ENTITLEMENT IN MATHEMATICS

Nikolaj Jang Pedersen

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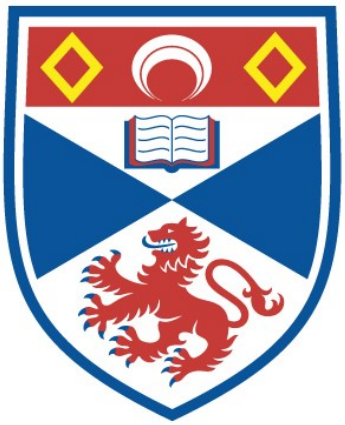

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Entitlement in mathematics

Submitted to the University of St. Andrews in fulfillment of the requirement for the Ph.D. in Philosophy

Nikolaj Jang Pedersen

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Contents

Abstract

This first half of this thesis investigates the epistemological foundations of mathematical theories, with special attention devoted to two questions: (1) how can we have a warrant for the satisfiability and consistency of mathematical theories, and (2) given we conceive of mathematical judgement as objective – as being concerned with a realm of abstract entities – can we have a warrant for thinking that such a realm of entities exists? In Chapter 2, two kinds of mathematical scepticism are developed. The regress sceptic argues that we can have a warrant for accepting neither the satisfiability nor the consistency of a mathematical theory. The I-II-III sceptic maintains that there can be no warrant for thinking that a realm of abstract entities exists if mathematical judgement is conceived as being objective. The notions of entitlement of cognitive project and entitlement of substance – recently introduced into the literature by Crispin Wright – are invoked to respond to the mathematical regress and I-II-III sceptic. This is done in Chapters 3 and 4. The distinctive feature of an entitlement is its non-evidential nature. What is relevant is not the presence of positive evidence, but rather the absence of sufficient countervailing evidence. The second half of the thesis explores and develops certain aspects of this proposal. Chapter 5 develops the notion of entitlement of cognitive project by investigating two of its three defining clauses. Chapter 6 draws a picture of a wider philosophical framework of which entitlement can be regarded an integrated part. In so doing entitlement is discussed in light of the internalism/externalism distinction and the distinction between monism and pluralism about epistemic value. Chapter 7 tables two fundamental challenges to the entitlement proposal – firstly, whether entitlement is an epistemic notion of warrant at all, and secondly, whether the notion of rationality associated with it is epistemic in nature or of some other kind?

Introduction

The topic of this thesis is the epistemological foundations of mathematical theories, with special attention devoted to two questions, a certain strategy for addressing them, and issues relating to this strategy. The questions are these:

(QUESTION 1) How can we have a warrant for the satisfiability and consistency of mathematical theories?

(QUESTION 2) Given we conceive of mathematical judgement as objective – as being concerned with a realm of abstract entities – can we have a warrant for thinking that such a realm of entities exists?

and the strategy is this:

(ENTITLEMENT) The satisfiability and consistency of mathematical theories can be warranted non-evidentially, as can the existence of a realm of abstract mathematical entities given a certain conception of mathematical judgement.

where the target notion of non-evidential warrant is referred to as 'entitlement'. The issues to be addressed in relation to the entitlement strategy are outlined below.

The thesis is divided into four parts. The first part consists of Chapters 1 and 2 and is devoted to preliminaries. In Chapter 1, we give the axioms of arithmetic, standard set theory,

simple type theory, and New Foundations. Chapter 2 contains a development of two kinds of mathematical scepticism. According to the mathematical regress sceptic, (QUESTION 1) reflects an unfounded optimism. The question should not be *how* we can have a warrant for satisfiability and consistency propositions, but rather if we can have such a warrant *at all* – the sceptical answer being that we cannot. The sceptical master thought is that, whenever a subject is warranted in believing a proposition, the subject's warrant consists, in part, in a warrant for some other proposition. That is, warrant always involves additional support or evidence. The master thought is brought to work in two regress arguments, one targeted at satisfiability propositions, the other at consistency propositions. The mathematical regress sceptic concludes that there can be no warrant for satisfiability and consistency propositions. Having rehearsed the two regress arguments, we go on to spell out an argument aimed at a broadly realist conception of the realm of mathematics, and, in particular, the tenability of an affirmative answer to (QUESTION 2). We do so on the model of what is referred to as 'I-II-III arguments' in the literature. Throughout the chapter similarities and differences between mathematical scepticism and (certain kinds of) scepticism concerning the empirical world are highlighted.

The second part of the thesis consists of Chapters 3 and 4 and gives a detailed account of how entitlement can be employed to respond to the sceptical arguments presented in the first part of the thesis. Chapter 3 introduces Crispin Wright's notion of entitlement of cognitive project, according to which one can be defeasibly warranted in accepting a proposition provided that there is no sufficient reason to believe it untrue. It is suggested that the notion delivers an effective response to mathematical regress scepticism. Chapter 4 introduces Wright's notion of entitlement of substance, according to which one can be non-evidentially warranted – i.e. entitled – to the basic ontology of a certain conception of a tract of reality provided there is no sufficient reason to believe it incoherent. The notion is used to deliver a response to the 'substance sceptic' who maintains that (QUESTION 2) cannot be given an affirmative answer.

Part three of the thesis consists of Chapters 5 and 6 and raises a number of issues pertaining to the further development and understanding of entitlement. In Chapter 5, we table two awkward wrinkles left by what Wright says about entitlement of cognitive project. The first awkward wrinkle is that it is not clear that entitlements of cognitive project are really defeasible,

though this is supposed to be one of their characteristic features. The second awkward wrinkle is that the status of P as an entitlement appears to be incompatible with the view that genuine progress can be made with respect to P, i.e. that it is possible to improve the epistemic standing of P. I distinguish between three ways of understanding entitlement of cognitive project and show how the proposed distinction enables us to remove the two awkward wrinkles. Chapter 6 discusses what to make of entitlement in light of a central distinction in contemporary epistemology – that between internalist and externalist conceptions of warrant. It is argued that warrant for satisfiability and consistency propositions should not be construed along externalist lines if it is to deliver a potent response to mathematical regress scepticism, conceived as an attack on rational claims to warrant (rather than possession of warrant). By contrast, or exclusion, we take this to recommend internalism about satisfiability and consistency warrants. In arguing this point, a picture is drawn of a wider philosophical framework of which the entitlement proposal can be regarded as an integrated part.

Part four of the thesis consists of Chapter 7. In Chapter 7, I raise two fundamental challenges to the entitlement proposal, *viz.* whether entitlement is an *epistemic* notion of warrant at all, and related thereto, whether the notion of rationality associated with entitlement is epistemic. The chapter develops a response to these worries centered around an idea – merely gestured at by Wright – that entitlement of cognitive project is an epistemic kind of warrant because, when P is an entitlement, trust in P is a dominant strategy with respect to promotion of epistemic value. We formulate a criticism of the proposal by arguing, firstly, that the prospects of success for this line of approach are bound to depend on the underlying theory of epistemic value, and secondly, that although any reasonable theory of epistemic value should count error-avoidance among its epistemic values, doing so in the context of the dominance argument threatens to undermine it. We formulate an amended version of the dominance argument, the basic idea being that trusting entitlements maximizes expected utility. The chapter ends on a critical note by pointing out that the expected utility approach rests on a controversial assumption to the effect that the probability of P's being the case (P an entitlement) exceeds the probability of $\neg P$'s being the case, but that this is an assumption that cannot be granted in a discussion with the sceptic. The aim of this last chapter is not to try to undermine the entitlement proposal,

but rather to point to some very fundamental issues that need to be addressed in future work.

In sum, the aims of this thesis are:

1. to develop, in considerable detail, mathematical regress scepticism and mathematical I-II-III scepticism;

2. to extend the application of the notions of entitlement of cognitive project and entitlement of substance by invoking them to respond to respectively mathematical regress scepticism and I-II-III scepticm;

3. to further the development of the notion of entitlement of cognitive project and to draw a picture of a wider philosophical framework of which the entitlement proposal can be regarded as an integrated part;

4. to present two fundamental challenges to the entitlement proposal and offer a discussion of them which, while suggesting that there is considerable work to do on behalf of the entitlement proposal, also provides a basic framework within which further discussion may fruitfully be pursued.

Chapter 1

Technical preliminaries

Before turning to mathematical scepticism, let us rehearse some technical preliminaries. The axioms of standard arithmetic (PA) and set theory (ZFC) will be stated, as will the axioms of simple type theory (STT) and Quine's New Foundations (NF).

1.1 Peano arithmetic

The language of first-order logic is well-known to anyone who has done a course in logic. It contains the sentential connectives ($\vee$, $\neg$, $\wedge$, $\rightarrow$, and $\leftrightarrow$), quantifiers ($\forall$ and $\exists$), the identity relation ($=$), and first-order variables ($x_1, x_2, \ldots$).

The language of first-order Peano arithmetic – or 'PA^1', for short – is the language of first-order logic together with the *non-logical* arithmetical vocabulary $\mathcal{L}_{PA} = \{\mathbf{0}, s, +, \times\}$. On their intended interpretation these items are respectively taken to be 0, i.e. zero, the successor function, addition, and multiplication. First-order Peano arithmetic, PA^1, is the conjunction of axioms (1)–(6) below supplemented by the induction schema (IND^1). An informal gloss is stated after each formal statement of an axiom (or axiom schema):

(1) $(\forall x)(\mathbf{0} \neq s(x))$, i.e. 0 is not the successor of any natural number.

(2) $(\forall x)(\forall y)(s(x) = s(y) \rightarrow x = y)$, i.e. any two natural numbers with the same successor are identical.

(3) $(\forall x)(x + \mathbf{0} = x)$, i.e. the result of adding any natural number to zero is the natural number itself.

(4) $(\forall x)(\forall y)(x + s(y) = s(x + y))$, i.e. for any two natural numbers x and y, the sum of x and the successor of y is identical to the successor of the sum of x and y.

(5) $(\forall x)(x \times \mathbf{0} = \mathbf{0})$, i.e. the result of multiplying any natural number by zero is zero.

(6) $(\forall x)(\forall y)(x \times s(y) = x \times y + x)$, i.e. for any two natural numbers x and y, x multiplied by the successor of y is identical to the result of multiplying x and y and adding x.

(IND^1) For any wff Φ, $(\Phi(\mathbf{0}) \wedge (\forall x)(\Phi(x) \rightarrow \Phi(s(x)))) \rightarrow (\forall x)\Phi(x)$, i.e. for any wff Φ, if Φ holds of zero and Φ holds of the successor of any natural number provided Φ holds of the natural number itself, then Φ holds of all natural numbers.

Second-order Peano arithmetic, PA^2, is the conjunction of (1)–(6) above and the axiom of induction obtained from (IND^1) by removing all occurrences of Φ and replacing them with X and prefixing the whole formula with a second-order universal quantifier:

(IND^2) $(\forall X)((X\mathbf{0} \wedge (\forall x)(Xx \rightarrow Xs(x))) \rightarrow (\forall x)Xx)$, i.e. for any property X, if X holds of zero and X holds of the successor of any natural number provided X holds of the natural number itself, then X holds of all natural numbers.

Below we will often omit superscripts and simply speak of 'Peano arithmetic'. It will be flagged

if it makes a difference whether induction is formulated using first- or second-order resources.

A few remarks. First, the question whether an item is logical or non-logical turns on what meaning it is assigned under interpretations of the language. An item is logical just in case its meaning is fixed across interpretations of the language; otherwise it is non-logical. For instance, '$\wedge$' and '$\forall$' pick out respectively conjunction and the universal quantifier under every interpretation of the language, while the interpretation of the non-logical items may vary from interpretation to interpretation.[1]

Second, the set of items in the language can, of course, be reduced without loss of expressive power. Any set of connectives containing negation and disjunction, conjunction, or implication can express the other connectives; and the quantifiers are interdefinable by use of negation: $(\forall x)\phi =_{df.} \neg(\exists x)\neg\phi$, and $(\exists x)\phi =_{df.} \neg(\forall x)\neg\phi$. In the following, we shall forego lexical purity for notational convenience and make use of the full range of items listed above – knowing, as we do, that we can increase lexical purity if we want to.

1.2 Standard set theory: ZFC

Sets are collections of objects that share a given property. For instance, the set of even numbers is the collection of numbers that have the property of being divisible by two with zero remainder, and the set of primes is the collection of numbers that have the property of having only themselves and one as divisors.[2] We use the language of set theory to talk about such collections. The language of first-order set theory is the language of first-order logic supplemented by a non-logical relation – MEMBERSHIP (formalized '$\in$') – which holds between a set and the objects in it. These objects are the members of the set. To use the preceding examples, the number four

[1] However, for second-order arithmetic, there is a sense in which the interpretation of the non-logical items will remain fixed as the second-order axioms pin down a unique structure 'up to isomorphism'. See Appendix A for a statement of some of the basic results of the model theory of PA (and ZFC).

[2] However, as anyone familiar with the set-theoretic paradoxes is aware, not every property can determine a set.

The notion of a property was used to introduce the notion of a set. There is, of course, an extensive literature on the nature of properties, and indeed, an extensive literature on the question whether properties should be included in our ontology at all. I am aware of these issues, but it goes beyond the scope of this thesis to take a well-supported stance on either of them. Here I shall simply assume that there is *a* defensible notion of property. That said, disagreement over the nature of properties is compatible with agreement over what properties – whatever their nature might be – determine sets.

is a member of the set of even numbers, but not a member of the set of primes. It satisfies the property that determines the former set, but fails to satisfy the property that determines the latter: four divided by two is exactly two, but four has one, two, and four as divisors.

First-order Zermelo-Fraenkel set theory with choice ('ZFC[1]', in short) is often referred to as 'standard set theory'. The axioms of the theory are as follows, with a natural language gloss following the formal statement:

Axiom of extensionality:

$(\forall x)(\forall y)((\forall z)(z \in x \leftrightarrow z \in y) \leftrightarrow x = y)$, i.e. any sets x and y are identical just in case they have the exact same members.

Empty set axiom:

$(\exists x)(\forall y) y \notin x$, i.e. there is a set without any members. (By extensionality, the set satisfying the empty set axiom is unique. Here we shall follow standard practice and introduce the constant '$\emptyset$' to denote the empty set.)

Axiom of pairing:

$(\forall x)(\forall y)(\exists z)(\forall w)(w \in z \leftrightarrow w = x \vee w = y)$, i.e. for any sets x and y there is a set z containing exactly x and y as members.

Axiom of union:

$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\exists w)(w \in x \wedge z \in w))$, i.e. for any set x there is a set y containing exactly the members of members of x. (The union of a set x is written '$\cup x$'. By extensionality the union of a set is unique.)

Axiom schema of separation:

$(\forall x_1) \ldots (\forall x_n)(\forall y)(\exists z)(\forall w)(w \in z \leftrightarrow w \in y \wedge \phi(w))$, where ϕ does not contain z free and $x_1 \ldots x_n$ are the free variables of ϕ; i.e. for any set y there is a set z containing exactly the members of y which satisfy the formula ϕ.

Power set axiom:

$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\forall w)(w \in z \rightarrow w \in x))$, i.e. for any set x there is a set y containing

exactly the subsets of x. (z is a subset of x just in case every member of z is a member of x, i.e. just in case $(\forall w)(w \in z \rightarrow w \in x)$, notated as '$z \subseteq x$'.)

Axiom of infinity:

$(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \rightarrow y^+ \in x))$, i.e. there is a set which contains $\emptyset$ and is closed under successor, where $y^+ = y \cup \{y\}$ (i.e. $\cup\{y, \{y\}\}$).

Axiom of foundation:

$(\forall x)(x \neq \emptyset \rightarrow (\exists y)(y \in x \wedge y \cap x = \emptyset))$, i.e. any non-empty set x has a member y with which it has no member in common, i.e. their intersection is empty. (The intersection of x and y is defined as follows: $(\forall z)(z \in x \cap y \leftrightarrow z \in x \wedge z \in y)$.)

Axiom schema of replacement:

$(\forall x_1) \ldots (\forall x_n)(\forall y)((\forall z)(\forall w)(\forall v)(z \in y \wedge \varphi(z, w) \wedge \varphi(z, v) \rightarrow w = v) \rightarrow (\exists u)(\forall t)(t \in u \leftrightarrow (\exists s)(s \in y \wedge \varphi(s, t))))$, i.e. for any functional formula φ, if the domain constitutes a set, then so does the range.

Axiom of choice:

$(\forall x)(((\forall y)(y \in x \rightarrow (\exists z)(z \in y)) \wedge (\forall w)(\forall v)((w \in x \wedge v \in x \wedge w \neq v) \rightarrow w \cap v = \emptyset)) \rightarrow (\exists u)(\forall y)(y \in x \rightarrow (\exists z)(z \in y \wedge u \cap y = \{z\})))$, i.e. for every set x the members of which are non-empty and pairwise disjoint (i.e. have no member in common), there is a set containing exactly one member from each member of x.

First-order Zermelo-Fraenkel set theory, ZF^1, is ZFC^1 minus choice. First-order Zermelo set theory, Z^1, is ZF^1 minus replacement.[3]

[3] The axiomatic set theory presented by Zermelo in his 1908 paper 'Investigations in the foundations of set theory I' ([159]) is, with a number of modifications, Z. First, Zermelo included an axiom of elementary sets according to which: (i) there is a set that has no members, (ii) for any object x of the domain there is a set with x as its sole member (i.e. the singleton of x), and (iii) for any objects x and y of the domain, there is a set containing exactly x and y as members. (i) and (iii) appear as separate axioms in Z – respectively empty set and pairing – while (ii) follows from pairing and extensionality. By pairing any x and y have a pair set. The singleton axiom follows by applying the axiom of extensionality in the special case where $x = y$. The singleton axiom also follows from the combination of power set and separation. Consider x and apply the power set to it. x is a member of $\mathcal{P}(x)$. Apply separation by letting letting ϕ be $y = x$ (i.e. the property of being identical to x).

Second, the axiom of foundation was not explicitly formulated until later. See Miramanoff [97] and von Neumann [100], published in respectively 1917 and 1925. Zermelo added the axiom to his axiom system and it thus appears in the list of axioms in his 1930 paper, Zermelo [160].

Third, Zermelo included the axiom of choice in the list of axioms.

The axiom (schema) of replacement is needed to develop the full theory of ordinals in its standard set-theoretic formulation, due to von Neumann. The ordinal number 0 is taken to be $\emptyset$, 1 is taken to be $\{0\} = \{\emptyset\}$, 2 is identified with $\{0,1\} = \{\emptyset, \{\emptyset\}\}$, 3 with $\{0,1,2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ and so forth to $\omega = \{0,1,2,\ldots\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots\}$. For any ordinal α, its successor $\alpha + 1$ is simply $\alpha \cup \{\alpha\}$, and a limit ordinal λ is taken to be the union of its predecessors, i.e. $\cup\{\beta : \beta < \lambda\}$. In general, a von Neumann ordinal is identical to the set of its predecessors.[4] The axiom of replacement is needed to ensure the existence of limit ordinals greater than ω. However, the axiom is not relied on often outside set theory (which I take to include the theory of von Neumann ordinals).[5]

The replacement schema implies the separation schema. Here is the proof: consider any $\phi(s)$. Let $\psi(t,s) =_{df.} t = s \wedge \phi(s)$. $\psi(t,s)$ is functional on any set y, i.e. for any any set y, it is the case that for every $z \in y$, there is at most one w such that $\psi(z,w)$. So, consider a set y and substitute $\psi(t,s)$ for $\varphi(s,t)$ in the replacement schema. Since $\psi(t,s)$ is functional on y – i.e. satisfies the antecedent – there is a set u such that $t \in u$ just in case there is some $s \in y$ such that $\psi(s,t)$. Given the definition of $\psi(t,s)$, that is to say that there is a set that contains exactly those members of y which satisfy $\phi(s)$.

In cases where x is finite the axiom of choice is dispensable in light of the other axioms. Here is a sketch of a proof: suppose that x has n members that are all non-empty (for $n \in \omega$). Consider any $y \in x$. Since $y \neq \emptyset$, there is some $z \in y$. Let $\phi(w) =_{df.} w = z$ and use this to apply separation to y. This yields the existence of $\{z\}$. Applications of the pairing axiom and the axiom of union deliver the existence of the choice set for x. The idea will not work for infinite sets, because we cannot write down infinitely many instances of separation.

However, the axiom of choice *is* dispensable in some infinite cases. For instance, one can

[4] It might not be entirely obvious that a limit ordinal is identical to the set of its predecessors. Proof of this fact can be found in standard textbooks. Cf., e.g., Enderton [40], Chapter 7.

[5] Concerning the existence of limit ordinals, there is a model of Z which does not contain the second limit ordinal $\omega + \omega$. However, any model of ZF contains $\omega + \omega$ and any other ordinals 'accessible' via the operations licensed by the ZF axioms. See Appendix A.

Concerning the reliance on replacement, there is a result due to Martin to the effect that every Borel game is determined which involves non-eliminable use of replacement. Nevertheless, while replacement often provides more elegant proofs, in most cases outside set theory alternative, slightly more tedious versions of the proof could be given without reliance on the axiom.

For a brief overview of the history of the axiom of replacement, see Hallett [67], Section 8.2.

define a function $f : (P(\omega) - \{\emptyset\}) \rightarrow \omega$ which shows that there is a choice set for any set of arbitrary non-empty pairwise disjoint subsets of the natural numbers, whether finite or infinite. Just let f be such that it picks out the least member of any non-empty $y \subseteq \omega$. On the other hand, there is no way of *specifying* a function that will pick numbers from arbitrary non-empty sets of reals. In such cases the axiom of choice ensures that there is indeed a choice function which does so although we cannot specify it.

On a historical note, Zermelo's use of the axiom of choice in his 1904 proof of the well-ordering theorem caused quite a stir in the mathematical community at the time. The formulation of choice employed in the proof is the one stated here. (See Zermelo [157]. Zermelo uses 'distinguished element' to refer to the element picked out by the choice function and 'covering' for the choice set.) §1 of Zermelo [158] – published in 1908 – contains a new proof of the well-ordering theorem, which invokes the same version of choice, but rests on weaker assumptions about well-ordering and ordinals than those made in the earlier proof. §2 contains Zermelo's response to criticisms of his use of choice that had come from various corners since his first proof.

1.3 Second-order resources

PA^1 includes induction as an axiom schema. ZFC^1 includes two axiom schemas, the axiom schemas of separation and replacement. For every appropriate formula ϕ each of the schemas delivers an individual axiom. Second-order PA and ZFC include axioms proper rather than schemas. Adherents of second-order theories often cite their greater expressive power as one of the main attractions.

Consider separation. Often separation is stated informally as follows: for any given property and any set x, there is a set y that has as members exactly the members of x that have the property in question. On a face-value reading, this formulation of separation involves quantification over properties. In formal languages, properties are represented by predicate letters. In first-order theories we cannot, however, quantify into predicate position, and so, there is no way of capturing the face-value reading of separation. Instead first-order set theory uses a schema.

Can the schema of separation fully capture the idea expressed by the informal formulation of separation? Arguably not. Consider ω. Every set determines a property,[6] and since there are uncountably many subsets of ω, there are uncountably many properties true of arbitrary sets of natural numbers. $P(\omega)$ has as members all subsets of ω. By the informal statement of separation, since $P(\omega)$ is a set, for any property true of some natural numbers, there should be a set containing exactly those numbers as members. By our reasoning, there should be uncountably many of these. Can the separation schema ensure this? It would seem not. Because the separation schema will only deliver the subsets of ω that are definable by first-order means. Since the language of first-order set theory is denumerable, only denumerably many definitions can be formulated in the language, and therefore, there are only denumerably many first-order definable subsets of ω. Hence, not all properties can be captured by the separation schema, and so, the schema cannot capture the idea expressed by the informal formulation of separation.

Replacement can be stated informally as follows: for any function, if its domain is a set, then so is its range. As with separation, the first-order attempt to express this idea is given by a schema – and it fails to do so fully for the same reason. Consider ${}^{\omega}P(\omega)$, the set of all functions $f : \omega \rightarrow P(\omega)$. Since the domain of these functions is a set, by the informal statement of replacement, its range should be as well. Now, there are uncountably many functions of the kind in question, yet, as before, the replacement schema cannot cover them all. Because since the language of first-order set theory is denumerable, only denumerably many formulae can be formulated in the language, and so, there can at most be denumerably many functional formulae to feed into the replacement schema.[7]

The language of second-order logic is obtained by extending the language of first-order logic. Predicate and function variables are added together with quantifiers that bind these types of variable:

(i) Predicate variables:

– one-place: $X^1_1, X^1_2, X^1_3, \ldots$

[6] This should not be conflated with the converse – paradoxical – claim that every property determines a set.

[7] To see that there are uncountably many functions in ${}^{\omega}P(\omega)$ it suffices to observe that $P(\omega)$ has uncountably many members, and that, for each of these members, there is a function mapping every member of ω onto it.

– two-place: $X^2_1, X^2_2, X^2_3, \ldots$

– $\ldots$

(ii) Function variables:

– one-place: $f^1_1, f^1_2, f^1_3, \ldots$

– two-place: $f^2_1, f^2_2, f^2_3, \ldots$

– $\ldots$

(iii) Second-order quantifier: $\forall^2$ (binding either a predicate or function variable)

Then add the following syntactic rules:

(iv) If $t_1, \ldots, t_n$ is sequence of n terms and X is an n-place predicate variable, then $X(t_1, \ldots, t_n)$ is a wff.

(v) $t_1, \ldots, t_n$ is sequence of n terms and f is an n-place function variable, then $f(t_1, \ldots, t_n)$ is a term.

(vi) If $\Phi(t_1, \ldots, t_n)$ is a wff, then $(\forall X)\Phi(t_1, \ldots, t_n)$ is a wff.

Usually, context allows us to use $X, Y, Z, \ldots$ and $f, g, h, \ldots$ for respectively predicate and function variables. In addition, context usually also allows us to omit sub- and superscripts. The second-order existential quantifier can be defined as follows: $(\exists X)\phi =_{df.} \neg(\forall X)\neg\phi$, and similarly, for existential quantification over functions.

It would thus seem that changing the background logic would help facilitate a formulation of separation and replacement more faithful to the informal statement of these axioms than the one supplied by the first-order schemas. Stated in second-order terms, separation and replacement are axioms proper:

Separation: $(\forall X)(\forall y)(\exists z)(\forall w)(w \in z \leftrightarrow w \in y \wedge Xw)$

Replacement: $(\forall f)(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\exists w)(w \in x \wedge z = f(w)))$

Second-order Zermelo set theory, Z^2, has as axioms the axioms of first-order Z, but with the separation schema replaced by the proper second-order axiom. Second-order Zermelo-Fraenkel set theory, ZF^2, is obtained from first-order ZF by replacing the schemas of separation and replacement by the second-order axioms. As with first-order ZF, we do not strictly speaking need separation since it can be derived from replacement.

The view implicit in what been said above is this: the first-order schemas in question are approximations at best. They fail to capture what they are meant to capture, as a matter of principle. To accomplish this second-order resources (or something equivalent, e.g. plural quantification) are needed. This view has been endorsed by number of authors, e.g., Shapiro.[8] One of the most cited advocates of the view is Boolos who contends that '... the principle of set-theoretic induction and the separation (Aussonderung) principle virtually cry out for second-order formulation ...' – and goes on to say:

> It is, I think, clear that our decision to rest content with a set theory formulated in the first-order predicate calculus with identity ... must be regarded as a compromise, as falling short of saying all that we might hope to say. Whatever our reasons for adopting Zermelo-Fraenkel set theory in its usual [first-order] formulation may be, we accept this theory because we accept a stronger theory consisting of a *finite* number of principles, among them some for whose complete expression second-order formulas are required. We ought to be able to formulate a theory that reflects our beliefs. (Boolos [14], p. 65)

The idea is that if we adopt first-order ZF, it is because we already accept second-order ZF – at least tacitly. First-order set theory is thus in some sense parasitic upon second-order set theory.

The status of second-order logic is a controversial issue, or at least the status of second-order logic *with standard semantics* is.[9] Typically what causes the big fuss is the interpretation of the second-order quantifiers as ranging over the full power set of the first-order domain. That is, on the standard semantics for second-order logic, the second-order variables range over *arbitrary subsets* of the first-order domain. This is supposed to be problematical because the notion of

[8] Shapiro [126]. The idea that plural devices can be used to interpret (monadic) second-order quantification was introduced by Boolos. Cf. Boolos [14] and [15].

[9] For technical background on second-order logic with standard semantics, cf. Shapiro [126], Chapter 4.

an arbitrary subset is.[10] The source of concern often seems to be epistemological in nature: we have *no idea* what the arbitrary subsets of a set are when the set is infinite. We have an epistemic grip on what the arbitrary subsets of a set is in the finite case, because they could be listed. However, we lose our grip on the notion in the infinite case – at least so the complaint goes. People critical of the notion of an arbitrary subset typically endorse a notion of subset specifiable by certain means, formulae with first-order parameters being a popular candidate.

Fortunately, much of what I will say in this thesis will not rely on any appeal to second-order logic, and hence, will not rely on second-order logic enjoying some specific status. I shall simply flag if, at some point in an argument, I need any specific assumptions about the standing of second-order logic and ask the reader to adjust her confidence in the conclusions I draw so it accords with her confidence that the requisite assumption about second-order logic can be made good.

A word on choice. Standard second-order deductive systems include a choice principle of the following form:

$$(\forall X^{n+1})[(\forall x_1)\dots(\forall x_n)(\exists y)X^{n+1}(x_1\dots x_n,y)\rightarrow(\exists f^n)(\forall x_1)\dots(\forall x_n)X^{n+1}(x_1\dots x_n,f(x_1\dots x_n))]$$

i.e. for every X^{n+1} if, for all $x_1\dots x_n$ there is a y such that $X^{n+1}(x_1\dots x_n,y)$, then there is an f^n such that, for all $x_1\dots x_n$, X^{n+1} holds of $x_1\dots x_n$ and $f(x_1\dots x_n)$.[11] The axiom of choice, in its set-theoretic formulation, follows from this choice principle. Thus, when dealing with second-order set theory, choice is usually counted among the axioms, and we shall do so here.

1.4 Simple type theory

We now proceed to simple type theory (which we shall sometimes abbreviate as 'STT').

The distinctive feature of the language of STT is the idea of a *typed variable*. The idea is that variables belong to exactly one level in a hierarchy of levels, and that the type associated with a variable indicates what level it belongs to. More specifically, we introduce denumerably

[10] See Jané [76], and, more recently, [77] for an incarnation of a worry of this kind.
[11] Shapiro [126], p. 67.

many types – or levels – and denumerably many variables of each type. In writing, each variables will come with a superscript as well as a subscript, the former indicating the type and the latter which variable of the given type we are dealing with:

- $v_1^1, v_2^1, \ldots$
- $v_1^2, v_2^2, \ldots$

$\vdots$

For the purposes of this thesis, we can allow ourselves to use letters from the back of the alphabet $(x, y, z, \ldots)$ and to omit subscripts.

The change in language is accompanied by two following syntactic rules (where $i, j \in \omega$):

[$\in$-RULE]:
$x^i \in y^j$ is well-formed just in case $i = j - 1$, i.e. $x^i \in y^j$ is well-formed just in case the type of the first variable is exactly one less than the type of the second variable. Likewise for constants.

[$=$-RULE]:
$x^i = y^i$ is well-formed just in case $i = j$, i.e. $x^i = y^i$ is well-formed just in case the two variables have the same type. Likewise for constants.

Occurrences of the same variable are assigned the same type.

STT has two axioms, both formulated in such a way as to incorporate the syntactic restrictions just stated:

Axiom schema of extensionality:
$(\forall x^i)(\forall y^i)((\forall z^{i-1})(z^{i-1} \in x^i \leftrightarrow z^{i-1} \in y^i) \leftrightarrow x^i = y^i)$

Axiom schema of comprehension:
$(\exists y^{i+1})(\forall x^i)(x^i \in y^{i+1} \leftrightarrow \phi(x^i))$, where $\phi(x^i)$ does not contain y^{i+1} free.

Note that both are axiom schemas. The axiom of extensionality has denumerably many instances, one for each variable type. Comprehension allows for variation in two respects, *viz.*

with respect to the type and with respect to ϕ. As with extensionality, we get an instance of comprehension for each variable type. Each of these instances in turn have an instance for each formula $\phi(x^i)$.

1.5 New Foundations

We now turn to Quine's New Foundations (NF).[12] As can be verified by consulting Quine's original paper, part of his motivation for introducing New Foundations was the reduplication of sets and theorems which takes place in simple type theory.[13]

Quine spelled out the restriction on comprehension through the notion of *stratification:*

> **Stratification:** a formula is *stratified* provided there is an assignment of types to all variables when ϕ is spelled out in primitive notation (so, with $\in$, $=$, and the language of logic) such that the assignment stratifies all the atomic subformulae of ϕ simultaneously (occurrences of the same variable are assigned the same type). An assignment of types to variables stratifies
>
> - a formula of the form '$x \in y$' provided two types i and j are assigned x and y such that $i + 1 = j$.
> - a formula of the form '$x = y$' provided the same type i is assigned to x and y.

Having introduced the idea of stratification, one can make the following proposal:

- Only instances of the comprehension schema are allowed in which the formula is stratified.

Note that the subformulae of a formula ϕ being stratified is a necessary, but not sufficient condition for ϕ itself being stratified. Consider, e.g., $x \in y \wedge y \in x$. Both $x \in y$ and $y \in x$ are stratified, but there is no way of assigning levels to both formulae at the same time and respecting the conditions imposed. This is why simultaneous stratification of sub-formulae is required for stratification.

[12] NF was introduced in 1937 in Quine [117]. A thorough investigation of NF-style set theories is given in Forster [49]. See also Holmes [74].

[13] Quine [117], p. 79. The reduplication occurs because a given theorem still holds if each of the types occurring in it are raised by 1.

Like STT, NF has two axioms:

Axiom schema of extensionality:

$(\forall x)(\forall y)((\forall z)(z \in x \leftrightarrow z \in y) \leftrightarrow x = y)$

Axiom schema of comprehension:

$(\exists y)(\forall x)(x \in y \leftrightarrow \phi(x))$, where $\phi(x)$ is a stratified formula that does not contain y free.

Unlike in STT, the axiom of extensionality is not typed, and unlike in STT, comprehension is not explicitly typed. However, given the restriction to stratified formulae, comprehension can perhaps be regarded as being 'implicitly typed'.

Chapter 2

Mathematical scepticism

This chapter is devoted to the task of developing two kinds of mathematical scepticism. First, we will develop a line of thought according to which there can be no warrant for thinking that a mathematical system is respectively consistent and satisfiable. The strategy will be to get the sceptical argument going by appeal to a kind of regress argument familiar from mainstream epistemology. The mathematical regress argument has not, to my knowledge, been discussed much within the philosophy of mathematics. The core of the argument is an entirely general thought about the nature of evidential warrant – *viz.* that it involves a notion of additional support – and stands in just as much need of being addressed in the philosophy of mathematics as any other area where the question can sensibly be asked whether all warrant is evidential. Second, we will formulate a mathematical version of so-called I-II-III scepticism, a kind of scepticism according to which there can be no warrant for certain fundamental ontological propositions – in the mathematical case that there is a realm of entities which our mathematical judgements concern.

My motivation for introducing and developing mathematical scepticism is not that I am a mathematical sceptic and believe that there can be no warrant for propositions concerning the consistency or satisfiability of mathematical systems or the existence of a realm of mathematical entities. However, I do take the sceptical arguments to teach us something: that such warrant cannot be evidential. In Chapters 3 and 4, we will explore the prospects of invoking non-evidential notions of warrant to meet the sceptical challenges.

2.1 The standard approach: axioms first

We know that $3 + 3 = 6$, that $2 \times 2 = 4$, and that every natural number has a successor. We also know that there is an empty set and that, for every two sets, there is a set containing exactly those sets as members. In other words, we have mathematical knowledge. One of the main tasks in the philosophy of mathematics is to answer the question what this knowledge consists in and how it can be acquired. If, as is common, one takes warranted belief to be a crucial component of knowledge, a considerable part of the response to this question will be a response to the question how we can acquire warranted mathematical beliefs and what mathematical warrant consists in. Here we shall focus on mathematical warrant rather than knowledge and shall do so for two reasons. First, we will follow most authors by taking warranted belief to be a necessary (but not sufficient) condition for knowledge. Thus, in order to give an account of mathematical knowledge we need, in any case, to give an account of mathematical warrant. Second, many of the interesting issues pertaining to knowledge are rooted in issues having to do with warrant.

Though there is much variety with respect to the details of accounts of mathematical knowledge and warrant, most authors agree on where to start. The standard approach is to focus on the axioms. If the task is to give an account of arithmetical knowledge, focus on the axioms of arithmetic. For set theory, start with the set-theoretic axioms. And so forth. There is a good reason for this. If we can account for how we acquire a warranted belief in the axioms, logic will do the rest of the work for us because, by logic, we get an account of all the propositions that follow from the axioms. Let A be a set of axioms. The theory of A, T_A, is A together with every proposition that follows from A. Thus, every theorem of T_A can be traced back to the axioms through some chain of inference. The idea is, roughly, that whatever warrants the axioms warrants the conclusions drawn from them through some appropriate inference, and so, ultimately, warrants theorems established by some chain of inference. The warrant for the axioms is said to *transmit* across the relevant inferences in these cases.

Two points are worth recording. First, it does not hold in general that warrant transmits from warranted premises to the conclusion of a valid inference. The inference has to be 'appropriate', as indicated above. In the next section, we will say more about what it takes for an

inference to meet this criterion. Second, we have to be careful when we say that 'logic will do the rest of the work'. According to Gödel's first incompleteness theorem, for any sufficiently strong mathematical theory T, there is some statement S in the language of T such that neither $T \vdash S$ nor $T \vdash \neg S$, but where S is nonetheless intuitively true. So, logic cannot do 'the rest of the work' if by this we mean that warrant is transmitted from the axioms to *every truth* in the language of the relevant theory. Not every truth is derivable from the axioms. This is why above it was said that, if we can account for the warrantability of the axioms, logic delivers an account for all the propositions that *follow from* them. 'Follow from' is here meant to be neutral between semantic and deductive consequence (in standard notation: $\vDash$ and $\vdash$ respectively). The reason is that whether we are speaking in terms of logical or deductive consequence, warrant will transmit across an inference from warranted premises to a conclusion provided that the inference is of an appropriate kind.[1]

2.2 Transmission of warrant

On the picture given in the previous section – the 'transmission picture' – whatever is warranted by appeal to the axioms is warranted inferentially. This leaves it entirely open whether or not the axioms themselves are warranted through inference. We will ultimately argue that they are not, but will leave the matter aside for now and instead introduce two requirements which, in general, must be satisfied in order for transmission of warrant to take place across an inference. We will return to the axioms in Section 2.4.

Some beliefs are warranted through inference. Suppose that I believe that Smith is wearing a black sweater. I have a perceptual warrant for this belief, acquired as a result of exercise of a properly functioning perceptual apparatus in a conducive environment. From the belief in question I acquire a belief in the existential that there is someone wearing a black sweater – a belief which is warranted inferentially. The idea is that my belief in the existential inherits the

[1] If we suppose that T is complete in the sense that whenever α is a semantic consequence of T, it is a syntactic consequence of T as well, the (first) Gödel incompleteness theorem yields the result that there is some statement S in the language of T such that neither $T \vDash S$ nor $T \vDash \neg S$. If T is not complete, it might be that there is some sentence S such that neither $T \vdash S$ nor $T \vdash \neg S$, but also that, for every sentence S, $T \vDash S$ or $T \vDash \neg S$. In other words, there can be theories which are Gödel incomplete, but semantically complete in the sense of semantically implying S or $\neg S$ for any statement S. An example of such a theory is second-order Peano arithmetic.

epistemic standing – that of being warranted – of the belief from which it was inferred. Not all inferences cater for transmission of warrant, however. So, let us consider an inference from warranted premises $A_1, \ldots, A_n$ to the conclusion B and ask what needs to be the case in order for an epistemic subject to be able to acquire a warrant for B on the basis on an inference from (the warranted premises) $A_1, \ldots, A_n$? That is, let us ask what it takes for an inference to be 'appropriate' for warrant-transmission?

Validity is a necessary, but not sufficient condition for transmission of warrant. It is necessary, because of the relationship between warranted belief and truth. To have a warranted belief in some proposition P is to hold a warranted attitude towards its truth. So suppose that some epistemic subject has a warranted belief that P, that Q follows from P through some rule of inference, but that the inference is not valid. Then P's truth does not guarantee the truth of Q, and hence, the idea that the inference will take us from a warranted attitude towards the truth of P to a warranted attitude towards the *truth* of Q can no longer be sustained. Therefore, validity is a necessary condition for warrant transmission. Validity is not sufficient. If it were, there would be, for any proposition P, an inference with P as its conclusion which would transmit warrant provided the premise was warranted. Just consider the inference from P to P. This is certainly valid: the truth of the premise guarantees the truth of the conclusion; there is no interpretation that makes P both true and false.[2] Many other inferences would do as well, were validity sufficient. For instance, the inference of P from the conjunction of P and Q – Q some other proposition – is also valid.

However, why is it that these inferences (and others with them) do not cater for warrant transmission? It is because the premise *cannot be warranted independently and antecedently of the conclusion.* In other words, in order to have a warrant for the premise we already need to have a warrant for the conclusion. Therefore, we cannot acquire a warrant for the conclusion on the basis of an inference from the premise. That is, there is nothing for us to gain or learn

[2] The inference is also valid in the dialethic system LP, the *logic of paradox.* In LP, there are three truth values: true, false, and both true and false. Thus, there are interpretations on which a proposition can be both true and false. However, the validity of the inference of P from P is not undermined. Validity in LP is defined in terms of designated values: an inference is valid just in case (necessarily) the conclusion is designated if the premises are. The designated values of LP are true and both true and false, and hence, on interpretations where P is both true and false, the premise as well as the conclusion comes out designated.

epistemically.

These considerations lead us to the following transmission principle[3]:

(FAILURE) Suppose that there is a valid inference from $A_1 \ldots A_n$ to B, and that $A_1 \ldots A_n$ are warranted. Then the warrant for $A_1 \ldots A_n$ is not transmitted to B in case one of $A_1 \ldots A_n$ cannot be warranted antecedently and independently of B.

Informed by (FAILURE), we will now say a little about the axioms and the relationship they must bear to propositions established on basis of them in order to cater for transmission of warrant to these propositions.

The axioms need to be warranted; otherwise there is no warrant to transmit to the mathematical statements that follow from them. In addition, the inferences relied on need to be of the right kind. We have just seen that validity is not sufficient for transmission of warrant from warranted premises to the conclusion of an inference. By the transmission principle stated above, the premises must be such that they can be warranted antecedently and independently of the conclusion. I take it that this requirement is usually met when propositions and theorems are established on the basis of the axioms (and definitions) of mathematical theories. Usually, the background logic is taken to be classical logic. The inference rules of classical logic can be shown to be valid.[4] *And*, importantly, the axioms are such that they can be warranted antecedently and independently of the propositions established on the basis of them, as required by (FAILURE).

We have not yet done anything to address the question whether the axioms themselves are

[3] Wright [147], pp. 36–37; Wright [151], p. 57; Wright [154], p. 172; Davies [32], p. 221.

[4] A qualification. Classical logicians and adherents of alternative logics, such as intuitionistic and relevance logic, do agree that certain rules of inference are valid. E.g, no one takes issue with the inference of B from the conjunction of A and B. However, the intuitionist does call into question the validity of double negation elimination, and the relevance logician takes issue with disjunctive syllogism.

Here we will allow ourselves to assume that the background logic is classical as our goal is not to shed light on the debate between adherents of different logics, but rather to say something about the epistemology of mathematics on the assumption that the logic stays fixed. It should be noted, though, that what has been said about transmission of warrant applies whatever logic is taken to be the background logic of the mathematical theory worked in. If the logic is intuitionistic or relevant, this does not remove the requirement that transmission of warrant from warranted premises to conclusion only obtains provided the relationship between premises and conclusion is right.

warranted inferentially or non-inferentially. According to what was said above, *if* the axioms are warranted inferentially, (FAILURE) must be respected. Just like the axioms must stand in the right kind of relationship to every proposition established by appeal to them in order for warrant transmission to obtain, whatever warrants the axioms must stand in the right kind of relationship to the axioms. In Chapter 3, however, we will explore the suggestion that the axioms of PA and ZFC are warranted non-inferentially.

2.3 Scepticism about the empirical world

In this section, we will provide an account of two kinds of scepticism concerning the empirical world. We will do so in order to set the scene for the rest of the chapter where we will turn to mathematical scepticism. It will be instructive to have empirical scepticism as background when introducing and developing mathematical scepticism so as to notice similarities and differences between them.

Following Wright [154], let us characterize the notion of a *cornerstone* as follows:

(COR) A certain proposition – or a specific type of proposition – is a *cornerstone* for a given region of thought just in case the proposition (or type of proposition) is such that, if we had no warrant for it, we could not rationally claim warrant for any belief in a proposition of that region.

The basic idea is that a proposition is a cornerstone for a given region of thought provided that it is sufficiently important or integral to that region, meaning that the proposition has to be integrated with the other propositions of the region in such a way that rational claim to warrant for these other propositions is bound up with, or hinges on, whether or not there is a warrant for the cornerstone.

Having introduced the notion of a cornerstone, Wright observes that many well-known sceptical challenges fit the following two-step template:

(STEP 1) An argument to the effect that a certain proposition C we typically accept is a cornerstone for a given range of thought.

(STEP 2) An argument to the effect that we have no warrant for C.

If (STEP 1) and (STEP 2) are found compelling, then – given (COR) – a certain kind of higher-order cognitive achievement is beyond our reach:

(CON) We cannot rationally claim warrant for *any* belief in the relevant region of thought.

Note the focus on rational claims to warrant rather than possession of warrant. What the characterization of the notion of a cornerstone commits one to is not that the absence of cornerstone warrant is incompatible with possession of warrant for belief in any ordinary proposition, but rather that rational *claim* to such warrant is.[5]

Wright discusses two kinds of scepticism concerning the empirical world, so-called Cartesian and I-II-III scepticism. Below we give a brief account of the arguments employed by the Cartesian and I-II-III sceptic respectively.

Cartesian arguments. (STEP 1) of Cartesian arguments is a case for the claim that it is a cornerstone for a wide range of beliefs ordinarily held about the empirical world that we are not victims of systematic cognitive error or disablement. Familiar examples from the literature include the following propositions: that I am not having a vivid, coherent dream; that I am not

[5] In Wright's own words:

> ...what is put in doubt by sceptical argument is – of course – not our *possession* of any knowledge or justified belief – not if knowledgeability, or justification, are conceived as constituted in aspects of the external situation in which we come to a belief. (How indeed could armchair ruminations show anything about that?) What is put in doubt is rather our right to *claim* knowledge and justified belief. (Wright [154], p. 210.)

In addition, see p. 169.

hallucinating; that I am not being deceived by a malicious, omnipotent demon; that I am not a brain in a vat.[6]

This seems reasonable enough. If we have no warrant for these cornerstones, we cannot rationally claim to have a warrant for any belief about the empirical world. To illustrate suppose that I had no warrant for any of the propositions mentioned above, and suppose, furthermore, that I have a visual experience of what seems to be a tree in front of me. Now, on the basis of this visual experience, I form a belief that there is a tree in front of me. Can I rationally *claim* to have a warrant for the belief in question? It would seem not. Because the relevant propositions – that I am not a brain in a vat, etc. – concern something integral to my investigation, *viz.* whether my experience represents a world which is, by and large as I take it to be – and so, whether the attendant circumstances are suitable for arriving at beliefs about the world at all.

Here we should remind ourselves that this should not be taken to amount to the view that, absent a warrant for a cornerstone of a given region of thought, one cannot possess a warrant for any belief from that region. It means that one cannot rationally *claim* such a warrant.

Concerning (STEP 2), since the relevant cornerstones ('I am not now dreaming', 'I am not a brain in vat', etc.) are empirical, the sceptic maintains that warrant for such cornerstones must be given by appropriate empirical evidence. This empirical evidence has to be collected through execution of an empirical procedure. The sceptic insists that the following principle must be respected:

> (PROPER) 'evidence acquired as the result of an empirical procedure cannot rationally be regarded as any stronger than one's independent grounds for supposing that the procedure in question has been executed properly.' (Wright [154], p. 168.)

(PROPER) is a minimizing principle. It says that the strength of evidence takes the 'minimum value' among the independently acquired reasons for supposing that the procedure by which the evidence has been acquired has been executed properly. The principle is brought to work in

[6] Sample references: Descartes [34], and Chapter 1 in Putnam [116].

(STEP 2) of the sceptical challenge:

Consider the proposition that I am not a brain in a vat, one of the cornerstones targeted by Cartesian scepticism. The sceptic maintains that, since this is an empirical proposition, evidence supporting it has to be collected by executing some empirical procedure. Suppose that I hold that I have a warrant for a belief in the proposition in question in virtue of what I take to be perception of my two hands. By (PROPER), my evidence cannot rationally be regarded as any stronger than the independent grounds for thinking that the procedure has been properly executed – and so, for thinking that the procedure was executed *in the first place*. That is, my warrant for the belief that I am not a brain in a vat cannot rationally be regarded as any stronger than my independent grounds for believing that I perceived – rather than vat-perceived – my two hands. To obtain a warrant for (a belief in) the cornerstone proposition I thus already need a warrant for that very proposition, and so, due to vicious circularity, I can never acquire a warrant for (a belief in) the cornerstone in question.[7]

I-II-III arguments. The other group of arguments which fit the two-step template are what has been labeled 'I-II-III arguments'. These arguments differ from Cartesian arguments by making no appeal to cognitive error, but focusing instead on *structural* features of attempts to supply cornerstone warrants. The basic point which the I-II-III sceptic makes is that the best such attempts are constituted by arguments, which, as a matter of principle, fail to transmit warrant due to a violation of the principle (FAILURE) from Section 2.2.

The structure of I-II-III arguments is this:

	I	My current evidence is in all respects as if P
So,	II	P
		If P, then C
So,	III	C

where C is a cornerstone of a given region of thought, and P is an ordinary proposition of that

[7] Cf. Wright [154], pp. 168–169. The reasoning is put in terms of a specific cornerstone, but can be straightforwardly modified to apply to other cornerstones.

region. The step from I to II is defeasible, and the conditional part of II is supposed to be a piece of philosophical theorizing – a conditional that gets in place due to a conceptual connection between P and C.

A prominent instance of the I-II-III template is Moore's famous 'proof' (or at least something reasonably similar to it) of the existence of an external world:

[MOORE]:

	I	My current experience is in all respects as if I have two hands.
So,	II	I have two hands.
		If I have two hands, then there is an external world.
So,	III	There is an external world.

Did Moore succeed? The inference from II to III is certainly valid – it is just an ordinary modus ponens step. However, does the support offered by I for II transmit to III? Not so according to the I-II-III sceptic.[8] Here is why:

The type I proposition is thought to constitute maximal – the best, and indeed only possible – evidence for the type II proposition. However, the transition from I to II is defeasible *and* held hostage to the availability of appropriate information. In particular, in order for the move from I to II to be warranted the type III proposition has to be in the pool of information on which the type I proposition depends – and it has to be so in a warranted fashion. For if there is no warrant for the type III proposition that there is an external world, it hardly seems plausible to claim that any type I proposition can warrant – by serving as evidence – the corresponding type II proposition that I have two hands. Absent a warrant for the proposition that there is an external world, my sensory experience being in all respects *as if* I have two hands does not

[8] For Moore's 'proof', see Moore [98]. As Wright notes, the reasoning generalizes to other minds and the past (Wright [154], p. 171):

I	X's behaviour and physical condition are in all respects as if she was in pain.	It seems to me that I remember it being the case that it was sunny yesterday.
II	X is in pain. If X is in pain, then there are other minds.	It was sunny yesterday. If it was sunny yesterday, then there is a past.
III	There are other minds.	There is a past.

On I-II-III scepticism, cf. also Wright [149], [151], and [152].

carry any evidential weight. There is no warrant that my experiences represent anything of the kind they are thought to represent.[9]

But if the type III proposition has to be in place for the move from I to II to be warranted, how is the type III proposition warranted? Here the I-II-III sceptic maintains that the only way to acquire a warrant for a type III proposition is by inference from a type II proposition. However, now there is a vicious circle: the warrantability of the transition from I to II rests on III, which in turn rests on II. The putative support offered by I for II cannot reasonably be accepted antecedently and independently of the conclusion. So, by (FAILURE), I-II-III arguments will fail to transmit warrant, and there can be no warrant for any type III proposition.

