible forms depending on the length N. These forms are given by

(i) Cases $N = 3n-2$ and $N = 3n-1$:

$$\det T(\lambda) = \begin{cases} \lambda^{\frac{1}{2}(g+1)}\,(\sigma\,\delta + \lambda)^{\frac{1}{2}(g+1)} & \text{if} \quad N = \text{even}\,, \\ \lambda^{\frac{1}{2}(g+2)}\,(\sigma\,\delta + \lambda)^{\frac{1}{2}g} & \text{if} \quad N = \text{odd}\,. \end{cases} \tag{6.12a}$$

(ii) Case $N = 3n$:

$$\det T(\lambda) = \begin{cases} \lambda^{\frac{1}{2}(g+2)}\,(\sigma\,\delta + \lambda)^{\frac{1}{2}(g+2)} & \text{if} \quad N = \text{even}\,, \\ \lambda^{\frac{1}{2}(g+3)}\,(\sigma\,\delta + \lambda)^{\frac{1}{2}(g+1)} & \text{if} \quad N = \text{odd}\,. \end{cases} \tag{6.12b}$$

For convenience we explicitly investigate the traces of $T(\lambda)$ and $T^2(\lambda)$ and the genus of spectral curve (6.2) for periods $N = 2$ to $N = 12$. The following table 6.1 describes the traces and the number of genera g for each period.

Periods	Trace (T) and Trace (T^2)	Genus
$N=2$	$\mathrm{tr}\, T = I_0\ ,\quad \mathrm{tr}\, T^2 = 6\,\epsilon\,\lambda + I_0'$	1
$N=3$	$\mathrm{tr}\, T = 3\,\lambda + I_0\ ,\quad \mathrm{tr}\, T^2 = 3\,\lambda^2 + \lambda\, I_1' + I_0'$	1
$N=4$	$\mathrm{tr}\, T = 6\,\epsilon\,\lambda + I_0\ ,\quad \mathrm{tr}\, T^2 = \lambda^2\, I_2' + \lambda\, I_1' + I_0'$	3
$N=5$	$\mathrm{tr}\, T = \lambda\, I_1 + I_0\ ,\quad \mathrm{tr}\, T^2 = \lambda^3\, I_3' + \lambda^2\, I_2' + \lambda\, I_1' + I_0'$	4
$N=6$	$\mathrm{tr}\, T = 3\,\lambda^2 + \lambda\, I_1 + I_0$ $\mathrm{tr}\, T^2 = 3\,\lambda^4 + \lambda^3\, I_3' + \lambda^2\, I_2' + \lambda\, I_1' + I_0'$	4
$N=7$	$\mathrm{tr}\, T = \lambda^2\, I_2 + \lambda\, I_1 + I_0$ $\mathrm{tr}\, T^2 = \lambda^4\, I_4' + \lambda^3\, I_3' + \lambda^2\, I_2' + \lambda\, I_1' + I_0'$	6
$N=8$	$\mathrm{tr}\, T = \lambda^2\, I_2 + \lambda\, I_1 + I_0$ $\mathrm{tr}\, T^2 = 24\,\epsilon\,\lambda^5 + \lambda^4\, I_4' + \lambda^3\, I_3' + \lambda^2\, I_2' + \lambda\, I_1' + I_0'$	7
$N=9$	$\mathrm{tr}\, T = 3\,\lambda^3 + \lambda^2\, I_2 + \lambda\, I_1 + I_0$ $\mathrm{tr}\, T^2 = 3\,\lambda^6 + \lambda^5\, I_5' + \lambda^4\, I_4' + \lambda^3\, I_3' + \lambda^2\, I_2' + \lambda\, I_1' + I_0'$	7
$N=10$	$\mathrm{tr}\, T = 15\,\epsilon\,\lambda^3 + \lambda^2\, I_2 + \lambda\, I_1 + I_0$ $\mathrm{tr}\, T^2 = \lambda^6\, I_6' + \lambda^5\, I_5' + \lambda^4\, I_4' + \lambda^3\, I_3' + \lambda^2\, I_2' + \lambda\, I_1' + I_0'$	9
$N=11$	$\mathrm{tr}_1 = \lambda^3\, I_3 + \lambda^2\, I_2 + \lambda\, I_1 + I_0$ $\mathrm{tr}_2 = \lambda^7\, I_7' + \lambda^6\, I_6' + \lambda^5\, I_5' + \lambda^4\, I_4' + \lambda^3\, I_3' + \lambda^2\, I_2' + \lambda\, I_1' + I_0'$	10
$N=12$	$\mathrm{tr}_1 = 3\,\lambda^4 + \lambda^3\, I_3 + \lambda^2\, I_2 + \lambda\, I_1 + I_0$ $\mathrm{tr}_2 = 3\,\lambda^8 + \lambda^7\, I_7' + \lambda^6\, I_6' + \lambda^5\, I_5' + \lambda^4\, I_4' + \lambda^3\, I_3' + \lambda^2\, I_2' + \lambda\, I_1' + I_0'$	10

Table 6.1: Brief description for periods $N = 2, \ldots, 12$ of BSQ model.

Having set up the general structure for the monodromy matrix as polynomial function of the spectral parameter and invariants, our attention turns to the technique of separation of variables for integrable mappings of BSQ type.

6.3 Separation of variables

The separation of variables method for integrable mappings of KdV type was discussed in section 4.2. Following the same prescription, we apply this technique to finite-dimensional integrable mappings of BSQ model in this section. By taking a slightly different approach, we also present a new machinery for defining the auxiliary spectrum. Moreover, the canonicity of the separated variables is verified with the use of Poisson brackets between the entries of the monodromy matrix and the r-matrix technique.

6.3.1 Normalization and spectral curve

The separation variables μ_j are obtained as the poles of the associated Baker-Akhiezer function. The Baker-Akhiezer function is the eigenvector

$$T(\lambda)\,\psi(\lambda) = \eta(\lambda)\,\psi(\lambda)\;, \tag{6.13}$$

of the monodromy matrix $T(\lambda)$ corresponding to the eigenvalue $\eta(\lambda)$ of the spectral curve

$$\det\left(T(\lambda) - \eta\mathbb{I}\right) = 0\;,$$

provided that a normalization vector $\vec{\alpha}$ of the eigenvector $\psi(\lambda)$ is fixed

$$\vec{\alpha}\,\psi = 1\;. \tag{6.14}$$

In our case, the proper normalization vector is

$$\vec{\alpha} = (1, 0, 0)\;. \tag{6.15}$$

From the linear problem (6.13) and normalization equation (6.14) we obtain

$$\vec{\alpha}\,T\,\psi = \eta \quad \text{and} \quad \vec{\alpha}\,T^2\,\psi = \eta^2\;. \tag{6.16}$$

Thus, we obtain

$$\psi = \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\,T(\lambda) \\ \vec{\alpha}\,T^2(\lambda) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \eta \\ \eta^2 \end{pmatrix}. \tag{6.17}$$

We now introduce the polynomial $\mathcal{F}$ by

$$\mathcal{F}(\lambda) := \det \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\,T(\lambda) \\ \vec{\alpha}\,T^2(\lambda) \end{pmatrix} = \vec{\alpha}\cdot\left(\vec{\alpha}\,T \times \vec{\alpha}\,T^2\right), \tag{6.18}$$

where the $(\times)$ denotes the cross product. This polynomial has to vanish on $\lambda = \mu_j$, i.e. $\mathcal{F}(\mu_j) = 0$, since the roots μ_j are the poles of the Baker-Akhiezer function $\psi(\lambda)$ given in equation (6.17). In terms of the entries of the monodromy matrix (6.1), the polynomial in equation (6.18) is written as

$$\mathcal{F}(\lambda) = B^2 F - B\,C\,(E - J) - C^2 H\,. \tag{6.19}$$

The polynomial $\mathcal{F}(\lambda)$ in (6.18) defines the *auxiliary spectrum for periodic problems of equations of BSQ type*. The auxiliary spectrum $\mathcal{F}(\lambda)$ has the following factorization

$$\mathcal{F}(\lambda) = \mathcal{F}_g \prod_{j=1}^{g} (\lambda - \mu_j)\,, \tag{6.20}$$

in which the $\mathcal{F}_g$ takes one of the following three distinct possible forms:

$$\mathcal{F}_g = \begin{cases} 1 & \text{if} \quad N = 3n - 2\,, \\ -1 & \text{if} \quad N = 3n - 1\,, \\ \mathcal{C} & \text{if} \quad N = 3n\,, \end{cases} \tag{6.21}$$

where $\mathcal{C}$ is a Casimir.

There is a significant relation between the derivative Q and the discriminant R of the spectral curve (6.2). In fact, this result holds for any cubic equation, as expanded upon in the following statement:

Lemma 6.3.1. *Given a cubic equation*

$$\eta^3 + a_1\,\eta^2 + a_2\,\eta + a_3 = 0 \quad \textit{where} \quad a_1, a_2, a_3 \in \mathbb{R}, \tag{6.22}$$

and define $\mathrm{p} := a_2 - a_1^2/3$ *. The discriminant* R *of equation (6.22) is given by*

$$R = \frac{1}{108}\, Q^2\,(Q + 3\,\mathrm{p}) \tag{6.23}$$

if and only if Q *is the derivative of equation (6.22), i.e.*

$$Q = 3\,\eta^2 + 2\,a_1\,\eta + a_2\,. \tag{6.24}$$

Proof

To prove this statement, let us first write the cubic equation (6.22) as an incomplete cubic form (Cardano formula):

$$\widetilde{\eta}^3 + \mathrm{p}\,\widetilde{\eta} + \mathrm{q} = 0\,, \tag{6.25}$$

in which

$$\widetilde{\eta} := \eta + \frac{1}{3}\,a_1\,, \quad \mathrm{p} = a_2 - \frac{1}{3}\,a_1^2 \quad \text{and} \quad \mathrm{q} := \frac{2}{27}\,a_1^3 - \frac{1}{3}\,a_1\,a_2 + a_3\,. \tag{6.26}$$

The roots $\widetilde{\eta}$ are then obtained by

$$\widetilde{\eta} = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} \sqrt[3]{\frac{-\mathrm{q}}{2} + \sqrt{R}} + \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} \sqrt[3]{\frac{-\mathrm{q}}{2} - \sqrt{R}}\,, \tag{6.27}$$

in which

$$R := \left(\frac{\mathrm{p}}{3}\right)^3 + \left(\frac{\mathrm{q}}{2}\right)^2, \tag{6.28}$$

and where the parameter ω is the root of unity given by $\omega \equiv \exp{(2\pi i/3)}$.

