COMPUTATIONAL TECHNIQUES IN FINITE SEMIGROUP THEORY

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Computational techniques in finite semigroup theory

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Acknowledgements

I looked forward to many aspects of life as a PhD student: the opportunities to travel to new places, the freedom to set my own schedule, and the independence to develop my mathematical interests. On these points, my expectations matched reality. There were also less-positive things that I did not expect, such as how my motivation could not be taken for granted. I can confirm that doing a PhD is not easy, and I would like to thank those of you who have helped me along the way: my friends, family, colleagues, and funders.

In my first year as a PhD student, my mother became ill, and died. That time, and the subsequent time for adjusting, has been very difficult. I regret that my PhD, ostensibly an exciting stage in my life, has coincided with such sadness. More than anything, I am grateful to have had, and to continue to have, the love and support of so many people, particularly my five siblings: Laura, Will, Poppy, Holly, and Alan. I'm so lucky to have you.

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Abstract

A semigroup is simply a set with an associative binary operation; computational semigroup theory is the branch of mathematics concerned with developing techniques for computing with semigroups, as well as investigating semigroups with the help of computers. This thesis explores both sides of computational semigroup theory, across several topics, especially in the finite case.

The central focus of this thesis is computing and describing maximal subsemigroups of finite semigroups. A maximal subsemigroup of a semigroup is a proper subsemigroup that is contained in no other proper subsemigroup. We present novel and useful algorithms for computing the maximal subsemigroups of an arbitrary finite semigroup, building on the paper of Graham, Graham, and Rhodes from 1968. In certain cases, the algorithms reduce to computing maximal subgroups of finite groups, and analysing graphs that capture information about the regular $\mathscr{J}$-classes of a semigroup. We use the framework underpinning these algorithms to describe the maximal subsemigroups of many families of finite transformation and diagram monoids. This reproduces and greatly extends a large amount of existing work in the literature, and allows us to easily see the common features between these maximal subsemigroups.

This thesis is also concerned with direct products of semigroups, and with a special class of semigroups known as Rees 0-matrix semigroups. We extend known results concerning the generating sets of direct products of semigroups; in doing so, we propose techniques for computing relatively small generating sets for certain kinds of direct products. Additionally, we characterise several features of Rees 0-matrix semigroups in terms of their underlying semigroups and matrices, such as their Green's relations and generating sets, and whether they are inverse. In doing so, we suggest new methods for computing Rees 0-matrix semigroups.

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Preface

The overarching theme of this thesis is *computational semigroup theory*, which includes both the study of semigroups via computers, and the development of the theoretical results and tools that facilitate this study. These aspects reinforce each other: the development of better tools enables a more diverse range of computations, whilst experimentation with computers assists the identification of further advances in semigroup theory. This thesis contains elements of both of these aspects of computational semigroup theory.

The use of computers is well-established in many branches of mathematics, especially in applied mathematics and statistics. Software can be employed to solve equations, model systems, perform simulations, and analyse data. This should not be surprising, since computers are inherently mathematical machines. The fundamental utility of computers is that they perform calculations that are far too difficult or time-consuming for a human to perform. Moreover, given correct programming, computers are essentially guaranteed to give correct results.

In addition, computers are increasingly being used for pure mathematical research, in areas such as semigroup theory. For example, the Semigroups [101] and Digraphs [10] packages for GAP [58] can be used to visualise particular features of a finite semigroup, such as the egg-box diagram of a $\mathscr{D}$-class, the Hasse diagram of its partial order of $\mathscr{J}$-classes, or a picture of its Cayley digraphs. For those researchers, teachers, or students who 'think in pictures', as in [64, p. xii], these kinds of visualisations are especially valuable.

Crucially, mathematical software for semigroups allows a researcher to construct and analyse examples of semigroups in far greater number, and of far greater complexity, than would be possible to consider otherwise. Software can be used in this way to produce instructive examples to include in a paper or talk, for instance, or to demonstrate concepts to a collaborator or student. By performing computational experiments on a wide range of examples, a mathematician can learn about the properties of various semigroups, and perhaps spot patterns and develop conjectures, or find counterexamples to pre-existing conjectures. Several results in Chapter 5 were first observed in this way. This kind of computational experimentation is particularly powerful when conducted in combination with pre-computed libraries of data. For example, the Smallsemi [32] package for GAP contains information about every semigroup (up to isomorphism and anti-isomorphism) containing at most 8 elements, and can be used to exhaustively search through these semigroups. Similar applications exist in many other branches of pure mathematics.

When computing with semigroups, there is often an 'obvious' way of computing a property, but this is rarely the most appropriate way. Roughly speaking, one of the main aims of theoretical research in computational semigroup theory is to exploit the structural properties of a semigroup to reformulate a certain problem in terms of features that are more feasible to compute. As a very straightforward example, consider the problem of deciding whether a finite semigroup is commutative. By definition, a semigroup S is commutative if and only if $xy = yx$ for all $x, y \in S$. It follows that an algorithm that directly applies the definition to test the commutativity of a finite semigroup S would perform $|S|(|S| - 1)$ multiplications in the worst case. However, a semigroup is commutative if and only if its generators commute, and a semigroup S is often defined by a generating set X that contains far fewer elements than S. In such cases, the commutativity of S can be tested by performing at most $|X|(|X| - 1)$

multiplications, which can be accomplished much more quickly.

Computational semigroup theory is closely related to *computational group theory*, which is defined analogously. This is a much more mature area of research: many books have been published on the topic of computational group theory in the last few decades [73, 118, 119], whereas there are none, as yet, on the topic of computational semigroup theory. Since every group is a semigroup, it might appear on first glance that computational semigroup theory encompasses computational group theory. However, as with the more general branches of group theory and semigroup theory themselves, these topics are distinct, although linked. In semigroup theory, we consider a particular problem to be solved if it can be reduced to a problem in group theory. However, many other problems in semigroup theory turn out to be solved via a reduction to a combinatorial problem, or to a graph-theoretic problem. We take the same approach when computing with semigroups: for example, the techniques that we present in Chapter 4 for computing maximal subsemigroups reduce, in some cases, to computing maximal subgroups in groups, or to computing maximal cliques in graphs. In this thesis, we regard algorithms in computational group theory as being given, and we do not concern ourselves with their inner workings.

Some of the earliest research on the topic of computing with semigroups was that of Forsythe [53], who enumerated the 126 semigroups of order 4 with the help of a computer, followed by that of Cannon [18, 19] and Perrot [106]. In the subsequent decades, researchers have continued to study semigroups with the use of computers and study techniques for computing with semigroups. This has led to the development and publication of a range of software, with diverse approaches and foci. There are numerous stand-alone pieces of software for computing with semigroups, such as AUTOMATE [21]; SEMIGROUPE [107] (which implements the techniques described in [54]); LIBSEMIGROUPS [102]; and SGPWIN [99]. The major computer algebra systems, like GAP [58], MAGMA [13], and SAGEMATH [122], include some functionality for semigroups, which is supplemented by specialised packages such as the NUMERICALSGPS [24], KBMAG [72], SMALLSEMI [32], and SEMIGROUPS [101] packages for GAP, for instance. Computer code implementing some of the ideas of the second, third, and fourth chapters of this thesis is available in the current version, or in an upcoming version, of the SEMIGROUPS package for GAP; most of the examples contained in this thesis were constructed by the use of the SEMIGROUPS package; and many of the ideas presented herein arose through experimentation with this software.

We briefly summarise the contents of this thesis. In Chapter 1, we give the necessary definitions, notions, and preliminary results that are required in the thesis.

In Chapter 2, we consider generating sets for direct products of semigroups from the perspective of computation, building on several results from the literature. More specifically, we describe generating sets, for arbitrary direct products of semigroups, that do not necessarily contain every element of the direct product. We also discuss techniques for directly computing generating sets of this kind in the case that the direct product is finitely generated and the factors are either finite or finitely presented. The crucial step in this procedure is an algorithm to find a non-trivial factorization of an element over some generating set, or to prove that none exists. The main ideas presented in this chapter, in the case of direct products of finite semigroups, have been implemented in the development version of the SEMIGROUPS [101] package for GAP [58], and will be included in an upcoming released version.

In Chapter 3, we present results relating to the computation of Rees 0-matrix semigroups, particularly those that are finite. There are well-established techniques for computing with finite regular Rees 0-matrix semigroups over groups. However, a Rees 0-matrix semigroup may be defined over an arbitrary semigroup, and there is far less research on the topic of computing with Rees 0-matrix semigroups over semigroups that are not groups. We characterise certain features and properties Rees 0-matrix semigroups in terms of their matrices and underlying semigroups. Given the ability to compute with the underlying semigroups, we discuss how these characterisations may be used to compute whether a Rees 0-matrix semigroup is inverse or regular, or to determine generating sets or the Green's structure in certain cases.

We focus on maximal subsemigroups of finite semigroups in the remainder of the thesis. Much of the research detailed in these chapters has been published in two research papers [35,45] in *Journal of Algebra*, in collaboration with several co-authors, but is more fully exposited here.

In Chapter 4, we put forth an algorithm for computing maximal subsemigroups in an arbitrary finite semigroup, building on the description of maximal subsemigroups given by Graham, Graham, and Rhodes in [61]. In order to do so, we first describe algorithms for computing maximal subsemigroups of an arbitrary finite regular Rees 0-matrix semigroup over a group. The algorithms described in this chapter are fully implemented in the Semigroups package for GAP. We also deduce theoretical results from the general solution that allow us to describe certain maximal subsemigroups of finite monoids.

In Chapter 5, we exploit the techniques described in Chapter 4 in order to describe and count the maximal subsemigroups of a wide range of families of finite transformation and diagram monoids. In particular, we describe the maximal subsemigroups of many monoids of order- or orientation-preserving or -reversing partial transformations, along with the maximal subsemigroups of the Motzkin, Brauer, Jones and partition monoids, and several further monoids. This work unifies and greatly extends many results in the literature concerning the maximal subsemigroups of particular transformation semigroups. Many of the results presented in Chapter 5 arose through experimentation with the Semigroups package for GAP.

Several open problems are posed throughout the main text of this thesis; they are collected in Appendix A for easy reference. In order to aid the reader, a table of notation is included at the end of this thesis on page 191, along with an index on page 195. In the original digital version of this document, many instances of notation serve as hyperlinks to their corresponding entries in the table of notation.

Chapter 1

Introduction

The purpose of this chapter is to introduce the standard ideas, definitions, and results that are used throughout this thesis. We introduce additional and more specialised notions in situ when it is more appropriate to do so.

1.1 Sets, relations, and functions

A *partition* $\mathcal{P}$ of a set X is a set of non-empty pairwise-disjoint subsets of X, called *parts*, whose union is X. If $\mathcal{P}$ is a partition of a set X, then a *transversal* of $\mathcal{P}$ is a subset of X that contains exactly one element from each part of $\mathcal{P}$.

The *Cartesian product* of an ordered list of sets $X_1, X_2, \ldots, X_n$ is the set of n-tuples

$$X_1 \times X_2 \times \cdots \times X_n = \big\{(x_1, x_2, \ldots, x_n) : x_i \in X_i \text{ for all } i \in \{1, \ldots, n\}\big\}.$$

A *relation* is just a subset of the Cartesian product $X \times Y$, for some sets X and Y. Let $\rho \subseteq X \times Y$ be a relation. If $x \in X$ and $y \in Y$, then we often write $x \; \rho \; y$ to denote that $(x, y) \in \rho$; this is particularly common for partial orders, and for the Green's relations on a semigroup, which are defined later. The *inverse* of ρ, denoted ρ^{-1}, is the relation $\big\{(y, x) : (x, y) \in \rho\big\} \subseteq Y \times X$. If $\tau \subseteq Y \times Z$ is a relation, then the *composition* of ρ and τ is the relation

$$\rho \circ \tau = \big\{(x, z) \in X \times Z : (x, y) \in \rho \text{ and } (y, z) \in \tau \text{ for some } y \in Y\big\}.$$

A *relation on a set* X is a subset of $X \times X$. Let X be a set and let ρ be a relation on X. Then ρ is *reflexive* if $\{(x, x) : x \in X\} \subseteq \rho$, ρ is *symmetric* if $\rho = \rho^{-1}$, ρ is *antisymmetric* if $\rho \cap \rho^{-1} \subseteq \{(x, x) : x \in X\}$, and ρ is *transitive* if $\rho \circ \rho \subseteq \rho$.

An *equivalence relation*, or an *equivalence*, is a relation on a set that is reflexive, symmetric, and transitive. If ρ is an equivalence on the set X, then the *equivalence class* of a point $x \in X$ is the set $\{y \in X : (x, y) \in \rho\}$. The equivalence classes of an equivalence relation on a set form a partition of that set. If $Y \subseteq X$ is a union of equivalence classes of some equivalence ρ on X, then we denote the set of these classes by Y/ρ.

A *partial order* on a set is a relation on the set that is reflexive, antisymmetric, and transitive. A *partially-ordered set* is a set with a partial order; unless otherwise specified, the partial order is denoted by $\leq$. When using the symbol $\leq$ for a partial order, as usual we write $x < y$ to denote that $x \leq y$ and $x \neq y$. Let X be a partially-ordered set, and let $x, y \in X$. We say that x is *less than* y, and that y is *greater than* x, if $x < y$. A point $x \in X$ is *maximal* if $x \leq y$ implies that $x = y$, for all $y \in X$, and x is *minimal* if $y \leq x$ implies that $x = y$, for all $y \in X$. A partially-ordered set may contain multiple maximal or minimal elements, or, when the set is infinite, possibly none. If $x < y$ and there is no element $z \in X$ such that $x < z < y$, then we say that x *is covered by* y in the partial order.

In this thesis, $|X|$ denotes the cardinality of the set X, and $\varnothing$ denotes the empty set; $\mathbb{Z}$ denotes the integers, $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers, and $\mathbb{N}_0$ is the set of non-negative integers $\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$. For a real number x, $\lceil x \rceil$ denotes the least integer greater than or equal to x, and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Let X and Y be sets. A *partial function* $f : X \longrightarrow Y$ is a subset of the Cartesian product $X \times Y$ where, for any $x \in X$, there is at most one $y \in Y$ such that $(x, y) \in f$. In particular, a partial function is a relation. The *domain* of a partial function $f : X \longrightarrow Y$ is the set

$$\mathrm{dom}(f) = \big\{x \in X \,:\, (x, y) \in f \text{ for some } y \in Y\big\} \subseteq X.$$

If $x \in \mathrm{dom}(f)$, then the *image of x under f* is the unique $y \in Y$ such that $(x, y) \in f$. We most commonly write a partial function to the right of its arguments, so that $(x)f$, or simply xf, denotes the image of x under f. The *image* of f is the set

$$\mathrm{im}(f) = \big\{xf \,:\, x \in \mathrm{dom}(f)\big\} \subseteq Y,$$

and the *kernel* of f is the equivalence on $\mathrm{dom}(f)$ given by

$$\ker(f) = \big\{(x_1, x_2) \in \mathrm{dom}(f) \times \mathrm{dom}(f) \,:\, x_1 f = x_2 f\big\}.$$

A partial function $f : X \longrightarrow Y$ is *injective* if its inverse relation f^{-1} defines a partial function $Y \longrightarrow X$, which we call the *inverse* of f. A partial function $f : X \longrightarrow Y$ is *surjective* if $\mathrm{im}(f) = Y$, and f is a *function* if $\mathrm{dom}(f) = X$. Another name for a function is an *operation*. A function is *bijective*, and is called a *bijection*, when it is both injective and surjective.

Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be partial functions. The *composition* of f and g, denoted $f \circ g$, or more usually fg, is the partial function $X \longrightarrow Z$ given by the composition of f and g as relations. In particular, if f and g are functions and $x \in X$, then $(x)fg = (xf)g$. Note that we compose functions from left to right.

A *partial transformation* is a partial function $\{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$, for some $n \in \mathbb{N}$. The *degree* of a partial transformation $\{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is n. If α is a partial transformation of degree n, then we define $\mathrm{rank}(\alpha)$, the *rank* of α, to be $|\mathrm{im}(\alpha)|$. A *transformation* is a partial transformation that is a function, and a *permutation* is a bijective transformation. A *partial permutation* is an injective partial transformation.

We may write a partial transformation of degree n in *two-line notation*. This uses a $2 \times n$ matrix, where the i^{th} entry in the first row contains the number i, and the i^{th} entry in the second row contains if when $i \in \mathrm{dom}(f)$, otherwise it contains a dash. For example, if f is the partial transformation of degree 8 whose domain is $\{2, 4, 6, 8\}$, and where $if = i/2$ for any $i \in \mathrm{dom}(f)$, then f is written in two-line notation as

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ - & 1 & - & 2 & - & 3 & - & 4 \end{pmatrix}.$$

Furthermore, we often write permutations in *disjoint cycle notation*; see [77, Chapter 1] for a definition of this notation, and for the standard terminology relating to permutations.

1.2 Graphs and digraphs

A *graph* is a pair (V, E), where V is any set of *vertices*, and E is a set of 2-subsets of V, called *edges*. Let $\Gamma = (V, E)$ be a graph, and let $u, v \in V$ be distinct vertices of Γ. We say that u *is adjacent to* v, and vice versa, and that u *and* v *are adjacent*, when $\{u, v\} \in E$. A vertex u

is a *neighbour* of a vertex v if u and v are adjacent. The *degree* of a vertex is the number of neighbours that it has; a vertex in a graph with degree 0 is *isolated*.

A *clique* of a graph is a set of mutually adjacent vertices, and an *independent subset* is a set of mutually non-adjacent vertices. The cliques of a graph are partially-ordered by inclusion, as are the independent subsets. Thus a clique of a graph is *maximal* if it is properly contained in no clique of the graph and, similarly, an independent subset is *maximal* if it is properly contained in no independent subset of the graph. A graph is *bipartite* if its vertices can be partitioned into two independent subsets. A graph (V, E) is a *complete bipartite graph* if $\{U', V'\}$ is a partition of V, and $E = \big\{\{u, v\} : u \in U' \text{ and } v \in V'\big\}$.

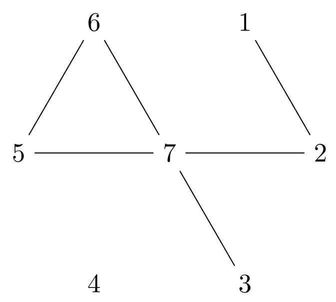

Figure 1.1: Let $V = \{1, \ldots, 7\}$, and let $E = \{\{1, 2\}, \{2, 7\}, \{3, 7\}, \{5, 6\}, \{5, 7\}, \{6, 7\}\}$. This figure gives a visualisation of the graph (V, E), which has seven vertices and six edges. The set of neighbours of the vertex 7 is $\{2, 3, 5, 6\}$, and so it has degree 4. The vertex 4 is isolated. The pair $\{5, 6\}$ is a clique, however it is not a maximal clique, since it is properly contained in the maximal clique $\{5, 6, 7\}$. The set $\{1, 3, 4, 5\}$ is a maximal independent subset.

A *digraph* is a pair (V, E), consisting of a set of *vertices* V, and a set of *edges* $E \subseteq V \times V$. In other words, the set of edges of a digraph is a relation on the set of vertices. Let (u, v) be an edge of a digraph. Then v is an *out-neighbour* of u, and u is an *in-neighbour* of v, and the edge is a *loop* if $u = v$. A *sink* is a vertex in a digraph with no out-neighbours, and a *source* is a vertex in a digraph with no in-neighbours. A *colouring* of a graph or digraph (V, E) is simply a function $V \longrightarrow \mathbb{Z}$.

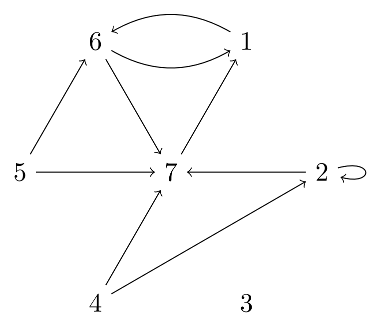

Figure 1.2: Let $E = \{(1, 6), (2, 2), (2, 7), (2, 4), (4, 7), (5, 6), (5, 7), (6, 1), (6, 7), (7, 1)\}$, and let $\Gamma = (\{1, \ldots, 7\}, E)$. Then Γ is a digraph with seven vertices and ten edges. The in-neighbours of the vertex 6 are 1 and 5, and the out-neighbours are 1 and 7. The vertices 3, 4, and 5 are sources of Γ, and 3 is the only sink. There is a single loop in Γ, at vertex 2.

A *path* in a graph (V, E) is a finite non-trivial sequence $(v_0, \ldots, v_n)$, for some $n \in \mathbb{N}$, where $v_i \in V$ for all $i \in \{0, 1, \ldots, n\}$, and $\{v_{i-1}, v_i\} \in E$ for all $i \in \{1, \ldots, n\}$. A *path* in a digraph is defined analogously. If $(v_0, \ldots, v_n)$ is a path in a graph or digraph, then the path is said to be a *path from v_0 to v_n*. A vertex v is *reachable* from a vertex u in a graph or digraph Γ if either $u = v$, or there exists a path from u to v in Γ. A *cycle* is a path from a vertex to itself. The *length* of the path or cycle $(v_0, \ldots, v_n)$ is n.

Two vertices of a graph are contained in the same *connected component* of the graph if one is reachable from the other, and two vertices of a digraph are contained in the same *strongly connected component* if one is reachable from the other, *and vice versa*. The vertex set of the graph or digraph is partitioned by its connected components or strongly connected components, respectively. A graph is *connected* if its vertices form a single connected component, and a digraph is *strongly connected* if it has a single strongly connected component. A graph without cycles is called a *forest*, and a connected forest is called a *tree*. A digraph without cycles is *acyclic*. Let $\Gamma = (V, E)$ be a graph. A *spanning forest* of Γ is any forest $\Gamma' = (V, E')$, such that $E' \subseteq E$ and such that the connected components of Γ' coincide with those of Γ. Any spanning forest of Γ has $|V| - n$ edges, where n is the number of connected components of Γ. A *spanning tree* is a spanning forest of a connected graph.

Example 1.3. Let $\Gamma = (V, E)$ be the graph defined in Figure 1.1. Then $(1, 2, 7, 5, 6)$ is a path from 1 to 6 in Γ, and the path $(5, 6, 7, 5)$ is a cycle in Γ. No vertex other than 4 itself is reachable from the vertex 4. The two connected components of Γ are $\{4\}$ and $V \setminus \{4\}$. If $E' = E \setminus \{\{5, 6\}\}$, then the graph (V, E') is a spanning forest of Γ.

Example 1.4. Let $\Delta = (V, E)$ be the digraph defined in Figure 1.2. Then $(4, 2, 7, 1, 6)$ is a path in Δ from 4 to 6, and so 6 is reachable from 4 in Δ. However, 4 is not reachable from 6, and so 4 and 6 are contained in different strongly connected components of Δ. The strongly connected components of Δ are $\{1, 6, 7\}$, and the singletons $\{2\}$, $\{3\}$, $\{4\}$, and $\{5\}$. The paths $(2, 2)$ and $(1, 6, 1)$ are cycles, and so Δ is not acyclic.

Let $\Gamma = (V, E)$ be a graph. The *complement* of Γ is the graph with vertices V and edges

$$\big\{\{u, v\} : u, v \in V,\ u \neq v,\ \text{and } \{u, v\} \notin E\big\}.$$

Thus a clique of a graph is an independent subset of the complement, and an independent subset is a clique of the complement. If $V' \subseteq V$, then the *induced subgraph* of Γ on V' is the graph $\big(V', \{\{u, v\} \in E : u, v \in V'\}\big)$. If $\mathcal{P}$ is a partition of V, then the *quotient of* Γ *by* $\mathcal{P}$ is the graph with vertex set $\mathcal{P}$, where two distinct parts $A, B \in \mathcal{P}$ are adjacent in the quotient if and only if there exists $u \in A$ and $v \in B$ such that $\{u, v\} \in E$. An *isomorphism* from a graph (V, E) to a graph (V', E') is a bijection $\phi : V \longrightarrow V'$, where $\{u, v\} \in E$ if and only if $\{u\phi, v\phi\} \in E'$, for all $u, v \in V$. The *complement*, *induced subdigraph*, and *quotient* of a digraph, and isomorphisms between digraphs, are defined analogously.

1.3 Groups and semigroups

In this section, we introduce the notation, definitions, and terminology from semigroup theory and group theory that we require in this thesis. We predominantly follow the notation of [76], which is prevalent in the literature. Throughout this thesis, we tend to refer the reader to [76, 110] for proofs of standard results, although there are many other well-regarded introductory books on semigroup theory, such as [22, 67, 71].

A *semigroup* is a set S with an associative operation $S \times S \longrightarrow S$. The *order* of a semigroup is its cardinality. Examples of semigroups include the real numbers $\mathbb{R}$ with their usual multiplication, the set of all $n \times n$ (for some $n \in \mathbb{N}$) matrices over a semiring with matrix multiplication, and the set of all relations on a set with composition of relations.

Unless otherwise stated, the associative operation is denoted by juxtaposition; more precisely, for elements $x, y \in S$, the juxtaposition xy of x and y denotes the image of the pair (x, y) under the operation. We often use the terminology of multiplication for such images. For example, xy is often referred to as the *product* of the elements x and y, and a subset A of S is *closed under multiplication* if $ab \in A$ for all $a, b \in A$. A semigroup S is *commutative* if $xy = yx$ for all $x, y \in S$. *Associativity* is the property that

$$(xy)z = x(yz) \quad \text{for all } x, y, z \in S.$$

Because a semigroup is associative, we may unambiguously form the product of any finite number of elements in a semigroup. If $x \in S$ and $k \in \mathbb{N}$, then x^k denotes the product $\underbrace{x \cdots x}_{k \text{ times}}$.

A *homomorphism* between semigroups S and T is a function $S \longrightarrow T$ that preserves the multiplication of S. In other words, a homomorphism is a function $\phi : S \longrightarrow T$ that satisfies $(xy)\phi = (x\phi)(y\phi)$ for all $x, y \in S$. An *anti-homomorphism* is a function that reverses the multiplication of S. An *isomorphism* of semigroups is a bijective homomorphism, and two semigroups S and T are *isomorphic* if there is an isomorphism between them; in this case, we write $S \cong T$. An *anti-isomorphism* is a bijective anti-homomorphism. Isomorphic semigroups have the same semigroup-theoretic properties; anti-isomorphic semigroups are left-right duals of each other. An injective homomorphism is called an *embedding*; if S and T and semigroups and $\phi : S \longrightarrow T$ is an embedding, then $S \cong \mathrm{im}\phi$.

An equivalence ρ on a semigroup S is a *left congruence* if it is compatible with left multiplication (i.e. $(sx, sy) \in \rho$ for all $(x, y) \in \rho$ and $s \in S$), a *right congruence* if it is compatible with right multiplication, and a *congruence* if it is both a left congruence and a right congruence. Given a semigroup S and a congruence ρ on S, we may form the *quotient semigroup* S/ρ of S by ρ. This consists of the equivalence classes of ρ, where the product of $A, B \in S/\rho$ is the equivalence class that contains ab, where $a \in A$ and $b \in B$ are arbitrary.

Let X be any set. The *free semigroup* over X, denoted X^+, is the semigroup consisting of all non-empty sequences over X with the operation of concatenation. A *semigroup presentation* is a pair $\langle X \mid R\rangle$, where X is a set of *generators*, and R is a subset of $X^+ \times X^+$, called *relations*. The semigroup defined by a presentation $\langle X \mid R\rangle$ is the quotient X^+/ρ, where ρ is the least congruence on X^+ (with respect to containment) that contains R. When X and R are both finite, the semigroup defined by $\langle X \mid R\rangle$ is called a *finitely presented semigroup*.

We name many kinds of semigroups and semigroup elements; see [104] for a comprehensive description of special kinds of semigroup. An element x of a semigroup is *idempotent*, and is called *an idempotent*, if $x^2 = x$ (that is, *idempotent* is both an adjective and a noun). Every finite semigroup contains an idempotent, but the natural numbers with addition is an example of an infinite semigroup that contains no idempotents. The set of idempotents contained in a subset X of a semigroup is denoted by $E(X)$. A semigroup in which every element is idempotent is called a *band*.

A *multiplicative zero* of a semigroup S, often just called a *zero* of S, is an element $0 \in S$ such that $0x = 0 = x0$ for all $x \in S$. We use the notation 0_S to denote the multiplicative zero of the semigroup S, provided that it has one. A *zero semigroup* is a semigroup S with zero such that $xy = 0_S$ for all $x, y \in S$. A *left identity* of a semigroup S is an element $e \in S$ such that $ex = x$ for all $x \in S$, and a *right identity* of S is defined analogously. A semigroup in which every element is a left identity is called a *right zero semigroup*, and a semigroup in which every element is a right identity is called a *left zero semigroup*. An *identity* is an element that is both a left identity and a right identity. Note that multiplicative zeroes, and left and right identities, are idempotent, and that a semigroup contains at most one zero and at most one identity. Let S be a semigroup and let $x \in S$. If $s \in S$ satisfies $sx = x$, then s is a *relative left identity for x*, and if $xs = x$, then s is a *relative right identity for* x. Such an element need not be idempotent.

If S is a semigroup, then we denote by S^0 the semigroup $S \cup \{0\}$, where 0 is a new element not contained in S, and the multiplication on S is extended to a multiplication on S^0 by defining $0x = 0 = x0$ for all $x \in S^0$. In other words, 0 is a multiplicative zero for S^0. We call S^0 the *semigroup formed by adjoining a zero to* S.

A semigroup that contains an identity is called a *monoid*. To any semigroup S we associate the monoid S^1. If S is itself a monoid, then we define $S^1 = S$. On the other hand, if S does not contain an identity, then we adjoin a new element 1_S, and we extend the multiplication on S to a multiplication on $S^1 = S \cup \{1_S\}$ by defining $1_S x = x = x 1_S$ for all $x \in S^1$. For any semigroup S, we write 1_S to denote the identity of the monoid S^1. Let S be a monoid. An element $u \in S$ is a *unit* of S if there exists some (necessarily unique) *inverse* element $u^{-1} \in S$ such that $uu^{-1} = 1_S = u^{-1}u$. A monoid in which every element is a unit is called a *group*. The subset of units in a monoid is a group, and is called the *group of units* of the monoid.

1.3.1 Subsemigroups, ideals, and generation

Let S be a semigroup, and let A and B be subsets of S. We denote the *set product* of A and B by $AB = \{ab : a \in A \text{ and } b \in B\}$. If $x \in S$, then xA denotes $\{x\}A$ and Ax denotes $A\{x\}$. This notation can be extended to the product of an arbitrary finite number of sets and elements in an obvious way. For example, if $A, B \subseteq S$ and $x \in S$, then AxB denotes the set $\{axb : a \in A,\, b \in B\}$. In this thesis, we say that a semigroup is *surjective* if $S = S^2$.

A *subsemigroup* of a semigroup is a subset that is closed under the same multiplication, i.e. a subset T such that $T^2 \subseteq T$. If U and V are subsets of a semigroup, then we write $U \leq V$ to denote that U is a subsemigroup of V. A *submonoid* of a semigroup is a subsemigroup that is itself a monoid, and a *subgroup* of a semigroup is any subsemigroup that is itself a group.

Let X be any subset of a semigroup S. The *subsemigroup of* S *generated by* X, denoted $\langle X \rangle$, is the intersection of all subsemigroups of S that contain X. Equivalently, $\langle X \rangle$ is the least subsemigroup of S, with respect to the partial order defined by containment, that contains X, and it consists of all finite products of elements in X. The set X is said to be a *generating set for* $\langle X \rangle$. If $X_1, X_2, \ldots, X_m$ is any collection of subsets of S, and $x_1, x_2, \ldots, x_n$ is any collection of elements of S, then we use the notation $\langle X_1, X_2, \ldots, X_m, x_1, x_2, \ldots, x_n \rangle$, or some rearrangement of this, for the subsemigroup of S generated by the union

$$X_1 \cup X_2 \cup \cdots \cup X_m \cup \{x_1, x_2, \ldots, x_n\}.$$

The generating sets of a semigroup are partially ordered by containment, and the cardinalities of the generating sets are also naturally ordered. A generating set is minimal with respect to containment if it does not properly contain another generating set, and minimal with respect to cardinality if its cardinality is least possible. The *rank* of a semigroup S, denoted $\operatorname{rank}(S)$, is the least cardinality of a generating set for S, and a semigroup S is *finitely generated* if it has a finite generating set, or equivalently, if $\operatorname{rank}(S) \in \mathbb{N}_0$. A semigroup is *monogenic* if it has rank 1.

Let S be a semigroup and let $X \subseteq S$. We write $F(X)$ for the subsemigroup $\langle E(X) \rangle$ generated by the idempotents in X. In particular, $F(S)$ is the *idempotent generated subsemigroup* of S.

Let I be a subset of a semigroup S. Then I is a *left ideal* if $SI \subseteq I$, a *right ideal* if $IS \subseteq I$, and a *two-sided ideal*, more usually called simply an *ideal*, if it is both a left ideal and a right ideal. Certainly any kind of ideal is a subsemigroup. Let $x \in S$ be arbitrary. The *principal left ideal generated by* x is the left ideal $S^1x = Sx \cup \{x\}$, and the *principal right ideal generated by* x is the right ideal $xS^1 = xS \cup \{x\}$. The *principal ideal generated by* x is the ideal S^1xS^1. If $X \subseteq S$, then *the ideal of* S *generated by* X is the intersection of all ideals of S that contain X; it is the union of the principal ideals $\{S^1xS^1 : x \in X\}$. The left and right ideals of a semigroup generated by a subset are defined analogously.

A subsemigroup or ideal of a semigroup is *proper* if it is not equal to the semigroup.

At various points throughout this thesis, we are concerned with describing generating sets for certain semigroups, and finding upper bounds on their ranks. The following lemma, and its subsequent corollary, will be useful in some of these cases.

Lemma 1.5. *Let S be a semigroup and let $X \subseteq S^0$. The following hold:*

(i) *X generates S^0 if and only if $X = X' \cup \{0\}$, for some generating set X' of S.*

Moreover, if X generates S^0, then

(ii) *with respect to cardinality, X generates S^0 minimally if and only if $X \setminus \{0\}$ generates S minimally; and*

(iii) *with respect to containment, X generates S^0 minimally if and only if $X \setminus \{0\}$ generates S minimally.*

Proof. Suppose that X generates S^0, and let $x \in S$. Then $x \in S^0 = \langle X\rangle$, and since $x \neq 0$, it follows that $x \in \langle X \setminus \{0\}\rangle$. Therefore $S = \langle X \setminus \{0\}\rangle$, and certainly $0 \in X$, since a product of non-zero elements of S^0 is contained in the subsemigroup S. Conversely, if $S = \langle X'\rangle$, then

$$S \lneq \langle X' \cup \{0\}\rangle \leq S^0 = S \cup \{0\},$$

and so $X' \cup \{0\}$ generates S^0. Therefore (i) holds.

By (i), the generating sets of S are in an inclusion-preserving one-to-one correspondence with the generating sets of S^0, via the addition or removal of 0. Therefore, it is clear that the minimality of generating sets is preserved under this correspondence, and so (ii) and (iii) hold. □

Corollary 1.6. *Let S be any semigroup. Then* $\operatorname{rank}(S^0) = \operatorname{rank}(S) + 1$.

1.3.2 Rees 0-matrix semigroups and completely 0-simple semigroups

A semigroup with no proper ideals is called *simple*, and a semigroup S is *completely simple* if it is simple and there exists an idempotent $e \in E(S)$ such that $ef = fe = f \Rightarrow e = f$ for all $f \in E(S)$. A non-trivial semigroup with zero S is *0-simple* if it is not a zero semigroup of order 2, and its only ideals are $\{0_S\}$ and S. A semigroup is *completely 0-simple* if it is 0-simple, and there exists a non-zero idempotent $e \in S$ such that, for all $f \in E(S)$,

$$ef = fe = f \neq 0 \Rightarrow e = f.$$

Every finite simple semigroup is completely simple, and every finite 0-simple semigroup is completely 0-simple, but in general these concepts are distinct.

Completely simple and completely 0-simple semigroups can be constructed as Rees matrix and Rees 0-matrix semigroups, respectively. Let T be any semigroup, let I and Λ be non-empty sets, and let $P = (p_{\lambda,i})_{\lambda\in\Lambda, i\in I}$ be a $\Lambda \times I$ matrix with entries in T. Then the $I \times \Lambda$ *Rees matrix semigroup over T with matrix P* is the semigroup $\mathscr{M}[T;\, I,\, \Lambda;\, P] = (I \times T \times \Lambda)$, with multiplication defined by

$$(i,\, t,\, \lambda)\,(j,\, u,\, \mu) = (i,\, tp_{\lambda,j}u,\, \mu) \quad \text{for all} \quad (i,\, t,\, \lambda),\, (j,\, u,\, \mu) \in I \times T \times \Lambda. \tag{1.7}$$

If we instead permit the matrix P to contain entries of T^0 (i.e. P can also contain 0 as an entry), then the $I \times \Lambda$ *Rees 0-matrix semigroup over T with matrix P* is the semigroup $\mathscr{M}[T;\, I,\, \Lambda;\, P] = (I \times T \times \Lambda) \cup \{0\}$, with multiplication defined by

$$\begin{aligned} & (i,\, t,\, \lambda)\,(j,\, u,\, \mu) = \begin{cases} 0 & \text{if } p_{\lambda,j} = 0, \\ (i,\, tp_{\lambda,j}u,\, \mu) & \text{if } p_{\lambda,j} \neq 0, \end{cases} \\ \text{and} \qquad & (i,\, t,\, \lambda)\,0 = 0\,(i,\, t,\, \lambda) = 0^2 = 0, \end{aligned} \tag{1.8}$$

for all $(i,\, t,\, \lambda), (j,\, u,\, \mu) \in I \times T \times \Lambda$. Clearly, Rees matrix and Rees 0-matrix semigroups are closely related; indeed, one may regard a Rees matrix semigroup $\mathscr{M}[T;\, I,\, \Lambda;\, P]$ as the subsemigroup $(I \times T \times \Lambda)$ of the Rees 0-matrix semigroup $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$. This similarity allows many results concerning Rees matrix semigroups to be deduced from corresponding results for Rees 0-matrix semigroups.

A Rees matrix or 0-matrix semigroup $\mathscr{M}[T;\, I,\, \Lambda;\, P]$ or $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ is finite if and only if the underlying semigroup T and the sets I and Λ are finite. Note that the nature of the elements of I and Λ is not relevant to the multiplication of the semigroup; I and Λ merely act as indexing sets. Therefore we may assume without loss of generality that $I \cap \Lambda = \varnothing$.

Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. We define the *Graham-Houghton graph of S*, introduced in [62, 75], to be the bipartite graph $\big(I \cup \Lambda,\, \big\{\{i,\, \lambda\} : p_{\lambda,i} \neq 0\big\}\big)$.

The original interest in Rees matrix and 0-matrix semigroups stems from their connection with completely simple and completely 0-simple semigroups. By the Rees-Suschkewitsch Theorem [76, Theorem 3.3.1], a semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup $\mathscr{M}[G;\, I,\, \Lambda;\, P]$, where G is a group, and by the Rees Theorem [76, Theorem 3.2.3], a semigroup is completely 0-simple if and only if it is isomorphic to a Rees 0-matrix semigroup $\mathscr{M}^0[G;\, I,\, \Lambda;\, P]$, where G is a group and P contains a non-zero entry in each row and each column. In this thesis, when we define $\mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ to be a *regular Rees 0-matrix semigroup over a group*, we require that G be a group and that P be a $\Lambda \times I$ matrix over G^0 that contains at least one non-zero entry in each of its rows and columns. We study Rees 0-matrix semigroups over arbitrary semigroups in Chapter 3.

1.3.3 Green's relations and regularity

One of the most fundamental ways of understanding a semigroup is by analysing its *Green's relations* [66]. These are five equivalences that can be defined on any semigroup. Let S be a semigroup. Green's $\mathscr{L}$-relation on S is defined by $x \mathrel{\mathscr{L}} y$ if and only if $S^1x = S^1y$ for all $x, y \in S$. Green's $\mathscr{L}$-relation is a right congruence. Green's $\mathscr{R}$-relation is the dual of $\mathscr{L}$, and is a left congruence. Green's $\mathscr{H}$-relation is the intersection $\mathscr{L} \cap \mathscr{R}$, and Green's $\mathscr{D}$-relation is the composition $\mathscr{L} \circ \mathscr{R}$, which is equal to the composition $\mathscr{R} \circ \mathscr{L}$ [76, Proposition 2.1.3]. Green's $\mathscr{J}$-relation is defined by $x \mathrel{\mathscr{J}} y$ if and only if $S^1xS^1 = S^1yS^1$ for all $x, y \in S$. On many kinds of semigroups, including finite semigroups, Green's $\mathscr{D}$- and $\mathscr{J}$-relations are equal; see for instance [110, Corollary A.2.5] and [76, Proposition 2.1.4]. Indeed, the Green's relations on any group G are all equal to $G \times G$, and all of the Green's relations on any commutative semigroup coincide. Figure 1.9 shows the containment of the Green's relations on a semigroup.

The following lemma, concerning finite semigroups, is used repeatedly in this thesis.

Lemma 1.10 ([110, Theorem A.2.4])**.** *Let S be a finite semigroup and let $x, y \in S^1$. Then*

$$x \mathrel{\mathscr{J}} yx \text{ if and only if } x \mathrel{\mathscr{L}} yx, \qquad \text{and} \qquad x \mathrel{\mathscr{J}} xy \text{ if and only if } x \mathrel{\mathscr{R}} xy.$$

In other words, every finite semigroup is stable*, in the sense of* [110, Definition A.2.1].

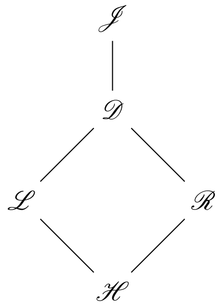

Figure 1.9: A Hasse diagram displaying the containment of the Green's relations on any semigroup. The inclusions shown here are not necessarily strict: for example, $\mathscr{D} = \mathscr{J}$ on any finite semigroup, and all of the Green's relations on any commutative semigroup are equal.

Certainly $\mathscr{L} \subseteq \mathscr{J}$ for any semigroup; by Lemma 1.10, in a finite semigroup, when a left multiple of an element is $\mathscr{J}$-related to that element, then it is also $\mathscr{L}$-related. The analogous statement holds for right multiples and Green's $\mathscr{R}$-relation.

Let S be a semigroup and let $\mathscr{K} \in \{\mathscr{H}, \mathscr{L}, \mathscr{R}, \mathscr{D}, \mathscr{J}\}$ be one of Green's relations on S. When it is necessary to emphasise that $\mathscr{K}$ is defined the semigroup S, we write $\mathscr{K}^S$ instead of $\mathscr{K}$. The semigroup S is said to be $\mathscr{K}$-trivial if $\mathscr{K}$ is the equality relation on S, i.e. if $\mathscr{K} = \{(x, x) : x \in S\}$. Let $x, y \in S$. We say that x and y are *$\mathscr{K}$-related* if $x \mathrel{\mathscr{K}} y$. The *$\mathscr{K}$-class* of x is the equivalence class of x in $\mathscr{K}$. This is denoted by K_x, or occasionally by K_x^S in order to emphasise the semigroup S on which $\mathscr{K}$ is defined. We refer to equivalence classes of Green's relations as *Green's classes*.

The following result, known as Green's Lemma, is fundamental to the study of semigroups.

Lemma 1.11 (Green's Lemma [76, Lemmas 2.2.1, 2.2.2])**.** *Let S be a semigroup, and $x, y \in S$.*

(i) *If $x \mathrel{\mathscr{L}} y$, and $u, v \in S^1$ satisfy $ux = y$ and $vy = x$, then the functions $\lambda_u : R_x \longrightarrow S$ and $\lambda_v : R_y \longrightarrow S$, defined by $a\lambda_u = ua$ and $b\lambda_v = vb$, give mutually inverse bijections from R_x to R_y, and R_y to R_x, respectively, that preserve Green's $\mathscr{L}$-relation.*

(ii) *If $x \mathrel{\mathscr{R}} y$, and $s, t \in S^1$ satisfy $xs = y$ and $yt = x$, then the functions $\rho_s : L_x \longrightarrow S$ and $\rho_t : L_y \longrightarrow S$, defined by $a\rho_s = as$ and $b\rho_t = bt$, give mutually inverse bijections from L_x to L_y, and L_y to L_x, respectively, that preserve Green's $\mathscr{R}$-relation.*

Let S be a semigroup. We define partial orders on the $\mathscr{L}$-, $\mathscr{R}$-, or $\mathscr{J}$-classes of S, in terms of containment of the corresponding left, right, or two-sided ideals. More specifically, for any $x, y \in S$, $L_x \leq L_y$ if and only if $S^1x \subseteq S^1y$; $R_x \leq R_y$ if and only if $xS^1 \subseteq yS^1$; and $J_x \leq J_y$ if and only if $S^1xS^1 \subseteq S^1yS^1$. The group of units of a monoid is an $\mathscr{H}$-class, and in a finite monoid the group of units is also the unique maximal $\mathscr{J}$-class in the partial order of $\mathscr{J}$-classes of the monoid. We require the following lemma, which relates the partial orders of Green's classes of a semigroup with multiplication.

Lemma 1.12. *Let S be a semigroup, and let $x_1, x_2, \ldots, x_n \in S$ for some $n \in \mathbb{N}$.*

(i) *If $i \in \{1, \ldots, n\}$, then $L_{x_1 \cdots x_n} \leq L_{x_i \cdots x_n}$.*

(ii) *If $i \in \{1, \ldots, n\}$, then $R_{x_1 \cdots x_n} \leq R_{x_1 \cdots x_i}$.*

(iii) *If $i, j \in \{1, \ldots, n\}$ and $i \leq j$, then $J_{x_1 \cdots x_n} \leq J_{x_i \cdots x_j}$.*

Proof. The statements (i) and (ii) are dual, so we prove only (i) and (iii).

(i). Let $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned} S^1(x_1 \cdots x_n) &= S^1(x_1 \cdots x_{i-1})(x_i \cdots x_n) \\ &= \left(S^1(x_1 \cdots x_{i-1})\right)(x_i \cdots x_n) \\ &\subseteq S^1(x_i \cdots x_n), \end{aligned}$$

and so $L_{x_1 \cdots x_n} \leq L_{x_i \cdots x_n}$ by definition of the partial order of $\mathscr{L}$-classes of S.

(iii). Let $i, j \in \{1, \ldots, n\}$ with $i \leq j$. Then

$$\begin{aligned} S^1(x_1 \cdots x_n)S^1 &= S^1(x_1 \cdots x_{i-1})(x_i \cdots x_j)(x_{j+1} \cdots x_n)S^1 \\ &= \left(S^1(x_1 \cdots x_{i-1})\right)(x_i \cdots x_j)\left((x_{j+1} \cdots x_n)S^1\right) \\ &\subseteq S^1(x_i \cdots x_j)S^1, \end{aligned}$$

and the result follows by the definition of the partial order of $\mathscr{J}$-classes of S. □

A notion intimately connected to Green's relations is that of regularity. An element x in a semigroup S is *regular* if there exists an element $y \in S$ such that $xyx = x$. If additionally $yxy = y$, then x and y are mutually *inverse*. A semigroup that contains only regular elements is called a *regular semigroup*. Every regular element has an inverse [76, Section 2.3]; a regular element is $\mathscr{D}$-related to all of its inverses. Any $\mathscr{D}$-class, and hence any $\mathscr{L}$-, $\mathscr{R}$-, or $\mathscr{H}$-class, of a semigroup S contains either no regular elements, and is called *non-regular*, or contains only regular elements, and is said to be *regular* [76, Proposition 2.3.1].

If J is a $\mathscr{J}$-class of a semigroup, then the *principal factor* of J is the semigroup $J^* = J \cup \{0\}$, where $0 \notin J$ and the multiplication $*$ on J^* is given by

$$x * y = \begin{cases} xy & \text{if } xy \in J, \\ 0 & \text{if } xy \notin J, \end{cases} \qquad \text{and} \qquad 0 * x = x * 0 = 0 * 0 = 0, \tag{1.13}$$

for all $x, y \in J$. By [76, Theorem 3.1.6], the principal factor of a $\mathscr{J}$-class either is a zero semigroup, or is 0-simple. In particular, if S is a finite semigroup and J is a $\mathscr{J}$-class of S, then either J is a non-regular $\mathscr{D}$-class and J^* is a zero semigroup, or J is a regular $\mathscr{D}$-class and J^* is isomorphic to a regular Rees 0-matrix semigroup over a group.

A regular $\mathscr{L}$-class contains some idempotents [76, Propositions 2.3.1], which are relative right identities for any of its elements, and similarly a regular $\mathscr{R}$-class contains some idempotents, which are relative left identities for the element of the $\mathscr{R}$-class [76, Propositions 2.3.2 & 2.3.3]. An $\mathscr{H}$-class that contains an idempotent is a subgroup [76, Corollary 2.2.6]; otherwise, an $\mathscr{H}$-class H satisfies $H^2 \cap H = \varnothing$ [76, Theorem 2.2.5]. In semigroup theory, a *maximal subgroup* of a semigroup is an $\mathscr{H}$-class of the semigroup that is a subgroup. However, this is inconsistent with the notion of a *maximal subgroup* from group theory: in this context, a maximal subgroup of a group is a proper subgroup that is contained in no other proper subgroup of the group. In this thesis, we exclusively use the term *maximal subgroup* in the group-theoretic sense; we call an $\mathscr{H}$-class of a semigroup that is a subgroup a *group $\mathscr{H}$-class*.

We use the following lemma repeatedly in this thesis.

Lemma 1.14. *Let T be a finite subsemigroup of a semigroup S that has non-empty intersection with each $\mathscr{H}$-class of S. Then T contains every idempotent of S.*

Proof. Let $e \in E(S)$, and let $x \in T \cap H_e^S$ be arbitrary. Since H_e^S is a subgroup of S and T is finite, it follows that x has finite order in H_e^S, or in other words, $e = x^k \in T$ for some $k \in \mathbb{N}$. □

Multiplication within a $\mathscr{D}$-class is particularly systematic, as shown in the following results.

Lemma 1.15 ([76, Proposition 2.3.7])**.** *Let S be a semigroup and let $(x, y) \in \mathscr{D}^S$. Then*

$$xy \in R_x \cap L_y \text{ if and only if } L_x \cap R_y \text{ is a group.}$$

Corollary 1.16. *Let x and y be elements of a finite semigroup.*

(i) *Suppose that $x \mathscr{R} y$. Then $xy \in R_x = R_y$ if and only if H_x is a group.*

(ii) *Suppose that $x \mathscr{L} y$. Then $xy \in L_x = L_y$ if and only if H_y is a group.*

Proof. If $x \mathscr{R} y$, then $R_x = R_y$, and so Lemma 1.15 becomes $xy \in R_y \cap L_y$ if and only if H_x is a group. Thus, it remains to prove that $xy \in L_y$ whenever $xy \in R_y$. If $xy \in R_y$, then $xy \mathscr{J} y$, and so $xy \mathscr{L} y$ by Lemma 1.10. This proves (i); the proof of (ii) is dual. □

Whenever an element $x \in S$ has a unique inverse y, we define $x^{-1} = y$ to be the *inverse* of x. This notation is consistent with the notation u^{-1} introduced earlier for a unit u of a monoid. An *inverse semigroup* is a semigroup in which each element has a unique inverse. In particular, any group is an inverse semigroup. There are many other characterisations of inverse semigroups; for instance, a semigroup is inverse if and only if it is regular and its idempotents commute; which occurs if and only if each of its $\mathscr{L}$-classes and each of its $\mathscr{R}$-classes contains exactly one idempotent [76, Theorem 5.1.1]. See [85] for a thorough treatment of inverse semigroups.

A *regular $*$-semigroup*, as introduced in [105], is a semigroup S with an operation $* : S \longrightarrow S$ that satisfies $(x^*)^* = x$, $(xy)^* = y^*x^*$, and $x = xx^*x$ for all $x, y \in S$. In particular, the $*$ operation is an anti-isomorphism. A regular $*$-monoid is simply a regular $*$-semigroup that is a monoid. In Chapter 5, we describe the maximal subsemigroups of several families of finite regular $*$-monoids. Any regular $*$-semigroup is a regular semigroup, since x^* is an inverse of the element x. An idempotent x of a regular $*$-semigroup is called a *projection* if $x^* = x$. In a regular $*$-semigroup, the $\mathscr{R}$-class of an element x contains the unique projection xx^*, and the $\mathscr{L}$-class of x contains the unique projection x^*x. Since a regular $*$-semigroup is anti-isomorphic to itself via the $*$ operation, this gives a correspondence between left multiplication and right multiplication. In particular, if S is a regular $*$-semigroup, and $x, y \in S$, then $x \mathscr{L} y$ if and only if $x^* \mathscr{R} y^*$. Since $x \mathscr{D} x^*$, the number of $\mathscr{L}$-classes is equal to the number of $\mathscr{R}$-classes in any $\mathscr{D}$-class of a regular $*$-semigroup. Any inverse semigroup S can be thought of as a regular $*$-semigroup in which every idempotent is a projection, by defining $x^* = x^{-1}$ for all $x \in S$. Conversely, any regular $*$-semigroup in which every idempotent is a projection is inverse.

1.3.4 Group theory and group actions

A group is a monoid consisting of units. In this thesis we require several group-theoretic notions. If G is a group, and V is a subgroup of G, and $g \in G$, then *the right coset of V in G defined by g* is the set Vg. Two right cosets Vg and Vh of V in G are equal if and only if $gh^{-1} \in V$. The right cosets of V in G, $\{Vg : g \in G\}$, form a partition of G, and the *index* of V in G, denoted $[G : V]$, is the number of cosets of V in G, if there are finitely many, or ∞.

Another important concept in group theory is that of *conjugation*. Two subgroups U and V of a group G are *conjugate* if there exists $g \in G$ such that $g^{-1}Ug = V$. Conjugate subgroups are isomorphic, and the collection of subgroups of a group is partitioned by its conjugacy classes. A *normal subgroup* of a group is a subgroup that is closed under conjugation, i.e. a subgroup whose conjugacy class is trivial. In any group G, the group G itself and the trivial subgroup $\{1_G\}$ are normal subgroups. The *normalizer* of a subgroup V in a group G, denoted $N_G(V)$, is the least normal subgroup of G, with respect to containment, that contains V.

If G is a group and X is a set, then a *right action* of G on X is a function $\psi : X \times G \longrightarrow X$ where $(x,\, 1_G)\,\psi = x$ and $((x,\, g)\,\psi,\, h)\,\psi = (x,\, gh)\,\psi$ for all $x \in X$ and $g, h \in G$. We usually write $x \cdot g$ instead of $(x,\, g)\,\psi$, when the right action ψ is clear from the context. The relation $\sim$ on X defined by $x \sim y$ if and only if $x \cdot g = y$ for some $g \in G$ is an equivalence, since $x \cdot 1_G = x$, $x \cdot g = y$ if and only if $y \cdot g^{-1} = x$, and $x \cdot g = y$ and $y \cdot h = z$ implies that $x \cdot gh = z$, for all $x, y, z \in X$ and $g, h \in G$. The equivalence classes of this equivalence are called *orbits*, and so a right action of a group on a set partitions that set into orbits. When a right action has only one orbit, we say that the action is *transitive*, and we say that the group acts *transitively.* A left action of a group on a set, and its orbits, are defined analogously.

1.3.5 Semigroups of partial transformations

Composition of relations is associative, and therefore, so is composition of partial functions. It follows that we may create semigroups from relations or partial functions on a set. Let $n \in \mathbb{N}$. We define $\mathcal{PT}_n$, the *partial transformation monoid of degree* n, to be the monoid consisting of all partial transformations of degree n, with composition of partial functions. We also define the following submonoids of $\mathcal{PT}_n$:

- $\mathcal{T}_n = \{\alpha \in \mathcal{PT}_n \,:\, \mathrm{dom}(\alpha) = \{1,\, \ldots,\, n\}\}$, the *full transformation monoid of degree* n;
- $\mathcal{I}_n = \{\alpha \in \mathcal{PT}_n \,:\, |\mathrm{im}(\alpha)| = |\mathrm{dom}(\alpha)|\}$, the *symmetric inverse monoid of degree* n; and
- $\mathcal{S}_n = \{\alpha \in \mathcal{PT}_n \,:\, \mathrm{im}(\alpha) = \{1,\, \ldots,\, n\}\}$, the *symmetric group of degree* n.

In other words, $\mathcal{T}_n$ consists of all transformations of degree n, $\mathcal{I}_n$ consists of all partial permutations of degree n, and $\mathcal{S}_n$ consists of all permutations of degree n. Note that the symmetric group $\mathcal{S}_n$ is the group of units of $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$. We define id_n, the *identity permutation of degree* n, to be the identity of each of these monoids; id_n fixes each point $i \in \{1,\, \ldots,\, n\}$. Full transformation monoids, symmetric inverse monoids, and symmetric groups can also be defined on arbitrary sets, including on infinite sets, however these will not be required in this thesis.

By Cayley's theorem, every group is isomorphic to a subgroup of some symmetric group; moreover, every finite group G is isomorphic to a subgroup of $\mathcal{S}_{|G|}$. Thus, in a sense, the study of permutation groups is the study of all groups. Semigroups of partial transformations, especially semigroups of transformations, are similarly foundational in semigroup theory. By an analogue of Cayley's theorem for groups, every semigroup is isomorphic to a subsemigroup of some full transformation monoid [76, Theorem 1.1.2]; indeed, any finite semigroup S is isomorphic to a subsemigroup of $\mathcal{T}_{|S|+1}$. Symmetric inverse monoids play the analogous role for inverse semigroups, by the Vagner-Preston theorem [76, Theorem 5.1.7]. In particular, any finite semigroup can be realised by transformations of degree n, for some $n \in \mathbb{N}$, and similarly any finite inverse semigroup can be realised by partial permutations of some degree m. See [57] for a detailed study of finite semigroups of partial transformations.

1.4 Computational semigroup theory

We end this chapter by introducing several additional concepts from computational semigroup theory, in order to provide the context in which the mathematical results of this thesis are presented. To reiterate, computational semigroup theory is both the investigation of semigroups with the help of computers, and the development of the tools that facilitate this study. In this section, we informally discuss the notions of algorithms, which we use to define and communicate techniques in computation, and computational complexity, which we use to classify the efficiency of an algorithm. We also briefly explain some of the main paradigms used for computing with semigroups, their aims, and their relative advantages and disadvantages.

1.4.1 Algorithms and complexity

In a casual sense, an *algorithm* is a method, or procedure, that describes how to solve any instance of a particular collection of problems in a finite number of steps. In other words, an algorithm describes a solution to a computational problem. See, for example, Algorithm 1.17, which uses *pseudocode* to define an algorithm for determining whether a given number is contained in a given list of numbers. A computer program is a specific implementation of an algorithm in some programming language.

A Turing machine [126] is one of the earliest abstract models of computation. Although the definition of a Turing machine is not important for the purposes of this thesis, this abstraction is crucial to the foundations of computer science. According to the widely-accepted *Church-Turing thesis*, Turing machines are 'universal' for computation: a problem that can be solved by some computer can also be solved by a Turing machine. Viewed in this context, an algorithm is a specification for defining a Turing machine that solves a particular problem, in such a way that given any instance of the problem, the Turing machine will terminate with the correct solution after a finite number of steps.

Algorithm 1.17 A simple algorithm for testing membership in a list of natural numbers.

Input: A finite list L consisting of natural numbers, and a natural number $x \in \mathbb{N}$.
Output: TRUE if x is an entry of L; else FALSE.

```
for n ∈ L do
    if n = x then
        return TRUE.
return FALSE.
```

For some computational problems, numerous diverse algorithms have been developed to solve the problem. For instance, bubble sort, insertion sort, quick sort, and merge sort are well-known and significantly different algorithms for sorting a list into ascending order.

The execution of a computer program requires time and space (the memory used to store intermediate results). The resources of any given computer are finite, and different algorithms typically require differing amounts of these resources when implemented and executed: the amount of time required by an algorithm is called its *time complexity*, and the amount of space required is the *space complexity*. We therefore require strategies for comparing algorithms and analysing their resource requirements. This helps us to choose the most suitable algorithm for a particular computation, and to determine whether a new algorithm is better than an existing one. In this thesis, we focus on the time complexity of algorithms. See [59, 128] for a much more thorough discussion of algorithms and their complexities than the one that follows.

An obvious way to study the performance of an implementation of an algorithm is to execute the algorithm on a carefully chosen range on inputs, and to measure the resources required to produce the output. There are good reasons to study algorithms in this way: it is easy to do, and it often demonstrates the real-life behaviour that a user might find in practice. Indeed, this is the approach we take in Chapter 4 for analysing the performance of algorithms for computing maximal subsemigroups of finite semigroups. Furthermore, by executing the same examples with competing algorithms, it is possible to fairly compare the performance of these algorithms on the same inputs.

However, there are problems with this approach. Perhaps, by some fluke, the algorithms demonstrate atypical performance on the range of examples that we choose to examine. For many computational tasks, there are an infinite number of possible inputs, and so it is impossible to test them all. Additionally, the specific implementation of the algorithm being tested may be poor, leading to unsatisfactory performance that obscures the potential of the algorithm.

We therefore desire a more systematic and mathematical way of measuring and comparing the resource requirements of algorithms. An algorithm defines a sequence of steps, each of which has some basic time cost. Therefore, if we can calculate the number of steps that are required for a given input, then we can use the known costs to describe the time requirements of the algorithm in terms of its input. However, this is hard to do exactly, and so we generally attempt to give upper bounds on the resource requirements of an algorithm, given the input.

We use *big O notation* [128, Chapter 1] to relate the resource requirements of an algorithm to its inputs, and thereby to describe its time complexity. Let X be a set of elements with some notion of size, and let f and g be functions $X \longrightarrow \mathbb{R}$. We say that the function f is $O(g)$ if there exists a positive constant c such that $0 \leq (x)f \leq c \cdot (x)g$ for all sufficiently large $x \in X$. If X is the set of possible inputs to some algorithm, and $g : X \longrightarrow \mathbb{R}$ is a function, then we say that the time complexity of the algorithm is $O(g)$ if we can show that the theoretical function $f : X \longrightarrow \mathbb{R}$, describing the time required to run the algorithm on the input X, on some computer and in some unit of measurement, is $O(g)$. Note that we may ignore the details of the specific computer used and the specific units of time, since big O notation disregards constant factors. See Example 1.18 for an analysis of the time complexity of Algorithm 1.17.

Example 1.18. Let L be a finite list of natural numbers, and let $x \in \mathbb{N}$. Suppose that it takes some constant length of time c (in some units) to test the equality of two natural numbers. Consider using the algorithm described in Algorithm 1.17 to test whether x is an entry of L. In the best case, the first element of L is equal to x, and the algorithm terminates with the correct answer after only one comparison. However, if x does not appear until the final entry of L, or if x is not an entry of L, then the algorithm does not determine this until every entry of L has been compared against x. It follows that, in the worst, case Algorithm 1.17 requires $c \cdot |L|$ units of time in order to produce its output. We can therefore say that the time complexity of Algorithm 1.17 is $O(|L|)$, or $O(n)$, where $n = |L|$.

Analysing time complexity in this way is often difficult, but it provides a useful way of comparing the requirements of different algorithms, and of understanding how the requirements of an algorithm grow as the input grows. If the time complexity of one algorithm is $O(n)$, and the time complexity of a competing algorithm is $O(1)$, then roughly speaking, we can expect that doubling the size of the input will double the length of time taken to compute with the first algorithm, but we can expect that it will not affect the time taken to compute with the second algorithm. We may therefore consider the second algorithm to be more time-efficient. More concretely, sorting a finite list L using the merge sort algorithm has time complexity $O(|L| \log |L|)$, whereas sorting L using the insertion sort algorithm has time complexity $O\big(|L|^2\big)$, which suggests that the merge sort algorithm is more time-efficient.

In this thesis, we do not formally describe the algorithms that we present with big O notation, however we use this notion of complexity in order to provide some background information and context to the problems that we approach. Algorithms in computational semigroup theory tend to have high time complexity, at least in comparison with algorithms in computational group theory. Despite this, we still find that it is possible to execute algorithms in computational semigroup theory on a wide range of interesting and useful examples.

1.4.2 Data structures for semigroups

In order to compute with a given semigroup, it is first necessary to store the definition of the semigroup on a computer in some way. In other words, we must choose a structure for defining the semigroup to a computer. A semigroup is a set with an associative binary operation, and so to specify a semigroup, it is necessary to unambiguously and clearly define its set of elements and a way of calculating the product of any two elements in the set.

One of the simplest ways of specifying a finite semigroup to a computer is by a *multiplication table*. If S is a finite semigroup of order n, then the multiplication table of S is an $n \times n$ array whose rows and columns are labelled by the elements of S; if $x, y \in S$, then the entry in the row labelled by x and the column labelled by y is the product of x and y in S. Thus the multiplication table encodes both the elements and the operation of the semigroup. See Table 1.19 for an example of a multiplication table that defines a semigroup with four elements.

$\circ$	x	y	z	t
x	x	y	z	t
y	y	x	z	t
z	z	t	z	t
t	t	z	z	t

Table 1.19: The multiplication table of a semigroup isomorphic to the full transformation monoid of degree 2 (see Section 1.3.5), which contains four elements, including an identity.

Multiplication tables provide a useful data structure in many respects. Given sufficient space, any finite semigroup can be given by a multiplication table. Furthermore, the multiplication table of a semigroup contains the elements of the semigroup and complete information about their products. Therefore, producing the set of elements and multiplying elements is simply a case of looking up the information in whatever data structure it is stored. The Smallsemi [32] package for GAP [58], which is a library containing every semigroup of order 8 or less, stores its semigroups via multiplication tables.

The most significant downside of giving a semigroup by its multiplication table is that storing a multiplication table requires a relatively large amount of memory, which can easily exceed the capabilities of a computer. Let S be a finite semigroup with n elements. In order to be able to distinguish between the elements of S on a computer, it is necessary to use at least k bits to identify each element, where $2^k \geq n$. In other words, we must choose $k \geq \log n$. It follows that storing the multiplication table of S on a computer, without using compression techniques, requires at least $n^2 \log n$ bits. At present, the memory on a personal computer is typically about 16 gigabytes, which is exceeded by the amount of memory required to store the multiplication table of a semigroup with just 100,000 elements. We wish to compute with finite semigroups that contain far more elements than this, and sometimes even with infinite semigroups, and a multiplication table is not appropriate for these cases.

All finite semigroups and many infinite semigroups may be given as the semigroup defined by a finite presentation $\langle X \mid R \rangle$. In certain cases, it is possible to develop techniques that directly manipulate the generators X and the relations R in order to compute properties of the semigroup that the presentation defines. We briefly describe a method for computing non-trivial factorizations of elements in a finitely presented semigroup in Chapter 2, however finitely presented semigroups are not the focus of this thesis. Computing with these kinds of semigroups involves many difficulties, not least because it has been shown that for many semigroup-theoretic properties, there exists no algorithm for determining whether that property holds in an arbitrary finitely presented semigroup; see [95, 108].

We can often define a semigroup by the way in which it relates to some other known semigroup. For example, a finite Rees 0-matrix semigroup can be specified to a computer by its underlying semigroup, in some form, and its matrix. A more general-purpose technique along these lines is to specify a semigroup as the subsemigroup of some other semigroup defined by a finite set of generators: the elements are the products of the generators, and the product of two elements is the same as the product in the parent semigroup. The parent semigroup may be given explicitly: for example, a semigroup could be defined as the subsemigroup of some Rees 0-matrix semigroup generated by a particular set of elements. On the other hand, the parent semigroup may be given implicitly: a semigroup could be given as the subsemigroup generated by a particular set of transformations of degree n, and the semigroup may be understood to be a subsemigroup of the full transformation monoid of degree n. The algorithms that we present in Chapter 4 for computing the maximal subsemigroups of an arbitrary finite semigroup require a semigroup defined by a generating set.

For some semigroups, such as left- or right-zero semigroups, the set of all elements of the semigroup is the only generating set. For some other semigroups, such as direct products, it is not immediately apparent how to specify any generating set other than the set of all elements. On the other hand, a generating set for a semigroup may be readily available and may be much smaller than the order of the semigroup, and thereby permit a compact way of storing the semigroup. For example, the full transformation monoid of degree n is generated by two permutations and any transformation of rank $n-1$, and many infinite semigroups are finitely generated, such as the natural numbers $\mathbb{N}$ under addition. The NumericalSgps [24] package for GAP [58] enables the creation and computation of finitely generated subsemigroups of $\mathbb{N}$, known as *numerical semigroups*.

1.4.3 Finite semigroups specified by generating sets

In this thesis, we predominantly focus on the computation of finite semigroups specified by generating sets. For a typical algorithm that requires a semigroup given in this fashion, the time complexity of the algorithm is given in terms of the number of generators that define the semigroup, amongst other factors, such as, perhaps, the order of the semigroup and the specific type of elements by which it is given. Therefore, from a computational perspective, we desire a generating set for any given semigroup that is as small as is reasonably practicable. Driven by this motivation, a significant portion of the research in this thesis is dedicated to the problem of describing relatively small generating sets for certain kinds of semigroups.

A fundamental prerequisite of many algorithms in computational semigroup theory, including the algorithms presented in Chapter 4, is the ability to compute the Green's structure of a finite semigroup from its generating set. Broadly speaking, there are two main approaches to computing with a finite semigroup defined by generating set, and thereby describing its Green's structure. There are those techniques that involve exhaustively enumerating the semigroup, i.e. producing and storing every element of the semigroup in some useful data structure, and there are those that produce a data structure describing the Green's structure of the semigroup without necessarily enumerating the semigroup. We briefly describe these approaches below.

Exhaustive enumeration

The most well-known techniques for exhaustively enumerating a semigroup are those that construct the left or right Cayley digraphs of the semigroup. Let S be a finite semigroup with a generating set X. The *left Cayley digraph of S with respect to X* is the digraph (S, E), where

$$E = \{(s, xs) : s \in S,\ x \in X\},$$

and the *right Cayley digraph of S with respect to X* is the digraph $(S, \{(s, sx) : s \in S,\ x \in X\})$. When constructing a Cayley digraph, we label an edge with the set of generators corresponding to that edge. In this way, the left and right Cayley digraphs of a semigroup encode left and right multiplication in that semigroup. See Figure 1.20 for an example.

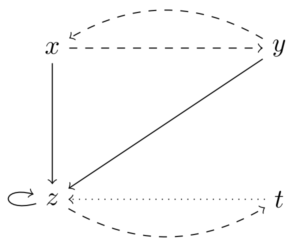

Figure 1.20: A picture of the right Cayley digraph of the semigroup defined by the multiplication table in Table 1.19, with respect to its generating set $\{y, z\}$. The dashed edges correspond to the generator y; the solid edges correspond to the generator z; and the dotted edge corresponds to both generators. The strongly connected components of this right Cayley digraph are $\{x, y\}$ and $\{z, t\}$, and so these are the $\mathscr{R}$-classes of the associated semigroup.

It is straightforward to see that the strongly connected components of a left Cayley digraph of S are the $\mathscr{L}$-classes of S, and the strongly connected components of a right Cayley digraph of S are its $\mathscr{R}$-classes. Furthermore, if (S, E) and (S, E') are left and right Cayley digraphs of S with respect to some generating set, then the strongly connected components of the digraph $(S, E \cup E')$ are the $\mathscr{J}$-classes of S, and the partial orders of the $\mathscr{L}$-, $\mathscr{R}$-, and $\mathscr{J}$-classes of S are given by the quotients of these digraphs by their strongly connected components, respectively. There are several well-known algorithms, such as those of Tarjan [121] and Gabow [55], for finding the strongly connected components of a digraph. See [80] and the references therein for more information about Cayley digraphs of semigroups.

A naive algorithm for computing the right Cayley digraph of a semigroup S with respect to a generating set X is given in Algorithm 1.21. This algorithm has time complexity $O(|S||X|)$. Note that the left Cayley digraph of S with respect to X can be deduced from the right Cayley digraph of S with respect to X, and vice versa: if $x \in X$ and $s \in S$, then there exist generators $y_1, \ldots, y_n \in X$ such that $s = y_1 \cdots y_n$, and $xs = (xy_1 \cdots y_{n-1}) \cdot y_n$, turning left multiplication by a generator into right multiplication.

The most famous and sophisticated algorithm for exhaustively enumerating a semigroup S defined by a generating set X, and obtaining the left and right Cayley digraphs of S with respect to X, is the Froidure-Pin Algorithm [54], which has time complexity $O(|S||X|)$. Although the Froidure-Pin Algorithm has the same time complexity as Algorithm 1.21, in practice it is much quicker to use the Froidure-Pin Algorithm to compute a Cayley digraph, since it deduces many of the edges of the digraph from previous calculations, and thereby avoids many unnecessary multiplications. The Froidure-Pin Algorithm was originally implemented by Pin in Semigroupe [107]. More recently, the algorithm has been extended in various ways, including by the development of a parallel version [79], which is implemented in libsemigroups [102]. The methods in libsemigroups [102] can be applied to a wide range of semigroups defined generating sets, including semigroups of transformations or partial permutations, semigroups of partitioned binary relations [96], and semigroups of boolean matrices.

Algorithm 1.21 A basic algorithm to compute the right Cayley digraph of a finite semigroup.

Input: A finite generating set X for a finite semigroup.
Output: The right Cayley digraph (V, E) of that semigroup with respect to X.

$V \leftarrow X$
$E \leftarrow \varnothing$
for $s \in V$, $x \in X$ **do**
  if $sx \notin V$ **then**
    $V \leftarrow V \cup \{sx\}$
    $E \leftarrow E \cup \{(s, sx)\}$
  Add x to the label of the edge (s, sx)
return (V, E).

A major limitation to algorithms that involve exhaustively enumerating a semigroup is memory. Consider a semigroup that consists of transformations of degree n. A straightforward approach to storing a transformation, as a list of images of each point, requires at least nk bits, where $k \geq \log n$. Therefore, fully enumerating and storing the elements of a transformation semigroup S of degree n in this simple way therefore requires at least $|S| \cdot n \log n$ bits: to illustrate, the full transformation monoid of degree 10 requires at least 38 gigabytes of memory with this approach. In order to compute large semigroups, we therefore require different techniques.

Techniques that are not necessarily exhaustive

There are a number of algorithms given in the literature, and implemented in software, that produce a data structure describing the Green's relations of a finite semigroup defined by a generating set without necessarily enumerating the whole semigroup. Lallement and McFadden [83] developed such an algorithm for computing with transformation semigroups; this was adapted by Konieczny [81] for semigroups of boolean matrices. Further techniques for computing with transformation semigroups were developed and implemented in the Monoid [89] package by Linton and co-authors [90, 91]. More recently, these approaches have been significantly extended and generalised by Mitchell and co-authors in [37]; this work applies to subsemigroups of arbitrary finite regular semigroups. The techniques in [37] have been fully implemented in the Semigroups [101] package for GAP [58], which allows for the non-exhaustive computation of semigroups given by a wide range of generating sets, including semigroups of matrices over finite fields, subsemigroups of regular Rees 0-matrix semigroups over groups, and semigroups given by generating sets consisting of transformations, partial permutations, or partitions (in the sense of Section 5.3.1), amongst other types.

The unifying approach of the above techniques is that they exploit certain actions of a semigroup (defined analogously to group actions) that are inherent when the semigroup is given in a particular form; these actions are connected to the Green's relations of the semigroup. These techniques construct groups from these actions, related to the group $\mathscr{H}$-classes of the semigroup, that allow certain computations about the semigroup to be reduced to computations about these groups. In doing so, these algorithms can take advantage of mature and efficient non-exhaustive algorithms from computational group theory, such as the Schreier-Sims algorithm [73, Section 4.4.2].

The above algorithms describe the Green's structure of a semigroup, in part, by enumerating representatives of the $\mathscr{L}$- and $\mathscr{R}$-classes of a semigroup, rather than by enumerating the $\mathscr{L}$- and $\mathscr{R}$-classes themselves. Therefore, for a semigroup with relatively large $\mathscr{J}$-classes (and with small numbers of $\mathscr{L}$- and $\mathscr{R}$-classes in comparison to the number of elements), the Green's structure can be determined without storing every element of the semigroup in memory. For example, it takes 15 seconds to compute the number of $\mathscr{R}$-classes of the full transformation monoid of degree 11 using the SEMIGROUPS package for GAP on a 2.66 GHz Intel Core i7 processor with 8GB of RAM, even though it would be difficult to store all 11^{11} transformations of degree 11 on a computer with such little memory. On the other hand, this means that computing $\mathscr{J}$-trivial semigroups with these methods involves exhaustive enumeration, in a fashion that is far less efficient than the Froidure-Pin Algorithm.

While there are non-exhaustive techniques for computing with many types of finite semigroups given by generating sets, they do not currently apply to all such semigroups.

At various points in this thesis, we require the ability to compute the Green's structure of a given finite semigroup. In these instances, the details of how this is achieved, whether exhaustively or otherwise, are not important, and we do not concern ourselves with this problem further.

Chapter 2

Generating sets for direct products of semigroups

2.1 Introduction

The *direct product* of a finite list of semigroups $S_1, \ldots, S_n$ is the semigroup consisting of the Cartesian product $S_1 \times \cdots \times S_n$ with component-wise multiplication. In other words, the multiplication on $S_1 \times \cdots \times S_n$ is given by

$$(s_1, \ldots, s_n)\,(t_1, \ldots, t_n) = (s_1t_1, \ldots, s_nt_n) \tag{2.1}$$

for all $(s_1, \ldots, s_n)$, $(t_1, \ldots, t_n) \in S_1 \times \cdots \times S_n$. The semigroups $S_1, \ldots, S_n$ are called the *factors* of the direct product $S_1 \times \cdots \times S_n$. Direct products of many other kinds of algebraic structures are defined analogously.

In this chapter, we build on the results of [2, Chapter 3] and [112,113] to describe generating sets for direct products of arbitrary semigroups which do not necessarily contain every element of the semigroup. These works predominantly concern the finite generation and presentability of direct products of semigroups, and describing generating sets for (and thereby finding upper bounds on the ranks of) certain kinds of direct products. Here, we particularly focus on computing relatively small generating sets for direct products of finite semigroups. The research detailed in this chapter was motivated by a desire to improve the performance of the Semigroups package [101] for GAP [58]. In version 3.0.0 of this software, creating a direct product of semigroups that are not all monoids required a brute-force search for a generating set, which had disappointing performance. For example, constructing a generating set for the direct product $(\mathcal{T}_4 \setminus \mathcal{S}_4) \times (\mathcal{T}_4 \setminus \mathcal{S}_4)$ took roughly 6 seconds on a 2.66 GHz Intel Core i7 processor with 8GB of RAM, while constructing a generating set for the direct product $(\mathcal{T}_5 \setminus \mathcal{S}_5) \times (\mathcal{T}_5 \setminus \mathcal{S}_5)$ did not terminate after several minutes. In the current development version of the Semigroups package, which implements some of the techniques described in this chapter, this latter computation takes roughly 10 milliseconds on the same hardware.

In certain instances, computing with a direct product can be reduced to performing independent computations in each of the factors. For example, the order of the direct product $S_1 \times \cdots \times S_n$ is equal to $|S_1| \cdots |S_n|$, and the set of idempotents of $S_1 \times \cdots \times S_n$ is $E(S_1) \times \cdots \times E(S_n)$. Furthermore, a direct product is regular, or inverse, or commutative if and only if each of its factors is regular, or inverse, or commutative, respectively. Similar statements hold for many other semigroup-theoretic properties.

Computing independently in the factors rather than in the direct product itself is advantageous for several reasons. For finite semigroups, the factors of a direct product are often much smaller than the direct product itself, and consequently they are much cheaper to compute with. To illustrate this, note that it is typically easier to enumerate three semigroups of order 1000 than it is to enumerate their direct product, which contains a billion elements. Additionally, the independence of the computations in the factors allows them to be carried out in parallel, which potentially saves time.

On the other hand, for some other questions, it is not immediately apparent how or whether any reduction to the factors can be achieved. For example, it is not necessarily the case that the Green's classes of a direct product are products of the Green's classes of the factors; any direct product involving $\mathbb{N}$ is $\mathscr{J}$-trivial, for instance.

Open Problem 2.2. Develop techniques for computing the Green's structure of a direct product of finite semigroups in terms of the Green's structures and other semigroup-theoretic properties of the factors.

Similarly, the maximal subsemigroups of a direct product do not necessarily correspond in an obvious way to maximal subsemigroups of the factors; see Example 2.4 for a demonstration of this. The maximal subgroups of direct products of groups can be given in terms of the maximal subgroups and the maximal normal subgroups of the factors (see for example [123, Lemma 1.3]) but an analogous characterisation for semigroups has not yet been found.

Open Problem 2.3. Investigate how the maximal subsemigroups of a direct product of finite semigroups relate to the maximal subsemigroups of the factors.

Example 2.4. Let $\mathcal{T}_2$ denote the full transformation monoid of degree 2. The maximal subsemigroups of $\mathcal{T}_2$ are its group of units $\mathcal{S}_2$ and the set $\mathcal{T}_2 \setminus \{(1\ 2)\}$, where $(1\ 2)$ is the non-identity permutation of degree 2; see Theorem 5.7. However, there are five maximal subsemigroups of the direct product $\mathcal{T}_2 \times \mathcal{T}_2$, three of which correspond to maximal subgroups of its group of units, which is the Klein four-group $\mathcal{S}_2 \times \mathcal{S}_2$, along with the sets

$$(\mathcal{T}_2 \times \mathcal{T}_2) \setminus ((\mathcal{T}_2 \setminus \mathcal{S}_2) \times \mathcal{S}_2) \quad \text{and} \quad (\mathcal{T}_2 \times \mathcal{T}_2) \setminus (\mathcal{S}_2 \times (\mathcal{T}_2 \setminus \mathcal{S}_2)).$$

These computations were performed with the Semigroups [101] package, which uses the algorithms presented in Chapter 4. Note that the maximal subsemigroups of $\mathcal{T}_2 \times \mathcal{T}_2$ are not direct products of maximal subsemigroups of $\mathcal{T}_2$. A correspondence between the maximal subsemigroups of $\mathcal{T}_2$ and the maximal subsemigroups of $\mathcal{T}_2 \times \mathcal{T}_2$ is not immediately apparent.

When it is possible to compute independently in the factors, this fact still has to be observed, and proven, and the corresponding algorithm has to be implemented, before it can be exploited in a computational algebra system. Where there do not exist specialised computational methods for direct products of semigroups, either because such algorithms are not known, or are simply not implemented, we instead rely on more general-purpose methods.

As discussed in Section 1.4, many algorithms for computing with semigroups require a semigroup defined by a finite generating set. Therefore, in order to compute effectively with finitely generated direct products, we require the ability to construct finite generating sets for them, which is anyway inherently interesting.

The only generating set given by the definition of a direct product is the set of all elements. This presents a problem, especially when the direct product is infinite, because it is impossible to store an infinite set of elements. When the direct product is finite, using the set of all elements is still undesirable: as discussed in Section 1.4.3, the time complexities of algorithms that require a generating set are typically given in terms of the size of the generating set, amongst other factors. For example, using the Froidure-Pin algorithm to compute the right Cayley digraph of a finite semigroup S with respect to a generating set X has time complexity $O(|S||X|)$. Roughly speaking, this means that the larger the generating set for a given semigroup, the more time that is required to compute with the semigroup. Furthermore, the cardinality of a direct product is the product of the cardinalities of the factors. Even if the factors themselves are finite and relatively small, storing in memory the set of all elements in the product can easily exceed the capabilities of a computer. In the finite case, when possible, we aim to compute generating sets that are significantly smaller than the set of all elements.

For monoids defined by generating sets, it requires no calculation to specify a generating set for the direct product that is typically much smaller than the order of the direct product. In more detail, let $n \in \mathbb{N}$, and for each $i \in \{1, \ldots, n\}$, let S_i be a monoid with generating set X_i. It is straightforward to see that the direct product $S_1 \times \cdots \times S_n$ is generated by the set

$$\bigcup_{i=1}^{n} \big\{ \big(1_{S_1}, \ldots, 1_{S_{i-1}}, x, 1_{S_{i+1}}, \ldots, 1_{S_n}\big) : x \in X_i \big\},$$

which contains at most $|X_1| + \cdots + |X_n|$ elements. By choosing the generating sets X_i to have minimal cardinality amongst the generating sets for S_i, and noting that for each i, the monoid S_i is a homomorphic image of $S_1 \times \cdots \times S_n$, it follows that

$$\max\{\operatorname{rank}(S_1), \ldots, \operatorname{rank}(S_n)\} \leq \operatorname{rank}(S_1 \times \cdots \times S_n) \leq \operatorname{rank}(S_1) + \cdots + \operatorname{rank}(S_n). \tag{2.5}$$

The bounds given in (2.5) are tight. If $p_1, \ldots, p_n$ are distinct primes and S_i is a cyclic group of order p_i for each i, then the direct product $S_1 \times \cdots \times S_n$ is cyclic. Therefore, the lower bound in (2.5) can be obtained. On the other hand, if each S_i is a cyclic group of some common order $k > 1$, then $\operatorname{rank}(S_1 \times \cdots \times S_n) = n$, and so the upper bound can also be obtained. It follows by (2.5) that a direct product of monoids is finitely generated if and only if each of its factors is finitely generated.

