On the Gruenberg - Kegel Graph of Integral Group Rings of Finite Groups

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ABSTRACT. The prime graph question asks whether the Gruenberg - Kegel graph of an integral group ring $\mathbb{Z}G$, i.e., the prime graph of the normalised unit group of $\mathbb{Z}G$, coincides with that one of the group G. In this note we prove for finite groups G a reduction of the prime graph question to almost simple groups. We apply this reduction to finite groups G whose order is divisible by at most three primes and show that the Gruenberg - Kegel graph of such groups coincides with the prime graph of G.
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1. Introduction

Let G be a group. Its prime graph $\Pi(G)$ is defined as follows. The vertices of $\Pi(G)$ are the primes p for which G has an element of order p. Two different vertices p and q are joined by an edge provided there is an element of G of order $p \cdot q$. If R is a group ring then its Gruenberg - Kegel graph $\Gamma(R)$ is just the prime graph $\Pi(V(R))$ of its normalised unit group $V(R)$.

The integral group ring of G is denoted by $\mathbb{Z}G$. Its normalised unit group $V(\mathbb{Z}G)$ is the subgroup of the unit group $U(\mathbb{Z}G)$ which consists of all units with augmentation 1, called normalised units. The subject of this article is $\Gamma(\mathbb{Z}G)$ in the case when G is a finite group.

The question is whether $\Gamma(\mathbb{Z}G)$ coincides with that of $\Pi(G)$ [28, Problem 21]. This question, known as the "Prime Graph Question" (PQ) may be regarded as a weak version of the first Zassenhaus conjecture (ZC-1) which says that each torsion unit of $V(\mathbb{Z}G)$ is conjugate within $\mathbb{Q}G$ to an element of G, where G is considered in a natural way as subgroup of $V(\mathbb{Z}G)$ and its elements are then called trivial units of $\mathbb{Z}G$. The conjecture (ZC-1) is certainly one of the major open questions for integral group rings and if it is valid for a specific group G it provides of course a positive answer to the prime graph question for $V(\mathbb{Z}G)$. However, only for specific classes (mainly specific soluble ones) of finite groups (ZC-1) is known. It is known that (PQ) holds for all soluble groups [29]. In this article we show that (PQ) holds as well for large classes of finite insoluble groups.

In Section 2 we reduce the study of the prime graph question to the study of nonabelian composition factors and their automorphism groups. Theorem 2.1

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shows that (PQ) holds provided G does not map onto an almost simple group which is a counterexample to (PQ). This gives many computer calculations made in the last years with respect to simple groups (see e.g. [9, 11, 23]) a new value. These calculations are not only examples. The verification of (PQ) for a single simple group establishes (PQ) for infinitely many groups. Moreover with respect to sporadic simple groups the computational proof of (PQ) might be the only way. We note that not only simple groups have to be checked but also their automorphism groups have to be examined which has been done up to now only very rarely.

As an application, we prove in Section 3 that for finite groups whose nonabelian composition factors have order divisible by only three primes the prime graph question has a positive answer.

For this it is shown with the aid of computational algebra that the prime graph question, or even in some cases (ZC-1), has a positive answer for all almost simple groups G of this type. For the most cases these computer calculations rest on two independent implementations of the HeLP algorithm (an algorithm initially developed by I. B. S. Passi and I. Luthar, and later enhanced by M. Hertweck). One of them (unpublished) is due to V. Bovdi and the second author. The second one is due to A. Bächle and L. Margolis and is available in the HeLP package [5] which is redistributed together with the computational algebra system GAP [38]. We like to point out that Theorem 3.1 has been completed by a recent result of A. Bächle and L. Margolis for torsion units of integral group rings of the groups M_{10} and $\mathrm{PGL}(2,9)$ which is based on a new algorithmic method, the so-called lattice method [33]. This method is also an essential tool in the remarkable analysis of $\Gamma(\mathbb{Z}G)$ for almost simple groups G whose order is divisible by exactly four primes [6, 7]. The number of such almost simple groups however is infinite and generic character tables of series of almost simple groups enter the picture.

With the aid of Theorem 2.1 and generic character tables it follows that the prime graph question has a positive answer for all finite groups whose composition factors are either of prime order or isomorphic to $\mathrm{PSL}(2,p^f)$, where $f \leq 2$ and $p \geq 5$ is a prime (cf. Corollary 3.2).