Let us now start by showing that Q is the expression given in (6.24). Consider the relation of the discriminant R, i.e. equation (6.23). By solving this equation for Q, one obtains

$$Q = -\mathrm{p} + \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} \Gamma_+ + \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} \Gamma_-\,, \tag{6.29}$$

in which

$$\Gamma_+ := \sqrt[3]{54\,R - \mathrm{p}^3 + \sqrt{108\,R\,(27\,R - \mathrm{p}^3)}}\,,$$
$$\Gamma_- := \sqrt[3]{54\,R - \mathrm{p}^3 - \sqrt{108\,R\,(27\,R - \mathrm{p}^3)}}\,.$$

Using equation (6.28), we can rewrite equation (6.29) in the following form

$$Q = -\mathrm{p} + 3 \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} \sqrt[3]{\left(\frac{\mathrm{q}}{2} + \sqrt{R}\right)^2} + 3 \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} \sqrt[3]{\left(\frac{\mathrm{q}}{2} - \sqrt{R}\right)^2}\,. \tag{6.30}$$

In fact, the right-hand side of this equation, using equation (6.27), can be written as

$$3\,\tilde{\eta}^2 + \mathrm{p}\,,$$

and hence we have

$$Q = 3\,\tilde{\eta}^2 + \mathrm{p}\,. \tag{6.31}$$

From this, and using equation (6.26), one obtains

$$Q = 3\,\eta^2 + 2\,a_1\,\eta + a_2\,.$$

For the other direction, let $Q = 3\,\tilde{\eta}^2 + \mathrm{p}$, we then have

$$\begin{aligned} Q^2\,(Q + 3\,\mathrm{p}) &= \left(3\,\tilde{\eta}^2 + \mathrm{p}\right)^2\left(3\,\tilde{\eta}^2 + 4\,\mathrm{p}\right) = 27\,\tilde{\eta}^6 + 54\,\mathrm{p}\,\tilde{\eta}^4 + 27\,\mathrm{p}^2\,\tilde{\eta}^2 + 4\,\mathrm{p}^3 \\ &= 27\,(\mathrm{p}\,\tilde{\eta} + \mathrm{q})^2 - 54\,\mathrm{p}\,\tilde{\eta}\,(\mathrm{p}\,\tilde{\eta} + \mathrm{q}) + 27\,\mathrm{p}^2\,\tilde{\eta}^2 + 4\,\mathrm{p}^3 \\ &= 27\,\mathrm{q}^2 + 4\,\mathrm{p}^3 = 108\left[\left(\frac{\mathrm{p}}{3}\right)^3 + \left(\frac{\mathrm{q}}{2}\right)^2\right] \\ &= 108\,R\,. \end{aligned}$$

Hence, the lemma follows. □

Upon lemma 6.3.1, for the spectral curve (6.2) we have $a_1 = -\mathrm{tr}\,T$, $a_2 = S$ and $a_3 = -\det T$. Thus, we can write the spectral curve (6.2) in the Cardano form:

$$\tilde{\eta}^3 + \mathrm{p}\,\tilde{\eta} + \mathrm{q} = 0\,, \tag{6.32}$$

in which

$$\tilde{\eta} := \eta - \frac{1}{3}\,\mathrm{tr}\,T\,, \quad \mathrm{p} := S - \frac{1}{3}\,(\mathrm{tr}\,T)^2 \quad \text{and} \quad \mathrm{q} := \frac{1}{3}\,S\,\mathrm{tr}\,T - \frac{2}{27}\,(\mathrm{tr}\,T)^3 - \det T\,. \tag{6.33}$$

The eigenvalues $\tilde{\eta}$ are then obtained by

$$\tilde{\eta} = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} \sqrt[3]{\frac{-\mathrm{q}}{2} + \sqrt{R}} + \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} \sqrt[3]{\frac{-\mathrm{q}}{2} - \sqrt{R}}\,, \tag{6.34}$$

in which

$$R := \left(\frac{\mathrm{p}}{3}\right)^3 + \left(\frac{\mathrm{q}}{2}\right)^2. \tag{6.35}$$

The top coefficients of the discriminant (6.35) for genus g have grading in terms of the spectral parameter λ:

$$R(\lambda) = \begin{cases} \frac{1}{4}\,\lambda^{2(g+1)} + \cdots & \text{if} \quad N = 3n-2 \;\text{ or }\; N = 3n-1\,, \\ \frac{1}{4}\,\lambda^{2(g+1)} C^2_{k_l-1} D^2_{k_l} H^2_{k_l-1} + \cdots & \text{if} \quad N = 3n\,. \end{cases} \tag{6.36}$$

The corresponding expression for Q is given by

$$Q = 3\,\eta^2 - 2\,\mathrm{tr}\,T\,\eta + S\,. \tag{6.37}$$

At $\lambda = \mu$ the quantity Q in equation (6.37) reads in terms of the labelling of the entries of the monodromy matrix as

$$Q := \frac{CH}{B}\,(2A - E - J) + (A-J)(E-J) - BD - CG + 2FH\,. \tag{6.38}$$

We will now seek in the next section to investigate the characteristic polynomial of the monodromy matrix (6.2) and explore a novel relation like (6.18), which also defines the auxiliary spectrum for periodic problems.

6.3.2 Auxiliary spectrum and characteristic polynomial

Recalling the discriminant R of the spectral curve (6.2), in terms of the roots η_1, η_2 and η_3 of the characteristic polynomial $\det\,(T - \eta\mathbb{I}) = 0$, we can write the R as follows

$$R = \frac{-1}{108}\,(\eta_1 - \eta_2)^2\,(\eta_2 - \eta_3)^2\,(\eta_1 - \eta_3)^2\,. \tag{6.39}$$

We note that $\mathrm{tr}\,T$, S (i.e. the sum of the principal 2×2 minors of T) and $\det T$ can also be written in terms of the roots η_j, namely

$$\mathrm{tr}\,T = \eta_1 + \eta_2 + \eta_3\,, \tag{6.40a}$$

$$S = \eta_1\,\eta_2 + \eta_2\,\eta_3 + \eta_1\,\eta_3\,, \tag{6.40b}$$

$$\det T = \eta_1\,\eta_2\,\eta_2\,. \tag{6.40c}$$

In fact, there is a fundamental relation given by

$$-T^3 + \mathrm{tr}\,(T)\,T^2 - S\,T + \det(T)\,\mathbb{I} = 0\,, \tag{6.41}$$

which is guaranteed by the Cayley-Hamilton theorem [27]. In other words, if η_1, η_2 and η_3 are the roots of the characteristic polynomial $\det(T - \eta\mathbb{I}) = 0$, then T obeys the matrix identity

$$(T - \eta_1\mathbb{I})(T - \eta_2\mathbb{I})(T - \eta_3\mathbb{I}) = 0 \,. \tag{6.42}$$

However, there is a remarkable result that is slightly stronger than Cayley and holds for any 3×3-matrix,

$$(T - \eta_2\mathbb{I})(T - \eta_3\mathbb{I}) = (T - \eta_1\mathbb{I})^{\text{cof}} \,, \tag{6.43}$$

where $(T - \eta_1\mathbb{I})^{\text{cof}}$ means the cofactor matrix of $(T - \eta_1\mathbb{I})$, that is, adj$(T - \eta_1\mathbb{I})$. The proof of equation (6.43) is given in appendix C.

As a consequence of equation (6.41), S and $\det T$ can be written in terms of $\operatorname{tr} T$, $\operatorname{tr} T^2$ and $\operatorname{tr} T^3$ as

$$S = \frac{1}{2}\left[(\operatorname{tr} T)^2 - \operatorname{tr} T^2\right] , \tag{6.44a}$$

$$\det T = \frac{1}{6}(\operatorname{tr} T)^3 - \frac{1}{2}\operatorname{tr} T \operatorname{tr} T^2 + \frac{1}{3}\operatorname{tr} T^3 \,. \tag{6.44b}$$

It can in fact be verified that the vector $(T - \eta\mathbb{I})^{\text{cof}}$ also defines the auxiliary spectrum for periodic problems of BSQ-type equations. Specifically, we have the following statement:

Proposition 6.3.1. *The constraint leading to the auxiliary spectrum in (6.18), namely* $\mathcal{F}(\lambda) = 0$ *on* $\lambda = \mu_j$ *can also be written as a vector identity:*

$$\vec{\alpha}\,(T - \eta\mathbb{I})^{cof} = 0 \,, \tag{6.45}$$

for any eigenvalue η *of the characteristic polynomial* $\det(T - \eta\mathbb{I}) = 0$.

Proof

Recall the polynomial $\mathcal{F}(\lambda)$ from equation (6.18). At the separation variable μ_j the row vectors $\vec{\alpha}$, $\vec{\alpha}\,T$ and $\vec{\alpha}\,T^2$ are linearly dependent since $\mathcal{F}(\mu_j) = 0$. This means that

$$p\,\vec{\alpha} + q\,\vec{\alpha}\,T + r\,\vec{\alpha}\,T^2 = 0 \,,$$

for some p, q and r. Thus, this implies

$$\vec{\alpha}\,T^2 = \mathrm{r}_0\,\vec{\alpha} + \mathrm{r}_1\,\vec{\alpha}\,T \,, \tag{6.46}$$

for some r_0 and r_1. Multiplying the Cayley's equation (6.41) by the normalization vector $\vec{\alpha}$, we obtain

$$\begin{aligned}\vec{\alpha}\,T^3 &= \mathrm{tr}\,T\,\vec{\alpha}\,T^2 - S\,\vec{\alpha}\,T + \det T\,\vec{\alpha}\\ &= \mathrm{tr}\,T\,(\mathrm{r}_0\,\vec{\alpha} + \mathrm{r}_1\,\vec{\alpha}\,T) - S\,\vec{\alpha}\,T + \det T\,\vec{\alpha}\\ &= (\mathrm{tr}\,T\,\mathrm{r}_0 + \det T)\,\vec{\alpha} + (\mathrm{tr}\,T\,\mathrm{r}_1 - S)\,\vec{\alpha}\,T\,. \end{aligned} \tag{6.47}$$

By multiplying equation (6.46) by T and subsequently using the acquired equation together with equations (6.46) and (6.47), we obtain

$$(\mathrm{r}_0\,\mathrm{r}_1 - \mathrm{tr}\,T\,\mathrm{r}_0 - \det T)\,\vec{\alpha} + \big(\mathrm{r}_0 + \mathrm{r}_1^2 - \mathrm{tr}\,T\,\mathrm{r}_1 + S\big)\,\vec{\alpha}\,T = 0\,. \tag{6.48}$$

Since $\vec{\alpha}$ and $\vec{\alpha}\,T$ are independent in general position, we then have

$$\mathrm{r}_0\,(\mathrm{r}_1 - \mathrm{tr}\,T) = \det T \quad \text{and} \quad \mathrm{r}_0 + \mathrm{r}_1^2 - \mathrm{tr}\,T\,\mathrm{r}_1 + S = 0\,.$$