The same is not true for semigroups in general, however. Let $\mathbb{N}$ denote the semigroup consisting of the natural numbers with addition. This semigroup is monogenic, since $\mathbb{N} = \langle 1 \rangle$; indeed, $\mathbb{N}$ is the unique infinite monogenic semigroup, up to isomorphism. Despite this, the direct product $\mathbb{N} \times \mathbb{N}$ is not finitely generated. To see this, let X be an arbitrary generating set for $\mathbb{N} \times \mathbb{N}$, and let $n \in \mathbb{N}$ be arbitrary. By definition, there exists a sequence of generators $(x_1, y_1), \ldots, (x_m, y_m) \in X$ such that

$$\begin{aligned}(1, n) &= (x_1, y_1) + \cdots + (x_m, y_m)\\ &= (x_1 + \cdots + x_m,\ y_1 + \cdots y_m).\end{aligned}$$

In particular, $1 = x_1 + \cdots + x_m$. But 1 cannot be written as the sum of two or more natural numbers. It follows that $m = 1$, and so $(1, n) \in X$. Since X and n were arbitrary, we conclude that any generating set for $\mathbb{N} \times \mathbb{N}$ contains an infinite number of elements. Indeed, Corollary 2.20 implies that $\big\{(1, n), (n, 1) : n \in \mathbb{N}\big\}$ is the unique minimal generating set for $\mathbb{N} \times \mathbb{N}$, with respect to containment.

The lower bound in (2.5) holds for direct products of semigroups in general, since each factor is a homomorphic image of the direct product. Therefore, if a direct product is finitely generated, then its factors are finitely generated. However, we have seen that the converse does not necessarily hold. The problem of determining when the direct product of two semigroups is finitely generated was solved in [113]. This work was extended to the direct product of any finite number of semigroups by Araújo [2, Theorem 3.39]; we restate this in the following proposition. Recall that a semigroup S is *surjective* if $S^2 = S$.

Proposition 2.6 ([2, Theorem 3.39]; see [113, Remark 2.6]). *Let $S_1, \ldots, S_n$ be semigroups. The direct product $S_1 \times \cdots \times S_n$ is finitely generated if and only if each semigroup S_i is finitely generated, and either:*

(i) *S_i is surjective for all i; or*

(ii) *S_i is finite for all i; or*

(iii) *S_i is infinite for some i, and S_j is finite and surjective for all $j \neq i$.*

In [113, Proposition 2.5] and [2, Chapter 3.1], the authors describe generating sets for the direct product of two surjective semigroups; in doing so, they obtain upper bounds on the ranks of direct products of this kind. Moreover, the bound $\operatorname{rank}(S \times T) \leq 2\operatorname{rank}(S)\operatorname{rank}(T)$ for arbitrary surjective semigroups S and T, given in [2, Corollary 3.6], is tight. As mentioned in [113, Remark 2.6], and as we discuss later, these results can be used to construct finite generating sets of the first kind described in Proposition 2.6. However, the authors do not directly address the problem of finding generating sets for direct products of the second or third kinds given in Proposition 2.6.

The main result of this chapter is Theorem 2.17, which describes a generating set for an arbitrary direct product of semigroups which does not, in general, contain every element of the direct product. Moreover, when the factors are defined by finite generating sets and the direct product is finitely generated, then the generating set given in Theorem 2.17 is finite. We discuss techniques for constructing the generating set described by Theorem 2.17 in certain cases.

This chapter is organised as follows. In Section 2.2, we introduce the concepts of decomposable and indecomposable elements in a semigroup. These notions prove to be important when constructing generating sets for direct products. In Section 2.3, we discuss methods for computing the indecomposable elements in certain kinds of semigroup, and for obtaining non-trivial factorizations of decomposable elements. We present the main results of this chapter in Section 2.4. In Section 2.5, we restrict our attention to describing generating sets for direct products of two finitely generated surjective semigroups. In some cases, these generating sets can be smaller than the generating sets described by the more general results of Section 2.4.

2.2 Decomposable and indecomposable elements

In this section, we introduce the concepts of decomposable and indecomposable elements in a semigroup, as in [113], which are closely linked to the generation of direct products. We present several results concerning these kinds of elements which are important for the later parts of this chapter. In Section 2.3, we discuss how to compute the indecomposable elements in a finite or finitely presented semigroup, and we discuss how to obtain non-trivial factorizations of the decomposable elements in such semigroups.

Let S be a semigroup, and let $s \in S$ be arbitrary. We say that the element s is *decomposable* if there exist elements $u, v \in S$ such that $s = uv$, or equivalently, if $s \in S^2$. The element s is *indecomposable* if it is not decomposable. Therefore, an indecomposable element of a semigroup is one that cannot be written as a product of two elements in the semigroup. The set of decomposable elements in S is S^2, and so the set of indecomposable elements in S is $S \setminus S^2$. Note that a surjective semigroup is one in which each of its elements is decomposable.

Any monoid S is surjective, since $S = 1_S S \subseteq S^2$, as is any regular or idempotent generated semigroup. Conversely, non-trivial zero semigroups are not surjective: if x is a non-zero element in a zero semigroup S, then $uv = 0_S \neq x$ for all $u, v \in S$. Other examples of semigroups that are not surjective are free semigroups, and monogenic semigroups that are not groups.

As discussed in [78, Section 6] and [113, Section 2], any generating set for a semigroup S contains the set of indecomposable elements $S \setminus S^2$, since an indecomposable element cannot be written as a product of other elements. It follows that the set of indecomposable elements in a finitely generated semigroup is finite. By the same token, if the set of indecomposable elements in a semigroup is infinite, then the semigroup is not finitely generated. It is for this reason that the conditions relating to surjective semigroups appear in Proposition 2.6.

Let $S_1, \ldots, S_n$ be semigroups. If an element $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ is decomposable, then, by definition, there exist elements $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ in the direct product such that

$$(s_1, \ldots, s_n) = (x_1, \ldots, x_n)(y_1, \ldots, y_n).$$

In particular, $s_i = x_i y_i$ for each i, and so each s_i is decomposable. It is clear that the converse also holds. To summarise, an element of a direct product is decomposable if and only if each of its components is decomposable, and so a direct product is surjective if and only if each of its factors is surjective. Therefore, computing whether a direct product is surjective, and computing the decomposable elements in a direct product, can be reduced to computing independently in the factors.

It follows that an element $(s_1, \ldots, s_n)$ is indecomposable if and only if s_i is indecomposable for some i. Since a generating set for a semigroup contains the indecomposable elements, we conclude that every generating set for the direct product $S_1 \times \cdots \times S_n$ contains the set of all indecomposable elements

$$\bigcup_{i=1}^{n} \big(S_1 \times \cdots \times S_{i-1} \times (S_i \setminus S_i^2) \times S_{i+1} \times \cdots \times S_n\big). \tag{2.7}$$

The cardinality of this set gives a lower bound on the rank of the direct product. If the sets in this union are finite, then the cardinality of the union may be found with the principle of inclusion-exclusion. In $\mathbb{N} \times \mathbb{N}$, the set of indecomposable elements $(\{1\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1\})$ is infinite, and so the semigroup is not finitely generated, as shown above.

The notions of decomposable and indecomposable semigroup elements are closely related to factorizations. A *factorization* of an element s over a set A is a sequence of elements of A whose product is s, and a factorization is *non-trivial* if the sequence consists of two or more elements. Let S be a semigroup with a generating set X, and let $s \in S$ be arbitrary. If $s = uv$ for some $u, v \in S$, then a non-trivial factorization of s over X can be obtained by concatenating a factorization of u over X with a factorization of v over X. On the other hand, if s has a non-trivial factorization over X, i.e. if $s = x_1 \cdots x_m$ for some $x_1, \ldots, x_m \in X$ with $m \geq 2$, then $s = x_1(x_2 \cdots x_m) \in S^2$. Therefore, an element $s \in S$ is decomposable in S if and only if there exists a non-trivial factorization of s over X.

Another connection between factorizations and decomposable elements is given by the following lemma, which we use repeatedly in later results.

Lemma 2.8. *Let S be a semigroup generated by a set X, and let $s \in S$. Then there exist arbitrary long factorizations of s over $S^2 \cap X$, or s can be expressed as a product in X that includes an indecomposable element.*

Proof. Assume that if $s = x_1 \cdots x_m$ for any $x_1, \ldots, x_m \in X$, then $x_1, \ldots, x_m \in S^2 \cap X$. Let $x_1, \ldots, x_m \in X$ be such that $s = x_1 \cdots x_m$. Then since x_m is decomposable, there exist elements $y_1, \ldots, y_n \in X$, $n \geq 2$, such that $x_m = y_1 \cdots y_n$, and hence $s = x_1 \cdots x_{m-1} y_1 \cdots y_n$ is a strictly longer factorization of s over $S^2 \cap X$. ☐

2.3 Computing indecomposable elements and non-trivial factorizations

In this section, we discuss techniques for computing the indecomposable elements of a semigroup, and for computing non-trivial factorizations of decomposable elements, in finite and finitely presented semigroups. These steps are required to construct the generating sets described in the main theorem of this chapter. In particular, to apply the results of Theorem 2.17 to direct products involving a semigroup $S = \langle X \rangle$, we require the ability, for each decomposable generator $s \in S^2 \cap X$, to construct elements $a, b \in S$ such that $s \in aX \cap Xb$. Such elements can be constructed from any non-trivial factorization $x_1 \cdots x_m$ of s over X, by defining $a = x_1 \cdots x_{m-1}$ and $b = x_2 \cdots x_m$. The indecomposable elements of S are those elements in X for which there does not exist a non-trivial factorization over X.

The indecomposable elements of a semigroup can also be described in terms of the partial order of $\mathscr{J}$-classes of the semigroup, as is done in the following lemma.

Lemma 2.9. *Let S be a semigroup and let $s \in S$ be arbitrary. Then s is indecomposable if and only if $\{s\}$ is a maximal $\mathscr{J}$-class of S and $s^2 \neq s$.*

Proof. ($\Rightarrow$) Suppose that s is indecomposable. Certainly $s^2 \neq s$. Suppose that $J_s \leq J_t$ for some $t \in S$. Then $s = utv$ for some $u, v \in S^1$, and since s is indecomposable, it follows that $u = v = 1_S$, and $s = t$. Thus $J_s = \{s\}$ is a maximal $\mathscr{J}$-class of S.

($\Leftarrow$) Suppose that s is decomposable and that J_s is a maximal $\mathscr{J}$-class of S. Then $s = uv$ for some $u, v \in S$, and $J_s \leq J_u$ and $J_s \leq J_v$ by Lemma 1.12. By the maximality of $J_s = \{s\}$, it follows that $J_u = J_v = \{s\}$. In particular, $s = u = v$, and s is idempotent. ☐

In a finite semigroup, Green's $\mathscr{D}$- and $\mathscr{J}$-relations coincide [76, Proposition 2.1.4]. A maximal $\mathscr{J}$-class in a finite semigroup is hence either a regular $\mathscr{D}$-class or a non-regular $\mathscr{D}$-class consisting of a single non-idempotent element. Thus we obtain the following corollary to Lemma 2.9.

Corollary 2.10 (cf. Lemma 2.2 in [112])**.** *Let S be a finite semigroup. Then S is surjective if and only if every maximal $\mathscr{J}$-class of S is regular.*

Hence, to determine whether a finite semigroup is surjective, or to find the indecomposable elements in a finite semigroup, it suffices to find its partial order of $\mathscr{J}$-classes, to find the maximal elements in this partial order, and to test which of them consists of a single non-idempotent; see Algorithm 2.11. If S is a finite semigroup to which the techniques of [37] apply, then the partial order of $\mathscr{J}$-classes of S can be found by the algorithm described in [37, Algorithm 14], which does not necessarily exhaustively enumerate the elements of the semigroup. The partial order of $\mathscr{J}$-classes of a finite semigroup can also be constructed from the quotient of the union of the left and right Cayley digraphs by its strongly connected components.

Algorithm 2.11 Compute the indecomposable elements in a finite semigroup using the partial order of $\mathscr{J}$-classes.

Input: A finite semigroup S with a generating set.
Output: The set A of indecomposable elements in S.

1: $A \leftarrow \varnothing$
2: $\mathfrak{J} \leftarrow$ the maximal $\mathscr{J}$-classes of S ▷ [37, Algorithm 14], or via the Cayley digraphs of S
3: **for** $J_x \in \mathfrak{J}$ **do**
4:     **if** $|J_x| = 1$ and $x^2 \neq x$ **then**
5:         $A \leftarrow A \cup \{x\}$
6: **return** A.

However, when computing with the Cayley digraphs of a semigroup, there are more straightforward ways of finding indecomposable elements; see Algorithm 2.12. We formulate the following in terms of right Cayley digraphs, however analogous statements hold concerning left Cayley digraphs. Let S be a finite semigroup with a generating set X, and let Γ be the right

Cayley digraph of S with respect to X. If $s = x_1 \cdots x_n$ is a non-trivial factorization of s over X, then $(x_1, x_1x_2, \cdots, x_1 \cdots x_n = s)$ is a path in Γ, of length at least 1, from a generator to s. In particular, s has an in-neighbour in Γ. Therefore, if s has no in-neighbour in Γ, then there exists no non-trivial factorization of s, and s is indecomposable. Conversely, if s has an in-neighbour u in Γ, then there exists a generator $x \in X$ that corresponds to the edge (u, s), i.e. $s = ux$. Therefore, a non-trivial factorization of s over X can be found by appending x to any factorization of u over X. It follows that the indecomposable elements of a semigroup are the sources in its right Cayley digraph, with respect to any generating set.

A factorization of an element $u \in S$ over X can be calculated from Γ by searching for a path in Γ from a generator to u. Finding such a path can be achieved by performing a breadth- or depth-first search in Γ, starting at the vertex of each generator. The factorization is then given by the relevant generator, followed by the generators corresponding to the edges in the path. Note that a path may have length 0, which gives a trivial factorization.

Algorithm 2.12 Compute a non-trivial factorization for an element in a finite semigroup, or determine that the element is indecomposable, by using the right Cayley digraph.

Input: A finite semigroup S with a generating set X, and an element $s \in S$.
Output: A non-trivial factorization of s over X, if one exists, else FAIL.

```
1: Γ ← the right Cayley digraph of S with respect to X        ▷ [54]; Algorithm 1.21
2: if s is a source of Γ then
3:     return FAIL.
4: u ← an in-neighbour of s in Γ
5: x ← a generator in X corresponding to the edge (u, s)
6: [x_1, ..., x_n] ← a (possibly trivial) factorization of u over X    ▷ See main text
7: return [x_1, ..., x_n, x].
```

For a finite semigroup $S = \langle X \rangle$ to which the techniques of [37] apply, we can obtain non-trivial factorizations of decomposable elements over X without constructing either the left or right Cayley digraphs of S, and without necessarily enumerating the semigroup. We briefly outline a method for doing so below; see Algorithm 2.13 for pseudocode describing this method.

Algorithm 13 in [37] gives a method for factorizing an arbitrary element of S over X. However, when $x \in X$, this algorithm may return the trivial factorization.

Let $x \in X$ be arbitrary. If x has a relative right identity, then a relative right identity for x can be constructed from the data structure used in [37]. This step is complicated to describe, and we do not include details here, however we note that this step has been implemented in an upcoming version of the Semigroups package [101] for GAP [58], by the author. Given a relative right identity f for x, it follows that $x = xx_1 \cdots x_n$, where $f = x_1 \cdots x_n$ is any factorization of f over X. If x has a relative left identity, then a non-trivial factorization of x over X may be obtained analogously.

Suppose that x has neither a relative left identity nor a relative right identity in S. If $y \in R_x \setminus \{x\}$, then there exist elements $u, v \in S$ such that $y = xu$ and $x = yv$; in particular, $x = yv = x(uv)$, and x has a relative right identity. Therefore $R_x = \{x\}$, and similarly $L_x = \{x\}$. We conclude that $R_x = L_x = J_x = \{x\}$. It follows that x is contained in any transversal of the set of $\mathscr{R}$-classes of S. Therefore, in the data structure returned by [37, Algorithm 11], either we find a generator $x' \in X$ and a representative r of an $\mathscr{R}$-class of S such that $x = x'r$, or we determine that R_x is a maximal $\mathscr{R}$-class of S. In the first case, we obtain a non-trivial factorization of x over X by first factorizing r over X. In the second case, it is straightforward to show that $\{x\}$ is also a maximal $\mathscr{L}$-class of S, and therefore a maximal $\mathscr{J}$-class of S, too. It follows by Lemma 2.9 that x is indecomposable, or equivalently, that there exists no non-trivial factorization of x over X.

Algorithm 2.13 Compute a non-trivial factorization for an element in a finite semigroup, or determine that the element is indecomposable, by using the data structure described in [37].

Input: A finite semigroup S with a generating set X, and an element $s \in S$.
Output: A non-trivial factorization of s over X, if one exists, else FAIL.

$[x_1, \dots, x_n] \leftarrow$ a factorization of s over X ▷ [37, Algorithm 13]
if $n > 1$ **then** ▷ If $n = 1$, then $s \in X$
    return $[x_1, \dots, x_n]$.
else if s has a relative left identity **then**
    $e \leftarrow$ a relative left identity for s
    $[y_1, \dots, y_m] \leftarrow$ a factorization of e over X
    return $[y_1, \dots, y_m, s]$.
else if s has a relative right identity **then**
    $f \leftarrow$ a relative right identity for s
    $[y_1, \dots, y_m] \leftarrow$ a factorization of f over X
    return $[s, y_1, \dots, y_m]$.
$\mathfrak{R} \leftarrow$ a set of $\mathscr{R}$-class representatives for S ▷ [37, Algorithm 11]
for $x \in X$, $r \in \mathfrak{R}$ **do**
    if $s = xr$ **then**
        $[y_1, \dots, y_m] \leftarrow$ a factorization of r over X
        return $[x, y_1, \dots, y_m]$.
return FAIL.

Of course, it is not possible to construct an algorithm to enumerate the indecomposable elements of an arbitrary semigroup, since an infinite semigroup (for example, an infinite zero semigroup) may contain an infinite number of indecomposable elements. However, as mentioned above, the set of indecomposable elements of a semigroup is contained in any generating set. It follows that the problem of enumerating the indecomposable elements in a semigroup defined by a finite generating set is a more appropriate problem to attempt to solve.

One way of specifying an infinite semigroup is by a semigroup presentation. The elements of a semigroup defined by a presentation are equivalence classes of finite sequences over the generators. The sequences in an equivalence class give the factorizations of the element over the generators. Therefore, an element has a non-trivial factorization if and only if its equivalence class contains a non-trivial sequence. It follows that a generator has a non-trivial factorization in the semigroup if and only if the presentation relates that generator to a non-trivial sequence. We clarify this observation in the following lemma.

Lemma 2.14. *Let S be the semigroup defined by the presentation $\langle X \mid R \rangle$, and suppose that R contains no relations of the form $x = y$, for any $x, y \in X$. Then for each $x \in X$, the generator of S corresponding to x is decomposable if and only if $x = w$ or $w = x$ belongs to R, for some $w \in X^+$. Moreover, w gives a non-trivial factorization of x over X.*

It follows that if S is a semigroup defined by a finite presentation of the kind described in Lemma 2.14, then the indecomposable generators, and non-trivial factorizations of the decomposable generators, can be obtained by searching the relations according to Lemma 2.14. Note that relations of the form $x = y$, for generators x and y, can be easily removed from any finite presentation to yield an equivalent presentation.

2.3.1 Corresponding features in the Semigroups package for GAP

The techniques described in this section have been implemented by the author in the development version of the Semigroups package [101] for GAP [58].

If S is a finite semigroup, or a semigroup defined by a finite presentation, then the command `IndecomposableElements(S);` returns a list of the indecomposable elements in S. If S is already known to be a surjective semigroup (because it is known to be a monoid or a regular semigroup, for example), then `IndecomposableElements` returns an empty list. Otherwise, if S is defined by a finite presentation, then the function `IndecomposableElements` uses Lemma 2.14 to compute the indecomposable elements. Otherwise, `IndecomposableElements` uses Lemma 2.9 to compute the indecomposable elements via the partial order of $\mathscr{J}$-classes of S. The command `IsSurjectiveSemigroup(S);` returns `true` if `IndecomposableElements(S);` returns an empty list, and `false` otherwise.

If S is a finite semigroup, X is the generating set of S given by `GeneratorsOfSemigroup(S);` and $s \in S$, then `NonTrivialFactorization(S, s);` returns a list defining a non-trivial factorization of s over X, if one exists, else it returns `fail`. If $s \in S \setminus X$, then any factorization of s over X is non-trivial. In this case, the function `NonTrivialFactorization` delegates to the pre-existing Semigroups package function `Factorization`. Otherwise, the function `NonTrivialFactorization` executes a version of either Algorithm 2.12 or Algorithm 2.13, as appropriate. Suppose that `NonTrivialFactorization(S, s);` returns a list L of length $n > 1$. Then the non-trivial factorization of s over X is given by

$$s = X[L[1]] \cdot X[L[2]] \cdots X[L[n]].$$

2.4 Arbitrary direct products of semigroups

As proved by Araújo [2, Theorem 3.39] and repeated in Proposition 2.6, the direct product of semigroups $S_1 \times \cdots \times S_n$ is finitely generated if and only if each semigroup S_i is finitely generated, and either:

(i) S_i is surjective for all i; or

(ii) S_i is finite for all i; or

(iii) S_i is infinite for some i, and S_j is finite and surjective for all $j \neq i$.

Given an arbitrary collection of semigroups whose direct product is finitely generated, the proof of this result could, in principle, be used to obtain a finite generating set for the direct product. However, it appears that this result was not proven with the aim of computing such finite generating sets. In this section, we build on this work by directly describing finite generating sets for any finitely generated direct product, with the aim of computing generating sets that are, in some sense, relatively small.

The following result of Araújo, which was an improvement to [113, Proposition 2.5 and Corollary 2.7], describes a generating set for the direct product of two surjective semigroups. The main theorem of this section was inspired by this result and its proof.

Proposition 2.15 ([2, Proposition 3.5 and Corollary 3.6]). *Let $S = \langle X \rangle$ and $T = \langle Y \rangle$ be surjective semigroups. For each $x \in X$, choose $b_x \in S$ such that $x \in X b_x$, and for each $y \in Y$, choose $a_y \in T$ such that $y \in a_y Y$. Define $A = \{a_y : y \in Y\}$ and $B = \{b_x : x \in X\}$. Then the set $(X \times A) \cup (B \times Y)$ generates $S \times T$. In particular,* $\operatorname{rank}(S \times T) \leq 2\operatorname{rank}(S)\operatorname{rank}(T)$.

Proof. Let $(s,\, t) \in S \times T$ be arbitrary, and let $s = x_1 \cdots x_{m-1}x_m$ and $t = y_1y_2 \cdots y_n$ be non-trivial factorizations of s and t over X and Y, respectively. We may iteratively replace the generator that follows x_{m-1} in the factorization of s by a product xb, for some $x \in X$ and $b \in B$. Similarly, we may replace the generator in $y_1y_2 \cdots y_n$ that precedes y_2 by a product ay, for some $a \in A$ and $y \in Y$. In particular, if we repeat this process n and m times, respectively, we conclude that

$$s = x_1 \cdots x_{m-1}x'b_1b_2 \cdots b_n \quad \text{and} \quad t = a_1 \cdots a_{m-1}a_my'y_2 \cdots y_n$$

for some $x' \in X$ and $b_1, \ldots, b_n \in B$, and for some $y' \in Y$ and $a_1, \ldots, a_m \in A$. Therefore

$$(s,\, t) = (x_1,\, a_1) \cdots (x_{m-1},\, a_{m-1})\, (x',\, a_m)\, (b_1,\, y')\, (b_2,\, y_2) \cdots (b_n,\, y_n) \in (X \times A)^m (B \times Y)^n,$$

and so $(X \times A) \cup (B \times Y)$ generates $S \times T$. Since $|A| \leq |Y|$ and $|B| \leq |X|$ by construction, it follows that $|(X \times A) \cup (B \times Y)| \leq 2|X||Y|$. The upper bound on $\operatorname{rank}(S \times T)$ is obtained by choosing generating sets X and Y of minimal cardinality. □

The example in [113, Example 2.8] presents a pair of surjective semigroups, of arbitrary finite ranks, whose direct product reaches the upper bound given in Proposition 2.15. In this sense, the bound given in Proposition 2.15 is optimal.

As noted in [113, Remark 2.6], since a direct product is surjective if and only if each of its factors is surjective, we may use Proposition 2.15 to describe generating sets for arbitrary direct products of surjective semigroups, i.e. for any direct product of type (i) from Proposition 2.6. More specifically, given surjective semigroups $S_1, \ldots, S_n$ defined by finite generating sets, we may find a finite generating set for $S_1 \times \cdots \times S_n$ by first using Proposition 2.15 to construct a finite generating set for $S_1 \times S_2$, which is itself surjective. We may then use Proposition 2.15 again to construct a finite generating set for $(S_1 \times S_2) \times S_3$, and so on, iterating in this way until complete. We may use this observation in an inductive argument to obtain $2^{n-1} \cdot \operatorname{rank}(S_1) \cdots \operatorname{rank}(S_n)$ as an upper bound for the rank of the direct product of any collection of surjective semigroups $S_1, \ldots, S_n$. However, for $n > 2$, this bound is not tight; we improve on this bound in Corollary 2.18.

Any direct product of type (ii) is finite, and so from the point of view of finite generation there is nothing to be solved, since the semigroup itself defines a finite generating set. However, for the purposes of computation, smaller generating sets are usually better. Using the set of all elements for computation is not satisfactory, in general. Indeed, the rank of a direct product of finite semigroups may be much smaller than its cardinality, even when the semigroups are not all surjective; see Example 2.16.

Example 2.16. Let $n \in \mathbb{N}$, $n \geq 2$, let $S = \langle x \rangle$ be a monogenic semigroup of order n that is not a group, and let $\mathcal{I}_1$ be the symmetric inverse monoid of degree 1. Note that S is not surjective, and that $\mathcal{I}_1$ is an inverse monoid of order 2 with identity element id_1. Therefore, the order of $S \times \mathcal{I}_1$ is $2n$. If $(x^m,\, y) \in S \times \mathcal{I}_1$, then $(x^m,\, y) = (x,\, y)\, (x,\, \mathrm{id}_1)^{m-1}$. It follows that $\{x\} \times \mathcal{I}_1$ is a generating set for $S \times \mathcal{I}_1$ that contains only two elements.

As far as we are aware, there are no results in the literature directly describing finite generating sets for direct products of type (iii).

The main result of this section is the following, which describes a generating set for an arbitrary direct product of semigroups. This theorem can be used, in combination with the techniques described in Section 2.3, to construct a generating set for a finitely generated direct product of finite or finitely presented semigroups. The proof of this theorem, which appears towards the end of this section, relies on several forthcoming lemmas.

Theorem 2.17. *Let $n \in \mathbb{N}$. For each $i \in \{1, \ldots, n\}$, let S_i be a semigroup generated by a set X_i, and for each $x \in S_i^2 \cap X_i$, choose $a_x, b_x \in S_i$ such that $x \in a_x X_i \cap X_i b_x$, and define $A_i = \{a_x : x \in S_i^2 \cap X_i\}$ and $B_i = \{b_x : x \in S_i^2 \cap X_i\}$. Let*

$$\Gamma = \bigcup_{i=1}^{n} \Big(\big(B_1 \times \cdots \times B_{i-1} \times (S_i^2 \cap X_i) \times A_{i+1} \times \cdots \times A_n\big) \\ \cup \big(S_1 \times \cdots \times S_{i-1} \times (S_i \setminus S_i^2) \times S_{i+1} \times \cdots \times S_n\big) \Big).$$

Then the direct product $S_1 \times \cdots \times S_n$ is generated by Γ.

Suppose that the generating sets X_i in the statement of Theorem 2.17 are finite. For each $i \in \{1, \ldots, n\}$, the sets

$$B_1 \times \cdots \times B_{i-1} \times (S_i^2 \cap X_i) \times A_{i+1} \times \cdots \times A_n$$

contain at most $|X_1| \cdots |X_n|$ elements, since $|A_i| \leq |X_i|$ and $|B_i| \leq |X_i|$ by construction. In particular, these sets are finite when the generating sets X_i are finite. Furthermore, the sets

$$S_1 \times \cdots \times S_{i-1} \times (S_i \setminus S_i^2) \times S_{i+1} \times \cdots \times S_n\}$$

comprise the indecomposable elements of the direct product, and are therefore contained in any generating set for the direct product. Hence these sets are finite when the direct product is finitely generated. In particular, Γ is finite when the generating sets X_i are finite and the direct product is finitely generated. In the worst case, the generating set Γ from the statement of Theorem 2.17 contains at most $n \cdot |X_1| \cdots |X_n|$ redundant elements; see Corollary 2.18.

When $n = 2$ and S_1 and S_2 are surjective, Theorem 2.17 implies that $S_1 \times S_2$ is generated by $(X_1 \times A_2) \cup (B_1 \times X_2)$. Therefore, Proposition 2.15 is a special case of Theorem 2.17.

Let $S_1, \ldots, S_n$ be a collection of finitely generated semigroups whose direct product is finitely generated, and suppose that each semigroup S_i is defined by a finite generating set X_i. By Theorem 2.17, in order to construct a generating for the direct product $S_1 \times \cdots \times S_n$, it suffices to determine, for each $i \in \{1, \ldots, n\}$, which generators in X_i are decomposable and which are indecomposable, and for each decomposable generator x, to find elements $a_x, b_x \in S_i$ such that $x \in a_x X_i \cap X_i b_x$. In order to reduce the size of the sets $\{a_x : x \in S_i^2 \cap X_i\}$ and $\{b_x : x \in S_i^2 \cap X_i\}$, and hence to reduce the size of the corresponding generating set, the elements a_x and b_x should be chosen carefully. For example, if S_i has a left identity e, then it would be sensible to choose $a_x = e$ for all $x \in S_i^2 \cap X_i$. More generally, the elements a_x and b_x can be constructed from non-trivial factorizations, as discussed in Section 2.3.

Note that the sets A_1 and B_n are not required by the generating set described in Theorem 2.17, and therefore they are not required to be constructed in any algorithm which implements the result given in Theorem 2.17.

Methods for computing a generating set for the direct product of an arbitrary collection of finite semigroups, based on Theorem 2.17, have been implemented by the author in the development version of the Semigroups package [101] for GAP [58]. These methods will be included in a future release of this software. With these new methods, the function `DirectProduct` may be applied to any collection of finite semigroups to produce a semigroup that is isomorphic to their direct product. If the arguments are all semigroups of partial permutations, then their direct product is returned as a semigroup of partial permutation; similarly, if the factors are all semigroups of partitions (in the sense of Section 5.3.1), or if the factors are all semigroups of *partitioned binary relations* (defined in [96]), then their direct product is returned as a semigroup of the same kind. Otherwise, the direct product is returned as a transformation semigroup. Given a direct product P of the factors $S_1, \ldots, S_n$ and some $i \in \{1, \ldots, n\}$, the command `Embedding(P, i)`; returns an embedding of the i^{th} factor S_i into P, and the command `Projection(P, i)`; returns a projection of P onto its i^{th} factor S_i.

We may use Theorem 2.17 to describe bounds on the ranks of arbitrary direct products.

Corollary 2.18. *Let $S_1, \ldots, S_n$ be semigroups, and let m denote the cardinality of*

$$\bigcup_{i=1}^{n} (S_1 \times \cdots \times S_{i-1} \times (S_i \setminus S_i^2) \times S_{i+1} \times \cdots \times S_n).$$

Then

$$m \leq \operatorname{rank}(S_1 \times \cdots \times S_n) \leq m + n \cdot \operatorname{rank}(S_1) \cdots \operatorname{rank}(S_n).$$

In particular, if each semigroup S_i is surjective, then

$$\operatorname{rank}(S_1 \times \cdots \times S_n) \leq n \cdot \operatorname{rank}(S_1) \cdots \operatorname{rank}(S_n).$$

Proof. Note that m is the number of indecomposable elements of the direct product $S_1 \times \cdots \times S_n$. Therefore, the first inequality holds. The second inequality can be shown to hold by counting the size of the generating set given in Theorem 2.17. For each $i \in \{1, \ldots, n\}$, choose X_i to be a generating set for S_i of minimal cardinality, and define sets A_i and B_i as in Theorem 2.17. Then the generating set is the union of (2.7), which contains the m indecomposable elements, along with the set

$$\bigcup_{i=1}^{n} (B_1 \times \cdots \times B_{i-1} \times X_i \times A_{i+1} \times \cdots \times A_n).$$

Since $|A_i| \leq |X_i|$ and $|B_i| \leq |X_i|$ for each i by construction, it follows that

$$|B_1 \times \cdots \times B_{i-1} \times X_i \times A_{i+1} \times \cdots \times A_n| \leq |X_1| \cdots |X_n| = \operatorname{rank}(S_1) \cdots \operatorname{rank}(S_n),$$

for all $i \in \{1, \ldots, n\}$. Therefore, the second inequality holds. If each semigroup S_i is surjective, then $m = 0$, and so the final inequality follows from the second. □

By adapting the example used in [113, Example 2.8], we may show that the upper bound given by Corollary 2.18 for the rank of a direct product of surjective semigroups is tight.

Example 2.19 (cf. [113, Example 2.8])**.** Let $n \in \mathbb{N}$. For each $i \in \{1, \ldots, n\}$, let $m_i \in \mathbb{N}$, and define a semigroup

$$S_i = \begin{cases} C_{i,1} & \text{if } m_i = 1, \\ C_{i,1} \cup \cdots \cup C_{i,m_i} \cup \{0\} & \text{if } m_i > 1, \end{cases}$$

where each $C_{i,j}$ is a cyclic group of order 2, and the product of elements within distinct groups, or involving 0, is defined to be 0. It is straightforward to see that S_i is a commutative inverse semigroup, and that the unique minimal generating set of S_i consists of the non-identity element of each $C_{i,j}$. In particular, S_i is surjective and $\operatorname{rank}(S_i) = m_i$. It follows by Corollary 2.18 that

$$\operatorname{rank}(S_1 \times \cdots \times S_n) \leq n \cdot \operatorname{rank}(S_1) \cdots \operatorname{rank}(S_n) = n \cdot m_1 \cdots m_n.$$

For each $i \in \{1, \ldots, n\}$, choose $j_i \in \{1, \ldots, m_i\}$ arbitrarily. Note that $C_{1,j_1} \times \cdots \times C_{n,j_n}$ is a direct product of n cyclic groups of order 2, and so

$$\operatorname{rank}(C_{1,j_1} \times \cdots \times C_{n,j_n}) = n.$$

But $(S_1 \times \cdots \times S_n) \setminus (C_{1,j_1} \times \cdots \times C_{n,j_n})$ is an ideal of $S_1 \times \cdots \times S_n$, which implies that any generating set for $S_1 \times \cdots \times S_n$ contains a generating set for $C_{1,j_1} \times \cdots \times C_{n,j_n}$. Since there are m_i choices for each j_i, it follows that any generating set for $S_1 \times \cdots \times S_n$ contains at least $n \cdot m_1 \cdots m_n$ elements. Therefore

$$\operatorname{rank}(S_1 \times \cdots \times S_n) = n \cdot \operatorname{rank}(S_1) \cdots \operatorname{rank}(S_n).$$

We also deduce the following corollary to Theorem 2.17.

Corollary 2.20. *Let S be a semigroup and suppose that $S = \langle S \setminus S^2 \rangle$. Then, with respect to containment, any direct product involving S has a unique minimal generating set, which consists of the indecomposable elements in the direct product. In particular, $S \setminus S^2$ is the unique minimal generating set for S, with respect to containment.*

Proof. By applying Theorem 2.17 to a collection of semigroups involving S with its generating set $S \setminus S^2$, we deduce that any direct product involving S is generated by its indecomposable elements. On the other hand, any generating set contains the indecomposable elements. $\square$

It follows by Corollary 2.20 that the generating set for the semigroup $S \times \mathcal{I}_1$ described in Example 2.16 is the unique minimal generating set of $S \times \mathcal{I}_1$, with respect to containment.

In order to prove Theorem 2.17, we first prove the following two technical lemmas. We note that the set Γ from the statement of the theorem contains $B_1 \times \cdots \times B_{i-1} \times X_i \times A_{i+1} \times \cdots \times A_n$ for each i, since $X_i = (S_i^2 \cap X_i) \cup (S_i \setminus S_i^2)$. We use this observation in the following proofs.

Lemma 2.21. *For all $i \in \{1, \ldots, n-1\}$ and $k \in \mathbb{N}_0$,*

$$S_1 \times \cdots \times S_{i-1} \times \big((S_i \setminus S_i^2) \underbrace{B_i \cdots B_i}_{k \text{ times}} \big) \times S_{i+1} \times \cdots \times S_n \subseteq \langle \Gamma \rangle. \tag{2.22}$$

Proof. Let $i \in \{1, \ldots, n-1\}$ be arbitrary. Note that if S_i is surjective, then the set $S_i \setminus S_i^2$, and consequently the product of sets in (2.22), is empty and so there is nothing to prove. Otherwise, we prove by induction that (2.22) holds for all $k \in \mathbb{N}_0$. Certainly (2.22) holds when $k = 0$, since Γ contains the indecomposable elements of $S_1 \times \cdots \times S_n$. Suppose that (2.22) holds for some $k = l$, where $l \in \mathbb{N}_0$, and let

$$\begin{aligned} \mathbf{s} &= (s_1, \ldots, s_{i-1}, s_i b_{i,1} \cdots b_{i,l+1}, s_{i+1}, \ldots, s_n) \\ &\in S_1 \times \cdots \times S_{i-1} \times \big((S_i \setminus S_i^2) \underbrace{B_i \cdots B_i}_{l+1 \text{ times}} \big) \times S_{i+1} \times \cdots \times S_n \end{aligned}$$

be arbitrary. We aim to show that $\mathbf{s} \in \langle \Gamma \rangle$.

If s_j is indecomposable for any $j \neq i$, then $\mathbf{s}$ is indecomposable, and $\mathbf{s} \in \Gamma$. Suppose instead that s_j is decomposable for each $j \neq i$. Thus we may decompose s_j as a product $u_j x_j$, for some $u_j \in S_j$ and $x_j \in X_j$. Since we have assumed that (2.22) holds for $k = l$, it follows that

$$\mathbf{u} = (u_1, \ldots, u_{i-1}, s_i b_{i,1} \cdots b_{i,l}, u_{i+1}, \ldots, u_n) \subseteq \langle \Gamma \rangle.$$

If x_j is indecomposable for any $j \neq i$, then $\mathbf{x} = (x_1, \ldots, x_{i-1}, b_{i,l+1}, x_{i+1}, \ldots, x_n)$ is indecomposable. Therefore $\mathbf{x} \in \Gamma$, and so $\mathbf{s} = \mathbf{ux} \in \langle \Gamma \rangle$, as required. Otherwise, x_j is decomposable, and is therefore contained in $A_j X_j \cap X_j B_j$, for each $j \neq i$. Hence for each $j \in \{1, \ldots, n-1\} \setminus \{i\}$, we may express x_j as a product $x_j' b_j$, for some $x_j' \in X_j$ and $b_j \in B_j$, and we may express x_n as a product $a_n x_n'$, for some $x_n' \in X_n$ and $a_n \in A_n$. Since we have assumed that (2.22) holds for $k = l$, it follows that

$$\mathbf{v} = \big(u_1 x_1', \ldots, u_{i-1} x_{i-1}', s_i b_{i,1} \cdots b_{i,l}, u_{i+1} x_{i+1}', \ldots, u_{n-1} x_{n-1}', u_n a_n \big) \subseteq \langle \Gamma \rangle,$$

and by definition,

$$\mathbf{b} = (b_1, \ldots, b_{i-1}, b_{i,l+1}, b_{i+1}, \ldots, b_{n-1}, x_n') \in B_1 \times \cdots \times B_{n-1} \times X_n \subseteq \Gamma.$$

Therefore $\mathbf{s} = \mathbf{vb} \in \langle \Gamma \rangle$. Since $\mathbf{s}$ was arbitrary, it follows that (2.22) holds for $k = l + 1$. By induction, (2.22) holds for all $k \in \mathbb{N}_0$. $\square$

Lemma 2.23. $X_1 \times \cdots \times X_n \subseteq \langle \Gamma \rangle$.

Proof. If $n = 1$, then $X_1 = \Gamma$, and so we assume for the remainder of this proof that $n > 1$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$ be arbitrary. It suffices to show that $\mathbf{x} \in \langle \Gamma \rangle$. If x_i is indecomposable for some i, then $\mathbf{x}$ is itself indecomposable, and we are done. Otherwise, x_i is decomposable for each i, i.e. $x_i \in S_i^2 \cap X_i$. In order to show that $\mathbf{x} \in \langle \Gamma \rangle$ in this case, we introduce the following sets, which are defined in terms of the sets X_i, A_i, and B_i from the statement of Theorem 2.17. For each $i \in \{1, \ldots, n\}$ and $k \geq 2$, we define

$$U_{i,k} = \begin{cases} X_1 B_1^{k-1} & \text{if } i = 1, \\ A_i^{k-1} X_i & \text{if } k \leq i, \\ A_i^{i-1} X_i B_i^{k-i} & \text{if } 1 < i < k, \end{cases}$$
$$\text{and} \quad \mathbf{C}_i = B_1 \times \cdots \times B_{i-1} \times X_i \times A_{i+1} \times \cdots \times A_n \subseteq \Gamma.$$

Note that $U_{1,k} \times \cdots \times U_{n,k} = \mathbf{C}_1 \cdots \mathbf{C}_{k-1} \cdot \big(B_1 \times \cdots \times B_{k-1} \times X_k \times \cdots \times X_n\big)$ for $2 \leq k < n$, and that $U_{1,n} \times \cdots \times U_{n,n} = \mathbf{C}_1 \cdots \mathbf{C}_n$. By finding values of k for which $x_i \in U_{i,k}$ for all i, we aim to show that $\mathbf{x}$ is contained in a product of subsets of Γ, including those of the form $\mathbf{C}_i$.

Note that $U_{1,2} = X_1 B_1$ and $U_{i,2} = A_i X_i$ for all $i \geq 2$. Since we have assumed that x_i is decomposable for each i, and since $S_i^2 \cap X_i \subseteq (A_i X_i \cap X_i B_i)$ by construction of the sets A_i and B_i, it follows that $x_i \in U_{i,2}$ for all i. However, it is not necessary that $x_i \in U_{i,k}$ for all $k > 2$.

Let i be arbitrary. By definition, $x_i \in U_{i,k}$ for some k if and only if x_i can be expressed as a product of length k consisting of $\min\{i, k\} - 1$ elements of A_i, followed by a generator in X_i, followed by $k - \min\{i, k\}$ elements of B_i. Suppose that $x_i \in U_{i,k}$ for some k, and let $a_1 \cdots a_m x' b_1 \cdots b_q$ be a corresponding expression of length k. If x' is decomposable, then $x' \in A_i X_i \cap X_i B_i$. In this case, by expressing x' as a product of length 2 in either $A_i X_i$ or $X_i B_i$, as appropriate, it follows that x_i can be expressed as a product of length $k+1$ consisting of $\min\{i, k+1\} - 1$ elements of A_i, followed by a generator in X_i, followed by $(k+1) - \min\{i, k+1\}$ elements of B_i. In other words, $x_i \in U_{i,k+1}$. Hence if $x_i \in U_{i,k} \setminus U_{i,k+1}$ for some k, and $a_1 \cdots a_m x' b_1 \cdots b_q$ is a corresponding expression for x_i of length k, then $x' \in S_i \setminus S_i^2$.

We define a function $f : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ by

$$f(i) = \min\left(\{n\} \cup \{k \in \mathbb{N} : x_i \notin U_{i,k+1}\}\right).$$

Note that $f(i) \in \{2, \ldots, n\}$ for all i. The above argument implies that, for any $i \in \{1, \ldots, n\}$ with $f(i) < n$,

$$x_i \in \begin{cases} (S_1 \setminus S_1^2) B_1^{f(1)-1} & \text{if } i = 1, \\ A_i^{f(i)-1}(S_i \setminus S_i^2) & \text{if } f(i) \leq i, \\ A_i^{i-1}(S_i \setminus S_i^2) B_i^{f(i)-i} & \text{if } 1 < i < f(i). \end{cases} \tag{2.24}$$

Fix $j \in \{1, \ldots, n\}$ such that $f(j) = \min\big\{f(i) : i \in \{1, \ldots, n\}\big\}$. It follows that

$$x_i \in U_{i,k} \text{ for all } i \in \{1, \ldots, n\} \text{ and } 2 \leq k \leq f(j). \tag{2.25}$$

The remainder of the proof is split into four cases, according to the values of j and $f(j)$.

Case 1: $f(j) = n$. By (2.25), $x_i \in U_{i,n}$ for all i. It follows that

$$\mathbf{x} \in U_{1,n} \times \cdots \times U_{n,n} = \mathbf{C}_1 \cdots \mathbf{C}_n \subseteq \langle \Gamma \rangle.$$

Case 2: $f(j) \leq j$ and $f(j) < n$. By (2.24) and (2.25), it follows that $x_j \in A_j^{f(j)-1}(S_j \setminus S_j^2)$ and $x_i \in U_{i,f(j)}$ for all $i \in \{1, \ldots, n\}$. Therefore

$$\begin{aligned} \mathbf{x} &\in U_{1,f(j)} \times \cdots \times U_{j-1,f(j)} \times \big(A_j^{f(j)-1}(S_j \setminus S_j^2)\big) \times U_{j+1,f(j)} \times \cdots \times U_{n,f(j)} \\ &\subseteq \mathbf{C}_1 \cdots \mathbf{C}_{f(j)-1} \cdot \big(S_1 \times \cdots \times S_{j-1} \times (S_j \setminus S_j^2) \times S_{j+1} \times \cdots \times S_n\big) \subseteq \langle \Gamma \rangle. \end{aligned}$$

Case 3: $j = 1$ and $f(j) < n$. By (2.24), $x_1 \in (S_1 \setminus S_1^2)B_1^{f(1)-1}$. Therefore, by Lemma 2.21,

$$\mathbf{x} \in \big((S_1 \setminus S_1^2)B_1^{f(1)-1}\big) \times S_2 \times \cdots \times S_n \subseteq \langle \Gamma \rangle.$$

Case 4: $1 < j < f(j) < n$. By (2.24) and (2.25), it follows that $x_j \in A_j^{j-1}(S_j \setminus S_j^2)B_j^{f(j)-j}$ and $x_i \in U_{i,j}$ for all $i \in \{1, \ldots, n\}$. Therefore

$$\begin{aligned}\mathbf{x} &\in U_{1,j} \times \cdots \times U_{j-1,j} \times \big(A_j^{j-1}(S_j \setminus S_j^2)B_j^{f(j)-j}\big) \times U_{j+1,j} \times \cdots \times U_{n,j} \\ &= \mathbf{C}_1 \cdots \mathbf{C}_{j-1} \cdot \big(B_1 \times \cdots \times B_{j-1} \times (S_j \setminus S_j^2)B_j^{f(j)-j} \times X_{j+1} \times \cdots \times X_n\big) \\ &\subseteq \mathbf{C}_1 \cdots \mathbf{C}_{j-1} \cdot \big(S_1 \times \cdots \times S_{j-1} \times (S_j \setminus S_j^2)B_j^{f(j)-j} \times S_{j+1} \times \cdots \times S_n\big) \subseteq \langle \Gamma \rangle,\end{aligned}$$

by Lemma 2.21.

In all cases, $\mathbf{x} \in \langle \Gamma \rangle$. Since $\mathbf{x}$ was arbitrary, it follows that $X_1 \times \cdots \times X_n \subseteq \langle \Gamma \rangle$. □

We may now prove Theorem 2.17, using Lemmas 2.21 and 2.23.

Proof of Theorem 2.17. The statement is clearly true when $n = 1$, so suppose that $n > 1$. Certainly $\langle \Gamma \rangle \leq S_1 \times \cdots \times S_n$. Therefore it remains to prove that the reverse inclusion holds, so let $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ be arbitrary. We split into two cases.

In the first case, assume that, for all $i \in \{1, \ldots, n\}$, every expression for s_i in S_i includes only decomposable elements. For each index $i \in \{1, \ldots, n\}$, by Lemma 2.8, s_i can be expressed as an arbitrarily long product in $S_i^2 \cap X_i$. Choose any such expression that is non-trivial, and define $l(i) > 1$ to be the length of this expression. Let $i \in \{1, \ldots, n\}$ be arbitrary. Then $s_i = x_1 x_2 \cdots x_{l(i)}$ for some decomposable generators $x_1, x_2, \ldots, x_{l(i)} \in S_i^2 \cap X_i$. Since $(S_i^2 \cap X_i) \subseteq A_i X_i$, we may replace x_1 by ay, for some $a \in A_i$ and $y \in X_i$, which gives a new expression $ayx_2 \ldots x_{l(i)}$ for s_i. It follows by assumption that y is decomposable. We may then replace y by $a'y'$, for some $a' \in A_i$ and $y' \in S_i^2 \cap X_i$, and we may continue in this fashion indefinitely. In this way, we can obtain an expression for s_i consisting of any number of elements of A_i, followed by $l(i)$ elements of $S_i^2 \cap X_i$. Additionally or alternatively, we may replace $x_{l(i)}$ by zb, for some $z \in S_i^2 \cap X_i$ and $b \in B_i$, and we may then replace z by $z'b'$, for some $z' \in S_i^2 \cap X_i$ and $b' \in B_i$, and so on. It follows that, for any $i \in \{1, \ldots, n\}$ and $k, m \in \mathbb{N}$, we may express s_i as an element of the sets

$$A_i^k (S_i^2 \cap X_i)^{l(i)}, \quad (S_i^2 \cap X_i)^{l(i)} B_i^m, \quad \text{and} \quad A_i^k (S_i^2 \cap X_i)^{l(i)} B_i^m.$$

In particular, if $\mathbf{C}_i = \big(B_1 \times \cdots \times B_{i-1} \times (S_i^2 \cap X_i) \times A_{i+1} \times \cdots \times A_n\big)^{l(i)}$ for each i, then

$$(s_1, \ldots, s_n) \in \mathbf{C}_1 \cdots \mathbf{C}_n \subseteq \langle \Gamma \rangle.$$

For the final case, assume that, for some $i \in \{1, \ldots, n\}$, there exists an expression for s_i that includes an indecomposable element. For each index $i \in \{1, \ldots, n\}$ where this is the case, we may choose an expression for s_i in X_i that includes an element of $(S_i \setminus S_i^2)$. Choose j from these indices such that the expression for s_j has minimal length amongst these expressions. By Lemma 2.8, for the indices $i \in \{1, \ldots, n\}$ for which we have not already chosen an expression, we may express s_i as a product in $S_i^2 \cap X_i$ that is strictly longer than the expression for s_j. For $i \in \{1, \ldots, n\}$, let $l(i)$ be the length of the expression for s_i. Note that, by construction, $l(j) = \min \big\{l(i) : i \in \{1, \ldots, n\}\big\}$. For each $i \in \{1, \ldots, n\}$ and $p \in \{1, \ldots, l(i)\}$, let $x_i^p \in X_i$ denote the p^{th} term in the expression for s_i, i.e. $s_i = x_i^1 x_i^2 \cdots x_i^{l(i)}$. Choose any $k \in \{1, \ldots, l(j)\}$

such that x_j^k is indecomposable, and define $r = l(j) - k$. Then for any $i \in \{1, \ldots, n\}$,

$$s_i = x_i^1 \cdots x_i^{k-1} \left(\prod_{\nu=k}^{l(i)-r} x_i^\nu \right) x_i^{l(i)-r+1} \cdots x_i^{l(i)}.$$

In particular, this gives an expression for s_i as a product of $l(j)$ elements of S_i, where each term, except for possibly the k^{th} term, is contained in X_i. We use these expressions to give $(s_1, \ldots, s_n)$ as a product of length $l(j)$ in $\langle \Gamma \rangle$:

$$\begin{aligned}(s_1, \ldots, s_n) = \ & (x_1^1, \ldots, x_{j-1}^1, x_j^1, x_{j+1}^1, \ldots, x_n^1) \\ & \cdots \\ & (x_1^{k-1}, \ldots, x_{j-1}^{k-1}, x_j^{k-1}, x_{j+1}^{k-1}, \ldots, x_n^{k-1}) \\ & \left(\prod_{\nu=k}^{l(1)-r} x_1^\nu, \ldots, \prod_{\nu=k}^{l(j-1)-r} x_{j-1}^\nu, x_j^k, \prod_{\nu=k}^{l(j+1)-r} x_{j+1}^\nu, \ldots, \prod_{\nu=k}^{l(n)-r} x_n^\nu \right) \\ & \left(x_1^{l(1)-r+1}, \ldots, x_{j-1}^{l(j-1)-r+1}, x_j^{k+1}, x_{j+1}^{l(j+1)-r+1}, \ldots, x_n^{l(n)-r+1} \right) \\ & \cdots \\ & \left(x_1^{l(1)}, \ldots, x_{j-1}^{l(j-1)}, x_j^{l(j)}, x_{j+1}^{l(j+1)}, \ldots, x_n^{l(n)} \right).\end{aligned}$$

Each term in this product, except for possibly the k^{th} term, is contained in $X_1 \times \cdots \times X_n$, which is a subset of $\langle \Gamma \rangle$ by Lemma 2.23. Finally, the k^{th} term

$$\left(\prod_{\nu=k}^{l(1)-r} x_1^\nu, \ \ldots, \prod_{\nu=k}^{l(j-1)-r} x_{j-1}^\nu, \ x_j^k, \prod_{\nu=k}^{l(j+1)-r} x_{j+1}^\nu, \ \ldots, \prod_{\nu=k}^{l(n)-r} x_n^\nu \right)$$

is contained in $S_1 \times \cdots \times S_{j-1} \times (S_j \setminus S_j^2) \times S_{j+1} \times \cdots \times S_n \subseteq \Gamma$, since we have chosen k so that x_j^k is indecomposable. Therefore $(s_1, s_2, \ldots, s_n) \in \langle \Gamma \rangle$, as required. □

2.5 Two finitely generated surjective semigroups

In the final section of this chapter, we restrict our attention to describing generating sets for direct products of pairs of finitely generated surjective semigroups. Generating sets for direct products of this kind are given in Proposition 2.15, and generating sets for direct products involving an arbitrary finite number of surjective semigroups are given in Theorem 2.17. Moreover, the generating sets given by these results have minimal cardinality in certain instances; see Example 2.19. Nonetheless, a generating set for a direct product that is specified by Proposition 2.15 or Theorem 2.17 may be significantly larger than necessary; see Example 2.26. This suggests that it may be possible to improve upon Proposition 2.15 and Theorem 2.17 in some cases.

Example 2.26. Let $m, n \in \mathbb{N}$ be arbitrary, let S be a right zero semigroup of order m, and let T be a left zero semigroup of order n. Note that S and T are finite surjective semigroups that have no proper generating subsets. Using the notation of Proposition 2.15, we find that the only choices for the sets B and A are the semigroups S and T themselves, respectively. Therefore, the only generating set for $S \times T$ that can be directly obtained from Proposition 2.15 is set of all elements, which has cardinality mn. However, $S \times T$ is an $n \times m$ rectangular band (see [76, Theorem 1.1.3]), and so the minimal cardinality of a generating set for $S \times T$ is $\max\{m, n\}$. Note that $\max\{m, n\} \leq mn$, and indeed $\max\{m, n\} < mn$ when $m, n > 1$.

Let $S = \langle X \rangle$ and $T = \langle Y \rangle$ be surjective semigroups. Roughly speaking, the fundamental idea behind Proposition 2.15 is to extend the generating sets X and Y to generating sets $X \cup B$ and $Y \cup A$ in such a way that, given any pair $(s, t) \in S \times T$, s and t can be systematically expressed as equal-length products over $X \cup B$ and $A \cup Y$, respectively. By doing so, it is possible to describe a generating set for $S \times T$ in terms of the sets $X \times A$ and $B \times Y$.

In the main result of this section, Theorem 2.31, we present an alternative approach (in the case that the generating sets X and Y are finite) to extending X and Y. We do so by taking into account the Green's structure of the semigroups. More specifically, the set A requires either one or two elements per maximal $\mathscr{R}$-class of T, and the set B requires requires either one or two elements per maximal $\mathscr{L}$-class of S. This allows us to give an upper bound on $\operatorname{rank}(S \times T)$ that involves the ranks of S and T as well as the Green's structure of S and T; see Corollary 2.33. For certain semigroups, a generating set described by Theorem 2.31 may be smaller than a generating set described by Proposition 2.15; see Example 2.32.

Our approach also suggests a direction for further improvements to Theorem 2.17, and, consequently, improved methods for computing direct products. However, in this section, we do not address the problem of constructing generating sets computationally.

In order to prove the main result of this section, we require the following lemmas, which we use when describing the Green's structure of a finitely generated surjective semigroup.

Lemma 2.27. *Let S be a semigroup generated by a set X, and let $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{J}\}$. If K is a maximal $\mathscr{K}$-class of S, then $K \cap X \neq \varnothing$. In particular, if S is finitely generated, then the number of maximal $\mathscr{K}$-classes of S is finite.*

Proof. We prove the result for $\mathscr{L}$; the proof for $\mathscr{R}$ is dual, and the proof for $\mathscr{J}$ is similar. Suppose that L is a maximal $\mathscr{L}$-class of S, let $x \in L$ be arbitrary, and let $x = x_1 \cdots x_n$ be a factorization of x over X. It follows that $L \leq L_{x_n}$ by Lemma 1.12(i), and so $x_n \in L$ by the maximality of L. $\square$

Lemma 2.28. *Let S be a semigroup with a finite generating set X, and let $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{J}\}$. For every $\mathscr{K}$-class K of S, there exists a maximal $\mathscr{K}$-class K' of S such that $K \leq K'$.*

Proof. Again, we prove only the result concerning $\mathscr{L}$. Let $L \in S/\mathscr{L}$ and $x \in L$ be arbitrary, and let $x = x_1 \cdots x_n$ be a factorization of x over X. It follows that $L \leq L_{x_n}$ by Lemma 1.12(i). If L_{x_n} is maximal, then we are done. Otherwise, there exists an $\mathscr{L}$-class $L' \in S/\mathscr{L}$ such that $L_{x_n} < L'$, and again, $L' \leq L_{x'}$ for some generator $x' \in X$. Therefore $L_{x_n} < L_{x'}$, which implies that $x \neq x_n$. Continuing in this way, either we find a maximal $\mathscr{L}$-class L_y for some $y \in X$ such that $L \leq L_y$, or we obtain an arbitrarily long strictly ascending chain of $\mathscr{L}$-classes of S, each of which contains a distinct element of X. Since X is finite, the latter is not possible. $\square$

In the following lemma, we show that every element in a finitely generated surjective semigroup can be written as a left multiple of an element of a maximal $\mathscr{L}$-class, and as a right multiple of an element of a maximal $\mathscr{R}$-class.

Lemma 2.29. *Let S be a surjective semigroup with finite generating set X, and let Λ and Ω be transversals of the sets*

$$\big\{L \cap X \,:\, L \text{ is a maximal } \mathscr{L}\text{-class of } S\big\} \quad \text{and} \quad \big\{R \cap X \,:\, R \text{ is a maximal } \mathscr{R}\text{-class of } S\big\},$$

respectively. Then $S = S\Lambda = \Omega S$.

Proof. Certainly $S\Lambda \subseteq S$ and $\Omega S \subseteq S$. We prove that $S \subseteq S\Lambda$; the proof that $S \subseteq \Omega S$ is dual. Let $s \in S$ be arbitrary. Then $s = ux$ for some $u \in S$ and $x \in X$, since s is decomposable. By Lemmas 2.27 and 2.28, there exists a maximal $\mathscr{L}$-class $L_{x'}$ of S, for some $x' \in X$, such that $L_x \leq L_{x'}$. By construction, there exists some $y \in L_{x'} \cap \Lambda$. Therefore $L_x \leq L_y$, i.e. $x \in S^1 y$. In particular, $s = ux \in SS^1\Lambda = S\Lambda$. □

In the main theorem of this section, we require relative left or right identities of the elements of the sets Λ and Ω from Lemma 2.29. Certainly any regular element x has both a relative left identity and a relative right identity, since if x' is an inverse of x, then $x = (xx')x = x(x'x)$. In the following lemma, we show that elements of maximal $\mathscr{L}$-classes and elements of maximal $\mathscr{R}$-classes of a surjective semigroup have relative left and right identities, respectively, regardless of regularity.

Lemma 2.30. *Let S be a surjective semigroup. Then every element in a maximal $\mathscr{L}$-class of S has a relative left identity, and every element in a maximal $\mathscr{R}$-class of S has a relative right identity.*

Proof. Let L be a maximal $\mathscr{L}$-class of S, and let $l \in L$. Since S is surjective, there exist elements $a, b \in S$ such that $l = ab$. It follows by Lemma 1.12(i) that $L \leq L_b$, and so the maximality of L implies that $b \in L$. Therefore $b = ul$ for some $u \in S^1$. It follows that $l = ab = (au)l$, i.e. au is a relative left identity for l. The statement concerning $\mathscr{R}$ is dual. □

We may now state and prove the main result of this section.

Theorem 2.31. *Let S and T be surjective semigroups with finite generating sets X and Y, respectively. Let Λ be a transversal of $\{L \cap X : L \text{ is a maximal } \mathscr{L}\text{-class of } S\}$, and let Ω be a transversal of $\{R \cap Y : R \text{ is a maximal } \mathscr{R}\text{-class of } T\}$. For each regular $l \in \Lambda$, choose a relative right identity e_l for l, and for each non-regular $l \in \Lambda$, choose a relative left identity f_l for l. Similarly, for each regular $r \in \Omega$, choose a relative left identity u_r for r, and for each non-regular $r \in \Omega$, choose a relative right identity v_r for r. Define*

$$B = \{e_l \,:\, l \in \Lambda \text{ and } l \text{ is regular}\} \cup \{f_l,\ l \,:\, l \in \Lambda \text{ and } l \text{ is non-regular}\}$$

and

$$A = \{u_r \,:\, r \in \Omega \text{ and } r \text{ is regular}\} \cup \{v_r,\ r \,:\, r \in \Omega \text{ and } r \text{ is non-regular}\}.$$

Then $S \times T$ is generated by the set $\Psi = (X \times A) \cup (B \times Y)$.

Proof. That the sets $\Lambda \subseteq X$ and $\Omega \subseteq Y$ are finite and non-empty follows by Lemmas 2.27 and 2.28; the relative left and right identities exist by Lemma 2.30 and the preceding discussion.

It suffices to show that $S \times T \subseteq \langle \Psi \rangle$. Let $(s,\, t) \in S \times T$. By Lemma 2.29, $s \in S\Lambda$ and $t \in T\Omega$. Therefore, $s = x_1 \cdots x_m l$ for some $x_1, \ldots, x_m \in X$ and $l \in \Lambda$, and similarly $t = r y_1 \cdots y_n$ for some $y_1, \ldots, y_n \in Y$ and $r \in \Omega$. Since S and T are surjective semigroups, we may assume without loss of generality that $m, n > 1$. The proof that $(s,\, t) \in \langle \Psi \rangle$ concludes with four cases, according to the regularity of l and r.

Case 1: l and r are regular.

$$\begin{aligned}(s,\, t) &= \left(x_1 \cdots x_m l e_l^{n+1},\ u_r^{m+1} r y_1 \cdots y_n\right) \\ &= (x_1,\, u_r) \cdots (x_m,\, u_r)\,(l,\, u_r)\,(e_l,\, r)\,(e_l,\, y_1) \cdots (e_l,\, y_n) \in \langle \Psi \rangle.\end{aligned}$$

Case 2: l is regular but r is not.

$$\begin{aligned}(s,\, t) &= (x_1 \cdots x_m l e_l^{n},\ r v_r^{m} y_1 \cdots y_n) \\ &= (x_1,\, r)\,(x_2,\, v_r) \cdots (x_m,\, v_r)\,(l,\, v_r)\,(e_l,\, y_1) \cdots (e_1,\, y_n) \in \langle \Psi \rangle.\end{aligned}$$

Case 3: r is regular but l is not.

$$\begin{aligned}(s,\, t) &= (x_1 \cdots x_m f_l^n l,\ u_r^m r y_1 \cdots y_n)\\ &= (x_1,\, u_r) \cdots (x_m,\, u_r)\,(f_l,\, r)\,(f_l,\, y_1) \cdots (f_l,\, y_{n-1})\,(l,\, y_n) \in \langle \Psi \rangle.\end{aligned}$$

Case 4: neither l nor r is regular.

$$\begin{aligned}(s,\, t) &= \big(x_1 \cdots x_m f_l^{n-1} l,\ r v_r^{m-1} y_1 \cdots y_n\big)\\ &= (x_1,\, r)\,(x_2,\, v_r) \cdots (x_m,\, v_r)\,(f_l,\, y_1) \cdots (f_l,\, y_{n-1})\,(l,\, y_n) \in \langle \Psi \rangle. \qquad \square\end{aligned}$$

Example 2.32. Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. We define $S = \mathcal{T}_n \setminus \mathcal{S}_n$ to be the semigroup consisting of all transformations of degree n that are not permutations, and let X be the generating set for S consisting of all $n(n-1)$ idempotents of rank $n-1$. We also define T to be a $2 \times n$ rectangular band, generated by a set Y consisting of n elements. Note that S and T are finite regular semigroups; S has n maximal $\mathscr{L}$-classes, and T has 2 maximal $\mathscr{R}$-classes. Therefore, a generating set for $S \times T$ specified by Theorem 2.31 contains at most $2|X| + n|Y| = 2n(n-1) + n^2 = n(3n-2)$ elements. On the other hand, choosing $A = Y$ and $B = X$, it follows by Proposition 2.15 that $(X \times A) \cup (B \times Y) = X \times Y$ is a generating set for $S \times T$; this set contains $n^2(n-1)$ elements. It follows that, at least for $n \geq 4$, a generating set given by Proposition 2.15 can be larger than the largest generating set given by Theorem 2.31. Although it is possible to construct smaller generating sets for $S \times T$ with Proposition 2.15, it is not clear from the statement of the proposition that this is the case.

By applying Theorem 2.31 to finitely generated surjective semigroups defined by generating sets of minimal cardinality, we obtain the following corollary.

Corollary 2.33. *Let S and T be finitely generated surjective semigroups. Define l to be the number of maximal $\mathscr{L}$-classes of S that are regular, and l' to be the number of maximal $\mathscr{L}$-classes of S that are not. Similarly, define r to be the number of maximal $\mathscr{R}$-classes of T that are regular, and r' to be the number of maximal $\mathscr{R}$-classes of T that are not. Then*

$$\operatorname{rank}(S \times T) \leq \operatorname{rank}(S) \cdot (r + 2r') + (l + 2l') \cdot \operatorname{rank}(T).$$

In particular, if every maximal $\mathscr{L}$-class of S and every maximal $\mathscr{R}$-class of T is regular, then

$$\operatorname{rank}(S \times T) \leq \operatorname{rank}(S) \cdot r + l \cdot \operatorname{rank}(T).$$

A finite semigroup contains at least one maximal $\mathscr{J}$-class, and a maximal $\mathscr{J}$-class in a finite surjective semigroup is a regular $\mathscr{D}$-class by Corollary 2.10; such a $\mathscr{D}$-class consists of regular maximal $\mathscr{L}$- and $\mathscr{R}$-classes. It follows that, in a finite surjective semigroup, there exists at least one regular maximal $\mathscr{L}$-class and at least one regular maximal $\mathscr{R}$-class.

Let S be a finitely generated semigroup. By Lemma 2.27, the number of maximal $\mathscr{L}$-, $\mathscr{R}$-, or $\mathscr{J}$-classes of S is at most $\operatorname{rank}(S)$. Therefore, if S is a finitely generated semigroup each of whose l maximal $\mathscr{L}$-classes is regular, and T is a finitely generated semigroup each of whose r maximal $\mathscr{R}$-classes is regular, then (since S and T are surjective) the upper bound on the rank of $S \times T$ given by Corollary 2.33 is no worse than the bound given by Corollary 2.18 and [2, Corollary 3.6]. More specifically,

$$\operatorname{rank}(S \times T) \leq \operatorname{rank}(S) \cdot r + l \cdot \operatorname{rank}(T) \leq 2\operatorname{rank}(S)\operatorname{rank}(T).$$

There are obvious analogues to Theorem 2.31 and Corollary 2.33 that involve the maximal $\mathscr{R}$-classes of S and the maximal $\mathscr{L}$-classes of T, rather than the maximal $\mathscr{L}$-classes of S and the maximal $\mathscr{R}$-classes of T, where S and T are finitely generated surjective semigroups.

Open Problem 2.34. Generalise Theorem 2.31 and Corollary 2.33 for direct products of an arbitrary number of finitely generated surjective semigroups. In particular, develop an upper bound for the rank of such a direct product that involves the ranks and the numbers of maximal $\mathscr{L}$- and $\mathscr{R}$-classes of the factors.

We end this chapter by posing the following easy-to-state problem concerning direct products of semigroups. As far as we are aware, this problem has not yet been considered in the literature.

Open Problem 2.35. Let S be an arbitrary finite semigroup. Develop practical methods for computing whether there exist non-trivial semigroups T and U such that $S \cong T \times U$; given this, develop methods for finding such semigroups T and U when they exist.

Chapter 3

Rees 0-matrix semigroups over a semigroup

3.1 Introduction

Given a semigroup T, non-empty sets I and Λ, and a $\Lambda \times I$ matrix P with entries in $T^0 = T \cup \{0\}$, the Rees 0-matrix semigroup $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ is the set $(I \times T \times \Lambda) \cup \{0\}$ with multiplication defined, as in (1.8), by

$$\begin{aligned} &(i,\, t,\, \lambda)\,(j,\, u,\, \mu) = \begin{cases} 0 & \text{if } p_{\lambda,j} = 0, \\ (i,\, tp_{\lambda,j}u,\, \mu) & \text{if } p_{\lambda,j} \neq 0, \end{cases} \\ \text{and} \qquad &(i,\, t,\, \lambda)\, 0 = 0\, (i,\, t,\, \lambda) = 0^2 = 0, \end{aligned}$$

for all $(i,\, t,\, \lambda),\ (j,\, u,\, \mu) \in I \times T \times \Lambda$. In this chapter, we investigate various semigroup-theoretic properties of Rees 0-matrix semigroups defined over arbitrary semigroups, with a particular emphasis on developing methods for computing with these kinds of semigroups.

The original interest in Rees 0-matrix semigroups stems from their deep association with the completely 0-simple semigroups. By the Rees Theorem [76, Theorem 3.2.3], a semigroup is completely 0-simple if and only if it is isomorphic to a regular Rees 0-matrix semigroup $\mathscr{M}^0[G;\, I,\, \Lambda;\, P]$, where G is a group. The principal factor of a $\mathscr{J}$-class in any finite semigroup is either a zero semigroup, or is a completely 0-simple semigroup [76, Theorem 3.1.6]. In essence, therefore, completely 0-simple semigroups, and correspondingly regular Rees 0-matrix semigroups over groups, can be thought of as the building blocks of finite semigroups. Because of this connection, many problems in finite semigroup theory reduce to, or at least involve, problems with Rees 0-matrix semigroups. For example, as noted in [65], any minimal generating set for a finite semigroup necessarily contains minimal generating sets for the principal factors of each of its maximal $\mathscr{J}$-classes. Therefore, in order to develop techniques for describing minimal generating sets of an arbitrary finite semigroup, we first require this problem to be solved for finite regular Rees 0-matrix semigroups over groups; this topic has been extensively researched, principally by Gray and Ruškuc, see [63,65,114] for more information. Rees 0-matrix semigroups are also central to the problem of computing maximal subsemigroups of finite semigroups, which is the topic of the next chapter. More precisely, the algorithm presented in the next chapter for computing maximal subsemigroups of an arbitrary semigroup relies, amongst other things, on an algorithm for computing the maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group. Indeed, the main purpose of Section 3.2 is to provide some necessary background information and results about these kinds of Rees 0-matrix semigroups.

Since regular Rees 0-matrix semigroups over groups are so well-understood, and of such importance to finite semigroup theory, there are well-developed methods for computing with these kinds of semigroups in the computational algebra system GAP [58]. Moreover, in GAP, if it is possible to construct a semigroup T and a $\Lambda \times I$ matrix P with entries in T^0, then it is possible the construct the Rees 0-matrix semigroup $\mathscr{M}^0[T; I, \Lambda; P]$ in GAP, even when T is not a group. This naturally leads one to experiment computationally with Rees 0-matrix semigroups over semigroups that are not groups, many of which turn out to exhibit a rich structure. However, since there is very little research on the topic of computing with Rees 0-matrix semigroups over semigroups that are not groups, there are few specialised methods for computing with them. Instead, GAP uses generic methods from computational semigroup theory, which often perform much more slowly than one might expect. The research in this chapter arose from performing experiments with Rees 0-matrix semigroups in GAP, and identifying areas where the theory, and associated computational methods, could be improved.

As with a direct product of semigroups, a Rees 0-matrix semigroup is defined in terms of its elements. This brings certain advantages for computation: for example, it allows us to easily perform membership checking by computing in the underlying semigroup and index sets, since an arbitrary triple (a, b, c) is contained in the Rees 0-matrix semigroup $\mathscr{M}^0[T; I, \Lambda; P]$ if and only if $a \in I$, $b \in T$, and $c \in \Lambda$. Moreover, the order of a Rees 0-matrix semigroup $\mathscr{M}^0[T; I, \Lambda; P]$ is $|I| \cdot |T| \cdot |\Lambda| + 1$. On the other hand, a Rees 0-matrix semigroup does not necessarily come with a proper generating subset. In the current version of GAP, unless the underlying semigroup is a group, the default method for computing a generating set of a Rees 0-matrix semigroup returns the set of all elements or all non-zero elements, depending on whether the matrix contains zero. This is obviously undesirable. In Section 3.4, we address this difficulty by describing relatively small generating sets for certain Rees 0-matrix semigroups.

When computing with a Rees 0-matrix semigroup $\mathscr{M}^0[T; I, \Lambda; P]$, we aim to formulate the relevant problem in terms of the underlying semigroup T, the index sets I and Λ, and the matrix P. In other words, wherever possible, we wish to make use of the representation of the Rees 0-matrix semigroup in question. Since all of the information about the Rees 0-matrix semigroup $\mathscr{M}^0[T; I, \Lambda; P]$ is encoded in T, I, Λ, and P, this goal seems sensible, provided that it is feasible to compute with these components. Perhaps most importantly, we wish to avoid having to exhaustively enumerate every element of a Rees 0-matrix semigroup, especially since we can represent and compute with Rees 0-matrix semigroups that are far too large to exhaustively enumerate and store in memory, such as those defined over large groups. This is because representing a Rees 0-matrix semigroup $\mathscr{M}^0[T; I, \Lambda; P]$ on a computer only requires representing the underlying semigroup T, and storing a matrix that contains $|I| \cdot |\Lambda|$ elements. In particular, we can represent certain infinite Rees 0-matrix semigroups, whereas the elements of an infinite semigroup can never be enumerated by a computer.

As a general rule of thumb, if it is not feasible to compute with a semigroup T, then it is not reasonable to expect to perform a related computation with a Rees 0-matrix semigroup whose underlying semigroup is T. Therefore, throughout this chapter, when we consider a computation with a Rees 0-matrix semigroup, we implicitly assume that its components are such that they may be readily computed.

This chapter is structured as follows.

In Section 3.2, we introduce several concepts and results that are required in later parts of this chapter, and the next. In particular, we discuss isomorphisms and normalizations of Rees 0-matrix semigroups, and we describe an algorithm for computing a generating set for the idempotent generated subsemigroup of a finite Rees 0-matrix semigroup over a group.

In Section 3.3, we consider the Green's structure of Rees 0-matrix semigroups over arbitrary semigroups. In our attempt to classify those Rees 0-matrix semigroups whose Green's structure can be deduced from the index sets and the Green's structure of the underlying semigroup, we introduce the notions of row-regular and column-regular matrices. These notions prove to be important for the whole of this chapter.

In Section 3.4, we prove results concerning generating sets of Rees 0-matrix semigroups. We give relatively small generating sets for certain kinds of Rees 0-matrix semigroups over monoids, and we give generating sets for Rees 0-matrix semigroups with row-regular or column-regular matrices. We also characterise some of the indecomposable elements of a Rees 0-matrix semigroup, in terms of the matrix and the indecomposable elements of the underlying semigroup.

Finally, in Section 3.5, we completely classify when an arbitrary Rees 0-matrix semigroup is regular, inverse, or a monoid, and we discuss the properties of such Rees 0-matrix semigroups.

Although the topic of this chapter is Rees 0-matrix semigroups, Rees matrix semigroups (without the zero) are not forgotten. Many results concerning Rees matrix semigroups can be deduced from results concerning Rees 0-matrix semigroups. For example, the obvious analogues of the results of Section 3.3 hold for Rees matrix semigroups. Furthermore, if $S = \mathscr{M}[T;\, I,\, \Lambda;\, P]$ is a Rees matrix semigroup, then the corresponding Rees 0-matrix semigroup $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ is isomorphic to S^0, and so by Lemma 1.5, the generating sets of S are in an inclusion-preserving one-to-one correspondence with the generating sets of $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$. This observation allows the results of Section 3.4 to be adapted for Rees matrix semigroups, for instance.

3.2 Preliminaries

In this section, we present some results about Rees 0-matrix semigroups that will be useful in later parts of this thesis. In Section 3.2.1, we discuss isomorphisms of Rees 0-matrix semigroups over groups and monoids. This topic allows us to introduce the notion of *normalizations* of such Rees 0-matrix semigroups. This is a vital concept required for the algorithms presented in Section 4.3, which concern the maximal subsemigroups of finite regular Rees 0-matrix semigroups over groups. In Section 3.2.2, we describe an algorithm for computing a small generating set for the idempotent generated subsemigroup a finite Rees 0-matrix semigroup over a group. This has inherent interest, and is again useful for the algorithms presented in Section 4.3.

3.2.1 Isomorphisms and normalizations of Rees 0-matrix semigroups

In general, it is not straightforward to determine whether two arbitrary Rees 0-matrix semigroups are isomorphic, or to construct isomorphisms between isomorphic Rees 0-matrix semigroups. For example, consider the Rees 0-matrix semigroups

$$S_1 = \mathscr{M}^0[\mathcal{T}_1;\, \{1',\, 2'\},\, \{1,\, 2\};\, P] \quad \text{and} \quad S_2 = \mathscr{M}^0[\mathcal{T}_2;\, \{1'\},\, \{1\};\, Q],$$

where $\mathcal{T}_n$ denotes the full transformation monoid of degree n (see Section 1.3.5), and P and Q are 2×2 and 1×1 matrices, respectively, that consist entirely of zeroes. Then S_1 and S_2 are zero semigroups of order 5, and are therefore isomorphic. However, this is perhaps counter-intuitive, since the underlying semigroups $\mathcal{T}_1$ and $\mathcal{T}_2$ are non-isomorphic, and their index sets have different cardinalities. Describing isomorphisms between arbitrary Rees 0-matrix semigroups is beyond the scope of this thesis; in this section, we restrict our attention to isomorphisms of Rees 0-matrix semigroups over groups or monoids.

Rees 0-matrix semigroups over groups

As shown in the following result, the difference between isomorphic regular Rees 0-matrix semigroups over groups is much smaller than the general case: their underlying groups are required to be isomorphic, and their respective index sets are necessarily in bijective correspondence. There is, however, significant possible variation between their matrices.

Proposition 3.1 ([76, Theorem 3.4.1]). *Let $S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ and $S' = \mathscr{M}^0[H;\, J,\, M;\, Q]$ be regular Rees 0-matrix semigroups over groups. Then $S \cong S'$ if and only if there exists an isomorphism of groups $\theta : G \longrightarrow H$, bijections $\psi : I \longrightarrow J$ and $\chi : \Lambda \longrightarrow M$ such that $p_{\lambda,i} \neq 0$ whenever $q_{\lambda\chi,i\psi} \neq 0$, and elements $u_i,\, v_\lambda \in H$ such that*

$$p_{\lambda,i} \neq 0 \quad \Rightarrow \quad p_{\lambda,i}\theta = v_\lambda \cdot q_{\lambda\chi,i\psi} \cdot u_i$$

for all $i \in I$ and $\lambda \in \Lambda$.

This characterisation can be used, for instance, to help enumerate the completely 0-simple semigroups of a given order up to isomorphism. For our purposes, this classification is particularly useful since it suggests the possibility of *normalizing* a regular Rees 0-matrix semigroup over a group. In essence, every such Rees 0-matrix semigroup is isomorphic to a Rees 0-matrix semigroup where the entries of the matrix satisfy certain convenient properties. This idea was first explored by Graham; indeed, some authors say that a Rees 0-matrix semigroup has *Graham normal form* if its matrix satisfies the properties described by Graham in [62, Section 4]. The topic has subsequently been considered by several authors, including in [76, Section 3.4], [64, Theorem 2.58], and [110, Theorem 4.13.34]; we refer the interested reader to these works for a more thorough treatment of the topic than the one given here. The results given in the following proposition are sufficient for our purposes. Recall from Section 1.3.2 that the Graham-Houghton graph of the Rees 0-matrix semigroup $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$, where $(p_{\lambda,i})_{\lambda\in\Lambda, i\in I}$, is the graph $\big(I \cup \Lambda,\, \big\{\{i,\, \lambda\} \,:\, p_{\lambda,i} \neq 0\big\}\big)$; we assume that $I \cap \Lambda = \varnothing$.

Proposition 3.2 ([62, Section 4]). *Let $S' = \mathscr{M}^0[G;\, I,\, \Lambda;\, Q]$ be a finite regular Rees 0-matrix semigroup over a group G, and let $\{I_1,\, \ldots,\, I_n\}$ and $\{\Lambda_1,\, \ldots,\, \Lambda_n\}$ be partitions of I and Λ, respectively, such that $I_k \cup \Lambda_k$ is a connected component of the Graham-Houghton graph of S' for each k. Then there exists a matrix $P : \Lambda \times I \longrightarrow G^0$ such that $S' \cong S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ and the following hold for each k:*

(i) *there exist indices $i_k \in I_k$ and $\lambda_k \in \Lambda_k$ such that $p_{\lambda_k,i_k} = 1_G$;*

(ii) *the non-zero entries of the sub-matrix $P_k : \Lambda_k \times I_k \longrightarrow G^0$ of P generate the group $G_k = \big\{g \in G \,:\, (i_k,\, g,\, \lambda_k) \in F(S)\big\}$.*

We refer to a finite regular Rees 0-matrix semigroup $S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$, over a group G, whose matrix that satisfies properties (i) and (ii) in Proposition 3.2 as being *normalized*, and a *normalization* of a Rees 0-matrix semigroup is an isomorphism to a normalized Rees 0-matrix semigroup. If S is normalized, then by [62, Theorem 2], each connected component of the Graham-Houghton graph of S corresponds to a regular Rees 0-matrix semigroup

$$\mathscr{M}^0[G_k;\, I_k,\, \Lambda_k;\, P_k] = (I_k \times G_k \times \Lambda_k) \cup \{0\}, \tag{3.3}$$

and the idempotent generated subsemigroup of S is the union of these subsemigroups

$$F(S) = \{0\} \cup \bigcup_{k=1}^{n} (I_k \times G_k \times \Lambda_k)\,. \tag{3.4}$$

In Section 4.3, we simplify the task of describing the maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group by assuming (without loss of generality, according to Proposition 3.2) that the semigroup is normalized. Therefore, in order to implement the algorithms described in Chapter 4, we require the ability to compute a normalization of an arbitrary finite regular Rees 0-matrix semigroup $S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$, where G is a group. This can be done in a straightforward manner; we briefly and roughly describe the method that has been implemented in the SEMIGROUPS package [101] for GAP [58], by the author. A normalization of the Rees 0-matrix semigroup S can be found with the SEMIGROUPS package by applying the function `Normalization` to S.

First, it is necessary to calculate the connected components of the Graham-Houghton graph of S; in doing so, we obtain the sets I_k and Λ_k from the statement of Proposition 3.2. Finding the connected components of a graph is a classical problem in graph theory, and can be solved with a standard depth- or breath-first search algorithm; see [117, Section 4.1]. For a graph (V, E), an algorithm of this kind typically has time complexity $O(|V| + |E|)$, and so for the Graham-Houghton of S, this is at most $O(|I| \cdot |\Lambda|)$.

The remaining step consists of multiplying the entries in each row of P by the inverse of the first non-zero entry in that row, and multiplying the entries in each column of P by the inverse of the first non-zero entry in that column; this step can be accomplished with time complexity $O(|I| \cdot |\Lambda|)$. It is not especially difficult to see that, by choosing the rows and columns in a suitable order, the first non-zero entry in each row and column of the resulting normalized matrix is 1_G. In particular, Proposition 3.2(i) holds. Furthermore, when the first non-zero entry in each row and column of the matrix is 1_G, it can be shown that Proposition 3.2 also holds; proving this is difficult, and is the main result of [62, Section 4].

Rees 0-matrix semigroups over monoids

The classification of isomorphisms between regular Rees 0-matrix semigroups over groups given in Proposition 3.1 exploits the fact that the elements of the matrix are units. Since a monoid contains units, this suggests the possibility of partially generalising this proposition to include Rees 0-matrix semigroups over monoids that are not necessarily groups. We present and prove a generalisation of the converse implication of Proposition 3.1 below. In later parts of this chapter, we use Proposition 3.5 to introduce the notion of normalizing certain Rees 0-matrix semigroups over monoids; see Sections 3.4.2 and 3.5.3.