Doing things a bit backwards, this – the conclusion that there can be no warrant for any type III proposition – was (STEP 2) of the sceptical template. (STEP 1) is an argument to the effect that type III propositions are cornerstones relative to type II propositions. The argument is this: type II propositions can only be warranted through type I propositions. However, absent a warrant for the type III proposition, a type I proposition cannot count as evidence for any type II proposition. Thus, absent a warrant for the type III proposition, there can be no rational claim to warrant for any type II proposition. For in that case it cannot rationally be claimed that there is a type I proposition which supports a type II proposition.[10]

2.4 Moving on to mathematical scepticism

If sound, the upshot of Cartesian and I-II-III arguments is that a certain kind of higher-order cognitive achievement is beyond our reach: we cannot rationally claim warrant for any proposition belonging to a region of thought of which a cornerstone has been put under attack. Having introduced scepticism about the empirical world, the remainder of this chapter is devoted to mathematical scepticism. As announced, we will develop two kinds of argument, starting with

[9] The idea is that the warrantability of the move from I to II has as a presupposition that there is indeed the kind of domain which the experiences involved in I are thought to represent, i.e. that it rests on the presupposition that they do not misrepresent. See Wright [154], p. 172. An opposing view can be found in Burge and Pryor. Both hold that type I propositions by themselves support type II propositions. See Burge [21] and Pryor [112].

[10] Since Wright is keen to stress that scepticism, properly construed, should be regarded as an attack on rational claims to warrant rather than possession of warrant, it is interesting to note that he sometimes slips and speaks in terms of warrant-*possession*. A particularly clear example can be found on p. 172 of Wright [154].

a mathematical regress argument and then proceeding, towards the end of the chapter, to formulate a mathematical I-II-III argument.

The mathematical regress argument is targeted at a specific class of propositions. Let T be a mathematical theory. Read '$Sat(T)$' and '$Con(T)$' respectively as shorthand for 'T is satisfiable' and 'T is consistent'. Propositions that assert the satisfiability of some theory will be referred to as 'satisfiability propositions', and the question whether T is satisfiable will be referred to as 'the satisfiability question (for T)'. Similarly for consistency.

Consistency and satisfiability propositions (and questions) are interesting from an epistemological perspective. We will offer support for this view in Section 2.6. We will do so by proposing that the consistency and satisfiability propositions of a given theory T are cornerstones for theorizing in T. Given (COR), that is to say that if we do not have a warrant for the consistency and satisfiability propositions of a given theory T, we cannot rationally claim warrant for belief in any proposition established in T. We will then proceed to formulate the mathematical regress argument. There will be both points of similarity and points of divergence with the two kinds of scepticism concerning the empirical world which we considered above.

2.5 Syntax and semantics: consistency and satisfiability

We have just announced that we will take on the task of (i) supporting the claim that $Sat(T)$ and $Con(T)$ are cornerstones for T-theorizing, and (ii) discussing a sceptical line of argument meant to raise doubts concerning the warrantability of satisfiability and consistency propositions. Before we take on these two tasks, let us say a little about satisfiability and consistency so as to appreciate the difference between the two notions.

We understand satisfiability and consistency in the standard way. That is, in general:

Definition 1 (Satisfiability) A set of sentences Γ is satisfiable just in case there is a model M such that $M \vDash \Gamma$, i.e. just in case there is an interpretation that makes every sentence α of Γ true.

Definition 2 (Consistency) A set of sentences Γ is consistent just in case there is no sentence α of Γ such that $\Gamma \vdash \alpha \wedge \neg\alpha$, i.e. just in case no contradiction is derivable from Γ.

We take a theory T to be a set of axioms, A, together with whatever follows from them. Thus, in terms of semantic consequence, $T = \{\alpha : A \models \alpha\}$, and, in terms of deductive consequence, $T = \{\alpha : A \vdash \alpha\}$.[11] The satisfiability and consistency question for a theory thus reduces to the question whether its axioms are respectively satisfiable and consistent.

Satisfiability is a semantic, or model-theoretic, notion, whereas consistency is a syntactic, or deductive, notion. The former concerns structures: $Sat(T)$ says that T has a model, i.e. that there is a structure on which the T-axioms are satisfied; there is something it is about. The latter concerns symbol manipulation in the sense that $Con(T)$ says that there is no manipulation of symbols in accordance with the rules of deduction of T that will allow us to deduce a formula of the form $\alpha \wedge \neg\alpha$ for some wff α in the language of T. Given the soundness and completeness of first-order theories, the two notions coincide extensionally in the first-order case. That is, for any set of sentences Γ, $Sat(\Gamma)$ if and only $Con(\Gamma)$.[12] However, still, the two notions should be kept apart. One reason is that, as just pointed out, $Sat(\Gamma)$ and $Con(\Gamma)$ express significantly different thoughts. Another reason is that the two notions do not coincide for every theory. Second-order theories (with standard semantics) are not complete, and so, the semantic and syntactic notions come apart in terms of range of application.

2.6 Mathematical cornerstones

Let us now consider the proposal that the satisfiability and consistency propositions for a theory T are cornerstones for T-theorizing.

Satisfiability. $Sat(T)$ is a necessary condition for the good standing of T. For suppose that T

[11] There is no need to take T to be respectively $A \cup \{\alpha : A \models \alpha\}$ and $A \cup \{\alpha : A \vdash \alpha\}$, because $\alpha \models \alpha$ and $\alpha \vdash \alpha$ for every (wff) α – including, of course, every axiom.

[12] The completeness theorem says that, if $Con(\Gamma)$, then $Sat(\Gamma)$. The soundness theorem is the converse: if $Sat(\Gamma)$, then $Con(\Gamma)$. Often the completeness theorem is stated as follows: if $\Gamma \models \alpha$, then $\Gamma \vdash \alpha$ – the corresponding statement of the soundness theorem being: if $\Gamma \vdash \alpha$, then $\Gamma \models \alpha$. The two statements of each of the theorems are equivalent. Cf. Enderton [39], Chapter 2.

is unsatisfiable. In that case it has no model, i.e. there is no structure on which the language of T can be interpreted so as to make the T-axioms true. In other words, T fails to delineate a subject-matter. This is bad news. For I take it that one of the central aims of mathematical theorizing – stronger: theorizing in general – is to learn truths about some subject-matter. This is a purpose that unsatisfiable theories cannot serve. They semantically imply any formula of the language in which they are couched. In this sense they tell us a lot. However, unfortunately, they tell us too much, and, what is more, they do so in the wrong kind of way. When reasoning in a mathematical theory, we would like the arguments used to establish theorems to be sound, i.e. valid and with true premises. For on the assumption that an argument is valid, the assumption of the truth of the premises guarantees the truth of the conclusion (in the relevant model). However, an unsatisfiable theory T cannot cater for a single sound argument of the form $T \vDash \alpha$ – α some statement in the language of T. That would require T to be *true* in a model, but it cannot be.[13]

The above line of reasoning makes plausible that $Sat(T)$ is important to T-theorizing. However, what we need to argue to establish that $Sat(T)$ is a cornerstone is that, absent a warrant for $Sat(T)$, there can be no rational claim to warrant for belief in any T-proposition. Thus, we are not quite done yet. But we can reason as follows:

Suppose that we are working in a theory T, have no warrant for $Sat(T)$, and that we have established a statement P as a theorem of T.[14] Can we rationally claim warrant for believing P? Arguably not. As argued above $Sat(T)$ is a necessary condition for the good standing of T. Hence, absent a warrant for $Sat(T)$, we have no warrant for thinking that T is in good standing. In particular, we have no warrant for thinking that T is about anything, and, thus, when we establish P as a theorem in T, we have no warrant for thinking that there is a structure which

[13] In speaking of 'structure' and 'subject-matter' above I do not mean to presuppose a realist view. There will be a crucial difference between satisfiable and unsatisfiable theories, irrespective of one's philosophical leanings with respect to mathematics. In the discussion of mathematical I-II-III scepticism we will, however, operate with an explicitly realist notion of subject-matter. Context will make the shift clear.

[14] When discussing scepticism about the empirical world, non-factive terminology was employed so as to not prejudge the issue against the sceptic. For instance, we wrote 'I have a visual experience of what *seems* to be a tree in front of me' (emphasis added) rather than 'I have a visual experience of a tree in front of me'. Now, somebody might suggest that care should also be taken with 'theorem' and 'established', the thought being that they are factive – so that, if P is a theorem or established, then P is true. Not so. That P is a 'theorem' and 'established' in T are intended to mean merely that P can be shown to follow from the axioms of T, which does not imply that P is true in the sense of there being a subject-matter which is says something about.

P says something about. Absent a warrant for $Sat(T)$, there is no warrant for thinking that reasoning in T has any evidential force. For if there is no warrant for thinking that T is about anything, there is no warrant for thinking that reasoning in T should bear positively on the question whether P. P is supposed to say something about the subject-matter of T – yet, what we lack is exactly a warrant to think that it has one.

The above considerations can be summarized as follows: there is a lack of warrant for $Sat(T)$. This lack of warrant implies that it cannot rationally be claimed that T-theorizing can result in evidence for any theorem P that can be established in T. It cannot rationally be claimed that T-theorizing results in evidence, because there is no warrant for thinking that there is anything that T is about, and so, no warrant for thinking that there is anything that T-propositions say something about. Now, suppose that in this kind of situation I claim to have a warrant for a belief in a theorem P that I have established in T. Then it seems fair to summarize my epistemic situation as follows:

(CLAIM-S) *I claim to warrantedly believe P, but cannot rationally claim to have any evidence for P on the basis of T-theorizing.*

What we are interested in now is whether my claim to warrant is rational. It is not. The warrant I claim to have is supposed to have been acquired through a proof carried out in T. The idea is that P is warranted evidentially – more specifically, inferentially – through the proof. However, as the second half of (CLAIM-S) reminds us, I cannot rationally claim to have any evidence for P on the basis of T-theorizing. But since the claimed warrant is supposed to be *evidential*, this means that I cannot rationally claim to have warrant for my belief in P.

This supports the view that $Sat(T)$ is a cornerstone for T-theorizing, i.e. that no rational claim to warrant for belief in any T-theorem can be made absent a warrant for $Sat(T)$.

As in our discussion of scepticism about the empirical world, it should be stressed that cornerstones relate to rational claims to warrant rather than possession of warrant. $Sat(T)$'s status as a cornerstone does *not* imply that there can be no possession of warrant for belief in any T-proposition in the absence of a warrant for $Sat(T)$. Rather, what is incompatible is the absence

of such warrant and a rational claim to warrant for a belief in a T-proposition. For instance, lacking a warrant for $Sat(ZF)$ I cannot rationally claim a warrant for my belief that every (von Neumann) ordinal is a transitive set well-ordered by $\in$, reached through a standard proof by transfinite induction in ZF. This is because the lack of warrant for $Sat(ZF)$ implies that there is no warrant for thinking that ZF-theorizing counts as evidence. However, on the other hand, it may very well be that I have a warrant for my belief that every (von Neumann) ordinal is a transitive set well-ordered by $\in$, arrived at in the manner indicated. In other words, what has been said above is meant to be compatible with the view that it suffices for such a warrant that ZF *is* satisfiable, and that the proof through which I arrived at the belief is correct. In the discussion of scepticism about the empirical world, the analogous point was that we can possess warrant for empirical beliefs though we lack a warrant for cornerstone propositions (I am not a brain in a vat, etc.) The motivation for this is in large part to accommodate those with externalist sympathies. That is, to satisfy those who think that possession of warrant for ordinary beliefs about the empirical world is a matter of such beliefs having been acquired via the exercise of a reliable method in a cooperative environment, and so, denies the relevance of cornerstone warrant to the possession of warrant for ordinary beliefs. In the mathematical case, the cooperation of the environment and the exercise of a reliable method correspond to respectively $Sat(T)$ and the correctness of the proof given. More will be said about internalism and externalism in Chapter 6.

Consistency. Consistency, like satisfiability, is a necessary condition for a theory T to be in good standing. Suppose that T is inconsistent. Then there would be some α such that $T \vdash \alpha \wedge \neg\alpha$. Now, it is a (derived) rule that anything follows from a contradiction, i.e. $\alpha \wedge \neg\alpha \vdash \beta$. So, if $T \vdash \alpha \wedge \neg\alpha$, then $T \vdash \beta$, for any statement β that can be formulated in the language of T. Suppose that T is ZF. If ZF was inconsistent, it would prove the statement that every von Neumann ordinal is a transitive set well-ordered by $\in$, as it should. However, unfortunately, it would also prove the negation of this statement. Indeed, for *any* statement of the language of ZF, ZF would prove it and its negation. In other words, the inconsistency of ZF would have a rather trivializing effect.

This suggests that $Con(T)$ is important to T-theorizing. However, to establish that $Con(T)$

is a cornerstone for T-theorizing, we still need to argue that there is no such thing as rational claim to warrant for belief in a T-proposition without a warrant for $Con(T)$. The reasoning supporting this conclusion is similar to the line of reasoning rehearsed for the corresponding claim concerning $Sat(T)$.

As before, let us suppose that we are working in theory T, that we have no warrant for $Con(T)$, and that we have deduced P in T. We now ask if we can rationally claim a warrant for believing P? Arguably not. Absent a warrant for $Con(T)$ we have no warrant for thinking that T is in good standing, in the sense that we have no warrant for thinking that T is not trivial. ($Con(T)$ says that there is no α such that $T \vdash \alpha \wedge \neg\alpha$, so we may reasonably take $Con(T)$ to say that T is not trivial.) Thus, when we derive some proposition P in T, perhaps with some effort, we have no warrant for thinking that we have not done so in a trivial theory. That is, in a theory T in which there is a very easy derivation of P simply because any statement that can be formulated in the language is derivable in the theory. Now, in a derivation, earlier steps serve as evidence for later ones. They do so in the following way: rules of derivation tell us that, given a formula (or formulae) of a certain form, we can derive another formula of a certain form. Having derived a formula β from, say, $\alpha \wedge \beta$ and being asked what licensed the move to β, we would cite the formula $\alpha \wedge \beta$ together with the elimination rule for $\wedge$. The formula arrived at by application of a rule of derivation is warranted by the particular input(s) given and the rule itself. However, absent a warrant for $Con(T)$, there is no warrant for thinking that derivations from the axioms of T do not trivially warrant everything. But T's providing a trivial warrant for everything should be taken as a reductio of the claim that T provides a proper warrant for anything at all. This suggests that, absent a warrant for $Con(T)$, there is no warrant for thinking that anything can be warranted by being derived from the axioms of T.

Now, suppose that I claim to have a warranted belief in P on the basis of a derivation of P in T and that I have no warrant for $Con(T)$. Then my epistemic situation can reasonably be characterized as follows:

(CLAIM-C) *I claim to warrantedly believe P, but cannot rationally claim to have any evidence for P on the basis of derivations in T.*

What is of interest to us now is whether the claim to warrant in the first half of (CLAIM-C) is rational. It is not. The warrant I claim is supposed to be evidential and to have been acquired on the basis of a derivation in T. The idea, as we have seen, is that the formula – or formulae – taken as input when a rule is applied serve as evidence – or warrant – for the output formula together with the applied rule itself. However, as the second half of (CLAIM-C) reminds us, I cannot rationally claim to have any evidence for P on the basis of derivations in T. But since the warrant claimed is supposed to be *evidential*, this means that I cannot rationally claim a warrant to believe P.

The above argument shows that, absent a warrant for $Con(T)$, no warrant can rationally be claimed for any proposition derived in T. Given the characterization of a cornerstone, this shows that $Con(T)$ is a cornerstone for propositions derivable in T. As with empirical cornerstones and $Sat(T)$, we stress that $Con(T)$ – regarded as a cornerstone – concerns *claims* to warrant rather than possession of warrant.

2.7 Level of generality

An immediate question concerns the level of generality involved when we are dealing with respectively empirical and mathematical cornerstones. How general are these propositions, considered as cornerstones? That is, how wide is the region of thought with which they are associated?

One noticeable thing about the two kinds of scepticism about the empirical world, which we accounted for earlier, is the sweeping generality of the attack. The cornerstones called into question are relevant to empirical thinking *as such*. If they are not warranted, we cannot rationally claim warrant for belief in any empirical proposition, across the board. The candidate mathematical cornerstones – $Sat(T)$ and $Con(T)$ – we will be considering appear to be more local in character.

Suppose, for instance, that the theory we are considering is ZFC. The satisfiability and consistency propositions for this theory do not appear to concern mathematical thinking in general. The assumption that we have no warrant for the satisfiability or consistency of ZFC does not seem to force us to hold that we cannot rationally claim warrant for any mathematical proposition. We might not have a warrant for either proposition, but still happily be doing arithmetic, and, what is more, be rationally claiming warrant for arithmetical beliefs.

It is tempting to think that $Sat(ZFC)$ and $Con(ZFC)$ would only be relevant to mathematical thinking in general on the assumption that some version of set-theoretic reductionism is true. That is, on the assumption that all mathematical entities are sets, and so, that numbers, say, are certain kinds of sets. The reason why it might be tempting to think so is that it seems natural to suppose that, if numbers are not sets, then how could the satisfiability and consistency questions for ZFC – i.e. questions about sets – be relevant to arithmetic? Arithmetic, after all, concerns numbers. Though tempting, this line of thought is too quick. To see this let us note that each of ZFC and arithmetic contains resources that allow for a representation of every formula of the language of the other. That is, for every ZFC-formula ϕ_{ZFC} there is a formula of arithmetic ϕ_A that represents it, and vice versa. ZFC-formulae are representable in the language of arithmetic, because we can arithmetize the syntax of set theory. This is *not* to say that, for every ZFC-formula ϕ_{ZFC} which follows from the ZFC-axioms, there is a corresponding arithmetical representation ϕ_A which follows from the axioms of arithmetic. Remember that ZFC is a stronger theory than arithmetic. Regarding the representation of arithmetical formulae by the use of set-theoretic means, the notions of classical mathematics can be faithfully represented in set theory. In particular, this means that the notion of natural number can be given a faithful set-theoretic representation, together with the successor relation, the less than relation, and the operations of addition, multiplication, and exponentiation.[15] By 'faithful representation' is not merely meant that we can give a syntactic representation of arithmetical statements using the resources of set theory. The representations are called 'faithful', because the set-theoretic surrogate notion of natural number together with the mentioned relations and operations can be shown to behave as they should – i.e. in accordance with the axioms of arithmetic. This

[15] See, e.g., Enderton [40], Chapter 4.

means that, for every statement ϕ_A of arithmetic such that ϕ_A follows from the axioms of arithmetic, there is a set-theoretic statement ϕ_{ZFC} which faithfully translates it and follows from the ZFC axioms. Let us call a set-theoretic statement of the above kind an *arithmetical ZFC-representative*. Observe that the existence of arithmetical representatives is entirely compatible with numbers not being sets, i.e. with set-theoretic reductionism being false.

We can now see how the satisfiability and consistency question for set theory can be relevant to the corresponding question for arithmetic. For if an arithmetical ZFC-representative ϕ_{ZFC} were to render ZFC unsatisfiable (or inconsistent), there would be a corresponding arithmetical statement ϕ_A which would render arithmetic unsatisfiable (or inconsistent). So, whether or not numbers are sets, the satisfiability (or consistency) question for set theory could be relevant to that of arithmetic. This is not to say, that the unsatisfiability (or inconsistency) of set theory would inevitably imply the unsatisfiability (or inconsistency) of arithmetic. Bear in mind that set theory is a stronger theory than arithmetic. Even if we had a warrant for believing $\neg Sat(ZFC)$ or $\neg Con(ZFC)$, arithmetic could still be satisfiable or consistent and in perfectly good standing, with rational claims to warrant being made.

Having seen that $Sat(ZFC)$ and $Con(ZFC)$ are not cornerstones of mathematical thinking in general, let us ask more narrowly whether $Sat(ZFC)$ and $Con(ZFC)$ are cornerstones for *set-theoretic* thinking in general? They would be, were ZFC and extensions thereof our only set theories. This is not the case, however. As indicated in Chapter 1, there are other set theories around, and indeed, many more than those mentioned there. The assumption that we have no warrant for the satisfiability or consistency of ZFC does not force us to hold that we cannot rationally claim warrant for any set-theoretic proposition. Absent a warrant for either $Sat(ZFC)$ or $Con(ZFC)$, we might very well still happily be doing, say, simple type theory (STT) and be rationally claiming warrants for the theorems established by means of the theory. The reasoning supporting this claim is similar to the reasoning rehearsed above for ZFC and arithmetic. Each of ZFC and STT has the resources to represent statements formulated in the language of the other theory. For every STT-formula ϕ_{STT} that follows from STT, there is a ZFC-formula ϕ_{ZFC} that represents ϕ_{STT} and follows from ZFC. However, again, ZFC is the

stronger theory – so the converse does not hold.[16]

As before, this tells us two things. Firstly, that $Sat(ZFC)$ and $Con(ZFC)$ may be relevant to respectively $Sat(STT)$ and $Con(STT)$, and secondly, that they need not be. Regarding the former point: if a type-theoretical ZFC-representative ϕ_{ZFC} renders ZFC unsatisfiable (or inconsistent), then there is a type-theoretical statement ϕ_{STT} that renders STT unsatisfiable (or inconsistent). Regarding the second point: the lack of a warrant for $Sat(ZFC)$ and $Con(ZFC)$ need not imply lack of warrant for $Sat(STT)$ and $Con(STT)$ – to wit, even if ZFC were to turn out to be unsatisfiable (or inconsistent), STT could still be satisfiable (or consistent) and in perfectly good standing, with rational claims to warrant being made.

The upshot of our above discussion is that there is a difference in terms of generality between satisfiability and consistency propositions, on the one hand, and the cornerstones attacked by the sceptic about the empirical world on the other. The empirical cornerstones – Cartesian as well as type III – which we encountered in Section 2.3 concern empirical thinking in general, while the satisfiability and consistency propositions for a mathematical theory T do not. Absent a warrant for an empirical cornerstone, there can be no rational claim to warrant for belief in *any* proposition about the empirical world. This is not the case for $Sat(T)$ and $Con(T)$. In particular, as we have seen, for ZFC and arithmetic, a warrant for $Sat(ZFC)$ or $Con(ZFC)$ could be absent without our rational claim to warrant for arithmetical beliefs being undermined.

There is an additional respect in which $Sat(T)$ and $Con(T)$ differ from empirical cornerstones. When facing the sceptic about the empirical world, whether we are right or wrong is an all or nothing matter. If we are wrong about not being envatted brains, our experiences fail, in a very dramatic way, to hook us up to a world which is, by and large, as we take it to be.

[16] The idea is that we can represent types as ranks, i.e. ordinals. (For a definition of the notion of rank, see Appendix A.) So, when we consider extensionality for type i, we consider extensionality for all the sets of the corresponding rank. Likewise for comprehension.

What about statements that seem to conflict with ZFC, e.g., those asserting the existence of a universal set for every type i? In STT, a universal set of type i contains everything at type $i-1$. Following the above directives, in ZFC, the statement that there is a universal set of type i is thus interpreted as follows: consider the rank that corresponds to i and the rank corresponding to $i-1$. At the former rank there is a set that contains every set of the latter rank.

It should be noted that we have to assume that there are $\aleph_0$ individuals of type 1 in order for STT to ensure the existence of an infinite set. If we want to interpret the theory resulting from the addition of this assumption to STT in ZFC, we cannot use the initial ω-segment of the cumulative hierarchy, but need to go up to rank ω (i.e. start at V_ω) so as to ensure that there are enough objects to make the interpretation work.

Indeed, they do quite the opposite: they misrepresent reality. If we are wrong about empirical cornerstones, the world is nothing like we take it to be. Matters appear to be different for satisfiability and consistency propositions. Though these kinds of proposition indicate, in each their respective way, that the relevant theory is correct, their failure may be somewhat local compared to the case of scepticism about the empirical world. An unsatisfiable set of axioms A need not get everything wrong. Like experiences will misrepresent reality in scenarios where our belief that we are not envatted brains is wrong, reasoning in a theory with an unsatisfiable set of axioms will also misrepresent mathematical reality. There is no structure on which every axiom in A is satisfied, and so, A fails to delineate a subject-matter. However, while this is so, there might be a proper subset $A* \subset A$ which *is* satisfiable. That is, there might be a set of axioms $A*$ which is quite similar to A and is satisfiable on some structure and succeeds in delineating a subject-matter. This is the sense in which an unsatisfiable set of axioms A need not get everything wrong.

Let us give an example. Let $PA*$ be the set of axioms that results from adding to the axioms of PA the statement $(\exists x)(x \neq 0 \wedge \neg(\exists y)x = s(y))$. On the intended reading, the added statement says that there is a number which is not 0 and not identical to the successor of any number. $PA*$ is not satisfiable since $(\forall x)(x \neq 0 \rightarrow (\exists y)x = s(y))$ – a consequence of PA – and $(\exists x)(x \neq 0 \wedge \neg(\exists y)x = s(y))$ clash. Thus, there is no structure which satisfies all the axioms of $PA*$, whence $PA*$ fails to delineate a subject-matter. Nonetheless, there is a set of axioms quite like $PA*$ which is satisfiable. One such set is the PA axioms. The PA axioms are satisfied on the natural numbers 0, 1, 2, ...

Above it has been argued that $Sat(T)$ and $Con(T)$, the suggested mathematical cornerstones, differ from empirical cornerstones in two respects. First, they are less general than empirical cornerstones. Second, while the falsity of an empirical cornerstone would imply that our experiences generally misrepresent, the unsatisfiability (or inconsistency) of a theory T need not imply that it gets nothing right. This is not meant to suggest that there could not be mathematical cornerstones which stand to mathematical thinking as the empirical cornerstones attacked in standard sceptical arguments stand to empirical thinking; and also, it is not meant to suggest that there are no scenarios in which we are wrong about not being brains in vats,

but where our experiences do not generally misrepresent. The purpose of this section is just to point out two differences between $Sat(T)$ and $Con(T)$ – regarded as cornerstones – and the cornerstones usually attacked in sceptical arguments regarding the empirical world.[17]

2.8 Regress scepticism

In the previous section, we offered support for the view that, for a given mathematical theory T, $Sat(T)$ and $Con(T)$ are cornerstones of T-theorizing. This gives us a good reason to be interested in $Sat(T)$ and $Con(T)$ from an epistemological point of view. For their status as cornerstones makes them crucial to a certain kind of higher-order cognitive achievement with respect to T, *viz.* that of rationally claiming warrant for our T-beliefs.

Though we have supported the status of $Sat(T)$ and $Con(T)$ as cornerstones, the question how they are warranted – if indeed they are so – remains. In this section, we will rehearse a familiar regress argument meant to lead to a disturbing trilemma. We give it in a general formulation and then proceed to run it against respectively satisfiability and consistency propositions.

The general regress argument: there is an old, familiar argument which appears to pose a disturbing trilemma for epistemologists. The master thought behind the argument is that, whenever an epistemic subject is warranted in believing a proposition, then this warrant consists, in part, by the subject being warranted in believing certain other propositions. It is not essential to the regress argument that there be more than one other proposition. One will suffice. If taken on board, this commits us to is the view that warrant always involves *additional support* – or evidence – for the relevant proposition. This commitment is the key to the regress argument.

The argument runs like this: consider a subject S with a warranted belief that P_1. By the master thought, S's warrant for believing P_1 consists partly in S's being warranted in believing

[17] Regarding the former point, a candidate for a cornerstone of a more general kind is the proposition that there is no mathematical reality. More on this when we discuss mathematical I-II-III scepticism. Regarding the latter, there seems to be a way to modify the usual sceptical scenarios in such a way that our experiences do not generally misrepresent. We indicate how to do so for the evil demon argument. Suppose that rather than an all-powerful, evil demon we have an all-powerful, not entirely evil demon. The demon is not entirely evil, because she alternates between deceiving us and letting us experience the world as it is (and makes sure to manipulate our experiences and memory in such a way that the switches back and forth go by unnoticed by us). In that case our experiences do not misrepresent in general. Sometimes they represent the world as it is, other times not.

at least one other proposition – let it be P_2. Since S has a warrant for P_2, the master thought kicks in again: S's warrant for P_2 consists, in part, in S's being warranted in believing some other proposition P_3. And so on. Therefore, the sceptic maintains, there is no such thing as P_1's being warranted.

Much is at stake here. The reasoning is supposed to be entirely general, and, as such, should be applicable to a purported warrant for any particular proposition. Suppose, for example, that I warrantedly believe that there is a tree in front of me. According to the master thought, my warrant for so believing consists, in part, of me being warranted in believing at least one other proposition – that my eyes are functioning properly, say. However, applying the master thought again, this warrant requires warrant for yet another proposition – let us just suppose that the eye doctor recently wrote a report in which she declared my eye sight to be working properly. But warrant for this proposition needs to be supported by a warrant for another proposition – and so it goes without end. Therefore, the sceptic concludes, there is no such thing as me being warranted in believing that there is a tree in front of me.

We had better find a way of responding to the regress argument, lest we are willing to grant the sceptic her conclusion. However, reflection on the various responses available appears to leave us with a disturbing trilemma – with the choice of having to embrace one of three unattractive options[18]:

(i) Accept that there is an infinite regress. Warrant for a proposition P_1 requires a warrant for another proposition P_2, which in turn requires warrant for another proposition P_3, and so forth.

(ii) Stop the regress at some specific proposition P_n.

(iii) Allow circularity, i.e. allow loops so warrant for a proposition P_n can be part of what warrants another proposition P_m, while warrant for P_m is also part of what warrants P_n, perhaps by there being a finite series of warranted propositions $P_k, P_{k+1}, \ldots, P_{k+l}$ such that part of what warrants P_k is a warrant for P_m, part of what warrants any proposition

[18] The trilemma is often referred to as 'Agrippa's trilemma' and is often rehearsed in the epistemological literature. See, e.g., Williams [142], Section 2.4.

in the series is a warrant for the preceding proposition, and part of what warrants P_n is a warrant for P_{k+l}.

Each option appears unattractive. If we go for the first option, we have to take on an non-terminating chain of justificatory tasks to acquire a warrant for a belief in P_1. However, epistemic subjects with finite capacities like ours can never complete infinitely many such tasks, and so, we can never acquire a warrant for a belief in P_1. The second option is unattractive, if unqualified. In the absence of a rationale for stopping the regress at a particular proposition P_n, the decision to do so will be arbitrary and *ad hoc*. In other words, we seem to have ended up with some kind of dogmatism. The third option is worrisome because of considerations on vicious epistemic circularity. If there is a loop, or circle, in the chain of warrants, the good standing of some proposition P_n will rest on some proposition P_m whose good standing in turn rests on that of P_n. This is viciously circular from an epistemic perspective, or at least so the argument goes. An example of a (small) circle would be this: suppose that I believe that my eye sight is functioning properly and that my belief (in that proposition) is warranted in part by my warrant for believing that the eye doctor recently wrote a report in which she declared my eye sight to be working properly. However, part of the warrant for this belief is the warrant for my belief that my eye sight is functioning properly. But now we have a circle: the warrant for my belief that my eye sight is functioning properly rests on my warrant for believing the eye doctor recently wrote a report in which she declared my eye sight to be working properly, but, on the other hand, my warrant for this latter belief rests on a warrant for the former.

What should we say in response to the trilemma urged on us by the above line of thought? The first option is the strategy that has received least attention in the literature. It should not be assumed too swiftly that the idea that warrant for P_1 might involve an infinite regress is incoherent, but I will not discuss the view here.[19] Much work has been done to explore the ramifications of going for the second or third option. Foundationalists block the regress. The details of foundationalist proposals vary, but the key thought is that there is a certain class of beliefs for which there is immediate warrant.[20] Coherentists allow for circles – which, typically,

[19] The most elaborate development of the view is due to Peter Klein. A recent exchange concerning the 'infinitist' option can be found in Klein [80] and Ginet [58].

[20] Traditionally, the idea of immediate warrant is spelled out in such a way that a belief which is thus warranted

are much more elaborate than the mock example given above. Roughly, the coherentist idea is that a belief, if warranted, is so by being a member of a comprehensive system of beliefs that stand in various justificatory relations and offer mutual support for each other.[21]

Recall that the master thought behind the regress argument was that warrant is always evidential. It always involves support from another proposition. In the next chapter, we will deny the sceptical master thought and investigate the proposal that some propositions can be warranted non-evidentially. There is only one way of doing this, *viz.* by going for the second – i.e. foundationalist – strategy. On both the first and the third – i.e. infinitist and coherentist – option the sceptical master thought is, to some extent, taken on board. Warrant for any proposition requires warrant for at least one other proposition. According to the infinitist, there is an infinite series of propositions, while the coherentist accommodates the requirement of additional support by allowing propositions to support each other mutually.

Let it be said, I shall not spend any time arguing against infinitist or coherentist responses to the regress argument. When I say that I will reject the sceptical master thought that all warrant is evidential, it is thus not because I have elaborate philosophical arguments to support my rejection and for preferring the accompanying view over infinitism or coherentism. Still, I hope that a detailed development of a view that rejects the sceptical master thought is sufficiently interesting by itself to merit our attention.

2.9 Mathematical regress scepticism

Though we have supported the status of $Sat(T)$ and $Con(T)$ as cornerstones, the question how they are warranted – if indeed they are so – remains. In the next four sections, we will develop two regress arguments suggesting that they are not. For this purpose, we will rely on the sceptical master thought – that all warrant is evidential – together with some technical

is so in virtue of the proposition believed having certain epistemic properties – e.g. being self-evident, infallible, incorrigible, indubitable, or in some sense basic. These properties are supposed to remove the need for warrant being supplied (at least in part) by *something else*. A classic example of the foundationalist view is given by Descartes in his *Meditations*. For more recent incarnations of foundationalism, see Alston [2], Chapters 1–3; Audi [4], Chapters 3, 4, 10, 12; Lewis [89], Pollock and Cruz [108], Chapter 2.

[21] Some of the most significant contributions to the literature on coherentism are due to BonJour. He has developed and defended the view, as well as attacked it. See BonJour [11] and [12]. See also Lehrer [87] and [88], and Thagard [140].

machinery – Gödel's second incompleteness theorem, to be specific.

As far as I know, no thorough attempt has been made in the literature to bring these results to work in the hands of a regress sceptic.[22] We will undertake such an attempt here and supplement it by a discussion of what the argument can teach us about the nature of warrant for satisfiability and consistency propositions (unless we want to grant the sceptic that we have no such warrant). Given the wide interest and extensive literature on the regress argument in mainstream epistemology, it is somewhat striking how little interest epistemologically oriented philosophers of mathematics have paid to the topic.

2.10 Gödel's incompleteness theorem and relative consistency

All the regress argument needs to get a grip is the master thought that all warrant is evidential. In mathematics, this thought has at least some initial pull. For isn't mathematics all about *proving* things? With the paradigmatic kind of mathematical evidence being proof, the specific case of the sceptical master thought that might seem at least initially plausible amounts to this: warrant for, i.e. proof of, any given proposition requires warrant for – or, again, proof of – another proposition.

Consider Peano arithmetic. (The axioms were stated in Chapter 1.) No paradox has ever emerged to show this theory inconsistent, and historically, there have never really been any severe doubts concerning its consistency. Yet can an affirmative answer to the consistency question for PA be established conclusively? The answer is 'no,' if the measure for conclusiveness is a consistency proof in the system itself, one of its subsystems, or another system whose consistency strength is no greater than that of PA.[23]

This is a lesson of Gödel's second incompleteness theorem:

[22] The most elaborate treatment in the existing literature is, to my knowledge, found in McNaughton [94] and [95]. Certain passages of Lakatos [86] also gesture in the direction of the mathematical sceptical challenges I will formulate in this chapter.

[23] A theory T_1 is said to be of greater consistency strength than another theory T_2 just in case the consistency of T_1 implies that of T_2, but not vice versa. T_1 and T_2 have the same consistency strength just in case the consistency of the one implies the other, and vice versa.

Theorem 1 (Gödel's second incompleteness theorem) *If T is (i) consistent, (ii) recursively enumerable, and (iii) powerful enough to express elementary arithmetic, then there is a statement Con(T) in the language of T stating the consistency of T which is not provable in T.*

PA, whether taken in its first- or second-order formulation satisfies (ii) and (iii). First-order PA is recursively enumerable, as is the set of the axioms of second-order PA and its deductive consequences. Both can express elementary arithmetic. Consequently, on the widely granted assumption that PA is consistent, Gödel's second incompleteness theorem applies. Therefore, $Con(PA)$ cannot be proved in PA itself, let alone in any of its subsystems or another system of consistency strength no greater than that of PA.[24]

Gödel's theorem is a limitative result. It tells us that there are certain things we can never achieve. Let us, however, for a moment imagine the impossible – that we could indeed prove PA consistent in PA, one of its subsystems, or another system of consistency strength no greater than that of PA itself. Even then, it would be reasonable to ask what this proof would be good for? A tempting response is this: a consistency proof has an epistemological pay off. A proof gives us a good reason to believe the conclusion, and hence, if a system is proved consistent, we have a warrant to believe it consistent. Bearing in mind what was said earlier about inferential warrant – of which warrant by proof is a sub-case – there is an immediate rejoinder to this thought: warrant does not magically appear when we reach the last step of a proof. Proof – indeed, inference in general – can only yield warrant for a belief in the statement following the last step of the reasoning through transmission of warrant from the premises. That is, in order for inference to serve as the vehicle of warrant acquisition, the premises have to be in good epistemic standing.

Hence, if the tempting response is to be plausible, it must be shown that PA, some subsys-

[24] The assumption that PA proves $Con(PA)$ obviously contradicts the incompleteness theorem. In the subsystem case, suppose that PA_{SUB} is a subsystem of PA. Then, any statement φ provable from PA_{SUB} is provable from PA too. Hence, if PA_{SUB} proves $Con(PA)$, so does PA itself. Contradiction. A system $T*$ of consistency strength no greater than PA will either be of less consistency strength than PA or of the same consistency strength as PA. If it is less, then $Con(T*)$ does not imply $Con(PA)$, and so, $T*$ cannot prove $Con(PA)$. If $T*$ is of the same consistency strength as PA, then $Con(T*)$ implies $Con(PA)$ and vice versa. Thus, if $T*$ proves $Con(PA)$, $T*$ proves $Con(T^*)$. This cannot be the case if $T*$ is sufficiently strong, i.e. strong enough to represent elementary arithmetic.

tem, or a system of consistency strength no greater than it is in good epistemic standing. If, for the first case, this means that we have a warranted belief in PA, then a consistency proof would not add anything. Consider, then, the case where a subsystem of PA is used. This idea matches up with one of the main goals of the original Hilbert programme, *viz.* to provide consistency proofs by finitary means. For PA the goal was thus to prove the theory consistent in one of its finitary subsystems. Now, if this could be accomplished, it would seem natural to suppose that there would be a significant epistemological pay off, and similarly, for a system whose consistency strength does not exceed that of PA itself. It is not easy to spell out what such an epistemological pay off would consist in exactly. However, a system of any of the two kinds mentioned might be thought more 'epistemically tractable' than PA, somehow epistemically safer or bound to give away fewer hostages. For a finitary theory, as opposed to an infinitary one, this might be because such a theory is less computationally complex. Fortunately, for our present purposes, we do not have to work out these details. Because the time has come to jump back into reality, reminding ourselves that, by Gödel's theorem, the consistency of PA can never be proved in PA itself, let alone any finitary subsystem thereof, or a system whose consistency strength is no greater than that of PA. The point to be kept in mind from our impossible fiction is this: the conclusion of an inference can only enjoy a positive epistemic standing provided the premises do.[25]

Gödel's theorem applies to any theory which is (i) consistent, (ii) recursively axiomatizable, and (iii) strong enough to represent elementary arithmetic. The theorem thus applies to many theories considered to be of great mathematical interest – arithmetic and set theory just to mention two. As said, Gödel's theorem is a limitative result. It tells us that there are certain things that we can never hope to achieve – in particular, that $Con(T)$ cannot be warranted through a proof in T itself, a finitary subsystem thereof, or some theory of consistency strength no greater than that of T.

In light of this, a common approach to the consistency question for a given mathematical theory T_1 is to try to prove T_1 consistent relative to another theory T_2, i.e. to show that

[25] Furthermore, the inference needs to be of the sort that will transmit warrant. See Section 2.2.

If $Con(T_2)$, then $Con(T_1)$

Two remarks. First, typically, T_2 is a supersystem of T_1 or a much stronger system in which the axioms of T_1 can be interpreted and shown to be theorems. By a 'supersystem of T_1' is meant a theory that includes all the axioms of T_1 and some extra ones. An example of a supersystem of PA is PA + 'PA is consistent'. An example of a supersystem of Z is ZF. Examples of systems stronger than PA in which interpretations of the arithmetical axioms can be established as theorems are ZF and ZFC. Second, we remind ourselves that consistency and satisfiability are extensionally equivalent for first-order theories. This fact is exploited in many relative consistency proofs, where what is established is the model-theoretic result that, if $\mathcal{T}_2$ is satisfiable/has a model, then $\mathcal{T}_1$ is satisfiable/has a model. By completeness (and soundness), if $\mathcal{T}_2$ is consistent, then $\mathcal{T}_1$ is consistent.[26]

It is not difficult to provide relative consistency proofs. Here is a straightforward template for a theory T_1: let T_2 be T_1 + 'T_1 is consistent.' (A consistency proof of PA relative to PA + 'PA is consistent' is an instance of this template.) Of course, what we must remember here is that the positive epistemic standing conferred upon a theory by a relative consistency proof, if any, can be no stronger than the epistemic standing of the theory relative to which it is shown consistent. This must also be borne in mind when we are dealing with less trivial relative consistency proofs involving PA and ZF, say.[27]

2.11 The regress: relative consistency

We have now given enough details concerning Gödel's second incompleteness theorem and its relation to consistency to run the regress argument.[28]

Recall that the limitations imposed by Gödel's second incompleteness theorem have made

[26] Many relative consistency proofs in set theory are obtained in this manner. That is, by proving that if some set theory has a model, then so does some other set theory, and then exploiting completeness (and soundness) to get the relative consistency proof. See, e.g., Kunen [84]. It should be noted, however, that the reliance on the meta-theoretic properties of the system is often implicit.

[27] There is another kind of consistency proof (for theories T_1 and T_2) which establishes $Con(T_1)$ as a *theorem* of T_2. In such a proof T_2 is used as a meta-theory in which the a sentence expressing the consistency of T_1 can be expressed and proved. This differs from relative consistency proofs of the kind mentioned in the main texts where T_1 and T_2 have the same meta-theory.

[28] See also McNaughton [94] and [95].

relative consistency proofs the standard approach to the consistency question for mathematical theories. That is, supposing that T_1 is sufficiently powerful to express elementary arithmetic, the standard approach is to try to show that, if $Con(T_2)$, then $Con(T_1)$ – where T_2 is some other mathematical theory. This, however, invites a regress. What we are after is a warrant for a belief in $Con(T_1)$. One immediate observation is that, in case we appeal to a relative consistency proof, the warrant for $Con(T_1)$ is held hostage to there being a warrant for $Con(T_2)$. However, a proof of T_1's being consistent relative to T_2 does nothing to establish *that*. We might bring yet another theory T_3 into play and show that if $Con(T_3)$, then $Con(T_2)$. It is not clear that this improves the situation, though, because as before, the consistency of one theory is held hostage to that of the other. We might bring into play a theory T_4, but the pattern repeats itself.

In the above argument the talk of warrant for a consistency proposition for a given theory being 'held hostage' to a warrant for that of another signals the application of the sceptical master thought. That is, the thesis that warrant for any given proposition requires additional support, in the form of warrant for some other proposition.

We seem to be stuck with a number of unattractive options. Embracing an infinite series of relative consistency proofs is not a live option, at least not if we are interested in there being some epistemological gain with respect to the theory T_1 whose consistency we were initially interested in. One concern is that we cannot undertake an infinite number of projects. This has to do with our cognitive limitations. However, even if we supposed that we could take on an infinite number of tasks, there would be another concern. As said, given a proof that T_1 is consistent relative to T_2, the consistency of T_1 is held hostage to that of T_2. It was also observed that T_2 has to be of a greater consistency strength than T_1. So, at least *prima facie*, there is a sense in which T_2 is as risky as T_1 from an epistemological point of view, if not more.[29] Circularity is not a live option either. By Gödel's theorem, there can be no loops on a series of relative consistency proofs (assuming that the systems are consistent).

[29] This is not to say that there are no cases where where nothing can be gained from a relative consistency proof in which a theory T_1 is proved consistent relative to a strictly stronger or equally strong theory T_2. Arguably, something would be gained epistemologically if a new theory were to be proved consistent relative to ZF. I am not quite sure how to spell out what the epistemological gain is in purely deductive terms. However, assuming soundness and completeness, let us try in more model-theoretic-sounding terms. The basic point must be that, since the model theory of ZF is fairly well-investigated, such a proof offers insights into at least one kind of structure that satisfies the axioms of the new theory.

Given that embracing an infinite regress and allowing circularity are not live options, we are left with the option of blocking the regress at some point, unless we want to grant the mathematical regress sceptic her conclusion. In effect, blocking the regress at some point amounts to there being some consistency proposition which is not warranted by proof, and so, amounts to a rejection of the sceptical master thought, as applied to consistency propositions. Above it was stressed that the strategy of blocking the regress at some proposition is undesirable if unqualified. This point is still valid. If we want to block the regress at some specific proposition, something should be said to motivate this. After all, the specific stopping point is epistemologically significant. It marks a line between a range of propositions that are evidentially warranted and one that is not, and there should be features of the special – or basic – proposition by reference to which this difference can be accounted for. This issue will be one of the things that will draw our attention in Chapter 3.

One might wonder why we should worry about consistency propositions at all? Why are they epistemologically interesting? The answer to these questions is that they are cornerstones of reasoning in mathematical theories – a thesis which was supported earlier in the chapter. It follows from the characterization of a cornerstone (and the status of consistency propositions as cornerstones) that a certain kind of higher-order cognitive achievement is impossible in a theory T without a warrant for the consistency proposition of T. Absent such a warrant no warrant can rationally be claimed for belief in any T-proposition.

2.12 The regress: relative satisfiability

If we are in a first-order setting, we can invoke completeness to get the following result:

Theorem 2 *If T is (i) satisfiable, (ii) recursively axiomatizable, and (iii) powerful enough to express elementary arithmetic, then there is a statement Sat(T) in the language of T stating the satisfiability of T which does not follow semantically from T.*

This is a limitative result. It shows us that there are certain things we can never achieve. More specifically, provided that T_1 is strong enough to represent elementary arithmetic, we can never establish the satisfiability of T_1 in T_1 itself, let alone one of its subsystems or a system of satisfiability strength no greater than that of T_1 itself.[30] The best we can do is to establish a relative satisfiability result:

$$\text{If } Sat(T_2), \text{ then } Sat(T_1)$$

where T_1 is the theory whose satisfiability we were initially interested in, and T_2 a theory of satisfiability strength greater than that of T_1.

We can now run a regress argument targeted at relative satisfiability propositions, taking on board the sceptical master thought that warrant for a proposition always involves warrant for some other proposition.

Let T_1 be a theory that satisfies conditions (i)-(iii) of the theorem stated at the beginning of the section and whose satisfiability we are initially interested in. Furthermore, suppose any warrant for the satisfiability of a mathematical theory is given by proof. In that case the best we can do is to show T_1 satisfiable relative to another theory T_2 of satisfiability strength greater than that of T_1. Warrant for $Sat(T_1)$ thus involves appeal to some other proposition, *viz.* $Sat(T_2)$. Now, appeal to a relative satisfiability proof does not improve the epistemic standing of $Sat(T_1)$. The warrant for $Sat(T_1)$ is held hostage to a warrant for $Sat(T_2)$, and the relative satisfiability proof does nothing to establish this – by assumption, we need a proof. Suppose that we bring in another theory T_3 and prove T_2 satisfiable relative to it. Does this improve the situation? No, because, as before, the warrant for $Sat(T_2)$ is held hostage to a warrant for another proposition, $Sat(T_3)$. If we try to appeal to yet another theory T_4, the pattern repeats itself.

The unattractive options remain the same. The three candidates are infinite regress, blocking the regress at some proposition, or allowing circles – unless we want to opt for an even more unattractive option by granting defeat to the sceptic and adopting the view that there can be no such thing as warrant for the satisfiability of a mathematical theory, provided that it is sufficiently strong.

[30] T_1 is of greater satisfiability strength than T_2 just in case $Sat(T_1)$ implies $Sat(T_2)$, but not vice versa. T_1 and T_2 are said to be of the same satisfiability strength just in case $Sat(T_1)$ implies $Sat(T_2)$ and vice versa.

Somebody might wonder why we should care about satisfiability propositions at all, and so, why we should care about the sceptical regress argument. The answer is that $Sat(T)$ is a cornerstone for T, as argued earlier in the chapter. By the characterization of a cornerstone, this means that there can be no rational claim to warrant for (a belief in) any proposition arrived at through T-theorizing absent a warrant for $Sat(T)$. This, it is recalled, is basically because without such a warrant there is no warrant for thinking that T concerns anything at all.[31]

2.13 I-II-III scepticism considered as O-scepticism

We have developed two mathematical regress arguments, targeted at respectively consistency and satisfiability propositions. We now move on to I-II-III scepticism. In this section, we will add to the characterization of I-II-III scepticism given earlier by suggesting that it should be understood as involving an attack on the objectivity of the relevant region of thought. The next section goes on to present a mathematical I-II-III argument.

On the assumption that the propositions attacked by the Cartesian sceptic and the I-II-III sceptic are cornerstones, both kinds of sceptic challenge our rational claim to warrant for (belief in any) propositions of the relevant region of thought. However, reflection on the details of I-II-III arguments invites the further suggestion that they endanger a feature ordinarily taken to be characteristic of these regions and our interaction with them, *viz.* their objectivity.

Consider Moore's proof. Here the target conclusion is not just that there is a world, but that there is an *external* world. Solipsists, idealists, and realists all agree on the former, but only realists buy into the latter. In other words, the listed parties disagree about the fundamental character of the world, and, additionally, the nature of our interaction with it. The realist holds that there is a constitutively mind-independent world, and that our experience of it is objective. Solipsists and idealists disagree: granted, there is a world, but it is nothing like we ordinarily

[31] In running the regress, we invoked the completeness of first-order logic. If T is a second-order theory and the semantics is taken to be Henkin semantics, we can still run a regress in terms of relative satisfiability by appealing to the theorem stated at the beginning of the section. This is because second-order logic with Henkin semantics has the same meta-theoretic properties as first-order logic. In particular, it is complete. (See Shapiro [126] for details.) However, matters get more complicated if the semantics is taken to be standard semantics – not just technically, but also philosophically. As indicated in Section 1.3., the philosophical status of second-order logic with standard semantics has been subject to considerable discussion.

take it to be. It is constitutively mind-dependent, and our experience of it is to be understood accordingly – meaning that it is not objective, in the sense of being of, or about, something which is constitutively independent of us.

Thus, it does not fully capture what is at stake with I-II-III scepticism to say that the I-II-III sceptic is out to challenge our rational claim to warrant for belief in propositions of the relevant region of thought (i.e. the ones which the type III proposition under attack is deemed to be a cornerstone of). What is at stake with the type III proposition is not just the existence of something, but the existence of something *with a certain feature*.[32] For the purposes of making this clear, the following piece of terminology will be useful:

> **O-cornerstone:** A proposition P is an *O-cornerstone* for a given region of thought, R, just in case, if we had no warrant for it, we could not rationally claim warrant for making any objective R-judgement.

For the sake of generality the characterization of an O-cornerstone has been phrased in terms of objective *judgements*. I take it that anything we are willing to consider an objective region of thought, or a domain of discourse, is also something whose subject-matter we take ourselves to make judgements about.

As seen in Section 2.3, type III propositions are cornerstones relative to the type II propositions of the relevant region of thought. If, in addition, type III propositions are O-cornerstones, the I-II-III sceptic attacks not just our rational claim to warrant for belief in (type II) R-propositions, but also our right to rationally claim warrant for making any *objective* judgements on R-matters. Without a warrant for thinking that there is an external world, I cannot rationally claim to have a warrant for objectively judging that there is, say, a chair in front of me. Similarly, absent a warrant for thinking that there is a substantive past, I cannot rationally claim to have warrant for judging objectively that I had two cups of coffee yesterday.

However, are type III propositions O-cornerstones? As indicated, the intended notion of objectivity is such that a judgement is objective just in case it purports to be about something – or to concern some subject-matter – which is constitutively mind-independent. For a region

[32] The suggestions to be made are implicit in what Wright goes on to say in his response to I-II-III scepticism. However, as a matter of exposition, it is worth making them explicit in setting out the sceptical challenge.

of thought R, the objectivity of R-judgements, if any, thus flows from the character of the subject-matter of R. If it is constitutively mind-independent, or independent of our making, R-judgements are objective; otherwise not. The propositions that there is an external world and that there is a substantive past are meant to say precisely that there is a constitutively mind-independent subject-matter for respectively our thinking about the empirical world and the past. Given this understanding of objectivity, type III propositions are O-cornerstones in the sense defined. Absent a warrant for the externality of the world and substantiality of the past, we cannot rationally claim to make any objective judgements about the empirical world and the past respectively. For without warrant for the type III proposition we have no warrant for thinking that we make judgements about a constitutively mind-independent subject-matter, and so, no warrant for thinking that we are making an objective judgement.[33]

2.14 Mathematical I-II-III scepticism

What should we make of what has been said so far about I-II-III arguments in the context of mathematics? One thing to ask ourselves is if we can formulate a mathematical I-II-III argument, and, if so, whether we can also take the mathematical I-II-III sceptic to be targeting the objectivity of mathematical theorizing. As shall become clear, the mathematical I-II-III argument is formulated within a broadly realist setting.