2. The reduction to almost simple groups

An almost simple group of type S is a subgroup of the automorphism group of a finite non-abelian simple group S which contains $\mathrm{Inn(S)} \cong S$. A group G is called almost simple if it is almost simple of type S for some simple group S. Clearly each finite simple group S defines its own family of almost simple groups parametrised by the subgroups of $\mathrm{Out(S)}$. It may happen that this family just consists of S. The object of this section is the following result which reduces the prime graph question to the study of almost simple groups.

Theorem 2.1. *Let G be a finite group. Assume that for each almost simple group X which occurs as image of G the prime graph question has a positive answer. Then the prime graph question has a positive answer for G.*

For $u \in V(\mathbb{Z}G)$ and $x \in G$ by $\varepsilon_x(u)$ we denote, as usual, the partial augmentation of u with respect to the conjugacy class x^G. If G is a finite group then $\pi(G)$ denotes the set of primes dividing the order of G, $\exp(G)$ denotes the exponent of G, and $o(g)$ denotes the order of $g \in G$.

Proposition 2.2. *Let N be a normal subgroup of E and $G = E/N$. Assume that $\Pi(V(\mathbb{Z}G)) = \Pi(G)$. Let $q \in \pi(E) \setminus \pi(N)$ and $p \in \pi(E)$ with $q \neq p$. Then $V(\mathbb{Z}E)$ has elements of order $p \cdot q$ if and only if E has elements of this order.*

Proof. The proof in one direction is obvious. So assume that $V(\mathbb{Z}E)$ has a unit u of order $p \cdot q$ but E does not have an element of this order. Denote by ϕ be the reduction from E onto G and let $\hat{\phi}$ be the homomorphism from $V(\mathbb{Z}E)$ to $V(\mathbb{Z}G)$ induced from ϕ.

If $\hat{\phi}(u)$ has order $p \cdot q$, by the assumption that $\Pi(V(\mathbb{Z}G)) = \Pi(G)$ we get an element of the order $p \cdot q$ in G. But then E has an element of this order. It follows that $\hat{\phi}(u^p) = 1$ or $\hat{\phi}(u^q) = 1$.

Let v be a non-trivial torsion unit of $\mathbb{Z}E$. By [34, Theorem 2.7] we get that $\varepsilon_x(v) = 0$ provided $o(x)$ is divisible by a prime not dividing the order of v. Moreover by S.D. Berman [8] and G. Higman [27, 35] the 1-coefficient of a non-trivial torsion unit with augmentation 1 has to be zero. It follows that orders of torsion elements of prime order in the kernel of $\hat{\phi}$ divide the order of N. Because of the assumption that q does not divide the order of N, we must have $\hat{\phi}(u^q) = 1$.

For a group X and a prime r denote by $X(r)$ the set of all elements of X whose order is a positive power of r. Write u as

$$u = \sum_{g \in E(p)} a_g g + \sum_{g \in E(q)} b_g g + \sum_{g \in R} c_g g,$$

where $R = E \setminus (E(p) \cup E(q))$. Because E has no elements of the order $p \cdot q$, by [34] we conclude that $\varepsilon_g(u) = 0$ if $g \in R$. From this it follows that $\sum_{g \in R} c_g = 0$. Clearly

$$\hat{\phi}(u) = \sum_{g \in E(p)} a_g \phi(g) + \sum_{g \in E(q)} b_g \phi(g) +$$

$$+ \sum_{\substack{g \in R, \\ \phi(g) \in G(p)}} c_g \phi(g) + \sum_{\substack{g \in R, \\ \phi(g) \in G(q)}} c_g \phi(g) + \sum_{\substack{g \in R, \\ \phi(g) \in R'}} c_g \phi(g),$$

where $R' = G \setminus (G(p) \cup G(q))$. By the above, $\hat{\phi}(u)$ has order q, hence

$$\sum_{g \in E(p)} a_g + \sum_{\substack{g \in R, \\ \phi(g) \in G(p)}} c_g + \sum_{\substack{g \in R, \\ \phi(g) \in R'}} c_g$$

is the sum of all partial augmentations of $\hat{\phi}(u)$ with respect to elements of G whose order is not a positive power of q. Because $\varepsilon_x(u) = 0$ for each $x \in R$ and because ϕ maps conjugacy classes onto conjugacy classes, we see that $\sum_{g \in E(p)} a_g = 0$. Similarly we get

$$(*) \qquad \sum_{g \in E(q)} b_g = 1$$

using additionally the fact that the augmentation of $\hat{\phi}(u)$ is equal to 1.