Proposition 3.5 (cf. [76, Theorem 3.4.1])**.** *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ and $S' = \mathscr{M}^0[U;\, J,\, M;\, Q]$ be Rees 0-matrix semigroups, where T and U are monoids, and $P = (p_{\lambda,i})_{\lambda\in\Lambda, i\in I}$ and $Q = (q_{\mu,j})_{\mu\in M, j\in J}$. If there exists an isomorphism $\theta : T \longrightarrow U$, bijections $\psi : I \longrightarrow J$ and $\chi : \Lambda \longrightarrow M$ such that $p_{\lambda,i} \neq 0$ whenever $q_{\lambda\chi,i\psi} \neq 0$, and units $u_i,\, v_\lambda \in U$ such that*

$$p_{\lambda,i} \neq 0 \quad \Rightarrow \quad p_{\lambda,i}\theta = v_\lambda \cdot q_{\lambda\chi,i\psi} \cdot u_i$$

for all $i \in I$ and $\lambda \in \Lambda$, then the function $\phi : S \longrightarrow S'$ defined by

$$0\phi = 0 \quad \text{and} \quad (i,\, t,\, \lambda)\,\phi = (i\psi,\, u_i(t\theta)v_\lambda,\, \lambda\chi)$$

for all $(i,\, t,\, \lambda) \in I \times T \times \Lambda$ is an isomorphism, and in particular, $S \cong S'$.

Proof. To show that ϕ is an homomorphism, let $x, y \in S$ be arbitrary. If $x = 0$ or $y = 0$, then $(xy)\phi = 0\phi = 0 = (x\phi)(y\phi)$. Suppose that $x = (i,\, t,\, \lambda)$, $y = (j,\, s,\, \mu) \in S \setminus \{0\}$. If $p_{\lambda,j} = 0$, then also $q_{\lambda\chi,j\psi} = 0$, and so again $(xy)\phi = 0\phi = 0 = (x\phi)(y\phi)$. Otherwise,

$$\begin{aligned}(x\phi)(y\phi) &= (i\psi,\, u_i(t\theta)v_\lambda,\, \lambda\chi)\,(j\psi,\, u_j(s\theta)v_\mu,\, \mu\chi)\\ &= (i\psi,\, u_i(t\theta)v_\lambda q_{\lambda\chi,j\psi}u_j(s\theta)v_\mu,\, \mu\chi)\\ &= (i\psi,\, u_i(t\theta)(p_{\lambda,j}\theta)(s\theta)v_\mu,\, \mu\chi)\\ &= (i\psi,\, u_i(tp_{\lambda,j}s)\theta v_\mu,\, \mu\chi)\\ &= (i,\, tp_{\lambda,j}s,\, \mu)\,\phi\\ &= (xy)\phi,\end{aligned}$$

and so ϕ is a homomorphism.

If $(i,\, t,\, \lambda)\, \phi = (j,\, s,\, \mu)\, \phi$, then in particular $i\psi = j\psi$ and $\lambda\chi = \mu\chi$. Since the functions ψ and χ are bijective, it follows that $i = j$ and $\lambda = \mu$. In addition, $u_i(t\theta)v_\lambda = u_i(s\theta)v_\lambda$. Using the fact that u_i and v_λ are units in U, we deduce that $t\theta = s\theta$. But θ is also bijective, which implies that $t = s$, and so ϕ is injective. Let $(i,\, t,\, \lambda) \in S' \setminus \{0\}$ be an arbitrary non-zero element, and define $j = i\psi^{-1}$ and $\mu = \lambda\chi^{-1}$. Then

$$\left(j,\, u_j^{-1}(t\theta^{-1})v_\mu^{-1},\, \mu\right)\phi = (i,\, t,\, \lambda)\,,$$

and so ϕ is surjective, and therefore bijective. In conclusion, ϕ is an isomorphism. □

In modern computer algebra systems such as GAP [58] and Magma [13], there are well-developed and practical methods for testing whether arbitrary finite groups are isomorphic, and for finding isomorphisms between isomorphic finite groups, see [20], for example. We do not concern ourselves with the details of how these tools function; however, because these tools exist, it is feasible to implement methods that build on Proposition 3.1 to test whether arbitrary finite regular Rees 0-matrix semigroups over groups are isomorphic, and to find isomorphisms between those semigroups that are isomorphic. Indeed, such functionality is available in the Semigroups [101] package for GAP. However, there do not currently exist practical methods for testing whether arbitrary finite monoids are isomorphic, or for finding isomorphisms between them. Because of this, Proposition 3.5 does not currently suggest a practical approach to computing isomorphisms between Rees 0-matrix semigroups over monoids.

3.2.2 The idempotent generated subsemigroup of a Rees 0-matrix semigroup

In this section, we consider the problem of quickly computing a small generating set for the idempotent generated subsemigroup of a finite Rees 0-matrix semigroup over a group. It appears to be difficult to find generating sets of minimal cardinality, see [64, Theorem 3.27], but doing so is not our goal. The generating set that we describe is small, in the sense that it does not necessarily include every idempotent of the semigroup, and it contains at most twice the minimal number of generators. The main result of this section is Proposition 3.7.

The work in this section was motivated by the algorithms presented in Chapter 4. In particular, in Algorithm 4.32, we describe a procedure for computing the maximal subsemigroups of an arbitrary finite regular Rees 0-matrix semigroup S over a group that intersect every $\mathscr{H}$-class of S non-trivially. The maximal subsemigroups that are returned by this algorithm are defined by generating sets. Crucially, for the purposes of this section, each of these sets includes a generating set for the idempotent generated subsemigroup of S.

As discussed in Section 1.4, the time complexities of many algorithms in computational semigroup theory are given in terms of the size of the generating set of the semigroup in question. Roughly speaking, this means that, given a semigroup S defined by a generating set, the smaller the generating set for S, the faster it is to compute with S. Therefore, in order to facilitate speedy computation with the maximal subsemigroups returned by Algorithm 4.32, it is desirable for these maximal subsemigroups to be defined by generating sets that are as small as is reasonably possible. One way to address this problem is to endeavour to produce a small generating set for the corresponding idempotent generated subsemigroup.

An obvious choice of a generating set for the idempotent generated subsemigroup of a finite Rees 0-matrix semigroup over a group is the set of all idempotents in the semigroup. However, the number of generators that are actually required can be smaller than this; see Example 3.6 for a very simple demonstration of this fact.

Example 3.6. Let $S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup, where G is the trivial group, $I = \{1',\, 2'\}$, $\Lambda = \{1,\, 2\}$, and

$$P = \begin{pmatrix} 1_G & 1_G \\ 1_G & 0 \end{pmatrix}.$$

There are four idempotents in S, and $(2',\, 1_G,\, 2)$ is the unique non-idempotent element. Since $(2',\, 1_G,\, 2) = (2',\, 1_G,\, 1)\,(1',\, 1_G,\, 2)$, it is clear that the idempotent generated subsemigroup of S is S itself. However, the pair of idempotents $\{(1',\, 1_G,\, 2)\,,\, (2',\, 1_G,\, 1)\}$ generates S.

In the proof of [64, Proposition 3.13], Gray observes that, for a finite idempotent generated Rees 0-matrix semigroup S over a group, a spanning tree of the Graham-Houghton graph of S corresponds to a generating set for S that consists of idempotents. Given (3.4), it is clear that a normalized finite Rees 0-matrix semigroup over a group G is idempotent generated if and only if its Graham-Houghton graph is connected, and its non-zero matrix entries generate the group G. Indeed, this result is stated in [64, Theorem 3.1]. Furthermore, in [64, Theorem 2.13], Gray proves that the minimal cardinality of a generating set for a finite idempotent generated Rees 0-matrix semigroup $\mathscr{M}^0[G;\, I,\, \Lambda;\, P]$, where G is a group, is $\max\{|I|,\, |\Lambda|\}$.

However, in Algorithm 4.32, we require a generating set for the idempotent generated subsemigroup of *any* finite regular Rees 0-matrix semigroup over a group, i.e. not only those that are idempotent generated. In Proposition 3.7, we generalise Gray's ideas concerning spanning trees in order to solve this problem.

Proposition 3.7. *Let $S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ be a finite Rees 0-matrix semigroup over a group G, and let $(I \cup \Lambda,\, E)$ be a spanning forest for the Graham-Houghton graph of S. Then the set*

$$X = \big\{ \big(i,\, p_{\lambda,i}^{-1},\, \lambda\big) \,:\, \{i,\, \lambda\} \in E \big\} \cup \{0\}.$$

generates $F(S)$, the idempotent generated subsemigroup of S. In particular, the smallest cardinality of a generating set of idempotents for $F(S)$ is at most $|I| + |\Lambda| - n + 1$, where n is the number of connected components of the Graham-Houghton graph of S.

Proof. Note that, since E is a subset of the edges of the Graham-Houghton graph of S, if $\{i,\, \lambda\} \in E$, then $p_{\lambda,i} \neq 0$. Hence the set X is well-defined. Furthermore, since X consists of idempotents, it follows that $\langle X \rangle \leq F(S)$. Therefore, to show that X generates $F(S)$, it suffices to show that $\langle X \rangle$ contains every idempotent of S.

Certainly X contains 0. The non-zero idempotents of S are the elements $\big(i,\, p_{\lambda,i}^{-1},\, \lambda\big)$, for each non-zero matrix entry $p_{\lambda,i}$, so let $p_{\lambda,i}$ be any such entry. If $\{i,\, \lambda\} \in E$, then $\big(i,\, p_{\lambda,i}^{-1},\, \lambda\big) \in X$ and we are done, so suppose otherwise. Since $\{i,\, \lambda\}$ is an edge in the Graham-Houghton graph of S, in particular the vertices i and λ are in the same connected component. Therefore, they are connected in the spanning forest, and so there exists a path

$$(i,\, \mu_1,\, j_1,\, \mu_2,\, \ldots,\, j_n,\, \lambda)$$

in the spanning forest from i to λ, for some $n \in \mathbb{N}$, $\mu_1,\, \ldots,\, \mu_n \in \Lambda$, and $j_1,\, \ldots,\, j_n \in I$. Consider the product of elements

$$x = \big(i,\, p_{\mu_1,i}^{-1},\, \mu_1\big)\big(j_1,\, p_{\mu_2,j_1}^{-1},\, \mu_2\big) \cdots \big(j_n,\, p_{\lambda,j_n}^{-1},\, \lambda\big). \tag{3.8}$$

Since $(i,\, \mu_1,\, j_1\, \ldots,\, j_n,\, \lambda)$ is a path in the spanning forest, it follows that each factor in (3.8) is contained in X, and the product itself is non-zero. In particular,

$$x = \big(i,\, p_{\mu_1,i}^{-1} p_{\mu_1,j_1} p_{\mu_2,j_2}^{-1} \cdots p_{\mu_n,\lambda_n} p_{\lambda,j_n}^{-1},\, \lambda\big) \in \langle X \rangle.$$

Each power of x has the form (i, g, λ), for some $g \in G$. However, S is finite, and so some power x^k of x is idempotent. In particular, $\left(i, p_{\lambda,i}^{-1}, \lambda\right) = x^k \in \langle X \rangle$, and so X generates $F(S)$.

If n is the number of connected components of the Graham-Houghton graph of S, then any spanning tree for the Graham-Houghton graph contains $|I| + |\Lambda| - n$ edges. Therefore the generating set X consists of $|I| + |\Lambda| - n + 1$ idempotents. It follows that a minimal generating set of idempotents for $F(S)$ contains at most this many elements. □

Note that Proposition 3.7 applies to any finite Rees 0-matrix semigroup over a group, including those that are not regular, and to those where the Graham-Houghton graph is not connected, as can be seen in the following example.

Example 3.9. Let $S = \mathscr{M}^0[\mathcal{S}_3; I, \Lambda; P]$ be a Rees 0-matrix semigroup over the symmetric group of degree 3, where $I = \{1', 2', 3', 4'\}$ and $\Lambda = \{1, 2, 3, 4, 5\}$, and P is given by

$$P = \begin{pmatrix} (1\ 3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & (2\ 3) & (1\ 2) & (1\ 2) \\ 0 & (1\ 3\ 2) & 0 & (2\ 3) \\ 0 & (1\ 3) & (1\ 2) & (1\ 2\ 3) \end{pmatrix}.$$

The Graham-Houghton graph of S, and a spanning forest, is shown in Figure 3.10. Since the Graham-Houghton graph has three connected components, there are $|I| + |\Lambda| - 3 = 6$ edges in any spanning forest, and so any generating set for $F(S)$ obtained with Proposition 3.7 contains seven elements. The generating set corresponding to the spanning forest Figure 3.10 is

$$\left\{0,\ (1', (1\ 3), 1)\,,\ (2', (2\ 3), 3)\,,\ (3', (1\ 2), 3)\,,\ (4', (1\ 2), 3)\,,\ (4', (2\ 3), 4)\,,\ (4', (1\ 3\ 2), 5)\right\}.$$

In total, there are ten idempotents in S. It is easy to see that, in this case, 0 can be expressed as a product of the non-zero generators, and is therefore redundant.

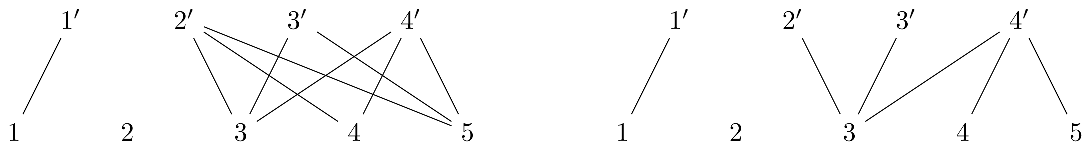

Figure 3.10: The graphs of the Rees 0-matrix semigroup S described in Example 3.9. On the left is the Graham-Houghton graph of S, with its vertices corresponding to I on the top row, and its vertices corresponding to Λ on the bottom. On the right is a spanning forest of the Graham-Houghton graph.

Let $S = \mathscr{M}^0[G; I, \Lambda; P]$ be a finite Rees 0-matrix semigroup over a group G. We may compare the generating set X for $F(S)$ described by Proposition 3.7 with the set of all idempotents of S. Let k and n be the number of edges and the number of connected components of the Graham-Houghton graph of S, respectively. As noted in the proof of Proposition 3.7, any spanning tree for the Graham-Houghton graph of S contains $|I| + |\Lambda| - n$ edges, which, by definition, is less than or equal to k. Therefore $|X| = |I| + |\Lambda| - n + 1 \leq k + 1$, and $k + 1$ is the number of idempotents of S. Note that the maximum possible value of k is $|I| \cdot |\Lambda|$, which occurs when P contains only non-zero entries. It follows that $k + 1$ can be much larger than $|X|$, and so in some cases, the generating set X may contain relatively few idempotents in comparison to the total number of idempotents of S.

In [64, Chapter 3], Gray is principally concerned with describing idempotent generating sets of minimal cardinality for certain kinds of finite regular Rees 0-matrix semigroups over groups; see especially [64, Theorem 3.27]. However, the relevant criteria for computing such minimal generating sets do not lend themselves well to computation. As is often the case, finding an optimal solution is significantly more difficult than finding a solution that is nearly optimal.

The generating set given by Proposition 3.7 does not necessarily give a generating set of smallest cardinality; nor does it necessarily even give an *idempotent* generating set of smallest cardinality; see Example 3.11, for instance. However, in a certain sense, the generating set described by Proposition 3.7 is close to being optimal, which we now discuss.

Let $S = \mathscr{M}^0[G; I, \Lambda; P]$ be a finite Rees 0-matrix semigroup where G is a group, let I' be the subset of indices $i \in I$ such that $p_{\lambda,i} \neq 0$ for some $\lambda \in \Lambda$, and let $\Lambda' \subseteq \Lambda$ be defined analogously. Clearly, any generating set for $F(S)$ contains at least one generator for each $i \in I'$, and at least one generator for each $\lambda \in \Lambda'$. In particular, any generating set for $F(S)$ contains at least $\max\{|I'|, |\Lambda'|\}$ elements (see also [114, Lemma 3.1]). On the other hand, the number of vertices of the Graham-Houghton graph that are contained in non-trivial connected components is $|I'| + |\Lambda'|$, and so the number of edges in any spanning forest is $|I'| + |\Lambda'| - m$, where m is the number of non-trivial connected components. If $m = 0$, then the only idempotent of S is 0, and the generating set $\{0\}$ given in Proposition 3.7 is minimal. Otherwise $m > 0$; in this case the number of elements in the generating set given in Proposition 3.7 is $|I'| + |\Lambda'| - m + 1$, and

$$|I'| + |\Lambda'| - m + 1 \leq |I'| + |\Lambda'| \leq 2 \cdot \max\{|I'|, |\Lambda'|\}.$$

In particular, the generating set contains at most two times as many elements as is necessary.

Example 3.11. Let S be the Rees 0-matrix semigroup defined in Example 3.6. The generating set given in Example 3.6 contains two elements, which are idempotents. However, the Graham-Houghton graph of S is itself a tree, and so the generating set described in Proposition 3.7 contains four elements. Therefore, this generating set is not minimal.

Importantly, as well as being reasonably small, the generating set described in Proposition 3.7 can be computed quickly; a simple description of an algorithm that implements this result is given in Algorithm 3.12.

Algorithm 3.12 Compute a generating set for the idempotent generated subsemigroup of a finite Rees 0-matrix semigroup over a group.

Input: A finite Rees 0-matrix semigroup $S = \mathscr{M}^0[G; I, \Lambda; P]$ over a group G.
Output: A generating set for the idempotent generated subsemigroup of S.

$X \leftarrow \varnothing$
$\Gamma \leftarrow$ the Graham-Houghton graph of S
$E \leftarrow$ the edges of a spanning forest of Γ
for each edge $\{i, \lambda\} \in E$ **do**
    $X \leftarrow X \cup \{(i, p_{\lambda,i}^{-1}, \lambda)\}$
if the induced subgraph of Γ on its non-isolated vertices is empty or complete bipartite **then**
    $X \leftarrow X \cup \{0\}$
return X.

Computationally, the only non-trivial step required by Algorithm 3.12 is the construction of a spanning forest for the Graham-Houghton graph of S. As with finding the connected components of a graph, which was discussed in Section 3.2.1, this can be done by using a standard depth- or breath-first search. Such algorithms have worst-case time complexity $O(|I| \cdot |\Lambda|)$, since $|I| \cdot |\Lambda|$ is the maximum number of edges in the Graham-Houghton graph of S.

The final part of Algorithm 3.12, on line 6, tests whether the induced subgraph of the Graham-Houghton graph of S on its non-isolated vertices is empty or is a complete bipartite graph. If this induced subgraph is empty, then S has no non-zero idempotents, and so certainly 0 is required in any generating set. Similarly, if the induced subgraph is complete bipartite, then the product of any number of non-zero idempotents is non-zero, and again 0 is required to be an element of the generating set. On the other hand, suppose that the induced subgraph is neither empty nor a complete bipartite graph. In this case, there exist non-isolated vertices $i \in I$ and $\lambda \in \Lambda$ such that $\{i, \lambda\}$ is not an edge of the Graham-Houghton graph, and there exist non-zero idempotents $\left(j, p_{\lambda,j}^{-1}, \lambda\right)$ and $\left(i, p_{\mu,i}^{-1}, \mu\right)$ in S whose product is 0. In particular, 0 is generated by the non-zero idempotents of S, and so 0 need not be a member of the generating set created by Algorithm 3.12. For example, given the Rees 0-matrix semigroup from Example 3.9, Algorithm 3.12 returns a generating set with six elements that does not contain 0.

Algorithm 3.12 has been implemented in the Semigroups package [101] for GAP [58] by the author. The idempotent generated subsemigroup of a finite Rees matrix or 0-matrix semigroup over a group can be computed with the function `IdempotentGeneratedSubsemigroup`.

For a finite Rees 0-matrix semigroup over an arbitrary semigroup, it is not clear how to develop algorithms for computing the idempotents, or a relatively small generating set for the idempotent generated subsemigroup, in terms of the underlying semigroup and the Graham-Houghton graph. We therefore end this section by posing the following problems.

Open Problem 3.13. Let S be a finite Rees 0-matrix semigroup over an arbitrary finite semigroup T, and assume that any necessary semigroup-theoretic properties of T are known *a priori*. Give algorithms for counting and listing the idempotents of S, using the properties of T and the Graham-Houghton graph of S.

Open Problem 3.14. Let S be a finite Rees 0-matrix semigroup over an arbitrary finite semigroup T, and assume that any necessary semigroup-theoretic properties of T are known *a priori*. Give an algorithm for constructing a generating set for $F(S)$ that contains at most a constant multiple of $\operatorname{rank}(F(S))$ elements.

3.3 The Green's structure of a Rees 0-matrix semigroup

As discussed in Chapter 1, one of the most important strategies for studying a semigroup is to analyse its Green's relations. Moreover, computing the Green's structure of a semigroup is a prerequisite for many further algorithms in computational semigroup theory, such as computing its maximal subsemigroups; see Chapter 4. Therefore, we wish to develop a framework for computing the Green's structure of an arbitrary Rees 0-matrix semigroup.

Of course, for any finite Rees 0-matrix semigroup $S = \mathscr{M}^0[T; I, \Lambda; P]$, the Green's structure on S can be computed by constructing and analysing the left and right Cayley digraphs of S, given sufficient time and memory. However, the order of S is $|I| \cdot |T| \cdot |\Lambda| + 1$, which may be significantly larger than the cardinality of any of I, T, or Λ. In particular, even when I, T, and Λ are relatively small, it may not be possible to store the elements of S in memory, which is required when constructing Cayley digraphs.

Furthermore, since S is defined by its underlying semigroup, index sets, and matrix, the structure of S is surely encoded in these components. Computing the Cayley digraphs of S ignores this; it would be sensible to attempt to take as much of this information into account as possible. In particular, we aim to be able to test for the Green's equivalence of elements of a Rees 0-matrix semigroup by analysing the underlying semigroup, the index sets, and the matrix, and without having to enumerate every element of the semigroup.

In any semigroup-with-zero S, the multiplicative zero 0_S forms a $\mathscr{J}$-class of S; it is the unique minimal element in the partial order of $\mathscr{J}$-classes of S. In particular, the Green's classes of 0 in a Rees 0-matrix semigroup are given immediately, and require no further computation. Therefore, it remains to describe the Green's relations on the non-zero elements of a Rees 0-matrix semigroup. For a regular Rees 0-matrix semigroup over a group, this is especially straightforward, and is stated in the following result.

Lemma 3.15. *Let $S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ be a regular Rees 0-matrix semigroup over a group G, and let $x = (i,\, g,\, \lambda)$, $y = (j,\, h,\, \mu) \in S \setminus \{0\}$ be arbitrary non-zero elements. The following hold:*

(i) *$x \mathscr{L}^S y$ if and only if $\lambda = \mu$;*

(ii) *$x \mathscr{R}^S y$ if and only if $i = j$;*

(iii) *$x \mathscr{H}^S y$ if and only if $i = j$ and $\lambda = \mu$; and*

(iv) *$x \mathscr{D}^S y$.*

Thus the Green's relations of a regular Rees 0-matrix semigroup $S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$, where G is a group, are defined very simply in terms of its index sets. The non-zero $\mathscr{L}$-classes of S are the sets $I \times G \times \{\lambda\}$ for each $\lambda \in \Lambda$, the non-zero $\mathscr{R}$-classes of S are the sets $\{i\} \times G \times \Lambda$ for each $i \in I$, the non-zero $\mathscr{H}$-classes of S are the sets $\{i\} \times G \times \{\lambda\}$ for each $i \in I$ and $\lambda \in \Lambda$, and the $\mathscr{D}$-classes of S are $\{0\}$ and $S \setminus \{0\}$. However, this is not true in general for an arbitrary Rees 0-matrix semigroup over a group, let alone for an arbitrary Rees 0-matrix semigroup, as demonstrated by the following example.

Example 3.16. Let $S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup over a group G, and suppose there exists $i \in I$ such that $p_{\mu,i} = 0$ for all $\mu \in \Lambda$. Then $x \cdot (i,\, g,\, \lambda) = 0$ for all $x \in S$, $g \in G$, and $\lambda \in \Lambda$. In particular, the $\mathscr{L}$-class of any element of $\{i\} \times G \times \Lambda$ is trivial.

The following lemma and its immediate corollary show a broad connection between the Green's relations on a Rees 0-matrix semigroup, its index sets, and the Green's relations on its underlying semigroup.

Lemma 3.17. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup, and let $x = (i,\, t,\, \lambda)$, $y = (j,\, u,\, \mu) \in S \setminus \{0\}$ be arbitrary non-zero elements. The following hold;*

(i) *if $x \mathscr{L}^S y$, then $t \mathscr{L}^T u$ and $\lambda = \mu$;*

(ii) *if $x \mathscr{R}^S y$, then $t \mathscr{R}^T u$ and $i = j$;*

(iii) *if $x \mathscr{H}^S y$, then $t \mathscr{H}^T u$, $i = j$, and $\lambda = \mu$; and*

(iv) *if $x \mathscr{D}^S y$, then $t \mathscr{D}^T u$.*

Proof. If $x = y$, then certainly $t \mathscr{L}^T u$ and $\lambda = \mu$. Therefore, to prove (i), suppose that $x \mathscr{L}^S y$ but that $x \neq y$. By definition, there exist $\alpha = (k,\, v,\, \nu)$, $\beta = (l,\, w,\, \eta) \in S$ such that $x = \beta y$ and $y = \alpha x$. Thus $(i,\, t,\, \lambda) = (l,\, wp_{\eta,j}u,\, \mu)$ and $(j,\, u,\, \mu) = (k,\, vp_{\nu,i}t,\, \lambda)$. In particular, $\lambda = \mu$, and $t = (wp_{\eta,j})u$ and $u = (vp_{\nu,i})t$ – i.e. $t \mathscr{L}^T u$.

Note that (ii) is the dual of (i), and (iii) follows from (i) and (ii) since $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$.

To prove (iv), recall that $\mathscr{D} = \mathscr{L} \circ \mathscr{R}$. Therefore, if $x \mathscr{D}^S y$, then there exists an element $z = (k,\, v,\, \nu) \in S$ such that $x \mathscr{L}^S z \mathscr{R}^S y$. Then $t \mathscr{L}^T v$ by (i), and $v \mathscr{R}^T u$ by (ii). Thus $t \mathscr{D}^T u$, as required. $\square$

Corollary 3.18. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. The following hold:*

(i) $|S/\mathscr{L}| \geq |T/\mathscr{L}| \cdot |\Lambda| + 1$*;*

(ii) $|S/\mathscr{R}| \geq |T/\mathscr{R}| \cdot |I| + 1$*;*

(iii) $|S/\mathscr{H}| \geq |T/\mathscr{H}| \cdot |I| \cdot |\Lambda| + 1$*; and*

(iv) $|S/\mathscr{D}| \geq |T/\mathscr{D}| + 1$.

Note that the additional Green's class counted in Corollary 3.18 in each instance is $\{0\}$.

Given two elements of an arbitrary Rees 0-matrix semigroup, Lemma 3.17 can be used to rule out the Green's equivalence of these elements, in certain cases. For example, a pair of elements with different first components are not $\mathscr{R}$-related, and a pair of elements are not $\mathscr{D}$-related if their middle components are not $\mathscr{D}$-related.

Ideally, we wish to produce an analogue of Lemma 3.15 that holds for an arbitrary finite semigroup, and which classifies the Green's equivalence of elements precisely in terms of separate conditions on the components of the elements. In full generality, the converse statements of Lemma 3.15 do not necessarily hold; see Example 3.19. In general, therefore, such an approach cannot simply involve only the underlying semigroup and the index sets; consideration must also be taken of the matrix.

Example 3.19. Let $\mathcal{T}_2$ denote the full transformation monoid of degree 2 (see Section 1.3.5), let $f \in \mathcal{T}_2$ be the constant transformation with $\mathrm{im}(f) = \{1\}$, and let P be a matrix whose sole entry is f. Define $S = \mathscr{M}^0[\mathcal{T}_2;\, \{i\},\, \{\lambda\};\, P]$. Since $1_{\mathcal{T}_2}$ and f are not $\mathscr{D}$-related in $\mathcal{T}_2$, it follows by Lemma 3.17 that $\big(i,\, 1_{\mathcal{T}_2},\, \lambda\big)$ and $\big(i,\, f,\, \lambda\big)$ are not $\mathscr{D}$-related in S. On the other hand, $1_{\mathcal{T}_2}$ and $(1\ 2)$ are $\mathscr{R}$-related in $\mathcal{T}_2$, where $(1\ 2)$ is the non-identity permutation of degree 2 written in disjoint cycle notation, but $\big(i,\, 1_{\mathcal{T}_2},\, \lambda\big)$ and $\big(i,\, (1\ 2),\, \lambda\big)$ are not $\mathscr{R}$-related in S. This is because any product involving f is a constant transformation, and so if $t \in \mathcal{T}_2$ is arbitrary, then

$$\big(i,\, 1_{\mathcal{T}_2},\, \lambda\big)\big(i,\, t,\, \lambda\big) = \big(i,\, ft,\, \lambda\big) \neq \big(i,\, (1\ 2),\, \lambda\big).$$

Therefore, any characterisation of the Green's relations on S requires detailed reference to P.

The characterisation of the Green's relations on a regular Rees 0-matrix semigroup over a group stated in Lemma 3.15, which is given in terms of its index sets, relies on the fact that the matrix of the Rees 0-matrix semigroup contains a non-zero entry in each row and column. As shown in Example 3.16, without this assumption, the characterisation does not necessarily hold. The matrix of the Rees 0-matrix semigroup considered in Example 3.19 contains a non-zero entry in each row and column, but nevertheless, Green's $\mathscr{R}$-relation on the semigroup is not classified by Lemma 3.17(ii) and its converse. In order to obtain a more general classification of Green's relations on certain Rees 0-matrix semigroups, that is analogous to Lemma 3.15, we need to impose more complicated conditions on the matrix than simply requiring the existence of non-zero entries. Finding appropriate conditions is the topic of the next section.

3.3.1 Rees 0-matrix semigroups with row- or column-regular matrices

In this section, we aim to develop conditions to impose on the matrix of an arbitrary Rees 0-matrix semigroup, so that its Green's relations may be simply characterised in terms of its index sets and in terms of the Green's relations on its underlying semigroup. Moreover, these conditions on the matrix should be readily computable, as should the resultant characterisation of the Green's relations. With this in mind, we introduce the concept of row-regular and column-regular matrices. The reason for this choice of terminology will become apparent in Section 3.5.1, where we prove that a Rees 0-matrix semigroup over an arbitrary semigroup is regular if and only if its underlying semigroup is regular, and its matrix is both row-regular and column-regular. Let s be an element of a semigroup S. Recall that a relative left identity for s is an element $e \in S$ such that $es = s$; a relative right identity for s is defined analogously.

Definition 3.20 (Row-regular matrix)**.** Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. The matrix P is *row-regular* if, for all $\lambda \in \Lambda$ and $t \in T$, there exists $i \in I$ such that $p_{\lambda,i}T$ contains a relative right identity for t.

Definition 3.21 (Column-regular matrix)**.** Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. The matrix P is *column-regular* if, for all $i \in I$ and $t \in T$, there exists $\lambda \in \Lambda$ such that $Tp_{\lambda,i}$ contains a relative left identity for t.

In order for the notions of row- and column-regularity to be useful for our purposes, it must be practical to check whether or not a given matrix is row- or column-regular. Given only the definition, it is not immediately clear how to do this – on the face of it, testing whether the matrix P of a Rees 0-matrix semigroup $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ is row-regular entails, for each $t \in T$ and $\lambda \in \Lambda$, searching for a relative right identity for t in as many as $|I|$ right ideals of T.

In order to provide more useful reformulations of row-regularity, we prove the following lemma, which holds for any Rees 0-matrix semigroup over a finite semigroup.

Lemma 3.22. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup, where T is finite. The following are equivalent:*

(i) *The matrix P is row-regular.*

(ii) *If $\lambda \in \Lambda$ and L is a maximal $\mathscr{L}$-class of T, then there exists $i \in I$ such that $E(L \cap p_{\lambda,i}T) \neq \varnothing$.*

(iii) *If $\lambda \in \Lambda$ and L is any maximal $\mathscr{L}$-class of T, then the right ideal of T generated by the non-zero entries of P in row λ contains an idempotent of L.*

Proof. (i) $\Rightarrow$ (ii). Let $\lambda \in \Lambda$, and let L_x be a maximal $\mathscr{L}$-class of T with representative $x \in L$. By assumption, there exists $i \in I$ such that $p_{\lambda,i}T$ contains a relative right identity s for x. In other words, $xs = x$, and so $xs^k = x$ for all $k \in \mathbb{N}$. Note that $s^k \in p_{\lambda,i}T$, since $p_{\lambda,i}T$ is a right ideal of T and is therefore a subsemigroup.

Since T is finite, there exists $n \in \mathbb{N}$ such that s^n is idempotent. It follows by Lemma 1.12(i) that $L_x = L_{xs^n} \leq L_{s^n}$. Since L_x is a maximal $\mathscr{L}$-class of T, we conclude that $L_x = L_{s^n}$, and hence that $s^n \in E(L_x \cap p_{\lambda,i}T)$, as required.

(ii) $\Rightarrow$ (iii). Let $\lambda \in \Lambda$, let L be a maximal $\mathscr{L}$-class of T, and let K be the right ideal of T generated by the non-zero entries of P in row λ. By assumption, there exists $i \in I$ such that $E(L \cap p_{\lambda,i}T) \neq \varnothing$. Therefore $p_{\lambda,i} \neq 0$, which implies that $p_{\lambda,i} \in K$. Since K is a right ideal, $p_{\lambda,i}T \subseteq K$, and so $E(L \cap K) \neq \varnothing$, i.e. K contains an idempotent from L, as required.

(iii) $\Rightarrow$ (i). Let $\lambda \in \Lambda$ and let $t \in T$. Since T is finite, there exists a maximal $\mathscr{L}$-class L of T such that $L_t \leq L$. By assumption, the right ideal K of T generated by the set $\{p_{\lambda,i} : i \in I,\ p_{\lambda,i} \neq 0\}$ contains an idempotent e of L. Since $L_t \leq L = L_e$, it follows that $t = ue$ for some $u \in T^1$. Thus $te = ue^2 = ue$, and e is a relative right identity for t. It remains to prove that there exists $i \in I$ such that $e \in p_{\lambda,i}T$. The ideal K is equal to the union of the principal right ideals $p_{\lambda,i}T^1$, for each $i \in I$ with $p_{\lambda,i} \neq 0$. Thus there exists $i \in I$ such that $e \in p_{\lambda,i}T^1 = p_{\lambda,i}T \cup \{p_{\lambda,i}\}$. If $e = p_{\lambda,i}$, then $e = e^2 = p_{\lambda,i}^2 \in p_{\lambda,i}T$, as required. Otherwise, if $e \neq p_{\lambda,i}$, then there is nothing to prove. $\square$

The following lemma is a direct analogue of Lemma 3.22, and is stated without proof.

Lemma 3.23. *Let $S = \mathscr{M}^0[T; I, \Lambda; P]$ be a Rees 0-matrix semigroup, where T is finite. The following are equivalent.*

(i) *The matrix P is column-regular.*

(ii) *If $i \in I$ and R is a maximal $\mathscr{R}$-class of T, then there exists $\lambda \in \Lambda$ such that $E(R \cap Tp_{\lambda,i}) \neq \varnothing$.*

(iii) *If $i \in I$ and R is any maximal $\mathscr{R}$-class of T, then the left ideal of T generated by the non-zero entries of P in column i contains an idempotent of R.*

For a Rees 0-matrix semigroup over a monoid, the conditions for row-regularity and column-regularity can be reformulated as follows. Note that, unlike Lemmas 3.22 and 3.23, the following result applies to infinite Rees 0-matrix semigroups as well as those that are finite.

Lemma 3.24. *Let $S = \mathscr{M}^0[T; I, \Lambda; P]$ be a Rees 0-matrix semigroup over a monoid T. Then:*

(i) *the matrix P is row-regular if and only if each row of P contains an element from the $\mathscr{R}^T$-class of 1_T; and*

(ii) *the matrix P is column-regular if and only if each column of P contains an element from the $\mathscr{L}^T$-class of 1_T.*

Proof. We prove only (i), since (ii) is dual.

($\Rightarrow$) Let $\lambda \in \Lambda$. Since P is row-regular and $1_T \in T$, there exist $i \in I$ and $u \in T$ such that $1_T p_{\lambda,i} u = 1_T$. But 1_T is the identity of T, which implies that $1_T p_{\lambda,i} u = p_{\lambda,i} u$. Therefore $p_{\lambda,i} \cdot u = 1_T$ and $1_T \cdot p_{\lambda,i} = p_{\lambda,i}$, i.e. $p_{\lambda,i} \mathrel{\mathscr{R}^T} 1_T$.

($\Leftarrow$) Let $\lambda \in \Lambda$ and $t \in T$ be arbitrary. By assumption, there exists $i \in I$ such that $p_{\lambda,i} \mathrel{\mathscr{R}^T} 1_T$. Thus there exists $u \in T$ such that $p_{\lambda,i} u = 1_T$. In particular, $p_{\lambda,i} u \in p_{\lambda,i} T$ is a relative right identity for t. □

Corollary 3.25. *Let $S = \mathscr{M}^0[T; I, \Lambda; P]$ be a Rees 0-matrix semigroup, where T is a finite monoid. Then:*

(i) *the matrix P is row-regular if and only if each row of P contains a unit in T; and*

(ii) *the matrix P is column-regular if and only if each column of P contains a unit in T.*

Proof. In a finite monoid T, the group of units is the unique maximal $\mathscr{J}$-class in the partial order of $\mathscr{J}$-classes of T. Therefore, an element is $\mathscr{R}$-related to 1_T if and only if it is $\mathscr{L}$-related to 1_T if and only if it is a unit. □

If a matrix is both row-regular and column-regular, then we simply say that it is *regular*. Traditionally, the matrix of a Rees 0-matrix semigroup over a group is said to be *regular* if it contains a non-zero entry in every row and every column; a Rees 0-matrix semigroup over a group is regular if and only if its matrix is regular. By Lemma 3.24, the notion of regularity given in Definition 3.26 extends this traditional meaning. More specifically, the matrix of a Rees 0-matrix semigroup over a group is row-regular and column-regular if and only if it contains a non-zero entry in every row and every column.

Definition 3.26 (Regular matrix)**.** Let $S = \mathscr{M}^0[T; I, \Lambda; P]$ be a Rees 0-matrix semigroup. The matrix P is *regular* if it is both row-regular and column-regular.

Let $S = \mathscr{M}^0[T; I, \Lambda; P]$ be a Rees 0-matrix semigroup, and let $x = (i, t, \lambda) \in S \setminus \{0\}$ be arbitrary. By definition, if P is row-regular, then t has a relative right identity $p_{\lambda,j} u$ for some $j \in I$ and $u \in T$; thus the element (j, u, λ) is a relative right identity for x. The analogous statements hold when P is column-regular. In other words, the following lemma holds.

Lemma 3.27. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. The following hold:*

(i) *if P is row-regular, then each element of S and T has a relative right identity; and*

(ii) *if P is column-regular, then each element of S and T has a relative left identity.*

In particular, if P is row- or column-regular, then S and T are surjective semigroups.

A particular consequence of the previous lemma is that if T is an arbitrary semigroup, then it is not necessarily possible to construct a row- or column-regular matrix over T^0. For example, if T is not surjective, then it is not possible to construct a row- or column-regular matrix over T^0.

In Algorithm 3.31, we make use of the following observation. Let $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup, where T is finite. If P is row-regular, then Lemma 3.22(ii) implies that each maximal $\mathscr{L}$-class of T contains an idempotent; similarly, if P is column-regular, then Lemma 3.23(ii) implies that each maximal $\mathscr{R}$-class of T contains an idempotent. Since an $\mathscr{L}$- or $\mathscr{R}$-class of a semigroup contains an idempotent if and only if it is regular, we obtain the following corollary to Lemmas 3.22 and 3.23.

Corollary 3.28. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup, where T is finite.*

(i) *If the matrix P is row-regular, then every maximal $\mathscr{L}$-class of T is regular.*

(ii) *If the matrix P is column-regular, then every maximal $\mathscr{R}$-class of T is regular.*

On the other hand, the underlying semigroup of a Rees 0-matrix semigroup $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ is not itself required to be regular in order for the matrix to be row- or column-regular: if T is a non-regular monoid and every row or column of P contains a unit of T, then P is row- or column-regular, respectively, by Lemma 3.24.

In Algorithm 3.31, we use the formulation of row-regularity given by Lemma 3.22(iii) to present an algorithm for determining whether the matrix of a finite Rees 0-matrix semigroup is row-regular. This algorithm assumes the ability to compute the maximal $\mathscr{L}$-classes of a finite semigroup (the sources in the quotient of the left Cayley digraph by its strongly connected components), the ability to test these $\mathscr{L}$-classes for regularity and to find their idempotents [37, Section 5.4]; and the ability to compute a right ideal of a finite semigroup. The right ideal generated by a set consists of the vertices in the right Cayley digraph of the semigroup that are reachable from the generators of the ideal. Therefore, the right ideal can be constructed by performing a depth- or breadth-first search in the Cayley digraph from each of these generators. There do not currently exist well-developed methods for computing left or right ideals, or the partial orders of $\mathscr{L}$- and $\mathscr{R}$-classes, of finite semigroups without computing Cayley digraphs.

Open Problem 3.29. Let S be a finite semigroup to which the techniques of [37] apply. Building on the results of [37], develop methods for computing the Green's structure of an arbitrary left or right ideal of S that do not necessarily exhaustively enumerate the ideal.

Open Problem 3.30. Let S be a finite semigroup to which the techniques of [37] apply. Building on the results of [37], develop methods for computing the partial orders of $\mathscr{L}$- and $\mathscr{R}$-classes of S without necessarily exhaustively enumerating S.

In the following proposition, we characterise the Green's relations of a Rees 0-matrix semigroup with a row- or column-regular matrix in terms of its index sets, and in terms of the Green's relations on its underlying semigroup. In particular, Proposition 3.32 states that the Green's relations on a Rees 0-matrix semigroup with a regular matrix are characterised by the statements of Lemma 3.17 and their converses. Corollary 3.33 follows immediately from this.

Algorithm 3.31 Compute whether a finite Rees 0-matrix semigroup has a row-regular matrix.

Input: A finite Rees 0-matrix semigroup $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$, where $P = (p_{\lambda,i})_{\lambda\in\Lambda, i\in I}$.
Output: TRUE if P is row-regular; else FALSE.

$\mathfrak{L} \leftarrow$ the maximal $\mathscr{L}$-classes of T
if some $\mathscr{L}$-class in $\mathfrak{L}$ is not regular **then**
  return FALSE ▷ Corollary 3.28.
for $\lambda \in \Lambda$ **do**
  $A \leftarrow \{p_{\lambda,i} \,:\, i \in I,\ p_{\lambda,i} \neq 0\}$ ▷ The non-zero entries of P in row λ.
  $U \leftarrow$ the right ideal of T generated by A
  for $L \in \mathfrak{L}$ **do**
    if $U \cap E(L) = \varnothing$ **then**
      return FALSE
return TRUE

Proposition 3.32. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup, and let $x = (i,\, t,\, \lambda)$ and $y = (j,\, u,\, \mu)$ be arbitrary non-zero elements of S. The following hold:*

(i) *if P is column-regular, then $x \mathrel{\mathscr{L}^S} y$ if and only if $t \mathrel{\mathscr{L}^T} u$ and $\lambda = \mu$;*

(ii) *if P is row-regular, then $x \mathrel{\mathscr{R}^S} y$ if and only if $t \mathrel{\mathscr{R}^T} u$ and $i = j$;*

(iii) *if P is regular, then $x \mathrel{\mathscr{H}^S} y$ if and only if $t \mathrel{\mathscr{H}^T} u$, $i = j$, and $\lambda = \mu$; and $x \mathrel{\mathscr{D}^S} y$ if and only if $t \mathrel{\mathscr{D}^T} u$.*

Proof. By Lemma 3.17, it remains to prove the converse implication of each part.

Suppose that the matrix P is column-regular, and that $\lambda = \mu$ and $t \mathrel{\mathscr{L}^T} u$. By the definition of Green's $\mathscr{L}$-relation, there exists $a \in S^1$ such that $at = u$. Since P is column-regular, there exists $\nu \in \Lambda$ such that $Tp_{\nu,i}$ contains a relative left identity for t. In particular, there exists $v \in T$ such that $vp_{\nu,i}t = t$. Therefore $(j,\, av,\, \nu) \in S$, and

$$\begin{aligned}(j,\, av,\, \nu)\, x &= (j,\, avp_{\nu,i}t,\, \lambda)\\ &= (j,\, at,\, \lambda)\\ &= (j,\, u,\, \lambda) = y.\end{aligned}$$

A symmetric argument shows that $zy = x$ for some $z \in S$. Therefore $x \mathrel{\mathscr{L}^S} y$, and (i) holds.

Note (ii) is dual to (i), and since $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$, the statement concerning $\mathscr{H}$ in (iii) holds by (i) and (ii). Suppose that P is regular, and that $t \mathrel{\mathscr{D}^T} u$. Since $\mathscr{D} = \mathscr{L} \circ \mathscr{R}$, there exists an element $v \in T$ such that $t \mathrel{\mathscr{L}^T} v \mathrel{\mathscr{R}^T} u$. Thus, by (i) and (ii), $x = (i,\, t,\, \lambda) \mathrel{\mathscr{L}^S} (j,\, v,\, \lambda) \mathrel{\mathscr{R}^S} (j,\, u,\, \mu) = y$. That is, $x \mathrel{\mathscr{D}^S} y$, and (iii) holds. □

Corollary 3.33. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. The following hold:*

(i) *if P is column-regular, then $|S/\mathscr{L}| = |T/\mathscr{L}| \cdot |\Lambda| + 1$;*

(ii) *if P is row-regular, then $|S/\mathscr{R}| = |T/\mathscr{R}| \cdot |I| + 1$; and*

(iii) *if P is regular, then $|S/\mathscr{H}| = |T/\mathscr{H}| \cdot |I| \cdot |\Lambda| + 1$ and $|S/\mathscr{D}| = |T/\mathscr{D}| + 1$.*

Let S be a Rees 0-matrix semigroup with a row-regular matrix. By using Proposition 3.32 and Corollary 3.33, if we may easily compute Green's $\mathscr{R}$-relation on T, then we may easily test for $\mathscr{R}$-equivalence of elements of S, and count the number of $\mathscr{R}$-classes of S. If S instead has a column-regular matrix, then this is true for Green's $\mathscr{L}$-relation; and if S has a regular matrix, then this is also true for Green's $\mathscr{H}$- and $\mathscr{D}$-relations.

It is natural to consider whether there are Rees 0-matrix semigroups that have neither row-regular nor column-regular matrices, but whose Green's relations are nevertheless characterised by the statements of Lemma 3.17 and their converses. In Example 3.35, we provide such a semigroup. Furthermore, by Lemma 3.34, Green's $\mathscr{H}$-relation on any Rees 0-matrix semigroup over an $\mathscr{H}$-trivial semigroup is characterised as in the statement of Proposition 3.32(iii). Therefore, the most general converse of Proposition 3.32 does not hold.

Lemma 3.34. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup, where T is $\mathscr{H}$-trivial, and let $x = (i,\, t,\, \lambda)$ and $y = (j,\, u,\, \mu)$ be arbitrary non-zero elements of S. Then $x \mathrel{\mathscr{H}^S} y$ if and only if $t \mathrel{\mathscr{H}^T} u$ and $i = j$ and $\lambda = \mu$.*

Proof. The direct implication holds by Lemma 3.17. On the other hand, since T is $\mathscr{H}$-trivial, if $t \mathrel{\mathscr{H}^T} u$ and $i = j$ and $\lambda = \mu$, then in fact $x = y$, so certainly $x \mathrel{\mathscr{H}^S} y$. $\square$

Example 3.35. Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$, where T is any $\mathscr{J}$-trivial semigroup, $|I| = |\Lambda| = 1$, and the sole entry of P is 0. Then S is a zero semigroup, since the product of any two elements in S is 0. In particular, S is $\mathscr{J}$-trivial. Let $\mathscr{K} \in \{\mathscr{H},\, \mathscr{L},\, \mathscr{R},\, \mathscr{D},\, \mathscr{J}\}$ be any Green's relation, and let $x = (i,\, t,\, \lambda)$, $y = (j,\, u,\, \mu) \in S \setminus \{0\}$ be arbitrary non-zero elements. Then

$$\begin{aligned}
x \mathrel{\mathscr{K}^S} y &\Leftrightarrow (i,\, t,\, \lambda) \mathrel{\mathscr{K}^S} (j,\, u,\, \mu) && \\
&\Leftrightarrow (i,\, t,\, \lambda) = (j,\, u,\, \mu) && \text{since } S \text{ is } \mathscr{J}\text{-trivial} \\
&\Leftrightarrow t = u && \text{since } |I| = |\Lambda| = 1 \\
&\Leftrightarrow t \mathrel{\mathscr{K}^T} u && \text{since } T \text{ is } \mathscr{J}\text{-trivial.}
\end{aligned}$$

In particular, since $|I| = |\Lambda| = 1$, it follows that $x \mathrel{\mathscr{L}^S} y$ if and only if $t \mathrel{\mathscr{L}^T} u$ and $\lambda = \mu$; that $x \mathrel{\mathscr{R}^S} y$ if and only if $t \mathrel{\mathscr{R}^T} u$ and $i = j$; that $x \mathrel{\mathscr{H}^S} y$ if and only if $t \mathrel{\mathscr{H}^T} u$, $i = j$, and $\lambda = \mu$; and that $x \mathrel{\mathscr{D}^S} y$ if and only if $t \mathrel{\mathscr{D}^T} u$. Therefore, the Green's relations on S are characterised by the statements of Lemma 3.17 and their converses, but the matrix P is neither row-regular nor column-regular.

Rees 0-matrix semigroups over $\mathscr{H}$-trivial semigroups, and the Rees 0-matrix semigroups that were considered in Example 3.35, are somewhat atypical. Indeed, in Proposition 3.36, we provide a partial converse to Proposition 3.32, which applies to Rees 0-matrix semigroups with non-trivial index sets. Since one could argue that a 'typical' Rees 0-matrix semigroup has non-trivial index sets, Proposition 3.36 suggests that row-regularity and column-regularity are suitable conditions to require, if one desires to consider Rees 0-matrix semigroups whose Green's structure may be readily computed. This opinion is further reinforced in Section 3.5.1, where we prove that a Rees 0-matrix semigroup is regular if and only if its underlying semigroup is regular, and its matrix is row-regular and column-regular.

Proposition 3.36. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. The following hold:*

(i) *if $|I| > 1$, and Green's $\mathscr{L}$-relation on S is characterised by*

$$x \mathrel{\mathscr{L}^S} y \quad \textit{if and only if} \quad t \mathrel{\mathscr{L}^T} u \textit{ and } \lambda = \mu,$$

for all non-zero elements $x = (i,\, t,\, \lambda)$ and $y = (j,\, u,\, \mu)$ in S, then P is column-regular;

(ii) *if* $|\Lambda| > 1$*, and Green's* $\mathscr{R}$*-relation on* S *is characterised by*

$$x \mathrel{\mathscr{R}^S} y \quad \textit{if and only if} \quad t \mathrel{\mathscr{R}^T} u \textit{ and } i = j,$$

for all non-zero elements $x = (i, t, \lambda)$ *and* $y = (j, u, \mu)$ *in* S*, then* P *is row-regular; and*

(iii) *if* $|I| > 1$ *and* $|\Lambda| > 1$*, and Green's* $\mathscr{D}$*-relation on* S *is characterised by*

$$x \mathrel{\mathscr{D}^S} y \quad \textit{if and only if} \quad t \mathrel{\mathscr{D}^T} u,$$

for all non-zero elements $x = (i, t, \lambda)$ *and* $y = (j, u, \mu)$ *in* S*, then* P *is regular.*

Proof. (i). Let (i, t, λ) be an arbitrary non-zero element of S. Since $|I| > 1$, there exists an index $j \in I \setminus \{i\}$ that is different from i. By assumption $(i, t, \lambda) \mathrel{\mathscr{L}^S} (j, t, \lambda)$. Therefore, there exist $\mu \in \Lambda$ and $u \in T$ such that

$$(j, u, \mu)\,(i, t, \lambda) = (j, t, \lambda)\,.$$

In particular, $up_{\mu,i}t = t$, and so $Tp_{\mu,i}$ contains a relative left identity for t. Since $i \in I$ and $t \in T$ were chosen arbitrarily, it follows that P is column-regular.

The proof of (ii) is dual to the proof of (i), and is therefore omitted.

(iii). Let (i, t, λ) be an arbitrary non-zero element of S. Since $|I| > 1$ and $|\Lambda| > 1$, there exist indices $j \in I\setminus\{i\}$ and $\mu \in \Lambda\setminus\{\lambda\}$. By assumption, $(i, t, \lambda) \mathrel{\mathscr{D}^S} (j, t, \mu)$, and so there exists an element $x \in S \setminus \{0\}$ such that $(i, t, \lambda) \mathrel{\mathscr{L}^S} x \mathrel{\mathscr{R}^S} (j, t, \mu)$. By Lemma 3.17, $x = (j, u, \lambda)$ for some $u \in T$. Since $(i, t, \lambda) \mathrel{\mathscr{L}^S} (j, u, \lambda)$, there exist indices $\gamma, \delta \in \Lambda$ and elements $v, w \in T$ such that

$$(j, v, \gamma)\,(i, t, \lambda) = (j, u, \lambda) \quad \text{and} \quad (i, w, \delta)\,(j, u, \lambda) = (i, t, \lambda)\,.$$

By multiplying, we find that $vp_{\gamma,i}t = u$ and $wp_{\delta,j}u = t$, and so

$$(wp_{\delta,j}v)p_{\gamma,i}t = t.$$

In particular, $Tp_{\gamma,i}$ contains a relative left identity for t. It follows that P is column-regular. The proof that P is row-regular follows similarly from the fact that $(j, t, \mu) \mathrel{\mathscr{R}^S} (j, u, \lambda)$. Therefore P is regular. □

3.4 Generating sets for Rees 0-matrix semigroups

As with direct products of semigroups, which were the topic of Chapter 2, a Rees 0-matrix semigroup is defined only in terms of its elements, and so it does not naturally come with a proper generating subset. Of course, if the matrix of a Rees 0-matrix semigroup S contains 0 as an entry, then it is obvious that $S \setminus \{0\}$ is a proper generating subset of S. Beyond this, however, it is not immediately apparent how to describe any smaller generating set for S. Sometimes this is not possible: any Rees 0-matrix semigroup with a matrix containing only 0 is a zero semigroup, and so for any such semigroup, the set of all non-zero elements is the only proper generating subset. However, for many Rees 0-matrix semigroups, it is possible to find generating sets that contain relatively few elements in comparison with the order of the semigroup: see Example 3.37 for an example of a Rees 0-matrix semigroup over a non-group, with thirteen elements, that has a minimal-cardinality generating set with only five elements.

Example 3.37. Let T be the semigroup $\{1, 2, 3\}$, with the operation of greatest common divisor. Clearly T is a commutative band. Let $I = \{1', 2'\}$, $\Lambda = \{1, 2\}$, and

$$P = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix},$$

and define S to be the Rees 0-matrix semigroup $\mathscr{M}^0[T; I, \Lambda; P]$. Then S contains 13 elements. It may be verified computationally that the set $X = \{0, (1', 2, 2), (1', 3, 1), (2', 2, 1), (2', 3, 2)\}$ is the unique minimal generating set of S. In particular, $\operatorname{rank}(S) = 5$.

In this section, we describe relatively small generating sets for several kinds of Rees 0-matrix semigroups. The motivation for proving such results is the same as that behind Chapter 2. To repeat: many, perhaps most, algorithms in computational semigroup theory require a generating set, and so to compute most efficiently with a Rees 0-matrix semigroup, we require a generating set. As a rule of thumb, the smaller the generating set for a given semigroup, the faster it is to compute with that semigroup. Therefore, when it is possible and reasonably practical, we would like to describe and compute generating sets that contain relatively few elements in comparison with the order of the semigroup.

The finitely generated and presented Rees 0-matrix semigroups were classified by Ayik and Ruškuc in [5]: the Rees 0-matrix semigroup $\mathscr{M}^0[T; I, \Lambda; P]$ is finitely generated if and only if T is finitely generated and the sets I, Λ, and $T \setminus K$ are all finite, where K is the ideal of T generated by the non-zero entries of P.

Significant attention has been given to the problem of describing small and minimal generating sets of Rees 0-matrix semigroups over groups, particularly by Gray and Ruškuc; indeed, this topic featured in each of their PhD theses [64,115]. In [115, Chapter 4, Theorem 2.1], given a minimal generating set for the group G, Ruškuc describes a generating set for a regular Rees 0-matrix semigroup $\mathscr{M}^0[G; I, \Lambda; P]$, where G is a group, that contains $\operatorname{rank}(G) + |I| + |\Lambda| - 1$ elements. In [114, Theorems 3.9 and 4.6] Ruškuc gives a formula for the rank of a finite regular Rees 0-matrix semigroup over a group whose Graham-Houghton graph is connected. Gray and Ruškuc describe a formula for the rank of an arbitrary regular Rees 0-matrix semigroup over a group in [63, Theorems 7.1], and in [64, Thereom 2.68], Gray gives the rank of an arbitrary finite (not necessarily regular) Rees 0-matrix semigroup over a group.

Our focus is not on finding generating sets that are minimal. Describing generating sets for Rees 0-matrix semigroups whose underlying semigroups are not groups has not been considered in the literature. Currently, with the computational algebra system GAP [58], it is possible to construct the Rees 0-matrix semigroup $S = \mathscr{M}^0[T; I, \Lambda; P]$, given a semigroup T and a matrix $P : \Lambda \times I \longrightarrow T^0$. However, unless T is a group, GAP uses the set of all non-zero elements of S as the generating set of S whenever one is required, or the set of all elements, when the matrix P contains only non-zero entries. For an infinite Rees 0-matrix semigroup, therefore, it is not currently possible to compute a generating set in GAP, even though the semigroup may be finitely generated. When S is finite, it is theoretically possible to enumerate the set of all elements, given sufficient time and space, but doing so is clearly undesirable in general.

In Section 3.4.1, we present some results concerning the decomposable and indecomposable elements of Rees 0-matrix semigroups. Since a generating set of a semigroup contains its indecomposable elements, this allows us to provide lower bounds on the ranks of certain Rees 0-matrix semigroups. In Section 3.4.2, we provide a generating set for any Rees 0-matrix semigroup $\mathscr{M}^0[T; I, \Lambda; P]$, where T is a monoid and P contains a unit. This generating set contains $|I| + |\Lambda| - 1$ more elements than a given generating set for T. In Section 3.4.3, we present results concerning generating sets of Rees 0-matrix semigroups with row-regular or column-regular matrices.

Before we proceed, we first note that the following inequality holds for any Rees 0-matrix semigroup $\mathscr{M}^0[T; I, \Lambda; P]$. This is because, by the definition of multiplication in a Rees 0-matrix semigroup, any generating set contains at least one member of $\{i\} \times T \times \Lambda$, for each $i \in I$, and at least one member of $I \times T \times \{\lambda\}$, for each $\lambda \in \Lambda$.

Lemma 3.38. *For any Rees 0-matrix semigroup* $S = \mathscr{M}^0[T; I, \Lambda; P]$, $\operatorname{rank}(S) \geq \max\{|I|, |\Lambda|\}$.

3.4.1 Decomposable and indecomposable elements

As discussed in Section 2.2, an element s in a semigroup S is decomposable if $s \in S^2$, and indecomposable if $s \notin S^2$. By Lemma 2.9, the indecomposable elements of a semigroup are those elements of maximal $\mathscr{J}$-classes (in the partial order of $\mathscr{J}$-classes) that consist of a single non-idempotent element. The relevance of decomposable and indecomposable elements to this section is due to the observation that any generating set for a semigroup S contains its set of indecomposable elements $S \setminus S^2$. Therefore, by describing the indecomposable elements of a Rees 0-matrix semigroup S, we find elements that are contained in any generating set for S; in doing so, we deduce a lower bound on $\operatorname{rank}(S)$.

By Lemma 3.27, a Rees 0-matrix semigroup with a row- or column-regular matrix is surjective. However, as explored in the following lemma, an arbitrary Rees 0-matrix semigroup may contain a large number of indecomposable elements.

Lemma 3.39. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup, let K be the ideal of T generated by the non-zero elements of P, and let $(i,\, t,\, \lambda) \in S \setminus \{0\}$ be an arbitrary non-zero element. The following hold:*

(i) *if $(i,\, t,\, \lambda)$ is a decomposable element of S, then $t \in K$;*

(ii) *if $(i,\, t,\, \lambda)$ is a decomposable element of S, then t is a decomposable element of T;*

(iii) *if every non-zero entry of P has a relative left identity and a relative right identity in T, then $(i,\, t,\, \lambda)$ is decomposable if and only if $t \in K$.*

Proof. (i). If $x = (i,\, t,\, \lambda)$ is decomposable, then there exist $\alpha = (j,\, u,\, \mu)\,,\ \beta = (k,\, v,\, \nu) \in S$ such that $x = \alpha\beta$. In particular, $p_{\mu,k} \neq 0$, and by equating the middle components, we find that $t = up_{\mu,k}v \in Tp_{\mu,k}T \subseteq K$.

(ii). As shown in the proof of (i), if $x = (i,\, t,\, \lambda)$ is decomposable, then there exist $u, v \in T$ and a non-zero matrix entry $p_{\mu,k} \in P$ such that $t = up_{\mu,k}v$. In particular, t is decomposable.

(iii). By (i), it remains to prove the converse implication. Suppose that $t \in K$. Since K is generated as an ideal by the non-zero elements of P, there exist $j \in I$ and $\mu \in \Lambda$ such that $p_{\mu,j} \neq 0$ and $t \in T^1p_{\mu,j}T^1$. Thus $t = ap_{\mu,j}b$ for some $a, b \in T^1 = T \cup \{1_T\}$. By assumption, $p_{\mu,j}$ has a relative left identity $l \in T$ and a relative right identity $r \in T$. Thus $t = (al)p_{\mu,j}(rb) \in Tp_{\mu,j}T$. Define $\alpha = (i,\, al,\, \mu) \in S$ and $\beta = (j,\, rb,\, \lambda) \in S$. Then $x = \alpha\beta$, and x is decomposable. □

Note that Lemma 3.39(i) implies that the number of elements in $T \setminus K$ is finite when S is finitely generated (using the notation of the lemma). This was previously shown in [5, Main Theorem], which classifies the finitely generated Rees 0-matrix semigroups.

We obtain the following immediate corollary from the contrapositive of Lemma 3.39(ii). This gives a lower bound on the number of indecomposable elements, and hence the rank, of any Rees 0-matrix semigroup.

Corollary 3.40. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. The number of indecomposable elements of S, and also $\operatorname{rank}(S)$, is at least $|I| \cdot |T \setminus T^2| \cdot |\Lambda|$.*

Example 3.41. Let $S = \mathscr{M}^0[\mathbb{N};\, I,\, \Lambda;\, P]$ be any Rees 0-matrix semigroup over the natural numbers with addition. Since 1 is an indecomposable element of $\mathbb{N}$, it follows by Lemma 3.39(ii) that $I \times \{1\} \times \Lambda$ is a set of indecomposable elements of S.

Lemma 3.39(iii) classifies the decomposable and indecomposable elements of any Rees 0-matrix semigroup where each non-zero entry of the matrix has a relative left identity and a relative right identity. Therefore, for any such Rees 0-matrix semigroup, we may provide a formula for the number of indecomposable elements. This applies, for instance, to Rees 0-matrix semigroups over regular semigroups, and Rees 0-matrix semigroups over monoids.

Corollary 3.42. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup where every non-zero entry of P has a relative left identity and a relative right identity in T, let A be the set of non-zero entries of P, and define*

$$\mathfrak{J} = \big\{J \in S/\mathscr{J} \,:\, J \not\leq J_x \text{ for any } x \in A\big\}.$$

Then the number of indecomposable elements of S is

$$|I| \cdot |\Lambda| \cdot \sum_{J \in \mathfrak{J}} |J|.$$

In particular, $\operatorname{rank}(S)$ *is at least this number.*

Proof. Let K be the ideal of T generated by A. By Lemma 3.39(iii), an element $(i,\, t,\, \lambda)$ is indecomposable if and only if $t \notin K$. Since K is generated as an ideal by A, $t \notin K$ if and only if $t \notin T^1xT^1$ for any $x \in A$, which, by definition of the partial order of $\mathscr{J}$-classes of a semigroup, is true if and only if $J_t \not\leq J_x$ for any $x \in A$. To summarise, $(i,\, t,\, \lambda)$ is indecomposable if and only if $J_t \in \mathfrak{J}$. Note that the multiplicative zero of a Rees 0-matrix semigroup is decomposable. ☐

Example 3.43. Let $n \in \mathbb{N}$, let $S = \mathscr{M}^0[\mathcal{T}_n;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup over the full transformation monoid $\mathcal{T}_n$ of degree n, and suppose that P contains a transformation of rank $n-1$, but no permutations. Since $\mathcal{T}_n$ is a monoid, every non-zero entry of P has a relative left and right identity in $\mathcal{T}_n$. Every ideal of $\mathcal{T}_n$ is principal, and the ideals form a chain, described by $\mathcal{T}_n\; x\; \mathcal{T}_n \subseteq \mathcal{T}_n\; y\; \mathcal{T}_n$ if and only if $\operatorname{rank}(x) \leq \operatorname{rank}(y)$, for all $x, y \in \mathcal{T}_n$. Therefore the ideal of $\mathcal{T}_n$ generated by the non-zero elements of P is the set $\mathcal{T}_n \setminus \mathcal{S}_n$ of all transformations with rank at most $n-1$. By Lemma 3.39(iii), the set of indecomposable elements of S is given by

$$S \setminus S^2 = I \times \mathcal{S}_n \times \Lambda.$$

3.4.2 Rees 0-matrix semigroups over monoids

Given that the ranks of Rees 0-matrix semigroups over groups have been described by Gray and Ruškuc, a natural extension is to consider the ranks, and minimal generating sets, in the case that the underlying semigroup is a monoid. In this short section, we present results relating the generating sets, and thereby the ranks, of certain Rees 0-matrix semigroups over monoids to those of the monoids themselves.

Lemma 3.44 (cf. [115, Chapter 4, Theorem 2.1])**.** *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup, where T is a monoid and P contains a unit in T. Let X be any generating set for T, and fix $i \in I$ and $\lambda \in \Lambda$ such that $p_{\lambda,i}$ is a unit in T. Then S is generated by the set*

$$Y = \big\{\big(i,\, xp_{\lambda,i}^{-1},\, \lambda\big) \,:\, x \in X\big\} \cup \big\{\big(i,\, 1_T,\, \mu\big) \,:\, \mu \in \Lambda \setminus \{\lambda\}\big\} \cup \big\{\big(j,\, 1_T,\, \lambda\big) \,:\, j \in I \setminus \{i\}\big\} \cup \{0\}.$$

Proof. Certainly $\langle Y \rangle \subseteq S$, so it remains to prove the converse inclusion. Let $t \in T$ be arbitrary. Since X generates T and $tp_{\lambda,i} \in T$, there exists a sequence $x_1,\, x_2,\, \ldots,\, x_n$ of elements of X such that $x_1x_2\cdots x_n = tp_{\lambda,i}$. Therefore

$$\begin{aligned}
\big(i,\, x_1p_{\lambda,i}^{-1},\, \lambda\big)\big(i,\, x_2p_{\lambda,i}^{-1},\, \lambda\big)\cdots\big(i,\, x_np_{\lambda,i}^{-1},\, \lambda\big) &= \big(i,\, x_1(p_{\lambda,i}^{-1}p_{\lambda,i})x_2(p_{\lambda,i}^{-1}p_{\lambda,i})\cdots x_np_{\lambda,i}^{-1},\, \lambda\big)\\
&= \big(i,\, x_1x_2\cdots x_np_{\lambda,i}^{-1},\, \lambda\big)\\
&= \big(i,\, tp_{\lambda,i}p_{\lambda,i}^{-1},\, \lambda\big)\\
&= \big(i,\, t,\, \lambda\big).
\end{aligned}$$

In particular, $\big(i,\, t,\, \lambda\big) \in \langle Y\rangle$, and since $t \in T$ was arbitrary, it follows that $\{i\}\times T\times\{\lambda\} \subseteq \langle Y\rangle$. In particular, note that $\big(i,\, 1_T,\, \lambda\big) \in \langle Y\rangle$. Let $\big(j,\, t,\, \mu\big) \in S\setminus\{0\}$ be an arbitrary non-zero element of S. By the previous arguments, $\big(i,\, p_{\lambda,i}^{-1}tp_{\lambda,i}^{-1},\, \lambda\big) \in \langle Y\rangle$, and so

$$\big(j,\, 1_T,\, \lambda\big)\big(i,\, p_{\lambda,i}^{-1}tp_{\lambda,i}^{-1},\, \lambda\big)\big(i,\, 1_T,\, \mu\big) = \big(j,\, 1_T(p_{\lambda,i}p_{\lambda,i}^{-1})t(p_{\lambda,i}^{-1}p_{\lambda,i})1_T,\, \mu\big) = \big(j,\, t,\, \mu\big).$$

In particular, $\big(j,\, t,\, \mu\big) \in \langle Y\rangle$. Since $0 \in Y$ by definition, the result follows. □

By choosing the generating set X in the statement of Lemma 3.44 to be a generating set of least cardinality, and by using Lemma 3.38, we deduce the following corollary to Lemma 3.44.

Corollary 3.45 (cf. [115, Chapter 4, Corollary 2.3])**.** *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup, where T is a monoid and P contains a unit. Then*

$$\max\{|I|,\, |\Lambda|\} \leq \operatorname{rank}(S) \leq \operatorname{rank}(T) + |I| + |\Lambda| - 1.$$

Although the problem of providing a formula for the rank of an arbitrary Rees 0-matrix semigroup seems rather intractable, we pose the following problem, whose solution could be a next step in the generalisation of the work of Gray and Ruškuc.

Open Problem 3.46. Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be an arbitary finite Rees 0-matrix semigroup where T is a monoid and P contains a unit. Give a formula that describes $\operatorname{rank}(S)$ in terms of $\operatorname{rank}(T)$, the semigroup-theoretic properties of T, and the matrix P.

3.4.3 Rees 0-matrix semigroups with row- or column-regular matrices

In Section 3.3.1, we defined row- and column-regular matrices of Rees 0-matrix semigroups. We showed that, in essence, the Green's structure of a Rees 0-matrix semigroup with a row-regular or column-regular matrix can be calculated in terms of the index sets of the matrix and the Green's structure on the underlying semigroup. This suggests that such semigroups are an important class of Rees 0-matrix semigroups, a viewpoint that will be bolstered by Section 3.5.1, which concerns regular Rees 0-matrix semigroups. In the following results, we describe generating sets for these kinds of semigroups.

Proposition 3.47. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup where P is row-regular, and let X be a generating set for T. Fix $\mu \in \Lambda$, and for each $x \in X$, choose $i_x \in I$ and $t_x \in T$ such that $p_{\mu,i_x}t_x$ is a relative right identity for x. Then S is generated by the set:*

$$\Gamma = \big\{(i,\, x,\, \mu) \,:\, i \in I,\ x \in X\big\} \cup \big\{(i_x,\, t_x x,\, \lambda) \,:\, x \in X,\ \lambda \in \Lambda\big\} \cup \{0\}.$$

Proof. Note that, for each $x \in X$, the index $i_x \in I$ and the element $t_x \in T$ exist since P is assumed to be row-regular; to reiterate, $xp_{\mu,i_x}t_x = x$ for each $x \in X$. Certainly $\Gamma \subseteq S$ and $0 \in \Gamma$, so it remains to prove that $S\setminus\{0\} \subseteq \langle\Gamma\rangle$.

Let $s = (i,\, t,\, \lambda) \in S\setminus\{0\}$ be an arbitrary non-zero element of S. Since $t \in T$ and T is generated by X, we can write t as a product in the generators X. Indeed, since S is surjective by Lemma 3.27, it follows by Lemma 2.8 that there exist generators $x_1, x_2, \ldots, x_n \in X$, with $n \geq 4$, such that $t = x_1\cdots x_n$. Thus $x_j p_{\mu,i_{x_j}} t_{x_j} = x_j$ for all $j \in \{1,\, \ldots,\, n\}$. It follows that

$$\begin{aligned} s = (i,\, t,\, \lambda) &= (i,\, x_1x_2\cdots x_{n-1}x_n,\, \lambda) \\ &= \Big(i,\, x_1(p_{\mu,i_{x_1}}t_{x_1})\cdot x_2(p_{\mu,i_{x_2}}t_{x_2})\cdots x_{n-1}(p_{\mu,i_{x_{n-1}}}t_{x_{n-1}})\cdot x_n,\, \lambda\Big) \\ &= \big(i,\, x_1,\, \mu\big)\big(i_{x_1},\, t_{x_1}x_2,\, \mu\big)\cdots\big(i_{x_{n-2}},\, t_{x_{n-2}}x_{n-1},\, \mu\big)\big(i_{x_{n-1}},\, t_{x_{n-1}}x_n,\, \lambda\big) \\ &\in \langle\Gamma\rangle. \end{aligned}$$

□

Example 3.48. Let $S = \mathscr{M}^0[T; I, \Lambda; P]$ be the Rees 0-matrix semigroup defined in Example 3.37, where T is the set $\{1, 2, 3\}$ with the operation of greatest common divisor, $I = \{1', 2'\}$, $\Lambda = \{1, 2\}$, and

$$P = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

The right ideal of T generated by the non-zero entries $\{2, 3\}$ of each row of P is T itself. Since every element of T is idempotent, it follows that this ideal contains an idempotent of each maximal $\mathscr{L}$-class of T. In particular, P is row-regular by Lemma 3.22(iii). Therefore, we may use Proposition 3.47 to describe a generating set for S. Let $X = \{2, 3\}$. Then X is a generating set for T. Fix $\mu = 1$. Since each element of T is idempotent, it follows that $p_{\mu,1'}2$ is a relative right identity for 2, and $p_{\mu,2'}3$ is a relative right identity for 3. In other words, if we define $i_2 = 1'$ and $t_2 = 2$, and $i_3 = 2'$ and $t_3 = 3$, it follows by Proposition 3.47 that S is generated by

$$\big\{0, (1', 2, 1), (1', 3, 1), (2', 2, 1), (2', 3, 1), (1', 2, 2), (2', 3, 2)\big\},$$

which contains seven elements. As noted in Example 3.37, $\operatorname{rank}(S) = 5$.

There is an obvious result that is dual to Proposition 3.47, which concerns generating sets of Rees 0-matrix semigroups with column-regular matrices. We state this result for completeness.

Proposition 3.49. *Let $S = \mathscr{M}^0[T; I, \Lambda; P]$ be a Rees 0-matrix semigroup where P is column-regular, and let X be a generating set for T. Fix $j \in I$, and for each $x \in X$, choose $\lambda_x \in \Lambda$ and $t_x \in T$ such that $t_x p_{\lambda_x,j}$ is a relative left identity for x. Then S is generated by the set:*

$$\Gamma = \big\{(j, x, \lambda) : x \in X,\ \lambda \in \Lambda\big\} \cup \big\{(i, xt_x, \lambda_x) : i \in I,\ x \in X\big\} \cup \{0\}.$$

By choosing a generating set X for T of minimal cardinality, we may use Propositions 3.47 and 3.49 to obtain an upper bound on the rank of a Rees 0-matrix semigroup with a row- or column-regular matrix. In combination with Lemma 3.38, we obtain the following corollary.

Corollary 3.50. *Let $S = \mathscr{M}^0[T; I, \Lambda; P]$ be a Rees 0-matrix semigroup where P is row- or column-regular. Then*

$$\max\{|I|, |\Lambda|\} \leq \operatorname{rank}(S) \leq \operatorname{rank}(T)(|\Lambda| + |I|) + 1.$$

3.5 Special kinds of Rees 0-matrix semigroups

In the final section of this chapter, we present results that allow us to characterise several properties of Rees 0-matrix semigroups. In particular, we classify the Rees 0-matrix semigroups that are regular, are inverse, or are monoids. Moreover, these classifications are given in terms of the underlying semigroup and the matrix of the Rees 0-matrix semigroup, which may be readily computed.

3.5.1 Regular Rees 0-matrix semigroups

In this section, we show that the regular Rees 0-matrix semigroups are those with regular matrices constructed from regular semigroups. As stated in the following lemma, the underlying semigroup of a regular Rees 0-matrix semigroup is regular.

Lemma 3.51. *Let $S = \mathscr{M}^0[T; I, \Lambda; P]$ be a Rees 0-matrix semigroup. If S is a regular semigroup, then T is a regular semigroup.*

Proof. Let $t \in T$, $i \in I$, and $\lambda \in \Lambda$ be arbitrary, and define $x = (i, t, \lambda)$. Since S is a regular semigroup, there exists a non-zero element $y = (j, u, \mu) \in S \setminus \{0\}$ such that $x = xyx$. By equating the middle components of x and xyx, we find that $t = t(p_{\lambda,j}up_{\mu,i})t$, and so t is a regular element of T. Since t was arbitrary, the semigroup T is regular. □

However, the converse of Lemma 3.51 does not hold, in general; for example, any Rees 0-matrix semigroup whose matrix consists of zeroes is a non-trivial zero semigroup, and is not regular, irrespective of the regularity of the underlying semigroup.

In Section 3.3.1, we introduced the notions of row-regular and column-regular matrices, in an attempt to classify those Rees 0-matrix semigroups whose Green's structure can be 'read off' from the index sets and the Green's structure of the underlying semigroup. In the following theorem, which is the main result of this section, we show that the regularity of a Rees 0-matrix semigroup is closely related to the row- and column-regularity of its matrix.

Theorem 3.52. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. Then S is a regular semigroup if and only if T is a regular semigroup and the matrix P is regular.*

Proof. ($\Rightarrow$) By Lemma 3.51, it remains to prove that the matrix P is both row-regular and column-regular. We prove that P is row-regular; the proof that P is column-regular is dual. Let $\lambda \in \Lambda$ and $t \in T$ be arbitrary. Fix $j \in I$, and define $x = (j, t, \lambda) \in S$. Since S is regular, there exists $y = (i, u, \mu) \in S$ such that $x = xyx$, and so $t = tp_{\lambda,i}up_{\mu,j}t$. Thus $p_{\lambda,i}up_{\mu,j}t$ is a relative right identity for t, and $p_{\lambda,i}(up_{\mu,j}t) \in p_{\lambda,i}T$, as required.

($\Leftarrow$) Certainly 0 is a regular element, so let $x = (i, t, \lambda) \in S \setminus \{0\}$ be an arbitrary non-zero element of S. Since P is row-regular, there exist $j \in I$ and $a \in T$ such that $p_{\lambda,j}a$ is a relative right identity for t; since P is column-regular, there exist $\mu \in \Lambda$ and $b \in T$ such that $bp_{\mu,i}$ is a relative left identity for t. By assumption, T is a regular semigroup, and so there exists $u \in T$ such that $t = tut$. Define $y = (j, aub, \mu) \in S$. Then

$$\begin{aligned}
xyx &= \big(i,\, t,\, \lambda\big)\big(j,\, aub,\, \mu\big)\big(i,\, t,\, \lambda\big) \\
&= \big(i,\, (tp_{\lambda,j}a)u(bp_{\mu,i}t),\, \lambda\big) \\
&= \big(i,\, tut,\, \lambda\big) \\
&= \big(i,\, t,\, \lambda\big) = x.
\end{aligned}$$

In other words, x is a regular element of S, and so S is regular. □

Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. By Theorem 3.52, we can determine whether S is regular solely by determining whether its underlying semigroup and matrix are regular. For a finite Rees 0-matrix semigroup, row-regularity of the matrix can be tested with Algorithm 3.31, and an analogous algorithm can be used to test for column regularity.

Throughout this chapter, we have often referred to the fact that a Rees 0-matrix semigroup over a group is regular if and only if each row and column of its matrix contains a non-zero element. This is easy to verify directly; it also follows from Theorem 3.52 and Lemma 3.24.

When the underlying semigroup of a Rees 0-matrix semigroup is a finite monoid, we may reformulate Theorem 3.52 in the following way.

Corollary 3.53. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup, where T is a finite monoid. Then S is regular if and only if T is regular and every row and every column of P contains a unit of T.*

Proof. By Corollary 3.25, the matrix P is regular if and only if each row and each column of P contains a unit in T. The result follows by Theorem 3.52. □

Example 3.54. Let T be the semigroup $\{-1, 1\}$ with the operation of min. Then T is a monoid with identity element 1 and multiplicative zero -1. Let I and Λ be index sets with $|I| = 3$ and $|\Lambda| = 2$, and let

$$P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

By Corollary 3.53, the Rees 0-matrix semigroup $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ is not a regular semigroup, since the second column of P does not contain a unit. On the other hand, $\mathscr{M}^0[T;\, I,\, \Lambda;\, Q]$ is a regular semigroup, since T is regular and each row and column of Q contains the identity element 1.

3.5.2 Rees 0-matrix monoids and inverse monoids

In this section, we prove that a Rees 0-matrix semigroup S over a semigroup T is a monoid if and only if T is a monoid and S is isomorphic to T^0. Since not every monoid has a multiplicative zero, and certainly not every monoid is isomorphic to a monoid with zero adjoined, it follows that not all monoids can be represented as Rees 0-matrix semigroups. Moreover, we find that the only possible way to construct a Rees 0-matrix monoid is rather trivial. We also find that analogous statements hold for inverse Rees 0-matrix monoids.

We first prove the main result concerning Rees 0-matrix monoids.

Proposition 3.55. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. Then S is a monoid if and only if $|I| = |\Lambda| = 1$, T is a monoid, and the sole entry of P is a unit in T.*

Proof. ($\Rightarrow$) Any Rees 0-matrix semigroup has a non-zero element by definition, and so

$$0 \neq 1_S = (i,\, t,\, \lambda) \in (I \times T \times \Lambda).$$

Let $x = (j,\, u,\, \mu) \in (I \times T \times \Lambda)$ be arbitrary. Then

$$\begin{aligned} (j,\, u,\, \mu) &= 1_S x \\ &= (i,\, t,\, \lambda)(j,\, u,\, \mu) \\ &= (i,\, tp_{\lambda,j}u,\, \mu)\,, \end{aligned}$$

and so $i = j$. Since j was arbitrary, it follows that $I = \{i\}$. Similarly, the equation $x1_S = x$ implies that $\Lambda = \{\lambda\}$. The sole matrix entry $p_{\lambda,i}$ is non-zero, since 1_S is idempotent. In particular, $p_{\lambda,i} \in T$. Let $u \in T$ be arbitrary. Then $1_S\,(i,\, u,\, \lambda) = (i,\, u,\, \lambda) = (i,\, u,\, \lambda)\,1_S$, which implies that $(tp_{\lambda,i})u = u = u(p_{\lambda,i}t)$. Therefore $tp_{\lambda,i}$ is a left identity for T, and $p_{\lambda,i}t$ is a right identity for T, and so T is a monoid with identity $tp_{\lambda,i} = p_{\lambda,i}t = 1_T$. Furthermore, $p_{\lambda,i}$ is a unit, with inverse $p_{\lambda,i}^{-1} = t$.

($\Leftarrow$) Suppose that $I = \{i\}$ and $\Lambda = \{\lambda\}$. Define $e = \big(i,\, p_{\lambda,i}^{-1},\, \lambda\big) \in S$, and let $x \in S$ be arbitrary. If $x = 0$, then certainly $ex = x = xe$. Otherwise, $x = (i,\, t,\, \lambda)$ for some $t \in T$, and

$$\begin{aligned} ex &= \big(i,\, p_{\lambda,i}^{-1},\, \lambda\big)\big(i,\, t,\, \lambda\big) \\ &= \big(i,\, p_{\lambda,i}^{-1}p_{\lambda,i}t,\, \lambda\big) \\ &= \big(i,\, 1_T t,\, \lambda\big) = x. \end{aligned}$$

Similarly, $xe = x$. It follows that e is the identity element of S. $\square$

Note that by Lemma 3.24, a Rees 0-matrix monoid has a regular matrix.

According to the following lemma, we may normalize a Rees 0-matrix semigroup over a monoid; that is, we may assume up to isomorphism that its matrix entry is the identity element of its underlying monoid.

Corollary 3.56. *Let $S = \mathscr{M}^0[T;\, \{i\},\, \{\lambda\};\, P]$ be a Rees 0-matrix monoid, and let Q be the $\{\lambda\} \times \{i\}$ matrix with unique entry 1_T. Then $S \cong \mathscr{M}^0[T;\, \{i\},\, \{\lambda\};\, Q]$.*

Proof. In Proposition 3.5, choose θ, ψ, and χ to be identity functions, and choose $u_i = p_{\lambda,i}$ and $v_\lambda = 1_T$. The result follows. □

For a normalized Rees 0-matrix monoid $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$, the function $\phi : S \longrightarrow T^0$ given by $0\phi = 0$ and $(i,\, t,\, \lambda)\,\phi = t$ for all $(i,\, t,\, \lambda) \in S\backslash\{0\}$ is clearly an isomorphism. Therefore, we deduce the following result from Corollary 3.56.

Corollary 3.57. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. Then S is a monoid if and only if T is a monoid and $S \cong T^0$.*

Given the preceding results concerning Rees 0-matrix monoids, it should perhaps not be surprising that the analogous results hold for inverse Rees 0-matrix monoids. Thus we present the following proposition and corollary, which are analogous to Proposition 3.55 and Corollary 3.57.

Proposition 3.58. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a Rees 0-matrix semigroup. Then S is an inverse monoid if and only if $|I| = |\Lambda| = 1$, T is an inverse monoid, and the entry of P is a unit in T.*

Proof. ($\Rightarrow$) By Proposition 3.55, it remains to prove that T is inverse. By Corollary 3.57, $S \cong T^0$, and so T^0 is an inverse monoid, i.e. every element of T^0 has a unique inverse. The sole inverse of any multiplicative zero is itself, and so every element of T has a unique inverse in T. Thus T is an inverse subsemigroup of T^0.