In trying to formulate a mathematical I-II-III argument it is helpful to think in analogy with Moore's proof. Moore's proof was formulated in terms of experience. Earlier we took it that we judge on R-matters for any region of thought R, and so, that judgement is a kind of interaction associated with any region of thought. Bearing this in mind, let us try to formulate a mathematical I-II-III argument in terms of (arithmetical) judgement:

[33] I have explicitly linked objectivity of judgement to mind-independence, or more specifically, mind-independent *subject-matter*. I am not, as a matter of principle, opposed to the idea that objectivity can be accounted for without appeal to a mind-independent subject-matter. (Here Yablo [156] offers some interesting considerations.) However, for our present purposes – a discussion I-II-III arguments – it does seem to me that we need the notion of objectivity adopted here. As shall transpire in the next section, constitutive mind-independence is what gives I-II-III scepticism a grip.

I Arithmetical judgements are in all respects as if there are numbers, abstract and not located in space-time.

So, II There are numbers, abstract and not located in space-time.

If there are numbers – abstract and not located in space-time – then there is a realm of abstract, non-spatio-temporal arithmetical entities – numbers.

So, III There is a realm of abstract, non-spatio-temporal arithmetical entities – numbers.

A few words on arithmetical judgements being *in all respects as if* there are numbers. Again it will be useful to think in analogy with the external world case. Experience is in all respects as if there is an external world, because experience impresses itself upon us as being of something independent of our making, something 'out there'. In a similar fashion, according to at least one of the major views on the scene (i.e. object-based realism), arithmetical judgements are in all respects as if they concern entities that are not of our making. That is, objects which are – in a metaphorical sense – 'out there'; objects that are abstract and non-spatio-temporally located entities. The phenomenology of arithmetical judgement is one of *aboutness*. Arithmetical problems are problems about numbers – conceived as mind-independent entities – and not just about what can be derived from a given set of axioms. In particular, *open* problems like Goldbach's Conjecture and the Twin Prime Hypothesis concern such mind-independent entities and have definite answers, determined by their properties.[34]

[34] The realist might supplement this by considerations of the following kind:

When I judge that $2 + 3 = 5$, I do not appear to be making a judgement about anything spatio-temporally located. Numbers are not the kind of thing I can bump into when walking down the street. (See, e.g., McGee [92].) I might meet *two* of my friends, rent *three* DVD's, or buy *five* chocolate chip cookies. This does not mean, however, that I have been spatio-temporally located in such a way that, were I to have taken a few steps to the side, I would have bumped into a number. Or, that had the driver of one of the cars passing by lost control of the vehicle and crashed into me, he would at the same time have crashed into one of the natural numbers, perhaps bringing its existence to an end and leaving, *per impossible*, a gap in the sequence of natural numbers. Numbers are necessary existents, and any view on their nature has to respect this. I suppose that what has just been said is compatible with numbers being spatio-temporally located, on the assumption that there are spatio-temporally located things that exist necessarily. Someone who believes in God (with the properties usually ascribed to Him) and understands 'God is everywhere' literally, might take God to be an example of a spatio-temporally located, necessary existent.

A note on 'abstract and non-spatio-temporally located'. Perhaps some are inclined to think that ascribing the property of being abstract to an object excludes the possibility that it is spatio-temporally located, and so, that

Having commented on 'being in all respects as if', let us now ask why the above argument should fail to transmit warrant? Recall that the idea behind I-II-III arguments is that the type III proposition has to be warranted evidentially, and that the I-II-III argument gives the structure of the best attempts to support it. In the case of the external world, since the type III proposition is empirical, the evidence is taken to be obtained through experience, i.e. the mode of interaction that puts us into contact with the external world. Similarly, for the arithmetical case, we assumed that the the mode of interaction that puts us into contact with the realm of numbers is judgement, and so, the thought goes, evidential warrant for the type III proposition has to be acquired on the basis of judgement.

The support provided by I for II is defeasible. Our arithmetical judgements being in all respects as if there are numbers – conceived as abstract, non-spatio-temporally located entities – does not entail that there are any such entities. The warrantability of a transition from our arithmetical judgements being in all respects as if P to P itself – i.e. the step from I to II – is held hostage to there being a warrant for thinking that there are arithmetical states of affairs to serve as truth-conferrers for the arithmetical propositions which we judge with respect to. However, these truth-conferrers are provided by the realm of numbers. Hence, the warrantability of the transition from I to II is held hostage to there being a warrant for thinking that there is a realm of numbers, i.e. to there being a warrant for the type III proposition.

The upshot of this is that the type II proposition cannot be warranted antecedently and independently of the type III proposition. By the transmission principle (FAILURE) – stated in Section 2.2 – warrant for II cannot transmit to III, and so, according to the sceptic, attempts to support III will all be subject to a principled failure.

It is worth saying a bit about the notion of 'cognitive distance' and how it ties in with the

it is altogether redundant to note of an object that it is not spatio-temporally located once it has been labeled 'abstract'. I do not think that this is so, at least not if we adopt an understanding of abstractness that can be found in Dummett [35], pp. 503–505. According to Dummett an object is *purely abstract* just in case its existence does not depend on the existence of anything concrete. An impurely abstract object is an abstract objects whose existence does depend on the existence of something concrete. An example is the set of my shoes. It is an abstract object, but nevertheless its existence depends on something concrete with spatio-temporal location, namely its members – my shoes. This falls short of saying that the set of my shoes itself is spatio-temporally located. If a set is taken to be spatio-temporally located if its members are, then the set of my shoes is located in space-time. However, this conditional might be denied by some. Even so, the existence of impure abstract entities will be so intimately tied to the existence of concrete things located in space-time that it is not altogether redundant to speak of abstract objects being non-spatio-temporally located.

I-II-III argument. What gives the sceptical argument bite is the 'cognitive distance' between the type I proposition and the type II proposition. Sense experience represents empirical matters as being in a certain way – in Moore's proof, that I have two hands. Arithmetical judgement represents arithmetical matters as being in a certain way – in the example, that there are numbers. However, the representational content of I – sense experience and arithmetical judgement – is such that it does not *by itself* warrant a transition to II. The transition is held hostage to there being a collateral warrant for the type III proposition, i.e. a warrant for thinking that there are truth-conferrers for the kinds of states of affairs that respectively sense experience and arithmetical judgement are supposed to represent.

Here it is significant that the framework within which the I-II-III argument is cast is broadly realist, understood as a thesis about mind-independence. For it is this feature of the framework that makes the transition from the representational step I to the factual step II defeasible and hostage to warrant for the type III proposition. The realist holds that the states of affairs which respectively our sense experiences and arithmetical judgements supposedly represent are constitutively mind-independent. This means that there is room for *error*. Sense experience and arithmetical judgement may misrepresent the tract of reality that they are supposed to put us into contact with. To warrantedly take it that they represent correctly we need to have a warrant for thinking that there is a reliable connection between our vehicles of representation – here experience and judgement – and the propositions they are taken to support. However, as just seen, to have a warrant for this correctness assumption a collateral warrant for the type III proposition is needed. As also seen, this means that there can be no route to warrant for a type III proposition through transition from I and II. Thus, the transmission failure in the I-II-III argument can, to a large extent, be traced to the 'cognitive distance' that the broadly realist framework drives in between I and II.[35]

The undesirable consequence of the sceptical challenge, if successful, is that we cannot ra-

[35] I have formulated the mathematical I-II-III argument in terms of judgement. Realists of the Gödelian stripe will here invoke some faculty for which 'quasi-perception' might be an appropriate label, and which gives us (quasi-)experience of mathematical entities like sets and numbers. For these realists, we could run the argument in terms of (quasi-)experience, and the argument should be very – if not entirely – similar to the I-II-III argument targeted at the existence of an external world. For a detailed development of a Gödel-inspired view, see Maddy [90]. For Gödel, cf. [63] and [64].

tionally claim warrant for mathematical type II propositions, conceived as describing a realm of abstract, non-spatio-temporally located objects. This follows from the type III proposition's status as a cornerstone relative to type II propositions. In addition, we cannot rationally claim to be making objective arithmetical judgements, conceived as judgements about a constitutively mind-independent subject-matter. This follows from type III propositions counting as O-cornerstones.

Before we move on to the conclusion of this chapter, let me point to a difference between mathematical regress scepticism and I-II-III scepticism. In formulating the regress argument targeted at satisfiability propositions, we spoke of structures, of there being something which the theory is about, and of subject-matter. Admittedly, this might *sound* realist. However, it was stressed that these ways of speaking should not be taken to carry any realist implications. If there were such implications, it might mistakenly be thought that it is part of the set up of the regress scepticism that only realist views are targeted. This is not so. What is needed to run the mathematical regress arguments is the sceptical master thought – that all warrant is evidential – together with a technical result, Gödel's theorem. And taking these on board does not require realism. On the other hand, I-II-III scepticism about some region of thought R specifically targets realism with respect to R. This kind of scepticism does not have much force against other views. For, as seen, it is the cognitive distance between I and II that gives I-II-III scepticism a grip, and to secure this cognitive distance, the truth-conferrers for type II propositions need to be independent of our making. That is, they need to be understood realistically.

2.15 Conclusion

In this chapter, we have introduced and developed mathematical regress scepticism and mathematical I-II-III scepticism. Both kinds of scepticism incorporate the assumption that all warrant is evidential.

The mathematical regress sceptic brought the assumption to work by demanding that satisfiability and consistency propositions be warranted by proof (this being the paradigmatic kind

of evidence in mathematics). Gödel's second incompleteness theorem was applied to generate regresses which, according to the regress sceptic, show that there can be a warrant for neither consistency propositions nor satisfiability propositions of theories of a certain strength. The mathematical I-II-III sceptic brought the assumption to work by demanding that the proposition that there is a realm of mathematical entities, conceived as abstract and non-spatio-temporally located, should be warranted evidentially. She then proceeded to argue that our best attempts to provide such a warrant – typified by a I-II-III argument – will misfire as a matter of principle, due to transmission failure.

If these arguments are sound, both mathematical regress scepticism and mathematical I-II-III scepticism have an immensely undesirable consequence, *viz.* that there can be no rational claim to warrant for a wide range of ordinary mathematical propositions. In addition, since type III propositions are O-cornerstones, I-II-III scepticism jeopardizes rational claim to objective mathematical judgements. However, as noted at the end of the previous section, regress scepticism and I-II-III scepticism differ in terms of what assumptions are made about the framework within which they are formulated. I-II-III scepticism only has force against realist views about the relevant region of thought, while there is no such restriction on regress scepticism.

Chapter 3

Entitlement of cognitive project

In this chapter, we will investigate a line of response to scepticism that rejects the sceptical master thought that all warrant is evidential. However, to have an effective response to respectively regress and I-II-III scepticism, this rejection needs to be supplemented by an appropriate notion of non-evidential warrant. We give an account of two non-evidential notions of warrant – entitlement of cognitive project and entitlement of substance – introduced by Wright to respond to Cartesian scepticism and I-II-III scepticism about the empirical world. The application of these notions as a response to mathematical regress and I-II-III scepticism is spelled out in detail in respectively this chapter and the next.

3.1 Wright's response to scepticism: entitlement

Let us return to the two kinds of scepticism about the empirical world discussed in Chapter 2. Recall that both of them fit a certain two-step template. The first step of this template was an argument to the effect that a certain proposition C we typically accept is a cornerstone for a given range of thought. The second step was an argument to the effect that we have no warrant for C. Given the characterization of the notion of a cornerstone, these two steps combined deliver an unfortunate conclusion, namely that we cannot rationally claim warrant for belief in *any* proposition of the relevant region of thought.

Wright's response to the sceptical challenges starts with the observation that, even if suc-

cessful, what they deliver falls short of what is needed to establish the second step.[1] While it is conceded that the sceptical arguments show that cornerstone warrant cannot be earned in the sense of being evidentially warranted, it is denied that this shows that there can be no such thing as cornerstone warrant. The sceptic implicitly assumes that evidential warrant is the only kind of warrant there is. If this assumption is granted and it is granted that the sceptical arguments undermine our best attempts to acquire this kind of warrant, the sceptical arguments *do* establish that we have no cornerstone warrant. However, Wright rejects the assumption. There can be non-evidential warrant.

The notion of entitlement is an attempt to spell out such a kind of warrant. The idea is this:

> Suppose there were a type of rational warrant which one does not have to *do any specific evidential work* to earn: better, a type of rational warrant whose possession does not require the existence of evidence – in the broadest sense, encompassing both *a priori* and empirical considerations – for the truth of the warranted proposition. Call it *entitlement*. If I am entitled to accept P, then my doing so is beyond rational reproach even though I can point to no cognitive accomplishment in my life, whether empirical or *a priori*, inferential or non-inferential, whose upshot could reasonably be contended to be that I had come to know that P, or had succeeded in getting evidence justifying P. (Wright [154], pp. 174–175)[2]

The notion of entitlement thus invites a response to the sceptic that takes issue with the second step of the sceptical argument. While it is conceded that the sceptical arguments put cornerstones beyond evidential warrant, it is denied that this shows that cornerstones cannot be warranted.

[1] Wright [154], p. 169.

[2] Wright never explicitly mentions the deontological conception of warrant. However, it should be noted that the use of 'beyond rational reproach' might be taken to suggest that his notion of entitlement has a deontological aspect to it. For comparison, consider the following characterization of justification given by Carl Ginet, an adherent of the deontological conception: 'One is *justified* in being confident that p if and only if it is not the case that one ought not to be confident that p ; one could not be justly reproached for being confident that p.' (Ginet [56], p. 28) Ginet talks about justification, but suppose that we recast the definition more generally in terms of warrant and read 'justly' as 'rationally'. Then Wright's idea that an entitlement to P implies that acceptance of P is beyond rational reproach is very much in keeping with the deontological conception of warrant. More on the deontological conception of warrant in Chapter 6.

3.2 Entitlement of cognitive project

We now turn to the details of the entitlement proposal. In this section, Wright's characterization of the notion of entitlement of cognitive project will be given. In the next section, we briefly account for how this notion can be used to respond to the Cartesian sceptic.[3]

A terminological note before we proceed. From now on we will use 'justification' to designate evidential species of warrant and 'entitlement' will be used for non-evidential ones. The term 'warrant' will be used disjunctively for either justification or entitlement. Sometimes I shall allow myself to speak of a subject's 'justificatory status' when discussing whether the subject is warranted in the inclusive sense.

The notion of entitlement of cognitive project is characterized as follows[4]:

> **Entitlement of cognitive project:** A proposition P is an entitlement of a cognitive project if
>
> (i) P is a *presupposition* of the project, i.e. if to doubt P (in advance) – or weaker: being open-minded about P – would rationally commit one to doubting (or being open-minded about) the significance or competence of the project[5];
>
> (ii) we have no sufficient reason to believe that P is untrue; and
>
> (iii) the attempt to justify P would involve further presuppositions in turn of no more secure a prior standing ... and so on without limit; so that someone pursuing the relevant enquiry who accepted that there is nevertheless an onus

[3] Prior to Wright's principal publication on entitlement – Wright [154] – a notion of epistemic entitlement had already been developed in detail by Tyler Burge. There are similarities as well as differences between Burge and Wright's respective notions of entitlement. In this chapter, there will be the occasional reference to Burge's notion, but it goes beyond the scope of our present investigations to give a detailed account of it and its applications. For details, cf. Burge [16], [17], [18], [19], [21]. See also Peacocke [104]. Wright tables the idea of entitlement – without so labeling it – in a number of earlier writings. See Wright [147] and [149]. Wright [152] contains an explicit formulation of the entitlement proposal, but one less detailed than the one to be found in Wright [154]. In Wright [152] and [154], it is made clear that there are similarities with Wittgenstein's considerations on so-called 'hinge-propositions'. See Wittgenstein [144].

[4] Wright [154], pp. 191–192.

[5] The characterization of a presupposition given here differs from Wright's own explicit characterization, which is phrased only in terms of doubt. Doubt is a stronger attitude than open-mindedness, in the sense that to doubt that P is to hold a positive attitude towards its negation, while open-mindedness – as intended here – involves a positive attitude towards neither P nor its negation. In various passages, Wright seems to be implicitly relying on the slightly modified characterization of the notion of a presupposition, which has been provided here. See, e.g., Wright [154], p. 193.

> to justify P would implicitly undertake a commitment to an infinite regress of justificatory projects, each concerned to vindicate the presuppositions of its predecessors.

Cognitive projects are projects whose successful execution can be regarded as a cognitive achievement – perhaps usefully thought of as providing an answer to a question. We have cognitive projects pertaining to the empirical world, mathematics, logic – in general, any subject-matter with respect to which we can undertake an investigation and *learn* things. For instance, at a friend's party, I might engage in the project of figuring out what substance is in the punch bowl. Investigation of the matter – having a taste, say – could lead me to the answer that it is sangria. To take another example, I might wonder what the cardinality is of the union of a denumerable set each of whose members itself has denumerably many members. Doing some cardinal arithmetic tells me that it is $\aleph_0$. Or I might be keen to know whether the law of excluded middle is a theorem of classical logic. Applying various rules of the system, I learn that it is.

Consider clause (i). Suppose, for example, that I want to check the dimensions of my laptop by using a measuring tape. It is a presupposition of this project that my perceptual apparatus is functioning properly. Doubt (open-mindedness) about the proper functioning of my perceptual apparatus would rationally commit me to doubt (open-mindedness) about the project. That P is a presupposition of the cognitive project means that it is an unavoidable commitment of sorts: to doubt (being open-minded about) P would rationally commit one to doubting the very competence of the project. The attitude held towards P must thus be one that excludes doubt (and open-mindedness), and it will be an unavoidable commitment at least in this sense.

Clause (ii) can be regarded as a *default clause:* provided that proposition P is a presupposition of a given cognitive project and satisfies clause (iii), we are entitled to P *unless* there is sufficient reason for thinking it untrue. What is required for entitlement is not the presence of positive evidence, but rather the *absence* of countervailing evidence. This is why entitlement is a non-evidential species of warrant.

The first half of clause (iii) tells us that attempts to justify – i.e. provide an evidential warrant – for an entitlement P will give rise to an infinite regress of justificatory projects which will

inevitably involve presuppositions of no more secure a prior standing. That is, presuppositions that are either of a less or equally secure prior standing. The second half of clause (iii) says that, if the onus to provide an evidential warrant for P is granted, we are implicitly committed to a regress of the mentioned kind. In examples given by Wright, the infinite regress, besides involving presuppositions of no more secure a prior standing than P, involves presuppositions of the same general kind.[6] Consider the cognitive project of figuring out what the dimensions of my laptop are by using a measuring tape and an entitlement (and so, presupposition) of this project — say, that my perceptual apparatus is functioning properly. Now, suppose that I grant that there is an onus for me to provide a justification for this entitlement. How might I proceed? Perhaps I will go to the doctor to have my hearing, eye sight, and so forth checked. The doctor's findings might be communicated to me in a number of ways – verbally or in writing. The point now is that whichever way it is, acquiring the evidence that is supposed to deliver the justification draws in presuppositions of the very same kind as the one I set out to investigate. Whether I am told or read that my perceptual apparatus is functioning properly, I need to take it as a presupposition that my perceptual apparatus is functioning properly on that occasion. But that calls for a new investigation. And so on.

One more thing is worth noting about clause (iii).

With a bit of unpacking, the clause can be taken to imply that no attempt to justify an entitlement P can improve its epistemic standing. (Here we recall that justification is evidential warrant.) It seems plausible to suppose that the following minimizing principle holds for justification: a justification for a proposition cannot be any stronger than the weakest evidence supporting one of its presuppositions.[7] Intuitively, if we think of the infinite regress as a 'justificatory chain', the minimizing principle says that justification for an entitlement P can be no stronger than the weakest link in its justificatory chain. Since any attempt to justify P will involve presuppositions of no more secure a prior standing, this means that attempts to justify P can bring no improvement in its epistemic standing. Here it will be useful to return

[6] Wright [154], p. 189.

[7] Wright [154], p. 191. Wright goes on to deny the principle in its full generality, the exceptions being presuppositions that are warranted non-evidentially. This is compatible with what has been said here since we are talking about justification, i.e. evidential warrant. Indeed, as far as I can tell Wright himself buys into the minimizing principle for *justification*.

to the example given above. As seen, the attempt to justify that my perceptual apparatus is functioning properly – a presupposition of one of my cognitive projects – involves presuppositions of the same general kind. These can be of no more secure prior standing than the initial presupposition. They are, after all, of the same general kind. Thus, if the attempt to justify P is supposed to improve on its epistemic standing, it must be concluded that it is bound to fail.

3.3 Entitlement as a response to the sceptic

Wright is optimistic about employing the notion of entitlement of cognitive project to respond to Cartesian scepticism. He writes:

> Entitlement of cognitive project seems to promise well in addressing the challenge of Cartesian scepticism, or any variety of scepticism that works by trying to dislodge a cornerstone of our intellectual or cognitive competence. (Wright [154], p. 205)

The idea should be clear: the cornerstone propositions attacked by the sceptic are warranted in the sense of being entitlements of cognitive project. Thus, we have a non-evidential notion of warrant to back the rejection of the sceptical master thought that all warrant is evidential. This gives us the means to resist the sceptical conclusion that we cannot rationally claim warrant for belief in any proposition of the relevant region of thought.

Among the presuppositions of any given project will be a range of cornerstone propositions. On any particular occasion these will include:

- the proper functioning of the cognitive capacities needed to pursue the project;
- the suitability of the attendant circumstances for the effective function of these capacities;
- the integrity or good standing of the concepts involved.

The proper functioning of my perceptual apparatus is a cornerstone when I go about investigating the empirical world on the basis of perception. That I am not a brain in a vat, not dreaming, not hallucinating, etc. are cornerstones concerning the suitability of the attendant circumstances

for the effective function of my cognitive capacities. When I do arithmetic, the good standing of the concept of natural number is a cornerstone of my arithmetical projects.

The entitlement proposal is this: when engaging in a cognitive project on any particular occasion, we have an entitlement of cognitive project to the cornerstones of the project *absent* sufficient reason for thinking them untrue. We have a warrant to trust the cornerstones of the project and can discount the sceptical scenarios that are supposed to dislodge them. (On the assumption that trust in a proposition P is an attitude that excludes doubt and open-mindedness about P. More on this below.) Adopting the notion of entitlement thus allows us to resist the sceptical conclusion that we cannot rationally claim warrant for any belief in the region of thought the sceptic is attacking. There is such a thing as cornerstone warrant. It is just that it is non-evidential.

The entitlement proposal offers a concessive answer to the challenge raised by Cartesian scepticism. It is concessive, because it is granted that the sceptic has shown that there can be no evidential warrant for cornerstone propositions. What is wrong with the sceptical argument, though, is the assumption that warrant has to be earned through evidence; that evidential warrant is the only kind of warrant there is. It should not be found surprising that Wright takes issue with this assumption. If (like Wright) you want to hold that the sceptical argument shows that cornerstones cannot be evidentially warranted, while maintaining that they are nevertheless warranted, you *have* to reject the assumption and spell out a non-evidential kind of warrant which applies to cornerstones.

Clause (ii) turns the tables on the sceptic. It takes into account the lesson from the sceptical arguments: there can be no evidential warrant for cornerstones, in the sense of there being positive evidence supporting them. This is not to say that clause (ii) makes evidence irrelevant to the question whether or not a given proposition is an entitlement. Evidence can overthrow an entitlement. There might be a sufficient reason to think a candidate entitlement untrue, and so, clause (ii) might turn out not to be satisfied. Entitlements are defeasible. However, clause (ii) does make the presence of specific *positive* evidence supporting cornerstones irrelevant (and for good reasons given to us by the sceptic). Instead, as said, what is required is the absence of sufficient countervailing evidence.

Above the sceptical notion of warrant was glossed as requiring that cornerstone warrant be *earned*, that possessing a cornerstone warrant should be due to some cognitive achievement. Does the entitlement approach imply that cornerstone warrants come on the cheap? Is cornerstone warrant for nothing and foundations for cognitive endeavours for free?[8] Entitlement *is* for free in the sense of not requiring any positive evidence. This is clear from the characterization of the notion. However, entitlement to P is not for free in the sense that there are no conditions P has to meet. In particular, clause (i) – P's being a presupposition – tells us that P is something we *have* to take for granted, or trust, when engaging in a cognitive project, lest we want to rationally commit ourselves to doubting its significance.

Entitlement is characterized as a warrant to trust rationally.[9] No attempt will be made to go into details with the nature of trust, but it should be noted that it has to be such that trusting a cornerstone suffices to exclude doubt and open-mindedness about it. Here is why[10]: cornerstone propositions of a given region of thought are unavoidable presuppositions or commitments of any cognitive project undertaken in that region. For to doubt (being open-minded about) a cornerstone of a given project would rationally commit one to doubting (being open-minded about) the competence of that very project. Suppose that I want to engage in some empirical project – say, checking the number of USB ports of my laptop. In that case to doubt (be open-minded about) whether I am not a brain in a vat or that I am not now being deceived by an evil demon would rationally commit me to doubting the competence of my project, as doubt (open-mindedness) about any of these cornerstones will raise doubt (open-mindedness) about something integral to the investigation, *viz.* whether conditions are appropriate for it to be carried out. Doubt (open-mindedness) about a cornerstone thus morphs into doubt (open-

[8] The phrases 'for nothing' and 'for free' are taken from the title of Wright [154], 'Warrant for Nothing (and Foundations for Free?)'.

[9] Wright [154], p. 204 and p. 205.

Wright distinguishes between belief and acceptance. Belief is a species of acceptance. Trust is also a kind of acceptance, but contrasts with belief by not being evidentially controlled. For the purposes of the entitlement proposal it is crucial that trusting a proposition can be rational in the absence of positive evidence supporting it. Though the distinction between belief and acceptance raises several issues in the context of Wright's work on entitlement, the distinction has not been discussed so far. While acknowledging its importance, I wish to keep it that way. This is not because the distinction is unimportant, but rather because we can make the points we want to make without going into great detail.

For a brief discussion of the distinction between belief and acceptance, cf. section II of Wright [154]. Shah [124] and Shah and Velleman [125] are recent works of relevance.

[10] See Wright [154], p. 193.

mindedness) whether the attendant circumstances are really suitable for the execution of my project, and so, whether it is a competent project to engage in at all.[11]

Hence, if doubt (open-mindedness) was installed about a cornerstone of a region of thought and yet we engaged in projects within that region, it would be reasonable to say that we were being irrational and epistemically irresponsible in engaging in these projects. It would not amount to a formal inconsistency, but it would sound odd if someone were to say, 'I doubt, or am open-minded about, whether I am not a brain in a vat, but I will engage in an investigation of the tree in front of me anyway' – and, through her experiences, come to *believe*, say, that the tree is more than two meters tall. Doubt (open-mindedness) as to whether one is not a brain in a vat is exactly the kind of thing that calls into question the suitability of the attendant circumstances. If it is granted that a subject's credence in a proposition should match the evidence available for it, it thus seems that having the tree belief while doubting (being open-minded about) the cornerstone would be irrational or epistemically irresponsible, because the subject assigns a higher credence to the tree proposition than she should. She believes it – which means, among other things, that she is convinced of its *truth* – while harbouring a doubt (open-mindedness) that seems to undermine the weight of her evidence.

When engaging in a cognitive project, trust in its cornerstones might be regarded as rational by contrast: it would be irrational to doubt – or weaker: to be open-minded about – the cornerstones of the project. It is rational to trust the cornerstones of a project because its exclusion of doubt and open-mindedness is what makes it appropriate to regard it as a *cognitive* project. If one doubted or was open-minded about the cornerstones of a project, then – for the reasons given above – it could reasonably be asked what it is one could hope to learn from the project? A point made above was precisely that doubt (or open-mindedness) should call into question that there is anything to be learned. Doubt or open-mindedness about a cornerstone of a project removes the project from the sphere of projects that may properly be regarded as cognitive. Trust puts it back in.

One of the tasks that must be undertaken to develop the entitlement proposal further is to

[11] Note that in order for the sceptical arguments to install doubt (open-mindedness) about cornerstones we have to assume that the principled lack of cornerstone warrant (which the sceptic takes herself to have established) implies doubt (open-mindedness) about cornerstones.

say more about trust, rationality, entitlement, and their interrelations. Some considerations of relevance to the execution of this task are offered in Chapter 7.

3.4 Entitlement in mathematics

Above we have seen how Wright uses the notion of entitlement of cognitive project to respond to scepticism about the empirical world, as cashed out in the Cartesian argument discussed earlier. I will now turn to the task of spelling out how the notion can help to provide a response to the mathematical regress arguments rehearsed in Chapter 2.

The mathematical regress arguments were targeted against the warrantability of satisfiability and consistency propositions. We will counter the arguments by rejecting the sceptical master thought according to which warrant is always evidential. In the particular context in which the two regress arguments were formulated, this rejection amounts to a rejection of the idea that mathematical warrant is always based on proof. As announced, the strategy will be to invoke the notion of entitlement of cognitive project. The proposal suggests a default epistemology of sorts for the target class of propositions. When a proposition P is warranted as a matter of entitlement, the warrant held is a default warrant in the sense that P is entitled *unless* there is sufficient reason to think otherwise. That is, entitlement is the default position. Thus glossed, the proposal to be presented in effect says that satisfiability and consistency propositions can be warranted by default.

To my knowledge, there has been no published attempt to spell out the default proposal with respect to satisfiability and consistency. Perhaps the lack of such an attempt is explained by two things. First, that the interest in a default notion of warrant – entitlement of cognitive project, at any rate – flows partly from an interest in certain kinds of scepticism, *and*, second, not much explicit focus has been placed on what I have referred to as 'mathematical regress scepticism' above. However, although this is so, if mathematical scepticism raises a challenge that deserves to be met, the entitlement proposal – as applied to satisfiability and consistency propositions – should be of considerable interest.[12]

[12] Shapiro briefly considers a default notion of warrant towards the end of [129], but in a somewhat different setting than our present one. He discusses whether an externalist can make sense of the KK-principle, which

3.5 $Sat(T)$ as an entitlement

In Section 3.2, the characterization of entitlement of cognitive project was stated. In this section, we will add detail to the suggestion that $Sat(T)$ is an entitlement, for T some mathematical theory. In the next section we turn to $Con(T)$. PA is used as an example, but similar reasoning can be rehearsed for ZFC and its sub-theories (or more carefully, those sub-theories that have the resources to represent elementary arithmetic).

Consider PA. To add detail to the proposal that $Sat(PA)$ is an entitlement of cognitive project for PA-theorizing we need to see if $Sat(PA)$ meets clauses (i)–(iii) in the characterization of entitlement. That is, we need to ask whether $Sat(PA)$ is a presupposition of PA-projects, whether there is any sufficient reason to believe it untrue, and whether attempts to justify it give rise to an infinite regress of justificatory projects involving presuppositions of no more secure a prior standing?

Does $Sat(PA)$ satisfy clause (i), i.e. is $Sat(PA)$ a presupposition of PA-projects? It is, at least if I am right in taking the significance or competence of a cognitive project to be crucially bound up with the potential cognitive value of the project. That is, with whether we could learn anything from a successful execution of the project; whether it could teach us anything about the domain of arithmetic (however one might construe it).

In order for $Sat(PA)$ to be a presupposition of PA-projects it has to be the case that doubt (open-mindedness) concerning $Sat(PA)$ rationally commits one to doubt (open-mindedness) about the significance or competence of any PA-project. Recall that $Sat(PA)$ is a necessary

is usually conceived as internalist in spirit. In the mathematical case, Shapiro suggests that the externalist can make sense of the KK-principle by appeal to a notion of default warrant.

In a different context – a discussion of the neo-Fregean programme – Agustín Rayo observes that there has been no defence of the appropriateness of appeal to a default notion of warrant in the neo-Fregean programme. More specifically, there has been no defence of linguistic stipulations – like the stipulation of Hume's Principle – being successful by default. (Hume's Principle is the second-order principle according to which, for any concepts X and Y, the number of X's is identical to the number of Y's just in case X and Y are equinumerous.) See Rayo [120], p. 226–228. For a full articulation of Rayo's worry, see Rayo [119]. In an unpublished manuscript ([155]), Wright takes on the task of giving a detailed development of the view that Hume's Principle is warranted by default.

The broader literature on epistemology does contain a number of notions of warrant that can be considered as default notions. Reference has already been made to Burge and Peacocke (see Section 3.2). Harman's negative coherence theory is also worth mentioning. Details can be found in Harman [68], [69], [70], [71]. A brief exposition is given in Pollock and Cruz [108], pp. 80–85. Bach discusses default reasoning in Bach [6]. His *take-for-granted-principle* (p. 44) can be considered as capturing a default notion of warrant. Furthermore, it is worthwhile mentioning Hartry Field's notion of default reasonableness (Field [47]) and Peter Klein's defeasibility theory (Klein [78], pp. 137–166, and [79]).

condition for PA to delineate a subject-matter; if PA is unsatisfiable, there is nothing the theory is about. Thus, if we doubt (or are open-minded about) $Sat(PA)$, we are in effect doubting (or being open-minded about) the question whether PA concerns anything at all. Harbouring such a doubt (open-mindedness), we would have to doubt (be open-minded about) whether there is anything to learn from executing the project. For if there is no structure satisfying the axioms of PA, there is nothing which execution of the project could teach us something about. The project cannot properly be regarded as *cognitive*. Thus, doubt (open-mindedness) about $Sat(PA)$ rationally commits one to doubt (open-mindedness) about the significance or competence of PA-projects. (Again, as stressed in Chapter 2, the talk of structures, subject-matter, and so forth is not intended to carry any realist implications.)

Does $Sat(PA)$ satisfy clause (ii), i.e. is it the case that there is no sufficient reason for believing it untrue? Reflection on mathematical practice suggests that it does. Nobody appears to hold that there is any reason to doubt $Sat(PA)$, let alone sufficient reason to think it untrue. Among the theories of classical mathematics, arithmetic would seem to be among the most secure ones. Mathematicians happily do arithmetic, including induction.

So, we now need to address the question whether $Sat(PA)$ meets clause (iii), i.e. whether attempts to justify it give rise to an infinite regress of justificatory projects involving some presupposition of no more secure a prior standing than $Sat(PA)$ itself? To make a case for an affirmative answer to this question, we basically need to rerun the regress argument, as formulated in terms of relative satisfiability. What got the regress going was the sceptic's contention that, if there is to be any warrant for $Sat(PA)$, it has to be acquired on the basis of a proof. $Sat(PA)$ cannot be established in PA itself. The best we can do is a relative satisfiability result. The sceptic then argued as follows: a result which shows PA satisfiable relative to another theory T_1 does not by itself provide a warrant for $Sat(PA)$. For $Sat(PA)$ is held hostage to the satisfiability of T_1, and the relative satisfiability result does nothing to establish *that*. Another relative satisfiability result is thus needed, and this will involve another theory T_2 of satisfiability strength greater than that of T_1 – and so, greater than that of PA as well. But the satisfiability of T_1 is held hostage to that of T_2. And so on. Each of the relative satisfiability results brings in a further presupposition, the satisfiability of another theory. Every theory involved in the infinite

regress of satisfiability proofs will all be stronger than PA in terms of satisfiability strength, and so, of no more secure a prior standing than $Sat(PA)$. In other words, $Sat(PA)$ satisfies clause (iii) of the characterization of the notion of entitlement of cognitive project. The infinite regress of relative satisfiability proofs that results from attempts to justify $Sat(PA)$ does involve some presupposition of no more secure a prior standing than $Sat(PA)$.

We have now made a case for $Sat(PA)$ satisfying each of clauses (i)–(iii). If this is found compelling, then $Sat(PA)$ is an entitlement of cognitive project. That is, $Sat(PA)$ is warranted non-evidentially. It may be tempting to think, 'But, surely, for $Sat(PA)$ to be warranted we have to provide a *positive* reason for holding PA satisfiable.' Anyone who thinks thus is likely to be impatient with the entitlement proposal. The entitlement proposal would be a non-starter, were the presence of positive evidence a requirement on all warrant. As stressed several times, entitlement is a negative notion of warrant. What is required is not the presence of positive evidence, but instead the absence of sufficient countervailing evidence.

3.6 $Con(T)$ as an entitlement

To add detail to the proposal that $Con(PA)$ is an entitlement for theorizing in PA, as with $Sat(PA)$, we need to see if it meets clauses (i)–(iii) in the characterization of entitlement. That is, we need to ask whether $Con(PA)$ is a presupposition of PA-projects, whether there is sufficient reason to think it untrue, and whether attempts to justify it give rise to an infinite regress of justificatory projects involving some presupposition of no more secure a prior standing than $Con(PA)$ itself?

Is $Con(PA)$ a presupposition of PA-projects, i.e. does doubt (open-mindedness) about $Con(PA)$ rationally commit one to doubt (open-mindedness) about the significance or competence of PA-projects? It does. Recall that the consistency of a theory ensures that the theory is not trivial. To doubt (be open-minded about) $Con(PA)$ thus involves doubting (being open-minded about) its non-triviality. However, the significance or competence of a PA-project is intimately tied to the question whether or not PA is trivial. By a theory being trivial we mean that it proves $\alpha \wedge \neg\alpha$ – for some statement α in the language – and thus, by *ex falso quodlibet*,

proves β for any wff. β. If PA is trivial in this sense, there is no significance or competence to be associated with projects aimed at showing that something deductively follows from the axioms. For *anything* follows. To doubt or be open-minded about $Con(PA)$ thus rationally commits one to doubting or being open-minded about the significance or competence of any (deductive) PA-project.

Is there any sufficient reason to think that $Con(PA)$ is untrue, i.e. is there any sufficient reason to think that $PA \vdash \alpha \wedge \neg\alpha$ for some α? It would appear not. It seems safe to say that nobody seriously doubts the consistency of PA, let alone thinks that there is sufficient reason to believe that a contradiction is derivable from it. If there is one, we sure have not seen it yet. We can take it that $Con(PA)$ satisfies clause (ii), then.

To make a case for the claim that $Con(PA)$ meets clause (iii) we need to show that attempts to justify it give rise to an infinite regress of justificatory projects which involve some presupposition of no more secure a prior standing than $Con(PA)$ itself. We rely on a regress argument similar to the one appealed to in the case of $Sat(PA)$. Now, to get the regress running appeal was made to Gödel's second incompleteness theorem and the sceptical assumption that warrant for $Con(PA)$ has to be evidential, acquired on the basis of a proof. By Gödel's theorem, $Con(PA)$ cannot be proved in PA itself. Thus, the best we can do is a relative consistency proof, i.e. a proof that shows PA consistent relative to some other theory T_1. This result, however, does not yield a warrant for $Con(PA)$. For the consistency of PA is held hostage to the consistency of T_1, and the proof does nothing to establish *that*. To support the consistency of T_1 another relative consistency proof is needed. This will involve a further theory T_2. But now $Con(T_1)$ is held hostage to $Con(T_2)$, and the additional relative consistency proof does nothing to establish the latter. Yet another relative consistency proof is need for that. And so on. For each consecutive pair of theories in the series PA, $T_1, T_2, \ldots$, it is the case that the consistency strength of the second is greater than that of the first. Therefore, each relative consistency proof brings in an additional presupposition which is of no more secure a prior standing than our initial proposition, $Con(PA)$. Attempts to justify $Con(PA)$ thus involve some presupposition of no more secure a prior standing than itself, and hence, $Con(PA)$ satisfies clause (iii).

$Con(PA)$ seems to satisfy the clauses characterizing entitlement of cognitive project. The

upshot of this is that $Con(PA)$ is non-evidentially warranted. Again it is important to bear in mind that entitlement is a negative kind of warrant. There is no requirement that positive evidence be present; the absence of sufficient countervailing evidence suffices.

3.7 An entitlement to $Sat(NF)$ and $Con(NF)$?

An immediate question regarding the entitlement proposal in our current mathematical setting concerns its range of application. In the previous sections, PA was taken as an example. However, does the reasoning generalize? Does the entitlement proposal support the thesis that, for any mathematical theory T, $Sat(T)$ and $Con(T)$ are non-evidentially warranted by being entitled?

The answer to the question is 'no'. As should be clear, there are theories for which $Sat(T)$ and $Con(T)$ fail to satisfy clause (ii) in the characterization of entitlement – i.e. the requirement that there be no sufficient reason to believe the theory untrue. Think of naive set theory (here abbreviated 'NST'). This theory fails to meet clause (ii) in a rather rampant way, by being paradoxical. Just recall the paradoxes of Russell, Cantor, and Burali-Forti. Each of these paradoxes suffices to show that NST implies, whether we are speaking semantically or syntactically, a formula of the form '$\alpha \wedge \neg\alpha$'. Thus, NST can be shown to be unsatisfiable and inconsistent. Consequently, neither $Sat(NST)$ nor $Con(NST)$ can be an entitlement. Therefore, it is not the case that the satisfiability and consistency propositions of any theory whatsoever qualify as entitlements.

This much might be thought obvious. However, there are other cases where it is not clear what to say. In particular, it is worth asking about Quine's New Foundations (NF). Is there an entitlement to $Sat(NF)$ and $Con(NF)$? I am inclined to think that there is. Below I offer some reasons for and against this view. Clause (ii) is likely to be a matter of some dispute, given the widespread ZFC bias. However, before we discuss this clause, we will deal with clauses (i) and (iii).

Do $Sat(NF)$ and $Con(NF)$ satisfy clause (i), i.e. are they presuppositions of NF-projects? Yes, they are. To support this claim adopt the lines of argument used in the context of PA and substitute occurrences of 'PA' with 'NF'. Indeed, this will work for any mathematical theory

T. Therefore, in particular, $Sat(NF)$ and $Con(NF)$ meet clause (i). Though this part of the argument for the status of $Sat(PA)$ and $Con(PA)$ as entitlements generalizes, it should be borne in mind that it is only *part* of the argument. Hence, by itself, it does not suffice to show that $Sat(T)$ and $Con(T)$ are entitlements for any theory T.

Do $Sat(NF)$ and $Con(NF)$ satisfy clause (iii)? In other words, will an attempt to justify each of them commit one to an infinite regress of justificatory projects involving some presupposition of no more secure a prior standing? This is a slightly tricky question. The reason why it is tricky is that not much progress has been made with respect to the satisfiability and consistency questions for NF. It is by no means clear how to proceed. In the case of ZF, say, we have two kinds of information concerning the regress. The first kind of information is given by purely formal constraints on the satisfiability and consistency strength of the theories in the regress. Let T_1 be the initial theory and $T_2, T_3, \ldots$ the other theories, and let each of them be such that they can express elementary arithmetic. Then the theorem of Gödel tells us respectively that, for any pair of theories T_m and T_n (where $m < n$), the consistency strength of T_n has to be greater than that of T_m and (assuming that we are dealing with a first-order theory) that the satisfiability strength of T_n has to be greater than that of T_m. The second kind of information concerns the specific character of the theories in the regress. For ZF, the regress might involve, other than ZF itself, the theory obtained from ZF by adding the axiom that there is a strongly inaccessible cardinal, while the other theories are obtained by adding stronger and stronger large cardinal axioms. The model theory of theories in the ZF tradition is a fairly well-investigated matter, and so, in the satisfiability regress, we might have specific information about what kind of structures we are dealing with.[13] This is a great asset, and, presumably, one of the main reasons why ZFC is widely regarded as *the* set theory.

Regarding NF, we have the first kind of information. NF is strong enough to represent elementary arithmetic, and, as such, is subject to the formal constraints on consistency and

[13] If the satisfiability regress involves a 'base theory' and extensions thereof, some sense can be made of speaking of the same general kind of presupposition reappearing at every step of the regress. The satisfiability of the extended theories will all involve the satisfiability of the base theory as they can only be satisfiable if the base theory is. In this sense, the satisfiability of the base theory will be brought up as a presupposition at every step of the regress and can be said to force presuppositions of the same general kind. However, still, the presuppositions brought about by the extended theories are stronger.

satisfiability strength imposed by the Gödel result. Arguably, this suffices for the suggestion that attempts to justify $Sat(NF)$ or $Con(NF)$ commits one to an infinite regress of justificatory projects involving some presupposition of no more secure a prior standing – when the relevant standard for security is taken to be respectively consistency and satisfiability strength. However, when it was said above that is by no means clear how to proceed, what I had in mind was the second kind of information concerning the specific nature of the theories in the regress. A result due to Specker shows that the consistency and satisfiability questions for NF can be reduced to the corresponding questions for an extension of simple type theory (STT from Chapter 1). In the words of an authoritative source on set theory, this theory is, 'easier to work with than NF, at least to mathematicians used to ZF.'[14] Though this is far from what we have in the ZF case, it still seems to me that there is a basis for holding that attempts to justify $Sat(NF)$ and $Con(NF)$ will lead to an infinite regress of justificatory projects that will involve some presupposition of no more secure a prior standing.

Let us now turn to clause (ii), the perhaps most controversial one of clauses (i)–(iii). There is some overlap with our preceding discussion of clause (iii). Is there sufficient reason to believe $Sat(NF)$ and $Con(NF)$ untrue? Some people might think so. NF disproves the axiom of choice, i.e. $NF \vdash \neg AC$.[15] By soundness, $NF \models \neg AC$. Now, many people think that ZFC is *the* set theory, reflected by the fact that the theory is often referred to as 'standard set theory', and so, due to NF's clash with choice might believe that NF cannot have any models.[16] In other words, someone might be a 'set-theoretic monist' and maintain that the space of models is exhausted by the models of ZFC. In particular, then, the axiom of choice is true in every model there is. However, this means that $\neg Sat(NF)$ since $NF \models \neg AC$. Thus, on this picture, there is sufficient reason to believe $Sat(NF)$ untrue, and hence, $Sat(NF)$ cannot be an entitlement. If that is the case, there is, by soundness, also a sufficient reason to believe $Con(NF)$ untrue, and therefore,

[14] Fraenkel et al. [50], p. 166. For a presentation of Specker's result, see Forster [49], Section 2.3. The result is due to Specker [132] and [133].

[15] Specker [131].

[16] Forster says, 'Specker's equiconsistency theorem [i.e. the one cited earlier] is the foundation for all study of the models of NF, and his refutation of AC in NF is the source of all worry that there might not be any models to study.' (Forster [49], p. 23).

One could, however, hold that ZFC and NF – perhaps set theories in general – concern different universes of sets. These universes are equally real, but differ in terms of what is true of them. In NF universes the axiom of choice is false, while the axiom is true in all ZFC universes (though not in all ZF universes, as indicated above).

$Con(NF)$ cannot be an entitlement either.

The reliance on choice in the previous paragraph might be deemed questionable. Suppose, then, that we drop choice and consider ZF instead. As the results of Gödel and Cohen show, choice is independent of ZF.[17] Choice holds in some models of ZF, but fails in others. Thus, the space of models being exhausted by the models of ZF is compatible with $NF \models \neg AC$ and $Sat(NF)$ (and with $NF \vdash \neg AC$ and $Con(NF)$ if we invoke soundness and move to talk of consistency). Therefore, the status of ZF as *the* set theory – in the sense indicated – does not imply that $Sat(NF)$ or $Con(NF)$ cannot be entitlements.

However, still, there are features of NF that will be found *weird* by anyone who favours ZF. These include[18]:

- Induction only holds for stratified formulae. Among other things, this means that, in NF, we cannot prove that for every finite cardinal n, there are exactly n cardinal numbers less than n (provided NF is consistent).
- There is a set whose cardinality is not the same as that of the set of the singletons of its members. The universal set is an example of such a set.
- Cantor's theorem does not hold in general. There is a set y which is equinumerous to its power set – e.g., the universal set.
- The axiom of infinity is derivable in NF.

On the other hand, since 1937 – when Quine's original paper was published - no one has succeeded in showing NF unsatisfiable or inconsistent either. This should count for something. At the very least $Sat(NF)$ and $Con(NF)$ do not fail to meet clause (ii) conclusively by there being a proof of their negations. However, there might be less dramatic ways in which they could fail to satisfy clause (ii), which, after all, is phrased in terms of there not being a sufficient reason for thinking the target proposition false. As indicated above, someone might cite the axiom of choice or some of the 'weird facts' to support the view that the satisfiability and consistency propositions

[17] See, e.g., Kunen [84]. The axiom of choice holds in Gödel's constructible universe L. L is a model of $V = L$, and $V = L$ implies the axiom of choice. Forcing can be used to yield models in which choice fails.
[18] See Forster [49].

for NF do not meet clause (ii). However, it was suggested that the appeal to choice itself is contestable. Although choice might give us a reason of some sort for thinking $Sat(NF)$ and $Con(NF)$ false, it is not a *sufficient* reason, and the same could be said about the 'weird facts'.

At this point, the question to ask is whose perspective we have adopted? 'Somebody with a liking for ZF, and, most likely, ZFC' is the answer. As for entitlement, what the above considerations on clause (ii) show is thus that, for someone with a liking for ZF or ZFC, it might be questionable or simply not the case that clause (ii) is met. However, to determine whether $Sat(NF)$ and $Con(NF)$ can qualify as entitlements of cognitive projects, the relevant case to consider is *not* one in which there is such a bias. Let us instead take someone who takes NF to be *the* set theory. Such a subject will not be terribly impressed by the observation that NF supports some 'weird facts' and clashes with choice. Instead such a subject will focus on the fact cited above – that there has been no proof that NF is either unsatisfiable or inconsistent – and that, really, there is no sufficient reason to believe either $Sat(NF)$ or $Con(NF)$ untrue. In other words, for such a subject, $Sat(NF)$ and $Con(NF)$ will be entitlements of cognitive project.

Wright does not address – and so, does not take a stance on – the matter just discussed. However, I submit that entitlement of cognitive project, properly understood, should allow for different subjects having different – indeed, incompatible – entitlements. More on this in Section 5.5.

3.8 A qualification: working history

In this section, I suggest that a certain qualification should be made when it comes to our understanding of $Sat(T)$ and $Con(T)$, considered as entitlements. The qualification is that T should have a working history.

Suppose that you ask me what the concept of set amounts to. Having thought about the question for a minute or two, I write down fifteen principles, quite long and formulated in the language of first-order logic, supplemented by $\in$. I then inform you that these principles give meaning to the concept of set. Let us call the theory 'T_{15}'. Do I have an entitlement to $Sat(T_{15})$

and $Con(T_{15})$? Never mind the details of T_{15}. The question seems to be reasonable provided the axioms of the theory are all well-formed, which we will assume they are.

One train of thought suggests that I am, indeed, entitled to $Sat(T_{15})$ and $Con(T_{15})$ – that is, that these two statements meet clauses (i)–(iii). For clauses (i) and (iii), the reasoning is analogous to the reasoning in the NF case. (To get the regress going in clause (iii) it needs to be assumed that T_{15} is strong enough to represent elementary arithmetic.) Regarding clause (ii), it would seem that there is no sufficient reason to think any of $Sat(T_{15})$ and $Con(T_{15})$ false. After all, we have not proved anything significant in the theory yet, semantically or syntactically. This means that there is nothing supporting $Sat(T_{15})$ and $Con(T_{15})$. There could not be, on the assumption that T_{15} is sufficiently strong and the relevant standard is taken to be proof. However, nothing tells against $Sat(T_{15})$ or $Con(T_{15})$ either. Since the absence of countervailing evidence is what is relevant to entitlement, this suggests that clause (ii) is satisfied.[19]

What should we make of this? Should $Sat(T_{15})$ and $Con(T_{15})$ be admitted as entitlements? The consequences of doing so appear rather undesirable. The resulting range of applicability would be too wide for the notion to be interesting. For instance, could we not programme a computer to generate lists of well-formed statements in all currently known mathematical languages, take the lists generated as mathematical theories, and, by the reasoning above, be said to have an entitlement to the satisfiability and consistency of many of these theories?

Let us see what happens if we grant this. Then, as said, the notion of entitlement would apply to the satisfiability and consistency propositions of a lot of theories. Somebody might point out that the entitlement to some of these satisfiability and consistency propositions would be defeated. Supposing that the programme in question was generating lists of statements from the language of zero and successor together with the standard first-order resources, some of the lists generated would include $\neg(\exists x)\mathbf{0} = s(x)$ and $(\exists x)\mathbf{0} = s(x)$. The conjunction of the principles of such lists is neither satisfiable nor consistent. Once this is uncovered, the entitlement would be defeated. Now, if we wrote a more complicated computer programme, we could, of course,

[19] A qualification. It was said that we have proved nothing significant in the theory yet. Of course, there are certain statements that we can rest assured will follow from the axioms. These include the axioms themselves and logical truths like $(\forall x)x = x$ and $\phi \leftrightarrow \phi$ (ϕ any wff, logical truths follow semantically as well as syntactically from the empty set of premises). So, it is not true to say that T_{15} does not prove anything at all.

eliminate some of these lists. Yet there would still be enough lists – and entitlements – left for the worry to remain that, due its wide range of application, the notion of entitlement should not be considered to be an interesting and significant kind of warrant.

