Francesco Nava and Pasquale Schiraldi

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Differentiated Durable Goods Monopoly

A Robust Coase Conjecture*

Francesco Nava† and Pasquale Schiraldi†

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Abstract: The paper analyzes a durable goods monopoly problem in which multiple varieties can be sold. A robust Coase conjecture establishes that the market eventually clears, with profits exceeding static optimal market-clearing profits and converging to this lower bound in all stationary equilibria with instantaneous price revisions. Pricing need not be efficient, nor is it minimal (equal to the maximum of marginal cost and minimal value), and can lead to cross-subsidization. Conclusions nest both classical Coasian insights and modern Coasian failures. The option to scrap products does not affect results qualitatively, but delivers a novel motive for selling high cost products.

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†Economics Department, London School of Economics, f.nava@lse.ac.uk.

†Economics Department, London School of Economics, p.schiraldi@lse.ac.uk.

1 Introduction

While dynamic pricing problems without commitment are well understood when a single variety is sold in the market, multi-variety extensions thereof have often been cast as inconsistent with classical Coasian dynamics. Our main contributions robustly generalize classical Coasian results to environments in which multiple varieties can be produced and sold, integrating both classical Coasian insights as well as several more modern Coasian failures into a unified framework.

Nobel laureate Ronald Coase first brought the commitment problem of a durable good monopolist to the attention of the academic community (Coase 1972). A monopolist unable to commit to future prices, and having sold to high-value buyers, would be compelled to lower prices in order to trade with buyers who did not yet purchase. As a result, forward-looking buyers would be less inclined to pay high prices when expecting prices to fall. With frequent price revisions, Coase originally conjectured that the implied competition from future selves would entirely dissipate the seller's market power, leading to an opening price close to the marginal cost and to the competitive quantity being sold in a *twinkle of an eye.* Formal proofs of these statements appear in seminal papers by Stokey 1981, Fudenberg, Levine and Tirole 1985, Gul, Sonnenschein and Wilson 1986, and Ausubel and Deneckere 1989.

The present work considers the same environment originally studied by Coase, but it presumes that the monopolist can sell more than one variety of the durable good. Such a natural extension does not rule out any one of the three key ingredients required to obtain classical Coasian results: lack of commitment, deteriorating market conditions, and competition from future selves willing to cut prices in the wake of market deterioration. Because of this, our conclusions will not give rise to outright failures of classical Coasian dynamics, but rather will qualify their content. In multi-dimensional settings, the Coasian logic will still prevail, in that: (1) prices at which all consumers purchase a variety still commit the seller, since the incentives to lower prices subside upon clearing the market; and (2) almost all consumers still purchase a variety at the opening price in any stationary equilibrium when the time between offers is small. Yet, unlike with the one-variety case, these insights will not lead to efficiency, pricing at minimal valuations, or zero profits, because intratemporal price discrimination will make up for the lack of intertemporal price discrimination, thereby restoring some of the market power lost because of the seller's inability to commit.

Specifically, we consider a monopolist with constant marginal costs who sells two varieties of a durable good to a continuum of buyers with unit-demand for the product (results easily extend to any finite number of varieties). Buyers are privately informed of their value for each of the two varieties, and the distribution of values is represented by a measure that can exhibit an arbitrary correlation structure.[1] The time horizon is infinite. In every period, the monopolist sets a price for each variety sold, while the buyers, upon observing prices, choose which variety to purchase (if any). In the baseline setting, to favor comparability with classical results, the marginal cost of each variety is set to zero, and consumers permanently exit the market upon buying a product.

The analysis begins by characterizing the static problem of maximizing profits subject to clearing the market (that is, selling a variety to every consumer in the support of the measure). When multiple varieties are sold, optimal market-clearing profits are always strictly positive, as it is always possible to sell one variety for free (thereby clearing the market) while using the other variety to screen consumers and raise profits. In some instances, these profits can coincide with monopoly profits in the commitment case.[2] But in general, optimal market-clearing profits exceed the lowest value of the durable good (that is, the smallest value of the preferred variety) and consequently the lowest value of each of the two varieties.[3] Optimal market-clearing, however, often distorts consumption decisions as buyers purchase their least preferred variety only because it is sold at a cheaper price.

The analysis then extends classical Coasian dynamics to our multi-dimensional setting. Preliminary results establish that in any perfect Bayesian equilibrium of the dynamic game: (1) there is skimming, as the measure of buyers in the market at any point in time is a truncation of the original measure; and (2) the market clears instantaneously whenever the seller sets static market-clearing prices. The latter immediately implies that optimal market-clearing profits bound equilibrium profits in the dynamic game from below. Results also establish that stationary equilibria always exist, and that these equilibria can display mixing along the entire equilibrium path in order to conceal future discounts. As in classical Coasian settings, though, when the time between offers converges to zero, stationary equilibrium profits always converge to optimal market-clearing profits, and profits accrue by selling almost instantaneously to almost all buyers.

Our Coasian dynamics are reminiscent of the classical results for the one-variety case, with two distinct scenarios. In the gaps case (when the lowest value in the support exceeds the marginal cost), the market clears in finite time, equilibrium profits are positive and unique, and they converge to the lowest valuation as the time between offers converges to zero. In the no-gaps case (when the lowest value does not exceed the marginal cost), the market clears in infinite time, a Folk Theorem applies to equilibrium profits, and stationary equilibrium

[1] The set-up accommodates several commonly used designs such as vertical product differentiation (when consumers' valuations for the two products are positively correlated) and horizontal product differentiation (when the valuations are negatively correlated).

[2] For instance, this is often the case when varieties are horizontally differentiated.

[3] For instance, this always occurs when independent varieties share a common minimal valuation.

profits converge to zero. Multi-variety settings closely resemble the single-variety gaps case even when some buyers value both varieties at zero, although there are some significant differences. First, unlike in the one-variety case, the seller does not lose any bargaining power from lack of commitment as long as optimal market-clearing profits coincide with optimal profits. Moreover: equilibrium profits are positive at high frequencies of price-revision even when there are no gaps; having gaps no longer guarantees equilibrium uniqueness, as several market-clearing prices may be optimal; and, stationary equilibria may display mixing after the initial period as the seller attempts to conceal future discounts. As in the one-variety case, though, the assumption on gaps still determines the time it takes for the market to clear, which is finite with gaps, but not necessarily finite without.

The second part of the analysis extends the baseline model and contextualizes our contribution. First, it generalizes results to settings with positive marginal costs. Similar conclusions hold as in the zero marginal cost case, but equilibria may display cross-subsidization, with one variety being sold above its marginal cost and the other below. Then, results are extended to settings in which consumers remain in the market after purchasing a variety. This is done by suitably adjusting the definition of static market-clearing. Clearing the market in such settings requires setting prices such that: (1) all buyers purchase a variety; and (2) the marginal cost of either variety exceeds the value of switching between varieties for every buyer. Thus, allowing consumers to remain in the market can bound mark-ups relative to our earlier analysis, but it does not restore efficiency when marginal costs are positive. As high marginal cost varieties can enlarge the size of the market-clearing set, a novel rationale emerges for producing high cost varieties, given that such products favor intratemporal price discrimination by preventing future price cuts. The final extension also clarifies why Coasian dynamics should not be summarized as optimal market-clearing (or agreement), but rather as renegotiation-proof agreement.[4] The analysis concludes by relating to classical Coasian conclusions and to some well-known Coasian failures and by nesting these within our framework.

A key insight of the analysis relates optimal market-clearing to stationary pricing in the dynamic game when price revisions are frequent. This observation can be leveraged to deliver testable predictions about approximate equilibrium pricing in durable goods markets. In the online appendix, we exploit this insight to investigate the optimal design of product lines. We establish that optimal designs must involve horizontal product differentiation, and that in contrast to the one-variety case, volatility in valuations can occasionally benefit the seller. To the best of our knowledge, these are the first theoretical attempts at analyzing the incentives

[4] In classical bargaining settings, *agreement* refers to the seller trading with every buyer whose value exceeds the marginal cost. In multi-variety settings, agreement amounts to market-clearing, or equivalently, to the depletion of gains from trade.

to develop product lines in the context of a dynamic pricing model.[5]

A vast literature presents tactics to circumvent the monopolist's commitment problem. Such tactics typically involve preventing the market from fully deteriorating in order to allow the monopolist to sustain higher profits.[6] Most closely related to this paper are Board and Pycia 2014, Hahn 2006, Inderst 2008, Takeyama 2002, and Wang 1998. Three of these provide examples in which the monopolist successfully mitigates its commitment problem in multi-variety settings by selling vertically differentiated products (with two types of consumers, Hahn 2006 and Inderst 2008, and with two periods, Takeyama 2002). All three focus primarily on the possibility of strategically changing the quality of the goods (via upgrades or downgrades). Our framework is able to nest these conclusions since vertical product differentiation is a feasible design. Therefore, in our view, their results should not be interpreted as failures of the Coase conjecture. Rather, they display the essence of the Coasian insight, which is optimal market-clearing (or agreement), and not efficiency or minimal pricing.

Similarly, Board and Pycia 2014 shows that a durable goods monopolist never cuts its price if an outside option with strictly positive value is available for free. Their conclusions can also be nested within our framework. Indeed, because any price set by the monopolist clears the market when an outside option is freely available, the monopolist not undercutting its initial price would be consistent with the proposed extension of the Coase conjecture. Of course, by pricing the outside option, the monopolist would be able to achieve a higher profit, as both varieties would be optimally sold at positive prices when there are gaps (which is the case in their setting). Similar considerations apply to Wang 1998, which establishes an instantaneous clearing result (evocative of Board and Pycia 2014) in a two-type model. As before, these and other related results have been cast as Coasian failures. Yet in our interpretation, these observations capture features of multi-dimensional Coasian generalizations, and not failures thereof.

The rest of the paper is organized as follows. Section 2 introduces the model and the solution concepts. Section 3 characterizes optimal market-clearing profits when marginal costs are zero. Section 4 solves the dynamic pricing game and presents our Coasian results when marginal costs are zero. Section 5 extends conclusions to settings with positive marginal

[5]Seminal references for static design questions are Mussa and Rosen 1978, Deneckere and McAfee 1996, and Johnson and Myatt 2016. A stylized dynamic exercise in House and Ozdenoren 2008 establishes the optimality of mass products in one-variety settings.

[6]Seminal studies have shown that the market does not fully deteriorate: by renting the good, Bulow 1982; by introducing best-price provisions, Butz 1990; by introducing new versions of the durable good, Levinthaland and Purohit 1989, Waldman 1993 and 1996, Choi 1994, Fudenberg and Tirole 1998, and Lee and Lee 1998; with capacity constraints, Kahn 1986 and McAfee and Wiseman 2008; with entry of new buyers, Sobel 1991; with time-varying buyers' valuations, Biehl 2001, Deb 2011, and Garrett 2016; with time-varying costs, Ortner 2014; with depreciation, Bond and Samuelson 1984; and with discrete demand, Bagnoli, Salant and Swierzbinski 1989, Fehr and Kuhn 1995, and Montez 2013.

costs as well as to settings in which buyers remain active upon purchasing a variety. Section 6 relates to classical Coasian results and their failures, and then concludes. Proofs of lemmas and propositions appear in the appendix, Section 7. Results on the optimal design of varieties and proofs of remarks are deferred to the online appendix.

2 A Market with Differentiated Varieties

A monopolist produces and sells two varieties of a durable good, a and b. A unit measure of non-atomic consumers has unit-demand for the durable good. Time is discrete, the time-horizon is infinite, and all players discount the future by a common factor δ. Consumers are completely pinned down by their value profile $v = (v_a, v_b)$, where v_i denotes the value of consuming variety $i \in \{a, b\}$. Value profiles are private information of consumers. A measure $\mathcal{F}$, defined on the unit square $[0,1]^2$, describes the distribution of value profiles among buyers. Throughout, denote by F its associated cumulative distribution and by V its support.[7] To simplify parts of the discussion, some results require the measure $\mathcal{F}$ to be non-atomic.

Condition 1 *The market is said to be regular if $\mathcal{F}$ admits a density f satisfying $f(v) \in \left(\underline{f}, \overline{f}\right)$ for any $v \in V$, and if the support V is convex.*

Regularity implies that the measure $\mathcal{F}$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}$ on $[0,1]^2$, and that there exists a bounded and strictly positive density on the entire support V. Weaker, albeit more involved, conditions could be imposed to discipline the measure only on the relevant parts of the support. We opted for a stronger but more elegant condition, while qualifying its role throughout analysis.[8] Denote the marginal cumulative distribution of variety i by F_i, its support by V_i, and its density, when it exists, by f_i.

In the baseline setting, buyers have unit-demand for the product and exit the market upon purchasing either of the two varieties.[9] Thus, the final payoff of a buyer purchasing variety $i \in \{a, b\}$, at a price p_i, in date t simply amounts to $\delta^t (v_i - p_i)$, while the payoff of a buyer never purchasing a variety simply amounts to 0. The monopolist's marginal cost of producing variety $i \in \{a, b\}$ is constant and is denoted by $c_i \in [0, 1]$. Marginal costs are common knowledge. Units are produced when sold in order to minimize production costs, and the monopolist's payoff simply amounts to the present discounted value of future profits.

[7] The support identifies the smallest closed set whose complement has probability zero.

[8] Our regularity condition differs from the classical assumptions imposed on the single-variety case in Gul, Sonnenschein and Wilson 1986. Moreover, the two assumptions cannot be nested as their conditions are stronger but local, whereas we impose weaker conditions but on the entire support. Our condition is, instead, a natural extension of the assumptions in Fundenberg, Levine and Tirole 1984.

[9] We discuss which conclusions are affected by the permanent exit assumption in Section 4.

To keep the action set of the seller compact, the price of each variety at any date t is chosen from the interval $[\omega, 1]$, for some $\omega < 0$.[10]

Thus, in every period: the firm sets a price in $[\omega, 1]$ for each of the two varieties produced in order to maximize the expected present value of future profits; and consumers who have not previously purchased a product choose whether to buy either of the two varieties at current prices in order to maximize their expected present value.

