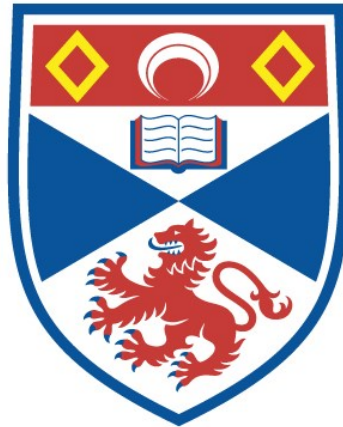# INVARIANCE AND INTENTIONALITY: NEW PERSPECTIVES ON LOGICALITY

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A Thesis Submitted for the Degree of MPhil
at the
University of St Andrews

2018

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Invariance and Intensionality: new perspectives on logicality.
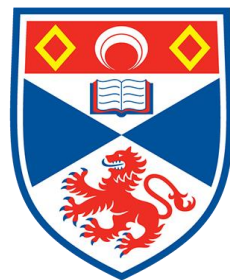
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This thesis is submitted in partial fulfilment for the degree of

Master of Philosophy (MPhil)

at the University of St Andrews

October 2018

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Abstract

What are logical notions? According to a very popular proposal, a logical notion is something invariant under some "transformation" of objects, usually permutations (Tarski 1966) or isomorphisms (Sher 1991). The first chapter is about extending "invariance" accounts of logicality to intensional notions, by asking for invariance under arbitrary permutations of both possible worlds and objects. I discuss the results one gets in this extended theory of invariance, and how to fix many technical issues.

The second chapter is about setting out a better theory of logicality. I discuss the limits of invariance frameworks, and the need for a theory of logicality with a more solid philosophical ground. I believe that the concept of information can play a major role in defining what logic is and what logical notions are. I spell out this intuition, by designing a new test for logicality. A notion is logical iff it behaves in a certain way, by checking only "structural aspects of information", and it does so under arbitrary transformations of its "informational inputs".

In the last chapter I explore some interesting features of my theory. I show how, contrary to standard invariance, in mine logical notions tend to stay persistent across different models of information. I also spell out an intermediate notion of quasi-logicality to make sense of the formality of "world-sensitive" notions: notions whose behaviour changes across worlds. I finally propose a case study: deontic modals. I discuss how one can argue for their quasi-logicality, in my framework. The dissertation is concluded with a technical appendix, in which I prove that my theory is a restriction of standard permutation invariance (at least for a class of items) when we model the space of information in a certain way: as a set of complete powersets of some sets.

Contents

INTRODUCTION

What is this dissertation about?

This dissertation is about formality and logicality. What does it mean to say that logical notions are formal? "Formal" is not a univocal concept, and before rushing into a discussion on what formal notions are, one should first cut off potential misunderstandings. For formality is applied to many things: theories, arguments, lexical notions, properties, etc. What is more, formality comes not just in different kinds, but in different shades, as well: one might say, for example, that a theory is more formal than another.

So, in what kind of formality are we interested in, in the scope of this work? The brief answer is this: in the formality of logical notions, exclusively. Thus, firstly, we take the formality we are interested in to apply to items of a formal language: it is "linguistic" formality. We are not interested in defining an attribute that applies to theories or arguments. We need to be careful, though, when making this claim, for a linguistic item, *per se*, is just a symbol without meaning. Yet, by logical "notions", we mean a linguistic item that is interpreted: so, formality here applies to items in a formal language that are already coupled with a meaning, given through an interpretation. Before we give a meaning to it, the item is just a symbol, and the very question of its formality cannot be raised. The general idea is that, in a formal language, a demarcation should be made between notions that are logical/formal, and notions that are not: conjunction and disjunction are formal, for example, while proper names for dogs are not.

Secondly, in the scope of this work, we are interested in the most formal notions: the logical ones. It may be that, in a language, formality is not an all-or-nothing attribute. For example, one might want to say that material implication is logical, while counterfactual implication is less formal, and yet more formal than a binary operator that takes ($\ulcorner 2+2 = 4 \urcorner$, $\ulcorner$Lemons are vegetables$\urcorner$) to $\ulcorner$Oranges are vegetables$\urcorner$ and everything else to $\bot$. We will say something about this topic in the last chapter but, generally, when we talk about "formality", we are referring to the formality shared by all and only the

logical notions. This is why, maybe unwisely, I will tend to use the term "logical" and "formal" as interchangeable, in the scope of this work.

Historical roots of this work.

Formality is a complex notion with a complex history. This is not a dissertation on the history of philosophy, yet I believe it would be unwise to approach the philosophical discussion of some concept without having a grasp, first, on how the contemporary discussion on it has been shaped into its current form. Novaes (2011), in her cunning article, distinguishes between two main ways logic is said to be formal:

i. 1-Formality interprets formal as "pertaining to forms": logic is formal because it "abstracts from matter".

ii. 2-formality, on the other hand, interprets "formal" as "pertaining to rules": logic is formal not because of what it is, but because of what it does. It is formal because it behaves formally, by defining the rules for reasoning.

In this dissertation, we will mainly talk about 1-Formality, since we are "taking on" from "invariance" theories of logicality, which are a kind of 1-formality theories. Let us talk briefly about 1-Formality. Its historical root, according to Novaes, is to be traced back to Aristotle's distinction between matter and form. The opposite of "formal", in 1-formality, is "material". Different choices of what "material" is lead to different interpretations of what 1-Formality means. For example, if we are focusing on the formality of "arguments", and we identify the "material" with some "lexical terms", we get to the old idea that an argument is formal because of the validity of its *schema*: the *schema* is the formal "structure" of the argument, and it is obtained when we substitute its "material" parts with place-holders (Novaes 2011, 307). Examples of this idea are to be found in Buridan, which suggests that the "material" part of the sentence are all its subjects and predicates, while the "formal" are all the other syncategorematic terms, like connectives and quantifiers (Buridan 1976, I.7.2, pp. 30 (7–12)).

By assuming that what "pertains to the form" are all the syncategorematic terms, Buridan is already making an important choice: what is the ground for it? Arguably, subjects and predicates, following the

Aristotelian tradition, correspond to substances and attributes. Thus, they are "material" because they are "too involved" with the world: they are "sensitive" to it, in a non-logical way. This suggests something interesting: the formality of linguistic items is not something that can be "tested" in the realm of language alone, but it is essentially linked to their meanings, which, in turn, are tied to what the world is. This "ontological" shift in the discussion on formality is even more apparent when one identifies "matter" with "identities of objects". Now, formal means "indifference to particulars" (MacFarlane 2000, 56). This is where invariance theories come into play. According to invariance theories, something is logical because it abstracts away from *objects*. To test if this is the case, we transform the objects in the model through which we are interpreting the linguistic term we are testing, and we check if the meaning of the term gets upset by some transformation. If this never happens, then the item is logical. This dissertation takes on from this tradition, so it is important to understand "how we got here". We are following a tradition that goes from Tarski onward, and that is closely related to the model-theoretic account of logical consequence and the problem of the demarcation of logical constants, as we will see in the first section of chapter 1.

CHAPTER 1. INVARIANCE AND INTENSIONALITY.

What are logical notions? The shape of the current debate on this issue has been heavily influenced by Tarski's work on logical consequence. Take the standard, Tarskian way of understanding logical consequence:

Tarski Logical Consequence Γ⊨A iff every model that makes Γ true makes A true, where Γ is a finitary set of sentences of the language.

What is a model? In a first order, extensional logic, a model is obtained when we interpret a language $\mathcal{L}$ over a domain of objects $\mathfrak{D}$, with an interpretation function that maps each element of the language to an appropriate set-theoretic construction from the domain: it maps each name to an object, each function of arity 1 to a subset of $\mathfrak{D}$, each 2-place relation to a subset of $\mathfrak{D}\times\mathfrak{D}$, each unary quantifier to a set of subsets of the powerset of $\mathfrak{D}$, and so on. A language plus a domain is called a frame. For every appropriate interpretation function we obtain a model for that frame. In a language $\mathcal{L}$, Γ⊨A iff every model that makes Γ true makes A true.

Now the question is: what is an "appropriate" interpretation function? Suppose Γ=P∧Q. We now want to say that P∧Q⊨P. To do that, we check that, for every reinterpretation of the premises and the conclusion, we still obtain a valid argument. That is, every model that makes the premises true makes the conclusion true. Note, however, that this is so only because we are keeping the interpretation of conjunction fixed. Connectives, however, are part of $\mathcal{L}$: why, then, should we keep conjunction fixed? Why not, for example, building a model in which conjunction is mapped onto disjunction? If such class of models were admissible we would end up with a counterexample to P∧Q⊨P: we just need to swap conjunction with disjunction and make P false.

The standard answer to the problem is saying that conjunction is logical, along with any other truth-functional operator, and so it must be kept constant through interpretations. This is where the logicality question becomes central to the model-theoretic account of logical consequence: having different sets of logical constants will result in having different results on what follows from what. Suppose we keep

every item in the language constant: then we would have just one model $\mathfrak{M}$. Thus, for every set of sentences Γ of $\mathcal{L}$, $\Gamma \vDash A$ iff either some sentence in Γ is false in $\mathfrak{M}$ or A is true in $\mathfrak{M}$. Suppose now that the set of logical constants is empty. Then very few arguments would be valid: for example, $\Gamma \vDash \Gamma$, for every Γ of $\mathcal{L}$.

1.1 PERMUTATION INVARIANCE

Logical consequence is central to what logic is and does, but to make it work, we need an account of logicality. We don't only need an elaborate, philosophical definition: we need something that is apt to formalization, so that we can build up a formal, precise test. This is where invariance come into play. When Tarski was developing his model-theoretic account of logical consequence in 1936, he was sceptical about the possibility of coming up with such a technical notion. However, in his famous paper in 1966, he provided a very elegant and precise test for logicality: permutation invariance. Although this idea was latter called "Tarskian", he was not the first to put forward such an idea: we find similar proposals in Mautner (1946) and Mostowski (1957:13). The proposal is quite simple: a logical notion must be "general" and "topic-neutral". 'Human' is not logical because it is sensitive to what humanity is, which is not something logic should be concerned with. On the other hand, 'identity' is general enough to be logical. To capture this idea, we can ask which notion remains invariant when we "swap" objects in the domain arbitrarily. We call logical whatever notion "survives" every such "permutation of the world onto itself" (Tarski 1966). The test gives plausible results: ⌜Being human⌝ is not logical, because when we swap me with my laptop, ⌜Being human⌝ gets assigned a different subset of the domain: one in which a non-human thing is present (my laptop). On the other hand, identity is logical because, no matter what objects we swap around, every object is self-identical, and so identity will still be mapped onto the set of all ordered pairs of an object with itself.

We can make this even clearer by talking in terms of "structure": being sensitive to "Humanity", in a logical framework, can be modelled as being sensitive to the membership between a thing and the set of humans. This membership relation creates a boundary, a structure in the logical space, because it

partitions things into two equivalence classes: the class of humans and the class of non-humans. To see whether a notion is indifferent to Humanity, we take some transformation that does not preserve this structure, like arbitrary permutations, where a permutation is just a "swapping" of the world onto itself. If we swap a human with a non-human, for example, we will "upset" the meaning of ⌜Being Human⌝, thereby showing that ⌜Being Human⌝ is not formal. If a notion is permutation-invariant, on the other hand, then it must be insensitive to these "worldly", boundaries of reality. The adequacy and efficiency of the proposal rests on the choice of transformation we look at. According to Tarski (1966), "permutations" are powerful enough to show abstraction from all the irrelevant "boundaries" of reality[1].

We can make the Tarski machinery more precise in this way: every item in a language $\mathcal{L}$ belongs to a precise type: a unique set-theoretic construction from $\mathfrak{D}$. A permutation of a set X is any automorphism of X; that is, any function from X to itself. $\sigma : X \to X$ and if a, b $\in$ X and a $\neq$ b, then $\sigma(a) \neq \sigma(b)$. (Lang 1970, 7.5). Now, given a permutation σ of $\mathfrak{D}$, call σ^Y the permutation of type Y induced by σ. In general, $\sigma^Y = \{\sigma^X | X \in Y\}$ and $\sigma(k) = \sigma^Y(k)$ if $Y \in \mathfrak{D}$. We can now define permutation invariance:

PI   An item k in type Y is permutation invariant iff, for every σ, $\sigma^Y(k) = k$.

That is, k is invariant if permutations do not affect it. Tarski's proposal is this:

Tarski   A item is logical iff it is permutation invariant, in every model.

To be fair, Tarski just says that the notion should be "permutation invariant". Yet, I guess that generalizing to every model makes things clearer. For the notion should be invariant *where*? In the actual world? In some model? And if so, in which one? Usually, if a notion is permutation invariant in one model, it is so in every model, but we can make up items that change meaning across models, like a notion that is mapped to identity in some model and to ⌜Loving⌝ in some other: this notion is permutation invariant in the first but not in the second model. I suppose that Tarski believed that invoking models in the definition of a logical constant was not a good idea, for the point of logical constants was to extract the appropriate interpretation function, which in turn defines the set of models.

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[1] On the other hand, Sher (1991) and McGee (1996) think arbitrary bijections between domains may do the job. Feferman (1999) employs an even more powerful transformation: homomorphisms. I discuss these solutions later on.

Yet, if we use models to define logical constants, the theory would be viciously circular. This is fair, but I am not necessarily worried by this because, even though my discussion "takes on" from Tarski's, we have different aims. Mine is not to ground what a model is in what a logical notion is, for I am just interested in a general definition of logicality. In fact, it seems to me that the Tarskian project will invariably fail, for invariance theories will ground what a logical notion is in what comes out invariant under some transformation of the domains of *models*. So, it seems clear to me that they are grounding logicality in model theory, not the other way around.

Let us now assess the main results of permutation invariance. This will also help shed light on the proposal itself. Which item comes out logical via PI and Tarski?

i. Individual constants.

No zero-place function, because $\sigma(k) = \sigma^Y(k)$ for every such k. This is good, for we do not usually think that names are logical. However, names for truth-values or empty names will probably be logical. The firsts are logical because, as a rule, we cannot "swap" truth-values: if we have, in our language, an individual constant that is mapped to some truth-value, that constant is automatically logical. Empty-names are logical because they usually get mapped to the empty-set, which is permutation invariant. This latter result may be problematic: what is the link between "emptiness" and logicality? Empty names are a can of worms I do not want to open. As a suggestion, I suppose that my favourite way to deal with this issue would be to take a "normative" stance, and do not allow empty names in the language[2].

ii. Unary relations.

We can easily see that every trivial and empty unary property come out logical: they are mapped to $\mathfrak{D}$ and $\emptyset$ respectively, which are the only permutation-invariant subsets of $\mathfrak{D}$.

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[2] This is just a suggestion, and I do not necessarily endorse it. For example, why should we curtail language to reach extensional adequacy? Should permutation invariance be enough to tell us that empty-names are not logical?

iii.   N-ary relations.

Again: every trivial or empty relation is logical. Both this result and the former are heavily discussed in the literature. Many think they are problematic. For example, take the predicate ⌜Being a male-widow⌝: it is mapped to the empty-set, so it is invariant. Yet, it should not be logical, because it is empty in virtue of the meaning of the non-logical terms "male" and "widow" (Gomez-Torrente 2002, 18). We can build similar examples with metaphysical necessity: ⌜Being a unicorn⌝, if Kripke (2013) is right, is necessarily empty, and yet it is not logical. We will see later on how we can try to deal with issues like this in an intensional, more powerful framework.

These results are interesting, in their own right, because they suggest that the reliability of the results we get depends on what a model is. When we are permuting objects in the domain, what are we doing, really? What does the domain represent? In his quite famous book, Etchemendy (1990) argued that models can be mainly conceptualized in two ways: either as reinterpretation of the language or as a change in the world we are interpreting the language on. According to the first, the "world" does not change: in every model, it is either the set of actual objects or restrictions of such domain. What changes are the meaning of the terms: for example, we obtain a model by imagining that ⌜Being a dog⌝ means ⌜Being a human⌝. This first conception of model is inadequate for permutation invariance. For then, any property that is "universal", like ⌜Being actually instantiated⌝ would be logical, for it would be mapped to the set of actual objects and any restriction of them. Yet, ⌜Being actually instantiated⌝ cannot be logical: it is not even something that things satisfy *a priori*.

According to the second conception, we keep the language fixed and change the "world" we are interpreting the language on. For permutation invariance to work, in this second conception models must somehow be "bigger" than the set of metaphysically possible worlds. For suppose this is not the case and suppose Kripke is right: unicorns are metaphysically impossible. Then, in each world, ⌜Being a unicorn⌝ would be mapped to ∅, and it would be invariant. Yet, it is not logical. The same goes for the "male-widow" issue: we need a world with male-widows, or otherwise this item will be invariant.

iv.   Quantifiers

Quantifiers are mapped to subsets of the powerset of $\mathfrak{D}$. For a quantifier is a function that takes propositional functions or n-tuples of propositional functions to True and False. Its extension, then, will be the set of all and only propositional functions taken to True. The existential quantifier, for example, takes every propositional function that is satisfied by at least one object to True, the rest to False. So, its extension will be the set of all non-empty subsets of $\mathfrak{D}$. The universal quantifier, on the other hand, will be mapped on the set of the maximal subsets of $\mathfrak{D}$: that is, $\{\mathfrak{D}\}$. Both are invariant under arbitrary permutations. For take the existential quantifier: the only way to make it invariant would be to "switch" the empty set with a non-empty. Yet, we saw that the empty set is invariant, so this is not possible. The same goes with $\forall$: since $\mathfrak{D}$ is invariant, so is $\{\mathfrak{D}\}$. What about other quantifiers? Well, for example, the complementation of $\forall$ and $\exists$ in $\wp\mathfrak{D}$ are logical: the quantifier whose extension is $\emptyset$ and $\wp\mathfrak{D} - \{\mathfrak{D}\}$, respectively (the ⌜Nothing⌝ and ⌜Not everything⌝). Most quantifiers are not logical, however. Take ⌜At least one human⌝: it takes all and only the predicates satisfied by at least one human to True. So, its extension is the set of all X⊆$\mathfrak{D}$ that contain at least one human. This unary quantifier will not be permutation invariant, because we can swap some humans with some other things, like dogs: now ⌜At least a human⌝ will have, in its extension, at least one set in which no human can be found.

Mostowski in the fifties proved that cardinality quantifiers are invariant under arbitrary bijections between models (1957). Cardinality quantifiers are items that are sensitive only to the number of elements of their extension. Examples are: ⌜At least 4 things or ⌜A transfinite number of things or ⌜Only 15 things⌝ etc. ⌜Exactly 4 objects⌝ has, as extension, the set Y = {X⊆$\mathfrak{D}$/|A| = 4}, where |X| is the cardinality of set X. Arbitrary permutations in X, being the set of all automorphisms of X, are a subset of arbitrary bijections between domains. Thus, permutations cannot affect the cardinality of the elements of $\wp\mathfrak{D}$, and so they cannot upset the extension of a quantifier that is sensitive only to the cardinality of sets. There has been a great deal of discussion on cardinality quantifiers. In his 1999 paper, Feferman puts forward a list of accusations to bijection-invariance (that applies to permutation-invariance, as well). Two of his points are especially useful to our discussion:

I.   The thesis assimilates logic to mathematics, specifically to set theory.

II.   No natural explanation is given by it of what constitutes the *same* logical operation over arbitrary basic domains (1999, 37).

The first accusation is related to things like cardinality quantifiers. According to Feferman, if cardinality quantifiers are logical, then the theory results in a confusion between logic and mathematics. Now one can "express the Continuum hypothesis and many other substantial mathematical propositions as logically determinate statements" (1999, 38). Bonnay (2008) shares similar worries[3].

> **Continuum Hypothesis** *the Continuum hypothesis (CH) states that the there is no set that is strictly "bigger" than the set of Integers and strictly "smaller" than the set of Reals. That is, there is no set X such that $|\mathbb{Z}|<|X|<|\mathbb{R}|$, where $|X|$ denotes the cardinality of X. As well know, it was proved by Godel (1940) and Cohen (1963, 1964) that the standard Zaermelo-Frankel theory, together with the axiom of choice, does not entail either the Continuum Hypothesis (CH) or its negation.*

What does Feferman mean when he says that the CH can be expressed as "logically determinate"? The choice of words is poor. As discussed in Griffiths & Paseau (2016), the threat may be that one will overgenerate logical truths. It is known that one can express CH in purely second-order logic vocabulary[4]. Feferman shows how one can turn a second-order quantifier into a first-order like operation across domains that is bijection-invariant (and thus permutation-invariant), and that is "cardinality-quantifier-like" (1999, 37-8). Consequently, so Feferman seems to think, CH will turn out logically true if true and logically false if false, and this proves that PI has conflated logic with mathematics[5].

   The second point hinges on cardinality, as well: it is unclear "what constitutes the *same* logical operation over arbitrary basic domains", because logical notions that act differently in domains with different cardinality are bijection invariant and thus logical, according to Tarski (and to "bijection-invariance theorists, like Sher (1991)). So, for example, a quantifier that is mapped to ∃ when $|\mathfrak{D}|>5$,

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[3] Bonnay's proposal is invariance under "potential isomorphisms" (2008). The test makes finite cardinality quantifier logical, but not higher quantifiers, so the Continuum Hypothesis problem is avoided.
[4] See, for example Shapiro (1991, 100-106).
[5] Griffiths & Paseau (2016) discuss thoroughly this claim, and try to deflate the issue. Also, if someone is a logicist, she may not be too unhappy with this result.

and to ∀ otherwise would be logical, on this view. This example makes explicit that cardinality, in bijection-invariance and in permutation invariance theories, is "logical structure": the quantifier above is clearly sensitive to it, and it turns out logical[6].

1.2   A BETTER FRAMEWORK TO WORK ON

Most of the standard examples of logical notions have been left out of our analysis: negation, conjunction, disjunction, implication etc. The problem is that it is harder to see what their extension is, when we see extensions as set-theoretic constructions from $\mathfrak{D}$. We can get around this in many ways. For example, one way is Lindstrom's (1966), quoted by Sher (1991, 68-71), in which connectives are general quantifiers, where a quantifier is a class of sequences consisting in the domain $\mathfrak{D}$ and a series of set-theoretical transformations on it. Lindstrom postulates that the only two 0-place relations are Truth T and Falsity F. So, for example, negation becomes the class of all sequences $\langle \mathfrak{D}, F \rangle$. The (classical) truth operator ⌜It is true that⌝, on the other hand, would be $\langle \mathfrak{D}, T \rangle$. Boolean connectives are sequences of form $\langle \mathfrak{D}, S_1, S_2 \rangle$, where S_n is any sentence of the language. So, for example, a disjunction is the class of sequences $\langle \mathfrak{D}, S_1, S_2 \rangle$ such that either S_1 or S_2 is true (sentences are mapped either to T or to F). As far as permutation-invariance goes, it is just assumed that truth-values are "fixed" and not "swappable" by permutations, so every Boolean connective, and indeed any sentential operator in general, will turn out automatically logical (more on this later).

The system works fine as long as we are in an extensional language, but it is unfit to model intensional notions, unsurprisingly. Since we will be dealing with them, I prefer to work in a type theoretic framework, with a categorial grammar, in the style of Van Benthem (1989) and MacFarlane (2000). In this framework, each type is a function from one type to another. In an extensional language, we start

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[6] My stance on these complex issues will be clearer after having sketched my theory, in chapter 2. I can say now that cardinality quantifiers are probably not logical, in my theory, and cardinality is not "logical structure", unless we state it explicitly.

with two basic types, O and V. O is the domain of objects, V is the set of truth-values. In general, we may define the set of all types inductively, as the smallest set $\mathbb{T}$ such that:

i. $\langle O,V \rangle \in \mathbb{T}$.

ii.   if X, Y$\in \mathbb{T}$ then $\langle X,Y \rangle \in \mathbb{T}$.

The interpretation of each typed item is this: elements of O are objects of the domain; V are truth-values; elements of type $\langle X,Y \rangle$ are elements of Y^X (functions from X to Y). O and V combine to form more complex types. Examples:

i. $\langle O,V \rangle$ are one place predicates, that yield a truth-value when given an object.

ii. $\langle \langle O,V \rangle,V \rangle$ are unary quantifiers, that yield a truth-value if fed with a predicate.

iii. $\langle V,V \rangle$ are unary sentential operators: functions from sentences to sentences.

With relations of arity n>1 we have different possibilities (MacFarlane 2000, 178). Take a binary relation R: it is a function from tuples to truth-values. We can either complicate the system, by adding a primitive way of chaining types (x, y), and take R to be of type $\langle (O,O),V \rangle$. The other alternative is to analyse R as a relational, unary property. It is of type $\langle O,\langle O,V \rangle \rangle$. This is the most popular option because we would have $\langle O,\langle O,V \rangle \rangle$ in our set of types anyway, and we can always transform a function from n-tuples to truth-values to a unary function. Take $\ulcorner$Love(x,y)$\urcorner$: when it is given an object a, it "gives out" the unary property $\ulcorner$Loving-a$\urcorner$. We can do the same for every n-ary function: in general, a function of arity n is equivalent to a unary function whose value is a function whose arity is n-1. I will take this second option.

With this system in place, we can ask which item is logical. Tarski's proposal, for instance, is to call a notion logical iff it is permutation invariant. In our type-system, this amounts to arbitrary permutations of O:

PI   Let σ be a permutation of type O, and let σ^y be the transformation on type Y induced by σ. An item k in Y is permutation-invariant iff for every permutation σ of O, $\sigma^y(k)=k$.

We can define the transformation of Y induced by σ inductively, on the complexity of Y. MacFarlane (2000, 183) offers an elegant definition:

- if Y=O, then for all k∈Y, $σ^Y(k) = σ(k)$

- if Y=V, then for all k∈Y, $σ^Y(k)= k$

- if Y=⟨X,Z⟩, for any types X, Z, then for all k∈Y, $σ^Y(k)= σ^Z ∘ k ∘ (σ^X)^{-1}$.

Here ∘ is composition: f ∘ g (a) = f(g(a)), for any f, g and a. f^{-1} is the inverse of function f: f(a) = b if and only if $f^{-1}(b)$ = a. Now, since this definition is not that obvious, let us check it with an example. Take an arbitrary permutation of O: σ(a) = b. We want to know what is the induced permutation in type ⟨O,V⟩. Take an item k in ⟨O,V⟩ such that k(a) = T. Following the inductive definition above, $σ^{⟨O,V⟩}(k)$ is the composite function $σ^V ∘ k ∘ (σ^O)^{-1}$. So, it is a function O → O → V → V. In particular, $(σ^X)^{-1}(b)$ = a, so it will go from a to b; then, since k(a) = T and σ(T) = T, it will end up in T. So, correctly, k goes from a to T, and its transformation goes from b to T: we have swapped 'a' with 'b' in k.