This implies

$$\mathrm{r}_1^3 - 2\,\mathrm{tr}\,T\,\mathrm{r}_1^2 + [S + (\mathrm{tr}\,T)^2]\,\mathrm{r}_1 - S\,\mathrm{tr}\,T + \det T = 0\,.$$

Writing this equation in the Cardano form,

$$\widetilde{\mathrm{r}}_1^3 + \mathrm{p}\,\widetilde{\mathrm{r}}_1 - \mathrm{q} = 0 \quad \text{where} \quad \widetilde{\mathrm{r}}_1 := \mathrm{r}_1 - \frac{2}{3}\,\mathrm{tr}\,T\,,$$

and in which the p and q are defined in (6.33), we obtain

$$\widetilde{\mathrm{r}}_1 = -\widetilde{\eta} \quad \Rightarrow \quad \mathrm{r}_1 = \mathrm{tr}\,T - \eta\,.$$

Hence, we have $\mathrm{r}_0 = -(\det T)/\eta$. Inserting the expressions acquired for r_0 and r_1 into (6.46), we obtain

$$\eta\,\vec{\alpha}\,T^2 + \eta\,(\eta - \mathrm{tr}\,T)\,\vec{\alpha}\,T + \det T\,\vec{\alpha} = 0\,. \tag{6.49}$$

By solving the characteristic polynomial (6.2) for $\det T$ and inserting the acquired expression for $\det T$ into (6.49), we obtain

$$\vec{\alpha}\,(T - \eta)(T + 2\,\eta - \mathrm{tr}\,T) + Q\,\vec{\alpha} = 0\,, \tag{6.50}$$

where the Q is defined in the lemma 6.3.1, in equation (6.37). Finally, using equation (6.43) and by direct calculations it is easy to see that equation (6.50) is equivalent to

$$\vec{\alpha}\,(T-\eta\mathbb{I})^{\text{cof}} = 0\,. \tag{6.51}$$

□

Upon this proposition, equation (6.45) leads in terms of the labelling of the entries of the monodromy matrix (6.1) to

$$\Big((E-\eta)(J-\eta)-FH,\;\; CH-B(J-\eta),\;\; BF-C(E-\eta)\Big) = (0,0,0)\,.$$

This leads to the following two simultaneous relations

$$\eta = E-\frac{BF}{C} = J-\frac{CH}{B}\,, \tag{6.52}$$

where η is a root of the characteristic polynomial (6.2). Observe that relation (6.52) leads to the constraint coming from the polynomial (6.19) on $\lambda=\mu_j$.

Next, it is interesting to verify further the canonicity of brackets between the separation variables. To do this calculation, one needs information about the Poisson brackets between the entries of the monodromy matrix.

6.3.3 Poisson brackets and canonical structures

For the system in question, we now proceed with verifying that the Poisson brackets between the separation variables have the canonical structure. To do this, we first need to compute the Poisson brackets between the entries of the monodromy matrix.

Let us recall the monodromy matrix $T(\lambda)$ from (6.1) and the vector $\alpha=(1,0,0)$. Introduce the two vectors $\beta=(0,1,0)$ and $\gamma=(0,0,1)$, we then have

$$B(\lambda)=\alpha\,T\,\beta^T,\qquad C(\lambda)=\alpha\,T\,\gamma^T,\qquad E(\lambda)=\beta\,T\,\beta^T,$$
$$F(\lambda)=\beta\,T\,\gamma^T,\qquad H(\lambda)=\gamma\,T\,\beta^T,\qquad J(\lambda)=\gamma\,T\,\gamma^T.$$

We can now give the Poisson brackets between the entries of the monodromy matrix (6.1), which follows a procedure similar to the KdV models discussed in chapter 4. Using equations (5.78) and (5.84) we extract the following Poisson brackets:

$$\{B(\lambda_1), B(\lambda_2)\} = \{C(\lambda_1), C(\lambda_2)\} = 0\,, \tag{6.53a}$$

$$\{F(\lambda_1), F(\lambda_2)\} = \{H(\lambda_1), H(\lambda_2)\} = \{J(\lambda_1), J(\lambda_2)\} = 0\,, \tag{6.53b}$$

$$\{E(\lambda_1), E(\lambda_2)\} = \frac{\lambda_1\, F(\lambda_1)H(\lambda_2) - \lambda_2\, F(\lambda_2)H(\lambda_1)}{\lambda_1\,\lambda_2}\,, \tag{6.53c}$$

$$\{B(\lambda_1), F(\lambda_2)\} = \frac{C(\lambda_1)E(\lambda_2) - C(\lambda_2)E(\lambda_1)}{\lambda_1 - \lambda_2} + \frac{1}{\lambda_2}\, C(\lambda_1)J(\lambda_2)\,, \tag{6.53d}$$

$$\{C(\lambda_1), H(\lambda_2)\} = \frac{\lambda_1\, B(\lambda_1)J(\lambda_2) - \lambda_2\, B(\lambda_2)J(\lambda_1)}{\lambda_1\,(\lambda_1 - \lambda_2)}\,, \tag{6.53e}$$

$$\{B(\lambda_1), E(\lambda_2)\} = \frac{B(\lambda_1)E(\lambda_2) - B(\lambda_2)E(\lambda_1)}{\lambda_1 - \lambda_2} + \frac{1}{\lambda_2}\, C(\lambda_1)H(\lambda_2)\,, \tag{6.53f}$$

$$\{B(\lambda_1), J(\lambda_2)\} = \frac{\lambda_1\, C(\lambda_1)H(\lambda_2) - \lambda_2\, C(\lambda_2)H(\lambda_1)}{\lambda_1\,(\lambda_1 - \lambda_2)}\,, \tag{6.53g}$$

$$\{C(\lambda_1), E(\lambda_2)\} = \frac{B(\lambda_1)F(\lambda_2) - B(\lambda_2)F(\lambda_1)}{\lambda_1 - \lambda_2}\,, \tag{6.53h}$$

$$\{C(\lambda_1), J(\lambda_2)\} = \frac{\lambda_1\, C(\lambda_1)J(\lambda_2) - \lambda_2\, C(\lambda_2)J(\lambda_1)}{\lambda_1\,(\lambda_1 - \lambda_2)}\,, \tag{6.53i}$$

$$\{B(\lambda_1), C(\lambda_2)\} = \frac{B(\lambda_2)C(\lambda_1) - B(\lambda_1)C(\lambda_2)}{\lambda_1 - \lambda_2}\,, \tag{6.53j}$$

$$\{E(\lambda_1), F(\lambda_2)\} = \frac{E(\lambda_2)F(\lambda_1) - E(\lambda_1)F(\lambda_2)}{\lambda_1 - \lambda_2} + \frac{1}{\lambda_2}\, F(\lambda_1)J(\lambda_2)\,, \tag{6.53k}$$

$$\{E(\lambda_1), H(\lambda_2)\} = \frac{\lambda_1\, E(\lambda_1)H(\lambda_2) - \lambda_2\, E(\lambda_2)H(\lambda_1)}{\lambda_1\,(\lambda_1 - \lambda_2)} - \frac{H(\lambda_1)J(\lambda_2)}{\lambda_1}\,, \tag{6.53l}$$

$$\{E(\lambda_1), J(\lambda_2)\} = \frac{\lambda_1\, F(\lambda_1)H(\lambda_2) - \lambda_2\, F(\lambda_2)H(\lambda_1)}{\lambda_1\,(\lambda_1 - \lambda_2)}\,, \tag{6.53m}$$

$$\{C(\lambda_1), F(\lambda_2)\} = \frac{C(\lambda_1)F(\lambda_2) - C(\lambda_2)F(\lambda_1)}{\lambda_1 - \lambda_2}\,, \tag{6.53n}$$

$$\{F(\lambda_1), H(\lambda_2)\} = \frac{\lambda_1\, E(\lambda_1)J(\lambda_2) - \lambda_2\, E(\lambda_2)J(\lambda_1)}{\lambda_1\,(\lambda_1 - \lambda_2)} - \frac{J(\lambda_1)J(\lambda_2)}{\lambda_1}\,, \tag{6.53o}$$

$$\{B(\lambda_1), H(\lambda_2)\} = \frac{\lambda_1\, B(\lambda_1)H(\lambda_2) - \lambda_2\, B(\lambda_2)H(\lambda_1)}{\lambda_1\,(\lambda_1 - \lambda_2)}\,, \tag{6.53p}$$

$$\{F(\lambda_1), J(\lambda_2)\} = \frac{\lambda_1\, F(\lambda_1)J(\lambda_2) - \lambda_2\, F(\lambda_2)J(\lambda_1)}{\lambda_1\,(\lambda_1 - \lambda_2)}\,, \tag{6.53q}$$

$$\{H(\lambda_1), J(\lambda_2)\} = \frac{H(\lambda_2)J(\lambda_1) - H(\lambda_1)J(\lambda_2)}{\lambda_1 - \lambda_2}\,. \tag{6.53r}$$

Equations (6.53) can be used to establish the canonicity of the separation variables, leading to the following statement:

Theorem 6.3.2. *The separation variables* (μ_j, η_j)*, where* $\eta_j = \eta(\mu_j)$*, for the BSQ system possess the following Poisson brackets:*

(i) $\{\eta_i, \eta_j\} = \{\mu_i, \mu_j\} = 0\,,$

(ii) $\{\eta_i, \mu_j\} = \delta_{ij}\eta_i\,.$

Proof

Let us start by showing that $\{\eta_i, \eta_j\} = \{\mu_i, \mu_j\} = 0$. In order to prove the statement $\{\eta_i, \eta_j\} = 0$, consider the relation (6.52), i.e.