Information Structure and Solution Concepts: Players observe the prices set by the monopolist in every previous period. A t-period seller-history, h^t, specifies for every period $s \in \{0, ..., t-1\}$ the prices that were set by the seller for each of the two varieties of the durable good. A t-period buyer-history for a player who has yet to purchase a variety, $\hat{h}^t$, consists of a history h^t followed by the prices announced by the monopolist at date t. Denote the set of t-period seller-histories by $H^t = [\omega, 1]^{2t}$ and the set of seller histories by $H = \cup_{t=0}^{\infty} H^t$. Similarly, denote the set of t-period buyer-histories by $\hat{H}^t = [\omega, 1]^{2t+2}$ and the set of buyer histories by $\hat{H} = \cup_{t=0}^{\infty} \hat{H}^t$.

As is customary in the literature, we impose measurability restrictions on joint consumers' strategies which require the set of consumers purchasing variety $i \in \{a, b\}$ at any possible history to be a measurable set. For a metric space X, denote by $\mathcal{P}(X)$ the set of all probability measures on $(X, \Omega(X))$, where $\Omega(X)$ denotes the Borel sigma-algebra. Similarly, denote by $\mathcal{P}^*(X)$ the set of all measures on $(X, \Omega(X))$. A behavioral pure strategy profile for buyers consists of a function $\alpha : \hat{H} \times V \to \{0, a, b\}$ such that $\alpha(\hat{h}, \cdot)$ is measurable for any $\hat{h} \in \hat{H}$. Action 0 is to be interpreted as the decision not to buy any product in the current period. Actions a and b respectively denote the decision to purchase variety a or b in the current period. Intuitively, α determines consumption decisions of buyers at every possible history. Behavioral mixed strategies at any history then consist of probability distributions over such measurable functions. A behavioral strategy profile for the monopolist consists of a function satisfying $\sigma : H \to \mathcal{P}([\omega, 1]^2)$, where σ determines the probability distribution over prices charged by the monopolist as a function of the history of play.

Any strategy profile $\{\sigma, \alpha\}$ generates a path of prices and sales which can be computed recursively. Given a mixed strategy profile $\{\sigma, \alpha\}$, let $\mathcal{D}_i(h^t) \in \mathcal{P}^*(V)$ denote the measure of consumers purchasing variety $i \in \{a, b\}$ at any buyer-history h^t, and let $D_i(h^t)$ denote its support. Consumers with value profile $v \in V$ are *active* at history h^t if they have not yet purchased a variety of the durable good. Formally, define the measure of *active buyers* $\mathcal{A}(h^t)$ at a given history h^t as

$$\mathcal{A}(E|h^t) = \mathcal{F}(E) - \textstyle\sum_{s=0}^{t-1} [\mathcal{D}_a(E|h^s) + \mathcal{D}_b(E|h^s)] \quad \text{for any} \quad E \in \Omega(V),$$

[10] Of course, despite $\omega < 0$, prices will be non-negative in equilibrium.

where h^s denotes the sub-history of length s of h^t. Let $A(h^t)$ denote the support of this measure. When clarity is not compromised, we omit the dependence on the history and we denote these measures and supports simply by $\mathcal{D}_i^t$, D_i^t, $\mathcal{A}^t$ and A^t. For any strategy profile $\{\sigma, \alpha\}$, let $\Pi(\sigma, \alpha|h)$ be the expected present value of profits generated after history h, and let $U(\sigma, \alpha|\hat{h}, v)$ be the expected present value of the surplus of an active buyer v who chooses not to buy any variety at history $\hat{h}$. When an equilibrium strategy is fixed, we omit the dependence on strategies and denote by $\Pi(h)$ the expected present value of profits and by $U(\hat{h}, v)$ the continuation value of player v.

A *perfect Bayesian equilibrium* (equivalently a PBE) consists of a mixed strategy profile $\{\sigma, \alpha\}$ and updated beliefs about the measure of active buyers satisfying the two standard requirements: that strategies are optimal given beliefs, and that beliefs are derived from strategies according to Bayes rule whenever possible.[11] To guarantee the existence of an equilibrium, players are allowed to mix at any stage of the game.

With the proposed information structure, buyers' deviations cannot be detected by the seller. In this respect, the paper is closest to the classical asymmetric information bargaining model in Fudenberg, Levine and Tirole 1985. Yet, rather than having a single buyer, our model preserves the durable goods interpretation by retaining a measure of buyers. Because buyers' deviations cannot be detected, no further refinements are invoked. If buyers' deviations were detectable, however, similar conclusions would hold for equilibria in which deviations by non-atomic subsets of buyers have no effect on future play.[12] In any such equilibrium, the path of play would still coincide with the path of play in a perfect Bayesian equilibrium of our model, given that players' strategies prescribe optimal behavior after all histories with non-atomic deviations. Consequently, unilateral deviations by non-atomic buyers would not affect the actions of the remaining consumers, their beliefs, or the actions of the monopolist.

In general, the buyers' equilibrium strategies depend on the entire history of play (as the entire history can affect beliefs about future prices). Ausubel and Deneckere 1989 show that a Folk Theorem can hold in this class of games, even when a single variety is for sale, if no additional restrictions are imposed on the solution concept.[13] As a similar logic applies to settings with multiple varieties, it is convenient to consider stationary equilibria in which the monopolist does not exploit changes in buyers' beliefs in order to commit to a given price

[11] By consistency, at any buyer history $\hat{h}^t$, buyers' beliefs about prices set at date $t+s$ amount to $\beta^1(\hat{h}^t) = \sigma(\hat{h}^t)$ for $s = 1$ and to $\beta^s(\hat{h}^t) = \int \sigma(\hat{h}^{t+s}) \prod_{r=1}^{s-1} d\beta^r(p^{t+r}|\hat{h}^{t+r-1})$ for $s \in \{2, 3, ...\}$.

[12] In classical durable goods settings (such as Gul, Sonnenschein and Wilson 1986 and Ausubel and Deneckere 1989), every deviation is detectable. To deal with the implied complications, a refinement is invoked restricting attention to equilibria in which deviations by subsets of active buyers with measure zero change neither the actions of the remaining buyers nor those of the seller.

[13] The Folk theorem holds in the single-variety case when some buyers do not value the good.

path. As is customary in the literature, therefore, the results on Coasian dynamics rely on a common class of Markovian equilibria. Define a *weak Markov equilibrium* (equivalently a WME) to be a perfect Bayesian equilibrium in which the strategy of active buyers depends only on the current price profile.[14] As in the single-variety case, the solution concept does not require that buyers' beliefs only depend on current prices (as such a restriction would compromise existence). For instance, beliefs may depend on the entire history of play when the seller deviates to setting prices that had been quoted in one of previous periods. Further, the solution concept does not impose restrictions on the seller's strategy which can depend on the entire history of play. Nevertheless, because buyers' decisions only condition on current prices, the price evolution will only depend on current prices on the equilibrium path when both varieties trade.

3 Optimal Market-Clearing

We begin by defining the set of static market-clearing prices and discussing some of its properties. Such prices will play an important role in the analysis of the dynamic pricing problem at hand. Sections 3 and 4 focus on the case in which *marginal costs equal zero.*

Throughout, when denoting by i a generic variety in $\{a, b\}$, we denote by $j \neq i$ the other variety. A *market-clearing price* is a price profile that clears the market when the seller commits to setting such prices for the infinite future. Equivalently, it is a price profile that clears the market in the static version of the model. Formally, the static demand $d_i(p)$ for a variety $i \in \{a, b\}$ given a price profile p satisfies

$$d_i(p) \in [\mathcal{F}(v_i - p_i > \max\{v_j - p_j, 0\}), \mathcal{F}(v_i - p_i \geq \max\{v_j - p_j, 0\})].$$

The demand equation does not impose tie-breaking assumptions for indifferent consumers.[15] The set of market-clearing prices M consists of those prices at which every consumer is willing to purchase at least one of the two varieties:

$$M = \left\{ p \in \mathbb{R}^2 \mid \max_{i \in \{a,b\}} \{v_i - p_i\} \geq 0 \text{ for all } v \in V \right\}.$$

Let $\underline{v}_i$ denote the *minimal value* for variety i in the support V. With only one variety, the highest market-clearing price always coincides with the minimal value in the support. With more than one variety, any price profile p in which one variety is sold at a price that does

[14] That is, a WME is a PBE in which, at any two histories $(p, h), (p, h') \in \hat{H}$, we have that $\alpha((p, h), v) = \alpha((p, h'), v)$ for all $v \in A(p, h) \cap A(p, h')$.

[15] When the market is regular, tie-breaking assumptions are entirely inconsequential.

not exceed its minimal value is a market-clearing price. Formally, $p_i \leq \underline{v}_i$ implies $p \in M$.

When the valuations of the two varieties are independently distributed, at least one of the two varieties must be sold at a price below its minimal value for the market to clear (Figure 1, Panel 1). This need not be the case in general, though. For instance, when the values of the two varieties display perfect negative correlation, market-clearing prices exist in which both varieties are sold at prices that strictly exceed their respective minimal values (Figure 1, Panel 3). For a consumer $v \in V$, define the *value of the durable good* as the value of the preferred variety (that is, $v_g = \max_{i \in \{a,b\}} v_i$). Then, the *minimal value of the durable good* amounts to

$$\underline{v}_g = \min_{v \in V} \max_{i \in \{a,b\}} v_i.$$

The minimal value of the durable good always exceeds the minimal value of each variety (that is, $\underline{v}_g \geq \max_{i \in \{a,b\}} \underline{v}_i$). Moreover, if $p \in M$, at least one variety must sell at a price smaller than $\underline{v}_g$ (that is, $\min_{i \in \{a,b\}} p_i \leq \underline{v}_g$), because the market clears.

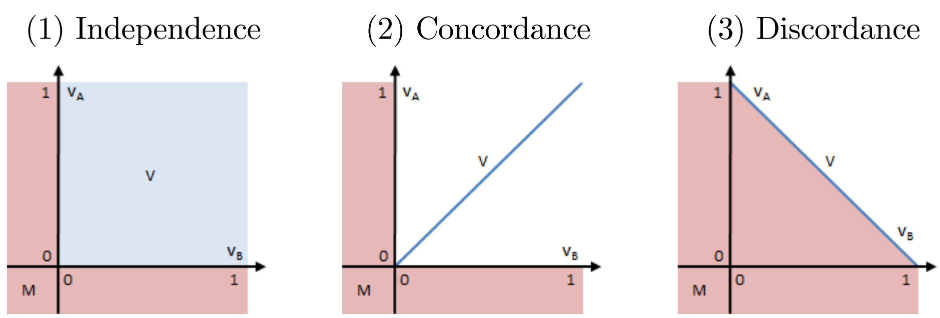

Figure 1: For three possible distributions: in pink, the set of market-clearing prices M; in blue, the support V.[16]

Optimal market-clearing prices, $\bar{p}$, are market-clearing prices that maximize static monopoly profits. Formally, an optimal market-clearing price is defined as a solution to the following static profit maximization problem:

$$\bar{p} \in \arg\max_{p \in M} [d_a(p)p_a + d_b(p)p_b]. \quad (1)$$

Optimal market-clearing prices may fail to exist when regularity is violated, as extrema may never be attained. Therefore, define the supremum of this problem as the *optimal market-clearing profit*, $\bar{\pi}$. Optimal market-clearing profits always exist, as profits are nonnegative and necessarily bounded above by 1. The next result bounds optimal market-clearing profits

[16] A distribution is *discordant* if its support is a decreasing set (that is, if $v_a > v'_a$ implies $v_b \leq v'_b$ for all $v, v' \in V$), and *concordant* if its support is an increasing set (that is, if $v_a > v'_a$ implies $v_b \geq v'_b$ for all $v, v' \in V$).

from below when more than one variety is for sale. The proof does not rely on assumptions on the measure $\mathcal{F}$ and includes scenarios in which the market is not regular. We say that *varieties are identical* if their values coincide for all buyers (that is, if $v_a = v_b$ for any $v \in V$). We say that *varieties are unranked* if not all buyers weakly prefer one variety to the other (that is, if for any i there is a $v \in V$ such that $v_i > v_j$).

Remark 1 *Optimal market-clearing profits:*
(1) weakly exceed $\underline{v}_g$*;*
(2) strictly exceed $\max_{i\in\{a,b\}} \underline{v}_i$ *if varieties are unranked;*
(3) equal $\min_{i\in\{a,b\}} \underline{v}_i$ *if and only if varieties are identical;*
(4) equal 0 *if and only if varieties are identical and* $(0,0) \in V$*;*
(5) strictly exceed $\underline{v}_g$ *if varieties are not identical,* $\underline{v}_a = \underline{v}_b$*, and* $(\underline{v}_a, \underline{v}_b) \in V$*.*

The monopolist can always clear the market by selling both varieties at price $\underline{v}_g$. So, optimal profits must weakly exceed the minimal value of the durable good, and consequently the minimal value of every variety. When varieties are unranked, the seller can raise higher profits while clearing the market by selling both varieties at prices that exceed the largest minimal value, one of them strictly so. When varieties are identical, optimal market-clearing profits must be equal to $\min_{i\in\{a,b\}} \underline{v}_i = \underline{v}_g$ as all buyers purchase the cheapest variety. But otherwise, profits always strictly exceed the smallest of the two minimal values, since market-clearing prices exist in which both varieties are sold, with one variety sold at a price that strictly exceeds the smallest minimal value. Because $\underline{v}_i \geq 0$ for any variety i, optimal market-clearing profits can therefore be equal to 0 if and only if varieties are identical and $(0,0) \in V$. By a similar logic, optimal market-clearing profits strictly exceed even the minimal value of the durable good when varieties are differentiated, minimal values coincide, and a single buyer has the smallest possible value for both varieties.[17]

Remark 1 hints at why the Coasian intuition about the seller eventually depleting the market does not necessarily lead to zero profits or to pricing at minimal values when differentiated varieties can be produced and sold. Although a monopolist lacking commitment may still have to clear the market, market-clearing no longer requires that profits coincide with minimal valuations. Indeed, when the market is regular, optimal market-clearing profits always strictly exceed the smallest minimal valuation; and under mild conditions, such profits strictly exceed even the minimal valuation of the durable good. Moreover, commitment profits can coincide with optimal market-clearing profits when price discrimination gains are small relative to the minimal value of the durable good (as in Panel 3 of Figure 2).