($\Leftarrow$) By Corollary 3.57, $S \cong T^0$. Since T is an inverse monoid, so too are T^0 and S. □

Corollary 3.59. *A Rees 0-matrix semigroup $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ is an inverse monoid if and only if T is an inverse monoid and $S \cong T^0$.*

We end this section about monoids by considering the rank of a Rees 0-matrix monoid, which represents an initial step towards addressing Open Problem 3.46.

Lemma 3.60. *Let $S = \mathscr{M}^0[T;\, \{i\},\, \{\lambda\};\, P]$ be a Rees 0-matrix monoid, and let X be a generating set for T. Then the set*

$$Y = \left\{\left(i,\, xp_{\lambda,i}^{-1},\, \lambda\right) \,:\, x \in X\right\} \cup \{0\}.$$

is a generating set for S. Furthermore, with respect to cardinality or containment, Y is a minimal generating set for S if and only if X is a minimal semigroup generating set for T. In particular, $\operatorname{rank}(S) = \operatorname{rank}(T) + 1$.

Proof. The fact that Y generates S follows by Lemma 3.44. Since $S \cong T^0$, the statements concerning minimality follows by Lemma 1.5, and the statement concerning $\operatorname{rank}(S)$ holds by Corollary 1.6. □

3.5.3 Inverse Rees 0-matrix semigroups

In Section 3.5.2, we learned that constructing an inverse Rees 0-matrix monoid is essentially the same as adjoining a zero to an inverse monoid. In this section, we consider inverse Rees 0-matrix semigroups more generally, and find that their classification is somewhat richer than the classification of inverse Rees 0-matrix monoids. We also describe an algorithm for computing whether a finite Rees 0-matrix semigroup is inverse, and we describe and count the idempotents of an inverse Rees 0-matrix semigroup.

First we give the well-known classification of inverse Rees 0-matrix semigroups over groups. Let G be a group, and let I be any non-empty set. The $I \times I$ *Brandt semigroup over* G, denoted $B(G, I)$, is the Rees 0-matrix semigroup $\mathscr{M}^0[G; I, I; P]$, where $P = (p_{i,j})_{i,j \in I}$ is the $I \times I$ matrix such that $p_{i,i} = 1_G$ and $p_{i,j} = 0$ when $i \neq j$. When $I = \{1, \dots, n\}$ for some $n \in \mathbb{N}$, we denote the Brandt semigroup $B(G, I)$ by $B(G, n)$, which we call the $n \times n$ *Brandt semigroup over* G. By Proposition 3.1, two Brandt semigroups $B(G, I)$ and $B(H, J)$ are isomorphic is and only if $G \cong H$ and $|I| = |J|$. In particular, if I is a finite set, then $B(G, I) \cong B(G, |I|)$.

The importance of Brandt semigroups is that they characterise the inverse completely 0-simple semigroups, as shown in the following result.

Proposition 3.61 ([76, Theorem 5.1.8])**.** *Let S be any semigroup. Then S is completely 0-simple and inverse if and only if S is isomorphic to the Brandt semigroup $B(G, I)$, for some group G and some non-empty set I.*

In particular, the principal factor of a $\mathscr{J}$-class of a finite inverse semigroup is isomorphic to the Brandt semigroup $B(G, n)$, for some group G and some $n \in \mathbb{N}$.

Given the description of isomorphisms of regular Rees 0-matrix semigroups over groups from Proposition 3.1, we may reformulate Proposition 3.61 in the following way.

Corollary 3.62. *Let $S = \mathscr{M}^0[G; I, \Lambda; P]$ be a Rees 0-matrix semigroup over a group G. Then S is inverse if and only if the matrix P contains exactly one non-zero entry in each row and each column.*

The following theorem, which is the main result of this section, generalises Corollary 3.62 to Rees 0-matrix semigroups over arbitrary semigroups, rather than just groups. The proof of the converse implication of Theorem 3.63 relies on the following property of inverse semigroups: if T is an inverse semigroup, and $x, y \in T$ are elements such that $x = xyx$, then $x = ey^{-1}$ for some idempotent $e \in E(T)$ [76, Proposition 5.2.1]. Indeed, if $x = xyx$, then $x^{-1}x$ and yx are clearly $\mathscr{L}$-related (and hence equal) idempotents. Idempotents are self-inverse, and so $x^{-1}x = x^{-1}y^{-1}$, which implies that

$$x = x(x^{-1}x) = x(x^{-1}y^{-1}) = ey^{-1},$$

where $e = xx^{-1}$ is an idempotent.

Theorem 3.63. *Let $S = \mathscr{M}^0[T; I, \Lambda; P]$ be a Rees 0-matrix semigroup. Then S is inverse if and only if T is an inverse monoid, the matrix P contains exactly one non-zero entry in each row and each column, and the non-zero matrix entries are units in T.*

Proof. ($\Rightarrow$) Since an inverse semigroup is regular, it follows by Theorem 3.52 that T is regular, and that the matrix P is regular. In particular, each row and column of P contains at least one non-zero entry. Let $i \in I$ be arbitrary, and let $\lambda, \mu \in \Lambda$ be indices such that $p_{\lambda,i}$ and $p_{\mu,i}$ are non-zero. By the regularity of T, there exist inverses $u, v \in T$ of $p_{\lambda,i}$ and $p_{\mu,i}$ respectively. The elements (i, u, λ) and (i, v, μ) are idempotent, and so they commute, because S is inverse. In particular, $\lambda = \mu$, and so there exists a unique index $\lambda \in \Lambda$ such that $p_{\lambda,i} \neq 0$. An analogous argument shows that any row of P contains exactly one non-zero entry. Therefore, without loss of generality, we may assume that $I = \Lambda$, and that the matrix P is diagonal, i.e. that the non-zero entries of P are the entries $p_{i,i}$ for each $i \in I$.

We aim to show that T is a monoid, and that the non-zero entries of P are units in T; to do this, we first show that these entries are $\mathscr{D}^T$-related, and then that they are $\mathscr{H}^T$-related.

Let $i \in I$ and $t \in T$ be arbitrary. Since P is row-regular, $p_{i,i}T$ contains a relative right identity for t, and since P is column-regular, $Tp_{i,i}$ contains a relative left identity for t. In other words, there exist elements $a, b \in T$ such that $ap_{i,i}t = t$ and $tp_{i,i}b = t$. In particular, $t \mathscr{L}^T p_{i,i}t$ and $t \mathscr{R}^T tp_{i,i}$. Since this holds for all $i \in I$ and $t \in T$, it follows that

$$p_{i,i} \mathscr{R}^T p_{i,i}p_{j,j} \mathscr{L}^T p_{j,j} \mathscr{R}^T p_{j,j}p_{i,i} \mathscr{L}^T p_{i,i} \tag{3.64}$$

for all $i, j \in I$. In particular, the non-zero matrix entries of P are $\mathscr{D}^T$-related.

By Lemma 1.15, the element $p_{i,i}p_{j,j}$ is contained in a group $\mathscr{H}^T$-class for all $i, j \in I$. Moreover, by choosing $i = j$ in (3.64), we find that $p_{i,i} \mathscr{H}^T p_{i,i}^2$ for all $i \in I$, and so the $\mathscr{H}^T$-class of each non-zero matrix entry is a group. Let $i, j \in I$ be arbitrary, let e be the idempotent such that $e \mathscr{H}^T p_{i,i}$ and let f be the idempotent such that $f \mathscr{H}^T p_{i,i}p_{j,j}$. Since e and f are $\mathscr{R}^T$-related idempotents, it follows that each is a relative left identity for the other, i.e. $ef = f$ and $fe = e$. Let u be the group-theoretic inverse of $p_{i,i}$ in its $\mathscr{H}^T$-class. Note that e is an identity for both $p_{i,i}$ and u, and that $p_{i,i}u = e = up_{i,i}$. Since $e \mathscr{R}^T f$ and $\mathscr{R}$ is a left congruence, it follows that $u = ue \mathscr{R}^T uf$. By Proposition 3.32, $(i, u, i) \mathscr{R}^S (i, uf, i)$. But these elements are idempotents, and since S is an inverse semigroup, which contains a unique idempotent in each $\mathscr{R}$-class, it follows that $u = uf$. Multiplying on the left by $p_{i,i}$, we deduce that $e = f$, and so in fact $p_{i,i} \mathscr{H}^T p_{i,i}p_{j,j}$. By an analogous argument, $p_{j,j} \mathscr{H}^T p_{i,i}p_{j,j}$, and so $p_{i,i} \mathscr{H}^T p_{j,j}$. Since i and j were chosen arbitrary, it follows that non-zero entries of P are $\mathscr{H}^T$-related, and that their $\mathscr{H}^T$-class is a group, whose idempotent we name e.

To show that e is the identity of T, let $t \in T$, $i \in I$ be arbitrary, and define $x = (i, t, i) \in S$. Then $x^{-1} = (i, u, i) \in S$ for some $u \in T$. Certainly x is an inverse of x^{-1}, but so too is $y = (i, te, i)$, since

$$\begin{aligned} x^{-1}yx^{-1} &= (i, up_{i,i}tep_{i,i}u, i) \\ &= (i, up_{i,i}tp_{i,i}u, i) \\ &= x^{-1}xx^{-1} = x^{-1}, \end{aligned}$$

and similarly $yx^{-1}y = y$. In addition, $z = (i, et, i)$ is an inverse of x^{-1}. Since x^{-1} has a unique inverse, it follows that $y = x = z$, and so $te = t = et$. Thus T is a monoid with identity e, and the non-zero entries of P are units in T.

Finally, to show that T is inverse, it suffices to show that the idempotents of T commute. Let $e, f \in E(T)$ and $i \in I$ be arbitrary, and let $x = \big(i, ep_{i,i}^{-1}, i\big), y = \big(i, fp_{i,i}^{-1}, i\big) \in S$. Then x and y are idempotents of S, which is an inverse semigroup, and so they commute. Therefore $xy = yx$, which implies that $efp_{i,i}^{-1} = fep_{i,i}^{-1}$, and so $ef = fe$.

($\Leftarrow$) It follows by Theorem 3.52 that S is regular, and so it remains to show that the idempotents of S commute. The multiplicative zero of S is an idempotent that commutes with every element of the semigroup. Since the matrix P contains precisely one non-zero entry in each row and column, we may assume without loss of generality that $I = \Lambda$, and that P is diagonal. Therefore, a non-zero element $(i, t, k) \in I \times T \times I$ is idempotent if and only if $i = k$ and $t = tp_{i,i}t$. It follows by the discussion preceding the statement of the theorem that $t = tp_{i,i}t$ if and only if $t = ep_{i,i}^{-1}$ for some idempotent $e \in E(T)$. Thus, any non-zero idempotent of S can be written as $\big(i, ep_{i,i}^{-1}, i\big)$, for some $i \in I$ and $e \in E(T)$. On the other hand, any element of this form is idempotent. It follows that

$$E(S) = \big\{\big(i, ep_{i,i}^{-1}, i\big) : i \in I,\ e \in E(T)\big\} \cup \{0\}.$$

Let $x = \big(i, ep_{i,i}^{-1}, i\big)$ and $y = \big(j, fp_{j,j}^{-1}, j\big)$ be arbitrary non-zero idempotents of S. If $i = j$, then

$$xy = \big(i, efp_{i,i}^{-1}, i\big) = \big(i, fep_{i,i}^{-1}, i\big) = yx,$$

since $p_{i,i}$ is a unit and the idempotents of T commute. Otherwise, $xy = yx = 0$. $\square$

Remark 3.65. Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be an inverse Rees 0-matrix semigroup, and let $x = (i,\, t,\, \lambda) \in S$ be an arbitrary non-zero element. Define $j \in I$ to be the unique index such that $p_{\lambda,j} \neq 0$, and define $\mu \in \Lambda$ to be the unique index such that $p_{\mu,i} \neq 0$. Then

$$x^{-1} = \left(j,\, p_{\lambda,j}^{-1} t^{-1} p_{\mu,i}^{-1},\, \mu\right).$$

As we did with Proposition 3.55 and Rees 0-matrix monoids, we may use Theorem 3.63 in conjunction with Proposition 3.5 to describe normalized inverse Rees 0-matrix semigroups.

Corollary 3.66. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be an inverse Rees 0-matrix semigroup, and let Q be a $I \times I$ matrix with 1_T on its leading diagonal, and 0 elsewhere. Then $S \cong \mathscr{M}^0[T;\, I,\, I;\, Q]$.*

Proof. In Proposition 3.5, choose θ to be the identity isomorphism on T, choose appropriate bijections $\psi : I \longrightarrow I$ and $\chi : \Lambda \longrightarrow I$ to diagonalise P, and choose $u_i = p_{\lambda,i}$ and $v_\lambda = 1_T$ for all $i \in I$ and $\lambda \in \Lambda$. The result follows. □

In Algorithm 3.67, we use Theorem 3.63 to describe a procedure for computing whether an arbitrary finite Rees 0-matrix semigroup is inverse. This assumes the ability to test whether the underlying semigroup is an inverse monoid, to compute its group of units, and to test membership in the group of units. This algorithm is used in the SEMIGROUPS package [101] for GAP [58], when the function `IsInverseSemigroup` is given a Rees 0-matrix semigroup.

Algorithm 3.67 Compute whether a finite Rees 0-matrix semigroup is inverse.

Input: A finite Rees 0-matrix semigroup $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$.
Output: TRUE if $\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ is inverse; else FALSE.

if T is not an inverse monoid **or** $|I| \neq |\Lambda|$ **then**
  return FALSE
$G \leftarrow$ the group of units of T
for $\lambda \in \Lambda$ and $i \in I$ **do**
  SEEN_I[i] $\leftarrow$ FALSE ▷ SEEN_I[i] $\Leftrightarrow$ a non-zero matrix entry $p_{\mu,i}$ has been found.
  SEEN_Λ[λ] $\leftarrow$ FALSE ▷ SEEN_Λ[λ] $\Leftrightarrow$ a non-zero matrix entry $p_{\lambda,j}$ has been found.
for $\lambda \in \Lambda$ **do**
  for $i \in I$ **do**
    if $p_{\lambda,i} \neq 0$ **then**
      if $p_{\lambda,i} \notin G$ **or** SEEN_I[i] **or** SEEN_Λ[λ] **then**
        return FALSE
      SEEN_I[i] $\leftarrow$ TRUE
      SEEN_Λ[λ] $\leftarrow$ TRUE
  if not SEEN_Λ[λ] **then**
    return FALSE
return TRUE

Proof of Algorithm 3.67. Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be a finite Rees 0-matrix semigroup, where P is a $\Lambda \times I$ matrix over T^0. By Theorem 3.63, S is inverse if and only if T is an inverse monoid, and each row and each column of P contains exactly one non-zero entry, which is a unit in T. Thus the matrix of an inverse Rees 0-matrix semigroup is square. Clearly Algorithm 3.67 returns TRUE when the input S is inverse, and so it remains to show that Algorithm 3.67 returns FALSE when the input S is inverse, that is, when one of the following holds:

1. T is not an inverse monoid, or

2. P contains any non-zero entry that is not a unit in T, or

3. P contains two or more non-zero entries in the same row or column, or

4. P contains only 0 in some row or column.

If T is not an inverse monoid, then the algorithm returns FALSE on line 1. If P has any non-zero non-unit entries, or if P contains two or more non-zero entries in some row or column, then this is detected on line 10. If P contains any row that consists entirely of zeroes, then this is detected on line 14. Finally, suppose that P has some column consisting entirely of zeroes. We may assume that no row or column of P contains two or more non-zero entries since this will be checked independently, as already discussed. Therefore, the number of non-zero entries of P is strictly fewer than $|I|$. If $|I| \neq |\Lambda|$, then the algorithm returns FALSE on line 2; if $|I| = |\Lambda|$, then it follows that some *row* of P also consists entirely of zeroes, and the algorithm returns FALSE on line 15, provided that it has not already done so. □

The proof of the converse implication of Theorem 3.63 involved a description of the idempotents of an inverse Rees 0-matrix semigroup. We state this description as the following lemma for the sake of completeness.

Lemma 3.68. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be an inverse Rees 0-matrix semigroup, and let $\sigma : I \longrightarrow \Lambda$ be the unique bijection such that $p_{i\sigma,i} \neq 0$ for each $i \in I$. Then the set of idempotents of S is given by*

$$F(S) = E(S) = \left\{\left(i,\, ep_{i\sigma,i}^{-1},\, i\sigma\right) \,:\, i \in I,\, e \in E(T)\right\} \cup \{0\}.$$

In particular, S contains $1 + |I| \cdot |E(T)|$ idempotents.

Proof. Given the proof of Theorem 3.63, it remains to count the idempotents of S. If two non-zero idempotents $x = \left(i,\, ep_{i\sigma,i}^{-1},\, i\sigma\right)$ and $y = \left(j,\, fp_{j\sigma,j}^{-1},\, j\sigma\right)$ of S are equal, then certainly $i = j$, and $ep_{i\sigma,i}^{-1} = fp_{i\sigma,i}^{-1}$. But the non-zero matrix entries of P are units, and so

$$e = ep_{i\sigma,i}^{-1}p_{i\sigma,i} = fp_{i\sigma,i}^{-1}p_{i\sigma,i} = f.$$

Therefore, there are $|I| \cdot |E(T)|$ non-zero idempotents of S, along with the zero. □

In the SEMIGROUPS [101] package, the functions `Idempotents` and `NrIdempotents` use the characterisation given in Lemma 3.68 to compute the idempotents, and the number of idempotents, respectively, of an inverse Rees 0-matrix semigroup.

On any inverse semigroup S may be defined a *natural partial order*. If $a, b \in S$, then $a \leq b$ in the natural partial order on S if and only if there exists $e \in E(S)$ such that $a = eb$. See [76, Section 5.2] for more information about the natural partial order on an inverse semigroup. In the following lemma, we use the description of the idempotents of an inverse Rees 0-matrix semigroup given by Lemma 3.68 to describe the natural partial order on such a semigroup.

Lemma 3.69. *Let $S = \mathscr{M}^0[T;\, I,\, \Lambda;\, P]$ be an inverse Rees 0-matrix semigroup, and let $x = (i,\, t,\, \lambda)$ and $y = (j,\, u,\, \mu)$ be arbitrary non-zero elements of S. Then $x \leq y$ in the natural partial order on S if and only if $i = j$, $\lambda = \mu$, and $t \leq u$ in the natural partial order on T.*

Proof. As in Lemma 3.68, let $\sigma : I \longrightarrow \Lambda$ be the bijection such that $p_{i\sigma,i} \neq 0$ for all $i \in I$.

($\Rightarrow$) Since $x \leq y$, it follows by Lemma 3.68 that there exists $k \in I$ and $e \in E(T)$ such that

$$(i,\, t,\, \lambda) = (k,\, ep_{k\sigma,k}^{-1},\, k\sigma)(j,\, u,\, \mu).$$

In particular, $p_{k\sigma,j} \neq 0$, which implies that $k = j$. Therefore $(i, t, \lambda) = (j, ep_{j\sigma,j}^{-1}p_{j\sigma,j}u, \mu)$, which implies that $i = j$, $\lambda = \mu$, and $t = eu$, i.e. $t \leq u$ in the natural partial order on T.

($\Leftarrow$) By assumption, there exists $e \in E(T)$ such that $t = eu$. By Lemma 3.68, $z = \big(i, ep_{i\sigma,i}^{-1}, i\sigma\big) \in E(S)$, and it is easy to verify that $x = zy$. Therefore $x \leq y$. $\square$

Chapter 4

Computing maximal subsemigroups of a finite semigroup

4.1 Introduction

A *maximal subsemigroup* of a semigroup S is a proper subsemigroup of S that is not contained in any other proper subsemigroup of S. The main purpose of this chapter is to present practical algorithms for computing the maximal subsemigroups of an arbitrary finite semigroup. These algorithms are based on the paper from 1968 of Graham, Graham, and Rhodes [61], and are implemented in the Semigroups package [101] for GAP [58]. Part of the research described in this chapter was conducted in collaboration with Casey R. Donoven and James D. Mitchell, and is published in [35]. Additionally, some of the results in this chapter, particularly most of those in Section 4.5, have been published in [45], in collaboration with Jitender Kumar, James East, and James D. Mitchell.

A closely related notion is that of a maximal subgroup of a group. We first address a potential source of confusion, since the term maximal subgroup has different and conflicting meanings in semigroup theory and group theory. In semigroup theory, *maximal subgroup* is a term meaning *group* $\mathscr{H}$*-class*. However, in this thesis, we exclusively use the meaning from group theory, where a *maximal subgroup* of a group G is a proper subgroup of G that is contained in no other proper subgroup of G.

Maximal subgroups of both finite and infinite groups have been extensively investigated in the literature. In part, this is because of the importance of maximal subgroups to other aspects of group theory. For example, the Frattini subgroup of a group is the intersection of its maximal subgroups, and gives information about the generating sets of the group. Maximal subgroups are also closely related to primitive permutation representations. If G is a group acting primitively on some set, then the stabilizer of any point is a maximal subgroup; on the other hand, if H is a maximal subgroup of a group G, then G acts primitively on the set of cosets of H in G. The maximal subgroups of the finite symmetric groups are described, in some sense, by the O'Nan-Scott Theorem [109, 116], given the Classification of Finite Simple Groups. See [6–8, 11, 14, 23, 92, 93, 100, 111] and the references therein for research concerning maximal subgroups of infinite groups.

If G is a finite group, then the subgroups of G are the non-empty subsemigroups of G. Therefore, except for the trivial group, the maximal subsemigroups of a finite group are its maximal subgroups, and so the notions of maximal subgroups and maximal subsemigroups are not really distinct in this case. The same does not hold in general for infinite groups; see Example 4.1. It follows that computing the maximal subgroups of a finite group is a special case of the problem we consider here. Indeed, in the algorithms that we present, certain cases reduce to the computation of maximal subgroups of particular group $\mathscr{H}$-classes of the given semigroup, even for semigroups that are not groups. There are well-developed techniques for finding the maximal subgroups of a finite group, such as those of Cannon and Holt [17] which are implemented in MAGMA [13], and those of Eick and Hulpke [47], which are implemented in GAP [58]; see also [3]. Therefore, in this thesis, we will suppose that it is possible to compute the maximal subgroups of any given finite group.

Example 4.1. Let $\mathbb{Z}$ denote the group of integers under addition. It is clear that $\mathbb{N}_0$ is a subsemigroup of $\mathbb{Z}$ that is not a group, as is $\mathbb{N}$. Indeed, $\mathbb{N}_0$ is a maximal subsemigroup of $\mathbb{Z}$, since if x an arbitrary negative integer, then $\langle \mathbb{N}_0,\ x\rangle = \mathbb{Z}$. Furthermore, since $\mathbb{N} \leq \mathbb{N}_0$ and $\mathbb{N}_0 \setminus \mathbb{N} = \{0\}$, it follows that $\mathbb{N}$ is a maximal subsemigroup of $\mathbb{N}_0$; see Lemma 4.3.

There are many papers in the literature relating to maximal subsemigroups of semigroups that are not groups. Maximal subsemigroups of infinite semigroups, as with maximal subgroups of infinite groups, are very different from their finite counterparts. Every non-empty finite semigroup has at least one maximal subsemigroup, and every proper subsemigroup of a finite semigroup is contained in a maximal subsemigroup of that semigroup. However, the same is not true in general for infinite semigroups: there exist infinite semigroups with proper subsemigroups that are contained in no maximal subsemigroups, and infinite semigroups with no maximal subsemigroups at all; see Example 4.2. Analogous statements hold for groups. See [38, 74, 86] and the references therein for information about maximal subsemigroups of infinite semigroups. There are also numerous papers in the literature about finding maximal subsemigroups of particular classes of finite semigroups; see for example [25–30, 68, 69, 82, 124, 129–131]. In Chapter 5, we classify the maximal subsemigroups of various families of finite transformation and diagram monoids, using the techniques presented in this chapter.

Example 4.2. Let $S = (1, \infty)$ be the semigroup consisting of the real numbers greater than 1 whose operation is the usual multiplication of real numbers. Certainly (x, ∞) is a proper subsemigroup of S for any $x \in S$. Let T be any proper subsemigroup of S. There exist elements $a, b \in S \setminus T$ with $a < b$. Let U be the union of T with the ideal of S consisting of all real numbers greater than or equal to b. Then U is a proper subsemigroup of S (since $a \notin U$) that properly contains T (since $T \subseteq U$ and $b \in U \setminus T$). It follows that every proper subsemigroup is properly contained in a proper subsemigroup of S, and so there are no maximal subsemigroups of S.

The most important paper on the topic of maximal subsemigroups of finite semigroups is arguably that of Graham, Graham, and Rhodes [61]. In this paper, the authors prove that every maximal subsemigroup of a finite semigroup has certain features, and that every maximal subsemigroup has one of a small number of types. This paper appears to have been overlooked for many years, and special cases of the results it contains have been repeatedly reproved. While the paper describes the possible forms of a maximal subsemigroup, it does not classify which forms arise in a given finite semigroup. Determining this is difficult, and until the publication of our paper [35], no practical mechanism for doing so had appeared in the literature.

The naive algorithm for computing the maximal subsemigroups of an arbitrary finite semigroup is to construct all proper subsemigroups of the semigroup, to find their partial order induced by containment, and then to find the maximal elements in that partial order. Computing every subsemigroup of a finite semigroup S could be accomplished by finding the subsemigroup generated by each subset of S. Alternatively, the subsemigroups of S can be found by first constructing each 1-generated subsemigroup $\langle x\rangle$, for $x \in S$, and then iteratively and systematically finding the subsemigroup generated by each distinct pair of subsemigroups, until all have been found. In general, given a finite semigroup S, the time complexity of such algorithms is at least

$O(2^{|S|})$, since there exist finite semigroups in which every subset is a subsemigroup. Finding all the subsemigroups of a semigroup is a difficult problem, which has been considered in the literature in some natural cases. To illustrate, there are 132 069 776 subsemigroups of the full transformation monoid of degree 4 up to conjugacy, and 3 161 965 550 in total [41, Table 2] (note that $\mathcal{T}_4$ contains only 256 elements). The exact values are not known for $\mathcal{T}_n$ when $n \geq 5$, although a large lower bound has been proven [16, Theorem 9.1]. This suggests that finding all subsemigroups is not a feasible general-purpose approach to finding maximal subsemigroups.

In this chapter, we present algorithms for computing the maximal subsemigroups of an arbitrary finite semigroup using the results of Graham, Graham, and Rhodes [61]. In order to solve this problem in general, we first develop algorithms for computing the maximal subsemigroups of an arbitrary finite regular Rees 0-matrix semigroup over a group. Certain aspects of the general solution reduce to other well-known computational problems, such as finding all maximal cliques in a graph, finding strongly connected components in a digraph, and, as mentioned above, computing the maximal subgroups of a group.

The algorithms that we present for computing the maximal subsemigroups of an arbitrary finite semigroup S require a description of the Green's structure of S. Throughout this chapter, we assume that we are able to compute the Green's structure of any finite semigroup; see Section 1.4 for more information about this problem. Therefore, if computing the Green's structure of S is not practical, perhaps because it requires too much time, or memory, then the algorithms presented here cannot be used to find the maximal subsemigroups of S. In many examples, we find that if it is practical to compute the Green's structure of S from its given generating set, then it is also practical to find the maximal subsemigroups of S using the algorithms we present. In such examples, the time taken to determine the Green's structure of S is roughly comparable to that taken to find its maximal subsemigroups, given the Green's structure. Further details can be found in Section 4.6.2.

This chapter is organised as follows. In Section 4.2, we provide some preliminary results concerning the form of a maximal subsemigroup of a finite semigroup, and we state the main results of Graham, Graham, and Rhodes [61] in Proposition 4.10. In Section 4.3, we describe algorithms for finding maximal subsemigroups in any finite regular Rees 0-matrix semigroup over a group. In Section 4.4, we build on these results by developing algorithms for finding the maximal subsemigroups of an arbitrary finite semigroup defined by a generating set. In Section 4.5, we present special cases of some of these techniques for finite monoids. These results are particularly useful for Chapter 5. In Section 4.6, we state the overall algorithm for computing maximal subsemigroups of finite semigroups, and analyse its performance experimentally. Note that we do not formally analyse the time complexities of the algorithms that we present.

4.2 The form of a maximal subsemigroup

In this section, we present some results concerning the form of a maximal subsemigroup of a finite semigroup. In particular, we restate the main proposition of Graham, Graham, and Rhodes from their 1968 paper [61]. We use these results when developing algorithms for arbitrary semigroups in the later parts of this chapter, and when describing the maximal subsemigroups of specific finite monoids in the next.

Lemma 4.3. *Let S be a semigroup and let M be a subsemigroup of S such that $|S \setminus M| = 1$. Then M is a maximal subsemigroup of S.*

Proof. Any proper subsemigroup of S lacks at least one element of S. Therefore, M is the only proper subsemigroups of S that contains M. □

Example 4.4. Let S be a non-empty left-zero semigroup. It is clear that any subset of S is a subsemigroup. In particular, the subset $S \setminus \{x\}$ is a subsemigroup of S for each $x \in S$, and is therefore a maximal subsemigroup by Lemma 4.3. Any other proper subsemigroup is properly contained in such a semigroup, and is therefore not maximal.

We find that the indecomposable elements of a semigroup give rise to maximal subsemigroups of the kind described in Lemma 4.3. Recall that an element x of a semigroup S is *decomposable* if $x \in S^2$, and *indecomposable* if not; see Section 2.2 for more details.

Lemma 4.5. *Let S be a semigroup, and let $x \in S \setminus S^2$ be an indecomposable element of S. Then $S \setminus \{x\}$ is a maximal subsemigroup of S. Moreover, if $S = \langle S \setminus S^2 \rangle$, then these are the only maximal subsemigroups of S.*

Proof. Suppose that $x \in S \setminus S^2$. Certainly $x \notin \langle S \setminus \{x\} \rangle$, since x cannot be written as a non-trivial product of elements in S. Therefore $S \setminus \{x\}$ is a subsemigroup of S; it is maximal by Lemma 4.3.

Suppose that S is generated by its indecomposable elements, and let M be a maximal subsemigroup of S. Since M is a proper subsemigroup, there exists an indecomposable element $x \in S \setminus M$. In particular, $M \subseteq S \setminus \{x\}$. But $S \setminus \{x\}$ is a maximal subsemigroup containing M, which implies that $M = S \setminus \{x\}$. □

Example 4.6. The number 1 is the only indecomposable element of the natural numbers $\mathbb{N}$ under addition. Moreover, $\mathbb{N} = \langle 1 \rangle$. By Lemma 4.5, the unique maximal subsemigroup of $\mathbb{N}$ is $\mathbb{N} \setminus \{1\} = \{2, 3, 4, \ldots\}$.

As shown in Lemma 2.9, if an element x in a semigroup S is indecomposable, then $\{x\}$ is a maximal $\mathscr{J}$-class of S. Therefore, a maximal subsemigroup M of S of the kind described in Lemma 4.5 is formed by removing a $\mathscr{J}$-class of S. In particular, M contains all other $\mathscr{J}$-classes of S. In the following lemma, we prove that a maximal subsemigroup of any semigroup S contains every $\mathscr{J}$-class of S, except for one. This result was stated and proved for finite semigroups in [61, Proposition (1)]. It follows that if S is $\mathscr{J}$-trivial, then every maximal subsemigroup of S is found by removing a $\mathscr{J}$-class, and has the form described in Lemma 4.3.

Lemma 4.7 (cf. [61, Proposition (1)]). *Let S be an arbitrary semigroup and let M be a maximal subsemigroup of S. There exists a $\mathscr{J}$-class of S such that $S \setminus M \subseteq J$.*

Proof. Let $x, y \in S \setminus M$ be arbitrary. Since M is a maximal subsemigroup of S, it follows that $S = \langle M,\ x \rangle$. Therefore $y \in \langle M,\ x \rangle \setminus M$, which implies that y can be expressed as a product in $M \cup \{x\}$ that involves x. In particular, $y = axb$ for some $a, b \in S^1$. By a symmetric argument, there exist elements $s, t \in S^1$ such that $x = syt$. Hence $x \mathscr{J} y$. □

In this thesis, we call a maximal subsemigroup whose complement is contained in a $\mathscr{J}$-class J a *maximal subsemigroup arising from J*. Note that if U is a subset of a semigroup S, and $J \in S/\mathscr{J}$, then $S \setminus U \subseteq J$ if and only if $S \setminus J \subseteq U$. By Lemma 4.7, the problem of computing every maximal subsemigroup in a semigroup can be approached by determining the maximal subsemigroups that arise from each of its $\mathscr{J}$-classes in turn, or indeed in parallel.

In the following lemma, we classify the maximal subsemigroups arising from a $\mathscr{J}$-class of a semigroup S in terms of the generating sets of S. This result is useful in this thesis when describing the maximal subsemigroups of a semigroup whose generating sets are well-understood. In particular, Lemma 4.8 is used in the proofs of Theorems 5.7, 5.49, and 5.60.

Lemma 4.8. *Let S be a semigroup, and let J be a $\mathscr{J}$-class of S. Suppose there exists an indexing set I and subsets $X_i \subseteq J$ for $i \in I$. Further suppose that no set X_i is contained in a* *different set X_j, and that $S = \langle S \setminus J,\ A\rangle$ if and only if $A \cap X_i \neq \varnothing$ for all $i \in I$. Then the maximal subsemigroups of S arising from J are precisely the sets $S \setminus X_i$ for each $i \in I$.*

Proof. Let $i \in I$ be arbitrary. We show that $S \setminus X_i$ is a subsemigroup of S; its maximality is then obvious. Let $x, y \in S \setminus X_i$. Since $S \setminus X_i$ does not generate S, but it contains $S \setminus J$ and an element $x_j \in X_j$ for each $j \in I \setminus \{i\}$, it follows that $xy \notin X_i$. Conversely, let M be a maximal subsemigroup of S arising from J. If $M \cap X_i \neq \varnothing$ for each i then, by assumption, $S = \langle M\rangle = M$, a contradiction. Thus $M \cap X_i = \varnothing$ for some i. In other words, $M \subseteq S \setminus X_i$. By the maximality of M in S, it follows that $M = S \setminus X_i$. □

In the remainder of this section, we restrict our attention to finite semigroups. By [61], a maximal subsemigroup of a finite semigroup S is either a union of $\mathscr{H}$-classes S, or it intersects every $\mathscr{H}$-class of S non-trivially; see Lemma 4.9. It is not known whether this statement holds more generally for infinite semigroups. The techniques that we develop for finding maximal subsemigroups of a finite semigroup differ significantly according to whether the desired maximal subsemigroup is a union of $\mathscr{H}$-class of the semigroup, or intersects each $\mathscr{H}$-class of the semigroup non-trivially.

Lemma 4.9 ([61, Proposition (2)]). *A maximal subsemigroup of a finite semigroup S either intersects every $\mathscr{H}$-class of S non-trivially, or is a union of $\mathscr{H}$-classes of S.*

Proof. Let M be a maximal subsemigroup of S. We will show that

$$T = \bigcup_{x \in M} H_x^S$$

is a subsemigroup of S. Since M is maximal, it then follows that either $T = M$, and M is a union of $\mathscr{H}$-classes of S, or that $T = S$, and M intersects every $\mathscr{H}$-class of S non-trivially.

Let $s, t \in T$ be arbitrary. Certainly $S \setminus J \subseteq M$ by Lemma 4.7, and $M \subseteq T$ by definition. Therefore, to show that T is a subsemigroup, it suffices to show that $st \in T$ whenever $st \in J$. So assume that $st \in J$, and that $s \in J$ or $t \in J$. In the first case, suppose that $s \in J$ and $t \in M$. Since $s \in J \cap T$, and a $\mathscr{J}$-class is a union of $\mathscr{H}$-classes, there exists by definition some $x \in M \cap J$ such that $s \mathscr{H}^S x$. By Lemma 1.10, $s \mathscr{R}\ st$, and so by Green's Lemma (Lemma 1.11), right multiplication by t defines an $\mathscr{H}$-class preserving bijection from L_s^S to L_{st}^S. In particular, $st \mathscr{H}^S xt \in M$, and so $st \in T$. A dual argument shows that $st \in T$ when $s \in M$ and $t \in J$. Finally, suppose that $s, t \in J$. Define x as above, and choose any $y \in M \cap J$ such that $t \mathscr{H}^S y$. By Lemma 1.10, $s \mathscr{R}\ st \mathscr{L}\ t$. By Lemma 1.15, $H_s^S H_t^S \subseteq H_{st}^S$. In particular, $xy \mathscr{H}^S st$, and since $xy \in M$, it follows that $st \in T$. □

The main result of Graham, Graham, and Rhodes [61] describes, in the finite case, the possible intersections of a maximal subsemigroup with the $\mathscr{J}$-class from which it arises. In Proposition 4.10, we include a slightly reformulated version of this main result. The principal difference is that the isomorphism ϕ in Proposition 4.10 is arbitrary, whereas the result in Graham, Graham, and Rhodes specifies only that there exists some isomorphism with the required properties. Proposition 4.10 follows from the proof in [61]. Proposition 4.10(a) is part (3) of the main proposition [61], and Proposition 4.10(b) is Case 1 of part (4). Proposition 4.10(c) is adapted from Case 2 of part (4) of the main proposition of [61], in order to render the statements (i)–(iv) mutually-exclusive.

Proposition 4.10 (Propositions (3) and (4) in [61]). *Let S be a finite semigroup, let M be a maximal subsemigroup of S, and let J denote the $\mathscr{J}$-class of S that contains $S \setminus M$. If J is regular, define $\phi : J^* \longrightarrow \mathscr{M}^0[G;\ I,\ \Lambda;\ P]$ to be any isomorphism from the principal factor of J to a regular Rees 0-matrix semigroup over a group G. Then exactly one of the following holds:*

(a) *J is non-regular and $M \cap J = \varnothing$;*

(b) *J is regular and*

$$(M \cap J)\phi \cup \{0\} \cong \mathscr{M}^0[H;\, I,\, \Lambda;\, Q],$$

where H is a maximal subgroup of G and Q is a $\Lambda \times I$ matrix over H^0.

In this case, $(M \cap J)\phi \cup \{0\}$ is a maximal subsemigroup of $\mathscr{M}^0[G;\, I,\, \Lambda;\, P]$;

(c) *J is regular and $(M \cap J)\phi$ is equal to one of the following:*

   (i) *$(I \times G \times \Lambda) \setminus (I' \times G \times \Lambda')$ for some proper non-empty subsets $I' \subsetneq I$ and $\Lambda' \subsetneq \Lambda$. In this case, $(M \cap J)\phi \cup \{0\}$ is a maximal subsemigroup of $\mathscr{M}^0[G;\, I,\, \Lambda;\, P]$;*

   (ii) *$I \times G \times \Lambda'$ for some proper non-empty subset Λ' of Λ;*

   (iii) *$I' \times G \times \Lambda$ for some proper non-empty subset I' of I;*

   (iv) *$\varnothing$.*

In the remainder of this chapter, we develop effective methods for determining the maximal subsemigroups of each possible form that arise in a finite semigroup, given its Green's structure. In order to be able to easily refer to these possible forms, we introduce the labels (M1)–(M5) as follows. Let S be any finite semigroup, and let M be a maximal subsemigroup of S. By Lemma 4.7, there exists a $\mathscr{J}$-class J of S such that $S \setminus M \subseteq J$. By Proposition 4.10, if $M \cap J$ is non-empty, then J is regular and precisely one of the following holds:

(M1) $M \cap J$ has non-empty intersection with every $\mathscr{H}$-class in J (Proposition 4.10(b));

(M2) $M \cap J = \bigcup_{A \in \mathfrak{L} \cup \mathfrak{R}} A$, where $\mathfrak{L}$ and $\mathfrak{R}$ are non-empty sets of $\mathscr{L}$- and $\mathscr{R}$-classes of J, respectively (Proposition 4.10(c)(i));

(M3) $M \cap J$ is a non-empty union of $\mathscr{L}$-classes in J (Proposition 4.10(c)(ii));

(M4) $M \cap J$ is a non-empty union of $\mathscr{R}$-classes in J (Proposition 4.10(c)(iii));

(M5) $M = S \setminus J$ (Proposition 4.10(a) and (c)(iv)).

A maximal subsemigroup of type (M1) intersects every $\mathscr{H}$-class of S non-trivially, whereas maximal subsemigroups of types (M2)–(M5) are unions of $\mathscr{H}$-classes of S; see Lemma 4.9. Note that a maximal subsemigroup of type (M5) can arise from either a regular or a non-regular $\mathscr{J}$-class; maximal subsemigroups of types (M1)–(M4) only arise from a regular $\mathscr{J}$-class.

In general, the collection of maximal subsemigroups arising from a particular regular $\mathscr{J}$-class J can have any combination of types (M1)–(M4). However, if $S \setminus J$ is a maximal subsemigroup of S, then it is the only maximal subsemigroup to arise from J, since a maximal subsemigroup of type (M1)–(M4) properly contains $S \setminus J$. In other words, there is at most one maximal subsemigroup of type (M5) arising from J, and its existence precludes the occurrence of maximal subsemigroups of types (M1)–(M4).

4.3 Finite regular Rees 0-matrix semigroups over groups

In this section, we describe techniques for computing the maximal subsemigroups of an arbitrary finite regular Rees 0-matrix semigroup over a group. This topic has inherent interest. As discussed in Section 3.1, a finite semigroup is 0-simple if and only if it is isomorphic to a regular Rees 0-matrix semigroup over a group, and in some sense, finite 0-simple semigroups are the building blocks of finite semigroups. Furthermore, we use the algorithms for finite regular Rees

0-matrix semigroups over groups when computing the maximal subsemigroups of an arbitrary finite semigroup, which is the topic of Section 4.4.

The following result, which classifies the maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group, is central to the algorithms presented in this section. This proposition is a slight reformulation of [61, Remark 1] and [62, Theorem 4].

Proposition 4.11 (cf. [62, Theorem 4] and [61, Remark 1]). *Let $S = \mathscr{M}^0[G; I, \Lambda; P]$ be a finite regular Rees 0-matrix semigroup over a group, and let M be any subset of S. Then M is a maximal subsemigroup of S if and only if one of the following conditions holds:*

(R1) $M = \{0\}$ *and* $|G| = |I| = |\Lambda| = 1$*;*

(R2) $M = I \times G \times \Lambda$ *and every entry of P is non-zero;*

(R3) $|\Lambda| > 1$, $M = S \setminus (I \times G \times \{\lambda\})$ *for some* $\lambda \in \Lambda$*, and for each* $i \in I$ *there exists* $\mu \in \Lambda \setminus \{\lambda\}$ *such that* $p_{\mu,i} \neq 0$*;*

(R4) $|I| > 1$, $M = S \setminus (\{i\} \times G \times \Lambda)$ *for some* $i \in I$*, and for each* $\lambda \in \Lambda$ *there exists* $j \in I \setminus \{i\}$ *such that* $p_{\lambda,j} \neq 0$*;*

(R5) $M = S \setminus \big((I \setminus X) \times G \times (\Lambda \setminus Y)\big)$*, where X and Y are proper non-empty subsets of I and Λ, respectively, and $X \cup Y$ is a maximal independent subset of the Graham-Houghton graph of S;*

(R6) *M is a subsemigroup of S isomorphic to a regular Rees 0-matrix semigroup of the form* $\mathscr{M}^0[H; I, \Lambda; Q]$*, where H is a maximal subgroup of G and Q is a $\Lambda \times I$ matrix over H^0.*

In [62, Theorem 4] and [61, Remark 1], the characterisation of the maximal subsemigroups of type (R1) is incorrectly stated. In more detail, if $S = \mathscr{M}^0[G; I, \Lambda; P]$ is a finite regular Rees 0-matrix semigroup where G is a group, then these results claim that $\{0\}$ is a maximal subsemigroup of S if and only if $|I| = |\Lambda| = 1$ and G is a simple cyclic group. However, if $I = \{i\}$ and $\Lambda = \{\lambda\}$ and G is non-trivial, then the set $\{0, (i, p_{\lambda,i}^{-1}, \lambda)\}$ is a proper subsemigroup of S that properly contains $\{0\}$, and so $\{0\}$ is not a maximal subsemigroup of S.

By Lemma 4.9, a maximal subsemigroup of a finite semigroup is either a union of $\mathscr{H}$-classes of the semigroup, or it intersects every $\mathscr{H}$-class of the semigroup non-trivially. Maximal subsemigroups of types (R1)–(R5) are subsemigroups that are unions of $\mathscr{H}$-classes, while maximal subsemigroups of type (R6) intersect every $\mathscr{H}$-class. The maximal subsemigroups of type (R6) are the most complicated to describe.

In Sections 4.3.1–4.3.4, we describe how to compute maximal subsemigroups of an arbitrary finite regular Rees 0-matrix semigroup over a group that have types (R1)–(R6). In each case, we also describe how to specify generating sets for these maximal subsemigroups. Often, this requires the construction of generating sets for related regular Rees 0-matrix semigroups over groups; as discussed in Section 3.4, constructing such generating sets is well-understood and straightforward. Although finding minimal generating sets for these kinds of semigroups is well-studied [63, 114], we do not claim that the generating sets in this section are minimal.

In Section 4.3.5, we demonstrate many of the results of Sections 4.3.1–4.3.4 by finding the maximal subsemigroups of a specific Rees 0-matrix semigroup.

Finally, in Section 4.3.6 we describe how to compute the maximal subsemigroups of any finite regular Rees 0-matrix semigroup that have type (R6) and that contain a particular subset of elements; see Algorithm 4.44. This is required by Algorithm 4.86 from Section 4.4, which describes a procedure for computing the maximal subsemigroups of an arbitrary finite semigroup.

4.3.1 Maximal subsemigroups of types (R1) and (R2)

Let $S = \mathscr{M}^0[G; I, \Lambda; P]$ be an arbitrary finite regular Rees 0-matrix semigroup over a group G. It is trivial to determine whether there exists a maximal subsemigroup of S of type (R1) or type (R2). For the former, we test whether G, I, and Λ are trivial, and if they are, then $\{0\}$ is a maximal subsemigroup of S. For the latter, we simply check whether 0 appears as an entry in the matrix P, and if it does not, then $S \setminus \{0\} = I \times G \times \Lambda$ is a maximal subsemigroup of S.

Example 4.12. Let $\mathcal{S}_2$ denote the symmetric group of degree 2, let I and Λ be index sets with two elements, and define

$$P = \begin{pmatrix} \mathrm{id}_2 & 0 \\ 0 & (1\ 2) \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} \mathrm{id}_2 & (1\ 2) \\ (1\ 2) & \mathrm{id}_2 \end{pmatrix}.$$

If $S = \mathscr{M}^0[\mathcal{S}_2; I, \Lambda; P]$, then S has no maximal subsemigroup of type (R2), since the matrix P contains 0. However, every entry of Q is non-zero, so if $T = \mathscr{M}^0[\mathcal{S}_2; I, \Lambda; Q]$, then $T \setminus \{0\}$ is the unique maximal subsemigroup of T of type (R2). Neither S nor T has a maximal subsemigroup of type (R1), since their underlying groups and index sets are non-trivial.

It is not complicated to describe generating sets for maximal subsemigroups of types (R1) and (R2). The subset $\{0\}$ is a trivial subsemigroup of S, and so its unique generating is $\{0\}$ itself. If $S \setminus \{0\}$ is a maximal subsemigroup of S, then by Lemma 1.5(i), a subset X of S generates S if and only if $X \setminus \{0\}$ generates $S \setminus \{0\}$. Therefore, a generating set for $S \setminus \{0\}$ can be obtained from any generating set for S by removing the zero element. See Section 3.4 for more information about finding generating sets for finite regular Rees 0-matrix semigroups over groups.

4.3.2 Maximal subsemigroups of types (R3) and (R4)

Let $S = \mathscr{M}^0[G; I, \Lambda; P]$ be an arbitrary finite regular Rees 0-matrix semigroup over a group G. In Proposition 4.11, the existence of maximal subsemigroups of types (R3) and (R4) is characterised in terms of the existence of certain non-zero entries in the matrix P. It is possible to test these conditions directly. However, for computational purposes, it is simpler to reformulate these conditions in terms of the Graham-Houghton graph of S.

Lemma 4.13. *Let $S = \mathscr{M}^0[G; I, \Lambda; P]$ be a finite regular Rees 0-matrix semigroup over a group G. The following hold:*

(i) *If $|\Lambda| > 1$ and $\lambda \in \Lambda$, then $S \setminus (I \times G \times \{\lambda\})$ is a maximal subsemigroup of S if and only if λ is not adjacent to a vertex of degree one in the Graham-Houghton graph of S.*

(ii) *If $|I| > 1$ and $i \in I$, then $S \setminus (\{i\} \times G \times \Lambda)$ is a maximal subsemigroup of S if and only if i is not adjacent to a vertex of degree one in the Graham-Houghton graph of S.*

Proof. We prove only (i), since (ii) is dual. Suppose that $|\Lambda| > 1$ and let $\lambda \in \Lambda$.

If $S \setminus (I \times G \times \{\lambda\})$ is a maximal subsemigroup of R, then it certainly has type (R3). Therefore, for each $i \in I$, there exists $\mu \in \Lambda \setminus \{\lambda\}$ such that $p_{\mu,i} \neq 0$. In other words, every vertex $i \in I$ in the Graham-Houghton graph of S that is adjacent to λ is also adjacent to some other vertex, and therefore has degree at least 2.

Conversely, suppose that every vertex $i \in I$ that is adjacent to λ in the Graham-Houghton graph of S has degree at least 2. Therefore, each such vertex is adjacent to some vertex $\mu \neq \lambda$. In other words, for each $i \in I$ there exists $\mu \in \Lambda \setminus \{\lambda\}$ such that $p_{\mu,i} \neq 0$. Therefore, by Proposition 4.11, the set $S \setminus (I \times G \times \{\lambda\})$ is a maximal subsemigroup of S of type (R3). □

Example 4.14. Define $S = \mathscr{M}^0[\mathcal{S}_3;\, I,\, \Lambda;\, P]$, where $\mathcal{S}_3$ denotes the symmetric group of degree 3, $I = \{1',\, 2',\, 3'\}$ and $\Lambda = \{1,\, 2,\, 3\}$, and

$$P = \begin{pmatrix} \mathrm{id}_3 & (1\ 3\ 2) & (1\ 2) \\ (1\ 3) & (2\ 3) & 0 \\ 0 & \mathrm{id}_3 & 0 \end{pmatrix}.$$

See Figure 4.15 for a picture of the Graham-Houghton graph of S. Note that $|I| > 1$ and $|\Lambda| > 1$. As discussed in the caption of Figure 4.15, 1 and $2'$ are the only vertices in the Graham-Houghton graph of S that are adjacent to vertices of degree one. Therefore, by Lemma 4.13, the sets

$$S \setminus (I \times G \times \{2\}) \quad \text{and} \quad S \setminus (I \times G \times \{3\}),$$

are the maximal subsemigroups of type (R3), and the maximal subsemigroups of type (R4) are

$$S \setminus (\{1'\} \times G \times \Lambda) \quad \text{and} \quad S \setminus (\{3'\} \times G \times \Lambda).$$

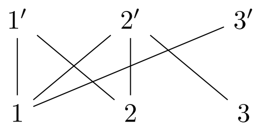

Figure 4.15: The Graham-Houghton graph of the Rees 0-matrix semigroup $S = \mathscr{M}^0[\mathcal{S}_3;\, I,\, \Lambda;\, P]$ from Example 4.14, where $I = \{1',\, 2',\, 3'\}$ and $\Lambda = \{1,\, 2,\, 3\}$. The only vertices of degree 1 in this graph are 3 and $3'$, which are adjacent to the vertices $2'$ and 1, respectively. There are four maximal independent subsets of the Graham-Houghton graph of S: $\{1,\, 2,\, 3\}$, $\{1',\, 2',\, 3'\}$, $\{1',\, 3',\, 3\}$, and $\{3',\, 2,\, 3\}$. Therefore, by Lemma 4.13, there is one maximal subsemigroup of type (R3) and one maximal subsemigroup of type (R4), and by Proposition 4.11, there are two maximal subsemigroups of S of type (R5).

A method that uses Lemma 4.13 to compute the maximal subsemigroups of S that have type (R3) is given in Algorithm 4.16. One of the steps involved is the construction of the Graham-Houghton graph of S, which requires only the identification of the non-zero entries of the matrix P. The only other substantive calculation involved in this algorithm occurs in line 7, which tests whether the vertex $i \in I$ has a unique neighbour $\lambda \in \Lambda$. Finding the neighbours of a vertex in a graph is a task that is fundamental to almost every algorithm in graph theory. Therefore, given any useful representation of the Graham-Houghton graph, finding the neighbours of i is trivially easy. Indeed, a graph is often represented on a computer by storing the set of neighbours of each vertex.

There is an obvious analogue of Algorithm 4.16 that describes a method for computing the maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group that have type (R4). Given the preceding discussion, therefore, it is easy to find the maximal subsemigroups of S of types (R3) and (R4).

If $T = S \setminus (I \times G \times \{\lambda\})$ is a maximal subsemigroup of S of type (R3), for some $\lambda \in \Lambda$, then we may regard T as the regular Rees 0-matrix semigroup $\mathscr{M}^0[G;\, I,\, \Lambda \setminus \{\lambda\};\, Q]$, where $Q = (p_{\mu,i})_{\mu \in \Lambda \setminus \{\lambda\}, i \in I}$ is the submatrix of P formed by removing row λ. Similarly, we can view a maximal subsemigroup of S of type (R4) as the regular Rees 0-matrix semigroup $\mathscr{M}^0[G;\, I \setminus \{i\},\, \Lambda;\, Q']$, for some $i \in I$ and for some submatrix Q' of P. There are well-developed techniques for finding generating sets of Rees 0-matrix semigroups over groups; see Section 3.4 for a discussion of this topic. In this way, by finding a generating set for a corresponding Rees 0-matrix semigroup, it is straightforward to construct a generating set for a maximal subsemigroup of type (R3) or (R4).

Algorithm 4.16 Maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group that have type (R3).

Input: A finite regular Rees 0-matrix semigroup $S = \mathscr{M}^0[G; I, \Lambda; P]$, where G is a group.
Output: The set $\mathfrak{M}$ of maximal subsemigroups of S of type (R3).

$\mathfrak{M} \leftarrow \varnothing$
if $|\Lambda| = 1$ **then**
  return $\mathfrak{M}$.
$\Gamma \leftarrow$ the Graham-Houghton graph of S
$\textsc{forbidden} \leftarrow \varnothing$
for each $i \in I$ **do**
  if some $\lambda \in \Lambda$ is the unique neighbour of i **then**
    $\textsc{forbidden} \leftarrow \textsc{forbidden} \cup \{\lambda\}$
for each $\lambda \in \Lambda \setminus \textsc{forbidden}$ **do**
  $\mathfrak{M} \leftarrow \mathfrak{M} \cup \big\{S \setminus (I \times G \times \{\lambda\})\big\}$ ▷ Lemma 4.13
return $\mathfrak{M}$.

4.3.3 Maximal subsemigroups of type (R5)

In this section, we discuss how to compute the maximal subsemigroups of an arbitrary finite regular Rees 0-matrix semigroup $S = \mathscr{M}^0[G; I, \Lambda; P]$, where G is a group, that have type (R5).

Certainly, the subsets I and Λ of the Graham-Houghton graph of S are maximal independent subsets. By Proposition 4.11, the remaining maximal independent subsets are in one-to-one correspondence with the maximal subsemigroups of type (R5). Therefore, computing these maximal subsemigroups is equivalent to computing the maximal cliques in the complement of the Graham-Houghton graph of S. This is a well-understood but computationally hard problem; see [15, 103, 125] for more information.

The methods in the Semigroups package [101] for GAP [58] for computing maximal subsemigroups of type (R5) use the clique-finding methods from the Digraphs package [10], which were written by the author. These clique-finding methods are based on the Bron-Kerbosch Algorithm [15], and use the automorphism group of the graph to break the symmetry of the search space. In many cases, it seems that the majority of the time taken to compute the maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group is spent calculating the maximal independent subsets of the Graham-Houghton graph. See Section 4.6.2, especially Figure 4.93, for more information about the performance of the methods that are implemented in Semigroups.

Example 4.17. Let $S = \mathscr{M}^0[\mathcal{S}_3; I, \Lambda; P]$ be defined as in Example 4.14; see Figure 4.15 for a picture of the Graham-Houghton graph of S. Two of the maximal independent subsets of the Graham-Houghton graph are I and Λ; the remaining maximal independent subsets are $\{1', 3', 3\} = (I \setminus \{2'\}) \cup (\Lambda \setminus \{1, 2\})$ and $\{3', 2, 3\} = (I \setminus \{1', 2'\}) \cup (\Lambda \setminus \{1\})$. Therefore, by Proposition 4.11, the maximal subsemigroup of S of type (R5) are the sets

$$S \setminus \big(\{2'\} \times G \times \{1, 2\}\big) \quad \text{and} \quad S \setminus \big(\{1', 2'\} \times G \times \{1\}\big).$$

An efficient way to specify a maximal subsemigroups of S that has type (R5) is by a generating set. Let X and Y be proper non-empty subsets of I and Λ, respectively, such that $X \cup Y$ is a maximal independent subset of the Graham-Houghton graph of S. The maximal subsemigroup T of S corresponding to this maximal independent subset is given by

$$\begin{aligned} T &= S \setminus ((I \setminus X) \times G \times (\Lambda \setminus Y)) \\ &= (X \times G \times Y) \cup (I \times G \times (\Lambda \setminus Y)) \cup ((I \setminus X) \times G \times \Lambda) \cup \{0\}. \end{aligned}$$

Let $Q = (p_{\lambda,i})_{\lambda \in \Lambda \setminus Y, i \in I}$ and $Q' = (p_{\lambda,i})_{\lambda \in \Lambda, i \in I \setminus X}$. We may regard $(I \times G \times (\Lambda \setminus Y)) \cup \{0\}$ as a regular Rees 0-matrix semigroup $\mathscr{M}^0[G; I, \Lambda \setminus Y; Q]$, and we may regard $((I \setminus X) \times G \times \Lambda) \cup \{0\}$ as the regular Rees 0-matrix semigroup $\mathscr{M}^0[G; I \setminus X, \Lambda; Q']$. As discussed in Section 3.4, it is straightforward to construct a small generating set for a regular Rees 0-matrix semigroup over a group.

Let A be any subset of T that contains generating sets for $(I \times G \times (\Lambda \setminus Y)) \cup \{0\}$ and $((I \setminus X) \times G \times \Lambda) \cup \{0\}$, and that contains any element of $X \times G \times Y$. Let $(i, g, \lambda) \in X \times G \times Y$ be arbitrary, and let (k, h, γ) be the unique element of $A \cap (X \times G \times Y)$. Since $k \in X$, there exists $\mu \in \Lambda \setminus Y$ such that $p_{\mu,k} \neq 0$, and since $\gamma \in Y$, there exists $j \in I \setminus X$ such that $p_{\gamma,j} \neq 0$. Noting that

$$\big(i,\, gp_{\mu,i}^{-1},\, \mu\big) \in (I \times G \times (\Lambda \setminus Y)) \subseteq \langle A \rangle, \quad \text{and} \quad \big(j,\, p_{\gamma,j}^{-1}h^{-1},\, \lambda\big) \in ((I \setminus X) \times G \times \Lambda) \subseteq \langle A \rangle,$$

it follows that

$$\big(i,\, g,\, \lambda\big) = \big(i,\, gp_{\mu,i}^{-1},\, \mu\big)\big(k,\, h,\, \gamma\big)\big(j,\, p_{\gamma,j}^{-1}h^{-1},\, \lambda\big) \in \langle A \rangle.$$

Therefore, A also contains a generating set for $(X \times G \times Y) \cup \{0\}$. In particular, $T = \langle A \rangle$.

In conclusion, to specify a generating set for T, it suffices to find a generating set for the two corresponding Rees 0-matrix semigroups described above, along with any element of $X \times G \times Y$.

4.3.4 Maximal subsemigroups of type (R6)

In this section, we develop an algorithm for computing the maximal subsemigroups of an arbitrary finite regular Rees 0-matrix semigroup over a group that have type (R6); these are the maximal subsemigroups that intersect every $\mathscr{H}$-class of the semigroup non-trivially. In this section, we assume that we are able to compute the maximal subgroups of any finite group.

As shown in Proposition 3.2, every finite regular Rees 0-matrix semigroup over a group can be normalized. See Section 3.2.1 for a discussion about computing normalizations of finite regular Rees 0-matrix semigroups over groups. Thus, without loss of generality, we assume throughout this section that $S = \mathscr{M}^0[G; I, \Lambda; P]$ is a normalized finite regular Rees 0-matrix semigroup over a group G. We fix the following notation, as used in Proposition 3.2. Let $I_1 \cup \Lambda_1, \ldots, I_n \cup \Lambda_n$ be the distinct connected components of the Graham-Houghton graph of S, where $I_k \subseteq I$ and $\Lambda_k \subseteq \Lambda$ for each k, and for each $k \in \{1, \ldots, n\}$, fix $i_k \in I_k$ and $\lambda_k \in \Lambda_k$ such that $p_{\lambda_k, i_k} = 1_G$, and define $G_k = \big\{g \in G : (i_k, g, \lambda_k) \in F(S)\big\}$.

By Proposition 4.11, a maximal subsemigroup of a finite regular Rees 0-matrix semigroup $\mathscr{M}^0[G; I, \Lambda; P]$ over a group G that has type (R6) is isomorphic to some regular Rees 0-matrix semigroup $\mathscr{M}^0[H; I, \Lambda; Q]$, where H is a maximal subgroup of G. More specifically, in [62, Theorems 3–4], Graham proves that if V is a maximal subgroup of G and there exist elements $g_1, \ldots, g_n \in G$ such that $G_k \leq g_k^{-1} V g_k$ for all k, then there exists a maximal subsemigroup of S that is isomorphic to $\mathscr{M}^0[V; I, \Lambda; Q]$, where Q is a matrix over V^0 that can be constructed from P. In this section, we develop these ideas in a way that allows us to describe and count, in an explicit and systematic way, the specific maximal subsemigroups of S that can be obtained from each maximal subgroup of G. We extend these techniques in Section 4.3.6, where we present an algorithm for computing the maximal subsemigroups of an arbitrary finite regular Rees 0-matrix semigroup over a group that have type (R6) and contain a given set.

In order to describe the Green's relations on a maximal subsemigroup of S of type (R6), we require the following lemma, which shows that any such subsemigroup is regular.

Lemma 4.18. *Let T be a subsemigroup of S that intersects every $\mathscr{H}$-class of S non-trivially. Then T is regular.*

Proof. Certainly 0 is a regular element, so let $x = (i,\, g,\, \lambda) \in T \setminus \{0\}$ be an arbitrary non-zero element of T. Since P contains a non-zero entry in every row and column, there exist indices $j \in I$ and $\mu \in \Lambda$ such that $p_{\lambda,j} \neq 0 \neq p_{\mu,i}$. Let $e = (j,\, p_{\lambda,j}^{-1},\, \lambda) \in E(S)$. Then $e \in T$ by Lemma 1.14. Certainly $xe = x$. Furthermore, if $y \in T$ is an arbitrary element of the $\mathscr{H}^S$-class $\{j\} \times G \times \{\mu\}$, then $yx \in \{j\} \times G \times \{\lambda\}$. Therefore, $(yx)^k = e$ for some $k \in \mathbb{N}$, by the finiteness of S. In particular, $x \mathscr{L}^T e$. It follows that T is regular. □

Let T be a subsemigroup of S. If T is a regular subsemigroup of S, then Green's $\mathscr{L}$-, $\mathscr{R}$-, and $\mathscr{H}$-relations on T are inherited from S [76, Proposition 2.4.2]. In particular, if T intersects every $\mathscr{H}$-class of S non-trivially, then by Lemmas 3.15 and 4.18, the following hold for all non-zero elements $x = (i,\, g,\, \lambda)$ and $y = (j,\, h,\, \mu)$ in T:

(i) $x \mathrel{\mathscr{L}^T} y$ if and only if $\lambda = \mu$;

(ii) $x \mathrel{\mathscr{R}^T} y$ if and only if $i = j$; and

(iii) $x \mathrel{\mathscr{H}^T} y$ if and only if $i = j$ and $\lambda = \mu$.

In particular, this characterises the Green's relations on any maximal subsemigroup of type (R6).

Arbitrary subsemigroups that intersect every $\mathscr{H}$-class non-trivially

In order to be able to easily specify the subsemigroups of S that intersect every $\mathscr{H}$-class of S non-trivially, we introduce the following notation. For any subgroup $V \leq G$ and any elements $g_1, g_2, \ldots, g_n \in G$, we define the following subset of S:

$$\textsc{sub}\big(V,\, g_1,\, g_2,\, \ldots,\, g_n\big) = \bigcup_{k,l \in \{1,\, \ldots,\, n\}} (I_k \times g_k^{-1} V g_l \times \Lambda_l) \cup \{0\}. \tag{4.19}$$

Note that $\textsc{sub}\big(V,\, g_1,\, g_2,\, \ldots,\, g_n\big)$ contains $|V|$ elements from each non-zero $\mathscr{H}$-class of S, since $|g^{-1}Vh| = |V|$ for all $g, h \in G$; see Corollary 4.24. Furthermore, we may assume without loss of generality that $g_1 = 1_G$; this observation is stated in the following lemma.

Lemma 4.20. $\textsc{sub}\big(V,\, g_1,\, g_2,\, \ldots,\, g_n\big) = \textsc{sub}\big(g_1^{-1}Vg_1,\, 1_G,\, g_1^{-1}g_2,\, \ldots,\, g_1^{-1}g_n\big)$.

In the following proposition, we show that any subsemigroup of S that intersects each $\mathscr{H}$-class of S non-trivially has the form $\textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$, for some subgroup V of G and elements $g_2, \ldots, g_n \in G$. It follows, therefore, that any maximal subsemigroup of S of type (R6) has such a form. In subsequent results, we classify the containment and equality of such semigroups, and we provide generating sets for them.

Proposition 4.21. *Let T be a subset of S that intersects each $\mathscr{H}$-class of S non-trivially. Then T is a subsemigroup of S if and only if there exists a subgroup V of G containing G_1, and for each $k \in \{2,\, \ldots,\, n\}$ there exists $g_k \in G$ such that $G_k \leq g_k^{-1}Vg_k$, and $T = \textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$.*

Proof. ($\Rightarrow$) For each $j \in I$ and $\mu \in \Lambda$, define $H_{j,\mu}$ to be the set $T \cap (\{j\} \times G \times \{\mu\})$, which is non-empty because $\{j\} \times G \times \{\mu\}$ is an $\mathscr{H}^S$-class of S. Since T is regular by Lemma 4.18, it follows that $H_{j,\mu}$ is an $\mathscr{H}^T$-class of T. Since $p_{\lambda_k,i_k} = 1_G$ for all $k \in \{1,\, \ldots,\, n\}$, it follows by Green's Lemma (Lemma 1.11) that

$$H_{j,\mu} = H_{j,\lambda_k}\,(i_k,\, h,\, \mu) = (j,\, h',\, \lambda_k)\, H_{i_k,\mu}, \qquad \text{and so} \qquad H_{j,\lambda_k} H_{i_k,\mu} = H_{j,\mu} \tag{4.22}$$

for all $j \in I$, $\mu \in \Lambda$, and for all $h, h' \in G$ such that $(i_k,\, h,\, \mu)$, $(j,\, h',\, \lambda_k) \in T$. These equations are used repeatedly throughout this proof.

We first define the subgroup V of G and the elements $g_2, \ldots, g_n$ required by the proposition. Since $p_{\lambda_1, i_1} = 1_G$, the $\mathscr{H}^T$-class H_{λ_1, i_1} is a group, and the function $\phi : H_{\lambda_1, i_1} \longrightarrow G$ given by $(i_1,\, g,\, \lambda_1) \mapsto g$ is an embedding. In particular, $H_{\lambda_1, i_1} = \{i_1\} \times V \times \{\lambda_1\}$, where $V = (H_{\lambda_1, i_1})\phi$ is a subgroup of G. Additionally, since T intersects each $\mathscr{H}^S$-class non-trivially, for each $k \in \{2, \ldots, n\}$ there exists some $g_k \in G$ such that $(i_1,\, g_k,\, \lambda_k) \in H_{i_1, \lambda_k}$. For convenience, we define $g_1 = 1_G$.

We next show that $G_k \leq g_k^{-1} V g_k$ for all k. Let $k \in \{1, \ldots, n\}$ be arbitrary. By (4.22),

$$H_{i_1, \lambda_k} = H_{i_1, \lambda_1} (i_1,\, g_k,\, \lambda_k) = \{i_1\} \times V g_k \times \{\lambda_k\},$$

and similarly, $H_{i_k, \lambda_1} = \{i_k\} \times U_k \times \{\lambda_1\}$ for some non-empty subset $U_k \subseteq G$. Again using (4.22), it follows that

$$H_{i_1, \lambda_1} = (i_1,\, g_k,\, \lambda_k)\, H_{i_k, \lambda_1} = \{i_1\} \times g_k U_k \times \{\lambda_1\},$$

but we have already shown that $H_{i_1, \lambda_1} = \{i_1\} \times V \times \{\lambda_1\}$. Therefore $U_k = g_k^{-1} V$, and by (4.22),

$$H_{i_k, \lambda_k} = H_{i_k, \lambda_1} H_{i_1, \lambda_k} = \{i_k\} \times g_k^{-1} V g_k \times \{\lambda_k\}. \tag{4.23}$$

By Lemma 1.14, T contains $E(S)$, and so $F(S) = \langle E(S) \rangle \leq T$. It follows that

$$\begin{aligned} G_k &= \big\{ g \in G : (i_k,\, g,\, \lambda_k) \in F(S) \big\} && \text{by definition; see Proposition 3.2(ii)} \\ &\leq \big\{ g \in G : (i_k,\, g,\, \lambda_k) \in T \big\} \\ &= g_k^{-1} V g_k, && \text{by (4.23)} \end{aligned}$$

as required.

Finally, since $0 \in T$, to show that $T = \textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$, it suffices to show that if $j \in I_k$ and $\mu \in \Lambda_l$ for some $k, l \in \{1, \ldots, n\}$, then $H_{j, \mu} = \{j\} \times g_k^{-1} V g_l \times \{\mu\}$.

Let $k \in \{1, \ldots, n\}$ and $j \in I_k$ be arbitrary. Since j and λ_k are contained in the same connected component of the Graham-Houghton graph of S, there exists a path in this graph from j to λ_k. Specifically, there exists an alternating sequence $(j = a_1,\, b_1,\, \ldots,\, a_m,\, b_m = \lambda_k)$ of indices from I_k and Λ_k, respectively, such that $p_{b_l, a_l} \neq 0$ and $p_{b_l, a_{l+1}} \neq 0$ for all possible l. Therefore $x = \prod_{l=1}^{m} \big(a_l,\, p_{b_l, a_l}^{-1},\, b_l\big) \in F(S) \leq T$. Indeed by Proposition 3.2(ii), $p_{b_l, a_l} \in G_k$ and $p_{b_l, a_{l+1}} \in G_k$ for all possible l, and so $x = (j,\, h,\, \lambda_k)$ for some $h \in G_k \leq g_k^{-1} V g_k$. By (4.22),

$$\begin{aligned} H_{j, \lambda_k} = x \cdot H_{i_k, \lambda_k} &= \{j\} \times h \cdot (g_k^{-1} V g_k) \times \{\lambda_k\} \\ &= \{j\} \times g_k^{-1} V g_k \times \{\lambda_k\}. \end{aligned}$$

It follows similarly that $H_{i_k, \mu} = \{i_k\} \times g_k^{-1} V g_k \times \{\mu\}$ for all $\mu \in \Lambda_k$.

Let $j \in I$ and $\mu \in \Lambda$ be arbitrary, and let $k, l \in \{1, \ldots, n\}$ be the unique indices such that $j \in I_k$ and $\mu \in \Lambda_l$. Then by applying the previous results with (4.22), we deduce that

$$\begin{aligned} H_{j, \mu} &= H_{j, \lambda_k} H_{i_k, \lambda_1} H_{i_1, \lambda_l} H_{i_l, \mu} \\ &= (\{j\} \times g_k^{-1} V g_k \times \{\lambda_k\})(\{i_k\} \times g_k^{-1} V \times \{\lambda_1\})(\{i_1\} \times V g_l \times \{\lambda_l\})(\{i_l\} \times g_l^{-1} V g_l \times \{\mu\}) \\ &= \{j\} \times (g_k^{-1} V g_k) \cdot (g_k^{-1} V) \cdot (V g_l) \cdot (g_l^{-1} V g_l) \times \{\mu\} \\ &= \{j\} \times g_k^{-1} V g_l \times \{\mu\}. \end{aligned}$$

($\Leftarrow$) Let $x = (a,\, f,\, \mu)$ and $y = (b,\, h,\, \gamma)$ be arbitrary non-zero elements of T. If $p_{\mu, b} = 0$, then certainly $xy = 0 \in T$. Otherwise, b and μ are contained in the same connected component of the Graham-Houghton graph of S, and so $b \in I_k$ and $\mu \in \Lambda_k$ for some $k \in \{1, \ldots, n\}$. Furthermore, since $T = \textsc{sub}\big(V,\, g_1 = 1_G,\, g_2,\, \ldots,\, g_n\big)$ and $x, y \in T$, there exist $l, m \in \{1, \ldots, n\}$ such that $a \in I_l$, $f \in g_l^{-1} V g_k$, $h \in g_k^{-1} V g_m$, and $\gamma \in \Lambda_m$. By Proposition 3.2(ii) $p_{\mu,b} \in G_k$, and by assumption, $G_k \leq g_k^{-1} V g_k$. Therefore,

$$\begin{aligned} f \cdot p_{\mu,b} \cdot h &\in (g_l^{-1} V g_k) \cdot G_k \cdot (g_k^{-1} V g_m) \\ &\subseteq (g_l^{-1} V g_k) \cdot (g_k^{-1} V g_k) \cdot (g_k^{-1} V g_m) \\ &= g_l^{-1} V^3 g_m \\ &= g_l^{-1} V g_m. \end{aligned}$$

It follows that

$$\begin{aligned} xy &= (a,\, f p_{\mu,b} h,\, \gamma) \\ &\in \{a\} \times g_l^{-1} V g_m \times \{\gamma\} \\ &\subseteq I_l \times g_l^{-1} V g_m \times \Lambda_m \\ &\subseteq \textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big) = T, \end{aligned}$$

as required, and so T is a subsemigroup of S. □

By Proposition 4.21, if T is a subsemigroup of S that intersects every $\mathscr{H}$-class of S non-trivially, then the set $V = \big\{g \in G : (i_1,\, g,\, \lambda_k) \in T\big\}$ is a subgroup of V, and T has the form $\textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$ for some elements $g_2, \ldots, g_n \in G$. We call a semigroup of the form $\textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$ a *subsemigroup of S arising from V*. The order of a subsemigroup of S arising from V is given in the following corollary to to Proposition 4.21, which is required in the proof of Theorem 4.28. The corollary holds since $|g^{-1} V h| = |V|$ for all $g, h \in G$.

Corollary 4.24. *Let T be a subsemigroup of S arising from V. Then every non-zero $\mathscr{H}$-class of T contains $|V|$ elements. In particular, $|T| = |I| \cdot |V| \cdot |\Lambda| + 1$, and if T contains a non-zero $\mathscr{H}$-class of S, then $T = S$.*

For the purposes of computation, it is useful to specify a semigroup by a generating set. Let V be a subgroup of G. By Proposition 4.21, any subsemigroup of S arising from V is equal to $\textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$, for some elements $g_2, \ldots, g_n \in G$. The subset $\textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$ is defined in (4.19) in terms of its elements, but it takes only the group V and the elements $g_2, \ldots, g_n$ to uniquely determine $\textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$. In the following lemma, we define a generating set for $\textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$ in terms of V, $g_2, \ldots, g_n$, and the idempotents of S.

Lemma 4.25. *Let V be a subgroup of G that contains G_1, and suppose there exist elements $g_2, \ldots, g_n \in G$ such that $G_k \leq g_k^{-1} V g_k$ for all $k \in \{2, \ldots, n\}$. For each $k \in \{2, \ldots, n\}$, define $x_k = \big(i_1,\, g_k,\, \lambda_k\big)$ and $y_k = \big(i_k,\, g_k^{-1},\, \lambda_1\big)$. Let E be a generating set for the idempotent generated subsemigroup of S, and let Y be a generating set for V. Then the set*

$$\big\{E,\ \{i_1\} \times Y \times \{\lambda_1\},\ x_2,\, \ldots,\, x_n,\, y_2,\, \ldots,\, y_n\big\}$$

generates $\textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$.

Proof. Let X denote the set defined in the lemma, and let $T = \textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$.

We first show that $X \subseteq T$. By the definition of T in (4.19), it is clear that T contains $\{i_1\} \times Y \times \{\lambda_1\}$ and the elements x_k and y_k for each k. By Proposition 4.21, T is a subsemigroup of S that intersects every $\mathscr{H}$-class of S non-trivially, and so T contains every idempotent of S by Lemma 1.14. Therefore T contains the idempotent generated subsemigroup of S, which contains E.

Next, we show that $T \subseteq \langle X \rangle$. Certainly $0 \in X$. Define $x_1 = y_1 = (i_1,\, 1_G,\, \lambda_1) \in X$ and $g_1 = 1_G$. By Proposition 4.21, T is a subsemigroup of S that intersects every $\mathscr{H}$-class of S

non-trivially, and the non-zero $\mathscr{H}$-classes of T are the sets $H_{j,\mu} = \{j\} \times g_k^{-1}Vg_l \times \{\mu\}$ for each $k, l \in \{1, \ldots, n\}$ and $j \in I_k$ and $\mu \in \Lambda_l$. Note that $H_{i_1,\lambda_1} = \langle \{i_1\} \times Y \times \{\lambda_1\} \rangle \subseteq \langle X \rangle$.

Let $k, m \in \{1, \ldots, n\}$, $j \in I_k$, and $\mu \in \Lambda_m$ be arbitrary. It remains to prove that $H_{j,\mu} \in \langle X \rangle$. By (4.22),

$$H_{i_k,\lambda_k} = y_k \cdot H_{i_1,\lambda_1} \cdot x_k \subseteq \langle X \rangle,$$

and similarly $H_{i_m,\lambda_m} \in \langle X \rangle$. Furthermore, the elements $(j,\, 1_G,\, \lambda_k)$ and $(i_m,\, 1_G,\, \mu)$ are idempotent generated by (3.4), and $F(S) = \langle E \rangle \leq \langle X \rangle$. Therefore, by repeatedly applying (4.22),

$$\begin{aligned} H_{j,\mu} &= H_{j,\lambda_k} H_{i_k,\lambda_1} H_{i_1,\lambda_m} H_{i_m,\mu} \\ &= \big((j,\, 1_G,\, \lambda_k) \cdot H_{i_k,\lambda_k} \big) \big(y_k \cdot H_{i_1,\lambda_1} \big) \big(H_{i_1,\lambda_1} \cdot x_m \big) \big(H_{i_m,\lambda_m} \cdot (i_m,\, 1_G,\, \mu) \big) \\ &\subseteq \langle X \rangle. \end{aligned}$$

□

See Section 3.2.2, especially Algorithm 3.12, for information about computing a small generating set for the idempotent generated subsemigroup of a finite Rees 0-matrix semigroup over a group. Such a generating set can be created easily from a spanning forest of the Graham-Houghton graph.

In the following results, we classify the containment, and therefore the equality, of two subsets $\textsc{sub}\big(U,\, 1_G,\, h_2,\, \ldots,\, h_n\big)$ and $\textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$ of S, in terms of the subgroups U and V, and the elements $h_2, \ldots, h_n$, and $g_2, \ldots, g_n$.

Lemma 4.26. *Let U and V be subgroups of G, and let $h_2, \ldots, h_n, g_2, \ldots, g_n \in G$ be arbitrary. Then $\textsc{sub}\big(U,\, 1_G,\, h_2,\, \ldots,\, h_n\big) \subseteq \textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$ if and only if $U \leq V$ and $Uh_k \subseteq Vg_k$ for all $k \in \{2, \ldots, n\}$.*

Proof. For convenience, we define $g_1 = h_1 = 1_G$. By Proposition 4.21,

$$\begin{aligned} &\textsc{sub}\big(U,\, 1_G,\, h_2,\, \ldots,\, h_n\big) \subseteq \textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big) \\ &\qquad \Leftrightarrow \big(I_k \times h_k^{-1}Uh_l \times \Lambda_l\big) \subseteq \big(I_k \times g_k^{-1}Vg_l \times \Lambda_l\big) \text{ for all } k, l \in \{1, \ldots, n\} \\ &\qquad \Leftrightarrow h_k^{-1}Uh_l \subseteq g_k^{-1}Vg_l \text{ for all } k, l \in \{1, \ldots, n\} \\ &\qquad \Leftrightarrow Uh_l \subseteq Vg_l \text{ for all } k \in \{1, \ldots, n\} \\ &\qquad \Leftrightarrow U \leq V \text{ and } Uh_l \subseteq Vg_l \text{ for all } k \in \{2, \ldots, n\}. \end{aligned}$$

□

Corollary 4.27. *Let U and V be subgroups of G, and let $h_2, \ldots, h_n, g_2, \ldots, g_n \in G$ be arbitrary. Then $\textsc{sub}\big(U,\, 1_G,\, h_2,\, \ldots,\, h_n\big) = \textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$ if and only if $U = V$ and $Vh_k = Vg_k$ for all $k \in \{2, \ldots, n\}$.*

Maximal subsemigroups that intersect every $\mathscr{H}$-class non-trivially

To find the maximal subsemigroups of $S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ that have type (R6), it suffices to find the maximal subsemigroups of S arising from each subgroup of G. As shown in the following theorem, which is the main result of this section, the maximal subsemigroups of S of type (R6) are precisely those subsemigroups of S that arise from maximal subgroups of G.

Theorem 4.28 (Maximal subsemigroups of type (R6)). *Let $S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ be a normalized finite regular Rees 0-matrix semigroup where G is a group, let $n \in \mathbb{N}$ denote the number of connected components of the Graham-Houghton graph of S, and let $I_1, \ldots, I_n$ and $i_1 \in I_1, \ldots, i_n \in I_n$, $\Lambda_1, \ldots, \Lambda_n$ and $\lambda_1 \in \Lambda_1, \ldots, \lambda_n \in \Lambda_n$, and $G_1, \ldots, G_n$ be defined as in Proposition 3.2. Let T be any subset of S that intersects every $\mathscr{H}$-class of S non-trivially. Then T is a maximal subsemigroup of S if and only if there exists a maximal subgroup V of G and elements $g_1 = 1_G, g_2, \ldots, g_n \in G$ such that $G_k \leq g_k^{-1}Vg_k$ for all k, and*

$$T = \bigcup_{k,l \in \{1,\, \ldots,\, n\}} (I_k \times g_k^{-1}Vg_l \times \Lambda_l) \cup \{0\}.$$

Proof. ($\Rightarrow$) By Proposition 4.21, $T = \text{SUB}(V, 1_G, g_2, \ldots, g_n)$ has the required form, and so it remains to prove that the subgroup V is maximal in G. Let K be any subgroup of G with $V \leq K \leq G$. Then $U = \text{SUB}(K, 1_G, g_2, \ldots, g_n)$ is a subsemigroup of S that contains T by Proposition 4.21 and Lemma 4.26. Since T is maximal, either $U = T$ or $U = S$. In the first case, Corollary 4.27 implies that $K = V$. Noting that $S = \text{SUB}(G, 1_G, g_2, \ldots, g_n)$, in the second case Corollary 4.27 implies that $K = G$. Therefore V is a maximal subgroup of G.

($\Leftarrow$) By Proposition 4.21, it remains to prove that the semigroup $T = \text{SUB}(V, 1_G, g_2, \ldots, g_n)$ is maximal in S. Let U be any subsemigroup of S that properly contains T. Then U intersects every $\mathscr{H}$-class of S non-trivially, and so by Proposition 4.21, $U = \text{SUB}(K, 1_G, h_2, \ldots, h_n)$ for some subgroup K of G, and for some elements $h_2, \ldots, h_n \in G$. By Lemma 4.26, $V \leq K$, but since $|T| < |U|$, it follows by Corollary 4.24 that $|V| < |K|$. Since V is a maximal subgroup of G, we deduce that $K = G$. Corollary 4.24 implies that $U = S$, and so T is maximal in S. $\square$

Given these results, we discuss how to compute the subsemigroups of S, and therefore the *maximal* subsemigroups of S, that arise from a given subgroup of G. Suppose that V is a subgroup of G that contains G_1. By Proposition 4.21 and Corollary 4.27, in order to find the subsemigroups of S that arise from V, it suffices to find an arbitrary transversal $\mathfrak{T}$ of the right cosets of V in G, and to find the sets

$$\{g \in \mathfrak{T} : G_k \leq g^{-1}Vg\}$$

for all $k \geq 2$. More explicitly, given any transversal $\mathfrak{T}$ of the right cosets of V in G, the subsemigroups of S arising from V are in one-to-one correspondence with the Cartesian product

$$\prod_{k=2}^{n} \{g \in \mathfrak{T} : G_k \leq g^{-1}Vg\}. \tag{4.29}$$

Note that for a normal subgroup V of G, $g^{-1}Vg = V$ for all $g \in G$. Therefore, if V is normal in G, $\mathfrak{T}$ is a transversal of the right cosets of V in G, and $k \geq 2$, then

$$\{g \in \mathfrak{T} : G_k \leq g^{-1}Vg\} = \begin{cases} \mathfrak{T} & \text{if } G_k \leq V, \\ \varnothing & \text{otherwise.} \end{cases} \tag{4.30}$$

Thus the sets $\{g \in \mathfrak{T} : G_k \leq g^{-1}Vg\}$ are particularly easy to compute when V is normal in G.

