

Relations between permutation representations in positive characteristic

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ABSTRACT

Given a finite group G and a field F , a G -set X gives rise to an $F[G]$ -permutation module $F[X]$. This defines a map from the Burnside ring of G to its representation ring over F . It is an old problem in representation theory, with wide-ranging applications in algebra, number theory, and geometry, to give explicit generators of the kernel $K_F(G)$ of this map, that is, to classify pairs of G -sets X, Y such that $F[X] \cong F[Y]$. When F has characteristic 0, a complete description of $K_F(G)$ is now known. In this paper, we give a similar description of $K_F(G)$ when F is a field of characteristic $p > 0$ in all but the most complicated case, which is when G has a subquotient that is a non- p -hypo-elementary (p, p) -Dress group.

1. Introduction

In the present paper, we study which finite G -sets X, Y , for a finite group G , give rise to isomorphic linear permutation representations over a field of positive characteristic. To explain the precise problem and the main result, we need to recall some terminology.

Let F be a commutative ring, and G a finite group. The *Burnside ring* $B(G)$ of G has one generator $[X]$ for every finite G -set X , and relations $[X] + [Y] = [Z]$ for all isomorphisms $X \sqcup Y \cong Z$ of G -sets, with multiplication being defined by $[X] \cdot [Y] = [X \times Y]$. Since every finite G -set is a finite disjoint union of transitive G -sets, and every transitive G -set is isomorphic to a set of the form G/H , where H is a subgroup of G , with G/H isomorphic to G/H' if and only if H is G -conjugate to H' , we deduce that as a group $B(G)$ is free abelian on the set of conjugacy classes of subgroups of G . We will therefore write elements Θ of $B(G)$ as linear combinations of subgroups of G , which are always understood to be taken up to conjugacy. We will also sometimes refer to these (representatives of) conjugacy classes of subgroups as *the terms of Θ* , so that if $\Theta \in B(G)$ and H is a subgroup of G , we may talk about *the coefficient of H in Θ* . The *representation ring* $R_F(G)$ of G over F has a generator $[M]$ for every finitely generated $F[G]$ -module M , and relations $[M] + [N] = [O]$ for all isomorphisms $M \oplus N \cong O$ of $F[G]$ -modules, with multiplication being defined by $[M] \cdot [N] = [M \otimes_F N]$, where G acts diagonally on the tensor product. This is not to be confused with the ring of Brauer characters of $F[G]$ -modules, which is also often denoted by $R_F(G)$, but which will not feature in our paper.

There is a natural map $B(G) \rightarrow R_F(G)$, sending the class of a G -set X to the class of the associated permutation representation $F[X]$. Let $K_F(G)$ denote the kernel of this map. Its elements will be referred to as *Brauer relations over F* , or, once the choice of F is understood, just as *relations*. It is easy to see that if F is a field, then the structure of $K_F(G)$ only depends on G and on the characteristic of F . A good understanding of Brauer relations over fields

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of different characteristics has many applications in number theory and geometry. Brauer and Kuroda were, independently, the first to systematically investigate this phenomenon when they used the non-triviality of $K_{\mathbb{Q}}(G)$ to derive interesting relations between class groups of number fields [10, 19]. Since then, Brauer relations have been found to give rise to many interesting relations between different invariants of number fields [1, 6, 21, 26], of elliptic and modular curves [13, 14, 22], and of Riemannian manifolds [3, 15, 24]. In these applications, F is usually taken to be a field, and one obtains interesting information already by analysing $K_{\mathbb{Q}}(G)$, but the sharpest results are typically achieved if one knows precisely for what primes p a given element of $B(G)$ is a relation over a field of characteristic p .

When F is a field of characteristic 0, a set of explicit generators of $K_F(G)$ for all G has been determined by the first author and Dokchitser in [2], following important advances due to Brauer himself [10], Langlands [20], Deligne [12], Snaith [23], Tornehave [25], and Bouc [9]. In contrast, almost nothing seems to be known about explicit generators of $K_F(G)$ when F is a field of positive characteristic. The main result of the present paper, Theorem 1.1, addresses that situation by making substantial progress towards a complete classification.

The standard approach to problems of this kind is to view an element of $K_F(G)$ as ‘uninteresting’ if it comes from a proper subquotient of G (see §2). Call such a relation *imprimitive*, and let $\text{Prim}_F(G)$ denote the quotient of $K_F(G)$ by the subgroup generated by the imprimitive relations. If one can find a set of generators for $\text{Prim}_F(G)$ for each finite group G , then one can give a complete description of $K_F(G)$: for every finite group G , every element of $K_F(G)$ is a linear combination of elements of the form $\text{Ind Inf } \Theta$ as Θ runs over generators of $\text{Prim}_F(U)$ for all subquotients U of G . Such a description turns out to be ideally suited for the applications in number theory and geometry mentioned above.

If p and q are prime numbers, then a finite group is called *q -quasi-elementary* if it has a normal cyclic subgroup of q -power index, equivalently if it is a split extension of a q -group by a cyclic group of order coprime to q ; a finite group is called *p -hypo-elementary* if it has a normal p -subgroup with cyclic quotient, equivalently if it is a split extension of a cyclic group of order coprime to p by a p -group; a group is called a *(p, q) -Dress group* if it has a normal p -subgroup with q -quasi-elementary quotient. A finite group is called *quasi-elementary* if it is q -quasi-elementary for some prime number q . The main result of the present paper is the following.

THEOREM 1.1. *Let F be either a field of characteristic $p > 0$, or a discrete valuation ring with finite residue field of characteristic p . Let G be a finite group, and suppose that $\text{Prim}_F(G)$ is non-trivial. Then:*

- (A) *the group G is not p -hypo-elementary, and in addition G satisfies one of the following conditions:*
 - (i) *the group $G = C \rtimes Q$ is quasi-elementary of order coprime to p , where C is cyclic and Q is a q -group for some prime number q , and either C is not of prime order, or Q does not act faithfully on C ;*
 - (ii) *there are a normal elementary abelian l -subgroup $W \cong (C_l)^d$ of G , where l is a prime number and $d \geq 1$ is an integer, and a (p, q) -Dress subgroup D of G , where q is a prime number, such that $G = W \rtimes D$, with D acting faithfully on W ; moreover, either D acts irreducibly on W , or $G = (C_l \rtimes D_1) \times (C_l \rtimes D_2)$, where D_1, D_2 are cyclic q -groups;*
 - (iii) *there is an exact sequence*

$$1 \rightarrow S^d \rightarrow G \rightarrow D \rightarrow 1,$$

where S is a non-abelian simple group, $d \geq 1$ is an integer, and D is a (p, q) -Dress group for a prime number q , such that the natural map $D \rightarrow \text{Out}(S^d)$ is injective and S^d is a minimal non-trivial normal subgroup;

- (iv) *the group G is a (p, p) -Dress group.*

Moreover,

- (B) in the cases (i)–(iii), the structure of $\text{Prim}_F(G)$ and a set of generators are as follows.
 - (i) One has $\text{Prim}_F(G) = \text{Prim}_{\mathbb{Q}}(G)$, and the latter is described by [2, Theorem A, case (4)].
 - (ii) If D is p -hypo-elementary, then $\text{Prim}_F(G) \cong \mathbb{Z}$, and otherwise $\text{Prim}_F(G) \cong \mathbb{Z}/q\mathbb{Z}$. In both cases, $\text{Prim}_F(G)$ is generated by Θ defined as follows:
 - (a) if $d = 1$ and $D = C_{mn} = C_m \times C_n$ is cyclic of order mn , where $m, n > 1$ are coprime integers, then $\Theta = G - C_{mn} + \alpha(C_n - C_l \rtimes C_n) + \beta(C_m - C_l \rtimes C_m)$, where α, β are any integers satisfying $\alpha m + \beta n = 1$;
 - (b) if $d = 1$ and $D = C_{q^{k+1}}$ is cyclic of order q^{k+1} , where $k \in \mathbb{Z}_{\geq 0}$, then $\Theta = C_{q^k} - qC_{q^{k+1}} - C_l \rtimes C_{q^k} + qG$;
 - (c) if $d \geq 2$, then $\Theta = G - D + \sum_{\substack{U \leq_G W \\ (W:U)=l}} (UN_D(U) - WN_D(U))$, where the sum runs over a full set of G -conjugacy class representatives of index l subgroups of W , and where $N_D(U)$ denotes the normaliser of U in D .
 - (iii) If D is p -hypo-elementary, then $\text{Prim}_F(G) \cong \mathbb{Z}$, and otherwise $\text{Prim}_F(G) \cong \mathbb{Z}/q\mathbb{Z}$. In both cases, $\text{Prim}_F(G)$ is generated by any relation of the form $G + \sum_{H \leq G} a_H H$, where a_H are integers.