[17] A model in which willingness to pay can be determined by budget constraints could deliver $\underline{v}_a = \underline{v}_b$ and $(\underline{v}_a, \underline{v}_b) \in V$.

A market-clearing price profile is said to be *efficient* if every buyer purchases its preferred variety. Formally, the set of efficient price profiles simply amounts to

$$M^* = \{p \in M \mid v_i \geq v_j \quad \Rightarrow \quad v_i - p_i \geq v_j - p_j \;\; \text{for all} \;\; v \in V\}\,.$$

We refer to such prices as efficient as they maximize utilitarian social welfare.[18] Any market-clearing price $p \in M$ such that $p_a = p_b$ is obviously always efficient. Furthermore, no other price can be efficient when $v_a = v_b$ for some $v \in V$. Although efficient market-clearing prices always exist, optimal market-clearing need not be efficient. For instance, when a single buyer has the smallest possible value for both varieties and the minimal values coincide, optimal market-clearing prices are necessarily inefficient, provided that varieties are differentiated.

Remark 2 *Optimal market-clearing prices are inefficient if:*

- *varieties are not identical,* V *is connected,* $\underline{v}_a = \underline{v}_b$*, and* $(\underline{v}_a, \underline{v}_b) \in V$*;*
- *varieties are unranked and* $V = [\underline{v}_a, \overline{v}_a] \times [\underline{v}_b, \overline{v}_b]$*.*

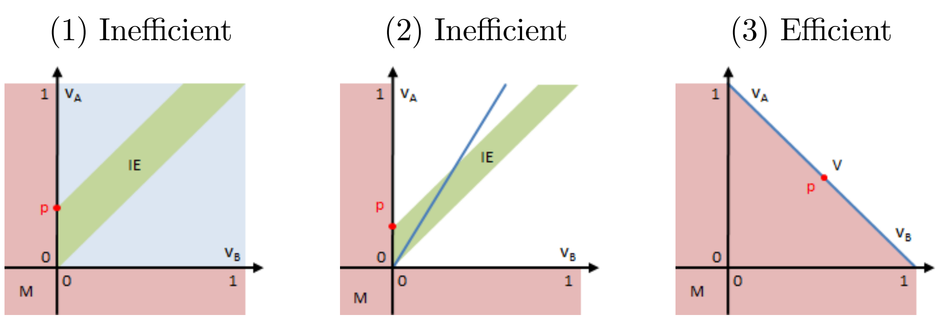

Figure 2: For three possible distributions: in red, optimal market-clearing prices; in green, buyers purchasing the inefficient variety at the optimal market-clearing price.

The inefficiency of optimal market-clearing prices is a generic phenomenon when the support is a Cartesian product. The result hints at why the generalizations of Coasian logic may not necessarily lead to efficiency if the main force disciplining dynamic pricing were shown to be optimal market-clearing. Of course, if varieties were identical, optimal market-clearing prices would be efficient simply because the market clears. This is not the case with differentiated products. Figure 2 depicts two instances of inefficiency in Panels 1 and 2, with independent and vertically differentiated products respectively, while Panel 3 shows why we had to assume $(\underline{v}_a, \underline{v}_b) \in V$ in order to guarantee frictions.

[18] As utility is transferrable and costs equal zero, an efficient price always maximizes surplus, since all buyers get to purchase their preferred variety.

4 Coasian Dynamics as Market-Clearing

This section extends classical Coasian results to settings with multiple varieties. It establishes that the market for the durable good must eventually clear even when multiple varieties can be sold. The intuition coincides with that of seminal Coasian results. As the monopolist cannot commit to future prices, the market must clear, or else clearing the market would become a profitable deviation as soon as the seller no longer expects to trade. In a multi-variety setting, however, market-clearing no longer implies that profits are minimal. Indeed, perfect Bayesian equilibrium profits always exceed optimal market-clearing profits, and they converge to such profits in any weak Markovian equilibrium as price revisions become arbitrarily frequent. These results highlight why lack of commitment and Coasian pricing do not necessarily lead to minimal-valuation pricing or efficiency, but only to market-clearing and agreement. When price revisions are frequent, the monopolist will simply choose the profit-maximizing way to supply all buyers. As in the single-variety case, having gaps will guarantee that all buyers are supplied in finite time. In contrast to the classical case, though, the market can clear in finite time even when there are no gaps.

We begin the analysis with a few preliminary results that unveil some important features of equilibrium strategies in this dynamic pricing game. As in the classical single-variety setting, the measure of active buyers must be a truncation of the original measure, and equilibrium play displays a specific form of top-down *skimming* of the market. In particular, at every possible history, a cutoff identifies the smallest value buyer who is willing to purchase a variety in the set $\{v \in \mathbb{R}^2 \mid v_a - v_b = k\}$ for all values of $k \in \mathbb{R}$. Thus, every subset of buyers with a given value difference k will be skimmed from the top down as a result of equilibrium play. To show this, we introduce a general notion of multidimensional truncation. We say that a measure $\mathcal{F}'$ is a *truncation* of $\mathcal{F}$ if for some set $A \subset \Omega(V)$,

$$\mathcal{F}'(E) = \mathcal{F}(E \cap A) \quad \text{for any} \quad E \in \Omega(V).$$

Lemma 1 *In any perfect Bayesian equilibrium, at any buyer-history h:*

(1) if buyer v strictly prefers to buy variety i, so does any active buyer v' such that

$$v'_i - v_i \geq \max\{0, v'_j - v_j\};$$

(2) if buyer v prefers to buy a variety, any active buyer $v' > v$ strictly prefers to buy if

$$\delta \max_{i \in \{a,b\}}\{v'_i - v_i\} < \min_{i \in \{a,b\}}\{v'_i - v_i\};$$

(3) if buyer v prefers not to buy, any active buyer $v' < v$ strictly prefers not to buy if

$$\delta \max_{i \in \{a,b\}}\{v_i - v_i'\} < \min_{i \in \{a,b\}}\{v_i - v_i'\};$$

(4) if the market is regular, the measure of active buyers is a truncation of $\mathcal{F}$.

The proof of the lemma is intuitive. If a buyer with value v was willing to purchase variety i at current prices, the same should hold for any active buyer $v' > v$ provided that the relative value for the two varieties is similar. In fact, by delaying, buyer v' could capture at most $\delta \max_i\{v_i' - v_i\}$ on top of the continuation value of buyer v. If so, however, buying now should be preferable, as buyer v' would capture $\min_i\{v_i' - v_i\}$ more surplus than v. This naturally follows, as delay costs are higher for high value consumers, and implies that the measure of active buyers must be a truncation whenever $\mathcal{F}$ is non-atomic. However, stronger notions of skimming would not apply. For instance, it is not in general the case that v buying a variety and $v' > v$ together imply that v' buys a variety. The active player set in the left panel of Figure 3 would violate this more stringent skimming requirement, as there are values v who purchase variety a and values $v' > v$ who do not purchase any variety. This occurs naturally in equilibrium when buyer v' prefers to wait to purchase good b at a lower price in the future. Still, whenever buyer v strictly prefers to purchase variety i, so do all the buyers v' who have a higher value for i, provided that the change in value for variety i exceeds that for variety j (equivalently, $v_i' - v_i \geq v_j' - v_j$). A similar logic also implies that if a buyer v does not buy any variety, neither does any buyer $v' < v$ with a similar relative value for the two varieties.

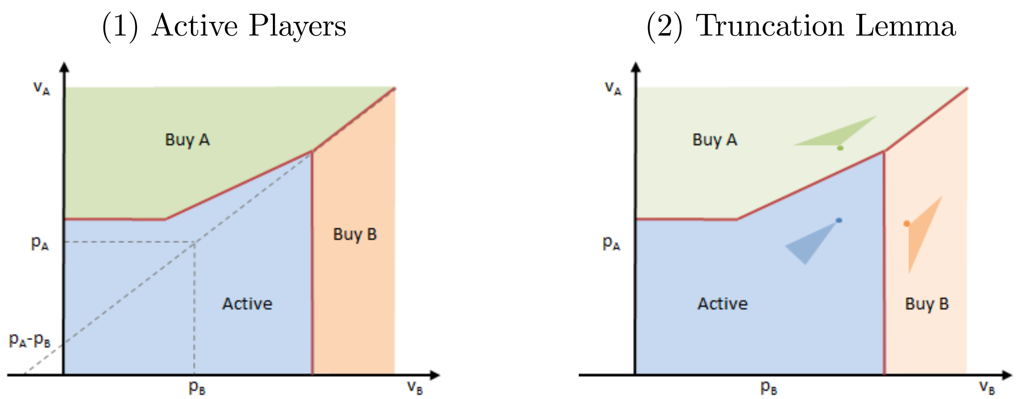

Figure 3: For a market clearing in period $t+1$: in blue A^{t+1}, the active buyer set; in green D_a^t, those who purchase variety a; and in orange D_b^t.

The left panel of Figure 3 depicts the set of types purchasing each of the two varieties, along with the active player set for a market that clears in the following period. The red lines identify buyers that are indifferent between purchasing different varieties at different dates.

The right panel depicts parts (1) and (3) of Lemma 1: if a dot belongs to one of the three regions, then the corresponding triangle must also belong to that region.

The next lemma relates features of equilibrium pricing to static market-clearing. The key observation establishes why static market-clearing prices must also clear the market in any equilibrium of the dynamic model. This immediately delivers two central conclusions. First, the monopolist never sets prices in the interior of the static market-clearing set. Second, optimal market-clearing profits bound profits from below in any equilibrium, at any history and for any possible discount factor. To state the result, let $\bar{M}$ denote the "interior" of the market-clearing price set M, or equivalently,

$$\bar{M} = \left\{ p \in \mathbb{R}^2 \mid \max_{i \in \{a,b\}} \{v_i - p_i\} > 0 \text{ for all } v \in V \right\}.$$

Given any history h and its associated active player set A, let $\bar{\pi}(A)$ denote the optimal market-clearing profit for the residual measure of buyers $\mathcal{F}(A)$. When the market is regular, $\bar{\pi}(A)$ simply amounts to

$$\bar{\pi}(A) = \max_{p \in M} \sum_{i \in \{a,b\}} p_i \mathcal{F}(v_i - p_i > v_j - p_j | A).$$

Lemma 2 *In any perfect Bayesian equilibrium, at any seller-history h:*

(1) all active buyers purchase a variety if prices are in $\bar{M}$;

(2) the monopolist never sets prices in $\bar{M}$;

(3) the present value of profits satisfies

$$\Pi(h) \geq \bar{\pi}(A).$$

Equilibrium profits must weakly exceed optimal market-clearing profits for any discount factor $\delta < 1$. As in the single-variety setting, the inability to commit to future prices (and the implied inability to intertemporally price discriminate forward-looking buyers) can still hurt the seller. Market-clearing and intratemporal price discrimination, however, shield the seller from further profit declines. In fact, even when the minimal value for the durable good equals zero ($\underline{v}_g = 0$), equilibrium profits cannot be competitive, and the allocation may be inefficient. This contrasts with a classical interpretation of the Coase conjecture for single-variety settings with no gaps, which requires equilibrium pricing to be approximately competitive and approximately efficient when buyers are arbitrarily patient. As the rest of the analysis clarifies, however, the Coasian logic persists to the extent that agreement and market-clearing still dictate equilibrium pricing. An immediate implication of Lemma 2 is that the seller cannot lose bargaining power because of its inability to commit to future prices when optimal market-clearing profits coincide with monopoly profits. The proof of Lemma 2 identifies the set of price profiles which are immediately accepted by all buyers regardless of their beliefs; it also establishes by contradiction that the closure of this set must include all static market-clearing prices because of consumer discounting.[19]

To grasp the full connection between known Coasian results and their failures, it is instructive to consider a few more observations. The next lemma establishes that the market must eventually clear, and that it must do so in finite time whenever the minimal value of the durable good $\underline{v}_g$ is strictly positive (call this the *gaps* case). The same result holds with a single variety when the smallest buyer's valuation is strictly positive. In contrast to the one-variety setting, though: the market clears in finite time even when the minimal value of both varieties equals zero, provided that $\underline{v}_g > 0$; and it may take infinite time to clear the market even when equilibrium profits are positive (which is generally the case, by Lemma 2 and Remark 1). The result, however, does not imply that the market will take infinitely many periods to clear whenever $\underline{v}_g = 0$ (call this the *no-gaps* case).

Lemma 3 *If the market is regular, in any perfect Bayesian equilibrium:*
(1) every buyer $v \in V$ purchases a variety as time diverges to infinity;
(2) if $\underline{v}_g > 0$, every buyer $v \in V$ purchases a variety in finite time.

The monopolist eventually sells to all buyers in any equilibrium, because otherwise, instantaneously clearing the market would be profitable whenever the measure of active buyers is close to a limit. When the measure of active buyers is small and $\underline{v}_g > 0$, the monopolist benefits from clearing the market instantaneously. This is the case because only buyers with similar valuations remain active as time elapses, and because the loss caused by discounting future profits outweighs any possible price discrimination when buyers are similar and $\underline{v}_g$ is strictly positive. However, despite equilibrium profits being positive in any support, the market does not need to clear instantaneously when $\underline{v}_g = 0$ even when all active buyers are similar. To see this, let $\delta = 0$ and consider a uniform measure on the support $V = [0, \varepsilon]^2$. In such settings, the market cannot clear instantaneously, regardless of the value of ε.[20] However, in contrast to the one-variety case, $\underline{v}_g = 0$ no longer implies that the market cannot clear instantaneously. In particular, the market would clear instantaneously if $\delta = 1/2$ and the measure was uniform on $V = [0, \varepsilon] \times [0, 1]^2$ for some sufficiently small ε. If so, the seller secures a payoff close to the optimal commitment profit by clearing the market instantaneously, but it necessarily loses some surplus due to discounting when deferring trade with some buyers of variety b.

[19] This is a common feature of many bargaining models, dating back to Rubinstein's 1982 seminal work.

[20] In particular, in any perfect Bayesian equilibrium, for any variety $i \in \{a, b\}$, the seller sets price $p_i^0 = \varepsilon/\sqrt{3}$ at date zero and price $p_i^t = p_i^{t-1}/\sqrt{3}$ at any date $t > 0$ on the equilibrium path.