Moreover, if V is an arbitrary subgroup of G (not necessarily containing G_1), and there exists $t \in G$ such that $t^{-1}Vt$ is a conjugate of V containing G_1, then we may use the set in (4.29) to find the subsemigroups of S that arise from $t^{-1}Vt$. In more detail, as argued above, if $\mathfrak{U}$ is an arbitrary transversal of the right cosets of $t^{-1}Vt$ in G, then the subsemigroups of S that arise from $t^{-1}Vt$ are in one-to-one correspondence with the Cartesian product

$$\prod_{k=2}^{n} \{g \in \mathfrak{U} : G_k \leq g^{-1}\left(t^{-1}Vt\right)g\}$$

However, if $\mathfrak{T}$ is a transversal of the right cosets of V in G, then the set $\{t^{-1}g : g \in \mathfrak{T}\}$ is a transversal of the right cosets of $t^{-1}Vt$ in G. In particular, given any transversal $\mathfrak{T}$ of the right cosets of V in G, the collection of all subsemigroups that arise from $t^{-1}Vt$ is in one-to-one correspondence with the Cartesian product

$$\prod_{k=2}^{n} \{t^{-1}g : g \in \mathfrak{T} \text{ and } G_k \leq (t^{-1}g)^{-1}\left(t^{-1}Vt\right)(t^{-1}g)\} = \prod_{k=2}^{n} \{t^{-1}g : g \in \mathfrak{T} \text{ and } G_k \leq g^{-1}Vg\}.$$

Let V be a representative of any conjugacy class of subgroups of G. By the previous arguments, to find all subsemigroups in S that arise from conjugates of V, it suffices to compute the set in (4.29), and to find the conjugates of V that contain G_1. Of course, if V is normal in G, then the only conjugate of V is V itself. In full generality, two conjugates $g^{-1}Vg$ and $t^{-1}Vt$ of V are equal if and only if g and t are representatives of the same right coset of $N_G(V)$ in G, where $N_G(V)$ is the normalizer of G in V. Therefore, to find the conjugates of V that contain G_1, we require any transversal $\mathfrak{U}$ of the right cosets of $N_G(V)$ in G; given $\mathfrak{U}$, the set

$$\left\{t \in \mathfrak{U} \,:\, G_1 \leq t^{-1}Vt\right\} \tag{4.31}$$

specifies the required conjugates. In particular, if $\mathfrak{G}$ is a transversal of the conjugacy classes of maximal subgroups of G, then by by finding the transversals $\mathfrak{T}$ and $\mathfrak{U}$ and the sets (4.29) and (4.31) for each representative $V \in \mathfrak{G}$, we find all maximal subsemigroups of S of type (R6). A method for finding the maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group that have type (R6), using the results presented in this section, is given in Algorithm 4.32.

Algorithm 4.32 Maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group that have type (R6).

Input: $S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P']$, a finite regular Rees 0-matrix semigroup over a group G.
Output: Generating sets for the maximal subsemigroups of S that have type (R6).

find the connected components $I_1 \cup \Lambda_1,\, \ldots,\, I_n \cup \Lambda_n$ of the Graham-Houghton graph of S
compute a normalization $\Psi : S \longrightarrow \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ ▷ Section 3.2.1 and [62, Section 4]
for each k, fix $i_k \in I_k$ and $\lambda_k \in \Lambda_k$ such that $p_{\lambda_k, i_k} = 1_G$ ▷ Proposition 3.2(i)
$G_k \leftarrow \left\langle \left\{ p_{\lambda,i} \,:\, i \in I_k,\ \lambda \in \Lambda_k,\ p_{\lambda,i} \neq 0 \right\} \right\rangle$ for all k ▷ Proposition 3.2(ii)
$E \leftarrow$ generators for the idempotent generated subsemigroup of S ▷ Algorithm 3.12
$\mathcal{C} \leftarrow$ a transversal of the conjugacy classes of maximal subgroups of G ▷ [17, 47]
$\mathfrak{M} \leftarrow \varnothing$
for $V \in \mathcal{C}$ **do**
  $Y \leftarrow$ a generating set for V
  $\mathfrak{T} \leftarrow$ a transversal of the right cosets of V in G
  $\mathfrak{U} \leftarrow$ a transversal of the right cosets of $N_G(V)$ in G
  $T_1 \leftarrow \{t \in \mathfrak{U} \,:\, G_1 \leq t^{-1}Vt\}$
  for $k \in \{2,\, \ldots,\, n\}$ **do**
    $T_k \leftarrow \{t \in \mathfrak{T} \,:\, G_k \leq t^{-1}Vt\}$
  for $t \in T_1,\, g_2 \in T_2,\, \ldots,\, g_n \in T_n$ **do**
    $X_1 \leftarrow \{i_1\} \times t^{-1}Yt \times \{\lambda_1\}$
    $X_2 \leftarrow \left\{ \left(i_1,\, t^{-1}g_2,\, \lambda_2\right),\, \ldots,\, \left(i_1,\, t^{-1}g_n,\, \lambda_n\right),\, \left(i_2,\, g_2^{-1}t,\, \lambda_1\right),\, \ldots,\, \left(i_n,\, g_n^{-1}t,\, \lambda_1\right) \right\}$
    $\mathfrak{M} \leftarrow \mathfrak{M} \cup \left\{ E \cup X_1\Psi^{-1} \cup X_2\Psi^{-1} \right\}$ ▷ Lemma 4.25
return $\mathfrak{M}$

By the foregoing discussion and (4.30), we may count the subsemigroups of S that arise from conjugates of a given subgroup of V, without having to construct the subsemigroups of S themselves; we may also obtain upper bounds on their number.

Lemma 4.33. *Let V be an arbitrary subgroup of G, let $\mathfrak{T}$ be an arbitrary transversal of the right cosets of G in V, and let $\mathfrak{U}$ be an arbitrary transversal of the right cosets of the normalizer $N_G(V)$ of V in G. Then the number of subsemigroups of S that arise from conjugates of V is*

$$m = \left|\left\{t \in \mathfrak{U} \,:\, G_1 \leq t^{-1}Vt\right\}\right| \cdot \prod_{k=2}^{n} \left|\left\{g \in \mathfrak{T} \,:\, G_k \leq g^{-1}Vg\right\}\right|.$$

In particular, $m \leq [G : N_G(V)] \cdot [G : V]^{n-1}$. If V is normal in G, then $m \in \left\{0,\, [G : V]^{n-1}\right\}$.

We may simplify the bounds from Lemma 4.33 in the case that V is a maximal subgroup of G. Since $N_G(V)$ is subgroup of G containing V, either $N_G(V) = V$ and $[G : N_G(V)] = [G : V]$, or $N_G(V) = G$ and $[G : N_G(V)] = 1$. We use this observation in combination with (4.30) to state Corollary 4.34. The bounds given in this corollary are tight; see Examples 4.35 and 4.36.

Corollary 4.34. *Let V be a maximal subgroup of G. If V is normal in G, then there are either 0 or $[G : V]^{n-1}$ maximal subsemigroups of S arising from V. Otherwise, there are at most $[G : V]^n$ maximal subsemigroups of S arising from conjugates of V.*

Example 4.35. Let $n \in \mathbb{N}$ be arbitrary, and let $S = B(\mathcal{S}_3, n)$ denote the $n \times n$ Brandt semigroup over $\mathcal{S}_3$; see Section 3.5.3 for the definition. Let $\mathcal{A}_3 = \langle (1\ 2\ 3) \rangle$ denote the alternating group of degree 3, which consists of all even permutations in $\mathcal{S}_3$, and let $\mathcal{C}_2 = \langle (1\ 2) \rangle$. Then $\mathcal{A}_3$ is a maximal subgroup of $\mathcal{S}_3$ that is normal, and $\mathcal{C}_2$ is a maximal subgroup of $\mathcal{S}_3$ with three conjugates. Since the subgroup G_k of $\mathcal{S}_3$ is trivial for each $k \in \{1, \ldots, n\}$, it follows by Lemma 4.33 that the number of maximal subsemigroups of S arising from conjugates of $\mathcal{A}_3$ is $[\mathcal{S}_3 : \mathcal{A}_3]^{n-1} = 2^{n-1}$, and the number of maximal subsemigroups of S arising from conjugates of $\mathcal{C}_2$ is $[\mathcal{S}_3 : \mathcal{C}_2]^n = 3^n$.

Example 4.36. Let $S = \mathscr{M}^0[\mathcal{S}_2; I, \Lambda; P]$, where $\mathcal{S}_2$ is the symmetric group of degree 2, I and Λ are any sets with two elements, and P is the $\Lambda \times I$ matrix given by

$$\begin{pmatrix} \mathrm{id}_2 & \mathrm{id}_2 \\ \mathrm{id}_2 & (1\ 2) \end{pmatrix}.$$

Then S is a normalized finite regular Rees 0-matrix semigroup over a group whose Graham-Houghton graph has a single connected component. Therefore the entries of P generate the subgroup G_1 of $\mathcal{S}_2$; see Proposition 3.2. But $G_1 = \mathcal{S}_2$, and no maximal subgroup of $\mathcal{S}_2$ contains $\mathcal{S}_2$. Therefore, by Lemma 4.33, no maximal subsemigroups of S have type (R6).

4.3.5 An example

In this section, we construct a specific Rees 0-matrix semigroup, and then demonstrate how the techniques of Sections 4.3.1–4.3.4 can be applied to calculate its maximal subsemigroups.

Let $S = \mathscr{M}^0[\mathcal{S}_5; I, \Lambda; P]$, where $\mathcal{S}_5$ is the symmetric group of degree 5, $I = \{1', \ldots, 6'\}$, $\Lambda = \{1, \ldots, 5\}$, and P is the $\Lambda \times I$ matrix

$$\begin{pmatrix} (1\ 2)(4\ 5) & 0 & 0 & 0 & 0 & 0 \\ 0 & (1\ 5\ 4\ 2\ 3) & (3\ 5) & 0 & 0 & 0 \\ 0 & (1\ 4\ 5)(2\ 3) & (2\ 5\ 3) & 0 & 0 & 0 \\ 0 & 0 & 0 & (1\ 5\ 2\ 3\ 4) & (1\ 2\ 5\ 3\ 4) & (1\ 2)(4\ 5) \\ 0 & 0 & 0 & (1\ 2)(3\ 5\ 4) & 0 & (1\ 3\ 2)(4\ 5) \end{pmatrix}.$$

Certainly $\mathcal{S}_5$ is a group, and S is regular since P contains a non-zero entry in each of its rows and columns. A diagram of the Graham-Houghton graph of S is shown in Figure 4.37.

We use the ideas discussed in Section 4.3.1 to find the maximal subsemigroups of S that have types (R1) or (R2). There is no maximal subsemigroup of type (R1), since the group $\mathcal{S}_5$ and the sets I and Λ are not all trivial – indeed, they are all non-trivial. In other words, $\{0\}$ is not a maximal subsemigroup of S. Since the matrix P contains 0, the subset $S \setminus \{0\}$ is not a subsemigroup of S, and so there is no maximal subsemigroup of S of type (R2).

We use Lemma 4.13 from Section 4.3.2 to find the maximal subsemigroups of S that have types (R3) or (R4). As discussed in Figure 4.37, the vertices in the Graham-Houghton graph of S that are adjacent to vertices of degree one are $1, 4 \in \Lambda$ and $1' \in I$. By Lemma 4.13(i),

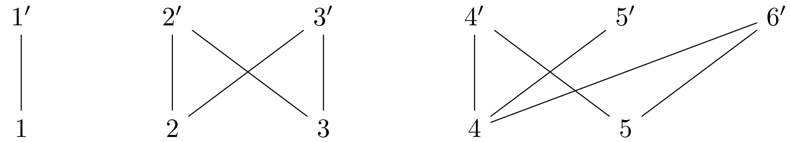

Figure 4.37: A diagram of the Graham-Houghton graph of S, whose three connected components are $\{1', 1\}$, $\{2', 3', 2, 3\}$, and $\{4', 5', 6', 4, 5\}$. The vertices of degree one in the graph are 1, 1′, and 5′, which are adjacent to the vertices 1′, 1, and 4, respectively. There are twelve maximal independent subsets, including I, Λ, $\{5', 1, 2, 3, 5\}$ and $\{1', 4', 5', 6', 2, 3\}$.

the maximal subsemigroups of S of type (R3) are the sets $S \setminus (I \times \mathcal{S}_3 \times \{\lambda\})$ for each $\lambda \in \Lambda \setminus \{1, 4\}$, and by Lemma 4.13(ii), the maximal subsemigroups of S of type (R4) are the sets $S \setminus (\{i\} \times \mathcal{S}_3 \times \Lambda)$ for each $i \in I \setminus \{i'\}$. In particular, there are three maximal subsemigroups of S that have type (R3), and five that have type (R4).

The maximal independent subsets of the Graham-Houghton graph of S are the index sets I and Λ, along with the sets:

$$\begin{array}{lllll} \{1', 4', 5', 6', 2, 3\}, & \{4', 5', 6', 1, 2, 3\}, & \{2', 3', 5', 1, 5\}, & \{5', 1, 2, 3, 5\}, & \{1', 5', 2, 3, 5\}, \\ \{2', 3', 4', 5', 6', 1\}, & \{1', 2', 3', 5', 5\}, & \{1', 2', 3', 4, 5\}, & \{1', 2, 3, 4, 5\}, & \{2', 3', 1, 4, 5\}. \end{array}$$

By the discussion in Section 4.3.3, these ten sets are in one-to-one correspondence with the maximal subsemigroups of 6 that have type (R5). If U is one of these sets, then the corresponding maximal subsemigroup of S is $S \setminus \big((I \setminus U) \times G \times (\Lambda \setminus U)\big)$. For example, if $U = \{1', 5', 2, 3, 5\}$, then the maximal subsemigroup of S of type (R5) that corresponds to U is $S \setminus \big(\{2', 3', 4', 6'\} \times G \times \{1, 4\}\big)$. In particular, there are ten maximal subsemigroups of S that have type (R5).

It remains to calculate the maximal subsemigroups of S that have type (R6). For this, we use the techniques described in Section 4.3.4. First, it is necessary to find a normalization of S, which is an isomorphism from S to a normalized Rees 0-matrix semigroup; see Section 3.2.1 for more detail about this step. One normalized Rees 0-matrix semigroup that is isomorphic to S is $T = \mathscr{M}^0[\mathcal{S}_5; I, \Lambda; Q]$, where

$$Q = \begin{pmatrix} \mathrm{id}_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{id}_5 & \mathrm{id}_5 & 0 & 0 & 0 \\ 0 & \mathrm{id}_5 & (1\ 4\ 2\ 3\ 5) & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_5 & \mathrm{id}_5 & \mathrm{id}_5 \\ 0 & 0 & 0 & \mathrm{id}_5 & 0 & (2\ 4)(3\ 5) \end{pmatrix}.$$

The Graham-Houghton graph of T is equal to the Graham-Houghton graph of S, which is shown in Figure 4.37. It is clear by inspection that the three connected components of this graph are the sets $I_1 \cup \Lambda_1 = \{1', 1\}$, $I_2 \cup \Lambda_2 = \{2', 3', 2, 3\}$, and $I_3 \cup \Lambda_3 = \{4', 5', 6', 4, 5\}$. To find the maximal subsemigroups of T that have type (R6), for each $k \in \{1, 2, 3\}$ we require the group G_k corresponding to the k^{th} connected component $I_k \cup \Lambda_k$ of the Graham-Houghton graph of S. Each such group is generated by the non-zero matrix entries of Q corresponding to the relevant connected component. Therefore,

$$G_1 = \{\mathrm{id}_5\}, \quad G_2 = \langle (1\ 4\ 2\ 3\ 5) \rangle, \quad \text{and} \quad G_3 = \langle (2\ 4)(3\ 5) \rangle.$$

For each maximal subgroup V of $\mathcal{S}_5$, we must find the maximal subsemigroups of T that arise from V. Up to conjugacy, the maximal subgroups of $\mathcal{S}_5$ are: the alternating group $\mathcal{A}_5$ consisting of all even permutations in $\mathcal{S}_5$; the symmetric group $\mathcal{S}_4$ consisting of all permutations in $\mathcal{S}_5$ that fix the point 5; a dihedral group $\langle(2\ 3),\ (1\ 2\ 3)(4\ 5)\rangle$ of order 12; and the group $\langle(1\ 2\ 3\ 5\ 4),\ (2\ 3\ 4\ 5)\rangle$. For each such maximal subgroup V, we require a transversal $\mathfrak{U}$ of the right cosets of $N_G(V)$ in G, a transversal $\mathfrak{T}$ of the right cosets of V in G, and the sets

$$\{g \in \mathfrak{U} : G_1 \leq g^{-1}Vg\} \quad \text{and} \quad \{g \in \mathfrak{T} : G_k \leq g^{-1}Vg\}, \quad \text{for all } k \in \{2, 3\}.$$

Case 1: $\mathcal{A}_5$. Note that $\mathcal{A}_5$ is a normal subgroup of $\mathcal{S}_5$ that contains the subgroups G_1, G_2, and G_3. In other words, every conjugate of $\mathcal{A}_5$ contains G_1, G_2, and G_3, and so by Corollary 4.34, there are four maximal subsemigroups of T (and hence S) that arise from $\mathcal{A}_5$.

Case 2: $\mathcal{S}_4$. Since G_2 contains a five-cycle, but there are no five-cycles in $\mathcal{S}_4$, it follows that $G_2 \not\leq g^{-1}\mathcal{S}_4 g$ for any $g \in G$. Therefore there are no maximal subsemigroups of T or S arising from conjugates of $\mathcal{S}_4$.

Case 3: $\langle(2\ 3),\ (1\ 2\ 3)(4\ 5)\rangle$. Since the group $\langle(2\ 3),\ (1\ 2\ 3)(4\ 5)\rangle$ contains no five-cycles, it follows that there are no maximal subsemigroups arising from its conjugates.

Case 4: $V = \langle(1\ 2\ 3\ 5\ 4),\ (2\ 3\ 4\ 5)\rangle$. Since V is not normal in $\mathcal{S}_5$, the normalizer $N_{\mathcal{S}_5}(V)$ is V itself. We choose the following transversal of the right cosets of V in $\mathcal{S}_5$:

$$\mathfrak{T} = \mathfrak{U} = \{\mathrm{id}_5,\ (4\ 5),\ (3\ 4),\ (3\ 5),\ (3\ 4\ 5),\ (3\ 5\ 4)\}.$$

Certainly, all six conjugates of V contain G_1, which is the trivial subgroup of $\mathcal{S}_5$. The only conjugate of V that contains G_2 is $(3\ 5\ 4)^{-1}V(3\ 5\ 4)$, and the two conjugates of V that contain G_3 are V itself and $(3\ 5)^{-1}V(3\ 5)$. Therefore, there are twelve $(= 6{\cdot}1{\cdot}2)$ maximal subsemigroups of T arising from conjugates of V, and twelve corresponding maximal subsemigroups of S.

Overall, there are 34 maximal subsemigroups of S: 3 of type (R3), 5 of type (R4), 10 of type (R5), and 16 of type (R6). Finding and constructing these maximal subsemigroups using the Semigroups package [101] for GAP [58], on a 2.66 GHz Intel Core i7 processor with 8GB of RAM, takes roughly 10 milliseconds.

4.3.6 Maximal subsemigroups of type (R6) that contain a given set

The culmination of this chapter is Algorithm 4.86, which describes a method for computing the maximal subsemigroups of an arbitrary finite semigroup. On line 20 of this algorithm, we require the maximal subsemigroups of the principal factor of a regular $\mathscr{J}$-class that contain a given set and intersect every $\mathscr{H}$-class of the principal factor non-trivially. Therefore, in order to use Algorithm 4.86, we require the ability to compute these kinds of maximal subsemigroups in the principal factor. It is straightforward to construct an isomorphism from the principal factor of a finite regular $\mathscr{J}$-class to some regular Rees 0-matrix semigroup over a group. Hence it suffices to describe an algorithm for computing the maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group that have type (R6), and that contain a given set, which is the topic of this section. We present an algorithm to solve this problem in Algorithm 4.44.

Throughout, we continue to use the notation and terminology from Section 4.3.4. More specifically, let $S = \mathscr{M}^0[G; I, \Lambda; P]$ denote a normalized finite regular Rees 0-matrix semigroup over a group G, let $n \in \mathbb{N}$ denote the number of connected components of the Graham-Houghton graph of S, and for each $k \in \{1, \ldots, n\}$, let I_k, Λ_k, i_k, λ_k, and G_k be defined as in Proposition 3.2. For a subgroup V of G and elements $g_1, \ldots, g_n \in G$, we again define the set $\textsc{sub}(V, g_1, g_2, \ldots, g_n)$ as in (4.19).

Let A be any subset of S. Note that any subsemigroup of S that intersects every $\mathscr{H}$-class of S non-trivially necessary contains 0, which is the unique element of the $\mathscr{H}^S$-class $\{0\}$. Therefore, we only concern ourselves with the non-zero elements of A.

One way to find the maximal subsemigroups of S of type (R6) that contain A is to use Algorithm 4.32 to construct *every* maximal subsemigroups of S that has type (R6), and then to use standard membership checking algorithms from computational semigroup theory to discard those maximal subsemigroups that do not contain A. However, such an approach is to be avoided when possible. By Corollary 4.34, the upper bound on the number of maximal subsemigroups arising from a given maximal subgroup of G is exponential in n. Therefore, the maximal subsemigroups of type (R6) can easily be far too numerous to construct, even when very few of them contain the set A. A more suitable solution would involve computing the desired maximal subsemigroups directly.

Let $T = \textsc{sub}\big(V,\, g_1 = 1_G,\, g_2,\, \ldots,\, g_n\big)$ be any subsemigroup of S that intersects every $\mathscr{H}$-class of S non-trivially; see Proposition 4.21. Observe that the elements of T are given in terms of the subgroup V of G and the group elements $g_2, \ldots, g_n \in G$; see (4.19). This suggests that we may also characterise whether T contains A in terms of these parameters. To that end, let $j, k, l \in \{1,\, \ldots,\, n\}$ with $k < l$, and define the sets

$$U_j = \big\{g \,:\, (i,\, g,\, \lambda) \in A \text{ for some } i \in I_j \text{ and } \lambda \in \Lambda_j\big\}, \text{ and} \tag{4.38}$$

$$U_{\{k,\,l\}} = \big\{g \,:\, (i,\, g,\, \lambda) \in A \text{ for some } i \in I_k \text{ and } \lambda \in \Lambda_l\big\} \cup \big\{g^{-1} \,:\, (i,\, g,\, \lambda) \in A \text{ for some } i \in I_l \text{ and } \lambda \in \Lambda_k\big\}. \tag{4.39}$$

These sets are required in the following lemma.

Lemma 4.40. *Let T be a subset of S that intersects each $\mathscr{H}$-class of S non-trivially. Then T is a (maximal) subsemigroup of S containing A if and only if there exists a (maximal) subgroup V of G and elements $g_1 = 1_G, g_2, \ldots, g_n \in G$ such that $T = \textsc{sub}\big(V,\, 1_G,\, g_2,\, \ldots,\, g_n\big)$, and:*

(i) *for all $k \in \{1,\, \ldots,\, n\}$, $(G_k \cup U_k) \subseteq g_k^{-1} V g_k$; and*

(ii) *for all $k, l \in \{1,\, \ldots,\, n\}$ with $k < l$, $U_{\{k,\,l\}} \subseteq g_k^{-1} V g_l$.*

Proof. By Proposition 4.21 and Theorem 4.28, T is a (maximal) subsemigroup of S if and only if there exists a (maximal) subgroup V of G and elements $g_1 = 1_G,\, g_2, \ldots, g_n \in G$ such that $G_k \leq g_k^{-1} V g_k$ for all $k \in \{1,\, \ldots,\, n\}$ and such that $T = \textsc{sub}\big(V,\, g_1,\, g_2,\, \ldots,\, g_n\big)$. It remains to prove that the remaining conditions hold in each case.

($\Rightarrow$) Let $k \in \{1,\, \ldots,\, n\}$ and let $g \in U_k$. Then $(i,\, g,\, \lambda) \in A \cap (I_k \times G \times \Lambda_k)$ for some $i \in I_k$ and $\lambda \in \Lambda_k$. But T contains A, and so $(i,\, g,\, \lambda) \in T \cap (I_k \times G \times \Lambda_k) = I_k \times g_k^{-1} V g_k \times \Lambda_k$. In particular, $g \in g_k^{-1} V g_k$. Since $g \in G$ was arbitrary, it follows that $U_k \subseteq g_k^{-1} V g_k$.

Let $k, l \in \{1,\, \ldots,\, n\}$ with $k < l$, and let $g \in U_{\{k,\,l\}}$. It suffices to show that $g \in g_k^{-1} V g_l$. Either $(i,\, g,\, \lambda) \in A \cap (I_k \times G \times \Lambda_l)$ or $(i,\, g^{-1},\, \lambda) \in A \cap (I_l \times G \times \Lambda_k)$, for some $i \in I$ and $\lambda \in \Lambda$. In the first case, since $A \subseteq T$, it follows that $g \in g_k^{-1} V g_l$, and we are done. In the second case, it follows that $g^{-1} \in g_l^{-1} V g_k$ and, by inversion, that $g \in g_k^{-1} V g_l$.

($\Leftarrow$) Let $x = (i,\, g,\, \lambda) \in A$, and fix $k, l \in \{1,\, \ldots,\, n\}$ such that $i \in I_k$ and $\lambda \in \Lambda_l$. It suffices to show that $g \in g_k^{-1} V g_l$. If $k = l$, then $g \in U_k$, and by assumption, $U_k \subseteq g_k^{-1} V g_k$. If $k < l$, then $g \in U_{\{k,\,l\}}$, and by assumption, $U_{\{k,\,l\}} \subseteq g_k^{-1} V g_l$. Finally, if $k > l$, then $g^{-1} \in U_{\{k,\,l\}}$, and by assumption, $U_{\{k,\,l\}} \subseteq g_l^{-1} V g_k$. Therefore, $g \in \big(g_l^{-1} V g_k\big)^{-1} = g_k^{-1} V g_l$. □

Throughout the rest of this section, let V be a subgroup of G, let $\mathfrak{U}$ be a transversal of the right cosets of $N_G(V)$ in G, and let $\mathfrak{T}$ be a transversal of the right cosets of V in G. Suppose there exist elements $g_1, \ldots, g_n \in G$ such that conditions (i) and (ii) of Lemma 4.40 hold. Then by Lemma 4.40, $\textsc{sub}\big(g_1^{-1} V g_1,\, 1_G,\, g_1^{-1} g_2,\, \ldots,\, g_1^{-1} g_n\big)$ is a subsemigroup of S that contains A and arises from $g_1^{-1} V g_1$. Conversely, suppose that $\textsc{sub}\big(t^{-1} V t,\, 1_G,\, h_2,\, \ldots,\, h_n\big)$ is a subsemigroup of S that contains A and arises from some conjugate of V, where $t, h_2, \ldots, h_n \in G$. Define $g_1 = t, g_2 = t \cdot h_2, \ldots, g_n = t \cdot h_n$. Then conditions (i) and (ii) of Lemma 4.40 hold.

By this observation and by Corollary 4.27, to compute all subsemigroups of S that contain A and arise from conjugates of V, it suffices to test each selection of elements $g_1 \in \mathfrak{U}$ and $g_2, \ldots, g_n \in \mathfrak{T}$ against the conditions in Lemma 4.40. Therefore, an algorithm for computing the maximal subsemigroups of S that contain A and have type (R6) can be created from Algorithm 4.32 by adding a test for these conditions after line 15. However, this approach leaves room for improvement, since it ignores the links between the elements $g_1, \ldots, g_n$ that are implied by Lemma 4.40(ii).

In order to take advantage of this information, we define the graph

$$\Gamma = \big(\{1, \ldots, n\}, \big\{\{k, l\} : U_{k,l} \neq \varnothing\big\}\big). \tag{4.41}$$

Let $\{D_1, \ldots, D_m\}$ be the set of connected components of Γ, numbered so that $1 \in D_1$. We reformulate Lemma 4.40 in the following corollary, which shows that the search for subsemigroups of S that contain A and arise from conjugates of V can be broken down into smaller independent searches that correspond to the connected components of Γ.

Corollary 4.42. *Let $g_1 \in \mathfrak{U}$ and $g_2, \ldots, g_n \in \mathfrak{T}$. Then $\textsc{sub}\big(g_1^{-1} V g_1,\, 1_G,\, g_1^{-1} g_2,\, \ldots,\, g_1^{-1} g_n\big)$ is a subsemigroup of S containing A if and only if for each $i \in \{1, \ldots, m\}$, the following hold:*

(i) *for all $k \in D_i$, $(G_k \cup U_k) \subseteq g_k^{-1} V g_k$; and*

(ii) *for all $k, l \in D_i$ with $k < l$, $U_{\{k,l\}} \subseteq g_k^{-1} V g_l$.*

Indeed, the connected components of Γ have further utility for our purposes. Let $g_1 \in \mathfrak{U}$ and $g_2, \ldots, g_n \in \mathfrak{T}$, and suppose that $\textsc{sub}\big(g_1^{-1} V g_1,\, 1_G,\, g_1^{-1} g_2,\, \ldots,\, g_1^{-1} g_n\big)$ is a subsemigroup of S that contains A. In the following lemma, we show that the choice of $g_1 \in \mathfrak{U}$ determines the value of g_k, for each $k \in D_1 \setminus \{1\}$, and that for $j \in \{2, \ldots, m\}$, the choice of $g_{\min(D_j)} \in \mathfrak{T}$ determines the value of g_k, for each $k \in D_j \setminus \{\min(D_j)\}$.

Lemma 4.43. *Let $g_1 \in \mathfrak{U}$ and $g_2, \ldots, g_n \in \mathfrak{T}$ be chosen arbitrarily, and suppose that the subset $\textsc{sub}\big(g_1^{-1} V g_1,\, 1_G,\, g_1^{-1} g_2,\, \ldots,\, g_1^{-1} g_n\big)$ is a subsemigroup of S containing A. The following hold.*

(i) *If $u \in U_{\{k,l\}}$, then $g_l \in \mathfrak{T}$ is the coset representative determined by $g_k u$ (when $k < l$) or $g_k u^{-1}$ (when $k > l$).*

(ii) *For all $i \in \{1, \ldots, m\}$ and $k \in D_i$, the value of g_k is determined by $g_{\min(D_i)}$.*

Proof. (i). Suppose that $k < l$. By Lemma 4.40(ii), $U_{\{k,l\}} \subseteq g_k^{-1} V g_l$, and so $g_k u \in V g_l$. Suppose that $k > l$. Then $U_{\{k,l\}} \subseteq g_l^{-1} V g_k$ by Lemma 4.40(ii), and so $g_k u^{-1} \in V g_l$.

(ii). If $k = \min(D_i)$ there is nothing to prove, so suppose that $k > \min(D_i)$. Since $\min(D_i)$ and k are connected in Γ, we may choose some path $(\min(D_i) = a_1, a_2, \ldots, a_q = k)$ in Γ from $\min(D_i)$ to k. By (i), $g_{a_2} \in \mathfrak{T}$ is the coset representative determined by $g_{\min(D_i)} u_2$, where $u_2 \in U_{\{\min(D_i), a_2\}}$ is arbitrary. By (i), $g_{a_3} \in \mathfrak{T}$ is the coset representative determined by $g_{a_2} u_3$, where $u_3 \in U_{\{a_2, a_3\}}$ (if $a_2 < a_3$) or $u_3^{-1} \in U_{\{a_2, a_3\}}$ (if $a_2 > a_3$) is arbitrary. But g_{a_2} is determined by $g_{\min(D_i)} u_2$, and so g_{a_3} is determined by $g_{\min(D_i)} u_2 u_3$. Continuing in this way, we may choose a sequence of elements $u_2, \ldots, u_q$, where either u_j or u_j^{-1} is contained in $U_{\{a_{j-1}, a_j\}}$ for each j, such that $g_k \in \mathfrak{T}$ is the unique coset representative of V in G determined by $g_{\min(D_i)} \cdot u_2 \cdots u_q$. $\square$

A method for finding the maximal subsemigroups of a normalized finite regular Rees 0-matrix semigroup over a group that have type (R6) and that contain a certain set is given in Algorithm 4.32. This algorithm relies on the preceding results, especially Corollary 4.42 and

Algorithm 4.44 Maximal subsemigroups of type (R6) that contain a given set.

Input: $S = \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$, a normalized finite regular Rees 0-matrix semigroup over a group, and a subset A of S.

Output: $\mathfrak{M}$, the maximal subsemigroups of S that have type (R6) and contain A.

$n \leftarrow$ the number of connected components of the Graham-Houghton graph of S
for each $k \in \{1, \ldots, n\}$, compute G_k as in Algorithm 4.32.
for each $j, k, l \in \{1, \ldots, n\}$ with $k < l$, construct U_j and $U_{\{k,\,l\}}$ ▷ (4.38) and (4.39)
$\mathcal{C} \leftarrow$ a transversal of the conjugacy classes of maximal subgroups of G ▷ [17, 47]
$\Gamma \leftarrow \big(\{1, \ldots, n\},\, \big\{\{k,\, l\} : U_{\{k,\,l\}} \neq \varnothing\big\}\big)$ ▷ The graph from (4.41)
find the connected components $D_1, \ldots, D_m$ of Γ, with $1 \in D_1$
$\mathfrak{M} \leftarrow \varnothing$
for $V \in \mathcal{C}$ **do**
    $T_i \leftarrow \varnothing$ for all $i \in \{1, \ldots, m\}$
    $\mathfrak{T} \leftarrow$ a transversal of the right cosets of V in G
    $\mathfrak{U} \leftarrow$ a transversal of the right cosets of $N_G(V)$ in G
    for $(i,\, g) \in (\{1\} \times \mathfrak{U}) \cup (\{2, \ldots, m\} \times \mathfrak{T})$ **do**
        $g_{\min(D_i)} \leftarrow g$
        for each $k \in D_i \setminus \{\min(D_i)\}$, fix $g_k \in \mathfrak{T}$ as determined by $g_{\min(D_i)}$ ▷ Lemma 4.43
        if $(G_j \cup U_j) \subseteq g_j^{-1} V g_j$ **and** $U_{\{k,\,l\}} \subseteq g_k^{-1} V g_l$ for all $j, k, l \in D_i$ with $k < l$ **then**
            $T_i \leftarrow T_i \cup \big\{\{(k,\, g_k) : k \in D_i\}\big\}$ ▷ T_i contains functions $D_i \longrightarrow G$
    for $\sigma_1 \in T_1, \ldots, \sigma_m \in T_m$ **do**
        $\sigma \leftarrow \sigma_1 \cup \cdots \cup \sigma_m$ ▷ σ defines a function $\{1, \ldots, n\} \longrightarrow G$
        $\mathfrak{M} \leftarrow \mathfrak{M} \cup \big\{\textsc{sub}\big((1\sigma)^{-1} V (1\sigma),\, 1_G,\, (1\sigma)^{-1}(2\sigma), \ldots, (1\sigma)^{-1}(n\sigma)\big)\big\}$ ▷ Corollary 4.42
return $\mathfrak{M}$.

Lemma 4.43. In Example 4.45, we demonstrate the application of Algorithm 4.44 to a small Rees 0-matrix semigroup.

Although we do not include the following optimisations in Algorithm 4.44, it is possible to use Lemma 4.43 to further prune the search space of subsemigroups of S that contain A and arise from conjugates of V. These techniques are implemented in the Semigroups [101] package. Let $\{k,\, l\}$ be any edge in Γ with $1 < k < l$. By Lemma 4.43(i), if $\textsc{sub}\big(g_1^{-1} V g_1,\, 1_G,\, g_1^{-1} g_2, \ldots, g_1^{-1} g_n\big)$ is a subsemigroup of S that contains A, then for any $u \in U_{\{k,\,l\}}$, the right cosets $V g_k u$ and $V g_l$ coincide. In particular, for all $u, v \in U_{\{k,\,l\}}$, it follows that $V g_k u = V g_k v$, and so $(g_k^{-1} V g_k) u = (g_k^{-1} V g_k) v$. In other words, g_k is required to be chosen so that every element of $U_{\{k,\,l\}}$ defines the same right coset of $g_k^{-1} V g_k$ in G. Moreover, if V is normal in G, then every element of $U_{\{k,\,l\}}$ defines the same right coset of V in G. It follows that, if V is a normal subgroup of G and the elements of some subset $U_{\{k,\,l\}}$ (with $k < l$) do not define the same right coset of V in G, then there are no subsemigroups of S that contain A and arise from V.

Example 4.45. Let $S = \mathscr{M}^0[\mathcal{S}_4;\, I,\, \Lambda;\, P]$, where $\mathcal{S}_4$ is the symmetric group of degree 4, $I = \{1', \ldots, 6'\}$, $\Lambda = \{1, \ldots, 6\}$, and P is the $\Lambda \times I$ matrix

$$\begin{pmatrix} \mathrm{id}_4 & \mathrm{id}_4 & 0 & 0 & 0 & 0 \\ \mathrm{id}_4 & (1\ 2)(3\ 4) & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_4 & \mathrm{id}_4 & 0 \\ 0 & 0 & 0 & \mathrm{id}_4 & (1\ 2\ 3\ 4) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{id}_4 \end{pmatrix}.$$

Let $A = \{0,\, (3',\, (1\ 3\ 2\ 4),\, 3)\,,\, (1,\, (1\ 4)(2\ 3),\, 3')\,,\, (6',\, (2\ 4\ 3),\, 4)\,,\, (6',\, \mathrm{id}_4,\, 6)\}$. In this example, we demonstrate how to compute the maximal subsemigroups of S that contain A and that have type (R6). The method that we use is essentially that described in Algorithm 4.44.

Notice that S is already normalized. The four connected components of the Graham-Houghton graph of S are $I_1 \cup \Lambda_1 = \{1',\, 2',\, 1,\, 2\}$, $I_2 \cup \Lambda_2 = \{3',\, 3'\}$, $I_3 \cup \Lambda_3 = \{4',\, 5',\, 4,\, 5\}$, and $I_4 \cup \Lambda_4 = \{6',\, 6\}$. For each $k \in \{1,\, 2,\, 3,\, 4\}$, we require the group G_k generated by the non-zero entries of P corresponding to the k^{th} connected component. These groups are

$$G_1 = \langle (1\ 2)(3\ 4) \rangle, \quad G_2 = G_4 = \{\mathrm{id}_4\}, \quad \text{and} \quad G_3 = \langle (1\ 2\ 3\ 4) \rangle.$$

We also require the sets of the form U_j and $U_{\{k,\, l\}}$, defined in (4.38) and (4.39). The only non-empty sets of these kinds are

$$U_2 = \{(1\ 3\ 2\ 4)\}, \quad U_4 = \{\mathrm{id}_4\}, \quad U_{\{1,\, 2\}} = \{(1\ 4)(2\ 3)\}, \quad \text{and} \quad U_{\{3,\, 4\}} = \{(2\ 3\ 4)\}.$$

Therefore, the connected components of the graph Γ from (4.41) are $\{1,\, 2\}$ and $\{3,\, 4\}$.

Finding the maximal subsemigroups of S that contain A and have type (R6) is equivalent to finding the maximal subsemigroups of S that contain A and that arise from conjugates of maximal subgroups of $\mathcal{S}_4$. Up to conjugacy, there are three maximal subgroups of $\mathcal{S}_4$: the alternating group $\mathcal{A}_4$ consisting of all even permutations in $\mathcal{S}_4$; the symmetric group $\mathcal{S}_3$ consisting of all permutations in $\mathcal{S}_4$ that fix the point 4; and $\mathcal{D}_4 = \langle (1\ 2\ 3\ 4),\ (1\ 2)(3\ 4) \rangle$, a dihedral group of order 8. By using the techniques in Section 4.3.4, it is possible to show that there are no maximal subsemigroups of S that arise from conjugates of $\mathcal{A}_4$ or $\mathcal{S}_3$, and that there are 27 maximal subsemigroups of S that arise from conjugates of $\mathcal{D}_4$. Therefore, it remains to find the maximal subsemigroups of S that contain A and arise from $\mathcal{D}_4$.

Since $\mathcal{D}_4$ is not a normal subgroup of $\mathcal{S}_4$, we choose

$$\mathfrak{U} = \mathfrak{T} = \{\mathrm{id}_4,\, (3\ 4),\, (2\ 3)\}$$

to be a transversal of the right cosets of $\mathcal{D}_4$ in $\mathcal{S}_4$. This transversal was computed with GAP [58].

We begin by considering the possible choices for g_1 and g_2, which correspond to the first connected component of Γ. By Lemma 4.43(ii), the choice of g_2 is determined by g_1. More specifically, by Lemma 4.43(i), g_2 is the representative in $\mathfrak{T}$ determined by $g_1 \cdot (1\ 4)(2\ 3)$. In each case, $g_1 = g_2$. The only conjugate of $\mathcal{D}_4$ that contains $G_2 \cup U_2 = \langle (1\ 3\ 2\ 4) \rangle$ is $(2\ 3)^{-1}\mathcal{D}_4(2\ 3)$, and so the remaining possible choice is $g_1 = g_2 = (2\ 3)$. Since $(2\ 3)^{-1}V(2\ 3)$ contains both $G_1 \cup U_1$ and $U_{\{1,\, 2\}}$, it follows by Corollary 4.42 that $g_1 = g_2 = (2\ 3)$ is the only valid choice corresponding to the first connected component of Γ.

Finally, we consider the possible choices for g_3 and g_4. The only conjugate of $\mathcal{D}_4$ that contains $G_3 \cup U_3 = \langle (1\ 2\ 3\ 4) \rangle$ is $\mathcal{D}_4$ itself. Therefore id_4 is the only possible choice for g_3, and by Lemma 4.43(i), this determines that $(3\ 4)$ is the only possibility for g_4. Since $(3\ 4)^{-1}\mathcal{D}_4(3\ 4)$ contains $G_4 \cup U_4 = \{\mathrm{id}_4\}$ and $U_{\{3,\, 4\}} \subseteq \mathrm{id}_4^{-1}\mathcal{D}_4(3\ 4)$, it follows that the selection of $g_3 = \mathrm{id}_4$ and $g_4 = (3\ 4)$ is the unique possibility corresponding to the second connected component of Γ.

In conclusion, by Corollary 4.42, $\textsc{sub}\big((2\ 3)^{-1}\mathcal{D}_4(2\ 3),\, 1_G,\, 1_G,\, (2\ 3),\, (2\ 4\ 3)\big)$ is the unique maximal subsemigroup of S that contains A and has type (R6). A generating set for this maximal subsemigroup can be constructed using Lemma 4.25.

Finding this solution with the Semigroups package [101] for GAP [58], which uses a version of Algorithm 4.44, takes roughly 10 milliseconds on a 2.66 GHz Intel Core i7 processor with 8GB of RAM. On the other hand, using Semigroups to find every maximal subsemigroup of S of type (R6), and discarding those that do not contain A, takes roughly 50 milliseconds.

4.4 Arbitrary finite semigroups

In this section, we develop a framework for computing the maximal subsemigroups of an arbitrary finite semigroup, building on the results of Section 4.3. We consider each of the possible kinds of maximal subsemigroups (M1)–(M5) separately in Sections 4.4.1–4.4.5, with the exception of maximal subsemigroups of types (M3) and (M4). Maximal subsemigroups of types (M3) and (M4) are dual, and are considered jointly in Section 4.4.4.

In the framework that we put forth, the calculation of maximal subsemigroups of types (M3) and (M4) requires the prior calculation of the maximal subsemigroups of type (M2), and for a regular $\mathscr{J}$-class, the calculation of maximal subsemigroups of types (M5) requires the prior calculation of the maximal subsemigroups of types (M1)–(M4).

Throughout Section 4.4, S denotes an arbitrary finite semigroup with a generating set X, J denotes an arbitrary $\mathscr{J}$-class of S whose principal factor is $J^* = J \cup \{0\}$, and X' denotes the set of generators $x \in X$ such that $J_x > J$ in the partial order of $\mathscr{J}$-classes on S.

We use the following lemma repeatedly to prove the forthcoming results. This lemma gives necessary and sufficient conditions for a subset T of S containing $S \setminus J$ to be a subsemigroup of S. Condition (i) requires that $T \cap J$ contain the elements of J that are generated by $S \setminus J$; condition (ii) requires that $(T \cap J) \cup \{0\}$ define a subsemigroup of J^*; and condition (iii) requires that $T \cap J$ be stabilized under left and right multiplication by $S \setminus J$, in some sense.

Lemma 4.46. *Let T be a subset of S such that $S \setminus T \subseteq J$. Then T is a subsemigroup of S if and only if*

(i) $\langle X' \rangle \subseteq T$;

(ii) *if $x, y \in T \cap J$, then $xy \in J$ implies that $xy \in T$; and*

(iii) *if $x \in T \cap J$ and $y \in \langle X' \rangle$, then $xy \in J$ implies that $xy \in T$, and $yx \in J$ implies that $yx \in T$.*

Proof. ($\Rightarrow$) The conditions hold since T is a subsemigroup of S that contains X'.

($\Leftarrow$) Let $x, y \in T$ be arbitrary. Since T contains $S \setminus J$, it suffices to show that $xy \in T$ whenever $xy \in J$, so suppose that $xy \in J$. We first prove that $x, y \in J \cup \langle X' \rangle$. Since X generates S, $x = x_1 \cdots x_n$, for some $n \in \mathbb{N}$ and $x_i \in X$. Note that $x_i \in J \cup X'$ for each i by Lemma 1.12. If $x_i \in X'$ for each i, then $x \in \langle X' \rangle$. Otherwise, $x_i \in J$ for some i; by Lemma 1.12, $J = J_{xy} \leq J_x \leq J_{x_i} = J$, and $x \in J$. Similarly, $y \in J \cup \langle X' \rangle$. If $x, y \in \langle X' \rangle$, then $xy \in \langle X' \rangle \subseteq T$ by (i). If $x, y \in J$, then $xy \in T$ by (ii). For the remaining cases, $xy \in T$ by (iii). □

4.4.1 Maximal subsemigroups that intersect every $\mathscr{H}$-class: (M1)

In this section, we consider those maximal subsemigroups of the finite semigroup S that arise from the exclusion of elements in a regular $\mathscr{J}$-class, and that intersect every $\mathscr{H}$-class of S non-trivially. In other words, we are considering maximal subsemigroups of type (M1). Throughout this section, we suppose that J is a regular $\mathscr{J}$-class of S.

By Proposition 4.10(b), if M is a maximal subsemigroup of S of type (M1) that arises from J, then $(M \cap J) \cup \{0\}$ is a maximal subsemigroup of the principal factor J^*. Recall that the principal factor J^* of a regular $\mathscr{J}$-class is a 0-simple semigroup; let $\mathscr{M}^0[G; I, \Lambda; P]$ be a regular Rees 0-matrix semigroup over a group that is isomorphic to J^*. It follows that a maximal subsemigroup of S that arises from J and has type (M1) gives a maximal subsemigroup of $\mathscr{M}^0[G; I, \Lambda; P]$ that has type (R6). The converse does not necessarily hold in general; in this section, we develop necessary and sufficient criteria to characterise the circumstances in which it does hold.

In particular, in Proposition 4.48 we characterise the maximal subsemigroups of type (M1) in terms of the maximal subsemigroups of J^* that have type (R6) and that contain a particular set. Computing such maximal subsemigroups was the topic of Section 4.3.6 and Algorithm 4.44. We use this algorithm in Algorithm 4.86 when computing maximal subsemigroups of type (M1)

We first describe the subsemigroups of S that contain $S \setminus J$ and intersect every $\mathscr{H}$-class of J non-trivially, before considering those that are maximal.

Lemma 4.47. *Let T be a subset of S containing $S \setminus J$, and suppose that T intersects every $\mathscr{H}$-class of S non-trivially. Let E be a set consisting of one idempotent from each $\mathscr{L}$-class of J. Then T is a subsemigroup of S if and only if*

(i) $EX' \cap J \subseteq T$*;*

(ii) *if $x, y \in T \cap J$, then $xy \in J$ implies $xy \in T$* (i.e. Lemma 4.46(ii) holds)*.*

Proof. ($\Rightarrow$) Since T is a subsemigroup, Lemma 4.46(ii) holds. Since T is finite and intersects every $\mathscr{H}$-class of S non-trivially, T contains every idempotent of S by Lemma 1.14. By definition, T contains $S \setminus J$, which contains X', and so $EX' \subseteq T$. In particular, $EX' \cap J \subseteq T$.

($\Leftarrow$) It suffices to show that the remaining conditions of Lemma 4.46 hold. In order to do this, we first show that the intersection $E\langle X'\rangle \cap J$ of the set product $E\langle X'\rangle$ with J is contained in T. Let $x \in E\langle X'\rangle \cap J$. By definition, there exist an idempotent $e_1 \in E$ and a sequence of generators $x_1, \ldots, x_n \in X'$ such that $x = e_1x_1 \cdots x_n$. Since e_1 and the product $e_1x_1 \cdots x_n$ are both members of J, it follows by Lemma 1.12 that the intermediate product $e_1x_1 \cdots x_k$ is a member of J for every $k \in \{1, \ldots, n\}$. Hence, by definition of E, for each $k < n$ there exists an idempotent $e_{k+1} \in E$ such that $e_{k+1} \mathrel{\mathscr{L}} e_1x_1 \cdots x_k$. In particular, $(e_1x_1 \cdots x_k)e_{k+1} = e_1x_1 \cdots x_k$ for each $k < n$, since an idempotent is a right identity for its $\mathscr{L}$-class. Therefore $x = \prod_{k=1}^{n} e_kx_k$. Furthermore, for each $k \in \{1, \ldots, n\}$ the element e_kx_k is contained in J since

$$J = J_{e_k} \geq J_{e_kx_k} \geq J_x = J.$$

Therefore $e_kx_k \in EX' \cap J \subseteq T$ By repeated application of condition (ii), it follows that $x = \prod_{k=1}^{n} e_kx_k \in T \cap J$. Since $x \in E\langle X'\rangle \cap J$ was arbitrary, it follows that $E\langle X'\rangle \subseteq T$.

Note that condition (ii) is equivalent to the statement that $(T \cap J) \cup \{0\}$ is a subsemigroup of the principal factor J^*. Since $(T \cap J) \cup \{0\}$ intersects every $\mathscr{H}$-class of J^* and J^* is finite, it follows by Lemma 1.14 that T contains every idempotent of J.

To prove that condition (i) of Lemma 4.46 holds, let $x \in \langle X'\rangle \cap J$. There exists an idempotent $f \in T \cap J$ in the $\mathscr{R}^S$-class of x, and so $fx = x$. By definition of E, there exists an idempotent $e \in E$ such that $e \mathrel{\mathscr{L}} f$, and so $x = fx = (fe)x = f(ex)$. Certainly $ex \in J$ by Lemma 1.12, and since $E\langle X'\rangle \cap J \subseteq T$, it follows that $ex \in T \cap J$. By assumption, $(T \cap J) \cup \{0\}$ is a subsemigroup of J^*, and so $x = f(ex) \in T$.

To prove that condition (iii) of Lemma 4.46 holds, let $x \in T \cap J$ and $y \in \langle X'\rangle$. First suppose that $xy \in J$. By assumption, there exists an idempotent $e \in E$ such that $x = xe$. Since $xy = x(ey) \in J$ it follows that

$$J = J_e \geq J_{ey} \geq J_{x(ey)} = J_{xy} = J$$

by Lemma 1.12, and so $ey \in J$. Furthermore, $ey \in E\langle X'\rangle \cap J \subseteq T$. Since $x, ey \in T \cap J$, $xy \in J$, and $(T \cap J) \cup \{0\}$ is a subsemigroup of J^*, it follows that $xy = x(ey) \in T \cap J$. Finally suppose that $yx \in J$. Since J is a regular $\mathscr{J}$-class, there exists an idempotent $f \in T \cap J$ such that $f(yx) = yx$. By definition of E, there exists $e \in E$ such that $fe = f$, and so $yx = f(yx) = (fe)yx = f(ey)x$. Note that $ey \in J$ since

$$J = J_e \geq J_{ey} \geq J_{feyx} = J_{yx} = J,$$

and $ey \in T$ since $E\langle X' \rangle \cap J \subseteq T$. Finally, $f, ey, x \in T \cap J$ and $(T \cap J) \cup \{0\}$ is a subsemigroup of J^*, and so $yx = f(ey)x \in T$. □

Given a description of the subsemigroups of S that contain $S \setminus J$ and intersect every $\mathscr{H}$-class of J non-trivially, it is straightforward to classify those subsemigroups that are maximal. We do this in the following proposition, which is the main result of this section.

Proposition 4.48 (Maximal subsemigroups of type (M1))**.** *Let S be a finite semigroup with generating set X, let J be a regular $\mathscr{J}$-class of S, and let $X' = \{x \in X : J < J_x\}$. Let T be any subset of S that intersects every $\mathscr{H}$-class of S non-trivially and contains $S \setminus J$, and let E be a set consisting of one idempotent from each $\mathscr{L}$-class of J. Then T is a maximal subsemigroup of S if and only if $(T \cap J) \cup \{0\}$ is a maximal subsemigroup of J^* containing $EX' \cap J$.*

Proof. ($\Rightarrow$) Let U be a subset of J such that $U \cup \{0\}$ is a maximal subsemigroup of J^* containing $(T \cap J) \cup \{0\}$. Then by Lemma 4.47, the set $M = (S \setminus J) \cup U$ is a proper subsemigroup of S containing T. Since T is maximal, it follows that $T = M$, and $U = T \cap J$.

($\Leftarrow$) Let M be a maximal subsemigroup of S containing T. By Lemma 4.47, $(M \cap J) \cup \{0\}$ is a proper subsemigroup of J^* containing $(T \cap J) \cup \{0\}$. Since the latter is a maximal subsemigroup of J^*, it follows that $T \cap J = M \cap J$, and hence $T = M$. □

Note that results dual to Lemma 4.47 and Proposition 4.48 hold if we replace EX' by $X'F$, where F is a set consisting of one idempotent from each $\mathscr{R}$-class of J.

We may use Proposition 4.48 to describe an algorithm to calculate the maximal subsemigroups of type (M1) arising from J. We first compute the set $EX' \cap J$ and a normalization $\Psi : J^* \longrightarrow \mathscr{M}^0[G; I, \Lambda; P]$ of the principal factor of J to some regular Rees 0-matrix semigroup over a group; see Section 3.2.1. We then search for the maximal subsemigroups of $\mathscr{M}^0[G; I, \Lambda; P]$ that contain $(EX' \cap J)\Psi$ and have type (R6). Finding such maximal subsemigroups was the topic of Section 4.3.6 and Algorithm 4.44. Therefore, Algorithm 4.44 can be used to compute the maximal subsemigroups of S that arise from J and have type (M1).

A generating set for any such maximal subsemigroup is given by a generating set for $\langle S \setminus J \rangle$, along with the preimage under Ψ of a generating set for the corresponding maximal subsemigroup of $\mathscr{M}^0[G; I, \Lambda; P]$ that has type (R6) (minus the element 0, if present). A generating set for $\langle S \setminus J \rangle$ is given by the union of $X \setminus J$ with a generating set for the ideal $\{x \in S : J_x < J\}$. Generating sets for maximal subsemigroups of type (R6) can be constructed by using Lemma 4.25.

In Chapter 5, we describe the maximal subsemigroups of some particular families of finite monoids. In several of these cases, the most difficult step is the calculation of the maximal subsemigroups that have type (M1). We present results tailored to these monoids in Section 4.5.2. However, in many instances in Chapter 5, the following lemma can be used to show that no maximal subsemigroups of type (M1) arise. Given the Green's structure of a semigroup, it is straightforward to check whether a given $\mathscr{J}$-class is $\mathscr{H}$-trivial. Therefore, part (a) of the lemma can be also be used more widely when searching for maximal subsemigroups of type (M1), in order to avoid unnecessary computation.

Lemma 4.49. *Let S be a finite semigroup, let J be a $\mathscr{J}$-class of S. If*

(a) *each $\mathscr{H}$-class of J is trivial, or*

(b) $J \subseteq \langle (S \setminus J) \cup E(J) \rangle$,

then there are no maximal subsemigroups of S of type (M1) that arise from J.

Proof. Let T be an arbitrary subsemigroup of S that contains $S \setminus J$ and intersects each $\mathscr{H}$-class of J non-trivially. If T is a maximal subsemigroup, then in particular, T is a proper subsemigroup. Thus, it suffices in each case to prove that T contains J. By definition, T contains an element from each $\mathscr{H}$-class of J. If each such $\mathscr{H}$-class is trivial, then T contains J, proving part (a). By Lemma 1.14, T contains every idempotent of S, and so T contains $E(J)$. Therefore, if $\langle (S \setminus J) \cup E(J) \rangle$ contains J, then T contains J, proving part (b). □

4.4.2 Graphs and digraphs for regular $\mathscr{J}$-classes

In this section, we introduce two graphs and two digraphs that can be constructed from any regular $\mathscr{J}$-class in a finite semigroup. Throughout this section, we suppose that J is a regular $\mathscr{J}$-class of the finite semigroup $S = \langle X \rangle$. Recall that $X' \subseteq X$ consists of those generators $x \in X$ such that $J_x > J$. We use the graphs associated with J to characterise the subsemigroups of S that arise from the exclusion of elements in J, and are unions of $\mathscr{H}$-classes of S.

In Sections 4.4.3–4.4.5, we build on this characterisation to give necessary and sufficient conditions, in terms of the properties of the associated graphs, for a subset of S containing $S \setminus J$ to be a maximal subsemigroup of type (M2), (M3), (M4), or (M5). This formulation in terms of graphs makes the problem of computing maximal subsemigroups of these types more tractable. In particular, we can take advantage of several well-known and mature algorithms from graph theory, such as those for computing strongly connected components in a digraph (see [55, 121] or [117, Section 4.2]) and finding all maximal cliques in a graph [15, 103, 125].

The following digraphs are central to the results in the forthcoming sections. We define $\Gamma_{\mathscr{L}}(S, J)$ to be digraph formed by removing the loops from the quotient of the digraph

$$\big(J/\mathscr{L},\ \big\{(L_a, L_b) \in J/\mathscr{L} \times J/\mathscr{L} \ :\ L_a x = L_b \text{ for some } x \in X'\big\}\big) \tag{4.50}$$

by its strongly connected components. Note in particular that $\Gamma_{\mathscr{L}}(S, J)$ is an acyclic digraph. We define a colouring π of $\Gamma_{\mathscr{L}}(S, J)$ so that any vertex V containing an $\mathscr{L}$-class that has non-empty intersection with $\langle X' \rangle$ has $\pi(V) = 1$, and every other vertex U in $\Gamma_{\mathscr{L}}(S, J)$ has $\pi(U) = 0$. The digraph $\Gamma_{\mathscr{R}}(S, J)$ is defined dually.

Open Problem 4.51. If A_L and A_R are arbitrary finite acyclic digraphs, does there exist a finite semigroup S with a regular $\mathscr{J}$-class J such that $\Gamma_{\mathscr{L}}(S, J) \cong A_L$ and $\Gamma_{\mathscr{R}}(S, J) \cong A_R$?

We require two additional graphs. The first graph, $\Delta(S, J)$, is isomorphic to a quotient of the Graham-Houghton graph of the principal factor of J. We define $\Delta(S, J)$ to have vertex set equal to the disjoint union of the vertices of $\Gamma_{\mathscr{L}}(S, J)$ and $\Gamma_{\mathscr{R}}(S, J)$. In the special case that J consists of a single $\mathscr{H}$-class, we follow the convention that the strongly connected component of $\mathscr{L}$-classes $\{J\}$ and the strongly connected component of $\mathscr{R}$-classes $\{J\}$ are distinct, so that $\Delta(S, J)$ contains two vertices. The edges of $\Delta(S, J)$ are defined as follows. If U is a vertex in $\Gamma_{\mathscr{L}}(S, J)$ and V is a vertex in $\Gamma_{\mathscr{R}}(S, J)$, then $\{U, V\}$ is an edge of $\Delta(S, J)$ if and only if the intersection of some $\mathscr{L}$-class in U with an $\mathscr{R}$-class in V is a group $\mathscr{H}$-class.

The second graph, $\Theta(S, J)$, has the same vertex set as $\Delta(S, J)$, and it contains the edge $\{U, V\}$ if and only if there is an element of $\langle X' \rangle$ in the intersection of some $\mathscr{L}$-class in U with an $\mathscr{R}$-class in V, or vice versa. Note that the colourings of $\Gamma_{\mathscr{L}}(S, J)$ and $\Gamma_{\mathscr{R}}(S, J)$ can be deduced from the edges of $\Theta(S, J)$. The graphs $\Delta(S, J)$ and $\Theta(S, J)$ are bipartite, since the vertices in $\Gamma_{\mathscr{L}}(S, J)$ and the vertices in $\Gamma_{\mathscr{R}}(S, J)$ form maximal independent subsets whose union is the whole vertex set.

Example 4.52. Define S to be the transformation semigroup of degree 7 generated by the set X, which consists of the following transformations:

$$x_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 1 & 5 & 5 & 5 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 1 & 3 & 5 & 5 & 5 \end{pmatrix},$$
$$x_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 3 & 1 & 2 & 5 & 5 & 5 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 4 & 2 & 3 & 5 & 5 & 5 \end{pmatrix},$$
$$x_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 3 & 4 & 5 & 5 & 6 \end{pmatrix}, \quad x_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 2 & 4 & 5 & 6 & 7 \end{pmatrix},$$
$$x_7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}, \quad x_8 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 4 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

Let J denote the $\mathscr{J}$-class J_{x_1}. The following calculations were performed with the GAP [58] package SEMIGROUPS [101]. The $\mathscr{J}$-class J is regular, and contains the generators x_1, x_2, x_3, and x_4. The remaining generators are contained in $\mathscr{J}$-classes that are greater than J in the $\mathscr{J}$-class partial order, and so $X' = \{x_5, x_6, x_7, x_8\}$. The sets of $\mathscr{L}$- and $\mathscr{R}$-classes of J are

$$J/\mathscr{L} = \{L_{x_1}, L_{x_3}, L_{x_4}, L_{x_1x_6}\} \quad \text{and} \quad J/\mathscr{R} = \{R_{x_1}, R_{x_2}, R_{x_3}, R_{x_8x_2}, R_{x_6x_2}, R_{x_7x_3}\}.$$

There are four strongly connected components of $\mathscr{L}$-classes in J, each of which consists of a single $\mathscr{L}$-class. Hence the digraph $\Gamma_{\mathscr{L}}(S, J)$ has four vertices, one for each strongly connected component. There are also four strongly connected components of $\mathscr{R}$-classes, two of which are singletons, and two of which are not; these strongly connected components form the vertices of $\Gamma_{\mathscr{R}}(S, J)$. See Figure 4.53 for a description of the edges of $\Gamma_{\mathscr{L}}(S, J)$ and $\Gamma_{\mathscr{R}}(S, J)$.

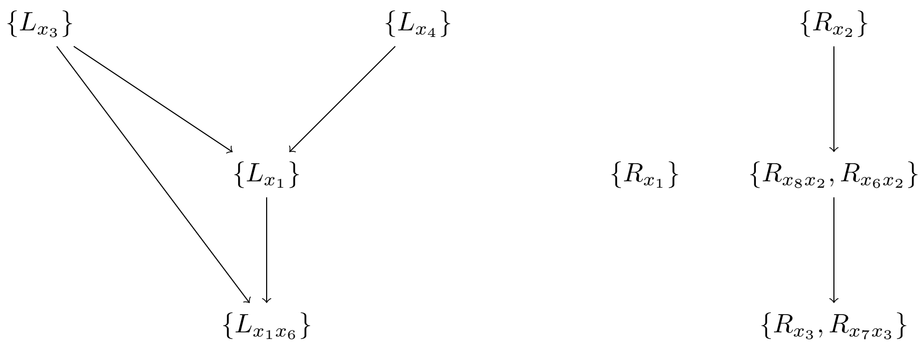

Figure 4.53: The digraphs $\Gamma_{\mathscr{L}}(S, J)$, left, and $\Gamma_{\mathscr{R}}(S, J)$, right, from Example 4.52.

Since $\Gamma_{\mathscr{L}}(S, J)$ and $\Gamma_{\mathscr{R}}(S, J)$ each have four vertices, it follows that the bipartite graphs $\Delta(S, J)$ and $\Theta(S, J)$ each have eight vertices. These graphs are shown in Figures 4.54 and 4.55. The set of edges of $\Delta(S, J)$ was determined by computation of the idempotents in J. There are four elements in $\langle X' \rangle \cap J$: $x_5^2 \in L_{x_1} \cap R_{x_3}$, $x_4x_1x_6 \in L_{x_1x_6} \cap R_{x_3}$, $x_7x_4x_1x_6 \in L_{x_1x_6} \cap R_{x_7x_3}$, and $x_7x_4x_1 \in L_{x_1} \cap R_{x_7x_3}$. The $\mathscr{L}$- and $\mathscr{R}$-classes of these elements determine the edges in the graph $\Theta(S, J)$, along with the colours of the vertices in $\Gamma_{\mathscr{L}}(S, J)$ and $\Gamma_{\mathscr{R}}(S, J)$. The vertices of $\Gamma_{\mathscr{L}}(S, J)$ with colour 1 are $\{L_{x_1}\}$ and $\{L_{x_1x_6}\}$, whilst the only vertex of $\Gamma_{\mathscr{R}}(S, J)$ with colour 1 is $\{R_{x_3}, R_{x_7x_3}\}$.

When the semigroup S and the $\mathscr{J}$-class J is obvious from the surrounding context, we abbreviate the names of the digraphs $\Gamma_{\mathscr{L}}(S, J)$ and $\Gamma_{\mathscr{R}}(S, J)$, and the graphs $\Delta(S, J)$ and $\Theta(S, J)$, to $\Gamma_{\mathscr{L}}$, $\Gamma_{\mathscr{R}}$, Δ, and Θ, respectively.

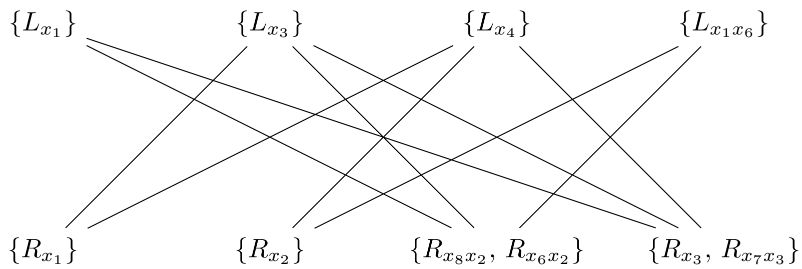

Figure 4.54: The graph $\Delta(S, J)$ from Example 4.52.

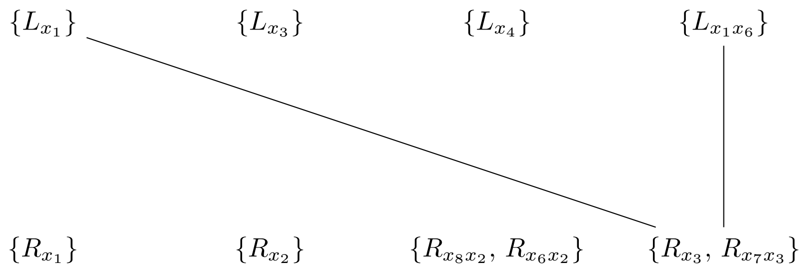

Figure 4.55: The graph $\Theta(S, J)$ from Example 4.52.

The digraphs $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$ and the graphs Δ and Θ can be constructed by applying graph algorithms to the left and right Cayley digraphs of S, with respect to its generating set X. The time complexity of finding $\Gamma_{\mathscr{L}}$, $\Gamma_{\mathscr{R}}$, Δ, and Θ by using the Cayley digraphs of S is $O(|S||X|)$. As discussed in Section 1.4, this is the same as the time complexity of determining the left and right Cayley digraphs of S themselves using the Froidure-Pin Algorithm [54]. However, in practice, finding $\Gamma_{\mathscr{L}}$, $\Gamma_{\mathscr{R}}$, Δ, and Θ using the Cayley digraphs of S is much quicker than determining the Cayley digraphs themselves. For certain types of semigroups, such as a semigroup generated by a set of transformations, the $\mathscr{J}$-class itself and the vertices and edges of $\Gamma_{\mathscr{L}}$, $\Gamma_{\mathscr{R}}$, and Δ can be determined without finding the Cayley digraphs of S and without necessarily exhaustively enumerating the semigroup; see [37] and see Section 1.4.3 for further details. In what follows, we assume that $\Gamma_{\mathscr{L}}$, $\Gamma_{\mathscr{R}}$, Δ, and Θ, and the colourings of $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$, are known *a priori*.

Note that the definitions of $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$ are given in terms of the subset X' of the generating set X. This suggests that the descriptions of these digraphs vary according to the chosen generating set for S. However, for our purposes, these variations are irrelevant.

By Green's Lemma [Lemma 1.11] and Lemma 1.10, if L_a and L_b are distinct $\mathscr{L}$-classes in J, then L_b is reachable from L_a in the digraph in (4.50) if and only if $sx \in L_b$ for some $s \in L_a$ and $x \in \langle S \setminus J \rangle$; see Lemma 4.56 for the proof of a closely related statement. In particular, it follows that the strongly connected components of the digraph in (4.50) do not depend on the chosen generating set for S, and so neither does the description of the vertices of $\Gamma_{\mathscr{L}}$. Similarly, the description of the vertices of $\Gamma_{\mathscr{R}}$ does not depend on X, and consequently the same is true of Δ and Θ. The definition of the edges of Δ is certainly independent of the generating set. Since $\langle S \setminus J \rangle \cap J = \langle X' \rangle \cap J$, it follows that the description of the edges of Θ, and the colourings of $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$, are independent of X, too.

On the other hand, in general, the descriptions of the edges of $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$ depend on X. However, in the forthcoming results, we are only concerned with whether a certain vertex is reachable from another in $\Gamma_{\mathscr{L}}$ or $\Gamma_{\mathscr{R}}$, and in particular, whether certain vertices are sources or sinks. The following lemma shows that this is independent of the chosen generating set.

Lemma 4.56. *The following hold.*

(i) *The vertex containing $L_b \in J/\mathscr{L}$ in $\Gamma_{\mathscr{L}}(S, J)$ is reachable from the vertex containing L_a if and only if $L_a = L_b$, or there exists $s \in L_a$ and $x \in \langle S \setminus J \rangle$ such that $sx \in L_b$.*

(ii) *The vertex containing $R_b \in J/\mathscr{R}$ in $\Gamma_{\mathscr{R}}(S, J)$ is reachable from the vertex containing R_a if and only if $R_a = R_b$, or there exists $s \in R_a$ and $x \in \langle S \setminus J \rangle$ such that $xs \in R_b$.*

Proof. We prove only (i), since (ii) is dual.

($\Rightarrow$) Suppose that $L_a \neq L_b$. It follows that L_b is reachable from L_a in the digraph (4.50). Therefore, there exists a path $(L_a = L_1, \ldots, L_n = L_b)$ in this digraph from L_a to L_b. By definition of (4.50), there exists a sequence of generators $x_1, \ldots, x_{n-1} \in X'$ such that $L_a x_1 \cdots x_{n-1} = L_b$. In particular, if $s \in L_a$ is arbitrary and $x = x_1 \cdots x_{n-1} \in \langle S \setminus J \rangle$, then $sx \in L_b$.

($\Leftarrow$) If $L_a = L_b$, then there is nothing to prove. Suppose there exists $s \in L_a$ and $x \in \langle S \setminus J \rangle$ such that $sx \in L_b$. Let $x = x_1 \cdots x_n$ be a factorization of x over $S \setminus J$. Note that

$$J = J_{sx} \leq J_x \leq J_{x_i} \text{ for each } i \in \{1, \ldots, n\}$$

by Lemma 1.12, and $x_i \notin J$. In particular, $x_i \in X'$ for each i. By Lemma 1.10, $s \mathrel{\mathscr{R}} sx_1 \cdots x_i$ for each $i \in \{1, \ldots, n\}$, and by Green's Lemma,

$$(L_a, L_a x_1, \cdots, L_a x_1 \cdots x_n = L_b)$$

is a path in (4.50) from L_a to L_b. The result follows. □

We may use the digraphs $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$ to understand the subsemigroups of S that contain $S \setminus J$ and that are unions of $\mathscr{H}$-classes of S. By Green's Lemma and Lemma 1.10, if T is a subsemigroup of S that contains $S \setminus J$, then $T \cap J$ contains an $\mathscr{L}$-class L if and only if $T \cap J$ contains every $\mathscr{L}$-class in every vertex of $\Gamma_{\mathscr{L}}$ that is reachable from the vertex containing L. The analogous statement holds for $\mathscr{R}$-classes. Furthermore, $T \cap J$ contains an $\mathscr{H}$-class H if and only if $T \cap J$ contains every $\mathscr{H}$-class that is the intersection of an $\mathscr{L}$-class and an $\mathscr{R}$-class that are contained in vertices that are reachable from the vertices containing the $\mathscr{L}$-classes and $\mathscr{R}$-classes containing H in $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$, respectively. This observation is fundamental to the forthcoming results, and is demonstrated in the following example.

Example 4.57. Let S and J be the semigroup and the $\mathscr{J}$-class, respectively, from Example 4.52. Suppose that T is a subsemigroup of S such that $S \setminus T \subseteq J$. By analysing the digraph $\Gamma_{\mathscr{L}}$, depicted in Figure 4.53, we see that if T contains the $\mathscr{L}$-class L_{x_4}, then T also contains the $\mathscr{L}$-classes L_{x_1} and $L_{x_1 x_6}$, since the vertices containing these $\mathscr{L}$-classes are reachable in the digraph from the vertex $\{L_{x_4}\}$. Likewise, if T contains the $\mathscr{R}$-class $R_{x_8 x_2}$, then by considering the digraph $\Gamma_{\mathscr{R}}$, depicted in Figure 4.53, we see that T also contains the $\mathscr{R}$-classes $R_{x_6 x_2}, R_{x_3}$, and $R_{x_7 x_3}$. If we consider these digraphs together, then we see that T contains the $\mathscr{H}$-class $L_{x_1} \cap R_{x_3}$ if and only if T also contains the $\mathscr{H}$-classes $L_{x_1} \cap R_{x_7 x_3}$, $L_{x_1 x_6} \cap R_{x_3}$, and $L_{x_1 x_6} \cap R_{x_7 x_3}$.

Suppose that T is a subset of S that contains $S \setminus J$, and suppose that there exist proper subsets $A \subsetneq J/\mathscr{L}$ and $B \subsetneq J/\mathscr{R}$ such that $T \cap J$ is the union of the Green's classes in A and in B. In other words, $T \cap J$ is a possibly empty union of $\mathscr{L}$-classes and/or $\mathscr{R}$-classes of J. In the following proposition, we give necessary and sufficient conditions for T to be a subsemigroup of S, in terms of the graphs and digraphs from this section.

Proposition 4.58. *Let S be a finite semigroup, and let T be a proper subset of S such that $S \setminus T$ is contained in a regular $\mathscr{J}$-class J of S. Suppose there exist proper subsets $A \subsetneq J/\mathscr{L}$ and $B \subsetneq J/\mathscr{R}$ such that $T \cap J$ is the union of the Green's classes in A and B. Then T is a subsemigroup of S if and only if the following hold:*

(i) *A and B are unions of vertices of $\Delta(S, J)$;*

(ii) *if $U \subseteq A$ is a vertex of $\Gamma_{\mathscr{L}}(S, J)$ and V is an out-neighbour of U, then $V \subseteq A$;*

(iii) *if $U \subseteq B$ is a vertex of $\Gamma_{\mathscr{R}}(S, J)$ and V is an out-neighbour of U, then $V \subseteq B$;*

(iv) *if $\{U, V\}$ is an edge in $\Theta(S, J)$, then $U \subseteq A \cup B$ or $V \subseteq A \cup B$;*

(v) *the vertices contained in $A \cup B$ form an independent subset of $\Delta(S, J)$.*

Proof. ($\Rightarrow$) As mentioned after Lemma 4.56, if $T \cap J$ contains an $\mathscr{L}$-class L, then $T \cap J$ contains every $\mathscr{L}$-class in every vertex of $\Gamma_{\mathscr{L}}$ that is reachable from the vertex containing L. An analogous statement holds for $\mathscr{R}$-classes and $\Gamma_{\mathscr{R}}$. Parts (i), (ii), and (iii) follow immediately from these observations.

If $\{U, V\}$ is an edge in Θ, then by definition, there exists an element $x \in \langle X' \rangle$ in the intersection of some $\mathscr{L}$-class L_x in U and some $\mathscr{R}$-class R_x in V. By Lemma 4.46, since T is a subsemigroup, $x \in T$ and so either $L_x \in A$ and U is contained in A; or $R_x \in B$ and V is contained in B. Therefore part (iv) holds.

If $A = \varnothing$ or $B = \varnothing$, then part (v) holds by the definition of Δ, so suppose otherwise. In order to reach a contradiction, suppose that $A \cup B$ does not form an independent subset of Δ. Therefore, there exists an $\mathscr{L}$-class $L \in A$ and an $\mathscr{R}$-class $R \in B$ such that $L \cap R$ is a group. Since A and B are proper subsets of Green's classes of J, we may choose $x \in L$ such that $R_x \notin B$, and we may choose $y \in R$ such that $L_y \notin A$. Note that $x, y \in T$. By Lemma 1.15, $xy \in R_x \cap L_y$. But $L_{xy} = L_y \notin A$ and $R_{xy} = R_x \notin B$, and so $xy \notin T$, contradicting the assumption that T is a subsemigroup. Therefore (v) holds.

($\Leftarrow$) It suffices to show that T satisfies the conditions (i), (ii), and (iii) of Lemma 4.46.

To verify that condition (i) of Lemma 4.46 holds, let $x \in \langle X' \rangle \cap J$. Then there exists an edge in Θ between the vertex U containing L_x and the vertex V containing R_x. By assumption (iv) of the proposition, either $U \subseteq A$ or $V \subseteq B$ (or both). In either case, it follows that $x \in T$.

For condition (ii), suppose that $x, y \in T \cap J$ and $xy \in J$. By Lemma 1.10, $xy \in R_x \cap L_y$, and so the $\mathscr{H}$-class $L_x \cap R_y$ is a group by Lemma 1.15. By assumption (v), the vertices contained in $A \cup B$ form an independent subset of Δ. Hence either $L_x \notin A$ or $R_y \notin B$. If $R_y \notin B$, then, since $T \cap J$ is a union of $\mathscr{L}$- and $\mathscr{R}$-classes of J and $y \in T \cap J$, we conclude that $L_y \in A$. By Lemma 1.10, $L_{xy} = L_y \in A$, and $xy \in T$. If $L_x \notin A$, then the proof is analogous.

To show that the final condition of Lemma 4.46 holds, let $x \in T \cap J$ and $y \in \langle X' \rangle$. Since $x \in T \cap J$, either $L_x \in A$ or $R_x \in B$. Suppose that $xy \in J$. In this case, note that Green's Lemma and Lemma 1.10 imply that $L_{xy} = L_y x$, and Lemma 1.10 implies that $R_{xy} = R_x$. If $L_x \in A$, then the vertex containing L_{xy} is reachable in $\Gamma_{\mathscr{L}}$ from the vertex containing L_x, and so $L_{xy} \in A$ and $xy \in T$. Otherwise, $R_x \in B$, and so $R_{xy} = R_x \in B$ and $xy \in T$. The proof that $yx \in J$ implies $yx \in T$ is similar. $\square$

Note that if the subset T from the statement of Proposition 4.58 is a maximal subsemigroup of S, then T has type (M2) when $A \neq \varnothing$ and $B \neq \varnothing$; T has type (M3) when $A \neq \varnothing$ and $B = \varnothing$; T has type (M4) when $A = \varnothing$ and $B \neq \varnothing$; and T has type (M5) when $A = \varnothing$ and $B = \varnothing$. We use Proposition 4.58 in the subsequent sections to obtain descriptions of the maximal subsemigroups of S that have types (M2)–(M5).

4.4.3 Maximal subsemigroups that are unions of $\mathscr{L}$- and $\mathscr{R}$-classes: (M2)

In this section, we discuss how to compute maximal subsemigroups that have type (M2), building on the work of Section 4.4.2. Throughout this section, we assume that J is a regular $\mathscr{J}$-class of the finite semigroup S. The following proposition, which is a slight adaptation of Proposition 4.58, gives necessary and sufficient conditions for the existence of a maximal subsemigroup of type (M2).

Proposition 4.59 (Maximal subsemigroups of type (M2))**.** *Let S be a finite semigroup, and let T be a proper subset of S such that $S \setminus T$ is contained in a regular $\mathscr{J}$-class J of S. Suppose that there exist proper non-empty subsets $A \subsetneq J/\mathscr{L}$ and $B \subsetneq J/\mathscr{R}$ such that $T \cap J$ is the union of the Green's classes in A and B. Then T is a maximal subsemigroup of S if and only if conditions (i)–(v) of Proposition 4.58 hold, and the independent subset of $\Delta(S, J)$ formed by the vertices contained in $A \cup B$ is maximal in $\Delta(S, J)$.*

Proof. ($\Rightarrow$) Let $\phi : J^* \longrightarrow \mathscr{M}^0[G; I, \Lambda; P]$ be an isomorphism from the principal factor of J to a Rees 0-matrix semigroup over a group. We may assume without loss of generality that $I = J/\mathscr{R}$ and $\Lambda = J/\mathscr{L}$, and that $L\phi = I \times G \times \{L\}$ and $R\phi = \{R\} \times G \times \Lambda$ for any $\mathscr{L}$-class L in J and for any $\mathscr{R}$-class R in J. Therefore

$$(M \cap J)\phi = (B \times G \times \Lambda) \cup (I \times G \times A) = (I \times G \times \Lambda) \setminus ((I \setminus B) \times G \times (\Lambda \setminus A)).$$

By Proposition 4.10, T is a maximal subsemigroup of type (M2), and $(M \cap J)\phi \cup \{0\}$ is a maximal subsemigroup of $\mathscr{M}^0[G; I, \Lambda; P]$ that has type (R5). By Proposition 4.11, $A \cup B$ is a maximal independent subset of the Graham-Houghton graph of $\mathscr{M}^0[G; I, \Lambda; P]$. Recall that Δ is a quotient of this graph. Therefore, the vertices of Δ contained in $A \cup B$ form a maximal independent subset of Δ, as required. The remaining conditions hold by Proposition 4.58.

($\Leftarrow$) By Proposition 4.58, it remains to prove that the subsemigroup T is maximal in S. Let M be a maximal subsemigroup of S that contains T. By the assumption that $T \cap J$ is a union of non-empty sets of $\mathscr{L}$- and $\mathscr{R}$-classes, M has type (M2). Let A' and B' be the sets of $\mathscr{L}$- and $\mathscr{R}$-classes of J, respectively, that are contained in M. Note that $A \subseteq A'$ and $B \subseteq B'$ since M contains T, and so $A \cup B \subseteq A' \cup B'$. By the forward implication, the vertices of Δ in $A' \cup B'$ form a maximal independent subset of Δ. But the vertices of Δ in $A \cup B$ form a maximal independent subset of Δ by assumption. Therefore $A = A'$ and $B = B'$, and so $T = M$, and T is maximal in S. $\square$

We use Proposition 4.59 to describe an algorithm for computing the maximal subsemigroups of S that have type (M2) and that arise from J.

The first step is to determine the maximal independent subsets of the bipartite graph Δ. As discussed in Section 4.3.3, this is equivalent to finding the maximal cliques in the complement of Δ. The Bron-Kerbosch Algorithm [15], which is implemented by the author in the Digraphs package [10] for GAP [58], is a recursive algorithm for finding the maximal cliques in any finite graph. Roughly speaking, the algorithm begins with a singleton clique, and recursively attempts to extend the given clique to a larger clique by adding another vertex.

We may use Proposition 4.58 to help to guide this recursion. By Proposition 4.58(ii) and (iii), we are only interested in those cliques K with following property: if $U \in K$ and V is an out-neighbour of U in $\Gamma_{\mathscr{L}}$ or $\Gamma_{\mathscr{R}}$, then $V \in K$. As such, the search tree can be pruned to exclude any branch starting at a clique containing a vertex U whose out-neighbours in $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$ do not extend the given clique, or where we have already discovered every maximal clique containing one of these out-neighbours. By executing the Bron-Kerbosch Algorithm in this modified way, we produce the maximal independent subsets of Δ, each of which corresponds to sets of Green's classes $A \subseteq J/\mathscr{L}$ and $B \subseteq J/\mathscr{R}$ that satisfy Proposition 4.58(i)–(iii) and (v).

The second step is then to check which of these sets A and B satisfy part (iv) of Proposition 4.58, which is routine. Given proper non-empty sets A and B satisfying all the conditions in Proposition 4.58, the final step is to specify a generating set for the corresponding maximal subsemigroup; see Proposition 4.61 for more details about this step.

Example 4.60. Let S be the transformation semigroup of degree 7, and let J be the $\mathscr{J}$-class of S, defined in Example 4.52. Consider the graph Δ, which is shown in Figure 4.54.

There are seven maximal independent subsets of Δ in total. Two of these correspond to all of the vertices of $\Gamma_{\mathscr{L}}$ and all of the vertices of $\Gamma_{\mathscr{R}}$, respectively. Three further maximal independent subsets correspond to sets A and B of $\mathscr{L}$-classes and $\mathscr{R}$-classes that either do not satisfy Proposition 4.58(ii) or do not satisfy Proposition 4.58(iii). For example, the maximal independent subset $Q = \{\{L_{x_1}\}, \{R_{x_1}\}, \{R_{x_2}\}\}$ does not satisfy Proposition 4.58(ii), since $\{L_{x_1x_6}\}$ is an out-neighbour of $\{L_{x_1}\}$ in $\Gamma_{\mathscr{L}}$, but $\{L_{x_1x_6}\} \notin Q$; see Figure 4.53. Furthermore, Q does not satisfy Proposition 4.58(iii), since $\{R_{x_8x_2}, R_{x_6x_2}\}$ is an out-neighbour of $\{R_{x_2}\}$ in $\Gamma_{\mathscr{R}}$ that is not contained in Q.