$$\eta = E - \frac{BF}{C} = J - \frac{CH}{B}\,, \tag{6.54}$$

where η is a root of the characteristic polynomial (6.2). Then, for $i \neq j$ we have

$$\{\eta_i C_i, \eta_j C_j\} = \{C_i E_i - B_i F_i, C_j E_j - B_j F_j\}\,, \tag{6.55}$$

where for simplicity of the notation, we have defined $C_i := C(\mu_i)$, $C_j := C(\mu_j)$, $E_i := E(\mu_i)$, $E_j := E(\mu_j)$ and so on. Equation (6.55) implies

$$\begin{aligned}\{\eta_i, \eta_j\} = {} & \frac{1}{C_i C_j}\left[\{C_i E_i, C_j E_j\} - \{C_i E_i, B_j F_j\} - \{B_i F_i, C_j E_j\} + \{B_i F_i, B_j F_j\}\right] \\ & - \frac{1}{C_i C_j}\left[C_i\{\eta_i, C_j\}\eta_j + \eta_i\{C_i, \eta_j\}C_j + \eta_i\{C_i, C_j\}\eta_j\right],\end{aligned}$$

and thus by applying the Leibniz rule we obtain

$$\begin{aligned}\{\eta_i, \eta_j\} = {} & \frac{1}{C_j}\left[\{E_i, C_j\}E_j + \{E_i, E_j\}C_j - \{E_i, B_j\}F_j - \{E_i, F_j\}B_j\right] - \frac{\eta_i\eta_j}{C_i C_j}\{C_i, C_j\} \\ & - \frac{\eta_i}{C_i C_j}\left[\{C_i, E_j\}C_j - \{C_i, B_j\}F_j - \{C_i, F_j\}B_j - \{C_i, 1/C_j\}B_j C_j F_j\right] \\ & - \frac{\eta_j}{C_i C_j}\left[C_i\{E_i, C_j\} - F_i\{B_i, C_j\} - B_i\{F_i, C_j\} - B_i C_i F_i\{1/C_i, C_j\}\right] \\ & + \frac{E_i}{C_i C_j}\left[\{C_i, C_j\}E_j + \{C_i, E_j\}C_j - \{C_i, B_j\}F_j - \{C_i, F_j\}B_j\right] \\ & + \frac{B_i}{C_i C_j}\left[\{F_i, B_j\}F_j + \{F_i, F_j\}B_j - \{F_i, C_j\}E_j - \{F_i, E_j\}C_j\right] \\ & + \frac{F_i}{C_i C_j}\left[\{B_i, B_j\}F_j + \{B_i, F_j\}B_j - \{B_i, C_j\}E_j - \{B_i, E_j\}C_j\right].\end{aligned}$$

Using equations (6.53) and subsequently computing the right-hand side of the latter equation, we obtain $\{\eta_i, \eta_j\} = 0$.

Now, in order to prove the statement $\{\mu_i, \mu_j\} = 0$, consider the polynomial $\mathcal{F}(\lambda)$ from (6.19),

$$\mathcal{F}(\lambda) = B^2 F - BC\,(E - J) - C^2 H\,. \tag{6.56}$$

Then, we have

$$\begin{aligned}
\{\mathcal{F}(\lambda_1), \mathcal{F}(\lambda_2)\} = {} & \{B^2(\lambda_1)F(\lambda_1), B^2(\lambda_2)F(\lambda_2)\} + \{C^2(\lambda_1)H(\lambda_1), C^2(\lambda_2)H(\lambda_2)\} \\
& - \{B^2(\lambda_1)F(\lambda_1), C^2(\lambda_2)H(\lambda_2)\} - \{C^2(\lambda_1)H(\lambda_1), B^2(\lambda_2)F(\lambda_2)\} \\
& + \{B(\lambda_1)C(\lambda_1)\,(E(\lambda_1) - J(\lambda_1))\,, B(\lambda_2)C(\lambda_2)\,(E(\lambda_2) - J(\lambda_2))\} \\
& - \{B^2(\lambda_1)F(\lambda_1), B(\lambda_2)C(\lambda_2)\,(E(\lambda_2) - J(\lambda_2))\} \\
& - \{B(\lambda_1)C(\lambda_1)\,(E(\lambda_1) - J(\lambda_1))\,, B^2(\lambda_2)F(\lambda_2)\} \\
& + \{C^2(\lambda_1)H(\lambda_1), B(\lambda_2)C(\lambda_2)\,(E(\lambda_2) - J(\lambda_2))\} \\
& + \{B(\lambda_1)C(\lambda_1)\,(E(\lambda_1) - J(\lambda_1))\,, C^2(\lambda_2)H(\lambda_2)\}\,.
\end{aligned}$$

As before, applying the Leibniz rule and subsequently using equations (6.53), we obtain

$$\{\mathcal{F}(\lambda_1), \mathcal{F}(\lambda_2)\} = 0\,. \tag{6.57}$$

Note that the calculations involved are very large and cannot be reproduced here, but have been verified by MAPLE. From equation (6.57) we have

$$\begin{aligned}
\{\mathcal{F}(\lambda_1), \mathcal{F}(\lambda_2)\} &= \{\mathcal{F}_g \prod_{i=1}^{g}(\lambda_1 - \mu_i), \mathcal{F}_g \prod_{j=1}^{g}(\lambda_2 - \mu_j)\} = \{\sum_{i=0}^{g} \lambda_1^i\, \mathcal{F}_i, \sum_{j=0}^{g} \lambda_2^j\, \mathcal{F}_j\} \\
&= \{\mathcal{F}_g \sum_{i=0}^{g-1} \lambda_1^i (-1)^{g-i} S_{g-i}(\boldsymbol{\mu}), \mathcal{F}_g \sum_{j=0}^{g-1} \lambda_2^j (-1)^{g-j} S_{g-j}(\boldsymbol{\mu})\} = 0\,,
\end{aligned}$$

in which $S_{g-i}(\boldsymbol{\mu})$ are the functions of symmetric products defined in (4.27) of chapter 4.

Further, we have

$$
\begin{aligned}
&\{\mathcal{F}_g \sum_{i=0}^{g-1} \lambda_1^i (-1)^{g-i} S_{g-i}(\boldsymbol{\mu}), \mathcal{F}_g \sum_{j=0}^{g-1} \lambda_2^j (-1)^{g-j} S_{g-j}(\boldsymbol{\mu})\} = 0 \\
\Rightarrow \quad & \sum_{j=1}^{g} \{\mu_j, \sum_{i=0}^{g-1} \lambda_1^i (-1)^{g-i} S_{g-i}(\boldsymbol{\mu})\} \prod_{\substack{l=1 \\ l\neq j}}^{g} (\lambda_2 - \mu_l) = 0 \\
\Rightarrow \quad & \sum_{i,j=1}^{g} \{\mu_i, \mu_j\} \prod_{\substack{k,l=1 \\ k\neq i, l\neq j}}^{g} (\lambda_1 - \mu_k)(\lambda_2 - \mu_l) = 0 \,.
\end{aligned}
$$

Thus, for all λ_1 and λ_2, we deduce

$$
\sum_{\substack{i,j=1 \\ i<j}}^{g} \{\mu_i, \mu_j\}(\mu_i - \mu_j) \prod_{\substack{k=1 \\ k\neq i,j}}^{g} (\lambda_1 - \mu_k)(\lambda_2 - \mu_k) = 0 \,.
$$

From above discussion we know that the discriminant $R(\lambda)$ cannot be zero. Thus, we have $\mu_i \neq \mu_j$ for all $i \neq j$, which implies that $\{\mu_i, \mu_j\} = 0$ must hold.

Finally, in order to prove the second statement $\{\eta_i, \mu_j\} = \delta_{ij}\eta_i$, let us recall the equations (6.54) and (6.56). Then, we have

$$
\{\eta_i C_i, \mathcal{F}(\lambda_2)\} = \{C_i E_i, \mathcal{F}(\lambda_2)\} - \{B_i F_i, \mathcal{F}(\lambda_2)\} \,,
$$

leading to

$$
\{\eta_i, \mathcal{F}(\lambda_2)\} = \frac{1}{C_i} \left[\{C_i E_i, \mathcal{F}(\lambda_2)\} - \{B_i F_i, \mathcal{F}(\lambda_2)\} - \eta_i \{C_i, \mathcal{F}(\lambda_2)\}\right].
$$

By computing the right-hand side of this equation and using equations (6.53), we arrive at the following form

$$
\begin{aligned}
\{\eta_i, \mathcal{F}(\lambda_2)\} = & \frac{C^2(\lambda_2)H(\lambda_2)}{\mu_i - \lambda_2}\left(\frac{B_i F_i}{C_i} - E_i\right) + \frac{B(\lambda_2)C(\lambda_2)F(\lambda_2)}{\mu_i - \lambda_2}\left(\frac{B_i(J_i - \eta_i)}{C_i} - H_i\right) \\
& + \frac{C^2(\lambda_2)E(\lambda_2)}{\mu_i - \lambda_2}\left(H_i - \frac{B_i(J_i - \eta_i)}{C_i}\right) + \frac{B(\lambda_2)C(\lambda_2)J(\lambda_2)}{\mu_i - \lambda_2}\left(E_i - \frac{B_i F_i}{C_i}\right) \\
& + \frac{\eta_i B(\lambda_2)}{\mu_i - \lambda_2}\left(B(\lambda_2)F(\lambda_2) - C(\lambda_2)E(\lambda_2)\right). \qquad (6.58)
\end{aligned}
$$

Since we have from equation (6.54) the fact

$$
C_i H_i - B_i(J_i - \eta_i) = 0 \;\text{ and }\; B_i F_i - C_i(E_i - \eta_i) = 0 \,,
$$

equation (6.58) can then be written as

$$\begin{aligned}\{\eta_i, \mathcal{F}(\lambda_2)\} &= \frac{\eta_i}{\mu_i - \lambda_2}\left[B^2(\lambda_2)F(\lambda_2) - B(\lambda_2)C(\lambda_2)\left(E(\lambda_2) - J(\lambda_2)\right) - C^2(\lambda_2)H(\lambda_2)\right] \\ &= \frac{-\eta_i}{\lambda_2 - \mu_i}\,\mathcal{F}(\lambda_2)\,.\end{aligned}$$

This equation implies

$$\begin{aligned} &\{\eta_i, \mathcal{F}_g \prod_{j=1}^{g}(\lambda_2 - \mu_j)\} = \frac{-\eta_i\,\mathcal{F}_g \prod_{j=1}^{g}(\lambda_2 - \mu_j)}{\lambda_2 - \mu_i} \\ \Rightarrow\quad &\{\eta_i, \mathcal{F}_g \sum_{j=0}^{g-1} \lambda_2^j (-1)^{g-j} S_{g-j}(\boldsymbol{\mu})\} = -\eta_i\,\mathcal{F}_g \prod_{\substack{j=1\\ i\neq j}}^{g}(\lambda_2 - \mu_j) \\ \Rightarrow\quad &\sum_{k=0}^{g-1} \lambda_2^k (-1)^{g-k} \sum_{j=1}^{g} \{\eta_i, \mu_j\} S_{g-1-k}(\boldsymbol{\mu})_{|\mu_j=0} = -\eta_i \prod_{\substack{j=1\\ i\neq j}}^{g}(\lambda_2 - \mu_j)\,. \end{aligned}$$

In fact, this equation can be posed in a matrix form, since λ_2 is arbitrary, as follows

$$\begin{pmatrix} \{\eta_i, \mu_1\} \\ \vdots \\ \vdots \\ \{\eta_i, \mu_g\} \end{pmatrix} = \begin{pmatrix} -S_{g-1}(\boldsymbol{\mu})_{|\mu_1=0} & \cdots & -S_{g-1}(\boldsymbol{\mu})_{|\mu_g=0} \\ \vdots & & \vdots \\ -S_1(\boldsymbol{\mu})_{|\mu_1=0} & \cdots & -S_1(\boldsymbol{\mu})_{|\mu_g=0} \\ -1 & \cdots & -1 \end{pmatrix}^{-1} \begin{pmatrix} -\eta_i\, S_{g-1}(\boldsymbol{\mu})_{|\mu_i=0} \\ \vdots \\ -\eta_i\, S_1(\boldsymbol{\mu})_{|\mu_i=0} \\ -\eta_i \end{pmatrix}.$$

By using the definition of the functions $S_k(\boldsymbol{\mu})$, $k = 1, \ldots, g-1$, it can be seen that this system of equations leads to $\{\eta_i, \mu_j\} = \eta_i$ when $i = j$ and $\{\eta_i, \mu_j\} = 0$ when $i \neq j$. □

The Dubrovin equations is the next object of interest through the rest of the chapter. We next consider the integration by means of continuous-time interpolating flow.