As in the single-variety case, it is possible to show that perfect Bayesian equilibria exist and that at least one of these equilibria is weakly Markovian. The next result proves directly the existence of a weak Markov equilibrium, which implies the existence of perfect Bayesian equilibria. The proof applies also to the no-gaps case, $\underline{v}_g = 0$.

Proposition 1 *If the market is regular, a weak Markov equilibrium exists.*

The proof strategy is classical and evocative of the single-variety case. When there are gaps, the equilibrium is finite, and thus backward induction and a suitable variant of the Kakutani-Fan-Glicksberg fixed point theorem suffices to establish existence. When there are no gaps, the equicontinuity of the equilibrium correspondence is further exploited to deliver the result. Stationary equilibria are not necessarily unique in multi-variety settings, since optimal market-clearing prices are not unique in general.

In contrast to classical results for the single-variety case, stationary mixed strategy equilibria may exist in which the seller randomizes along the equilibrium path (and not just in the initial period). This is the case because the monopolist may benefit from concealing future price reductions if buyers delay purchasing those varieties that are going to be more heavily discounted. At the end of this section, we display an instance of this phenomenon for a market that clears in two periods and in which the seller randomizes in the final period, upon clearing the market.

The final result about the baseline dynamic pricing game delivers a generalization of the classical Coasian insight to multi-variety settings. In any stationary equilibrium, the seller's profit must always converge to the optimal market-clearing profit as the discount factor converges to unity. As in the single-variety setting, patience deteriorates the seller's bargaining power and decreases its equilibrium profit. Because of Lemma 2 though, the inability to intertemporally price discriminate buyers does not fully erode the seller's bargaining power when more than one variety can be sold.

Proposition 2 *If the market is regular, profits converge to optimal market-clearing profits in any weak Markov equilibrium as δ converges to* 1.

The proof establishes that when the discount factor is close to 1, prices must be close to market-clearing after any real time T. Thus, profits will be close to market-clearing as patient consumers would wait any finite amount of time for a price reduction, and consequently varieties will only ever be sold at prices that are close to market-clearing. The intuition for this result is as follows. Consider a time period t in which the demand for both products is small. A possible deviation for the monopolist in period t consists of setting prices according to its mixed strategy in period $t+1$ rather than setting the equilibrium price p^t. Such a deviation would have three effects on the profit of the monopolist. First, it would reduce profits by lowering the price paid by those who were expecting to consume a variety i at date t and continue to do so. Second, it would increase profits by anticipating the stream of future revenue on all units to be sold at later stages. Third, it would have an ambiguous effect on profits by inducing some consumers to change their demand from one product to the other. The first effect, however, is small, as price changes must be small if a patient consumer is unwilling to wait one period to purchase the product. Similarly, the third effect must be small (if positive), because the set of buyers contemplating switching varieties is a small subset of those contemplating a purchase when price changes are small (by absolute continuity). Thus, for such a deviation not to be profitable, profits must be arbitrarily small after a finite time T. If so, prices must be close to market-clearing after date T, given that equilibrium profits exceed market-clearing profits by Lemma 2 and given that optimal market-clearing profits can be small only if the measure of active buyers is small by Remark 1. If buyers are patient, the latter implies that prices must be close to market-clearing from the beginning of the game for sales to take place before date T.

As in classical settings, a monopolist lacking commitment extracts no more than optimal market-clearing profits in any stationary equilibrium when price revisions can be arbitrarily frequent. However, stationary pricing without commitment only amounts to optimal market-clearing and global agreement, not to minimal pricing or efficiency.

On-Path Mixing Example: We conclude the section by constructing a stationary equilibrium in which the seller randomizes in the final period. Consider an atomic measure of buyers with support

$$V = (1,1) \cup \left\{ v \in [0,1]^2 \mid v_j = (1-v_i)/3 \text{ for any } v_i \in [1/4,1] \ \& \text{ any } i \in \{a,b\} \right\}.$$

The dark blue region in the left panel of Figure 4 depicts this support. Although the measure fails regularity, a similar conclusion would hold in the regular market in which the support is the convex hull of V (the light blue shaded region in the left plot of Figure 4) and in which almost all of the measure is on V. Consider the following joint distribution on V:

$$F(v) = \begin{cases} 1 & \text{if} \quad v_i = 1 & \& \quad v_j = 1 \\ (6v_i + 6v_j - 3)/10 & \text{if} \quad v_i \in [1/4,1] & \& \quad v_j \in [1/4,1) \\ (18v_j + 6v_i - 6)/10 & \text{if} \quad v_i \in [1/4,1] & \& \quad v_j \in [1-3v_i, 1/4] \\ 0 & \text{if} \quad \text{otherwise} \end{cases}.$$

Intuitively, such a distribution has $1/10$ of the measure on the atom at $(1,1)$, while $9/10$ of the measure is uniformly distributed on the other component of V.

Optimal market-clearing profits in this market amount to 0.272 (approximately) and can be secured via two symmetric market-clearing price profiles, with one variety sold at 5/12 and the other at 7/36. If instead the seller had the ability to commit to a price profile, it would optimally sell both varieties at a price of 13/24 and raise a profit of 0.352 (approximately). As in the one-variety case, the seller's bargaining power is diminished by its inability to commit to the price path (at least for sufficiently high values of δ). However, optimal market-clearing profits exceed both the minimal value of the durable good, as $\underline{v}_g = 0.25$, and the minimal value of each variety, as $\underline{v}_a = \underline{v}_b = 0$. Intratemporal price discrimination partly offsets the inability to intertemporally price discriminate.

(1) Support (2) Buying Regions

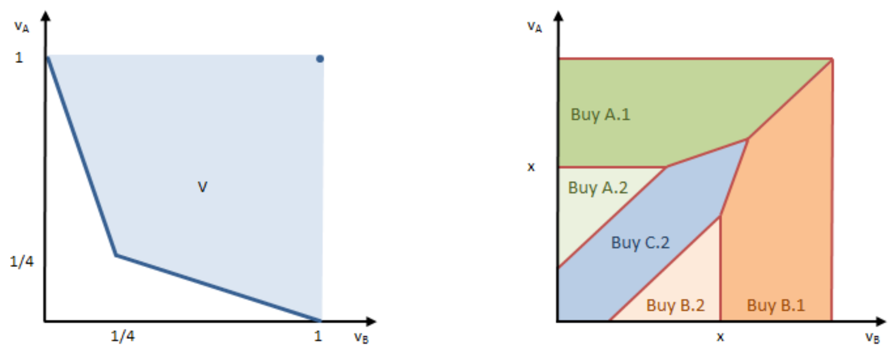

Figure 4: On the left, the support of a measure with late mixing; on the right, values partitioned into buying regions. In each region, the letter stands for the variety purchased (where c denotes the cheapest variety) and the number stands for the date of purchase.

When $\delta = 3/4$, the stationary equilibrium that maximizes the payoff of the seller in the dynamic pricing game entails stochastically clearing the market in exactly two periods. In this equilibrium, the monopolist sells both varieties in the first period at a price equal to 311/864, and it clears the market in the second period by setting one of two market-clearing price profiles, $(73/216, 143/648)$ and $(143/648, 73/216)$, with equal probability. By doing so, the seller secures a profit of approximately 0.292. The right panel of Figure 4 partitions values into buying regions for a mixed strategy equilibrium in which the market clears in two periods.

Mixed strategy equilibrium profits exceed the profit that the seller can secure in any stationary pure strategy equilibrium, which only amounts to approximately 0.288. The loss of profit stems from the following intuition. When $\delta = 3/4$, in all stationary equilibria, the market clears in two periods, and buyers with value $(1, 1)$ necessarily purchase in the first period. But if so, buyers with value $(1, 1)$ are unwilling to pay much more than the lowest market-clearing price set in the second period in any pure equilibrium, or the average market-clearing price set in the second period in any mixed equilibrium. For suitably chosen values of δ, this effect depresses the price that can be charged for the cheaper variety in the first period in a pure strategy equilibrium, implying that clearing the market stochastically can benefit the seller. The details of this example are reported in the online appendix.

5 Extensions: Costs and Market Exit

Positive Marginal Costs: The key insights discussed in the previous sections carry over to settings with strictly positive marginal costs. However, a few significant differences arise. In general, our notion of market-clearing only required that gains from trade be depleted. Consequently, market-clearing no longer requires selling to all buyers when marginal costs are positive. Rather, it requires selling a variety at the current prices to all buyers who value at least one variety more than its marginal cost. In particular, denote by V^+ the set of values with positive gains from trade:

$$V^+ = \left\{ v \in V \mid \max_{i \in \{a,b\}} \{v_i - c_i\} \geq 0 \right\}.$$

When marginal costs are positive, the set of market-clearing prices then amounts to

$$M^+ = \left\{ p \in \mathbb{R}^2 \mid \max_{i \in \{a,b\}} \{v_i - p_i\} \geq 0 \text{ for all } v \in V^+ \right\}.$$

A price p in the interior of the support V can now clear the market, but no price in the interior of V^+ clears the market. As displayed in Figure 5, any price $p \leq c$ clears the market even when it is interior to the support V. Market-clearing prices (even optimal ones) may display *cross-subsidization*, which amounts to selling one variety below marginal cost while selling the other above marginal cost. Figure 5 depicts such an instance.

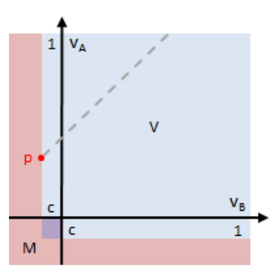

Figure 5: In pink, market-clearing prices outside V; in purple, market-clearing in the interior of V; and in blue, the rest of the support V.

When costs are positive, optimal market-clearing profits no longer need to be strictly positive. When the minimal valuation of each product within V^+ is much below its marginal cost, it may be hard to clear the market at a positive profit. The next remark provides a simple sufficient condition for optimal market-clearing profits to be strictly positive. As before, let $\underline{v}_i^+$ denote the *minimal value* for variety i in the support V^+. With costs, we say that *varieties are unranked* if not all buyers weakly prefer one variety to the other when prices equal marginal costs (that is, if for any i there is a $v \in V$ such that $v_i - c_i > v_j - c_j$).

Remark 3 *Optimal market-clearing profits are strictly positive if varieties are unranked and* $\underline{v}_i^+ \geq c_i$ *for some variety* $i \in \{a, b\}$.

The remark intuitively holds, because it is always possible to clear the market and make positive profits by setting prices $p_i = c_i$ and $p_j > c_j$ whenever varieties are unranked.

Lemma 2 immediately extends to settings with positive marginal costs, as its proof did not impose much discipline on the seller's preferences. Thus, even with positive costs, equilibrium profits remain bounded below by optimal market-clearing profits. Furthermore, as in Proposition 2, stationary equilibrium profits still converge to optimal market-clearing profits as δ converges to 1. As before, when δ is close to 1, the measure of active buyers will be arbitrarily small after any finite time T (for the seller not to profitably deviate by selling units sooner), and profits will not exceed by much the static optimal market-clearing profits (when buyers are patient). We summarize the two key Coasian observations in the following remark.

Remark 4 *If the market is regular, optimal market-clearing profits:*
(1) are a lower bound on perfect Bayesian equilibrium profits;
(2) coincide with the limit of weak Markov equilibrium profits as δ *converges to* 1.

Relaxing the Permanent Exit Assumption: It may seem that our interpretation of Coase's seminal result as market-clearing relies on the assumption (implicit in some of the literature) that buyers permanently exit the market upon purchasing a variety. Such an assumption is without loss: (i) if every buyer purchases its preferred variety; or (ii) if goods are consumed when purchased thereby dissipating their need forever; or (iii) if players commit to stay out of the market upon purchasing the good. The first scenario is not so uncommon when the measure is symmetric (for instance, for discordant symmetric distributions). In those circumstances, pricing in the baseline model may be efficient (as varieties are always sold at the same price) and may eventually clear the market while strictly exceeding minimal values. The second scenario is compelling for goods that are durable, but that are consumed once purchased (such as many services). After all, in these models, durability simply amounts to sales permanently depleting the demand for the good. In other markets, however, it may be more plausible to assume that buyers remain active in the market until they purchase their preferred variety. If so, they may scrap the variety they purchased in an earlier round once their preferred variety is sufficiently cheap.

To analyze this setting, we postulate that when a buyer $v \in V$ purchases any variety $i \in \{a, b\}$ of the durable good, their value for each variety transitions to

$$v_i' = 0 \text{ and } v_j' = v_j - v_i.$$

Thus, upon purchasing a variety, the value of that variety fully depletes, whereas the value for the other variety amounts to the difference between the two original values. The latter is natural, as the change in value from scrapping variety i to purchase j amounts to $v_j - v_i$.

As pointed out in the section on costs, our notion of market-clearing simply amounts to full depletion of gains from trade. Applying this definition to settings in which buyers remain active upon making a purchase changes the shape of the market-clearing set as follows:

$$M^* = \left\{ p \in M \mid v_i - p_i \geq v_j - p_j \;\Rightarrow\; c_j \geq v_j - v_i \text{ for all } v \in V^+ \right\}.$$

This definition states that a price clears the market if: (i) every buyer purchases a variety; and (ii) the marginal cost of supplying variety j to any buyer purchasing variety i exceeds their change in value. Therefore, the market must clear whenever the change in price is smaller than the cost of every variety, or formally

$$M^* \supseteq \left\{ p \in M^+ \mid -c_b \leq p_a - p_b \leq c_a \right\}.$$

Moreover, the latter must hold with equality whenever $c_i \leq \max_{v \in V^+} v_i - v_j$ for every variety i. Market-clearing prices will thus be efficient (as $p_a = p_b$) when the marginal cost of each product equals zero, but not otherwise. As before, provided that $v \geq c$ for all values in the support V, optimal market-clearing profits equal zero if and only if products are identical and there are no gaps (that is, $c \in V$). More generally, as in the previous part of the section, optimal market-clearing profits are strictly positive if varieties are unranked and $\underline{v}_i^+ \geq c_i$ for some variety $i \in \{a, b\}$.