The remaining two maximal independent subsets of Δ correspond to non-empty sets A and B that satisfy Proposition 4.58(i)–(iii). The first of these is $\{\{L_{x_1}\}, \{L_{x_1x_6}\}, \{R_{x_1}\}\}$, which corresponds to the sets $A_1 = \{L_{x_1}, L_{x_1x_6}\} \subseteq J/\mathscr{L}$ and $B_1 = \{R_{x_1}\} \subseteq J/\mathscr{R}$; the second corresponds to the sets $A_2 = \{L_{x_1x_6}\}$ and $B_2 = \{R_{x_1}, R_{x_3}, R_{x_7x_3}\}$.

It remains to test whether these subsets satisfy Proposition 4.58(iv). There are two edges in the graph Θ, as shown in Figure 4.55:

$$\{\{L_{x_1}\}, \{R_{x_3}, R_{x_7x_3}\}\}, \quad \text{and} \quad \{\{L_{x_1x_6}\}, \{R_{x_3}, R_{x_7x_3}\}\}.$$

For the first edge, $\{L_{x_1}\} \subseteq A_1$ and $\{R_{x_3}, R_{x_7x_3}\} \subseteq B_2$; for the second edge, $\{L_{x_1x_6}\} \subseteq A_1$ and $\{R_{x_3}, R_{x_7x_3}\} \subseteq B_2$. In other words, the sets A_1 and B_1, and the sets A_2 and B_2, satisfy Proposition 4.58(iv). Therefore, there are two maximal subsemigroups of S of type (M2) arising from J: the set consisting of $S \setminus J$ along with the union of the $\mathscr{L}$-classes in A_1 and the union of the $\mathscr{R}$-classes in B_1, and the set consisting of $S \setminus J$ along with the union of the $\mathscr{L}$-classes in A_2 and the union of the $\mathscr{R}$-classes in B_2.

Proposition 4.61. *Let T be a proper subsemigroup of a finite semigroup $S = \langle X \rangle$ that arises from a regular $\mathscr{J}$-class J of S. Suppose that there exist non-empty proper subsets $A \subsetneq J/\mathscr{L}$ and $B \subsetneq J/\mathscr{R}$ such that $T \cap J$ is the union of the Green's classes in A and in B. Then T is generated by any set consisting of:*

(i) *$X \setminus J$;*

(ii) *a semigroup generating set for the ideal $\{x \in S : J_x < J\}$ of S;*

(iii) *for each source U in $\Gamma_{\mathscr{L}}(S, J)$ contained in A, an element t such that $R_t \in B$ and $L_t \in U$;*

(iv) *a generating set for a group $\mathscr{H}$-class H_x, where $L_x \in A$;*

(v) *for each source V in the induced subdigraph of $\Gamma_{\mathscr{R}}(S, J)$ on the complement of B, an element $z \in L_x$ such that $R_z \in V$; and*

(vi) *for each source U in the induced subdigraph of $\Gamma_{\mathscr{L}}(S, J)$ on A, an element $y \in R_x$ such that $L_y \in U$;*

(vii) *a generating set for a group $\mathscr{H}$-class $H_{x'}$, where $R_{x'} \in B$;*

(viii) *for each source U' in the induced subdigraph of $\Gamma_{\mathscr{R}}(S, J)$ on B, an element $y' \in L_{x'}$ such that $R_{y'} \in U'$;*

(ix) *for each source V' in the induced subdigraph of $\Gamma_{\mathscr{L}}(S, J)$ on the complement of A, an element $z' \in R_{x'}$ such that $L_{z'} \in V'$.*

Proof. Let Y be a set of the kind described in the proposition. Clearly every element of Y is contained in T, and so $\langle Y \rangle$ is a subsemigroup of T. Furthermore, the inclusion of the generators in (i) and (ii) implies that $\langle Y \rangle$ contains $S \setminus J$. To show that $T \leq \langle Y \rangle$, let $a \in T \cap J$. By the definition of T, either $L_a \in A$ or $R_a \in B$.

First suppose that $L_a \in A$ and $R_a \notin B$. The vertex of $\Gamma_{\mathscr{L}}$ containing L_a is reachable from some vertex U that is a source of the induced subdigraph of $\Gamma_{\mathscr{L}}$ on A. Hence there exists an element $y \in Y$ of type (vi) such that $L_y \in U$ and $y \mathscr{R} x$, where x is an element in the group $\mathscr{H}$-class from part (iv). By Lemma 4.56, either $L_a = L_y$, or there exists $r' \in \langle S \setminus J \rangle \subseteq \langle Y \rangle$ such that $yr' \in L_a$ and, by Lemma 1.10, $yr' \in R_y = R_x$. In either case, there exists an element $r \in \langle Y \rangle$ such that $r \mathscr{R} x$ and $r \mathscr{L} a$. Likewise, by using a generator of type (v), there exists an element $s \in \langle Y \rangle$ such that $s \mathscr{L} x$ and $s \mathscr{R} a$. Since H_x is a group, it follows by Green's Lemma that $a \in H_a = sH_xr \subseteq \langle Y \rangle$. If $L_a \notin A$ and $R_a \in B$, then the proof that $a \in \langle Y \rangle$ is similar.

For the final case, suppose that $L_a \in A$ and $R_a \in B$. The vertex of $\Gamma_{\mathscr{L}}$ containing L_a is reachable from some vertex U that is a source of $\Gamma_{\mathscr{L}}$. If $U \subseteq A$, then by (iii) there exists an element $t \in Y$ such that $R_t \in B$ and $L_t \in U$. If $U \nsubseteq A$, then by the previous paragraph, every element t' such that $R_{t'} \in B$ and $L_{t'} \notin A$ is contained in $\langle Y \rangle$. In either case, there exists an element $t \in \langle Y \rangle$ such that $R_t \in B$ and $L_t \in U$.

If $a \mathscr{L} t$, then $t \in L_a \cap R_t$. Otherwise, by Lemmas 4.56 and 1.10, there exists an element $r' \in \langle S \setminus J \rangle \subseteq \langle Y \rangle$ such that $tr' \in L_a \cap R_t$. In either case, there exists $r \in \langle Y \rangle$ such that $r \in L_a \cap R_t$. By the regularity of J, there exists an idempotent $e \in R_t$, and since $R_e = R_t \in B$ and $A \cup B$ corresponds to an independent subset of Δ, it follows that $L_e \notin A$. By the arguments of the second paragraph, the $\mathscr{H}$-classes H_e and $H_s = L_e \cap R_a$ are both contained in $\langle Y \rangle$. Since H_e is a group, it follows by Green's Lemma and Lemma 1.15 that $a \in H_a = sH_er \subseteq \langle Y \rangle$.

Note that if $L_a \in A$ and $R_a \in B$, and there exists an idempotent $e \in J$ such that $L_e \notin A$ and $R_e \notin B$, then $a \in H_a = (L_a \cap R_e)(L_e \cap R_a)$. Therefore the generators in (iii) are redundant in this case, which occurs if and only if the complement of $A \cup B$ in $(J/\mathscr{L}) \cup (J/\mathscr{R})$ corresponds to a non-independent subset of Δ. □

4.4.4 Maximal subsemigroups that are unions of $\mathscr{L}$- or $\mathscr{R}$-classes: (M3)–(M4)

In this section, we build on the results of Sections 4.4.2 and 4.4.3 in order to describe the subsemigroups, and therefore the maximal subsemigroups, that can be obtained by the removal of either $\mathscr{L}$-classes or $\mathscr{R}$-classes from a regular $\mathscr{J}$-class of a finite semigroup. In other words, we describe how to compute maximal subsemigroups of types (M3) and (M4).

Consider Proposition 4.58. If the set of $\mathscr{R}$-classes B in the statement of this proposition is empty, then the subset T is a union of $\mathscr{L}$-classes of the semigroup, and the criteria for T to be a subsemigroup can be simplified. In more detail, conditions (iii) and (v) of Proposition 4.58 are immediately satisfied, as is the second part of condition (i). Furthermore, the reference to the set B in condition (iv) can be removed. By performing these simplifications, we obtain the following corollary to Proposition 4.58.

Corollary 4.62. *Let S be a finite semigroup with a regular $\mathscr{J}$-class J, and let T be a subset of S that contains $S \setminus J$. Suppose there exists a proper subset $A \subseteq J/\mathscr{L}$ such that $T \cap J$ is the union of the $\mathscr{L}$-classes in A. Then T is a subsemigroup of S if and only if A is a union of vertices of $\Gamma_{\mathscr{L}}(S, J)$, including every vertex with colour 1, and Proposition 4.58(ii) holds.*

We use this corollary in the proof of the following proposition.

Proposition 4.63 (Maximal subsemigroups of type (M3))**.** *Let S be a finite semigroup with a regular $\mathscr{J}$-class J, and let T be a subset of S that contains $S \setminus J$. Suppose there exists a proper* *non-empty subset $A \subsetneq J/\mathscr{L}$ such that $T \cap J$ is the union of the $\mathscr{L}$-classes in A. Then T is a maximal subsemigroup of S if and only if the complement of A is a source of $\Gamma_{\mathscr{L}}(S, J)$ with colour 0, and there is no maximal subsemigroup of S of type (M2) whose corresponding subset of $J/\mathscr{L}$ is A.*

Proof. ($\Rightarrow$) By Corollary 4.62, A is a union of vertices of $\Gamma_{\mathscr{L}}$. Since $\Gamma_{\mathscr{L}}$ is finite and acyclic, it follows that any induced subdigraph of $\Gamma_{\mathscr{L}}$ is finite and acyclic. Therefore, there exists a sink in the induced subdigraph of $\Gamma_{\mathscr{L}}$ on the vertices not contained in A. Note that this vertex is not necessarily a sink in $\Gamma_{\mathscr{L}}$ itself. Let A' be the subset of $J/\mathscr{L}$ formed from the union of A with the set of $\mathscr{L}$-classes contained in this sink, and define T' to be the subset of S such that $S \setminus T' \subseteq J$ and $T' \cap J$ is the union of the $\mathscr{L}$-classes in A'. Then either $A' = J/\mathscr{L}$, or, by Corollary 4.62, T' is a proper subsemigroup of S that properly contains T. Since T is maximal, it follows that $A' = J/\mathscr{L}$, and so the complement of A forms a single vertex of $\Gamma_{\mathscr{L}}$. This vertex is a source of $\Gamma_{\mathscr{L}}$ by condition (ii) of Proposition 4.58, and has colour 0 by Corollary 4.62.

Since T is a maximal subsemigroup of S of type (M3), it is not contained in a maximal subsemigroup of S of type (M2).

($\Leftarrow$) Let M be a maximal subsemigroup of S that contains T. Since T contains a union of $\mathscr{L}$-classes of J, M has type (M2) or (M3). In either case, it follows that M contains the $\mathscr{L}$-classes in A. However, A lacks only one vertex of $\Gamma_{\mathscr{L}}$, and since a maximal subsemigroup is a proper subsemigroup, it follows that M contains no additional $\mathscr{L}$-classes. Such a maximal subsemigroup of type (M2) does not exist by assumption, and so $M = T$ has type (M3). □

Analogues of Corollary 4.62 and Proposition 4.63 hold which concern subsemigroups that are unions of $\mathscr{R}$-classes of a finite semigroup. For completeness, we state the analogue of Proposition 4.63, which describes the maximal subsemigroups of type (M4).

Proposition 4.64 (Maximal subsemigroups of type (M4))**.** *Let S be a finite semigroup with a regular $\mathscr{J}$-class J, and let T be a subset of S that contains $S \setminus J$. Suppose there exists a proper non-empty subset $B \subsetneq J/\mathscr{R}$ such that $T \cap J$ is the union of the $\mathscr{R}$-classes in B. Then T is a maximal subsemigroup of S if and only if the complement of B is a source of $\Gamma_{\mathscr{R}}(S, J)$ with colour 0, and there is no maximal subsemigroup of S of type (M2) whose corresponding subset of $J/\mathscr{R}$ is B.*

We describe an algorithm that uses Proposition 4.63 to compute the maximal subsemigroups of type (M3) arising from a regular $\mathscr{J}$-class J of a finite semigroup S. An analogous algorithm, using Proposition 4.64, permits the computation of maximal subsemigroups of type (M4). The first step is to compute the maximal subsemigroups of S of type (M2) that arise from J, as described in Section 4.4.3. In doing so, we record the sets of $\mathscr{L}$-classes and $\mathscr{R}$-classes of J that correspond to each maximal subsemigroup of type (M2). The second step is to search for the sources of the digraph $\Gamma_{\mathscr{L}}(S, J)$ with colour 0. For each source, we test whether its complement occurs as the set of $\mathscr{L}$-classes of some maximal subsemigroup of type (M2), and discard if so. By Proposition 4.63, the complements of the remaining sources define the maximal subsemigroups of S that arise from J and have type (M3). The final step is to specify a generating set for each such maximal subsemigroup; these generating sets can be constructed by using Proposition 4.66.

Example 4.65. Let S and J be the transformation semigroup and the $\mathscr{J}$-class, respectively, from Example 4.52. The digraph $\Gamma_{\mathscr{L}}$, depicted in Figure 4.53, contains two sources with colour 0, $\{L_{x_3}\}$ and $\{L_{x_4}\}$. The complements of these vertices in $J/\mathscr{L}$ are $A_1 = \{L_{x_1}, L_{x_4}, L_{x_1x_6}\}$ and $A_2 = \{L_{x_1}, L_{x_3}, L_{x_1x_6}\}$, respectively. In Example 4.60, we found that there is no maximal subsemigroup of S of type (M2) whose set of $\mathscr{L}$-classes is equal to A_1 or A_2. By Proposition 4.63, there are two maximal subsemigroups of S of type (M3) arising from J: the set consisting of

$S \setminus J$ and the union of the $\mathscr{L}$-classes in A_1, and the set consisting of $S \setminus J$ and the union of the $\mathscr{L}$-classes in A_2.

In $\Gamma_{\mathscr{R}}$, the sources are $\{R_{x_1}\}$ and $\{R_{x_2}\}$, and they have colour 0. In Example 4.60, we found that there is no maximal subsemigroup of S of type (M2) whose set of $\mathscr{R}$-classes is equal to the complement of either of these sources. By Proposition 4.64, $S \setminus R_{x_1}$ and $S \setminus R_{x_2}$ are the maximal subsemigroups of S of type (M4) that arise from J.

The following proposition can be used to describe generating sets for maximal subsemigroups of type (M3). Generating sets for maximal subsemigroups of type (M4) are obtained analogously. The proof of Proposition 4.66 is similar to, but simpler than, the proof of Proposition 4.61, and is omitted.

Proposition 4.66. *Let T be a subsemigroup of a finite semigroup $S = \langle X \rangle$ such that $S \setminus T$ is contained in a regular $\mathscr{J}$-class J of S. Suppose that there exists a non-empty proper subset $A \subsetneq J/\mathscr{L}$ such that $T \cap J$ is the union of the $\mathscr{L}$-classes in A. Then T is generated by any set consisting of:*

(i) *$X \setminus J$;*

(ii) *a semigroup generating set for the ideal $\{x \in S : J_x < J\}$ of S;*

(iii) *a generating set for a group $\mathscr{H}$-class H_x, where $L_x \in A$;*

(iv) *for each source U in the induced subdigraph of $\Gamma_{\mathscr{L}}(S, J)$ on A, an element $y \in R_x$ such that $L_y \in U$; and*

(v) *for every source V in $\Gamma_{\mathscr{R}}(S, J)$, an element $z \in L_x$ such that $R_z \in V$.*

4.4.5 Maximal subsemigroups by removing a $\mathscr{J}$-class: (M5)

In this section, we discuss how to compute maximal subsemigroups of type (M5), which are formed by removing a $\mathscr{J}$-class from a finite semigroup. Recall that J is an arbitrary $\mathscr{J}$-class of the finite semigroup $S = \langle X \rangle$, and that $X' = \{x \in X : J < J_x\}$.

If the set of $\mathscr{L}$-classes A and the set of $\mathscr{R}$-classes B in the statement of Proposition 4.58 are both empty, then conditions (i)–(iii) and (v) of Proposition 4.58 are vacuously satisfied, which leaves only condition (iv). Thereby, we obtain the following corollary.

Corollary 4.67. *Suppose that J is regular. Then the subset $S \setminus J$ is a subsemigroup of S if and only if the graph $\Theta(S, J)$ has no edges.*

As mentioned previously, the existence of a maximal subsemigroup of type (M5) arising from J precludes the existence of maximal subsemigroups of types (M1)–(M4) arising from J, since any of these kinds of maximal subsemigroups properly contains the set $S \setminus J$. Therefore, $S \setminus J$ is a maximal subsemigroup of S if and only if $S \setminus J$ is a subsemigroup, and no maximal subsemigroups of types (M1)–(M4) arise from J. Alternatively, $S \setminus J$ is a maximal subsemigroup if and only if at least one maximal subsemigroup arises from J, but no maximal subsemigroups of types (M1)–(M4) arise from J. We use the following lemma to determine whether at least one maximal subsemigroup arises from J.

Lemma 4.68. *The following are equivalent:*

(i) *Some proper subsemigroup of S contains $S \setminus J$.*

(ii) *Some maximal subsemigroup of S arises from J.*

(iii) *Every generating set for S has non-empty intersection with J.*

(iv) $\langle S \setminus J\rangle \neq S$.

(v) $J \cap X \not\subseteq \langle X'\rangle$.

Proof. (i) ⇒ (ii) Let T be a proper subsemigroup of S that contains $S \setminus J$. Since J is finite, T is contained in a maximal subsemigroup of S, which arises from J.

(ii) ⇒ (iii) Let M be a maximal subsemigroup of S arising from J, and let A be any subset of S that is disjoint from J. Then $\langle A\rangle \leq \langle S \setminus J\rangle \leq M \neq S$, and so A does not generate S.

(iii) ⇒ (iv) Since $S \setminus J$ is disjoint from J, it follows that $\langle S \setminus J\rangle \neq S$.

(iv) ⇒ (v) (Contrapositive). Certainly $S \setminus J$ contains $X \setminus J$, and X' in particular. Therefore, if $J \cap X \subseteq \langle X'\rangle$, then $\langle S \setminus J\rangle$ contains X, which is a generating set for S, and so $\langle S \setminus J\rangle = S$.

(v) ⇒ (i) Since $\langle S \setminus J\rangle \cap J = \langle X'\rangle \cap J$, it follows that $\langle S \setminus J\rangle$ does not contain $J \cap X$. In particular, $\langle S \setminus J\rangle$ is a proper subsemigroup of S that contains $S \setminus J$. □

Note that when the generating set X is minimal with respect to containment, $J \cap X \subseteq \langle X'\rangle$ if and only if $J \cap X = \varnothing$. Therefore condition (v) can be tested more easily in this case.

We use the following proposition, which is the main result of this section, when computing maximal subsemigroups of type (M5).

Proposition 4.69 (Maximal subsemigroups of type (M5))**.** *Let $S = \langle X\rangle$ be a finite semigroup, let J be a $\mathscr{J}$-class of S with representative x, let $X' = \{s \in X : J < J_s\}$, and suppose that no maximal subsemigroups of types (M1)–(M4) arise from J. The following are equivalent:*

(i) *$S \setminus J$ is a maximal subsemigroup of S;*

(ii) *any of the conditions in Lemma 4.68 holds; and*

(iii) $x \notin \langle X'\rangle$.

Furthermore, if J is regular, then the following is also equivalent to the above conditions:

(iv) *the graph $\Theta(S, J)$ has no edges.*

Proof. The equivalence of (i), (ii), and (iv) has been established by the above discussion and Corollary 4.67. It remains to prove the equivalence of (iii).

(i) ⇒ (iii) Since $X' \subseteq S \setminus J$, it follows that $\langle X'\rangle \leq \langle S \setminus J\rangle = S \setminus J$. In particular, $x \notin \langle X'\rangle$.

(iii) ⇒ (ii) Since $\langle X'\rangle \cap J = \langle S \setminus J\rangle \cap J$, it follows that $x \notin \langle S \setminus J\rangle$. Therefore $\langle S \setminus J\rangle \neq S$, i.e. Lemma 4.68(iv) holds. □

By Proposition 4.10, a maximal subsemigroup arising from a non-regular $\mathscr{J}$-class has type (M5). Therefore, if J is non-regular, then the assumption concerning maximal subsemigroups of types (M1)–(M4) in the statement of Proposition 4.69 can be ignored.

In Algorithm 4.86, we use Proposition 4.69(ii) in combination with Lemma 4.68(v) to compute the maximal subsemigroups of type (M5) that arise from a regular non-maximal $\mathscr{J}$-class; see lines 17 and 27. For a non-regular non-maximal $\mathscr{J}$-class, we simply test whether Proposition 4.69(iii) holds. If $S \setminus J$ is a subsemigroup of S, then a generating set for $S \setminus J$ is given by the union of $X \setminus J$ with a semigroup generating set for the ideal $\{x \in S : J_x < J\}$ of S.

Maximal subsemigroups and minimal generating sets

Before continuing, we briefly elucidate a few connections between the minimal generating sets of a semigroup and the maximal subsemigroups that are formed by removing a $\mathscr{J}$-class.

Lemma 4.70. *Let $S = \langle X \rangle$ be a semigroup, suppose that $S \setminus J$ is a maximal subsemigroup of S for some $J \in S/\mathscr{J}$, and let $x \in J$ be arbitrary. Then $(X \setminus J) \cup \{x\}$ generates S.*

Proof. Let J' be an arbitrary $\mathscr{J}$-class of S. If $J' \not\leq J$, then, by the contrapositive of Lemma 1.12(iii), the elements of J' cannot be expressed involving elements of $X \cap J$. In other words, $J' \subseteq \langle X \setminus J \rangle$. Let $I = \{s \in S : J_s < J\}$. Since I is an ideal of S, the union $\langle (X \setminus J) \cup \{x\} \rangle \cup I$ is a subsemigroup of S that contains every $\mathscr{J}$-class of S, except for possibly J. Hence

$$T = \langle (X \setminus J) \cup \{x\} \rangle \cup I = \langle (X \setminus J) \cup \{x\} \rangle \cup (S \setminus J).$$

Since T properly contains the maximal subsemigroup $S \setminus J$, it follows that $T = S$. In particular, $T \cap J = J$. But $T \cap J = \langle (X \setminus J) \cup \{x\} \rangle \cap J$. Therefore $\langle (X \setminus J) \cup \{x\} \rangle$ contains J, and hence contains the generating set X. ☐

Corollary 4.71. *Let S be a semigroup that has a minimal generating set X with respect to containment, let J be a $\mathscr{J}$-class of S, and suppose that $S \setminus J$ is a maximal subsemigroup of S. Then $|J \cap X| = 1$.*

Since a finite semigroup has minimal generating sets, we may use these results to describe the minimal generating sets of a finite semigroup whose maximal subsemigroups have type (M5). The result implies that a finite $\mathscr{J}$-trivial semigroup has a unique minimal generating set [36].

Corollary 4.72 (cf. [36])**.** *Let S be a finite semigroup whose maximal subsemigroups have type (M5), and let $\mathfrak{J} = \big\{ J \in S/\mathscr{J} : S \setminus J \text{ is a maximal subsemigroup of } S \big\}$. Then the minimal generating sets of S (with respect to both cardinality and containment) are the transversals of $\mathfrak{J}$. In particular,* $\operatorname{rank}(S) = |\mathfrak{J}|$.

Proof. In a finite semigroup, a generating set that is minimal with respect to cardinality is necessarily minimal with respect to containment. The generating sets in the statement have common cardinality $|\mathfrak{J}|$. Therefore, if they comprise all generating sets that are minimal with respect to containment, then the two notions of minimality are equivalent in this case. By assumption and by Lemma 4.68, a minimal generating set for S (with respect to containment) contains only elements of $\mathscr{J}$-classes in $\mathfrak{J}$. By Lemma 4.70, the result follows. ☐

4.5 Arbitrary finite monoids

In Chapter 5, we describe and count the maximal subsemigroups of several families of particular finite monoids. For almost all of these monoids, the maximal subsemigroups arise either from the group of units, or from a $\mathscr{J}$-class that is covered (in the partial order of $\mathscr{J}$-classes) by the group of units. While the techniques of the previous section certainly apply to finite monoids, many of the cases that they treat do not arise for these kinds of $\mathscr{J}$-classes. Thus, the purpose of this section is to simplify some of the results of Section 4.4, under the assumption that the $\mathscr{J}$-class in question is either equal to, or is covered by, the group of units of a monoid.

4.5.1 Maximal subsemigroups from the group of units

The maximal subsemigroups of a finite monoid that are the easiest to describe are those that arise from its group of units. Such maximal subsemigroups exist by Lemma 4.68, since the non-units in a finite monoid form an ideal, and so every generating set of the monoid contains a unit. As shown in the following lemma, maximal subsemigroups of this kind can be calculated from the group of units in isolation, without reference to the remainder of the monoid.

Lemma 4.73. *Let S be a finite monoid with group of units G. Then the maximal subsemigroups of S arising from G are the sets $(S \setminus G) \cup U$, for each maximal subsemigroup U of G. In other words, if G is trivial, then the unique maximal subsemigroup of S arising from G is $S \setminus G$, which has type (M5); if G is non-trivial, then the maximal subsemigroups of S arising from G are the sets $(S \setminus G) \cup U$, for each maximal subgroup U of G, which have type (M1).*

Proof. Since $S \setminus G$ is an ideal of S, it follows that, for a subset U of G, $(S \setminus G) \cup U$ is a subsemigroup of S if and only if U is a subsemigroup of G. Since this correspondence between subsemigroups of S containing $S \setminus G$ and subsemigroups of G preserves inclusion, the result follows. Note that a *subsemigroup* of a finite group is a *subgroup*, unless it is empty; the only group to possess the empty semigroup as a maximal subsemigroup is the trivial group. □

4.5.2 Maximal subsemigroups from a $\mathscr{J}$-class covered by the group of units

Before presenting the main results of this section, we first state the following corollary to Lemma 4.8. This corollary is used several times in Chapter 5 to describe the maximal subsemigroups that arise from a $\mathscr{J}$-class covered by the group of units of a monoid.

Corollary 4.74. *Let S be a finite monoid with group of units G, and suppose there exists a non-empty subset A of $S \setminus G$ with the property that $S = \langle G,\ x\rangle$ if and only if $x \in A$. Then the maximal subsemigroups of S are those that arise from the group of units and $S \setminus A$.*

Proof. It remains to describe the maximal subsemigroups that do not arise from the group of units. Let $x \in A$. Since $S = \langle G,\ x\rangle$ and G is closed under multiplication, the principal ideal of S generated by x is $S \setminus G$. Since x was arbitrary, every element of A generates the same principal ideal, and so A is contained in some $\mathscr{J}$-class J of S. By Lemma 4.68, the remaining maximal subsemigroups of S arise from J. By applying Lemma 4.8 with $k = 1$ and $X_1 = A$, we find that the unique maximal subsemigroup in this case is $S \setminus A$. □

In the remainder of this section, we let S denote a finite monoid with generating set X, let G denote the group of units of S, and let J be a $\mathscr{J}$-class of S that is covered by the group of units; that is, if $J' \in S/\mathscr{J}$ and $J < J'$, then $J' = G$. Define

$$X' = \{x \in X \ :\ J < J_x\} = G \cap X.$$

The problem of finding the maximal subsemigroups of S that arise from J is simpler than the general case. This is because elements of $\mathscr{J}$-classes that are strictly greater than J are units, which generate no elements within J, and because, roughly speaking, the action of units on J by multiplication is easier to understand than the action of arbitrary elements.

Maximal subsemigroups of type (M5)

We first show that at least one maximal subsemigroup of S arises from a $\mathscr{J}$-class covered by the group of units. Since $X' = \{x \in X \ :\ J < J_x\} \subseteq G$, it follows that $\langle X'\rangle \cap J \subseteq G \cap J = \varnothing$, and so we may obtain the following corollary to Proposition 4.69.

Corollary 4.75 (Maximal subsemigroups of type (M5))**.** *Let S be a finite monoid with group of units G, and let J be a $\mathscr{J}$-class of S that is covered by G. Then $S \setminus J$ is a maximal subsemigroup of S if and only if no maximal subsemigroups of types (M1)–(M4) arise from J.*

Maximal subsemigroups of types (M2)–(M4)

Suppose that J is regular, and that the generating set X contains the group of units G. Therefore $X' = G$. The digraphs $\Gamma_{\mathscr{L}}(S, J)$ and $\Gamma_{\mathscr{R}}(S, J)$ and the graphs $\Delta(S, J)$ and $\Theta(S, J)$ were introduced in Section 4.4.2 in order to classify the maximal subsemigroups that arise from J and have types (M2)–(M5). In the present case, when J is a regular $\mathscr{J}$-class covered by the group of units, these graphs and digraphs have particularly straightforward descriptions.

By (4.50), $\Gamma_{\mathscr{L}}(S, J)$ is the digraph formed by removing the loops from the quotient of

$$\big(J/\mathscr{L},\ \big\{(L_x, L_y) \in J/\mathscr{L} \times J/\mathscr{L} \,:\, L_x g = L_y \text{ for some } g \in G\big\}\big) \tag{4.76}$$

by its strongly connected components. Let $L \in J/\mathscr{L}$, $x \in L$, and $g \in G$ be arbitrary. Since $xg \mathrel{\mathscr{R}} x$, it follows by Green's Lemma [Lemma 1.11] that $L_x g = L_{xg}$. Furthermore, $\mathscr{L}$ is a right congruence, and so $L_x = L_y$ implies that $L_{xg} = L_{yg}$. Therefore, the function $(J/\mathscr{L} \times G) \longrightarrow J/\mathscr{L}$ defined by $L \cdot g = Lg$ for all $L \in J/\mathscr{L}$ and $g \in G$, where $x \in L$ is arbitrary, gives a well-defined right action of G on $J/\mathscr{L}$. In other words, G acts on the $\mathscr{L}$-classes of J by right multiplication.

The strongly connected components of the digraph in (4.76) are simply the orbits of G under its right action on $J/\mathscr{L}$. Therefore, the digraph $\Gamma_{\mathscr{L}}(S, J)$ has no edges, and its vertices are the orbits of the right action of G on $J/\mathscr{L}$ by right multiplication. Analogously, the vertices of the digraph $\Gamma_{\mathscr{R}}(S, J)$ are the orbits of the left action of G on $J/\mathscr{R}$ by left multiplication, and the digraph has no edges. As before, the vertices of $\Delta(S, J)$ and $\Theta(S, J)$ are the disjoint union of the vertices of $\Gamma_{\mathscr{L}}(S, J)$ and $\Gamma_{\mathscr{R}}(S, J)$. Since $\langle X' \rangle \cap J = G \cap J = \varnothing$, it follows that $\Theta(S, J)$ has no edges, and that every vertex in $\Gamma_{\mathscr{L}}(S, J)$ and $\Gamma_{\mathscr{R}}(S, J)$ has colour 0.

The results in Section 4.4 concerning the existence of maximal subsemigroups of types (M2)–(M4) are formulated in terms of the vertices and edges of the graphs and digraphs mentioned above. In the present case, when J is a regular $\mathscr{J}$-class of a finite monoid covered by the units, the conditions that concern the edges or the colour of vertices in $\Gamma_{\mathscr{L}}(S, J)$, $\Gamma_{\mathscr{R}}(S, J)$, or $\Theta(S, J)$ are vacuously satisfied; the remaining conditions on these graphs and digraphs can be reformulated in terms of $\Delta(S, J)$.

We begin by reformulating Proposition 4.59, which concerns maximal subsemigroups of type (M2). This reformulation follows from the statement of Proposition 4.59, since the conditions in Proposition 4.58(ii)–(iv) concern only the edges of $\Gamma_{\mathscr{L}}(S, J)$, $\Gamma_{\mathscr{R}}(S, J)$, and $\Theta(S, J)$, and can therefore be ignored.

Proposition 4.77 (Maximal subsemigroups of type (M2))**.** *Let S be a finite monoid, let J be a regular $\mathscr{J}$-class of S covered by the group of units, and let T be a subset of S that contains $S \setminus J$. Suppose there exist proper non-empty subsets $A \subsetneq J/\mathscr{L}$ and $B \subsetneq J/\mathscr{R}$ such that $T \cap J$ is the union of the Green's classes in A and B. Then T is a maximal subsemigroup of S if and only if A and B are unions of vertices that form a maximal independent subset of $\Delta(S, J)$.*

By Proposition 4.77, the maximal subsemigroups of S of type (M2) arising from J are in one-to-one correspondence with the maximal independent subsets of $\Delta(S, J)$, excluding the set of all orbits of $\mathscr{L}$-classes, and the set of all orbits of $\mathscr{R}$-classes. We deduce the following.

Corollary 4.78. *The number of maximal subsemigroups of S of type (M2) arising from J is two less than the number of maximal independent subsets of $\Delta(S, J)$.*

We reformulate Proposition 4.63 as follows.

Proposition 4.79 (Maximal subsemigroups of type (M3))**.** *Let S be a finite monoid, let J be a regular $\mathscr{J}$-class of S covered by the group of units of S, and let T be a subset of S such that* *contains $S \setminus J$. Suppose there exists a proper non-empty subset $A \subsetneq J/\mathscr{L}$ such that $T \cap J$ is the union of the $\mathscr{L}$-classes in A. Then T is a maximal subsemigroup of S if and only if $(J/\mathscr{L}) \setminus A$ is a vertex in $\Delta(S,\, J)$ that is not adjacent to a vertex of degree one.*

Proof. ($\Rightarrow$) By Proposition 4.63, $(J/\mathscr{L}) \setminus A$ is a vertex in $\Delta(S,\, J)$. Suppose that $(J/\mathscr{L}) \setminus A$ is adjacent to a vertex $B' \subseteq J/\mathscr{R}$ of degree one. The vertices in $A \cup B'$ form an independent subset of $\Delta(S,\, J)$. This independent subset is contained in a *maximal* independent subset corresponding to $A \cup B$, for some non-empty proper subset $B \subsetneq J/\mathscr{R}$ that contains B'. By Proposition 4.77, a maximal subsemigroup of S corresponding to the maximal independent subset $A \cup B$ properly contains T. Therefore no such vertex B' exists.

($\Leftarrow$) Every vertex in $\Gamma_{\mathscr{L}}(S,\, J)$ is a source with colour 0, and so by Proposition 4.63 it suffices to show that no maximal subsemigroup of type (M2) contains T. Equivalently, by Proposition 4.77, if suffices to show that there exists no non-empty proper subset $B \subsetneq J/\mathscr{R}$ such that $A \cup B$ forms a maximal independent subset of $\Delta(S,\, J)$. Let $B' \subseteq J/\mathscr{R}$ be a vertex of $\Delta(S,\, J)$. Since J is regular, B' is adjacent to some orbit of $\mathscr{L}$-classes. If B' is adjacent to $(J/\mathscr{L}) \setminus A$, then by assumption, B' is also adjacent to some vertex in A. In any case, B' is adjacent to some vertex in A, and so $A \cup B'$ does not form an independent subset of $\Delta(S,\, J)$. In particular, there exists no subset $B \subsetneq J/\mathscr{R}$ such that $A \cup B$ forms a maximal independent subset of $\Delta(S,\, J)$. □

There is a natural dual to this proposition, which characterises the maximal subsemigroups of S that arise from J and that have type (M4) in terms of the orbits of $\mathscr{R}$-classes in $\Delta(S,\, J)$ that are not adjacent to vertices of degree one.

By the regularity of J, every vertex in $\Delta(S,\, J)$ has degree at least one. Therefore, by Proposition 4.79, the number of maximal subsemigroups of S of type (M3) is the number of orbits of $\mathscr{L}$-classes that are adjacent in $\Delta(S,\, J)$ only to orbits of $\mathscr{R}$-classes with degree at least two. In the case that every orbit of $\mathscr{R}$-classes has degree two or more in $\Delta(S,\, J)$, then the number of maximal subsemigroups of type (M3) is simply the number of orbits of $\mathscr{L}$-classes. The analogous statements hold for maximal subsemigroups of type (M4). By the same token, the existence of maximal subsemigroups is restricted when there is a single orbit of $\mathscr{L}$-classes or a single orbit of $\mathscr{R}$-classes (i.e. when the group acts transitively).

Lemma 4.80. *If G acts transitively on the $\mathscr{L}$-classes of J, then no maximal subsemigroups of types (M2) or (M3) arise from J. Similarly, if G acts transitively on the $\mathscr{R}$-classes of J, then no maximal subsemigroups of types (M2) or (M4) arise from J.*

Proof. Suppose that G acts transitively on the $\mathscr{L}$-classes of J. Since J is regular, for every $\mathscr{L}$-class L of J there exists an $\mathscr{R}$-class R of J such that $L \cap R$ is a group, and vice versa. Therefore there are no isolated vertices in $\Delta(S,\, J)$, and so each vertex of $\mathscr{R}$-classes is adjacent only to the unique vertex of $\mathscr{L}$-classes. It follows that the set of all $\mathscr{L}$-classes and the set of all $\mathscr{R}$-classes are the only maximal independent subsets of $\Delta(S,\, J)$. By Corollary 4.78, there are no maximal subsemigroups of type (M2) arising from J, and by Proposition 4.79, there are no maximal subsemigroups of type (M3). The proof of the second statement is dual. □

When S is a regular $*$-monoid, the $\mathscr{L}$-classes and $\mathscr{R}$-classes of a $\mathscr{J}$-class are in bijective correspondence via the $*$ anti-isomorphism, and so the graph $\Delta(S,\, J)$ is particularly easy to describe in this case.

Lemma 4.81. *Let S be a regular $*$-monoid and let J be a $\mathscr{J}$-class covered by the group of units G of S. A collection of $\mathscr{L}$-classes $\{L_{x_1},\, \ldots,\, L_{x_n}\}$ is a vertex of $\Delta(S,\, J)$ if and only if the collection of $\mathscr{R}$-classes $\{R_{x_1^*},\, \ldots,\, R_{x_n^*}\}$ is a vertex of $\Delta(S,\, J)$, and any pair of such vertices is adjacent in $\Delta(S,\, J)$.*

Proof. If L_x and L_y are $\mathscr{L}$-classes of J in the same vertex of $\Delta(S, J)$, then there exists $g \in G$ such that $L_x g = L_y$. Therefore

$$g^* R_{x^*} = g^* L_x^* = (L_x g)^* = L_y^* = R_{y^*},$$

and so R_{x^*} and R_{y^*} belong to the same vertex of $\Delta(S, J)$. By an analogous argument, it follows that the first statement holds. The second statement holds since, for any element $x \in J$, the $\mathscr{H}$-class $L_x \cap R_{x^*}$ contains the projection x^*x, and is therefore a group. In particular, the vertex of $\Delta(S, J)$ containing L_x is adjacent to the vertex of $\Delta(S, J)$ that contains R_{x^*}. ☐

The situation is further simplified when every idempotent of J is a projection, which occurs, for instance, when S is inverse.

Corollary 4.82. *Let S be a finite regular $*$-monoid with group of units G, and let J be a $\mathscr{J}$-class of S that is covered by G and whose only idempotents are projections. Suppose that $\{O_1, \ldots, O_n\}$ are the orbits of the right action of G on the $\mathscr{L}$-classes of J by right multiplication Then the maximal subsemigroups of S arising from J have types (M1), (M2), or (M5). A maximal subsemigroup of type (M2) is the union of $S \setminus J$ and the union of the Green's classes*

$$\{L : L \in O_i,\ i \in A\} \cup \{L^* : L \in O_i,\ i \notin A\},$$

where A is any proper non-empty subset of $\{1, \ldots, n\}$. In particular, there are $2^n - 2$ maximal subsemigroups of type (M2), and no maximal subsemigroups of types (M3) or (M4).

Proof. By definition of $\Delta(S, J)$, the vertices of $\mathscr{L}$-classes of $\Delta(S, J)$ are $\{O_1, \ldots, O_n\}$, and so by Lemma 4.81, the vertices of $\mathscr{R}$-classes are $\{\{L^* : L \in O_i\} : i \in \{1, \ldots, n\}\}$. Since every idempotent of J is a projection, for each $\mathscr{L}$-class L_x of J, the only group $\mathscr{H}$-class contained in L_x is $L_x \cap R_{x^*}$, and so the vertex containing L_x is only adjacent to the vertex containing R_{x^*}. Therefore the edges of $\Delta(S, J)$ are $\{O_i,\ \{L^* : L \in O_i\}\}$ for each $i \in \{1, \ldots, n\}$. In particular, each vertex of $\Delta(S, J)$ has degree one, and it follows from Proposition 4.79 and its dual that no maximal subsemigroups of types (M3) or (M4) arise from J. Furthermore, given the description of $\Delta(S, J)$, it is clear that a maximal independent subset of $\Delta(S, J)$ is formed by choosing any one vertex from each of the n edges, and so there are 2^n maximal independent subsets. The description and number of maximal subsemigroups of type (M2) follows by Proposition 4.77 and Corollary 4.78. ☐

Maximal subsemigroups of type (M1)

A few of the monoids that we consider in Chapter 5 exhibit maximal subsemigroups of type (M1) that arise from a regular $\mathscr{J}$-class covered by the group of units. The results in Section 4.4.1 neatly characterise the existence of maximal subsemigroups of type (M1) in terms of certain maximal subsemigroups of the relevant principal factor. However, while this is very useful in general, especially from the point of view of computation, we desire more specialised tools to easily describe the maximal subsemigroups that arise in the particular examples included in Chapter 5. In Proposition 4.85, we present a result that will be useful for these cases. To prove this proposition, we require the following definition.

Definition 4.83. Let S be a finite regular $*$-monoid with group of units G, and let $A \subseteq S$. Then the *setwise stabilizer of A in G*, $\mathrm{Stab}_G(A)$, is the subgroup $\{g \in G : Ag = A\}$ of G.

Let S and G be defined as in Definition 4.83, and let $A \subseteq S$. Note that $\mathrm{Stab}_G(A)$ is defined to be the set of elements of G that stabilize A on the *right*. However, with $A^* = \{a^* : a \in A\}$, the set of elements of G that stabilize A on the *left* is equal to $\mathrm{Stab}_G(A^*)$, since

$$\{g \in G : gA = A\} = \{g \in G : A^* g^* = A^*\} = \mathrm{Stab}_G(A^*)^* = \mathrm{Stab}_G(A^*)^{-1} = \mathrm{Stab}_G(A^*).$$

Thus, for a subset A of S that satisfies $A^* = A$, such as for the $\mathscr{H}$-class of a projection,

$$\mathrm{Stab}_G(A) = \{g \in G \,:\, Ag = A = gA\}.$$

In Proposition 4.85, we require the set $e\mathrm{Stab}_G(H_e) = \big\{es \,:\, s \in \mathrm{Stab}_G(H_e)\big\}$, where e is a projection of the regular $*$-monoid S, and the $\mathscr{J}$-class J_e is covered by G. Any submonoid of S that contains both e and G also contains $e\mathrm{Stab}_G(H_e)$. In particular, every maximal subsemigroup of type (M1) arising from J_e contains G and all idempotents in J_e by Lemma 1.14, and hence contains $e\mathrm{Stab}_G(H_e)$. A stronger result, necessary for the proof of Proposition 4.85, is given by the following lemma.

Lemma 4.84. *Let S be a finite monoid with group of units G, let $e \in E(S)$, and let T be a submonoid of S that contains both e and G. Then the set $e\mathrm{Stab}_G(H_e^S)$ is a subgroup of H_e^T.*

Proof. Since e is an idempotent, $H_e^T = T \cap H_e^S$. Clearly $e\mathrm{Stab}_G(H_e^S) \subseteq eG \subseteq T$. Let $g \in \mathrm{Stab}_G(H_e^S)$. Then $eg \in H_e^S$ by definition, and so $e\mathrm{Stab}_G(H_e^S) \subseteq H_e^S$. Thus $e\mathrm{Stab}_G(H_e^S) \subseteq T \cap H_e^S = H_e^T$, and the subset is non-empty since $e = e1_S \in e\mathrm{Stab}_G(H_e^S)$. Since S is finite, it remains to show that $e\mathrm{Stab}_G(H_e^S)$ is closed under multiplication. Let $g, g' \in \mathrm{Stab}_G(H_e^S)$. Since $eg \in H_e^S$ and e is the identity of H_e^S, it follows that $(eg)e = eg$. Thus

$$(eg)(eg') = (ege)g' = (eg)g' = e(gg') \in e\mathrm{Stab}_G(H_e^S). \qquad \square$$

We use these notions and results to state and prove the following proposition.

Proposition 4.85. *Let S be a finite regular $*$-monoid with group of units G, let J be a $\mathscr{J}$-class of S that is covered by G, and let H_e^S be the $\mathscr{H}$-class of a projection $e \in J$. Suppose that G acts transitively on the $\mathscr{R}$-classes or the $\mathscr{L}$-classes of J, and that J contains one idempotent per $\mathscr{L}$-class and one idempotent per $\mathscr{R}$-class (i.e. every idempotent of J is a projection). Then the maximal subsemigroups of S arising from J are either:*

(a) $(S \setminus J) \cup GUG = \langle S \setminus J,\, U\rangle$, *for each maximal subgroup U of H_e^S that contains* $e\mathrm{Stab}_G(H_e^S)$ (type (M1)), *or*

(b) $S \setminus J$, *if no maximal subsemigroups of type* (M1) *exist* (type (M5)).

Proof. Since S is a regular $*$-monoid, G acts transitively on the $\mathscr{L}$-classes of J if and only if G acts transitively on the $\mathscr{R}$-classes of J. Hence there are no maximal subsemigroups of types (M2), (M3), or (M4) arising from J, by Lemma 4.80. By Corollary 4.75, it remains to describe the maximal subsemigroups of type (M1).

Let U be a maximal subgroup of H_e^S that contains $e\mathrm{Stab}_G(H_e^S)$, and define $M_U = (S \setminus J) \cup GUG$. To prove that M_U is a maximal subsemigroup of S, we first show that M_U is a proper subset of S, then that it is a subsemigroup, and finally that it is maximal in S. Since G acts transitively on the $\mathscr{L}$- and $\mathscr{R}$-classes of J and M_U contains $S \setminus J$, it follows that the set M_U intersects every $\mathscr{H}$-class of S non-trivially. Once we have shown that M_U is a subsemigroup, it will obviously follows that M_U is generated by $(S \setminus J) \cup U$, since $G \subseteq S \setminus J$.

To prove that M_U is a proper subset of S, it suffices to show that $GUG \cap H_e^S \subseteq U$. Let $x \in GUG \cap H_e^S$. Since $x \in GUG$, we may write $x = \alpha u \beta$ for some $\alpha, \beta \in G$ and $u \in U$. Since $u, \alpha u \beta \in H_e^S$, it is straightforward to show that $\alpha u,\, u\beta \in H_e^S$. Thus

$$\alpha H_e^S = \alpha(uH_e^S) = (\alpha u)H_e^S = H_e^S, \quad \text{ and } \quad H_e^S\beta = (H_e^S u)\beta = H_e^S(u\beta) = H_e^S.$$

In other words, α and β stabilize H_e^S on the left and right, respectively. Thus $\alpha, \beta \in \mathrm{Stab}_G(H_e^S)$, and

$$x = ex = e\alpha u\beta = e\alpha u e\beta \in \big(e\mathrm{Stab}_G(H_e^S)\big)U\big(e\mathrm{Stab}_G(H_e^S)\big) \subseteq U^3 = U.$$

In order to show that M_U is a subsemigroup, it suffices to show that $xy \in M_U$ whenever $x, y \in G \cup GUG$, because $S \setminus (G \cup J)$ is an ideal of S. If $x \in G$ and $y \in G$, then certainly $xy \in G$. If $x \in G$ and $y \in GUG$, then $xy \in G^2UG = GUG$ and $yx \in GUG^2 = GUG$. For the final case, assume that $x, y \in GUG$ and that $xy \in J$. By definition, $x = \alpha u \beta$ and $y = \sigma v \tau$ for some $\alpha, \beta, \sigma, \tau \in G$ and $u, v \in U$. It suffices to show that $\beta\sigma \in \mathrm{Stab}_G(H_e^S)$, because then

$$\begin{aligned} xy = (\alpha u \beta)(\sigma v \tau) = \alpha(ue)\beta\sigma v\tau = \alpha u(e\beta\sigma)v\tau &\in GU\left(e\mathrm{Stab}_G(H_e^S)\right)UG \\ &\subseteq GU^3G = GUG. \end{aligned}$$

Since H_e^S is a group containing u and v, it follows that $u^*u = vv^* = e$. Thus

$$e\beta\sigma e = u^*u\beta\sigma vv^* = u^*\alpha^{-1}(\alpha u\beta\sigma v\tau)\tau^{-1}v^* = u^*\alpha^{-1}(xy)\tau^{-1}v^*.$$

Together with $xy = \alpha u(e\beta\sigma e)v\tau$, it follows that $e\beta\sigma e \in J$. By Lemma 1.10, $e\beta\sigma e \in R_e^S$. Since the elements $e\beta\sigma$ and e, and their product $e\beta\sigma e$, are all contained in R_e^S, Corollary 1.16 implies that $H_{e\beta\sigma}^S$ is a group. By assumption, R_e^S contains only one group $\mathscr{H}$-class, which is H_e^S. Thus $e\beta\sigma \in H_e^S$, and so $H_e^S\beta\sigma = (H_e^Se)\beta\sigma = H_e^S(e\beta\sigma) = H_e^S$, i.e. $\beta\sigma \in \mathrm{Stab}_G(H_e^S)$, as required.

Let M be a maximal subsemigroup of S that contains M_U. By Proposition 4.10, $M \cap H_e^S$ is a maximal subgroup of H_e^S, and the intersection of M with any $\mathscr{H}$-class of J contains exactly $|M \cap H_e^S|$ elements. Since $M \cap H_e^S$ contains U, the maximality of U in H_e^S implies that $U = M \cap H_e^S$. Since the group G acts transitively on the $\mathscr{L}$- and $\mathscr{R}$-classes of J, the intersection of GUG with any $\mathscr{H}$-class of J contains $|U|$ elements. Thus $|M| \leq |M_U|$, and $M = M_U$.

Conversely, suppose that M is a maximal subsemigroup of S of type (M1) arising from J. By Proposition 4.10, $U = M \cap H_e^S = H_e^M$ is a maximal subgroup of H_e^S, and U contains $e\mathrm{Stab}_G(H_e^S)$ by Lemma 4.84. Since M contains G, U, and $S \setminus J$, it contains the maximal subsemigroup $M_U = (S \setminus J) \cup GUG$. But M is a proper subsemigroup, and so $M = M_U$. $\square$

4.6 The algorithm

In this section, we describe the overall algorithm for computing the maximal subsemigroups of any given finite semigroup S represented by a generating set X. This algorithm is composed of the procedures described in Sections 4.3 and 4.4 and is fully implemented in the Semigroups package [101] for GAP [58]; the underlying algorithms for graphs and digraphs are implemented in the Digraphs [10] package for GAP. Pseudocode is given in Algorithm 4.86. We describe the corresponding functionality of the Semigroups package in Section 4.6.1, and we analyse the performance of these functions on a range of examples in Section 4.6.2.

We first compute the $\mathscr{J}$-classes of S that contain an element of X. If a particular $\mathscr{J}$-class J has empty intersection with X, then by Lemma 4.68(iv), there is no maximal subsemigroup of S arising from J. A $\mathscr{J}$-class of S that intersects X non-trivially does not necessarily give rise to maximal subsemigroups. However, as discussed after the proof of Lemma 4.68, if the generating set X is minimal, with respect to containment, then a maximal subsemigroup arises from a $\mathscr{J}$-class if and only if that $\mathscr{J}$-class has non-empty intersection with X.

Maximal and non-maximal $\mathscr{J}$-classes are treated separately.

Suppose that J is a maximal $\mathscr{J}$-class. Note that every maximal $\mathscr{J}$-class J intersects X non-trivially, since the subset $S \setminus J$ is an ideal of S. If $|J| = 1$, then by Lemma 4.3, the only maximal subsemigroup to arise from J is $S \setminus J$, which has type (M5). Suppose that $|J| > 1$. If follows that J is necessarily regular, and so the principal factor $J^* = J \cup \{0\}$ is a 0-simple semigroup. We may therefore compute a normalization Ψ from J^* to a regular Rees 0-matrix semigroup over a group $\mathscr{M}^0[G;\, I,\, \Lambda;\, P]$; see Section 3.2.1. Since $S \setminus J$ is an ideal, it is straightforward to see that a maximal subsemigroup of S arising from J has the form $(S \setminus J) \cup U$, where $U \subseteq J$

Algorithm 4.86 Maximal subsemigroups of a finite semigroup defined by generating set.

Input: $S = \langle X \rangle$, a finite semigroup with generating set X.
Output: $\mathfrak{M}$, the maximal subsemigroups of S.

$\mathfrak{M} \leftarrow \varnothing$
if $S = \varnothing$ **then**
  return $\mathfrak{M}$.
for $J_x \in \{J \in S/\mathscr{J} \,:\, J \cap X \neq \varnothing\}$ **do**
  $X' \leftarrow \{y \in X \,:\, J_y > J_x\}$
  if $X' = \varnothing$ **then** ▷ J_x is a maximal $\mathscr{J}$-class of S
    if $|J_x| = 1$ **then**
      $\mathfrak{M} \leftarrow \mathfrak{M} \cup \{S \setminus \{x\}\}$ ▷ Lemma 4.3
    else ▷ J_x is necessarily regular
      compute a normalization $\Psi : J_x^* \longrightarrow \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ ▷ Section 3.2.1
      $\mathfrak{X} \leftarrow$ maximal subsemigroups of $(J_x^*)\Psi$ of types (R3)–(R6) ▷ Sections 4.3.3–4.3.4
      for $M \in \mathfrak{X}$ **do**
        $\mathfrak{M} \leftarrow \mathfrak{M} \cup \{(S \setminus J_x) \cup (M \setminus \{0\})\Psi^{-1}\}$
  else ▷ J_x is a non-maximal $\mathscr{J}$-class
    if J_x is non-regular and $x \notin \langle X' \rangle$ **then**
      $\mathfrak{M} \leftarrow \mathfrak{M} \cup \{S \setminus J_x\}$ ▷ Type (M5), Proposition 4.69(iii)
    else if J_x is regular and $J_x \cap X \not\subseteq \langle X' \rangle$ **then** ▷ Lemma 4.68(v)
      compute a normalization $\Psi : J_x^* \longrightarrow \mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ ▷ Section 3.2.1
      $E \leftarrow \{$one idempotent from each $\mathscr{L}$-class of $J_x\}$
      $\mathfrak{X} \leftarrow$ maximal subsemigroups of $(J_x^*)\Psi$ of type (R6) containing $(EX')\Psi$ ▷ Alg. 4.44
      for $M \in \mathfrak{X}$ **do**
        $\mathfrak{M} \leftarrow \mathfrak{M} \cup \{(S \setminus J_x) \cup (M \setminus \{0\})\Psi^{-1}\}$ ▷ Type (M1), Proposition 4.48
      compute $\Gamma_{\mathscr{L}}(S,\, J_x)$, $\Gamma_{\mathscr{R}}(S,\, J_x)$, $\Delta(S,\, J_x)$, and $\Theta(S,\, J_x)$ ▷ Section 4.4.2
      $\mathfrak{M} \leftarrow \mathfrak{M}\ \cup$ maximal subsemigroups of type (M2) ▷ Section 4.4.3
      $\mathfrak{M} \leftarrow \mathfrak{M}\ \cup$ maximal subsemigroups of types (M3) and (M4) ▷ Section 4.4.4
      if no maximal subsemigroups of types (M1)–(M4) arise from J_x **then**
        $\mathfrak{M} \leftarrow \mathfrak{M} \cup \{S \setminus J_x\}$ ▷ Type (M5), Proposition 4.69(ii)
return $\mathfrak{M}$.

and $U \cup \{0\}$ is a maximal subsemigroup of J^*. In other words, M is a maximal subsemigroup of S arising from J if and only if $(M \cap J)\Psi \cup \{0\}$ is a maximal subsemigroup of $\mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ of type (R3), (R4), (R5), or (R6). Therefore, it suffices to compute the maximal subsemigroups of $\mathscr{M}^0[G;\, I,\, \Lambda;\, P]$ of these kinds, using the techniques of Section 4.3.

Suppose that J is a non-maximal $\mathscr{J}$-class of S that intersects X non-trivially. We first compute the set $X' = \{x \in X \,:\, J < J_x\}$. If J is non-regular, then by Proposition 4.69(iii), $S \setminus J$ is a maximal subsemigroup of type (M5) if and only if $x \notin \langle X' \rangle$, where x is an arbitrary element of J. No other kinds of maximal subsemigroup arise from a non-regular $\mathscr{J}$-class. If J is regular, then by Lemma 4.68, maximal subsemigroups of S arise from J if and only if $J \cap X \not\subseteq \langle X' \rangle$. If this condition holds, then we compute the maximal subsemigroups of types (M1)–(M4), as discussed in Sections 4.4.1, 4.4.3, and 4.4.4. If no such maximal subsemigroups exist, then by Proposition 4.69(ii), $S \setminus J$ is a maximal subsemigroup of S of type (M5).

Open Problem 4.87. Develop tools for computing subsemigroups that are maximal with respect to some property, such as maximal commutative or maximal regular subsemigroups of a finite semigroup; or maximal inverse subsemigroups of a finite inverse semigroup.

4.6.1 Relevant functionality in the Semigroups package for GAP

The techniques described in this chapter for computing maximal subsemigroups of a finite semigroup have been implemented by the author in the Semigroups package [101] for GAP [58]. The functions in the Semigroups package can be applied to any non-empty finite semigroup that can be constructed in GAP, subject to the time and space constraints of the computer. There are technical difficulties in GAP that arise when computing with an empty semigroup. For this reason, the Semigroups package finds only *non-empty* maximal subsemigroups. Note that only a semigroup of order 1 has the empty semigroup as a maximal subsemigroup.

Let S to be a non-empty finite semigroup. The command `MaximalSubsemigroups(S);` returns a list of the non-empty maximal subsemigroups of S. The behaviour of the function `MaximalSubsemigroups` may be customised by providing, as an additional argument, a record `opts` that describes these customisations via its components. The following independent customisations are available, and may be used together in any combination:

- If `opts.number` is `true`, then `MaximalSubsemigroups(S, opts);` returns the number of maximal subsemigroups of S that it finds, rather than constructing the maximal subsemigroups. It can be computationally expensive to find a generating set for a maximal subsemigroup (see Propositions 4.61 and 4.66). Therefore, in many cases, it can be much quicker to count the maximal subsemigroups of S directly than it is to construct them.

  Note that the command `NrMaximalSubsemigroups(S);` is equivalent to the command `MaximalSubsemigroups(S, rec(number := true));`

- If `opts.contain` is a subset of S, then the command `MaximalSubsemigroups(S, opts);` returns the maximal subsemigroups of S that contain this set of elements. Note that this is often much more efficient that computing all maximal subsemigroups of S and then retaining only those that contain the desired set. In particular, a version of Algorithm 4.44 is used in this instance when S is a regular Rees 0-matrix semigroup over a group.

- If `opts.D` is a $\mathscr{D}$-class of S (or, equivalently, a $\mathscr{J}$-class of S, since S is finite), then `MaximalSubsemigroups(S, opts);` finds the maximal subsemigroups of S that arise from D. Since Algorithm 4.86 independently searches for the maximal subsemigroups arising from each $\mathscr{D}$-class, the maximal subsemigroups that arise from D can be computed without finding maximal subsemigroups that arise from any other $\mathscr{D}$-class.

- If S is a regular Rees matrix or Rees 0-matrix semigroup over a group, and `opts.types` is a subset of `[1 .. 6]`, then `MaximalSubsemigroups(S, opts);` finds only those maximal subsemigroups of S of the types amongst (R1)–(R6) that correspond to `opts.types`. For instance, if `opts.contain` is `[1, 3, 4, 6]`, then `MaximalSubsemigroups(S, opts);` finds the maximal subsemigroups of S that have types types (R1), (R3), (R4), or (R6).

The following GAP code demonstrates how to perform a custom maximal subsemigroups computation with the Semigroups package. This code produces the answer 24, which is the number of maximal subsemigroups of S that arise from D_x and contain the set $\{x, y\}$.

```
S := SingularTransformationMonoid(5);
x := Transformation([2, 2, 3, 4, 5]);
y := Transformation([5, 4, 1, 2, 1]);
opts := rec(number := true, contain := [x, y], D := DClass(S, x));
MaximalSubsemigroups(S, opts);
```

Further details and examples are included in the manual of the Semigroups [101] package.

4.6.2 Performance measurements

In this section, we experimentally investigate the performance of the algorithms presented in this chapter. The computations described in this section were run using the Semigroups package [101] for GAP [58], on a 2.66 GHz Intel Core i7 processor with 8GB of RAM.

Given a semigroup S represented by a set of generators, we compare the time taken to compute the Green's structure of S with that taken to find the maximal subsemigroups of S, given the Green's structure. As argued previously, if it is not possible to compute the Green's structure of a given semigroup within the limitations of the hardware, then there is little of significance that can be computed about that semigroup. Therefore, this comparison is appropriate. Additionally, we include the length of time that was spent finding maximal cliques or maximal subgroups of group $\mathscr{H}$-classes during the computation of the maximal subsemigroups, when these times are not negligible. We analyse the performance of computing the maximal subsemigroups of the following semigroups.

- For each $n \in \{2, \ldots, 11\}$, we compute the maximal subsemigroups of the full transformation monoid $\mathcal{T}_n$, as defined in Section 1.3.5. There are n^n elements and n $\mathscr{J}$-classes in $\mathcal{T}_n$, and $\operatorname{rank}(\mathcal{T}_n) = 3$ when $n > 2$. See Section 5.2.3 for a description of the maximal subsemigroups of $\mathcal{T}_n$ in general, and see Table 5.8 for their number.

- For each $n \in \{11, \ldots, 20\}$, we calculate the maximal subsemigroups of the inverse monoid $\mathcal{PORI}_n$, which consists of all partial permutations of degree n that preserve or reverse the cyclic orientation of $\{1, \ldots, n\}$; see Section 5.2.1 for a precise definition. There are
$$1 + n\binom{2n}{n} - \frac{n^2(n^2 - 2n + 3)}{2}$$
elements and $n + 1$ $\mathscr{J}$-classes in $\mathcal{PORI}_n$, and $\operatorname{rank}(\mathcal{PORI}_n) = 3$ when $n > 2$. See Section 5.2.8 and Table 5.37 for the maximal subsemigroups of $\mathcal{PORI}_n$.

- We compute the maximal subsemigroups of the *Jones monoid* $\mathcal{J}_n$, as defined in Section 5.3.1, for each $n \in \{6, \ldots, 20\}$. If $n > 1$, then $\operatorname{rank}(\mathcal{J}_n) = n - 1$, and the order of $\mathcal{J}_n$ is the n^{th} Catalan number [43, Section 9], $\frac{1}{n+1}\binom{2n}{n}$. The number of $\mathscr{J}$-classes in $\mathcal{J}_n$ is $\lceil n/2 \rceil$. See Section 5.3.7 and Table 5.59 for the maximal subsemigroups of $\mathcal{J}_n$, in general.

- We compute the maximal subsemigroups of 100 transformation semigroups of degree 9, each generated by 9 transformations chosen uniformly at random from $\mathcal{T}_9$. See Table 4.88 for some data about these semigroups.

We chose to analyse the performance of the algorithms with $\mathcal{T}_n$, $\mathcal{PORI}_n$, and $\mathcal{J}_n$ because these monoids are well-studied in the literature, because their internal representations in the Semigroups package are different, and because the methods implemented in the Semigroups package exhibit different behaviour for these monoids. In Figures 4.89, 4.90, and 4.91, we compare the time taken to compute the partial order of the $\mathscr{J}$-classes of these monoids with the time taken to find the maximal subsemigroups, given the partial order of $\mathscr{J}$-classes. The partial order of $\mathscr{J}$-classes of a semigroup is a fundamental part of its Green's structure, and was computed with the method described in [37, Algorithm 14], which is implemented in the Semigroups package. In each case, the largest value of n considered was the largest for which the Green's structure could be computed within the limitations of the given hardware.[1]

For $S \in \{\mathcal{T}_n, \mathcal{PORI}_n\}$, the time taken to compute the maximal subsemigroups of S is dominated neither by the time taken to compute maximal cliques, nor by the time taken

[1] Note that $|\mathcal{T}_{11}| = 285\ 311\ 670\ 611$, $|\mathcal{PORI}_{20}| = 2\ 756\ 930\ 503\ 801$, and $|\mathcal{J}_{20}| = 6\ 564\ 120\ 420$.

to compute maximal subgroups; rather, the majority of the time is spent constructing and processing the digraphs $\Gamma_{\mathscr{L}}(S, J)$ and $\Gamma_{\mathscr{R}}(S, J)$ and the graphs $\Delta(S, J)$ and $\Theta(S, J)$ from Section 4.4, where J is the $\mathscr{J}$-class of S consisting of elements of rank $n-1$. For these monoids, the graphs $\Delta(S, J)$ have 2 vertices, and so it can be immediately deduced that there are no maximal cliques from which maximal subsemigroups may arise. On the other hand, for the Jones monoids, the majority of the time in the computation of maximal subsemigroups is spent searching for maximal cliques, as is shown in Figure 4.91.

The monoids $\mathcal{T}_n$, $\mathcal{PORI}_n$, and $\mathcal{J}_n$ are atypical examples of semigroups: for instance, they are all regular monoids, and each of their ideals is principal. Because of this, we opted to include some data relating to 'random' semigroups in order to demonstrate what is perhaps more typical behaviour of the algorithms for computing maximal subsemigroups. It is not clear what constitutes a reasonable notion of randomness, however, we believe that the notions used here are somewhat meaningful for the discussion here. The random semigroups were chosen to be generated by nine transformations of degree nine, because semigroups of this kind approach the limit of what is possible to compute with the given hardware.

	Min	Max	Mean	Median	Standard deviation
Size	125333	85014449	5657333	2484319	9865603
Number of $\mathscr{J}$-classes	8	89858	5336	2550	10187
Number of maximal subsemigroups	9	640	25	13	65
Time for maximal subsemigroups (ms)	8	272132	15998	5003	37944

Table 4.88: Information about the 100 transformation semigroups of degree 9 that we consider, each of which was generated by by 9 transformations that were chosen uniformly at random.

Figure 4.92 compares the time taken to compute the maximal subsemigroups of the 100 random transformation semigroups described above with the time taken to compute their partial orders of $\mathscr{J}$-classes. Each point on the horizontal axis of this graph corresponds to one of these semigroups. The points are sorted along the horizontal axis in increasing order, according to the ratio of the time taken to compute the maximal subsemigroups to the time taken to compute the partial order of its $\mathscr{J}$-classes. While there are some instances where computing the maximal subsemigroups is several orders of magnitude slower than computing the Green's structure, for the majority, the times taken are roughly comparable.

For each $n \in \{1, \ldots, 20\}$, we also analysed the performance of computing the maximal subsemigroups of 100 regular Rees 0-matrix semigroups $\mathscr{M}^0[G; I, \Lambda; P]$, where in each instance, $|I| = |\Lambda| = n$, G was a permutation group chosen uniformly at random from the representatives of conjugacy classes of subgroups of $\mathcal{S}_{10}$, the number of connected components of the Graham-Houghton graph was chosen uniformly from $\{1, \ldots, n\}$, and the entries of the matrix P were chosen randomly, subject to this constraint. The entries of P were not chosen uniformly, nor do we claim that these Rees 0-matrix semigroups represent a uniform sample of such semigroups. The algorithms presented in Section 4.3, as implemented in the Semigroups [101] package, were used to compute these maximal subsemigroups.

The Green's structure of a regular Rees 0-matrix semigroup over a group can be determined immediately from its definition, and so the time taken to determine this was not used for comparison. In Figure 4.93, for each dimension considered, the mean time taken to compute the maximal subsemigroups of the 100 semigroups is shown alongside the mean time spent computing maximal cliques as part of the calculation. For the Rees 0-matrix semigroups we considered, it appears that the time taken to compute maximal cliques approaches the time taken to compute maximal subsemigroups, as the dimension increases; maximal cliques correspond to maximal subsemigroups of type (R5). The mean time spent computing maximal subgroups, which is a step in the computation of maximal subsemigroups of type (M1) was negligible in these examples.

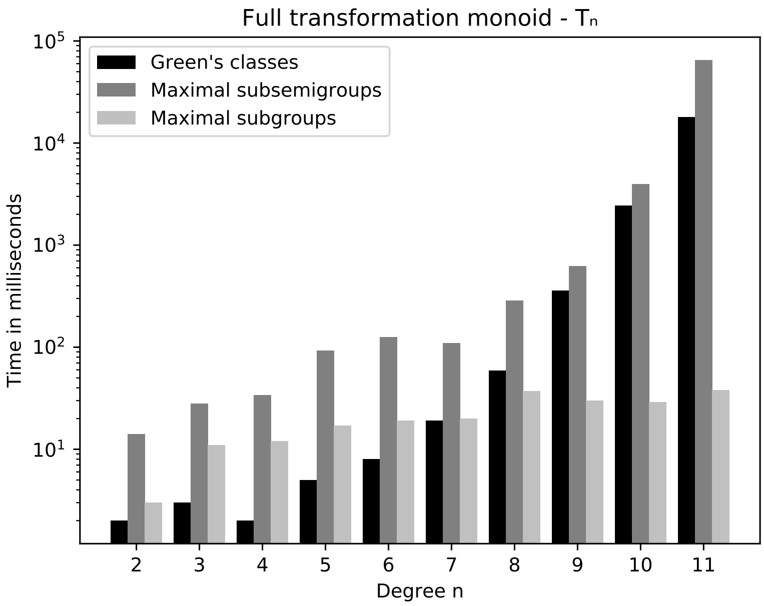

Figure 4.89: For $n \in \{2, \ldots, 11\}$, this graph compares the time taken to compute the partial order of $\mathscr{J}$-classes of $\mathcal{T}_n$ with both the time taken to compute its maximal subsemigroups, and the time spent during this finding the maximal subgroups of group $\mathscr{H}$-classes in $\mathcal{T}_n$.

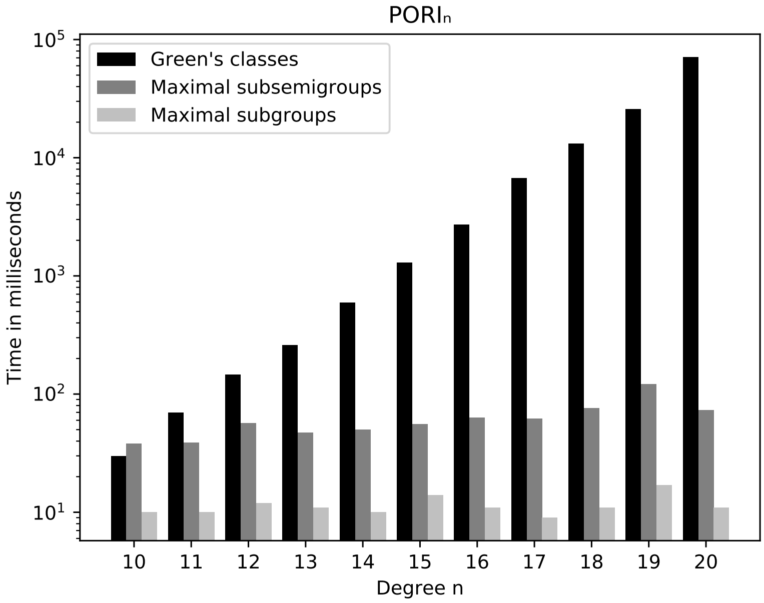

Figure 4.90: For $n \in \{10, \ldots, 20\}$, this graph compares the time taken to compute the partial order of $\mathscr{J}$-classes of $\mathcal{PORI}_n$ with both the total time taken to compute its maximal subsemigroups, and the amount of time that was spent as part of this finding maximal subgroups of group $\mathscr{H}$-classes in $\mathcal{PORI}_n$. As n increases, it appears that the time taken to compute the partial order of $\mathscr{J}$-classes of $\mathcal{PORI}_n$ increases much more rapidly than the time taken to compute the maximal subsemigroups of $\mathcal{PORI}_n$, given the partial order of $\mathscr{J}$-classes.

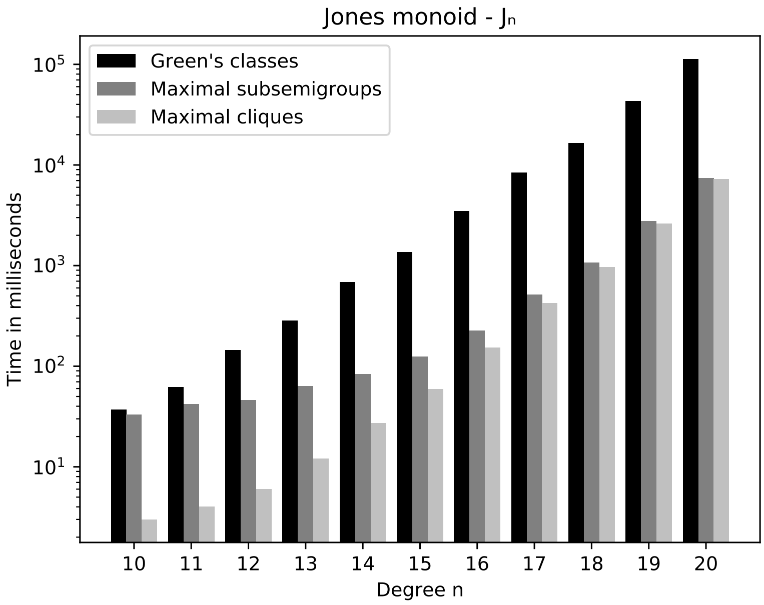

Figure 4.91: For $n \in \{10, \ldots, 20\}$, this graph compares the time taken to compute the partial order of $\mathscr{J}$-classes of $\mathcal{J}_n$ with the total time taken to compute its maximal subsemigroups, and the time spent as part of this in computing maximal cliques. It appears that, as n increases, the length of time spent computing maximal cliques approaches the total time taken.

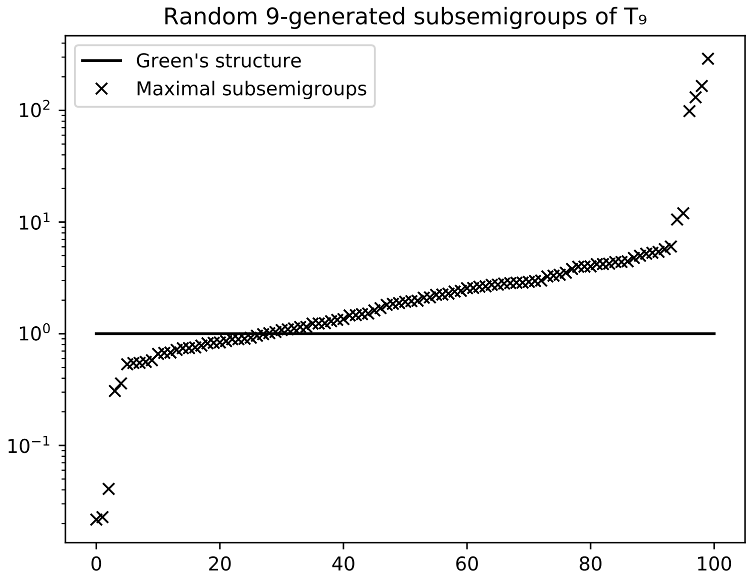

Figure 4.92: A graph showing, for 100 random 9-generated subsemigroups of $\mathcal{T}_9$, the ratio of the time taken to compute the maximal subsemigroups with the time taken to compute the partial order of $\mathscr{J}$-classes. Each cross represents a semigroup. If a cross sits above the horizontal line, then more time was spent computing the maximal subsemigroups of the corresponding semigroup than was spent computing its partial order of $\mathscr{J}$-classes. For most of these semigroups, the times taken for both steps was roughly comparable.

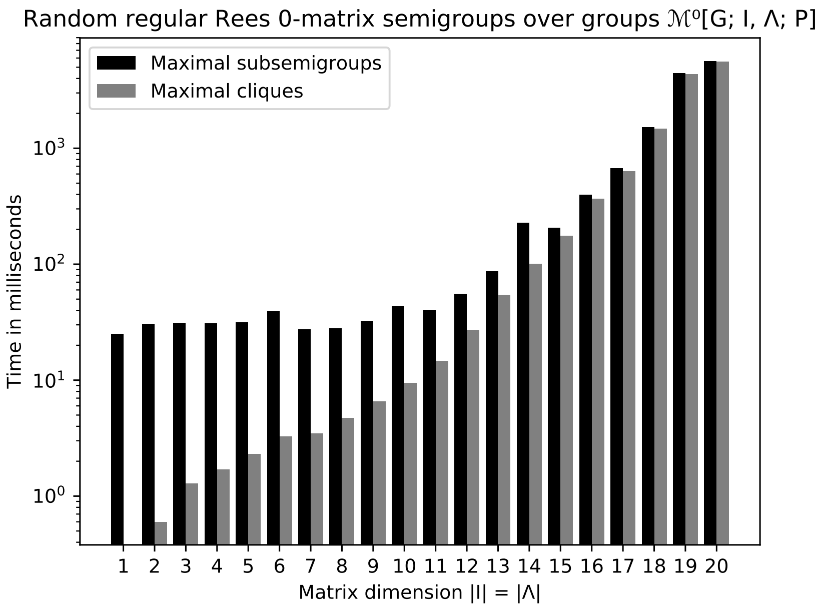

Figure 4.93: For each matrix dimension $|I| = |\Lambda| \in \{1, \ldots, 20\}$, this graph compares the mean time taken to compute the maximal subsemigroups of 100 random regular Rees 0-matrix semigroups $\mathscr{M}^0[G; I, \Lambda; P]$ with the mean time taken to compute the maximal cliques in the duals of their Graham-Houghton graphs. As the matrix dimension increases, it seems that the time spent computing maximal cliques approaches the total time for maximal subsemigroups.

Chapter 5

Maximal subsemigroups of finite transformation and diagram monoids

5.1 Introduction

In this chapter, we describe and count the maximal subsemigroups of several families of well-known finite transformation monoids and diagram monoids, using the tools provided in Chapter 4. The research underpinning this chapter was conducted in collaboration with James East, Jitender Kumar, and James D. Mitchell. A paper containing the results of this chapter is available on arXiv [44]; a shorter version of that paper, containing the most significant subset of these results, is published in the *Journal of Algebra* [45]. The research presented in this chapter is included with the permission of the co-authors.

Having obtained various theoretical results in Chapter 4 that concern the maximal subsemigroups of an arbitrary finite semigroup or monoid, a natural continuation of this work is to apply these results and techniques to study the maximal subsemigroups of particular semigroups.

The problem of describing or counting the maximal subsemigroups of certain transformation semigroups and monoids has been studied extensively in the literature. Much of this research has been led by I. Dimitrova, V. H. Fernandes, and co-authors; see [25–31, 56, 69], and the references therein. The techniques used to prove these results were typically rather disparate, and did not lend themselves well to generalisation. However, we can use the framework described in Chapter 4 in order to recover these results in a unified way, and moreover, to prove new results. Our strategy also allows us to clearly see the common features in the maximal subsemigroups of similar monoids. We especially make use of the tools of Section 4.5, which concern the maximal subsemigroups of an arbitrary finite monoid.

The same techniques can also be applied to describe the maximal subsemigroups of many other finite semigroups and monoids. To further demonstrate the utility of this framework, we describe the maximal subsemigroups of various families of diagram monoids. Diagram monoids are submonoids of the partition monoid, and they are currently well-studied in the literature; see [34, 39, 40, 42, 43, 46, 51, 70, 94, 98]. However, with the exception of those of the dual symmetric inverse monoid [94], the maximal subsemigroups of these diagram monoids had not been described in the literature.

Some of the results in this chapter were obtained with the help of the Semigroups package [101] for GAP [58], using the tools for computing maximal subsemigroups described in Section 4.6.1. More specifically, by applying the function `MaximalSubsemigroups` to several finite monoids from a particular family, it was often straightforward to analyse the results and make a conjecture about their general description.

This chapter is structured as follows. In Section 5.1.1, we present some preliminary results concerning the maximal subgroups of the groups of units of the monoids that appear in the later sections. In Section 5.2, we study partial transformation monoids. We define various families of monoids that consist of order- and orientation-preserving and -reversing partial transformations in Section 5.2.1, and we provide some prerequisite information about them in Section 5.2.2. We describe the maximal subsemigroups of these monoids in Sections 5.2.3–5.2.8. We study monoids of partitions in Section 5.3. This section is structured similarly: in Sections 5.3.1 and 5.3.2, we define and present prerequisite results about the diagram monoids whose maximal subsemigroups we subsequently classify in Sections 5.3.3–5.3.7. In particular, we classify the maximal subsemigroups of the partition, Jones, Motzkin, and Brauer monoids, and several related monoids. In Section 5.4, we present Table 5.62. This table gives a summary of the information in this chapter. More specifically, for each of the families of monoids considered in this chapter, the table lists the corresponding sequence of the numbers of maximal subsemigroups, with references to the theorems where the maximal subsemigroups are described.

5.1.1 Maximal subgroups of cyclic, dihedral, and symmetric groups

It follows by Lemma 4.73 that, in order to classify the maximal subsemigroups of the monoids in this chapter, we require a description of the maximal subgroups of their groups of units. The groups of units that appear in this chapter are cyclic, dihedral, and symmetric groups. The conjugacy classes of maximal subgroups of the finite symmetric groups are described in [87] and counted in [88]; see [120, A066115] for the sequence of their number. In general, however, there is no known simple formula for the number of *all* maximal subgroups of a finite symmetric group. Throughout this chapter, we use the notation s_k to denote the number of maximal subsemigroups of the symmetric group of degree k [120, A290138]. For the maximal subgroups of the cyclic and dihedral groups, we present the following well-known results. The proofs of these results require descriptions of the subgroups of the cyclic and dihedral groups. These descriptions are straightforward to prove.

Lemma 5.1. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $G = \langle \alpha \rangle$ be a cyclic group of order n. The maximal subgroups of G are the subgroups $\langle \alpha^p \rangle$, for each prime divisor p of n. In particular, the total number of maximal subgroups G is the number of prime divisors of n.*

Proof. The subgroups of G are $\langle \alpha^d \rangle$, for each positive divisor d of n, and $\langle \alpha^d \rangle \leq \langle \alpha^{d'} \rangle$ if and only if d' divides d. Therefore, the maximal subgroups of G are the subgroups $\langle \alpha^d \rangle$, where $d > 1$ is a divisor of n whose positive divisors are 1 and d: these are the prime divisors of n. □

Some of the families of transformation monoids considered in this chapter have trivial groups of units, or groups of units of order 2. In the first case, by Lemma 4.73, no further work is required. In the second case, since a group of order 2 is cyclic, it follows by Lemma 5.1 that its unique maximal subgroup is trivial, consisting of the identity element of the group.

The following lemma was stated incorrectly in [45, Lemma 2.5], but appears correctly below.

Lemma 5.2. *Let $n \in \mathbb{N}$, $n \geq 3$, and let $G = \langle \sigma,\ \rho \rangle$ be a dihedral group of order $2n$, where σ has order 2, ρ has order n, and $\sigma\rho = \rho^{-1}\sigma$. The maximal subgroups of G are the subgroups $\langle \rho \rangle$ and $\langle \rho^p,\ \sigma\rho^i \rangle$, for each prime divisor p of n and each integer $i \in \{0, \ldots, p-1\}$. In particular, the number of maximal subgroups of G is one more than the sum of the prime divisors of n.*

Proof. The subgroups of G are $\langle \sigma^d \rangle$ and $\langle \sigma^d,\ \sigma\rho^i \rangle$, for each positive divisor d of n and each integer $i \in \{0, \ldots, d-1\}$. Any subgroup of the form $\langle \rho^d \rangle$ is contained in $\langle \rho \rangle$, which is maximal since its index in G is 2. The subgroup $\langle \rho^d,\ \sigma\rho^i \rangle$ is not contained in $\langle \rho \rangle$, and is contained in

$\langle \rho^{d'}, \sigma\rho^{i'} \rangle$ if and only if d' divides d and $i = i'$. Therefore, the maximal subgroups of G in this second form are the subgroups $\langle \rho^{d}, \sigma\rho^{i} \rangle$, where d is a prime divisor of n and $0 \leq i \leq d-1$. $\square$

5.2 Partial transformation monoids

In this section, we describe and count the maximal subsemigroups of several well-known families of partial transformation monoids. We define these monoids in Section 5.2.1, and present some prerequisite results and background information about them in Section 5.2.2; the main results are given in Sections 5.2.3–5.2.8. For certain families, the descriptions of their maximal subsemigroups, and the corresponding proofs, are alike. In such cases, we collect the results concerning these families, in order to highlight their similarities.

5.2.1 Definitions

As defined in Section 1.3.5, $\mathcal{PT}_n$ is the monoid of all partial transformations of degree n under composition of partial functions; its submonoid $\mathcal{T}_n$ consists of all transformations of degree n, and its inverse submonoid $\mathcal{I}_n$ consists of all partial permutations of degree n. In Sections 5.2.3–5.2.8, we describe and count the maximal subsemigroups of these monoids, as well as those of a collection of other submonoids of $\mathcal{PT}_n$, which we define here. These other monoids consist of partial transformations that preserve or reverse the usual order, or orientation, of $\{1, \ldots, n\}$.