6.4 Dubrovin equations for interpolating flows

Continuous Dubrovin equations [36, 46] are one of the principal objects of interest in the Hamiltonian approach to finite-gap integration through separation of variables. In this section we look for these equations by introducing continuous-time evolutions for the auxiliary spectrum where we consider $\operatorname{tr} T(\lambda)$ and $\operatorname{tr} T^2(\lambda)$ of the monodromy matrix (6.1) as the generating Hamiltonian of the flow.

Let us now consider the time-flow generated by the $\operatorname{tr} T(\lambda)$ and $\operatorname{tr} T^2(\lambda)$, for fixed value of λ, respectively, namely

$$\dot{T}(\lambda') = \partial_{t_{1\lambda}} T(\lambda') = \operatorname{tr}_1\{T_1(\lambda),\, T_2(\lambda')\} = [\mathrm{M}(\lambda,\lambda'), T(\lambda')] \;, \tag{6.59a}$$

$$\overset{\star}{T}(\lambda') = \partial_{t_{2\lambda}} T(\lambda') = \operatorname{tr}_1\{T_1^2(\lambda),\, T_2(\lambda')\} = 2\,[\mathrm{N}(\lambda,\lambda'), T(\lambda')] \;, \tag{6.59b}$$

in which

$$\mathrm{M}(\lambda,\lambda') := \operatorname{tr}_1(s_{\bar{1}2} - r_{\bar{1}2})\, T_1 = \frac{1}{\lambda}\sum_{i=1}^{2} \operatorname{tr}_1(E_{i,3}\, T_1) \otimes E_{3,i} - \frac{1}{\lambda-\lambda'}\, T_1(\lambda) \;, \tag{6.60a}$$

$$\mathrm{N}(\lambda,\lambda') := \operatorname{tr}_1(s_{\bar{1}2} - r_{\bar{1}2})\, T_1^2 = \frac{1}{\lambda}\sum_{i=1}^{2} \operatorname{tr}_1(E_{i,3}\, T_1^2) \otimes E_{3,i} - \frac{1}{\lambda-\lambda'}\, T_1^2(\lambda) \;. \tag{6.60b}$$

In equations (6.59), for simplicity of the notation, we denote

$$\operatorname{tr}_1 := \operatorname{tr} \otimes \mathbb{I}\;, \quad T_1 \equiv \overset{1}{T} := T(\lambda) \otimes \mathbb{I}\;, \quad T_2 \equiv \overset{2}{T} := \mathbb{I} \otimes T(\lambda')\;.$$

Using equation (6.59a), we have

$$\begin{aligned}
\partial_{t_{1\lambda}} T^2(\lambda') &= \dot{T}(\lambda')\, T(\lambda') + T(\lambda')\, \dot{T}(\lambda') \\
&= [\mathrm{M}(\lambda,\lambda'), T(\lambda')]\, T(\lambda') + T(\lambda')\, [\mathrm{M}(\lambda,\lambda'), T(\lambda')] \\
&= \mathrm{M}(\lambda,\lambda')\, T^2(\lambda') - T^2(\lambda')\, \mathrm{M}(\lambda,\lambda') \\
&= [\mathrm{M}(\lambda,\lambda'), T^2(\lambda')] \;,
\end{aligned}$$

whereas using equation (6.59b), we have

$$\partial_{t_{2\lambda}} T^2(\lambda') = 2\,[\mathrm{N}(\lambda,\lambda'), T^2(\lambda')] \;.$$

Recall the auxiliary spectrum discussed in section 6.3, that is

$$\mathcal{F}(\lambda) = \det \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\, T(\lambda) \\ \vec{\alpha}\, T^2(\lambda) \end{pmatrix} = \mathcal{F}_g \prod_{j=1}^{g} (\lambda - \mu_j) \;. \tag{6.61}$$

Then, the following relation

$$\partial_{t_{1\lambda}} \log \det \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\, T(\lambda') \\ \vec{\alpha}\, T^2(\lambda') \end{pmatrix} = \partial_{t_{1\lambda}} \operatorname{tr} \log \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\, T(\lambda') \\ \vec{\alpha}\, T^2(\lambda') \end{pmatrix}, \tag{6.62}$$

holds in general, cf. ref. [4], since $\mathcal{F}(\lambda) \neq 0$. By considering the generating Hamiltonian of the flow generated by $\operatorname{tr} T(\lambda)$, from equation (6.62) we have

$$\begin{aligned}
\partial_{t_{1\lambda}} \log \mathcal{F}_g \prod_{j=1}^{g} (\lambda' - \mu_j) &= \operatorname{tr} \left\{ \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\, T(\lambda') \\ \vec{\alpha}\, T^2(\lambda') \end{pmatrix}^{-1} \partial_{t_{1\lambda}} \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\, T(\lambda') \\ \vec{\alpha}\, T^2(\lambda') \end{pmatrix} \right\} \\
&= \frac{1}{\mathcal{F}_g \prod_{j=1}^{g} (\lambda' - \mu_j)} \operatorname{tr} \left\{ \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\, T \\ \vec{\alpha}\, T^2 \end{pmatrix}^{\text{cof}} \begin{pmatrix} 0 \\ \vec{\alpha}\, [\mathrm{M}, T] \\ \vec{\alpha}\, [\mathrm{M}, T^2] \end{pmatrix} \right\} \\
&= \frac{1}{\mathcal{F}_g \prod_{j=1}^{g} (\lambda' - \mu_j)} \operatorname{tr} \left\{ \begin{pmatrix} \vec{\alpha}\, T \times \vec{\alpha}\, T^2 \\ \vec{\alpha}\, T^2 \times \vec{\alpha} \\ \vec{\alpha} \times \vec{\alpha}\, T \end{pmatrix}^{T} \begin{pmatrix} 0 \\ \vec{\alpha}\, [\mathrm{M}, T] \\ \vec{\alpha}\, [\mathrm{M}, T^2] \end{pmatrix} \right\}.
\end{aligned}$$

This implies

$$\sum_{j=1}^{g} \frac{-\dot{\mu}_j}{\lambda' - \mu_j} = \frac{1}{\mathcal{F}_g \prod_{j=1}^{g} (\lambda' - \mu_j)} \left(\vec{\alpha}\, [\mathrm{M}, T] \cdot (\vec{\alpha}\, T^2 \times \vec{\alpha}) + \vec{\alpha}\, [\mathrm{M}, T^2] \cdot (\vec{\alpha} \times \vec{\alpha}\, T) \right).$$

Applying the residue theorem [104] to the latter equation, one obtains

$$\dot{\mu}_j = \frac{-1}{\mathcal{F}_g \prod_{i \neq j}^{g} (\mu_j - \mu_i)} \left(\vec{\alpha}[\mathrm{M}, T] \cdot (\vec{\alpha}\, T^2 \times \vec{\alpha}) + \vec{\alpha}[\mathrm{M}, T^2] \cdot (\vec{\alpha} \times \vec{\alpha}\, T) \right)_{|\lambda' = \mu_j}. \tag{6.63}$$

By using the equation for the auxiliary spectrum (6.19) at the separation variables μ_j, we write the expression given by the numerator on the right-hand side of equation (6.63) as

$$\vec{\alpha}\, [\mathrm{M}, T] \cdot (\vec{\alpha}\, T^2 \times \vec{\alpha}) + \vec{\alpha}\, \left[\mathrm{M}, T^2\right] \cdot (\vec{\alpha} \times \vec{\alpha}\, T) = \frac{Q(\mu_j)}{(\lambda - \mu_j)} \left(B_1 C_2 - B_2 C_1 \right),$$

where $Q(\mu_j)$ is defined in (6.37). Thus, the equations (6.63) are then written as

$$\dot{\mu}_j = \frac{-Q(\mu_j)}{\mathcal{F}_g \prod_{i \neq j}^{g} (\mu_j - \mu_i)} \left(\frac{B_1 C_2 - B_2 C_1}{\lambda - \mu_j} \right), \tag{6.64}$$

which define the so-called *Dubrovin equations* for the time-flow generated by $\operatorname{tr} T(\lambda)$.

Now, by considering the generating Hamiltonian of the flow generated by $\operatorname{tr} T^2(\lambda)$, similarly to the standard treatment of the first flow, we find the differential equations for the second flow:

$$\overset{\star}{\mu}_j = \frac{-2}{\mathfrak{F}_g \prod_{i\neq j}^{g}(\mu_j - \mu_i)} \Big(\vec{\alpha}[\mathrm{N}, T] \cdot (\vec{\alpha}\, T^2 \times \vec{\alpha}) + \vec{\alpha}[\mathrm{N}, T^2] \cdot (\vec{\alpha} \times \vec{\alpha}\, T) \Big)_{|_{\lambda' = \mu_j}} . \tag{6.65}$$

Once again, using equation (6.19) on $\lambda = \mu_j$ we write the expression given by the numerator on the right-hand side of this equation as

$$\vec{\alpha}\,[\mathrm{N}, T] \cdot (\vec{\alpha}\, T^2 \times \vec{\alpha}) + \vec{\alpha}\,\big[\mathrm{N}, T^2\big] \cdot (\vec{\alpha} \times \vec{\alpha}\, T) = \frac{Q(\mu_j)}{(\lambda - \mu_j)} \Big((\vec{\alpha} \times \vec{\alpha}\, T_1^2) \cdot \vec{\alpha}\, T_2 \Big) ,$$

where $(\vec{\alpha} \times \vec{\alpha}\, T_1^2) \cdot \vec{\alpha}\, T_2 = [C_1 H_1 + B_1\,(A_1 + E_1)]\, C_2 - B_2\,[C_1\,(A_1 + J_1) + B_1 F_1]$. Thus, the Dubrovin equations for the second flow are given by

$$\overset{\star}{\mu}_j = \frac{-2\, Q(\mu_j)}{\mathfrak{F}_g \prod_{i\neq j}^{g}(\mu_j - \mu_i)} \left(\frac{[C_1 H_1 + B_1\,(A_1 + E_1)]\, C_2 - B_2\,[C_1\,(A_1 + J_1) + B_1 F_1]}{\lambda - \mu_j} \right) . \tag{6.66}$$

In fact, equations (6.64) and (6.66) generalize the Dubrovin equations that are presented in [33, 34] in the sense that the time-flows we consider here depend on a parameter λ, which generate the entire hierarchy of BSQ flows considered in [33]. What remains to be done is to show that the coefficients are expressible as a symmetric function of the μ's and invariants. This approach will be a subject of further investigation.