Even in this setting, equilibrium profits remain bounded below by static optimal market-clearing profits, since the argument establishing Lemma 2 readily applies to all prices in M^*. As was the case with permanent exit, stationary equilibrium profits remain uniquely pinned down by static optimal market-clearing profits when price revisions are instantaneous. The intuition is again similar to that of Proposition 2, and it relies on the measure of switchers remaining small when players are patient and prices are close to market-clearing. We summarize these conclusions in the following remark, which is proven in the online appendix.

Remark 5 *If buyers remain active and the market is regular, optimal market-clearing profits:*
(1) are a lower bound on perfect Bayesian equilibrium profits;
(2) coincide with the limit of weak Markov equilibrium profits as δ converges to 1*.*

When marginal costs equal zero and players remain active, limiting stationary equilibria are efficient, as all buyers eventually purchase the preferred variety when prices belong to M^* (this is the case in the top three panels of Figure 6). In such efficient limiting stationary equilibria, varieties are not necessarily sold at their minimal value, but rather at the minimal value of the durable good. For instance, with discordant valuations, there can be scenarios in which pricing is efficient and in which every variety is sold above its minimal value (see the top right panel of Figure 6).

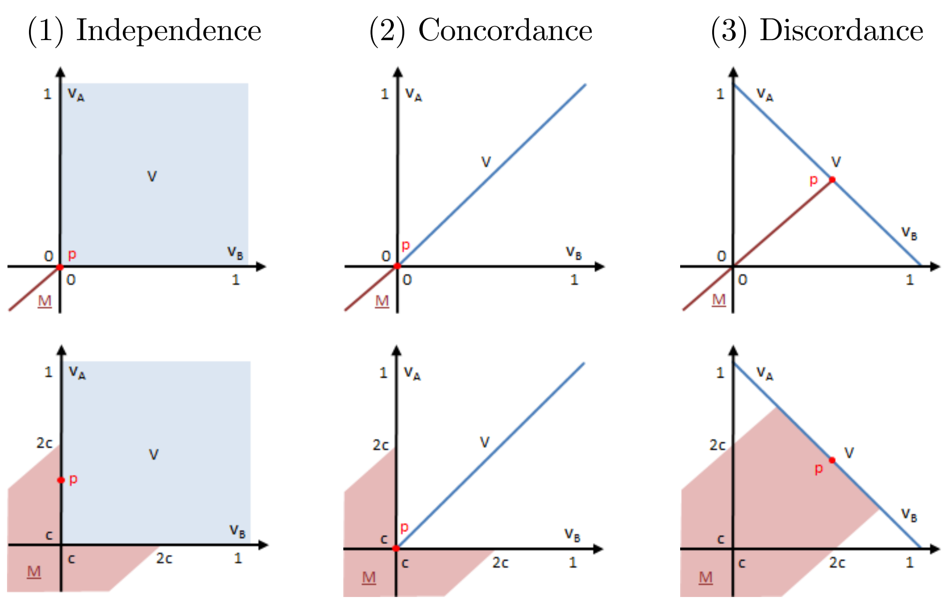

Figure 6: Market-clearing set without exit, $\underline{M}$, for three possible distributions. The panels at the top focus on $c_a = c_b = 0$, and those at the bottom on $c_a = c_b = c > 0$.

When marginal costs are positive though, results are closer to the baseline setting in which buyers commit to exit. The seller retains its ability to intratemporally price discriminate, and efficiency is seldom obtained since marginal costs prevent the seller from undercutting in order to supply buyers who were initially sold the inefficient variety (this is the case in the bottom three panels of Figure 6). This logic offers a novel rationale for selling high marginal cost varieties, namely, intratemporal price discrimination in durable goods markets.

6 Classical Results and Coasian Failures

Relationship to Classical Coasian Results: The paper discussed a dynamic monopoly problem in which multiple varieties of a product were produced and sold. It extended classical conclusions on equilibrium pricing and established that in any robust generalization of Coasian results, static optimal market-clearing would play a role similar to that of the minimal valuation in the single-variety case. This insight considerably simplified the analysis of the dynamic game and enabled meaningful generalizations of classical Coasian dynamics. With more than one variety, intratemporal price discrimination was shown to make up at least in part for the absence of intertemporal price discrimination caused by the lack of commitment. Although the paper was presented for two varieties, analogous conclusions would be obtained with more than two varieties.

The table below summarizes classical contributions on dynamic monopoly pricing with one variety, highlighting which conclusions are specific to this scenario and which generalize to multi-dimensional settings. In the table, "OMC" stands for optimal market-clearing profit, "WME" means there exists a weak Markov equilibrium with the desired property, "Both" means time to clear can be finite or infinite, and "Minimal Limit Profits" means all goods are traded at their minimal value in V in the limit as $\delta \to 1$.

Number of Varieties	Single		Multiple		
Gaps	No	Yes	No		Yes
OMC	0	+	0	+	+
Market Clearing	Yes	Yes	Yes	Yes	Yes
Bound on PBE Profit	OMC	OMC	OMC	OMC	OMC
Limit WME Profit	OMC	OMC	OMC	OMC	OMC
Time to Clear	Infinite	Finite	Infinite	Both	Finite
Efficiency	Yes	Yes	WME	Rare	Rare
Minimal Limit Profits	WME	Yes	WME	No	No
PBE Late Mixing	No	No	–	Yes	Yes
PBE Uniqueness	No	Yes	No	No	No

We would like to argue that the three consistent phenomena across Coasian settings are: (i) eventual market-clearing; (ii) optimal market-clearing providing a lower bound on equilibrium profits; and (iii) optimal market-clearing identifying stationary equilibrium profits. Thus, one could consider these three aspects as the essence of the Coase conjecture. Other phenomena, meanwhile, are not robust, in that they depend on the specific assumptions invoked on the durable goods environment. These phenomena include: the time it takes for the market to clear; the efficiency of equilibrium pricing; whether goods are eventually sold at minimal valuations; whether mixing can take place after the initial period; and equilibrium uniqueness.

Multiplicity of perfect Bayesian equilibria contrasts with the uniqueness result obtained in the single-variety case with gaps. Multiplicity naturally arises when more than one variety is for sale because optimal market-clearing prices need not be unique. This observation alone does not imply that a Folk Theorem holds as in Ausubel and Deneckere 1989. Their seminal contribution shows how to construct a Folk theorem when a single variety is sold and there are no gaps. In such settings, the seller can extract the full static monopoly surplus by following a strategy with a slow price descent. Such a strategy is incentive compatible for the seller if consumers' beliefs about future prices revert to the stationary equilibrium path upon observing any deviation. However, for the slow price descent to be incentive compatible, the stationary equilibrium limit profit must equal 0, and thus there must be no gaps. In multi-variety settings, it is unclear whether the no-gaps assumption would suffice to deliver a full-fledged Folk theorem, as equilibrium profits may be strictly positive even when there are no gaps.[21] If a Folk theorem were to hold, our analysis would identify the lowest perfect Bayesian equilibrium profit and the stationary limit payoff.

As usual, it is possible to interpret our setting as a two-player model of bargaining with one-sided incomplete information in which the uninformed party always proposes. In this interpretation, varieties would amount to alternative prospects that the proposer could offer to the receiver to screen their type. If so, our conclusions would establish that the uninformed party regains some bargaining power by statically screening consumers, since it can extract surplus even if it has to agree with every possible type of the informed player. Our bargaining interpretation of the Coase conjecture would then amount to immediate agreement in limiting stationary equilibria and would essentially coincide with optimal market-clearing. If players were to stay in the market upon purchasing the product, agreement would have to be renegotiation-proof to guarantee that no player would want to switch varieties at any price exceeding marginal cost.

Approximating stationary equilibrium profits (with frequent price revisions) with optimal market-clearing may not just amount to a theoretical curiosity. Instead, such an approximation could in principle deliver a concrete stepping-stone to inform applied research on durable goods pricing and to develop product design implications for such markets.

Relationship to Some Coasian Failures: Our analysis is closely related to some known violations of the classical Coase conjecture. Board and Pycia 2014 considers a durable goods monopoly problem in which buyers have the option to commit to stay out of the market by taking an outside option. The outside option amounts to a second variety of the durable

[21] We defer the full-blown analysis of non-stationary equilibria to future work.

good that must be sold at a price of zero. They consider settings in which the value of the outside option is strictly positive for all players (there are gaps) and independent of the value of the durable good (the left plot of Figure 7 depicts such an environment). Their main contribution establishes that the monopolist sets a strictly positive price for the durable good and never undercuts the initial price as the market clears at once. In our setting, their result holds immediately by Lemma 2. Since the price of the outside option is zero, any price for the durable good is a market-clearing price. Thus, the monopolist would never undercut. Furthermore, this holds even without gaps, and even with an arbitrary correlation structure. Of course, setting the price of the outside option to zero would be suboptimal in our environment, as any such price profile would belong to the interior of the market-clearing price set. Still, in our view, Board and Pycia's novel contribution should not be classified as a failure of the Coase conjecture. Rather, our results aim to highlight that the essence of the Coasian intuition is market-clearing, and not necessarily minimal pricing, zero profits, or efficiency. Similar considerations apply to Wang 1998, who establishes a result evocative of Board and Pycia in the context of a two-type model.

Likewise, Hahn 2006 expands on classical conclusions by showing that selling damaged products can increase the profit of a durable goods monopolist. A damaged product acts like a second variety with a lower value. In particular, their analysis considers settings in which the valuations of the two varieties are binary and perfectly correlated (the right plot of Figure 7 depicts such an environment). Similar conclusions hold in our setting independently of the joint measure of valuations. However, these are again not failures of the Coase conjecture, but rather its essence, as limit profits again amount to optimal market-clearing profits in our formulation of the problem. Other similarly classified Coasian failures fit this bill.

(1) Outside Options (2) Vertical Differentiation

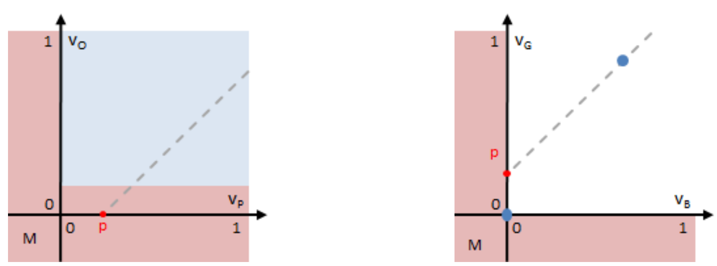

Figure 7: The left plot depicts the environment studied in Board and Pycia 2014; the right plot depicts alleged Coasian failures with vertical differentiation.

Our conclusions hold even when no buyer values one of the two varieties. In essence, if the monopolist could pay buyers a penny (or any small amount) to permanently exit the market, the Coasian profit would no longer amount to the smallest valuation in the support. Instead, the monopolist would be able to approximately extract the full monopoly profit as any price would clear the market. Hence, Coasian dynamics would be inessential if the seller was able trade products of low value that would commit the buyers to exit market. If the penny did not deplete buyers' demand for the other variety, however, the seller's payoff would instead amount to the smallest valuation (provided that producing the penny was costless). In such scenarios, high marginal cost varieties would be necessary in order to prevent undercutting by future selves and to sustain positive profits, as shown in Section 5.

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7 Appendix

Proof Lemma 1. Consider any PBE and any buyer-history $h^t \in \hat{H}^t$. To establish (1), observe that, since v strictly prefers buying variety i,

$$v_i - p_i > \max\{v_j - p_j, \delta U(h^t, v)\},$$

where $U(h^t, v)$ denotes the equilibrium continuation value of player v at date $t+1$ after history h^t. As buyer v can mimic the strategy of buyer v' from period $t+1$ onwards (by accepting and rejecting the very same offers) and since $v'_j - v_j < v'_i - v_i$, it follows that

$$U(h^t, v') - U(h^t, v) \leq \sum_{s=0}^{\infty} \delta^s \left[\sum_{k\in\{a,b\}} \alpha_k^s(h^t, v')\,(v'_k - v_k)\right] \leq v'_i - v_i,$$

where $\alpha_j^s(h^t, v')$ denotes the probability conditional on h^t that variety j is purchased by v' at time $t+s+1$. But if so, buyer v' strictly prefers buying variety i, and part (1) follows, since

$$\begin{aligned} v'_i - p_i = v_i - p_i + (v'_i - v_i) &> \max\{v_j - p_j, \delta U(h^t, v)\} + (v'_i - v_i) \\ &\geq \max\{v'_j - p_j, \delta U(h^t, v')\}. \end{aligned}$$

To prove (2), similarly observe that, since buyer v weakly prefers to buy a variety,

$$\max_{i\in\{a,b\}}\{v_i - p_i\} \geq \delta U(h^t, v).$$

As buyer v can mimic the strategy of buyer v' from period $t+1$ onwards, it follows that

$$U(h^t, v') - U(h^t, v) \leq \max_{i\in\{a,b\}} \{v'_i - v_i\}.$$

But if $\delta \max_i\{v_i - v'_i\} < \min_i\{v_i - v'_i\}$, then buyer v' strictly prefers buying a variety, since

$$\begin{aligned} \max_i\{v'_i - p_i\} &\geq \max_i\{v_i - p_i\} + \min_i\{v'_i - v_i\} \\ &> \delta U(h^t, v) + \delta \max_{i\in\{a,b\}} \{v'_i - v_i\} \geq \delta U(h^t, v'). \end{aligned}$$

Similarly, to prove (3), observe that, since buyer v weakly prefers not to buy any variety,

$$\max_{i\in\{a,b\}}\{v_i - p_i\} \leq \delta U(h^t, v),$$

As buyer v' can mimic the strategy of buyer v from period $t+1$ onwards, it follows that

$$U(h^t, v) - U(h^t, v') \leq \max_{i\in\{a,b\}} \{v_i - v'_i\}.$$

But if so, buyer v' strictly prefers not buying any variety, as

$$\begin{aligned} \max_i\{v'_i - p_i\} &\leq \max_i\{v_i - p_i\} - \min_i\{v_i - v'_i\} \\ &< \delta U(h^t, v) - \delta \max_i\{v_i - v'_i\} \leq \delta U(h^t, v'). \end{aligned}$$

Also, note that (2) immediately implies (4). This follows because on any ray $v_a = v_b + k$ there exists a cut-off valuation identifying the marginal buyer, and because indifferent consumers have measure zero when the market is regular. ■