1.3 EXTENDING INVARIANCE TO INTENSIONAL OPERATORS

How do we treat sentential operators? For we have already said that paradigmatic examples of logicality, such as negation and conjunction, are sentential operators. Since their form has no occurrence of type O, they are all trivially invariant and thus logical. This is fine, as long as we confine ourselves in an extensional language, because in such a system the only sentential operators one can model are the truth-functional, which are all plausibly logical. Yet, what happens if we want to say something about more complex sentential operators, i.e. non-truth functional operators? Well, there are operators that are not truth-functional, and not logical, at all. For example, take $\mathcal{Z}$: it is a unary sentential operator that takes P: ⌜Pizza is healthy⌝ to ⊥ and everything else to ⊤. $\mathcal{Z}$ is too complex to be modelled in our system, so the theory is just silent on such items. Yet, if we try to model it, it would come out trivially invariant, because it would be of form ⟨V,V⟩. Another example: $\mathcal{X}$ takes every sentence S to its 'conjunction-operator' ⌜∧S⌝, with the exemption of P': ⌜Oranges are vegetables⌝; here, the output is the 'disjunction-

13

operator', ⌜∨S'⌝[7]. $\mathcal{X}$ is not to be modelled in our extensional system; yet, again, we would like to say that it is not logical. Yet, if we do model it, it turns out trivially invariant. By "trivially" I mean that the test gives its verdict not because $\mathcal{Z}$ or $\mathcal{X}$ are somehow "special" operators, but only because they are sentential operators.

The problem is that operators like $\mathcal{X}$ and $\mathcal{Z}$ are sensitive to the identity of propositions in a non-logical way. Yet, the system, being extensional, equates every proposition with its truth-value, making true (and false) propositions undistinguishable. This is not a feature of our system only: it is usually just harder to acknowledge, given the popular construction of sentential functions as generalized quantifiers. For example, in Lindstrom's framework (1966), we are forced to "respect" the identity of truth-values, and the system identifies sentences extensionally.

Of course, this does not show that standard invariance theories make things like $\mathcal{Z}$ or $\mathcal{X}$ logical, but rather that they are just silent on those items, and structurally unfit to deal with them. If we take invariance as a theory whose only aim is to give a technical notion of logicality, designed for the restricted domain of extensional languages, then its inability to handle non-extensional operators should not worry us. Yet, it should worry us if we see invariance as a sound, philosophical theory of what logical notions are, for the concept of "logical notion" is *not* restricted to extensional logics: it is far more general. We have just showed how two non-extensional items, $\mathcal{Z}$ and $\mathcal{X}$, are *not* logical.

Granted that one wants invariance to be a general theory, we need a way to model intensionality in our framework. To do so we need more "structure" and complexity. Semantics for non-extensional operators "extensionalize" them, by interpreting these notions over some algebraic structure, like a set of worlds/times and some relations on them. How does this work in type theory? I will show how to model modal notions, but similar results can be obtained for other class of items, like temporal operators in tense logic. First, we change the inductive definition of types:

i.   O,V are types.

ii.   If X and Y are types, then ⟨X,Y⟩ is a type.

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[7] MacFarlane (2000, 207-9) makes similar examples.

iii.   If X is a type, then ⟨W,X⟩ is a type.

'W' is "worlds". It is not a type by itself. The only basic types are still O and V. In this way, we do not create funky items like ⟨⟨O,V⟩,W⟩, that we do not need[8]. We can now accommodate richer notions in our theory:

- ⟨W,V⟩ are propositions: functions from worlds to truth-values.

- ⟨W,⟨O,V⟩⟩ are predicates that may change extensions across worlds.

- ⟨W,O⟩ are individual concepts: for each world, they yield an object.

Now, a second problem comes for standard invariance theories. Before, we said that items like $\mathcal{Z}$ or $\mathcal{X}$ are simply non-representable, so it was best to say that the theory was just silent on their formality. Now, however, we do have a mean to model non-truth-functional notions. And yet, if we follow standard invariance and permute objects arbitrarily, $\mathcal{Z}$ and $\mathcal{X}$ do come out logical. The reason is that their form is ⟨⟨W,V⟩,⟨W,V⟩⟩ and ⟨⟨W,V⟩,⟨⟨W,V⟩,⟨W,V⟩⟩⟩: O does not appear in it, and so they are both trivially invariant. To fix this, the first requirement that comes to mind is that logical notions should be invariant under permutations of worlds, as well (McCarthy 1981; Van Benthem 1989).

The way we define permutations on worlds and objects is just an extension of the old definition: for any k of type Y, the permutation induced in k is defined like this:

- if Y=O, then for all k∈Y, $\sigma^Y(k) = \sigma^O(k)$

- if Y=W, then for all k∈Y, $\sigma^Y(k) = \sigma^W(k)$

- if Y=V, then for all k∈Y, $\sigma^Y(k) = k$

- if Y=⟨X,Z⟩, for any types X, Z, then for all k∈Y, $\sigma^Y(k) = \sigma^Z \circ k \circ (\sigma^X)^{-1}$.

The new definition of permutation invariance is:

PI2   Let σ^O and σ^W be some permutations of O and W respectively, and let σ^y be the transformation on type Y induced by such σ^O and σ^W. An item k in Y is permutation-invariant iff for all σ^O and σ^W, $\sigma^y(k)=k$.

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[8] This is in the style of Montague's universal grammar (1970, 1973). The other way is to take O, V and W as basic and to have just one formation rule: you then simply disregard the items you do not need (e.g. Gallin 1975).

1.3.1 Problems of size

Things get messy, however, when we consider worlds with restricted or augmented domains. Take the logical predicate F: ⌜Being something⌝. It is an element of $\langle W,\langle O,V\rangle\rangle$. Take two worlds w_1 and w_2 with different domains $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively. When given w_1, F yields the $\langle O,V\rangle$ whose extension is $\mathfrak{D}_1$. Define a permutation $\sigma^w(w_1) = w_2$. Now for w_1, F yields the $\langle O,V\rangle$ whose extension is $\mathfrak{D}_2$. F is not invariant and not logical, via PI2. The same argument can be run with $\forall$: $\forall$ in w_1 is the element which takes every predicate whose extension is $\mathfrak{D}_1$ to True, and everything else to False. In w_2, it takes every predicate whose extension is $\mathfrak{D}_2$ to True, and everything else to false. By swapping w_1 with w_2, we show that $\forall$ is not invariant.

The issue stems from the fact that O, in modal settings, is the set of all possible objects, and worlds may have non-trivial subsets of O as domains: thus, logical items lose their "universality". We need to be careful here in distinguishing between two kinds of issues. The "extensional" universal quantifier $\forall$ and the "extensional" logical predicate ⌜Being something⌝ are still logical, and their logicality is not called into question. What is at stake, here, is rather the logicality of their intensional, modal counterparts of type $\langle W, \langle\langle O, V\rangle, V\rangle\rangle$ and $\langle W, \langle O, V\rangle\rangle$, respectively. For example, with quantifiers, we are asking whether some "intensional quantifier" is logical. Such items, in modal logics with variable domains, will not logical via PI2. This is bad because the logicality of at least a class of intensional quantifiers is desirable, even though not all of them are logical. For example, take the modal $\forall$ defined over the set of worlds in which there are purple cows[9]. It is unwise to let this quantifier be logical, because it is indirectly sensitive to purple-cow-ness: when we swap a purple cows world with a world with no purple cows, we are arguably upsetting this quantifier. On the other hand, it seems sensible to suppose that the modal $\forall$ defined over the set of all and only logically possible worlds is logical. Yet it will not be, along with any other modal quantifier, if we let PI2 in place and we let domains vary.

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[9] Here I am assuming that, if an item k is defined only over some set X, it is undefined for every set that contains some $y \notin X$. So, if we swap an $x \in X$ with some $y \notin X$, we always upset k.

Having said that, let us look at how to amend this. I can see three options, here: (i) we keep the domain fixed across worlds (very inelegant, yet suitable for "necessitists"[10]), (ii) we restrict the definition of types, (iii) we disregard permutations that upset the membership of an object to a world. I will follow the latter way. PI2 becomes:

PI3 Let σ^W be a permutation between worlds with the same domains, σ^0 a permutation of O that respects the membership of objects to worlds, and let σ^y be the transformation on type Y induced by such σ^W and σ^0. An item k in Y is permutation-invariant iff for all σ^0 and σ^W, $\sigma^y(k)=k$.

Basically, now we just look at arbitrary permutations of worlds with the same domain and, for each world, all and only the permutations of objects that exist in that world[11]. PI3 takes care of items like $\mathcal{Z}$ (just swap ⌜Pizza is healthy⌝ with some other proposition).

1.3.2 Invariant modal notions

What modal operators are invariant under PI3? Let us focus on unary operators, for simplicity. The only invariant elements of ⟨W,V⟩ are the true in every world and true in no world, which we label ⊤ and ⊥, respectively. We can build up this table:

Argument	Value							
⊥	⊤	⊤	⊤	⊤	⊥	⊥	⊥	⊥
⊤	⊥	⊥	⊤	⊤	⊥	⊥	⊤	⊤
⟨W,V⟩−{⊥,⊤}	⊤	⊥	⊤	⊥	⊤	⊥	⊤	⊥
Interpretation	¬□	¬◊	Trivial	¬Conting	Conting	Empty	◊	□

Table 1

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[10] People that think that necessarily everything is necessarily something (Williamson 2013).
[11] I feel like I should say more on this artificial restriction on the group of permutations. What is its philosophical justification? I must admit that I do not have any reasonable answer to this worry. It seems clear to me that something should be done to stop the failure of logicality for universal quantifiers defined over the set of logically possible worlds. This is a way to do it. I am not sure about the philosophical consequences we can extract from this choice, and I do not know if this is to be taken as a philosophically deep fact or just as a glitch in the system.

⟨W,V⟩−{⊥,⊤⟩ is every element of type ⟨W,V⟩ except ⊤ and ⊥. The first column sets the input (the first ⟨W,V⟩), each other column is one full combination of outputs. The first line explains the meaning of the results: "conting." is contingency. In a Kripke relational semantics, Conting., □ and ◊ would be the modal operators obtained from an accessibility relation R:w→w that is "universal": such that, for every w,w'∈W, R(w,w').

Definition of Kripke Model. *A Kripke frame is a tuple ⟨W, R⟩, where W is a set of worlds and R ⊆ W×W. Let language ℒ be the smallest set generated from a set of atomic sentences plus Boolean connectives, enriched with an operator □. Let $\mathcal{V}$ be an "atomic" valuation function that maps each atomic formula to a subset of W. ⟨W, R, V, ⊨⟩[12] is a Kripke model for ℒ, where ⊨ is a relation from W to ℒ defined inductively:*

i. *w⊨ A iff w ∈ $\mathcal{V}$(A), where A is atomic*

ii. *w⊨ ¬φ iff w⊭ A*

iii. *w⊨ φ ∧ ψ iff w⊨ φ and w⊨ ψ.*

iv. *w⊨ φ ∨ ψ iff not both w⊭ φ and w⊭ ψ.*

v. *w⊨ φ → ψ iff either w⊨ φ or w⊭ ψ.*

vi. *w⊨ □φ iff ∀w'(Rww', w'⊨ φ)*

A formula A is valid in 𝔐 iff for all w ∈ W, w⊨ A. One can show that any Kripke model will make valid some theorems and rules, like:

- *K : □(P → Q) → (□P → □Q)*

- *Necessitation: ⊨ A then ⊨ □A.*

- *Distribution of □ over conjunction: □(A ∧ B) → (□A ∧ □B)*

---

[12] It is also common to remove $\mathcal{V}$ and let ⊨ make the atomic valuation.

For modal logics that do not have these axioms, the so called "non-normal" modal logics, one needs

a different semantics, like Neighbourhood semantics or Algebraic semantics[13]*. To have more*

axioms, on the other hand, one can add properties to R.

In general, we obtain four invariant operators for every permutation-invariance operation. Example: the identity operation is permutation-invariant. Call '=' the output of the identity operation:

Argument	Value			
⊥	⊤	⊤	⊥	⊥
⊤	⊤	⊥	⊥	⊤
⟨W,V⟩−{⊥,⊤}	=	=	=	=
Interpretation				Identity

Table 2

The last line says that every element of ⟨W,V⟩−{⊥,⊤} is mapped onto itself. The last column is identity, the others are invariant combinations obtained from identity. "Identity" corresponds to the truth-operator ⌜It is true that⌝, at least if the semantics is classical. The fact that such operator is logical is good, for we usually think that "Truth" is essential to logic. In Frege's words, "Logic has much the same relation to truth as physics has to weight or heat" (Ged. 58).

"Cardinality operators" are items like ⌜In at least five worlds⌝ or ⌜In 25≤n≤60 worlds⌝. Such operators are sensitive only to the cardinality of the extension of the proposition they receive as input. E.g. ⌜In at least three worlds⌝ takes every A whose extension X has cardinality $|X|≥3$ to ⊤, and everything else to ⊥. As already said, Mostowski (1957) proved that cardinality is invariant under isomorphic structures, and the group of all permutations are a subset of all isomorphic structures (they are all the bijections from a set to itself), so cardinality is permutation-invariant, as well. Thus, for every such operator, there will be four invariant-operators. ⌜In at least three worlds⌝, for example, takes ⊥ to ⊥ and ⊤ to ⊤ (granted that $|W|>3$). So, as before, we obtain three different variations, switching the output ⊥ with ⊤ and vice-

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[13] See Pacuit (2017) for a discussion on Neighbourhood Semantics and Chagrov & Zakharyaschev (1997) for algebraic semantics.

versa. These variations correspond to Boolean combinations of cardinality operations (e.g: if we change the output of ⊥ into ⊤ in ⌜In at least three worlds⌝, we obtain: ⌜In at least three worlds or in none⌝).

Benthem has this suggestion:

Benthem's Proposition. "Among n-ary operations on sets, the only permutation invariant ones are those defined at each tuple of arguments by some Boolean combination" (1989, 318).

He does not have a detailed proof, but a sketchy one: when one tries not to follow any Boolean combination, one can easily come up with a permutation that upsets the operation. So, for example, take this diagram:
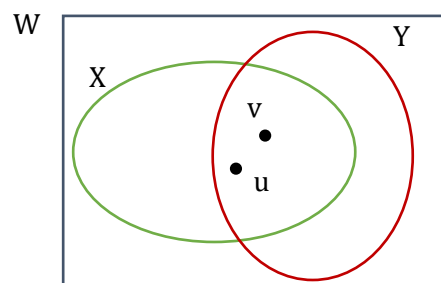
Image 1

Take a binary operation F(X,Y) that is not defined through some Boolean combination. Then, you could have, for example, that F(X,Y) contains u ∈ X∩Y and yet lacks v ∈ X∩Y. Yet, we can now define a permutation σ(v) = u. σ⟨W,V⟩(X) = X and σ⟨W,V⟩(Y) = Y, and yet σ⟨W,V⟩(F(X,Y)) ≠ F(X,Y)[14].

I would like to suggest that the inverse of Benthem's proposition is less controversial: all Boolean operations are permutation-invariant.

Inverted Benthem's proposition. Among n-ary operations on sets, those defined at each tuple of arguments by some Boolean combination are permutation-invariant.

If we model propositions as sets of possible worlds, we can model the intensional counterparts of extensional Boolean operators as operation defined on sets of worlds (while the extensionals are defined over truth-values): disjunction is union of sets, conjunction is intersection, negation is complementation. These are clearly permutation-invariant. Thus, the "intensional" counterpart of the

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[14] Benthem makes an example with X−Y (1989, 319).

extensional ∨, ∧ and →, along with any other Boolean combination of Boolean operators one can devise, are all invariant under PI3.

1.4 OVERGENERATION?

The new intensional system may still be susceptible to counterexamples like $\mathcal{Z}$, in which operators behave differently when given necessary/impossible propositions. Take $\mathcal{H}$: it yields ⌜2+2=4⌝ when given ⌜Socrates is human⌝, ⌜2+2=5⌝ otherwise. Granted that all these propositions are equated to ⊥ or ⊤, $\mathcal{H}$ will be invariant. These issues rely on the assumption that W is the set of all metaphysically/mathematically possible worlds. If so, the system equates every metaphysically/mathematically necessary and impossible proposition with their intensions, making them invariant. To deal with items like this, one needs to make them not invariant. One very intuitive way to do that is to extend W enough so that the truth-sets of such propositions come out different. One then demands invariance under arbitrary permutations of this extended W. Yet, how big should W be?

1.4.1 A Lower and Upper Bound

We need W "bigger" than the set of metaphysically possible worlds, for two reasons. First, now the item in ⟨W, V⟩ that takes every w to True, namely ⊤, is logical. If W is the set of metaphysically possible worlds, any metaphysical truth would be logical. This seems wrong: intuitively, what we want is that ⊤ (which is mapped to W) represents a logical truth, because ⊤ is a formal notion (it is invariant). The second reason is that, otherwise, the account over-generates. Suppose the physicalist is right: it is metaphysically necessary that everything is physical. Take the unary function G: ⌜Being physical⌝. If W is restricted to metaphysical possibility, and the physicalist is right, then G would be logical, because regardless of how we permute objects or worlds, G will be invariant (it will be mapped to the function from any world to the domain of that world, which is invariant under arbitrary, domain preserving permutations). Yet, do we want ⌜Being physical⌝ to be a logical notion? I don't think we do.

MacFarlane claims that operators defined through a "universal" relation between worlds, in a Kripke relational semantics, are invariant (2000, 217). MacFarlane is right: if R = W×W then, no matter how we permute W, we will never upset R. There are other invariant modal operators: modal operators definable through (i) an empty accessibility relation (empty extension), obtainable by adding axiom □⊥, or (ii) a "solipsistic" accessibility relation (R is just reflexive), obtainable when we add axiom □A↔A. (Novaes 2014, 92): now both possibility and necessity collapse into the identity operation described in table 2. How one should interpret these results? Novaes (2014) complained that this new account discards S4-operators, which are widely used in logic and do not necessarily seem less "formal" than S5 operators. This is a first worry, which however I will not dwell upon, for the moment (more on this in section 3.4). What about S5 operators? Well, S5-□ just means that the accessibility relation through which we define □ is an equivalence relation. Such equivalence relation will be invariant only if it is "universal", and mapped to W×W. So, instead of asking: "Are S5 operators logical?" we should ask "Which S5 operator, if any, is logical?". The answer depends on which set is W. It is sensible to suppose that at least the "logical" S5-necessity should be logical[15]. To ensure its logicality we should have, as a rule, that the elements of W are all and only the logically possible worlds: in this way the only invariant, "universal" S5-necessity is logical necessity, and thus we achieve a lower and an upper bound for W.

This move takes care of the ⌐Being physical⌐ problem since, unless we think that ⌐Everything is physical⌐ is a logical truth: now we will now have, in W, a metaphysically impossible and yet logically possible world in which something is not physical. It is also effective against counterexamples like $\mathcal{H}$, unless they are not counterexamples: if mathematical truths are logical, then we can build $\mathcal{H}$-ish invariant connectives that change meaning when given different mathematical truths. Yet, if mathematical truths are really logical truths, I don't see why such items should not be logical, as they will be sensitive only to the identity of logical truths. If they are not logical truths, then there will be a world that falsify them. If we equate W with the logically possible worlds (W^L), then we can also take care of some other objections to invariance, like the "male-widow issue" discussed in section 1.1.

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[15] I am assuming that logical necessity is an S5-operator. For a defence of this, see Burgess (1999).

Proof 1.1: ⌜Being male-widow⌝ is of form $\langle W,\langle O,V\rangle\rangle$. ⌜Nothing is a male-widow⌝ is not a logical truth, so there is a w_k in W such that ⌜Being a male-widow⌝ (BMW) has a non-empty extension X. Define a permutation $\sigma^w(w_k)=@$. In the actual world there are no male widows so, when fed with @, BMW gives out a predicate whose extension is $\emptyset$; yet $\sigma^{\langle W,\langle O,V\rangle\rangle}$(BMW), when given @, yields a predicate whose extension is $X \neq \emptyset$. So $\sigma^{\langle W,\langle O,V\rangle\rangle}$(BMW) $\neq$ BMW.

I suppose that setting W as the set of *logically* possible world is somehow disappointing. For it solves all the issues by *fiat*, so to speak. It also makes impossible to ground what is logically possible in what logical notions are, for we are grounding what comes out as logical notion in what is logically possible. Yet, we are not trying to do that. In fact, I believe that this result comes with no surprise: we should expect that what is logical in the modal space depends on the boundaries of what is logically possible.

We have a strong corroboration for this upper-bound: we cannot extend W beyond W^L for otherwise the system heavily under-generates[16]. Take for example the intensional, "objectual" identity defined over the set of logical possibilities: this item is sensibly a good candidate for logicality. Yet, if we add some logically impossible world w_i to W, in which something is not identical to itself, logical identity would not be invariant.

Proof 1.2. Take the element in type $\langle W,\langle O,\langle O,V\rangle\rangle\rangle$ that is logical identity ($=^L$), and suppose $w_i \vDash$ $a\neq a$. When fed with w_i and a, $=^L$ yields the $\langle O,V\rangle$ whose extension is $\emptyset$. Define a permutation $\sigma^O(a)=b$, where b is such that b=b in w_i. When fed with w_i and b, $=^L$ now yields the $\langle O,V\rangle$ whose extension is {b}. So, there is some permutation for which $\sigma^{\langle W,\langle O,\langle O,V\rangle\rangle\rangle}$('$=^L$') $\neq$ '$=^L$'.

1.4.2. Hyper-logicality

To be fair, we have two ways of ensuring the logicality of $=^L$: we can let W be as big as we want and restrict the permutations we look at to all the permutations "inside" W^L, or we can restrict W to W^L. Both ways ensure the logicality of logical necessity. In fact, the two solutions are equivalent for items defined strictly in W^L, and yet they come apart when we look at the logicality of items defined over some strict

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[16] One might say that there are no impossible worlds, so the problem does not arise. I want to stay neutral on this. For a defence of impossible worlds, see Priest (2005). For a discussion, see Berto (2013).

superset of W^L. Let us call these notions "hyper-logical". If we restrict W to W^L, then hyper-logical notions are simply not representable in our system; in such a situation, it is up to us to decide what to make of their logicality. We have two options:

i. PI3 is simply silent on hyper-logical items.

ii. PI3 can say something on Hyper-logical items: they are logical iff their partial image in W^L is.

If, on the other hand, we decide to restrict the group of permutations, we can represent hyper-logical items and test their logicality directly. The sensible option then is:

iii. PI3 is effective for hyper-logical items.

We can easily show that options (ii) and (iii) are equivalent. Suppose we take route (iii). Take all the admissible bijections σ: $W^L \rightarrow W^L$ and all the admissible permutations of objects (that is, all the domain-preserving permutations of W^L and all the permutations of objects inside each w in W^L). Call invariance under such group of permutations invariance^{WL}. Options (iii) says that hyper-logical items are logical iff they are invariant^{WL}. Suppose now that we take option (ii): $W^L = W$. Take some hyper-logical item k: such item must have form $\langle W^N, Y \rangle$, with $W^N \supset W^L$. The image of k under W^L is the result of restricting the function f: $W^N \rightarrow Y$ to $W^L \rightarrow Y$, where the restricted function f' is such that f'(w) = f(w) for all w in W^L. This image is invariant iff it is invariant under all the domain-preserving bijections $W^L \rightarrow W^L$ and under all the admissible permutations of objects in W^L. Unsurprisingly, these is exactly invariance^{WL}. Thus, the image of k under W^L is invariant iff k is invariant^{WL}.

The real difference, then, is between options (i) vs options (ii) and (iii). What is the best route to take? Take the modal identity $=^{PC}$ defined over W^{PC}, where W^{PC} is the union of W^L and the set of logically impossible yet conceivable worlds in which there are purple cows. Such $=^{PC}$ is indirectly sensitive to purple-cow-ness: if we swap some logically impossible world with purple cows with another impossible one with no purple cows, we would upset $=^{PC}$[17]. W^{PC} is a superset of W^L, and its image under W^L is the invariant and logical $=^L$, so $=^{PC}$ would be invariant, if we follow option (ii) or (iii). Thusly, my suggestion

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[17] Since they are impossible worlds, it may be that, in them, there are and there are not purple cows. I am assuming that such worlds will not make both ⌜There are purple cows⌝ and its complement true.

is to follow route (i) and admit that invariance is silent on hyper-logical notions. So again, it is best to equate W with W^L.

1.5   WOMBAT-DISJUNCTION AND OTHER STRANGE ITEMS

In the literature, permutation invariance has been debated a lot. One popular objection is that "permutations" are not powerful enough to show that an item is not sensitive to some logically-extraneous structure. Take the "wombat-disjunction" (WD) example:

Wombat-Disjunction $=_{df}$ ∨ iff there are wombats in the domain, ∧ otherwise[18].

Wombat-disjunct satisfies permutation-invariance, but we don't want it to be logical, because it is sensitive to "wombat-ness", which is not a logical structure (McGee 1996). The problem with wombat-disjunct is that every permutation will rearrange the objects *within* the domain, without allowing comparisons *across* domains. So, it does not matter if we swap all the wombats with some dogs: the operator will still be assigned disjunction, because the rule through which it is defined forces us to look at the domain in general, and the fact that we have swapped every wombat with some dog does not make the wombats "go away": they are still in the domain of the model. Similar issues can be built for any type of item in the system:

Wombat-quantifier $=_{df}$ ∃ iff there are wombats in the domain, ∀ otherwise.

Wombat-relation $=_{df}$ R iff there are wombats in the domain, G otherwise, for some permutation invariant n-ary relations R and G.

The standard solution, embraced both by McGee (1996) and Sher (1991), is to look at some stronger transformation: all isomorphic transformations between domains. In our case, these are all and only the bijections between domains, and therefore the stronger test is usually called "bijection-invariance". In general, two models $\mathfrak{M}$ and $\mathfrak{M}'$ are isomorphic iff there is a bijection f: $\mathfrak{D}_\mathfrak{M}{\to}\mathfrak{D}_{\mathfrak{M}'}$ satisfying:

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[18] Here we are assuming that domains vary between models, for extensional languages. Otherwise, we would get that, if the cardinality of $\mathfrak{D}$ is n, ⌜There are n things⌝ would be logically true (Etchemendy 1990).

i. For each n-placed relation R of $\mathfrak{M}$ and the corresponding relation R' of $\mathfrak{M}'$, $R(x_1 \ldots x_n)$ if and only if $R'(f(x_1) \ldots f(x_n))$, for all $x_1 \ldots, x_n$ in $\mathfrak{D}_\mathfrak{M}$.

ii. For each constant x of $\mathfrak{M}$ and the corresponding constant x' of $\mathfrak{M}'$, $f(x) = x'$. (Chang & Keisler 1990, 21).

Now, the WD issue is solved: take a model $\mathfrak{M}_1$ with a domain with wombats $\mathfrak{D}_1$. WD is assigned disjunction. Define a bijection between $\mathfrak{D}_1$ and an equinumerous domain $\mathfrak{D}_2$ of some $\mathfrak{M}_2$ in which every wombat is replaced with some wolf. Now wombat-disjunct in $\mathfrak{D}_1$ is assigned a different item: conjunction, and thus it is not invariant.

Bijection invariance   Where π is a bijection from X to Y, where X and Y are arbitrary domains of the same cardinality, call π^Z the variation of type Z induced by π. An item k in Z is logical iff for every π, $\pi^Z(k)=k$.

My suggestion is that WD is quite ambiguous: to which domain is it referring? In extensional frameworks we have just one domain for each model, but in our intensional systems we have plenty. If we understand "in the domain" as "in a domain of some world", then the problem with WD is that its interpretation-rule is such that it does not have the same output for every world one is valuating it from: if one is valuating it from a world that is an element of the truth-set of ⌜There are wombats⌝, then the output is disjunction; it is conjunction otherwise. So, WD is not invariant: we just need to swap a world with wombats with another without wombats.