6.5 Discrete-time evolution

In this section the derivation mechanism of discrete motion equations in terms of the auxiliary spectrum (discrete Dubrovin equations) is discussed. However, we focus on the class of discrete integrable systems obtained by considering the “vertical shift evolution”, where the case of diagonal shift evolution is left as the subject of future study. It is not yet clear how to manage the final expression for the discrete Dubrovin equations, but by considering the discrete-time map we are able to make some steps towards that goal.

Let us recall the monodromy matrix from equation (6.1),

$$T(\lambda) = \prod_{n=1}^{\overset{\curvearrowleft}{N}} V_n(\lambda) := \begin{pmatrix} A(\lambda) & B(\lambda) & C(\lambda) \\ D(\lambda) & E(\lambda) & F(\lambda) \\ G(\lambda) & H(\lambda) & J(\lambda) \end{pmatrix} , \tag{6.67}$$

which we assume to evolve under a discrete-time map according to

$$\widehat{T}(\lambda) = M(\lambda)\, T(\lambda)\, M(\lambda)^{-1} , \qquad M(\lambda) = \begin{pmatrix} x & 1 & 0 \\ y & z & 1 \\ \lambda & 0 & 0 \end{pmatrix} , \tag{6.68}$$

in which x, y and z are some dynamical variables that are independent of the spectral parameter λ. Now, from the discrete-time evolution (6.68) we have the following discrete relations for its entries

$$\widehat{A} = x\,B - z\,(x\,C + F) + E\,, \tag{6.69a}$$
$$\widehat{B} = x\,C + F\,, \tag{6.69b}$$
$$\lambda\,\widehat{C} = x\,A - x\,(x\,B + E) + (x\,z - y)(x\,C + F) + D\,, \tag{6.69c}$$
$$\widehat{D} = y\,B - z\,(y\,C - E + z\,F + J) + H\,, \tag{6.69d}$$
$$\widehat{E} = y\,C + z\,F + J\,, \tag{6.69e}$$
$$\lambda\,\widehat{F} = y\,A - x\,(y\,B + z\,E + H) + (x\,z - y)(y\,C + z\,F + J) + z\,D + G\,, \tag{6.69f}$$
$$\widehat{G} = \lambda\,(B - z\,C)\,, \tag{6.69g}$$
$$\widehat{H} = \lambda\,C\,, \tag{6.69h}$$
$$\widehat{J} = A - x\,B + (x\,z - y)\,C\,. \tag{6.69i}$$

Expanding equations (6.69f) and (6.69i) in powers of λ, for the lowest coefficients we obtain the equations

$$D_0 = xE_0 - (xz - y)F_0 \quad \text{and} \quad A_0 = xB_0 - (xz - y)C_0\,. \tag{6.70}$$

Solving this system of equations, we obtain the following expressions for x and $xz - y$,

$$x = \frac{A_0F_0 - C_0D_0}{B_0F_0 - C_0E_0}\,, \qquad xz - y = \frac{A_0E_0 - B_0D_0}{B_0F_0 - C_0E_0}\,. \tag{6.71}$$

By considering the elementary symmetric function $\widehat{\mathcal{F}}(\lambda)$ of the zeroes $\widehat{\mu}_1, \ldots, \widehat{\mu}_g$, we can extract from the set of equations (6.69) the following relation

$$\begin{aligned}\lambda\,\widehat{\mathcal{F}}(\lambda) = x\,\{F\,[(A-J)(A-E) - BD - FH] - C\,[D\,(A-2E+J) - 2FG\,]\} \\ - x^2\,[BF\,(2A-E-J) - C\,(A-E)(E-J) - C\,(BD+CG-2FH)] \\ + x^3\,\mathcal{F}(\lambda) + F\,[D\,(A-J) + FG\,] - CD^2\,, \end{aligned} \tag{6.72}$$

where $\mathcal{F}(\lambda)$ is the elementary symmetric function of the zeroes $\mu_1, \ldots, \mu_g$.

Alternatively, using proposition 6.3.1, which tells that the symmetric function $\mathcal{F}(\lambda) = 0$ can also be written as the vector $\vec{\alpha}\,(T - \eta\mathbb{I})^{\text{cof}} = 0$, we can also extract from the set of equations (6.69) another system of equations, namely

$$\begin{aligned}\lambda\,[\widehat{B}\,\widehat{F} - \widehat{C}(\widehat{E}-\eta)] = (xz-y)\,\{x\,[BF - C(E-\eta)] - F(A-\eta) + CD\} \\ - x^2\,[CH - B(J-\eta)] - x\,[(A-E)(J-\eta) - CG + FH] \\ - D(J-\eta) + FG\,, \end{aligned} \tag{6.73a}$$

$$\widehat{C}\,\widehat{H} - \widehat{B}(\widehat{J}-\eta) = x\,[BF - C(E-\eta)] - F(A-\eta) + CD\,, \tag{6.73b}$$

$$\begin{aligned}(\widehat{E}-\eta)(\widehat{J}-\eta) - \widehat{F}\widehat{H} = x\,\{CH - B(J-\eta) - z\,[BF - C(E-\eta)]\} \\ - z\,[CD - F(A-\eta)] + (A-E)(J-\eta) - CG\,. \end{aligned} \tag{6.73c}$$

We consider equations (6.73) together with equations (6.71) and (6.72) as an implicit form of discrete Dubrovin equations for the mapping of BSQ type. On the basis of comparison with the derivation steps of discrete Dubrovin equations for the KdV system expounded in chapter 4, we believe that by using the latter implicit equations, in addition to the facts (6.37) and (6.43), we will be able to derive an explicit form of discrete Dubrovin equations for the BSQ system. This, in fact, requires to rewrite the equations in terms of invariants and auxiliary spectrum; the derivation of these equations is left as a subject for future work.

6.6 Summary

In this chapter we have considered the BSQ model, where we have demonstrated the characteristic behaviour of the monodromy matrix as polynomial function of the spectral parameter and applied some techniques to the model, such as separation of variables, r-matrix structure and finite-gap integration.

By using the Weierstrass gap sequence method, we introduced a novel form for the genus of characteristic polynomial of the monodromy matrix, which depends on the length of chain on the staircase. Based on the genus, we proposed a general form for the monodromy matrix that has a natural grading in terms of the spectral parameter, leading to three different cases that are characterized by the values of the top coefficients.

The method of separation of variables was successfully applied to the mappings of BSQ type. Moreover, we uncovered a new formulation for the elementary symmetric function that leads to the separation variables. This new formulation revealed some interesting results: there is, in general, an important relation between the discriminant and derivative of a cubic equation. There is also a fundamental relation that is stronger than Cayley [27] and holds for any 3×3-matrix.

Finally, we presented the continuous Dubrovin equations for the BSQ model, which generalize the Dubrovin equations that are presented in [33] for the BSQ hierarchy. It is not yet clear how to express the coefficients in the Dubrovin equations in terms of the symmetric function of the separation variables and invariants, but this outstanding problem is left for future work. We also discussed the discrete Dubrovin equations and set the principal steps for deriving these equations. The full analysis of the derivation of these equations still needs to be performed and this is also left as a subject for future study.

Chapter 7

Conclusions

We have come to the end of the story. We conclude by summarizing the results and suggesting possible directions for future work.

7.1 Summary of results

This thesis deals with the subjects of discrete-time integrable systems and geometric numerical integration. Specifically, it is concerned with integrable lattice equations and modified Hamiltonian equations.

Chapter 1 was mainly a review, but it also contains a few topics that were recapitulated in such a way as to focus on the features required for the rest of the thesis. More precisely, an introduction to classical canonical maps was given. We respectively used the Bäcklund transformation and continuum limits as the main tools to link the continuous KdV equation to its discrete version and vice versa. We showed the procedure of constructing integrable mappings (generalized McMillan maps) as periodic reductions of the lattice KdV equation. The notion of "integrability" was defined for discrete-time systems, followed by an investigation of systems adhering to this definition. Yoshida's method, which is based on the Baker-Campbell-Hausdorff (BCH) series, was also reviewed, followed by a discussion of the convergence of the modified Hamiltonians. Finally, we gave the outline of the thesis supported by simplified diagrams.

In chapter 2 we pulled together some largely known facts about lattice KdV models, but also showed some new insights, such as families of integrable mappings that are viewed as an application of the symplectic Euler method, as well as families of commuting integrable mappings that arise by exploiting the multi-dimensional consistency of the linearized lattice KdV equation. Most of the results in this chapter have already appeared in the literature, but we covered them in order to establish the notations and develop the tools needed for the subsequent chapters.

The next chapters in the thesis contain new material. The simplicity of the linear models explored in chapter 2 makes them helpful examples of the modified Hamiltonian. Thus, as the starting point of chapter 3 we considered the linear system; the discrete maps and their commuting maps led to a novel perspective on the discrete harmonic oscillators. By applying the BCH series to such systems with one and two degrees of freedom, it was found that family of closed-form modified Hamiltonians are in involution with respect to the variables of the original maps. We next turned to the nonlinear model derived from the lattice KdV and mKdV equations, the parent equations of the linear lattice models. In general, it is well known that common numerical schemes applied to Hamiltonians of the Newtonian form $H = p^2/2 + V(q)$, where $\partial V(q)/\partial q$ is nonlinear, have a divergent expansion for the modified Hamiltonian. However, an example of a non-Newtonian Hamiltonian system, which arises from the simplest case reduction of the nonlinear lattice KdV equation, was presented in this chapter. The modified Hamiltonian was found to have a closed-form expression in terms of an elliptic integral, indicating that the expansion of the modified Hamiltonian obtained from Yoshida's approach does converge. Additionally, we presented another example of a Hamiltonian system that displayed an implicit scheme dependence on the time step by considering a staircase reduction on the lattice mKdV equation. In this case, we also obtained a closed-form expression in terms of an elliptic integral and compared it with the expansion that we got from Yoshida's method.