Consider

$$u^q = \sum_{g \in E(p)} \alpha_g g + \sum_{g \in E(q)} \beta_g g + \sum_{g \in R} \gamma_g g.$$

Because u^q has order p, we see as before that $\sum_{g \in E(q)} \beta_g = 0$ and $\gamma_1 = 0$.

Denote as usual by $[\mathbb{Z}E, \mathbb{Z}E]$ the additive commutator of $\mathbb{Z}E$, i.e., the abelian subgroup generated by $\{xy - yx \mid x, y \in E\}$. Because q is a prime, an easy calculation shows that

$$u^q \equiv \sum_{g\in E(p)} a_g g^q + \sum_{g\in E(q)} b_g g^q + \sum_{g\in R} c_g g^q \mod [\mathbb{Z}E, \mathbb{Z}E] + q\mathbb{Z}E.$$

As g^q is a q-element if and only if g is a q-element, it follows that

$$\sum_{g\in E(q)} b_g \equiv \sum_{g\in E(q)} \beta_g \mod q.$$

Consequently,

$$\sum_{g\in E(q)} b_g \equiv 0 \mod q.$$

This contradiction to $(*)$ completes the proof. □

Corollary 2.3. *Let G be a group and N be a normal subgroup of G. Then if $\Pi(N)$ is a complete graph and $\Pi(V(\mathbb{Z}\,G/N)) = \Pi(G/N)$ then $\Pi(V(\mathbb{Z}G)) = \Pi(G)$.*

Proof. By Proposition 2.2, we have only to consider different primes dividing the order of N. By the assumption, these primes are connected in $\Pi(G)$ and therefore also in $\Pi(V(\mathbb{Z}G))$. □

Remarks. Instead of [34, Theorem 2.7] we could have used the stronger result [23, Theorem 2.3] which says that partial augmentations of a torsion unit of $\mathbb{Z}G$ are non-zero only for conjugacy classes of G whose representative has order dividing the order of u. It is clear that this and modifications of the arguments for Proposition 2.2 will lead to stronger results on the possible orders of torsion units in integral group rings. However in this article we restrict ourselves on (PQ).

The special case when N is a p-group in Corollary 2.3 gives a proof of [29, Proposition 4.3]. In the case of soluble groups even more is known.

In [24, Theorem] it is shown that even (SIP-C), i.e., that all finite cyclic subgroups of $V(\mathbb{Z}G)$ are isomorphic to one of G, holds provided G is soluble. Note that (SIP-C) has been posted as research problem 8 in [37] and the abbreviation (SIP-C) stands for the subgroup isomorphism problem for cyclic subgroups. A closer look shows that the proof given in [24] actually shows the following.

Proposition 2.4. *Let G be a finite group. Assume that N is a soluble normal subgroup of G and that for each torsion unit $u \in V(\mathbb{Z}G)$ there exists a $g \in G/N$ such that $o(u) = o(g)$ and $\varepsilon_g(u) \neq 0$ then the same holds for torsion units of $\mathbb{Z}G$. In particular (SIP-C) is valid for $\mathbb{Z}G$.*

Corollary 2.5. *Let G be a finite Frobenius group. Then (SIP-C) is valid for $\mathbb{Z}G$. More precisely for each torsion unit $u \in V(\mathbb{Z}G)$ there exists $g \in G$ such that u and g have the same order and $\varepsilon_g(u) \neq 0$.*

Proof. From the structure theorems on Frobenius groups it follows that a finite Frobenbius group is soluble or a soluble extension of S_5 or $\mathrm{SL}(2,5)$. In the first case the corollary follows from [24, Theorem], In the second case by [17] for $\mathrm{SL}(2,5)$ and [32] for S_5 the Zassenhaus conjecture (ZC-1) holds. This shows that the condition on the partial augmentations in Proposition 2.4 is fulfilled and the result follows. □

Proposition 2.2 permits also statements on some almost simple groups.