Here are two observations which suggest that things are not so bad after all. First, suppose that we bite the bullet and grant that entitlements come as easily as the above line of thought would have it. Even so, it does not follow that the theories whose satisfiability and consistency we would be entitled to would be of any mathematical interest. By assumption, we have not really proved anything in these theories yet, and so, do not know anything about how they relate to other thoroughly investigated theories or whether the theory, by itself, will result in lots of worthwhile mathematics. Thus, granting the satisfiability and consistency propositions under consideration the status of entitlements does not commit one to the view that the theories involved are in any sense important or mathematically interesting.

This first observation points to the second: that an established practice or working history of a theory is relevant to the question whether the satisfiability and consistency propositions for the theory are candidates for being entitlements. One way of cutting down the number of entitlements would be to enforce a requirement to the effect that only the satisfiability and consistency propositions of theories with an established practice or working history get to be candidates for being entitlements. This strikes me as plausible. It does so for the simple reason that without an established practice or working history, clause (ii) seems to be slightly out of place. For in that case there will not be much that can be recognized as something that could overthrow a satisfaction of clause (ii). In other words, clause (ii) will be met in a somewhat trivial manner. This is exactly what was exploited above, in the train of thought aimed to show that there are many, many entitlements.

This should be distinguished from another plausible thought, *viz.* that the theories with an established practice are the ones we should really be interested in. Why should we care about some theory which has not proved its worth, either by shedding light on other already established theories or on its own by serving as a framework for worthwhile mathematics? Remember that a proposition P is an entitlement of cognitive project relative to a class of projects. When P is a satisfiability or consistency proposition for some theory T, the class of projects is given

by the T-projects. If T is without an established practice and a working history, the class of T-projects is very narrow. However, once T has been worked in, matters might change. A wealth of theorems might be established in T, revealing links to other mathematical disciplines or generating results concerning T interesting in their own right. Perhaps numerous attempts to show T unsatisfiable or inconsistent fail. All of these aspects contribute to the significance of the theory, in the sense that it gets increasingly ingrained in epistemic practice and a larger number of epistemic projects is associated with it as a consequence thereof.

3.9 Pre-axiomatic mathematics and entitlement

So far we have discussed entitlement as applied to *axiomatic* mathematics. However, one might reasonably wonder if the notion of entitlement can be applied to non-axiomatic mathematics. I think it can. Even at the pre-axiomatic stage mathematics is carried out by the use of methods. These methods may very well be fairly local in their application, applicable to only very specific kinds – or a very restricted class – of operations, computations, or what have we. Considered as a whole the class of methods of pre-axiomatic mathematics might be less systematic and simple than a later axiomatic system. However, this does not mean that the entitlement idea cannot be applied. One natural suggestion is to consider individual methods and take their good standing to be an entitlement of the computations, operations, etc. they are used to carry out.

3.10 Conclusion

The Cartesian sceptic challenged the warrantability of the cornerstones of our empirical thinking. If her argument is compelling, the price we have to pay is high: there can be no rational claim to warrant for belief in any of the propositions about the empirical world we ordinarily take ourselves to have a warrant for. The entitlement proposal started by rejecting the sceptical master thought that all warrant is evidential. This rejection needed backing by a non-evidential notion of warrant. We followed Wright in taking entitlement of cognitive project as a promising candidate for responding to the Cartesian sceptic. It was then suggested that the notion of entitlement of cognitive project also promises well as a response to the mathematical regress

arguments presented in Chapter 2. Endorsing the sceptical master thought, the mathematical regress sceptic maintained that warrant for satisfiability and consistency propositions has to be given by *proof*, the paradigmatic kind of mathematical evidence. In this chapter, we rejected this thought and explored the proposal that such propositions can be warranted non-evidentially, as a matter of entitlement of cognitive project. PA was used as an example. The reasoning can be extended to standard set theory. Furthermore, the question was raised whether entitlement of cognitive project yields an entitlement to $Sat(NF)$ and $Con(NF)$. I tentatively suggested that it does.

Chapter 4

Entitlement of substance

We will now turn to the task of responding to mathematical I-II-III scepticism. In the first section of the chapter, we briefly say something about what is required of an adequate response to I-II-III scepticism and indicate why entitlement of cognitive project will not do the job. In the second section, we introduce the notion of entitlement of substance, a non-evidential notion of warrant gestured at towards the end of Wright [154] and proposed to have the potential to deliver a satisfactory response to I-II-III scepticism. In addition, given how little Wright says about the notion, we take on the task of putting a definite characterization of the notion on the table. As we shall see, the characterization to be offered differs from what Wright explicitly allows in one crucial respect. In the third section, we employ the notion to respond to mathematical I-II-III scepticism.

4.1 Responding to I-II-III scepticism

Recall that entitlement of cognitive project was supposed to respond to Cartesian scepticism. The Cartesian sceptic aimed to show that we have no warrant for thinking that we are not subject to systematic cognitive error, because we have no warrant for some cornerstone of our thinking about the empirical world. The sceptical conclusion was that we cannot rationally claim warrant for our ordinary empirical beliefs. In the mathematical case, entitlement of cognitive project was invoked to respond to the mathematical regress sceptic. The mathematical regress

sceptic aimed to show that we have no warrant for thinking that our mathematical theorizing is not systematically misguided in virtue of there not being anything our mathematical theories are about (satisfiability), or their being trivial (consistency). The sceptical conclusion was that we cannot rationally claim warrant for our ordinary mathematical beliefs, whether obtained by model-theoretic or proof-theoretic reasoning.

Recall that the I-II-III sceptic argues that our best attempts to support type III propositions will fail as a matter of principle, due to transmission failure. The transition from I to II cannot be warranted antecedently and independently of III. Reflection on the character of type III propositions suggests that, properly understood, I-II-III scepticism should be taken to have two aspects. The type III proposition has an existential and an attributional part. In the case of Moore's proof, the existential part says that there is a world, and the attributional part says of this world that it has a certain character, *viz.* that it is external. Thus, in attacking type III propositions, the sceptic does not just target a proposition concerning the existence of some subject-matter, but the existence of a subject-matter with certain features – and, crucially, whatever follows about the nature of our interaction with this subject-matter from its having these features. We made this point in Chapter 2 by characterizing type III propositions as O-cornerstones, i.e. as propositions integral to the conception of judgement within the relevant region as objective.

How are we best to respond to the I-II-III sceptic? The beginnings of a response to the sceptic emerges when we deny the thesis on which the sceptic implicitly relies in her argument, *viz.* that a type III proposition, if warranted, has to be so evidentially. To back our denial we need to supply a non-evidential notion of warrant. Though promising as a response to Cartesian scepticism, there is reason to doubt that the notion of entitlement of cognitive project can effectively be employed to respond to I-II-III scepticism. The reason is that type III propositions do not seem to satisfy clause (iii) of the characterization of entitlement of cognitive project. That is to say, granting that there is an onus to justify a type III proposition does not appear to give rise to an infinite regress of justificatory projects involving some presupposition of no more secure a prior standing. Instead, if such an onus is accepted for a type III proposition, it would be

unclear how to even proceed with a justification, absent some prior response to the sceptic.[1] In other words, while accepting an onus to justify a Cartesian cornerstones gives rise to an infinite regress of the kind required by clause (iii), accepting an onus to justify a type III proposition simply leaves one wondering how to get started.

Thus, we still need a response to I-II-III scepticism.

4.2 Entitlement of substance

In this section, we will respond to I-II-III scepticism by invoking the notion of entitlement of substance, sketched in Wright [154]. First an account will be provided of the little Wright says on the matter, and I then make an attempt to extract a definite proposal from it.

Entitlement of substance, as applied to the empirical world, goes hand in hand with a specific view of the world, our place in it, and our experiential interaction with it:

> Certainly, all *our* actual thought and activity is organized under the aegis of a distinction between states of affairs accessible to us at our own cognitive station and others that lie beyond, and it is difficult to form any clear concept of how things might be otherwise. There is however a well-known train of thought, popularly understood as Kantian and given body by Strawson's classic discussion in Chapter 2 of *Individuals* ... which argues, in effect, that cognitive locality goes with the very idea of our experience as being of an *objective* world, of a reality that stands independent of it. More specifically, it is only via a conception of the possibility of states of affairs and processes occurring *un*perceived that sense can be given to the idea that experience informs us of a reality not of its making. But that conception calls in turn for a conception of a way, or ways, in which states of affairs and processes can elude the awareness of a thinker, which – according to the Kantian train of thought – in turn necessarily involves some *dimension* of variation of locality – the idea of a situation obtaining, in the most abstract sense, *elsewhere* – and hence a conception of that dimension of variation. And now *that* conception in turn arguably demands some notion of the make-up – substance – of a state of affairs suitable to allow it to be situated 'elsewhere'. (Wright [154], p. 202)

Wright here emphasizes the *cognitive locality* of our thought and activity. We, as subjects, are always located at a specific place in the world, with a range of states of affairs accessible to us,

[1] This is the line taken by Wright [154], p. 197.

and some beyond our reach or 'elsewhere'. The view paired with the idea of cognitive locality is broadly realist: our experiences connect us to a mind-independent reality, and, in particular, the states of affairs that are not accessible to us at our 'cognitive station' are part of such reality.

Wright highlights the Kantian idea that cognitive locality is part of the very notion of our experiences putting us in connection with an objective reality, meaning a reality that is constitutively independent of experience. It is only by conceiving of our investigation of the world as cognitively local that we can sustain the idea that it puts us in contact with an objective reality. This Kantian conception of our situation in, and interaction with, the world points to a notion of entitlement of substance, an entitlement to 'have a view about the most basic categories of stuff and things the world contains' (p. 205). Here is a sketch of the idea:

> ... it does not seem altogether fanciful that a developed (Kantian or Strawsonian) metaphysics might teach us that to operate any scheme of thought rich enough to recognise objective experience – rich enough to allow for experience of states of affairs whose existence is constitutively independent of experience – must involve a grasp of the idea of particular states, events and processes existing outside the thinker's cognitive locality, and hence some conception of dimension(s) of locality and an appropriately coordinated conception of substance. If so, then the mere conception of ourselves as capable of experiences *of* a world cannot escape *some* conception of substance: of the nature of what fundamentally constitutes the kinds of states of affairs that can be situated 'elsewhere'. Assuming that conceiving of experience as objective is somehow independently mandated ... a somewhat minimal notion of entitlement of substance might then emerge: since some conception of one's cognitive locality and of the substance of states of affairs that are elsewhere is essential to any objective conception of experience – and since (suppose) so conceiving of experience is independently warranted or unavoidable – a thinker is entitled to the basic ontology involved in an otherwise coherent conception of what kind of thing might obtain at other localities. (Wright [154], p. 203)

The key idea in the Kant-inspired line of reasoning is that conceiving of our interaction with the world in a certain way can, with some philosophical unpacking, be shown to have certain ontological implications. In the particular case rehearsed by Wright, conceiving of experience as objective has certain ontological implications. The philosophical unpacking that needs to be done is, roughly, this: experience has to be conceived as experience *of* something – here states

of affairs, events, or processes. Furthermore, experience can only be conceived as objective if we take on board the thought that it is cognitively local, that there are states of affairs which obtain 'elsewhere' or are constitutively mind-independent. To sustain the notion of cognitive locality a certain conception of substance is required, *viz.* one that will support the idea of states of affairs obtaining elsewhere or in a constitutively mind-independent way.

The beginning of the first quote of this section suggests that Wright takes the received view on experience to be that it is regarded as objective. By the above line of reasoning, this has certain commitments – ultimately, a certain notion of substance. However, is there a warrant for thinking that a substance, or stuff or range of things, of the required kind exists? The proposal on offer is that there is, as a matter of entitlement.

The sketched notion of entitlement of substance is referred to as 'minimal'. It is so in two respects. First, the target notion of entitlement is supposed to concern only *basic* ontology – i.e. the fundamental categories which make up the furniture of the world – and second, in the paragraph following the one just quoted, it is said that entitlement of substance 'falls short of requiring a *specific* ontology'. For the case of the empirical world, applying the notion of entitlement of substance will thus not deliver an entitlement to the existence of, say, footballs. (If it did, it might just have turned out that 'Football World' is not only the name of a sports shop in Dundee, but also an appropriate label for the world we inhabit.) It will not deliver an entitlement to take everything in the empirical world to be made up of atoms either. The former because footballs cannot – even by a long shot – be counted among the basic categories of things; the latter, not because the category of atom is not basic enough, but because it is too specific.

If I understand Wright correctly, the use of 'minimal' is in important respects supposed to indicate that entitlement of substance only concerns very general features of the relevant substance. To be explicit, I take the notion of entitlement of substance to be entitling in two respects. First, there is an *existential* entitlement – an entitlement to the existence of substance, or the basic stuff of which the world is made. Second, there is an *attributional* entitlement – an entitlement to an attribution of some very general features of the substance whose existence there is an entitlement to. The generality aspect of the second 'sub-entitlement' is crucial, and

exactly where the atom suggestion goes wrong.

Wright is not entirely clear on this matter. He sometimes appears to hold that it is only the ontology – i.e. the existence of substance – that is an entitlement.[2] On the other hand, he also says that entitlement of substance is an entitlement to 'have a view about the most basic categories of stuff and thing the world contains'.[3] This might be taken to suggest that the entitlement is merely attributional. As just seen, I treat entitlement of substance as involving both aspects. My reason for so doing is that, if – among other things – entitlement of substance is supposed to help sustain a certain conception of our interaction with the world, mere existence will not do. For such a conception involves the thought that what we interact with has a certain character – and for this the attributional part of entitlement is needed.

Entitlement of substance is meant to deliver *just enough* to sustain a certain conception of the world and our interaction with it – nothing more, nothing less. Wearing broadly realist glasses this means that, in the empirical case, entitlement of substance provides an entitlement to the existence of some stuff – or substance – which, ultimately, makes up the world (this is the existential part), *and* an entitlement to take this stuff to be robust, or mind-independent, in character (this is the attributional part).

We will now move on to the task of extracting an explicit characterization of the notion of entitlement of substance from what has been said so far. As shall transpire, the characterization provided will differ from the one sketched by Wright in a crucial respect – *viz.* what is required concerning coherence. However, it will be proposed that the characterization to be given here is closer to the spirit of the entitlement strategy than what Wright explicitly allows.

Let us start by introducing some terminology that will prove useful:

> **Conception of substance:** a conception of substance C_R for a region of thought R is a conception of the general nature, or character, of the substance, or stuff, that fundamentally constitutes R-states of affairs.
>
> **Conception of interaction:** a conception of interaction CI_R for a region of thought R is a conception of the character of a subject's investigation of, or interaction with, R-

[2] Wright [154], p. 203.
[3] Wright [154], p. 205.

states of affairs. (Conceptions of substance for a region of thought constrain conceptions of interaction.)

World view: a world view $W_{C_R-CI_R}$ for a region of thought R generated by a conception of substance C_R is C_R together with a conception of interaction CI_R which is compatible with the constraints imposed by C_R. That is, $W_{C_R-CI_R}$ is a view on the general nature of the substance that fundamentally constitutes R-states of affairs and the character of our investigation of, and interaction with, it.

Objective region of thought: an objective region of thought is any region of thought R with an associated conception of substance C_R and conception of interaction CI_R according to which R-experiences are conceived as being cognitively local and objective – meaning, among other things, that the notion of there being states of affairs obtaining 'elsewhere' is endorsed.

Basic ontology of a conception of substance: A basic ontology O_{C_R} of a conception of substance C_R for a region of thought R is a substance, stuff or range of things that realize C_R, or of which C_R is true.

With these notions in place, we can characterize the notion of entitlement of substance. Note that Wright explicitly refers to *objective* regions of thought, i.e. regions of thought that deal in states of affairs that are constitutively mind-independent. Here is the characterization:

Entitlement of substance: let O_{C_R} be a basic ontology of a conception of substance C_R for an objective region of thought R. Then a thinker is entitled to O_{C_R}, i.e. has an entitlement of substance to accept that the basic ontology of the world view $W_{C_R-CI_R}$ (i.e. the world view generated by C_R) exists just in case:

(a) Kantian presupposition: the conception of substance C_R is a Kantian presupposition for conceiving of R-experiences as being objective. That is, C_R is essential to the conception of R-experiences as objective; it is only via C_R that the conception of R-experiences as objective is possible; only via C_R can sense be made of states of affairs obtaining elsewhere.

(b) Default: there is no sufficient reason for thinking the conception of substance C_R incoherent.

Here the assumption that R is objective involves a conception of R-experiences as objective, and so, a conception of them being cognitively local. This in turn relies on the idea of states of affairs obtaining elsewhere or being constitutively mind-independent.

Clause (a) characterizes the conception of substance C_R as an unavoidable commitment for somebody who wants to conceive of R-experiences as objective. It is so in the sense that it is not possible to conceive of R-experiences as objective without it. Clause (b) is a default condition. It imposes a coherence constraint, but one that can be met by default, i.e. by the *absence* of sufficient reason for thinking that the relevant conception of substance is incoherent.

Now, to repeat, here is a gloss on the kind of thought that the target notion of entitlement tries to capture: certain regions of thought are considered objective, and, in particular, for such regions experience is taken to be objective. It requires a certain kind of world view to vindicate the objectivity of such regions – and, in particular, the conception of experience as objective. For on such a conception experience has to be conceived as being of something within the subject's cognitive locality, but with other states of affairs obtaining elsewhere. This enforces the need for the world to be of a nature that can sustain the idea of states of affairs being elsewhere. Here the notion of entitlement of substance – a two-part affair – is meant to kick in. First, it delivers an entitlement to the existence of a substance – stuff or range of things – and, second, it yields an entitlement to take this substance to have certain general features that will support a world view of the required kind.

A number of clarifying points and comments are worthwhile making. I will make five:

First, it is implicitly assumed that we have R-experiences. This is an assumption we can allow ourselves.

Second, in stating the characterization of entitlement of substance, we do not take R-experiences to *be* objective. This is what an entitlement of substance will deliver a warrant for thinking, but not what we start out assuming. Rather the starting point is a certain *conception* of R-experiences, *viz.* that they are conceived of as being objective. Given this conception of

R-experiences philosophical unpacking shows that it brings on a commitment to a certain kind of substance, a warrant for whose existence is delivered as a matter of entitlement (of substance). With this warrant in place, we have a warrant to a substance that will sustain the claim that R-experiences *are* objective. That is, we go from conceiving of R-experiences as having a certain property (being objective) to having a warrant for thinking that they have that property.

Third, one might reasonably wonder if, for any given region of thought R, there is really just one conception of substance C_R for that region – the idea being that C_R is a 'maximally general' conception of the substance of R that subsumes any other conception of the substance of R which might sustain the objectivity of R-experiences. The talk of C_R being 'essential' and its being a necessary condition for making sense of states of affairs obtaining elsewhere suggest that there has to be such a maximally general conception. Given the understanding of objectivity adopted in Chapter 2, what this maximally general conception has to be able to sustain is the constitutive mind-independence of R-states of affairs. If there is such a conception, we can refer explicitly to it – rather than to *a* conception of substance – in the characterization of entitlement of substance.

Fourth, it should be stressed that the characterization given above differs from what Wright himself says in the passage quoted earlier. In sketching the notion of entitlement of substance, he imposes the requirement that conception of substance should be coherent.[4] It would be desirable to have entitlement of cognitive project and entitlement of substance unified by the feature that they are both non-evidential in character and can be regarded as default notions of warrant. However, we do not get such unification if we stick to Wright's own formulation. The coherence requirement is a positive rather than negative requirement and makes Wright's own characterization of entitlement of substance dissimilar to what I take to be the key thought behind the entitlement proposal, *viz.* that we can sometimes be non-evidentially warranted. Hence, to obtain unification in this respect, I have chosen to formulate the coherence requirement so that it is a default, or negative, clause. Adopting this clause, entitlement of substance – like entitlement of cognitive project – is a non-evidential, default notion of warrant. What matters

[4] Wright's focus on coherence in a matter of ontology is by no means new to the literature. Indeed, some authors take coherence as a criterion of existence in mathematics. One such author is Shapiro. See Shapiro [127], p. 95 and p. 105.

is not the presence of specific positive evidence supporting the coherence of C_R, but rather the absence of (sufficient) countervailing evidence to think it incoherent.

Fifth, the characterization of entitlement given was stated in terms of objective *experiences*. We can drop this restriction and obtain a more general characterization by talking instead about objective *judgements*. I take it that anything we are willing to consider an objective region of thought, or a domain of discourse, is also something whose subject-matter we take ourselves to make judgements about. On the assumption that our source of mathematical beliefs and knowledge is not experience, we need this generalization if we are to apply entitlement of substance in a mathematical setting. (This generalization is analogous to the generalization made when introducing the notion of an Q-cornerstone in Section 2.14.)

Having offered a proper characterization of the notion of entitlement of substance, let us briefly indicate how it can be employed to respond to I-II-III scepticism. In the first section of this chapter, it was observed that I-II-III scepticism has two aspects – one existential, the other attributional. With Moore's proof what is up for debate is not just that there is a world, but that there is an external world. Hence, to adequately deal with I-II-III scepticism, any proposed response must block both aspects of the challenge.

Entitlement of substance has the potential to do so. The notion is supposed to be entitling in exactly the two respects attacked by the sceptic. An entitlement of substance is an entitlement to take it that a substance with certain general features exists. Let us apply the notion to the external world case.

Recall that the I-II-III sceptic attacks a *world view* for a given region of thought R, i.e. a combination of a certain conception of the substance of R and our interaction with it. The region of thought we are concerned with here is our thinking about the empirical world. The relevant conception of substance is one according to which the world is external, in the sense of being constitutively mind-independent. The conception is paired with a conception of our experiences as being objective. To determine whether we have an entitlement to the basic ontology of the conception in question we need to check if it is a Kantian presupposition of our conception of experience as objective, and whether it satisfies the default condition concerning coherence. (Recall that the basic ontology of a conception is a substance, stuff or a range of things that

realize it or of which it is true.) We deal with the clauses in turn:

The conception of the empirical world as external is a Kantian presupposition of the conception of experience as objective. That is, this conception of the empirical world is essential to our conception of experience (of it) as objective. Objectivity, as we have understood it, is linked to mind-independence. Conceiving of experience as objective involves conceiving it as being *of* something, and crucially, something which is not of our making, i.e. something constitutively mind-independent. Only by taking the world to be external can we sustain this idea. Without ascribing this feature to the world we could not properly conceive of the empirical world as being 'out there', as allowing for states of affairs obtaining 'elsewhere' and not being of our making.

The conception of the empirical world as external also satisfies the default clause. There is no sufficient reason to think it incoherent. There may be difficult questions that one has to provide answers to, e.g. *exactly* how perception works. There are two things to say to this. First, any thesis about the general nature of some significant tract of reality is bound to be confronted with a range of difficult questions that need answering. Second, and perhaps more to the point, there being difficult questions addressed at a certain conception of the world does not mean that there is reason – let alone sufficient reason – to think it incoherent.

In sum, then, the existence of a substance for which 'external world' is an appropriate label is an entitlement of substance. The I-II-III sceptic proceeded on the assumption that the type III proposition – here that there is an external world – can only be warranted evidentially. We rejected this assumption and now have a non-evidential notion of warrant, entitlement of substance, to back our rejection. As said, it is our entitlement to take it that the relevant substance has a certain feature that sustains the objectivity of our interaction with the region. The feature in our present case is, we recall, that the world is *external*.[5]

[5] For accounts of perceptual warrant that imply a different take on I-II-III arguments than Wright's, see Burge [21] and Pryor [110] and [112]. On Burge and Pryor's views, the type I proposition *by itself* yields a perceptual warrant for the type II proposition. Unlike Wright, there is no requirement of collateral warrant for the type III proposition.

4.3 Entitlement of (mathematical) substance

Applied to the mathematical case, the entitlement proposal suggests that we can be non-evidentially warranted in accepting that there is a realm of numbers, conceived as abstract, non-spatio-temporal entities. In the discussion of Cartesian scepticism, the notion of entitlement of cognitive project was used to give body to the idea of non-evidential warrant. In our discussion of I-II-III scepticism, we will employ the notion of entitlement of substance.[6]

The response goes like this: the mathematical I-II-III sceptic launches her attack in a specific context, one in which mathematical reality is conceived in a realist fashion and our judgements about it conceived so as to accord with such a conception. In particular, our mathematical judgements are regarded as being objective. The I-II-III sceptic argues that there can be no warrant for thinking that mathematical reality, realistically construed, exists, and thus, that our conception of mathematical judgement as objective cannot be sustained. However, we can maintain that we have a non-evidential warrant for thinking that there is such a mathematical reality *provided* that this reality satisfies the two conditions on a basic ontology qualifying as an entitlement of substance. It does.

To see that this is so, we need to set out the antecedent part of the characterization of entitlement of substance. For arithmetic, the realist world view combines a conception of substance C_R according to which the stuff, or substance, that constitutes arithmetical states of affairs is abstract and without spatio-temporal location, and a conception of interaction CI_R according to which arithmetical judgement is objective. Taken together C_R and CI_R makes a world view $W_{C_R-CI_R}$. The basic ontology O_{C_R} of $W_{C_R-CI_R}$ is a substance, some stuff or range of things that realize C_R or make C_R true. In other words, the basic ontology of the realist (arithmetical) world view is an abstract, non-spatio-temporally located substance or some stuff or range of things that fit this description.[7]

[6] An immediate worry about this proposal is that it seems to deliver an entitlement of substance to some species of mathematical platonism. Having noted that I am aware of this worry, I will allow myself to bracket it until the end of the section. However, let me put the main point now: if anything, there is only a *conditional* entitlement to platonism.

[7] The talk of 'substance' or 'stuff' might be thought to suggest that we are dealing with something that is spatio-temporally located. I do not intend to use the terms in such a way here, as should be clear from our present discussion.

Having set out the antecedent of the characterization, we need to check if the two clauses are satisfied, i.e. whether C_R is a Kantian presupposition for conceiving of arithmetical judgements as objective, and whether C_R meets the default clause for coherence. First, is C_R essential to the conception of arithmetical judgements as objective? That is, is it only by taking on board the conception of the substance or stuff that constitutes arithmetical states of affairs as abstract and without spatio-temporal location that we can conceive of arithmetical judgement as objective, understood as supporting the idea of arithmetical states of affairs obtaining 'elsewhere'? As stressed earlier, what appears to be crucial to make sense of the idea of states of affairs obtaining elsewhere is that they are constitutively mind-independent. Thus, how strong a case can be made for the Kantian presupposition clause being met depends on how strongly it can be argued that conceiving of the stuff or substance that constitutes arithmetical states of affairs as abstract and without location in space-time is the only way to secure the required constitutive mind-independence.

Recall that C_R is supposed to be maximally general. Bearing this in mind, one may reasonably wonder whether every conception of the substance of arithmetic that secures the constitutive mind-independence of arithmetical states of affairs is one according to which the substance or range of things that make up these states of affairs are abstract and not located in space-time. Arguably, this is so for every brand of platonism. On the other hand, any brand of constructivism fails to make the substance of arithmetic constitutively mind-independent by tying mathematical existence to the operations that can be executed by some (more or less idealized) subject, and so, need not be considered here. Positions like Field-style nominalism are harder to reach a verdict on. On Field's account of mathematics, mathematical entities – or rather, their nominalistic counterparts – are accounted for by appeal to space-time points and regions, i.e. certain *physical* entities.[8] Provided that we are realists about the physical world, it would thus seem that one can maintain that the substance of arithmetic caters for the mind-independence of arithmetical states of affairs (because this substance is physical and such substances are constitutively mind-independent), but at the same time hold that it is neither abstract nor without spatio-temporal location.

[8] Field [46].

I am not sure that I have an effective response to this. However, one thing to note is that it is not clear that Field-style nominalism may be relevantly invoked when we are discussing *mathematical* I-II-III arguments. After all, according to this kind of nominalist, there are no mathematical entities as traditionally understood. There are only nominalistic analogues, and these are all physical entities. Thus, if anything, the Field-style nominalist might more appropriately be brought up in a discussion of the I-II-III argument as formulated about the external world.[9]

Supposing that we can meet the potential worry just raised, we proceed to ask if the second condition is met, i.e. whether there is no sufficient reason for thinking C_R incoherent. Sure, there are issues and problems pertaining to the conception of the substance of arithmetic as abstract and without space-time location – most famously, perhaps, the kind of epistemological worry exemplified by one horn of Benacerraf's much celebrated dilemma: given that mathematical entities are abstract and outside the causal nexus, how can we give a plausible account of our knowledge of them?[10] No doubt there are other problems and issues. However, there being a problem with a view or conception is one thing, there being a sufficient reason to think it incoherent quite another. As far as I know, there is no such reason.

If the rehearsed reasoning is found to have something to it, maintaining that we have an entitlement of substance to the basic ontology of the particular conception of the substance discussed is a viable option. The basic thought is that, provided that the platonist conception is essential to sustaining the objectivity of arithmetical judgement (and there is no sufficient reason to think it incoherent), then we have an entitlement to the existence of a substance that realizes the conception, i.e. a substance according to which the tract of reality with which arithmetic deals is abstract and not located in space-time. The idea behind the entitlement

[9] Here is a suggestion for a I-II-III argument:

	I	My experience is in all respects as if $2 + 2 = 4$.
So,	II	$2 + 2 = 4$.
		If $2 + 2 = 4$, then there is a realm of numbers.
So,	III	There is a realm of numbers.

Here the realm of numbers is conceived nominalistically as being given by space-time points. Without a warrant for there being a realm of numbers, the transition from I to II is not warranted. Because in that case there is no warrant for thinking that the experience mentioned in I represents anything at all.

[10] Benacerraf [7].

of substance proposal is that philosophizing can help unpack certain ontological commitments, or theses, from a conception of our interaction with a region of thought as being objective – and that we can be warranted, as a matter of entitlement of substance, to taking it that these commitments are met.[11]

It is worth making a number of points on what to make, and, crucially, what not to make of the suggestion that there can be a non-evidential warrant for thinking that the basic ontology of a conception of substance exists.

First, as was stressed when the notion was introduced, an entitlement of substance is unspecific. We are only entitled to take it that some substance with very general features exists.

Second, the default clause connects coherence and existence. However, it does *not* make the absence of sufficient reason to think incoherent a criterion of existence for mathematical substances – just as the default clause for entitlement of cognitive project does not make the absence of countervailing reason to believe P untrue (necessary and) sufficient for the truth of P, when P is an entitlement.

Third, one can draw a distinction between two questions that an epistemology of axioms might address. First, there is the question in what sense, if any, the axioms are first principles? Second, there is a question about rationality – how can it be rational to accept that the axioms are true or satisfiable? The paradigmatic, traditional understanding of first principles can be found in Euclid and Descartes. According to this understanding, what is characteristic of a first principle is its possession of some special epistemic property – being obvious, self-evident, or indubitable, say – which is grasped immediately, perhaps through some special faculty like the 'light of reason' or 'rational intuition'. Entitlement of substance, as well as of cognitive project, does not tell us anything about any such matters, and, indeed, special epistemic properties

[11] There are similarities in spirit between the application of entitlement of substance and indispensability arguments. The indispensability argument can be summarized as follows: scientific theories are true. Scientific theories can only be true, if the mathematical entities needed for their truth exist. So, the mathematical entities needed for the truth of scientific theories exist. In other words, an ontological thesis – the existence of mathematical objects – is taken to follow from its indispensability to the task of accounting for the truth of certain theories. Somewhat analogously, according to the entitlement of substance proposal, a warrant for an ontological thesis – that the basic ontology of a conception of substance exists – is taken to flow, partly, from its indispensability to operating a certain scheme of thought according to which our interaction with the relevant tract of reality is objective. The indispensability argument has most famously been advocated by Quine and Putnam. See Quine [118] and Putnam [114].

and faculties are ascribed no justificatory significance on the entitlement proposal. (Because the entitlement proposal introduces a notion of *warrant*, but without appealing to any such properties or faculties.) Thus, if providing an account of the status of axioms as first principles (if taken to be so) is thought to be a task for special epistemic property/faculty epistemology, it is also a task that the entitlement proposal will not be of any help in addressing. However, it does seem of relevance to the second question. One suggestion is that it is rational to take axioms to be true or satisfiable in order to sustain certain aspects of epistemic practice. For entitlement of cognitive project, it is rational to trust that $Sat(T)$ when engaging in a T-project, because not doing so would undermine any significance of the project. For entitlement of substance, it is rational to take it that a substance with certain features exists, because not doing so would clash with important features of our scheme of thought – in particular, that our interaction, be it experience or judgement, is objective.

Fourth, one might worry that entitlement of substance delivers too much. In both the case of the external world and of the mathematical realm, entitlement of substance delivers an entitlement to some brand of realism. However, should a notion of warrant, properly so-called, really recommend a specific philosophical doctrine? Shouldn't the notion be at the service of individuals with other philosophical leanings? In the mathematical case the answer to this worry is that entitlement of substance delivers at most a conditional recommendation of platonist realism. This should be clear from the 'package deal character' of the notion. An entitlement of substance to a realist basic ontology comes as a part of a larger realist package, and you do not get an entitlement to a realist basic ontology unless you buy into the whole package. The crucial point here is what specific conceptions of substance and interaction come as part of the package. As seen earlier, it is the conception of our interaction with the relevant tract of reality as objective that recommends realism. Without this conception, there is no such recommendation. For instance, if I am an idealist, there is no way that the notion of entitlement of substance will force realism about the empirical world upon me, and similarly, there is no way that a constructivist will be compelled to take up realism about mathematics.

4.4 Conclusion

In Chapter 2, we formulated two kinds of mathematical scepticism, regress scepticism and I-II-III scepticism. Chapter 3 responded to the regress sceptic by invoking the notion of entitlement of cognitive project, and in this chapter we responded to I-II-III scepticism by appealing to the notion of entitlement of substance. It has been worthwhile doing so. The entitlement approach is an interesting one, but also one whose philosophical significance it is difficult to assess at this stage. To a large extent this is because Wright's take on entitlement is a very recent addition to the literature, and there has not yet been much discussion of potential applications and problems. By suggesting – and spelling out – an application of the entitlement proposal in a mathematical setting, I hope that this chapter and the two preceding it can be regarded as a contribution to the discussion of the application aspect.

Chapter 5

Three ways of understanding entitlement of cognitive project

In the three previous chapters, we have been concerned with two kinds of mathematical scepticism – regress and I-II-III – and the application of respectively the notions of entitlement of cognitive project and substance as a response to these. This chapter is devoted to developing the notion of entitlement of cognitive project further. In particular, it is reasonable to ask how we are to understand clauses (ii) and (iii). When it is said in clause (ii) that there should be no sufficient reason to believe P untrue – P an entitlement – what domain of reasons for belief is it relevant to consider? And what class or classes of methods and procedures are relevant to determining whether clause (iii) is met? I start this chapter by bringing attention to two awkward wrinkles concerning entitlement of cognitive project left by the little Wright explicitly says about how to understand the notion, especially when it comes to clauses (ii) and (iii). I proceed to introduce a distinction between three different ways of understanding entitlement of cognitive project and give examples corresponding to these. It is then shown how the distinction can help us to remove the two awkward wrinkles. I conclude the chapter by considering a number of issues, the discussion of which will help to shed further light on the notion of entitlement of cognitive project.

5.1 Two awkward wrinkles

We now introduce two awkward wrinkles concerning the notion of entitlement of cognitive project left by what Wright says about the notion.

Awkward wrinkle 1: are entitlements of cognitive project really defeasible? According to Wright, entitlements of cognitive project are defeasible. This is a consequence of entitlement being a default notion of warrant and is signalled by the presence of the 'unless' in saying that P is an entitlement unless there is sufficient reason to believe it untrue. The idea must be that, though there is no sufficient reason to believe P untrue – and so, clause (ii) is satisfied – such a reason may nevertheless present itself and undermine the satisfaction of clause (ii). However, as shall transpire, the problem here is to give an account of how to understand 'sufficient reason' in such a way that defeasibility is ensured.

Frege's Basic Law V can be used as an example:

$$\text{(BLV)} \qquad (\forall X)(\forall Y)(Ext(Y) = Ext(X) \leftrightarrow (\forall x)(Xx \leftrightarrow Yx))$$

i.e. for all concepts X and Y, the extension of X is identical to that of Y just in case X and Y are co-extensional. One might think that Frege was entitled to rely on (BLV), *but* that his entitlement was defeated, because it turned out to be inconsistent.[1] Frege had an entitlement to (BLV) in engaging with his projects in the *Grundgesetze*, but lost it, because the inconsistency implied a failure to meet clause (ii) of the characterization of entitlement of cognitive project.

[1] In Wright [153] and [155], the soundness of basic logical laws – like modus ponens – is taken to be an entitlement of cognitive project. In Wright [150], pp. 57–58, (BLV) – formulated as a pair of rules – is discussed, but not in the context of an investigation of entitlement. However, still, in light of the application of entitlement to logical rules, the present example might be found apt.

It should be noted that, strictly speaking, (BLV) is a special case of Frege's own statement of Basic Law V in §20 of the *Grundgesetze*. Frege's own statement is in terms of courses-of-values rather than extensions: for all functions f and g, the course-of-values of f is the same as that of g just in case f and g yield the same value for every argument. However, in the literature, (BLV) is often referred to as 'Basic Law V'. This, I suspect, is due to Russell's letter in which he alerted Frege of the inconsistency, and, perhaps even more so, due to Frege's own letter back to Russell (Russell [122] and Frege [55]). Russell put the inconsistency in terms of the predicate w, the predicate of being a predicate that cannot be predicated of itself. He does not mention Basic Law V, but talks instead of §9 of the *Begriffsschrift*, the paragraph on functions. Frege himself realized that there was trouble with Basic Law V of his *Grundgesetze*. In his letter to Russell, he starts out talking about Basic Law V, as stated in terms of courses-of-values, and then switches to talk of extensions at the end of the letter. This switch also indicates the switch to the special case of the law given here as (BLV).

This example raises a number of interesting questions about entitlement of cognitive project and how to properly understand it. The crucial question here is what to make of the idea of there being no sufficient reason to believe untrue in clause (ii). If we wish to retain the idea that Frege was initially entitled to (BLV), but that his entitlement was defeated, there are certain paths we cannot take. One might think that logical facts are eternal and entirely independent of us. In particular, one might think that it is a logical fact that (BLV) is inconsistent, and that this is entirely independent of us; that it has always been, is, and will always be so. In addition, one might take it that the sufficient reason – if there is one – need not be within the 'epistemic field' of the subject, i.e. that a defeater need not be accessible to the subject. However, if we take all of this on board, we get the following verdict on (BLV) and its candidacy for being an entitlement: Frege did not have an entitlement to (BLV) before he became aware of its being inconsistent. There was always a defeater around (*viz.* its inconsistency), and so, (BLV) never satisfied clause (ii), and hence, was never an entitlement. Perhaps Frege could claim to be entitled to (BLV), but when realizing its inconsistency, he would, at the same time, realize that his claim was incorrect and that he never really had an entitlement.

The view on logical facts and reasons for belief relied on in the above line of reasoning do have adherents, and may be found not all that far-fetched.[2] The reasoning leaves room to manoeuvre, though. Still, we need to do the manoeuvring in order to maintain that entitlements of cognitive project are defeasible. This is a task we will pursue in the next section.

Awkward wrinkle 2: is an entitlement to P compatible with genuine progress with respect to P?

Suppose that a subject S has an entitlement of cognitive project to P. P can only be an entitlement if it can be ruled out that there can be justification – i.e. evidential warrant – for P. This is a consequence of clause (iii) according to which any attempt to justify P will lead to an infinite regress of justificatory projects involving some presupposition of no more secure a prior standing. As seen earlier, the involvement of some presupposition of no more secure a prior

[2] An expression of the kind of realist conception of logical facts appealed to is present in various of Frege's writings. See, e.g., Frege [54]. A realist conception of logic is also often ascribed to the early Wittgenstein. Cf. Wittgenstein [143].

standing in the infinite regress of justificatory projects implies that attempts to justify P cannot improve its epistemic standing. Consequently, the status of a proposition P as an entitlement appears to be incompatible with the view that genuine progress can be made with respect to P. However, do we really want entitlement of cognitive project to be understood in this way – do we really want to say that, as a matter of principle, there can *never* be any progress with respect to P?

5.2 Three ways of understanding entitlement

In the previous section we stated two awkward wrinkles concerning the notion of entitlement of cognitive project. In this section, we will distinguish between three ways of understanding entitlement, which will enable us to eliminate the awkward wrinkles.

What is crucial to get clear on is how to understand clauses (ii) and (iii). Each of them can be read in two significantly different ways – one relative, the other absolute (where P is a candidate entitlement and S the potentially entitled subject)[3]:

Information-and-method-relative understanding of entitlement:

- The requirement that there is no sufficient reason to believe P untrue is met just in case there is no sufficient reason for S to believe P untrue relative to the information S has.
- The requirement that attempts to justify P give rise to an infinite regress of justificatory projects involving presuppositions of no more secure a prior standing is met just in case such a regress results when attempts are made to justify P by reliance or employment of the capacities, methods, and procedures in S's current repertoire.

[3] Here I would like to thank Robbie Williams and Crispin Wright for making very helpful suggestions during an Arché Mathematics Seminar.

Absolutist understanding of entitlement:

- The requirement that there is no sufficient reason to believe P untrue is met just in case there is, absolutely speaking, no sufficient reason for S to believe P untrue.
- The requirement that attempts to justify P give rise to an infinite regress of justificatory projects involving presuppositions of no more secure a prior standing is met just in case such a regress results no matter what capacities, methods, and procedures S can rely on or employ in attempts to justify P.

The information-and-method-relative reading tells us that whether or not a subject is entitled is a matter that turns on features of the current epistemic situation of the subject. (In the sequel I will often use 'relativistic' or 'relativized' instead of 'information-and-method-relative'.) As for clause (ii), whether or not there is sufficient reason to believe P untrue is a matter of what is included in S's 'epistemic field' or 'epistemic baggage' – i.e. what information S has access to. As for clause (iii), the attempts to justify P of relevance are those executed by reliance or employment of capacities, methods, and procedures currently available to the subject. On this reading, the infinite regress of justificatory projects might be a function of the restricted class of methods and procedures in the repertoire of the subject. On the other hand, on the absolutist reading the answer to the question whether a subject is entitled is independent of her current epistemic situation. Whether or not there is sufficient reason to believe P untrue is independent of what information is available to S. The absolutist reading thus requires that reasons for belief that go beyond a subject S can be of relevance to the question whether S is entitled to a proposition P. The infinite regress of justificatory projects must result from attempts to justify P executed by appeal to not just the methods and procedures in S's repertoire, but any methods and procedures *whatsoever*.

It might be tempting to think that the difference between the two readings derives from a difference in what *kinds* of things are counted as reasons for belief. It need not be so. It is possible to give a uniform account of the range of things that are counted as reasons for belief, information-relative or absolute. Once this account has been provided, the absolutist reading can be taken to incorporate the thesis that the domain of reasons relevant to the question whether a

subject S is entitled to a proposition P includes everything that is counted as a reason for belief on the proposed account. Let this domain be D. We can then take the relativistic reading to incorporate the thesis that the justificatorily relevant range of reasons for belief is a sub-domain of D. The restriction results from an enforcement of the requirement that the justificatorily relevant reasons be such that they are included in the information S has.

What was said in the previous paragraph was not put in very specific terms. Let me try to do so now. If there is to be a uniform account of reasons for belief and the relativistic and absolutist readings are to relate to each other in the manner just indicated, reasons for belief have to be such that they are intimately tied to information. The most obvious way to accommodate this is to take them to *be* the kind of thing that codes or carries information. One candidate is belief. However, if reasons are beliefs, then it would be difficult to make sense of the absolutist reading – if this reading is meant to sustain an understanding of entitlement on which the domain of reasons for belief is supposed to potentially outrun the 'epistemic field' of the subject. Instead let us try to see where we end up by taking reasons to be propositions, i.e. the kind of thing that belief is an attitude towards. Propositions code or carry information by having propositional content.[4]

On the absolutist reading we can take the domain of reasons for belief to be all the propositions of the relevant region of thought. This yields that there is (absolutely) no sufficient reason to believe a proposition P untrue just in case there is no proposition (possibly non-atomic) which is true and whose truth suffices for the untruth of P. Running along these lines, it could be that, for some subject S, for all she believes, clauses (i)-(iii) are met for a proposition P, and so, for all she believes, she is entitled to P – and yet, in fact, she is not so entitled, because unbeknownst to her some other proposition Q is true and the truth of Q suffices for the untruth of P.[5]

The approach to clause (ii) just described makes entitlements non-defeasible. Or rather, it does on a certain assumption – that the region of thought we are concerned with is such that, for

[4] For present purposes, we do not, as far as I can tell, need to make any specific assumptions about the nature of content.

[5] If, as we have assumed, we are in a classical setting, this way of understanding absolute entitlement will rule out the possibility of any false proposition qualifying as an entitlement, absolutely speaking. For if P is false, then there is a (non-atomic) proposition whose truth suffices for the untruth of P, *viz.* $\neg P$.

every proposition of the region, it is a determinate whether or not the proposition is true and this determinacy is constitutively mind-independent. For a subject S and a candidate entitlement P (satisfying clauses (i) and (iii)), it may be that, for all S believes, there is no sufficient reason to believe P untrue. Given the assumption just flagged (and taking propositions as reasons for belief), either there is, absolutely speaking, no sufficient reason to believe P untrue or there is. If the former is the case, then S is (absolutely) entitled to P, whereas, if the latter, S is not (absolutely) entitled to P.[6]

As the first awkward wrinkle reminds us, Wright is keen to stress that entitlements of cognitive project are defeasible. This suggests that Wright does *not* intend clause (ii) to be read in the absolutist way. Still, it should be noted that, on the absolutist reading, one can still maintain the idea that a subject's right to *claim* entitlement is governed by what information she has and the method and procedures in her repertoire, while whether she is (absolutely) entitled or not is independent of what information she has access to and what methods and procedures she can employ. Though entitlements themselves are not defeasible on the absolutist picture, the right to claim entitlement can be regarded as such. Consider a proposition P and a subject

[6] Let me say a bit to qualify the need for the determinacy assumption. The assumption has two parts. One posits determinacy of truth-value, the other that this determinacy be of a certain nature or have a certain source. The legal domain seems to be a plausible example of a region where the assumption is not met. Most laws are incomplete. They typically fail to determine completely what is and what is not lawful. Not every 'legal proposition' is determinately true or false. Without determinacy, it can be indeterminate whether or not there is, absolutely speaking, sufficient reason to believe P untrue (P some proposition). It might be that there are lots of propositions whose truth would imply the untruth of P – and so, would be a sufficient reason to believe P untrue – but its simply being indeterminate whether or not these propositions are true. In that case the absolute reading fails to support non-defeasibility. However, not because entitlements are defeasible, but rather because it is indeterminate whether they are. That is to say, there are cases where it is indeterminate whether, absolutely speaking, there is sufficient reason to believe some proposition P untrue.

Without the region of thought being constitutively mind-independent, we do not get non-defeasibility either – even on the assumption that, for every proposition, it is determinate whether it is true or not. Consider the legal domain again. Let us just suppose that the law is complete, i.e. that it completely determines what is lawful and what is not, and so, for every legal proposition, the law tells us whether it is true or not. However, we do not get non-defeasibility from this. For though, by assumption, the law determines a truth-value for every legal proposition, it does not do so stably. The law is mind-dependent. It can be changed. Specific laws can be amended or otherwise modified, or dropped. New laws can be added to the existing law. Suppose that current law says that some legal proposition P is true. Well, the law-makers can change the law so P is not true. Today's crime might not be a violation of the law tomorrow.

Without stability, it may be that, absolutely speaking, there is no sufficient reason to believe a proposition P untrue. However, if the law is somehow modified, this could change. Some proposition Q, whose truth suffices for the untruth of P, could be true according to the modified law. In that case there would, absolutely speaking, be sufficient reason to believe P untrue. Thus, if we have a region of thought which is constitutively mind-dependent, there could, absolutely speaking, be an entitlement to some proposition P, which was defeated due to a change in the facts of the relevant region.

S with a right to claim entitlement to P. Then the idea is that adding a piece of information whose truth would undermine the entitlement to P or a method or procedure the employment of which would generate such a piece of information undermines S's right to claim entitlement to P.

Let us move on to the relativistic reading of entitlement. The range of reasons regarded as relevant to the question whether there is sufficient reason to believe P untrue is, on the relativistic reading, a sub-domain of the range of reasons appealed to on the absolutist reading. Let these two domains be D_R and D_A. (Also, remember we adopted the proposal that reasons for belief are propositions, and that propositions code, or carry, information.) D_R is the sub-domain of D_A obtained by taking the propositions of D_A that are available to S, i.e. are part of the information she has.

Here is how to characterize D_R by imposing restrictions on D_A. Among the propositions of D_A, there are the ones that are within the 'cognitive range' of S – the propositions which S has the cognitive resources to grasp or understand. 'Cognitive resources' is meant to refer to the capacities, methods, and procedures of S. Among these propositions we certainly find those propositions of D_A to which S holds some doxastic attitude as, e.g., belief, disbelief, open-mindedness, or doubt.[7] However, we should include more propositions than these. For, at any given time, a subject holds doxastic attitudes to only a few propositions, and so, the propositions within S's cognitive reach would, by this standard, be few. In addition, we should count those propositions to which the subject S would hold some doxastic attitude if S were to exercise some capacity like introspection or memory, and also, any proposition P to which S would hold a doxastic attitude if she were to exercise or employ all her capacities, skills, methods and procedures.

What doxastic attitude S holds to a cognitively reachable proposition can change over time. In particular, such a change in attitude takes place when S learns something new, realizes that something she believes is false, when the evidence supporting her belief that P is suddenly counterbalanced by new evidence and she turns open-minded about P, etc. Note that the

[7] Disbelieving P is here taken to be believing $\neg P$. Being open-minded about a proposition P is understood as neither belief nor disbelief that P, and doubting P is understood as being weaker than disbelieving P, but stronger than open-mindedness about P.

counterfactual clause by which the cognitively reachable propositions are partially delineated is compatible with there being some such propositions which S has never considered. One way in which S can learn something new is to come to believe a proposition she never previously considered by exercising certain of her capacities, methods or procedures.

Formulated generically in terms of the doxastic attitudes mentioned above, let me state some ways attitude-shifts can happen. I include the case where S comes to hold a doxastic attitude to some proposition she never considered before:

- S can come to believe, disbelieve, doubt, or be open-minded about a proposition she never previously considered.
- S can come to believe a proposition which she previously disbelieved, doubted, or was open-minded about, and conversely, S can come to disbelieve, doubt, or be open-minded about a proposition she previously believed;
- S can come to disbelieve a proposition which she previously believed, was open-minded about, or doubted, and conversely, S can come to believe, be open-minded about or doubt a proposition she previously disbelieved;
- S can come to be open-minded about a proposition she previously believed, doubted, or disbelieved, and conversely, S can come to believe, doubt, or disbelieve a proposition she was previously open-minded about.
- S can come to doubt a proposition she previously believed, was open-minded about, or disbelieved, and conversely, S can come to believe, be open-minded about, or disbelieve a proposition she previously doubted.

Suppose, e.g., that some subject S is doing a course in set theory. At a certain stage, S is posed as an exercise to prove that every von Neumann ordinal is a (pure) transitive set well-ordered by $\in$. S does the proof and comes to believe that, indeed, every von Neumann ordinal is a set of the mentioned kind, but S had not considered the matter prior to the proof. This is an example of a subject's coming to believe a proposition she previously held no doxastic attitude towards.

It is straightforward to give examples of the other kinds of transitions, but we shall not occupy ourselves with this task here.

The propositions included in S's information – D_R – are certain propositions of those cognitively reachable by S (which, in turn, are among the ones in D_A, the absolutist domain of reasons for belief). A proposition is included in D_R just in case S would believe it, were she to exercise her capacities, methods, and procedures to reach a verdict on the question whether P. Notice that this is compatible with D_R including propositions which S at a given point has never considered, is open-minded about, doubts, or disbelieves. The basic idea is that D_R include those propositions that S would believe at the end of her enquiry, employing her cognitive capacities and the methods and procedures in her repertoire.[8]

Let us see how clause (ii) comes out when paired with this characterization of D_R, the domain of reasons of belief for S.