In chapter 4 we dealt with systems obtained by extending the reduction of the lattice KdV equation to higher dimensions, where we restricted ourselves to two-dimensional periodic reduction, i.e. two-degrees-of-freedom systems. Such systems can be treated by certain mathematical techniques, such as separation of variables, r-matrix structure and finite-gap integration, in order to arrive at the action-angle variables. We used specific techniques of the monodromy matrix and Lax pair in order to employ the method of separation of variables in the KdV system. Further, we verified the canonicity of the separated variables by using the information about the Poisson brackets between the entries of the monodromy matrix provided by the classical r-matrix structure. Studying the generalized KdV maps for diagonal evolution reduction, we uncovered a novel expression for the discrete Dubrovin equations, which are different from the discrete Dubrovin equations derived in [80], which are the equations for the dynamics of the auxiliary spectrum associated with the mappings in the vertical direction. Moreover, we established the continuous-time evolution generated by the invariants, where we constructed an interpolating flow for the discrete map by considering the trace of the monodromy matrix as the generating Hamiltonian of the flow, leading to the continuous Dubrovin equations. Defining the associated momenta variables with the auxiliary spectrum variables, we uncovered a novel expression for the second companion equations of the Dubrovin equations for the conjugate variables. As a consequence, we derived the Hamiltonian of the system in terms of the separated variables similar to the one in the Toda model; cf. chapter 5 in [113], but supplementing it with the relevant kinetic term. Having obtained the canonical transformations between the original, separated and action-angle variables, combined with the description of the dynamical transformations of these variables, we illustrated the schemes for the system of two degrees of freedom. The main result is a novel formulation of a closed-form modified Hamiltonian for the system of of two degrees of freedom, which indicates that the expansion of the modified Hamiltonian obtained from the method of Yoshida [128] must converge in this integrable case.

In chapter 5 we studied the Boussinesq (BSQ) model, derived the lattice BSQ equation from the Bäcklund transformation (BT) and presented the Lax pair of the latter as well as the relevant r-matrix structure. The BT for the BSQ equation was given by Chen [30] in a way that it does not contain any BT parameters. However, in the light of the present study, we introduced a proper parameter-dependent BT for the scalar form that leads to the discrete BSQ equation in nine-point stencil form, as originally given by Nijhoff et al. [87]. A staircase reduction on the lattice BSQ equation gave rise to multi-dimensional families of mappings that turned out to be integrable. Expressing the mappings and invariants in terms of proper reduced variables required us to work with a representation in terms of a big Lax pair (dual Lax) form instead of the local Lax pair. Thus, beginning from the local Lax pair we showed the derivation of the big Lax matrix for the mappings of BSQ type. Having derived the mappings and their local and big Lax pairs, we illustrated the general schemes for periods one and two reduction. We encoded the Poisson bracket structure of the Lax matrix with the r-matrix, which led to the discrete version of the non-ultralocal Poisson bracket structure [87]. Further, the r-matrix structure can be used to prove the involutivity of the invariants and the preservation of the Poisson bracket structure under the mapping.

In the final chapter, we analyzed the BSQ mappings studied in chapter 5. We used the Weierstrass gap sequence method in order to find a novel expression for the genus of a spectral curve that is characterized by the length of the chain on the staircase in the lattice. We set up a general formulation of the monodromy matrix, which has a natural grading in terms of the spectral parameter that depends on the number of genera. This structure was found to have three cases associated with the values of the top coefficients. To our knowledge, this is the first structure for the monodromy matrix of the BSQ model proposed in such a way. We applied the technique of separation of variables to the BSQ mappings, where the separated coordinates appear as the poles of the properly normalized eigenvector of the corresponding monodromy matrix. Further, we set up new general schemes of describing the symmetric function of the separation variables. By investigating the characteristic polynomial of the spectral curve we revealed an interesting result that uncovers a significant relation between the discriminant and derivative of a cubic equation. Moreover, for any 3×3-matrix a remarkable result was uncovered, which is slightly stronger than the Cayley-Hamilton theorem [27]. In fact, this result does not hold only for the 3×3-matrix case, but can also be extended to any $N \times N$-matrix. The continuous Dubrovin equations were presented for the BSQ hierarchy in [33]. Our approach, however, provided an alternative scheme for deriving such equations for time-flows which depend on a parameter λ, and hence generate the entire hierarchy of BSQ flows considered in [33]. Finally, we discussed the equations of the discrete motion for the vertical evolution reduction in terms of the auxiliary spectrum, that is, the discrete Dubrovin equations. By considering the discrete-time map, we made the first steps towards deriving the relevant discrete Dubrovin equations, but the derivation is more complicated in the BSQ case; we obtained an implicit description of the dynamics of the auxiliary spectrum, but more work is needed to derive an explicit form of the equations. The derivation of the explicit form of these equations is left to future work.

We see the research described in this thesis as an early investigation into the structures of modified Hamiltonian on the discrete level of multiple-degrees-of-freedom systems: many important questions remain, but perhaps these findings will be a useful starting point. The discrete integrable systems theory is a new way of understanding the link between integrals (invariants) and modified Hamiltonians from a variational perspective, even suggesting new ways of thinking about modified Hamiltonians themselves as convergent series to variational systems of degrees-of-freedom. By investigating one and two degrees of freedom systems for discrete KdV models, we uncovered the connection between the integrals and the relevant modified Hamiltonians. However, extending the research of modified Hamiltonian for higher class of lattice models, such as lattice BSQ models, is a challenging problem, due to the difficulty of carrying out the required mapping reduction, integrals, techniques of separation of variables and finite-gap integration, and Dubrovin equations. Nevertheless, investigating the BSQ models we have made some steps towards that goal.

7.2 Future work

I believe the goal of a PhD thesis is not only to solve problems, but also to pose novel ones as an outgrowth of the work. Therefore, in this section we discuss some open problems as directions for the future.

The one-dimensional discrete-time systems of nonlinear KdV models, which we have investigated in chapter 3, give rise to examples of Hamiltonian systems of one degree of freedom, where the modified Hamiltonian, as given by Yoshida's approach, is written in closed-form expression. One future direction is to exploit the multi-dimensional consistency property of the nonlinear KdV system to derive commuting integrable mappings. It is certainly of interest from the mathematical point of view, in that it provides us an insight into the basic structures underlying modified Hamiltonian expansions. The ideas can also be applied to the two-dimensional evolution of the higher period staircase reduction, which we studied in chapter 4. This may give us a new way of understanding the structure of the modified Hamiltonians that are given as power series in the time step by Yoshida's method based on the BCH series.

For the BSQ system in chapter 5, on the one hand, the connection between the KdV mappings for period $P = 2$ and the BSQ mappings for period $P = 1$ is not obvious, although both mappings correspond to elliptic curves [87]. As a direction for the future, looking for a certain transformation that reveals this connection may be of interest. On the other hand, the corresponding Hamiltonian structures of the BSQ mappings that we derived in section 5.4.4 of chapter 5, as far as we are aware, can be established. Having derived the Hamiltonian systems, our attention turns to the modified Hamiltonian equations. Studying such equations for the BSQ models definitely gives us an advantage to obtain good insights into the essential structures behind the BCH formula. For the moment these goals are hampered by the sheer complexity of the system and would require several machineries.

Another future direction is to derive the discrete Dubrovin equations for the class of discrete-time systems to which the BSQ-type mappings (also the modified BSQ and Schwarzian BSQ mappings, e.g. in [56, 81, 108]) belong. In the thesis we set up the general scheme and took some initial steps towards the equations of the discrete motion for the vertical evolution in terms of the auxiliary spectrum, but there is more to be done to obtain an explicit form of the equations. Finding an explicit formula for the discrete Dubrovin equations of the vertical evolution is an object of our interest. It would also be interesting to further explore the equations of the discrete motion for the diagonal evolution in terms of the auxiliary spectrum, which we expect to be used to prove the convergence of the modified Hamiltonian for higher-degrees-of-freedom systems.

Lastly, we remark that for the modified Hamiltonians which are constructed by the well-known BCH series, the precise relation between the integrability of mappings and the structure of the BCH series remains mysterious. We can conjecture that a deeper understanding of the BCH series may arise from the convergence analysis of the expansion for the modified Hamiltonian. We also remark that the interplay between discrete-time integrable systems and geometric numerical integration occurs in a wider range of settings than the specific integrable models we considered here. In both geometric integration in numerical analysis and quantum field theory, mathematical structures have been discovered and studied which provide insights into the BCH series, such as the rooted-trees expansions [28,29,52,62,78], Butcher groups [15,19,52,53] and Hopf algebras [25,31,35] (for more sources and historical review on the discussion of these insights and the context of BCH series, we refer to [26, 38, 76]). These insights have never been directly applied, to our knowledge, to the context of discrete integrable systems. Therefore, we may conjecture that techniques arising from the theory of discrete integrable system combined with the techniques arising from the areas of numerical analysis and Hopf algebras may shed light on the structure of the BCH series. As a direction for the future, establishing connections between rooted-trees expansions and the modified Hamiltonians discovered in the light of this thesis may be of interest.

Appendix A

Dubrovin equations for KdV mappings

Here we look at the derivation of the Dubrovin equations (4.55) by taking a slightly different approach from the one expounded in section 4.5 of chapter 4. Let us remind the reader of the Dubrovin equations for the mapping of KdV type derived in section 4.5. These equations are given by a first-order system of coupled ordinary differential equations

$$\dot{\mu}_j = \frac{\sqrt{R(\mu_j)}\,B(\lambda)}{(\lambda-\mu_j)\,B_g\prod_{i\neq j}(\mu_j-\mu_i)}\,. \tag{A.1}$$

Consider the time-flow generated by $\mathrm{tr}\,T(\lambda)$, which is the trace of the monodromy matrix $T(\lambda)$, namely

$$\dot{T}(\lambda') = \partial_{t_\lambda}T(\lambda') = \mathrm{tr}_1\{T_1(\lambda),\, T_2(\lambda')\} = [\mathrm{M}(\lambda,\lambda'), T(\lambda')]\,,$$

in which

$$\mathrm{M}(\lambda,\lambda') := \frac{1}{\lambda-\lambda'}\begin{pmatrix} A(\lambda) & B(\lambda) \\ \lambda'\,C(\lambda)/\lambda & D(\lambda)\end{pmatrix}.$$

Now, by considering the auxiliary spectrum (4.8),

$$B(\lambda) = \det\begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\,T(\lambda)\end{pmatrix} = B_g\prod_{j=1}^{g}(\lambda-\mu_j),$$

we then have

$$\partial_{t_\lambda}\log\,\det\begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\,T(\lambda')\end{pmatrix} = \partial_{t_\lambda}\,\mathrm{tr}\log\begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\,T(\lambda')\end{pmatrix}.$$