In this way, we also end up with a theory that may be more powerful than Bijection-Invariance. Take this item:

$\mathfrak{U}$: ⌜More than the number of humans⌝.

This is a quantifier, whose extension is $\{X_{X \subseteq \mathfrak{D}}: |X|= \text{number of humans}\}$[19]. Do we want this quantifier to be logical? After all, it is sensitive to how many humans there are, which is hardly relevant to logic. Yet, it is bijection-invariant, being sensitive only to cardinality. Yet, if there had been more humans, it would

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[19] Similar examples are presented in McCarthy (1981, 1987) and Hanson (1997). Sher (1991) criticised them. For a recent discussion, see Sagi (2015).

have had a different extension: logical operators should not be "contingent" in this way. In our framework, on the other hand, since the number of humans is contingent, there will be a logically possible world with another number of humans, so the quantifier is not PI3 and not logical.

To avoid a potential misunderstanding here, it is quite essential to make a distinction: when we are dealing with models for extensional languages, we should not confuse Wombat-disjunct with another, similar operator:

Wombat-disjunct$_2$(WD2) $=_{df} \vee$ iff there are (actually) wombats, $\wedge$ otherwise.

This second operator is different: its meta-language is talking about the presence of wombats in the *actual world*, not in the domain of some model. WD2 is indeed bijection-invariant: it is disjunction in every "extensional" model. The "original" wombat-disjunct, on the other hand, is defined through a rule that talks about models and domains: WD is disjunction if there are wombats *in the domain*. A similar, subtle distinction is to be made between $\mathfrak{U}$: $\ulcorner$More than the numbers of humans$\urcorner$ and $\mathfrak{U}_2$, defined as

$$\mathfrak{U}_2 =_{df} \{X_{X \subseteq \mathfrak{D}}: |X| > \text{number of humans in } \mathfrak{D}_\mathfrak{M}\}.$$

While the first quantifier is bijection-invariant, in Sher's system, this second is not, because the number of humans is not.

Proof 1.3. Take a model $\mathfrak{M}$ with domain $\mathfrak{D}_0$. Take $A_0 = \{x/\ x \in \mathfrak{D}_0\ \&\ x \text{ is human}\}$. For any domain $\mathfrak{D}$ of any model, $\mathfrak{U}_2$ is mapped in that model to $\{X_{X \subseteq \mathfrak{D}}: |X| > |A_0|\}$. Define a bijection $\pi(\mathfrak{D}_0) = \mathfrak{D}_1$, with $A_1 = \{x/\ x \in \mathfrak{D}_1\ \&\ x \text{ is human}\}$ and $|A_1| \neq |A_0|$. Now $\mathfrak{U}_2$ is mapped to $\{X_{X \subseteq \mathfrak{D}}: |X| > |A_1|\}$. So, $\pi^{\langle\langle 0,V\rangle,V\rangle}(\mathfrak{U}_2) \neq \mathfrak{U}_2$.

We may take $\mathfrak{U}$ and $\mathfrak{U}_2$ and WD and WD2 to have different ways of being "contingent": $\mathfrak{U}$ and WD2 are "world" contingent, while $\mathfrak{U}_2$ and WD are "model" contingent. The first two change extensions across worlds, the seconds change extensions across models. Now, this distinction may be empty, if we take models to be worlds, for example. Something similar is mentioned in relation to $\mathfrak{U}$ by Sagi (2015). She rightfully highlights the fact that the satisfaction-conditions for these items are not clear: are these conditions defined in the "object language"? That is, in the signature of the model? If so, it must be

valuated through models, and will be invariant only if "the number of humans" is, which clearly is not. If they are not defined in the object language but in some meta-language, then we must be more precise: are we talking about what might be the case, what is actually the case or what is the case in the model we are handling?

The conclusion I want to draw is this: my impression is that the distinction between $\mathfrak{U}$ and $\mathfrak{U}_2$ and WD and WD2 may stop being relevant, when we make the language powerful enough to handle intensionality. For, in our system, we can model modal notions. $\mathfrak{U}$ is not permutation invariant now, because the number of humans is not the same across worlds. A similar argument works for WD, as we saw. So, the system may be able to handle them without resorting to bijections. Having said that, we could still build up a Wombat-Disjunction-ish item that survives arbitrary permutations in our intensional type theory:

Wombat-Disjunction$_3$ $=_{df}$ $\vee$ iff, in $\mathfrak{M}$, $\ulcorner$There are wombats$\urcorner = \emptyset$; $\wedge$ otherwise.

Here wombat-disjunction$_3$ is mapped to disjunction if wombats are logically impossible. Now, it might be that in no model wombats are logically impossible. Yet, do we want this item to be logical? After all, it is sensitive to wombat-ness, in an indirect way. Yet, it is PI3 invariant. One might say that this item is hyper-logical, and thus the theory is silent on it. I would rather conclude that this seems like a glitch in the system, rather than a big objection. Yet, it still suggests that it would be better to switch to a bijection-invariance framework. I will still stick to permutation-invariance, however, for two reasons: it is technically easier, and it has the same results, for items that do not weirdly shift interpretation across models like this one.

CHAPTER 2: A NEW PERSPECTIVE ON FORMALITY.

This chapter is about the limits of standard permutation invariance and how to fix them. I will try to sketch a novel theory of logicality, with a better philosophical foundation.

In section 2.1 I will discuss the limits of standard invariance as a way of spelling out what formality is. I put forward two main criticisms. The first is that invariance tests are not explanatory: the fact that an item is invariant is usually not a reason, by itself, to explain its logicality. Rather, the authors tend to draw from some external, independent theory of logicality the justification for the results they reach. The second criticism is that what comes out invariant will invariably depend on the algebraic structure of the system at hand. Different semantics will result in different invariant items. Yet, logical notions should not vary across different systems in this way.

In section 2.2, I will introduce my theory. The general idea is that the concept of information can play a major role in defining what is logical and what is not. Logical notions are notions that are sensitive only to structural aspects of information. To be logical, they must satisfy two requirements:

I.   They must perform an operation that checks only structural aspects of information. I call this "formal computability".

II.   They must preserve formal computability under arbitrary permutations of their inputs.

In the remaining sections I set out my theory in the following steps: in section 2.3 I define the concept of formal operation, where a formal operation is an operation that is sensitive only to structural aspects of information. I will put forward some candidates. In section 2.4 I set out the technical system for "formal computability". In section 2.5 I define what is the basic formal system for describing information, and how to transform it. In section 2.6 I finally suggest two different versions of "formal-computability invariance". I argue that something is a logical notion iff it is formal-computability invariant. The next chapter will show the usefulness of the new theory, and its ties to the "old" permutation invariance.

2.1 What permutation-invariance is missing.

Why should we think that permutation-invariance is a useful test for formality? What is its implicit, philosophical ground? What Tarski (1966), Mautner (1946) and Mowstowski (1957) had in mind, probably, was a technical way of implementing a popular and old idea: logic is concerned with the most abstract and general concepts, and thus logical notions must be the most abstract and general notions. So, if a notion is sensitive to some specific "structure" of the world, then it is not "general" enough to be logical. Arbitrary auto-morphisms of the universe will ensure that all such extraneous "sensitivities" are discarded. At least, this was the initial idea.

The following popular improvements did not change the core presupposition of the theory: that logical notions "abstract" away from matter. This is clearly reflected by the way the Tarskian account has been revised over the years: the main issue was that its choice of transformation may not ensure enough abstraction: bijection invariance takes care of even more "extraneous structure", and it is better suited as a test for logicality (Sher 1991), even though, again, it may not be powerful enough (McGee 1996). Strong-homomorphism invariance is even a stricter test, and it was designed to abstract away from cardinality, as well: logic should be even more general than mathematics (Feferman 1999). In this way, identity goes out of the picture, as well, because it is not homomorphism-invariant: it is not abstract enough.

2.1.1 Standard Invariance is not explanatory

The best way to show the limits of standard invariance accounts is to show the limits of the picture we have just drawn. For it may be true that logic is the most abstract subject of all, but it is still about something, in some sense. By that, I mean that logic is still concerned with "structures" of reality: it is not the absence of structure. Thus, a more powerful test may not mean a more accurate one. Intuitively, the test should not just "abstract away" from everything, but from all and only what is not relevant to logic. And what is not relevant to logic, if not what is not logical structure? Therefore, the test should check that a notion is sensitive only to logical structure, by abstracting away from all and only non-logical structures. Yet, how can we be sure of the accuracy of the test? Well, we first need a theory that

tell us what logical structure is. Then, we need to build up a test that mirrors this theory. In this way, the very fact that an item "passes" the test is a sufficient justification for its logicality.

Do we find such strategy in the literature on invariance? Not quite. Invariance theorists do not usually give any such explanatory test. Rather, what they seem to do is this: they take an independently obtained extension as *desideratum*, and "build up" the invariance test so that it happens to roughly pick out the extension they have chosen "before-hand". So, for example, Lindenbaum & Tarski (1935) show that the standards connectives definable in simple type theory (that of Principia Mathematica) are all permutation invariant. McGee (1996) generalized this result and proved that an operation is permutation invariant if and only if it is describable in a language with, as only primitives, paradigmatically logical items: identity, substitution, disjunction, negation and existential quantification. So, these authors have already a rough idea on the what the results of the test should be, and they seem to "swap around" the order of explanation: it is not that what passes the test is logical because it passes the test, but rather the test is accurate because it gives out the "right" results, that we already know to be right from independent sources. The same goes with Feferman (1999); at the very beginning of his paper he says: "I have been moving more and more to the position that the first order predicate logic has a privileged role in our thought" (31-2). He has an objective in mind: making logical all and only first order predicate notions. This objective is independently motivated, and its ground is not in homomorphism invariance. He then builds up the test so that it happens to pick out, roughly, only first order predicate notions, identity excluded. Even in the most recent debates we find something similar: Bonnay (2008) takes partial isomorphisms as the "right" transformations to look at, because they avoid the Continuum Hypothesis problem, by making logical only "finite" cardinality quantifiers. There is no special story about why logicality should be reflected by partial isomorphisms invariance. Rather, there is a problem that is quite independent of invariance: we want logical notions not to include a particular class of items. We then devise the test "bottom-up", so that it does not pick out that class. Again, it is not that what passes the test is logical *because* it passes the test, but rather the test is accurate because it gives out what we planned to be the right results.

The problem with this strategy, as rightly noted by Novaes, is that it is a kind of "reverse engineering" (2014, 87). There is a price to pay, if one devises a test by reverse engineering the already obtained results: one loses explanatory power. A theory of logicality should not just be extensionally adequate, but also explanatory: it should tell us why these are exactly the right notions to pick out. Yet, if one starts from the results already, no test will tell you that. Suppose I disagree with Feferman: first-order logic is not the only "genuine" logic, and cardinality quantifiers are logical. Feferman will reply with some objections, like the issue with the Continuum Hypothesis discussed in section 1.1. Yet, none of the arguments he will offer will hinge upon homomorphisms. That is, he will never say that his are the "right" logical notions *because* they are homomorphism invariant. There is not "special", explanatory link between homomorphisms and logicality: they just happen to pick out what Feferman already thought to be the right extension for logical constants, in the first place.

This issue becomes even more serious in intensional frameworks, where our intuitions on what counts logical are less clear-cut. Again, Novaes made a good point about S4 operators: they are not permutation invariant, and yet why should they be not logical?

> *The point is not that the S4 modal operators should necessarily be counted as logical; rather, the point is what independent motivations would justify that the S4 modal operators do not count as logical, whereas their counterparts interpreted on universal frames do. What is the fundamental, philosophical difference between these two cases besides the fact that they are interpreted on different structures? (2014, 94).*

I am inclined to agree. Given that the test itself does not reflect and it is not "build up" from a criterion of logicality, it is not persuasive to say that S4 operators are not logical *because* they do not pass the test. Why should we trust that permutations or bijections or homomorphisms will "take care" *only* of non-logical structure? Maybe they undergenerate. After all, these tests have been devised through "reverse engineering", so that the test itself is not really part of the explanation of why its results are logical. Moreover, the extension the test was designed to pick is usually specific to a particular logical framework: extensional logic. With such conditions in place, how can we be sure that, when applying the same test in intensional frameworks, the results will be adequate? Thus, here comes the question:

what independent ground do we have for making S4 operators not-logical, if invariance is not *per se* explanatory?

To sum up: if invariance theories are theories of what logical notions are, they need to tell us two things: which notion is logical and why. Invariance theorists usually have already in mind a rough idea of the extension of logical notions, and they build up the test "bottom-up", so that it gives out that extension. This is problematic, because now the test is not explanatory. What one should do is rather this: she should first argue for an independently motivated theory of logicality, and then build up a test that mirrors the theory, and see what results she gets. Any result will automatically have a better philosophical ground, given that the test itself mirrors the underlying theory: the fact that the item passes the test is itself a corroboration for its logicality. Moreover, we can be more confident that, when we apply the test in different frameworks, we will get plausible results.

2.1.2   Invariant items are impersistent

I believe it is desirable that something should "come out" logical regardless of the semantic system we choose to model it. Call this characteristic "persistence":

Persistence. An item is persistent if and only if its logicality persists across different semantical systems.

Logical items, in the standard invariance theories, are helplessly impersistent: their logicality strictly depend on how we model our notions, and on the formal properties of the system at hand. For obviously, what is invariant under arbitrary transformations of a system depends on the features of the system. I will give an example of such behaviour. Suppose that, when we model propositions, instead of using types W and V, we take them as elements of a primitive type P of propositions. We then define arbitrary permutations on P. Take conjunction: it takes every two true propositions to their meet: the smallest proposition that makes them both true. In this "propositional semantics" conjunction is not permutation invariant: we just need to swap their meet with some other proposition. We can do it because elements

in P are "primitive"[20]. Thus, if we apply permutation invariance in this new system, we upset what is invariant, and thus we change what is logical. This should not be: what is logical should be more independent of the semantics choices we make, and the test should ideally "track" what is logical across different semantics.

Sher realized this problem, when she says:

> When it comes to sentential connectives, we can regard their formality as based on "not distinguishing the identity of propositions." Intuitively, sentential connectives are formal iff they distinguish only patterns of propositions possessing truth values and nothing else. The interpretation of logical connectives as (denoting) Boolean truth functions reflects just this intuition (1996, 678).

Sher seems to have in mind a model like our P. However, she also seems to be contradicting herself: first she says that formal notions should not distinguish the "identity of propositions", suggesting that we should set arbitrary automorphisms of P, at least. Immediately after, however, she says that sentential connectives are formal/logical if they do distinguish something: the "alethic structure" of P. So, they are not invariant under arbitrary permutations of P. So, what should we do? Maybe Sher is suggesting that we should limit the transformations the test is looking at: we should not "upset" the alethic structure. Yet, why is that? If we decided that permutation/bijection invariance is the "right" test, and we apply it in this system, we should not preserve alethic distinctions, because alethic distinctions are not preserved under arbitrary permutations of propositions. This very fact should be evidence that alethic distinctions are *not* logical structure, since we trusted that the test itself should preserve all and only logical structure. Yet, again, given that bijection invariance is not really part of the explanation of the logicality of the results, Sher is happy to limit the class of transformations one should look at. Her justification is that we have the intuition that logical connectives are Boolean functions. Again, as invariance theorists usually do, she is taking before-hand a favourite set of logical notions and she is reverse-engineering the test so that it picks out that extension.

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[20] I discuss this in more details in section 3.2.

To sum up: in a "propositional" semantics like the one just described, permutation invariance, along with any stricter test, heavily under-generates, while in a modal semantics we saw that it has some plausible results. Yet, why should we "prefer" a modal semantics to a propositional, when we apply a test for logicality? Since logical notions are logical independently of the system we choose to model them, the results of the test, too, should be persistent across different semantical systems. Failure of persistence is also tied to the fore-mentioned problem of "trusting" the test when we apply it to intensional notions. If one reverse-engineers the test in a particular semantics, then the test will probably have "plausible" results only in that semantics. When we "switch" to another system, we may be forced to make additional assumptions to force the right results, for example by limiting the class of permutations.

2.2 A Way out? Information first

My idea is that we can ground invariance theories only after we have discussed what logic is. One should start from a criterion of logicality, and build up the test so that it clearly mirrors the criterion. The first thing to do, then, is asking: "What is logical structure?". Now, it seems clear to me that one cannot tell what logical structure is if not by sketching first what logic is. Only then, she will be able to suggest some plausible candidates. What is logic, then? This is a question that is so complicated and, at the same time so important, that it surely deserves another dissertation. And yet, I somehow have now to answer it, if I want to conclude this one[21]. In some sense, I find myself in a conundrum, and the only way out, as far as I can see, is to make some assumptions. I need to assume a particular stance on what logic is, and build up from there. If someone has a radically different view, she will be rightfully disappointed. I hope that there will be some "overlapping consensus" on what logical structure is, even though everyone starts from different perspectives on what logic is.

The general framework I will be assuming for my theory can be summarized as follows: logic is about information, in a special way. Logic is the most abstract study of "controlled" information exchange.

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[21] I am sure I will be able to discuss this more thoroughly in my future doctoral research.

Information is everywhere, and in some sense, everything is "about" information. Biology is about a specific kind of information: information about biological structures. Medicine, too, is about a specific kind of information: information about how our bodies work, and maybe about how to make them work properly. Yet, logic is somehow special in this respect, because it pays attention only to the structural aspects of it. For example, the fact that a piece of information is "contained" in another piece of information, or the fact that two pieces of information have some information in common. From this point of view, logic is blind to most specific properties of information. For example, it does not "pay attention" to the fact that a piece of information is about dogs, while another is not; or to the fact that a piece of information is believed by a certain person, and not by some other; or, again, to the fact that a piece of information is critical to my survival, while another is entirely useless.

Logic is insensitive to what information is *about*. However, this does not entail that logic is not concerned with something: it is about the structural aspects of information. If information conveys content, one can put it in this way: logic is not about the content of information, but about information itself. Let us give an example. Take the information conveyed by the proposition ⌜Fido is a black dog⌝ and that conveyed by ⌜Fido is the dog walking down the street⌝, call them infon1 and infon2, respectively. They have many properties, like being about the same object, and being about Fido. Yet, logic is totally blind to that: the only thing it registers is, for example, that infon1 and infon2 have *some information* in common, whatever that is. Logic abstracts away from *what* they have in common, and from what they are *about* (their content); what is left is their mere structure: the fact that there is a communality of information between them.

To sum up, I will argue that a notion is logical iff it shares the same sensitivities and insensitivities that logic, as a whole, possesses. What logic is sensitive to are all and only the structural aspects of information. Thus, a notion is logical iff it is sensitive only to the structural aspects of information. For example, a notion that is "sensitive" to usefulness to survival cannot be logical: not having this property is necessary for logicality. On the other hand, a notion that is sensitive only to communalities of information is logical: this is a sufficient condition for logicality. Formal notions are "formal" for two reasons:

i.   Because they behave formally: they handle information only by testing its structural aspects.

ii.   Because they abstract away from the non-structural aspects of information.

The first is a sort of 2-formality requirement: logicality is about what logical notions do. The second is more like a 1-formality requirement: logical notions are formal because they abstract away from matter. In my theory, these two reasons are mirrored by the two requirements for logicality I will spell out: logical notion must behave in a certain way, and they must behave in that way regardless of the pieces of information they are fed with. The first bit will be implemented by suggesting some basic "computation" that we may take as describing a "formal operation". The notion's behaviour must be describable through a combination of such basic computations. The second bit will be handled by transforming arbitrarily the information content the notion is taking as input for its computation: the notion must be computable by a formal operation under arbitrary permutations of inputs.

2.2.1   Digression: MacFarlane's theory.

MacFarlane (2000) builds up a theory that shares some similarities with mine. His proposal consists in defining a set of structures that we take as "intrinsic"[22] and should not be disrupted by permutations. We "save" the structure by delimiting the group of permutations we look at, for each type: all and only the intrinsic-structure preserving permutations. His general definition is:

MarcFarlane PI A semantic value k in type Z is permutation-invariant iff for all intrinsic structure-preserving permutations σ^A, σ^B, σ^C, etc., of the basic types A, B, C, etc., $\sigma^Z (k) = k$ (2000, 210).

I do agree with MacFarlane on the limits of standard invariance accounts. I also partially agree on the solution: I agree on the necessity of specifying a set of "intrinsic/logical" structures. Yet, I am not sure this will be sufficient. MacFarlane, on the other hand, shares with standard-invariance theorists the opinion that our chosen transformation-invariance will also identify the logical behaviours: all and only

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[22] I don't like the label because "intrinsic" does not seem relevant. In what sense the partition between human and non-human things is not "intrinsic" to type O? Yet, it is not logical. I prefer to call it by its obvious name: "logical structure".

the transformation-invariant ones. My strategy may be more complicated, but I think it checks in a more reliable way if the item is performing some logical operation, and it has therefore more explanatory power. For my theory is two-fold: there is a "logical-behaviour" bit and an "abstraction" bit. The "logical behaviour" bit should be done by specifying a set of "acceptable" formal behaviours. Then, the "abstraction bit" is done by ensuring that such behaviour is not "by chance", but really general: the operator should act formally under arbitrary permutations of informational inputs. This have consequences: for example, in my theory cardinality quantifiers are not logical, because they fail to represent a formal operation (more in section 3.3). In MacFarlane's, they are logical simply because they are invariant.

2.3 The Structure of Information

The theory I will sketch in this dissertation puts information first and tries to extract logical structure from information-structure. Logic is the study of controlled information exchange. Logical notions are all and only the notions that are sensitive only to the structural aspects of information. This is tested in two steps: the notion must check only structural aspects of information, and it should do so under arbitrary permutations of the information-content it receives as input. To make clear how we are going to implement this idea in our system, we need to specify first what information is, by putting forward some candidates for "structural aspects of information". These candidates will determine what operations formal notions can perform, and define what "formal-computability" means.

What are formal operations, then? I will write down a list of them. The list will inevitably be incomplete. Since logic is about the structural aspects of information, an operation is formal iff it can be performed by being "sensitive" only to structural aspects of information. So, for example, the operation that takes every sentence about humans to truth and everything else to false is not a logical operation because, to perform it, the notion must be "aware" of the non-structural partitions, in the space of information, between what is human and what is not. The question then is: what are the structural aspects of information? The followings seem to me interesting candidates, worth discussing:

i. "Information-containment".

There is a clear enough sense in which information is ordered by a notion of "containment": an infon contains another iff it conveys all that the other conveys. So, for example, the information ⌜There are red and blue roses⌝ conveys the information that ⌜There are red roses⌝. "Containment" is probably just a metaphor: it is not essentially set-theoretic, but we call it "containment" because it helps us understand what is going on. Formally, this relation is a partial order in the space of information: it is reflexive, transitive and antisymmetric. It is reflexive, because every infon contains all the information it has. It is antisymmetric because if two infons "conveys each other", they must contain the same information, and if so, since infons are nothing but clusters of information, they are just the same infon. Transitivity is reminiscent of the so called "Xerox principle" (Dretske 1981, 57): if α carries some information β, and β carries some information γ, then α carries γ, as well. This seems to be inherent and essential to our very concept of information. "Containment" goes by many names. In situation-theory, for example, it is called "persistence" (Barwise and Perry 1983).

The corresponding operation for "containment" is, maybe unoriginally, the "information-containment-operation": the operation that checks if one piece of information conveys another. Containment, even though it is not essentially set-theoretic, will usually be modelled as the subset relation, in the scope of this dissertation. For example, a proposition P contains proposition Q iff $[\![Q]\!] \subseteq [\![P]\!]$, where $[\![X]\!]$ is the truth-set of X. In set-theoretic terms, containment is always the subset relation and not the membership-relation (more on this in section 3.1).

ii. Trivial and Empty information.

In the system of information, we usually assume that there is a *minimum* and a *maximum* for every "type" of information, defined by containment $\sqsubseteq$:

- k is the *maximum* of a certain type of information Y = k is the element such that $\forall x_{\in Y}(x \sqsubseteq k \ \& \ \forall z_{\in Y}(k \sqsubseteq z \rightarrow k=z))$

- k is the *minimum* of Y = k is the element such that $\forall x_{\in Y}(k \sqsubseteq x \ \& \ \forall z_{\in Y} (z \sqsubseteq k \rightarrow k=z))$.

Information may come in different types and, always, elements in different types are not linked by containment. For example, if infon(a) answers the question "Which number is even?" and infon(b) conveys the information: "There are roses", these two infons do not bear any containment relation: one is about objects, one is propositional. One may be modelled as a set of objects: the set of even numbers, and the other, being propositional, may be modelled as a set of worlds. Yet, each of them has a unique *maximum* and *minimum*, in their type: a "trivial" information that contains every other infon of that type, and a null information that is contained in all the infons of that type. These are structural aspects of the space of information, and we will usually refer to them as 1 and 0. What they mean depends on the kind of information we are representing, in general. For example, in the set of truth-values, they are Truth and Falsity, in the modal space, they are W^L and the empty set; in the "objectual space", the set of all objects O and the empty set. This structural aspect is linked with the popular importance of alethic distinctions in logic. Truth matters to logic: in my theory, this is reflected by the fact that Truth and Falsity are structural aspects of information, being the maximum and minimum of the alethic space. The corresponding operation a formal notion can perform is this: at any point of the computation, it can "write down" 1 or 0, whatever they mean.

iii.   Communalities of information.

Logic registers what is common between two infons, in the most abstract sense. That is, it does not register how two pieces of information have something in common: it does not say, for example, whether they are both about cats. It just says that they have *some* information in common, whatever that information might be. This "structure" in the logical space is fundamental enough, and a plausible candidate for logicality. The main formal operations mirrored by such structure are two. The first is the operation that takes a set of infons Σ and gives, as output, the meet $\bigwedge\Sigma$ (the infon that has all and only the information common to every member of Σ). The second is the operation that takes a set of infons Σ and gives, as output, the join $\bigvee\Sigma$ (the smallest infon that has all the information of every member of Σ). This structure is probably reducible to containment, since we can define meet and join through it:

- $\bigwedge\{x, y\} = z_{\in \jmath}/z \sqsubseteq x \ \& \ z \sqsubseteq y \ \& \ \forall i_{\in \jmath}((i \sqsubseteq x \ \& \ i \sqsubseteq y) \rightarrow (i \sqsubseteq z))$.
- $\bigvee\{x, y\} = z_{\in \jmath}/x \sqsubseteq z \ \& \ y \sqsubseteq z \ \& \ \forall i_{\in \jmath}((x \sqsubseteq i \ \& \ y \sqsubseteq i) \rightarrow (z \sqsubseteq i))$.

Meet and join are the intensional counterpart of conjunction and disjunction, respectively, when we are modelling information as sets of possible worlds[23]. However, meet and join can take on different forms when the space of information is not Boolean.

iv.   The place of an infon in the space of information.

This is one of the most structural aspects of information. The correlated formal operation is complementation: the operation that takes an infon α to the largest infon it excludes: α^c. The idea is that to understand what an infon carries is to know what it excludes. It is tightly correlated with containment: α^c contains everything that is excluded by α.

v.   Compatibility of information.

Compatibility is one of the most general boundaries in the space of information. It is related to complementation: the complement of an infon is the biggest infon incompatible with it. Its correspondent operation checks whether two infons are compatible with each other. Usually, compatibility is reducible to other structures. So, for example, α is compatible with β iff their meet is not empty, or if the complement of α does not contain β.