Recall that, for an entitlement P and a subject S, the relativistic reading of clause (ii) demands that there be no sufficient reason for S to believe P untrue relative to her information, i.e. that there be no sufficient reason for S to believe P untrue in D_R. Given the way we have cashed out D_R this amounts to the requirement that there be no proposition Q (possibly non-atomic) whose truth suffices for P's being untrue and which S would believe, were she to exercise her capacities and all the methods and procedures in her repertoire to reach a verdict as to whether Q.

Notice that whether clause (ii) is satisfied on this reading is quite independent of S's current attitudes to the cognitively reachable propositions. This is a feature that the relativistic reading shares with the absolutist one.[9]

[8] There is some similarity here with Wright's notion of superassertibility, which is characterized as follows: 'A statement is superassertible ... if and only if it is, or can be, warranted and some warrant for it would survive arbitrarily close scrutiny of its pedigree and arbitrarily extensive increments to or other forms of improvement of our information.' (Wright [148], p. 48)

[9] In Chapter 3, we discussed the question whether there is an entitlement to the satisfiability and consistency of any list of well-formed axioms when the axioms are written down. The thought was that there might be, because there is evidence neither for nor against the satisfiability and consistency of such a list of axioms right when it has been written down. The importance of a working history was emphasized, and the question was answered in the negative. Such 'entitlements' would meet clause (ii) in a rather trivial or uninteresting way. This can be regarded as consonant with the relativistic approach in that a theory's having a working history represents an attempt to reach a verdict on propositions of the theory, to improve on the set of attitudes held to its propositions. The absolutist approach to entitlement, on the other hand, appears to make little room for working history to be significant to whether or not clause (ii) is satisfied. On that approach (and given the

Notice also that, on the relativistic approach, a proposition Q's yielding a failure of clause (ii) for some candidate entitlement P does not imply that Q is true. Clause (ii) can thus fail to be met because of some *false* proposition being in D_R, the domain of reasons for belief of the subject. Remember that clause (ii) – relativistically understood – demands that there is no proposition in D_R whose truth suffices for the untruth of candidate entitlement. Here the use of 'whose truth' falls short of implying that the proposition is true. This is a crucial difference compared to the absolutist proposal on which propositions that undermine clause (ii) will always be true. The difference can be put as follows: on the absolutist reading, what undermines clause (ii) – if it is undermined – is a fact, or a proposition made true by a fact, while, on the relativistic reading, clause (ii) is undermined by something the subject would *take to be* a fact, were she to investigate the matter fully by use of her capacities and methods and procedures currently in her repertoire. This is compatible with the underminer not being true (or there not being a fact making it true), as well as with its being so.

What about clause (iii)? As indicated, if P is an entitlement on the relativistic proposal, attempts to justify P might lead to an infinite regress of justificatory projects (involving presuppositions of no more secure a prior standing) as a function of the restricted class of methods and procedures in the repertoire of the entitled subject.[10] On the other hand, according to the absolutist reading, the class of methods and procedures relevant to clause (iii) comes without restrictions: attempts to justify P lead to an infinite regress of the required sort by employment of any methods and procedures *whatsoever*.

An interesting question is whether entitlements are defeasible on the relativistic reading. We have seen that entitlements are not defeasible on the absolutist reading. Defeasibility on the relativistic reading would thus mark a significant difference between the two. However, I will now argue that, strictly speaking, entitlements in the relativized sense are not defeasible. Here is the argument that entitlements in the relativized sense are not defeasible, strictly speaking:

Consider a subject S entitled to P relative to some domain of reasons for belief D_R and a

determinacy assumption), clause (ii) will be met or not independently of whether it has an elaborate working history. The facts by themselves settle whether there is a proposition (not necessarily atomic) which is true and whose truth suffices for the untruth of the candidate entitlement.

[10] It might also be that there are cases in which the class of methods and procedures available to the subject leaves her with no idea how to attempt to justify P, or perhaps the subject has options, but just very poor ones.

fixed class of methods and procedures CM. Recall that D_R includes every proposition Q which S would believe if she were to fully investigate the question whether Q by exercising her capacities and the methods and procedures of CM. Thus, P's status as an entitlement is determined by reference to the best qualified beliefs S can (counterfactually) come to hold given her current cognitive repertoire. In this sense there is no room for improvement or fluctuation with respect to what propositions are in D_R. For this reason entitlements relativized to a given domain of reasons and class of methods and procedures will, strictly speaking, be non-defeasible. Whether there is sufficient reason to believe a candidate entitlement P untrue is determined by D_R, and since D_R is stably fixed, so is the question whether or not clause (ii) is met.

There is, however, a way in which it is natural to say that such relativized entitlements are defeasible. What I have in mind is this: consider again a subject S entitled to P relative to some domain of reasons for belief D_R and a fixed class of methods and procedures CM. Consider some domain of reasons for belief D_{R*} which corresponds to another, *improved* class of methods and procedures CM^* obtained from CM through modification of its methods or procedures or addition of new methods or procedures. Now, it may be that D_{R*} – an improvement, or refinement, compared to D_R – includes a proposition Q whose truth suffices for the untruth of P. A consequence of this is that P is not an entitlement relative to D_{R*} and CM^*.

It is natural in this case to say that the initial entitlement – held relative to D_R and CM – is defeated, because the class of methods and procedures CM^* is *superior* to CM. However, the relativistic reading of entitlement does not allow us to say this. As just argued, entitlements are non-defeasible on this reading. However, we can express the idea that entitlements are defeasible by using the relativistic notion in conjunction with the notion of an 'open-ended entitlement':

Open-ended entitlement:

Let S be a subject with a cognitive project C. Let D_R and CM be a domain of reasons for belief and a corresponding class of methods and procedures and let D_{R*} and CM^* be another domain of reasons and class of methods and procedures, where CM^* is obtained by improving CM through modification of its methods or procedures or addition of new methods or procedures. Suppose that S is entitled to a proposition P relative to D_R and

CM. Then we say that P is an *open-ended* entitlement of cognitive project relative to D_R and CM just in case clause (i) of the characterization of entitlement of cognitive project is met and

(ii) There is no domain of reasons for belief D_{R*} and a corresponding class of improved methods and procedures CM^* such that D_{R*} includes a proposition Q (not necessarily atomic) whose truth suffices for the untruth of P.

(iii) Attempts to justify P by reliance or employment of the capacities, methods, and procedures of any improved class of methods and procedures CM^* give rise to an infinite regress of justificatory projects involving presuppositions of no more secure a prior standing.

To repeat, an entitlement relative to D_R and CM is an entitlement in the open-ended sense exactly when there is no improvement of D_R and CM which defeats P (i.e. on which there is sufficient reason to believe P false) and the regressive justificatory structure will result by attempts to justify P by any improved class of methods and procedures. Since clauses (ii) and (iii) range open-endedly over improved domains of reasons for belief and classes of methods and procedures, the open-ended entitlements are the relative entitlements that survive arbitrary improvement.[11]

It is crucial to stress that an extended class of methods and procedures CM^* being an *improvement* over the base class CM is not meant to suggest that it necessarily does better in terms of delivering truths when the relevant region of thought is investigated. CM^* may contain methods or procedures that lead to false findings. By an extended class of methods and procedures being an improvement over the base class is meant that the extended class is an improvement from the perspective of the subject – meaning that, by her lights, the addition of the new methods and procedures put her in a better position to explore the relevant region of thought (she has new ways of investigating it, can reach verdicts on questions that were

[11] Note that we can also define the notion of a 'bounded entitlement' by restricting the open-endedness proposal to some specific improved domain of reasons and class of methods and procedures. That is, let D_{R*} and CM^* be an improved domain and a corresponding class of methods and procedures. Then we can say that P is an entitlement of cognitive project relative to D_R and CM and bounded by D_{R*} and CM^* just in case clauses (ii) and (iii) are satisfied when read with the specific pair D_{R*}-CM^*.

previously unanswered, etc.).[12]

In this setting, the defeasibility of an entitlement P can be expressed as P's being an entitlement relative to some domain of reasons for belief D_R and class of methods and procedures CM, but P's failing to be an open-ended entitlement relative to D_R and CM (or its failing to be an entitlement relative to D_R and CM and bounded by some improved pair D_{R*}-CM^*). Note that P's being an entitlement might be defeated – i.e. might fail to be an open-ended entitlement – due to an 'early improvement' of CM. That is, it might be that adding, say, one new method to CM yields the resources to support a proposition Q whose truth suffices for the untruth of P.

We have now made sense of how relativized entitlements can be defeated by appeal to open-ended entitlements. However, what about open-ended entitlements themselves – are they defeasible? This depends on whether you think it is determinate what classes of methods and procedures the open-ended 'schemas' (ii) and (iii) appeal to. If it is, then either clauses (ii) and (iii) will be met or they will not, and there is no room for genuine defeasibility. That is, if P is an open-ended entitlement, it will stay that way, and, similarly, if it is not, it will never become one. If it is not determinate, things might be otherwise. The issue is a deep and difficult one – and interesting – but I do not think that I need to resolve it for present purposes. Whether or not we think that it is determinate what classes of methods and procedures the open-ended clauses (ii) and (iii) appeal to, the notion of an open-ended entitlement can be adopted. It is just that people whose views differ over what to make of open-endedness will understand the notion differently. And, crucially, whether or not it is one or the other, I think we have the machinery to remove the two awkward wrinkles presented in the first section of the chapter.

[12] The characterization of open-ended entitlements as 'the relative entitlements that survive arbitrary improvement' might have been found to signal a great similarity with Wright's notion of superassertibility. Though there is a superficial similarity, there is also a deep difference. The notion of improvement Wright appeals to in the characterization of superassertibility is one that is intimately tied to truth. The notion of improvement in play in the characterization of open-ended entitlement is different. It bears no intimate relation to truth. Rather, as indicated, an extended class of methods and procedures CM^* is labeled an 'improvement' from the perspective of the subject. It was also indicated that CM^*'s being an improvement over some base class CM is compatible with methods or procedures in CM^* leading to false results about the region of thought investigated by the subject.

5.3 Examples

Now that we have introduced the absolute, relativized, and open-ended ways of understanding entitlement of cognitive project, let us try to give some examples which will serve to highlight the differences and similarities between them.

First, let us return to the Frege-Basic Law V example mentioned in the first section of this chapter. It might be tempting to think that Frege was entitled to rely on Basic Law V, but that his entitlement was defeated when be became aware of its inconsistency. Not so. He did not hold an absolutist entitlement to BLV, because of the fact that it was inconsistent. (Here there seems to be a realist background assumption about logical facts in play, but see Section 5.5, B.) He did not hold a relativized or an open-ended entitlement either. To see this it suffices to show the former, as the notion of an open-ended entitlement is an extension of the notion of an relativized entitlement. Recall that the domain of reasons on the relativized approach was defined as the propositions which the subject would believe were she to work her capacities and current methods and procedures to their fullest to investigate the relevant region of thought. Frege certainly had the methods to establish the inconsistency of Basic Law V and would have believed it inconsistent – independently of his interaction with fellow logicians – had he worked his capacities, methods, and procedures to their fullest. Frege's domain of reasons for belief, understood in the relativized sense, thus contained an underminer for satisfaction of clause (ii). Therefore, Frege was not relativistically entitled to BLV, and so, not open-endedly entitled to it either.

A little adjustment of the Frege scenario yields an example in which there is a relativized entitlement, but neither an open-ended nor an absolute entitlement. Consider a subject, Frege*, who works in a theory T. Let us just suppose that Frege* has an entitlement to $Con(T)$ relative to his information (i.e. domain of reasons for belief) and the class of methods and procedures CM at his service. But consider now another subject, Russell*, who is able to derive a contradiction from T by using a class of methods CM^* containing, in addition, to Frege*'s methods and procedures a newly developed method. Russell* writes Frege* to inform him of the inconsistency.

In this scenario, Frege* has, by assumption, an entitlement to $Con(T)$ relative to his current information and class of methods and procedures. However, he does not have an open-ended entitlement to $Con(T)$, because there is an improved class of methods and procedures, *viz.* CM^*, which generates a domain of reasons for belief that includes a sufficient reason to believe $Con(T)$ false. That is, a subject who were to fully investigate T with CM^* would hold the belief that a contradiction can be derived from T, and so, would believe a proposition whose truth is sufficient for the untruth of $Con(T)$. (Here, then, we also have an example of defeasibility in the sense suggested earlier, i.e. $Con(T)$'s being an entitlement, relativistically understood, but failing to be so open-endedly.)

What about a subject's having an open-ended entitlement to a proposition P, but not having an absolute entitlement to P? Absolute entitlements are always true, while the notion of an open-ended entitlement does not exclude the possibility that such an entitlement is false. This observation gives us what we need for a proposition P's being an open-ended entitlement, but failing to be an absolute entitlement. Consider a subject S with a domain of reasons for belief D_R and a class of current methods and procedures CM relative to which some proposition P is an entitlement. Let us suppose that S is also entitled open-endedly to P, i.e. let us suppose that, for any improved domain of reasons for belief D_{R*} and class of methods and procedures CM^*, there is no proposition in D_{R*} whose truth suffices for the untruth of P, and every attempt to justify P by appeal to the methods and procedures of CM^* gives rise to an infinite regress of justificatory projects involving presuppositions of no more secure a prior standing. That said, given the lines we have been running along so far, the domain of reasons for belief generated by any improved class of methods and procedures CM^* might fail to include some proposition Q which is true and whose truth suffices for the untruth of P. In case there is such a proposition, P will not be an absolute entitlement.

Now, can a subject S have an absolute entitlement which is not also an open-ended entitlement? Presumably, she can. Return to the Frege* example given earlier. Frege* had an entitlement to $Con(T)$ relative to his current domain of reasons for belief and class of methods and procedures. However, as seen, Frege* did not have an open-ended entitlement to $Con(T)$. For the domain of reasons for belief generated by the class of methods and procedures that

includes the new method developed by Russell*, $Con(T)$ is not an entitlement because the addition of the new method to T enables us to derive a contradiction from T. In order for $Con(T)$ to be an absolute entitlement – but not an open-ended one – something has to go wrong in the reasoning meant to show that a contradiction is derivable from T. Presumably, we do not want to take issue with the logic used in the derivation. However, the new method relied on might lead to false results.

5.4 Removing the awkward wrinkles

Wright [154] does not offer much in terms of how to understand clauses (ii) and (iii) of the characterization of entitlement of cognitive project. Hopefully, the preceding section has made some progress on that matter. We will now make use of what has been said in addressing the two awkward wrinkles stated at the beginning of the chapter.

Removing the first awkward wrinkle:
The first awkward wrinkle posed the question whether entitlements of cognitive project are really defeasible. Recalling the distinctions made, the answer is that relativistic and absolute entitlements are not. However, a proposition P can be an entitlement relative to a certain domain of reasons for belief and a class of methods, but fail to be an open-ended entitlement. That is, P can fail to be an entitlement relative to some improved pair of a domain of reasons for belief and corresponding class of methods and procedures, and in this sense be regarded as being defeated by this pair.

Removing the second awkward wrinkle:
The second awkward wrinkle amounted to this: clause (iii) of the characterization of entitlement of cognitive project implies that any attempt to justify an entitlement cannot improve its epistemic standing, and so, the status of a proposition P as an entitlement of cognitive project is incompatible with the view that any genuine progress can be made with respect to P. Again, what enables us to remove the wrinkle is that there can be cases where P is an entitle-

ment in the relativistic sense, but fails to be an open-ended entitlement. It is not, however, as straightforward to remove the second wrinkle as the first.

What needs to be observed is that there is more than one way in which a proposition P can be an entitlement in the relativized sense, but fail to be an open-ended entitlement. First, there is the way already familiar to us – a failure to meet clause (ii) understood open-endedly. It may be that there is no sufficient reason to believe P untrue relative to some domain of reasons D_R and a corresponding class of methods and procedures CM, but that there is such a reason relative to an improved domain of reasons for belief D_{R^*} and corresponding class of methods and procedures CM^*. We appealed to this kind of failure above in our attempt to accommodate defeasibility. Second, clause (iii) might be satisfied relative to a certain class of methods and procedures, but fail to be so open-endedly. That is to say, it is <u>not</u> the case that attempts to justify P give rise to an infinite regress of justificatory projects involving some presupposition of no more secure a prior standing. Or, to put it more plainly, in attempting to justify P we can do better than such an infinite regress. (So, unlike with the first awkward wrinkle, the failure to satisfy the relevant clause does not imply that the epistemic standing of P gets worse, but, rather, that it improves.) There are two ways in which this can be the case: there is some improved class of methods and procedures relative to which attempts to justify P will improve its epistemic standing either

(a) because such attempts lead to an infinite regress of justificatory projects, but involve only further presuppositions of more secure a prior standing, or

(b) because there is no infinite regress of justificatory projects, but a finite chain of justificatory projects which (i) involves only presuppositions of more secure a prior standing than P, and (ii) stops at a presupposition Q which does not itself raise further presuppositions.

It seems fair to say that a failure of clause (iii) in the sense of (a) improves the epistemic standing of P. After all, the further presuppositions raised by attempts to justify P are all of more secure a prior standing than P itself. As seen, the regress Wright has in mind results because any attempt to vindicate P brings up presuppositions of the same general kind as P itself – and these presuppositions, if anything, are most naturally regarded as enjoying an equally secure

prior standing. To get an infinite regress of justificatory projects involving only presuppositions of more secure a prior standing than P the sort of regress appealed to by Wright will not do. We need a regress that brings up presuppositions of a different kind than P itself.

The game we are playing with the sceptic is one that challenges us to improve the epistemic standing of P. The sceptical contention is that this is impossible because we will get into an infinite regress that will involve some presupposition of no more secure a prior standing than P itself, *and* the justification for P will be no stronger than that of the presupposition in the regress with the least secure standing. (Recall here that a minimizing, or 'weakest link', principle is in play: attempts to justify P cannot confer an epistemic standing on P which is greater than the epistemic standing of the weakest of the presuppositions involved in such attempts.) However, if *all* the further presuppositions involved in the regress are of more secure a prior standing than P, the evidence provided by the initial attempt to justify P (i.e. the first step in the regress) will improve the epistemic standing of P.

Now on to (b). The reason why the epistemic standing of P is improved in the (b) case is the same as in the (a) case: every presupposition involved in the chain of justificatory projects is of more secure a prior standing than P. However, there is a difference in terms of cardinality. The justificatory chain is finite. There is no regress, because one of the justificatory projects raises a presupposition which does not raise further presuppositions. That is, a presupposition which is of more secure a prior standing than P, but whose standing needs no further presuppositions.

Thus, genuine progress can be made with respect to an entitlement P in the sense that P is an entitlement relative to some class of methods and procedures, but fails to meet clause (iii) relative to some improved class of methods and procedures.

Entitlement of cognitive project, as characterized in Wright [154], comes with two awkward wrinkles. In this section, we have shown how to eliminate them – thereby, hopefully, contributing to the further development of the notion.

5.5 Further considerations

In this section, we will do four things. First, we will discuss the question whether entitlement of cognitive project leaves conceptual space for different – potentially incompatible – world views. I maintain that it does when it is taken in either the relativized or open-ended sense. Second, we will briefly touch on the question whether the absolutist reading can be embraced by people with anti-realist sympathies. Third, we will return to clause (ii). In particular we will briefly discuss what is meant by 'sufficient reason'. Fourth, I will flag some questions concerning clause (iii) that need to be considered to fully develop the notion of entitlement of cognitive project.

A. Different world views.

The conception of the domain of reasons on the absolutist approach appears to be realist in nature, and, indeed, is explicitly so if we take on board the determinacy assumption that implies that absolute entitlements are non-defeasible. (However, see Part B of this section.) On this picture, for any candidate entitlement P and a subject S, P either satisfies clause (ii) or it does not. If it does not, it might be due to the truth of some proposition Q (whose truth implies the untruth of P) which S has never conceived of and does not even have the conceptual resources to grasp. The absolutist approach goes hand in hand with the thesis that there is a fixed class of entitlements for each region of thought, *viz.* the ones that latch on to the facts. These entitlements will be the cornerstones of the 'true world view', the only one catered for on the absolutist conception.

Things are rather different according to the relativistic understanding of entitlement. Here the propositions included in the domain of reasons for belief were delineated by appeal to counterfactually held beliefs (after full investigation of the propositions by employment of the subject's capacities and current methods and procedures). Recall that truth was not necessary for a proposition to be in the relativistic domain of reasons for belief. The notion of an open-ended entitlement can be regarded as a strengthening of the relativistic notion, the strengthening consisting in the understanding of clauses (ii) and (iii) as holding open-endedly, i.e. no matter how we improve our methods and procedures. On the open-ended approach, truth is not a necessary

condition for a proposition being in the domain of reasons for belief either. It is compatible with both the relativistic and the open-ended approach that entitlements are false. On the other hand, on the absolutist picture, entitlements are always true. Otherwise it could not satisfy the requirement that there be, absolutely speaking, no reason to believe it untrue. (If we assume that an absolute entitlement P is false, there will be true propositions that suffice for the untruth of P, and hence, are sufficient reasons for believing P untrue. In that case clause (ii) would not be satisfied.)

The relativistic and open-ended notions of entitlement can accommodate different world views. To start with the relativistic notion, the way to see this is to remind ourselves that the relativistic domain of reasons for belief, D_R, was characterized as including the propositions the subject would believe, were she to exercise her capacities and current methods and procedures to their fullest. In addition to a fixed class of methods and procedures this characterization implicitly appeals to the current doxastic attitudes of the subject. The current methods and procedures for investigating some region of thought might differ for two subjects S_1 and S_2, and so might the class of propositions to which they currently hold some doxastic attitude. If they hold a doxastic attitude to the same proposition, which attitude they hold might differ too. (Let us just suppose that subjects have the same basic cognitive capacities – sense-experience, memory, introspection, etc.)

What propositions D_R includes will be a function of what doxastic attitudes the subject holds initially and the class of her current methods and procedures. Consider two subjects S_1 and S_2 that differ with respect to these two parameters. Now, the possibility that they converge on the same domain of reasons for belief cannot be excluded. Two mathematicians investigating some mathematical domain might differ with respect to what propositions they are doxastically related to and what methods and procedures they have at their service. One way in which this can happen is by their adoption of different axiom systems for investigating the domain in question. However, if they carry out as much work as they can in their respective theories, relying on all their relevant capacities, methods, and procedures, they might find that their theories – though initially seemingly different – turn out to be equivalent. In such a case the subjects will converge on the same domain of reasons for belief. Their domains of reasons for

belief will thus respectively undermine and support the same entitlement candidates – the former by the two subjects believing some proposition (possibly non-atomic) whose truth is sufficient for the untruth of the entitlement candidate, and the latter 'negatively' by there *not* being such a proposition.[13]

On the other hand, for two subjects that differ in terms of initial doxastic attitudes and methods and procedures, the domain of reasons for belief might respectively undermine and support different entitlement candidates. This is the possibility of relevance to the claim that the relativistic understanding of entitlement can accommodate different world views, by allowing two subjects to have different entitlements. Returning to ZFC and NF, consider two mathematicians S_1 and S_2, where S_1 accepts ZFC and S_2 accepts NF. ZFC and NF represent different set-theoretic world views. On the ZFC picture, there is sufficient reason to believe NF false. If S_1 were to exercise her capacities and current class of methods and procedures – those of ZFC – to their fullest, she would, among other things, believe that there is no universal set. Now, S_2, were she to exercise her capacities and current class of methods and procedures – those of NF – to their fullest, would believe that there is a universal set, among other things. ('$x = x$' can be stratified and thus plugged into comprehension in NF, delivering the existence of a universal set.) Here S_1's domain of reasons for belief includes a proposition whose truth is sufficient for the untruth of $Sat(NF)$, and accordingly, S_1's domain of reasons for belief undermines $Sat(NF)$'s entitlement candidacy. But, on the other hand, S_2's domain of reasons for belief includes a proposition whose truth suffices for the untruth of $Sat(ZFC)$. As a consequence, S_2's domain of reasons for belief undermines $Sat(ZFC)$'s entitlement candidacy.

Are S_1 and S_2 entitled to respectively $Sat(ZFC)$ and $Sat(NF)$ relative to their domains of reasons for belief and class of current methods and procedures. The case for clause (i) carries over from Chapter 3. S_1 and S_2's domains of reasons for belief D_{S_1} and D_{S_2} support respectively $Sat(ZFC)$ and $Sat(NF)$'s meeting clause (ii). They do so negatively, by there not being any sufficient reason in the D_{S_1} to believe $Sat(ZFC)$ untrue and there not being any such reason in D_{S_2} to believe $Sat(NF)$ untrue. As before, the domains of reasons for belief are identified with the beliefs the subject would hold, were she to exercise her capacities and class of current

[13] By an entitlement candidate being undermined is not meant that it is an entitlement and that it is defeated.

methods and procedures to their fullest. As regards clause (iii), this clause appears to be satisfied too. Attempts to support $Sat(ZFC)$ and $Sat(NF)$ by employment of their respective current class of methods and procedures give rise to an infinite regress of justificatory projects involving presuppositions of no more secure a prior standing than these propositions themselves. For any theory ZFC can be shown satisfiable relative to has to be of greater satisfiability strength than ZFC itself, and thus, of no more secure a prior standing. Likewise for NF. In other words, attempts to justify $Sat(ZFC)$ and $Sat(NF)$ give rise to infinite regresses whose nature is such that they cannot improve the epistemic standing or pedigree of these propositions (even if we assume that we can execute an infinity of justificatory projects).

In sum, the relativistic notion of entitlement can accommodate different world views. Incompatible propositions can be entitlements, for different subjects. This last qualification is important since, given what has been said above, the assumption that the *same* subject can be entitled to two incompatible propositions is incoherent. Each of the incompatible propositions will be in the domain of reasons for belief, but, since they are incompatible, each of them is a sufficient reason to believe the other one untrue. Therefore, a pair of incompatible propositions cannot satisfy clause (ii) on the same domain of reasons for belief, and so, cannot both be entitlements of the same subject.

Also, notice that it is crucial that relativized entitlements need not be true. If truth was enforced as a requirement on a proposition's being a relativized entitlement, even the accommodation of *different* subjects having incompatible entitlements would be incoherent.[14]

Let us move on to the open-ended notion of entitlement. Our considerations will be an extension of the ones already offered in relation to relativized entitlement.

As seen above, the open-ended notion is obtained from the relativized notion by incorporating the idea that refinement, or improvement, of the methods and procedures of the subject should be taken into consideration in determining whether clauses (ii) and (iii) are met. The domain of reasons for belief determined by some improved class of methods and procedures $CM*$ is a modification of the domain determined by the initial class CM. The addition of new

[14] Here again on the assumption that there is one set-theoretic universe and that the subjects are not simply theorizing about distinct, but equally real set-theoretic universes.

methods or procedures might expand the cognitive reach of the subject, i.e. expand the class of propositions that the subject can grasp. The obvious way this can happen is by the introduction of a new method or procedure allowing the subject to introduce notions previously not present in the theory. Such an expansion in cognitive reach is likely to also bring with it an expansion in the domain of reasons for belief by the subject's adopting some attitude to the propositions added to her cognitive reach. The change in the domain of reasons for belief can also change by the subject's changing her attitude towards a proposition she already holds a doxastic attitude towards. At the beginning of the 20th century, it is reasonable to suppose that the domains of reasons for belief of set theorists included open-mindedness concerning the question whether the satisfiability or consistency of arithmetic, ZFC, or some other reasonably strong mathematical theory could be established in the theory itself or some finitary system. However, with the development of the powerful method of arithmetization of syntax – or coding – and the limitative results of Gödel, the attitude of open-endedness changed to disbelief (or, equivalently, belief in the negations of the relevant propositions). Or consider CH. In the couple of decades following Hilbert's 1900 lecture, mathematicians might have been open-minded as to whether CH could be shown to follow from ZF, the axiomatic set theory that gradually became dominant in the wake of the set-theoretic paradoxes.[15] Gödel's result that the constructible universe – a model of ZF – satisfies CH sustained open-mindedness, perhaps with mathematicians finding it more credible that CH could be shown to follow semantically (and so, by soundness, syntactically) from the axioms of standard set theory.[16] However, add to the methods and procedures available at the time of Gödel's result the method of forcing later developed by Cohen, and things change. As Cohen's result shows, there are models of ZF in which CH fails. Accordingly, the domain of reasons for belief generated by that class of methods and procedures has to be such that open-mindedness about CH following from ZF is replaced by disbelief (or belief in the negation of the claim that it does).

We have seen that the relativized reading of entitlement leaves conceptual space for different subjects to have incompatible entitlements. The open-ended notion of entitlement does so too.

[15] It would probably be a bit of an anachronism to say 'follow semantically or syntactically' as the semantics/syntax distinction had not been clearly articulated at that point in time.

[16] Around this time, the semantics/syntax distinction had come into use.

The domain of reasons that determines whether or not clause (ii) is satisfied is a function of the initial doxastic attitudes held by the subject, the current methods and procedures of the subject, and improvements of these together with the addition of new methods and procedures.

Consider two subjects S_1 and S_2 working in different mathematical theories T_1 and T_2. Let their initial domains of reasons for belief be respectively D_{S_1} and D_{S_2} and let their initial classes of methods and procedures be CM_{S_1} and CM_{S_2}. We can suppose that CM_{S_1} and CM_{S_2} differ, partly due to T_1 and T_2 differing in terms of what methods of proof and definition they support. We can suppose that D_{S_1} and D_{S_2} overlap, i.e. that S_1 and S_2 would believe some of the same propositions were they to exercise their capacities and current methods and procedures to their fullest. However, we will suppose that S_1 and S_2 differ with respect to which doxastic attitude they would hold to at least some propositions. It is compatible with this that $Sat(T_1)$ and $Sat(T_2)$ satisfy clauses (i)-(iii) for respectively S_1 and S_2, with clauses (ii) and (iii) being satisfied open-endedly, and so, S_1 being entitled open-endedly to $Sat(T_1)$ and S_2 to $Sat(T_2)$. In particular, this means that no matter how the respective domains of S_1 and S_2 are improved and no matter how their classes of methods and procedures are refined or expanded, there will be no sufficient reason for S_1 to believe $Sat(T_1)$ untrue and there will be no sufficient reason for S_2 to believe $Sat(T_2)$ untrue, and any attempt to justify each of these propositions by any improved class of methods and procedures will give rise to an infinite regress of justificatory projects involving some presupposition of no more secure a prior standing.

We can add to this that S_1 and S_2 initially differ with respect to a proposition P. P can be shown to be true in every model of T_1 and is believed by S_1, but disbelieved by S_2, as P contradicts a theorem of T_2. This is so through arbitrary refinement of the methods and procedures at the service of S_1 and S_2 (and their corresponding domains of reasons for belief). Thus, any refinement $D_{S_1}*$ of D_{S_1} contains a sufficient reason to believe $Sat(T_2)$ untrue, *viz.* P. Hence, $Sat(T_2)$ cannot be an entitlement for S_1. Conversely, any refinement $D_{S_2}*$ of D_{S_2} contains a sufficient reason to believe $Sat(T_1)$ untrue, *viz.* $\neg P$. Hence, $Sat(T_1)$ cannot be an entitlement for S_2. In sum, we have two subjects S_1 and S_2 with incompatible entitlements. (Recall that $Sat(T_1)$ and $Sat(T_2)$ are incompatible because $Sat(T_1)$ implies that P is true, but $\neg P$ is a theorem of T_2, and so, is implied by $Sat(T_2)$.)

It should be noted that there is a difference between what we have said about incompatible entitlements in the relativized sense and what was just said about incompatible open-ended entitlements. In the former case, a *specific example* was given: $Sat(ZFC)$ and $Sat(NF)$ are entitlements relative to the domains of reasons for belief and class of current methods and procedures of respectively someone who accepts ZFC and someone who accepts NF. In the open-ended case we gave no specific example of a pair of incompatible entitlements where the entitlement to each member of the pair was held by a different subject. Rather certain assumptions were made concerning two subjects, their theories, and the relationship between these with respect to domains of reasons for belief and current and improved classes of methods and procedures. It was then said that these assumptions left conceptual space for the two subjects to have incompatible entitlements. This difference is traceable to the nature of open-ended entitlements. It is by no means clear that a specific example of an open-ended entitlement can be given, let alone a pair of incompatible ones. For how can we be sure that arbitrary improvements or refinements of the domain of reasons of belief and the class of methods and procedures at the service of the subject(s) do not undermine clauses (ii) and (iii) of the characterization of open-ended entitlements? Perhaps the best we can do is to *claim* that we have provided an example of an open-ended entitlement, or a pair of such entitlements. This kind of claim will be defeasible. The addition of new methods or procedures to a class of methods and procedures, or refinement of existing ones, could lead to a realization that clause (ii) or (iii) is not satisfied after all.

B. Anti-realism and the absolutist reading: a happy marriage?

Our treatment of the absolutist reading of entitlement has been very realist in spirit. For a given entitlement candidate, we have talked about *the facts* determining whether or not clause (ii) is met, absolutely speaking, and it was said that it might be *beyond our ken* which of the two it is. This way of speaking may make some people feel uneasy. In particular, someone with anti-realist sympathies can be expected to be impatient with the idea that there is such a thing as a subject's being absolutely entitled.

Such a person might wish to abandon the absolutist reading of entitlement, focusing entirely on the relativistic and open-ended readings. It might be objected that an exclusive focus on

these two readings is illicit, because, after all, the domains of reasons for belief with which they operate were characterized by appeal to the absolutist domain of reasons for belief. (For a given region of thought first the absolute domain of reasons for belief was characterized, and then the relativized domain of reasons for belief was characterized by imposing certain restrictions on it. On the open-ended reading, there was a range of relativized domains of reasons.) This is not a very strong objection. It is correct that they were initially introduced as being obtained from the absolutist domain of reasons in some way or the other. However, the appeal to the absolutist domain of reasons is dispensable. It is more interesting to note that the absolutist reading is not essentially realist. I have chosen to adopt propositions as reasons for belief and as that in terms of which it is determined whether or not clause (ii) is satisfied. On the absolutist reading the facts determine what propositions are true, and, in turn, the true propositions determine whether or not clause (ii) is satisfied for a given entitlement candidate. The observation to be made here is that the notion of fact or truth can be made to accord with one's philosophical outlook. The facts can be taken to be mind-dependent – rather than mind-independent – and truth can be conceived as being epistemically constrained.[17]

C. Sufficient reason for believing untrue.

Clause (ii) requires that there be no sufficient reason to believe proposition P untrue in order for P to be an entitlement. What exactly is meant by 'sufficient' in the clause? Tough question. The notion of sufficient reason can be notoriously difficult to get to grips with. Sufficient reason for belief is a case in hand. Still, let us try to say at least a little about the matter.

Belief is a kind of attitude that is controlled by evidence. Thus, whether there is sufficient reason for believing P untrue should be taken to be tied to whether there is sufficient evidence for holding this belief. In our previous discussion some clear cases of sufficient reason for believing P untrue have emerged when P is either a satisfiability or consistency proposition. For a mathematical theory T, there is a sufficient reason to believe $Sat(T)$ untrue if there is a proof

[17] The thought that truth is epistemically constrained is often expressed as follows:

(EC) Any true proposition is feasibly known.

The literature on (EC) is extensive. Key items include many of Dummett's writings, together with contributions by Tennant and Wright, among others. Sample references: Dummett [36], [37], [38]; Tennant [139]; Wright [148].

of $\neg Sat(T)$. That is, if $\alpha \wedge \neg\alpha$ can be shown to follow semantically from T (for some wff. α). Similarly, there is a sufficient reason to believe $Con(T)$ untrue if there is a proof of $\neg Con(T)$, i.e. if $\alpha \wedge \neg\alpha$ can be deduced from T. We typically take proof to give conclusive reason – or evidence, as it were – for belief. However, of course, the difficult cases to get clear on are those where there is inconclusive reason to believe P untrue, but where there may nevertheless be sufficient reason for so believing.

One way to go would be to use subjective probability as a measure for sufficiency, either directly by phrasing clause (ii) in such a way that it requires the subject to assign a subjective probability to $\neg P$ which is greater than a certain real in the unit interval, or indirectly by saying that there is sufficient reason to believe P untrue just in case the subject assigns such a subjective probability to $\neg P$. An attractive feature of this proposal is that subjective probability is usually taken to be determined by the evidence available to the subject, and so, makes for a happy fit with clause (ii) being spelled out in terms of belief (rather than some other kind of acceptance).

The use of subjective probability strikes me as somewhat promising, but, also, it should be clear that the key task remains to be executed – *viz.* to specify some $x \in [0; 1]$ such that there is sufficient reason to believe P untrue just in case $x < p(\neg P)$. What about letting x be 0.5? If a subject S takes it that $0.5 < p(\neg P)$, S certainly has *more* reason to believe P untrue than not to do so. However, it is not plausible to suppose that there is sufficient reason to believe P untrue whenever such a probability assignment is made. For $p(\neg P)$ could be given by some real greater than – but also arbitrarily close to – 0.5.

The question of how to fix the sub-interval of [0; 1] in which $p(\neg P)$ yields sufficient reason to believe $\neg P$ untrue is non-trivial. I will make no specific suggestion here concerning what sub-interval to settle for. I do, however, think that we have some handle on when there is sufficient reason to believe P untrue. Remember that the propositions we are interested in are candidate entitlements, and that entitlements are presuppositions of cognitive projects. Bearing this in mind, let me express sympathy with the following claim: a subject S is (rationally) unwilling to engage in a cognitive project of which P is a presupposition just in case S has sufficient reason to believe P untrue. The idea here is that there is a link between the willingness of a subject

to engage in a project and the probability that the subject assigns to its presuppositions being met (which is taken to be linked to reasons).[18]

Applying the above suggestion to the mathematical case, we get that a subject S has sufficient reason to believe $Sat(T)$, or $Con(T)$, untrue just in case S is unwilling to engage in mathematical projects of which $Sat(T)$, or $Con(T)$, is a presupposition. This is pretty unspecific, but, I submit, likely to be as good as it gets given the complexity of mathematical practice. There are many cases in which mathematicians have believed that there was sufficient reason to believe $Sat(T)$ or $Con(T)$ untrue. However, it is difficult to give a clear account of what exactly it takes for there to be such a reason. Sometimes the reason has been outright unsatisfiability or inconsistency, other times something else. Note, though, that in each case mathematicians have stopped working in the relevant theory, or, in our present jargon, they have stopped engaging in cognitive projects in the theory.

D. Secure prior standing.

Before we conclude this chapter, let us return to clause (iii). Clause (iii) requires that attempts to justify P – P an entitlement – lead to an infinite regress of justificatory projects involving further presuppositions of no more secure a prior standing. There is a number of issues that need to be addressed in order to reach an adequate understanding of the clause in question, in particular concerning the notion of secure prior standing:

> **Degree of security issues.** How is the degree of security of a proposition determined? In the mathematical case is the degree of security of consistency and satisfiability propositions given entirely by respectively their consistency and satisfiability strength (as defined earlier)? Do we need a more generally applicable measure of degree of security?

[18] It might be helpful to think in analogy with actions guided by testimony. Suppose that Bob is passing through an unfamiliar neighbourhood and asks a stranger for directions. Bob might trust the stranger and let the testimony guide his actions, i.e. follow the directions given to him. However, it might also be that Bob is reluctant to follow these directions. In the latter case, Bob's reluctance is likely to be explained by a lack of trust, which, in turn, is likely to be explained by Bob's thinking that there is sufficient reason – or evidence – to believe that what the stranger says is untrue.

Some of what has been said here may sound similar to things that are part of a common account of degree of belief in terms of betting behaviour. This should hardly be surprising as there are clear affinities between, on the one hand, betting behaviour and, on the other, the willingness of subjects to engage in cognitive projects or rely on the testimony of others.

It would seem that we need a more generally applicable measure of degree of security. One reason for thinking so is that empirical considerations might be relevant to satisfiability and consistency questions for mathematical theories. For instance, our best scientific theories might suggest that certain mathematical theories have a model in the physical universe. Or the indispensability of certain mathematical theories in empirical science might be taken to be an argument for their truth. In both cases it would seem that there is evidence on offer for the satisfiability of the relevant mathematical theory (and so, by soundness, its consistency too). However, once we move to empirical theories, a non-mathematical or non-logical way of determining the degree of security might be found natural or needed. This is partly because it is not clear how to proceed purely in terms of satisfiability or consistency strength, but also – and perhaps more importantly – because no purely mathematical way of determining the degree of epistemic security will be entirely adequate when we start considering empirical theories and data.

5.6 Conclusion

Wright says very little himself about how to understand clauses (ii) and (iii) of the characterization of entitlement of cognitive project. However, more should be said if we want to reach a proper understanding of the notion. In this chapter, I hope to have made some progress on this matter.

We started the chapter by presenting two awkward wrinkles concerning entitlement of cognitive project left by what is explicitly said about the notion in Wright [154]. We then proceeded to distinguish between three ways of understanding the notion: the relativistic, the absolute, and the open-ended way. Examples were given of how the distinction works, and the distinction was applied to remove the two awkward wrinkles presented at the outset. In both cases, we removed the wrinkle by considering a case in which a proposition P is an entitlement of cognitive project relative to a domain of reasons for belief and a certain class of methods and procedures, but fails to be an open-ended entitlement. The first wrinkle – to account for the supposed defeasibility of entitlements – was removed by considerations on clause (ii). It may be that there is no sufficient reason to believe P untrue relative to some domain of reasons for belief and class of methods

and procedures, but that there is nevertheless such a reason in the domain of reasons for belief corresponding to some improved class of methods and procedures. In that case the latter domain of reasons defeats the entitlement to P. The second awkward wrinkle – how to account for genuine progress – was removed by considerations on clause (iii). It may that attempts to justify P leads to an infinite regress of justificatory projects involving some presupposition of no more secure a prior standing relative to a base class of methods and procedures, but that no such regress results for some improved class of methods and procedures, and that, as a result thereof, the epistemic standing of P can be regarded as having been improved.

Lastly, some further considerations were offered on four issues. First, it was argued that the relativized and open-ended ways of understanding entitlement of cognitive project leave conceptual space for different world views, in the sense that different subjects can be relatively or open-endedly entitled to incompatible propositions. It was also argued that the absolute understanding of entitlement differs from the other two in this respect. Second, it was proposed that, contrary to first impressions, the absolute understanding of entitlement does not presuppose realism – that, if desired, it can be embedded into a philosophical framework which operates with an epistemically constrained notion of truth. Third, I offered some considerations on what it takes for there to be sufficient reason to believe P untrue. Fourth, we tabled a number of issues concerning the notion of security in clause (iii) of the characterization of entitlement.

Chapter 6

Scepticism, internalism and cognitive responsibility

In this chapter, I will try to present a wider philosophical framework of which the entitlement proposal can be regarded as an integrated part. More specifically, an attempt will be made to draw such a picture by offering some considerations on the relationship between epistemic value, cognitive responsibility and the potency of answers to the sceptical challenge by appeal to respectively internalist and externalist notions of warrant. The chapter has six sections. In the first section, we state one way of making the distinction between internalist and externalist notions of warrant. In the second section, we see that Wrightian entitlement is intended to be understood along internalist lines, and, given the statement of the internalism/externalism distinction adopted, spell out what this amounts to. In the third section, we extract and briefly comment on a number of theses concerning the internalism/externalism distinction, epistemic value, and scepticism that are either implicit in Wright or mentioned, but not developed any further. It is worthwhile doing so as it will help in working out the details of the wider philosophical framework of which the entitlement proposal is a part. In the fourth section, we will discuss monism and pluralism about epistemic value and relate the distinction to the debate between internalists and externalists. I will suggest that a common view on the internalism/externalism debate is misconceived. According to this view – the 'exclusion view' – warrant is either in-

ternalist or externalist. Against this I will maintain, as have others, that both internalists and externalists delineate proper notions of warrant. Pluralism about epistemic value is a natural companion of this view, the 'inclusion view', since it allows us to say that both the features highlighted by internalists and those highlighted by externalists track genuine epistemic values. The inclusion view is compatible with the thesis that internalist and externalist notions of warrant can provide answers to different epistemological problems. In particular, in the fifth section of the chapter, it will be argued that construing the notion of entitlement along externalist lines leaves it impotent as a response to the sceptical challenge, considered as a challenge to our rational claims to warrant. This can be taken as an indirect argument for internalism about cornerstone warrant. In the course of presenting the argument, it will be suggested that there is no support to be drawn for internalism *in general* from an emphasis of the deontological features of warrant – contra a prominent argument for internalism – but also that, in the specific case of cornerstone warrant, there may be something to the thought. In the sixth section, we will briefly consider what sense, if any, the entitlement proposal delivers foundations.

The chapter comes with an appendix in which I discuss a potential problem for internalism. Basically, the worry is that an internalist understanding of entitlement – and other species of warrant more generally – would make the possession of entitlements a privilege restricted to philosophically informed subjects.

6.1 Internalism and externalism

An internalist is, roughly, someone who adheres to the thesis that a subject's justificatory status with respect to a certain belief – more generally, doxastic attitude – supervenes on a class of facts to which the subject has special access. By appeal to this characterization, we can characterize an externalist (roughly) as someone who denies this thesis. The characterizations just given are rough, because we need to say something about what is meant by 'special access'. We will follow many authors in taking special access to the relevant facts to be a matter of whether one can access these facts by reflection alone. Here reflection is meant to cover a priori reasoning,

introspection, and memory.[1]

With terminology fixed, we shall take a weak internalist to be someone who adheres to the following thesis (Pryor [111], p. 104, but with slight modifications – see below):

(W-INTERNALISM) Whether a subject S is warranted in accepting P supervenes on facts which S is in a position to access by reflection alone.

An externalist is someone who denies (W-INTERNALISM). Thus, for instance, according to a weak internalist whether I am warranted in accepting $Sat(PA)$ supervenes on facts which I am in a position to access by reflection alone, meaning that I can access them through a priori reasoning, introspection, or memory. The externalist denies this.

Since (W-INTERNALISM) is a thesis about warrant – and so, entitlement – it has been formulated in terms of acceptance rather than belief. Recall that the appropriate attitude to pair with entitlement is trust, i.e. a mode of acceptance that is not belief.

(W-INTERNALISM) should be distinguished from moderate and strong internalism, the following stronger theses (Pryor [111], p. 105):

(M-INTERNALISM) A subject S who is warranted in accepting P is always in a position to acquire a warrant for accepting that this is so.

(S-INTERNALISM) A subject S who is warranted in accepting P is always warranted in accepting that this is so.

Let us take the notion of warrant in (M-INTERNALISM) and (S-INTERNALISM) to be that captured by (W-INTERNALISM): whether a subject is warranted supervenes on facts accessible to her through reflection alone. The reason why (M-INTERNALISM) and (S-INTERNALISM) are stronger than (W-INTERNALISM) is that each of them requires higher-order reflective awareness. (W-

[1] It should be noted that some authors give an alternative characterization of internalism in terms of mental states. According to this characterization, the factors that are relevant to whether a subject is warranted in believing a certain proposition are all internal to the subject's mental life. Sometimes internalism thus characterized is referred to as 'mentalism'. See Conee and Feldman [26], p. 233.

INTERNALISM) demands only that the facts upon which a subject's justificatory status supervenes are ones that are accessible to her purely by reflection.[2] A subject need not possess a higher-order warrant for an attitude concerning her own justificatory status at the first-order level.

This distinction can be put by saying that (M-INTERNALISM) and (S-INTERNALISM) support a version of the WW-principle, while (W-INTERNALISM) does not. The WW-principle is the S4 axiom $\Box P \rightarrow \Box\Box P$ with the box read as a warrant operator, indexed to a subject S: $W_S AP \rightarrow W_S(W_S AP)$. This follows directly from the characterization of the two theses – indeed, (M-INTERNALISM) and (S-INTERNALISM) simply are statements of the WW-principle, read informally.

For example, the weak internalist might hold that warrant for arithmetical beliefs supervene on facts about what can and what cannot be proved in Peano arithmetic. These facts, the weak internalist might continue, are ones that we can access through reflection alone. E.g., we can reason a priori to prove that $2 + 7 = 9$. If we have already done so, we can search our memory for the details of the proof, or perhaps simply recollect *that* we have done the proof, but without going into detail. The supervenience thesis commits the weak internalist to holding that there can be no variation with respect to a subject's justificatory status without there being variation in the justificatorily relevant facts. To use the example given, she is committed to the view that there can be no variation with respect to her warrant for believing that $2 + 7 = 9$ without there being any variation with respect to the justificatorily relevant facts about proofs accessible to her by reflection. However, there is no requirement to the effect that she must be able to reflect on these facts and thereby acquire a warrant for believing that she is warranted in believing that $2 + 7 = 9$, or be in a position to acquire such a warrant. In other words, the WW-principle is not supported.

On the other hand, both (M-INTERNALISM) and (S-INTERNALISM) require that the subject somehow enjoys a higher-order access to her own justificatory status. (M-INTERNALISM) does so by requiring that, if she is warranted in accepting P, then she should be in a position to acquire a warrant for accepting that this is so. Using the example from above, the moderate internalist

[2] Alston has defended the view that the internality requirement involves higher-order states. See Alston [2], p. 221.

will maintain that a subject should be in a position to acquire a warrant for her warrantedly believing that $2 + 7 = 9$ if she warrantedly believes that $2 + 7 = 9$. The strong internalist will insist that the subject possess the higher-order warrant in question.

As we have just seen, the characterization of warrant endorsed by the weak internalist does not commit her to the WW-principle, whether understood strongly or moderately. Weak internalism is compatible with each of moderate and strong internalism, but also with their negations. It is thus reasonable to ask what might tempt one to hold that warrant supervenes on reflectively accessible facts and that some version of the WW-principle holds. The way to answer this question is to consider what class (or classes) of facts are taken to be accessible through reflection.

Suppose that S is a subject that warrantedly believes P, and let 'F_1' denote the class of facts reflectively accessible to S upon which the warrant supervenes. Does S have a warrant for her warrant to believe P? Well, she does if there is a class of facts F_2 upon which such a warrant supervenes. The following proposal suggests that there is always such a class of facts: S can reflectively access the F_1-facts. This is a *fact*, and, moreover, one which is reflectively accessible to S – the thought being that, since the F_1-facts are reflectively accessible to S, the (F_2-)fact that this is so should itself be reflectively accessible to S. To return to our arithmetical example above, the F_1-facts are facts about what can be proved and what cannot be proved in PA. So, S's belief that $2 + 7 = 9$ is warranted, because of the fact that $2 + 7 = 9$ is provable in PA and this fact is accessible to S through reflection (i.e. through a priori reasoning, introspection, or memory). The relevant F_2-fact here is the fact that S can reflectively access facts about what can be proved and what cannot be proved in PA, and this F_2-fact is accessible to S through reflection. This is what S's warrant for her warrant for believing that $2 + 7 = 9$ consists in.

What differentiates the moderate and strong internalist is whether S should merely be in a position to reflectively access the F_2-facts or is required to access them. In this particular case – and with the assumption that the F_1-facts concern proofs – moderate internalism is fairly plausible, but strong internalism is unduly demanding. Given that S can reflect on what can be proved and what cannot be proved in PA (i.e. the F_1-facts), it seems plausible that she is in a position to reflectively access the (F_2-)fact that she can access facts about what can be proved

and what cannot be proved in PA. On the other hand, it is rather implausible to suppose that she reflectively accesses this fact.[3]

Above an externalist was defined as anyone who denies (W-INTERNALISM), i.e. anyone who denies that whether a subject S is warranted in accepting P supervenes on facts which S is in a position to access by reflection alone. We can deny (W-INTERNALISM) in at least two ways, the first weaker than the second:

(W-EXTERNALISM) Some of the facts upon which S's justificatory status supervenes are such that S is not in a position to access them through reflection alone.

(S-EXTERNALISM) All of the facts upon which S's justificatory status supervenes are such that S is not in a position to access them through reflection alone.

Externalism has drawn most attention with respect to perceptual warrant and knowledge. The idea here is, roughly, that a subject's justificatory status is determined by conditions external to the subject, typically external conditions on the relevant cognitive faculties and the environment in which the subject is situated. Again, somewhat roughly, this might be spelled out as the demand that the subject's cognitive faculties be functioning properly and that the surrounding environment be cooperative.[4] If this strikes you as right, an additional thought that might drive

[3] The analogue of the WW-principle for knowledge is the KK-principle, which is the S4 axiom with the box interpreted as a knowledge operator, again indexed to a subject: $K_S P \rightarrow K_S K_S P$. Like the WW-principle, the KK-principle can be read in a number of ways depending on whether we read '$K_S P$' as 'S is in a position to know P' or 'S knows that P'.

Traditionally, the KK-principle has been taken to be an internalist principle. Perhaps this is because it is thought natural to suppose that if what determines whether we have knowledge or are warranted is simply a matter of what is reflectively accessible to the subject, then, surely, the KK principle holds, because the subject can reflect on what is reflectively accessible. Though this is a tempting thought, it would be to go overboard to take the KK-principle to be a defining feature of internalism. Just because it appears easy to make sense of the principle from an internalist perspective does not make it essentially internalist. There is no principled reason why the externalist could not adopt the KK-principle, giving the K-operator an externalist reading.