Explicit formulae for relations as in Theorem 1.1(B)(iii) can be derived from [7, 8].

Let us briefly sketch the main ingredients of the proof and the structure of the paper. In Section 2, we recall the basic formalism of Brauer relations and results from the literature that we will need in the rest of the paper. The most important one of these is Theorem 2.7, which places tight restrictions on the possible quotients of a finite group G for which $\text{Prim}_F(G)$ is non-trivial. For example, it states that if G is a finite group for which $\text{Prim}_F(G)$ is non-trivial, then there exists a prime number q such that all proper quotients of G are (p, q) -Dress groups. If moreover G itself is not a (p, q') -Dress group for any prime number q' , then $\text{Prim}_F(G)$ is cyclic, and is generated by any relation of the form $\Theta = G + \sum_{H \leq G} a_H H$, where $a_H \in \mathbb{Z}$. This almost immediately implies the conclusions of the theorem when G is not soluble — see part (iii) of the conclusion.

In Section 3, we turn our attention to soluble groups. First, we prove in Theorem 3.2 that if q is a prime number different from p , and G is a (p, q) -Dress group with a non-trivial normal p -subgroup, then $\text{Prim}_F(G)$ is trivial. We then analyse the consequences of Theorem 2.7 for soluble groups, which leads, in Theorem 3.4, to a proof of part (A) of Theorem 1.1.

To prove part (B) of the theorem, it then remains to exhibit explicit relations of the form $\Theta = G + \sum_{H \leq G} a_H H$ for groups G appearing in the theorem that are not (p, q) -Dress for any prime number q , and to separately deal with (p, q) -Dress groups that do not have a non-trivial normal p -subgroup, that is, that are q -quasi-elementary. The main difference between the case we are treating in this paper and the case of F having characteristic 0, which was treated in [2], is that we do not have character theory at our disposal. Instead, to prove that an element of $B(G)$ is a relation, we use Conlon’s induction theorem, Theorem 2.4, so we are led to computing fixed points of various G -sets under all p -hypo-elementary subgroups of G . Section 4 is devoted to these somewhat technical calculations, and Proposition 4.1 and Theorem 4.2 furnish the final ingredients for the proof of Theorem 1.1. The whole proof is summarised at the end of Section 4.

We remark that for a full classification of Brauer relations in positive characteristic, one would also need to determine the structure and generators of $\text{Prim}_F(G)$ for groups G that are (p, p) -Dress groups. That problem is left open in this work.

Notation

Throughout the rest of the paper, we fix a prime number p , and F will denote either a field of characteristic p , or a local ring with finite residue field of characteristic p ; G will always denote

a finite group; $O_p(G)$ is the p -core of G , defined as the intersection of all its p -Sylow subgroups; for a prime q , $O^q(G)$ will denote the minimal normal subgroup of G of q -power index; $\text{Aut}(G)$ denotes the automorphism group of G , and $\text{Out}(G)$ denotes the outer automorphism group of G , that is, the quotient of $\text{Aut}(G)$ by the subgroup of inner automorphisms.

If H, U are two subgroups of G , and $g, x \in G$, then we will write ${}^g x = gxg^{-1}$ and ${}^g U = gUg^{-1}$; the normaliser of H in G will be denoted by $N_G(H)$; the commutator $[H, U]$ is the subgroup of G generated by $\{[h, u] = huh^{-1}u^{-1} : h \in H, u \in U\}$. The commutator $[H, U]$ is trivial if and only if every element of H commutes with every element of U .

If H is a subgroup of G , then a (left) transversal for G/H is a set $T \subseteq G$ such that G is a disjoint union $G = \bigsqcup_{g \in T} gH$.

The Frattini subgroup $\Phi(G)$ of G is defined as the intersection of all maximal subgroups of G . If l is a prime, and W is an l -group, then the Frattini subgroup of W is equal to $[W, W]W^l$. It has the property that a normal subgroup N of W contains the Frattini subgroup if and only if W/N is an elementary abelian l -group. It also has the property that every element of $\Phi(W)$ is a ‘non-generator’, meaning that every generating set of W remains a generating set if all elements of $\Phi(W)$ are omitted.

If R is any set of prime numbers, then an R -Hall subgroup of G is a subgroup whose order is a product of primes in R , and whose index is not divisible by any prime in R . Hall’s theorem says that if G is soluble, then for every set R of prime numbers, an R -Hall subgroup of G exists, any two R -Hall subgroups are conjugate, and every subgroup of G whose order is a product of prime numbers in R is contained in some R -Hall subgroup [16, Theorems 3.13, 3.14, Problem 3C.1]. If q is a prime number, we will say ‘ $(-q)$ -Hall subgroup’, when we mean an R -Hall subgroup for R being the set of all prime numbers except for q .

2. Basic properties and induction theorems

Let G be a finite group, let H be a subgroup of G , let N be a normal subgroup of G , and let π denote the quotient map $G \rightarrow G/N$. There are maps

$$\begin{aligned} \text{Ind}_{G/H} : \quad & \text{K}_F(H) && \rightarrow && \text{K}_F(G), \\ & \sum_{U \leq H} n_U U && \mapsto && \sum_U n_U U; \\ \\ \text{Res}_{G/H} : \quad & \text{K}_F(G) && \rightarrow && \text{K}_F(H), \\ & \sum_{U \leq G} n_U U && \mapsto && \sum_{U \leq G} \sum_{g \in H \backslash G/U} n_U ({}^g U \cap H); \\ \\ \text{Inf}_{G/N} : \quad & \text{K}_F(G/N) && \rightarrow && \text{K}_F(G), \\ & \sum_{\bar{U} \leq G/N} n_{\bar{U}} \bar{U} && \mapsto && \sum_{\bar{U} \leq G/N} n_{\bar{U}} \pi^{-1}(\bar{U}), \end{aligned}$$

induced by the natural induction, restriction, and inflation maps, respectively, on the Burnside rings.

LEMMA 2.1. *Let G be a finite group, let N be a normal subgroup of G , and let q be a prime number. Then:*

- (a) *if G is q -quasi-elementary, then so is G/N ;*
- (b) *if G is a (p, q) -Dress group, then so is G/N .*

Proof. (a) Let C be a normal cyclic subgroup of G of q -power index. Then, CN/N is a normal cyclic subgroup of G/N of q -power index.

(b) The p -core of G/N contains $O_p(G)N/N$, so $(G/N)/O_p(G/N)$ is a quotient of $G/O_p(G)$. The assertion therefore follows from part (a). \square

DEFINITION 2.2. Given a G -set X and a subgroup U of G , define $f_U(X)$ to be the number of fixed points in X under the action of U . Extended linearly, f_U defines a ring homomorphism $B(G) \rightarrow \mathbb{Z}$.

Let G be a finite group, let \mathcal{C} and \mathcal{P} denote full sets of representatives of conjugacy classes of cyclic, respectively, p -hypo-elementary subgroups of G .