Let $n \in \mathbb{N}$, and let α be a partial transformation of degree n. Clearly, there is a unique way to write $\operatorname{dom}(\alpha) = \{i_1, \ldots, i_k\}$, where $i_j \in \{1, \ldots, n\}$ for each j, and $i_1 < \cdots < i_k$. We say that α is *order-preserving* if $i_1\alpha \leq \cdots \leq i_k\alpha$, and *order-reversing* if $i_1\alpha \geq \cdots \geq i_k\alpha$. Similarly, we say that α is *orientation-preserving* if there exists at most one $l \in \{1, \ldots, k-1\}$ such that $i_l\alpha > i_{l+1}\alpha$, and α is *orientation-reversing* if there exists at most one $l \in \{1, \ldots, k-1\}$ such that $i_l\alpha < i_{l+1}\alpha$. In a certain informal sense, an orientation-preserving or -reversing partial transformations preserves or reverses the 'cyclic ordering' of its domain. Note that an order-preserving partial transformation is orientation-preserving, and similarly, an order-reversing partial transformation is orientation-reversing. Rules for the composition of such partial transformations are shown in Table 5.3.

$\circ$	**Order-preserving**	**Order-reversing**
Order-preserving	Order-preserving	Order-reversing
Order-reversing	Order-reversing	Order-preserving

$\circ$	**Orientation-preserving**	**Orientation-reversing**
Orientation-preserving	Orientation-preserving	Orientation-reversing
Orientation-reversing	Orientation-reversing	Orientation-preserving

Table 5.3: Rules describing the composition of order-preserving/-reversing partial transformations, and the composition of orientation-preserving/-reversing partial transformations. It follows that we may create subsemigroups of $\mathcal{PT}_n$ that consist of such partial transformations.

Given these notions, and the rules described in Table 5.3, it is natural to define the largest submonoids of $\mathcal{PT}_n$ that consist of such elements. These monoids are:

- $\mathcal{PO}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is order-preserving}\}$,
- $\mathcal{POD}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is order-preserving or -reversing}\}$,

- $\mathcal{POP}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is orientation-preserving}\}$, and
- $\mathcal{POR}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is orientation-preserving or -reversing}\}$.

We also define their largest submonoids consisting of transformations, and their largest inverse submonoids consisting of partial permutations. That is, we define their intersections with $\mathcal{T}_n$:

- $\mathcal{O}_n = \mathcal{PO}_n \cap \mathcal{T}_n$,
- $\mathcal{OD}_n = \mathcal{POD}_n \cap \mathcal{T}_n$,
- $\mathcal{OP}_n = \mathcal{POP}_n \cap \mathcal{T}_n$, and
- $\mathcal{OR}_n = \mathcal{POR}_n \cap \mathcal{T}_n$;

and we define their intersections with $\mathcal{I}_n$:

- $\mathcal{POI}_n = \mathcal{PO}_n \cap \mathcal{I}_n$,
- $\mathcal{PODI}_n = \mathcal{POD}_n \cap \mathcal{I}_n$,
- $\mathcal{POPI}_n = \mathcal{POP}_n \cap \mathcal{I}_n$, and
- $\mathcal{PORI}_n = \mathcal{POR}_n \cap \mathcal{I}_n$.

These monoids have been studied extensively in the literature; see for example [28, 31], and the references therein, where the notation that we use for these monoids was introduced. In Sections 5.2.4–5.2.8, we classify the maximal subsemigroups of these monoids.

We may easily describe the permutations, and therefore the largest subgroups of $\mathcal{S}_n$, that satisfy each of these properties. Recall that the symmetric group $\mathcal{S}_n$ is the group consisting of all permutations of degree n; $\mathcal{S}_n$ is the group of units of $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$. The identity permutation of degree n, id_n, is the unique order-preserving permutation of degree n; it is the identity element of each of the monoids that we define above. There is also a unique order-reversing permutation of degree n, which we write in disjoint cycle notation as $(1\ n)(2\ n{-}1)\cdots(\lfloor(n{+}1)/2\rfloor\ \lceil(n{+}1)/2\rceil)$. A permutation is orientation-preserving if and only if it is a power of the n-cycle $(1\ 2\ \ldots\ n)$, and an orientation-reversing permutation is the product of an orientation-preserving permutation with the unique order-reversing permutation. We define

$$\mathcal{C}_n = \langle (1\ 2\ \ldots\ n) \rangle \tag{5.4}$$

to be the subgroup of $\mathcal{S}_n$ consisting of all orientation-preserving permutations; it is cyclic of order n, and so its maximal subgroups are given by Lemma 5.1. We also define

$$\mathcal{D}_n = \langle (1\ 2\ \ldots\ n),\ (1\ n)(2\ n-1)\cdots(\lfloor(n+1)/2\rfloor\ \lceil(n+1)/2\rceil) \rangle. \tag{5.5}$$

to be the group of orientation-preserving or -reversing permutations of degree n. When $n \geq 3$, $\mathcal{D}_n$ is dihedral of order $2n$, and so its maximal subgroups are given by Lemma 5.2. Note that $\mathcal{C}_2 = \mathcal{D}_2 = \langle (1\ 2) \rangle$.

Thus we may describe the groups of units of the monoids that we have defined. The groups of units of $\mathcal{PO}_n$, $\mathcal{O}_n$, and $\mathcal{POI}_n$ are the trivial group $\{\mathrm{id}_n\}$; the groups of units of $\mathcal{POD}_n$, $\mathcal{OD}_n$, and $\mathcal{PODI}_n$ are $\langle (1\ n)(2\ n-1)\cdots(\lfloor(n+1)/2\rfloor\ \lceil(n+1)/2\rceil) \rangle$; the groups of units of $\mathcal{POP}_n$, $\mathcal{OP}_n$, and $\mathcal{POPI}_n$ are $\mathcal{C}_n$; and the groups of units of $\mathcal{POR}_n$, $\mathcal{OR}_n$, and $\mathcal{PORI}_n$ are $\mathcal{D}_n$.

The maximal subsemigroups of several of these monoids have previously been described in the literature, including those of $\mathcal{OP}_n$ [28, Theorem 1.6], $\mathcal{OR}_n$ [28, Theorem 2.6], $\mathcal{POI}_n$ [56, Theorem 2], and $\mathcal{PODI}_n$ [29, Theorem 4]. The maximal subsemigroups of $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$ are well-known folklore. Furthermore, the maximal subsemigroups of the singular ideal $\mathcal{O}_n \setminus \{\mathrm{id}_n\}$ of $\mathcal{O}_n$ were described in [25, Theorem 2], and those of the singular ideal $\mathcal{PO}_n \setminus \{\mathrm{id}_n\}$ of $\mathcal{PO}_n$ were described in in [31, Theorem 1]. Additionally, the maximal subsemigroups of the singular ideal of $\mathcal{OD}_n$ were found in [69, Theorem 2], but since the group of units of $\mathcal{OD}_n$ is non-trivial, this is a fundamentally different problem than finding the maximal subsemigroups of $\mathcal{OD}_n$ itself. Until the publication of the research detailed in this section (in [45]), the maximal subsemigroups of $\mathcal{OD}_n$, $\mathcal{POD}_n$, $\mathcal{POP}_n$, $\mathcal{POPI}_n$, $\mathcal{POR}_n$, and $\mathcal{PORI}_n$ remained open problems.

Nevertheless, we describe the maximal subsemigroups of all the monoids from this section. The purpose of reproving the known results is to demonstrate that they may all be obtained in a largely unified manner, using the tools described in Chapter 4. Furthermore, the descriptions of the maximal subsemigroups of some of these monoids, such as those of $\mathcal{PO}_n$ and $\mathcal{POD}_n$, and those of $\mathcal{O}_n$ and $\mathcal{OD}_n$, are closely linked, and so it is instructive to present them together.

5.2.2 Preliminaries

Let $n \in \mathbb{N}$, $n \geq 2$. In order to describe the maximal subsemigroups of the submonoids of $\mathcal{PT}_n$ from Section 5.2.1, we require some additional facts and notation. We first require a description of the Green's relations on $\mathcal{PT}_n$. These are given by:

- $\alpha \mathscr{L} \beta$ if and only if $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$,
- $\alpha \mathscr{R} \beta$ if and only if $\ker(\alpha) = \ker(\beta)$, and
- $\alpha \mathscr{J} \beta$ if and only if $\mathrm{rank}(\alpha) = \mathrm{rank}(\beta)$,

for $\alpha, \beta \in \mathcal{PT}_n$; see [57, Theorem 4.5.1] for a proof. Note that $\ker(\alpha) = \ker(\beta)$ implies that $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$, since the kernel of a partial transformation is defined to be an equivalence on its domain. Each of the submonoids of $\mathcal{PT}_n$ defined in Section 5.2.1 is regular, and so it follows easily that the Green's relations on these monoids are characterised in the same way.

By Lemma 4.68, we also require generating sets for these monoids. Any order-reversing partial transformation α is the product of some order-preserving partial transformation and the permutation $\sigma = (1\ n)(2\ n-1)\cdots(\lfloor (n+1)/2 \rfloor\ \lceil (n+1)/2 \rceil)$, since $\alpha = (\alpha\sigma^{-1})\sigma$, and $\alpha\sigma^{-1}$ is order-preserving. Similarly, any orientation-preserving partial transformation is the product of some order-preserving partial transformation with some permutation in $\mathcal{C}_n$. The analogous statement holds for orientation-reversing partial transformations. Therefore, if S is any of the monoids of order- or orientation-preserving or -reversing partial transformations defined in Section 5.2.1, then S is generated by its units and its order-preserving elements. However, $\mathcal{O}_n$ is generated by id_n and its idempotents of rank $n-1$ [1], and $\mathcal{POI}_n$ is generated by id_n and its elements of rank $n-1$ [48, Lemma 2.7]. It follows that S is generated by its units and its order-preserving elements of rank $n-1$. The monoids $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$ are generated by $\mathcal{S}_n$ and their elements of rank $n-1$ [57, Theorems 3.1.3–3.1.5].

Therefore, by Lemma 4.68, in order to describe the maximal subsemigroups of any of the monoids from Section 5.2.1, we must find those maximal subsemigroups that arise from the group of units, and those that arise from the $\mathscr{J}$-class containing elements of rank $n-1$. The results of Section 4.5.1 apply in the former case, those of Section 4.5.2 apply in the latter case.

To describe the maximal subsemigroups arising from the $\mathscr{J}$-class consisting of elements of rank $n-1$, we require the following notation for the Green's classes of this $\mathscr{J}$-class. We define

$$J_{n-1} = \left\{\alpha \in \mathcal{PT}_n \,:\, \mathrm{rank}(\alpha) = n-1\right\}$$

to be the $\mathscr{J}$-class of $\mathcal{PT}_n$ that consists of all partial transformations of rank $n-1$. A partial transformation of rank $n-1$ lacks exactly one element from its image, and is either a partial permutation that also lacks exactly one element from its domain, or is a transformation with a unique non-trivial kernel class, which contains two points. Thus for distinct points $i, j \in \{1, \ldots, n\}$, we define the Green's $\mathscr{L}$- and $\mathscr{R}$-classes of J_{n-1}:

- $L_i = \{\alpha \in J_{n-1} : i \notin \mathrm{im}(\alpha)\}$, which is an $\mathscr{L}$-class of J_{n-1};
- $R_i = \{\alpha \in J_{n-1} : i \notin \mathrm{dom}(\alpha)\}$, which is an $\mathscr{R}$-class of J_{n-1} of partial permutations; and
- $R_{\{i,j\}} = \{\alpha \in J_{n-1} : (i, j) \in \ker(\alpha)\}$, which is an $\mathscr{R}$-class of J_{n-1} of transformations.

An $\mathscr{H}$-class of the form $L_i \cap R_j$ is a group if and only if $i = j$, and an $\mathscr{H}$-class of the form $L_i \cap R_{\{j,k\}}$ is a group if and only if $i \in \{j, k\}$ [57, Theorem 2.7.2].

We observe that the non-trivial kernel class of an order-preserving or -reversing transformation of rank $n-1$ has the form $\{i, i+1\}$ for some $i \in \{1, \ldots, n-1\}$, and that the non-trivial kernel class of an orientation-preserving or -reversing transformation of rank $n-1$ has the same form, or is equal to $\{1, n\}$. Any non-empty subset of $\{1, \ldots, n\}$ appears as the image of some partial transformation in each of the monoids defined in Section 5.2.1.

Let S be one of the submonoids of $\mathcal{PT}_n$ defined in Section 5.2.1. By the preceding discussion, it follows that the set $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of S, that the $\mathscr{L}$-classes of $J_{n-1} \cap S$ are the sets of the form $L_i \cap S$, and that the $\mathscr{R}$-classes of $J_{n-1} \cap S$ are those non-empty sets of the forms $R_i \cap S$ and $R_{\{i,j\}} \cap S$, for distinct $i, j \in \{1, \ldots, n\}$.

Often, the principal obstacle to describing the maximal subsemigroups of S that arise from $J_{n-1} \cap S$ is to determine the maximal independent subsets of the graph $\Delta(S, J_{n-1} \cap S)$, which was introduced in Section 4.4.2; see also Section 4.5.2 for more information. In every case in this section, the relevant $\mathscr{J}$-class is $J_{n-1} \cap S$, and so we henceforth refer to this graph as simply $\Delta(S)$. Whenever we present a picture of $\Delta(S)$, such as in Figure 5.16, we label an $\mathscr{L}$-class as L_i, rather than as $L_i \cap S$, and so on, in order to avoid cluttering the image. This approach also has the advantage of highlighting the similarities between the graphs of related monoids, and allows us to more easily see that some graphs may be obtained as induced subgraphs of others.

To describe $\Delta(S)$, we must calculate the left and right actions of the group of units G of S on the $\mathscr{R}$-classes and on the $\mathscr{L}$-classes of $J_{n-1} \cap S$, respectively. The following lemma shows that these actions correspond to natural right actions of G on points and pairs in $\{1, \ldots, n\}$.

Lemma 5.6. *Let S be one of the submonoids of $\mathcal{PT}_n$ defined in Section 5.2.1, and let G be its group of units.*

(a) *Let $\Omega \subseteq \{1, \ldots, n\}$. Then Ω is an orbit of G on $\{1, \ldots, n\}$ if and only if $\{L_i \cap S : i \in \Omega\}$ is an orbit of the right action of G on $(J_{n-1} \cap S)/\mathscr{L}$.*

(b) *Let $\Omega \subseteq X = \{i : R_i \cap S \neq \varnothing\}$. Then Ω is an orbit of G on X if and only if $\{R_i \cap S : i \in \Omega\}$ is an orbit of the left action of G on the $\mathscr{R}$-classes of $J_{n-1} \cap S$ that contain partial permutations.*

(c) *Let $\Omega \subseteq Y = \{\{i, j\} : i \neq j,\ R_{\{i,j\}} \cap S \neq \varnothing\}$. Then Ω is an orbit of G on Y if and only if $\{R_{\{i,j\}} : \{i, j\} \in \Omega\}$ is an orbit of the left action of G on the $\mathscr{R}$-classes of $J_{n-1} \cap S$ that contain transformations.*

Proof. To prove part (a), let $i, j \in \{1, \ldots, n\}$ and let $\alpha \in L_i \cap S$. Then, with respect to the right action of G on $\{1, \ldots, n\}$,

$$\begin{aligned} i \text{ and } j \text{ lie in the same orbit} \quad &\Leftrightarrow \quad i\sigma = j \text{ for some } \sigma \in G \\ &\Leftrightarrow \quad j \notin \mathrm{im}(\alpha\sigma) \text{ for some } \sigma \in G \\ &\Leftrightarrow \quad \alpha\sigma \in L_j \cap S \text{ for some } \sigma \in G \\ &\Leftrightarrow \quad (L_i \cap S)\sigma = L_j \cap S \text{ for some } \sigma \in G. \end{aligned}$$

The proof of part (b) is similar to the proof of part (a). To prove part (c), let $\{i,\, j\}, \{k,\, l\} \in Y$ be arbitrary, and let $\alpha \in R_{\{i,\,j\}} \cap S$. Then, with respect to the right action of G on Y,

$$\begin{aligned} \{i,\, j\} \text{ and } \{k,\, l\} \text{ lie in the same orbit } &\Leftrightarrow \{k\sigma,\, l\sigma\} = \{i,\, j\} \text{ for some } \sigma \in G \\ &\Leftrightarrow k\sigma\alpha = l\sigma\alpha \text{ for some } \sigma \in G \\ &\Leftrightarrow \sigma\alpha \in R_{\{k,\,l\}} \cap S \text{ for some } \sigma \in G \\ &\Leftrightarrow \sigma(R_{\{i,\,j\}} \cap S) = R_{\{k,\,l\}} \cap S \text{ for some } \sigma \in G. \quad \square \end{aligned}$$

The right actions of the groups $\{\mathrm{id}_n\}$, $\langle (1\ n)(2\ n-1) \cdots (\lfloor (n+1)/2 \rfloor\ \lceil (n+1)/2 \rceil) \rangle$, $\mathcal{C}_n$, $\mathcal{D}_n$, and $\mathcal{S}_n$ (as defined in Section 5.2.1) on the points, and pairs of points in $\{1, \ldots, n\}$, are easy to understand, since a permutation of degree n is defined in terms of its action on $\{1, \ldots, n\}$. Therefore, if S is one of the monoids mentioned in Section 5.2.1, then the actions of its group of units on the $\mathscr{R}$- and $\mathscr{L}$-classes of $J_{n-1} \cap S$ may be readily understood via Lemma 5.6.

5.2.3 $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$

To start, we describe and count the maximal subsemigroups of $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$. Of course, the maximal subsemigroups of the these monoids are well-known folklore, and so the following result is by no means novel. However, we include this theorem and the subsequent discussion for completeness, and to begin to show the utility of the results presented in Chapter 4.

Theorem 5.7. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. The following hold.*

(a) *The maximal subsemigroups of $\mathcal{PT}_n$ are the sets:*

  (i) $(\mathcal{PT}_n \setminus \mathcal{S}_n) \cup U$*, where U is a maximal subgroup of $\mathcal{S}_n$* (type (M1))*;*

  (ii) $\mathcal{PT}_n \setminus \{\alpha \in \mathcal{T}_n : \mathrm{rank}(\alpha) = n-1\}$ (type (M4))*; and*

  (iii) $\mathcal{PT}_n \setminus \{\alpha \in \mathcal{I}_n : \mathrm{rank}(\alpha) = n-1\}$ (type (M4)).

(b) *The maximal subsemigroups of $\mathcal{T}_n$ are the sets:*

  (i) $(\mathcal{T}_n \setminus \mathcal{S}_n) \cup U$*, where U is a maximal subgroup of $\mathcal{S}_n$* (type (M1))*; and*

  (ii) $\mathcal{T}_n \setminus \{\alpha \in \mathcal{T}_n : \mathrm{rank}(\alpha) = n-1\}$ (type (M5)).

(c) *The maximal subsemigroups of $\mathcal{I}_n$ are the sets:*

  (i) $(\mathcal{I}_n \setminus \mathcal{S}_n) \cup U$*, where U is a maximal subgroup of $\mathcal{S}_n$* (type (M1))*; and*

  (ii) $\mathcal{I}_n \setminus \{\alpha \in \mathcal{I}_n : \mathrm{rank}(\alpha) = n-1\}$ (type (M5)).

In particular, for $n \geq 2$, there are $s_n + 2$ maximal subsemigroups of $\mathcal{PT}_n$, and $s_n + 1$ maximal subsemigroups of both $\mathcal{T}_n$ and $\mathcal{I}_n$.

	2	3	4	5	6	7	8	9	10	11	12	n
$\mathcal{PT}_n$	3	6	10	24	55	186	355	1378	3979	363906	396500	$s_n + 2$
$\mathcal{T}_n$, $\mathcal{I}_n$	2	5	9	23	54	185	354	1377	3978	363905	396499	$s_n + 1$

Table 5.8: The numbers of maximal subsemigroups of the monoids $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$, for $n = 2, \ldots, 12$, along with the general formulae. Recall that s_n denotes the number of maximal subgroups of the symmetric group $\mathcal{S}_n$ [120, A290138]. See Theorem 5.7 for a description of the maximal subsemigroups of these monoids.

In the proof of this theorem, we make use of the following well-known facts [57, Theorems 3.1.3–3.1.5]. A subset of $\mathcal{T}_n$ or $\mathcal{I}_n$ is a generating set if and only if the subset contains a generating set for $\mathcal{S}_n$, and any transformation or partial permutation, respectively, of rank $n-1$. In addition, a subset of $\mathcal{PT}_n$ is a generating set for $\mathcal{PT}_n$ if and only if it contains a generating set for $\mathcal{T}_n$ and a generating set for $\mathcal{I}_n$.

Proof. Since $\mathcal{S}_n$ is the group of units of $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$, it follows by Lemma 4.73 that, in each case, the maximal subsemigroups arising from the group of units are those described in the statement of the theorem, and that there are s_n of them. It remains to describe the maximal subsemigroups of each monoid that arise from the $\mathscr{J}$-class of rank $n-1$. By Corollary 4.74, the unique maximal subsemigroup of $\mathcal{T}_n$ or $\mathcal{I}_n$ to arise from its $\mathscr{J}$-class of rank $n-1$ has type (M5). By using Lemma 4.8 with $k=2$, $X_1 = J_{n-1} \cap \mathcal{T}_n$, and $X_2 = J_{n-1} \cap \mathcal{I}_n$, we find that the two maximal subsemigroups of $\mathcal{PT}_n$ arising from J_{n-1} are those described. $\square$

For each of these monoids, the descriptions of the maximal subsemigroups that arise from the $\mathscr{J}$-class of rank $n-1$ can also be obtained with the techniques and results from Section 4.5.2. Certainly, these monoids are each generated by their units and their idempotents of rank $n-1$, and so by Lemma 4.49(b), no maximal subsemigroups of type (M1) arise from this $\mathscr{J}$-class. The right action of $\mathcal{S}_n$ on the $\mathscr{L}$-classes of J_{n-1} given by right multiplication is transitive. On the other hand, there are two orbits under the left action of $\mathcal{S}_n$ on the $\mathscr{R}$-classes of J_{n-1}: one of these contains the $\mathscr{R}$-classes of transformations, and the other contains the $\mathscr{R}$-classes of partial permutations. Thus we may deduce the definitions of $\Delta(\mathcal{PT}_n)$, $\Delta(\mathcal{T}_n)$, and $\Delta(\mathcal{I}_n)$; pictures of these graphs are given in Figures 5.9, 5.10, and 5.11, respectively.

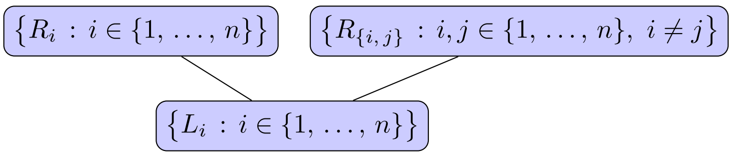

Figure 5.9: The graph $\Delta(\mathcal{PT}_n)$. The group of units $\mathcal{S}_n$ acts transitively on the $\mathscr{L}$-classes of J_{n-1}, and so there is a single vertex corresponding to this orbit. However, $\mathcal{S}_n$ has two orbits under its action on the $\mathscr{R}$-classes of J_{n-1}, and so there are two vertices that contain $\mathscr{R}$-classes.

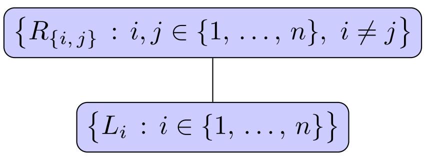

Figure 5.10: The graph $\Delta(\mathcal{T}_n)$. Note that this graph may be obtained as the induced subgraph of $\Delta(\mathcal{PT}_n)$ on its orbits of Green's classes that contain transformations.

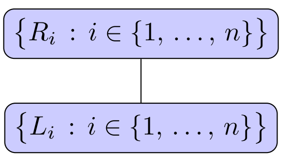

Figure 5.11: The graph $\Delta(\mathcal{I}_n)$. Note that this graph may be obtained as the induced subgraph of $\Delta(\mathcal{PT}_n)$ on its orbits of Green's classes that contain partial permutations.

By Lemma 4.80, there are no maximal subsemigroups of any of these monoids of types (M2) or (M3), and there are no maximal subsemigroups of $\mathcal{T}_n$ or $\mathcal{I}_n$ of type (M4). Therefore, by Corollary 4.75, the only maximal subsemigroup of $\mathcal{T}_n$ or $\mathcal{I}_n$ to arise from the $\mathscr{J}$-class of rank $n-1$ has type (M5). However, by the dual of Proposition 4.79, there are two maximal subsemigroups of $\mathcal{PT}_n$ of type (M4) that arise from J_{n-1}, which are formed by removing either the partial permutations, or the transformations, of degree $n-1$.

5.2.4 $\mathcal{PO}_n$ and $\mathcal{POD}_n$

The maximal subsemigroups of $\mathcal{PO}_n$ were described in [31, Theorem 1]. The maximal subsemigroups of $\mathcal{POD}_n$ had not been described in the literature prior to [45]. Using our approach, we find that the maximal subsemigroups of $\mathcal{POD}_n$ are closely linked to those of $\mathcal{PO}_n$.

The main results of this section are the following theorems; to state them, we first reiterate the notation and background information given in Section 5.2.2. Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary, and let $S \in \{\mathcal{PO}_n, \mathcal{POD}_n\}$. Then $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of S, the $\mathscr{L}$-classes of $J_{n-1} \cap S$ are the sets $\{L_i \cap S : i \in \{1, \ldots, n\}\}$, and the $\mathscr{R}$-classes are the sets $\{R_i \cap S : i \in \{1, \ldots, n\}\}$ and $\{R_{\{i,\, i+1\}} \cap S : i \in \{1, \ldots, n-1\}\}$. We also use the fact that $\mathcal{PO}_n$ is idempotent generated [60, Theorem 3.13], and that $\mathcal{POD}_n$ is generated by $\mathcal{PO}_n$ and the order-reversing permutation $(1\ n)(2\ n-1)\cdots(\lfloor (n+1)/2 \rfloor\ \lceil (n+1)/2 \rceil)$.

Theorem 5.12. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. The maximal subsemigroups of $\mathcal{PO}_n$ are:*

(a) $\mathcal{PO}_n \setminus \{\mathrm{id}_n\}$ (type (M5))*;*

(b) *the union of $\mathcal{PO}_n \setminus J_{n-1}$ and the union of the sets in*

$$\{L_i \cap \mathcal{PO}_n : i \in A\} \cup \{R_i \cap \mathcal{PO}_n : i \notin A\} \cup \{R_{\{i,\, i+1\}} \cap \mathcal{PO}_n : i, i+1 \notin A\},$$

where A is any non-empty proper subset of $\{1, \ldots, n\}$ (type (M2)) *and;*

(c) $\mathcal{PO}_n \setminus R$*, where R is any $\mathscr{R}$-class in $J_{n-1} \cap \mathcal{PO}_n$* (type (M4)).

In particular, for $n \in \mathbb{N}$, there are $2^n + 2n - 2$ maximal subsemigroups of $\mathcal{PO}_n$.

Theorem 5.13. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. The maximal subsemigroups of $\mathcal{POD}_n$ are:*

(a) $\mathcal{POD}_n \setminus \{(1\ n)(2\ n-1)\cdots(\lfloor (n+1)/2 \rfloor\ \lceil (n+1)/2 \rceil)\}$ (type (M1))*;*

(b) *the union of $\mathcal{POD}_n \setminus J_{n-1}$ and the union of the sets in*

$$\{(L_i \cup L_{n-i+1}) \cap \mathcal{POD}_n : i \in A\} \cup \{(R_i \cup R_{n-i+1}) \cap \mathcal{POD}_n : i \notin A\} \\ \cup \{(R_{\{i,\, i+1\}} \cup R_{\{n-i,\, n-i+1\}}) \cap \mathcal{POD}_n : i, i+1 \notin A\},$$

where A is any non-empty proper subset of $\{1, \ldots, \lceil n/2 \rceil\}$ (type (M2))*;*

(c) $\mathcal{POD}_n \setminus (R_i \cup R_{n-i+1})$*, for* $i \in \{1, \ldots, \lceil n/2 \rceil\}$ (type (M4))*; and*

(d) $\mathcal{POD}_n \setminus (R_{\{i,\, i+1\}} \cup R_{\{n-i, n-i+1\}})$*, for* $i \in \{1, \ldots, \lfloor n/2 \rfloor\}$ (type (M4)).

In particular, for $n \in \mathbb{N}$, there are $2^{\lceil n/2 \rceil} + n - 1$ maximal subsemigroups of $\mathcal{POD}_n$.

The most substantial step involved in the proofs of these theorems is the description of the maximal independent subsets of $\Delta(\mathcal{PO}_n)$ and $\Delta(\mathcal{POD}_n)$.

Since $\mathcal{PO}_n$ has a trivial group of units, the orbits of $\mathscr{L}$-classes and $\mathscr{R}$-classes of $J_{n-1} \cap \mathcal{PO}_n$ are singletons, and $\Delta(\mathcal{PO}_n)$ is isomorphic to the Graham-Houghton graph of the principal factor of $J_{n-1} \cap \mathcal{PO}_n$. A picture of $\Delta(\mathcal{PO}_n)$ is shown in Figure 5.15.

	1	2	3	4	5	6	7	8	9	10	11	12	n
$\mathcal{PO}_n$	2	6	12	22	40	74	140	270	528	1042	2068	4118	$2^n + 2n - 2$
$\mathcal{POD}_n$	2	3	6	7	12	13	22	23	40	41	74	75	$2^{\lceil n/2 \rceil} + n - 1$

Table 5.14: The numbers of maximal subsemigroups of the monoids $\mathcal{PO}_n$ and $\mathcal{POD}_n$, for $n = 1, \ldots, 12$, along with the general formulae. Note that the number of maximal subsemigroups of $\mathcal{PO}_n$ equals the number of maximal subsemigroups of $\mathcal{POD}_{2n-1}$. See Theorems 5.12 and 5.13 for descriptions of the maximal subsemigroups of $\mathcal{PO}_n$ and $\mathcal{POD}_n$, respectively.

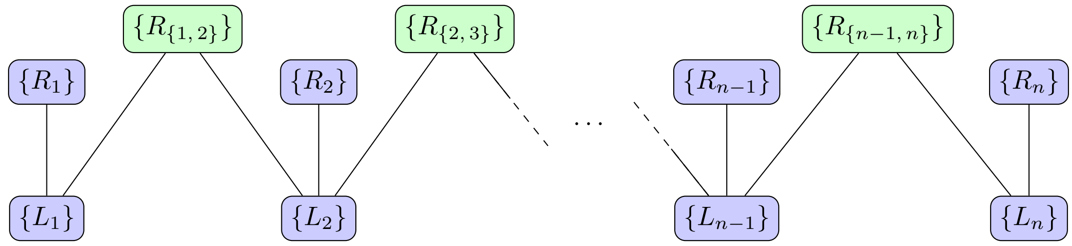

Figure 5.15: A visualisation of the graph $\Delta(\mathcal{PO}_n)$. The group of units of $\mathcal{PO}_n$ is trivial, and so $\Delta(\mathcal{PO}_n)$ is isomorphic to the Graham-Houghton graph of the principal factor of $\mathcal{PO}_n \cap J_{n-1}$. The vertices of degree 1 in $\Delta(\mathcal{PO}_n)$ are those orbits whose $\mathscr{R}$-classes are composed of partial permutations.

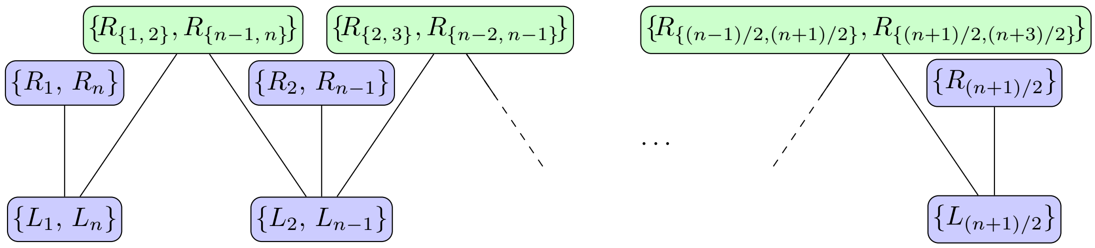

Figure 5.16: The graph $\Delta(\mathcal{POD}_n)$, when n is odd. Note that the orbits $\{L_{(n+1)/2}\}$ and $\{R_{(n+1)/2}\}$ are singeltons, and that $\Delta(\mathcal{POD}_n) \cong \Delta(\mathcal{PO}_{(n+1)/2})$; see Figure 5.15. The vertices of degree 1 in the graph are those orbits that contain $\mathscr{R}$-classes composed of partial permutations.

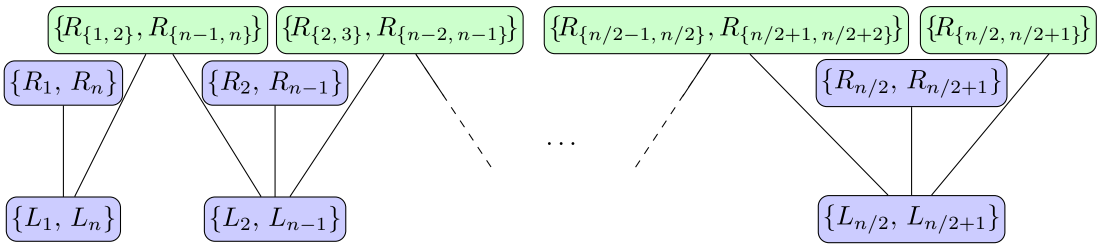

Figure 5.17: The graph $\Delta(\mathcal{POD}_n)$, when n is even. Note that the orbit $\{R_{\{n/2,n/2+1\}}\}$ is a singleton; this orbit and the orbits of $\mathscr{R}$-classes of partial permutations are the vertices of degree 1 in the graph.

The group of units of $\mathcal{POD}_n$ is generated by $(1\ n)(2\ n-1)\cdots(\lfloor (n+1)/2 \rfloor\ \lceil (n+1)/2 \rceil)$, the unique order-reversing permutation. Since $\langle (1\ n)(2\ n-1)\cdots(\lfloor (n+1)/2 \rfloor\ \lceil (n+1)/2 \rceil) \rangle$ has $\lceil n/2 \rceil$ orbits on the set $\{1, \ldots, n\}$, it follows by Lemma 5.6 that there are $\lceil n/2 \rceil$ corresponding orbits of $\mathscr{L}$-classes and $\lceil n/2 \rceil$ orbits of $\mathscr{R}$-classes of partial permutations. Furthermore, there are $\lfloor n/2 \rfloor$

orbits of $\langle(1\ n)(2\ n-1)\cdots(\lfloor(n+1)/2\rfloor\ \lceil(n+1)/2\rceil)\rangle$ on the set $\big\{\{i,i+1\} : i\in\{1,\ldots,n-1\}\big\}$, and these orbits correspond to $\lfloor n/2\rfloor$ orbits of $\mathscr{R}$-classes of transformations. Due to the division by 2 in each case, the properties of $\Delta(\mathcal{POD}_n)$ depend on the parity of n. A picture of $\Delta(\mathcal{POD}_n)$ is shown in Figure 5.16 for odd n, and in Figure 5.17 for even n; see these pictures for a description of the edges of this graph. When n is odd, the graph $\Delta(\mathcal{POD}_n)$ is isomorphic to $\Delta(\mathcal{PO}_{\lceil n/2\rceil})$.

Given descriptions of $\Delta(\mathcal{PO}_n)$ and $\Delta(\mathcal{POD}_n)$, we establish the following lemmas.

Lemma 5.18. *Let K be any subset of then vertices of the graph $\Delta(\mathcal{PO}_n)$. Then K is a maximal independent subset of $\Delta(\mathcal{PO}_n)$ if and only if*

$$K=\big\{\{L_i\cap\mathcal{PO}_n\} : i\in A\big\}\cup\big\{\{R_i\cap\mathcal{PO}_n\} : i\notin A\big\}\cup\big\{\{R_{\{i,\,i+1\}}\cap\mathcal{PO}_n\} : i,i+1\notin A\big\}$$

for some subset A of $\{1,\ldots,n\}$.

Proof. ($\Rightarrow$) Suppose that K is a maximal independent subset of $\Delta(\mathcal{PO}_n)$. There exists a subset $A\subseteq\{1,\ldots,n\}$ of indices such that $\big\{\{L_i\cap\mathcal{PO}_n\} : i\in A\big\}$ is the collection of $\mathscr{L}$-class vertices in K. Since a vertex of the form $\{R_i\cap\mathcal{PO}_n\}$ is adjacent in $\Delta(\mathcal{PO}_n)$ only to the vertex $\{L_i\cap\mathcal{PO}_n\}$, it follows by the maximality of K that $\{R_i\cap\mathcal{PO}_n\}\in K$ if and only if $i\notin A$. Similarly, since an orbit of the form $\{R_{\{i,\,i+1\}}\cap\mathcal{PO}_n\}$ is adjacent in $\Delta(\mathcal{PO}_n)$ only to the orbits $\{L_i\cap\mathcal{PO}_n\}$ and $\{L_{i+1}\cap\mathcal{PO}_n\}$, it follows that $\{R_{\{i,\,i+1\}}\cap\mathcal{PO}_n\}\in K$ if and only if $i\notin A$ and $i+1\notin A$. We have considered all vertices of $\Delta(\mathcal{PO}_n)$, and so K has the required form.

($\Leftarrow$) It is easy to verify that K is a maximal independent subset of $\Delta(\mathcal{PO}_n)$. $\square$

The following lemma is very similar to Lemma 5.18, and is therefore stated without proof.

Lemma 5.19. *Let K be any collection of vertices of the graph $\Delta(\mathcal{POD}_n)$. Then K is a maximal independent subset of $\Delta(\mathcal{POD}_n)$ if and only if K is equal to*

$$\begin{aligned}\big\{\{L_i\cap\mathcal{POD}_n, L_{n-i+1}\cap\mathcal{POD}_n\} : i\in A\big\}\cup\big\{\{R_i\cap\mathcal{POD}_n, R_{n-i+1}\cap\mathcal{POD}_n\} : i\notin A\big\}\\ \cup\big\{\{R_{\{i,\,i+1\}}\cap\mathcal{POD}_n, R_{\{n-i,\,n-i+1\}}\cap\mathcal{POD}_n\} : i,i+1\notin A\big\},\end{aligned}$$

for some subset A of $\{1,\ldots,\lceil n/2\rceil\}$.

We may now prove the main results of this section.

Proof of Theorems 5.12 and 5.13. The group of units of $\mathcal{PO}_n$ is trivial, and the group of units of $\mathcal{POD}_n$ has order 2. By Lemma 4.73, the maximal subsemigroup that arises from the group of units in each instance is the one described.

Let $S\in\{\mathcal{PO}_n,\ \mathcal{POD}_n\}$. Since $\mathcal{POD}_n$ is generated by $\mathcal{PO}_n$ and the order-reversing permutation, and since $\mathcal{PO}_n$ is idempotent generated, it follows by Lemma 4.49(b) that there are no maximal subsemigroups of type (M1) arising from $J_{n-1}\cap S$. It follows directly from Proposition 4.77, and Lemmas 5.18 and 5.19, that the maximal subsemigroups of type (M2) are those described in the theorem. There exist vertices of degree 1 in $\Delta(S)$: these are orbits of $\mathscr{R}$-classes, and each orbit of $\mathscr{L}$-classes is adjacent to a vertex of degree 1. Thus by Proposition 4.79 and its dual, there are no maximal subsemigroups of type (M3) arising from S, but each orbit of $\mathscr{R}$-classes can be removed to provide a maximal subsemigroup of type (M4). There are $2n-1$ maximal subsemigroups of this type for $\mathcal{PO}_n$, and there are n of this type for $\mathcal{POD}_n$. By Corollary 4.75, there is no maximal subsemigroup of S of type (M5). $\square$

5.2.5 $\mathcal{O}_n$ and $\mathcal{OD}_n$

The maximal subsemigroups of the singular ideal $\mathcal{O}_n \setminus \{\mathrm{id}_n\}$ of $\mathcal{O}_n$ were incorrectly described and counted in [130]: the formula $n^2 - 2n + 2$ presented for the number of maximal subsemigroups is correct for $2 \leq n \leq 5$, but gives only a lower bound when $n \geq 6$; see Table 5.28. A correct description, although no formula for the sequence of the numbers, was later given in [25, Theorem 2]. Since the group of units of $\mathcal{O}_n$ is trivial, the maximal subsemigroups of its singular ideal correspond in an obvious way to the maximal subsemigroups of $\mathcal{O}_n$. Furthermore, the maximal subsemigroups of the singular ideal $\mathcal{OD}_n \setminus \langle (1\, n)(2\, n{-}1) \cdots (\lfloor (n{+}1)/2 \rfloor\, \lceil (n{+}1)/2 \rceil) \rangle$ of $\mathcal{OD}_n$ were described in [69, Theorem 2]. However, the group of units of $\mathcal{OD}_n$ acts on $\mathcal{OD}_n$ non-trivially, and so the correspondence between the maximal subsemigroups of the singular ideal and the maximal subsemigroups of $\mathcal{OD}_n$ itself is lost. Thus [69] solves an essentially different problem than the description of the maximal subsemigroups of $\mathcal{OD}_n$.

We summarise the information about $\mathcal{O}_n$ and $\mathcal{OD}_n$ from Section 5.2.2. Let $n \in \mathbb{N}$, $n \geq 2$, and let $S \in \{\mathcal{O}_n, \mathcal{OD}_n\}$. Then S is a regular monoid, the set $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of S, the set of $\mathscr{L}$-classes of $J_{n-1} \cap S$ is $\big\{L_i \cap S : i \in \{1, \ldots, n\}\big\}$, and the set of $\mathscr{R}$-classes of $J_{n-1} \cap S$ is $\big\{R_{\{i,\, i+1\}} \cap S : i \in \{1, \ldots, n-1\}\big\}$. Additionally, $\mathcal{O}_n$ is generated by its idempotents of rank $n - 1$ [1], and $\mathcal{OD}_n$ is generated by $\mathcal{O}_n$ and the order-reversing permutation [49].

Since $\mathcal{O}_n = \mathcal{PO}_n \cap \mathcal{T}_n$, we may identify a Green's class of $J_{n-1} \cap \mathcal{O}_n$ with the Green's class of $J_{n-1} \cap \mathcal{PO}_n$ that contains it, so that $L_i \cap \mathcal{O}_n$ corresponds with $L_i \cap \mathcal{PO}_n$, and $R_{\{i,\, i+1\}} \cap \mathcal{O}_n$ corresponds with $R_{\{i,\, i+1\}} \cap \mathcal{PO}_n$. In this way, we obtain $\Delta(\mathcal{O}_n)$ as the induced subgraph of $\Delta(\mathcal{PO}_n)$ on those orbits of Green's classes that contain transformations. In other words, the definition of $\Delta(\mathcal{O}_n)$ is contained in that of $\Delta(\mathcal{PO}_n)$. The graph $\Delta(\mathcal{O}_n)$ contains n singleton orbits of $\mathscr{L}$-classes, and $n - 1$ singleton orbits of $\mathscr{R}$-classes; see Figure 5.20 for a picture.

In a similar way, $\Delta(\mathcal{OD}_n)$ may be obtained as the induced subgraph of $\Delta(\mathcal{POD}_n)$ on its orbits of Green's classes that contain transformations. This graph contains n vertices; a picture of $\Delta(\mathcal{OD}_n)$ is shown in Figure 5.21 for odd n, and in Figure 5.17 for even n.

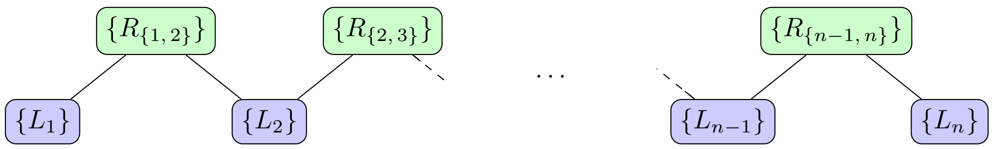

Figure 5.20: The graph $\Delta(\mathcal{O}_n)$, which is a path graph of order $2n - 1$. Since the group of units of $\mathcal{O}_n$ is trivial, $\Delta(\mathcal{O}_n)$ is isomorphic to the Graham-Houghton graph of the principal factor of $J_{n-1} \cap \mathcal{O}_n$. Note that $\Delta(\mathcal{O}_n)$ may be obtained as the induced subgraph of $\Delta(\mathcal{PO}_n)$ on its orbits of Green's classes that contain transformations; see Figure 5.15. The vertices of degree 1 in $\Delta(\mathcal{O}_n)$ are $\{L_1\}$ and $\{L_n\}$.

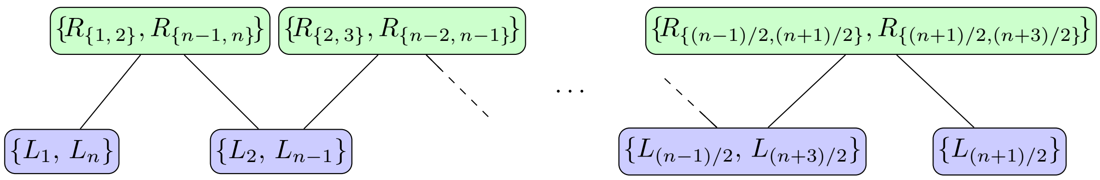

Figure 5.21: The graph $\Delta(\mathcal{OD}_n)$, when n is odd. This is a path graph of order n. Note that $\Delta(\mathcal{OD}_n) \cong \Delta\big(\mathcal{O}_{n+1/2}\big)$, and that $\Delta(\mathcal{OD}_n)$ may be obtained as the induced subgraph of $\Delta(\mathcal{POD}_n)$ on its orbits of Green's classes that contain transformations; see Figure 5.16. The vertices of degree 1 in $\Delta(\mathcal{OD}_n)$ are $\{L_1, L_n\}$ and $\{L_{(n+1)/2}\}$.

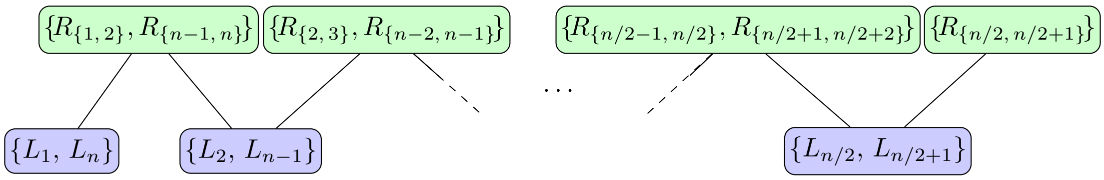

Figure 5.22: The graph $\Delta(\mathcal{OD}_n)$, when n is even. This is a path graph of order n. Note that $\Delta(\mathcal{OD}_n)$ may be obtained as the induced subgraph of $\Delta(\mathcal{POD}_n)$ on its orbits of Green's classes that contain transformations; see Figure 5.17. The vertices of degree 1 in $\Delta(\mathcal{OD}_n)$ are $\{L_1, L_n\}$ and $\{R_{\{n/2,\, n/2+1\}}\}$.

For $k \in \mathbb{N}$, define the *path graph* of order k to be the graph with vertex set $\{1, \ldots, k\}$, and edge set

$$\big\{\{i,\, i+1\} \,:\, i \in \{1, \ldots, k-1\}\big\}.$$

The vertices of degree 1 in the path graph of order k are the end-points, 1 and k. It is easy to see that $\Delta(\mathcal{O}_n)$ is isomorphic to the path graph of order $2n-1$, via the isomorphism that maps the orbit $\{L_i \cap \mathcal{O}_n\}$ to the vertex $2i-1$, and maps the orbit $\{R_{\{i,\,i+1\}} \cap \mathcal{O}_n\}$ to the vertex $2i$. Similarly, $\Delta(\mathcal{OD}_n)$ is isomorphic to the path graph of order n. We can describe and count the number of maximal independent subsets of a path graph, and therefore the maximal independent subsets of $\Delta(\mathcal{O}_n)$ and $\Delta(\mathcal{OD}_n)$, by using the following results.

Lemma 5.23. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary, let Γ be the path graph of order n, and let U be a subset of the vertices of Γ. Then U is a maximal independent subset of Γ if and only if the following conditions hold:*

(a) *the least vertex in U is either 1 or 2; and*

(b) *for each $i \in U \cap \{1, \ldots, n-1\}$, $i+1 \notin U$; and*

(c) *for each $i \in U \cap \{1, \ldots, n-2\}$, exactly one of $i+2$ and $i+3$ is contained in U.*

Proof. Since vertices in Γ are adjacent if and only if they are consecutive, U is an independent subset of Γ if and only if (b) holds. It is easy to verify that an independent subset U of Γ is maximal when conditions (a) and (c) hold. Conversely, if U satisfies (b) but contains neither 1 nor 2, then $U \cup \{1\}$ is an independent subset properly containing U, and U is not maximal. Similarly, suppose that U satisfies (b) and contains some $i \in \{1, \ldots, n-2\}$, but contains neither $i+2$ nor $i+3$. Then since U also does not contain $i+1$, it follows that $U \cup \{i+2\}$ is an independent subset properly containing U, and U is not maximal. Thus, if U is a maximal independent subset of Γ, then conditions (a) and (c) hold. □

There are two special maximal independent subsets of a path graph: the subset of all even vertices, and the subset of all odd vertices. These maximal independent subsets correspond to the subset of all orbits of $\mathscr{L}$-classes, and the subset of all orbits of $\mathscr{R}$-classes, of $\Delta(\mathcal{O}_n)$ and $\Delta(\mathcal{OD}_n)$. These are the unique maximal independent subsets that do not give rise to maximal subsemigroups of $\mathcal{O}_n$ and $\mathcal{OD}_n$ of type (M2): see Proposition 4.77 and Corollary 4.78.

Corollary 5.24. *The number of maximal independent subsets of the path graph is counted by the sequence $(A_n)_{n\in\mathbb{N}}$, defined by*

$$A_1 = 1,\ \ A_2 = A_3 = 2,\ \ \textit{and } A_n = A_{n-2} + A_{n-3} \textit{ for } n \geq 4. \tag{5.25}$$

Note that A_n is the $(n+6)^{\text{th}}$ term of the Padovan sequence [120, A000931], which satisfies the same recurrence, with different initial values.

Proof. For $n \in \mathbb{N}$, define Γ_n to be the path graph of order n, and let $f(n)$ denote the number of maximal independent subsets of Γ_n. It is straightforward to verify that $f(1) = 1$, and $f(2) = f(3) = 2$, so suppose that $n \geq 4$. By Lemma 5.23, if U is a maximal independent subset of Γ_{n-3}, then $U \cup \{n-1\}$ is a maximal independent subset of Γ_n, and if U is a maximal independent subset of Γ_{n-2}, then $U \cup \{n\}$ is a maximal independent subset of Γ_n. Thus distinct maximal independent subsets of Γ_{n-3} and Γ_{n-2} give rise to distinct maximal independent subsets of Γ_n, and $f(n) \geq f(n-2) + f(n-3)$. Conversely, if U is a maximal independent subset of Γ_n, then by Lemma 5.23, U contains either $n-1$ or n. If $n-1 \in U$, then $n-2 \notin U$, which implies that $U \setminus \{n-1\}$ is a maximal independent subset of Γ_{n-3}. Otherwise, $U \setminus \{n\}$ is a maximal independent subset of Γ_{n-2}. Thus $f(n) \leq f(n-2) + f(n-3)$, and $f(n) = A(n)$. ☐

We may now describe and count the maximal subsemigroups of $\mathcal{O}_n$ and $\mathcal{OD}_n$.

Theorem 5.26. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. The maximal subsemigroups of $\mathcal{O}_n$ are:*

(a) $\mathcal{O}_n \setminus \{\mathrm{id}_n\}$ (type (M5))*;*

(b) *the union of $\mathcal{O}_n \setminus J_{n-1}$ and the union of the Green's classes in*

$$\{L_{(i+1)/2} \cap \mathcal{O}_n \,:\, i \in A,\ i \text{ is odd}\} \cup \{R_{\{i/2,\,(i/2)+1\}} \cap \mathcal{O}_n \,:\, i \in A,\ i \text{ is even}\},$$

where A is a maximal independent subset of the path graph of order $2n-1$ that contains both odd and even numbers, as described in Lemma 5.23 (type (M2))*;*

(c) $\mathcal{O}_n \setminus L$*, where L is any $\mathscr{L}$-class in $J_{n-1} \cap \mathcal{O}_n$* (type (M3))*; and*

(d) $\mathcal{O}_n \setminus R_{\{i,\,i+1\}}$*, where $i \in \{2, \ldots, n-2\}$* (type (M4)).

In particular, for $n \geq 3$ there are $A_{2n-1} + 2n - 4$ maximal subsemigroups of $\mathcal{O}_n$, where A_{2n-1} is as defined in (5.25).

Theorem 5.27. *Let $n \in \mathbb{N}$, $n \geq 3$, be arbitrary. The maximal subsemigroups of $\mathcal{OD}_n$ are:*

(a) $\mathcal{OD}_n \setminus \{(1\ n)(2\ n-1) \cdots (\lfloor (n+1)/2 \rfloor\ \lceil (n+1)/2 \rceil)\}$ (type (M1))*;*

(b) *the union of $\mathcal{OD}_n \setminus J_{n-1}$ and the union of the sets in*

$$\{(L_{(i+1)/2} \cup L_{n+1-(i+1)/2}) \cap \mathcal{OD}_n \,:\, i \in A,\ i \text{ is odd}\} \\ \cup \{(R_{\{i/2,\,(i/2)+1\}} \cup R_{\{n-(i/2), n+1-(i/2)\}}) \cap \mathcal{OD}_n \,:\, i \in A,\ i \text{ is even}\},$$

where A is a maximal independent subset of the path graph of order n that contains both odd and even numbers, as described in Lemma 5.23 (type (M2))*;*

(c) $\mathcal{OD}_n \setminus (L_i \cup L_{n-i+1})$*, where* $\begin{cases} i \in \{1, \ldots, (n+1)/2\} & \text{if } n \text{ is odd,} \\ i \in \{1, \ldots, n/2 - 1\} & \text{if } n \text{ is even} \end{cases}$ (type (M3))*; and*

(d) $\mathcal{OD}_n \setminus (R_{\{i,\,i+1\}} \cup R_{\{n-i, n-i+1\}})$*, where* $\begin{cases} i \in \{2, \ldots, (n-3)/2\} & \text{if } n \text{ is odd,} \\ i \in \{2, \ldots, n/2\} & \text{if } n \text{ is even} \end{cases}$ (type (M4)).

In particular, for $n \geq 4$, there are $A_n + n - 3$ maximal subsemigroups of $\mathcal{OD}_n$, where A_n is as defined in (5.25).

	1	2	3	4	5	6	7	8	9	10	11	12	n
$\mathcal{O}_n$	1	3	6	11	18	29	47	77	128	216	369	636	$A_{2n-1}+2n-4$
$\mathcal{OD}_n$	1	2	3	4	6	8	11	14	18	23	29	37	A_n+n-3

Table 5.28: The numbers of maximal subsemigroups of the monoids $\mathcal{O}_n$ and $\mathcal{OD}_n$, for $n = 1, \ldots, 12$, along with the general formulae. The sequence $(A_n)_{n\in\mathbb{N}}$ is defined in (5.25). Note that the number of maximal subsemigroups of $\mathcal{O}_n$ equals the number of maximal subsemigroups of $\mathcal{OD}_{2n-1}$. See Theorems 5.26 and 5.27 for descriptions of the maximal subsemigroups of $\mathcal{O}_n$ and $\mathcal{OD}_n$, respectively.

Proof of Theorems 5.26 and 5.27. The group of units of $\mathcal{O}_n$ is trivial, and the group of units of $\mathcal{OD}_n$ has order 2. By Lemma 4.73, the maximal subsemigroup that arises from the group of units in each instance is the one described.

Since $\mathcal{O}_n$ is generated by id_n and its idempotents of rank $n-1$, and since $\mathcal{OD}_n$ is generated by these idempotents and its non-identity permutation, it follows by Lemma 4.49(b) that there are no maximal subsemigroups of type (M1) arising from the $\mathscr{J}$-class consisting of elements of rank $n-1$. Since $\Delta(\mathcal{O}_n)$ and $\Delta(\mathcal{OD}_n)$ are path graphs of length $2n-1$ and n, respectively, it follows by Proposition 4.77 and Lemma 5.23 that the maximal subsemigroups of type (M2) are those described in the theorems. By Corollary 4.78 and Corollary 5.24, the number of maximal subsemigroups of type (M2) is $A_{2n-1}-2$ for $\mathcal{O}_n$, and A_n-2 for $\mathcal{OD}_n$.

Let $S \in \{\mathcal{O}_n, \mathcal{OD}_n\}$. By Corollary 4.75, and since $n \geq 4$, there is no maximal subsemigroup of S of type (M5). To describe the maximal subsemigroups of types (M3) and (M4), it suffices to identify the two vertices of $\Delta(S)$ that are adjacent to the end-points of $\Delta(S)$. From this, the description of the maximal subsemigroups of types (M3) and (M4) follows from Proposition 4.79 and its dual. In particular, the total number of both types of maximal subsemigroups is two less than the number of vertices of $\Delta(S)$. □

5.2.6 $\mathcal{POI}_n$ and $\mathcal{PODI}_n$

The maximal subsemigroups of $\mathcal{POI}_n$ are described and counted in [56, Theorem 2], and those of $\mathcal{PODI}_n$ are described and counted in [29, Theorem 4]. Extensive additional information about $\mathcal{POI}_n$ may be found in [48, 56]. We reprove these results in this section, for completeness.

We recall the following information from Section 5.2.2. Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary, and let $S \in \{\mathcal{POI}_n, \mathcal{PODI}_n\}$. Then S is an inverse monoid, and $J_{n-1} \cap S$ is a $\mathscr{J}$-class of S. Since $\mathcal{POI}_n$ is generated by id_n and its elements of rank $n-1$ [48, Lemma 2.7], and since $\mathcal{PODI}_n$ is generated by $\mathcal{POI}_n$ and the permutation $(1\ n)(2\ n-1)\cdots(\lfloor(n+1)/2\rfloor\ \lceil(n+1)/2\rceil)$, it follows by Lemma 4.68 that the maximal subsemigroups of S arise from its group of units and from its $\mathscr{J}$-class $J_{n-1} \cap S$. By definition, $\mathcal{POI}_n = \mathcal{PO}_n \cap \mathcal{I}_n$, and $\mathcal{PODI}_n = \mathcal{POD}_n \cap \mathcal{I}_n$, and so given the description of the Green's classes of $\mathcal{PO}_n$ and $\mathcal{POD}_n$ from Section 5.2.4, it follows that

$$(J_{n-1} \cap S)/\mathscr{L} = \{L_i \cap S : i \in \{1, \ldots, n\}\}, \text{ and } (J_{n-1} \cap S)/\mathscr{R} = \{R_i \cap S : i \in \{1, \ldots, n\}\}.$$

In Theorems 5.29 and 5.30, we describe the maximal subsemigroups of these inverse monoids.

Theorem 5.29. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. The maximal subsemigroups of $\mathcal{POI}_n$ are:*

(a) $\mathcal{POI}_n \setminus \{\mathrm{id}_n\}$ (type (M5)); *and*

(b) *the union of $\mathcal{POI}_n \setminus J_{n-1}$ and the union of*

$$\{L_i \cap \mathcal{POI}_n : i \in A\} \cup \{R_i \cap \mathcal{POI}_n : i \notin A\},$$

where A is any non-empty proper subset of $\{1, \ldots, n\}$ (type (M2)).

In particular, for $n \geq 2$, there are $2^n - 1$ maximal subsemigroups of $\mathcal{POI}_n$.

Proof. The inverse monoid $\mathcal{POI}_n$ is $\mathscr{H}$-trivial, and so by Lemma 4.49(a) there are no maximal subsemigroups of type (M1), and by Lemma 4.73, the unique maximal subsemigroup arising from the group of units is formed by removing id_n. Since the group of units of $\mathcal{POI}_n$ is trivial, its action on the $\mathscr{L}$-classes of $J_{n-1} \cap \mathcal{POI}_n$ is also trivial. Therefore, there are n singleton orbits. The $\mathscr{R}$-class $R_i \cap \mathcal{POI}_n$ is equal to $(L_i \cap \mathcal{POI}_n)^{-1}$, and so by Corollary 4.82, the maximal subsemigroups of $\mathcal{POI}_n$ arising from $J_{n-1} \cap \mathcal{POI}_n$ are those of type (M2) described in the theorem. □

Theorem 5.30. *Let $n \in \mathbb{N}$, $n \geq 3$, be arbitrary. For $i, j \in \{1, \ldots, n\}$, define $\alpha_{i,j}$ to be the order-preserving partial permutation with domain $\{1, \ldots, n\} \setminus \{i\}$ and image $\{1, \ldots, n\} \setminus \{j\}$, and define $\beta_{i,j}$ to be the order-reversing partial permutation with this domain and image. The maximal subsemigroups of $\mathcal{PODI}_n$ are:*

(a) $\mathcal{PODI}_n \setminus \{(1\ n)(2\ n-1) \cdots (\lfloor (n+1)/2 \rfloor\ \lceil (n+1)/2 \rceil)\}$ (type (M1));

(b) $(\mathcal{PODI}_n \setminus J_{n-1}) \cup I_A$, *where n is even,*

$$I_A = \big\{\alpha_{i,j},\ \beta_{i,n-j+1},\ \beta_{n-i+1,j},\ \alpha_{n-i+1,n-j+1} \ :\ i, j \in A \text{ or } i, j \notin A\big\}$$
$$\cup \big\{\beta_{i,j},\ \alpha_{i,n-j+1},\ \alpha_{n-i+1,j},\ \beta_{n-i+1,n-j+1} \ :\ i \in A,\ j \notin A \text{ or } i \notin A,\ j \in A\big\},$$

and A is any subset of $\{2, \ldots, n/2\}$ (type (M1)); *and*

(c) *the union of $\mathcal{PODI}_n \setminus J_{n-1}$ and the union of the sets in*

$$\big\{(L_i \cup L_{n-i+1}) \cap \mathcal{PODI}_n \ :\ i \in A\big\} \cup \big\{(R_i \cup R_{n-i+1}) \cap \mathcal{PODI}_n \ :\ i \notin A\big\},$$

where A is any non-empty proper subset of $\{1, \ldots, \lceil n/2 \rceil\}$ (type (M2)).

In particular, for $n \geq 3$ there are $3 \cdot 2^{(n/2)-1} - 1$ maximal subsemigroups of $\mathcal{PODI}_n$ when n is even, and $2^{(n+1)/2} - 1$ when n is odd.

	1	2	3	4	5	6	7	8	9	10	n
$\mathcal{POI}_n$	2	3	7	15	31	63	127	255	511	1023	$2^n - 1$
$\mathcal{PODI}_n$	2	2	3	5	7	11	15	23	31	47	$\begin{cases} 3 \cdot 2^{(n/2)-1} - 1 & \text{even } n \\ 2^{(n+1)/2} - 1 & \text{odd } n \end{cases}$

Table 5.31: The numbers of maximal subsemigroups of the monoids $\mathcal{POI}_n$ and $\mathcal{PODI}_n$, for $n = 1, \ldots, 10$, along with the general formulae. Note that the number of maximal subsemigroups of $\mathcal{POI}_n$ equals the number of maximal subsemigroups of $\mathcal{PODI}_{2n-1}$. See Theorems 5.29 and 5.30 for descriptions of the maximal subsemigroups of $\mathcal{POI}_n$ and $\mathcal{PODI}_n$, respectively.

Proof. Since the group of units of $\mathcal{PODI}_n$ is generated by the permutation $(1\ n)(2\ n-1) \cdots (\lfloor (n+1)/2 \rfloor\ \lceil (n+1)/2 \rceil)$, which has order two, it follows by Lemma 4.73 that the unique maximal subsemigroup arising from the group of units of $\mathcal{PODI}_n$ is formed by removing this non-identity permutation.

The graph $\Delta(\mathcal{PODI}_n)$ may be obtained as the induced subgraph of $\Delta(\mathcal{POD}_n)$ on those orbits of Green's classes that contain partial permutations. In particular, the orbits of the group of units $\langle(1\ n)(2\ n-1)\cdots(\lfloor(n+1)/2\rfloor\ \lceil(n+1)/2\rceil)\rangle$ on the $\mathscr{L}$-classes of $J_{n-1}\cap\mathcal{PODI}_n$ are the sets $\{L_i\cap\mathcal{PODI}_n,\ L_{n-i+1}\cap\mathcal{PODI}_n\}$, for each $i\in\{1,\ldots,\lceil n/2\rceil\}$. By Corollary 4.82, the maximal subsemigroups of $\mathcal{PODI}_n$ that arise from $J_{n-1}\cap\mathcal{PODI}_n$ are those $2^{\lceil n/2\rceil}-2$ maximal subsemigroups of type (M2) described in the statement of the theorem, as well as any maximal subsemigroups of type (M1).

It remains to describe the maximal subsemigroups of $\mathcal{PODI}_n$ that arise from $J_{n-1}\cap\mathcal{PODI}_n$ and that have type (M1); suppose that M is such a maximal subsemigroup. By Proposition 4.10(b), the intersection of M with each $\mathscr{H}$-class of $J_{n-1}\cap\mathcal{PODI}_n$ is non-empty, and each of these intersections has some common size, $q\geq 1$. Since an $\mathscr{H}$-class in $J_{n-1}\cap\mathcal{PODI}_n$ contains two elements, and M is a proper subsemigroup of $\mathcal{PODI}_n$ that lacks only elements from $J_{n-1}\cap\mathcal{PODI}_n$, it follows that $q=1$. In other words, the intersection of M with each $\mathscr{H}$-class of $J_{n-1}\cap\mathcal{PODI}_n$ contains a single element. For $i,j\in\{1,\ldots,n\}$, let $\delta_{i,j}$ denote the unique element of M that is contained in the $\mathscr{H}$-class $\mathcal{PODI}_n\cap(L_i\cap R_j)=\{\alpha_{i,j},\ \beta_{i,j}\}$ of $\mathcal{PODI}_n$. In other words, $M\cap(L_i\cap R_j)=\{\delta_{i,j}\}$.

Since M contains γ, it follows that $\delta_{i,j}\in M$ if and only if $\delta_{i,j}\cdot\gamma\in M$, $\gamma\cdot\delta_{i,j}\in M$, and $\gamma\cdot\delta_{i,j}\cdot\gamma\in M$. In particular, $\alpha_{i,j}\in M$ if and only if $\alpha_{i,j},\ \beta_{i,n-j+1},\ \beta_{n-i+1,j},\ \alpha_{n-i+1,n-j+1}\in M$, and $\beta_{i,j}\in M$ if and only if $\beta_{i,j},\ \alpha_{i,n-j+1},\ \alpha_{n-i+1,j},\ \beta_{n-i+1,n-j+1}\in M$. For odd n, this leads to the statement that $\alpha_{(n+1)/2,(n+1)/2}\in M$ if and only if $\beta_{(n+1)/2,(n+1)/2}\in M$, contradicting the fact that M contains exactly 1 element in each $\mathscr{H}$-class of $J_{n-1}\cap\mathcal{PODI}_n$. Hence n is even.

Given these observations, in order to describe M, it suffices to specify $\delta_{i,j}$ for each $i,j\in\{1,\ldots,n/2\}$. Indeed, our description can be even more concise. We observe that $\delta_{i,i}=\alpha_{i,i}$, since M contains every idempotent of $\mathcal{PODI}_n$. This implies that $\delta_{i,j}\cdot\delta_{j,i}=\delta_{i,i}=\alpha_{i,i}$, and so $\delta_{i,j}=\alpha_{i,j}$ if and only if $\delta_{j,i}=\alpha_{j,i}$. Furthermore, $\delta_{i,j}=\delta_{i,1}\delta_{1,j}$. Thus, to specify $\delta_{i,j}$ for each $i,j\in\{1,\ldots,n\}$, it suffices to specify $\delta_{1,i}$ for each $i\in\{2,\ldots,n/2\}$. Let $A=\big\{i\in\{2,\ldots,n/2\}:\delta_{1,i}=\beta_{1,j}\big\}$. A routine calculation shows that $M=(\mathcal{PODI}_n\setminus J_{n-1})\cup I_A$, where I_A is the set defined in the statement of the theorem.

Conversely, for an even number $n\geq 4$ and a subset $A\subseteq\{2,\ldots,n/2\}$, it is tedious, but routine, to verify that $(\mathcal{PODI}_n\setminus J_{n-1})\cup I_A$ is a subsemigroup of $\mathcal{PODI}_n$; by construction, it intersects every $\mathscr{H}$-class of $\mathcal{PODI}_n$ non-trivially. Any maximal subsemigroup of $\mathcal{PODI}_n$ that contains $(\mathcal{PODI}_n\setminus J_{n-1})\cup I_A$ has type (M1), and so by the preceding arguments, we see that it is equal to $(\mathcal{PODI}_n\setminus J_{n-1})\cup I_A$. Thus $(\mathcal{PODI}_n\setminus J_{n-1})\cup I_A$ is a maximal subsemigroup of $\mathcal{PODI}_n$ of type (M1).

For two subsets $A,A'\subseteq\{2,\ldots,n/2\}$, it is clear from the definitions that $I_A=I_{A'}$ if and only if $A=A'$. Thus there are $2^{(n/2)-1}$ maximal subsemigroups of type (M1) when n is even, and none when n is odd. □

5.2.7 $\mathcal{POP}_n$, $\mathcal{POR}_n$, $\mathcal{OP}_n$, and $\mathcal{OR}_n$

The maximal subsemigroups of $\mathcal{OP}_n$ were described in [28, Theorem 1.6], and the maximal subsemigroups of $\mathcal{OR}_n$ were described in [28, Theorem 2.6]. Prior to the publication of [45], no description of the maximal subsemigroups of $\mathcal{POP}_n$ and $\mathcal{POR}_n$ had appeared in the literature.

To state the results of this section, we require the following notation and information from Section 5.2.2. Let $n\in\mathbb{N}$, $n\geq 2$, be arbitrary, and let $S\in\{\mathcal{POP}_n,\ \mathcal{POR}_n,\ \mathcal{OP}_n,\ \mathcal{OR}_n\}$. Then $J_{n-1}\cap S$ is a regular $\mathscr{J}$-class of S. Since $\mathcal{O}_n$ and $\mathcal{PO}_n$ are each generated by $\{\mathrm{id}_n\}$ and their idempotents of rank $n-1$ [60], it follows that S is generated by its units and idempotents of rank $n-1$.

The $\mathscr{L}$-classes contained in $J_{n-1}\cap S$ are the intersections $L_i\cap S$, for each $i\in\{1,\ldots,n\}$. If

$S \in \{\mathcal{POP}_n, \mathcal{POR}_n\}$, then the $\mathscr{R}$-classes of $J_{n-1} \cap S$ are the sets $R_i \cap S$, for each $i \in \{1, \ldots, n\}$, and $R_{\{i, i+1\}} \cap S$, for each $i \in \{1, \ldots, n-1\}$, along with the set $R_{\{1, n\}} \cap S$. If $S \in \{\mathcal{OP}_n, \mathcal{OR}_n\}$, then $S \leq \mathcal{T}_n$, and so the $\mathscr{R}$-classes of $J_{n-1} \cap S$ are the sets $R_{\{i, i+1\}} \cap S$ for each $i \in \{1, \ldots, n-1\}$ and $R_{\{1, n\}} \cap S$. The group of units of $\mathcal{POP}_n$ and $\mathcal{OP}_n$ is $\mathcal{C}_n$, defined in (5.4), and the group of units of $\mathcal{POR}_n$ and $\mathcal{OR}_n$ is $\mathcal{D}_n$, defined in (5.5).

The following theorems are the main results of this section.

Theorem 5.32. *Let $n \in \mathbb{N}$, $n \geq 3$, be arbitrary, let $S \in \{\mathcal{POP}_n, \mathcal{POR}_n\}$, and let G be the group of units of S. The maximal subsemigroups of S are:*

(a) $(S \setminus G) \cup U$, *where U is a maximal subgroup of G* (type (M1))*; and*

(b) $S \setminus \{\alpha \in S \cap \mathcal{T}_n : \operatorname{rank}(\alpha) = n - 1\}$ (type (M4))*; and*

(c) $S \setminus \{\alpha \in S \cap \mathcal{I}_n : \operatorname{rank}(\alpha) = n - 1\}$ (type (M4)).

In particular, for $n \geq 3$, there are $|\mathbb{P}_n| + 2$ maximal subsemigroups of $\mathcal{POP}_n$, and $3 + \sum_{p \in \mathbb{P}_n} p$ maximal subsemigroups of $\mathcal{POR}_n$, where $\mathbb{P}_n$ is the set of primes that divide n.

Proof. By Lemmas 4.73, 5.1, and 5.2, the maximal subsemigroups that arise from the group of units are those described in the statement of the theorem; there are $|\mathbb{P}_n|$ such maximal subsemigroups of $\mathcal{POP}_n$, and $1 + \sum_{p \in \mathbb{P}_n} p$ such maximal subsemigroups of $\mathcal{POR}_n$.

Let $S \in \{\mathcal{POP}_n, \mathcal{POR}_n\}$, and let G be the group of units of S. Since $\mathcal{PO}_n$ is idempotent generated, and $S = \langle \mathcal{PO}_n, G \rangle$, it follows by Lemma 4.49(b) that there are no maximal subsemigroups of type (M1) arising from $J_{n-1} \cap S$.

The remainder of the proof is similar to the discussion in Section 5.2.3 after the proof of Theorem 5.7. The group of units G of S acts transitively on the $\mathscr{L}$-classes of $J_{n-1} \cap S$, and so there are no maximal subsemigroups of types (M2) and (M3) by Lemma 4.80. On the other hand, G has two orbits on the set of $\mathscr{R}$-classes of $J_{n-1} \cap S$: it transitively permutes the $\mathscr{R}$-classes of transformations, and it transitively permutes the $\mathscr{R}$-classes of partial permutations. By the dual of Proposition 4.79, the two maximal subsemigroups of S of type (M4) are found by removing either the partial permutations, or the transformations, of rank $n - 1$. By Corollary 4.75, there is no maximal subsemigroup of type (M5). ☐

Theorem 5.33. *Let $n \in \mathbb{N}$, $n \geq 3$, be arbitrary, let $S \in \{\mathcal{OP}_n, \mathcal{OR}_n\}$, and let G be the group of units of S. The maximal subsemigroups of S are:*

(a) $(S \setminus G) \cup U$, *where U is a maximal subgroup of G* (type (M1))*; and*

(b) $S \setminus \{\alpha \in S : \operatorname{rank}(\alpha) = n - 1\}$ (type (M5)).

In particular, for $n \geq 3$, there are $|\mathbb{P}_n| + 1$ maximal subsemigroups of $\mathcal{OP}_n$, and $2 + \sum_{p \in \mathbb{P}_n} p$ maximal subsemigroups of $\mathcal{OR}_n$, where $\mathbb{P}_n$ is the set of primes that divide n.

Proof. Let $S \in \{\mathcal{OP}_n, \mathcal{OR}_n\}$, and let G be the group of units of S. The description and number of the maximal subsemigroups arising from G follows by the same arguments that were used in the proof of Theorem 5.32. Let $\alpha \in J_{n-1} \cap S$ be arbitrary, and let ε be an arbitrary idempotent of $\mathcal{O}_n$ of rank $n - 1$. Since G acts transitively on the $\mathscr{L}$- and $\mathscr{R}$-classes of $J_{n-1} \cap S$, there exist permutations $\sigma, \tau \in G$ such that $\sigma\alpha\tau \in H_{\varepsilon}^{S}$. Therefore $\varepsilon = (\sigma\alpha\tau)^k$ for some $k \in \mathbb{N}$. Since ε was chosen arbitrarily, it follows that $\langle G, \alpha \rangle$ contains every idempotent of S of rank $n - 1$. But S is generated by its units and idempotents of rank $n - 1$, and so $S = \langle G, \alpha \rangle$. The description of the remaining maximal subsemigroup follows by Corollary 4.74. ☐

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	n
$\mathcal{POP}_n$	2	3	3	3	3	4	3	3	3	4	3	4	3	4	4	$\lvert\mathbb{P}_n\rvert + 2$
$\mathcal{OP}_n$	1	2	2	2	2	3	2	2	2	3	2	3	2	3	3	$\lvert\mathbb{P}_n\rvert + 1$
$\mathcal{POR}_n$	2	3	6	5	8	8	10	5	6	10	14	8	16	12	11	$3 + \sum_{p\in\mathbb{P}_n} p$
$\mathcal{OR}_n$	1	2	5	4	7	7	9	4	5	9	13	7	15	11	10	$2 + \sum_{p\in\mathbb{P}_n} p$

Table 5.34: The numbers of maximal subsemigroups of the monoids $\mathcal{POP}_n$, $\mathcal{OP}_n$, $\mathcal{POR}_n$, and $\mathcal{OR}_n$, for $n = 1, \ldots, 15$, along with the general formulae. Recall that $\mathbb{P}_n$ denotes the set of primes that divide n. See Theorems 5.32 and 5.33 for descriptions of the maximal subsemigroups of these monoids.

5.2.8 $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$

The maximal subsemigroups of the inverse monoids $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$ exhibit maximal subsemigroups of type (M1) arising from a $\mathscr{J}$-class covered by the group of units, and to which we can apply Proposition 4.85.

Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary, and let $S \in \{\mathcal{POPI}_n, \mathcal{PORI}_n\}$. Then $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of S consisting of partial permutations, and S is generated by its units and its elements of this $\mathscr{J}$-class. By definition, $\mathcal{POPI}_n = \mathcal{POP}_n \cap \mathcal{I}_n$ and $\mathcal{PORI}_n = \mathcal{POR}_n \cap \mathcal{I}_n$. Therefore, the group of units of $\mathcal{POPI}_n$ is $\mathcal{C}_n$ and the group of units of $\mathcal{PORI}_n$ is $\mathcal{D}_n$, and it follows from Section 5.2.7 that the $\mathscr{L}$-classes and $\mathscr{R}$-classes of $J_{n-1} \cap S$ are $\big\{L_i \cap S : i \in \{1, \ldots, n\}\big\}$ and $\big\{R_i \cap S : i \in \{1, \ldots, n\}\big\}$, respectively.

We describe the maximal subsemigroups of $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$ in the following theorems.

Theorem 5.35. *Let $n \in \mathbb{N}$, $n \geq 3$, be arbitrary, and define the partial permutation of degree n*

$$\zeta_n = \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 & n \\ 2 & 3 & \cdots & n-1 & 1 & - \end{pmatrix}.$$

For $k \in \mathbb{N}$, let $\mathbb{P}_k$ denote the set of all primes that divide k. Then the maximal subsemigroups of $\mathcal{POPI}_n$ are the sets:

(a) $(\mathcal{POPI}_n \setminus \mathcal{C}_n) \cup U$*, where U is a maximal subgroup of $\mathcal{C}_n$* (type (M1))*; and*

(b) $\langle \mathcal{POPI}_n \setminus J_{n-1},\ \zeta_n^p \rangle$*, for each $p \in \mathbb{P}_{n-1}$* (type (M1)).

In particular, for $n \geq 3$, there are $|\mathbb{P}_n| + |\mathbb{P}_{n-1}|$ maximal subsemigroups of $\mathcal{POPI}_n$.

Proof. By Lemmas 4.73 and 5.1, the $|\mathbb{P}_n|$ maximal subsemigroups arising from the group of units $\mathcal{C}_n$ are those stated in the theorem. Since $\mathcal{POPI}_n$ is inverse, its idempotents are projections, and certainly $\mathcal{C}_n$ acts transitively on the $\mathscr{L}$-classes and $\mathscr{R}$-classes of $J_{n-1} \cap \mathcal{POPI}_n$. Define

$$H = H_{\zeta_n}^{\mathcal{POPI}_n} = \big\{\alpha \in \mathcal{POPI}_n : \mathrm{dom}(\alpha) = \mathrm{im}(\alpha) = \{1, \ldots, n-1\}\big\}.$$

Then H is a group $\mathscr{H}$-class in the $\mathscr{J}$-class $J_{n-1} \cap \mathcal{POPI}_n$. Note that H is a cyclic group of order $n-1$, generated by the partial permutation ζ_n, with identity $e = \zeta_n^{n-1}$. Since the conditions of Proposition 4.85 are satisfied, we may apply its conclusions. Therefore, the maximal subsemigroups that arise from $J_{n-1} \cap \mathcal{POPI}_n$ are the subsemigroups $\langle \mathcal{POPI}_n \setminus J_{n-1},\ U\rangle$, for each maximal subgroup U of H that contains $e \cdot \mathrm{Stab}_{\mathcal{C}_n}(H)$ (see Definition 4.83), or $\mathcal{POPI}_n \setminus J_{n-1}$, if no such maximal subgroups exist. The setwise stabilizer $\mathrm{Stab}_{\mathcal{C}_n}(H)$ is equal to the pointwise stabilizer $\{\sigma \in \mathcal{C}_n : n\sigma = n\}$ of n in $\mathcal{C}_n$, which is trivial. Therefore, any maximal subgroup of H gives rise to a maximal subsemigroup of $\mathcal{POPI}_n$; by Lemma 5.1, the maximal subgroups of H are $\langle \zeta_n^p \rangle$ for each $p \in \mathbb{P}_{n-1}$, and as $n \geq 3$, the result follows. $\square$

Theorem 5.36. *Let $n \in \mathbb{N}$, $n \geq 4$, be arbitrary, and define the partial permutations*

$$\zeta_n = \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 & n \\ 2 & 3 & \cdots & n-1 & 1 & - \end{pmatrix}, \quad \textit{and} \quad \tau_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n-1 & n-2 & \cdots & 1 & - \end{pmatrix}.$$

For $k \in \mathbb{N}$, let $\mathbb{P}_k$ denote the set of all primes that divide k. Then the maximal subsemigroups of $\mathcal{PORI}_n$ are the sets:

(a) $(\mathcal{PORI}_n \setminus \mathcal{D}_n) \cup U$*, where U is a maximal subgroup of $\mathcal{D}_n$* (type ((M1)))*; and*

(b) $\langle \mathcal{PORI}_n \setminus J_{n-1},\ \zeta_n^p,\ \tau_n \rangle$*, for each $p \in \mathbb{P}_{n-1}$* (type (M1)).

In particular, for $n \geq 4$, there are $1 + |\mathbb{P}_{n-1}| + \sum_{p \in \mathbb{P}_n} p$ maximal subsemigroups of $\mathcal{PORI}_n$.

Proof. By Lemmas 4.73 and 5.2, and since $n \geq 4$, the description of the maximal subsemigroups arising from the group of units $\mathcal{D}_n$ is as stated in the theorem, and there are $\sum_{p \in \mathbb{P}_n} p + 1$ such maximal subsemigroups.

Since $\mathcal{PORI}_n$ is inverse, each of its idempotents is a projection, and $\mathcal{D}_n$ acts transitively on the $\mathscr{L}$-classes and $\mathscr{R}$-classes of $J_{n-1} \cap \mathcal{PORI}_n$. Therefore we may use Proposition 4.85 to describe the maximal subsemigroups of $\mathcal{PORI}_n$ that arise from $J_{n-1} \cap \mathcal{PORI}_n$. Define e to be the idempotent partial permutation with domain and image $\{1, \ldots, n-1\}$, and define

$$H = H_e^{\mathcal{PORI}_n} = \big\{\alpha \in \mathcal{PORI}_n \,:\, \operatorname{dom}(\alpha) = \operatorname{im}(\alpha) = \{1, \ldots, n-1\}\big\}.$$

Then H is a group $\mathscr{H}$-class of $\mathcal{PORI}_n$ contained in $J_{n-1} \cap \mathcal{PORI}_n$. Note that since $n \geq 4$, H is a dihedral group of order $2(n-1)$, and it is generated by the partial permutations ζ_n and τ_n. An element of $\mathcal{D}_n$ belongs to the setwise stabilizer $\operatorname{Stab}_{\mathcal{D}_n}(H)$ if and only if it fixes the point n. Thus $\operatorname{Stab}_{\mathcal{D}_n}(H)$ contains only the identity permutation id_n, which fixes every point in $\{1, \ldots, n\}$, and the permutation $(1\ n-1)(2\ n-2)\cdots(\lfloor n/2 \rfloor\ \lceil n/2 \rceil)$ that fixes n and reverses the order of $\{1, \ldots, n-1\}$. In particular,

$$e \cdot \operatorname{Stab}_{\mathcal{D}_n}(H) = \big\{e \cdot h \,:\, h \in \operatorname{Stab}_{\mathcal{D}_n}(H)\big\} = \{e, \tau_n\}.$$

Since any subgroup of H contains e, it follows from Proposition 4.85 that the maximal subsemigroups arising from $\mathcal{PORI}_n \cap J_{n-1}$ are $\langle \mathcal{PORI}_n \setminus J_{n-1}, U \rangle$, for each maximal subgroup U of H that contains τ_n. By Lemma 5.2, the maximal subgroups of H are $\langle \zeta_n \rangle$ and the subgroups $\langle \zeta_n^p,\ \tau_n \zeta_n^i \rangle$, where $p \in \mathbb{P}_{n-1}$ and $0 \leq i \leq p-1$. Thus the maximal subgroups of H that contain τ_n are those in the latter form where $i = 0$. It follows that the maximal subsemigroups arising from $\mathcal{PORI}_n \cap J_{n-1}$ are those stated in the theorem, and that there are $|\mathbb{P}_{n-1}|$ such maximal subsemigroups. □

	1	2	3	4	5	6	7	8	9	10	11	12	n
$\mathcal{POPI}_n$	2	2	2	2	2	3	3	2	2	3	3	3	$\lvert\mathbb{P}_n\rvert + \lvert\mathbb{P}_{n-1}\rvert$
$\mathcal{PORI}_n$	2	2	5	4	7	7	10	4	5	9	14	7	$1 + \lvert\mathbb{P}_{n-1}\rvert + \sum_{p \in \mathbb{P}_n} p$

Table 5.37: The numbers of maximal subsemigroups of the monoids $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$, for $n = 1, \ldots, 12$, along with the general formulae. Note that $\mathbb{P}_n$ denotes the set of primes that divide n. See Theorems 5.35 and 5.36 for descriptions of the maximal subsemigroups of $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$, respectively.