Corollary 2.6. *Let G be an almost simple group of type S. Assume that there is a prime $q \in \pi(G)$ such that $q \notin \pi(S)$. Then for each $p \in \pi(S)$ the primes p and q are connected in $\Gamma(\mathbb{Z}G)$ if and only if they are connected in $\Pi(G)$.*

Proof. By CFSG Schreier's conjecture is valid, i.e., Out(S) is soluble [16, p.(ii) and p.(xv)]. Thus G/S is soluble and therefore $\Gamma(\mathbb{Z}G/S) = \Pi(G/S)$. So we may apply Proposition 2.2 and we get immediately the result. □

Proof of Theorem 2.1. Clearly Theorem 2.1 holds when G itself is almost simple.

Thus we may apply induction on the length of a chief series. If G has a minimal normal subgroup N which is an abelian p-group, then we see by Proposition 2.2 that the Theorem is valid provided it holds for G/N.

Suppose that all minimal normal subgroups of G are perfect. Let $p, q \in \pi(G)$ be different primes and suppose that $V(\mathbb{Z}G)$ has an element of order $p \cdot q$. By Proposition 2.2 it follows that G has an element of order $p \cdot q$ provided $p \in \pi(G) \setminus \pi(N)$ or $q \in \pi(G) \setminus \pi(N)$ for at least one minimal normal subgroup N. So assume that p and q divide the order of each minimal normal subgroup of G.

If G has at least two different minimal normal subgroups N_1, N_2 then obviously G has elements of order $p \cdot q$ because $N_1 \times N_2$ is a subgroup of G. The same argument applies if G has a minimal normal perfect subgroup which is not simple because such a subgroup is a direct product of at least two copies of isomorphic simple groups.

Thus the only case which remains is that G has a unique minimal normal subgroup which is a non-abelian simple group S and p and q divide the order of S. But this case holds by assumption. □

Finally we give the following addendum to Theorem 2.1.

Proposition 2.7. *Let G be a finite group. Assume that G has a normal subgroup N such that G/N is the direct product of two almost simple groups. Then (PQ) holds for $\mathbb{Z}G$.*

Proof. For each normal subgroup $M \subset N$ the quotient G/M satisfies the hypothesis of the proposition. Consider now a minimal counterexample and induct on a chief series of G through N. If G/M maps onto G/N then G/M cannot be almost simple because G/N has two non-abelian composition factors. But if G/M is almost simple it has only one non-abelian composition factor using again that by CFSG Schreier's conjecture holds. Thus a minimal counterexample is the direct product $G_1 \times G_2$ of two almost simple groups of type S_1, S_2 rsp. As in the proof of Theorem 2.1 we see by Proposition 2.2 that primes p and q providing a counterexample to (PQ) both divide $|S_1|$ and $|S_2|$. But then these primes are connected in $\Pi(S_1 \times S_2)$ and therefore in $\Pi(G)$. □

A typical example for Proposition 2.7 is a wreath product of type $H \wr Q$, where Q is the direct product of A_5 and $\mathrm{PSL}(2, 7)$. Theorem 2.1 of course also shows that for finite groups which have no almost simple images, e.g. $\mathrm{A}_5 \wr Q$ where Q is an arbitrary soluble group, (PQ) has a positive answer.

3. Groups of orders divisible by three primes only and further applications

The goal of this section is the following result.

Theorem 3.1. *Let G be a finite group such that the order of every almost simple image of G is divisible by exactly three different primes. Then $\Gamma(\mathbb{Z}G) = \Pi(G)$.*

Proof. By Theorem 2.1 it suffices to prove this for all almost simple groups whose order is divisible by exactly three different primes.

By CFSG, see [18, p.72] and [26], the simple groups of order divisible by exactly three different primes are

$$\mathrm{PSL}(2,5) \cong \mathrm{A}_5, \mathrm{PSL}(2,7), \mathrm{PSL}(2,8), \mathrm{PSL}(2,9) \cong \mathrm{A}_6$$
$$\mathrm{PSL}(2,17), \mathrm{PSL}(3,3), \mathrm{PSP}(3,4) \cong \mathrm{U}(4,2), \mathrm{U}(3,3).$$

Their outer automorphism group has order 2, except the case that $OutA_6 \cong C_2 \times C_2$.