So, to sum up, the hypothesis I will entertain is that the behaviour of a formal item will be describable by a combination of the formal operations just mentioned. The most important structure is containment. The list is provisional and up for upgrades. In the scope of this dissertation, I will focus just on those candidates. We can already gain insights on the reasons why some of the most striking examples of logical notions are logical. Take disjunction and conjunction, for example: why are they formal? The information-theory explains this by saying that they are logical because they are sensitive only to "communalities of information". We can check this by looking at the operation they perform: it is one instance of the information-structural operation that takes two pieces of information to their join and to their meet, respectively. Of course, we are still lacking the "abstraction" part: we want to be sure that

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[23] In extensional languages, the information in a proposition is either T or F. Thus, for example, conjunction is still the meet, because it checks what is the information common to some propositions: either all of it or none.

disjunction and conjunction perform the information-structural operation regardless of the informational inputs they receive. This will be developed in section 2.6.

2.4   Formal computability

We said that the behaviour of a logical notion should be describable by a combination of formal operations. To make all of this more precise we need a clear definition of "being describable by formal operations". Call this characteristic "formal-computability" (FC). When is a notion formally-computable? First, let us do it in the "informal" way. We can describe a formal operation by specifying its inputs and its rules of transformation. Inputs, intuitively, are countable sequences of infons (α, β, γ, δ, …) called input-sequences. When we define an operation we also specify the length of the sequences it takes as input. We will label |s| the length of a sequence s: if we take a sequence to be a function from natural numbers to the set of elements of the sequence, then |s| is the cardinality of the range[24]. Where x, y and z are variables for infons in the input sequence, the basic rules of a formal operation are the following:

1. Check $x \sqsubseteq y$

2. Calculate $\bigwedge\{x, y, …, z\}$

3. Calculate $\bigvee\{x, y, …, z\}$

4. Calculate x^c

5. Check $\beta^c \not\sqsubseteq \alpha$.

6. Write x, where x is an element of the input sequence.

The first rule-type is "information-containment". Rule (2) and (3) are operations that register "communalities of information": meet and join. Rule (4) is about what a piece of information excludes: it takes you to the relative complement of x in the type of information of x. (5) is compatibility. Rule (6) "recalls" the inputs.

---

[24] In the scope of this dissertation, by "range" I always mean the image of the domain in the codomain, and not the codomain as a whole.

Definition 1 A formal-operation is whatever deterministic computation with a unique output can be done by taking as input sequences of infons, and setting any appropriate sequence of rules of type (1) ... (6).

The F-operation must have a single infon as output, for every fully specified input. The F-operation, thus, will be a function. To compute relations, we can have, as a rule, that an n-ary relation R is formal iff its unique valuation $\mathcal{V}_R$ is, where its valuation is the function $\mathcal{V}_R$: (A×B×, ..., ×N) → {0, 1} such that $\mathcal{V}_R$(a, b, ..., n) = 1 iff (a, b, ..., n) ∈ R, 0 otherwise. The sequence of rules must be "appropriate", in the sense that it must picture a meaningful operation, and not just a bunch of actions that are not related to one another (more on this later). The F-operation must be deterministic, as well: there is only one possible output. So, the following cannot be an F-operation:

Fake F-operation Input: sequences of type (0, 1, α, β).

Rules:

i. check α⊑β.

ii. If yes print 0 or 1.

This is not admissible, because the operation can "choose" to follow one disjunct or the other. This also takes care of potential trivializations, as we will see: we want logical items to act always in the same F-computational way, regardless of the infons they receive as input. Suppose some k is F-computable by X only when it takes as arguments (a, b, ...n) and by Y only with arguments (o, p, ..., z). Then k is arguably "aware" of the specific identities of the arguments and should not be logical. Yet, using indeterministic operations, one could just make up the new formal-operation X ∨ Y to describe the behaviour of k, bypassing the uniqueness requirement.

Now we can have a first, rough definition of F-computability:

Definition 2. Take an item k and a formal operation X. k is computable by X iff, for any tuple (s,y) ∈ k, there is a tuple (s', y) ∈ X, where s and s' are sequences, and s has only elements of s'.

Basically, for some item k to be computable by an operation X, it must be that, for any instance of k, there is an instance of the formal operation (FO) that takes, as input, the argument k has taken and maybe something else, and gives out the same output. More technically, there must be a "composition function" Comp: $FO^{-1} \circ k$ such that Comp(s) = s' iff k(s) = FO(s') and s has only elements of s'.

A first problem with this definition is that some operation may compute k in non-formal ways. That is, it may be that, even though the formal operation can simulate k, it does so in a way that is sensitive to some irrelevant structure. For example, take a quantifier like ⌜Every dog⌝. It takes predicates true of every dog to 1 and everything else to 0. "Dog-ness" is not a structural aspect of information, but we can still "sneak it in", in the way we let a formal operation compute the quantifier. Take Compatibility, for example: for every X such that (Every dog)(X) = 1, set Compatibility (X, O). For every other Y, set Compatibility (X, ∅). Now ⌜Every dog⌝ is computed by compatibility: whenever the output is True, we are checking the compatibility of the predicate with the trivial predicate, and whenever the output is False, we are checking the compatibility with the empty predicate (that is never satisfied). To avoid this trick, we can ask that the FO does the same thing, for every instance of the computation. A way to ensure this is by complicating the requirements for the composition function to hold.

Definition 2'. Take an n-ary item k and a formal operation X. K is computable by X iff there exists a function Comp: $X^{-1} \circ k$ such that:

a) For every s and s', Comp(s) = s' iff k(s) = X(s')

b) For every s, Comp(s) is the sequence such that $(b_m)_{1 \leq m \leq n} = (a_m)^s$ and $(b_m)_{m>n} = k$, where $(a_x)^s$ is the x-th element of s.

The definition says that the FO, in order to compute some K(a, b, ..., n), must take a sequence that is the exact copy of (a, b, ..., n) up to n, and then, if it takes more elements, these are constants: they are the same for every instance of the computation. Now the trick we used before is not available, anymore: the FO cannot check the compatibility with the empty set for some elements, and with the trivial set with some others.

There are more complications ahead, however. We can show that, using only rules of type (6), any n-ary function whose output is some constant k is F-computable.

Proof 2.1. Let K be an arbitrary n-ary function whose range is a singleton. Define this formal operation U (Universal-operation):

  i. inputs: any sequence of length n+1.
  ii. Rule: write (a_{n+1}).

Suppose K(a, b, ..., n) = p, for any (a, b, ..., n). U(a, b, ..., n, p) = p, because (a_{n+1}) is p. Moreover, (a, b, ..., n, p) is the sequence such that $(b_m)_{1 \leq m \leq n}$ = (a, b, ..., n) and $(b_m)_{m > n}$ = p, for any argument of U. So U computes K for any (a, b, ..., n).

This is not as catastrophic as it looks, for the fact that such items are formally computable does not entail that they are so regardless of the specific content they are fed with. Thus, the theory is not bluntly trivialized by this result. In fact, when we introduce the invariance part, we can show that such items will never be logical unless their output is 1 or 0 (the *maximum* or the *minimum* of the type of information their inputs have, respectively).

It is time to implement a better framework to work on. For we have still no clear and precise idea about what counts as an "admissible" formal operation: what does it mean, exactly, that the sequence of rules should be "admissible"? We just need to make our framework more precise. For that, I need some very simple automata theory. I will use Turing machines, even though they may be an overkill here, since I suspect that the computation we are after can be done by simpler automata[25].

Definition of a Turing Machine [26]*(Hopcroft et al. 2001: 8) A Turing Machine can be informally described as an infinite tape with a head that can read what is written in each cell of the tape. The head can read a symbol on a cell, move to the next cell, (over)write something on it, and move either left, right or not move at all. Mathematically, it is defined with a tuple of seven elements (S,*

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[25] I am inclined to think that pushdown automata, which are less powerful automata, can do the job. Finite state automata may not be enough, though.

[26] There are other equivalent definitions of Turing Machines, with slightly different instructions. For example, Turing himself used instructions like "In tuple ⟨s,a⟩, print symbol b and move right/left/null (1937).

Σ, Γ, δ, s_0, b, F). S is a set of linearly ordered states: the "cells" of the tape. Σ is the input alphabet of the machine: the set of symbols the automaton can write. Γ is the set of tape symbols: it is always a (sometimes improper) superset of Σ. s_0 is the initial state. F is the set of final states, with F⊆S. "b" is the blank symbol: the initial symbol written on any cell. Finally, δ is the transition function S×Γ → S×Σ×D. It defines the instruction the Turing Machine can take. D means "direction": it can be left, right or null, usually referred to as -1, +1, 0. Informally, the instructions have always this meaning: when in state s you read x, go in the next state q, write y and move right/left/don't move. A Turing Machine is deterministic iff for any tuple S×Γ, there is only one instruction.

It is important to make something clear from the start: for our purposes, we do not need to make the Turing Machine do the actual work (it may probably be not even possible). By that, I mean that it is not up to the Turing machine to actually check whether α is contained in β, or to calculate the meet Λ{x, y, ..., z} of infons x, y, ...z. What we will need the machine to do is just register the result of the test, and act accordingly, by finally printing the right output. So, technically, it is more correct to say that there are three actors in play, here: the item whose formality we are testing, a formal operation and a Turing machine that registers what the formal operation is doing. However, to make things easier and more concise, I will just speak as if the Turing machine and the formal operation are the same thing.

For our purposes, we do not need D, so we can take any instruction to have, as D, the null-direction 0. Usually, I will just omit D all-together. As a rule, every Turing Machine will have printed, in its initial state, the input sequence. That is, if the input-sequence is (a, b, c), then on s_0, at the start of the computation, we find symbol (a, b, c)[27]. This rule is not necessary, but it makes things easier. The rules (1)-(6) above must be replaced by Turing instructions. We will model them as actions on couples of ⟨state, symbol⟩. Informally, we will say something like: when in state (s,a), perform action [.]. Atomic actions, then, are functions defined over tuples S×Σ, which correspond to arguments and values of δ. That is, [..](s,a) = (s', b) iff δ(s, a) = (s', b, 0). Compounded actions are sequences of simple actions. For

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[27] Note that the input is an ordered n-tuple.

example, [.][..](s,a) means: first, apply action [..] to (s,a), obtaining (s', a'); then, apply action [.] to the resulting (s', a'). For our purposes, the following will be all and only the tests our machine can perform:

1. $[1]$ $(s_n, a) = (s_{n+1}, 1)$.

2. $[0]$ $(s_n, a) = (s_{n+1}, 0)$.

3. $[\sqsubseteq]$ $(s_n, (x,y)) = [1]$ $(s_n, (x,y))$ iff $x \sqsubseteq y$, $[0]$ $(s_n, (x,y))$ otherwise. That is, in state n read two symbols x and y, move to the next state s_{n+1} and print 1 iff $x \sqsubseteq y$, print 0 otherwise.

4. $[\wedge]$ $(s_n, a) = (s_{n+1}, \wedge \{x/ x \in a\})$. That is, in state n read a = (x, y, ..., z) move to the next state s_{n+1} and print the meet of x, y, ..., z.

5. $[\vee]$ $(s_n, a) = (s_{n+1}, \vee \{x/ x \in a\})$. That is, in state n read a = (x, y, ..., z), move to the next state s_{n+1} and print the join of x, y, ..., z.

6. $[\neg]$ $(s_n, x) = (s_{n+1}, x^c)$. That is, in state n read x, move to the next state s_{n+1} and print the complement of x.

7. $[C]$ $(s_n, a) = [\wedge]$ (s_n, a) iff $\wedge \{x/ x \in a\} = 0$; $[1]$ $[\wedge]$ (s_n, a) otherwise. That is, in state n read a = (x, y, ..., z), go to the next state and print their meet. If the meet is not 0, go to the next state and print 1.

8. $[input, b]$ $(s_n, a) = (s_{n+1}, b)$, where b is some sequence whose elements are in the input sequence or on some state s_m with $m \leq n$.

As a rule, the set Σ of symbols a formal TM can write is defined inductively:

- If z ∈ input-sequence, or z=1 or z=0, then z ∈ Σ

- If $z = \wedge \{x, y..., n\}$ or $\vee \{x, y..., n\}$ or x^c, or $x \sqsubseteq z$ or $z \sqsubseteq x$, then z ∈ Σ, where x, y, ..., n ∈ Σ.

Basically, Σ is the closure under meet, join, complementation and containment of the set of elements of the input-sequence plus 1 and 0.

Now, to (re)define formal computability, we just write:

Formal Turing Machine A Formal Turing Machine is any deterministic Turing Machine whose only possible moves are combinations of actions of type (1)-(8).

Formal Computability K is Formally-Computable $=_{df}$ there is an FTM such that there exists a function Comp: $FTM^{-1} \circ K$ such that:

a) For every s and s', Comp(s) = s' iff K(s) = FTM(s')

b) For every s, Comp(s) is the sequence such that $(b_m)_{1 \leq m \leq n} = (a_m)^s$ and $(b_m)_{m > n} = k$, where $(a_x)^s$ is the x-th element of s).

Some comments on the action-types. Action (7) says that, to register computability between x and y, the machine will first write on the next cell the meet of x and y. If it is empty, the machine has nothing else to do, since the join is 0, which is also the result of the compatibility test. If it is not empty, then the machine writes 1.

Rule (8) is needed to let the machine work on some element of the input, or to "add" some transformed element to some element in the input. To see why we need (8), suppose we want to model the operation that checks if two items are contained in each other: call it "symmetric containment". This is a formal operation, in our sense. Yet, without (8) we cannot compute it, for we would need to apply the containment test twice: first to (a,b) then to (b,a), where (a,b) is the input sequence. Yet, suppose a and b are contained in each other. Then the result of applying [⊑] first to (a,b) will be 1. So, now the machine reads 1 and does not know how to reiterate the test: we thus recall the inputs, using an instance of (8), and then re-apply the containment test. Symmetric containment is thus formally computable by this FTM:

- Input: sequences (x, y)
- F = {(s_1, 0), s_3}.
- Instructions:
    - [⊑](s_0, input).
    - [input, (y,x)] (s_1, 1).
    - [⊑] (s_2, (y,x)).

Here, the FTM first checks if x is in y. If not, it halts, printing 0. If yes, it goes on printing (y, x). It then checks if y is in x. If not, it halts printing 0; if yes, it halts printing 1. A graph makes this very intuitive:

Table 3

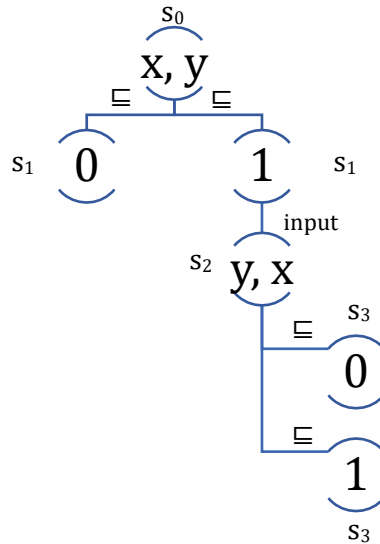Something similar happens for material implication. $A \rightarrow B$ is equivalent to $\neg A \vee B$. Without (8), we could not compute material implication in this way. Here is how we do it:

- Input: sequences (x, y)

- $F = \{s_4\}$.

- Instructions:

  - $[input\ (x)](s_0, input)$.

  - $[\neg]\ (s_1, x)$.

  - $[input\ (\neg x, y)]\ (s_2, (\neg x))$.
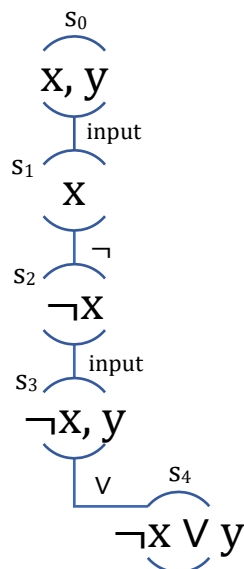
  - $[\vee]\ (s_3, (\neg x, y))$.

Table 4

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2.5 How to model information

Now that we have a formal machinery for modelling formal operations, we need to say what is information, in our system, to implement the second requirement: that a formal notion is formally computable regardless of the infons it is fed with. The most general and comprising way of modelling information is this: we just assume a primitive set of infons $\mathcal{I}$, and define some relations on it. The algebra we impose on the set will determine the formal properties of the system. In general, I will require a minimal structure on $\mathcal{I}$, because I want to stay as neutral as possible on how to model information. The minimal requirements, in the scope of this dissertation, have already been put forward in the discussion on formal operations. On $\mathcal{I}$, we define a partial order of "containment": $i \sqsubseteq i'$ iff the information contained in i is contained in i'. We will also suppose that there is a 0 and a 1, for every "type" of information in $\mathcal{I}$: for every type Y, there is a unique element that is such that every $i \in Y$ are contained in it, and a unique element that is contained in every $i \in Y$. With containment, we can define meet and join:

- $\bigwedge\{x, y\} = z_{\in \mathcal{I}}/z \sqsubseteq x \;\&\; z \sqsubseteq y \;\&\; \forall i_{\in \mathcal{I}}((i \sqsubseteq x \;\&\; i \sqsubseteq y) \rightarrow (i \sqsubseteq z))$.

- $\bigvee\{x, y\} = z_{\in \mathcal{I}}/x \sqsubseteq z \;\&\; y \sqsubseteq z \;\&\; \forall i_{\in \mathcal{I}}((x \sqsubseteq i \;\&\; y \sqsubseteq i) \rightarrow (z \sqsubseteq i))$.

The meet of x and y is the biggest infon that is contained in x and y, the join of x and y is the smallest infon that contains x and y. We can also let the complement of i be the biggest i incompatible with i. That is, i^c is the biggest element such that $i \wedge i^c = 0$. I will not assume any other property. In particular, I will not assume that, for any two infons, there is a join and a meet. I will not assume that for any infon i, $i \vee i^c = 1$, either. So, I will not settle whether $(\mathcal{I}, \leq)$ forms a lattice or a complemented lattice.

Definition Lattices (Birkhoff 1948). A lattice (A, ≤) is a partially ordered set such that every two elements have a join and a meet. A lattice is distributed when join and meet distribute over each other. A lattice is complete if any of its subsets has a meet and a join. A lattice is complemented when it is distributed and every element has a complement in A (that is, for any i ∈ A, there exists

a i^c such that $i \lor i^c = 1$ and $i \land i^c = 0$, where 1 and 0 are the maximum and the minimum of A,

respectively). A complete, distributed and complemented lattice is a Boolean algebra.

How can the space of information not be a lattice? Well, it may be that two infons do not bear any containment relation, and that there is no "super-infon" that contains them, because they are different "kinds" of information. I will give such an example in chapter 3, in section 3.1. Depending on how we are modelling information, however, each subset of $\mathcal{I}$ closed under type membership may form a Boolean Algebra, or some weaker algebra. In the Appendix I show that, if one takes these classes to be the complete powersets of some sets, then they form a Boolean Algebra. In an interesting paper, Barwise & Etchemendy show how to work with a Heyting Algebra of infons (1990). A Heyting Algebra is weaker than a Boolean Algebra, because its elements may have only pseudo-complements: it is not the case that $i \lor i^c = 1$, for any i[28].

2.5.1   Transformation of infons

Since we want to stay neutral on the nature of infons, there will be no unique way to transform them. There will rather be different levels of "transformation" of information, based on how we are modelling it. By defining Formal Computability invariance, I will just talk generally of "transformation of information". When I give proofs, I am assuming that permutation invariance ensures enough abstraction, and that information is modelled in a way akin to what we have done in the last chapter: we build infons as set theoretical construction from the primitives sets W, O and V of worlds, objects and truth-values, respectively. So, for example, if an item takes sentences as arguments, then we permute its inputs by permuting worlds, thereby inducing a permutation in the set of sentences I discuss this system thoroughly in section 3.1, in the next chapter.

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[28] Heyting Algebras were used by Heyting himself to give an algebraic, semantic theory for intuitionistic logic. See, for example, Heyting (1966). Heyting was a student of Brower himself: the father of intuitionism.

2.6 FORMAL COMPUTABILITY-INVARIANCE

The fact that the behaviour of some item in the language is describable through a formal operation is not enough to make it formal: it is only a necessary condition. Take this example: S is ⌜In every cat-world⌝, where a cat-world is any world in which the proposition ⌜There are cats⌝ is true. S is "fixated" with the existence of cats in a non-logical way. Yet, we can show that this item is formally computable:

Proof 2.2. Define the following FTM [⊒]:

- Input: sequences (x,y)

- $F = \{s_2\}$.

- Instructions:

    ○ [input (y, x)] (s_0, input)

    ○ [⊑] $(s_1, (y, x))$.

Set y = [set of all cat-worlds]. Set x = A, where A is some arbitrary sentential argument of the operator. [⊒] computes S for A. For $S(A) = 1$ iff A is true in all cat-worlds, iff the set of all cat-worlds is a subset of the truth-set of A[29]. A was arbitrary, so, for all sentential arguments φ, [⊒] computes S, when given the appropriate sequence (φ, [set of all cat-worlds])[30].

This result shows how little, on its own, formal computability can do. The next step, then, will be to ensure that the item is not F-computable "by accident", so to speak, and just for some specific pieces of information, but generally. To ensure the formality of some notion k, k must be formally-computable regardless of the specific content it is fed with. This is where permutation-invariance comes in handy. The way to implement it, though, will be tricky. In standard invariance tests, we are asking invariance of the "identity" of the notion under arbitrary permutations. Suppose we ask identity plus formal computability. An item is be logical iff it is invariant and formally computable. In this case, our theory

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[29] With S, the output for φ is really the proposition ⌜ In every cat-world: φ ⌝. However, I take that this proposition will be either true in every world or in none. For, if φ is true in every A-world, then this fact will not "change" from world to world. So, its truth-set will be either ⊤ or ⊥; that is, 1 or 0.

[30] S is computable in this way because it is equivalent to a Kripke modal operator defined over an accessibility relation R such that R(w, w') iff w' is a cat-world. As we will see in the next chapter, any such Kripke modal operator is formally computable.

would be a contraction of permutation invariance. I do not follow this theory, however, mainly because we would have different sets of logical notions in different systems, based on how we are modelling information. For, clearly, what is invariant under automorphisms of a system will depend on the features of the system. In different systems we will end up with different invariant items (more of this in section 3.2).

What if we drop invariance of the "identity" of the item and, instead, we ask invariance under F-computability? Then, an item k of type Y is formally-computable invariant (FCI or FC-invariant, for short) iff:

   i.   k is F-computable
   ii.   For every admissible permutation of information content σ, the resulting $σ^Y(k)$ is still F-computable.

Yet, this will not do, because the requirement is vacuous. It is easy to prove, for instance, that $\mathcal{S}$ would turn out FC-invariant, with this definition. For suppose we use the permutation-invariance system set out in the last chapter, and we look at permutations of worlds. Take all $σ^{\langle\langle W,V\rangle\langle W,V\rangle\rangle}(\mathcal{S})$. For some, $\mathcal{S}$ is invariant (that is, $σ^{\langle\langle W,V\rangle\langle W,V\rangle\rangle}(\mathcal{S}) = \mathcal{S}$) . If so, trivially it keeps its formal-computability. When it is not invariant, it is because one transformed the set of all cats-words (call it C). Take some σ' that does that, and suppose $σ'^{\langle W,V\rangle}(C)=C'$. Now the result of the permutation $\mathcal{S}$' takes any proposition φ such that $⟦φ⟧ \supseteq$ C' to ⊤, and everything else to ⊥, where $⟦φ⟧$ is the truth-set of φ. By setting inputs (φ, C'), the resulting $\mathcal{S}$' is still F-computable by [⊒], for any sentential argument φ. So, this requirement would make $\mathcal{S}$ logical.

We need another way of capturing the blindness to specific pieces of information that items like $\mathcal{S}$ lack. Intuitively, $\mathcal{S}$ is F-computable only because we have a way of "choosing" the infons before-hand (that is, before "passing them" to the formal-computation). Here is a suggestion: take the item whose formality we are testing and the formal operation the item is supposed to be performing. Suppose we induce a transformation in the *inputs* the two are receiving. What we do then is look if, under arbitrary transformations of *inputs*, the formal operation and the item still behave in the same way, by printing the same output. If so, then we may say that the item is logical, because it keeps its formal computability

regardless of how we transform its inputs. For each instance of the computation, each couple of inputs "track" each other, under arbitrary transformations. This is the kind of invariance I will try to model, in the next sections.

2.6.1 Formal-Computability Invariance Defined

To get to the technical definition of FC-invariance, however, we first need a definition of sequence-permutation. The problem is that formal operations are taking as input sequences of infons, and we have yet to explain how permutations affect sequences. First, let us add a type for sequences S:

Sequence An item in S is any sequence of elements x, y, ..., z of any type X, Y, ..., Z.

Take an arbitrary permutation σ of information. Call σ^S the transformation induced in type S by σ. It is defined as follows:

Sequence permutation Take any n-ary sequence S = (a, b, ..., n) of elements of types A, B, ..., N respectively. Take a permutation σ of information. Let σ^A, σ^B, ..., σ^N be the transformation induced by σ in types A, B, ..., N respectively, and σ^S the transformation of sequences induced by σ. $\sigma^S(S) =$ ($\sigma^A(a)$, $\sigma^B(b)$, ..., $\sigma^N(N)$).

We have now all the means necessary to define FC-invariance. I have two suggestions for FC-invariance. The first takes an item to be FC-invariant if there is a formal operation that computes it under arbitrary permutations of inputs. The second defines FC-invariance as preservation of computability under arbitrary permutations of inputs. I will call the first FC-invariance, the second strong FC-invariance.

FC-Invariance An item *k* is FC-invariant iff there is a FTM that computes k under arbitrary permutations of inputs.

Strong FC-Invariance An item *k* is strongly FC-invariant iff it is computable by some FTM, and for any FTM that computes k, FTM still computes it under arbitrary permutation of inputs.

The second implies the first but not vice-versa. For the first, it can be that some (but not all) formal Turing Machines compute k, yet do not compute it anymore when we permute their arguments. To satisfy the stronger invariance, this cannot be the case: all FTM that compute k must compute it under arbitrary permutations of their arguments.

Using the definition of permutation of sequences, and the definition of computability, we can give a precise definition for both kinds of invariance. Start with the second. To be strongly FC-invariant means that an item must keep its F-computability for every permutation of the input-sequence induced by arbitrary permutations of information. That is:

Strong FC-Invariance An item k of arity n is strongly FC-invariant $=_{df}$ There is an FTM such that there exists a function Comp: $FTM^{-1} \circ K$ such that:

a) For every s and s', Comp(s) = s' iff K(s) = FTM(s').

b) For every s, Comp(s) is the sequence such that $(b_m)_{1 \leq m \leq n} = (a_m)^s$ and $(b_m)_{m>n} = k$, where $(a_x)^s$ is the x-th element of s.

c) Let σ be a permutation of information and let σ^S be the transformation induced in type S by such permutation. For every FTM that computes k, for every s and s', if k(s) = FTM(s') then, for every σ, $k(\sigma^S(s)) = FTM(\sigma^S(s'))$.

Informally, we can describe what is happening with an image. Suppose there are some people in different rooms. They all have some instructions to follow, which consist in receiving some inputs, performing some tests, and finally printing a specific, unique output. Each person represents the behaviour of some operation: person1 is the item K whose formality we are testing and person-2, person-3 ... person-N are all the formal operations K is supposed to be performing "blindly", regardless of the information it is fed with. To ensure that K is really "blind" to "specificities" of content, we permute infons arbitrarily. We then "feed" the transformed infons to all the various people. If, under arbitrary permutations, everyone still "tracks" everyone else, by printing the same output given the correspondent transformed input, then we say that K is strongly FC-invariant.