Proof Lemma 2. To prove the result, it suffices to show that, in any PBE, all consumers accept any price in $\bar{M}$ at any information set. Suppose this were not the case. Select any equilibrium, and let P denote the set of prices that will be accepted by all buyers in any possible subgame:

$$P = \left\{ p \in \mathbb{R}^2 \;\middle|\; \max_{i \in \{a,b\}} \{v_i - p_i\} > \delta U((h,p),v) \;\text{ for all }\; (h,v) \in H \times V \right\}.$$

By contradiction, suppose that $\bar{M}$ is not contained in P (that is, $\bar{M} \backslash P \neq \emptyset$). Observe that $p \in P$ whenever $\min_{i \in \{a,b\}} p_i < -1$. To show the latter, observe that the proof of Lemma 1 implies that the buyers' value functions at any buyer-history $\hat{h} \in \hat{H}$ are non-decreasing in v and have modulus of continuity less than 1, since for $v' \geq v$,

$$U(\hat{h}, v') - U(\hat{h}, v) \leq \max_{i \in \{a,b\}} \{v'_i - v_i\}.$$

But then, in any PBE we have that $U(\hat{h}, v) \leq 1$ for all $v \in V$. This in turn implies that all buyers strictly prefer to purchase a variety of the durable good when $\min_{i \in \{a,b\}} p_i < -1$, as

$$\max_{i \in \{a,b\}} \{v_i - p_i\} > 1 > \delta U(\hat{h}, v) \;\text{ for all }\; \hat{h} \in \hat{H}. \tag{2}$$

As $\min_{i \in \{a,b\}} p_i < -1$ implies $p \in P$, for any $\varepsilon > 0$ there is a price $\hat{p} \in \bar{M} \backslash P \neq \emptyset$ such that:

$$\begin{array}{ll} \text{(i)} & p \leq \hat{p} - (\varepsilon, 0) \;\text{ implies }\; p \in P; \\ \text{(ii)} & p \leq \hat{p} - (0, \varepsilon) \;\text{ implies }\; p \in P. \end{array}$$

To find such a price $\hat{p}$, let $\tilde{p}_a = \inf_{q \in \bar{M} \backslash P} q_a$, and for some $\eta \in (0, \varepsilon)$, let

$$\tilde{p}_b = \inf_{q \in \bar{M} \backslash P} q_b \;\text{ s.t. }\; q_a \leq \tilde{p}_a + \eta,$$

where $\min_{i \in \{a,b\}} \tilde{p}_i \geq -1$ by (2). Then set a $\hat{p}$ to be any price in $\bar{M} \backslash P$ such that $\hat{p} \leq \tilde{p} + (\eta, \eta)$. Such a price must exist by definition of $\tilde{p}$ for all sufficiently small η. Moreover, (i) holds since $p \in P$ when $p_a \leq \hat{p}_a - \varepsilon < \tilde{p}_a$, by definition of $\tilde{p}_a$; while (ii) holds since $p \in P$ for any $p_a \leq \hat{p}_a$ when $p_b \leq \hat{p}_b - \varepsilon$, by definition of $\tilde{p}_b$.

But, when ε is sufficiently small,

$$\max_{i \in \{a,b\}} \{v_i - \hat{p}_i\} > \delta \max_{i \in \{a,b\}} \{v_i - \hat{p}_i + \varepsilon\} \quad \Leftrightarrow \quad \varepsilon < \frac{1-\delta}{\delta} \max_{i \in \{a,b\}} \{v_i - \hat{p}_i\}.$$

If so, all consumers would accept $\hat{p}$ at any seller-history $h \in H$. If a type were to reject an offer, they could agree no sooner than tomorrow, and the most they could expect any one price to drop is ε as any further drop would lead to acceptance by all buyers. Thus, for all $v \in V$,

$$\max_{i\in\{a,b\}} \{v_i - \hat{p}_i + \varepsilon\} \geq U((h, \hat{p}), v).$$

But, this in turn would imply that

$$\max_{i\in\{a,b\}} \{v_i - \hat{p}_i\} > \delta \max_{i\in\{a,b\}} \{v_i - \hat{p}_i + \varepsilon\} \geq \delta U((h, \hat{p}), v) \text{ for any } h \in H.$$

As $\hat{p} \notin P$, the latter contradicts the definition of P and consequently establishes (1) and (2). Because every consumer buys when prices belong to $\bar{M}$, the seller can secure a payoff arbitrarily close to the optimal market-clearing profits $\bar{\pi}(A) > 0$ (where $A = A(h)$ denotes the active player set associated with history h) by choosing a price in $\bar{M}$. Part (3) then follows. ■

Proof Lemma 3. To prove (1), fix a PBE. Let $A^t = A(h^t)$ denote the support of the measure of active players associated with a history $h^t \in H$ of length t. Suppose that there exists a history $h^s \in H$ with $\mathcal{F}(A^s) > 0$ such that

$$\mathcal{F}(A^s) - \mathcal{F}(A^t) < \eta$$

for any active player set A^t that may arise with positive probability at any date $t > s$ as a result of equilibrium play after history h^s. At such a history, the equilibrium profit of the seller must be bounded by $\Pi(h^s) < \eta$, as no variety is ever sold at a price higher than 1 (the highest value in the initial support). As optimal market-clearing profits are strictly positive whenever $\mathcal{F}(A^s) > 0$ though, $\bar{\pi}(A^s) > \eta$ for η sufficiently small. But if so, a contradiction would emerge as the seller would prefer to immediately clear the market:

$$\Pi(h^s) < \eta < \bar{\pi}(A^s).$$

Thus, when $\eta < \bar{\pi}(A^s)$, there always exists a continuation-history h^t that occurs with positive probability on the equilibrium path such that

$$\mathcal{F}(A^s) - \mathcal{F}(A^t) > \eta.$$

The latter however implies that for any $\varepsilon > 0$, there exists a T sufficiently long such that, at any history $h^T \in H$ consistent with equilibrium play,

$$\mathcal{F}(A^T) < \varepsilon.$$

To prove that the market always clears in finite time when $\underline{v}_g > 0$, consider any sequence of sets $\{A^t\}_{t=0}^{\infty}$ satisfying $A^{t+1} \subseteq A^t \subseteq V$. Denote by A^{∞} the limit of this sequence, $A^{\infty} = \cap_{t=0}^{\infty} A^t$. We begin by establishing a preliminary result, proven in the online appendix, which shows that price discrimination gains must become small for at least one of the two varieties.

Remark 6 *If A^t satisfies Lemma 1 for any $t \geq 0$ and $\mathcal{F}(A^{\infty}) = 0$, then for all $\varepsilon > 0$, there exists a T sufficiently large such that $|v_i - v_i'| \leq \varepsilon$ for some i and for all $v, v' \in A^T$.*

For any set $A \subseteq V$, define $\underline{v}_g(A) = \min_{v \in A} v_g$ and $\overline{v}_g(A) = \max_{v \in A} v_g$. Fix an equilibrium. Consider any infinitely long history h^{∞} consistent with equilibrium play. Next we establish that $\overline{v}_g(A^{\infty}) = \underline{v}_g(A^{\infty})$. If this were not the case, the previous arguments would imply that $\overline{v}_j(A^{\infty}) = \overline{v}_g(A^{\infty})$, where j denotes the variety with non-negligible price discrimination gains. But, if so, for any $\varepsilon > 0$ there must exist a t sufficiently large and a sub-history h^t of h^{∞} with $A^t = A(h^t)$ such that:

$$\text{(a)}\ \left|\overline{v}_j(A^t) - \overline{v}_g(A^{\infty})\right| < \varepsilon;\ \text{(b)}\ \left|\overline{v}_i(A^t) - \underline{v}_i(A^t)\right| < \varepsilon;\ \text{(c)}\ 1 - F_j(\overline{v}_g(A^{\infty})|A^t) < \varepsilon.$$

If so, however, a contradiction emerges for ε sufficiently small, since

$$\begin{aligned}
\Pi\left(h^t\right)/\mathcal{F}(A^t) &\leq (1 - F_j(\overline{v}_g(A^{\infty})|A^t))\overline{v}_j(A^t) + F_j(\overline{v}_g(A^{\infty})|A^t)\overline{v}_i(A^t) \\
&\leq \varepsilon\overline{v}_j(A^t) + (1-\varepsilon)\overline{v}_i(A^t) \leq \varepsilon\overline{v}_g(A^{\infty}) + (1-\varepsilon)\underline{v}_i(A^t) + \varepsilon \\
&< \max_{p_j}(1 - F_j(p_j|A^t))p_j + F_j(p_j|A^t)\underline{v}_i(A^t) \leq \bar{\pi}(A^t)/\mathcal{F}(A^t).
\end{aligned}$$

The first inequality holds as the seller instantaneously sells all products at the highest possible value, given that buyers with $v_j < \overline{v}_j(A^{\infty})$ never purchase variety j by Lemma 1. The second and third inequalities are immediate consequences, respectively, of (c) and $\overline{v}_j(A^t) > \overline{v}_i(A^t)$, and of (a) and (b). The fourth inequality relies on the fact that when ε is sufficiently small, the optimal $p_j \in (\underline{v}_i, \overline{v}_g(A^{\infty}))$, as by choosing such a price it is possible to bound profits away from $\underline{v}_i(A^t)$. The final inequality is trivial and relies on the fact that we have checked for maximized profits only on a subset of M. Thus, for any $\varepsilon > 0$, there exists a T^* such that $\overline{v}_g(A^s) - \underline{v}_g(A^s) < \varepsilon$ for all $s \geq T^*$.

As $\underline{v}_g > 0$, by Lemma 1 we know that $\underline{v}_g(A) > 0$ for any $A \subseteq V$. Next, we show that whenever $\overline{v}_g(A) - \underline{v}_g(A) \leq \varepsilon$, the seller prefers to immediately clear the market. Denote the residual surplus given A by

$$S(h) = \textstyle\int_A \max\{v_a, v_b\}\, dF(v|A) \leq \overline{v}_g(A)\mathcal{F}(A).$$

If so, there exists an $\alpha \in (0,1)$ such that

$$\Pi(h) < \alpha S(h) + \delta(1-\alpha)S(h)$$

as the seller cannot instantaneously extract the full surplus with a linear price. Since we also have that $\bar{\pi}(A) \geq \underline{v}_g(A)\mathcal{F}(A)$ by Remark 1, it follows that

$$\begin{aligned}\Pi(h) - \bar{\pi}(A) &< \alpha S(h) + \delta(1-\alpha)S(h) - \underline{v}_g(A)\mathcal{F}(A) \\ &= (\alpha+\delta-\alpha\delta)S(h) - \underline{v}_g(A)\mathcal{F}(A) \leq \mathcal{F}(A)\left[(\alpha+\delta-\alpha\delta)\overline{v}_g(A) - \underline{v}_g(A)\right] \\ &= \mathcal{F}(A)\left[\gamma\overline{v}_g(A) - \underline{v}_g(A)\right] \leq \mathcal{F}(A)\left[\gamma\varepsilon - (1-\gamma)\underline{v}_g(A)\right],\end{aligned}$$

where $\gamma = \alpha+\delta-\alpha\delta \in (0,1)$ for all $\alpha, \delta < 1$. However, we cannot have that $\Pi(h) < \bar{\pi}(A)$, by Lemma 2, and so the market clears instantaneously whenever

$$\varepsilon \leq \underline{v}_g(A)\,(1-\gamma)\,/\gamma.$$

Thus, if $\underline{v}_g > 0$, there exists a period T^* such that all buyers purchase a variety before date T^* in any PBE, given that $\overline{v}_g(A^s) - \underline{v}_g(A^s) < \varepsilon$ for all $s \geq T^*$. ■

Proof Proposition 1. To begin, we assume that there are gaps, and so $\underline{v}_g > 0$. If so, by Lemma 3, in any PBE there exists a time T such that a measure $\mathcal{F}(V)$ of buyers purchases a variety of the durable good before date T. For all values of $z \in \mathbb{N}$, we inductively construct an equilibrium, and we prove its existence in a corresponding game in which, from some date z onwards, the seller must forever set prices in M. Then we argue that this establishes the existence of a WME even in unrestricted games, provided that $T < z$. Finally, we establish that the proof generalizes to the $\underline{v}_g = 0$ case by equicontinuity.

Let $\mathcal{K}(V) = \{A \subseteq V \mid A \text{ is non-empty and compact}\}$. Let $s \in \{0, ..., z\}$ denote the number of periods before prices must belong to M. While proving existence, we allow buyers' beliefs (and thus the seller's strategy) to depend on the time to market-clearing, s. We then show that this is without loss as current prices fully pin down the time it takes for the market to clear. When $s = 0$, the seller must instantaneously set $p \in M$. For any mixed strategy set by the seller $\rho \in \mathcal{P}(M)$, denote the expected payoff of a buyer with value v when $s = 0$ by

$$U^0(\rho, v) = \int_M \max_i \{v_i - p_i\}\, d\rho(p).$$

Fix any active player set $\underline{A} \in \mathcal{K}(V)$. Denote demand for variety $i \in \{a, b\}$ when $s = 0$ by

$$d_i^0(p|\underline{A}) = \mathcal{F}(v_i - p_i > \max\{v_j - p_j, 0\}|\underline{A}).$$

With a minor abuse of notation, let $pd^0(p|\underline{A}) = p_a d_a^0(p|\underline{A}) + p_b d_b^0(p|\underline{A})$. By Lemma 1, when the market is regular, the seller's beliefs about the measure $\mathcal{A}$ of active buyers are fully pinned down by the support A of the measure $\mathcal{A}$. Denote the best response of the seller who believes that only players in $\underline{A}$ are active when $s = 0$ by

$$B^0(\underline{A}) = \arg\max_{\rho \in \mathcal{P}(M)} \int_M pd^0(p|\underline{A}) d\rho(p).$$

Let $\Pi^0(\underline{A}) = \bar{\pi}(\underline{A})$ denote the value of this program, or the profit that the seller would make if the market had to clear and only buyers in $\underline{A}$ were active.