This implies

$$\partial_{t_\lambda} \log B_g \prod_{j=1}^{g} (\lambda' - \mu_j) = \mathrm{tr} \left\{ \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\, T(\lambda') \end{pmatrix}^{-1} \partial_{t_\lambda} \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\, T(\lambda') \end{pmatrix} \right\}$$

$$= \frac{1}{B_g \prod_{j=1}^{g} (\lambda' - \mu_j)} \mathrm{tr} \left\{ \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\, T \end{pmatrix}^{\mathrm{cof}} \begin{pmatrix} 0 \\ \vec{\alpha}\, [\mathrm{M}, T] \end{pmatrix} \right\}$$

$$= \frac{1}{B_g \prod_{j=1}^{g} (\lambda' - \mu_j)} \mathrm{tr} \left\{ \begin{pmatrix} B(\lambda') & 0 \\ -A(\lambda') & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \vec{\alpha}\, [\mathrm{M}, T] \end{pmatrix} \right\},$$

which leads to

$$\sum_{j=1}^{g} \frac{-\dot{\mu}_j}{\lambda' - \mu_j} = \frac{1}{B_g \prod_{j=1}^{g} (\lambda' - \mu_j)} \left(\frac{B(\lambda')\,[A(\lambda) - D(\lambda)] - B(\lambda)\,[A(\lambda') - D(\lambda')]}{\lambda - \lambda'} \right).$$

By applying the residue theorem, we then obtain

$$\dot{\mu}_j = \frac{1}{B_g \prod_{i \neq j}^{g} (\mu_j - \mu_i)} \left(\frac{B(\lambda)\,[A(\mu_j) - D(\mu_j)]}{\lambda - \mu_j} \right), \tag{A.2}$$

which is the Dubrovin equations (A.1) since $A(\mu_j) - D(\mu_j) = \sqrt{R(\mu_j)}$.

Appendix B

Weierstrass gap sequence

Here we will go over the steps how to use the mechanism of the Weierstrass gap sequence in order to determine the number of genus for a specific period of reduction. Let us apply this technique for two different periods of reduction, say $N = 4$ and $N = 6$.

Recall the spectral curve (6.2), i.e.

$$\det\left(T(\lambda) - \eta\mathbb{I}\right) = -\eta^3 + \operatorname{tr} T(\lambda)\,\eta^2 - S(\lambda)\,\eta + \det T(\lambda) = 0 \,. \tag{B.1}$$

By eliminating the term corresponding to order η^2, we write an incomplete cubic equation in $\widetilde{\eta}$,

$$\det\left(T(\lambda) - \widetilde{\eta}\mathbb{I}\right) = \widetilde{\eta}^3 + p\,\widetilde{\eta} + q = 0 \,, \tag{B.2}$$

in which

$$\widetilde{\eta} := \eta - \frac{1}{3}\operatorname{tr} T \,, \quad p := S - \frac{1}{3}\left(\operatorname{tr} T\right)^2 \,, \quad q := \frac{1}{3}\, S \operatorname{tr} T - \frac{2}{27}\left(\operatorname{tr} T\right)^3 - \det T \,.$$

For the case of $N = 4$ $(P = 2)$, the associated monodromy matrix is given by

$$T_4(\lambda) = V_4\, V_3\, V_2\, V_1 \,,$$

where V_j are given in (5.31a). Thus, the corresponding curve reads in terms of $\lambda, \widetilde{\eta}$ as

$$f(\lambda, \widetilde{\eta}) = \widetilde{\eta}^3 + \left(r_{10}\,\lambda^2 + r_7\,\lambda + r_4\right)\widetilde{\eta} + \left(-\lambda^4 + r_9\,\lambda^3 + r_6\,\lambda^2 + r_3\,\lambda + r_0\right) \,, \tag{B.3}$$

where r_j are functions of some dynamical variables $\mathrm{u}, \mathrm{v}, \mathrm{w}$. According to the Weierstrass gap sequence, the curve (B.3) corresponds to the following sequence (observe that in (B.3) the highest order of λ is 4):

0	1	2	3	4	5	6	7	8	9	10	11	12
1	$\sqcup$	$\sqcup$	λ	$\widetilde{\eta}$	$\sqcup$	λ^2	$\lambda\,\widetilde{\eta}$	$\widetilde{\eta}^2$	λ^3	$\lambda^2\,\widetilde{\eta}$	$\lambda\,\widetilde{\eta}^2$	$\lambda^4, \widetilde{\eta}^3$.

This sequence shows that there are 3 gaps. Hence, the spectral curve (B.1) defines trigonal curve of genus $g = 3$. In the case of $N = 6$ $(P = 3)$, the corresponding curve is given in terms of $\lambda, \widetilde{\eta}$ by

$$h = \widetilde{\eta}^3 + (r_{13}\,\lambda^3 + r_{11}\,\lambda^2 + r_8\,\lambda + r_5)\,\widetilde{\eta} + (r_{15}\,\lambda^5 + r_{12}\,\lambda^4 + r_9\,\lambda^3 + r_6\,\lambda^2 + r_3\,\lambda + r_0)\,. \quad \text{(B.4)}$$

Again, according to the Weierstrass gap sequence we have

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	$\sqcup$	$\sqcup$	λ	$\sqcup$	$\widetilde{\eta}$	λ^2	$\sqcup$	$\lambda\,\widetilde{\eta}$	λ^3	$\widetilde{\eta}^2$	$\lambda^2\,\widetilde{\eta}$	λ^4	$\lambda^3\,\widetilde{\eta}$	$\lambda^2\,\widetilde{\eta}^2$	$\lambda^5, \widetilde{\eta}^3$,

which shows that there are 4 gaps, and thus the spectral curve (B.1) defines trigonal curve of genus $g = 4$.

Appendix C

Cayley-Hamilton theorem and proof of equation (6.43)

In this appendix we will show how to derive the relation (6.43) starting from Cayley's theorem (6.42). For convenience, the relation (6.43) is given by

$$(T - \eta_2 \mathbb{I})(T - \eta_3 \mathbb{I}) = (T - \eta_1 \mathbb{I})^{\text{cof}} \ . \tag{C.1}$$

Now, let us consider the Cayley's theorem (6.42), i.e.

$$(T - \eta_1 \mathbb{I})(T - \eta_2 \mathbb{I})(T - \eta_3 \mathbb{I}) = 0 \ ,$$

where η_1, η_2 and η_3 are the roots of the characteristic polynomial $\det(T - \eta\mathbb{I}) = 0$. Hence, for any η not a root of the characteristic polynomial we have

$$\begin{aligned}
& (T - \eta_1 \mathbb{I})(T - \eta_2 \mathbb{I})(T - \eta_3 \mathbb{I}) = (T - (\eta - \eta + \eta_1)\mathbb{I})(T - \eta_2 \mathbb{I})(T - \eta_3 \mathbb{I}) = 0 \\
\Rightarrow \quad & (T - \eta \mathbb{I})(T - \eta_2 \mathbb{I})(T - \eta_3 \mathbb{I}) + (\eta - \eta_1)(T - \eta_2 \mathbb{I})(T - \eta_3 \mathbb{I}) = 0 \\
\Rightarrow \quad & (T - \eta_2 \mathbb{I})(T - \eta_3 \mathbb{I}) + (\eta - \eta_1)(T - \eta \mathbb{I})^{-1}(T - \eta_2 \mathbb{I})(T - \eta_3 \mathbb{I}) = 0 \ .
\end{aligned}$$

Note that $(T-\eta\mathbb{I})^{-1}(T-\eta_2\mathbb{I}) = \mathbb{I}+(\eta-\eta_2)(T-\eta\mathbb{I})^{-1}$ in the latter equation, and hence we have

$$
\begin{aligned}
&(T-\eta_2\mathbb{I})(T-\eta_3\mathbb{I}) + (\eta-\eta_1)(T-\eta_3\mathbb{I}) + (\eta-\eta_1)(\eta-\eta_2)(T-\eta\mathbb{I})^{-1}(T-\eta_3\mathbb{I}) = 0 \\
\Rightarrow\ &(T-\eta_2\mathbb{I})(T-\eta_3\mathbb{I}) + (\eta-\eta_1)(T-\eta_3\mathbb{I}) + (\eta-\eta_1)(\eta-\eta_2)\mathbb{I} \\
&\qquad\qquad + (\eta-\eta_1)(\eta-\eta_2)(\eta-\eta_3)(T-\eta\mathbb{I})^{-1} = 0 \\
\Rightarrow\ &T^2 + (\eta-\eta_1-\eta_2-\eta_3)\,T + \big(\eta^2 - (\eta_1+\eta_2+\eta_3)\,\eta + \eta_1\,\eta_2 + \eta_1\,\eta_3 + \eta_2\,\eta_3\big)\,\mathbb{I} \\
&= -(\eta-\eta_1)(\eta-\eta_2)(\eta-\eta_3)(T-\eta\mathbb{I})^{-1} \,.
\end{aligned}
$$

Subsequently, let $\eta \longrightarrow$ root of the characteristic polynomial, we then have

$$
\begin{aligned}
&\det\,(T-\eta\mathbb{I})\,\big[T^2 + (\eta - \operatorname{tr}T)\,T + (\eta^2 - \operatorname{tr}T\,\eta + S)\,\mathbb{I}\big] \\
&\qquad\qquad = -(\eta-\eta_1)(\eta-\eta_2)(\eta-\eta_3)(T-\eta\mathbb{I})^{\mathrm{cof}} \,.
\end{aligned}
$$

Setting $\eta = \eta_1 + \rho$ and expanding up to linear terms in the small parameter ρ, we obtain

$$
\begin{aligned}
&\rho\,\big(-3\,\eta_1^2 + 2\operatorname{tr}T\,\eta_1 - S\big)\,\big(T^2 + (\eta_1 - \operatorname{tr}T)\,T + (\eta_1^2 - \operatorname{tr}T\,\eta_1 + S)\,\mathbb{I}\big) \\
&= -\rho\,\big(3\,\eta_1^2 - 2\operatorname{tr}T\,\eta_1 + S\big)\,(T-\eta_1\mathbb{I})^{\mathrm{cof}} \\
\Rightarrow\ &T^2 + (\eta_1 - \operatorname{tr}T)\,T + (\eta_1^2 - \operatorname{tr}T\,\eta_1 + S)\,\mathbb{I} = (T-\eta_1\mathbb{I})^{\mathrm{cof}} \\
\Rightarrow\ &T^2 - (\eta_2+\eta_3)\,T + (\eta_2\,\eta_3)\,\mathbb{I} = (T-\eta_1\mathbb{I})^{\mathrm{cof}} \\
\Rightarrow\ &(T-\eta_2\mathbb{I})(T-\eta_3\mathbb{I}) = (T-\eta_1\mathbb{I})^{\mathrm{cof}} \,.
\end{aligned}
$$

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