PHD

Asymptotic pattern formation in second and higher order quasilinear parabolic equations

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Award date:
2004

Awarding institution:
University of Bath

Asymptotic Pattern Formation in Second and Higher Order Quasilinear Parabolic Equations

submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

November 2004

Abstract

This thesis describes the large-time behaviour of partial differential equations by studying their similarity solutions. The aim of this work is to try to extend the known theory for the heat equation and the semilinear heat equation to more complicated models. The particular models considered are as follows:
(i) We consider the porous medium equation (PME)

$$u_t = \Delta(|u|^{m-1}u) \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \;\; u(x,0) = \hat{u}(x) \text{ in } \mathbb{R}^N,$$

with continuous, compactly supported initial data $\hat{u}$. The PME admits various similarity solutions of the form $u_k(x,t) = t^{-\alpha_k}\psi_k(x/t^{\beta_k})$, $k = 0,1,2,\ldots$. The nonlinear eigenfunction subset $\Phi = \{\psi_k\}$ is shown to be *evolutionary complete*, i.e. describes the asymptotics of arbitrary global C_0-solutions of the PME. This evolution completeness holds in one dimension and in radial geometry in $\mathbb{R}^N$.
(ii) We consider the PME with absorption

$$u_t = \Delta u^m - u^p \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \text{ with } m,p > 1.$$

It is known that its global L^1-solutions change their large-time behaviour at the critical absorption exponent $p_0 = m + \frac{2}{N}$. We extend this by showing that, provided $u(x,t)$ changes sign, there exists an infinite sequence of critical exponents $\{p_k\}$ that generate a countable subset of different non-self-similar asymptotic patterns. These results are extended to the dual PME with absorption

$$u_t = |\Delta u|^{m-1}\Delta u - |u|^{p-1}u \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \text{ with } m > 1.$$

(iii) We consider the semilinear parabolic equation of reaction-diffusion type

$$u_t = -(-\Delta)^m u + |u|^{p-1}u \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \text{ with exponents } p > 1, m > 2$$

and initial data $\hat{u} \in L^q(\mathbb{R}^N)$, $q \geq 1$. This is an extension of the semilinear heat equation that is known to exhibit non-uniqueness for $p > 1 + \frac{2q}{N}$. We show that non-uniqueness occurs for the higher order parabolic equations if $p > 1 + \frac{2mq}{N}$ by describing a discrete subset of similarity solutions $u_*(x,t) = t^{-1/(p-1)}V(x/t^{1/2m})$. We also establish the existence of radially symmetric similarity profiles for p close to the bifurcation exponents $p_l = 1 + \frac{2m}{N+l}$, $l = 0,2,\ldots$, and prove that the p-bifurcation branches remain in the range $p < p_S = \frac{N+2m}{(N-2m)_+}$.

Acknowledgements

The first acknowledgement must go to God for his everlasting love and support. Without him I would not have even begun this course of study, but more importantly, without him I would have no hope. About two thousand years ago God the Father sent his Son Jesus to earth to die on my behalf, so that I might have a relationship with God. Without Jesus' sacrifice, my sin would have kept me from knowing God for all eternity and I would now be living a life without hope. God also sent his Holy Spirit to guide me through this life and I am grateful for his council in all things. In regard to this thesis, am grateful for the Holy Spirit's help in solving many difficult problems and proving many Theorems and Lemmas. In regard to my life, I am grateful for his help in keeping me on God's path. Without God, I am nothing and can do nothing.

I also acknowledge my husband Damien for his encouragement, love, prayers and very patient support through all the times we have experienced together. He is a continual reminder of God's goodness to me. I thank my family: Mum and Dad (both sets), Adam, Heidi, Benjamin, Elisha and Simon for their encouragement, for keeping my feet on the ground and for cheering my up with frequent visits when we both had the time. Much gratitude goes to my discipleship group: Haydn, Muriel, Johanna and Nathaniel Wilkins; James, Sally and Abigail Taylor; John and Alyssa Miller; John and Katherine Lush; Hywel and Claire Carr and Maurice and Penny Sharp, for their continued prayer support, practical help and the provision of nice food every Wednesday night.

My thanks go to my supervisor Victor Galaktionov for his guidance throughout this thesis and his extensive knowledge of mathematics that he was so willing to share. I would also like to thank my internal supervisor Chris Budd for his advice and support for the duration of this Ph.D. and prior to that. I thank my office mates Bradley Bonner, Andrew Hill, Patrick Lechner, André Léger, Simone Mandica and David Simpson for their friendship and my good friend Jennifer Wright for forcing me to take coffee breaks.

Finally, this thesis was written for the glory of God. I have endeavoured to honour him in all things and I dedicate this thesis to him.

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Chapter 1

Introduction

In this thesis we study four different nonlinear models that are examples of second and higher order parabolic partial differential equations. Our main tool is constructing their similarity solutions and using them to extend the results to general solutions by using various mathematical techniques.