What is crucial, in the definition, is the scope of the transformations: we do not write "$\sigma^Y(K)$" but "$K(\sigma^S(a, b, ..., n))$": we are not directly permuting the item we are testing, but only its arguments. That is, if an item K is such that $\{((a, b), b), ((b, c), c) ((c, d), d)\} \in K$, then the permutations will not affect this. What we rather do is this: we first check whether the item is formally computable. If so, there will be a function "Comp", as described above. Suppose K is F-computable by some FTM in this way:

Domain K	FTM(x)	Domain FTM
• (a, b)	• b	• (a, b, k)
• (b, c)	• c	• (b, c, k)
• (c, b)		• (c, b, k)
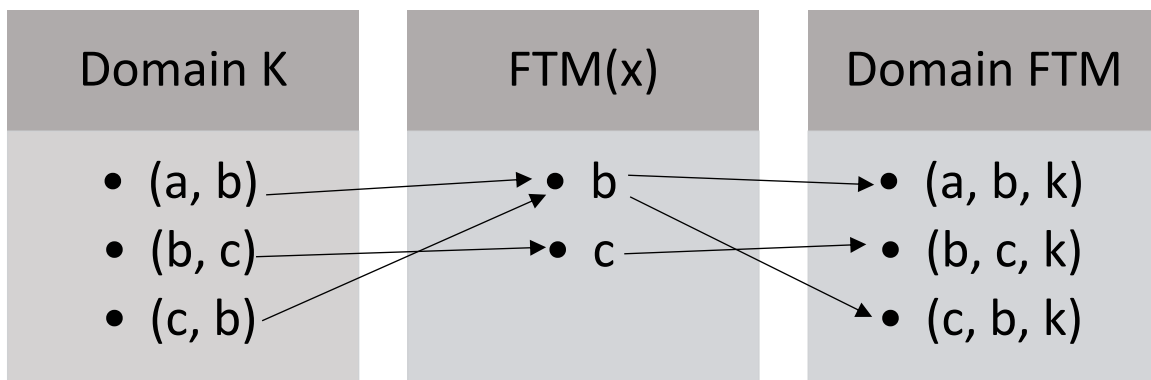
Table 5

This is a partial graph of how the computation works, and it is to be read like this: K takes (a, b) to b, and the FTM takes (a, b, k) to b. Accordingly, Comp takes (a, b) to (a, b, k)

This was the first step. We now permute some objects: we can swap a with c, for example. We then look at what permutation is induced in the relevant domain and range of the computation function. In this case, $\sigma^S(a, b) = (c, b)$ and $\sigma^S(a, b, k) = (c, b, k)$. Now we "feed" K with $\sigma^S(a, b)$, obtaining b. We feed the transformed input-sequence $\sigma^S(a, b, k)$ to FTM and we check if FTM, too, prints b. If yes, then the formal-computability is preserved for that instance; if not, the item is not FCI. In this case, it is preserved, because they both print b, so $k(\sigma(a, b)) = FTM(\sigma^S(a, b, k))$. I believe this system implements an intuitive notion of "blindness to specific information content". This is what we need for the invariance part of the theory.

We can make this even clearer by proving that the non-logical item S above is not strongly-FCI.

Proof 2.3. Define again the same FTM [⊒], that computes $\mathcal{S}$ for arbitrary arguments φ when we set

as inputs (φ, C), where C = ⌜There are cats⌝. Take a φ such that $[\![C]\!] \subseteq [\![φ]\!]$ and φ is not 1. Define a

permutation σ(w) = w', with w ∈ $[\![C]\!]$ but w' ∉ $[\![φ]\!]$. $σ^{\langle W,V \rangle}$(C) = C', and $σ^{\langle W,V \rangle}$(φ)= φ', with $[\![C']\!] \subseteq [\![φ']\!]$

but $[\![C]\!] \nsubseteq [\![φ']\!]$. [⊒]($σ^S$(φ, C)) = [⊒](φ', C') = 1. Yet, $\mathcal{S}(σ^{\langle W,V \rangle}$(φ)) = $\mathcal{S}$(φ') = 0. So, there is a

permutation σ for which $\mathcal{S}$(φ) = [⊒](φ, C) and yet $\mathcal{S}(σ^{\langle W,V \rangle}$(φ)) ≠ [⊒]($σ^S$(φ, C)).

Informally, the two people here are simulating $\mathcal{S}$ and the FTM [⊒], respectively. When we give a φ that

is true in every C-world for person1 and (φ, C) for person2, the two can simulate each other's behaviour.

Yet, to see if this behaviour is general, we now swap an element w in C with w', that is neither an C-world

nor a φ-world. This induce a permutation both in C and φ (because C ⊆ $[\![φ]\!]$), obtaining C' and φ'. We

now give φ' to person1: she will simulate $\mathcal{S}$ by checking if φ' is true in every cat-world, and the test will

be negative, for in φ' there is w', which is not a cat-world. Person2, on the other hand, is following the

rules above, and will check if C' is contained in φ', and the result is positive: she will print 1 as before.

We have found a permutation for which the two people behave differently, so $\mathcal{S}$ is not FC-invariant.

We must note, here, that $\mathcal{S}$ is not "aware" of the permutation induced in C, and so it does not change

behaviour. One might complain that, since the permutation changed C, $\mathcal{S}$ should change accordingly.

However, again, we have ensured that this does not happen because of the way we set the definition of

FC-invariance: we are permuting only the inputs the operators take, not the operator itself. This is

essential: $\mathcal{S}$ would change its behaviour only had it been in the scope of the transformation. The graph

we used before can make things more intuitive:

Domain $\mathcal{S}$	$[\sqsupseteq](s)$	Domain $[\sqsupseteq]$
• (φ) • (φ')	• 0 • 1	• (φ, C) • (φ', C) • (φ', C')
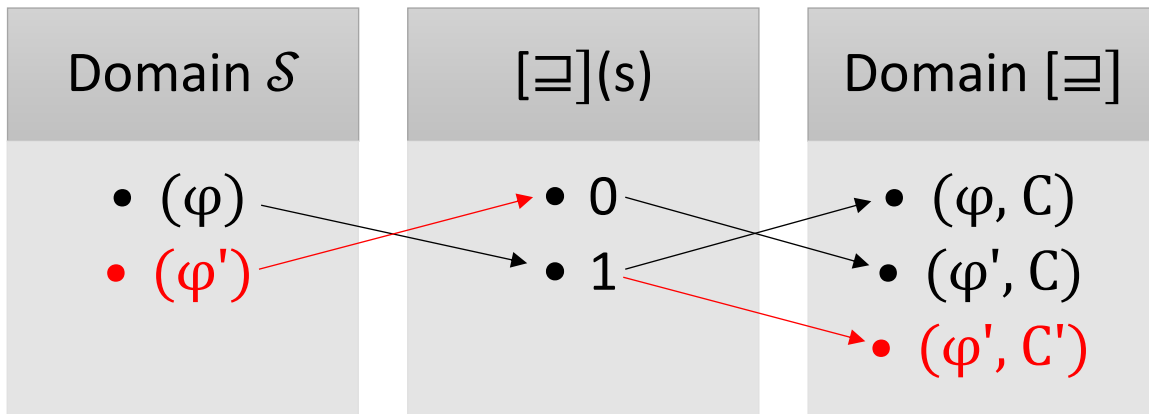
Table 6

$\mathcal{S}$ and $[\sqsupseteq]$ fail to preserve computability for φ and (φ, C), because the transformed arguments, indicated in red, cannot be "linked" by the composition function: the red lines fail to link them because they do not give out the same value.

The way we define simple FC-Invariance is similar:

FC-Invariance An item k of arity n is FC-invariant $=_{df}$ There is an FTM such that there exists a function Comp: FTM^{-1} ∘ K such that:

    a) For every s and s', Comp(s) = s' iff K(s) = FTM(s').

    b) For every s, Comp(s) is the sequence such that $(b_m)_{1 \leq m \leq n} = (a_m)^s$ and $(b_m)_{m > n}$ = k, where $(a_x)^s$ is the x-th element of s.

    c) Let σ be a permutation of information and let σ^S be the transformation induced in type S by such permutation. For every s, s', if k(s) = FTM(s') then, for every σ, $k(\sigma^S(s)) = FTM(\sigma^S(s'))$.

So, to test the FC-invariance of some item K we do this:

    i. Firstly, we check if there is a FTM that computes it.

ii.   We then permute information arbitrarily. For any permutation, we then feed the transformed arguments to K and FTM, and register the corresponding results we get. If they are still the same, then it means that K is FCI.

I do not think that I should choose between one of these two definitions, for they both implement my general idea of formality. Therefore, I will settle for the weaker version:

Logicality An item k is logical iff it is FC-invariant.

This completes the basic exposition of the theory. I will now give some examples of FCI items:

i.   Conjunction, disjunction, negation are instances of meet, join and complementation, both in an extensional and in an intensional language. For example, in an extensional language, the information a proposition contains is either 1 or 0. In such a case, conjunction calculates the smallest infon two propositions both contain: if they are both 1, it is 1; if one is 0, it is 0. In an intensional language, conjunction calculates the intersection of the two propositions in the modal space.

ii.   The identity function is computed by the FTM whose instructions are empty. That is, by the FTM such that $F = \{s_0\}$ and instructions: $\emptyset$. Since, by default, the input is written on s_0, the output is the input.

iii.   N-ary functions that take everything to $\top$ can be computed by the FTM whose instructions are [1] (s_0, input). Any n-ary function that takes everything to $\bot$ is computed by [0] (s_0, input). Any other function whose output is some constant k not equal to 1 or 0 will never be FCI[31].

iv.   The valuation of $\neq$ is computed by the complementation of symmetric containment. Input: sequences (x,y). $F = \{(s_2, 1), s_4\}$. Instructions:

> ○ $[\sqsubseteq](s_0, (x, y))$.
>
> ○   [1] (s_1, 0).
>
> ○   [input, (y, x)] (s_1, 1).

---

[31] This is easy to prove. Take an n-ary K whose output is k, with $k \neq 1$ and $k \neq 0$. K(a, b, ..., n) = [input (k)] (a, b, ..., n, k). Yet, k is not invariant: there is a $\sigma^S(k) \neq k$. So, $K\sigma^S$(a, b, ..., n) = k, but [input (k)] σ^S(a, b, ..., n, k) = $\sigma^S(k) \neq k$.

○ $[\sqsubseteq]$ $(s_2, (y, x))$.

○ $[0]$ $(s_3, 1)$

○ $[1]$ $(s_3, 0)$.
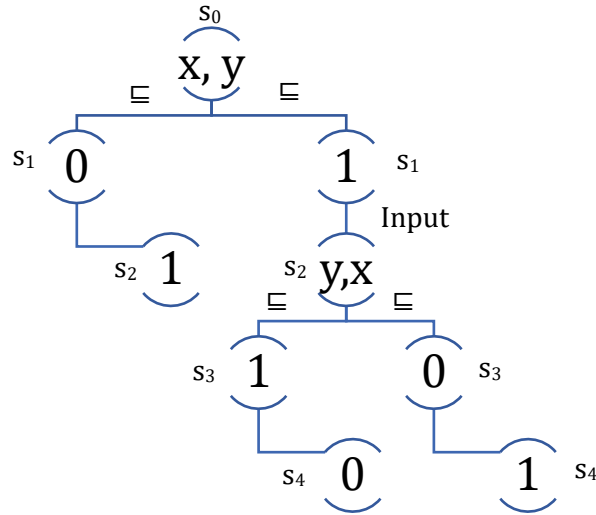
Table 7

v.   Universal and existential quantifiers are FCI, but cardinality quantifiers are not FCI. More on this in the next chapter, in section 3.3.

vi.   Logical necessity is FCI[32]. It is computed by various operations. Examples:

     a. Containment of $\top$ in φ.

     b. the operation that checks if the meet of φ and $\top$ is $\top$.

     c. the operation that checks if the join of φ and $\top$ is $\top$.

vii.   Logical possibility is FCI. For every way logical necessity is computed, there is a computation of logical possibility via the transformation rule $\Box\varphi$ iff $\neg\Diamond\neg\varphi$. So, for example, instead of (a) we have the complementation of the containment test of $\top$ in the complement of φ.

viii. Empty necessity and possibility, where "empty necessity" is defined in a Kripke structure through an empty R.

ix. $\top$ and $\bot$ are computed by containment, when given input $(\top, \top)$ and $(\bot, \bot)$, respectively.

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[32] We are still assuming logical necessity to be S5. As I will show in section 3.4 of the next chapter, these ways of computing necessity operators fail for some weaker logics.

CHAPTER 3: APPLICATIONS OF FORMAL-COMPUTABILITY-INVARIANCE.

This chapter explores some applications of formal computability invariance. In section 3.1 I apply formal-computability invariance to a very plausible model of information, which I call modal/object system: information about objects are taken to be set-constructions from the set of objects, and propositional information is a built up from the modal space. This system has some interesting features: in the Appendix, I show how formal-computability-invariance sometimes collapses into permutation-

invariance, at least for a class of operations. This happens when we model information in a certain way: if the set of infons is made up from the complete powersets of some sets. In the object/modal system infons are usually built up from the powersets of W and $\mathfrak{D}$, so we do have the behaviour described in the appendix.

For some systems, like the one just described, if an n-ary operation on the set of infon is formally computable invariant, then it is permutation invariant. Yet, when we change our way of modelling information we may lose this result. I will show how this may happen in section 3.2, in which I set up a model of information in which some non-permutation invariance items are formally-computable invariant. This shows something interesting: what is invariant under automorphisms of the space of information strictly depends on how we are modelling information. On the other hand, what is formally-computable invariant is more "stable" and tends to be preserved across different models. This is something desirable, because what is logical should ideally be independent of our chosen model. I call this feature "persistence".

In section 3.3 I prove that the existential and the universal quantifiers are formally-computable invariant. I also show how, even if something is permutation-invariant in some systems, there may not be any system in which it is formally-computable invariant. I argue that this is the case for cardinality quantifiers: they are not formally-computable, at all. Thus, the theory is not a conservative extension of

permutation invariance, generally. Cardinality quantifiers are usually regarded as a problematic case of logicality, so I do not take this as a bad result.

In section 3.4 I introduce a relaxed concept of logicality we may call "quasi-logicality". This concept can be applied to the study of the formality of a class of modal operators which I call "world-sensitive": they change behaviour across different worlds. In section 3.5 I try to show how my theory may help shed light on the formality of some interesting notions that have never been discussed in the literature, so far. My case study will be deontic operators: is Ought logical, or at least quasi-logical? It depends: I argue that a radical form of contextualism may make Ought more formal than people guess.

3.1 The object/modal system of information.

My definition of logicality does not specify a particular system of information. I want now to set up a very intuitive model of information that can be applied to my theory. We take infons to be set theoretical constructions from worlds and objects. Transformation of information is induced by arbitrary permutations on W and O. There are philosophical reasons in favour of this solution. Firstly, it is very intuitive. Suppose I am in a class, and I want to know which student is male. This "infon" can be modelled as the set of all male students in the class. With propositional information, we have a "modal" account: the infon of ⌜There are dogs⌝ is the set of all worlds in which there are dogs.

When we model a "propositional" infon as a set of worlds, I am following what Floridi calls a "modal" approach to information (2011, 31). In a modal approach, an infon x is usually defined as the set of possible worlds excluded by x[33]. For take a logical truth: its truth-set is the biggest possible in the system, and yet logical truth are the least informative. So, its information content must be its complement, not its truth-set. However, since it is just easier to go the other way around, I will normally take x to be the set of possible worlds not excluded by x[34]. I do this for two reasons: one is that the two accounts are

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[33] I don't know who was the first to suggest this way of modelling information. Floridi (2011, 111) says that Popper (1935) was "one of the first" to suggest this kind of approach.

[34] For a similar account, see Hanson (1990). Hanson, however, tends to use "situations", which can be partial worlds.

equivalent for most systems, because they usually lead to exactly the same conclusions (we have just swapped each infon with its complement, in its correspondent type). The second is that it is just more intuitive: now the informational content of a proposition is just its truth-set. A good reason for adopting a "modal approach to information" lies in its "affinity" with the way we model beliefs in epistemic logic. For example, take the information contained in a proposition: the more information it has, the more logical space is *excluded* by the proposition (i.e. the "bigger" its complement is). In epistemic logic: the more you know, the less worlds in the epistemic space will be accessible to you. Knowledge is a matter of "subtraction" of the epistemic space[35]. The two limit cases are propositions whose truth-set is W and whose truth-set is $\emptyset$. In the first case, $W^C = \emptyset$: the proposition carries no information. In the second case, $\emptyset^C = W$: the proposition carries maximal information.

   It is worth nothing that this leads to two well-known "oddities", that I will just mention, here. First, since W, for us, is the set of all logically possible worlds, every logical truth will have no information-content whatsoever. To foster the analogy between modal approaches to information and epistemic logic, this first problem is akin to the issue of logical omniscience: if Δ is a set of logical truths, then any agent knows each member of Δ. (Hendricks & Symons 2015). Secondly, a contradiction has maximal information content: this second issue is usually referred to in the literature as the *Bar-Hillel-Carnap Paradox*, mainly because Bar-Hillel and Carnap (1953) were one of the first to discuss it. The idea is that a contradiction is "too much" to be true. In our system W is 1 and $\emptyset$ is 0, since W and $\emptyset$ are the maximal and minimal element, respectively, of the set of propositions, and so W is $\top$ and $\emptyset$ $\bot$. However, we must keep in mind that, informationally speaking, things are swapped around: W is the minimal element and 0 the maximal.

   By adopting this system, we do not need to change our type theoretic framework a lot. "Objectual information" will be modelled as sets of objects, "propositional information" as sets of worlds: their type will be $\langle O, V \rangle$ or $\langle W, V \rangle$, respectively, since their extension is uniquely determined by a function. However, this is not enough, for an infon may be about propositional infons, or about "objectual" infons. Suppose I want to model the infon that answers the following question: "Which proposition is about dogs?". This

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is not a propositional infon, but an infon about propositions. The simplest way to answer this question is to show the list of all the sentences about dogs. This "list" is uniquely determined by a function from propositions to truth-values: its type is $\langle\langle W,V\rangle V\rangle$. Something similar can be raised for properties. Take the infon that answers the following question: "Which ice-cream flavour do you prefer?". This is a question about properties: the answer is a list of flavour-properties, and its type is $\langle\langle O,V\rangle V\rangle$. One can go higher, if she fancies, by asking questions like "Among all the possible ice-cream flavours preferences one can have, which are the best?". The answer will be a list of lists of flavour-properties so, type-theoretically, the infon is a higher-order unary quantifier $\langle\langle\langle O,V\rangle V\rangle,V\rangle$.

Since we are dealing with the same system of chapter 2, the same results apply: the only permutation-invariant infons are O and $\emptyset$ for $\langle O,V\rangle$ and $\top, \bot$ for $\langle W,V\rangle$. For sets of sentences $\langle\langle W,V\rangle V\rangle$, invariant infons are items like $\{\top\}, \{\top, \bot\}, \{\bot\}, \{X_{\in\langle W,V\rangle}/ |X| = n\}$, etc. For short, we will call $\top$ 1 and $\bot$ 0. We are also still equating W with the set of all logically possible worlds, and we are assuming that such set contains metaphysically impossible worlds. We are also assuming that, for any truth-set, there is a corresponding proposition.

Information, usually, is at least a subset of some set, never an element: normally, there is no infon that is mapped to some specific object, for example, even though, for any object, there is the infon that is the singleton of that object. However, we need single objects and single worlds as pieces of information, for otherwise we would never be able to compute any first-level item. So, I will assume that any element of level 0 are "special" infons: any such infon contains and is contained only in itself. One could object to this assumption, however, so I need to say more to justify it. For one may say that an infon α of type O is "contained" in any infon that is mapped to a set X such that $X \in \alpha$. For example, why is Fido not contained in "dog-ness"? After all, it is an element of it and a set contains all its elements. The reason is that the "containment" we are interested in is "information containment", and this relation is never modelled as the membership relation but, at most, as the subset-relation. This is very important, so I want to argue for this in detail. Information-Containment is always a "horizontal" relation: it is between elements of the same type. Membership, on the other hand, is "vertical": it is (or rather it should be) always between elements on different "levels" (Russell's paradox docet). In the object/modal system, a type represents

a "kind" of information: e. g. ⟨W, V⟩ is propositional information and ⟨⟨W, V⟩, V⟩ is information "about" propositions. We said that an infon contains another iff it conveys all that the other conveys. Since elements of different types conveys different kinds of information, two infons of different kinds cannot be such that one conveys all the information that the other conveys. In other terms: we assign a type to an infon on the basis of the kind of information it conveys. For example, if an infon conveys propositional information, this is sufficient to assign it to some ⟨W,V⟩. Suppose we think containment is sometimes modelled by membership. Take a propositional infon α. There is at least an infon β of type ⟨⟨W, V⟩, V⟩ that contains α. Since $\alpha \in \beta$, then $\alpha \sqsubseteq \beta$, then β must have all the information α has, which is propositional, so β has propositional information and, therefore, it is of type ⟨W,V⟩. Yet, it is not, so it cannot be that $\alpha \sqsubseteq \beta$, so 'membership' does not model containment.

The object/modal system has interesting algebraic properties. Since "containment" is never membership but at most subset-relation, any two elements of different types are "incomparable": they are not linked by containment. Thus, containment is not a total order. The space of information is not a lattice, either. Take a set of worlds and a set of objects: not only they are not comparable, but there is also no infon that contains both. Thus, they surely have no join. However, take the subclass of all the propositional information of "level 1": all the subsets of W^L. In Appendix A I show that this is a Boolean algebra. In general, in the Appendix I show that any system that models information from the complete powerset of some set, like this one, will make Formal Computability Invariance collapse into standard Permutation Invariance, at least for n-ary operations defined over the Boolean algebra. This behaviour may not be preserved under extension of the list of formal operations: if we were to add some operations to the list that are not permutation-invariant in this system, we would lose this result. Many formal operations like complementation, disjunction, conjunction and containment, are all permutation-invariant in the object/modal system, as I show in the Appendix: they are all Boolean and therefore invariant, via Inverted Benthem's Proposition.

3.2 A simpler system of information.

I will now propose another plausible model of information with different features. This is the simplest way to model information I can think of: we start with primitive set of infons $\mathcal{I}$. The types of items in this set are contextually determined by the type of items we are testing. So, for example, if we are looking at sentential operators, $\mathcal{I}$ may comprise the set of all sentences. We add new types if needed. On $\mathcal{I}$, we define a partial oder of containment $\sqsubseteq$, with all and only these additional rules:

- if $a \in Y$ and $b \in Y$, then $a \sqsubseteq b$ or $b \sqsubseteq a$.

- if $a \in Y$ and $b \in X$, with $X \neq Y$, then $a \not\sqsubseteq b$ and $b \not\sqsubseteq a$

We then define the transformation of information inductively. Where Y is a type and σ^Y is the transformation induced in type Y, $\sigma^Y = \{\sigma^X | X \in Y\}$, unless k is such that $\forall x_{\in Y}(x \sqsubseteq k \& \forall z_{\in Y} (k \sqsubseteq z \to k=z))$ or $\forall x_{\in Y}(k \sqsubseteq x \& \forall z_{\in Y} (z \sqsubseteq k \to k=z))$. In that case, $\sigma^{\mathcal{I}}(k) = k$. So, basically, we look at arbitrary automorphisms in each type in $\mathcal{I}$, with the exception of each *maximum* and *minimum*, for each kind, which remain constant. We need this exception for otherwise the system "crashes": it is the most minimal preservation condition I can think of[36].

In this framework, we have lost "structure", so to speak. By losing "structure", the resulting transformations are more powerful. This may be counterintuitive, so I will give an example. Take two sentences S and S' such that $[\![S]\!] \subset [\![S']\!]$ and S and S' are not $\bot$ or $\top$. Suppose we are in the object/modal system: propositions are sets of worlds. There is no permutation for which $\sigma^{\langle W,V \rangle}(S) = S'$. One way to see this is by looking at their cardinality: $[\![S]\!]$ is strictly contained in $[\![S']\!]$, it is finite (if not, pick a finite one), and so its cardinality must be strictly less than $[\![S']\!]$: since for any permutation, $\sigma^{\langle W,V \rangle}(S)$ has the same cardinality of S, there is no permutation for which $\sigma^{\langle W,V \rangle}(S) = S'$. On the other hand, in this simpler system, we can swap S with S', because they are not identical to either $\bot$ or $\top$. This has interesting

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[36] More on this in a few paragraphs.

consequences on the system: we can prove that, unlike the previous system, almost all canonical logical operations will not be permutation invariant.

Proof 3.1. Take containment. It is permutation invariant iff (A⊑B iff $\sigma^J(A) \sqsubseteq \sigma^J(B)$), for any σ. Define a permutation $\sigma^J(A) = C$, with $\bigwedge\{C, A\} = 0$. A ⊑ B and yet it is not the case that $\sigma^J(A) \sqsubseteq \sigma^J(B)$. Similar proofs work with meet, join, complementation.

We can swap A with C without "changing" B because we are just taking items in *J* as primitives, without any "internal structure", apart from the *maximum* and *minimum*. If we had modelled sentences as sets of worlds, this would not have been possible. In this simpler system, containment is not the subset relation. Another interesting feature is that 1 and 0 will not be invariant, unless we postulate it. We need them invariant, however, for otherwise we "blow" up the system. For example, take logical necessity: it is FCI via [⊒] when set with (φ, ⊤). Yet, if we could swap ⊤ with some other elements of propositional information, logical necessity would not be formally computable invariant.

One might complain that I am "cheating", because I am "keeping fixed" the *maximum* and *minimum*. What I have to say in my defence is that what I care about, here, is just that the resulting system is still a reasonable model for information. This fact alone is sufficient to show that our theory does not "collapse" into permutation-invariance: some items that will not be permutation-invariant under this system will be formally-computable-invariant. The simplest examples are the formal operations themselves. In general, if we are applying FCI to any FTM, trivially, FTM(s) = FTM(s) and, for every permutation σ, FTM($\sigma^S(s)$) = FTM($\sigma^S(s)$). So FTM is logical. Given that meet, join, complementation and compatibility-test are all formal operations, trivially they will be FC-invariant, because they compute themselves under arbitrary permutations of inputs. Yet, in this simpler system, they are not permutation-invariant. Consequently, my theory in this system is *not* a contraction of PI, and it is clearly more extensionally adequate: PI heavily under-generates logical notions, in this model. This also shows something interesting. One of the problems with invariance theories we discussed was about the "impersistence" of invariant notions: what comes out invariant depends on the features of the system at hand. Yet, we said that logical notions should be persistent:

Persistence. An item is persistent if and only if its logicality persists across different systems of information.

As I have shown, in the object-modal system we mostly get plausible results, when we apply permutation-invariance. Yet, if we switch to simpler systems, like this one, the result may be catastrophic: here intensional conjunction, disjunction and negation are not invariant. Yet, they still retain their formal-computability invariance. In fact, since they satisfy trivially the FC-invariance definition, they will be persistent across any system of information. This suggests that most FC-invariant items will be persistent, as opposed to invariant items. However, not all FCI items will be invariant in any system of information: if we had not kept the *maxima* and *minima* fixed, we could have made logical necessity not FCI, in this system. Thus, we may suggest that all FCI items persist across all systems with the minimal condition of keeping the *maxima* and *minima* fixed, while the FTM themselves are persistent across any system whatsoever. This is still better than invariance, even though persistence is not perfectly general.