In the table below we summarise details about these groups. We also include in the table the isomorphism type of Out(S). In the third column we indicate with (ZC-1) when the 1st Zassenhaus conjecture is established, with (SIP-C) when the finite cyclic subgroups of $V(\mathbb{Z}S)$ are isomorphic to subgroups of S and with (PQ) that the prime graph question has an affirmative answer. The fourth column contains the references.

	Out(S)		
$\mathrm{A}_5 \cong \mathrm{PSL}(2,5)$	C_2	(ZC-1)	[31]
$\mathrm{PSL}(2,7)$	C_2	(ZC-1)	[22]
$\mathrm{PSL}(2,8)$	C_3	(ZC-1)	[20, 30]
$\mathrm{A}_6 \cong \mathrm{PSL}(2,9)$	$C_2 \times C_2$	(ZC-1)	[25]
$\mathrm{PSL}(2,17)$	C_2	(ZC-1)	[20]
$\mathrm{PSL}(3,3)$	C_2	(PQ)	[3]
$\mathrm{PSP}(3,4) \cong \mathrm{U}(4,2)$	C_2	(PQ)	[30]
$\mathrm{U}(3,3)$	C_2	(SIP-C)	see below

For the full automorphism groups of these simple groups the results are as follows.

$\mathrm{Aut}(\mathrm{A}_5) \cong S_5$	(ZC-1)	[32]
$\mathrm{Aut}(\mathrm{PSL}(2,7)) \cong \mathrm{PGL}(2,7)$	(ZC-1)	[30]
$\mathrm{Aut}(\mathrm{PSL}(2,8)) \cong \mathrm{P\Gamma L}(2,8)$	(SIP-C)	[30]
$\mathrm{Aut}(\mathrm{A}_6) \cong \mathrm{P\Gamma L}(2,9)$	(SIP-C)	see below
$\mathrm{Aut}(\mathrm{PSL}(2,17)) \cong \mathrm{PGL}(2,17)$	(ZC-1)	[30]
$\mathrm{Aut}(\mathrm{PSL}(3,3))$	(PQ)	[30]
$\mathrm{Aut}(\mathrm{U}(4,2))$	(PQ)	[30]
$\mathrm{Aut}(\mathrm{U}(3,3))$	(SIP-C)	see below

For the three subgroups of index two of $\mathrm{Aut}(\mathrm{A}_6)$ the results are as follows.

S_6	(SIP-C)	[30]
$\mathrm{PGL}(2,9)$	(SIP-C)	[4]
M_{10}	(SIP-C)	[4]

The detailed report on calculations required to complete both tables above was given in [30]; therefore, we will omit them here and will only summarise known information in the table below which presents results about possible orders and partial augmentations of normalised torsion units in integral group rings for the groups listed above.

- Column 2 contains the name of the character table in the GAP character table library provided by the GAP package CTblLib [12].
- Column 3 lists orders (of normalised torsion units) for which the rational conjugacy is known, either as an immediate consequence of [22, Proposition 3.1] or using the HeLP method.
- Column 4 lists orders of elements of G with remaining non-trivial tuples of partial augmentations. In each entry of the form M(T+N), M means the order, and T and N mean the number of respectively trivial and non-trivial admissible tuples of partial augmentations that are produced by the HeLP method.

- Column 5 lists orders of elements of G which were omitted as not relevant to (PQ) (for some groups it was possible to cover all or most of orders, though). A dash (—) means that no orders were omitted.
- Column 6 lists orders dividing $\exp(G)$ that do appear neither in G nor in $V(\mathbb{Z}G)$, eliminated using the HeLP method (except units of order 6 in A_6 which are eliminated in [25], and of order 6 in $\mathrm{PGL}(2,9)$ and M_{10} eliminated in [4]).
- Column 7 lists some orders with remaining non-trivial tuples of partial augmentations that still have to be eliminated for positive answers on (SIP-C) for some groups.
- Column 8 lists further divisors of $\exp(G)$ needed for the complete account on torsion units of $V(\mathbb{Z}G)$ but omitted as not relevant to (PQ).