Shapiro [129] discusses this point in considerable detail. In the present context, it is interesting to note that he argues that the seemingly only way for the externalist to make sense of the KK-principle is to take it that the external conditions on knowledge are met by *default*, in the sense that there is no reason to doubt that they are met.

[4] Prominent externalists include Goldman and Plantinga. See Goldman [59] and [60]; Plantinga [107] and

you to embrace either (W-EXTERNALISM) or (S-EXTERNALISM) would seem to be this: facts concerning the function of cognitive faculties and the surrounding environment are external to the subject. Therefore, they are not the kind of facts that can be accessed through reflection alone. If facts about the function of cognitive faculties and about the surrounding environment are thought to exhaust the supervenience base for a subject's justificatory status, then this thought seems to recommend (S-EXTERNALISM). If the kinds of facts just mentioned are not thought exhaustive of the supervenience base, the thought recommends only (W-EXTERNALISM).

What is interesting for our present purposes is to ask what externalism about mathematical warrant would amount to. As indicated above, the key idea behind externalism is the proper function, or reliability, of the methods and capacities exercised by the subject and the cooperativeness of the world or the surrounding environment. To formulate externalism in a mathematical setting we need to look for something that will bear witness to this idea.

For 'methods' perhaps various methods of proof and definition will do – e.g., proof by induction and definition by recursion. If the good standing – i.e. 'reliability' or 'proper function' – of these methods are thought relevant to the justificatory status of a subject, the internalist will insist that it should be accessible to the subject through reflection. The externalist will deny this, maintaining that it suffices that these methods *do* enjoy a good, or proper, standing, whether or not the subject is in a position to reflectively access this fact. Regarding 'cognitive capacities', I am not sure that I can give a plausible example. However, the intellect or reason might do. The tentative suggestion on behalf of the mathematical externalist is thus that whether or not our intellect or reason – i.e. the cognitive capacities exercised in doing mathematics – functions properly is relevant to the justificatory status of a subject, *and* that this fact need not be reflectively accessible to the subject. Regarding the cooperativeness of the world, or the environment, satisfiability of the theory worked in will do. A dramatic way the world can fail to cooperate is by the theory being unsatisfiable. In that case there is no structure that satisfies the axioms of the theory. The theory cannot be said to be about anything, and so, does not delineate a subject-matter. According to the externalist, worldly cooperation – here: satisfiability – is relevant to determining the justificatory status of a subject, but there is no

[106].

requirement to the effect that facts concerning satisfiability should be reflectively accessible to the subject. It suffices that the theory worked in *is* satisfiable.

The proposal just sketched on behalf of the mathematical externalist is that there is a range of facts concerning the functioning of mathematical methods, mathematical capacities, and the cooperativeness of mathematical reality (whatever that might be). These facts determine whether a given subject's attitude towards a mathematical proposition is warranted or not, but there is no requirement to the effect that the subject should be able to access them through reflection.

Now, it might be tempting to think that externalism in mathematics is not a live option, because mathematics would appear to be a paradigmatic example of a purely a priori discipline, and, therefore, any fact relevant to the justificatory status of a subject should be reflectively accessible to the subject. If we take this on board, an immediate rejoinder to the externalist view just sketched is that the facts concerning the functioning of methods and capacities, and the cooperativeness of reality are of a kind that are reflectively accessible to a mathematical subject.

But to win this exchange decisively the internalist must present an argument to support the enforcement of a requirement of reflective accessibility, or internality, on warrant. Such arguments have, indeed, been presented in the literature, a familiar strategy being to try to motivate internalism by appeal to the deontological conception of warrant, cognitive responsiblity, and cognate notions.. We will bracket the discussion for now, but will return to it in Section 6.5.

6.2 Wright entitlement is internalist

This section marks the beginning of the discussion whether entitlement is to be understood along internalist or externalist lines.

In the following passage, Wright suggests that entitlement is to be understood along internalist lines:

> ...entitlements, it appears, in contrast with any broadly externalist conception of warrant, are

> essentially recognisable by means of traditionally internalist resources – *a priori* reflection and self-knowledge – and are generally independent of the character of our actual cognitive situation in the wider world – indeed, are designed to be so. (Wright [154], p. 209)

Let us try to spell out the idea that entitlement of cognitive project is internalist in the sense of (W-INTERNALISM). This means that the facts upon which a subject S's entitlement to a proposition P supervenes should be reflectively accessible to S. The first thing we need to get clear on is what range of facts might reasonably be taken to determine whether or not a subject S (with a certain cognitive project) is entitled to a proposition P. A natural range of candidate facts is given by clauses (i)-(iii) – or more specifically, whether each of them is met. If we adopt weak internalism, we get that a subject S with a cognitive project C is entitled to P just in case

- P is a presupposition of C, i.e. to doubt (or be open-minded about) P would rationally commit S to doubt (or be open-minded about) the competence or significance of C;
- there is no sufficient reason to think P untrue;
- attempts to justify P would lead to an infinite regress of justificatory projects involving some presupposition of no more secure a prior standing than P;

  and

- the facts that determine whether these clauses are met are reflectively accessible to S.

(W-INTERNALISM) thus requires that the entitled subject has a concept of these facts obtaining, and so, a concept of a proposition P's being a presupposition of a cognitive project, there being no sufficient reason to think P untrue, and attempts to justify P giving rise to a justificatory regress of the stated kind. This is the case whichever way entitlement of cognitive project is understood – i.e. whether it is taken in the relativized, open-ended, or absolute sense. As we shall see, entitlement of cognitive project can only reasonably be held to be internalist on the relativized and the open-ended ways of understanding the notion.[5] The task of supporting this

[5] Internalism about entitlement – warrant more generally – might be thought to demand a degree of philosophical sophistication which most subjects lack, and hence, to make possession of entitlement a rather exclusive privilege. I touch on this problem in Appendix B.

claim is what will occupy us for the remainder of this section.

A. The relativistic reading: internalism.

The relativistic reading of entitlement of cognitive project can be taken to be internalist, meaning that the facts that determine whether clauses (i)-(iii) are met are reflectively accessible to the subject. Throughout P is an entitlement for a region of thought R, and S a subject who is entitled, in the relativized sense, to P.

For clause (i) we reason as follows: P is a presupposition of a cognitive project just in case to doubt (or be open-minded about) P rationally commits one to doubt (or be open-minded) about the significance or competence of the project. The aim of executing an R-project is to answer the question whether Q for some R-proposition Q. The entitlement candidates all concern the good standing of supposed R-evidence: if S doubts or is open-minded about P, she cannot rationally claim that the execution of an R-project leads to evidence that will determine whether Q. Hence, doubt or open-mindedness about P rationally commits S to doubt or be open-minded about the significance of the relevant project. In other words, P is a presupposition of R-projects. This is something S can work out by reflection – more specifically, the mere understanding of P suffices for this purpose. In particular, S's understanding of $Sat(T)$ and $Con(T)$ – the mathematical entitlements of cognitive project considered – suffices for her to work out that each of them is a presupposition of respectively semantic (or model-theoretic) and syntactic (or proof-theoretic) T-projects.

On to clause (ii). The relativized domain of reasons for belief was characterized as those propositions that S would believe were she to exercise her current capacities, methods, and procedures to their fullest. Clause (ii) is satisfied just in case there is no sufficient reason to believe P untrue in this domain. The facts that determine whether this is the case are these: whether, for each proposition in the domain, it is a sufficient reason to believe P untrue. There are at least two readings of what it takes for these facts to be reflectively accessible to S. On a strong reading, this requires S to be able to reflectively access what propositions she would believe were she to exercise her capacities, methods, and procedures to their fullest and figure out whether, among these propositions, there is a sufficient reason to believe P untrue. On

another – weaker – reading, the subject need not be able to reflectively access what propositions she would believe were she to exercise her capacities, methods, and procedures to their fullest; rather, what must be the case is that, for any proposition she would believe were she to exercise capacities, methods, and procedures to their fullest, she would be able to tell, or recognize, through reflection whether it undermines clause (ii).

If we go for the strong reading, it is implausible to suppose that it is reflectively accessible to S whether clause (ii) is met. One way to see this is to note that, if the region of thought we are concerned with is our thinking about the empirical world, S's methods will include empirical methods. Now, there is no way that S can tell purely through reflection what propositions she would believe were she to exercise these methods to their fullest. However, I suggest we understand the reflective accessibility requirement in accordance with the weaker of the two readings. What this reading requires is that S grasps the concepts of clause (ii) – here the concept of a sufficient reason to believe a proposition untrue – and that, when S acquires a belief in a proposition as a result of exercising her capacities, methods, or procedures, she is able to tell through reflection whether it undermines clause (ii).

(I am aware that I am here invoking a fairly weak reading of what it takes for the facts that determine whether clause (ii) is met to be reflectively accessible. On the other hand, it does not seem to me that it can be made stronger without jeopardizing the plausibility of maintaining that these facts are reflectively accessible to S.)

What about clause (iii)? There are certain facts about what capacities, methods, and procedures a subject S has. She can reflectively access these facts. E.g., she can recall past experiences that tell her that she has certain perceptual capacities, and she can introspect and tell that she is able to make inferences.[6] In other words, I take it to be reflectively accessible to the subject what capacities she has and what her current methods and procedures are. In that case it is also reflectively accessible to S whether attempts to justify P – P an entitlement candidate – lead to an infinite regress of justificatory projects involving some presupposition of no more secure a

[6] Some people might object to counting memories of perceptual experiences as reflectively accessible. Such people are likely be ones who insist that whatever is reflectively accessible must not in any way have relied on sense-experience. (This is something like the strong notion of reflective accessibility stated in Schmitt [123], p. 87.) It should be clear that I do not adhere to this view. E.g., whatever can be recalled is reflectively accessible, irrespective of its original source.

prior standing. Since she can reflectively access what capacities, methods, and procedures she has in her repertoire, she can, by understanding P, also work out through reflection whether attempts to justify P will lead to an infinite regress of the mentioned kind. The idea here is that this is a matter of the generality of P relative to the capacities, methods, and procedures of S, and that the subject can work this out by reflection. For instance, a subject with cognitive capacities like ours can, purely by grasping the proposition that my perceptual apparatus is functioning properly work out that it is sufficiently general that any attempt to justify it will give rise to a presupposition of the same general kind.

B. The open-ended reading: internalism.

The open-ended understanding of entitlement of cognitive project can also be taken to be internalist. The argument for clause (i) is the same as for the relativized reading. Clause (ii) works by extending the reasoning offered for the relativized reading, as does clause (iii).

So consider a 'base' domain of reasons for belief D_R and a corresponding class of methods and procedures CM and let D_{R^*} be an arbitrary improved domain of reasons for belief and CM^* a corresponding class of methods and procedures. Supposing that P is an entitlement relative to D_R and CM – which can be understood in an internalist fashion – can the subject S reflectively access the facts upon which it supervenes whether or not P satisfies clauses (ii) and (iii) open-endedly? Here an argument similar to the one appealed to in the discussion of clause (ii) for the relativized understanding of entitlement may be invoked.

It cannot reasonably be supposed that it is reflectively accessible to the subject S whether some improvement D_{R^*} of D contains a sufficient reason to believe P untrue. This follows from it not being reasonable to suppose that it is reflectively accessible to S whether there is such a reason in D_R, the base domain of reasons for belief. If it is not reflectively accessible to S whether a sufficient reason to believe P untrue is to be found among the propositions she would believe were she to exercise her capacities, current methods and procedures to their fullest, it is not reflectively accessible to her either whether this is the case for arbitrary improvements of the base class of methods and procedures. However, the weaker understanding of what it takes for the justificatorily relevant facts to be reflectively accessible to S – employed in our discussion of

the relativized case – can be employed again for every improved domain of reasons for belief D_{R^*}. What reflective accessibility, in the weak sense, requires is not that S can reflectively survey what methods and procedures are in arbitrary improvements of her current class methods and procedures, reflectively access what the corresponding domains of reasons for belief are, and then be able to tell whether any of these contain a sufficient reason to believe P untrue. Rather what is required is that, for each improved class of methods and procedures and proposition she would believe were she to exercise the capacities, methods, and procedures in that class to their fullest, she would be able to tell, or recognize, through reflection whether it undermines clause (ii). This requires S to grasp the concepts of clause (ii), understood open-endedly, and also that S is able to tell through reflection whether clause (ii) is undermined when she acquires a belief as a result of exercising her capacities, methods, or procedures in one of these improved classes.

The argument for clause (iii) is somewhat similar. What is required for the justificatorily relevant facts being reflectively accessible to S is not that S can reflectively access what methods and procedures will be in arbitrary improvements of her current class of methods and procedures and then tell whether attempts to justify P by use of the methods and procedures in any of these classes will give rise to an infinite regress of justificatory projects involving some presupposition of no more secure a prior standing than P itself. Rather what needs to be the case is that, for each of such improved class of methods and procedures, she would be able to work out, by reflection, whether an infinite regress of the required kind results from attempts to justify P by use of the methods and procedures in that class. Once she has an improved class of methods and procedures, we can rehearse the argument given for clause (iii) on the relativized reading.

C. The absolutist reading: externalism.

Regarding clause (i), we can reason in the same way we did in the relativized and open-ended senses. However, although this clause can be said to sustain reflective accessibility, clause (ii) cannot – even in the weak understanding of the requirement of reflective accessibility employed above. This suffices to show that absolute reading of entitlement cannot be regarded as being internalist in nature. It should be noted, however, that the consideration to be offered relies on the assumption that the absolute domain of reasons for belief is conceived of in a realist manner.

Thus, given what was said in Section 5.5 (B), room is left to manoeuvre.

The facts that determine whether clause (ii) – understood absolutely – is met cannot be said to be reflective accessible to the subject. On the absolute conception of the domain of reasons for belief, the domain of reasons include all the propositions of the relevant region of thought. This domain contains a sufficient reason to believe P untrue just in case there is a proposition which is true and whose truth suffices for the untruth of P. However, on the assumption of realism, whether the domain of reasons for belief contains such a reason is not reflectively accessible to S. There may be true propositions that S would never even consider, but which are nonetheless sufficient for believing some entitlement candidate P untrue. Such propositions cannot be said to be reflectively accessible to S.

D. Summary.

It was suggested that internalism about relativized and open-ended entitlement is plausible on a certain, not too strong, understanding of the requirement of reflective accessibility. It was also suggested that entitlement, understood absolutely, cannot be regarded as internalist.

6.3 Scepticism and the internalist/externalist distinction

Wright notes that people with externalist sympathies are likely not to be very receptive to the idea of entitlement, construed in an internalist fashion, playing a central role in the response to scepticism. According to such-minded people, the justificatory status of a subject is determined by features of our capacities and surrounding environment – ones that may be inaccessible through reflection to the subject. From this perspective, one may reasonably wonder why scepticism is any threat at all? After all, scepticism is a challenge formulated in the armchair, and armchair considerations would appear not to be of any relevance to what determines the justificatory status of a subject on the externalist picture.

The response to this kind of reservation is a reminder that the entitlement proposal was supposed to answer to sceptical challenges targeted, not at possession of warrant, but rather our right to *claim* to warrant – and *that* worry externalism cannot deal with adequately. According

to Wright,

> The *right to claim* knowledge, as challenged by scepticism, is something to be understood in terms of – and to be settled by – canons of intellectual integrity. The paradoxes of scepticism are paradoxes for the attempt at a systematic respect of those canons. They cannot be addressed by a position which allows that in the end thoroughgoing intellectual integrity is unobtainable, that all we can hope for is fortunate situation. There are, indeed, other good-circumstantial-qualities which our beliefs may have, even when good conscience fails but what is wanted is good conscience for the claim that this possibility is realised on the grand scale we customarily assume. (Wright [154], p. 211)

This quote invites the formulation of two theses. The first concerns the proper characterization of the challenge raised by scepticism and the second concerns the relation between intellectual integrity and the internalism/externalism distinction:

(CLAIMS AND INTEGRITY):
The sceptical challenge is targeted at rational claim to warrant rather than possession of warrant. The right to claim warrant is 'something to be understood in terms of – and to be settled by – canons of intellectual integrity'.

(INTEGRITY AND INTERNALISM):
Whether a subject's attitude to a proposition displays intellectual integrity is an internal matter.

One half of (CLAIMS AND INTEGRITY) can be extracted directly from the quote, namely that the sceptical challenge, properly understood, is an attack on the right to claim warrant. The other, negative half – that scepticism is not an attack on our possession of warrant – is given briefly before the quoted passage.[7] (INTEGRITY AND INTERNALISM) is not expressed directly in the quote. It is declared that positions which focus on 'fortunate situation', or 'good-circumstantial-qualities', are impotent to address the challenge raised by scepticism. Since externalists typically focus on features of this kind, it is reasonable, by contrast, to take Wright to be proposing internalism to be the way to respond to the sceptical challenge.

[7] Wright [154], p. 210. The passage in question was quoted in Section 2.3.

In the following sections, I will try to make good the second half of (CLAIMS AND INTEGRITY) – that the right to claim warrant is to be settled by considerations on intellectual integrity – and (INTEGRITY AND INTERNALISM). The attempts to do so will not be clearly separated. It should be recorded, though, that I will only defend a restricted version of (INTEGRITY AND INTERNALISM), the restriction being that the proposition has to be a cornerstone.

6.4 Monism and pluralism about epistemic value

We have seen in the previous section that Wright considers scepticism as being targeted at the right to claim warrant, and, crucially, that he thinks such a right is to 'be understood in terms of – and to be settled by – canons of intellectual integrity'. We have also seen that intellectual integrity, as Wright understands it, is to be distinguished from the reliability of the methods and capacities exercised by the subject and the cooperativeness of the surrounding environment.

In effect, Wright adopts a distinction between two kinds of epistemic value, associated with respectively intellectual integrity and situational provenance or 'circumstantial fortune'. The distinction is supposed to match a distinction between two kinds of moral value. He writes:

> While some metaethical views – classical utilitarianism, for instance – can be seen as driven by a sort of moral monism, it is intuitively plausible that there are at least two quite different kinds of virtue which an action may possess or lack: virtue of consequence (utility), and virtue of provenance – of conscience, or integrity – relating to the attitudinal states of the agent that determined her choice to act in that particular way. The two types of virtue are not, of course, independent – good conscience requires that one reckon with the foreseeable consequences of ones actions – and there seems no reason to expect that one should generally trump the other, still less to expect reducibility in either direction. I want to endorse a broad analogy with the ethics of belief: that we should allow a comparable kind of distinction between considerations of *intellectual integrity* and considerations to do with the *situational provenance* and other potentially fortunate or unfortunate aspects of the circumstances of a particular belief (for instance, its being the product of a reliable-truth-conducive-belief-forming mechanism.) Both are important – indeed, I would argue, indispensable. So those philosophers who have done so have been right to lay stress on notions of knowledge, or justification, which emphasise the second. But, again, there is no reason to expect either type of virtue to reduce to, or trump, the other. (Wright [154], pp. 210–211)

The distinction between the kind of epistemic value resulting in situations of intellectual integrity and situations of provenance – or worldly cooperation – is supposed to be analogous to the distinction between the value brought about by the consequences of an action and the value of an action measured by the intentions – or, more generally, the attitudes – of the acting agent. The passage signals that the moral values are not unrelated ('good conscience requires that one reckon with the foreseeable consequences of ones actions'), but also that the interrelation of the moral values does not support a reducibility thesis ('there seems no reason to expect that one should generally trump the other, still less to expect reducibility in either direction' and 'there is no reason to expect either type of virtue to reduce to, or trump, the other'). By analogy, reducibility should not be expected in either direction with respect to the epistemic values resulting from situations of intellectual integrity and situations of provenance, or worldly cooperation. Let us put the non-reducibility thesis down explicitly:

> (NON-REDUCIBILITY):
> The value of intellectual integrity is not reducible to the value resulting from situations of provenance, and the value resulting from situations of provenance is not reducible to the value of intellectual integrity.

It thus seems reasonable to pair the entitlement proposal with a pluralist view on epistemic value. There are (at least) two kinds of epistemic value, and neither kind is reducible to the other. Both kinds of value are characterized as 'important' and 'indispensable'. To relate this to the internalist/externalist distinction, (NON-REDUCIBILITY) suggests that it is *not* that philosophers with externalist sympathies get things wrong, or fail to delineate a legitimate notion of warrant – indeed, as seen in the quote, they 'have been right in stressing the importance of situational provenance'. Rather, externalist warrant falls short in the sense that there are certain questions it cannot adequately address. In particular, recalling what was said in the previous section, the entitlement proponent believes that externalism cannot satisfactorily address the sceptical challenge, regarded as a challenge to our right to claim warrant.

At this point, it is worthwhile drawing attention to two ways of looking at the internalism/externalism debate. On the one hand, we have what one might label 'the exclusion view', and, on the other, 'the inclusion view'. According to the exclusion view, the debate over internalism and externalism is exclusive. It is one or the other, but not both. Either internalism is right about warrant, or externalism is, and there is no conceptual space to allow both views to delineate proper notions of warrant. This is the predominant view amongst authors who have written extensively on the internalism/externalism debate.[8] According to the latter view, though the internalist and the externalist focus on different things, they nevertheless both track genuine epistemic qualities, or values, and delineate a proper notion of warrant. A subject who is warranted in an externalist fashion enjoys a positive epistemic standing, as does a subject who is warranted in an internalist fashion. They are different kinds of positive epistemic standing – signaled by their matching up with different kinds of value – but both sorts of standing earn the honorific label 'warrant'.

Given what has been said above, the entitlement proponent should endorse the inclusion view on the internalism/externalism debate. For the inclusion view is the only way to stably grant that both the externalist and the internalist are on to something (which, as seen, Wright thinks is the case). Note that the inclusion view is compatible with holding that the externalist is able to cope better with some problems than the internalist, and conversely, that the internalist is able to deal better than the externalist with other problems. In particular, it is compatible with the inclusion view that the internalist has a more compelling response in hand to sceptical challenges – regarded as challenging claims to warrant – than does the externalist.

This claim will be defended in due course. However, before we take on this task, let us say a bit about the contrast between monism and pluralism about epistemic value.

What things are epistemically valuable? The first thing that springs to mind is true belief. This should be fairly uncontroversial. But what else? A list of candidates – organized according to whether they concern single beliefs, systems of belief, or epistemic subjects – would probably include the following:

[8] Authors who invoke the argument from the deontological conception – to be discussed in the next section – typically buy into the exclusion view.

Single beliefs:

- truth
- significance or importance (bearing inferential relations to many other beliefs)
- evidential support
- warrant
- knowledge[9]

Systems of belief:

- coherence

Epistemic subjects:

- exercise of reliable methods, processes, procedures, cognitive faculties or powers
- cognitive responsibility
- understanding

I do not claim that this list of candidates is exhaustive. Neither do I claim that each of the listed candidates should be counted as an epistemic value. They are just candidates, after all.[10]

A not uncommon view is that, if the things listed are epistemically valuable, they are so solely in virtue of contributing towards reaching true beliefs. This suggests a distinction between things that are intrinsically of epistemic value, and things that are only epistemically valuable in an instrumental or derivative sense.

Let us distinguish between monism, unitarianism, and pluralism concerning epistemic value[11]:

(MONISM):

There is only one thing of epistemic value.

(UNITARIANISM):

There is more than one thing of epistemic value which share, or are derived from, a common theme. According to *strong unitarianism*, the epistemic values stand in a single relation

[9] If thought to be of epistemic value, knowledge would subsume other of the suggested epistemic values, *viz.* truth and warrant. Whether one thinks that knowledge is epistemically valuable over and above the epistemic value associated with truth and warrant will depend on one's view on warrant. If, like Plantinga, you think that warrant is whatever renders knowledge when added to true belief, it does not seem that there is any room for maintaining that knowledge possesses epistemic value over and above that associated with true, warranted belief.

[10] However, it is not too hard to find proponents of the epistemic value of the listed candidates in the contemporary literature. See, e.g., the essays in Fairweather and Zagzebski [42] and Steup [135].

[11] Here we follow Goldman [62]. Note, however, that Goldman casts the distinction in terms of virtues, while we use values.

to a common goal or value. According to *weak unitarianism*, there need not be a single relation between the epistemic values and the common goal or value.

(PLURALISM):

There is more than one thing of epistemic value and these things do not stand in a single relation to, and are not derived from, a common theme.

Given the centrality of truth, the perhaps most interesting (or, in any case, the most prominent) version of monism is veristic monism, according to which truth, or true belief, is the only thing of epistemic value. Analogously, the perhaps most interesting version of unitarianism is veristic unitarianism, according to which there is a range of epistemic values, but among which truth, or true belief, is the only intrinsic one. The other values are derivative. Whether we are dealing with a strong or weak version of unitarianism is determined by what the connection, or connections, between truth and the other values are taken to be.

Now, it might not be easy to draw a sharp distinction between monism and strong unitarianism. At least there would not seem to be much of a difference between, say, a veristic monist and a strong veristic unitarian who maintains that, in addition to truth, there is just one other epistemic value closely connected to truth. Similarly, it might not be straightforward to draw a clear line between weak unitarianism and pluralism. For instance, a weak unitarian who holds that there is a wide range of epistemic values related by an equally wide range of relations might not differ significantly from proponents of certain forms of pluralism. However, what matters here is that we can articulate clear-cut cases of, on the one hand, monism or strong unitarianism and, on the other, pluralism about epistemic value. (In Chapter 7 we will make use of veristic monism.)

Externalist writers typically lay stress on features that bear an intimate relation to truth – reliability or proper functioning of the methods and capacities exercised by the subject and the cooperativeness of the surrounding environment. If the exclusion view on the internalism/externalism debate is adopted it is thus natural to take externalism to be paired with veristic monism or some version of strong veristic unitarianism.[12] On the other hand, suppose

[12] In [62], Goldman – one of the prominent advocates of reliabilism – explicitly adopts a version of veristic unitarianism. See also David [28] and [29].

that someone adheres to the inclusion view by holding that there is both a legitimate externalist notion of warrant and also a legitimate internalist notion and that these notions of warrant are associated with two kinds of epistemic value E_1 and E_2, for which there is reducibility in neither direction. Such a person will be committed to pluralism.

It was suggested above that the entitlement proposal can be understood as incorporating pluralism about epistemic value, with provenance and intellectual integrity being identified as features associated with distinct epistemic values. The former is to be linked with external notions of warrant, and we may take it as simply another way of saying that the methods and capacities exercised are reliable and that the surrounding environment cooperates. Thus understood, the epistemic value resulting from situations of provenance can be taken to be truth.

Notions of warrant focusing on the externalist features were deemed impotent to address the sceptical challenge, the solution being located instead with an internalist notion of warrant with a proper emphasis on intellectual integrity. In the next section, the aim is to argue that cornerstone warrant understood along externalist lines will not do the job and take this as an indirect argument for internalism about such warrant.

It was suggested that pluralism about epistemic value makes for a happy fit with the entitlement proposal. Here no elaborate attempt will be made to defend pluralism. Quite simply,

I will assume pluralism about epistemic value in what follows.

The only support that will be offered for this assumption here is indirect, in the form of an argument against monism. The argument is very simple.[13] It starts by noting what may strike people as a platitude or a rather uncontroversial claim:

Knowledge is more epistemically valuable than mere true belief.

However, the typical monist about epistemic value cannot account for this. According to the typical monist – the veristic monist, that is – the only thing of epistemic value is true belief. But if that is so, knowledge cannot be of greater epistemic value than mere true belief. Knowledge is epistemically valuable, because it involves true belief (by factivity) – and *only* for that reason.

[13] The argument has probably been rehearsed a number of places. I came across it in DePaul [33].

Thus, whatever else knowledge may involve which mere true belief does not (warrant, say) is not relevant with respect to epistemic value, and we have a clash with the initial claim. Given the uncontroversial status of the claim, we cannot reasonably be monists about epistemic value, or so the argument goes.

I find the argument compelling. As noted, though, one thing it does not do is to provide a direct argument for pluralism. It works indirectly or by exclusion. That is, it recommends pluralism by leading to the conclusion that monism is untenable. I shall refrain from trying to provide a direct argument for pluralism. This is not meant to be an indication that I find the monism-pluralism question uninteresting and not worthy of attention. On the contrary, it strikes me as one of the most interesting issues in epistemology. However, given the many issues already on the agenda here, I refrain from attempting to provide an argument for pluralism since, no doubt, I would be unable to go into as much detail as the issue deserves.

6.5 Internalism, scepticism and cognitive responsibility

A not uncommon strategy for people with a liking for internalism is to try to support their view by appeal to the deontological conception of warrant. I find myself in agreement with a criticism launched by Goldman and do not think that the deontological conception of warrant supports internalism about warrant *in general*. In addition to the details of the argument not holding up to scrutiny, the argument itself is a manifestation of a misconceived view on the internalism/externalism debate – what was referred to as 'the exclusion view' earlier. The intended conclusion is that warrant properly so-called is to be understood in an internalist fashion. Having subscribed to the inclusion view, I cannot, however, agree with this. I will give an argument for an internalist understanding of *cornerstone* warrant, though. The argument will involve considerations on cognitive responsibility and may, in this respect, be thought to bear some similarity to the argument from the deontological conception. However, unlike the argument from the deontological conception, the argument to be offered will be indirect in that the conclusion supported is that externalist accounts of cornerstone warrant cannot adequately deal with scepticism, and so, by exclusion, we should go for some kind of internalism. This means that

the argument is also conditional because the argument only gets a grip on the assumption that an account of cornerstone warrant is meant to deliver a potent response to a specific problem, namely scepticism. Lastly, though deontological features of cornerstone warrant will play a role in the argument, what does the work is not the deontological conception by itself, but rather its relation to a feature that becomes particularly relevant in the context of scepticism (what will be called 'defensibility').

A. Internalism and the deontological conception of warrant.

According to the deontological conception of warrant, a subject's being warranted in holding some doxastic attitude (belief, say) to P consists in the subject's doing so in accordance with her epistemic duties, obligations, or requirements; i.e. in a epistemically or cognitively responsible, permissible, blameless, or non-culpable manner.[14]

It is not uncommon for people with internalist sympathies to invoke the deontological conception in an attempt to support internalism.[15] The argument goes, roughly, like this[16]:

> **First step.** Adopt the deontological conception of warrant. That is, take warrant to be understood in terms of some deontological notion like epistemic duty, obligation, or requirement, and some correlated notion like epistemic or cognitive responsibility, permissibility, blamelessness, or non-culpability.
>
> **Second step.** Derive a reflective accessibility requirement from the deontological conception. That is, show that adherence to this conception brings with it a commitment to the view that whether a subject's doxastic attitude towards a proposition P accords with epistemic duty, obligation, or requirement supervenes on facts that are accessible to the subject through reflection alone (and so, whether the subject holds this attitude in an

[14] Sources in which warrant is discussed in deontological terms include Alston [1]; BonJour [12]; Castaneda [23]; Chisholm [24]; Feldman [44] and [45]; Ginet [56]; Goldman [61]; Kornblith [81] and [82]; Moser [99]; Pollock [108]; Steup [134] and [136]; Wolterstorff [145]. Inclusion in the list is not meant to suggest that the author adheres to the deontological conception of warrant.

[15] See, e.g., Ginet [56], pp. 34–36, and BonJour [12]. Though Plantinga is not an adherent of internalism, he does take its main motivational source to be the deontological conception. See Plantinga [106], p. 15, pp. 24–25. Similarly for Goldman [61].

[16] Here I follow the exposition in Goldman [61], Section 1. See also Schmitt [123], pp. 90–91, and Greco [65], Section 2.1. The exposition given here differs in one crucial respect from Goldman's, though. The difference will be mentioned in a later footnote.

epistemically or cognitively responsible, permissible, blameless, or non-culpable manner supervenes on reflectively accessible facts).

Conclusion. Therefore, warrant is internalist.

This argument has not gone by unchallenged. One line of attack is to grant the first step, but deny the second, i.e. deny that the deontological conception implies a requirement of reflective accessibility. An argument of this sort has been developed by Goldman.[17] The thought meant to support the second step of the argument can be put as follows: a subject is warranted in holding a doxastic attitude towards P just in case that attitude accords with her epistemic duties, obligations, or requirements. A subject can only be said to have duties, obligations, or be subject to requirements if the subject is able to tell whether these are met. In order for a subject to be able to tell whether her duties, obligations, or requirements in force are met, the facts that determine whether, indeed, they are so must be readily available. However, this means that they have to be internal facts – i.e. reflectively accessible facts – as external facts are not readily available.[18] Goldman argues – convincingly, I think – that this is implausible. In particular, if we consider perceptual warrant, external facts would seem as natural a candidate for being justificatorily relevant as any internal fact – and could very well be readily available. For instance, suppose that I warrantedly believe that the temperature in the room is 20C, and that I have acquired this belief by a reliable procedure – consulting a thermometer on the wall, say. Running along externalist lines, we can say that my belief is warranted because it has been acquired through a reliable procedure. The fact that the procedure was reliable is not accessible through reflection alone, but it is nevertheless readily available to me.[19]

[17] Goldman [61].

[18] Goldman [61], p. 209; Conee and Feldman [26], p. 239.

[19] Now to the difference between my exposition of the internalist argument and that given by Goldman. According to Goldman, the internalist argument also incorporates what he calls the 'guidance conception of warrant', according to which warrant is such that it should guide our belief formation, i.e. it should help us decide what beliefs to adopt. It is not clear to me that the internalist argument cannot be formulated without the guidance conception, or perhaps rather, that adding it to the reasoning will make a stronger case for internalism.

In any case, the guidance conception brings problems of its own. According to an influential criticism by Alston, the guidance view presupposes doxastic voluntarism – i.e. the thesis that we can exercise control over doxastic attitudes and decide, or choose, to adopt them. However, this thesis is implausible unless modified. A basic point is that some doxastic attitudes – belief, in particular – are evidentially governed. Consider a case in which I have strong, or perhaps conclusive, evidence that P. Suppose, for instance, that I have proved in PA that $2 + 2 = 4$. In that case it seems implausible to maintain that I can simply choose, or decide, whether I want to believe,

I agree with Goldman in his criticism of the argument from the deontological conception. It is not plausible to suppose that viewing the concept of warrant as being deontological in nature will force upon us the view that it is to be understood in accordance with the doctrines of internalism. The internalist is mistaken in taking a requirement of ready availability to imply a requirement of reflective accessibility. Sometimes a subject can be warranted, properly so-called, without the justificatorily relevant facts – i.e. the supervenience base – being reflectively accessible to the subject.

Having granted this, let us try to restrict focus to cornerstone warrant and see if this makes a difference.

B. Responsibility, presuppositions and claims to warrant.

As seen in Chapter 3, Wright speaks of acceptance of entitlements being 'beyond rational reproach'. He also speaks of intellectual integrity and of warrant being acquired 'whenever investigation is undertaken in a *fully responsible* manner'.[20] This is deontological talk. Let us try to figure out what to make of it with focus restricted to cornerstone warrant.

The extent to which what I will say agrees with the internalist argument from the deontological conception is this: considerations on cognitive responsibility – a deontological notion – may be part of what recommends an internalist understanding of cornerstone warrant. However, as shall transpire, just as much work is done by the assumption that entitlement is supposed to respond to scepticism as by its supposed deontological nature. Thus, if anything, there is an argument in support of internalism about cornerstone warrant not from the deontological conception by itself, but from this conception together with a number of theses concerning the relation it bears to other notions central to the wider context in cornerstone warrant is brought up – namely a discussion of scepticism. Additionally, even if these wider considerations support an internalist approach to cornerstone warrant, they do not support internalism concerning all kinds of warrant, contra the spirit of the argument from the deontological conception.

disbelieve, or stay open-minded about whether $2 + 2 = 4$.

Alston's criticism appears in his [1]. Schmitt and Goldman find themselves in agreement with many of Alston's points. See Schmitt [123], Section 1, and Goldman [61], p. 210 respectively. For a defence of (restricted) versions of doxastic voluntarism, see Feldman [44] and [45]; Ginet [57]; Heil [73].

[20] Wright [154], p. 175, p. pp. 210–211.

These things said, I will now take on the task of making a case that there are interesting connections between cognitive responsibility and scepticism, rational claims to warrant, and presuppositions.

Recall that scepticism, as we have discussed it, is a challenge to rational *claims* to warrant, and that the entitlement proposal was introduced to salvage the right to make such claims. Recall also that there is an intimate connection between cornerstones of a given region of thought and rational claims to warrant for beliefs arrived at through execution of projects within this region. In Section 2.6, the connection was brought out as follows: consider a region of thought R, a cornerstone P of R, and an epistemic subject S. Suppose that S has no warrant for P, but arrives at a belief that Q through execution of an R-project, *and* that S claims a warrant for her belief that Q. Then S's epistemic situation can be summarized as follows:

S claims to warrantedly believe Q, but cannot rationally claim to have any evidence for Q.

Is S's claim to warrant rational? The answer was, 'No'. Absent a warrant for the cornerstone P, S cannot rationally claim to have any evidence for Q. However, since the execution of the R-project through which S arrived at Q was supposed to deliver an *evidential* warrant for Q, this means that S's claim to warrant is not rational.[21]

So far, so good. But how does internalism enter the picture? As we have seen earlier in the chapter, Wright suggests that internalism is invited when we bear in mind that the notion of entitlement is meant to be something by appeal to which we can answer the sceptic. The sceptic challenges rational claims to warrant rather than the possession of warrant, and, as also seen, externalism is impotent to address scepticism thus construed. I find myself in agreement with this way of looking at things. However, I also find that not much of an attempt has been made to support it. Wright merely gives an indication of how an argument might go by saying that 'The *right to claim* knowledge – as challenged by scepticism – is something to be understood in terms of – and to be settled by – canons of intellectual integrity.' The thought has to be that, firstly, there is an intimate connection between cognitive responsibility – or intellectual integrity – and whether a subject's engagement in a cognitive project can sustain the right to

[21] In Section 2.6, this was argued for the specific cases, where P was either a satisfiability or consistency proposition. However, the argument generalizes to any proposition that can be shown to be a cornerstone.

claim warrant for a belief arrived at through its execution – which, in turn, depends on warrant for its presuppositions; and, secondly, that cognitive responsibility, if it is to be relevant to the right to claims to warrant, is an internal matter.

I will now try to develop this thought. Note that rather than 'intellectual integrity' I will speak of 'cognitive responsibility'. Often I will leave 'cognitive' implicit and simply speak of 'responsibility'.

There is an intimate link between the right to claim warrant for belief in a proposition arrived at through execution of some project, there being warrant for the presuppositions of that project, and cognitive responsibility. Above we have already reminded ourselves that there is an intimate link between cornerstones – one kind of presupposition – and rational claim to warrant. What we need to show now is how cognitive responsibility fits into the picture. Cognitive responsibility relates to presuppositions in the following way:

It is part of the idea of a subject's undertaking a cognitive project in a fully responsible manner that the subject be warranted in thinking that the presuppositions of the project are met. Perhaps an analogy with a non-cognitive project will help. Suppose that I want to go mountain biking. In order to do so in a fully responsible manner I have to be warranted in thinking that the presuppositions for a successful execution of my project are met. Among these presuppositions will be that the bike is in good order. To engage in my project in a fully responsible manner I thus need to have a warrant for thinking that my bike is indeed in good order. Similarly, in order for my engagement in a cognitive project to exhibit full responsibility I need a warrant for thinking that its presuppositions are met, that the project can be successfully executed. For instance, if I want to establish a proposition in a mathematical theory T and take the proposition to tell me something about some mathematical structure, I need a warrant for thinking that T is satisfiable. If there is no warrant for thinking that T is satisfiable, there is no warrant for thinking that there is anything the theory is about, and so, no warrant for thinking that the presuppositions for a successful execution of the project are met.[22]

[22] Cognitive responsibility, as it relates to cornerstones, differs from the notion of responsibility in play in the mountain biking example in a crucial respect. A presupposition of my fully responsibly going for a mountain bike ride is that I have warrant for thinking that the bike is in good order. This is something I can check evidentially by taking the bike for a short test ride, systematically testing steering, the breaks, gear shifts, and other things that might fail during a more serious ride. However, we have granted the sceptic that there can be no evidential

Since we are already moving in deontological territory, it might here be natural to pair the talk of cognitive responsibility with talk of *obligations* or *requirements*. We can say that to exhibit full cognitive responsibility in engagement in a project the subject has to discharge, or meet, certain obligations or requirements – *viz.* having a warrant for thinking that the presuppositions of the project are met.

A relevant consideration here is what it takes to discharge these obligations or requirements. That is, to ask what it takes to have a warrant for thinking, or accepting, that the presuppositions are met. When the presupposition in question is an entitlement, the answer is this: the presuppositions are met 'by default', i.e. in case there is no sufficient reason to think that they are not. In a certain respect this makes the notion of obligation or requirement appropriate to the notion of entitlement similar to that in play in the legal domain. Members of a society are said to be obliged or required to live by the law. What does it take to do so? According to legal practice, one is a law-abiding citizen provided one is not found guilty of any crime. Additionally, it is an integral part of legal practice that anyone accused of a crime is *innocent until proven guilty.* On this conception, one can meet one's obligation or the requirement to live by the law by default – meaning that one does so provided there is no sufficient evidence that one has violated some law or regulation.[23] That is, one can be a 'legally responsible', or law-abiding, citizen by default.

C. Defensibility and the right to claim warrant.

We have now shed a bit of light on the relation between rational claims to warrant, cognitive responsibility and presuppositions, when the presuppositions considered are cornerstones. The argument for there being an intimate relation between rational claims to warrant and cornerstones was a lemma we allowed ourselves to import from Chapter 2. The basic thought was that rational claims to warrant are held hostage to there being a warrant for cornerstones. Corner-

warrant for cornerstones.

[23] Recall that, for entitlement of cognitive project, the question arose how one should conceive of the domain of reasons relevant to determining whether clause (ii) is met. Similarly, in the legal case, there will be a question of how to understand there being no sufficient evidence for thinking an individual guilty of any crime. Is it to be understood relative to some specific jury and the evidence presented to its members, in an absolute sense, or in an intermediate way? In practice, the relative understanding is in force, verdicts being made on the basis of what is counted as evidence by a specific jury. However, sometimes – in the cases of appeal or re-trial – something like an open-ended notion of sufficient countervailing evidence may be in play.

stones relate to cognitive responsibility by being a specific kind of presupposition and it being a requirement for fully responsible engagement in a cognitive project that its presuppositions be warranted. What we need to show is how internalism fits into this picture. As shall transpire, a crucial assumption in this context is that we want entitlement to respond to an argument that challenges our right to *claims* to warrant. More specifically, it will be suggested that it is a general feature of a rightful claim that the claimant (i.e. whoever makes the claim) can defend it.

Let us start out by discussing what appears to be a general feature of claims. The general feature I have in mind is this: claims are made in a dialectical context – more specifically, in a context where the claim might be questioned by an interlocutor. For instance, suppose that I see a job advertised which requires teaching competence in introductory meta-logic, and that I apply for the job, stating introductory meta-logic as one of my areas of competence. In that case my claim might be questioned by an interlocutor. Supposing that I get interviewed for the job, someone on the interviewing committee might very well invite me to give a detailed course description and ask questions whose answers require a grasp of the kind of logic I claim to be able to teach.

Some claims are made rightfully, others not. I would like to suggest that the right to claim warrant requires that the claimant be able to defend or back her claim, if requested to do so by an interlocutor. Let us call this feature 'defensibility', and accordingly, we will speak of 'defensible claims'. Thus, I only have a right to claim that I can teach introductory meta-logic provided that my claim is defensible. That is, only if I am able to make good my claim, do I have a right to make it. In the particular example given, this means that if asked for a detailed course description I will have to be able to provide one, and similarly, if asked a question whose answer requires a sketch of the Lindenbaum lemma, say, I will have to be able to respond. In other words, a successful defence of a claim consists in the claimant's presenting the grounds that sustain the claim in such a way that an interlocutor can recognize that the claimant has the right to make it. (Note that sometimes the interlocutor and the claimant may be one and the same.)[24]

[24] As far as I know, there has not been any extensive discussion of claims to knowledge and warrant recently.

Recall that scepticism is an attack on our claims to warrant. Applying the above considerations, to have the right to make these claims we have to be able to defend them. That is, we have to be able to present the grounds that sustain our claims to warrant to an interlocutor, if requested. It follows from what has been said earlier that our claims to warrant for ordinary propositions of a region of thought are sustained, in part at least, by there being a warrant for the cornerstones of that region. So, to maintain our right to claims to warrant for ordinary propositions we need to be able to defend these claims, and this involves being able to present the interlocutor with a warrant for the relevant cornerstones.

D. A conditional, indirect recommendation of internalism.

We now have the setting to make the link with internalism. However, first let us make a comment about Goldman's objection against the argument from the deontological conception, and then a comment about the character of the argument to be presented in favour of internalism about cornerstone warrant.

As indicated earlier, the argument for internalism from the deontological conception is not convincing. Goldman is right in saying that the internalist is too swift in moving from its being readily available to the subject whether she meets her duty or obligation to warrant being an internal matter. There are cases in which it is readily available to the subject whether or not she has met her duty or obligation, but where it is not a matter of reflective accessibility. In other words, the internalist mistake is to think that being readily available amounts to – or at least requires – being reflectively accessible.

I will now try to reach – or better, *conditionally recommend* – internalism via a different, and, note, indirect path. Unlike the argument from the deontological conception, the scope is not to say something about warrant in general, but only about cornerstone warrant. In addition – and this needs to be emphasized – the conclusion I want to support is very different. What I take the considerations to be rehearsed to support is this: if cornerstone warrant is to sustain our right to claim warrant, then it should not be construed along externalist lines. This falls short of showing that cornerstone warrant is internal. After all, it is just a conditional, and a

The view expressed in this paragraph is inspired by Warnock [141], a paper published in 1962.

conditional recommendation not to go for externalism at that. At most, then, the argument will issue a conditional recommendation of (some version of) internalism and do so by exclusion.

As argued above, the right to make a claim requires defensibility, i.e. that the claimant be able to defend her claim if pressed by an interlocutor. The defence consists in the claimant presenting the grounds that sustain the claim. In particular, to maintain the right to make claims to warrant for ordinary propositions of some region of thought, we have to be able to present the grounds that sustain these claims. Among these grounds are cornerstones of the region. The grounds that sustain the claim need to be readily available to the claimant. In particular, this means that cornerstone warrants need to be readily available to the claimant. Otherwise she would not be able to present them to the interlocutor. (The talk of the grounds sustaining a claim being readily available might be taken to indicate a similarity at this point with the argument from the deontological conception.)

We now consider cornerstones of the kinds we have been concerned with so far, i.e. empirical and mathematical ones.

Consider first empirical cornerstones like 'I am not a brain in a vat', 'I am not now having a vivid, coherent dream', and 'My cognitive capacities are functioning properly'. Let us suppose that these cornerstones are warranted by externalist means, i.e. as a result of having been acquired through a reliable method in a cooperative environment. Suppose, furthermore, that the externalist about cornerstone warrant maintains that this will sustain the right to make claims to warrant for belief in ordinary propositions about the empirical world. Bearing in mind what was said earlier, sustaining the right to make claims involves that the claimant can defend them. Thus, the externalist needs to defend her rational claim to warrant for belief in ordinary propositions about the empirical world. This involves presenting the grounds that sustain them, in particular a warrant for the cornerstones. However, can the subject with an externalist cornerstone warrant present this warrant to the sceptic in such a way that the sceptic can recognize that she has it, and that it gives her the right to claim warrant for her empirical beliefs? Arguably not. The externalist cornerstone warrant is delivered by the *fact* that the cornerstone beliefs were reliably acquired in a conducive environment. But how can the externalist present this fact to the sceptic in the required manner? It is an empirical fact,

but one that may obtain beyond the ken of the subject whose cornerstone warrant it delivers. In that case the subject can hardly be said to be able to present the fact in question to the sceptic. On the other hand, suppose that the subject is convinced that her cornerstone belief was acquired in a reliable manner in a conducive environment, and that she cites *evidence* for so thinking – say, certain perceptual experiences. In that case the sceptic will object. The sceptic purports to have shown exactly that attempts to supply evidence for cornerstones will fail, and so, attempts to run this kind of argument will hardly impress her. At most, it will make her run the sceptical argument one more time.[25]

This suggests that externalism about empirical cornerstones will not do.

Next, consider the kind of mathematical cornerstones we have been concerned with – say, $Sat(T)$ and $Con(T)$, for some mathematical theory T. As before, let us suppose that these cornerstones (of T-theorizing) are warranted in an externalist fashion, by T in fact being respectively satisfiable and consistent – i.e. by worldly cooperation – and by the beliefs having been acquired in a reliable manner. In addition, suppose that the externalist maintains that cornerstone warrant thus conceived will sustain the right to claim to warrant for beliefs arrived at through execution of T-projects. In order for this to be so the claimant must be able to defend these claims, and this requires that she be able to present the grounds that sustain them – in particular, a warrant for the mathematical cornerstones. Again, the crucial question is whether the subject with the presumed externalist warrant can present the warrant to the sceptic in such a manner that the sceptic can recognize that the subject has the right to make the claims in question. Arguably not, for reasons similar to the empirical case. The externalist cornerstone warrant is supposed to be supplied by the fact that $Sat(T)$ or $Con(T)$ obtain and the fact that the belief that $Sat(T)$ or the belief that $Con(T)$ has been acquired in a reliable fashion. However, how can the externally warranted subject present this fact to the sceptic in the demanded manner? As in the empirical case, the warrant-conferring fact(s) may obtain unbeknownst to the subject, and in that case will not be something that she can present to the sceptic to defend her claims to warrant. On the other hand, the subject might think that she has access to the

[25] This is not to suggest that it cannot be evidentially shown that certain external conditions are met. Rather what is taken for granted in the argument is that the sceptic takes herself to have shown that certain presuppositions of *sufficient generality* cannot be shown to be warranted evidentially.

(external) warrant-conferring fact and cite evidence for thinking that $Sat(T)$ or $Con(T)$ and for thinking that the way in which she acquired the belief that $Sat(T)$ or the belief that $Con(T)$ was reliable. The sceptic will object, though. She takes herself to have shown that $Sat(T)$ and $Con(T)$ cannot be warranted evidentially, and so, she will not like the move that the externally warranted subject makes.

In both cases, then, the supposed cornerstone warrant will either not be available to the subject so she can present it to the sceptic and thereby defend her rational claim to warrant for ordinary T-propositions, or she takes it to be available and tries to cite evidence for so thinking, but at the cost of invoking a move that the sceptic does not allow. This suggests that externalism about empirical and mathematical cornerstones is not a viable option. By contrast, we should adopt some kind of internalism.

Once more it should be stressed that the argument just presented is very restricted in several respects. First, it only concerns cornerstone warrant. Second, at most the argument yields a conditional recommendation: if you want cornerstone warrant to sustain the right to claims to warrant, it should not be understood along externalist lines. By contrast, we can take some version of internalism to be recommended – but, to stress, on the assumption that we are looking for a response to scepticism.

Also, the argument does not say anything about what specific kind of internalist cornerstone warrant to adopt. However, at this stage, we can broaden our perspective and take it to work in conjunction with things said earlier. The notion of entitlement of cognitive project happily offers itself if we want an account of cornerstone warrant, and, as has been argued in Section 6.2, this notion of warrant can be taken to be internalist when taken in the relativistic or open-ended sense.

6.6 Default foundations?

The notions of entitlement of cognitive project and of substance are meant to apply to mathematical cornerstones. Cornerstones seem foundational in character. If P is a cornerstone for a region of thought R, by the very characterization of a cornerstone, we cannot make rational

claims to warrant for belief in any R-proposition absent a warrant for P. It is thus worth asking if the entitlement proposal, as it has been developed here, delivers foundations in any interesting sense, and if so, in what sense exactly?

There is a familiar kind of foundations the entitlement proposal does not deliver. The entitlement proposal makes no play with the kind of epistemology to be found in the writings of at least three prominent figures that are traditionally conceived of as mathematical foundationalists: Euclid, Descartes, and Frege. In setting out their epistemology of axioms – not necessarily in a particularly systematic manner – each of these thinkers invokes some property of being obvious, self-evident, certain, or indubitable.[26] There are significant differences between their employment of these notions, but one thing that does make it reasonable to group them together epistemologically is their adoption of a faculty for which 'rational intuition' or 'the light of reason' seems a happy label. Each thinker takes the exercise of rational intuition to be characterized by a certain immediacy, axioms instantaneously striking the subject as obvious, self-evident or indubitable, *and* being warranted on the basis thereof.

Entitlement of cognitive project is meant to apply to axioms – or more specifically, their satisfiability and consistency – and entitlement of substance to the existence of basic mathematical substances with a certain feature. However, the epistemology that goes along with these kinds of entitlement is not one of immediacy, of these propositions having a special epistemic property instantaneously striking the subject or being grasped through a special faculty. No appeal is made to rational intuition or the light of reason.[27] It is important to note that this is <u>not</u> to say that, on the entitlement proposal, a satisfiability or consistency proposition cannot strike the subject as obvious, self-evident, certain, or indubitable. Rather, what is denied is that this has any epistemic force, i.e. that the subject's being in a certain mental state when reflecting or thinking about these propositions somehow contributes to their being warranted.