1.1 The porous medium equation

The porous medium equation (PME) $u_t = \Delta(u^m)$ with the exponent $m > 1$ is one of the simplest examples of a quasilinear evolution equation of parabolic type. It describes diffusion of liquids and gases in porous media as well as processes of electron and ion conductivity in plasma. Physically, the restriction $u \geq 0$ applies but mathematically it is interesting to allow negative values of u. In this case the PME must be redefined for it to remain parabolic: $u_t = \Delta(|u|^{m-1}u)$.

The Cauchy problem for the PME:

$$u_t = \Delta(|u|^{m-1}u) \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \ \ u(x,0) = \hat{u}(x) \text{ in } \mathbb{R}^N \tag{1.1}$$

does not have classical solutions for $\hat{u} \in L^1(\mathbb{R}^N)$ since the equation is only parabolic when $u > 0$ (it degenerates at the level $u = 0$). Finite propagation is a direct consequence of this and a solution with compactly supported initial data is compactly supported at all times. A free boundary separates the sets $\{u = 0\}$ and $\{u > 0\}$. Near such moving interfaces solutions are not smooth. The concept of a generalised solution needs to be introduced to ensure that the problem is well-posed in this class.

The PME became one of the more important equations of modern mathematical physics from the 1950s and was studied by many well known mathematicians and experts in mechanics. The investigation of source-type solutions began in the 1950s with papers by Zel'dovich and Kompaneetz [90], Barenblatt [6] and Pattle [73], which introduced the need for generalised solutions. The first study of existence and uniqueness in one dimension was in the paper by Oleinik, Kalashnikov and Yui-Lin' [72]. Since the 1970s there have been many new results including Aronson's paper [3] on regularity at the interface, Bénilan [9] on general well-posedness and semigroups, and Kamenomostskaya [59] on asymptotic behaviour. The study of the regularity of solutions and free boundaries was largely developed by Aronson, Caffarelli, Friedman and coworkers. Existence under optimal conditions is due to the combined efforts of Aronson, Caffarelli, Bénilan, Crandall, Pierre, Dahlberg and Kenig, amongst others. A full list of references and a presentation of some of the results can be found in the following books and papers: [7, 32, 27, 39, 50, 78].

1.2 The dual porous medium equation

The dual porous medium equation (DPME)

$$u_t = |\Delta u|^{m-1}\Delta u \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \text{ with } m > 1, \tag{1.2}$$

has not been studied in as much detail as the PME, though it can be reduced to the PME; see 4. The DPME is of great interest mathematically because it is a simple example of a fully nonlinear, degenerate parabolic equation. However, it also has practical uses; appearing in some problems in elasticity with damping as well as in problems of Bellman-Dirichlet type.

Bernis, Hulshof and Vazquez [11] were the first to treat the asymptotic behaviour of the DPME in detail. In this paper the nonnegative self-similar solutions were classified and shown to describe the large-time behaviour of all nonnegative solutions of the DPME defined in $Q = \{(x,t) : x \in \mathbb{R}, t > 0\}$ whose initial data are continuous and compactly supported. This property is analogous to that of fundamental solutions of diffusion equations.

1.3 The porous and dual porous medium equations with absorption

A natural extension to the two previous models is to consider the effects of an absorption term. In Chapter 4 we will study the PME with absorption:

$$u_t = \Delta(|u|^{m-1}u) - |u|^{p-1}u, \text{ with } m > 1, \ p > m,$$

which describes thermal propagation in absorbing media, and the DPME with absorption:

$$u_t = |\Delta u|^{m-1}\Delta u - |u|^{p-1}u, \text{ with } m, p > 1. \tag{1.3}$$

It has been known since 1980's that global L^1-solutions of the classical PME with absorption, change their large-time behaviour at the critical absorption exponent $p_0 = m + 2/N$ (also known as the Fujita critical exponent in blow-up theory). Work has been done in analysing such behaviour; see the references in the book [50]. The Fujita critical exponent for the DPME with absorption in $\mathbb{R} \times \mathbb{R}_+$ was calculated in [57].

1.4 A $2m$th order semilinear equation with reaction term

The semilinear parabolic equation:

$$u_t = -(-\Delta)^m u + |u|^{p-1}u, \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.4}$$

where $m \geq 2$ is an integer and $p > 1$ is a fixed exponent, is an extension of the semilinear heat equation:

$$u_t = \Delta u + u^p, \ (u \geq 0), \tag{1.5}$$

which occurs in combustion theory [89] and several other physical applications. This equation exhibits various evolution phenomena including blow-up; see the books [50, 78] and references therein. The question of local solubility and uniqueness of L^q-solutions was studied by Haraux and Weissler, amongst others, in the 1970s and 1980s. Weissler's first paper on the subject was published in 1979 [85].