5.3 Diagram monoids

In this section, we describe and count the maximal subsemigroups of the partition monoid, and several of its submonoids. We define these monoids in Section 5.3.1, and provide some additional necessary information about them in Section 5.3.2. We then classify the maximal subsemigroups of these monoids in Sections 5.3.3–5.3.7.

5.3.1 Definitions

Recall from Section 1.1 that an equivalence is a reflexive, symmetric, and transitive relation on a set, and recall that a partition of a set is a collection of non-empty disjoint subsets of that set whose union is the whole set. Since the equivalence classes of an equivalence form a partition, and vice versa, we may consider these notions to be interchangeable.

Let $n \in \mathbb{N}$ be arbitrary. A *partition of degree* n is an equivalence on the set

$$\{1, \ldots, n\} \cup \{1', \ldots, n'\}.$$

The equivalence classes of a partition of degree n are called *blocks*, and a block is called *transverse* if it contains points from both $\{1, \ldots, n\}$ and $\{1', \ldots, n'\}$. A *block bijection* is a partition of some degree whose blocks are all transverse, and a block bijection of degree n is *uniform* if each of its blocks contains an equal number of points of $\{1, \ldots, n\}$ and $\{1', \ldots, n'\}$.

Example 5.38. Define partitions γ and δ of degree 5 by

$$\gamma = \{\{1, 1'\}, \{2, 2', 4'\}, \{3\}, \{3', 5'\}, \{4, 5\}\}, \text{ and } \delta = \{\{1, 2, 2', 3', 5'\}, \{3, 1'\}, \{4, 5, 4'\}\}.$$

There are five blocks in γ, only two of which are transverse, and δ has three blocks, which are all transverse. Therefore δ is a block bijection, but γ is not. However, δ is not a uniform block bijection, since, for example, its block $\{4, 5, 4'\}$ contains two points from $\{1, \ldots, 5\}$, but only one point from $\{1', \ldots, 5'\}$.

The topic of this thesis is semigroup theory, and so of course we require an associative operation on partitions of the same degree. Let $n \in \mathbb{N}$ be arbitrary, and let α and β be partitions of degree n. Calculating the product $\alpha\beta$ involves several steps, and requires three auxiliary partitions, each one being a partition of a different set. From α, we create a partition $\alpha^\vee$ of the set $\{1, \ldots, n\} \cup \{1'', \ldots, n''\}$, by replacing every occurrence of each i' in α by i''. Similarly, by replacing each i in β by i'', we obtain from β a partition $\beta^\wedge$ of $\{1'', \ldots, n''\} \cup \{1', \ldots, n'\}$. We define $(\alpha\beta)^\diamond$ to be the smallest equivalence on $\{1, \ldots, n\} \cup \{1', \ldots, n'\} \cup \{1'', \ldots, n''\}$ that contains the relation $\alpha^\vee \cup \beta^\wedge$. In other words, since $\alpha^\vee \cup \beta^\wedge$ is reflexive and symmetric, $(\alpha\beta)^\diamond$ is the least transitive relation that contains $\alpha^\vee \cup \beta^\wedge$. Finally, the product $\alpha\beta$ is the intersection of $(\alpha\beta)^\diamond$ with $(\{1, \ldots, n\} \cup \{1', \ldots, n'\}) \times (\{1, \ldots, n\} \cup \{1', \ldots, n'\})$. See [70, Section 1] for a different approach to defining this multiplication.

This operation may be shown to be associative, and so the set $\mathcal{P}_n$ of all partitions of degree n forms a semigroup under this operation. The partition $\big\{\{i, i'\} : i \in \{1, \ldots, n\}\big\}$ is the identity element of this semigroup. We call this element the *identity partition of degree* n. Therefore $\mathcal{P}_n$ is a monoid, named the *partition monoid of degree* n. A *diagram monoid* is the name given to any submonoid of a partition monoid.

Example 5.39. Let γ and δ be the partitions of degree 5 defined in Example 5.38. We form the product $\delta\gamma$ as follows. First, we create the partitions

$$\delta^\vee = \{\{1, 2, 2'', 3'', 5''\}, \{3, 1''\}, \{4, 5, 4''\}\}, \text{ and}$$
$$\gamma^\wedge = \{\{1'', 1'\}, \{2'', 2', 4'\}, \{3''\}, \{3', 5'\}, \{4'', 5''\}\}.$$

The least equivalence on $\{1, \ldots, 5\} \cup \{1', \ldots, 5'\} \cup \{1'', \ldots, 5''\}$ that contains $\delta^{\vee} \cup \gamma^{\wedge}$ is

$$(\delta\gamma)^{\diamond} = \{\{1, 2, 4, 5, 2, 2', 4', 2'', 3'', 4'', 5''\}, \{3, 1', 1''\}, \{3', 5'\}, \{3''\}\};$$

its intersection with $\{1, \ldots, 5\} \cup \{1', \ldots, 5'\}$ is the product

$$\delta\gamma = \{\{1, 2, 4, 5, 2', 4'\}, \{3, 1'\}, \{3', 5'\}\}.$$

Let $n \in \mathbb{N}$ and $\alpha \in \mathcal{P}_n$. We define α^* to be the partition of degree n created from α by replacing the point i by i' in the block in which it appears, and by replacing the point i' by i, for all $i \in \{1, \ldots, n\}$. For example, if $\alpha = \{\{1, 3'\}, \{3\}, \{2, 1', 2'\}\}$, a partition of degree 3, then $\alpha^* = \{\{1', 3\}, \{3'\}, \{2', 1, 2\}\}$. For arbitrary partitions $\alpha, \beta \in \mathcal{P}_n$, the following hold:

$$(\alpha^*)^* = \alpha, \ \alpha\alpha^*\alpha = \alpha, \ \text{and} \ (\alpha\beta)^* = \beta^*\alpha^*.$$

In particular, $\mathcal{P}_n$ is a regular $*$-monoid, as defined in Section 1.3.3.

There is a canonical embedding $\psi : \mathcal{S}_n \longrightarrow \mathcal{P}_n$, where a permutation $\alpha \in \mathcal{S}_n$ is mapped to the partition $\big\{\{i, (i\alpha)'\} : i \in \{1, \ldots, n\}\big\}$. It is easy to verify that the units of $\mathcal{P}_n$ are those partitions whose blocks have the form $\{i, j'\}$, for some $i, j \in \{1, \ldots, n\}$. It follows that the image of ψ is the group of units of $\mathcal{P}_n$. Therefore, throughout this section, we identify the permutation group $\mathcal{S}_n$ with the group of units of $\mathcal{P}_n$. In particular, we use disjoint cycle notation to refer to units in $\mathcal{P}_n$, and we use id_n to denote the identity element of $\mathcal{P}_n$.

In order to introduce the notions of planar and annular partitions, which are necessary to define some of the monoids whose maximal subsemigroups we classify, we define a canonical ordering

$$n' < (n-1)' < \cdots < 1' < 1 < 2 < \cdots < n \tag{5.40}$$

on the set $\{1, \ldots, n\} \cup \{1', \ldots, n'\}$. A partition $\alpha \in \mathcal{P}_n$ is *planar* if there do not exist distinct blocks A and X of α, and points $a, b \in A$ and $x, y \in X$, such that $a < x < b < y$. More generally, if $\rho_n = (1\ 2\ \ldots\ n)$, then a partition $\alpha \in \mathcal{P}_n$ is said to be *annular* if $\alpha = \rho_n^k \cdot \beta \cdot \rho_n^l$ for some planar partition $\beta \in \mathcal{P}_n$ and for some indices $k, l \in \mathbb{N}_0$. A partition may be represented graphically; planar and annular partitions have special representations in this context. For more information about this, see [4, 70].

Let $n \in \mathbb{N}$ be arbitrary. In Sections 5.3.3–5.3.7, we classify the maximal subsemigroups of $\mathcal{P}_n$, and the following submonoids:

- $\mathcal{PB}_n = \{\alpha \in \mathcal{P}_n : \text{each block of } \alpha \text{ contains 1 or 2 points}\}$, the *partial Brauer monoid of degree* n, introduced in [97];
- $\mathcal{B}_n = \{\alpha \in \mathcal{P}_n : \text{each block of } \alpha \text{ contains 2 points}\}$, the *Brauer monoid of degree* n, introduced in [97];
- $\mathcal{I}_n^* = \{\alpha \in \mathcal{P}_n : \alpha \text{ is a block bijection}\}$, the *dual symmetric inverse monoid of degree* n, introduced in [52];
- $\mathfrak{F}_n = \{\alpha \in \mathcal{P}_n : \alpha \text{ is a uniform block bijection}\}$, the *uniform block bijection monoid of degree* n, also known as the *factorisable dual symmetric inverse monoid of degree* n, see [50] for more details;
- $\mathcal{PP}_n = \{\alpha \in \mathcal{P}_n : \alpha \text{ is planar}\}$, the *planar partition monoid of degree* n, introduced in [70];
- $\mathcal{M}_n = \{\alpha \in \mathcal{PB}_n : \alpha \text{ is planar}\}$, the *Motzkin monoid of degree* n, see [9] for more details;

- $\mathcal{J}_n = \{\alpha \in \mathcal{B}_n : \alpha \text{ is planar}\}$, the *Jones monoid of degree n*, also known as the *Temperley-Lieb monoid*, introduced in [84]; and

- $\mathcal{AJ}_n = \{\alpha \in \mathcal{B}_n : \alpha \text{ is annular}\}$, the *annular Jones monoid of degree n*, introduced in [4].

Each of these diagram monoids is closed under the * operation, and is therefore a regular $*$-semigroup. Furthermore, the monoids $\mathcal{I}_n^*$ and $\mathfrak{F}_n$, which consist of block bijections, are inverse. The groups of units of $\mathcal{PB}_n$, $\mathcal{B}_n$, $\mathcal{I}_n^*$, and $\mathfrak{F}_n$ are the symmetric group $\mathcal{S}_n$; the group of units of $\mathcal{AJ}_n$ is the cyclic group $\mathcal{C}_n$, as defined in (5.4); and the groups of units of $\mathcal{PP}_n$, $\mathcal{M}_n$, and $\mathcal{J}_n$ are the trivial group $\{\mathrm{id}_n\}$. See Figure 5.41 for a diagram showing the containment of these submonoids of $\mathcal{P}_n$, and their groups of units.

By [70], the planar partition monoid $\mathcal{PP}_n$ is isomorphic to the Jones monoid $\mathcal{J}_{2n}$. Therefore, we will not separately determine the maximal subsemigroups of $\mathcal{PP}_n$, since their descriptions can be deduced from the results in Section 5.3.7. The maximal subsemigroups of $\mathcal{I}_n^*$ were described in [94, Theorem 19], but we reprove this result for completeness. The maximal subsemigroups of the remaining monoids had not been classified until the publication of [45].

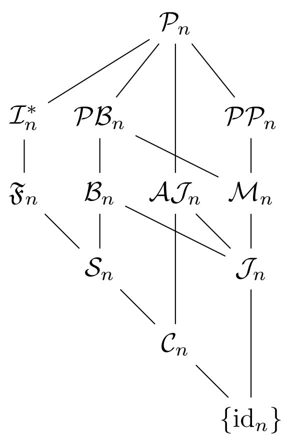

Figure 5.41: A Hasse diagram showing the containment of the diagram monoids defined in Section 5.3.1, along with their groups of units. We classify the maximal subsemigroups of these monoids in Sections 5.3.3–5.3.7.

5.3.2 Preliminaries

In order to prove the descriptions of the maximal subsemigroups of the diagram monoids defined in Section 5.3.1, we require some additional information about these monoids, which we introduce in this section.

Let $n \in \mathbb{N}$ be arbitrary. Define the function $\phi : \mathcal{PT}_n \longrightarrow \mathcal{P}_n$ such that if $\alpha \in \mathcal{PT}_n$, then the non-singleton blocks of $\alpha\phi$ are $\{i'\} \cup i\alpha^{-1}$, for each $i \in \mathrm{im}(\alpha)$. It is clear that ϕ is injective, since a partial transformation α is uniquely determined by its image and the pre-image $i\alpha^{-1}$ of each point i in $\mathrm{im}(\alpha)$. The restriction of ϕ to $\mathcal{T}_n$ gives an embedding, as does the restriction of ϕ to $\mathcal{I}_n$. However, ϕ is not a homomorphism in general. Nevertheless, this injective function gives us a means of regarding partitions as generalisations of partial transformations. Furthermore, planar and annular partitions generalise order- and orientation-preserving partial transformations: for any $\alpha \in \mathcal{PT}_n$, α is order-preserving if and only if $\alpha\phi$ is planar, and α is orientation-preserving if and only if $\alpha\phi$ is annular.

Let $\alpha \in \mathcal{P}_n$ be arbitrary. We require the following notions. The *rank* of α is the number of transverse blocks that it contains, and is denoted by $\operatorname{rank}(\alpha)$. The *kernel* of α, $\ker(\alpha)$, is the restriction of the equivalence α to $\{1, \ldots, n\}$. We also define $\operatorname{dom}(\alpha)$, the *domain* of α, to be the subset of $\{1, \ldots, n\}$ consisting of those points that are contained in a transverse block of α. Given these definitions, we define $\operatorname{coker}(\alpha) = \ker(\alpha^*)$ and $\operatorname{codom}(\alpha) = \operatorname{dom}(\alpha^*)$, the *cokernel* and *codomain* of α, respectively.

Example 5.42. Let ξ be the partition of degree 5 given by

$$\xi = \{\{1, 3\}, \{2, 4, 2'\}, \{5, 1', 3', 4'\}, \{5'\}\}.$$

Then the rank of ξ is two, since it contains two transverse blocks. The equivalence classes of the kernel of ξ are $\{\{1, 3\}, \{2, 4\}, \{5\}\}$, and the domain of ξ is $\{2, 4, 5\}$. Furthermore, $\operatorname{codom}(\xi) = \{1, 2, 3, 4\}$, and the equivalence classes of $\operatorname{coker}(\xi)$ are $\{\{1, 3, 4\}, \{2\}, \{5\}\}$.

For the majority of the monoids defined in Section 5.3.1, their Green's relations are completely determined by domain, kernel, and rank, as shown in the following lemma.

Lemma 5.43. *Let $n \in \mathbb{N}$, let $S \in \{\mathcal{P}_n, \mathcal{PB}_n, \mathcal{B}_n, \mathcal{I}_n^*, \mathcal{M}_n, \mathcal{J}_n\}$, and let $\alpha, \beta \in S$. Then:*

(a) *$\alpha \mathscr{R} \beta$ if and only if* $\operatorname{dom}(\alpha) = \operatorname{dom}(\beta)$ *and* $\ker(\alpha) = \ker(\beta)$*;*

(b) *$\alpha \mathscr{L} \beta$ if and only if* $\operatorname{codom}(\alpha) = \operatorname{codom}(\beta)$ *and* $\operatorname{coker}(\alpha) = \operatorname{coker}(\beta)$*; and*

(c) *$\alpha \mathscr{J} \beta$ if and only if* $\operatorname{rank}(\alpha) = \operatorname{rank}(\beta)$.

See [97], [127, Theorem 17], and [33, Theorem 2.4] for the proof of this lemma. For $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$, we define

$$J_k = \{\alpha \in \mathcal{P}_n : \operatorname{rank}(\alpha) = k\}$$

to be the $\mathscr{J}$-class of $\mathcal{P}_n$ that comprises the partitions of rank k.

If S is a regular $*$-semigroup and $x, y \in S$, then $x \mathscr{L} y$ if and only if $x^* \mathscr{R} y^*$, and so parts (a) and (b) of Lemma 5.43 are equivalent. For the uniform block bijection monoid $\mathfrak{F}_n$, parts (a) and (b) certainly hold, since $\mathfrak{F}_n$ is a regular submonoid of $\mathcal{P}_n$ [76, Proposition 2.4.2] (indeed, it is inverse). However, while part (c) does not hold for $\mathfrak{F}_n$ in general, it does hold for uniform block bijections of ranks n or $n-1$ [52, Section 3]. The description also holds for the annular Jones monoid $\mathcal{AJ}_n$.

It follows that, in general, if S is any of the diagram monoids defined in Section 5.3.1, then S has a unique $\mathscr{J}$-class J that is covered by the group of units of S. In the case that S is either the partition monoid or the Jones monoid, in order to determine the maximal subsemigroups of S that arise from this $\mathscr{J}$-class J, we require the graph $\Delta(S, J)$, as defined in Section 4.4.2 and further discussed in Section 4.5. Given a description of the $\mathscr{L}$-classes and $\mathscr{R}$-classes of J, to describe $\Delta(S, J)$, it remains to describe the left action of the group of units on the $\mathscr{R}$-classes of J by left multiplication. Since S is a regular $*$-monoid, a description of the right action of the group of units on the $\mathscr{L}$-classes of J by right multiplication is obtained as a consequence.

If $\alpha \in \mathcal{P}_n$ and $\sigma \in \mathcal{S}_n$, then the blocks are $\sigma\alpha$ are

$$(A \cap \{1, \ldots, n\})\sigma^{-1} \cup (A \cap \{1', \ldots, n'\}),$$

for each block A of α. Therefore,

$$\operatorname{dom}(\sigma\alpha) = \left\{i\sigma^{-1} : i \in \operatorname{dom}(\alpha)\right\} \quad \text{and} \quad \ker(\sigma\alpha) = \left\{\left(i\sigma^{-1}, j\sigma^{-1}\right) : (i, j) \in \ker(\alpha)\right\}. \quad (5.44)$$

Given this description and Lemma 5.43, the left action of a subgroup of $\mathcal{S}_n$ on the $\mathscr{R}$-classes of a particular $\mathscr{J}$-class is straightforward to determine.

5.3.3 The partition monoid $\mathcal{P}_n$

Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. In this section, we classify the maximal subsemigroups of the partition monoid of degree n. To do so, we require the following information about the Green's classes of $\mathcal{P}_n$ contained in its $\mathscr{J}$-class J_{n-1}. Let $\alpha \in J_{n-1}$. By definition, α contains $n-1$ transverse blocks. Since each transverse block contains at least two points, and since there are only $2n$ points in $\{1, \ldots, n\} \cup \{1', \ldots, n'\}$, there are few possible combinations for the kernel and domain for α. In particular, either $\ker(\alpha)$ is trivial and $\operatorname{dom}(\alpha) = \{1, \ldots, n\} \setminus \{i\}$ for some $i \in \{1, \ldots, n\}$, or $\operatorname{dom}(\alpha) = \{1, \ldots, n\}$ and $\{i, j\}$ is the unique non-trivial kernel class of α, for some distinct point $i, j \in \{1, \ldots, n\}$. By Lemma 5.43, these properties describe the $\mathscr{R}$-classes of J_{n-1}. Since the $\mathscr{L}$-classes and $\mathscr{R}$-classes of a regular $*$-semigroup correspond via the * operation, this also gives a description of the $\mathscr{L}$-classes of J_{n-1}. Thus, for distinct $i, j \in \{1, \ldots, n\}$, we make the following definitions:

- $R_i = \big\{\alpha \in J_{n-1} : \operatorname{dom}(\alpha) = \{1, \ldots, n\} \setminus \{i\}\big\}$, which is an $\mathscr{R}$-class of J_{n-1};
- $R_{\{i,j\}} = \big\{\alpha \in J_{n-1} : (i, j) \in \ker(\alpha)\big\}$, which is an $\mathscr{R}$-class of J_{n-1};
- $L_i = R_i^* = \big\{\alpha \in J_{n-1} : \operatorname{codom}(\alpha) = \{1, \ldots, n\} \setminus \{i\}\big\}$, which is an $\mathscr{L}$-class of J_{n-1};
- $L_{\{i,j\}} = R_{\{i,j\}}^* = \big\{\alpha \in J_{n-1} : (i, j) \in \operatorname{coker}(\alpha)\big\}$, which is an $\mathscr{L}$-class of J_{n-1}.

In general, it is slightly complicated to describe the group $\mathscr{H}$-classes of $\mathcal{P}_n$ in terms of kernels, domains, cokernels, and codomains. However, in the $\mathscr{J}$-class J_{n-1}, it is straightforward. An $\mathscr{H}$-class of the form $L_i \cap R_j$ is a group if and only if $i = j$, an $\mathscr{H}$-class of the form $L_i \cap R_{\{j,k\}}$ or $R_i \cap L_{\{j,k\}}$ is a group if and only if $i \in \{j, k\}$, and an $\mathscr{H}$-class of the form $L_{\{i,j\}} \cap R_{\{k,l\}}$ is a group if and only if $\{i, j\} = \{k, l\}$.

The main result of this section is the following theorem. In the proof, we use the fact that the ideal $\mathcal{P}_n \setminus \mathcal{S}_n$ is generated by its idempotents of rank $n-1$ [40, Section 6],

Theorem 5.45. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. The maximal subsemigroups of $\mathcal{P}_n$ are:*

(a) $(\mathcal{P}_n \setminus \mathcal{S}_n) \cup U$, *where U is a maximal subgroup of $\mathcal{S}_n$* (type (M1))*;*

(b) $\mathcal{P}_n \setminus \big\{\alpha \in \mathcal{P}_n : \operatorname{rank}(\alpha) = n-1$ *and* $\ker(\alpha)$ *is trivial*$\big\}$ (type (M4))*;*

(c) $\mathcal{P}_n \setminus \big\{\alpha \in \mathcal{P}_n : \operatorname{rank}(\alpha) = n-1$ *and* $\operatorname{dom}(\alpha) = \{1, \ldots, n\}\big\}$ (type (M4))*;*

(d) $\mathcal{P}_n \setminus \big\{\alpha \in \mathcal{P}_n : \operatorname{rank}(\alpha) = n-1$ *and* $\operatorname{coker}(\alpha)$ *is trivial*$\big\}$ (type (M3))*; and*

(e) $\mathcal{P}_n \setminus \big\{\alpha \in \mathcal{P}_n : \operatorname{rank}(\alpha) = n-1$ *and* $\operatorname{codom}(\alpha) = \{1, \ldots, n\}\big\}$ (type (M3)).

In particular, for $n \geq 2$, there are $s_n + 4$ maximal subsemigroups of $\mathcal{P}_n$.

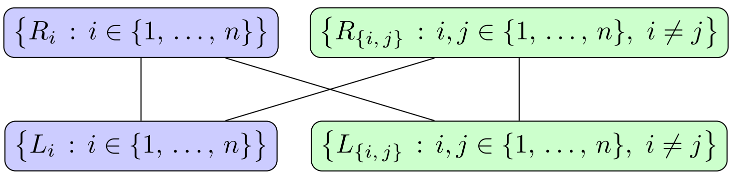

Figure 5.46: The graph $\Delta(\mathcal{P}_n, J_{n-1})$, which is a complete bipartite graph with four vertices. The two maximal independent subsets are the set of orbits of $\mathscr{L}$-classes, and the set of orbits of $\mathscr{R}$-classes. Each vertex in the graph has degree two.

Proof. Since $\mathscr{J}$-equivalence in $\mathcal{P}_n$ is determined by rank, and $\mathcal{P}_n$ is generated by its units and idempotents of rank $n-1$, it follows by Lemma 4.68 that the maximal subsemigroups of $\mathcal{P}_n$ arise from its group of units and its $\mathscr{J}$-class J_{n-1}. By Lemma 4.73, the s_n maximal subsemigroups that arise from the group of units are those stated in the theorem with type (M1).

By Lemma 4.49(b), there are no maximal subsemigroups of $\mathcal{P}_n$ arising from J_{n-1} with type (M1). Given (5.44), it is clear that $\mathcal{S}_n$ transitively permutes the $\mathscr{R}$-classes of J_{n-1} with trivial kernel, and it transitively permutes the $\mathscr{R}$-classes of J_{n-1} with domain $\{1, \ldots, n\}$. Hence there are two orbits of $\mathscr{R}$-classes under this right action, and so there are two orbits of $\mathscr{L}$-classes under the corresponding left action. We may therefore deduce the description of $\Delta(\mathcal{P}_n) = \Delta(\mathcal{P}_n, J_{n-1})$; see Figure 5.46. Since $\Delta(\mathcal{P}_n)$ is a complete bipartite graph, with only two maximal independent subsets, Corollary 4.78 implies that there are no maximal subsemigroups of type (M2). Each vertex of $\Delta(\mathcal{P}_n)$ has degree 2, and so by Proposition 4.79, there are two maximal subsemigroups of type (M3), formed by removing each orbit of $\mathscr{L}$-classes in turn. Similarly, there are two maximal subsemigroups of type (M4). $\square$

5.3.4 The Brauer monoid $\mathcal{B}_n$ and the uniform block bijection monoid $\mathfrak{F}_n$

Let $n \in \mathbb{N}$, $n \geq 2$. In this section, we describe and count the maximal subsemigroups of $\mathcal{B}_n$ and $\mathfrak{F}_n$. The main results of this section are the following theorems. In the proofs of these theorems, we exploit the following results. By [4], $\mathcal{B}_n$ is generated by its group of units $\mathcal{S}_n$ and any projection of rank $n-2$, and by [94, Section 5], $\mathfrak{F}_n$ is generated by it groups of units $\mathcal{S}_n$ and any projection of rank $n-1$.

Theorem 5.47. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. The maximal subsemigroups of $\mathcal{B}_n$ are:*

(a) $(\mathcal{B}_n \setminus \mathcal{S}_n) \cup U$, *where U is a maximal subgroup of $\mathcal{S}_n$* (type (M1))*; and*

(b) $\mathcal{B}_n \setminus \{\alpha \in \mathcal{B}_n : \operatorname{rank}(\alpha) = n-2\}$ (type (M5)).

In particular, for $n \geq 2$, there are $s_n + 1$ maximal subsemigroups of $\mathcal{B}_n$.

Theorem 5.48. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. The maximal subsemigroups of $\mathfrak{F}_n$ are:*

(a) $(\mathfrak{F}_n \setminus \mathcal{S}_n) \cup U$, *where U is a maximal subgroup of $\mathcal{S}_n$* (type (M1))*; and*

(b) $\mathfrak{F}_n \setminus \{\alpha \in \mathfrak{F}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M5)).

In particular, for $n \geq 2$, there are $s_n + 1$ maximal subsemigroups of $\mathfrak{F}_n$.

Proof of Theorems 5.47 and 5.48. By Lemma 4.68, the maximal subsemigroups of $\mathcal{B}_n$ arise from its group of units and its $\mathscr{J}$-class $J_{n-2} \cap \mathcal{B}_n$, and the maximal subsemigroups of $\mathfrak{F}_n$ arise from its group of units and from its $\mathscr{J}$-class $J_{n-1} \cap \mathfrak{F}_n$. In both cases, by Lemma 4.73, there is one maximal subsemigroup corresponding to each maximal subgroup of $\mathcal{S}_n$, as stated in the theorem, and so there are s_n such maximal subsemigroups.

To describe the remaining maximal subsemigroups of $\mathcal{B}_n$, let $\alpha \in J_{n-2} \cap \mathcal{B}_n$. The non-transverse blocks of α are $\{i, j\}$ and $\{k', l'\}$ for some $i, j, k, l \in \{1, \ldots, n\}$ with $i \neq j$ and $k \neq l$. Let $\tau \in \mathcal{S}_n$ be a permutation that contains the blocks $\{k, i'\}$ and $\{l, j'\}$. Therefore the non-transverse blocks of $\alpha\tau$ are $\{i, j\}$ and $\{i', j'\}$, and so $(\alpha\tau)^m$ is a projection of rank $n-2$ for some $m \in \mathbb{N}$. Thus $\langle \mathcal{S}_n, \alpha \rangle \supseteq \langle \mathcal{S}_n, (\alpha\tau)^m \rangle = \mathcal{B}_n$, since $\mathcal{B}_n$ is generated by $\mathcal{S}_n$ and any projection of rank $n-2$, and so $\mathcal{B}_n = \langle \mathcal{S}_n, \alpha \rangle$. By a very similar argument, $\mathfrak{F}_n = \langle \mathcal{S}_n, \beta \rangle$ for any uniform block bijection of rank $n-1$. By Corollary 4.74, in each case, the remaining maximal subsemigroups are those stated in the theorems. $\square$

5.3.5 The partial Brauer monoid $\mathcal{PB}_n$

Let $n \in \mathbb{N}$, $n \geq 2$. Recall that $\mathcal{I}_n$, the symmetric inverse monoid, is the monoid consisting of all partial permutations of degree n. The restriction to $\mathcal{I}_n$ of the injective map $\phi : \mathcal{PT}_n \longrightarrow \mathcal{P}_n$, defined at the start of Section 5.3.2, is an embedding of $\mathcal{I}_n$ into $\mathcal{P}_n$. In particular, $\mathcal{I}_n\phi$ is a submonoid of $\mathcal{PB}_n$ isomorphic to $\mathcal{I}_n$. Therefore, in this section, we will identify $\mathcal{I}_n$ with its image under ϕ, and regard $\mathcal{I}_n$ as a submonoid of $\mathcal{PB}_n$.

By [33, Proposition 3.16], $\mathcal{PB}_n$ is generated by its elements with rank at least $n-2$, and any generating set contains elements of ranks n, $n-1$, and $n-2$. Therefore, by Lemma 4.68, in order to describe the maximal subsemigroups of $\mathcal{PB}_n$, we require a description of the elements of $\mathcal{PB}_n$ with these ranks. Partitions of degree n and rank n are units, and the group of units of $\mathcal{PB}_n$ is $\mathcal{S}_n$.

Let $\alpha \in J_{n-1} \cap \mathcal{PB}_n$. By definition, α contains $n-1$ transverse blocks, which contain a point of $\{1, \ldots, n\}$ and a point of $\{1', \ldots, n'\}$. Therefore, the remaining blocks of α are the singletons $\{i\}$ and $\{j'\}$, for some $i, j \in \{1, \ldots, n\}$. In other words, α is the image of some partial permutation of rank $n-1$ under the embedding ϕ. Since α was arbitrary, and $\mathcal{I}_n \subseteq \mathcal{PB}_n$, it follows that $J_{n-1} \cap \mathcal{PB}_n = J_{n-1} \cap \mathcal{I}_n$.

Let $\alpha \in J_{n-2} \cap \mathcal{PB}_n$. Then α contains $n-2$ transverse blocks, and so a pair of points of $\{1, \ldots, n\}$ and a pair of points of $\{1', \ldots, n'\}$ are not contained in transverse blocks. Each one of these pairs forms either a block of size 2, or two singleton blocks. In particular, $\operatorname{dom}(\alpha)$ lacks two points i and j, and either $\ker(\alpha)$ is trivial, or $\{i, j\}$ is the unique non-trivial kernel class of α. A similar statement holds for the codomain and cokernel of α.

By [33, Proposition 3.15], $\mathcal{PB}_n = \langle \mathcal{S}_n,\ J_{n-1} \cap \mathcal{I}_n,\ J_{n-2} \cap \mathcal{B}_n \rangle$. As stated in the proof of Theorem 5.7, $\mathcal{I}_n$ is generated by $\mathcal{S}_n$ and any partial permutation of rank $n-1$, and as stated in the proof of Theorem 5.47, $\mathcal{B}_n$ is generated by $\mathcal{S}_n$ and any of its elements of rank $n-2$. Therefore, $\mathcal{PB}_n = \langle \mathcal{S}_n,\ \alpha,\ \beta \rangle$, for any $\alpha \in J_{n-1} \cap \mathcal{I}_n$ and $\beta \in J_{n-2} \cap \mathcal{B}_n$.

Theorem 5.49. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. The maximal subsemigroups of $\mathcal{PB}_n$ are:*

(a) $(\mathcal{PB}_n \setminus \mathcal{S}_n) \cup U$, *where U is a maximal subgroup of $\mathcal{S}_n$* (type (M1))*;*

(b) $\mathcal{PB}_n \setminus \{\alpha \in \mathcal{PB}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M5))*;*

(c) $\mathcal{PB}_n \setminus \{\alpha \in \mathcal{PB}_n : \operatorname{rank}(\alpha) = n-2$ *and* $\operatorname{coker}(\alpha)$ *is non-trivial*$\}$ (type (M3))*; and*

(d) $\mathcal{PB}_n \setminus \{\alpha \in \mathcal{PB}_n : \operatorname{rank}(\alpha) = n-2$ *and* $\ker(\alpha)$ *is non-trivial*$\}$ (type (M4)).

In particular, for $n \in \mathbb{N}$, there are $s_n + 3$ maximal subsemigroups of $\mathcal{PB}_n$.

Proof. By Lemma 5.43, Green's $\mathscr{J}$-relation in $\mathcal{PB}_n$ is determined by rank. Given the description of the generating sets of $\mathcal{PB}_n$, it follows by Lemma 4.68 that the $\mathscr{J}$-classes of $\mathcal{PB}_n$ from which there arise maximal subsemigroups are the group of units, $J_{n-1} \cap \mathcal{PB}_n$, and $J_{n-2} \cap \mathcal{PB}_n$. The group of units of $\mathcal{PB}_n$ is the symmetric group $\mathcal{S}_n$, and so by Lemma 4.73, the s_n maximal subsemigroups that arise in this case are those described.

Since the $\mathscr{J}$-class $J_{n-1} \cap \mathcal{PB}_n = J_{n-1} \cap \mathcal{I}_n$ is covered by the group of units, $\mathcal{PB}_n \setminus J_{n-1}$ is a subsemigroup of $\mathcal{PB}_n$. However, as stated above, $\mathcal{PB}_n = \langle \mathcal{S}_n,\ \alpha,\ \beta \rangle$, for any $\alpha \in J_{n-1} \cap \mathcal{I}_n$ and $\beta \in J_{n-2} \cap \mathcal{B}_n$. In particular, $\langle \mathcal{PB}_n \setminus J_{n-1},\ \alpha \rangle = \mathcal{PB}_n$ if and only if $\alpha \in J_{n-1} \cap \mathcal{PB}_n$. By Lemma 4.8, the only maximal subsemigroup of $\mathcal{PB}_n$ to arise from this $\mathscr{J}$-class has type (M5).

In order to determine the maximal subsemigroups of $\mathcal{PB}_n$ that arise from its $\mathscr{J}$-class of rank $n-2$, we define the subsets

$$X = \{\alpha \in \mathcal{PB}_n : \operatorname{rank}(\alpha) = n-2 \text{ and } \ker(\alpha) \text{ is non-trivial}\}, \text{ and}$$
$$X^* = \{\alpha^* : \alpha \in X\} = \{\alpha \in \mathcal{PB}_n : \operatorname{rank}(\alpha) = n-2 \text{ and } \operatorname{coker}(\alpha) \text{ is non-trivial}\}.$$

Note that X is a union of $\mathscr{R}$-classes of $J_{n-2} \cap \mathcal{PB}_n$, and X^* is a union of $\mathscr{L}$-classes of $J_{n-2} \cap \mathcal{PB}_n$. Let A be any subset of $J_{n-2} \cap \mathcal{PB}_n$ such that $(\mathcal{PB}_n \setminus J_{n-2}) \cup A$ generates $\mathcal{PB}_n$, and let $\alpha \in X$ be arbitrary. Then α can the written as a product $\alpha = \beta_1 \cdots \beta_k$ of some of these generators. The generators $\beta_1, \ldots, \beta_k$ all have rank at least $n-2$ by Lemma 1.12. Every element in $\mathcal{PB}_n$ of rank n and $n-1$ has a trivial kernel, and the subset of partitions with trivial kernel in $\mathcal{P}_n$ forms a subsemigroup. Thus, there exists some $r \in \{1, \ldots, k\}$ such that $\operatorname{rank}(\beta_r) = n-2$ and $\ker(\beta_r)$ is non-trivial – in other words, $\beta_r \in X$. A dual argument shows that $A \cap X^* \neq \varnothing$.

Conversely, let A be any subset of $J_{n-2} \cap \mathcal{PB}_n$ that intersects X and X^* non-trivially, and fix elements $\alpha \in A \cap X$ and $\beta \in A \cap X^*$. It is straightforward to see that there exists some permutation $\sigma \in \mathcal{S}_n$ such that $\alpha\sigma\beta$ has non-trivial kernel and cokernel, and has rank $n-2$. In particular, $\alpha\sigma\beta \in J_{n-2} \cap \mathcal{B}_n$. Since $\mathcal{PB}_n = \langle \mathcal{S}_n, \alpha, \beta \rangle$, for any $\alpha \in J_{n-1} \cap \mathcal{I}_n$ and $\beta \in J_{n-2} \cap \mathcal{B}_n$, it follows that $\langle \mathcal{PB}_n \setminus J_{n-2}, A \rangle = \mathcal{PB}_n$. By Lemma 4.8, the maximal subsemigroups of $\mathcal{PB}_n$ arising from $J_{n-2} \cap \mathcal{PB}_n$ are $\mathcal{PB}_n \setminus X^*$ and $\mathcal{PB}_n \setminus X$; these maximal subsemigroups have types (M3) and (M4), respectively. ☐

5.3.6 The dual symmetric inverse monoid $\mathcal{I}_n^*$

The maximal subsemigroups of the dual symmetric inverse monoid were first described by Maltcev [94, Theorem 19]. We reprove this result in the following theorem. See the same paper for much more information about $\mathcal{I}_n^*$, including the fact that for $n \geq 3$, $\mathcal{I}_n^* = \langle \mathcal{S}_n, \alpha \rangle$ if and only if $\alpha \in J_{n-1} \cap (\mathcal{I}_n^* \setminus \mathfrak{F}_n)$ [94, Proposition 16].

Theorem 5.50. *Let $n \in \mathbb{N}$, $n \geq 3$, be arbitrary. The maximal subsemigroups of $\mathcal{I}_n^*$ are:*

(a) $(\mathcal{I}_n^* \setminus \mathcal{S}_n) \cup U$, *where U is a maximal subgroup of $\mathcal{S}_n$* (type (M1)); *and*

(b) $\mathcal{I}_n^* \setminus \{\alpha \in \mathcal{I}_n^* : \operatorname{rank}(\alpha) = n-1$ *and α is not uniform*$\}$ (type (M1)).

In particular, for $n \geq 3$ there are $s_n + 1$ maximal subsemigroups of $\mathcal{I}_n^*$.

Proof. By Lemma 4.68, the maximal subsemigroups of $\mathcal{I}_n^*$ arise from its group of units and it's $\mathscr{J}$-class $J_{n-1} \cap \mathcal{I}_n^*$. The group of units of $\mathcal{I}_n^*$ is $\mathcal{S}_n$, and so by Lemma 4.73, the maximal subsemigroups arising from the group of units are those described. Using Corollary 4.74 with $X = J_{n-1} \cap (\mathcal{I}_n^* \setminus \mathfrak{F}_n)$, we find the description of the remaining maximal subsemigroup. Note that $\mathcal{I}_n^*$ is the monoid of all block bijections, and $\mathfrak{F}_n$ is the monoid of all uniform block bijections, and so the set X consists of the block bijections of rank $n-1$ that are not uniform. ☐

The maximal subsemigroup of $\mathcal{I}_n^*$ that arises from its $\mathscr{J}$-class $J_{n-1} \cap \mathcal{I}_n^*$ can also be found by using Proposition 4.85, since $\mathcal{S}_n$ acts transitively on the $\mathscr{R}$-classes of $J_{n-1} \cap \mathcal{I}_n^*$, and each of the idempotents in this $\mathscr{J}$-class is a projection.

	1	2	3	4	5	6	7	8	9	10	11	12	n
$\mathcal{P}_n$	2	5	8	12	26	57	188	357	1380	3981	363908	396502	$s_n + 4$
$\mathcal{PB}_n$	2	4	7	11	25	56	187	356	1379	3980	363907	396501	$s_n + 3$
$\mathcal{B}_n$, $\mathfrak{F}_n$, $\mathcal{I}_n^*$	1	2	5	9	23	54	185	354	1377	3978	363905	396499	$s_n + 1$

Table 5.51: The numbers of maximal subsemigroups of the monoids $\mathcal{P}_n$, $\mathcal{PB}_n$, $\mathcal{B}_n$, $\mathfrak{F}_n$, and $\mathcal{I}_n^*$, for $n = 1, \ldots, 12$, along with the general formulae. Recall that s_n denotes the number of maximal subgroups of the symmetric group $\mathcal{S}_n$ [120, A290138]. See Theorems 5.45–5.50 for the descriptions of the maximal subsemigroups of these monoids; see also Table 5.8.

5.3.7 The Jones monoid $\mathcal{J}_n$ and the annular Jones monoid $\mathcal{AJ}_n$

Let $n \in \mathbb{N}$ be arbitrary. In this section, we classify the maximal subsemigroups of the Jones and annular Jones monoids $\mathcal{J}_n$ and $\mathcal{AJ}_n$. Recall that $\mathcal{J}_n$ consists of all planar partitions in the Brauer monoid $\mathcal{B}_n$, and $\mathcal{AJ}_n$ consists of all annular partitions in $\mathcal{B}_n$; therefore $\mathcal{J}_n$ is a submonoid of $\mathcal{AJ}_n$. Since the planar partition monoid $\mathcal{PP}_n$ is isomorphic to $\mathcal{J}_{2n}$ [70], by determining the maximal subsemigroups of the Jones monoids, we also find those of the planar partition monoids.

First, we classify the maximal subsemigroups of $\mathcal{J}_n$. Suppose that $n \geq 2$. The Green's relations on $\mathcal{J}_n$ are given by Lemma 5.43, and it was proved in [12] that $\mathcal{J}_n$ is generated by $\{\mathrm{id}_n\}$ and its projections of rank $n-2$. Therefore, by Lemma 4.68, the maximal subsemigroups of $\mathcal{J}_n$ arise from its group of units, and from its $\mathscr{J}$-class $J_{n-2} \cap \mathcal{J}_n$. Since there are no elements of rank $n-1$ in $\mathcal{J}_n$, this $\mathscr{J}$-class is covered by the group of units, and so we may apply the results of Section 4.5.2 to describe the maximal subsemigroups that arise in this case.

To construct the graph $\Delta(\mathcal{J}_n) = \Delta(\mathcal{J}_n,\, J_{n-2} \cap \mathcal{J}_n)$, we require a description of the Green's classes of $J_{n-2} \cap \mathcal{J}_n$. Let $\alpha \in J_{n-2} \cap \mathcal{J}_n$ be arbitrary. Then α has $n-2$ transverse blocks, and these contain two points, and by planarity, the non-transverse blocks have the form $\{i,\, i+1\}$ and $\{j',\, (j+1)'\}$ for some $i, j \in \{1, \ldots, n-1\}$. By Lemma 5.43, the $\mathscr{J}$-class contains contains $n-1$ $\mathscr{R}$-classes, and $n-1$ $\mathscr{L}$-classes; the planarity of $\mathcal{J}_n$ implies that $\mathcal{J}_n$ is $\mathscr{H}$-trivial.

For $i \in \{1, \ldots, n-1\}$, we define

- $R_i = \big\{\alpha \in \mathcal{J}_n : \mathrm{rank}(\alpha) = n-2 \text{ and } \{i,\, i+1\} \text{ is a block of } \alpha\big\}$, which is an $\mathscr{R}$-class; and
- $L_i = \big\{\alpha \in \mathcal{J}_n : \mathrm{rank}(\alpha) = n-2 \text{ and } \{i',\, (i+1)'\} \text{ is a block of } \alpha\big\}$, which is an $\mathscr{L}$-class.

The intersection of the $\mathscr{L}$-class L_i and the $\mathscr{R}$-class R_j is a group if and only if $|i-j| \leq 1$. Since the group of units of $\mathcal{J}_n$ is trivial, its actions on the Green's classes of $J_{n-2} \cap \mathcal{J}_n$ are trivial. A picture of $\Delta(\mathcal{J}_n)$ is shown in Figure 5.52. The maximal independent subsets of $\Delta(\mathcal{J}_n)$ are described in Lemma 5.53, and counted in Corollary 5.55.

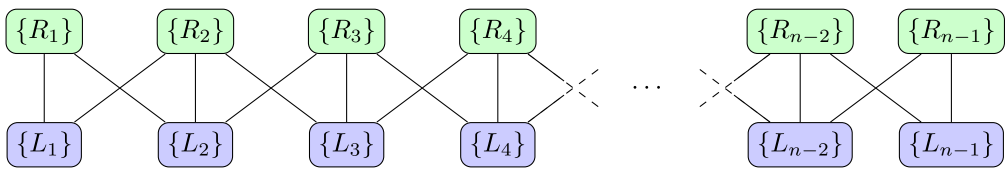

Figure 5.52: The graph $\Delta(\mathcal{J}_n,\, \mathcal{J}_n \cap J_{n-2})$. Since the group of units of $\mathcal{J}_n$ is trivial, its actions on its $\mathscr{L}$-classes and $\mathscr{R}$-classes are trivial, and so its orbits are singletons.

Lemma 5.53. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary, and let U be a subset of the vertices of $\Delta(\mathcal{J}_n)$. Then U is a maximal independent subset of $\Delta(\mathcal{J}_n)$ if and only if all of the following hold:*

(a) *U contains exactly one of $\{L_1\}$ and $\{R_1\}$; and*

(b) *U contains exactly one of $\{L_{n-1}\}$ and $\{R_{n-1}\}$; and*

(c) *if $\{L_i\} \in U$ for some $i \in \{1, \ldots, n-2\}$, then the vertex in $U \setminus \big\{\{L_i\}\big\}$ with the least index greater than or equal to i is either $\{L_{i+1}\}$ or $\{R_{i+2}\}$; and*

(d) *if $\{R_i\} \in U$ for some $i \in \{1, \ldots, n-2\}$, then the vertex in $U \setminus \big\{\{R_i\}\big\}$ with the least index greater than or equal to i is either $\{R_{i+1}\}$ or $\{L_{i+2}\}$.*

Proof. Suppose that U is a maximal independent subset of $\Delta(\mathscr{J}_n)$. We first show that condition (a) holds; the proof that condition (b) holds is analogous. Since $\{L_1\}$ and $\{R_1\}$ are adjacent in $\Delta(\mathscr{J}_n)$, they are not both contained in U; the same is true of $\{L_2\}$ and $\{R_2\}$. If $\{L_2\} \notin U$, then either $\{L_1\} \in U$, or $\{L_1\} \notin U$, and the maximality of U implies that $\{R_1\} \in U$. If instead $\{R_2\} \notin U$, then it follows similarly that $\{L_1\} \in U$ or $\{R_1\} \in U$.

To prove that condition (c) holds, let $i \in \{1, \ldots, n-2\}$ be arbitrary, and suppose that $\{L_i\} \in U$. Consider the vertex in $U \setminus \big\{\{L_i\}\big\}$ with least index greater than or equal to i; some such vertex exists, since U contains either $\{L_{n-1}\}$ or $\{R_{n-1}\}$, by (b). Certainly this vertex is neither $\{R_i\}$ nor $\{R_{i+1}\}$, since these are neighbours of $\{L_i\}$ in $\Delta(\mathscr{J}_n)$. If additionally $\{R_{i+2}\} \notin U$, then the maximality of U implies that $\{L_{i+1}\} \in U$. Noting that U does not contain both $\{L_{i+2}\}$ and $\{R_{i+2}\}$, it follows that the vertex in $U \setminus \big\{\{L_i\}\big\}$ with smallest index greater than or equal to i is either $\{L_{i+1}\}$ or $\{R_{i+2}\}$, as required. Condition (d) holds by an analogous argument.

Conversely, if U satisfies conditions (a)–(d) of the lemma, then it is straightforward to verify that U is an independent subset, and that it is maximal. $\square$

In order to count the maximal subsemigroups of $\Delta(\mathscr{J}_n)$, we introduce the famous *Fibonacci* sequence [120, A000045], $(F_n)_{n\in\mathbb{N}}$, which is defined by

$$F_1 = F_2 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3. \tag{5.54}$$

In [43, Theorem 9.9], it was shown that the ideal $\mathscr{J}_n \setminus \{\mathrm{id}_n\}$ of $\mathscr{J}_n$ has exactly F_n minimal idempotent generating sets. In the following corollary to Lemma 5.53, we show that the Fibonacci numbers also count the maximal independent subsets of $\Delta(\mathscr{J}_n)$.

Corollary 5.55. *Let $n \geq 2$. The number of maximal independent subsets of $\Delta(\mathscr{J}_n)$ is $2F_{n-1}$.*

Proof. The result may be verified directly for $n \in \{2, 3\}$, so suppose that $n \geq 4$. By the symmetry of $\Delta(\mathscr{J}_n)$, the maximal independent subsets that contain $\{L_1\}$ are in one-to-one correspondence with the maximal independent subsets that contain $\{R_1\}$. Therefore, we count $a(n)$, the number of maximal independent subsets of $\Delta(\mathscr{J}_n)$ that contain $\{L_1\}$; by Lemma 5.53(a), the total number of maximal independent subsets of $\Delta(\mathscr{J}_n)$ is $2a(n)$.

For $i \in \{1, 2\}$, define Λ_{n-i} be the induced subgraph of $\Delta(\mathscr{J}_n)$ on the vertices

$$\{\{L_{i+1}\}, \ldots, \{L_{n-1}\}, \{R_{i+1}\}, \ldots, \{R_{n-1}\}\}.$$

In other words, Λ_{n-1} is the graph formed from $\Delta(\mathscr{J}_n)$ be removing the vertices $\{L_1\}$ and $\{R_1\}$, and Λ_{n-2} is formed from Λ_{n-1} by additionally removing the vertices $\{L_2\}$ and $\{R_2\}$. It is obvious that Λ_{n-i} is isomorphic to $\Delta(\mathscr{J}_{n-i})$, and so the number of maximal independent subsets of Λ_{n-1} that contain $\{L_2\}$ is $a(n-1)$, and the number of maximal independent subsets of Λ_{n-2} that contain $\{R_3\}$ is $a(n-2)$.

Let U be a maximal independent subset of $\Delta(\mathscr{J}_n)$ that contains $\{L_1\}$. By Lemma 5.53, U contains either $\{L_2\}$ or $\{R_3\}$, but not both. If U contains $\{L_2\}$, then $U \setminus \big\{\{L_1\}\big\}$ is a maximal independent subset of Λ_{n-1} that contains $\{L_2\}$, while if U contains $\{R_3\}$, then $U \setminus \big\{\{L_1\}\big\}$ is a maximal independent subset of Λ_{n-2} that contains $\{R_3\}$. Hence $a(n) \leq a(n-1) + a(n-2)$. Conversely, each maximal independent subset of Λ_{n-1} that contains $\{L_2\}$, and each maximal independent subset of Λ_{n-2} that contains $\{R_3\}$, gives rise to a unique maximal independent subset of $\Delta(\mathscr{J}_n)$ containing $\{L_1\}$, by the addition of $\{L_1\}$. Hence $a(n) = a(n-1) + a(n-2)$. Since $a(2) = F_1$ and $a(3) = F_2$, we deduce that $a(n) = F_{n-1}$. $\square$

Note that the sets $\big\{\{L_1\}, \ldots, \{L_{n-1}\}\big\}$ and $\big\{\{R_1\}, \ldots, \{R_{n-1}\}\big\}$ are the only maximal independent subsets of $\Delta(\mathscr{J}_n)$ that do not contain both orbits of $\mathscr{L}$-classes and orbits of $\mathscr{R}$-

classes. By Proposition 4.77, these are the two maximal independent subsets of $\Delta(\mathcal{J}_n)$ that do not give rise to maximal subsemigroups of $\mathcal{J}_n$ of type (M2).

Theorem 5.56. *Let $n \in \mathbb{N}$, $n \geq 3$, be arbitrary. The maximal subsemigroups of $\mathcal{J}_n$ are:*

(a) $\mathcal{J}_n \setminus \{\mathrm{id}_n\}$ (type (M5))*;*

(b) *The union of $\mathcal{J}_n \setminus J_{n-2}$ and the union of the Green's classes in a maximal independent subset of $\Delta(\mathcal{J}_n)$ that contains both $\mathscr{L}$-classes and $\mathscr{R}$-classes* (type (M2))*;*

(c) *$\mathcal{J}_n \setminus L$, where L is any $\mathscr{L}$-class in $\mathcal{J}_n$ containing elements of rank $n-2$* (type (M3))*; and*

(d) *$\mathcal{J}_n \setminus R$, where R is any $\mathscr{R}$-class in $\mathcal{J}_n$ containing elements of rank $n-2$* (type (M4)).

In particular, for $n \geq 3$, there are $2F_{n-1} + 2n - 3$ maximal subsemigroups of $\mathcal{J}_n$, where F_{n-1} is the $(n-1)^{th}$ term of the Fibonacci sequence, as defined in (5.54).

Proof. The group of units of $\mathcal{J}_n$ is the trivial group $\{\mathrm{id}_n\}$, and so by Lemma 4.73, the unique maximal subsemigroup that arises from the group of units is formed by removing it.

It remains to describe the maximal subsemigroups that arise from the $\mathscr{J}$-class $J_{n-2} \cap \mathcal{J}_n$. The Jones monoid $\mathcal{J}_n$ is $\mathscr{H}$-trivial, and so by Lemma 4.49(a), $\mathcal{J}_n$ has no maximal subsemigroups of type (M1). The description of the maximal subsemigroups of type (M2) follows directly from Proposition 4.77 and Lemma 5.53, and their number $2F_{n-1} - 2$ follows by Corollaries 4.78 and 5.55. Since $n \geq 3$, each vertex of $\Delta(\mathcal{J}_n)$ has degree 2 or 3, and so by Proposition 4.79, any $\mathscr{L}$-class of rank $n-2$ can be removed to form a maximal subsemigroup of type (M3), and similarly, any $\mathscr{R}$-class of rank $n-2$ can be removed to form a maximal subsemigroup of type (M4). Hence there are $n-1$ maximal subsemigroups of each type. By Corollary 4.75, there is no maximal subsemigroup of type (M5). □

In the second part of this section, we classify the maximal subsemigroups of the annular Jones monoid $\mathcal{AJ}_n$. In order to do so, we require an understanding of its generating sets. Let $n \in \mathbb{N}$, $n \geq 2$, and define ρ_n to be the n-cycle $(1\ 2\ \ldots\ n)$. Recall from Section 5.3.1 that $\mathcal{C}_n = \langle \rho_n \rangle$ is the group of units of $\mathcal{AJ}_n$, and that a partition $\alpha \in \mathcal{P}_n$ is annular if $\alpha = \rho_n^k \beta \rho_n^l$, for some planar partition $\beta \in \mathcal{PP}_n$ and for some $k, l \in \mathbb{N}_0$. The elements of $\mathcal{AJ}_n$ are the annular partitions of degree n whose blocks contain two points.

Let $\alpha = \rho_n^k \beta \rho_n^l \in \mathcal{AJ}_n$ be arbitrary, where β is planar and $k, l \in \mathbb{N}_0$. Since ρ_n^k and ρ_n^l are units, it follows that $\beta = \rho_n^{-k} \cdot \alpha \cdot \rho_n^{-l} \in \mathcal{AJ}_n$, and so the blocks of β have size two. In other words, $\beta \in \mathcal{J}_n$. Therefore

$$\mathcal{AJ}_n = \left\{\rho_n^k \beta \rho_n^l \,:\, \beta \in \mathcal{J}_n \text{ and } k,\, l \in \{1, \ldots, n\}\right\} = \langle \mathcal{J}_n,\ \rho_n \rangle. \tag{5.57}$$

For each $i \in \{1, \ldots, n-1\}$, define ξ_i to be the projection in $\mathcal{AJ}_n$ of rank $n-2$ whose non-transverse blocks are $\{i,\, i+1\}$ and $\{i',\, (i+1)'\}$, and define ξ_n to be the remaining projection of rank $n-2$ in $\mathcal{AJ}_n$; its non-transverse blocks are $\{1,\, n\}$ and $\{1',\, n'\}$. For all $i, j \in \{1, \ldots, n\}$, it is straightforward to verify that $\xi_i = \rho_n^{j-i} \xi_j \rho_n^{i-j}$. Since $\mathcal{J}_n$ is generated by id_n and its projections of rank $n-2$ [12], it follows by this discussion and by (5.57) that $\mathcal{AJ}_n = \langle \rho_n,\ \xi_i \rangle$ for any $i \in \{1, \ldots, n\}$.

We proceed with the same technique that was used in the proof of Theorems 5.47 and 5.48. Let $\alpha \in J_{n-2} \cap \mathcal{AJ}_n$ be arbitrary. Working modulo n, there exist indices $i, j \in \{1, \ldots, n\}$ such that for all $k \in \mathbb{N}_0$, the non-transverse blocks of $\alpha \rho_n^k$ are $\{i,\, i+1\}$ and $\{(j\rho_n^k)',\, (j\rho_n^k+1)'\}$. Thus we may fix $k \in \mathbb{N}_0$ such that the non-transverse blocks of $\alpha \rho_n^k$ are $\{i,\, i+1\}$ and $\{i',\, (i+1)'\}$. Since $\mathcal{AJ}_n$ is finite, some power of $\alpha \rho_n^k$ is equal to the projection ξ_i. Therefore

$$\mathcal{AJ}_n \geq \langle \rho_n,\ \alpha \rangle \geq \langle \rho_n,\ \alpha \rho_n^k \rangle \geq \langle \rho_n,\ \xi_i \rangle = \mathcal{AJ}_n,$$

and so $\langle \rho_n,\ \alpha\rangle = \langle \mathcal{C}_n,\ \alpha\rangle = \mathcal{AJ}_n$. Since $\mathcal{AJ}_n$ contains no elements of rank $n-1$, any generating set for $\mathcal{AJ}_n$ contains elements of rank $n-2$. Therefore, it follows that $\langle \mathcal{C}_n,\ \alpha\rangle$ if and only if $\alpha \in J_{n-2} \cap \mathcal{AJ}_n$. This result is required in the proof of the following theorem.

Theorem 5.58. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. The maximal subsemigroups of $\mathcal{AJ}_n$ are:*

(a) $(\mathcal{AJ}_n \setminus \mathcal{C}_n) \cup U$, *where U is a maximal subgroup of $\mathcal{C}_n$* (type (M1))*; and*

(b) $\mathcal{AJ}_n \setminus \{\alpha \in \mathcal{AJ}_n : \operatorname{rank}(\alpha) = n-2\}$ (type (M5)).

In particular, for $n \geq 2$, there are $|\mathbb{P}_n| + 1$ maximal subsemigroups of $\mathcal{AJ}_n$, where $\mathbb{P}_n$ denotes the set of primes that divide n.

Proof. The group of units of $\mathcal{AJ}_n$ is $\mathcal{C}_n$, and so the description and number of the maximal subsemigroups that arise from the group of units follows Lemmas 5.1 and 4.73. Since $\mathcal{AJ}_n = \langle \mathcal{C}_n,\ \alpha\rangle$ if and only if $\alpha \in J_{n-2} \cap \mathcal{AJ}_n$, it follows by Corollary 4.74 that the sole remaining maximal subsemigroup is $\mathcal{AJ}_n \setminus J_{n-2}$. □

	1	2	3	4	5	6	7	8	9	10	11	12	13	n
$\mathcal{J}_n$	1	2	5	9	13	19	27	39	57	85	129	199	311	$2F_{n-1} + 2n - 3$
$\mathcal{AJ}_n$	1	2	2	2	2	3	2	2	2	3	2	3	2	$\lvert\mathbb{P}_n\rvert + 1$

Table 5.59: The numbers of maximal subsemigroups of the Jones and annular Jones monoids $\mathcal{J}_n$ and $\mathcal{AJ}_n$, for $n = 1, \ldots, 13$, along with the general formulae. The sequence $(F_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence, defined in (5.54), and $\mathbb{P}_n$ is the set of all primes dividing n. See Theorems 5.56 and 5.58 for descriptions of the maximal subsemigroups of $\mathcal{J}_n$ and $\mathcal{AJ}_n$, respectively.

5.3.8 The Motzkin monoid $\mathcal{M}_n$

In the final section of this chapter, we classify the maximal subsemigroups of the Motzkin monoid, $\mathcal{M}_n$, which consists of all planar partitions in $\mathcal{PB}_n$. We first present some prerequisite information about this monoid.

By Lemma 4.68, in order to describe the maximal subsemigroups of $\mathcal{M}_n$, it is necessary to find the $\mathscr{J}$-classes that intersect every generating set of $\mathcal{M}_n$ non-trivially. Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. By [33, Proposition 4.2], the Motzkin monoid is generated by its elements of rank at least $n-2$, and any generating set for $\mathcal{M}_n$ contains elements of ranks n, $n-1$, and $n-2$. By Lemma 5.43, Green's $\mathscr{J}$-relation on $\mathcal{M}_n$ is determined by rank, and so the maximal subsemigroups of $\mathcal{M}_n$ arise from the three $\mathscr{J}$-classes of $\mathcal{M}_n$ that correspond to these ranks.

We therefore require a description of the elements of $\mathcal{M}_n$ that have rank at least $n-2$. The unique element of rank n in $\mathcal{M}_n$ is id_n. An arbitrary element of rank $n-1$ in $\mathcal{M}_n$ has trivial kernel and cokernel, and is determined by the unique point that it lacks from its domain and the unique point that it lacks from its codomain. By Lemma 5.43, this describes the $\mathscr{L}$- and $\mathscr{R}$-classes of the $\mathscr{J}$-class $J_{n-1} \cap \mathcal{M}_n$. More specifically, two elements of rank $n-1$ in $\mathcal{M}_n$ are $\mathscr{R}$-related if they lack the same point from their domains, and $\mathscr{L}$-related if they lack the same point from their codomains. Furthermore, every idempotent in $J_{n-1} \cap \mathcal{M}_n$ is a projection: an element of rank $n-1$ in $\mathcal{M}_n$ is idempotent if and only if its domain and codomain are equal, and so every idempotent in $J_{n-1} \cap \mathcal{M}_n$ is fixed by the $*$ operation. Therefore, we may use Corollary 4.82 to describe the maximal subsemigroups that arise from this $\mathscr{J}$-class.

We use Lemma 4.8 to describe the maximal subsemigroups of $\mathcal{M}_n$ that arise from the $\mathscr{J}$-class $J_{n-2} \cap \mathcal{M}_n$. To apply this lemma, we require [33, Lemma 4.11]. In the case that $r = n-1$, this lemma states that one generating set for $\mathcal{M}_n$ consists of id_n, the elements of rank $n-1$, and the projections of rank $n-2$ that contain no singleton blocks. We also require [33, Lemmas 4.3 and 4.4], which states that the elements of rank $n-1$ in $\mathcal{M}_n$ generate those elements of rank $n-2$ whose non-transverse blocks are singletons.

Theorem 5.60. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. The maximal subsemigroups of $\mathcal{M}_n$ are:*

(a) $\mathcal{M}_n \setminus \{\mathrm{id}_n\}$ (type (M5));

(b) *The union*

$$\big(\mathcal{M}_n \setminus J_{n-1}\big) \cup \bigcup_{i\in A} \{\alpha \in \mathcal{M}_n \,:\, \mathrm{rank}(\alpha) = n-1 \text{ and } \{i\} \text{ is a block of } \alpha\}$$
$$\cup \bigcup_{i\notin A} \{\alpha \in \mathcal{M}_n \,:\, \mathrm{rank}(\alpha) = n-1 \text{ and } \{i'\} \text{ is a block of } \alpha\},$$

where A is any non-empty proper subset of $\{1, \ldots, n\}$ (type (M2));

(c) $\mathcal{M}_n \setminus \big\{\alpha \in J_{n-2} \,:\, \{i', (i+1)'\} \text{ is a block of } \alpha\big\}$ *for* $i \in \{1, \ldots, n-1\}$ (type (M3)); *and*

(d) $\mathcal{M}_n \setminus \big\{\alpha \in J_{n-2} \,:\, \{i, i+1\} \text{ is a block of } \alpha\big\}$ *for* $i \in \{1, \ldots, n-1\}$ (type (M4)).

In particular, for $n \geq 2$, there are $2^n + 2n - 3$ maximal subsemigroups of $\mathcal{M}_n$.

	1	2	3	4	5	6	7	8	9	10	11	12	n
$\mathcal{M}_n$	2	5	11	21	39	73	139	269	527	1041	2067	4117	$2^n + 2n - 3$

Table 5.61: The number of maximal subsemigroups of the Motzkin monoid $\mathcal{M}_n$, for $n = 1, \ldots, 12$, along with the general formula. See Theorem 5.60 for a description of the maximal subsemigroups of $\mathcal{M}_n$.

Proof. The group of units of $\mathcal{M}_n$ is the trivial group $\{\mathrm{id}_n\}$, and so by Lemma 4.73, $\mathcal{M}_n \setminus \{\mathrm{id}_n\}$ is the unique maximal subsemigroup to arise from the group of units.

Since the Motzkin monoid consists of planar partitions, it is $\mathscr{H}$-trivial. By Lemma 4.49(a), there are no maximal subsemigroups of $\mathcal{M}_n$ of type (M1). Given the earlier description of the $\mathscr{L}$- and $\mathscr{R}$-classes in $J_{n-1} \cap \mathcal{M}_n$, it follows by Corollary 4.82 that the maximal subsemigroups that arise from the $\mathscr{J}$-class of rank $n-1$ elements are those described in the theorem of type (M2), and that there are $2^n - 2$ of them.

It remains to describe the maximal subsemigroups that arise from the $\mathscr{J}$-class that consists of rank $n-2$ elements. For $i \in \{1, \ldots, n-1\}$, define the subsets

$$X_i = \{\alpha \in \mathcal{M}_n \,:\, \mathrm{rank}(\alpha) = n-2 \text{ and } \{i, i+1\} \text{ is a block of } \alpha\}, \text{ and}$$
$$X_i^* = \{\alpha^* \,:\, \alpha \in X_i\} = \{\alpha \in \mathcal{M}_n \,:\, \mathrm{rank}(\alpha) = n-2 \text{ and } \{i', (i+1)'\} \text{ is a block of } \alpha\}$$

of the $\mathscr{J}$-class $J_{n-2} \cap \mathcal{M}_n$. Note that X_i is an $\mathscr{R}$-class of $\mathcal{M}_n$, and X_i^* is an $\mathscr{L}$-class of $\mathcal{M}_n$.

Let A be any subset of $J_{n-2} \cap \mathcal{M}_n$ such that $(\mathcal{M}_n \setminus J_{n-2}) \cup A$ generates $\mathcal{M}_n$, and let $i \in \{1, \ldots, n-1\}$ and $\alpha \in X_i$ be arbitrary. Then α can be written as a product $\alpha = \beta_1 \cdots \beta_k$ of the generators in $(\mathcal{M}_n \setminus J_{n-2}) \cup A$; moreover, we may assume that $\mathrm{rank}(\beta_j) \in \{n-1, n-2\}$ for each β_j; see Lemma 1.12. If $\mathrm{rank}(\beta_1) = n-1$, then β_1, and therefore each of its right multiples, contains the singleton block $\{j\}$ for some $j \in \{1, \ldots, n\}$. However, α is a right multiple of β_1, and α contains no such block. Hence $\mathrm{rank}(\beta_1) = n-2 = \mathrm{rank}(\alpha)$. Lemmas 1.10 and 5.43 imply that $\ker(\alpha) = \ker(\beta_1)$, i.e. $\beta_1 \in A \cap X_i$. A dual argument shows that $A \cap X_i^* \neq \varnothing$.

Conversely, let A be any subset of $J_{n-2} \cap \mathcal{M}_n$ that intersects X_i and X_i^* non-trivially for all $i \in \{1, \ldots, n-1\}$. Let $i \in \{1, \ldots, n-1\}$ be arbitrary, and define ξ_i to be the projection of rank $n-2$ in $\mathcal{M}_n$ whose non-transverse blocks are $\{i, i+1\}$ and $\{i', (i+1)'\}$. We aim to show that $\xi_i \in \langle \mathcal{M}_n \setminus J_{n-2},\ A\rangle$. Fix $\alpha \in X_i \cap A$ and $\beta \in X_i^* \cap A$. Since α and β have rank $n-2$, $\operatorname{codom}(\alpha) = \left\{j_1' > j_2' > \cdots > j_{n-2}'\right\}$ and $\operatorname{dom}(\beta) = \left\{k_1 < k_2 < \cdots < k_{n-2}\right\}$, for some points $j_1, \ldots, j_{n-2}$ and $k_1, \ldots, k_{n-2} \in \{1, \ldots, n\}$, and using the canonical ordering defined in (5.40). Define γ to be the partition of degree n whose transverse blocks are $\{j_1, k_1'\}, \ldots, \{j_{n-2}, k_{n-2}'\}$, and whose non-transverse blocks are singletons. Then γ has rank $n-2$ and is planar. The non-transverse blocks of γ are singletons, and so by the earlier discussion, it follows that γ is a product of elements of rank $n-1$ in $\mathcal{M}_n$. In particular, $\gamma \in \langle \mathcal{M}_n \setminus J_{n-2},\ A\rangle$, and since clearly $\xi_i = \alpha\gamma\beta$, it follows that $x_i \in \langle \mathcal{M}_n \setminus J_{n-2},\ A\rangle$. Since $i \in \{1, \ldots, n-1\}$ was arbitrary, we deduce that $\langle \mathcal{M}_n \setminus J_{n-2},\ A\rangle$ contains every projection in $J_{n-2} \cap \mathcal{M}_n$ that contains no singleton blocks. Since $\mathcal{M}_n$ is generated by these projections and its elements of ranks $n-1$ and n, it follows that $\mathcal{M}_n = \langle \mathcal{M}_n \setminus J_{n-2},\ A\rangle$. By Lemma 4.8, the maximal subsemigroups of $\mathcal{M}_n$ arising from its $\mathscr{J}$-class of rank $n-2$ are the sets $\mathcal{M}_n \setminus X_i$ and $\mathcal{M}_n \setminus X_i^*$ for each $i \in \{1, \ldots, n-1\}$. $\square$

5.4 Table of results

Monoid	Group of units	Number of maximal subsemigroups	OEIS [120]	Result	Compare with
$\mathcal{POI}_n$	Trivial	$2^n - 1$	A000225	Theorem 5.29	[56, Theorem 2]
$\mathcal{PO}_n$		$2^n + 2n - 2$	A131520	Theorem 5.12	[31, Theorem 1]
$\mathcal{M}_n$		$2^n + 2n - 3$	A131898	Theorem 5.60	—
$\mathcal{O}_n$		$A_{2n-1} + 2n - 4$	A000931	Theorem 5.26	[25, Theorem 2]
$\mathcal{J}_n$		$2F_{n-1} + 2n - 3$	A290140	Theorem 5.56	—
$\mathcal{PP}_n$		$2F_{2n-1} + 4n - 3$	A290140	Theorem 5.56	—
$\mathcal{PODI}_{2n}$	Order 2	$3 \cdot 2^{n-1} - 1$	A052955	Theorem 5.30	[29, Theorem 4]
$\mathcal{PODI}_{2n-1}$		$2^n - 1$	A052955	Theorem 5.30	[29, Theorem 4]
$\mathcal{POD}_n$		$2^{\lceil n/2 \rceil} + n - 1$	A016116	Theorem 5.13	—
$\mathcal{OD}_n$		$A_n + n - 3$	A000931	Theorem 5.27	[68, Theorem 2]
$\mathcal{POPI}_n$	$\mathcal{C}_n$	$\lvert\mathbb{P}_n\rvert + \lvert\mathbb{P}_{n-1}\rvert$	A059957	Theorem 5.35	—
$\mathcal{POP}_n$		$\lvert\mathbb{P}_n\rvert + 2$	A083399	Theorem 5.32	—
$\mathcal{AJ}_n$		$\lvert\mathbb{P}_n\rvert + 1$	A083399	Theorem 5.58	—
$\mathcal{OP}_n$		$\lvert\mathbb{P}_n\rvert + 1$	A083399	Theorem 5.33	[28, Theorem 1.6]
$\mathcal{PORI}_n$	$\mathcal{D}_n$	$1 + \lvert\mathbb{P}_{n-1}\rvert + \sum_{p \in \mathbb{P}_n} p$	A290289	Theorem 5.36	—
$\mathcal{POR}_n$		$3 + \sum_{p \in \mathbb{P}_n} p$	A008472	Theorem 5.32	—
$\mathcal{OR}_n$		$2 + \sum_{p \in \mathbb{P}_n} p$	A008472	Theorem 5.33	[28, Theorem 2.6]
$\mathcal{T}_n$	$\mathcal{S}_n$	$s_n + 1$	A290138	Theorem 5.7	—
$\mathcal{I}_n$		$s_n + 1$	A290138	Theorem 5.7	—
$\mathcal{I}_n^*$		$s_n + 1$	A290138	Theorem 5.50	[94, Theorem 19]
$\mathfrak{F}_n$		$s_n + 1$	A290138	Theorem 5.48	—
$\mathcal{B}_n$		$s_n + 1$	A290138	Theorem 5.47	—
$\mathcal{PT}_n$		$s_n + 2$	A290138	Theorem 5.7	—
$\mathcal{PB}_n$		$s_n + 3$	A290138	Theorem 5.49	—
$\mathcal{P}_n$		$s_n + 4$	A290138	Theorem 5.45	—

Table 5.62: Information about the maximal subsemigroups of the monoids considered in this chapter, where $n \in \mathbb{N}$ is sufficiently large (usually $n \geq 2$ or $n \geq 3$; see the relevant theorem for the precise value). The maximal subsemigroups themeselves are described in the referenced theorems. For $k \in \mathbb{N}$, $\mathbb{P}_k$ is the set of primes that divide k; A_k is the k^{th} term of the sequence defined in (5.25); F_k is the k^{th} Fibonacci number, defined in (5.54) with $F_1 = F_2 = 1$; and s_k is the number of maximal subgroups of $\mathcal{S}_k$, the symmetric group of degree k.

For small values of n, these numbers of maximal subsemigroups may be verified by applying the function `NrMaximalSubsemigroups` (which is described in Section 4.6.1) from the Semigroups package [101] for GAP [58] to the appropriate monoid. For example, by executing

```
List([1 .. 10], n -> NrMaximalSubsemigroups(PORI(n)));
List([1 .. 10], n -> NrMaximalSubsemigroups(JonesMonoid(n))); or
List([1 .. 10], n -> NrMaximalSubsemigroups(PartialTransformationMonoid(n)));
```

one can produce a list of the number of non-empty maximal subsemigroups of $\mathcal{PORI}_n$, $\mathcal{J}_n$, or $\mathcal{PT}_n$, respectively, for $n = 1, \ldots, 10$.

Appendix A

Open problems

In this appendix, we collate the open problems that we posed elsewhere in the text.

Open Problem 2.2 (Page 40). Develop techniques for computing the Green's structure of a direct product of finite semigroups in terms of the Green's structures and other semigroup-theoretic properties of the factors.

Open Problem 2.3 (Page 40). Investigate how the maximal subsemigroups of a direct product of finite semigroups relate to the maximal subsemigroups of the factors.

Open Problem 2.34 (Page 58). Generalise Theorem 2.31 and Corollary 2.33 for direct products of an arbitrary number of finitely generated surjective semigroups. In particular, develop an upper bound for the rank of such a direct product that involves the ranks and the numbers of maximal $\mathscr{L}$- and $\mathscr{R}$-classes of the factors.

Open Problem 2.35 (Page 58). Let S be an arbitrary finite semigroup. Develop practical methods for computing whether there exist non-trivial semigroups T and U such that $S \cong T \times U$; given this, develop methods for finding such semigroups T and U when they exist.

Open Problem 3.13 (Page 68). Let S be a finite Rees 0-matrix semigroup over an arbitrary finite semigroup T, and assume that any necessary semigroup-theoretic properties of T are known *a priori*. Give algorithms for counting and listing the idempotents of S, using the properties of T and the Graham-Houghton graph of S.

Open Problem 3.14 (Page 68). Let S be a finite Rees 0-matrix semigroup over an arbitrary finite semigroup T, and assume that any necessary semigroup-theoretic properties of T are known *a priori*. Give an algorithm for constructing a generating set for $F(S)$ that contains at most a constant multiple of $\operatorname{rank}(F(S))$ elements.

Open Problem 3.29 (Page 73). Let S be a finite semigroup to which the techniques of [37] apply. Building on the results of [37], develop methods for computing the Green's structure of an arbitrary left or right ideal of S that do not necessarily exhaustively enumerate the ideal.

Open Problem 3.30 (Page 73). Let S be a finite semigroup to which the techniques of [37] apply. Building on the results of [37], develop methods for computing the partial orders of $\mathscr{L}$- and $\mathscr{R}$-classes of S without necessarily exhaustively enumerating S.

Open Problem 3.46 (Page 80). Let $S = \mathscr{M}^0[T; I, \Lambda; P]$ be an arbitary finite Rees 0-matrix semigroup where T is a monoid and P contains a unit. Give a formula that describes $\operatorname{rank}(S)$

in terms of $\operatorname{rank}(T)$, the semigroup-theoretic properties of T, and the matrix P.

Open Problem 4.51 (Page 118)**.** If A_L and A_R are arbitrary finite acyclic digraphs, does there exist a finite semigroup S with a regular $\mathscr{J}$-class J such that $\Gamma_{\mathscr{L}}(S, J) \cong A_L$ and $\Gamma_{\mathscr{R}}(S, J) \cong A_R$?

Open Problem 4.87 (Page 136)**.** Develop tools for computing subsemigroups that are maximal with respect to some property, such as maximal commutative or maximal regular subsemigroups of a finite semigroup; or maximal inverse subsemigroups of a finite inverse semigroup.

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Table of notation

This table lists some of the mathematical notation that is used in this thesis. In any row, the first column gives an example of some notation, the second column gives a brief description of what the notation represents, and the third column gives the page number where the notation is defined or first used in the main text. In the original digital version of this document, many instances of such notation are hyperlinks to their corresponding entries in this table.

0_S	The multiplicative zero of the semigroup-with-zero S.	23
1_S	The identity element of the monoid S^1.	24
$\varnothing$	The empty set.	20
$\lvert X\rvert$	The cardinality of the set X.	20
X^+	The free semigroup over the set X.	23
$\langle X\rangle$	The subsemigroup of a semigroup generated by its subset X.	24
$\langle X \mid R\rangle$	The semigroup presentation with generators X and relations R.	23
$\lceil x\rceil$	The least integer greater than or equal to the real number x.	20
$\lfloor x\rfloor$	The greatest integer less than or equal to the real number x.	20
x^*	The image of x under the $*$ operation of a regular $*$-semigroup.	29
x^{-1}	The unique inverse of an element x in a semigroup.	29
x^n	The element $\underbrace{xx\cdots x}_{n \text{ times}}$.	23
$\mathcal{AJ}_n$	The annular Jones monoid of degree n.	167
$\mathcal{A}_n$	The alternating group, consisting of all even permutations of degree n.	108
$(A_n)_{n\in\mathbb{N}}$	The sequence $A_1 = 1$, $A_2 = A_3 = 2$, and $A_n = A_{n-2} + A_{n-3}$ for $n > 3$.	157
$B(G,\, I)$	The $I \times I$ Brandt semigroup over the group G.	85
$B(G,\, n)$	The $n \times n$ Brandt semigroup $B(G,\, \{1,\, \ldots,\, n\})$.	85
$\mathcal{B}_n$	The Brauer monoid of degree n.	166
$\mathcal{C}_n$	The cyclic group of order n generated by the n-cycle $(1\ 2\ \ldots\ n)$.	148
$\operatorname{codom}(\alpha)$	The codomain of the partition α.	168
$\operatorname{coker}(\alpha)$	The cokernel of the partition α.	168
$\mathscr{D}$	Green's $\mathscr{D}$-relation.	26
$\mathcal{D}_n$	The permutation group $\langle(1\ 2\ \ldots\ n),\ (1\ n)(2\ n-1)\cdots(\lfloor n/2\rfloor\ \lceil n/2\rceil)\rangle$, which is a dihedral group of order $2n$ when $n \geq 3$.	148
$\operatorname{dom}(\alpha)$	The domain of the partial transformation, or partition, α.	20
$\Delta = \Delta(S,\, J)$	The graph defined in §4.4.2, where S is a semigroup and $J \in S/\mathscr{J}$.	118
$\Delta(S)$	The graph $\Delta(S,\, J)$, where S is a monoid with a unique $\mathscr{J}$-class J covered by its group of units.	150
$E(X)$	The set of idempotents contained in a subset X of a semigroup.	23

$(F_n)_{n\in\mathbb{N}}$	The Fibonacci sequence [120, A000045] with $F_1 = F_2 = 1$.	174
$\mathfrak{F}_n$	The uniform block bijection monoid of degree n.	166
$F(X)$	The subsemigroup $\langle E(X)\rangle$ generated by the idempotents of X.	24
$\Gamma_{\mathscr{L}} = \Gamma_{\mathscr{L}}(S, J)$	A digraph representing the action of $S \setminus J$ on the $\mathscr{L}$-classes of J.	118
$\Gamma_{\mathscr{R}} = \Gamma_{\mathscr{R}}(S, J)$	A digraph representing the action of $S \setminus J$ on the $\mathscr{R}$-classes of J.	118
$[G : V]$	The index of the subgroup V in the group G.	29
$\mathscr{H}$	Green's $\mathscr{H}$-relation.	26
id_n	The identity permutation of degree n.	30
$\mathrm{im}(\alpha)$	The image of the partial function α.	20
$\mathcal{I}_n$	The symmetric inverse monoid of degree n.	30
$\mathcal{I}_n^*$	The dual symmetric inverse monoid of degree n.	166
J^*	The principal factor of the Green's $\mathscr{J}$-class J.	28
$\mathscr{J}$	Green's $\mathscr{J}$-relation.	26
J_k	The $\mathscr{J}$-class of $\mathcal{PT}_n$ or $\mathcal{P}_n$ consisting of all elements of rank k.	149
$\mathcal{J}_n$	The Jones monoid of degree n.	138
$\ker(\alpha)$	The kernel of the partial function, or partition, α.	20
$\mathscr{K}^S$	Green's $\mathscr{K}$-relation on the semigroup S, for $\mathscr{K} \in \{\mathscr{H}, \mathscr{L}, \mathscr{R}, \mathscr{D}, \mathscr{J}\}$.	27
K_x	The Green's $\mathscr{K}$-class of x in some semigroup.	27
K_x^S	The Green's $\mathscr{K}$-class of x in the semigroup S.	27
$\mathscr{L}$	Green's $\mathscr{L}$-relation.	26
$\mathscr{M}[T;\, I,\, \Lambda;\, P]$	The $I \times \Lambda$ Rees matrix semigroup over T with matrix P.	25
$\mathscr{M}^0[T;\, I,\, \Lambda;\, P]$	The $I \times \Lambda$ Rees 0-matrix semigroup over T with matrix P.	26
$\mathcal{M}_n$	The Motzkin monoid of degree n.	166
$\mathbb{N}$	The natural numbers $\{1, 2, 3, \ldots\}$.	20
$\mathbb{N}_0$	The non-negative integers $\{0, 1, 2, 3, \ldots\}$.	20
$N_G(V)$	The normalizer of the subgroup V in the group G.	29
$O(n^2)$	Big O notation, used to describe the complexity of an algorithm.	32
$\mathcal{OD}_n$	All order-preserving or -reversing transformations of degree n.	148
$\mathcal{O}_n$	All order-preserving transformations of degree n.	148
$\mathcal{OP}_n$	All orientation-preserving transformations of degree n.	148
$\mathcal{OR}_n$	All orientation-preserving or -reversing transformations of degree n.	148
$\mathcal{PB}_n$	The partial Brauer monoid of degree n.	166
$\mathcal{P}_n$	The partition monoid of degree n.	165
$\mathbb{P}_n$	The set of primes that divide $n \in \mathbb{N}$.	162
$\mathcal{PODI}_n$	All order-preserving or -reversing partial permutations of degree n.	148
$\mathcal{POD}_n$	All order-preserving or -reversing partial transformations of degree n.	147
$\mathcal{POI}_n$	All order-preserving partial permutations of degree n.	148
$\mathcal{PO}_n$	All order-preserving partial transformations of degree n.	147

$\mathcal{POPI}_n$	All orientation-preserving partial permutations of degree n.	148
$\mathcal{POP}_n$	All orientation-preserving partial transformations of degree n.	148
$\mathcal{PORI}_n$	All orientation-preserving or -reversing partial permutations of degree n.	138
$\mathcal{POR}_n$	All orientation-preserving or -reversing partial transformations in $\mathcal{PT}_n$.	148
$\mathcal{PP}_n$	The planar partition monoid of degree n.	166
$\mathcal{PT}_n$	The partial transformation monoid of degree n.	30
$\mathscr{R}$	Green's $\mathscr{R}$-relation.	26
$\mathbb{R}$	The real numbers.	22
$\operatorname{rank}(\alpha)$	The rank of the semigroup element α.	20
$\operatorname{rank}(S)$	The least cardinality of a generating set for the semigroup S.	24
S^0	The semigroup formed by adjoining a zero to the semigroup S.	24
S^1	For a semigroup S, if S is a monoid, then $S^1 = S$, else S^1 is the monoid formed by adjoining the identity element 1_S to S.	24
$\mathcal{S}_n$	The symmetric group of degree n.	30
s_n	The number of maximal subgroups of $\mathcal{S}_n$ [120, A290138].	146
$\operatorname{Stab}_G(A)$	The setwise stabilizer $\{g \in G : Ag = A\}$ of A in the group G.	133
$\textsc{sub}(V, g_1, \ldots, g_n)$	The subset of a normalized finite 0-simple semigroup defined in (4.19).	102
$\mathcal{T}_n$	The full transformation monoid of degree n.	30
$\Theta = \Theta(S,\, J)$	The graph defined in §4.4.2, where S is a semigroup and $J \in S/\mathscr{J}$.	118
$\mathbb{Z}$	The integers.	20

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