The best response correspondence $B^0(\underline{A})$ is upper-hemicontinuous[22] in $\underline{A}$ and has non-empty, compact, convex values by Berge's maximum theorem.[23] The theorem applies here because both $\mathcal{K}(V)$ and $\mathcal{P}(M)$ are Hausdorff, the objective function is continuous in both ρ and $\underline{A}$ (as $d_i^0(p|\underline{A})$ is continuous in $\underline{A}$ by regularity), and the solution belongs to $\mathcal{P}\left(M \cap [0,1]^2\right)$ which is non-empty and compact.[24] The convexity of the correspondence $B^0(\underline{A})$ follows from the linearity in ρ. For any $A \in \mathcal{K}(V)$, any $p \in [0,1]^2$, and any $\rho \in \mathcal{P}(M)$, let $\underline{A}^0(p, \rho|A)$ identify those buyers who prefer not to purchase a variety at price p when $s = 1$ if they expect prices to be drawn from ρ in the following period:

$$\underline{A}^0(p, \rho|A) = \left\{ v \in A \mid \max_i \left\{ v_i - p_i \right\} \leq \delta U^0(\rho, v) \right\}.$$

Next, observe that for any $A \in \mathcal{K}(V)$ and any $p \in [0,1]^2$, there exists a $\sigma^0 \in \mathcal{P}\left(M \cap [0,1]^2\right)$ such that

$$\sigma^0 \in B^0(\underline{A}^0(p, \sigma^0|A)). \tag{3}$$

The latter follows because $\underline{A}^0(p, \rho|A)$ is continuous in ρ, as $U^0(\rho, v)$ is linear and hence continuous in ρ. Moreover, $\underline{A}^0(p, \rho|A)$ is single-valued in the space $\mathcal{K}(A)$ as a function of ρ, and it is thus convex-valued as a function of ρ. Therefore, the correspondence $B^0(\underline{A}^0(p, \rho|A))$ has a closed graph and convex values, since B^0 is upper-hemicontinuous and has non-empty, compact, convex values. As $\mathcal{P}\left(M \cap [0,1]^2\right)$ is a non-empty, compact, convex subset of a locally convex Hausdorff space, the Kakutani-Fan-Glicksberg fixed point theorem applies.[25] Hence, equation 3 has a non-empty compact set of fixed points. For any initial price p quoted by the seller, these fixed points identify the seller's equilibrium strategy when $s = 0$ if in the previous period the price was p and the active player set was A. For any $p \in [0,1]^2$, label any such fixed point as $\sigma^0(p|A) \in \mathcal{P}(M)$. Moreover, if p and ρ are such that for some variety

[22] More specifically, the best response correspondence is upper-hemicontinuous when the Hausdorff metric is applied to its domain $\mathcal{K}(V)$.

[23] For the relevant statement of the Maximum Theorem see Aliprantis and Border 2006 page 570.

[24] $\mathcal{P}(M \cap [0,1]^2)$ is compact because $M \cap [0,1]^2$ is compact. See Aliprantis and Border 2006 page 513.

[25] For the statement of the relevant Fixed Point Theorem see Aliprantis and Border 2006 page 583.

$i \in \{a, b\}$,

$$v_i - p_i < \max\{v_j - p_j, \delta U^0(\rho, v)\} \text{ for all } v \in A,$$

then it is possible to reduce p_i to $p'_i < p_i$ while leaving the active player set unaffected. Hence, if $\sigma^0(p|A)$ is a fixed point at p, it is also a fixed point at such a price profile $p' = (p'_i, p_j)$.

Next, by induction, we show that if an equilibrium exists when prices must belong to M after $s-1$ periods, then an equilibrium also exists when prices must belong to M after s periods. Fix any $\underline{A} \in \mathcal{K}(V)$. Suppose that the seller sets price profile p when the market has to clear in at most s periods. Denote by $\beta^{s-1}(p|\underline{A})$ the buyers' beliefs about the distribution prices of in the following period. First, conjecture that an equilibrium exists in which the seller follows a strategy $\sigma^{s-1}(p)$ that is independent of $\underline{A}$ on the equilibrium path, and then verify that such an equilibrium indeed exists. If this were the case, buyers' beliefs would also be independent of $\underline{A}$ by consistency as $\beta^{s-1}(p) = \sigma^{s-1}(p)$ (where $\sigma^{s-1}(p)$ exists by induction hypothesis).

Given these beliefs, for any mixed strategy of the seller $\rho \in \mathcal{P}\left([0,1]^2\right)$, denote the expected payoff of buyer v when prices must belong to M in at most s periods by

$$U^s(\rho, v) = \int_{[0,1]^2} \max\{\max_i \{v_i - p_i\}, \delta U^{s-1}(\beta^{s-1}(p), v)\} d\rho(p).$$

Denote demand for any variety $i \in \{a, b\}$ when prices must belong to M in at most s periods by

$$d_i^s(p|\underline{A}) = \mathcal{F}(v_i - p_i > \max\{v_j - p_j, \delta U^{s-1}(\beta^{s-1}(p), v)\}|\underline{A}).$$

To maintain stationarity of the equilibrium strategy, select equilibria in which the seller only sets prices such that for each variety $i \in \{a, b\}$, there exists a buyer $v \in \underline{A}$ satisfying

$$v_i - p_i \geq \max\{v_j - p_j, \delta U^{s-1}(\beta^{s-1}(p), v)\}.$$

This is without loss when the measure is regular, because for any price p violating the condition, there is a price p' satisfying it which: (1) leads to the same set of fixed points (and so we can set $\beta^{s-1}(p'|\underline{A}) = \beta^{s-1}(p)$); and (2) raises the same profit for the seller, as $d^s(p|\underline{A}) = d^s(p'|\underline{A})$. Let $M^s(\underline{A}) \subseteq [0,1]^2$ denote the compact set of prices fulfilling this requirement.[26] In such equilibria, for any $p \in M^s(\underline{A})$ and for any truncation $A \in \mathcal{K}(V)$ fulfilling Lemma 1 such that $A \supseteq \underline{A}$, we have that

$$\underline{A}^{s-1}(p, \beta^{s-1}(p)|\underline{A}) = \underline{A}^{s-1}(p, \beta^{s-1}(p)|A).$$

[26] In the single-variety case, this would amount to the set of prices at which the active buyer with the highest value is indifferent between buying and not buying the durable good.

Hence, any price in $M^s(\underline{A})$ fully determines the active player set in the following period, and consequently buyers' beliefs about future prices do not depend on $\underline{A}$ when such prices are quoted. Clearly, buyers' beliefs must depend also on $\underline{A}$ even in a stationary equilibrium if prices outside $M^s(\underline{A})$ are quoted, because such prices no longer determine the active player set in the continuation game.

Given these beliefs, denote the best response of the seller who believes that only players in $\underline{A}$ are active when the market clears in at most s periods by

$$B^s(\underline{A}) = \arg\max_{\rho \in \mathcal{P}(M^s(\underline{A}))} \int_{[0,1]^2} p d^s(p|\underline{A}) + \delta \Pi^{s-1}(\underline{A}^{s-1}(p, \beta^{s-1}(p)|\underline{A}))d\rho(p).$$

Let $\Pi^s(\underline{A})$ denote the value of this program. The best response correspondence $B^s(\underline{A})$ is upper-hemicontinuous in $\underline{A}$ and has non-empty, compact, convex values by Berge's maximum theorem. As before, the theorem applies here because both $\mathcal{K}(V)$ and $\mathcal{P}(M^s(\underline{A}))$ are Hausdorff, the objective function is continuous in both ρ and $\underline{A}$ (as both $d_i^s(p|\underline{A})$ is continuous in $\underline{A}^{s-1}(p, \beta^{s-1}(p)|\underline{A})$), and the solution belongs to $\mathcal{P}(M^s(\underline{A}))$ which is non-empty and compact. The convexity of the correspondence $B^s(\underline{A})$ follows again from the linearity in ρ.

For any $A \in \mathcal{K}(V)$, any $\rho \in \mathcal{P}(M^s(A))$, and any $p \in [0,1]^2$, let $\underline{A}^s(p, \rho|A)$ identify those buyers who prefer not to purchase a variety at price p if they believe that prices will drawn from ρ in the following period and that the market clears in at most s periods:

$$\underline{A}^s(p, \rho|A) = \{v \in A \mid \max_i \{v_i - p_i\} \leq \delta U^s(\rho, v)\}.$$

As before, for any p and any A, there exists a $\sigma^s \in \mathcal{P}(M^s(A))$ such that

$$\sigma^s \in B^s(\underline{A}^s(p, \sigma^s|A)). \tag{4}$$

The latter follows because $\underline{A}^s(p, \rho|A)$ is continuous in ρ, as $U^s(\rho, v)$ is linear and hence continuous in ρ. $\underline{A}^s(p, \rho|A)$ is single-valued in the space $\mathcal{K}(V)$ as a function of ρ, and it is thus convex-valued in ρ. And again, the correspondence $B^s(\underline{A}^s(p, \rho|A))$ has a closed graph and convex values since B^s is upper-hemicontinuous and has non-empty, compact, convex values. As $\mathcal{P}(M^s(A))$ is a non-empty, compact, convex subset of a locally convex Hausdorff space, the Kakutani-Fan-Glicksberg fixed point theorem again applies. Thus, equation 4 has a non-empty compact set of fixed points. For any p, label any such fixed point as $\sigma^s(p|A)$. These fixed points identify the seller's equilibrium path pricing strategy after price p has been quoted at active player set A when the market has to clear in at most s periods. This concludes the inductive step and proves equilibrium existence. Of course, when $s = z$, prices are set so that $\sigma^z \in B^z(V)$.

In each of the constructed equilibria, the seller's strategy depends on the time to clearing s, on the active player set A, and on the price quoted in the previous period p, since such a price identifies buyer's beliefs. To construct a WME, we need the buyers' strategies to depend only on the current price. Since any price p quoted on the equilibrium path by the seller at s belongs to $M^s(A)$, the active player set in the following period is independent of A. Hence, the seller's strategy and the buyers' beliefs depend only on p and s, and not on A, on the equilibrium path. In particular, in any such equilibrium, for any history $h \in H^t$ of length $t \in \{0, ..., z-1\}$ and for any $p \in M^{z-t}(A(h))$, we have that $\sigma(h,p) = \sigma^{z-1-t}(p) = \beta^{z-1-t}(p)$. If, instead, a price $p \notin M^{z-t}(A(h))$, buyers need to consider $A(h)$ in order to identify a corresponding price $p' \in M^{z-t}(A(h))$ which raises the same profit to the seller and pins down their beliefs about the price evolution to $\beta^{z-1-t}(p')$. Nevertheless, the strategy of any active buyer in $A(h)$ coincides by construction at p and p'. In these equilibria, deviating at $s+1$ from setting a price p to setting a price p' in the support of $\sigma^s(p|A)$ leads to the same continuation play as if the seller had sold at price p at date $s+1$ followed by p' at s. Hence, skipping a period does not affect the active player set in the continuation game.

Provided that $z > T$, the seller's strategy and consequently the buyers' strategies can also be made independent of the time s it takes to clear the market. To show this, define

$$\bar{\Pi}^s(p) = pd^s(p|V) + \delta\Pi^{s-1}(\underline{A}^{s-1}(p, \sigma^{s-1}(p)|V)).$$

Next, let $X^0 = M$ and $X^s = \left\{p \in [0,1]^2 \setminus X^{s-1} \mid \bar{\Pi}^{s+1}(p) = \bar{\Pi}^s(p)\right\}$ for any $s \in \{1, ..., z\}$. For $z > T$, the collection $\{X^s\}_{s=0}^z$ partitions $[0,1]^2$ by Lemma 3. Intuitively, X^s identifies those prices at which the seller does not benefit from having $s+1$ periods rather than s periods to clear the market, but at which the seller would suffer by having to clear the market in fewer than s periods. Then, the strategy of the seller only depends on the price posted in the previous period, since for any history $h \in H$,

$$\sigma(h,p) = \sigma(p) = \begin{cases} \sigma^z & \text{if} \quad p = \emptyset \\ \sigma^s(p) & \text{if} \quad p \in X^{s+1} \end{cases}.$$

Naturally, the strategy is then an equilibrium of the restricted game, since: (i) the seller maximizes the present value of profits given its beliefs about the active player set; (ii) buyers maximize the present value of surplus given the expected pricing path; (iii) buyers' beliefs are consistent with the seller's strategy; and (iv) the seller's and buyers' beliefs about the active player set are correct (provided that a measure zero of buyers has deviated). The previous argument also establishes WME existence in the unrestricted game $z = \infty$ whenever $\underline{v}_g > 0$. In fact, the seller's strategy cannot be affected by the constraint $\sigma(h) \in \mathcal{P}(M)$ when $z \geq T$, because prices necessarily belong to M in at most T periods by Lemma 3, and because all buyers purchase when prices belong to M by Lemma 2. Furthermore, these equilibria are indeed weak Markov equilibria, as the buyers' strategies depend only on current prices p.

The final step deals with games in which $\underline{v}_g = 0$, and its proof can be found in the online appendix.