3.2.1 An objection to persistence

An objection to the persistence of logicality across systems in the FC-invariance theory can be raised with identity. Take the identity relation between propositions, for example. Since in the object/modal system propositions are functions from W to V, if two propositions have the same truth-set, they are the same proposition. In the simple system, on the other hand, we obtain a finer-grained model: propositions with the same truth-set can be different. One then could argue that persistence fails for identity. For take two *different* propositions with the *same* truth-set P and Q. In the simple system, identity is computed by symmetric containment. The information in P is different from the information in Q because P ≠ Q. Thus, in the simple system, when fed with (P, Q) symmetric containment will give out 0. If we switch to the object/modal system, however, P and Q are the same truth-set, and so have the same information. Thus, now symmetric containment, when fed with (P, Q), will give out 1, even though P ≠ Q, by assumption, and thus, when fed with (P, Q), identity gives out 0. So, symmetric containment does not compute identity anymore, and identity is not persistent.

I am not deeply troubled by this objection, because this is not a problem with my theory, but rather with the definition of a proposition in standard, modal logics. It is well known that, in such systems, necessarily equivalent propositions are identical, and this is often taken as an objection to the idea that propositions are sets of worlds, or functions from worlds to truth-values. This is an example by Merricks:

> Fermat's Last Theorem (FLT) is necessarily true. So, FLT is true in all possible worlds. So, the proposition that dogs bark is true in all and only those possible worlds in which the proposition that dogs bark and FLT is true is true. So, the set of possible worlds in which that dogs bark is true is identical with the set of possible worlds in which that dogs bark and FLT is true is true. So, if propositions are sets of possible worlds, the proposition that dogs bark is identical with the proposition that dogs bark and FLT is true (Merricks 2015, 88).

The conclusion is that the proposition that dogs bark and FLT is true is not identical to the proposition that dogs bark, therefore propositions are not sets of possible worlds (here, since we are looking at all logically possible worlds, to make the same example we would have to switch Fermat's last theorem with some logical truth). Now, this is a matter of debate, and it is not in the business of this dissertation to decide whether this is a good objection or not. It is interesting, however, because it does show an intrinsic limit to the fine-grained-ness of the object/modal system: in such system, necessarily coextensive propositions are identical. Therefore, it is unfair to assume, as the objection to persistence does, that there are two *different* propositions with the *same* truth-set, in this system. Indeed, symmetric containment does his job as it should be: when fed with P and Q, it will give out 1 because P *is* Q, in the modal system, and thus (P, Q) is part of the extension of =, contrary to what the objection says. It is not that symmetric containment fails to compute identity when we switch system. What happens is rather that identity changes extension through different systems, and symmetric containment changes, accordingly.

3.3 Quantifiers.

I want now to show how to apply my theory to quantifiers in an extensional language. In fact, we already know that this is possible because quantifiers are structurally similar to modal operators: a modal operator is like a quantifier that ranges over the modal space. Let us start with extensional, universal quantifier. It is FCI.

Proof 3.2. Start with a set of objects $\mathfrak{D}$. $\forall$ is a function $\wp\mathfrak{D} \to \{V, F\}$. It takes $\mathfrak{D}$ to T and everything else to F. Define again this FTM [$\sqsupseteq$]:

- Input: sequences (x,y)

- F = {s_2}.

- Instructions:

  - [input (y, x)] (s_0, input)

  - [$\sqsubseteq$] (s_1, (y, x)).

This is a reversed containment test. Here 0 and 1 are True and False. [$\sqsupseteq$] computes $\forall$ when we set input (X, $\mathfrak{D}$), where X is the argument of $\forall$. Feed [$\sqsupseteq$] with ($\mathfrak{D}$, $\mathfrak{D}$). Following the instructions, we obtain this graph:
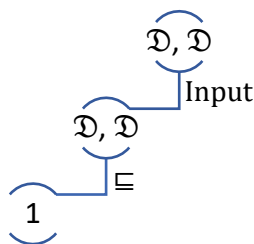

Table 8

So, correctly, when fed with ($\mathfrak{D}$, $\mathfrak{D}$) the output is True. Now Feed [$\sqsupseteq$] with any other (X, $\mathfrak{D}$).

X, 𝔇

Input

𝔇, X

⊑

0
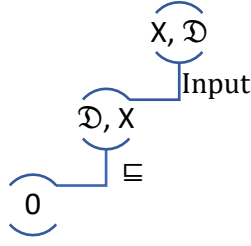
Table 9

Here, no proper subset of $\mathfrak{D}$ contains it, so the test is doomed to fail, for any X. So, correctly, the output is False. So, [⊒] computes ∀.

Now, for the permutation part, set arbitrary permutations in $\mathfrak{D}$. ∀(X) is either 0 or 1. The only invariant elements of the powerset $\wp\mathfrak{D}$ are $\mathfrak{D}$ and ∅. So, the sequence $(\mathfrak{D}, \mathfrak{D})$ is invariant as well. Since $(\mathfrak{D}, \mathfrak{D})$ is the only input for which [⊒] has value 1, for every σ, $\forall(\sigma^{\langle 0,V\rangle}(\mathfrak{D})) = [\sqsupseteq](\sigma^S(\mathfrak{D}, \mathfrak{D}))$ = 1. Also, obviously, for any other X ≠ $\mathfrak{D}$., $\forall(\sigma^{\langle 0,V\rangle}(X)) = [\sqsupseteq](\sigma^S(X, \mathfrak{D})) = 0$. For suppose this is not the case: then there must be an X such that $[\sqsupseteq]\sigma^S(X, \mathfrak{D}) = 1$ or $\forall(\sigma^{\langle 0,V\rangle}(X)) = 1$. Yet, this happens only if $\sigma^{\langle 0,V\rangle}(X) = \mathfrak{D}$, but then $\mathfrak{D}$ would not be invariant. Thus, for every σ, $\forall(\sigma^{\langle 0,V\rangle}(X)) = [\sqsupseteq](\sigma^S(X, \mathfrak{D}))$.

Since ∃ is ¬∀¬ we know it is FCI, as well. Yet, let us prove it anyway.

Proof 3.3. Define FTM [¬][⊑]:

- Input: sequences (x,y)
- F = {s_1}.
- Instructions:
  - [⊑](s_0, input).
  - [¬] (s_1, x).

This is the complementation of containment. ∃ is computed by [¬][⊑] when we feed it with (X, ∅): ∅ contains X if and only if X = ∅, so when X = ∅, the output is 0; 1 otherwise, which is exactly the behaviour of ∃. So, again, there is a FTM [¬][⊑] such that, for any argument X of ∃, there exists a sequence (X, ∅) such that ∃(X) = [¬][⊑](X, ∅).

For the permutation part, set again arbitrary permutations on $\mathfrak{D}$. $\emptyset$ is invariant, and so is $(\emptyset, \emptyset)$, which is the only input for which $[\neg][\sqsubseteq]$ is 0. Thus, under arbitrary permutations, $\exists(\sigma^{\langle 0,V\rangle}(\emptyset)) = [\neg][\sqsubseteq](\sigma^S(\emptyset, \emptyset)) = 0$. Since for any other argument, the output is 1, then under arbitrary permutations, it will still be 1, for the same reasons as before. Thus, for every σ and every X, $\exists(\sigma^{\langle 0,V\rangle}(X)) = [\neg][\sqsubseteq](\sigma^S(X, \emptyset))$.

I find these proofs interesting, because they give us a peculiar philosophical justification for the logicality of $\forall$ and $\exists$. They are logical because they reproduce an instance of information-containment and its complementation, respectively. The universal quantifier is logical because it can be computed by something that checks whether a piece of information contains the *maximum*. The existential quantifier is logical because it is like something that checks if the empty infon does not contain another infon. This behaviour is general, because it is preserved under arbitrary automorphisms of the domain and induced transformations of inputs that the quantifiers are receiving.

  Here comes the interesting part: cardinality quantifiers are permutation-invariant, and yet I cannot think of any way they could be formally-computable. We would need some combination of intersection, union, complementation, containment and compatibility that imitates the behaviour of cardinality quantifiers, but I cannot see how this can be possible. The only thing sets of cardinality 5 have in common with all and only the sets of cardinality 5 is their cardinality, I take it. There is no checking of containment, or communality of information these sets share. Being arguably not formally-computable at all, cardinality quantifiers cannot be formally-computable invariant, and therefore they are not-logical. This is not a bad result, I believe, because cardinality quantifiers are one of the main sources of criticism for both permutation invariance and bijection invariance theories. As already discussed in chapter 2, their logicality has been heavily criticised in Feferman (1999) and Bonnay (2008).

  At any rate, I would like to point out that my theory may score better than permutation or bijection invariance regardless of one's opinion on the logicality of cardinality quantifiers. For these items, even when they come out logical, they are mainly seen as "involuntary by-products" of the choice of transformation we take. It is not that Sher and Tarski wanted cardinality quantifiers to be logical: they just happened to come out that way because, unluckily, they turn out invariant. When we switch to

stricter transformations, like homomorphism invariance, they are not logical anymore (Feferman 1999). However, we have a price to pay: identity is not logical either! My theory, on the other hand, makes clear what it takes for cardinality quantifiers to be logical. The fact that they do not change under some transformation does not matter much. Do they or do they not perform an operation that, under our general conception of what logic is, counts as logical? No. Then they cannot be logical, no matter what invariance feature they might possess. However, if we were to change our underlying theory of what logic is, and thus extend or alter our list of basic formal operations, then cardinality quantifiers may turn out logical. For example, if we take the sizes of infons to be structural aspect of information, then cardinality would arguably be logical structure. We will then have some formal operation that is sensitive to cardinality that computes them. Since arbitrary permutations will not change the cardinality of an infon, under arbitrary permutations of arguments, this operation will still compute some cardinality quantifier. These considerations, I suggest, show that my theory is not on the wrong track. If someone wants these items to be logical, she does not need to come up with some choice of transformation that makes them invariant. Rather, she needs to show that cardinality matters to logic, and that is logical structure: this seems like the right way to go.

3.3.1 Objection

Take a language $\mathcal{L}_{\infty\infty}$ whose primitive notions are identity, substitution, disjunction, negation and existential quantification. McGee (1996) proved that an n-ary operation is definable in $\mathcal{L}_{\infty\infty}$ if and only if it is permutation invariant. The left-to-right part is not interesting, but the right-to-left is very important: it says that *any* PI operation is definable in that language. Since the language has, as primitives, paradigmatic examples of logical notions, and since an operation describable by a combination of logical notions is still a logical notion, McGee concludes that all permutation invariant items are logical notions. In my theory cardinality quantifiers are not FCI. Yet, there is a language, namely $\mathcal{L}_{\infty\infty}$, whose all primitives are FCI-items, in which we can define cardinality quantifiers. This raises a doubt: even if it was true that cardinality quantifiers are not formally-computable, is this a problem with cardinality quantifiers or rather with the machinery we have been using? Maybe it is just a technical

limitation of our computational system, built in the way we set it up, rather than an interesting feature of the theory.

Maybe so. In the next section I do raise doubts on how I set up the computational system. In fact, I do believe that it must be changed to allow for more expressive power (see 3.4.1). So, suppose this objection is right: then my theory, at least in the object/modal system, would probably collapse into PI. This may not be so bad, after all. My theory would not collapse into PI in *any* system, so it would not be, strictly speaking, just PI in another disguise. Secondly, my theory would still retain its explanatory power. For, contrary to permutation invariance, it says more about why conjunction or disjunction are logical: they are operations that checks what two infons have in common, and they reflect structural aspects of information. If, in the object-modal case, PI is just FCI, then this would give a philosophical ground for invariance theorists: permutation invariant items are all and only items which are intuitively describable by operations that check only structural aspects of information, and structural aspects of information, so I argued, are logical structure.

However, I am not sure that cardinality quantifiers do not come out F-computable due to quirks of the system. Or, better, these "quirks" may not be quirks at all, but sensible limitations. I will give an analogy: the sentence $\ulcorner\exists_{n>3}x(Fx)\urcorner$, where $\exists_{n>3}x$ is the cardinality quantifier $\ulcorner$At least three things$\urcorner$, can be rewritten employing paradigmatic logical notions only as $\ulcorner\exists x\exists y\exists z(x\neq y\neq z \ \& \ Fx \ \& \ Fy \ \& \ Fz)\urcorner$. Yet, this will not satisfy the critics of the logicality of cardinality quantifiers, in the slightest. They will still point out the problems related to these notions, which have not been addressed by the fact that we can substitute them with some combination of logical notions. This may suggest, by analogy, that something similar is happening with $\mathcal{L}_{\infty\infty}$: maybe it allows *too much*, even though one is employing logical notions only. The debate, thus, may still be open, and I could object that even though a notion is describable in McGee's language, there may be no logical operation that computes it.

3.4 World-Sensitivity and Quasi Logicality

   In the discussion on the logicality of modal operators, so far, we have always talked about a certain kind of items: items that are not world-sensitive. By that, I mean that their behaviour does not change from world to world. For example, take logical necessity: we assumed that it is an S5 modal operator, where an S5 modal operator, in a Kripke semantics, is defined through an accessibility relation that is transitive, reflexive and symmetric (Garson 2016). In such system, every world valuates modal statements the same. For example, if P is possible from some w, it is possible from any w' of the model. If something is world-sensitive, on the other hand, it will show some sort of "indexicality" that S5 operators lack. By "indexicality" I mean that it shifts behaviour across contexts (I will be more precise later on).

   Take a S4 modal operator, for example. In a Kripke semantics, it is defined through an accessibility relation that is reflexive, transitive but not symmetric. Thus, it may happen that some w is accessible from w', but w' is not accessible from w. So, across worlds, the set of accessible worlds changes, and thus the operator may change behaviour, as well. For example, so far, we computed Kripke modal operators through reverse containment. Call "modal context of w" the set $C = \{w'/ Rww'\}$. We compute $\Box P$ by checking if P contains C, and we compute $\Diamond P$ by checking if the complement of P does not contain C. The simplest way to check that S4 is world-sensitive is to look at C: it varies from world to world, so it is world-sensitive. Take a model with $W = \{w, u\}$ and an accessibility relation $R = \{(w, u), (w, w), (u, u)\}$. R satisfies S4. To valuate necessary statements before, we used $[\sqsupseteq]$ with inputs of type (φ, C), where C was constant. However, now containment fails to account for the S4 necessity just described. This happens for at least two, interconnected reasons. Firstly, C is not constant: in w, $[\![C]\!] = \{w, u\}$, but in u, $[\![C]\!] = \{u\}$. Finally, the value of $\Box\varphi$ is neither 1 nor 0, for some formula φ (1 is mapped to $\top$, with $[\![\top]\!] =$ W, and 0 to $\bot$, with $\bot = \emptyset$). Suppose $[\![\varphi]\!] = \{u\}$. Then, φ is necessary from u but not from w, so $[\![\Box\varphi]\!] = \{u\} \neq 1$ or 0. This is the most powerful way to spot world-sensitivity, since it is applicable to semantics that are not Kripkean (like Algebraic semantics, Neighbourhood semantics etc.[37]). If the operation we

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[37] As already mentioned, these semantics are useful for modal logics that cannot be captured through Kripke frames. For example, logics without necessitation (where P is a theorem, so is $\Box$P), or without

use to valuate modal statements does not change across worlds, then every world will "agree" on every modal statement. Thus, an item is world sensitive in $\mathfrak{M}$ if, for some modal formula ψ, $\psi \neq 1$ and $\psi \neq 0$. So, a Kripke modal operator is world-sensitive if this holds:

i.   Not for all $w \in \mathfrak{M}$, $C^w \neq k$, where k is some constant and $C^w = \{w'/ R(w, w')\}$.

The general definition however is this:

World-sensitivity: An n-ary sentential operator $\mathcal{X}$ is world-sensitive iff, for some $\mathfrak{M}$ for $\mathcal{X}$,

$\mathcal{X}(a, b, ..., n) \neq \top$ and $\mathcal{X}(a, b, ..., n) \neq \bot$, for any n-ary sequence of formulas (a, b, ..., n) of the language.

World-sensitivity generalizes (i), since (i) implies the condition set in the definition. Suppose $\mathcal{X}$ is a necessity operator □. Suppose C is not constant. Then there will be two worlds w and u such that $C^w \neq C^u$. Regardless, there will be at least a proposition P such that □P is true in w but not in u, and thus $\emptyset \subset [\![□P]\!] \subset W$ (C^w and C^u are either one a subset of the other or not. If they are, then take the P whose truth-set is the smaller context. If they are not, take the P whose truth-set is one of the contexts). The condition in the definition implies (i) in Kripke semantics, as well.

The fact that the definition of world-sensitivity is all about the extension of modal statements helps clarify what world-sensitivity is. I called it a form of indexicality, yet this was imprecise. An item is indexical if it shifts content through contexts (in this case, worlds). In Kaplanian terms, we take these items to have both a character and a content. A character is a function from circumstances of evaluation to contents, a content is the extension of some term (Kaplan 1989). For example, ⌜Today⌝ has a character that "takes you" from a context to a day: the day of the context. An item is indexical iff it has different contents in different contexts. Yet S4 modal operators are not indexical, in this sense. For take what is supposed to be their character: the function from context to contents. Since the modal is the relevant variable that makes the content "shift", we can equate the context with type W. S4 necessity is of type $\langle\langle W,V\rangle,\langle W,V\rangle\rangle$, thus its character is of type $\langle W,\langle\langle W,V\rangle,\langle W,V\rangle\rangle\rangle$. Yet, this function takes any w to the same

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K: □(P→Q)→(□P→□Q) etc. The definition will apply as long as, if some formula is mapped to 1, then it is necessary from every world, and if it is mapped to 0, it is impossible from every world.

content: to the same function from propositions to propositions. So, the content of S4 necessity does not shift, and it is not indexical. What is changing across worlds is rather the extension of statements that contain S4 necessity. MacFarlane has carefully distinguished these two kinds of sensitivity to context:

> **Use-sensitive.** *An expression (or content) is use-sensitive iff its extension (relative to a context of use and context of assessment[38]) depends on features of the context of use.*

> **Use-indexical.** *An expression is use-indexical iff it expresses different contents at different contexts of use (MacFarlane 2015, 79).*

S4 necessity is use-sensitive, but it is not use-indexical.

Being not world-sensitive is not a sufficient condition for logicality. For example, take a necessity operator defined over an accessibility relation that is such that, for any $w \in W$, Rwa, where 'a' is constant. This operator is not world-sensitive, but it is clearly not FCI. In fact, the already mentioned $\mathcal{S}$: ⌜In every cat-world⌝ is *not* world-sensitive and yet not logical. As one may suspect already, however, FCI usually implies world-insensitivity. This happens for two reasons, mainly. Firstly, the operator that is world-sensitive cannot be computed by an operation whose output is 1 or 0. The only way to compute it, then, would consist in some combination of meet, join and complementation, but this is usually impossible for things like necessity operators. Secondly, almost all the FTMs suggested so far for computing Kripke modal operators presupposed a fixed "modal context". To compute a Kripke world-sensitive operator we would have to change the final part of the sequence the FTM is taking, and thus the item would not fit the requirements for formal computability, for its input-sequences would not be such that $(b_2) = k$. Consequently, modal operators in logics weaker than S5 will usually not be FC-invariant. The same usually goes for any "context-variable" modal operators. There is an exception, though: take a Kripke model such that R(w, w), for any w: it is the "solipsistic" R already mentioned. In such models, necessity and possibility collapse into the identity function, which is FCI: to compute these operators we just need the empty FCI.

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[38] MacFarlane thinks some notions are "assessment-sensitive": they are sensitive to shifts in context of assessment, rather than context of "use".

3.4.1 Quasi-Formal-Computability

As already mentioned, Novaes criticizes standard invariance for making S4 modal operators not logical:

> *The point is not that the S4 modal operators should necessarily be counted as logical; rather, the point is what independent motivations would justify that the S4 modal operators do not count as logical, whereas their counterparts interpreted on universal frames do. What is the fundamental, philosophical difference between these two cases besides the fact that they are interpreted on different structures? (2014, 94).*

This is a fair point, which is linked to our discussion of standard invariance theories: they are usually not very explanatory, since they fail to specify what is "logical structure" and what is not, and they tend to reverse engineer the test so that it gives out the extension they have in mind for reasons independent of invariance *per se*. In such a situation, it is fair to ask: why is S4 accessibility relation not logical? Why should it be "below" the lower bound of logical structures, given that we did not specify any? Formal Computability Invariance, however, does account for the difference between a S5 modal operator defined over the set of logically possible worlds, and an S4 modal operator defined over the same set. The first is computable by an operation that checks only structural aspects of information, which are "logical structure", and this behaviour is preserved under transformations of the information the operator receives. The second, however, is not computable by any such operation, at all. Thus, S4 modal operators are never logical: either they are not defined over the set of all logically possible worlds (and thus will never be invariant), or, even if they are, they are not formally computable.

To be fair, though, Novaes may have a point: maybe our test is too restrictive. Yet in what sense can S4 operators be logical? We said that there are two problems: the context shifts and the output in neither 1 nor 0. Well, the second issue is arguably a limitation of our system, rather than a philosophical problem. For suppose we want to valuate $\Box\varphi$ in w, where $\Box$ is S4, and it is defined over W^L. We can use containment: we just need to check if $\{w'/Rww'\} \subseteq \llbracket\varphi\rrbracket$. If yes, the output is True (i.e. 1), otherwise False (i.e. 0). We may then design an FTM that "stores" the worlds for which the output is 1, and then print

the list of them: this will give you the extension of $\Box\varphi$. This improvement in the FTM will not jeopardize its "formality", I take it. However, I prefer another solution. In section 2.4, we have already established that relations are logical iff their "valuation" is, so we may do something similar here. Call the valuation of a n-ary sentential operator $\mathcal{X}$, relative to a model $\mathfrak{M} = \langle W, R, V, \vDash \rangle$ the set of instances of some FTM needed to evaluate, for each $w \in W$, formulas of the form $\mathcal{X}(a, b, ...n)$, where $(a, b, ..., n)$ is any n-ary sequence of well-formed sentences of the language (not only atomic). This operation will "follow" the $\vDash$ relation of the model:

1. **Valuation of $\mathcal{X}$ relative to $\mathfrak{M}$ ($\mathcal{V}_{\mathfrak{M}}^{\mathcal{X}}$).** Take any n-ary modal operator $\mathcal{X}$. Where S^n is the set of all n-ary sequences of sentences of the language, $\mathcal{V}_{\mathfrak{M}}^{\mathcal{X}}$ is the function $W \times S_n \to V$ such that, for any $(P, Q, ...N) \in S^n$, $\mathcal{V}_{\mathfrak{M}}^{\mathcal{X}}(w, (P, Q, ..., N)) = T$ iff $w \vDash \mathcal{X}(P, Q, ..., N)$; F otherwise.

This takes care of the first problem. There are other issues, though: suppose we want to compute some Kripkean world-sensitive operator through [$\sqsupseteq$]. We can't, for two reasons: (i) the sequences the FTM takes as arguments are not such that $(b_m)_{m>n} = k$, since the context shifts; (ii) the sequences would not match the input of containment, since the valuation has, as inputs, a world and a sentence, the other a set of worlds and a sentence. Yet, suppose we want to stick to our intuition that, in some sense, we are using containment to evaluate formulas at each world. We may try to model a notion of weak formal computability. A first guess could be this: we relax compatibility by dropping the assumption that $(b_m)_{m>n} = k$. Now the context can "shift", for we are just asking that the FTM takes, as input, a sequence that contains the argument of the item we are testing. For the other issue, we can make an exception:

Quasi-FC.1. An n-ary modal operator $\mathcal{X}$ defined over W^L is quasi-FC $=_{df}$ the valuation of $\mathcal{X}$ is weakly formally-computable. That is, for any model $\mathfrak{M}$ for $\mathcal{X}$ such that $W^L \in \mathfrak{M}$, for some FTM, there exists a function Comp: FTM$^{-1} \circ \mathcal{V}_{\mathfrak{M}}^{\mathcal{X}}$ such that:

d) For every (w, s_n) and s, $\text{Comp}(w, s_n) = s$ iff $\mathcal{V}_{\mathfrak{M}}^{\mathcal{X}}(w, s_n) = \text{FTM}(s)$.

e) For every (w, s_n), $\text{Comp}(w, s_n)$ is the sequence such that $(b_m)_{1 \leq m \leq n} = (a_m)^{sn}$, where $(a_x)^{sn}$ is the x-th element of s_n.

As it should be, quasi-formal computability does not imply formal computability[39]. However, this definition will not do because it is too weak. We can show that most unary operators whose output is 0 or 1 are weakly FC:

Proof 3.4. Define [⊑]:

- Input: sequences (x,y)

- F = {s_1}.

- Instructions: [⊑](s_0, input).

This is a simple containment test. Now define [C]:

- Input: sequences (x,y)

- F = {s_1}.

- Instructions: [C](s_0, input).

This is compatibility. Now take any unary sentential operator k whose output is either 0 or 1 and such that either for any w of $\mathfrak{M}$, $\mathcal{V}_{\mathfrak{M}}^k(w, \emptyset) = 1$ or, for any w of $\mathfrak{M}$, $\mathcal{V}_{\mathfrak{M}}^k(w, \emptyset) = 0$.

Suppose the former: in any w ∈ $\mathfrak{M}$, w ⊨ k(∅). When $\mathcal{V}_{\mathfrak{M}}^k(w, \varphi) = 1$, take [⊑] and set ($\varphi$, W). When $\mathcal{V}_{\mathfrak{M}}^k(w, \varphi) = 0$, take [⊑] and set ($\varphi$, ∅). [⊑] weakly computes $\mathcal{V}_{\mathfrak{M}}^k$. For take all the φ's such that $\mathcal{V}_{\mathfrak{M}}^k(w, \varphi) = 1$. $\mathcal{V}_{\mathfrak{M}}^k(w, \varphi) = $ [⊑](φ, W), because every φ is contained in the W, even ∅. On the other hand, for any φ such that $\mathcal{V}_{\mathfrak{M}}^k(w, \varphi) = 0$, $\mathcal{V}_{\mathfrak{M}}^k(w, \varphi) = $ [⊑](φ, ∅), for the only thing that is contained in ∅ is ∅, but we have assumed that $\mathcal{V}_{\mathfrak{M}}^k(w, \emptyset) = 1$.