So, what kind of foundation *does* the entitlement proposal deliver? Given the characterization of a cornerstone, in the case of entitlement of cognitive project, the answer has to be this: an entitlement to the cornerstones of a theory T yields a foundation for rational claims to warrant

[26] Indications of Euclid's leanings can be found throughout his *Elements*, and the same goes for the writings of Descartes. For Frege on foundations, let me refer to an excellent secondary source: Burge [20].

[27] This point is made in the context of logic in Wright [153].

for belief in T-propositions. For entitlement of substance, the answer has an extra part to it: an entitlement to the basic ontology of a certain conception of some mathematical realm yields a foundation for making rational claims to warrant for belief in propositions concerning the entities in that realm *and* for the objectivity of judgements concerning them. The foundations provided thus pertain to the rational claimability of warrant rather than possession of warrant, as emphasized earlier. In this sense what we get is higher-order foundations – foundations for rationally claiming our 'first-order' cognitive achievements. An entitlement of cognitive project to $Sat(T)$ supplies a foundation for rationally claiming warrant for beliefs arrived at through model-theoretic or semantic reasoning in T. An entitlement of cognitive project to $Con(T)$ supplies a foundation for rationally claiming warrant for beliefs arrived at through proof-theoretic or deductive reasoning in T. An entitlement of substance to the existence of a realm of numbers – conceived as abstract and non-spatio-temporally located – yields a foundation for rational claims to warrant for beliefs concerning such abstract and non-spatio-temporally located entities, as arrived at through mathematical judgement.

We should comment on a central feature of entitlement which, to some minds, may make it inappropriate to speak of entitlements delivering foundations. Entitlement, whether of cognitive project or of substance, is a default notion of warrant. A cornerstone P for a given region of thought R is an entitlement of cognitive R-projects provided that there is no sufficient reason to believe P untrue. The existence of a constitutively mind-independent basic ontology of a conception of substance for R is an entitlement of substance provided that there is no sufficient reason to think the conception incoherent.[28] Thus, to say that we get a warrant for (higher-order) foundations as a matter of entitlement is, in effect, to say that we get foundations by default. Similarly, by taking entitlement of cognitive project to mandate rational trust and entitlement of substance as sustaining that we can rationally take certain ontological theses for granted, the entitlement proponent appears to operate with a notion of rationality according to which one can rationally place trust in certain propositions and rationally take it for granted that a certain basic ontology exists by default.

[28] For both entitlement of cognitive project and substance we assume that the other clauses are met. Also, again, it should be stressed that the entitlement to a realist ontology does not just appear out of the blue. It is conditional on a certain conception of mathematical judgement.

6.7 Conclusion

In this chapter we have discussed entitlement in light of the distinction between internalism and externalism, considerations on epistemic value, cognitive responsibility, and scepticism. It was argued that entitlement of cognitive project is an internalist notion of warrant, when understood in the relativized or open-ended sense (as spelled out in Chapter 5). We also saw that Wright intends entitlement to be an internalist notion of warrant and extracted two theses, (CLAIMS AND INTEGRITY) and (INTEGRITY AND INTERNALISM), according to which scepticism is an attack on the right to claims to warrant and the right to make such claims is a matter of intellectual integrity (or cognitive responsibility), which is an internal matter. An attempt was made to draw a picture of a wider philosophical framework of which entitlement can naturally be regarded to be a part. (CLAIMS AND INTEGRITY) was a part of this framework, as was a restricted version of (INTEGRITY AND INTERNALISM). (It is not plausible in general that cognitive responsibility is an internal matter. However, when restricted to cornerstones, the thesis is plausible.) A further part of the framework was pluralism about epistemic value – something which makes for a happy fit with the entitlement proposal (and which Wright can also reasonably be taken to adhere to).

Setting out the wider philosophical framework in question resulted in various findings pertaining specifically to entitlement, but also of a more general kind. Concerning the latter, it was proposed that a common view on the internalism/externalism debate – the exclusion view – is misconceived. The debate between internalists and externalists should not be regarded as a debate over whether warrant is internal or external across the board. Rather, it can be granted that both sides of the debate delineate proper notions of warrant. However, it is compatible with this view – the inclusion view – that internalist and externalist notions of warrant are potent to address different problems. The inclusion view was spelled out with pluralism about epistemic value taken as background.

It was argued that there was an intimate relationship between warrant for presuppositions, the right to claim warrant, cognitive responsibility, and a feature of claims called 'defensibility'. A claim was said to be defensible provided the claimant can defend it in discussion with an interlocutor, i.e. provided that she is able to present the grounds that sustain the claim in such

a way that the interlocutor can recognize that she has the right to make it. For the particular case of cornerstone warrant, it was argued that externalism undermines the defensibility of claims to warrant that are meant to be supported by cornerstone warrants. This was taken to issue the following conditional recommendation: if an account of cornerstone warrant is to address scepticism – regarded as an attack on the right to claims to warrant – then it cannot be externalist. By exclusion, this was taken to support an adoption of an internalist account of cornerstone warrant. Not too surprisingly, it was pointed out that entitlement offers itself as a candidate.

The chapter finished by briefly discussing the question what sense, if any, can be made of the view that the entitlement proposal delivers foundations? The response given was that, by the very characterization of a cornerstone, entitlement supplies a certain kind of higher-order foundation – the right to claim our first-order cognitive achievements. It was pointed out that, given the nature of entitlement, we get this kind of foundation by default.

Chapter 7

Is entitlement an epistemic kind of warrant?

In this chapter, we will formulate two fundamental challenges to the entitlement proposal: is entitlement an epistemic kind of warrant at all, and, related thereto, is the notion of rationality associated with entitlement epistemic in character or of some other kind? This chapter develops a response to these worries centered around the idea – merely gestured at by Wright – that entitlement of cognitive project is an epistemic kind of warrant because, when P is an entitlement, trust in P is a dominant strategy with respect to promotion of epistemic value.

We formulate a criticism of the proposal by arguing, firstly, that the prospects of success for this line of approach are bound to depend on the underlying theory of epistemic value, and secondly, that although any reasonable theory of epistemic value should count error-avoidance among its epistemic values, doing so in the context of the dominance argument threatens to undermine it. We formulate an amended version of the dominance argument, the basic idea being that trusting entitlements maximizes expected utility. The chapter ends on a critical note by pointing out that the expected utility approach rests on a controversial assumption, which, in the context of a discussion of scepticism, cannot be granted.

7.1 A reminder: entitlement and trust

Before we present the challenges that will concern us in this chapter, let us remind ourselves that Wright adopts a distinction between belief and acceptance, where the latter is taken to be

a more general kind of attitude than the former.[1] Modes of acceptance other than belief include assumption, supposition, and trust. What is distinctive about belief is that it is controlled by evidence. Beliefs are formed, destroyed, and otherwise modified in light of evidence. This does not hold in general for acceptance. Consider, e.g., *assuming that P*. I can choose to work on the assumption that Euclidean geometry is true, though there is evidence that it is not. Or think of proof by reductio. If I work in classical logic and want to prove that the law of excluded middle is a theorem, it is a good strategy for me to assume its negation and treat it as true until I derive a contradiction.

Entitlement was characterized as a non-evidential species of warrant. We have seen that it is granted the sceptic that there can be no specific positive evidence supporting cornerstone propositions. Since belief is evidentially controlled, entitlement, *qua* cornerstone warrant, has to be a warrant to accept rather than a warrant to believe. More specifically, we have seen that the mode of acceptance associated with entitlement is *trust*. Trust, as Wright conceives it, contrasts with other modes of acceptance like assumption and supposition in that trust in P is not compatible with belief in $\neg P$, i.e. a disbelief that P. It is not compatible with doubt about P either, or even open-mindedness about P. Entitlement has to be paired with an attitude that excludes doubt and any attitude that implies doubt because to doubt – or weaker, to be open-minded about – a presupposition of a project would, by the characterization of presupposition, rationally commit one to doubting the competence of the project.

7.2 Two challenges: epistemic reasons and rationality

We will now formulate two fundamental challenges concerning entitlement.

There is an intimate relationship between warrants and reasons. Some might hold that a warrant is simply given by, or amounts to, a reason or a set of reasons. I do not want to make this assumption here. The worries to be presented can be formulated without it, and we shall instead make the following weaker assumption:

[1] Wright [154], Section II.

(WARRANT AND REASONS):

A subject's having a warrant of kind X for ϕ-ing that P requires that the subject has a reason, or reasons, of kind X for ϕ-ing that P,

where 'ϕ' is a place-holder for some propositional attitude. The instance of (WARRANT AND REASONS) of relevance to entitlement is this: a subject's having an epistemic warrant for trusting that P requires that the subject has an epistemic reason, or reasons, for trusting that P.[2]

A fundamental question about entitlement is whether it is an epistemic notion of warrant at all.[3] It is by no means clear that it is. (WARRANT AND REASONS) invites us to reflect on what epistemic reasons are involved in cases of respectively evidential and non-evidential warrant. There is an easy answer for the evidential case. Whenever we have an evidential warrant for believing P, we can simply take the evidence to be the epistemic reason demanded by (WARRANT AND REASONS). By definition, this answer is not available in the case of non-evidential warrant, and so, in particular, it is not available in the case of entitlement. But what is the answer then?

Now, it is not difficult to give reasons for accepting cornerstones, which, if warranted, are so as a matter of entitlement. Let the proposition that I am not a brain in a vat serve as our example. Suppose that someone tells me that I will receive a cheque of £1,000,000 if I accept that I am not a brain in a vat. In this situation, I certainly have a reason to accept that I am not a brain in a vat. However, it seems uncontroversial to say that, whatever kind of reason it is, it is not an epistemic one.

The proponent of entitlement, conceived as an *epistemic* kind of warrant, faces the following challenge:

[2] In order to hold that possession of an epistemic warrant for ϕ-ing that P not just requires, but amounts to the subject having an epistemic reason or a set of reasons for ϕ-ing that P, one has to say something further about the reason or reasons involved. A subject's merely having an epistemic reason to ϕ-ing that P does not suffice for the subject to be epistemically warranted in believing P. Consider belief. Suppose that Smith visits Scotland for the first time, and while out for a walk in the Highlands, sees first one black sheep, then a bit later one more, and so on up to five. Now, Smith has some evidence, and thus, some epistemic reason to believe that sheep in Scotland are black. However, this reason – the evidence – does not amount to a warrant. One proposal would be to speak in terms of *sufficient* reason for warrant.

[3] A worry of this sort is articulated in Pritchard [109].

(CHALLENGE 1):

Provide a characterization of epistemic reasons and show that, when P is an entitlement, epistemic reasons are present for trusting P, as required by the relevant instance of (WARRANT AND REASONS).

It is important for the entitlement proponent to respond to the challenge. If no epistemic reasons can be pointed to, by (WARRANT AND REASONS), entitlement cannot be an epistemic kind of warrant. Entitlement is certainly intended to be an epistemic notion of warrant.[4] However, it is by no means clear what makes it so. One reason for wanting entitlement to be epistemic in nature is that it is introduced as a response to scepticism. Scepticism surely appears to be an attack on our epistemic practice – more specifically, on our right to claim to warrant for a wide range of beliefs we hold. If entitlement was not epistemic in character, we would in effect be invoking a non-epistemic notion of warrant to respond to an epistemic challenge. There would be something slightly misdirected about that.

There is a related challenge concerning rationality. When a subject has a warrant of kind X for ϕ-ing that P, the subject is thought to be X-rational in ϕ-ing that P. (Here 'X' is a place-holder for 'epistemic', 'pragmatic', etc.) The idea is that X-rationality flows from the X-warrant possessed:

(WARRANT AND RATIONALITY):

When a subject has an X-warrant for ϕ-ing that P, the subject is X-rational in ϕ-ing that P because of her X-warrant for ϕ-ing that P.

The instance of (WARRANT AND RATIONALITY) relevant to entitlement is this: when a subject has an epistemic entitlement to trust that P, the subject is epistemically rational in trusting that P because of her epistemic entitlement to trust P. Given (WARRANT AND REASONS), (WARRANT AND RATIONALITY) suggests, quite plausibly, that there is a link between reasons and rationality.

[4] This is suggested by the title of the session at which Wright [154] and Davies's response to Wright ([32]) were presented. The title was 'On *Epistemic* Entitlement' (emphasis added).

Let us see how this squares with respectively evidential and non-evidential warrant. As before, there seems to be no problem in the case of justification (i.e. evidential warrant). When an epistemic subject is justified in believing P, evidence which is strong enough to yield – or, depending on your views, constitute – a warrant is present. In light of this evidence it is rational for the subject to believe that P. Suppose, say, that I prove (in ZF) the proposition that every transitive set well-ordered by membership is an ordinal and form a belief in the proposition as a result thereof. In that case my proof delivers evidence which supplies, or constitutes, an epistemic warrant for believing the proposition, and I am epistemically rational in believing this, *because* of the proof; because of the evidence available.

In the non-evidential case, matters do not seem to be as straightforward. The advocate of entitlement seems to face a second challenge:

> (CHALLENGE 2):
>
> Provide an account of what the rationality of trusting P consists in when P is warranted as a matter of entitlement and show that the rationality in play is epistemic in nature.

This challenge is related to the first. A successful response to (CHALLENGE 1) will deliver an answer to (CHALLENGE 2) as well, to the extent that it is granted that there is an intimate relationship between warrant and reasons.

7.3 The non-rationality of not trusting entitlements

It might be natural to suppose that one is somehow epistemically blameworthy or irresponsible if one believes without possessing evidence sufficient for warrant. Wright is keen to stress that this point does not transfer when instead of belief we talk of entitled acceptances:

> If I am entitled to accept P, then my doing so is beyond rational reproach even though I can point to no cognitive accomplishment in my life, whether empirical or *a priori*, inferential or non-inferential, whose upshot could reasonably be contended to be that I had come to know that P, or had succeeded in getting evidence justifying P. (Wright [154], pp. 174–175)

But this is not all. According to Wright, when P is an entitlement, accepting – or, to be specific, trusting – P is not only beyond rational reproach – *not* trusting P if engaging in the project is

not rational. Why? Because presuppositions of cognitive projects are unavoidable commitments of these projects. They are so in the sense that, by the characterization of a presupposition, doubting a presupposition of a cognitive project rationally commits one to doubting the competence of the project.

However, even if trusting P is a necessary condition for rational engagement in projects for which P is a presupposition, one might wonder whether it is sufficient. Wright clearly thinks it is, characterizing entitlement as 'rational trust.'[5] Note that although we grant Wright that entitlement is rational trust, the two challenges from the previous section remain unaddressed. That is, a case remains to be made for the claims that entitlement is an epistemic kind of warrant and that the rationality that goes with it is epistemic in nature.

7.4 Epistemic values, reasons and rationality

We will now explore a potential line of response to the challenges raised above. The core idea is gestured at in the following passage:

> If a cognitive project is indispensable, or anyway sufficiently valuable to us – in particular, if its failure would at least be no worse than the costs of not executing it, and its success would be better ... then we are entitled to – may help ourselves to, take for granted – the original presuppositions without any specific evidence in their favour. (Wright [154], p. 192)

Here the perhaps most straightforward way to understand the talk of 'worse', 'success', and 'better' is in terms of values, conceived as ends, and means to achieve these. The basic idea thus seems to be that trusting entitlements is a dominant strategy with respect to bringing about epistemic value; that they are the best means for reaching certain ends.[6]

Some stage-setting is needed to develop this idea into a response to (CHALLENGE 1) and (CHALLENGE 2). More specifically, something needs to be said that relates epistemic reasons and epistemic values, and, in addition, something needs to be said about what things are of

[5] Wright [154], p. 194, 205.

[6] Wright [154] also contains considerations on a notion of entitlement referred to as 'strategic entitlement'. The difference is that the dominance idea figures explicitly in the characterization of strategic entitlement, while it figures neither in clause (i) nor in clauses (ii) or (iii) of the characterization of entitlement of cognitive project.

epistemic value. We will undertake the first task in the remainder of this section. The second task was briefly touched upon in Section 6.4 and a bit more will be said below.

Let us turn to epistemic reasons and values. I will adopt a characterization of epistemic reasons offered by Foley in his book on epistemic rationality. The characterization seems ideal for the purposes of developing the dominance idea as it ties epistemic reasons intimately to epistemic values. A generalized formulation of the characterization reads as follows:

> (REASONS AND VALUE):
> If X is a Y-valuable end and bringing about Z promotes X, then, all else being equal, one has a Y-reason to bring about Z.

In particular, if X is an epistemically valuable end and bringing about Z promotes X, then, all else being equal, one has an epistemic reason to bring about Z.[7] The plan is to offer some suggestions as to what might be regarded as being epistemic values and assess the prospects of arguing by reference to one or more of these values that, when P is an entitlement, one has epistemic reason to trust P because it promotes epistemic value, and that this is what makes such trust epistemically rational.

Now, (REASONS AND VALUE) is likely to trigger some questions. Below I give brief responses to questions of a clarificatory or critical nature:

Question: what is meant by 'promote'?. Answer: 'promote' is to be taken in a fairly wide sense. I take Z's promoting X to include (at least) the following cases: (i) Z causes X, (ii) Z entails X, (iii) bringing about Z increases the likelihood of X occurring, (iv) bringing about Z will not do worse with respect to X than not doing Z and may do better, and (v) bringing about Z promotes X if anything does. If Z promotes X in the sense captured by (iv), bringing about Z is said to be a *dominant strategy* with respect to X. As indicated earlier, dominance is what shall concern us later.

Question: should it not be added that it has to be known – or perhaps believed – that bringing about Z promotes X? For instance, suppose that Bill Gates decides to spend the day by London Bridge with a sign saying, 'Say 'Hi Bill!'', and that, for every person who passes by and says

[7] See Section 1.1 of Foley [48].

'Hi Bill', he will donate £1,000,000 to charity. Furthermore, suppose that I pass by and am not aware of this. Do I have a moral reason to say, 'Hi Bill!' though I do not know – or believe – that so doing will promote happiness which, we may assume, is a moral good? For an epistemic case, suppose that little Bob is a child prodigy in mathematics whom I get to ask yes or no questions on difficult questions in functional analysis (which I do not myself know the answers to). Do I have an epistemic reason to believe the answers he gives though I do not know – or believe – that so doing will lead to true beliefs which, we may assume, is an epistemic good? Given certain assumptions to be made in our treatment of the dominance argument, the subject can be assumed to have beliefs about what promotes epistemic value.

*Question: is it not the case that (*REASONS AND VALUE*) has obvious counterexamples? Suppose, as is plausible, that having true beliefs is epistemically valuable and consider the following scenario: I go to visit an oracle who, on the day of my visit, happens to be very grumpy. Now, because the oracle is grumpy, she does not want to speak – so, she does not tell me any truths which is unfortunate, because that is why I came to visit. However, she makes it clear that, if I buy her a big block of chocolate, she will state truths for two hours straight. So, I have an epistemic reason to buy the oracle chocolate.* Answer: I bite the bullet, conditionally. *If* you adhere to (REASONS AND VALUE), then there is an epistemic reason to buy the oracle chocolate. In case one is sympathetic to (REASONS AND VALUE), but, at the same time, takes the oracle example to teach us something, perhaps the lesson is that (REASONS AND VALUE) cannot be taken in full generality, but needs to be restricted somehow. One suggestion is to restrict the thesis to reasons for adopting certain *attitudes* to propositions.

*Question: is it not the case that (*REASONS AND VALUE*) fails to take into consideration situations in which I have to bring about something which is not of epistemic value or reduces epistemic value in order to bring about something else of epistemic value? Recall the oracle example. Instead of demanding chocolate suppose that the oracle tells me that she will talk for two hours straight if I believe that the Earth is flat.* Answer: though some things – like believing falsely that the Earth is flat – are not of epistemic value themselves, they *can* be subsumed under (REASONS AND VALUE) provided we construe promotion of X in such a way that whether or not Z promotes X (which is Y-valuable) does not only depend on whether Z itself promotes

X directly, but also on how the consequences of Z do in this respect. In that way Z's own lack of – or reduction in – Y-value can be outweighed by the Y-value of its consequences.

*Question: is it not problematic that there are many things one can easily bring about that will promote epistemic value, and so, according to (*REASONS AND VALUE*) have an epistemic reason to bring about these things? Example: one can go to the supermarket and start counting how many bananas they have, the number of coke cans left on the shelves, etc. thereby arriving at a number of true – and so, epistemically valuable – beliefs.* Answer: granted, one can have an epistemic reason to do so. However, there can still be stronger reasons to do other things. For instance, a set theorist who has been working on a proof of a result which, if true, would have many immediate consequences within set theory – and perhaps other mathematics theories as well – would have a stronger reason to work on the proof than to go to the supermarket and start counting various kinds of groceries.

7.5 The dominance argument

We will now take on the task of developing the dominance argument in detail. As announced, the basic idea is to do so by cashing out dominance in terms of dominance with respect to promotion of epistemic value. If the dominance idea can be made to work, a response to (CHALLENGE 1) and (CHALLENGE 2) might very well be forthcoming. For, in that case we can say, by appeal to (REASONS AND VALUE), that there is an epistemic reason to trust P – P an entitlement – because it is a dominant strategy with respect to promotion of epistemic value, and then go on to cite this reason in accounting for the rationality of trusting P.

The dominance argument will be formulated in two versions. The first version incorporates veristic monism. A second, slightly more sophisticated, version of the argument will be given. What is interesting for our present purposes is that it seems to matter how much emphasis is put on truth when we turn to the dominance argument. It will be argued that if truth takes centre stage – i.e. if we adhere to veristic monism or certain kinds of veristic unitarianism – the dominance argument goes through. (Veristic monism was introduced in Section 6.4.) However, it will also be suggested that such views are implausible, because they fail to count error-avoidance

as an epistemic value, *and* that taking error-avoidance into consideration threatens to undermine the dominance argument.

7.6 Preliminaries

Before we move on to the two versions of the dominance argument, it is worth saying a bit about the table which we will use to run the argument and what assumptions will be made. Here is the table, with the computations of epistemic value still to be added:

	1. P is the case	2. $\neg P$ is the case
Trust P and execute		
Not trust P and not execute		

'P' refers to an entitlement. Vertically we have the different actions or strategies. Horizontally we have whether or not P is the case, i.e. whether or not the world cooperates.

The following assumptions will be made:

1. The subject we consider is a rational subject, deliberating which strategy is optimal. This is taken to involve that the subject is aware of what the epistemic values are; that the subject engages in projects only if she is not committed to doubting that they can be successfully executed; that the subject will engage in projects whenever she trusts that the presuppositions for their success are met.

2. When computing the overall epistemic value of combinations of the different strategies and the states of the world, we will do so on the basis of a *class* of projects rather than a single project. The projects in the class are the projects for which P is a presupposition.

3. Many beliefs are formed in the case of execution (one for each project executed). Only a few beliefs are formed in the case of non-execution.

4. The target notion of execution is to be such that it is compatible with the idea that, say, a brain in a vat can execute projects, or more generally, that projects can be executed in cases

where the relevant entitlement is false. In the brain in a vat case, the execution of a project could be taken to amount to a series of states that are qualitatively indistinguishable from the states someone who is not a brain in a vat has when engaging in the project.

It is worth making a number of further comments. As regards 3, in setting up the tables, we shall use 'few' and 'many'. For our present purposes, there is no need for a precise cardinal measure. What is important is that occurrences of 'few' and 'many' across columns are assumed to give the same number, and that, within a column, the number picked out by 'few' is smaller than that picked out by 'many'.

Someone might suggest that there should be a row for non-trust and execution, and, additionally, one for trust and non-execution. However, we have left these rows out as the subject is assumed to be rational. Not trusting that P and in engaging projects of which it is a presupposition clashes with this assumption. It is not rational for a subject to engage in a project unless she trusts that its presuppositions are met. For without this being so, she is rationally committed to doubting that the project can be successfully executed. Trusting P and not engaging clashes with the rationality assumption as well. Whenever the subject trusts that the presuppositions of a project are met, she is assumed to engage.

I have allowed myself to leave out an 'all else being equal' parameter. One might want to include it, because there are many ways execution of a project can go wrong. E.g., lighting conditions might not be optimal when I try to determine the colour of a jumper, or I might stumble and fall flat on my face when I try to move across my office to count how many books there are in one of my book cases. We could include the parameter if we wanted, but it would not make a difference to the dominance argument. So, I have chosen to leave it out to keep things simple.[8]

[8] However, since it is a substantial omission I will try to qualify it. The reason is that an all else being equal parameter will have the same impact for every row in a column, and so, will preserve the relative ordering of the outcomes (of the various strategies) within a column. Since the question of which strategy is dominant (if any) is answered by comparing strategies – i.e. rows – within the columns of the table, this means that the parameter under consideration would have no significance with respect to the dominance question.

If we wanted to include 'all else being equal' it would have to be understood in such a way that an entitlement's being false is compatible with all else being equal. For example, in the vat scenario, all else being equal could be taken to amount to nothing breaking the series of states which are qualitatively indistinguishable from the states a non-envatted subject goes through when engaging in some project. This would happen if, e.g., an envatted brain was going through a series of states qualitatively indistinguishable from the states a non-envatted subject

7.7 First formulation of the argument

According to veristic monism, truth is the only thing of (intrinsic) epistemic value. The view is usually regarded as one of the serious contenders in the discussion of epistemic value.[9] The first version of the dominance argument takes veristic monism as the underlying theory of epistemic value.

Here is the table on which the first version of the argument is based ('T' abbreviates 'true'):

	1. P	2. $\neg P$
Trust P and execute	Many T beliefs, T acceptance of P	Few T beliefs
Not trust P and not execute	Few T beliefs	Few T beliefs

So, which is the optimal strategy – trust or non-trust? The answer is that trust is as it dominates non-trust:

First dominance argument: trust and execution is the dominant strategy. In column 2 it does no worse than non-trust and non-execution, but in column 1 it does better.

7.8 A response to (CHALLENGE 1) and (CHALLENGE 2)

Let us return to the question of whether the dominance idea implicit in Wright can be developed into a response to (CHALLENGE 1) and (CHALLENGE 2). We recall that (WARRANT AND REASONS) demanded that, whenever there is an epistemic warrant, there is an epistemic reason, and that (CHALLENGE 1) invited us to provide a characterization of epistemic reasons and show that, whenever there is an entitlement, there is an epistemic reason. (CHALLENGE 2) invited us to support the claim that trust in P is rational when P is an entitlement, and further, that the rationality in play is epistemic.

is in when checking what colour some object is and suddenly being put in a state qualitatively indistinguishable from the state a non-envatted subjected is in when light conditions change drastically.

[9] See Goldman [62]; David [29] and [28].

The framework chosen to develop the dominance idea was a means-end framework. Given this framework and the dominance argument, it seems that we now have a response to the two challenges on behalf of the friend of entitlement:

When P is an entitlement, we have an epistemic reason to trust P because doing so promotes epistemic value. As we have seen, it is a dominant strategy. This answers (CHALLENGE 1). As for (CHALLENGE 2), the response flows directly from the response to (CHALLENGE 1). When we trust P as a matter of entitlement, it is epistemically rational to do so, because we have an epistemic reason to trust P. The reason is that it promotes epistemic value.

7.9 Does the dominance argument really work?

All of this seems fine and dandy. If truth is really the only thing of epistemic value, then the (first formulation of the) dominance argument can be made to work. But that's a big 'if'.

I have a serious misgiving about the formulation of the dominance argument rehearsed above. The misgiving is this: the argument seems to misrepresent the relevant epistemic situation by missing out one crucial thing of epistemic value, *viz.* error-avoidance. There is a simple reason why error-avoidance should be counted as an epistemic value alongside truth (and possibly others). If truth – and only truth – was epistemically valuable, there would be an easy way to maximize epistemic value – namely, to believe every proposition. (According to (REASONS AND VALUE), one would have an epistemic reason to believe every proposition.) But, if one believed every proposition, one would have many false beliefs, and surely, having false beliefs is not epistemically valuable – quite the opposite: *avoiding* such beliefs is epistemically valuable.[10]

What this misgiving suggests is that the first formulation of the dominance argument does not stand up to scrutiny. The table on which the dominance reasoning is based grossly misrepresents the outcomes. Error-avoidance is of epistemic value and this should be reflected in the table. So, the table should look like this ('F' abbreviates 'false'):

[10] We cannot take avoiding a false belief to be a special case of believing something true in the sense that, if P is false and we avoid believing P, then we believe that $\neg P$ is true. Avoiding a false belief does not imply believing its negation to be true – one could be agnostic.

	1. P is the case	2. $\neg P$ is the case
Trust P and execute	Many T beliefs, Few F beliefs, T acceptance of P,	Few T beliefs, Many F beliefs, F acceptance of P
Not trust P and not execute	Few T beliefs Few F beliefs	Few T beliefs Few F beliefs

Does trust dominate non-trust? No, it does not:

Dominance breakdown: trust and execution has the best outcome in column 1. However, it is by no means clear that trust and execution does no worse than non-trust and non-execution in column 2. Indeed, if it is granted that having a true belief and avoiding a false one are equally epistemically valuable, then trust and execution does *worse* than non-trust and non-execution in column 2.

Thus, the answer to (CHALLENGE 1) and (CHALLENGE 2) which the first formulation of the dominance argument so happily offered is no longer available. There is no epistemic reason to trust an entitlement P in the sense that so doing is a dominant strategy with respect to promotion of epistemic value, and, hence, a subject cannot be said to be epistemically rational in trusting P, because there is an epistemic reason – dominance – to do so.

Before we proceed to the next section, let me record a dialectical point which I hope will prevent a certain misunderstanding. I am aware that many people have misgivings about the Foley-style characterization of epistemic reasons given in Section 7.4. The source of these misgivings is probably a strong feeling – perhaps backed by philosophical argument – that epistemic reasons are not properly characterized within a means-end framework – and, as a result, if the only support entitlement can be given is provided by a dominance argument, then entitlement is not an epistemic kind of warrant. Now, rather than balk at the employment of the Foley-style characterization, such people should applaud it. Because what the considerations offered in this section suggest is that *even if the means-end characterization of epistemic reasons is granted, it*

is not clear that the dominance argument goes through.

7.10 Expected utility to the rescue?

One reaction to the discussion of the dominance idea provided so far might be that something crucial is missing from the framework, *viz.* probability. For isn't probability always something a reflective subject takes into consideration when deliberating what to do or what to believe? For instance, suppose that my friends have told me that they might be at a certain bar tonight, and that now, at night, I am trying to decide whether or not to go to the bar in question. Part of what will make me go one way or the other is the probability I associate with respectively my friends being there and their not being there. If I find it highly improbable that they will be there, that will count against my going, while, on the other hand, if I think that they will almost certainly be there, this will encourage me to go.

These things said, why might probability be a relevant parameter to take into consideration in our discussion of entitlement? Well, the basic thought is that, if the probability of $\neg P$ is sufficiently low, then so should its significance be with respect to determining whether trusting P does better than not trusting P in terms of promotion of epistemic value. For the purpose of being able to accommodate this thought it would be natural to switch to talk of *expected utility*.

The first thing to do is to return to the notion of promotion associated with (REASONS AND VALUE). Earlier we saw that the notion was intended to be such that Z promotes X if bringing about Z is a dominant strategy with respect to X. The *expected utility approach* starts with the suggestion that we should also include as instances of promotion cases where bringing about Z maximizes expected utility, and, in particular, that trust in entitlements is rational exactly because it promotes epistemic value in the sense of maximizing expected epistemic value.

To spell out this idea in detail we need to say how the expected utility of an action is calculated. Let $A_1 \ldots A_n$ be the possible courses of action. Let $O_1 \ldots O_k$ be the possible outcomes. Assign each pair of an action A_i and an outcome O_j a value, $V(A_i, O_j)$. Lastly, we need a probability $p(O_j)$ for each possible outcome O_j.

We will use '$U(A_i, O_j)$' to denote the expected value of an action A_i and an outcome O_j.

The expected value of the action-outcome pair (A_i, O_j) is calculated by taking the product of the probability of $p(O_j)$ and the value of (A_i, O_j), i.e.

$$U(A_i, O_j) = p(O_j) \times V(A_i, O_j)$$

The expected utility of A_i – $U(A_i)$ – is calculated by aggregating the expected value of A_i for each possible outcome, i.e. let $O_1 \ldots O_k$ be the possible outcomes, then

$$U(A_i) = \sum_{j \leq k} U(A_i, O_j) = \sum_{j \leq k} (p(O_j) \times V(A_i, O_j))$$

Given the courses of action $A_1 \ldots A_n$, an action A_i maximizes expected utility just in case $U(A_i) > U(A_j)$ for all $1 \leq j \leq n$ $(j \neq i)$.

Here we note that the kind of probability in play is *subjective*. When we speak of the probability of P, what is intended is the probability which the subject associates with P. The subjective probability which an agent associated with P is standardly taken to be determined by the evidence available to the agent.[11]

7.11 The plunging strategy

In this section, we shall consider what will be called the 'plunging strategy'. As we shall see, trusting an entitlement P can be shown to maximize expected utility on the assumption that the probability of P is higher than that of $\neg P$. The plunging strategy is to maintain that we can help ourselves to this assumption, that we can 'plunge' the probability of sceptical scenario. The plunging strategy seems initially attractive. After all, don't sceptical scenarios always strike people as far-fetched or unlikely?

Let us try to apply the expected utility machinery to the issues that have occupied us so far. The various strategies are trusting P and executing and not trusting P and not executing. The possible outcomes are the possible states of the world, i.e. P's being the case and $\neg P$'s being the case. We have two things left – determining a value for each action-outcome pair and assigning each of P and $\neg P$ a probability. Neither of these will be easy, but, obviously, we need

[11] For an overview of the main interpretations of probability, see Hájek [66].

to say something to get things started.

Bearing in mind our considerations from earlier, it should be clear why it is not easy to determine a value for each action-outcome pair. First, it presupposes that we have a list of epistemic values. Second, it presupposes that we have some handle on the values – not necessarily in terms of an absolute, cardinal measure, but at least a relative ordering. Though it is a controversial issue what things are of epistemic value, we do not, in our present context, have to spend any sleepless nights thinking about it. Recall that we are considering the expected utility approach, because we want to see if it might succeed where the dominance approach failed. So we already have a list of the epistemic values that are considered relevant, *viz.* the ones used to state the worry about dominance. Something similar applies regarding the second question concerning what handle we have on the epistemic values. Above we assumed that truth and falsity cancel out. So, we will go ahead and make this assumption in what follows.

Regarding the question what probabilities to assign respectively to P and $\neg P$, it should not come as a surprise to anyone that the assessment of the expected utility approach will depend on what the appropriate response is to this particular question. A preliminary observation is that, since P and $\neg P$ are mutually exclusive, standard probability theory tells us that their joint probability must be 1, where 1 is used to signify that something is certain. That is, $p(P) + p(\neg P) = 1$. From this it follows that $p(P) = 1 - p(\neg P)$ and $p(\neg P) = 1 - p(P)$. We thus know that if the probability of P is high, the probability of $\neg P$ is low, and on the other hand, if the probability of P is low, the probability of $\neg P$ is high. Without getting too much ahead of ourselves, if the expected utility approach is going to offer any improvement over the dominance approach, it must at least be the case that $p(P) > p(\neg P)$.

We are looking at the following table:

	1. P is the case (O_1)	2. $\neg P$ is the case (O_2)
Trust P and execute (A_1)	Many T beliefs, Few F beliefs, T acceptance of P,	Few T beliefs, Many F beliefs, F acceptance of P
Not trust P and not execute (A_2)	Few T beliefs Few F beliefs	Few T beliefs Few F beliefs

– with the probabilities yet to be assigned to P and $\neg P$.

In order for trust and execution (A_1) to maximize expected utility we need:

$$U(A_1) > U(A_2)$$

i.e.

$$U(A_1, O_1) + U(A_1, O_2) > U(A_2, O_1) + U(A_2, O_2)$$

i.e.

$$p(O_1) \times V(A_1, O_1) + p(O_2) \times V(A_1, O_2) > p(O_1) \times V(A_2, O_1) + p(O_2) \times V(A_2, O_2)$$

Now, the reason that plunging can help here is the way that expected utility is calculated – i.e. by the use of multiplication and adding *across* rows.

As for multiplication, what the plunging strategy exploits is a basic feature of multiplication, *viz.* that the closer $p(\neg P)$ is to 0, the smaller the value of $p(\neg P) \times y$ – and so, the smaller the significance of column 2.[12] To see the significance of calculating expected utility by adding *across* rows, suppose that we were still within a game-theoretic setting and we wanted to find a dominant strategy, but that this time we wanted the strategy to be dominant with respect to expected value rather than value. In that case the plunging move would be ineffective. For (assuming that $p(\neg P) > 0$) the expected values in column 2 would inherit the ordering of the values to which we applied the multiplication operation, whatever these values might be. In other words, since the problems that column 2 raises rest entirely on the ordering *within* the column, whatever worries there are about values will transfer when we rephrase in terms of expected values. Hence, for the plunging move to be significant, it is crucial that we do not just move from asking for a dominant strategy in terms of value to asking for one in terms of

[12] We here assume that $p(P)$ and $p(\neg P)$ are always multiplied with finite values. Now, there *are* arguments in the literature which involve infinite expected utility. Perhaps the most famous example is Pascal's Wager, where believing in God's existence is usually taken to bring eternal – and so infinite – reward if God exists. I do not deny that there are cases in which it is appropriate to reason with infinite expected utility. What I do deny, however, is that the argument we are concerned with here is one of them. This is not a controversial assumption to make, lest I be much mistaken about the range of the epistemic gains and loses we are in for when engaged in our usual epistemic practices.

expected values. We also need to switch from talk of dominant strategies to talk of maximization of expected utility.

7.12 A response to (CHALLENGE 1) and (CHALLENGE 2)

Suppose that the plunging strategy works. That is, suppose that we can assign a probability to $\neg P$ that is low enough to reduce the significance of column 2 – the column that made the dominance reasoning break down – to such an extent that the expected utility of trusting P and executing exceeds that of not trusting P and not executing. Then we can recover an answer to (CHALLENGE 1) and (CHALLENGE 2). To the former we respond: entitlement is an epistemic kind of warrant, because it promotes epistemic value in the sense of maximizing expected epistemic utility. To the latter we say: when we trust P as a matter of entitlement, it is epistemically rational to do so, because we have an epistemic reason to trust, *viz.* promotion of epistemic value in the sense of maximization of expected epistemic utility.

7.13 Expected utility works if and only if dominance does

As we have just seen, we can recover an answer to (CHALLENGE 1) and (CHALLENGE 2) if the plunging strategy works. In this section, it will be argued that it cannot. The basic point of the argument is that a prerequisite for the plunging strategy to work is that $p(P) > p(\neg P)$, but that this should not be granted in the specific context we are in – a discussion of scepticism.

By the lights of someone who likes the entitlement idea, the probability of the sceptical scenario should not be plunged. The argument is quite simple.[13] First, we need to remind ourselves that an entitlement proponent has granted the sceptic that her arguments show that there can be no evidential warrant for cornerstones – or stronger, that there can be no evidence supporting cornerstones. Second, we observe that it is standard to take subjective probability, or credence, to be regulated by evidence. The subjective probability which one associates with a given proposition should accord with the evidence available.

Consider some entitlement P. By the first point, there can be no positive evidence to

[13] Here I am indebted to Elia Zardini for a number of helpful suggestions.

support it. Thus, $p(P)$ should not be greater than 0.5. By clause (ii) of the characterization of entitlement, there is no sufficient reason to believe P untrue. So, $p(P)$ should not be less than 0.5. The only way both of these requirements can be accommodated is by $p(P) = 0.5$. However, since $p(P) + p(\neg P) = 1$, this means that $p(P) = p(\neg P) = 0.5$. Hence, the probability assignment needed to argue that trust and execution maximizes expected utility is not the probability assignment that should be adopted, because this assignment requires that $p(P) > p(\neg P)$. In other words, the probability of the sceptical scenario should not be plunged.

This is not all. It will now be argued that expected utility offers no improvement over dominance when the objective is to overcome the worry tabled against dominance earlier. We will make a case for this by showing that, given the probability assignment $p(P) = p(\neg P) = 0.5$, the expected utility approach will issue the exact same recommendations as the dominance approach.

So, let $p(P) = p(\neg P) = 0.5$. We have seen that switching to expected utilities will do nothing to distort the ordering of the entries *within* a column. Hence, if dominance worked prior to calculating the expected values, then dominance can be rerun in terms of expected values. On the other hand, if dominance failed prior to the switch to expected values, it will also do so if we try to run dominance in terms of expected values. Thus, moving to expected values cannot make a difference with respect to dominance. The idea then was that the switch from values to expected utility – i.e. expected values multiplied by probabilities – should be accompanied by a move from a search for a dominant strategy to a strategy that will maximize expected utility. In that case we can, as seen, have cases where dominance considerations fail to issue a recommendation (whether attempted in terms of value or expected value), but where there is a strategy that maximizes expected utility.

We have seen that involving probabilities will not make a difference with respect to dominance. In particular, then, the assignment $p(P) = p(\neg P) = 0.5$ will not make a difference with respect to dominance. This means that, since dominance did not work before the probability assignment, it will not work after if we try to run it in terms of expected values. However, this specific assignment has a further, for our purposes, very important feature, *viz. that the maximization of expected utility line does not work either.* This is a corollary of the fact that

we can show that, given $p(P) = p(\neg P) = 0.5$, dominance considerations issue the exact same recommendations as considerations on expected utility. To see that this is so it suffices to note that, not only does $p(P) = p(\neg P) = 0.5$ preserve the ordering of the entries *within a column*, it also preserves the ratio between any two entries for a given action *across rows*. That is to say, we claim that

$$\frac{V(A_i, O_1)}{V(A_i, O_2)} = \frac{0.5 \times V(A_i, O_1)}{0.5 \times V(A_i, O_2)}$$

for $i = 1$ or $i = 2$. The equality can be verified by routine computation:

$$\frac{0.5 \times V(A_i, O_1)}{0.5 \times V(A_i, O_2)} = \frac{0.5}{0.5} \times \frac{V(A_i, O_1)}{V(A_i, O_2)} = 1 \times \frac{V(A_i, O_1)}{V(A_i, O_2)} = \frac{V(A_i, O_1)}{V(A_i, O_2)}$$

From this it follows that, when we start aggregating the expected value of an action for the various outcomes, each entry will have the same significance as before we multiplied with the relevant set of probabilities. (Note that $p(P) = p(\neg P) = 0.5$ is the only probability assignment which has this feature, as $p(P) + p(\neg P) = 1$.) Consequently, considerations on dominance and considerations on expected utility will issue the exact same recommendations – in which case the switch to expected utility does not offer any improvement over dominance reasoning.

7.14 Conclusion

We have discussed two of the most fundamental challenges to the entitlement proposal, *viz.* whether entitlement is an epistemic notion of warrant at all, and related thereto, whether the kind of rationality associated with entitlement is epistemic in nature.

It was suggested that Wright can be understood as gesturing at a response to this fundamental challenge in a brief passage which seems to suggest that trusting P – P an entitlement – is a dominant strategy with respect to promotion of epistemic value. A means-ends framework was adopted as a natural framework for developing the dominance idea and exploring the prospects of responding to the two challenges by reference to it. The dominance argument was formulated in two versions. The first version was formulated with veristic monism as the underlying theory of epistemic value, while the second version added error-avoidance to the list of values. It was

argued that though the first, monistic version of the dominance argument works, the reasoning breaks down once error-avoidance is taken into consideration.

One thing the considerations offered in this chapter show is that whether the dominance argument runs or not will depend on one's underlying theory of epistemic value. That is, on what things are taken to be of epistemic value. This conclusion invites a procedural point: in order to use the dominance idea to defend the status of entitlement as an epistemic kind of warrant, the advocate of entitlement should first give a theory of epistemic value and *then* proceed to the dominance argument.

Having raised these worries, we discussed whether a switch to expected utility might offer any improvement over dominance reasoning with respect to responding to the two challenges we set out to explore, i.e. (CHALLENGE 1) and (CHALLENGE 2). It was shown that it will indeed do so provided that the probability of the sceptical scenarios can be plunged. However, it was argued that, by her own lights, the proponent of entitlement cannot justify such a probability assignment. Instead the probability assignment should be such that the sceptical and non-sceptical scenarios are assigned equal probabilities. Since $p(P) + p(\neg P) = 1$, there is only one such assignment, *viz.* $p(P) = p(\neg P) = 0.5$. It was then shown that, given this assignment, dominance reasoning and expected utility reasoning issue the exact same recommendations. Hence, whatever undermines dominance reasoning undermines reasoning in terms of expected utility, and so, the latter cannot offer any improvement over the former.

We conclude that neither the dominance nor the expected utility approach works, as it stands. While I am somewhat sympathetic to the idea that entitlement is an epistemic kind of warrant – indeed, I would like to defend the view – I am thus also convinced that it will take careful thought to meet (CHALLENGE 1) and (CHALLENGE 2). Here I cannot take on the task of doing so, but will have to remain content with having identified a task to be taken on in future work.

Chapter 8

Conclusion

In the Introduction, I set out the goals of the thesis as follows:

1. to develop, in considerable detail, mathematical regress scepticism and mathematical I-II-III scepticism (Chapter 2);

2. to extend the application of the notions of entitlement of cognitive project and entitlement of substance by invoking them to respond to respectively mathematical regress scepticism and I-II-III scepticm (Chapters 3 and 4);

3. to further the development of the notion of entitlement of cognitive project and to draw a picture of a wider philosophical framework of which the entitlement proposal can be regarded as an integrated part (Chapters 5 and 6, Appendix B);

4. to present two fundamental challenges to the entitlement proposal and offer a discussion of them which, while suggesting that there is considerable work to do on behalf of the entitlement proposal, also provides a basic framework within which further discussion may fruitfully be pursued (Chapter 7).

I hope to have reached these goals, or at least to have gone some way towards doing so. Let us summarize the main points of our discussion.

In the Introduction, we started by asking (i) how we can have a warrant for the satisfiability and consistency of mathematical theories, and (ii) given we conceive of mathematical judgement as objective – as being concerned with a realm of abstract entities – can we have a warrant for

thinking that such a realm of entities exists? The mathematical regress sceptic argued that neither $Con(T)$ nor $Sat(T)$ can be warranted (provided that T is sufficiently strong), while the mathematical I-II-III sceptic made a case that a certain kind of attempt to justify the existence of a mathematical realm is bound to fail if mathematical judgement is conceived as being objective. The sceptical arguments were formulated in considerable detail in Chapter 2, with sceptical arguments concerning the empirical world serving as background. If successful, they have the undesirable consequence that we cannot rationally claim warrant for belief in a wide range of propositions we ordinarily take ourselves to be warranted in believing. Thus, unless we want to give up the right to make such claims, we have to answer the sceptic.

The beginnings of a response emerged with the rejection of the sceptical master thought that warrant for any proposition involves warrant for some other proposition – something assumed by the regress as well as the I-II-III sceptic in setting out her argument. This rejection needed backing by an appropriate notion of warrant. The notions of entitlement of cognitive project and entitlement of substance were invoked for this purpose in Chapters 3 and 4, the former as a response to mathematical regress scepticism, the latter as a response to mathematical I-II-III scepticism. The distinctive feature of entitlement, whether of cognitive project or of substance, is that it is a non-evidential, or default, notion of warrant. This was what turned the tables on the sceptic. The sceptic maintained that warrant for any proposition always involves warrant for some other proposition – that, properly construed, warrant is always evidential. Against this the entitlement proposal suggested that for a certain class of propositions – the cornerstones – what is relevant is not the presence of positive evidence, but rather the absence of sufficient countervailing evidence.

Having applied the entitlement strategy to respond to mathematical scepticism, we set out to further develop the entitlement proposal. Chapter 5 started by spelling out two awkward wrinkles left by what Wright explicitly says about how to understand clauses (ii) and (iii) in the characterization of the notion of entitlement of cognitive project. We distinguished between three ways of understanding the notion – relativized, open-ended, and absolute. It was then shown how the two awkward wrinkles could be removed once this distinction is taken on board.

Chapter 6 drew a picture of (at least part of) a wider philosophical framework of which the entitlement proposal can be regarded as an integrated part. It was argued that entitlement of cognitive project (when understood in the relativized or open-ended sense) is internalist in nature. Furthermore, it was suggested that pluralism about epistemic value makes for a happy fit with the entitlement proposal. The distinction between monism and pluralism was linked to another distinction, that between the 'exclusion view' and the 'inclusion view' on the internalism/externalism debate. According to the former view, warrant is either one or the other, across the board. I suggested that this view is misconceived, and that the inclusion view gets things right by granting that internalists as well as externalists characterize proper notions of warrant. Pluralism was taken to be a natural companion of the inclusion view, because it enables us to hold that notions of warrant proposed by both sides of the debate match up with genuine, distinct epistemic values. It was noted, however, that the inclusion view is compatible with an internalist notion of warrant dealing more adequately with a specific philosophical problem than externalist notions. Towards the end of the chapter, a conditional, indirect argument was given for internalism about cornerstone warrant. The argument was conditional, because it only has force if we are interested in giving an account of cornerstone warrant which enables us to provide a potent response to scepticism, regarded as a challenge to our right to claim warrant. It was indirect, because what was supported was not the conclusion that we need to be internalists about cornerstone warrant to deal with scepticism, but rather that externalism will not do the job. However, by exclusion, we took the argument to recommend some version of internalism about cornerstone warrant – and here, perhaps to no great surprise, it was suggested that entitlement is a good candidate. The conditional, indirect argument drew on considerations on the relationship between cognitive responsibility, presuppositions, claims to warrant, and the defensibility of such claims.

A worry concerning internalism about entitlement – perhaps internalism more generally – was developed separately in Appendix B. The worry was essentially that internalism sets a high standard for the possession of entitlements, with philosophically informed subjects seemingly being the only ones who possess warrants of this kind. Various strategies for dealing with the problem were discussed, but very briefly.

Chapter 7 tabled two fundamental challenges to the entitlement proposal. The first worry was whether entitlement is an *epistemic* notion of warrant at all. The second, related, worry was whether the notion of rationality associated with entitlement is epistemic. Wright himself appears to gesture at an idea which, under a certain interpretation, may be thought to deliver a response to these two challenges, *viz.* that trust in P – when P is an entitlement – is a dominant strategy with respect to epistemic value. This thought was developed in some detail and it was shown that the dominance argument goes through if we assume veristic monism, i.e. that truth, or true belief, is the only thing of epistemic value. We offered the following criticism of the argument: any reasonable theory of epistemic value should list error-avoidance as a value, which veristic monism does not. However, if we add this value, the dominance argument breaks down. A slightly more sophisticated argument was formulated, the hopeful observation being that it can be shown that trust and execution will maximize expected utility provided we can plunge the (subjective) probability of the sceptical scenario. Against this, it was argued that the setting we are in – a discussion of scepticism – not only blocks the probability of the sceptical scenario from being less than that of the non-sceptical one, but mandates a probability assignment that makes the expected utility approach issue the exact same recommendations as the dominance argument. Though the spirit of the chapter was mainly critical, it is hoped that the way the discussion was framed provides a setting within which further discussion of the two challenges can be conducted.

The entitlement proposal is interesting and worthy of attention. Hopefully, what has been said in this thesis sheds some light on various of its aspects by suggesting that entitlement of cognitive project and entitlement of substance can be applied in a mathematical setting, by discussing issues pertaining to the question how to understand the notion of entitlement of cognitive project and by outlining a wider philosophical framework of which it can be regarded as a part. However, while the entitlement proposal should be looked upon with great interest, one should also do so with some reservation. A case still needs to be made that it can cash in on all its promises, among other things that it has given us a basis for providing a response to the sceptic by appeal to an *epistemic* notion of warrant that comes with an *epistemic* notion of rationality. As said in the previous chapter, there is still considerable work to be done by the

friend of entitlement.

However, if the lines along which we conducted the discussion in Chapter 7 are thought to be the right ones, we are not left entirely without a clue how to proceed. To adequately address the challenges raised in that chapter, one strategy is to try to give a plausible theory of epistemic value that will make the dominance argument work. Veristic monism supported the dominance argument, but was found implausible as a theory of epistemic value. Truth together with error-avoidance was found more plausible, but led to a breakdown in dominance. So, if the dominance thought is to be explored any further, the first question to ask should be what, other than truth and error-avoidance, is to be counted as being epistemically valuable?

With a view to helping the entitlement proposal, we need to look for a feature, or features, that hold true of the trust-execution strategy, but not of the non-trust-non-execution strategy – and, crucially, which can be argued to be associated with some kind of epistemic value. One obvious candidate is *epistemic activity*. In the trust and execution case the subject explores the relevant tract of reality, while in the non-trust and non-execution case she does, well, nothing. In the former case the subject has many projects and acquires a wealth of beliefs by executing them, while she has no projects in the latter and, accordingly, barely forms any beliefs about the world. This invites a line of investigation that will make a case that a subject's having projects by itself is valuable, even in cases where execution of these projects may fail completely in terms of bringing about true beliefs.