G	Character table name in CTblLib	(ZC-1)	order(#) in G	Not considered orders in G	No orders in $V(\mathbb{Z}G)$	order(#) in $V(\mathbb{Z}G)$	Not considered in $V(\mathbb{Z}G)$
1	2	3	4	5	6	7	8
PSL(2,5)	`A5`	2, 3, 5	—	—	6, 10, 15	—	—
PSL(2,7)	`PSL(2,7)`	2, 3, 4, 7	—	—	6, 14, 21	—	—
PSL(2,8)	`PSL(2,8)`	2, 3, 7, 9	—	—	6, 14, 21	—	—
A_6	`A6`	2, 3, 4, 5	—	—	6, 10, 15	—	—
PSL(2,17)	`PSL(2,17)`	2, 3, 4, 8, 9, 17	—	—	6, 34, 51	—	—
PSL(3,3)	`PSL(3,3)`	2, 4, 13	$3(2+3), 6(1+43)$	4, 8	26, 39	12(8)	—
U(4,2)	`U4(2)`	5	$2(2+1), 3(4+3), 4(2+11)$	6, 9, 12	10, 15	—	18
U(3,3)	`U3(3)`	2, 7	$3(2+1)$	4, 6, 8, 12	14, 21, 24	—	—
S_5	`S5`	2, 3, 4, 5, 6	—	—	10, 12, 15	—	—
PGL(2,7)	`L3(2).2`	2, 3, 4, 6, 7, 8	—	—	12, 14, 21	—	—
PΓL(2,8)	`PSL(2,8).3`	2, 3, 7, 9	$6(2+20)$	—	14, 18, 21	—	—
PΓL(2,9)	`A6.2^2`	2, 3, 5, 10	$4(3+2), 6(1+3), 8(2+1)$	—	12, 15, 20	—	—
PΓL(2,17)	`L2(17).2`	2, 3, 4, 6, 8, 9, 16, 17, 18	—	—	12, 34, 51	—	—
Aut(PSL(3,3))	`L3(3).2`	2, 4, 8, 13	$3(2+3)$	6, 12	26, 39	—	24
Aut(U(4,2))	`U4(2).2`	5, 10	$2(4+6), 3(3+3), 9(1+1)$	4, 6, 8, 12	15	—	18, 20, 24
Aut(U(3,3))	`U3(3).2`	2, 4, 7	$3(2+1), 6(2+36), 8(2+1), 12(3+9)$	—	14, 21, 24	—	—
S_6	`S6`	3, 5	$2(3+1), 4(2+4), 6(2+14)$	—	10, 12, 15	—	—
PGL(2,9)	`A6.2_2`	2, 3, 4, 5, 8, 10	—	—	6, 15, 20	—	—
M_{10}	`M10`	2, 3, 4, 5	$8(4)$	—	6, 10, 15	—	—

Let $G = \mathrm{Aut}(\mathrm{U}(3,3)) = \mathrm{U}(3,3).2$. To prove (SIP-C) for this group, we need to show that there are no units of order 14, 21 and 24 in $V(\mathbb{Z}G)$. Since $\mathrm{U}(3,3) \subseteq \mathrm{U}(3,3).2$, this will also prove (SIP-C) for $\mathrm{U}(3,3)$.

For torsion units of orders 14 and 21 we will use the method of (p,q)-constant characters, introduced in [10]. Below we provide the table containing the data for the constraints on partial augmentations, from which the proof can be derived. This table uses the notation from [11]; in particular, ν_k denotes the sum of partial augmentations of u with respect to conjugacy classes of elements of order k in G, $\xi(C_k)$ denotes the value of the character ξ on elements of order k, and the notation $(\chi_i)_{[p]}$ indicates that the p-Brauer character is used. Using this table, one can write constraints on ν_p and ν_q of the form

$$\mu(u, \zeta^l, \xi) = \tfrac{1}{pq}\left(m_1 + \nu_p m_p + \nu_q m_q\right) \geq 0, \tag{1}$$

where ζ is the pq^{th} primitive root of unity, and show that for $o(u) \in \{14, 21\}$ these systems of constraints have no solutions.

$o(u)$	p	q	ξ	$\xi(C_p)$	$\xi(C_q)$	l	m_1	m_p	m_q
14	2	7	$\xi = (\chi_1 + \chi_6 + \chi_8)_{[3]}$	3	0	0	38	18	0
						1	32	3	0
						7	32	-18	0
21	3	7	$\xi = (\chi_3 + \chi_{10})_{[*]}$	0	-1	0	21	0	-12
						1	28	0	-1
						7	21	0	6

For torsion units of order 24 we need to consider 24624 cases determined by possible partial augmentations for torsion units of order 2, 3, 4, 6, 8 and 12 (each having 2, 3, 3, 38, 3 and 12 cases respectively). Using computer, in each of these cases we obtained a contradiction by finding a pair of i, j such that the multiplicity of ζ^i as eigenvalue of u in the representation affording χ_j is not an integer.