THEOREM 2.3 (Artin's Induction Theorem). *The ring homomorphism*

$$\prod_{U \in \mathcal{C}} f_U : B(G) \rightarrow \prod_{U \in \mathcal{C}} \mathbb{Z}$$

has image of finite additive index, and its kernel is precisely equal to $K_{\mathbb{Q}}(G)$.

Proof. See the proof of [5, Theorem 5.6.1]. \square

THEOREM 2.4 (Conlon's Induction Theorem). *The ring homomorphism*

$$\prod_{U \in \mathcal{P}} f_U : B(G) \rightarrow \prod_{U \in \mathcal{P}} \mathbb{Z}$$

has image of finite additive index, and its kernel is precisely equal to $K_F(G)$.

Proof. See [11, § 81B] or [5, § 5.5–5.6]. \square

COROLLARY 2.5. *The group $K_F(G)$ is free abelian of rank equal to the number of conjugacy classes of non- p -hypo-elementary subgroups of G .*

COROLLARY 2.6. *There exists a Brauer relation over F of the form $a_G G + \sum_{U \in \mathcal{P}} a_U U \in K_F(G)$, where $a_G, a_U \in \mathbb{Z}$.*

Proof. If G is p -hypo-elementary, then the statement is empty, so assume that G is not p -hypo-elementary. By Theorem 2.4, the set consisting of $F[G/G]$ and of $F[G/U]$ for $U \in \mathcal{P}$ is linearly dependent in $R_F(G)$. On the other hand, it is easy to see that if the elements U_i of \mathcal{P} are ordered in non-descending order with respect to size, then the matrix $(f_{U_i}(U_j))_{U_i, U_j \in \mathcal{P}}$ is triangular with non-zero entries on the diagonal, so by Theorem 2.4 the set $\{F[G/U] : U \in \mathcal{P}\}$ is linearly independent in $R_F(G)$. This proves the corollary. \square

Corollary 2.6 is also often referred to as Conlon's Induction Theorem. The following theorem is the basic structure result on Prim_F .

THEOREM 2.7. *Let G be a finite group that is not a (p, q) -Dress group for any prime number q . Then, the following trichotomy holds:*

- (a) *if all proper quotients of G are p -hypo-elementary, then $\text{Prim}_F(G) \cong \mathbb{Z}$;*
- (b) *if there exists a prime number q such that all proper quotients of G are (p, q) -Dress groups, and at least one of them is not p -hypo-elementary, then $\text{Prim}_F(G) \cong \mathbb{Z}/q\mathbb{Z}$;*
- (c) *if there exists a proper quotient of G that is not a (p, q) -Dress group for any prime number q , or if there exist distinct prime numbers q_1 and q_2 and, for $i = 1$ and 2 , a*

proper quotient of G that is a non- p -hypo-elementary (p, q_i) -Dress group, then $\text{Prim}_F(G)$ is trivial.

In cases (a) and (b), $\text{Prim}_F(G)$ is generated by any relation in which G has coefficient 1.

Proof. See [4, Theorem 1.2]. □

COROLLARY 2.8. *Let G be a finite group, and suppose that $\text{Prim}_F(G)$ is non-trivial. Then, G is an extension of the form*

$$1 \rightarrow S^d \rightarrow G \rightarrow D \rightarrow 1,$$

where S is a finite simple group, $d \geq 0$ is an integer, and D is a (p, q) -Dress group for some prime number q . Moreover, if $d \geq 1$ and S is not cyclic, then the canonical map $D \rightarrow \text{Out}(S^d)$ is injective, and S^d has no proper non-trivial subgroups that are normal in G . In this case, $\text{Prim}_F(G) \cong \mathbb{Z}$ if D is p -hypo-elementary, and $\text{Prim}_F(G) \cong \mathbb{Z}/q\mathbb{Z}$ otherwise.

Proof. If G is a (p, q) -Dress group for some prime number q , then the assertion is clear, so suppose that it is not. The group G has a chief series, so there exists a normal subgroup $W \cong S^d$ of G , where S is a simple group and $d \geq 1$. By Theorem 2.7, the quotient G/W is a (p, q) -Dress group for some prime number q .

Now suppose that S is not cyclic. Let K be the kernel of the map $G \rightarrow \text{Aut}(S^d)$ given by conjugation. The centre of S^d is trivial, so $K \cap S^d = \{1\}$. If K is non-trivial, then G/K is a proper quotient that is not soluble, and in particular not a (p, q) -Dress group, contradicting Theorem 2.7. So G injects into $\text{Aut}(S^d)$, and thus $G/S^d = D$ injects into $\text{Out}(S^d)$. Similarly, if $N \triangleleft G$ is a proper subgroup of S^d , then G/N is not soluble, and in particular not a (p, q) -Dress group, contradicting Theorem 2.7. Finally, the description of $\text{Prim}_F(G)$ is given by Theorem 2.7. □

3. Main reduction in soluble groups

In this section, we analyse $\text{Prim}_F(G)$ for soluble groups G . The main results of the section are Theorem 3.2, concerning (p, q) -Dress groups, and Theorem 3.4, which gives necessary conditions on a soluble group G for $\text{Prim}_F(G)$ to be non-trivial. The first of these is proved by comparing the consequences of Conlon’s Induction Theorem for G and for its subquotients, while the second is derived from a careful analysis of the implications of Theorem 2.7 for soluble groups.

LEMMA 3.1. *Let q be a prime number different from p , and let $G = P \rtimes (C \rtimes Q)$ be a (p, q) -Dress group, where P is a p -group, Q is a q -group, and C is a cyclic group of order coprime to pq . Let \mathcal{S} be a full set of G -conjugacy class representatives of subgroups of P . For each $U \in \mathcal{S}$, let N_U be a $(-p)$ -Hall subgroup of $N_G(U)$, and let \mathcal{T}_U be a full set of N_U -conjugacy class representatives of subgroups of N_U . Then:*

- (a) *for every $U \in \mathcal{S}$, and all subgroups V_1, V_2 of N_U , V_1 and V_2 are N_U -conjugate if and only if they are $N_G(U)$ -conjugate;*
- (b) *for every subgroup H of G , there exists a unique $U \in \mathcal{S}$ and a unique $V \in \mathcal{T}_U$ such that H is G -conjugate to $U \rtimes V$.*

Proof. To prove part (a), let $U \in \mathcal{S}$, and $V_1, V_2 \leq N_U$, and suppose that there exists an element g of $N_G(U)$ such that ${}^gV_1 = V_2$. Since $N_G(U) = N_P(U) \rtimes N_U$, we may write $g = nu$, where $u \in N_P(U)$ and $n \in N_U$. Let $v \in V_1$. By assumption, ${}^gv \in V_2 \subseteq N_U$, so ${}^uv \in N_U$, and hence $[v, u] = v(uv^{-1}u^{-1}) \in N_U$. On the other hand, $N_P(U) = N_G(U) \cap P$ is normal in $N_G(U)$,

so $[v, u] = (vuv^{-1})u^{-1} \in N_P(U)$. Since $N_P(U) \cap N_U = \{1\}$, this implies that u and v commute. Since v was arbitrary, we deduce that u centralises V_1 , so that ${}^gV_1 = {}^nV_1 = V_2$, as claimed.

Now, we prove the existence statement of part (b). Let H be a subgroup of G , and let $U = H \cap P$. After replacing H with a subgroup that is G -conjugate to it if necessary, we may assume that $U \in \mathcal{S}$. We then have $H \leq N_G(U)$. Let V be a $(-p)$ -Hall subgroup of H , which is contained in a $(-p)$ -Hall subgroup of $N_G(U)$. Since all $(-p)$ -Hall subgroups of $N_G(U)$ are conjugate to each other, we may assume, after possibly replacing H with a subgroup that is $N_G(U)$ -conjugate to it, that V is contained in N_U , so that after possibly replacing H by a subgroup that is N_U -conjugate to it, we may assume that $V \in \mathcal{T}_U$, which concludes the proof of the existence statement.