Remark 7 *If $\underline{v}_g = 0$, a weak Markov equilibrium exists.*

The proof of the remark first constructs a sequence of games with $\underline{v}_g > 0$ that converges to the original game, then shows that the WME of the games in the sequence converge by equicontinuity to a weak Markovian equilibrium of the limit game in which $\underline{v}_g = 0$. ■

Proof Proposition 2. Fix a weak Markovian equilibrium $\{\sigma, \alpha\}$. We shall omit the dependence on $\{\sigma, \alpha\}$ to simplify notation. Let $\delta = e^{-r\Delta}$ and consider what happens when Δ converges to 0. As the buyers' strategies are stationary, denote by $\hat{U}(p, v)$ the equilibrium expected payoff of a buyer with value v when p was the last price quoted by the monopolist. As in the proof of Proposition 1, for any quoted price p, define the equilibrium path active buyer set as

$$\hat{A}(p) = \left\{ v \in V \mid \max_i \{v_i - p_i\} \leq \delta \hat{U}(p, v) \right\}.$$

For any price p^t, with a minor abuse of notation, denote by $D_i(p^t)$ the set of buyers who purchase variety i at such a price:

$$D_i(p^t) = \left\{ v \in \hat{A}(p^{t-1}) \mid v_i - p_i^t > \max\{v_j - p_j^t, \delta \hat{U}(p^t, v)\} \right\},$$

and denote by $d_i(p^t)$ the measure of this set. Let $\hat{\sigma}(p^t)$ denote the equilibrium mixed strategy of the seller when the set of active buyers is $\hat{A}(p^t)$, and let $\hat{E}[\cdot \mid p^t]$ denote the expectation with respect to this distribution. If $v \in D_i(p^t)$, buyer v prefers purchasing variety i immediately over purchasing the preferred variety tomorrow:

$$v_i - p_i^t \geq \delta \hat{E}[\max\{v_i - p_i^{t+1}, v_j - p_j^{t+1}\} \mid p^t].$$

Thus, the expected price reductions at histories with $d_i^t > 0$ satisfy

$$v_i(1 - \delta) \geq \hat{E}[\max\{p_i^t - \delta p_i^{t+1}, p_i^t - \delta p_j^{t+1} + \delta(v_j - v_i)\} \mid p^t]. \tag{5}$$

Let $\hat{\Pi}(p^t, p^{t-1})$ denote the present discounted value of equilibrium profits when the active player set is $\hat{A}(p^{t-1})$ and the price set by the seller is p^t:

$$\hat{\Pi}(p^t, p^{t-1}) = p^t d(p^t) + \delta \hat{E}\left[\hat{\Pi}(p^{t+1}, p^t) \mid p^t\right].$$

Because of the stationarity of buyers' strategies, the seller's present discounted value of equilibrium profits depends only on the distribution of active buyers (which is summarized by its support $\hat{A}(p^{t-1})$) and not on the entire history of play h^t. For a strategy to be an equilibrium, setting a price p^t in the support of $\hat{\sigma}(p^{t-1})$ at date t and selling according to $\hat{\sigma}(p^t)$ at date $t+1$ must be more profitable than selling according to $\hat{\sigma}(p^t)$ directly at date t. Formally,

$$\sum_i p_i^t d_i(p^t) + \delta \hat{E}\left[\hat{\Pi}(p^{t+1}, p^t) \mid p^t\right] \geq \hat{E}\left[\hat{\Pi}(p^{t+1}, p^{t-1}) \mid p^t\right]. \tag{6}$$

For any price p^{t+1} in the support of $\hat{\sigma}(p^t)$, denote by $K_i(p^{t+1})$ the set of buyers who were expected to purchase variety i at price p^t and that keep consuming the variety i at price p^{t+1}. Similarly, denote by $S_i(p^{t+1})$ the set of buyers who instead switch from variety i to variety j. Because the equilibrium is weak Markovian and the active player set depends only on the current prices p^{t+1}, these sets simplify to

$$\begin{aligned} K_i(p^{t+1}) &= \left\{v \in D_i(p^t) \mid v_i - v_j \geq p_i^{t+1} - p_j^{t+1}\right\}, \\ S_i(p^{t+1}) &= \left\{v \in D_i(p^t) \mid v_i - v_j \leq p_i^{t+1} - p_j^{t+1}\right\}, \end{aligned}$$

where indifference is unimportant by absolute continuity. Denoting the measures of the two sets by $k_i(p^{t+1})$ and $s_i(p^{t+1})$, respectively, condition (6) can then be rewritten as follows:

$$R = \sum_i \hat{E}[\underbrace{(p_i^t - p_i^{t+1})k_i(p^{t+1})}_{\text{Discrimination Gain}} + \underbrace{(p_i^t - p_j^{t+1})s_i(p^{t+1})}_{\text{Substitution Effect}} \mid p^t] \;\geq\; \underbrace{(1-\delta)\hat{E}\left[\hat{\Pi}(p^{t+1}, p^t) \mid p^t\right]}_{\text{Deferral Loss}}. \tag{7}$$

Fix any real time $\hat{T}$. The number of periods between 0 and $\hat{T}$ amounts to $\hat{T}/\Delta$ and diverges to infinity as $\Delta \to 0$. Because of this, for any value $\eta > 0$ there exists a Δ sufficiently small such that $d_a(p^t)+d_b(p^t) \leq \eta$ in almost every period $t \leq \hat{T}/\Delta$. In particular, $d_a(p^t)+d_b(p^t) > \eta$ for at most $1/\eta$ periods, as the market would clear by date $\hat{T}$ otherwise. Let H^* denote the set of histories of length $\hat{T}/\Delta$ that can occur with positive probability when players comply with the equilibrium strategies. We aim to show that for any $h \in H^*$ there exists a $p^t \in h$ such that the expected stationary equilibrium profit of the seller $\hat{E}\left[\hat{\Pi}(p^{t+1}, p^t) \mid p^t\right] \leq \kappa\eta$ for some constant $\kappa > 0$ independent of δ. The latter would imply that after any history in H^*, only a few players could be active, as $\hat{E}\left[\hat{\Pi}(p^{t+1}, p^t) \mid p^t\right]$ always exceeds optimal market-clearing profits by Remark 1, and as optimal market-clearing profits can be small only when the measure of active buyers is small by Lemma 2. But, if almost all buyers were to purchase before date $\hat{T}$, prices would necessarily be close to market-clearing by date $\hat{T}$ (as buyers would never pay more than their value for a product). By choosing $\hat{T}$ sufficiently small, the cost of delaying consumption by any real time $\hat{T}$ would then vanish, and hence no buyer would purchase a variety unless prices were close to the expected market-clearing price. The latter would then imply that the seller's initial profit would be close to a static market-clearing profit for any Δ sufficiently small.

To conclude, we show that profits must become small before date $\hat{T}/\Delta$ when Δ is small. Fix any history $h \in H^*$. The conclusion obviously holds when there exists a $p^t \in h$ such that $d_a(p^t) + d_b(p^t) = 0$, as $\hat{E}\left[\hat{\Pi}(p^{t+1}, p^t) \mid p^t\right] = 0$ by (7). So, begin by considering prices $p^t \in h$ such that $d_a(p^t) + d_b(p^t) \leq \eta$. The left hand side of (7) can be rewritten as:

$$R = \sum_i \hat{E}[p_i^t - p_i^{t+1} \mid p^t] d_i(p^t) + \hat{E}[(p_i^{t+1} - p_j^{t+1})(s_i(p^{t+1}) - s_j(p^{t+1})) \mid p^t].$$

By (5), we have that whenever $d_i(p^t) > 0$,

$$\hat{E}[p_i^t - p_i^{t+1} \mid p^t] \leq (1 - \delta).$$

Thus, at such a history, the desired conclusion would hold if for all δ there would exist some $\kappa' < K$ such that

$$\hat{E}[(p_i^{t+1} - p_j^{t+1})(s_i(p^{t+1}) - s_j(p^{t+1})) \mid p^t] \leq (1 - \delta)\eta\kappa',$$

as $\hat{E}\left[\hat{\Pi}(p^{t+1}, p^t) \mid p^t\right] \leq (2 + \kappa')\eta$. If instead the converse inequality held for all $\kappa' < K$, there would exist prices p^{t+1} in the support of $\hat{\sigma}(p^t)$ such that

$$(p_i^{t+1} - p_j^{t+1})(s_i(p^{t+1}) - s_j(p^{t+1})) > (1 - \delta)\eta K.$$

At such prices p^{t+1}, we would have that

$$\begin{aligned} p_i^{t+1} > p_j^{t+1} \quad &\Rightarrow \quad s_i(p^{t+1}) > (1 - \delta)\eta K, \\ p_i^{t+1} < p_j^{t+1} \quad &\Rightarrow \quad s_j(p^{t+1}) > (1 - \delta)\eta K. \end{aligned}$$

Moreover, at such prices, some players would necessarily switch their demand decision, as $s_i(p^{t+1}) - s_j(p^{t+1}) \neq 0$. If so, at p^{t+1} there would exist a type $\bar{v} \leq (1, 1)$ that would be indifferent between the two varieties (that is, $\bar{v}_i - p_i^{t+1} = \bar{v}_j - p_j^{t+1}$) and willing to purchase at the current price by Lemma 1.[27]

When such a type $\bar{v}$ exists, condition (5) implies that

$$(1 - \delta) \geq \hat{E}[\max\{p_i^t - \delta p_i^{t+1}, p_j^t - \delta p_j^{t+1} + (1 - \delta)(p_j^t - p_i^t)\} \mid p^t].$$

[27] If $s_i(p^{t+1}) > 0$, the latter would follow by taking any type $v \in S_i(p^{t+1})$ and then considering $\bar{v} = (\bar{v}_i, \bar{v}_i - p_i^{t+1} + p_j^{t+1})$.

Given that $(p_a^t - p_b^t) \in [-1,1]$ for the market not to clear, this in turn implies that

$$2(1-\delta) \geq \hat{E}[\max\{p_i^t - p_i^{t+1}, p_j^t - p_j^{t+1}\} \mid p^t].$$

With minor manipulations, the left hand side of (7) can be rewritten and bounded as follows:

$$R = \textstyle\sum_i \hat{E}[(p_i^t - p_i^{t+1})(k_i(p^t) + s_j(p^t)) \mid p^t] + (p_i^t - p_j^t)\hat{E}[s_i(p^{t+1}) - s_j(p^{t+1}) \mid p^t] \leq$$
$$\eta\hat{E}[\max\{p_i^t - p_i^{t+1}, p_j^t - p_j^{t+1}\} \mid p^t] + (p_i^t - p_j^t)\hat{E}[s_i(p^{t+1}) - s_j(p^{t+1}) \mid p^t].$$

Hence, either $\hat{E}\left[\hat{\Pi}(p^{t+1}, p^t) \mid p^t\right] \leq (2+\kappa')\eta$ for some $\kappa' < K$ (as desired), or we must have that

$$(p_i^t - p_j^t)\hat{E}[s_i(p^{t+1}) - s_j(p^{t+1}) | p^t] > (1-\delta)\eta K.$$

For the latter to be the case, it must be that $p_i^t - p_j^t \notin [-(1-\delta)K, (1-\delta)K]$, as by assumption we know that

$$\hat{E}[s_i(p^{t+1}) - s_j(p^{t+1}) | p^t] \in [-\eta, \eta].$$

Thus if $p_i^t > p_j^t$, there exist prices p^{t+1} in the support of $\hat{\sigma}(p^t)$ such that

$$s_i(p^{t+1}) > \frac{1-\delta}{p_i^t - p_j^t}\eta K.$$

However, at such prices we must have that

$$\frac{1-\delta}{p_i^t - p_j^t}\eta K < s_i(p^{t+1}) < \overline{f}\mathcal{L}(S_i^t(p^{t+1})) \leq \overline{f}\left(p_i^{t+1} - p_j^{t+1} - p_i^t + p_j^t\right)\mathcal{L}(D_i(p^t))$$
$$\leq (\overline{f}/\underline{f})\left(p_i^{t+1} - p_j^{t+1} - p_i^t + p_j^t\right)d_i(p^t) \leq (\overline{f}/\underline{f})\left(p_i^{t+1} - p_j^{t+1} - p_i^t + p_j^t\right)\eta,$$

where the second inequality holds by regularity as $s_i(p) \leq \overline{f}\mathcal{L}(S_i^t(p))$; the third holds by absolute continuity given that $S_i(p^{t+1})$ is a subset of $D_i(p^t)$ with height bounded by $\left(p_i^{t+1} - p_j^{t+1} - p_i^t + p_j^t\right)$; the fourth holds by $d_i(p^t) \geq \underline{f}\mathcal{L}(D_i(p^t))$; and the final inequality holds as $d_i(p^t) \leq \eta$. Thus, at such histories, the price difference $(p_i^t - p_j^t)$ increases by at least τ with strictly positive probability whenever the difference is positive,

$$p_i^{t+1} - p_j^{t+1} - p_i^t + p_j^t > \frac{\underline{f}(1-\delta)K}{\overline{f}(p_i^t - p_j^t)} = \tau,$$

and it similarly declines by at least τ whenever it is negative. Moreover, when $p_i^t > p_j^t$, the probability of such an increase in the price difference would necessarily satisfy

$$\begin{aligned}
&\Pr(p_i^{t+1} - p_j^{t+1} - p_i^t - p_j^t > \tau \mid p^t) \\
&\geq \Pr((p_i^t - p_j^t)(s_i(p^{t+1}) - s_j(p^{t+1})) > (1-\delta)\eta K \mid p^t) \\
&\geq \frac{(p_i^t - p_j^t)\hat{E}[s_i(p^{t+1}) - s_j(p^{t+1}) \mid p^t]/\eta K - (1-\delta)}{(p_i^t - p_j^t) - (1-\delta)} > 0,
\end{aligned}$$

by a simple variant of the Markov inequality. But the previous arguments imply that, whenever demand is small ($d_a(p^t) + d_b(p^t) \leq \eta$) and profits are large ($\hat{E}\left[\hat{\Pi}(p^{t+1}, p^t) \mid p^t\right] > (2+K)\eta$), the price difference $(p_i^t - p_j^t)$ has to increase with strictly positive probability by at least τ if it is positive, and it has to decline with strictly positive probability by at least τ if it is negative. If that were the case in every period in which demand was small (which is almost every period when Δ is small), the price difference would eventually fall outside the set $[-1, -(1-\delta)K] \cup [(1-\delta)K, 1]$ as the number of periods diverged to infinity:

$$\lim_{\Delta \to 0} \Pr\left(p_i^t - p_j^t \notin [-1, -(1-\delta)K] \cup [(1-\delta)K, 1] \text{ for some } t \leq \hat{T}/\Delta\right) = 1.$$

If so, the market would be close to clearing before date $\hat{T}/\Delta$ with probability 1, as $p_i^t - p_j^t \in [-(1-\delta)K, (1-\delta)K]$ would imply that profits were small, while $p_i^t - p_j^t \notin [-1, 1]$ would imply that one of the two prices was either smaller than the other or equal to zero (which implies market-clearing by the proof of Lemma 2). But if so, buyers would expect the market to almost clear before date $\hat{T}/\Delta$, and so profits would be small (that is, $\hat{E}\left[\hat{\Pi}(p^{t+1}, p^t) \mid p^t\right] \leq (2+K)\eta$) with probability 1 on the equilibrium path before date $\hat{T}/\Delta$. ■