For the proof of Theorem 3.1 it was a big advantage that there are only a finite number of finite simple groups whose order is divisible by exactly three primes. If three is replaced by four then this is no longer the case as mentioned in the introduction. In the meantime there is also an infinite series of almost simple groups known for which (PQ) holds. Thus the following is a consequence from Theorem 2.2.

Corollary 3.2. *Let G be a finite group. Assume that all composition factors of G are of prime order or isomorphic to $\mathrm{PSL}(2, p^f)$ with $f \leq 2$ and a prime $p \geq 5$. Then PQ holds for $\mathbb{Z}G$.*

Proof. By [23, Propositions 6.3 and 6.7] (PQ) folds for all simple groups $\mathrm{PSL}(2,p)$. By [33, Satz 2.4.2] (PQ) holds for all groups $PGL(2,p) = \mathrm{Aut}(\mathrm{PSL}(2,\mathrm{p}))$. The case for almost simple groups with socle $\mathrm{PSL}(2,p^2)$ is established in [6, Theorem A]. Thus the result follows from Theorem 2.2. □

There are further simple groups S known such that (PQ) is valid for $\mathbb{Z}G$ for all almost simple groups G of type S, e.g. when S is isomorphic to one of the five simple Mathieu groups or isomorphic to A_n, $n \leq 17$ [2, Theorem 5.1]. This has been checked with HeLP package as well as (for A_7 and A_8) with the unpublished implementation of the HeLP method by the 2nd author and V. Bovdi.

4. Reproducibility of results

Since part of the results described in Section 3 are obtained with the help of computer calculations, we are adding a special section to describe their reproducibility. Following ACM policy [1], we distinguish the ability to *replicate* the experiment by obtaining same results with the same software on a different computer from the ability to *reproduce* the experiment by obtaining same results using another software.

In this terminology, we believe that our results are reproducible. All calculations reported in the table in Section 3 were performed twice. One calculation used the unpublished implementation of the Hertweck-Luthar-Passi method developed by the 2nd author and V. Bovdi and used, for example, in [9, 11]. Another calculation used independent implementation of the HeLP method by A. Bächle and L. Margolis provided in the HeLP package [5], which is now a part of the distribution for the computational algebra system GAP. Both of these programs are written in GAP, but developed independently, and use different solvers: the former uses Minion [19], ECLiPSe [36] and two custom solvers written in GAP, while the latter uses Normaliz [15] by means of the GAP package NormalizInterface [21].

We compared both the final outcomes of calculations and their numerical characteristics such as e.g. orders that have to be analysed to reach certain conclusions, lists of admissible partial augmentations, etc., and detected no discrepancies; hence we are convinced in the reproducibility of our results.

The check was tedious - sometimes it was not enough just to compare lists of admissible partial augmentations straightforwardly. For example, for $\mathrm{PSL}(3,3)$ it happened that HeLP package eliminates the following cases of partial augmentations of elements of order 12:

$$\{(-1,-1,-2,1,4),(-1,0,-3,1,4),(-1,0,0,1,1),(1,0,0,1,-1),$$

$$(1,0,3,1,-4),(1,1,-1,1,-1),(1,1,2,1,-4)\},$$

which are not vanishing in calculations performed with the other implementation: all of them have non-zero partial augmentations for the class of elements of order 2, while HeLP returns only solutions where the partial augmentations for this class are zero. It happened that HeLP implements an additional constraint (so called "Wagner test", see [39]) unlike the other code. Furthermore, as well as HeLP authors, we tried to use it to check earlier results from e.g. [10], and also they agree.

Of course, one could go further and observe that both programmes derive their data from the same source which is the GAP character table library [12]. Here [14] discusses the reliability and reproducibility information contained in the Atlas of Finite Groups and reports its verification done using Magma, and [13] demonstrates how to compute in GAP the ordinary character tables of some Atlas groups, using character theoretic methods. This gives additional credibility and confidence in the correctness of our results.

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