Finally, we prove uniqueness. Let $U_1, U_2 \in \mathcal{S}$, and let $V_i \in \mathcal{T}_{U_i}$ for $i = 1, 2$ be such that $H_1 = U_1 \rtimes V_1$ is G -conjugate to $H_2 = U_2 \rtimes V_2$. Since U_i is the unique Sylow p -subgroup of H_i for $i = 1$ and 2 , this implies that U_1 and U_2 are G -conjugate; and since both are contained in P , and \mathcal{S} is assumed to be a complete set of distinct conjugacy class representatives, this implies that $U_1 = U_2$. Write $U = U_1$. We deduce that H_1 and H_2 are $N_G(U)$ -conjugate. Since V_i is a $(-p)$ -Hall subgroup of H_i for $i = 1$ and 2 , it follows that V_1 and V_2 are also $N_G(U)$ -conjugate, so by part (a), they are N_U -conjugate. Since \mathcal{T}_U is a full set of representatives of N_U -conjugacy classes, we have $V_1 = V_2$. \square

THEOREM 3.2. *Let q be a prime number different from p , and let G be a (p, q) -Dress group with non-trivial p -core. Then, $\text{Prim}_F(G)$ is trivial.*

Proof. We keep the notation of Lemma 3.1. In particular, we write $G = P \rtimes (C \rtimes Q)$, where P is a non-trivial p -group, Q is a q -group, and C a cyclic group of order coprime to pq .

For each $U \in \mathcal{S}$, identify N_U with UN_U/U via the quotient map, and consider the map

$$\iota_U = \text{Ind}_{G/UN_U} \text{Inf}_{UN_U/U} : B(N_U) \rightarrow B(G).$$

Let $I_U = \iota_U(K_F(N_U))$. Note that all $\Theta \in I_U$ are imprimitive, since either U is non-trivial, so that UN_U/U is a proper quotient, or N_U is a $(-p)$ -Hall subgroup of G , which is proper since the p -core of G is assumed to be non-trivial. We will now show that $\sum_{U \in \mathcal{S}} I_U = K_F(G)$.

First, we claim that each ι_U is injective. Inflation is always an injective map of Burnside rings, so it suffices to show that the induction map Ind_{G/UN_U} is injective on the image of $\text{Inf}_{UN_U/U}$. Let H_1 and H_2 be subgroups of UN_U containing U that are G -conjugate. Since the common p -core is U , they are then $N_G(U)$ -conjugate. Since each of their respective $(-p)$ -Hall subgroups is contained in a $(-p)$ -Hall subgroup of UN_U , and all $(-p)$ -Hall subgroups of UN_U are conjugate, we may assume, replacing H_1 and H_2 by UN_U -conjugate subgroups if necessary, that $H_i = UV_i$, where $V_i \leq N_U$ for $i = 1, 2$, and where V_1 is $N_G(U)$ -conjugate to V_2 . Lemma 3.1(a) then implies that V_1 and V_2 are also N_U -conjugate, so H_1 and H_2 are UN_U -conjugate, and injectivity of ι_U follows.

Next, we claim that the I_U for $U \in \mathcal{S}$ are linearly independent. Indeed, suppose that $\sum_{U \in \mathcal{S}} \Theta_U = 0$, where $\Theta_U \in I_U$. Let U be maximal with respect to inclusion subject to the property that $\Theta_U \neq 0$. Then, all terms of Θ_U contain U , while for all elements $U' \neq U$ of \mathcal{S} , all terms of $\Theta_{U'}$ are contained in $U'N_{U'}$, which does not contain U . Thus, for the sum to vanish, we must have $\Theta_U = 0$ — a contradiction.

A similar argument shows that $\sum_{U \in \mathcal{S}} I_U$ is saturated in $K_F(G)$: suppose that $\sum_{U \in \mathcal{S}} \Theta_U$ is divisible by some $n \in \mathbb{Z}_{\geq 2}$ in $K_F(G)$ for $\Theta_U \in I_U$, and consider $U \in \mathcal{S}$ that is maximal subject to the property that Θ_U is not divisible by n in $K_F(U)$, or, equivalently, in $B(U)$; then note that the above argument shows that for every subgroup H of G that contains U , the coefficient of H in $\Theta_{U'}$ is divisible by n for all elements $U' \neq U$ of \mathcal{S} , so its coefficient in Θ_U must also be divisible by n , so that in fact Θ_U is divisible by n in $B(G)$ — a contradiction.

To prove equality, it therefore only remains to compare the ranks of $\sum_{U \in \mathcal{S}} I_U$ and of $K_F(G)$. By linear independence and by Corollary 2.5, we have

$$\begin{aligned} \text{rank} \left(\sum_{U \in \mathcal{S}} I_U \right) &= \sum_{U \in \mathcal{S}} \text{rank } I_U \\ &= \sum_{U \in \mathcal{S}} \#\{\text{conjugacy classes of non-cyclic subgroups of } N_U\}, \end{aligned}$$

and by Lemma 3.1(b) and Corollary 2.6, this is equal to the rank of $K_F(G)$, which completes the proof. \square

LEMMA 3.3. *Let G be a finite group, and let W be an abelian normal subgroup with quotient D . Suppose that there exists a normal subgroup H of D such that $\gcd(\#H, \#W) = 1$ and such that no non-identity element of W is fixed under the natural conjugation action of H on W . Then, $G \cong W \rtimes D$.*

Proof. We may view W as a module under D . Since H and W have coprime orders, the cohomology group $H^1(H, W)$ vanishes, so the Hochschild–Serre spectral sequence gives an exact sequence

$$H^2(D/H, W^H) \rightarrow H^2(D, W) \rightarrow H^2(H, W).$$

The last term in this sequence also vanishes by the coprimality assumption, while the first term vanishes, since W^H is assumed to be trivial. Hence, $H^2(D, W) = 0$, and therefore the extension G of D by W splits. \square

THEOREM 3.4. *Let G be a finite soluble group, and suppose that $\text{Prim}_F(G)$ is non-trivial. Then G is one of the following:*

- (i) a quasi-elementary group $C \rtimes Q$ of order coprime to p , where C is cyclic and Q is a q -group for some prime number q , and either C is not of prime order, or Q does not act faithfully on C ,
- (ii) a semidirect product $G = W \rtimes D$, where $W = (C_l)^d$ for a prime number $l \neq p$ and an integer $d \geq 1$, and D is a (p, q) -Dress group for some prime number q , acting faithfully and irreducibly on W ,
- (iii) $G = (C_l \rtimes D_1) \times (C_l \rtimes D_2)$, where $l \neq p$ is a prime number, D_1, D_2 are cyclic q -groups for a prime number q that act faithfully on $C_l \times C_l$,
- (iv) a (p, p) -Dress group.

Proof. We begin by observing that if G is a (p, q) -Dress group for some prime number q , then the conclusion of the theorem holds. Indeed, if G is a (p, p) -Dress group, then this is clear. If, on the other hand, G is a (p, q) -Dress group for a prime number $q \neq p$, then it follows from Theorem 3.2 that G must have trivial p -core, so the order of G is coprime to p , which implies that all p -hypo-elementary subquotients of G are cyclic. By Theorems 2.3 and 2.4, we then have $\text{Prim}_F(G) = \text{Prim}_{\mathbb{Q}}(G)$, and it follows from [2, Theorem A, case (4)] that G satisfies the conditions of part (i) of the theorem. In particular, if G is quasi-elementary, then the conclusion of the theorem holds.

We will repeatedly use this observation without further mention.

By Corollary 2.8, G is an extension of the form

$$1 \rightarrow W = (C_l)^d \rightarrow G \rightarrow D \rightarrow 1, \tag{3.5}$$

where l is a prime number, $d \geq 0$ is an integer, and D is a (p, q) -Dress group for some prime number q . If $d = 0$ or $l = p$, then G is a (p, q) -Dress group, and we are done. For the rest of the proof, assume that $d \geq 1$ and $l \neq p$. We now consider several cases.