Now suppose the latter: $\mathcal{V}_{\mathfrak{M}}^k(w, \emptyset) = 0$, for any w of $\mathfrak{M}$. When $\mathcal{V}_{\mathfrak{M}}^k(w, \varphi) = 1$, take [C] and set ($\varphi$, W). When $\mathcal{V}_{\mathfrak{M}}^k(w, \varphi) = 0$, take [C] and set ($\varphi$, ∅). For all φ such that $\mathcal{V}_{\mathfrak{M}}^k(w, \varphi) = 0$, $\mathcal{V}_{\mathfrak{M}}^k(w, \varphi) = $ [C](φ, W), for [C](φ, W) = 1 iff $\wedge(\varphi, W) \neq \emptyset$, and this does not happen only for $\wedge(\emptyset, W)$, but we

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[39] The reverse may not hold, as well. It holds whenever an item k is computable by an FTM whose output is 0 or 1. For then its valuation is weakly computable by the same machine. Yet, if the output of the FTM that computes k is not 0 or 1, we cannot use the same FTM to compute k's valuation, since the range of $\mathcal{V}$ is 0 and 1. It can be, then, that there is no way to compute k's valuation, even though k is formally-computable, or even FCI. This problem suggests that we should change the machine so that it can "store" the results of the valuation, for each world, and then print the list of worlds for which $\mathcal{V}(w, s_n) = 1$. I will not do this here, however, for my focus will be Kripke modal operators only, for which we do not need such complication.

have assumed that $\mathcal{V}_{\mathfrak{M}}^{k}(w, \emptyset) = 0$. On the other hand, for all φ such that $\mathcal{V}_{\mathfrak{M}}^{k}(w, \varphi) = 0$, $\mathcal{V}_{\mathfrak{M}}^{k}(w, \varphi)$

$=[C](\varphi, \emptyset)$, for everything is incompatible with $\emptyset$, even $\emptyset$ itself and W: $\wedge(\emptyset, W) = \emptyset$ and $\wedge(\emptyset, \emptyset) =$

$\emptyset$. So, in any case, k is weakly formally computable either by $[\sqsubseteq]$ or by $[C]$.

So, any unary operator that does not change behaviour across worlds when it takes as input $\emptyset$ will be weakly formally computable. Should we fix this? Maybe. I think we should, because, intuitively, the valuation of a necessity operator in a Kripke structure is almost never checking if some sentence is compatible with the empty set. Rather, it checks if the sentence is true in the extension of R for that world. How can we strengthen our notion of computation, so that the FTM that computes the valuation of $\mathcal{X}$ really "simulates" it? We may ask that whenever the valuation does something different, so does the FTM that computes it. When does this happen? Valuations take, as arguments, a world and some sequence of sentences. A change in sequence is not interesting for us, so we will assume that switching sequence never changes what the valuation is doing. Changing world, however, may or may not change what the valuation is doing: the valuation of a Kripkean necessity operator, for example, does something different only when we change the extension of the accessibility relation. This is why logical necessity, for example, is not world-sensitive: the extension of the accessibility relation is constant, so the valuation is always doing the same thing.

The problem with Quasi FC.1 is now clearer: what the FTM is doing does not "track" changes in what the valuation is doing. For the fact that $\mathcal{V}_{\mathfrak{M}}^{\mathcal{X}}(w, \varphi) = \mathcal{V}_{\mathfrak{M}}^{\mathcal{X}}(v, \varphi)$ alone does not imply that, in these two instances, the valuation of k is doing "the same thing": it may be that $C^{w} \neq C^{v}$. To fix this, we ask that, for every "change" in what the valuation is doing, there is a correspondent "change" in what the FTM is doing. So, suppose we divide the behaviour of some item in "actions". Actions will partition the domain of the item: different kinds of inputs will require a different action. Actions of some K are individuated as subsets of the domain of K. Actions for FTMs are finely grained in this sense: suppose an FTM computes an n-ary item. Then, actions of the FTM are all the subsets of its domain whose elements are identical from n+1 up. For example, with Kripkean necessity operators, actions of the FTM are equinumerous with the set of C^{w}: there is one action for every "shift" in the extension of R.

I will call the set of actions of K $\mathcal{A}(K)$. A single action will be labelled by capital letters A, B, C etc. We can give this definition of quasi-FC:

Quasi-FC.2. An n-ary modal operator $\mathcal{X}$ defined over W^L is quasi-FC $=_{df}$ the valuation of $\mathcal{X}$ is weakly formally-computable. I.e., for any model $\mathfrak{M}$ for $\mathcal{X}$ such that $W^L \in \mathfrak{M}$, for some FTM, there exists a function Comp: $FTM^{-1} \circ \mathcal{V}_{\mathfrak{M}}^{\mathcal{X}}$ such that:

a) For every s_n and s, $Comp(w, s_n) = s$ iff $\mathcal{V}_{\mathfrak{M}}^{\mathcal{X}}(w, s_n) = FTM(s)$.

b) For every (w, s_n), $Comp(w, s_n)$ is the sequence such that $(b_m)_{1 \leq m \leq n} = (a_m)^{sn}$, where $(a_x)^{sn}$ is the x-th element of s_n.

c) There is an injection between $\mathcal{A}(\mathcal{V}_{\mathfrak{M}}^{\mathcal{X}})$ and $\mathcal{A}_{Comp}$, where $\mathcal{A}_{Comp} = \{A \in \mathcal{A}(FTM) / A \subseteq Comp(x)\}$.

From now on, when I speak of quasi-FC, I mean quasi-FC.2. Condition (c) looks at the domain and range of Comp, and divides them in actions. If there are at least as many actions in the range as there are in the domain, then (c) is satisfied. Now, for the most part, it will be impossible to use just $[\sqsubseteq]$ or $[C]$ for most unary operators, as in proof 3.4, because, in both cases, the set of actions of the range has just two elements: there are just two 2nd elements of any input of the FTM: either $\emptyset$ or W. If the valuation is doing at least three different things, there is not injection between the two sets. If not, then we may as well use that FTM to weakly compute the valuation.

It is hard to give a precise definition of quasi-formal computability in this way, since the notion of "actions" is somewhat vague. We have, however, a pretty clear definition of different actions for unary modal operators in a Kripke structure, which will be enough for our discussion. The set $\mathcal{A}(\mathcal{V}_{\mathfrak{M}}^{K})$ for some Kripkean modal operator K is the set of X such that:

i. $X \subseteq W \times S_1$

ii. If $(w, \varphi) \in X$ then $(v, \varphi') \in X$ iff $(R(w, w')$ iff $R(v, w'))$, for any $w, w', v \in W$.

Basically, $\mathcal{A}(\mathcal{V}_{\mathfrak{M}}^{K})$ is the set of sets of (w, φ) closed under identity of C^w. Thus, actions of $\mathcal{V}_{\mathfrak{M}}^{K}$ will be equinumerous with the set of C^w. In fact, for most Kripkean modal operators, the injection will be a

bijection. For, usually, for any way they are weakly-computable, the range of Comp is the set of (φ, C^w). Suppose it is so for K. Since $\mathcal{A}(\mathcal{V}_{\mathfrak{M}}^K)$ is equinumerous with the set of C^w, which is equinumerous with $\mathcal{A}_{\text{Comp}}$, there will be a bijection between $\mathcal{A}(\mathcal{V}_{\mathfrak{M}}^K)$ and $\mathcal{A}_{\text{Comp}}$[40].

3.4.2 Quasi-Logicality defined.

Now that we have a notion of quasi-formal-computability, we need a relaxed notion of invariance as well. However, asking for the invariance of the item's weakly formal computability may be too stringent. For, in this case, we are asking that, if some (w, φ) was computed by (φ, C^w), then *the same* (w, φ) must be computed by the same (φ, C^w) under permutations. They are tracking each other *individually*, so to speak. Yet in this way, most world-sensitive items do not come out quasi-logical, anyway. If our intent was to model a relaxed notion of logicality, this would not get us very far. We may relax the notion even more in this way: what if we ask that the "tracking" under permutations, rather than applying individually, for each argument and value of Comp, it applies "collectively"? Suppose an FTM weakly computes the valuation of K, and suppose we list all the instances of the FTM needed to weakly compute each instance of $\mathcal{V}_{\mathfrak{M}}^K$. My suggestion is that an item is quasi-logical iff, under arbitrary permutations, the whole transformed list of instances of FTM is still enough to weakly compute in the same way the whole transformed list of instances of $\mathcal{V}_{\mathfrak{M}}^K$. Before, we were asking that each couple of permuted inputs of the item and permuted inputs of the FTM "tracked" each other, by still giving out the same output. Now we are just asking that, under arbitrary permutations of inputs, for each permuted inputs of the item, the "work" needed to compute the resulting instance of the valuation can still be done by some member of the transformed list of instances of the FTM.

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[40] Interestingly, for modal operators defined over the "solipsistic" R, even though $\mathcal{V}_{\mathfrak{M}}^K(w, \varphi) = $ FTM (φ), with the empty FTM, this FTM cannot be used to weakly compute these items, because it performs the same action across any world, even though the items are world-sensitive. So, here we will need to use reversed containment.

Quasi-Logicality.1. An n-ary modal operator $\mathcal{X}$ is quasi-logical $=_{df}$ it is completely defined over W^L

and for any model $\mathfrak{M}$ for $\mathcal{X}$, for some FTM, there exists a function Comp:

FTM$^{-1} \circ \mathcal{V}^{\mathcal{X}}_{\mathfrak{M}}$ such that:

a) For every s_n and s, Comp(w, s_n) = s iff $\mathcal{V}^{\mathcal{X}}_{\mathfrak{M}}$(w, s_n) = FTM(s).

b) For every s_n, Comp(w, s_n) is the sequence such that $(b_m)_{1 \leq m \leq n} = (a_m)^{s_n}$, where $(a_x)^{s_n}$ is
    the x-th element of s_n.

c) There in an injection between $\mathcal{A}(\mathcal{V}^{\mathcal{X}}_{\mathfrak{M}})$ and $\mathcal{A}_{Comp} = \{A \in \mathcal{A}(FTM)/ A \subseteq Comp(x)\}$.

d) For every permutation of information σ, for every (w, s_n), there exists a (w', s_n') such
    that Comp(σ^S(w, s_n)) = σ^S(Comp(w', s_n')).

This is a noteworthy behaviour, I believe, which shows an interesting abstraction from content. Not only the item is computable by a formal operation but, no matter how we permute information, the transformed instances of the formal operation can compute in the same way the transformed instances of the item. It is also clear why this is still weaker than "pure" logicality: it may be that some transformed instance of the operation is computed by *some other* transformed instance of the item. The "tracking" which invariance consists of is not done for each couple of argument-value of Comp, but on a collective scale: the transformed domain of Comp is still computed by the transformed range.

More generally, since the item we are testing is usually *completely* defined, something interesting happens. By "completely" I mean that, if the item $\mathcal{X}$ is a "normal" n-ary operator, the valuation function will valuate any n-ary sequence of sentences of the language in each world of the model. That is, the valuation function will have, as domain, the complete cartesian product $W^L \times S_n$. This is as it should be. For suppose the item is not completely valuated. Then, somehow, the item is sensitive either to the identity of some specific logically possible world, or to the identity of some proposition. This may be a good reason to deny it the status of "logical" item, or even "quasi-logical" item.

The domain of $\mathcal{V}^{\mathcal{X}}_{\mathfrak{M}}$ is invariant under arbitrary automorphisms of W^L, because $W^L \times S_n$ is: for any (w, s_n), for any σ, there will be a (σ(w), $\sigma^S(s_n)$) $\in W^L \times S_n$, simply because σ(w) $\in W^L$ and $\sigma^S(s_n) \in S_n$, given that permutations cannot switch cardinality ($\sigma^S(s_n)$ cannot "end up" in some element of S_m, with m ≠ n)

and sentences, in the model, are mapped to subsets of W^L (so, transforming sentences means ending up in some other subset of W^L). Since the domain of $\mathcal{V}_{\mathfrak{M}}^{\mathcal{X}}$ is the domain of Comp, it must be that, for any (w, s_n), for any σ, $Comp(\sigma^S(w, s_n)) = Comp(w', s_n')$, for some (w', s_n') in the domain of Comp. So, necessarily, $Comp\ (\sigma^S(w, s_n)) = \sigma^S(Comp(w', s_n')$ for some (w', s_n') if and only if $\sigma^S(Comp(w', s_n'))$ is "already" in the range[41] of Comp; that is iff $\sigma^S(Comp(w', s_n')) = Comp\ (w'', s_n'')$, for some (w'', s_n''). This definition of quasi-logicality is thus equivalent to the previous one:

Quasi-Logicality.2. An n-ary item $\mathcal{X}$ is quasi-logical iff

    a) Its valuation is complete and defined over W^L.

    b) Its valuation is weakly FC.

    c) For any permutation σ, for any (w, s_n), $\sigma^S(Comp(w, s_n)) = Comp(w', s_n')$ for some (w', s_n').

Basically, the definition says that K is quasi-logical only if, no matter how you transform a value of the computation function, you still "end up" in one value of the range of the computation. More generally, K is quasi-logical only if the range of the computation function that weakly computes its valuation is invariant. For the range of Comp is invariant iff, for any σ, $\sigma^S(s) = s'$, with s, s' $\in$ Comp (x).

Now, take a modal operator in a Kripke semantics: for either of them, the arguments any FTM uses to compute an arbitrary instance of their valuation is always (φ, C^w), where C^w is the modal context of w. Therefore, when we transform (φ, C^w), we transform φ or C^w. If only the former, then we are obviously "ending up" in the valuation of the operator in w for $\sigma^{\langle W,V\rangle}(\varphi)$. If the latter, then we are switching context, and the only way for $\sigma^S(\varphi, C^w)$ not to "end up" outside the range of the computation function is that $\sigma^{\langle W,V\rangle}(C^w)$ is some C^u, where u is some world of the model. Therefore, another way to express condition (c) above for these modal operators is to say that, under arbitrary permutations, for any w $\in W^L$, there exists a v $\in W^L$ such that $\sigma^{\langle W,V\rangle}(C^w) = C^v$.

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[41] Again, by "range" I mean the image of the domain in the codomain, and not the codomain as a whole.

3.4.3 Quasi Logical Items

We can easily prove that any Kripkean modal operator that is not world-sensitive and not FCI is not Quasi-Logical.

> *Proof 3.5.* Take any □ defined in a Kripke structure. For any model $\langle W, R, V, \vDash \rangle$, for any w of the model, $C^w = C$, with C constant and not permutation invariant (otherwise the operator would be FCI). The valuation of □ is weakly computed by [⊒], when given (φ, C), for any sentential argument φ. Define a permutation such that $\sigma^{\langle W,V \rangle}(C) \neq C$. Now, for any (w, φ) there is no (w', φ') such that $\text{Comp}(\sigma^S(w, \varphi)) = \sigma^S(\text{Comp}(w', \varphi'))$, for $\sigma^S(\text{Comp}(w', \varphi')) = (\sigma^{\langle W,V \rangle}(\varphi), \sigma^{\langle W,V \rangle}(C))$, for all (w, φ), $\text{Comp}(w, \varphi) = (\varphi, C)$, and $(\varphi, C) \neq (\varphi, \sigma^{\langle W,V \rangle}(C))$. For any other way □ is computable, a similar proof applies. For ◊, a similar proof applies.

Thus, items like ⌜In any cat-world⌝ are not logical. This is a good start. We can also prove that any FCI unary modal operator we encountered so far is quasi-logical:

- The valuation of logical necessity is computed by [⊒] with, as domain, the set of (φ, W^L). W^L is invariant, so the range of the computation function is invariant, as well. Thus, logical necessity (and possibility, for the same reason) is quasi-logical.

- The valuation of "empty" necessity is computed by [⊒] with, as range for Comp, the set of $(\varphi, \emptyset)$. Again, this range is invariant, so the item is quasi-logical.

- The modal operator defined over the solipsistic relation R/R(w, w), for any $w \in W^L$ is quasi logical. For, again, we have that the item is weakly computable by reverse containment and the range of Comp is the set of $(\varphi, \{w\})$, for any φ and any $w \in W^L$, and this set is invariant.

What world-sensitive items are quasi-logical? Well, we can come up with some examples of items that are not FCI and yet quasi-logical. What we need is (i) that the item is completely evaluated over W^L, (ii) that this valuation is weakly-computable, and (iii) that the range of the FTM that weakly computes the item is invariant. I have some examples of quasi-logical and yet not FCI Kripkean necessity operators:

a. Take a Kripke necessity operator defined over W^L, with an R such that R(w, v) iff $w \neq v$. The valuation of this item is weakly computable via [⊒], for any (w, φ). Since, by the definition of

permutation, the complement of identity is preserved under arbitrary permutations, if $w \neq w'$ then $\sigma(w) \neq \sigma(w')$, for any w and w'. Thus, given that $C^w = W^L - \{w\}$, which is the extension of the complement of identity for w, we have that, for any σ, $\sigma^{\langle W,V \rangle}(C^w) = C^v$, for some $v \in W^L$. In particular, if $\sigma(w) = w$, then $\sigma^{\langle W,V \rangle}(C^w) = C^w$, and if $\sigma(w) = v$, then $\sigma^{\langle W,V \rangle}(C^w) = C^v$. Interestingly, this necessity operator is world-sensitive (C^w is not constant) and, even though R is nothing but the complement of the identity relation, and even though the necessity operator defined over the identity relation was FCI, this operator is not FCI, because it is not formally computable[42].

b. Take any bijection $\mathcal{b}$ between W^L and itself. Define an accessibility relation R on W^L such that $R(w, v)$ iff $\mathcal{b}(w) = v$. The corresponding necessity operator is not worldly-invariant, and is probably not FCI, with the exception of one case: the identity bijection $\mathcal{b}(w) = \{w\}$ (in that case, the item is the "solipsistic" operator). Yet, it is weakly computable by $[\sqsupseteq]$, for every (w, φ), when given the corresponding (φ, C^w). In particular, the range of Comp will be invariant, because it is the total cartesian product of $S_1 \times W^L_1$, where W^L_1 is the set of all singletons of elements of W^L and S_1 is the set of sentences of the language. Since cardinality is permutation invariant, W^L_1 is permutation invariant; S_1 is invariant, as well, being the total $\wp W^L$, so their cartesian product must be invariant, as well. Therefore, the so defined necessity operator is quasi-logical. Since bijections between W^L and itself are nothing but permutations of W^L, there is a different quasi-logical necessity operator for every permutation of W^L.

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[42] The operator is not PI, either. This is interesting: while FCI is preserved under complementation, it is not preserved for modal operators under complementation of their accessibility relation. The same goes for permutation invariance.

3.5 A case study: Deontic Modality and Logicality.

In this section, I want to show an application of my theory to a particular, well-discussed item of the language: deontic modality. Are deontic modals formal? And if so, in what sense? This is an interesting discussion, I believe, for it shows how the framework we are setting up has potential, under-discussed ramifications, given its general applicability. I will show how a contextualist Ought might be quasi-logical.

3.5.1 Standard Deontic Logic

Deontic modals have been extensively studied in linguistics and philosophy of language. There are numerous, very different semantical accounts of their meaning[43]. However, there is a system that is standard, in the literature. In fact, it is called "Standard Deontic Logic" (SDL)[44]. It is unclear who first thought of this application of modal logic to normative systems: SDL was probably heavily influenced by Von Wright's work on the application of modal logic, in the 50's (Von Wright 1951, 1953).

More technically, we obtain SDL when we take propositional logic and we add these axioms[45]:

(K) $\mathcal{O}(p \rightarrow q) \rightarrow (\mathcal{O}p \rightarrow \mathcal{O}q)$.

(N) if $\vdash p$ then $\vdash \mathcal{O}p$

(D) $\mathcal{O}p \rightarrow \neg\mathcal{O}\neg p$.

Sometimes we also to this minimal SDL this additional axiom:

(4) $\mathcal{O}(\mathcal{O}p \rightarrow p)$

The first is K: it says that 'Ought' distributes over implication. The second is Necessitation: whatever is a theorem, it ought to be the case. The third is the standard axiom D: it says that if something is necessary

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[43] For an overview of the current debate, see McNamara (2014). For an updated philosophical discussion, see Charlow, N., & Chrisman (2016). For a more logic-driven discussion, see Gabbay et al. (2013).
[44] See, for example, McNamara (2014) or Hilpinen & McNamara (2013).
[45] Here I am following McNamara's axiomatization (2014).

it is possible. Axiom (4) is not derivable through the minimal SDL, but it is desirable: it says that it ought to be that whatever ought to be the case is realised.

The semantics for SDL is quite simple: since we have both K and N, it is a normal modal logic, and it can be modelled via standard Kripke semantics. Let language $\mathcal{L}$ be the smallest set generated from a set of atomic sentences plus Boolean connectives, enriched with operators $\mathcal{O}$. Let $\mathcal{V}$ be an "atomic" valuation function that maps each atomic formula to a subset of W^L, where W^L is the set of all logically possible worlds. $\langle W^L, R, V, \vDash \rangle$ is a Kripke model for $\mathcal{L}$, where $\vDash$ is a relation from W^L to $\mathcal{L}$ defined inductively:

    i. $w \vDash A$ iff $w \in \mathcal{V}(A)$, where A is atomic

    ii. $w \vDash \neg\varphi$ iff $w \nvDash A$

    iii. $w \vDash \varphi \wedge \psi$ iff $w \vDash \varphi$ and $w \vDash \psi$.

    iv. $w \vDash \varphi \vee \psi$ iff not both $w \nvDash \varphi$ and $w \nvDash \psi$.

    v. $w \vDash \varphi \rightarrow \psi$ iff either $w \vDash \varphi$ or $w \nvDash \psi$.

    vi. $w \vDash \mathcal{O}\varphi$ iff $\forall w'(Rww', w' \vDash \varphi)$.

The conditions on R depends on the system we choose. In minimal SDL, we assume R to be serial:

    Seriality $\forall w(\exists v(Rwv))$.

Seriality implies that if something is necessary it is possible: it is axiom D. D is satisfied when, from every world, you can access some world. This prevents the clause for Ought to be trivially satisfied: if, for some w, no world is accessible, the system would make in w every φ a duty. (4), on the other hand, is equivalent to this condition:

    Shift-Reflexivity $\forall w(\forall v(Rwv \rightarrow Rvv))$.

What is the meaning of Rwv? Generally, it is thought to mean that, from w, v is (one of the) best worlds. So, something ought to be the case from w iff it is the case in every best world from w. Something is permissible iff it is true in at least one best-world from w. The fact that we are talking about "best" worlds suggests that we are implicitly "extracting" the accessibility relation out of an ethical ordering between worlds. So, suppose that, for any v of W^L, we define an indexed relation $\leq_v$ on W^L:

i. $w \leq_v w$ (reflexivity)

ii. $(w \leq_v w' \& w' \leq_v w'') \rightarrow w \leq_v w''$ (transitivity)

iii. $w \leq_v w'$ or $w' \leq_v w$ (Totality)

iv. $\exists w(\forall w'(w' \leq_v w))$ (Limit Assumption)

$\leq_v$ is thus a pre-order with an upper bound. The limit assumption corresponds to seriality: from any world, there is always a best-world, for any world-indexed ethical ordering (McNamara 2014). The idea is that it ought to be the case that P, from v, iff in every upper bound of $\leq_v$, P is the case[46] (we will discuss in the end a system without the Limit Assumption).

3.5.2 Is Standard Deontic Modality Formal?

Now that we have a decent grasp on the standard semantics for deontic modality, we may ask: if this was the right semantics for Ought, would Ought be logical? By sticking to standard invariance, the answer would just be: "Of course not". That is, unless R is either empty or if it collapses into logical possibility, or if it is the "solipsistic" necessity[47], but this is very unlikely: relevant worlds in which we evaluate what ought to be the case will arguably be a proper subset of W^L, so arbitrary permutations of worlds will affect deontic modals. Yet, one may ask: why do we permute worlds arbitrarily? Maybe the ethical ordering is logical structure and, if so, we should not "abstract away" from it. How can we establish if this is the case?

Standard invariance does not provide an answer to this, nor does it help providing it, because it does not offer an opinionated definition of logical structure we can apply, here. I believe that my theory can do a better job in justifying whatever conclusion we reach. Is ethical ordering logical structure? Yes, if and only if it is a structural aspect of information. If so, then there will be a corresponding formal operation that computes Ought, by being sensitive only to this logical structure. Personally, I do not

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[46] Even though, at first glance, a translation between SDL and a semantics with such ordering relation should be quite straightforward, it is not. An investigation on how to extract SDL out of an "ordering" semantics is in Goble 2003.

[47] If this was the case, it would be true in every world that the actual world is the best of all possible worlds. Pangloss would surely be happy to embrace this view without hesitation.

believe that ethical orderings are structural aspects of information, though: they do not seem to shape the structure of information as such. In any case, if someone wants to, she can try to give some arguments in this direction, and then Ought will come out logical. For then, no matter how we permute worlds, when we give the resulting input to $\mathcal{O}$, some FTM will still simulate its behaviour. This is the road one should take, I believe: it does not matter much if Ought is invariant, but rather if it behaves logically, and if it does so "blindly".

This is a first possibility: that the ethical ordering is logical structure. However, it is quite far-fetched. What if the ethical ordering is not a structural aspect of information? Then Ought is probably not logical. Yet, it may be quasi-logical. An item is quasi logical iff (i) it is completely defined over W^L, (ii) its valuation is weakly-computable and (iii) the range of the computation function that weakly computes its valuation is invariant. We already know that Ought, at least in SDL, is weakly-computable, because it is a necessity operator in a normal Kripke semantics. We just need some FTM like reversed containment $[\sqsupseteq]$: for every $\mathcal{V}^{\mathcal{O}}_{\mathfrak{M}}$, $[\sqsupseteq]$ computes $\mathcal{V}^{\mathcal{O}}_{\mathfrak{M}}(w, \varphi)$ when given (φ, C^w), where $C^w = \{w'/Rww'\}$. We can also safely assume that Ought is completely definable over W^L.

Now, for the invariance part, we really need to know more about how we are ordering worlds. For suppose that what is ethically the best is not world-sensitive: then the best-worlds is a constant set, and does not change from world to world. In this case, unless there is no best world, or any world is the best, or in every world the actual is the best, then Ought will not be quasi-logical. This is highly unlikely: it is more sensible to suppose that, for any world w, the w-best-worlds are a proper subset of W^L. Therefore, via proof 3.5, Ought is not quasi-logical: by swapping a world "inside" C^w with one "outside" C^w we will upset it. However, things are not that easy: by doing that, we will upset "Ought from w". Yet, it may still be that the valuation of Ought is weakly-FCI: we just need that, when transforming C^w, the resulting transformation is a C^v, for some $v \in W^L$. That is, for any permutation σ, for any (w, φ), $\sigma^S(\text{Comp}(w, \varphi)) = \text{Comp}(w', \varphi')$ for some (w', φ'), where Comp is the relevant computation function Comp: $[\sqsupseteq]^{-1} \circ \mathcal{V}^{\mathcal{O}}_{\mathfrak{M}}$.

How can this happen? It is hard, but not impossible. Let us give an example. Since we adopted the Limit Assumption, our ordering of worlds will have a unique set of best-world, for every w. Therefore, $C^v = \{w/\forall w'(w \nless_v w')\}$[48], where $<_w$ is the correspondent strict version of $\leq_w$:

i. $w \nless_v w$ (irreflexivity)

ii. $w <_v w' \rightarrow w' \nless_v w$ (a-symmetry)

iii. $(w <_v w' \ \& \ w' <_v w'') \rightarrow w <_v w''$ (transitivity)

We lose totality, for it may be that two worlds are "weighted" the same, so any two words are not such that one is strictly better than the other or vice-versa. Since the range of the computation function for $\mathcal{O}$, in this case, is the set of all the tuples $(\varphi, \{w'/\forall w'' \ (w' \nless_w w'')\})$, it follows that $\mathcal{O}$ is quasi-logical only if, for any indexed ethical linear order $<_w$, the set of its upper bounds is such that, for any permutation σ, $\sigma^{\langle W,V \rangle}(\{w'/\forall w'' \ (w' \nless_w w'') = \{u/\forall w'' \ (u \nless_v w'')\}$, for some $v \in W^L$. We can now give necessary and sufficient conditions for the quasi-logicality of $\mathcal{O}$:

Quasi-Logicality of $\mathcal{O}$ $\mathcal{O}$ is quasi-logical iff:

    a)   it is completely defined over W^L

    b)   It is weakly computable.

    c)   for any indexed ethical linear order $\leq_w$, the set of its upper bounds is such that, for any permutation σ, $\sigma^{\langle W,V \rangle} (\{w'/\forall w'' \ (w' \nless_w w'') = \{u/\forall w'' \ (u \nless_v w'')\}$, for some $v \in W^L$ (Abundance).