This strategy strikes me as having some promise, but, of course, a firmer verdict cannot be given until the matter has been subjected to thorough investigation. I cannot here undertake such an investigation, but will have to rest content to conclude having *indicated* a strategy for future exploration.

Appendix A

Standard models and categoricity

In this appendix, a bit of detail will be given concerning certain kinds of model of arithmetic and set theory, so-called 'standard models'.

Define the notion of an interpretation as follows:

Definition 3 (Interpretation) Let Γ be a collection of sentences formulated in a language $\mathcal{L}$. An interpretation of Γ is a structure $I = \langle d, i \rangle$, where d is a non-empty collection (the domain of Γ) and i is a function that interprets the $\mathcal{L}$-items occurring in Γ on d. □

Models are certain kinds of interpretations:

Definition 4 (Model) A model of a collection of sentences Γ is an interpretation of Γ under which every sentence of Γ is true. □

In particular, a model of PA is an interpretation that makes all the PA-axioms true, and similarly, a model of ZFC is an interpretation of the ZFC-axioms that make them all true.

The standard model of PA is defined as follows:

Definition 5 (Standard model of arithmetic) The standard model of arithmetic, $\mathcal{N}$, has the non-negative integers as its domain (including zero), interprets $\mathbf{0}$ as zero, $s(\mathbf{0})$ as one, $s(s(\mathbf{0})$ as two, and so forth; interprets $s(x)$ as the successor function; $+$ as addition; $\times$ as multiplication; and $=$ as identity.

Define the notion of categoricity as follows:

Definition 6 (Categoricity) A theory T is categorical if and only if any two models M and N of the theory are isomorphic, i.e. if there is a bijection between M and N which preserves structure.

The models of a categorical theory as thus completely alike with respect to structure, and accordingly, categorical theories are often said to pin down a unique structure 'up to isomorphism'.

Second-order PA is categorical, but first-order PA is not. (For a proof of the categoricity of PA^2, see Shapiro [126], pp. 82–83.)

We will now give a definition by reference to which a certain class of models of ZFC can be specified. Define V by recursion on the ordinals:

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha) \\ V_\lambda &= \cup_{\alpha<\lambda} V_\alpha \qquad \text{for } \lambda \text{ a limit ordinal} \end{aligned}$$

This definition can be given *in* ZFC (or ZF for that matter), i.e. in set theory. V is called the 'cumulative hierarchy', the idea being that sets accumulate through the V_α's as the recursion takes greater and greater ordinal inputs. The cumulative aspect of V is reflected by the fact that for any $\beta < \alpha$, $V_\beta \subset V_\alpha$. Thus, $V_0 \subset V_1 \subset V_2 \subset \ldots$[1]

The *rank* of a set x is the least ordinal α such that every member y of x is in V_α, or, equivalently, the least ordinal α such that $x \subseteq V_{\alpha+1}$. The rank of a set x thus tells us how many times the power set operation needs to be applied to get a V_α of which x is a subset. Here the expression 'how many times' cannot be understood as given by a cardinal number, because when the shift is made from the finite to the infinite, ordinals are grouped together in well-ordered classes (under $<$) of the *same cardinality*. Thus, e.g., in terms of cardinal measure, the power set operation is applied the same number of times to get all the sets at respectively V_5 and V_6, *viz.* $\aleph_0$-many times. Instead the number of times the power set operation needs to be applied should be understood in terms of which ordinal α in the recursion is such that x is a subset of V_α.

To specify the class of models mentioned we need to introduce the notion of a strongly

[1] The cumulative hierarchy had certainly appeared in print in papers by other mathematicians prior to the appearance of Zermelo's 1930 paper, [160]. Von Neumann introduced the hierarchy in his 1929 paper (von Neumann [101]), and it was implicit in Miramanoff [96]. However, while this is so, it is also the case that Zermelo's paper is the first place where the cumulative hierarchy was made the subject of extensive, systematic study.

inaccessible cardinal:

Definition 7 (Strongly inaccessible cardinal) A cardinal κ is strongly inaccessible just in case:

(i) $\kappa > \aleph_0$, i.e. κ is uncountable.

(ii) $2^\lambda < \kappa$ for any $\lambda < \kappa$.

(iii) κ cannot be represented as the supremum of fewer than κ smaller ordinals (i.e. if S is a set of ordinals less than κ and the cardinality of S is less than κ, then the ordinal of the least upper bound of S is less than κ).

The label 'inaccessible' is well-chosen in the sense that, glossed informally, a cardinal κ is strongly inaccessible just in case no application or combination of applications of the set-theoretic axioms suffices to establish its existence – that is, if it is inaccessible by applications of the set-theoretic axioms. (Recall that, in set theory, ordinals are treated as sets, and that cardinals are certain ordinals.) In particular, (ii) and (iii) ensure closure under the operations of power set and union.

The following result can be established:

Theorem 1 Any structure $\langle V_\kappa, \in \rangle$ – κ a strongly inaccessible cardinal – is a model of first-order ZFC.

PROOF (SKETCH) :
Consider some such structure and relativize the quantifiers of the ZFC-axioms to the structure. It can then be verified that each of the axioms is true on the structure. ■

Given this theorem, we can define the notion of a standard model:

Definition 8 (Standard model) A standard model of ZFC is any structure isomorphic to some $\langle V_\kappa, \in \rangle$, κ a strongly inaccessible cardinal. □

Two things are worth noting. First, any structure $\langle V_\kappa, \in \rangle$ is isomorphic to itself by the identity function, and so, any such structure with κ a strongly inaccessible is a standard model in the sense just defined. Second, the existence of a strongly inaccessible cardinal cannot be established in ZFC, on pain of contradicting Gödel's second incompleteness theorem.

First-order Z, ZF, and ZFC are not categorical. Indeed, no first-order theory with an infinite model is. This follows from the upwards Löwenheim-Skolem theorem, according to which any

first-order theory with an infinite model has a model in every infinite cardinality. In particular, if first-order Z, ZF, and ZFC have an infinite model, they have one in every infinite cardinality. Since there can be no bijection between structures with domains of different cardinality, these theories cannot be categorical.

While the upwards Löwenheim-Skolem theorem yields the result that first-order set theories have many big models, the downwards Löwenheim-Skolem theorem delivers the result that it has at least one small model as well. According to the downwards LS-theorem, if a first-order theory has an infinite model, then it has a model whose domain is at most denumerable. Applied to (first-order) Z, ZF, and ZFC, this means that these theories have models whose domain is countable.

What about second-order ZFC (ZFC^2)? Theorem 1 can still be established if, for 'first-order ZFC', we read 'second-order ZFC'.[2] This suffices to show that ZFC^2 is not categorical. Let κ and λ be respectively the first and second strongly inaccessible cardinal. By the second-order version of Theorem 1, each of $\langle V_\kappa, \in\rangle$ and $\langle V_\lambda, \in\rangle$ is a model of ZFC^2. These cannot be isomorphic to each other since V_κ and V_λ are of different cardinality, and there can no isomorphism between structures with non-equinumerous domains.

Although ZFC^2 is not categorical, it does do better than first-order ZFC. First-order ZF and ZFC have models of the form $\langle V_\theta, \in\rangle$, where θ is not a strong inaccessible. ZFC^2 does not have any models of this kind. This follows from the fact that the second-order version of Theorem 1 can, in fact, be strengthened. κ being a strong inaccessible is not just a sufficient condition for a structure (isomorphic to some) $\langle V_\kappa, \in\rangle$ to be a model of ZFC^2 – it is also necessary. In other words, the models of ZFC^2 are *exactly* those isomorphic to some $\langle V_\kappa, \in\rangle$, κ a strongly inaccessible cardinal. So, in the jargon, ZFC^2 has only standard models.

A model $M = \langle V_\kappa, \in\rangle$ is isomorphic to an initial segment of a model $N = \langle V_\lambda, \in\rangle$ just in case $\kappa \leq \lambda$ and there is an isomorphism f between M and some $V_\mu \subseteq V_\lambda$ with $\in$ in N restricted to V_μ. Note that any model is an initial segment of itself and is trivially isomorphic to itself by the identity function. We have the following result for ZFC^2:

[2] This result is essentially due to Zermelo [160]. However, as will be indicated below, we actually get a stronger result from Zermelo's paper.

Theorem 2 (Quasi-categoricity of ZFC^2) Let $M = \langle V_\kappa, \in \rangle$ and $N = \langle V_\lambda, \in \rangle$ be models of ZFC^2. Then one is isomorphic to an initial segment of the other.

PROOF :
By the construction of V_κ and V_λ. For details, cf. Zermelo [160].[3]

So, though ZFC^2 does not pin down a unique structure up to isomorphism, the structures the theory pins down are very structurally similar. That is, any two (distinct) models of ZFC^2 are structurally alike with respect to everything except the ordinal at which we stop in the recursive definition of V: if $\langle V_\kappa, \in \rangle$ and $\langle V_\lambda, \in \rangle$ are models of ZFC^2 and $\kappa < \lambda$, there is an initial segment of V_λ to which V_κ is isomorphic, and V_λ is what is obtained by carrying on the recursion to the ordinal λ instead of stopping at κ.

[3] It is worth mentioning that Zermelo works in a framework that allows urelements and actually proves a more general result than what has here been stated as Theorem 2. He shows that, for any two models of ZFC^2 with the same base, one is isomorphic to an initial segment of the other. Theorem 2 is the special case where there are no urelements and V_0 is simply the empty set.

Appendix B

The exclusive club problem

Wright intends his notion of entitlement to be understood along internalist lines. (See Chapter 6.) For comparison, we will cite a passage from Burge who has developed a notion of entitlement which is meant to be externalist in character. We will then go on to discuss a passage from Wright that seems to agree in certain important respects with what Burge says. We will use this observation as a platform to formulate a worry that I will call 'the exclusive club problem for internalism'. The worry is essentially that only philosophically informed subjects will have much in terms of entitlement if the notion is understood along internalist lines, as Wright intends. Or rather, a dilemma will emerge: either entitlement is understood along internalist lines – as Wright wants – but at the cost of making entitlement a philosopher's privilege, or entitlement is not a philosopher's privilege, but has to be understood in accordance with externalism, contra Wright's own intentions. The problem seems to be fair to raise against internalist accounts of warrant more generally, although I shall focus on entitlement in this appendix.[1] Three possible lines of response will be indicated, but not developed in any great detail.

Let us now turn to the passage from Burge mentioned above. Burge writes:

> The distinction between justification and entitlement concerns two types of epistemic warrant. Both justifications and entitlements are epistemic warrants with rational justificational force. But

[1] Granted, at this stage, the dilemma might strike people as an *ad hominem* against Wright. The second horn of the dilemma is not clearly undesirable. It is only undesirable to someone who, like Wright, has an interest in keeping entitlement internalist. However, as argued in Section 6.5, if entitlement is to deliver a response to scepticism, it should be construed along internalist lines. The dilemma thus seems to require attention from those who worry about scepticism and regard the entitlement proposal as a promising strategy for responding to the sceptic. In addition, as just indicated, the problem appears to have some bite against internalist accounts of warrant in general, and so, to the extent this is agreed to, is not an *ad hominem* against Wright.

> entitlements need not be understood by or even accessible to the individual subject, whereas justifications, in my narrow sense, involve reasons that individuals have and have access to. Entitlement is my partially externalist analog of the internalist notion, justification. I will leave open exactly what may count as an entitlement, although I do not believe that *any* entitlements are *mere* matters of reliability. What is important is that with an entitlement, a *full* reason, or a full expression of the warrant associated with a state or capacity need not be available to the individual.
>
> We are entitled to rely on our capacities for perception, memory, interlocution, deductive and inductive reasoning. Children are entitled to rely on particular perceptual beliefs even though they could not understand why. Perhaps only philosophers can explain why. Justifications, in my narrow sense, are available on reflection to the justified individuals. Justifications may be self-sufficient premises, or reasons that a person could 'in principle' call up. (Burge [19], p. 3)

In the quote, Burge characterizes justification as warrant based on reason, or reasons, accessible to the subject who possesses the warrant, and explicitly refers to it as 'internalist'. Entitlement, on the other hand, is explicitly characterized as the 'partially externalist analog of the internalist notion, justification', with no requirement to the effect that a 'full reason' or 'full expression of the warrant' be available to the individual. In another source, Burge says, 'entitlements are ... warrants that need not be understood by or even accessible to the subject' (Burge [16], p. 458).

Entitlement, as Burge conceives the notion, is weakly externalist in the sense of the thesis that was labeled '(W-EXTERNALISM)'. Some of the facts upon which entitlements supervene are such that they need not be accessible to the subject through reflection. It should be noted that Burge does not explicitly restrict the kind of access associated with internalism to be reflective. However, we can reasonably take him to endorse this restriction implicitly when he characterizes justification as being 'available on *reflection* to the justified individuals' (emphasis added).

Both Burge and Wright use 'warrant' disjunctively to mean either justification or entitlement. However, one fundamental difference between them concerns how they understand the notion of entitlement, and related thereto, how the distinction between justification and entitlement is drawn. We have just seen that Burge draws the distinction by reference to accessibility, i.e., roughly, as a distinction between an internalist and an (at least weakly) externalist notion of warrant. On the other hand, Wright draws the distinction in terms of evidence: justification is evidential warrant, while entitlement is non-evidential warrant. Crucially, it should be noted that, if Wright's way of drawing the distinction is adopted, we can still ask for each of justifica-

tion and entitlement whether they are to be understood along internalist or externalist lines – and, indeed, this is the question that occupied us in Chapter 6.[2]

Though Burge and Wright draw the distinction between justification and entitlement differently, the following (long-ish) passage from Wright heavily suggests that the Burgean and Wrightian notions of entitlement do share certain features:

> The question arises whether, in order to enjoy an entitlement to a particular proposition, one has oneself to accomplish the demonstration that there is such warrant – to recognise oneself that the case is one where trust is rational. In normal circumstances, and putting to one side special issues concerning testimony, a thinker's knowing or justifiably believing something requires that she herself have evidence for it sufficient to constitute knowledge or justified belief. By contrast we do not, at least in a wide class of cases, demand that before a thinker can justifiably infer in accordance with a principle of inference, she must herself have accomplished a justification for the use of that rule. At least in cases where a valid pattern of inference demands no special training but is followed by the 'light of natural reason', we will naturally credit a thinker with warrant to proceed as she does, even if she has given no thought to that way of proceeding and would not have the slightest idea how to answer if a request for explicit justification was made. If entitlement stands comparison in this matter with justified belief rather than warranted inference, then – in the present state of our understanding of these issues – no-one yet has ever had much in the way of entitlements. Roll on the day when we get these things straightened out, and can at last get some entitlement to our cornerstones, fend-off scepticism and start accumulating some knowledge! Clearly this is yet another issue for further attention, but entitlement had better prove to be 'for nothing' in this additional sense too – had better be comparable to rights of basic inference, as it were – if any but a few philosophers are to benefit from a vindication of the notion. The matter is deep and the comparison with basic logic suggestive. (Wright [154], pp. 204–205)[3]

According to Wright, a subject's being warranted in inferring in accordance with a given rule of inference does not require an ability to give a justification for the rule or that she even has an idea of how to provide one. One motivation for holding the view that entitlements better be comparable to rights of basic inference in this respect is the worry that, otherwise, being entitled would be a privilege restricted to philosophically informed individuals. It is clear from the quote that this is not how Wright wants entitlement of cognitive project to be construed.

[2] Wright himself observes that he and Burge draw the contrast between justification and entitlement differently. See Wright [153], p. 163, n. 5.

[3] A comment on the relationship between entitlement and knowledge suggested by the quote. Wright says, 'Roll on the day when we get these things straightened out, and can at last get some entitlement to our cornerstones, fend-off scepticism and start accumulating some knowledge!' This suggests that entitlement concerns *possession* of knowledge (and warrant), contra to what Wright himself is keen to stress – *viz.* that entitlement can help sustain rational *claims* to warrant. As should be clear from my treatment of the scepticism and the entitlement proposal, I take the official proposal to be that entitlement is of relevance to claims to, rather than possession of, warrant and knowledge.

Philosopher and non-philosopher alike are entitled to cornerstone propositions. In this respect Wright's notion of entitlement is meant to agree with that of Burge. Burge explicitly states that 'we' are entitled to rely on a range of capacities – perception, memory, interlocution, deductive and inductive reasoning, among others – where 'we' is meant to include philosopher and non-philosopher alike. This I take to be supported by his remark, 'Children are entitled to rely on particular perceptual beliefs even though they could not understand why'.

While non-philosopher and philosopher alike possess entitlements, it might – as Burge writes – very well be that only the philosopher can explain or articulate, why we do so or in what they consist. This suggests that possessing an entitlement should be distinguished from being able to explain, or articulate, why one does so. Being entitled does not require an understanding of the notion of entitlement, whereas explaining or articulating our entitlements does. Given Burge's (partially) externalist take on entitlement it is easy to hear this distinction, and see how a subject can be (Burge-)entitled without understanding the notion of entitlement. Since the notion is partially externalist, some of the facts upon which a subject's (Burge-)entitlement supervenes need not be reflectively accessible, and so, need not be such that the subject has the concepts required to reason about these facts. This also makes it clear why it can be that entitlement is not just an exclusive privilege for the philosophically sophisticated.[4]

As seen, Wright intends his notion of entitlement to have the same non-exclusive range of applicability. He also intends it to be understood in an internalist way, as seen in Section 6.2. It is not, however, clear that this makes for a happy fit. I will now try to say why one might think not.

To understand entitlement of cognitive project along internalist lines is to say that the facts upon which entitlement supervene are reflectively accessible to the subject, which, in turn, is to say that it is reflectively accessible to the subject for each of clauses (i)-(iii) whether the clause is met. To reflectively access whether some proposition P obtains one needs to have the concepts figuring in P. Thus, in particular, in order to reflectively access whether, for each of clauses (i)-(iii), the clause is met, the subject needs to have the concepts involved in the clauses. In the case of entitlement these include the concept of a proposition P's being a presupposition of

[4] See also Burge [22], a reply to Davies [31].

some cognitive project, the concept of a reason (and, in particular, of a sufficient countervailing reason), and the concepts of an infinite regress and of a presupposition's being of no more secure a prior standing.

To understand entitlement this way puts the standards quite high for subjects to be entitled. For who has a clear and ready grasp of these concepts? Probably not your average Joe. More likely we are talking about individuals with some kind of philosophical training – a couple of courses in contemporary epistemology, say. In other words, the answer to the question would seem to be, 'Not that many'. We can summarize the gist of these considerations in the form of a problem for internalism about entitlement:

> **The exclusive club problem for internalism about entitlement:**
>
> Suppose that we want to be internalists about entitlement. Then we face the following dilemma: either
>
> - we hold on to our internalist understanding of entitlement, but at the cost of restricting the possession of entitlements to individuals with a grasp of the concepts figuring in clauses (i)-(iii), or
> - we do not restrict the possession of entitlement to individuals with a grasp of the concepts figuring in (i)-(iii), but at the cost of giving up our internalist understanding of entitlement.

That is, the internalist about entitlement has to choose between making possession of entitlement a privilege restricted to the members of the exclusive Entitlement Club (which one is eligible for provided one has some degree of philosophical sophistication) or give up her internalist understanding of the notion.

It is worth commenting, briefly, on the scope of this kind of consideration. It seems to me that internalism about any kind of warrant involves some degree of sophistication in one way or another. Some internalists write in a way that makes this clear. For instance, Ginet writes, '...assuming that S has the relevant concepts, S can always tell whether or not he has justification for being confident that p' (Ginet [56], p. 36). Here the present worry can

be brought out simply by noting that S needs to have the relevant concepts in order for the assumption to be met and for it to make sense to talk about S being justified at all. So, while the exclusive club problem is presented specifically in terms of entitlement of cognitive project in this appendix, it appears to be a worry that can legitimately be raised for internalist notions of warrant more generally. In particular, this goes for entitlement of substance.[5]

Assuming that the internalist – and here I count Wright as one – wants to hold on to her understanding of entitlement, she seems to be stuck with the exclusive club problem. Looking at the problem in a wider perspective may be thought to make it particularly worrisome. Recall that the notion of entitlement of cognitive project was introduced as a response to scepticism concerning the empirical world. In Chapter 2, we formulated regress scepticism as cashed out in a mathematical setting. Chapter 3 suggested that entitlement might make for a satisfactory response. What the entitlement proposal was supposed to salvage was the rational claim to warrant for the wide range of ordinary propositions involved in the regions of thought attacked by the sceptic. However, if possession of entitlement is really an exclusive privilege, the rational claim to warrant for these wide ranges of ordinary propositions should be equally exclusive – leaving any subject without the requisite philosophical sophistication in the jaws of scepticism.

At this point, the friend of internalist entitlement will have to say something, one way or the other. Below I will indicate three ways one might try to address the problem. The first will be dismissed. The two others will be deemed more promising, but the task of discussing them fully will be left as a task to be taken on at some other occasion.

A. Being entitled vs. entitle-ing

Some epistemologists draw a distinction between being justified and justifying. Alston is particularly explicit in doing so:

[5] For entitlement of substance, what the entitled subject needs to grasp are the notions of a conception of substance for a region of thought R being essential to sustaining the conception of our interaction with R as objective and a (sufficient) reason for believing a conception of substance incoherent. Note also that, for both entitlement of cognitive project and substance, the kind of issue we are presently pursuing can be raised legitimately whether we are dealing with the relativized or open-ended understanding of either species of entitlement. What matters is that we are dealing with a way of understanding the target notion which goes with internalism. As seen in Chapter 6, the relativized and the open-ended way of understanding entitlement of cognitive project and substance both seem to make a fairly happy fit with internalism.

> We must clear out the way of confusion between one's *being* justified in believing that p, and one's *justifying* one's belief that p, where the latter involves one's *doing* something to show that p, or to show that one's belief was justified or to exhibit one's justification. (Alston [2], p. 82)

He adds that a subject can be justified in believing that p without being able to justify a belief in p. Audi expresses the same view, maintaining that 'one can have a justified belief even if, in response to someone who doubts this, one could not show that one does' (Audi [4], p. 145). Burge's distinction between a subject's being entitled and articulating or explaining her entitlement seems to be similar to the justified/justifying distinction. (This distinction was mentioned in passing earlier in this appendix.)

Being justified is a state the subject is in, while justifying is an activity. If, as before, we take justification to be evidential warrant, we can understand a subject's being justified as the subject's holding the belief that P on the basis of adequate evidence, while we can take a subject's justifying the belief that P to involve – or perhaps consist in – the subject's uncovering or presenting adequate evidence for P. One might impose the further condition that the subject believes that the evidence has been acquired in an appropriate way.

For example, suppose that S believes that there is a tree in front of her on the basis of a certain visual experience, and that she is justified in doing so, because the process through which it was acquired was reliable and the experience provides adequate evidence for the belief. Suppose also that S justifies her belief. Her so doing consists in her indicating that her evidence – the visual experience – was acquired in an appropriate manner, i.e. reliably, and that it is adequate. Now, to take on board the point made by Alston and Audi, S's being justified in believing that there is a tree in front of her does not require her being capable of justifying this belief. For S to justify her tree belief is more demanding than being justified in holding it. In order to be justified S just needs to have a visual experience of the right kind. On the other hand, justifying the belief requires S to, in some sense, make a case for its good epistemic standing.

This all seems plausible. However, wherein lies the potential relevance of the justified/justifying distinction to the exclusive club problem for internalism about entitlement of cognitive project?

Well, here is the thought: justifying a belief is more demanding than being justified in having the belief. In particular, it requires that the subject doing the justification has the concepts needed to make a case that the relevant belief is in good epistemic standing. The magic words were, of course, 'has the concepts'. The reason why the exclusive club problem emerged was exactly because being entitled demanded too much in terms of what concepts the entitled subject should possess. However, perhaps a case can be made that it is not really being entitled that is demanding, but rather entitle-ing, in analogy with the justified/justifying distinction. (Here 'entitle-ing' is adopted as a label for the activity that stands to being entitled as justifying stands to being justified.) If such a case can be made, the possession of entitlements might not be a privilege restricted to the philosophically sophisticated after all.

My immediate comment to this suggestion is this: do not get your hopes up too high. The entitled/entitle-ing distinction does not help. Being entitled (as a matter of cognitive project) supervenes on whether clauses (i)-(iii) are met, and given the internalist approach adopted, there is simply no avoiding that the requirement of reflective accessibility demands that the entitled subject grasp the concepts figuring in these clauses. Furthermore, it is not even clear that the other half of the distinction – entitle-ing – can be of any help. To see this ask what entitle-ing amounts to? If the relationship between being entitled and entitle-ing is to be analogous to that between being justified and justifying, entitle-ing is an activity engaged in by the subject. Justifying that P can be taken to consist in the subject's presenting the evidence which justifies P. As for entitle-ing that P, the perhaps most natural suggestion is to take it to amount to the subject's making a case that clauses (i)-(iii) are met for P, i.e. the subject's spelling out that P is an entitlement (of cognitive project). However, with respect to the exclusive club worry, focus on entitle-ing rather than being entitled leaves us no better off. For making a case that P is an entitlement – i.e. entitle-ing P – just as much involves having the concepts figuring in clauses (i)-(iii) as being entitled to P does (where this is understood as whether clauses (i)-(iii) are met being reflectively accessible to the subject).

I therefore conclude that the distinction between being entitled and entitle-ing – thought in analogy to that between being justified and justifying – cannot address the exclusive club problem.

B. Welfare state epistemology.

Let me indicate – but not explore in any great detail – how adoption of a thesis of division of epistemic labour might also be of relevance to the exclusive club problem. Here is the basic idea: for the various regions of thought, there are experts and non-experts, and for any of these regions, the average epistemic subject has warrants for many propositions belonging to it, but, for some of these, only courtesy of work done by the experts of that region.[6] Adopting this basic idea, one could say this: take the cornerstones put under attack by the sceptic to be among the propositions that subjects have courtesy warrant for due to work done by the experts. Philosophers – or individuals with some degree of philosophical sophistication – are, in the context of scepticism, the experts. Let us suppose that these individuals have entitlements to cornestones, and that entitlement is understood along internalist lines.

Two things should be noted. First, 'welfare state epistemology' might be a happy label for the thought that underlies the proposal. In a welfare state, individuals that are worse off can benefit from individuals that are better off. Similarly, in an epistemic community, non-experts can benefit from the epistemic work done by the experts.[7] Second, the welfare state proposal offers a concessive response to the exclusive club problem. It is granted that it is only individuals who possess the relevant range of concepts that have are entitled to cornerstone propositions. However, it is denied that this means that only philosophers have a *warrant* for these cornerstones. This immediately raises the question whether the warrant held by the non-philosopher is an entitlement or some other kind of warrant. Whichever of the two, the warrant has to be externalist in nature. It supervenes on environmental facts – the presence of certain experts with entitlements. (Still, the entitlements of the experts – the 'initial warrants' – are to be understood in an internalist manner.) Thus, the present proposal is not just concessive by granting that, strictly speaking, entitlements are restricted to the members of a fairly exclusive club, but also by granting that the social, or courtesy, warrant for cornerstones held by the

[6] There are similarities here with Putnam's thesis of division of linguistic labour. See Putnam [115].

[7] The analogy is not supposed to be perfect. For instance, in a welfare state, worse off individuals benefit from the better standing of other individuals through a law-governed redistribution of social goods. Though worse off individuals benefit from the better standing of other individuals in an epistemic community, this is not to be accounted for in terms of a *redistribution* of epistemic goods, let alone a law-governed one.

non-expert subjects of the epistemic community is externalist in nature.

Whether or not welfare state epistemology is tenable will depend on what can be said about how warrant is supposed to transfer from expert to non-expert. The following proposal will not do: the mere presence of experts with a warrant for some proposition P gives non-experts a warrant for believing P. This would leave it rather mysterious what the warrant of the non-expert consists in. Furthermore, it would seem that there are clear counterexamples to this proposal. For instance, set theorists are warranted in believing that use of the method of forcing can be used to show ZF consistent with the negation of CH. However, it seems utterly implausible to say that the mere presence of expert subjects with a warrant for so believing yields a warrant for any other member of the epistemic community to hold the same belief. Another, more promising proposal appeals to testimony. An expert's telling a non-expert that P might warrant the non-expert in believing P, perhaps on the assumption that the expert has a warrant for believing P herself.[8]

One reason to take welfare state epistemology seriously – but which is independent of its potential relevance to the exclusive club problem – is that it appears to fit well with our epistemic practices. At least sometimes, if pressed on a particular question, we cite or otherwise appeal to experts. This might be taken to suggest that we do think that, sometimes, our warrant for holding some belief depends on, or derives from, a warrant held by an expert. However, it should be stressed that this aspect of our epistemic practices is something that has to be accounted for whether one has internalist or externalist leanings, and, as such, should be kept distinct from the internalism/externalism question.

C. Invoking a modality.

According to the weakly internalist approach, the facts upon which a subject's entitlement supervenes are reflectively accessible to the subject. This can fail in at least two ways. One way is by the subject not having the concepts needed to reflectively access the justificatorily relevant facts. The subject might lack some competence or not master some method or procedure

[8] The question how warrant is transferred through testimony has generated considerable discussion in the literature. Hume [75] and Reid [121] include classic discussions of testimony. Recent contributions include Burge [16] and Faulkner [43].

through which such access is gained. Another way the justificatorily relevant facts might not be reflectively accessible to the subject is by these facts simply not being a reflectively accessible kind of fact. That is, a kind of fact not reflectively accessible to any subject, whatever her capacities and methods and procedures. An example of this kind of fact are facts about the character of the surrounding environment. It might be that lighting conditions are appropriate for making colour verdicts and that a subject has access to this fact. However, it is not plausible to suppose that this access is gained through reflection alone. In particular, it will involve the subject's exercising her perceptual capacities.

Here we are interested in the first kind of failure. It is this kind of failure that raises the question whether it is tenable to maintain that the warrant determined by the justificatorily relevant facts is internalist. (Also, for the second kind of failure, there are simply certain kinds of facts that could not plausibly be taken of a sort that are accessible purely through reflection.) As seen above, the exclusive club problem puts pressure on the idea that it is. We have just seen that drawing a distinction between being entitled and entitle-ing does not resolve the problem. However, perhaps relaxing the reflective accessibility requirement will. We can do so by giving it a modal reading. Instead of taking the reflective accessibility requirement to demand that the subject have, or actually grasp, the relevant range of concepts, it can be taken to demand that she be in a position to acquire these concepts and invoke a modality to cash out 'being in a position to acquire'. We can understand the reflective accessibility requirement as demanding that the subject has an implicit grasp of the relevant range of concepts or could acquire them were she to exercise the capacities and methods and capacities available to her to their fullest.

This strategy for dealing with the exclusive club problem strikes me as fairly promising, but I shall not attempt to offer a detailed development of it. That task will be left for some other occasion.

D. Summary.

The exclusive club problem for internalism about entitlement was formulated. The problem was basically that internalism seems to make the possession of entitlements a privilege restricted to philosophically informed individuals. We dismissed employment of the distinction between

being entitled and entitle-ing as a response to the problem. The welfare state idea was suggested to be of potential relevance to the exclusive club problem, but it was also suggested that, if anything, it delivers a concessive response to the problem: though non-experts have a warrant for cornerstones, only the 'experts' – i.e. the philosophically informed – hold *entitlements*, and non-expert warrants for cornerstones are externalist in nature. Lastly, it was indicated that appeal to a modality appears to be a promising route for the internalist to take. However, the task of exploring this matter further was left for another occasion.

Bibliography

[1] W. Alston: 'The Deontological Conception of Epistemic Justification', pp. 257–299 in J. Tomberlin (ed.): *Philosophical Perspectives* Vol. 2, 1988.

[2] W. Alston: *Epistemic Justification* (Ithaca: Cornell University Press), 1989.

[3] W. Alston: 'Internalism and Externalism in Epistemology', pp. 68–110 in Kornblith [83].

[4] R. Audi: *The Structure of Justification* (Cambridge: Cambridge University Press), 1993.

[5] R. Audi: 'Doxastic Voluntarism and the Ethics of Belief', pp. 93–111 in Steup [135].

[6] K. Bach: 'Default Reasoning: Jumping to Conclusions and Knowing When to Think Twice', pp. 37–58 in *Pacific Philosophical Quarterly* Vol. 65(1), 1984.

[7] P. Benacerraf: 'Mathematical Truth', pp. 661–679 in *Journal of Philosophy* Vol. 70, 1973.

[8] P. Benacerraf and H. Putnam (eds.): *Philosophy of Mathematics*, 2^{nd} edition (Cambridge: Cambridge University Press).

[9] P. Boghossian: 'How Are Objective Epistemic Reasons Possible?', pp. 1–40 in *Philosophical Studies* vol. 106, 2001.

[10] P. Boghossian and C. Peacocke (eds.): *New Essays on the A Priori* (Oxford: Oxford University Press), 2000.

[11] L. BonJour: 'The coherence theory of empirical knowledge', pp. 281–312 in *Philosophical Studies* Vol. 30, 1976.

[12] L. BonJour: *The Structure of Empirical Knowledge* (Cambridge, MA: Harvard University Press), 1985.

[13] G. Boolos: *Logic, Logic, and Logic* (Cambridge, Mass.: Harvard University Press), 1998.

[14] G. Boolos: 'To be is to be a value of a variable (or to be some values of some variables', pp. 54-72 in Boolos [13].

[15] G. Boolos: 'Nominalist platonism', pp. 73-87 in Boolos [13].

[16] T. Burge: 'Content Preservation', pp. 457–488 in *Philosophical Review*, Vol. 102(4), 1993.

[17] T. Burge: 'Our Entitlement to Self-Knowledge', pp. 91–116 in *Proceedings of the Aristotelian Society*, Vol. 96, 1996.

[18] T. Burge: 'Interlocution, Perception, and Memory', pp. 21–47 in *Philosophical Studies* Vol. 86, 1997.

[19] T. Burge: 'Computer Proof, A Priori Knowledge, and Other Minds', pp. 1–37 in *Philosophical Perspectives*, Vol. 12, 1998.

[20] T. Burge: 'Frege on Knowing the Foundations', pp. 305–348 in *Mind* Vol. 106, 1998.

[21] T. Burge: 'Perceptual Entitlement', pp. 503–548 in *Phillsophy and Phenomenological Research*, Vol. 67, 2003.

[22] T. Burge: 'Reply to Martin Davies', pp. 250–257 in Frápolli and Romero [51].

[23] H.-N. Castaneda: 'Knowledge and Epistemic Obligation', pp. 211–233 in *Philosophical Perspectives* Vol. 2, 1988.

[24] R. Chisholm: *Theory of Knowledge*, 2nd edition (Englewood Cliffs, N.J.: Prentice Hall), 1977.

[25] L. Code: *Epistemic Responsibility* (Hanover: Brown University Press), 1987.

[26] E. Conee and R. Feldman: 'Internalism Defended', pp. 231–260 in Kornblith [83].

[27] H. Dales and G. Oliveri: *Truth in Mathematics* (Oxford: Oxford University Press), 1998.

[28] M. David: 'Truth as the epistemic goal', pp. 151-169 in Steup [135].

[29] M. David: 'Truth as the Primary Epistemic Goal: A Working Hypothesis', pp. 296-312 in Steup and Sosa [137].

[30] M. Davies: 'Externalism and Armchair Knowledge', pp. 384–414 in Boghossian and Peacocke [10].

[31] M. Davies: 'Externalism, Self-Knowledge and Transmission of Warrant', pp. 105–130 in Frápolli and Romero [51].

[32] M. Davies: 'Epistemic Entitlement, Warrant Transmission, and Easy Knowledge', pp. 213–244 in *Proceedings of the Aristotelian Society Supplementary Vol. LXXVIII*, 2004.

[33] M. DePaul: 'Value Monism in Epistemology', pp. 170-183 in Steup [135].

[34] R. Descartes: *Meditations on first philosophy.*

[35] M. Dummett: *Frege: Philosophy of Language* (London: Duckworth), 1973.

[36] M. Dummett: *Elements of Intuitionism* (Oxford: Oxford University Press), 1977.

[37] M. Dummett: *Truth and Other Enigmas* (Cambridge, MA: Harvard University Press), 1978.

[38] M. Dummett: *The Seas of Language* (Oxford: Oxford University Press), 1993.

[39] H. Enderton: *A Mathematical Introduction to Logic*, 2^{nd} ed. (San Diego: Academic Press), 1972.

[40] H. Enderton: *Elements of Set Theory* (NY: Academic Press), 1977.

[41] W. Ewald (ed.): *From Kant to Hilbert: a source book in the foundations of mathematics* vol. 2 (Oxford: Oxford University Press), 1996.

[42] A. Fairweather and L. Zagzebski (eds.): *Virtue Epistemology: Essays on Epistemic Virtue and Responsibility* (Oxford: Oxford University Press), 2001.

[43] P. Faulkner: 'On the Rationality of our Responses to Testimony', pp. 353–370 in *Philosophical Studies* Vol. 131, 2002.

[44] R. Feldman: 'Epistemic Obligations', pp. 235–256 in J. Tomberlin (ed.): *Philosophical Perspectives* Vol. 2, 1988.

[45] R. Feldman: 'Voluntary Belief and Epistemic Evaluation', pp. 77–92 in Steup [135].

[46] H. Field: *Science without numbers* (Princeton: Princeton University Press), 1980.

[47] H. Field: 'Apriority as an Evaluative Notion', pp. 117–149 in Boghossian and Peacocke [10].

[48] R. Foley: *The Theory of Epistemic Rationality* (Cambridge, Mass.: Harvard University Press), 1987.

[49] T. Forster: *Set Theory with a Universal Set*, 2^{nd} edition (Oxford: Oxford University Press), 1995.

[50] A. Fraenkel et al.: *Foundations of Set Theory* (Amsterdam: North-Holland), 2^{nd} revised edition, 1984.

[51] M. J. Frápolli and E. Romero (eds.): *Meaning, Basic Self-Knowledge, and Mind: Essays on Tyler Burge* (Stanford: CSLI Publications), 2003.

[52] G. Frege: *Die Grundgesetze der Arithmetik* Vol. I (Jena: H. Pohle), 1893.

[53] G. Frege: *Die Grundgesetze der Arithmetik* Vol. II (Jena: H. Pohle), 1903.

[54] G. Frege: 'Der Gedanke', pp. 58–77 in *Beitr/"age zur Philosophie des deutschen Idealismus* I, 1918–1919.

[55] G. Frege: 'Letter to Russell', pp. 127–128 in van Heijenoort [72].

[56] C. Ginet: *Knowledge, Perception, and Memory* (Dordrecht: D. Reidel Pub. Co.), 1975.

[57] C. Ginet: 'Deciding to Believe', pp. 63–76 in Steup [135].

[58] C. Ginet: 'Infinitism Is Not the Solution to the Regress Problem', pp. 140–149 in Steup and Sosa [137].

[59] A. Goldman: 'What Is Justified Belief?', pp. 1–23 in G. Pappas (ed.): *Justification and Knowledge: New Studies in Epistemology* (Dodrecht: D. Reidel), 1979.

[60] A. Goldman: *Epistemology and Cognition* (Cambridge, MA: Harvard University Press), 1985.

[61] A. Goldman: 'Internalism exposed', pp. 115–133 in Steup [135].

[62] A. Goldman: 'The Unity of the Epistemic Virtues', pp. 30–49 in Fairweather and Zagzebski [42].

[63] K. Gödel: 'Russell's mathematical logic', pp. 447–469 in Benacerraf and Putnam [8].

[64] K. Gödel: 'What is Cantor's continuum problem?', pp. 470-485 in Benacerraf and Putnam [8].

[65] J. Greco: 'Justification Is not Internal', pp. 257–270 in Steup and Sosa [137].

[66] A. Hájek: 'Interpretations of Probability', in E. Zalta (ed.): *Stanford Encyclopedia of Philosophy* (Summer 2003 Edition), URL = http://plato.stanford.edu/archives/sum2003/entries/probability-interpret/.

[67] M. Hallett: *Cantorian set theory and limitation of size* (Oxford: Oxford University Press), 1984.

[68] G. Harman: *Thought* (Princeton: Princeton University Press), 1973.

[69] G. Harman: 'Reasoning, and Explanatory Coherence', pp. 151–182 in *American Philosophical Quarterly* Vol. 17, 1980.

[70] G. Harman: 'Positive vs. negative undermining in belief revision', pp. 39–49 in *Noûs* Vol. 18, 1984.

[71] G. Harman: *Change in View* (Cambridge, MA: MIT Press), 1986.

[72] J. v. Heijenoort: *From Frege to Gödel: A Source Book in Mathematical Logic 1879–1931* (Cambridge, Mass.: Hardvard University Press), 1967.

[73] J. Heil: 'Doxastic Agency', pp. 355-364 in *Philosophical Studies* Vol. 43, 1983.

[74] R. Holmes: *Elementary Set Theory with a Universal Set, Cahiers du Centre de logique*, Vol. 10 (Louvain-la-Neuve: Academia), 1998.

[75] D. Hume: *Enquiries Concerning Human Understanding and Concerning the Principles of Morals*, 3^{rd} edition, revised by P. Nidditch (Oxford: Clarendon Press), 1777.

[76] I. Jané: 'A Critical Appraisal of Second-Order Logic', pp. 67–86 in *History and Philosophy of Logic*, Vol. 14, 1993.

[77] I. Jané: 'Higher-order Logic Reconsidered', pp. 781–810 in S. Shapiro (ed.): *The Oxford Handbook for Logic and Philosophy of Mathematics.*

[78] P. Klein: *Certainty: A Refutation of Scepticism* (Minnesota: University of Minnesota Press), 1981.

[79] P. Klein: 'Warrant, Proper Function, Reliabilism, and Defeasibility', pp. 97–130 in Kvanvig [85].

[80] P. Klein: 'Infinitism Is the Solution to the Regress Problem', pp. 131–140 in Steup and Sosa [137].

[81] H. Kornblith: 'Justified Belief and Epistemically Responsible Action', pp. 33–48 in *Philosophical Review* Vol. 92, 1983.

[82] H. Kornblith: 'Epistemic Obligation and the Possibility of Internalism', pp. 231–248 in Fairweather and Zagzebski [42].

[83] H. Kornblith (ed.): *Epistemology: Internalism and Externalism* (Oxford: Blackwell), 2001.

[84] K. Kunnen: *Set Theory: An Introduction to Independence Proofs* (Amsterdam: North-Holland), 1980.

[85] J. Kvanvig (ed.): *Warrant in Contemporary Epistemology* (New York: Rowman and Littlefield), 1996.

[86] I. Lakatos: *Proofs and Refutations: The Logic of Mathematical Discovery* (Cambridge: Cambridge University Press), 1976.

[87] K. Lehrer: *Knowledge* (Oxford: Clarendon Press), 1974.

[88] K. Lehrer: 'The Coherence Theory of Knowledge', pp. 5–25 in *Philosophical Topics* Vol. 14(1), 1986.

[89] C. I. Lewis: 'The given element in empirical knowledge', pp. 168–175 in *Philosophical Review* Vol. 61, 1952.

[90] P. Maddy: *Realism in Mathematics* (Oxford: Oxford University Press), 1990.

[91] D. A. Martin: 'Mathematical Evidence', pp. 215-231 in Dales and Oliveri [27].

[92] V. McGee: 'How We Learn Mathematical Language', pp. 35–68 in *Philosophical Review*, Vol. 106 (1).

[93] D. McManus (ed.): *Wittgenstein and Skepticism* (London: Routledge), 2004.

[94] R. McNaughton: 'Axiomatic Systems, Conceptual Schemes, and the Consistency of Mathematical Theories', pp. 44-53 in *Philosophy of Science*, Vol. 21(1), 1954.

[95] R. McNaughton: 'Conceptual Schemes in Set Theory', pp. 66-80 in *Philosophical Review*, Vol. LXVI, 1957.

[96] D. Miramanoff: 'Les antinomies de Russell et de Burali-Foreti et le problemème fondamental de la théorie des ensembles, pp. 37–52 in *Enseignement mathématique*, Vol. 19, 1917.

[97] D. Miramanoff: 'Remarques sur la théorie des ensembles et les antinomies cantoriennes', pp. 209–217 in *Enseignement mathématique*, Vol. 19, 1917.

[98] G. E. Moore: *Philosophical Papers* (New York: Collier Books), 1962.

[99] P. Moser: *Empirical Justification* (Dordrecht: D. Reidel Publishing Company), 1985.

[100] J. von Neumann: 'An axiomatization of set theory', pp. 394–413 in Heijenoort [72].

[101] J. von Neumann: 'Über eine Widerspruchsfreiheitsfrage in der axiomatischen Mengenlehre', reprinted as pp. 494–508 in Neumann [102].

[102] J. von Neumann: *Collected Works*, Vol. 1 (New York: Pergamon), 1961.

[103] R. Nozick: *Philosophical Explanations* (Oxford: Oxford University Press), 1981.

[104] C. Peacocke: *The Realm of Reason* (Oxford: Oxford University Press), 2003.

[105] N. J. Pedersen: 'Entitlement in mathematics', forthcoming in *MATHNET 12: Newsletter of the Danish Network for the History and Philosophy of Mathematics.*

[106] A. Plantinga: *Warrant: The Current Debate* (Oxford: Oxford University Press), 1993.

[107] A. Plantinga: *Warrant and Proper Function* (Oxford: Oxford University Press), 1993.

[108] J. Pollock and J. Cruz: *Contemporary Theories of Knowledge*, 2^{nd} edition (New York: Rowman and Littlefield), 1999.

[109] D. Pritchard: 'Wittgenstein's *On Certainty* and Contemporary Anti-Scepticism', forthcoming in D. Moyal-Sharrock and W. H. Brenner (eds.): *Investigating On Certainty: Essays on Wittgenstein's Last Work* (London: Palgrave Macmillan).

[110] J. Pryor: 'The Skeptic and the Dogmatist', pp. 517–549 in *Noûs* Vol. 34, 2000.

[111] J. Pryor: 'Highlights of Recent Epistemology', pp. 95–124 in *British Journal for the Philosophy of Science* Vol. 52, 2001.

[112] J. Pryor: 'Is Moore's Argument an Example of Transmission-Failure?', in *Philosophical Perspectives.*

[113] J. Pryor: 'Is There Immediate Justification?', pp. 181–202 in Steup and Sosa [137].

[114] H. Putnam: *Philosophy of Logic* (New York: Harper Torchbooks), 1971.

[115] H. Putnam: 'The Meaning of 'Meaning", in K. Gunderson (ed.) *Language, Mind and Knowledge*, Minnesota Studies in Philosophy of Science Series, Vol. 7, 1975.

[116] H. Putnam: *Reason, Truth and History* (Cambridge: Cambridge University Press), 1981.

[117] W. v. O. Quine: 'New Foundations for Mathematical Logic', pp. 70–80 in *American Mathematical Monthly*, Vol. 44, 1937.

[118] W. v. O. Quine: 'On What There Is', pp. 21–38 in *Review of Metaphysics* Vol. 2, 1948.

[119] A. Rayo: 'Success by Default?', pp. 305–322 in *Philosophia Mathematica* Vol. 11, 2003.

[120] A. Rayo: 'Logicism Reconsidered', pp. 203–235 in Shapiro [128].

[121] T. Reid: 'An Inquiry into the Mind on the Principles of Common Sense', in W. H. Bart (ed.): *The Works of Thomas Reid* (Edinburgh: Maclachlan & Stewart), 1764.

[122] B. Russell: 'Letter to Frege', pp. 124–125 in van Heijenoort [72].

[123] F. Schmitt: *Knowledge and Belief* (London: Routledge), 1992.

[124] N. Shah: 'How Truth Governs Belief', *Philosophical Review* Vol. 12(4), 2003.

[125] N. Shah and D. Velleman: 'Doxastic Deliberation', forthcoming in *Philosophical Review.*

[126] S. Shapiro: *Foundations wihtout Foundationalism* (Oxford: Oxford University Press), 1991.

[127] S. Shapiro: *Philosophy of Mathematics: Structure and Ontology* (Oxford University Press), 1997.

[128] S. Shapiro (ed.): *The Oxford Handbook of Philosophy of Mathematics and Logic* (Oxford: Oxford University Press), 2005.

[129] S. Shapiro: 'Externalism, anti-realism, and the KK-thesis', forthcoming in D. DeVidi and T. Kenyon (eds.): *A Logical Approach to Philosophy: Essays in Honour of Graham Solomon.*

[130] E. Sosa: 'The Raft and the Pyramid', pp. 3–25 in *Midwest Studies in Philosophy* Vol. 5, 1980.

[131] E. Specker: 'The axiom of choice in Quine's new foundations for mathematical logic', pp. 972–975 in *Proceedings of the National Academy of Sciences of the USA* Vol. 39, 1953.

[132] E. Specker: 'Dualität', pp. 451–465 in *Dialectica* Vol 12, 1958.

[133] E. Specker: 'Typical ambiguity', in E. Nagel (ed.): *Logic, methodology and philosophy of science* (Paolo Alto: Stanford University Press), 1962.

[134] M. Steup: 'The Deontic Conception of Justification', pp. 65–84 in *Philosophical Studies* Vol. 53, 1988.

[135] M. Steup (ed.): *Knowledge, Truth, and Duty: Essays on Epistemic Justification, Responsibility, and Virtue* (Oxford: Oxford University Press), 2001.

[136] M. Steup: 'Epistemic Duty, Evidence, and Internality', pp. 134–148 in Step [135].

[137] M. Steup and E. Sosa: *Contemporary Debates in Epistemology* (Oxford: Blackwell), 2005.

[138] P. F. Strawson: *Individuals: An Essay in Descriptive Metaphysics* (London: Methuen), 1959.

[139] N. Tennant: *The Taming of the True* (Oxford: Oxford University Press), 1997.

[140] P. Thagard: *Coherence in Thought and Action* (Cambridge, MA: MIT Press), 2000.

[141] G. J. Warnock: 'Claims to Knowledge', pp. 19–32 in *Proceedings of the Aristotelian Society Supplement* Vol. 36, 1962.

[142] M. Williams: *Unnatural Doubts: Epistemological Realism and the Basis of Skepticism*, 2^{nd} edition (Princeton, NJ: Princeton University Press), 1996.

[143] L. Wittgenstein: *Tractatus Logico-Philosophicus* (London: Kegan Paul & Trubner), 1922.

[144] L. Wittgenstein: *On Certainty* (Oxford: Basil Blackwell), 1969.

[145] N. Wolterstorff: 'Can Belief in God Be Rational If It Has No Foundations?', in A. Plantinga and N. Wolterstorff (eds.): *Faith and Rationality* (Notre Dame: University of Notre Dame Press), 1983.

[146] N. Wolterstorff: 'Obligation, Entitlement, and Rationality', pp. 326–338 in Steup and Sosa [137].

[147] C. Wright: 'Facts and Certainty', pp. 429–472 in *Proceedings of the British Academy*, 71, 1985.

[148] C. Wright: *Truth and Objectivity* (Cambridge, MA: Harvard University Press), 1992.

[149] C. Wright: 'Cogency and question-begging: Some reflections on McKinsey's argument and Putnam's proof', pp. 140–163 in *Philosophical Issues* Vol. 10, 2000.

[150] C. Wright: 'On Basic Logical Knowledge', pp. 41–85 in *Philosophical Studies* Vol. 106, 2001.

[151] C. Wright: 'Some reflections on the acquisition of warrant by inference', pp. 57–77 in S. Nuccetelli (ed.): *New Essays on Semantic Externalism and Self-Knowledge* (Cambridge, MA: MIT Press), 2003.

[152] C. Wright: 'Wittgenstenian certainties', in McManus [93].

[153] C. Wright: 'Intuition, Entitlement and the Epistemology of Logical Laws', pp. 155–175 in *Dialectica*, Vol. 58(1), 2004.

[154] C. Wright: 'Warrant for Nothing (and Foundations for Free)?', pp. 167–212 in *Proceedings of the Aristotelian Society Supplementary Vol. LXXVIII*, 2004.

[155] C. Wright: 'An entitlement to Hume's Principle?', typescript.

[156] S. Yablo: 'Apriority and Existence', pp. 197–228 in Boghossian and Peacocke [10].

[157] E. Zermelo: 'Proof that every set can be well-ordered', pp. 139–141 in van Heijenoort [72].

[158] E. Zermelo: 'A new proof of the possibility of a well-ordering', pp. 183–198 in van Heijenoort [72].

[159] E. Zermelo: 'Investigations in the foundations of set theory I', pp. 199–215 in van Heijenoort [72].

[160] E. Zermelo: 'On boundary numbers and domains of sets: New investigations in the foundations of set theory', translation by M. Hallett in Ewald [41], pp. 1219-1233.