Case 1: $l \nmid \#D$. By the Schur–Zassenhaus theorem [16, Theorem 3.8], the short exact sequence (3.5) splits, so we have $G \cong W \rtimes D$, and we may view D as a subgroup of G . Let $N \triangleleft G$ be the centraliser of W in D .

Case 1(a): $N \neq \{1\}$ and D is p -hypo-elementary. The subgroup WN/N is normal in G/N . By Theorem 2.7, G/N is a (p, q) -Dress group for some prime number q . It follows that D/N is also normal in G/N , so $G/N = WN/N \times D/N$, so the commutator $[W, D]$ is contained in $N \leq D$. But also, since W is normal in G , this commutator is contained in W , so it is trivial. It follows that W commutes with D , and G is a (p, l) -Dress group.

Case 1(b): $N \neq \{1\}$ and D is not p -hypo-elementary. By Theorem 2.7, G/N is a (p, q) -Dress group. Since $l \nmid pq$, this implies that W must be cyclic, and, by the same argument as in case 1(a), it must commute with $O^q(D)$. It follows that G is a (p, q) -Dress group.

Case 1(c): $N = \{1\}$ and D acts reducibly on W . Let U be a proper non-trivial subgroup of W that is normal in G . Since $l \nmid \#D$, the $\mathbb{F}_l[D]$ -module W is semisimple, so there exists a subgroup V of W that is normal in G and such that $UV = W$ and $U \cap V = \{1\}$. By Theorem 2.7, both G/U and G/V are (p, q) -Dress groups. Since $l \nmid pq$, this implies that $V \cong W/U \cong C_l$ and $U \cong W/V \cong C_l$. Thus, $G \cong (U \rtimes D_1) \times (V \rtimes D_2)$, where D_1 acts faithfully on U , and D_2 acts faithfully on V , and in particular both are cyclic. It follows that $O_p(G/U)$ is of the form NU/U for a p -subgroup N of D_1 . For G/U to be a (p, q) -Dress group, the $(-q)$ -Hall subgroup of G/UN must be cyclic, which forces D_2 to be a q -group, and similarly for D_1 . This is case (iii) of the theorem.

Case 1(d): $N = \{1\}$ and D acts irreducibly on W . This is case (ii) of the theorem.

Case 2: $l \mid \#D$ and $G = W \rtimes D$. In this case, $N = \ker(D \rightarrow \text{Aut } W)$ is again a normal subgroup of G .

Case 2(a): $N \neq \{1\}$. By Theorem 2.7, the quotient G/N is a (p, q) -Dress group. Since D/N acts faithfully on W , no non-trivial subgroup of D/N can be normal in G/N . In particular, $O_p(G/N)$ must be trivial, so N contains $O_p(D)$, and G/N is in fact quasi-elementary, $G/N \cong C \rtimes Q$, where C is cyclic and Q is a q -group. By the same argument, C is an l -group. Now, if $q = l$, then G/N is an l -group, and G is an extension of an l -group by the (p, l) -Dress group N , hence is itself a (p, l) -Dress group. If $q \neq l$, then W must be cyclic, and must commute with $O_p(D)$, so $O_p(D)$ is normal in G , and $G/O_p(D)$ is q -quasi-elementary, whence G is a (p, q) -Dress group.

Case 2(b): $N = \{1\}$ and D acts reducibly on W . Let $U \leq W$ be a non-zero proper $\mathbb{F}_l[D]$ -subrepresentation of W . By Theorem 2.7, the quotient G/U is a (p, q) -Dress group.

Case 2(b)(i): $l \neq q$. Then, the l -Sylow subgroups of G/U must be cyclic. In particular, any l -Sylow subgroup C of D , which is non-trivial by assumption, acts trivially by conjugation on W/U . Since G is assumed to be a semi-direct product, the l -Sylow subgroup of G/U is a direct product of W/U and C , and therefore cannot be cyclic—a contradiction.

Case 2(b)(ii): $l = q$. Either G/U is an l -group, in which case so is G , and we are in case (i) of the theorem; or there exists a subgroup $C \leq D$ of order coprime to l such that CU/U is normal in G/U , and in particular C is normal in D . The $\mathbb{F}_l[C]$ -module W is then semisimple, so there exists a subgroup $V \leq W$ that is normalised by C , and such that $VU = W$ and $V \cap U = \{1\}$. Since CU/U is normal in G/U , and since W/U is also normal in G/U , CU/U and W/U commute, so we have $[C, V] \leq U$. But since V is normalised by C , we also have $[C, V] \leq V$, so C in fact centralises V . Thus, V is contained in W^C , which is a normal subgroup of G . If $W^C = W$, then $C \leq N$, contradicting the assumption that $N = \{1\}$. So W^C is a proper non-trivial subgroup of W . Since $l \nmid \#C$, there exists a non-trivial subgroup $U' \leq W$ such that $W = U'W^C$ and $U' \cap W^C = \{1\}$. In particular, $(U')^C = \{1\}$. By Theorem 2.7, the quotient G/W^C is (p, l) -Dress, so CW^C/W^C is contained in the normal subgroup $O^l(G/W^C) = O^l(D)W^C/W^C$.

It follows that $[C, U'] \leq W^C O^l(D)$. But since U' is normalised by C , we also have $[C, U'] \leq U'$. Since $U' \cap W^C O^l(D)$ is trivial, we deduce that C centralises U' —a contradiction.

Case 2(c): $N = \{1\}$, and D acts irreducibly on W . This is case (ii) of the theorem.

Case 3: $l \mid \#D$ and the extension of D by W is not split. By the Schur–Zassenhaus theorem, the preimage of $O_p(D)$ under the quotient map $G \rightarrow G/W$ is a split extension by W . Let P be a complement to W in this preimage. In other words, P is a subgroup of G that maps isomorphically onto $O_p(D)$ under the quotient map $G \rightarrow G/W$.

Case 3(a): $P = \{1\}$ and $l \neq q$. Then, the l -Sylow subgroup S of G is normal in G . If it is elementary abelian, then the extension of D by S splits by the Schur–Zassenhaus theorem [16, Theorem 3.8], and we are in Case 2 of the proof. Otherwise, the Frattini subgroup $\Phi = [S, S]S^l$ of S is non-trivial, and since it is a characteristic subgroup of S , it is normal in G . By Theorem 2.7, the quotient G/Φ is a (p, q) -Dress group, so the l -Sylow subgroup of G/Φ is cyclic. But since Φ consists of ‘non-generators’ of S , this implies that S itself is cyclic, so G is q -quasi-elementary.

Case 3(b): $P = \{1\}$ and $p \neq l = q$. Let C be a $(-l)$ -Hall subgroup of G . The assumptions on G imply that C is cyclic, and that D is of the form $C \rtimes Q$, where Q is a q -group. If $W^C = W$, then C is a normal subgroup of G , and G is q -quasi-elementary. If $W^C = \{1\}$, then Lemma 3.3 implies that the extension of D by W splits — a contradiction. So W^C is a non-trivial proper subgroup of W , which is normal in G , since C is normal in D . Since the order of C is coprime to l , the $\mathbb{F}_l[C]$ -representation W is semisimple, so there exists a subgroup U of W that is normalised by C , and such that $UW^C = W$, $U \cap W^C = \{1\}$. By Theorem 2.7, the quotient G/W^C is a (p, q) -Dress group. But it has trivial p -Sylow subgroup, so it is q -quasi-elementary, and CW^C/W^C is normal in G/W^C . Thus, $[C, U] \leq W^C$. But also, U is a C -subrepresentation, so $[C, U] \leq U$, whence we deduce that C centralises U , so that $W^C = W$ — a contradiction.