There is no reason to doubt that (a) is satisfied. We also know that $\mathcal{O}$ satisfies (b). Moreover, for any way $\mathcal{O}$ is weakly-computable, the range of Comp is a subset of $\wp W^L \times \wp W^L$: the set of (φ, C^w), with $C^w = \{w'/\forall w'' \ (w' \nless_w w'')\}$. What about (c)? Call this condition *Abundance*. Since the set of transformations induced by arbitrary permutations, for a subset X of W^L, is the set of sets of cardinality $|X|$, we have Abundance iff, for any C^w, any X of cardinality $|C^w|$ is some C^v, for some $v \in W^L$. Now we can prove that $\mathcal{O}$ is quasi-logical iff conditions (a), (b) and (c) are met:

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[48] $\{w/\forall w'(w \nless_v w')\} = \{w/\forall w'(w' \leq_v w)\}$, since everything is not strictly better than w iff everything is less-or-equal-to w.

Proof 3.6. Conditions (a) and (b) are part of the definition of quasi-logicality, and we already know that $\mathcal{O}$ meets them. For condition (c), first, suppose Abundance is not met. Then, for some indexed ethical linear order $\leq_v$, the set of its upper bounds C^v is not such that, for any permutation σ, $\sigma^{\langle W,V \rangle}$ ($\{w/\forall w'\ (w \nprec_v w') = \{w''/\forall w'\ (w'' \nprec_u w'\}$, for some $u \in W^L$. Since the set of transformations of C^v are all and only the subsets of W^L with cardinality $|C^v|$, it follows that there must an $A \subseteq W^L$ of cardinality $|C^v|$, such that $\nexists w \in W^L\ (C^w = A)$. $\mathcal{V}_{\mathfrak{M}}^{\mathcal{O}}$ is weakly computable by an FTM such that Comp has, as range, the set of all (φ, C^w). Define a set of permutation $\sigma_1, \sigma_2...$ such that $\sigma^{\langle W,V \rangle}(C^v) = A$ (we know this is possible because C^v and A have the same cardinality). There exists at least a (w, φ) such that $\sigma^S(\text{Comp}(w, \varphi)) \neq \text{Comp}(w', \varphi')$ for any (w', φ'). In particular, this happens for any (v, φ). For $\sigma^S(\text{Comp}(v, \varphi)) = \sigma^S(\varphi, C^v) = (\sigma^{\langle W,V \rangle}(\varphi), A)$, and, for any (w', φ'), Comp $(w', \varphi') \neq (\sigma^{\langle W,V \rangle}(\varphi), A)$, since for no w, $C^w = A$. Therefore, $\mathcal{O}$ is not quasi-logical. Ergo, if $\mathcal{O}$ is quasi-logical, then Abundance is met.

Now for the reverse, suppose Abundance is met. Then for any C^w, any X of cardinality $|C^w|$ is some $C^{w'}$, for some $w' \in W^L$. We know that $\mathcal{O}$ is weakly-computable, so there is a Comp $[\sqsupseteq]^{-1} \circ \mathcal{V}_{\mathfrak{M}}^{\mathcal{O}}$ such that $\text{Comp}(w, \varphi) = (\varphi, C^w)$. We know conditions (a) and (b) are met. Since Abundance is met, the range of Comp is such that, for any permutation, $\sigma^S(\text{Comp}(w, \varphi) = \text{Comp}(w', \varphi')$, since for any σ, $\sigma^{\langle W,V \rangle}(C^w) = C^{w'}$, for some $C^{w'}$. So, condition (c) is met and $\mathcal{O}$ is quasi-logical.

Now we just have to make sense of the Abundance condition. How can it be possibly met? My guess is that, in the framework we have been discussing so far, Abundance is highly unlikely. To see why, let us ask first why orders change from world to world. One reason is that, for example, if in w it is true that $\ulcorner\mathcal{O}$: John saves Maria$\urcorner$, all the upper-bounds of $\leq_w$ must be worlds in which Maria exists, at least. The set of upper-bounds for some other world v in which Maria does not exist cannot be the same, then, for otherwise in v it would be true that $\ulcorner\mathcal{O}$: John saves Maria$\urcorner$: the poor John ought to save a non-existent person! There can be other reasons why we need more than one order: for example, it may be that we have the same domain in w and w', and yet in w things develop so that it is better to do some action A in some situation, while in w' the same A would have catastrophic effects. So, w and w' will have different indexed orders: one with best-worlds in which A happens, and one with no best-A-worlds.

However, even so, it is very hard to make sense of such a plenitude of sets of upper-bounds, as Abundance asks. For we are changing the ordering relation across worlds because different things happen in different worlds, not because we are changing the criterion of preference with which we are building the ordering itself: that is kept constant across worlds, it just has different outputs, in different worlds, given different backgrounds of events. Now, suppose that the criterion we take is broadly hedonistic: what ought to be the case is what minimizes pain and maximises pleasure. Suppose some C^v has cardinality n. Take n worlds in W^L with the same domain, but with very different states of affairs. One is paradisiac: everyone is happy and we all live in peace; another is hellish: everyone suffers strenuously and perpetually, and war is everywhere; another is so and so: more like our world. There can hardly be a set of upper-bounds that contains these worlds, given the hedonistic criterion we have chosen in the first place. Now, suppose the criterion is more "deontological": duties are determined by compliance to general Laws of morality. Take a world in which everyone is very dutiful, according to our set of Laws, and another world in which people do the exact opposite. Hardly will we find a set of upper-bounds with both these two worlds in. What I mean is that, for any kind of preference criterion one can conjure up, it seems to me that we can always find a set of worlds that cannot be in any set of upper-bounds of the preference relation that we extracted from that criterion. So, there will be a series of permutations for any set of upper-bounds, such that the resulting transformed set is not the set of upper-bounds, for any world. If so, Abundance cannot be met, regardless of the preference choices we make, unless our criterion is empty, or a total, irrational mess (in that case, arguably, anything is a best-world or none is, respectively).

The fact that in this framework $\mathcal{O}$ is not logical does not entail that in no framework is. Suppose we take Ought to work in this way: given a certain preference criterion that is usually pragmatically determined by the context, $\mathcal{O}$ takes you to some set of duties Γ, given a certain situation[49]. $\mathcal{O}$ has, therefore, a character and a content. The character takes you from a preference criterion and a world to a content: a function from sentences to sentences. In SDL, the world-variable is apparent, but not the preference criterion. Yet, it seems intuitive to me that we need such variable. Take these examples:

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[49] Kratzer has a similar view on Ought, even though she uses a Neighbourhood modal logic to model it (1977).

i. "You ought to go to bed early, if you want to wake up early".

ii. Bob and Maria want to rob a bank. Bob says: "We ought to use masks to cover our faces".

iii. "You ought to help others" (said by a priest during mass).

In (i), the preference criterion is made explicit through a conditional. In (ii), the preference condition is pragmatically identified by the discourse. In (iii), the preference criterion is probably ethical/religious. In fact, for any situation, we can make the preference criterion explicit by prefixing a ⌜In view of X⌝ clause, as suggested by Kratzer (1977, 340). So, what the priest says in (iii) can be rewritten as something like:

iv. In view of what the Bible says, you ought to help others.

I argue that this is a sensible theory of how $\mathcal{O}$ works. Different contexts will result in different sets of duties, given different circumstances. We can associate to each context a preference criterion that determines an ordering indexed to worlds. So, in this framework, $\mathcal{O}$ is determined by a function from preference criteria to functions from worlds to orderings of worlds. To make things easier, we can think of $\mathcal{O}$ as changing across couples of ⟨pc, w⟩, where pc is a variable for preference criterion: the pc determines the indexed ordering, the world determines the index of the ordering, to extract the relevant best-worlds.

It seems to me that in this contextualist framework it may be that, for any random bunch of worlds we take from W^L, there is some possibly very weird preference criterion that makes this random set the upper-bound of its ordering relation, for some world. For the set of preference criteria we are taking is very big and wide: it is the set of all conceivable preference criteria, however queer they might seem to us. This set definitely outstrips the realm of the rational and the sensible, or of the coherent, for what matters. After all, what is a preference criterion? It is just a way of ranking different situations. Take any bunch of worlds X: they are the meet of some set of propositions. Take the truth of these propositions as what counts for the ranking, and you will extract a preference criterion whose sets of best worlds is exactly X. If I am right, then this Ought can be quasi-logical, for it is fair to assume that, for any sets of upper-bounds, for any transformations of these sets, the result is still a set of upper-bounds, according to some (possibly) very weird and yet conceivable preference criterion.

This contextual Ought would be quasi-logical even in more complicated systems. For example, one very influential alternative to SDL is a logic without the limit assumption:

$\exists w \, \forall w' \, w' \leq_v w$ (**Limit Assumption**)

The limit assumption has been subjected to a lot of criticism. For it seems conceivable that there is not upper limit to some ordering $\leq_w$: for any world, there could always be one that is slightly better. If so, the standard system would have the weird consequence of making seriality fail in w: no world is accessible from it, for the extension of R in w is the set of upper-bounds for $\leq_w$, which is empty. So, in w, everything is a duty, because the universal clause for $\mathcal{O}$ is trivially satisfied. To improve the system, Lewis (1973; 1974) suggested a new clause for Ought, that can be summarised as follows:

Lewis $\mathcal{O}$ $w \vDash \mathcal{O}$: φ iff $\forall w' \, u \leq_w w'$, $w' \vDash \varphi$, where u is some constant associated with $\leq_w$.

The idea is that every order comes with an associated "least respectable" world u from which we can start to evaluate the Ought clause. Ought: P iff in every world at least as good as the least acceptable u, P is the case. If there is an upper limit, the modified clause collapses into the old one, so this adjustment is conservative. If we do accept this adjustment, we can still make a case for the quasi-logicality of the contextualist Ought. Indeed, the new framework makes even more sense: for many queer preference criteria, there can be no upper-bound to what is the best world associated with that criterion.

CONCLUSION

Standard invariance theories claim that a notion is logical iff it is invariant under some transformation of the domain of objects. In this dissertation I first used a categorial grammar to extend standard invariance theories to intensional notions: a notion is logical iff it is invariant under arbitrary automorphisms of both the domain and the modal space. I then showed what I take to be the limits of standard invariance in this extended framework. I tried to sketch a novel theory of logicality, based on the idea that logic is about information, in a special way. Logical notions are notions that are sensitive only to the structural aspects of information. Something is logical if it performs an information-structure operation, and it does so under arbitrary transformations of informational inputs. The new theory preserves some *desiderata* of standard invariance, like the formality of paradigmatic examples of logical notions: conjunction, negation, material implication, existential quantifiers etc. Logical possibility and necessity are invariant, as well. Thus, the new theory achieves extensional adequacy. However, it is somehow more restrictive than permutation invariance: I believe that cardinality quantifiers are not logical, on my view. What is more, in my theory logical notions tend to stay persistent across many different models of information, while in standard invariance this is hardly the case. Finally, I tried to spell out a relaxed notion of quasi-logicality, because I believe that logicality is a kind of formality, and formality comes in shades. I concluded this work by applying my theory of quasi-logicality to deontic modals, showing how they might be quasi-logical, if a strong form of contextualism is true.

I am not entirely satisfied with my theory, so far. I hope that, in my future PhD dissertation, I will be able to dig a little deeper into the matter. I still believe that my theory may be too fixated on the algebraic structure of the system we are choosing to model information, and this may not be ideal. I already showed how "persistence" is not perfectly general, in my theory. How can we fix that? What should be the link between algebraic structure and logicality? It is also still unclear where my theory is situated. For example, my invariance requirement would be something like "invariance of computability under transformations of inputs": it would be interesting to see if there are some similarities between this requirement and other novel proposals, like potential isomorphisms invariance (Bonnay 2008) or invariance under bisimulations (Van Benthem & Bonnay 2008).

I also want to make a final point: the literature on invariance is usually focused solely on "extensional" notions. I ended up with a theory that is applicable to any intensional notion, as well. In fact, it is applicable to any notion that can be modelled in a Montague grammar. Still, there is much work to be done in non-classical frameworks: what about the logicality of non-classical notions, like intuitionistic negation and relevant implication? What about non-truth-functional logics, like dynamic logic? I highly doubt that standard invariance can be the answer, here. In fact, I suspect it may be catastrophic, if applied in non-classical frameworks. Yet, will my theory score any better, in its current form? I hope that, in my future work, I will be able to deal extensively on these undiscussed and yet important matters.

Appendix A: Formal-Computability Invariance in Boolean Algebras of Infons.

In this appendix, we will prove that, in a Boolean algebra of infons, "formal computability invariance" is a restriction of standard permutation-invariance, for n-ary operations on the set of infons.

__Definition of Boolean Algebra__ A Boolean algebra is a (A, +, ×, 0, 1), where A is a set, + and × are operations on A and 0 and 1 the minimal and maximal elements of A. To be Boolean + and × must satisfy these conditions, for any a, b, c, elements of A:

    i.   *a + b = b + a*

    ii.   *a × b = b × a*

    iii.   *(a + b) × c = (a × c) + (b × c)*

    iv.   *(a × b) + c = (a + c) × (b + c)*

    v.   *(a + b) + c = a + (b + c)*

    vi.   *(a × b) × c = a × (b × c)*

In short, + and × must be commutative, associative, and they must distribute over each other.

In general, we are interested in cases in which some subclass of $\mathcal{I}$, together with ∧ and ∨, forms a Boolean algebra. This generally happens when we are modelling information from the powerset of a set, as in the object/modal system of chapter 3. In the following, I will call $\mathcal{I}$ the set of all infons, and $\mathcal{I}_1, \mathcal{I}_2, ...,$ $\mathcal{I}_n$ all the "subclasses" of $\mathcal{I}$, where a subclass is any subset of $\mathcal{I}$ closed under membership to a type.

Theorem 1. When $\mathcal{I}_n$ is the powerset of some set X, $(\mathcal{I}_n, \sqsubseteq)$ forms a complemented distributive lattice, and thus a Boolean Algebra.

There is no need for a proof for this, really, because these are well known properties of any powerset of a set (e.g. Enderton 1977). Where $\mathcal{I}_n$ is the powerset $\wp X$ of some set X, information-containment is the subset relation ⊆ defined on $\wp X$. Meet and Join are intersection and union: A∩B is the meet of {A,B}, A∪B is the join of {A,B}. For every two elements of $\wp X$, there exists their intersection and union. Thus, $\wp X$ forms a lattice. This lattice is distributive, because union and intersection are distributive and, for any

three sets, clearly, A∩(B∪C)=(A∩B)∪(A∩C) and A∪(B∩C)=(A∪B)∩(A∪C). Moreover, for any A in $\mathcal{I}_n$, there is the complement A^C such that $A∪A^C = 1$ and $A∩A^C = 0$, where 1 and 0 are the greatest and lowest element of the lattice (in $\mathscr{p}X$, the greatest element is X and the lowest is ∅). In $\mathscr{p}X$, the complement of A is simply X – A. Thus, for every Y in $\mathscr{p}X$, Y∩(X−Y) = ∅ and Y∪(X−Y) = X, because Y⊆X. So, $\mathscr{p}X$ is a complemented distributive lattice. So, ($\mathscr{p}X$, ⊆) is a distributive complemented lattice, and thus a Boolean algebra.

Inverted Benthem's Proposition. Among n-ary operations on sets, those defined at each tuple of arguments by some Boolean combination are permutation-invariant.

Theorem 2 When $\mathcal{I}_n$ is the powerset of some set X, ∧, ∨ and ¬ are permutation invariant.

Proof A.1. Suppose $\mathcal{I}_n$ is some $\mathscr{p}X$. Then, for Theorem 1, ($\mathcal{I}$, ⊑) forms a Boolean Algebra. By Inv. Benthem's proposition, ∧, ∨ and ¬ are permutation invariant, because they are Boolean.

When $\mathcal{I}_n = \mathscr{p}X$, for some X, elements of $\mathcal{I}_n$, are systematically equivalent to functions from X to V. In particular, for every i ∈ $\mathcal{I}_n$, there exists a function f: X → V such that f(x)= T iff x ∈ i. Thus, there is a bijection between elements of $\mathcal{I}_n$ and elements of type ⟨X,V⟩. Let σ be a permutation of X and $σ^Y$ the induced transformation in type Y:

- if Y=X, then for all k∈Y, $σ^Y(k) = σ(k)$
- if Y=V, then for all k∈Y, $σ^Y(k)= k$
- if Y=⟨X,Z⟩, for any types X, Z, then for all k∈Y, $σ^Y(k)= σ^Z ∘ k ∘ (σ^X)^{-1}$.

An element k of type Y is permutation invariant iff ∀σ, $k = σ^Y(k)$.

Assumption 1: Suppose Y ∈ $\mathscr{p}X$. Define a permutation σ(a) = b, with a and b elements of X. $σ^{⟨X,V⟩}Y = (Y−\{a\})∪\{b\}$.

Theorem 3: When $\mathcal{I}_n$ is the powerset of some X, ⊆ is permutation invariant. That is, Z ⊆ Y iff $σ^{⟨X,V⟩}(Z) ⊆ σ^{⟨X,V⟩}(Y)$, for any permutation of X and any induced transformation $σ^{⟨X,V⟩}$.

Proof A.2. Take any two Z and Y of $\mathscr{p}X$ and suppose Z⊆Y. Then ∀x∈X (x ∈ Z → x ∈ Y). The only way to upset Z⊆Y would be to change some elements in Z so that something is an element of Z but not

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Y. Define a permutation $\sigma^X(a)=a'$, with $a \in Z$. If $a' \in Z$, then $\sigma^{\langle X,V\rangle}(Z) \subseteq \sigma^{\langle X,V\rangle}(Y)$. If $a' \notin Z$, by Assumption 1, $\sigma^{\langle X,V\rangle}(Z) = Z' = (Z-\{a\})\cup\{a'\}$. Given that $Z\subseteq Y$, $a \in Y$. If $a' \in Y$, as well, then $\sigma^{\langle X,V\rangle}(Y)=Y$, and $Z' \subseteq Y$. If $a' \notin Y$, then, by Assumption 1, $\sigma^{\langle X,V\rangle}(Y) = Y'$, where $Y'= (Y-\{a\})\cup\{a'\}$. But then $Z' \subseteq Y'$, since $a' \in Z'$ and $a' \in Y'$. Thus, for any permutation σ of X and any corresponding induced permutation in $\langle X,V\rangle$, if $Z\subseteq Y$ then $\sigma^{\langle X,V\rangle}(Z) \subseteq \sigma^{\langle X,V\rangle}(Y)$.

Now, for the reverse, suppose $Z\nsubseteq Y$. Then $\exists x_{\in X} (x \in Z \wedge x \notin Y)$. The only way to upset this would be to change all such x's, so that no element of Z is not an element of Y. Define a series of permutations $\sigma^X(a) = a'$, $\sigma^X(b) = b'$, ..., $\sigma^X(n) = n'$, where $\{a, b, ..., n\} = \{x \, / \, x \in Z \wedge x \notin Y\}$, and $\{a', b', ..., n'\}$ is a subset of both Z and Y. $\sigma^{\langle X,V\rangle}(Z)=Z$, for all the permutations are nothing but automorphisms of Z ($\{a, b, ..., n\}$ and $\{a', b', ..., n'\}$ are both subsets of Z). The transformation induced in Y is $\sigma^{\langle X,V\rangle}(Y)=Y'$, where $Y'=(Y-\{a, b, ..., n\})\cup\{a', b', ..., n'\}$, by Assumption 1. But then $Z \nsubseteq Y'$, for $\{a, b, ..., n\}$ is a subset of Z but not of Y', by assumption. So, if $Z \nsubseteq Y$ then $\sigma^{\langle X,V\rangle}(Z) \nsubseteq \sigma^{\langle X,V\rangle}(Y)$. Thus, $Z \subseteq Y$ iff $\sigma^{\langle X,V\rangle}(Z) \subseteq \sigma^{\langle X,V\rangle}(Y)$, for any permutation σ.

The proof itself comes with no surprise, since information-containment, in our system, is the order of the lattice, and it can be defined using intersection or union: $A\subseteq B$ iff $A\cap B=A$ and iff $A\cup B=B$.

Theorem 4 When $\mathcal{J}_n$ is the powerset of some X, Compatibility is permutation invariant. That is, C (x, y) iff C ($\sigma^{\langle X,V\rangle}(x)$, $\sigma^{\langle X,V\rangle}(y)$)., for any permutation of X and any induced transformation $\sigma^{\langle X,V\rangle}$.

Proof A.3. Take a binary relation of compatibility C defined over sets of elements of X. $C(Z,Y) =_{df} Z\cap Y\neq\emptyset$. Suppose there are some A and B such that $C(A,B)$ but $\neg C(\sigma^{\langle X,V\rangle}(A),\sigma^{\langle X,V\rangle}(B))$. This can be only if the permutations "switch" all the (a, b, ..., n) that are elements of both A and B with some elements (a', b', ..., n') that are not in A nor in B. Yet, via assumption 1, now $\sigma^{\langle X,V\rangle}(A) = (A-\{a, b, ..., n'\})\cup\{a', b', ..., n'\}$ and $\sigma^{\langle X,V\rangle}(B) = (B-\{a, b, ..., n'\})\cup\{a', b', ..., n'\}$. Since $\{a, b, ..., n\} \subseteq \cap\{\sigma^{\langle X,V\rangle}(A), \sigma^{\langle X,V\rangle}(B)\}$, we have $C(\sigma^{\langle X,V\rangle}(A), \sigma^{\langle X,V\rangle}(B))$, contrary to our assumption. So, for any Z and Y in $\wp X$, if $C(Z,Y)$ then $C(\sigma^{\langle X,V\rangle}(Z),\sigma^{\langle X,V\rangle}(Y))$.

For the reverse, suppose $\neg C(A,B)$ and yet $C(\sigma^{\langle X,V\rangle}(A), \sigma^{\langle X,V\rangle}(B))$. Then $A\cap B=\emptyset$ and $\sigma^{\langle X,V\rangle}(A) \cap \sigma^{\langle X,V\rangle}(B) \neq \emptyset$. Then there must be at least a permutation $\sigma(a) = a'$ such that $a' \in$

$\cap\{\sigma^{(X,V)}(A), \sigma^{(X,V)}(B)\}$. By assumption 1, $\sigma^{(X,V)}(A)=(A-\{a\})\cup\{a'\}$ and $\sigma^{(X,V)}(B)=(B-\{a\})\cup\{a'\}$. But then $A=(\sigma^{(X,V)}(A)-\{a'\})\cup\{a\}$ and $B=(\sigma^{(X,V)}(B)-\{a'\})\cup\{a\}$. So $a\in A\cap B$ and thus $C(A, B)$, contrary to our assumption. So, by *reductio,* $\neg C(\sigma^{(X,V)}(A), \sigma^{(X,V)}(B))$. So, if $C(\sigma^{(X,V)}(A), \sigma^{(X,V)}(B))$ then $C(A, B)$.

Corollary 1: Take any FTM, and the formal operation Ω it describes. When $\mathcal{I}_n$ is some powerset $\wp X$, Ω is permutation invariant.

Proof A.4. By Theorem 1-4, the formal operations of meet, join, information-containment, complementation and compatibility are permutation invariant. These are all the basic formal operations. Any formal operation is a combination of basic formal operations. Any combination of permutation-invariant operations is permutation invariant. Ω, being a formal operator, is a combination of permutation invariant operations, so it is permutation invariant, as well.

Corollary 2: When $\mathcal{I}_n$ is some powerset $\wp X$, an n-ary operator defined on X is FC-invariant only if permutation-invariant.

Proof A.5. Suppose $\mathcal{I}_n$ is some powerset $\wp X$. An n-ary operator K is FC-invariant iff there is an FTM such that there exists a function Comp: FTM$^{-1} \circ$ K such that:

a) For every s and s', Comp(s) = s' iff K(s) = FTM(s').

b) For every s, Comp(s) is the sequence such that $(b_m)_{1\leq m\leq n} = (a_m)^s$ and $(b_m)_{m>n} = k$, where $(a_x)^s$ is the x-th element of s.

c) Let σ be a permutation of information and let σ^S be the transformation induced in type S by such permutation. For every s, s', if $k(s) = $ FTM(s') then, for every σ, $k(\sigma^S(s)) = $ FTM($\sigma^S(s')$).

Suppose K is not permutation invariant. Then, there exists a series of permutations for which $K \neq \sigma^Y(K)$, where Y is the type for K. That is, there is a series of permutations $\sigma_1, \sigma_2, ..., \sigma_n$ such that, for some argument (a, b, ..., n) and value t of K, K(a, b, ..., n) = t but K(σ^S(a, b, ..., n)) $\neq \sigma^S$(t). K is either F-computable or not. If not, then it is not FCI. So, suppose K is F-computable by some FTM Ω. Then K(a, b, ..., n) = Ω (a, b, ..., n, ..., k) = t. By Corollary 1, Ω is permutation invariant. Thus, Ω(s) = s' iff $\Omega(\sigma^S(s)) = \sigma^S$(s'), for any permutation σ of X and any induced σ^S.

In particular, under $\sigma_1, \sigma_2, ..., \sigma_n$, $\Omega(\sigma^S(a, b, ..., n, ..., k)) = \sigma^S(t)$. Thus, $\Omega(\sigma^S(a, b, ..., n, ..., k)) = \sigma^S(t) \neq K(\sigma^S(a, b, ..., n))$. Hence, even if K is F-computable it is not FCI. So, in any case, if K is not permutation invariant, it is not FCI. Thus, K is FC-invariant only if permutation-invariant.

Here the schema of the proof. FC-invariance is (FC) = F-computability & (I) = persistence of computability under permutations. So, what we want is to prove is $\vdash$ FC&I $\rightarrow$ PI.

1, (1) ¬PI. Assumption.

$\vdash$ (2) FC ∨ ¬FC. Excluded middle.

3, (3) ¬FC. Assumption.

3, (4) ¬(FC&I). 3, meaning of &.

$\vdash$ (5) ¬FC→ ¬(FC&I). 3, 4, Conditional Proof.

6, (6) FC. Assumption.

1,6, (7) FC&¬PI. 1, 6, &-introduction.

1,6 (8) ¬(FC&I). 1, 6, proof explained above.

1 (9) FC→ ¬(FC&I). 6, 8, Conditional Proof.

1, (10) ¬(C&I). 2, 5, 9, Disjunction Elimination.

$\vdash$ (11) ¬P → ¬(C&I). 1, 10, Conditional Proof.

$\vdash$ (12) C&I → PI. 11, Logical equivalences.

Thus, in a Boolean algebra of infons, FC-invariance is a "contraction" of permutation invariance, for n-ary operations defined over the set of infons. However, this does not entail that my theory is a contraction of permutation invariance, in general, as I have already discussed in section 3.2.

QED

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