Case 3(c): $P \neq \{1\}$ and $W^P = W$. In this case, P is a non-trivial normal p -subgroup of G . By Theorem 2.7, the quotient G/P is (p, q) -Dress, therefore so is G itself.

Case 3(d): $P \neq \{1\}$ and $W^P \neq W$. By Lemma 3.3, the subgroup W^P is non-trivial. Moreover, since P is a normal subgroup of D , W^P is a normal subgroup of G . The \mathbb{F}_l -representation W of P is semisimple, so there exists a subgroup $U \leq W$ that is normalised by P and such that $UW^P = W$, $U \cap W^P = \{1\}$. By Theorem 2.7, the quotient G/W^P is a (p, q) -Dress group. We claim that $O_p(G/W^P)$ must be trivial. Indeed, $O_p(G/W^P)$ is necessarily of the form NW^P/W^P , where N is a subgroup of P that is normal in D . But then we have $[N, U] \leq W^P$, and also $[N, U] \leq U$, since U is a P -subrepresentation of W . Thus N centralises U , whence $W^N = W$. By Lemma 3.3, the assumption that the extension of D by W is non-split forces $N = \{1\}$.

Case 3(d)(i): $l \neq q$. Then the l -Sylow subgroup of G/W^P must be cyclic and normal in G/W^P . Since $W^P \neq W$, and since we assume that $l \mid \#D$, this implies that the l -Sylow subgroup S of G is normal in G and has an element of order strictly greater than l . Thus, the Frattini subgroup $\Phi = [S, S]S^l$ of S is non-trivial, and since it is a characteristic subgroup of S , it is normal in G . By Theorem 2.7, the quotient G/Φ is a (p, q) -Dress group, so the l -Sylow subgroup of G/Φ is cyclic. But that implies that the l -Sylow subgroup of G is also cyclic, and therefore $W \cong C_l$, contradicting the assumptions that $\{1\} \neq W^P \neq W$.

Case 3(d)(ii): $l = q$. Then, $p \neq q$, so the p -Sylow subgroup of the (p, q) -Dress group G/W^P must be normal in G/W^P , contradicting the observation that $O_p(G/W^P)$ is trivial.

This covers all possible cases, and concludes the proof. □

4. Explicit relations

In the present section, we prove Theorem 1.1. Proposition 4.1 below proves parts (B)(ii)(a) and (B)(ii)(b) of the theorem. The main remaining step is to prove that the element appearing

in part (B)(ii)(c) of the Theorem is indeed an element of $K_F(G)$, and that is achieved in Theorem 4.2. Most of the section is devoted to the proof of Theorem 4.2. With all the ingredients in place, the proof of Theorem 1.1 is assembled from them at the end of the section.

PROPOSITION 4.1. *Let $l \neq p$ be a prime number, and let $G = C_l \rtimes C$, where C is a non-trivial cyclic group, acting faithfully on C_l . Then, $\text{Prim } G \cong \mathbb{Z}$, and is generated by the following relation Θ :*

(i) *if $C \cong C_{mn}$, where $m, n > 1$ are coprime integers, then*

$$\Theta = G - C + \alpha(C_n - C_l \rtimes C_n) + \beta(C_m - C_l \rtimes C_m),$$

where α, β are any integers satisfying $\alpha m + \beta n = 1$;

(ii) *if $C \cong C_{q^{k+1}}$, where q is a prime number, and $k \in \mathbb{Z}_{\geq 0}$, then*

$$\Theta = C_{q^k} - qC - C_l \rtimes C_{q^k} + qG.$$

Proof. The hypotheses on G imply that all non-cyclic subquotients of G have trivial p -core, so a subquotient of G is cyclic if and only if it is p -hypo-elementary. It therefore follows from Theorems 2.3 and 2.4 that $K_F(G) = K_{\mathbb{Q}}(G)$, and $\text{Prim}_F(G) = \text{Prim}_{\mathbb{Q}}(G)$, and the result follows from [2, Theorem A, case 3a]. □

THEOREM 4.2. *Let $l \neq p$ and q be prime numbers, let $G = W \rtimes Q$, where $W = (C_l)^d$ with $d \in \mathbb{Z}_{\geq 2}$, and Q is a (p, q) -Dress group acting faithfully on W . Suppose that either Q acts irreducibly on W , or $d = 2$, and $G = (C_l \rtimes P_1) \times (C_l \rtimes P_2)$, where the P_i are q -groups acting faithfully on the respective factor of W . Then, the element*

$$\Theta = G - Q + \sum_{\substack{U \leq G^W \\ (W:U)=l}} (U N_Q(U) - W N_Q(U)),$$

of $B(G)$ is in $K_F(G)$, where the sum runs over a full set of G -conjugacy class representatives of index l subgroups of W .

The proof of the theorem will require some preparation.

Recall from Definition 2.2 that if X is a G -set, and U is a subgroup of G , then $f_U(X)$ denotes the number of fixed points in X under U , and that this extends linearly to a ring homomorphism $f_U : B(G) \rightarrow \mathbb{Z}$.

LEMMA 4.3. *Let G be a finite group, and let H and K be subgroups. Then, $f_K(H) = \#\{g \in G/H : K \subseteq {}^gH\} = \#\{g \in G/H : {}^gK \subseteq H\}$.*

Proof. By Mackey’s formula for G -sets, we have

$$\text{Res}_K(G/H) = \bigsqcup_{g \in K \backslash G/H} K / ({}^gH \cap K).$$

By definition, $f_K(H)$ is the number of singleton orbits under the action of K on G/H , so $f_K(H) = \#\{g \in K \backslash G/H : K \subseteq {}^gH\}$. An explicit calculation shows that the map $G/H \rightarrow K \backslash G/H$, $gH \mapsto KgH$ defines a bijection between $\{g \in G/H : K \subseteq {}^gH\}$ and $\{g \in K \backslash G/H : K \subseteq {}^gH\}$, which proves the first equality. The second equality is clear. □

LEMMA 4.4. *Let G be a finite group, let l be a prime number, and let \mathcal{K} be a field of characteristic l . Suppose that there exists a normal subgroup N of G such that $l \nmid \#N$ and G/N is a cyclic l -group. Then, for every $\mathcal{K}[G]$ -module M , we have $\dim_{\mathcal{K}} M^G = \dim_{\mathcal{K}} M_G$. Moreover, if M is an indecomposable $\mathcal{K}[G]$ -module, then this dimension is 0 or 1.*

Proof. Let M be a $\mathcal{K}[G]$ -module. We may, without loss of generality, assume that M is indecomposable. The element $e = (1/\#N) \sum_{n \in N} n \in \mathcal{K}[G]$ is a central idempotent, and we have $M^N = eM$. If $M^G = 0$, then it follows from the assumption that G/N is an l -group that $M^N = 0$ also. Since $l \nmid \#N$, the N -module M is semisimple, so $M_N = 0$ also, so *a fortiori* $M_G = 0$, and we are done.

Suppose that $M^G \neq 0$, so $eM \neq 0$. Since $M = eM \oplus (1 - e)M$, and M is indecomposable, it follows that $eM = M$, so that M is an indecomposable $\mathcal{K}[G/N]$ -module. Since G/N is a cyclic l -group, it follows from [17, 18] that the maximal semisimple submodule and the maximal semisimple quotient module of M are both simple. But the only simple $\mathcal{K}[G/N]$ -module is the trivial one, which completes the proof. \square

LEMMA 4.5. *Let l be a prime number, let $d \geq 1$ be an integer, let $G = W \rtimes Q$, where $W = (C_l)^d$ and Q is any subgroup of G . Let Θ be the element of $B(G)$ given by*

$$\Theta = G - Q + \sum_{\substack{U \leq_G W \\ (W:U)=l}} (U N_Q(U) - W N_Q(U)),$$

where the sum runs over a full set of G -conjugacy class representatives of index l subgroups of W . Then for every subgroup K of Q , we have $f_K(\Theta) = \#(W_K) - \#(W^K)$.

Proof. For $w \in W$, we have $K \leq {}^w Q$ if and only if $(w^{-1}kwk^{-1})k \in Q$ for all $k \in K$. Since the bracketed term is in W , this is equivalent to $w^{-1}kwk^{-1} = 1$ for all $k \in K$, that is, to $w \in W^K$. Since W forms a transversal for G/Q , it follows from Lemma 4.3 that

$$f_K(G) = 1, \tag{4.6}$$

$$f_K(Q) = \#\{w \in W : K \leq {}^w Q\} = \#(W^K). \tag{4.7}$$

We now calculate the remaining terms in $f_K(\Theta)$. Let $U \leq W$ be a subgroup of index l . Let $T \subseteq Q$ be a transversal for $G/W N_Q(U)$. Let $x \in W \setminus U$. Then, a transversal for $G/U N_Q(U)$ is given by $\{tx^m : t \in T, 0 \leq m \leq l - 1\}$. Applying Lemma 4.3, and noting that $K \leq {}^t Q$ for all $t \in T$, we have

$$f_K(W N_Q(U)) = \#\{t \in T : K \leq {}^t N_Q(U)\},$$

and

$$f_K(U N_Q(U)) = \#\{(t, m) \in T \times \{0, \dots, l - 1\} : K \leq {}^{tx^m} (U N_Q(U))\}.$$

To count that last number, we note that for all $k \in K$, and for all $y = tx^m$ in the above transversal, we have ${}^{y^{-1}}k = (x^{-m}t^{-1}ktx^m t^{-1}k^{-1}t)(t^{-1}kt)$, and of the two bracketed terms the first is in W , and is equal to $[x^{-m}, t^{-1}kt]$, while the second is in Q . It follows that we have $K \leq {}^y (U N_Q(U))$ if and only if $[x^{-m}, t^{-1}Kt] \leq U$ and $t^{-1}Kt \leq N_Q(U)$. If $m \neq 0$, then these conditions are equivalent to $[\langle x \rangle, t^{-1}Kt] \leq U$ and $t^{-1}Kt \leq N_Q(U)$, and in particular are independent of m . Partitioning the transversal $\{tx^m : t \in T, 0 \leq m \leq l - 1\} = T \sqcup \{tx^m : t \in T, 1 \leq m \leq l - 1\}$, we find that

$$\begin{aligned} f_K(U N_Q(U) - W N_Q(U)) &= (l - 1) \cdot \#\{t \in T : t^{-1}Kt \leq N_Q(U), [\langle x \rangle, t^{-1}Kt] \leq U\} \\ &= (l - 1) \cdot \#\{t \in T : K \leq N_Q({}^t U), K \text{ acts trivially on } W/{}^t U\}. \end{aligned}$$

As t runs over T , ${}^t U$ runs once over the G -orbit of U , since T is a transversal for $G/W N_Q(U) = G/N_G(U)$. It follows that if we take the sum of the above expression over a full set of

representatives U of G -conjugacy classes of index l subgroups of W , we obtain

$$\begin{aligned} & \sum_{\substack{U \leq G^W \\ (W:U)=l}} f_K(U N_Q(U) - W N_Q(U)) \\ &= (l - 1) \#\{\text{quotients of } W_K \text{ of order } l\} = \#(W_K) - 1. \end{aligned} \tag{4.8}$$

The result follows by combining equations (4.6)–(4.8). □

LEMMA 4.9. *Let $G = W \rtimes Q$ be a soluble group, where $W = (C_l)^d$ for some prime number $l \neq p$, so that W is naturally an $\mathbb{F}_l[G]$ -module, and let $K \leq G$ be a subgroup of the form $K = K_{l'} \rtimes \langle \gamma \rangle$, where $K_{l'}$ is contained in Q and is of order coprime to l , and $\gamma = wh$ is of order a power of l , with $w \in W$ and $h \in Q$. Suppose that K is not G -conjugate to any subgroup of Q . Then, there exists an $\mathbb{F}_l[K]$ -submodule U of W of index l not containing w . Moreover, for any such $U \leq W$, the group K acts trivially on W/U .*

Proof. First, we claim that $w \in W^{K_{l'}}$. Let $k \in K_{l'}$ be arbitrary. Since $K_{l'}$ is normal in K , we have $whkh^{-1}w^{-1} \in K_{l'} \subset Q$. But also, since $whkh^{-1}w^{-1} = [w, hkh^{-1}]hkh^{-1}$, and $hkh^{-1} \in Q$, it follows that $[w, hkh^{-1}] \in Q$. On the other hand, since $w \in W$, and W is normal in G , we also have $[w, hkh^{-1}] \in W$, hence $[w, hkh^{-1}] = 1$, or equivalently $kh^{-1}w^{-1}hk^{-1} = h^{-1}w^{-1}h$. We deduce that $h^{-1}wh \in W^{K_{l'}}$. But since $K_{l'}$ is normal in K , the subgroup $W^{K_{l'}}$ is a K -submodule of W , so that $w = \gamma(h^{-1}wh) \in W^{K_{l'}}$ also, as claimed.

Let $W^{K_{l'}} = N \oplus N'$ as a K -module, where N is an indecomposable K -module containing w . Since $K_{l'}$ acts trivially on N , we may view it as an $\mathbb{F}_l[\langle \gamma \rangle]$ -module. Let e_1, \dots, e_k be an \mathbb{F}_l -basis of N with respect to which γ acts in Jordan normal form. Then, we claim that w is not contained in the proper K -submodule L generated by e_1, \dots, e_{k-1} . Indeed, if it were, say $w = e_1^{\alpha_1} \cdots e_{k-1}^{\alpha_{k-1}}$ for $\alpha_1, \dots, \alpha_{k-1} \in \mathbb{Z}$, then the element $e_2^{-\alpha_1} \cdots e_k^{-\alpha_{k-1}}$ would conjugate wh to h and would commute with $K_{l'}$, thus conjugating K to a subgroup of Q , which contradicts the hypotheses on K . Thus, the submodule $U = L \oplus N'$ satisfies the conclusions of the lemma.

Finally, for any U satisfying those conclusions, $K_{l'}$ acts trivially on W/U , since it centralises $w \notin U$. Moreover, $K/K_{l'}$ is an l -group, so also acts trivially on W/U , since that quotient has order l . □

LEMMA 4.10. *Let $G = W \rtimes Q$, $K = K_{l'} \rtimes \langle \gamma \rangle$, $w \in W$, and $U \leq W$ be as in Lemma 4.9. Let \mathcal{S}_1 be the set of subgroups of W of index l that are normalised by K , do not contain w , and are different from U , and let \mathcal{S}_2 be the set of subgroups V of W of index l that are normalised by K , contain w , and such that K acts trivially on W/V by conjugation. Then, $\#\mathcal{S}_1 = (l - 1) \cdot \#\mathcal{S}_2$.*

Proof. Suppose that either of \mathcal{S}_1 or \mathcal{S}_2 is non-empty, let $U' \in \mathcal{S}_1 \cup \mathcal{S}_2$. Then, $U \cap U'$ is a K -submodule of W of index l^2 , and the $\mathbb{F}_l[K]$ -module $W/(U \cap U')$ has at least two distinct quotients of order l with trivial K -action, namely W/U and W/U' . It follows that the $\mathbb{F}_l[K]$ -module $W/(U \cap U')$ splits completely as a direct sum of two trivial $\mathbb{F}_l[K]$ -modules. Thus, there exist exactly $l + 1$ index l submodules of W containing $U \cap U'$, one of them equal to U , exactly one of them containing w , and thus in \mathcal{S}_2 , and $l - 1$ distinct elements of \mathcal{S}_1 . This proves that the map $\mathcal{S}_1 \rightarrow \mathcal{S}_2, U'' \mapsto \langle w \rangle(U \cap U'')$ is $(l - 1)$ to 1, and hence the lemma. □

LEMMA 4.11. *Let $G = W \rtimes Q$ be as in Lemma 4.9. Let $K = K_{l'} \rtimes \langle \gamma \rangle$ be a subgroup of G , where $K_{l'}$ is contained in Q and has order coprime to l , and γ has order a power of l . Let $U \leq W$ be a subgroup of index l , let $t \in Q$, let $x \in W \setminus U$, and let $m \in \{1, \dots, l - 1\}$. Then, the following are equivalent:*

- (i) for all $n \in \{1, \dots, l-1\}$, we have $K \leq^{tx^n} (U N_Q(U))$;
- (ii) we have $K \leq^{tx^m} (U N_Q(U))$;
- (iii) we have $K \leq {}^t N_G(U)$, $[{}^t \langle x \rangle, K] \leq {}^t U$, and $w \in {}^t U$.

Proof. We will first show that (ii) is equivalent to (iii). We clearly have $K \leq^{tx^m} (U N_Q(U))$ if and only if

- (a) $K_{l'} \leq^{tx^m} (U N_Q(U))$ and
- (b) $\gamma \in {}^{tx^m} (U N_Q(U))$.

First, we discuss (a). Let $k \in K_{l'}$. Then,

$$x^{-m} t^{-1} k t x^m = x^{-m} (t^{-1} k t) x^m (t^{-1} k t)^{-1} (t^{-1} k t),$$

where the last bracketed term is in Q , and the expression preceding it is in W and equals $[x^{-m}, t^{-1} k t]$. It follows that (a) is equivalent to $[{}^t \langle x^{-m} \rangle, K_{l'}] \leq {}^t U$ and $K_{l'} \leq {}^t N_Q(U)$. Moreover, since $(t^{-1} k t) x^m (t^{-1} k t)^{-1} \in W$, and W is abelian, we have $x^{-m} (t^{-1} k t) x^m (t^{-1} k t)^{-1} = (x(t^{-1} k t) x^{-1} (t^{-1} k t)^{-1})^{-m}$, so that $[{}^t \langle x^m \rangle, K_{l'}] \leq {}^t U$ if and only if $[{}^t \langle x \rangle, K_{l'}] \leq {}^t U$. In summary, (a) is equivalent to $[{}^t \langle x \rangle, K_{l'}] \leq {}^t U$ and $K_{l'} \leq {}^t N_Q(U)$.

We analyse condition (b) similarly. Write $\gamma = wh$, where $w \in W$ and $h \in Q$. Then, by the same calculation as before, (b) is equivalent to $[{}^t \langle x \rangle, \gamma] w \leq {}^t U$ and $h \in {}^t N_Q(U)$. But if $h \in {}^t N_Q(U) = N_Q({}^t U)$, then γ normalises ${}^t U$ and, having order a power of l , acts trivially on the quotient $W/{}^t U$, so that in this case $\gamma({}^t x)^{-1} \gamma^{-1} = ({}^t x)^{-1} u$ for some $u \in {}^t U$. We then have $[{}^t x, \gamma] w = uw$, and the condition that this is in ${}^t U$ is equivalent to $w \in {}^t U$, so condition (b) is equivalent to $[{}^t \langle x \rangle, \gamma] \leq {}^t U$ and $w \in {}^t U$. This proves the equivalence between (ii) and (iii).

Since the condition (iii) does not depend on m , this also proves the equivalence between (i) and (ii). □

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. By Theorem 2.4, the statement of the theorem is equivalent to the claim that for all p -hypo-elementary subgroups K of G , we have $f_K(\Theta) = 0$.

If K is a p -hypo-elementary subgroup of G , then either K is G -conjugate to a subgroup of Q ; or Hall's theorem implies that the $(-l)$ -Hall subgroup of K , which is necessarily normal in K , is conjugate to a subgroup of Q , so that, possibly after replacing with a conjugate subgroup, K is as in Lemmas 4.9 and 4.10.

If K is a p -hypo-elementary subgroup that is conjugate to a subgroup of Q , then by Lemma 4.5, we have $f_K(\theta) = \#W_K - \#W^K$, which is equal to 0 by Lemma 4.4.

Suppose that $K = K_{l'} \rtimes \langle \gamma \rangle$, where $K_{l'}$ is of order coprime to l and is contained in Q , and γ has order a power of l , and assume that K is not conjugate to a subgroup of Q . Then, we have $f_K(G) = 1$ and $f_K(Q) = 0$. Write $\gamma = wh$, where $w \in W$ and $h \in Q$. Let $U \leq W$ have index l , let $T \subseteq Q$ be a transversal for $G/W N_Q(U)$, and let $x \in W \setminus U$, so that a transversal for $G/U N_Q(U)$ is given by $\{tx^m : t \in T, 0 \leq m \leq l-1\}$. Then, by Lemma 4.3, we have

$$\begin{aligned} f_K(W N_Q(U)) &= \#\{t \in T : K \leq {}^t (W N_Q(U))\}, \\ f_K(U N_Q(U)) &= \#\{t \in T : K \leq {}^t (U N_Q(U))\} \\ &\quad + \#\{(t, m) \in T \times \{1, \dots, l-1\} : K \leq^{tx^m} (U N_Q(U))\}. \end{aligned}$$

For $t \in T$, the condition that $K \subseteq {}^t (W N_Q(U))$ and $K \not\subseteq {}^t (U N_Q(U))$ is equivalent to $K \leq {}^t N_G(U)$ and $w \notin {}^t U$. Combining these observations with Lemma 4.11, we have

$$\begin{aligned}
& f_K(UN_Q(U) - WN_Q(U)) \\
&= \#\{(t, m) \in T \times \{1, \dots, l-1\} : K \leqslant {}^{tx^m}(UN_Q(U))\} \\
&\quad - \#\{t \in T : K \leqslant {}^tN_G(U), w \notin {}^tU\} \\
&= (l-1)\#\{t \in T : K \leqslant {}^tN_G(U), [{}^t\langle x \rangle, K] \leqslant {}^tU, w \in {}^tU\} \\
&\quad - \#\{t \in T : K \leqslant {}^tN_G(U), w \notin {}^tU\}.
\end{aligned}$$

For $K \leqslant {}^tN_G(U) = N_G({}^tU)$, the condition $[{}^t\langle x \rangle, K] \leqslant {}^tU$ is equivalent to the condition that K acts trivially on the quotient $W/{}^tU$. Since T is a transversal for $G/N_G(U)$, it follows that as t runs over T , tU runs exactly once over the G -orbit of U . Hence, summing over representatives of G -orbits of hyperplanes of W , we deduce

$$\begin{aligned}
f_K(\Theta) &= 1 + (l-1)\#\{U \leqslant W : (W : U) = l, K \leqslant N_G(U), w \in U, (W/U)^K = W/U\} \\
&\quad - \#\{U \leqslant W : (W : U) = l, K \leqslant N_G(U), w \notin U\}.
\end{aligned}$$

By Lemma 4.10, this is equal to 0. \square

Proof of Theorem 1.1. Part (A) follows from Corollary 2.8 if G is not soluble, and from Theorem 3.4 if G is soluble. Part (B)(i) follows by combining Theorems 2.3 and 2.4. Suppose that G is as in part (A)(ii). If either $d > 1$ or D is not of prime power order, then G is not a (p, q') -Dress group for any prime number q' , while it is easy to see that all its proper quotients are (p, q) -Dress groups, so by Theorem 2.7 $\text{Prim}_F(G)$ has the claimed structure, and is generated by any relation in which G has coefficient 1. Thus, part (B)(ii)(a) follows from Theorem 4.1(i), while part (B)(ii)(c) follows from Theorem 4.2. The quasi-elementary case, part (B)(ii)(b) follows from Proposition 4.1(ii). Finally, part (B)(iii) follows from Corollary 2.8. \square

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