In it and its sequel [86] he studies the initial value problem associated with (1.4):

$$u'(t) = Au(t) + J(u(t)), \;\; t > 0, \;\; u(0) = \phi,$$

(where $u(t)$ is a curve in a Banach space E, A is the infinitesimal generator of a C_0 semi-group on E, and J is a nonlinear function on E or a subset of E) by means of the corresponding integral equation

$$u(t) = \mathrm{e}^{tA}\phi + \int_0^t \mathrm{e}^{(t-s)A} J(u(s))\,\mathrm{d}s.$$

They prove the existence of a semi-flow and thus prove the existence of a solution in L^p.

In later papers Haraux and Weissler study (1.5) directly. In their paper [54] they prove the following non-uniqueness result. If $p > 1 + 2q/N$ then there exists a non-trivial global solution of (1.5) in

$$C([0,\infty); L^q(\mathbb{R}^N)) \cap C^1((0,\infty); L^q(\mathbb{R}^N))$$

such that $u(0) = 0$. Thus they get at least three different solution curves, u, $-u$, and 0, emanating from the initial data 0. The particular u they construct is positive for $t > 0$. They also prove a local existence and uniqueness theorem showing that this non-uniqueness result is optimal in the sense that if $p < p_0$ then there is a unique solution that is local in time.

1.5 Sturm's theorems

A key ingredient in some of the analysis contained in this thesis is Sturm's Theorem concerning the evolution of zero sets of parabolic partial differential equations. In 1836 C. Sturm published two celebrated papers in the first volume of J. Liouville's Journal de Mathematique Pures et Appliquées. The first paper [79] on zeros of solutions $u(x)$ of second order ordinary differential equations such as

$$u'' + q(x)u = 0, \quad x \in \mathbb{R}, \tag{1.6}$$

very quickly exerted a great influence on the general theory of ODEs. Then and nowadays Sturm's oscillation, comparison and separation theorems can be found in most textbooks on ODEs with various generalisations to other equations and systems of equations. In general, such theorems classify and compare zeros and zero sets $\{x \in \mathbb{R} : u(x) = 0\}$ of different solutions $u_1(x)$ and $u_2(x)$ of (1.6) or solutions of equations with different continuous ordered potentials $q_1(x) \geq q_2(x)$.

The second paper [80] was devoted to the *evolution* analysis of zeros and zero sets $\{x : u(x,t) = 0\}$ for solutions $u(x,t)$ of partial differential equations of parabolic type, for instance,

$$u_t = u_{xx} + q(x)u, \quad x \in [0, 2\pi], \;\; t > 0, \tag{1.7}$$

with the same ordinary differential operator as in (1.6) and the Dirichlet boundary condition $u = 0$ at $x = 0$ and $x = 2\pi$ and given smooth initial data at $t = 0$. Two of Sturm's results on PDEs like (1.7) can be stated as follows:

First Sturm Theorem: nonincrease with time of the number of zeros (or sign changes) of solutions;

Second Sturm Theorem: a classification of blow-up self-focusing formations and collapses of multiple zeros.

We will refer to both of Sturm's Theorems together as the *Sturmian argument* on zero set analysis.

We are not going to use the second theorem so no further explanation is necessary. The complete statement of the first Theorem is as follows:

First Sturm Theorem Let $u(x,t)$ be a classical solution of a linear, uniformly parabolic ($a > 0$) equation with sufficiently smooth coefficients

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)$$

in $Q_T = (A, B) \times (0, T)$. If $u(x,t)$ does not change sign on the parabolic boundary of Q_T, then the number of zeros of $u(x,t)$ does not increase with time.

Unlike the classical Sturm theorems on zeros of solutions of second order ODEs, Sturm's evolution zero set analysis for parabolic PDEs did not attract much attention in the nineteenth century and, in fact, was forgotten for almost a century. It seems that G. Pólya (1933) [75] was the first person in the twentieth century to revive interest in the first Sturm Theorem for the heat equation. Since the 1930s the Sturmian argument has been rediscovered in part several times. For instance, a key idea of the Lyapunov monotonicity analysis in the famous KPP-problem, by A.N. Kolmogorov, I.G. Petrovskii and N.S. Piskunov (1937) [62] on the stability of travelling waves (TWs) in reaction-diffusion equations, was based on the first Sturm Theorem in a simple geometric configuration with a single intersection between solutions. This was separately proved there by the Maximum Principle.

From the 1980s the Sturmian argument for PDEs began to penetrate more and more into the theory of linear and nonlinear parabolic equations and was found to have several fundamental applications. These include asymptotic stability theory for various nonlinear parabolic equations, orbital connections and transversality of stable-unstable manifolds for semilinear parabolic equations as Morse-Smale systems, unique continuation theory, Floquet bundles and a Poincaré-Bendixson theorem for parabolic equations and problems of symplectic geometry and curve shortening flows. We refer to the books [43], [39] for a detailed exposition and history of this PDE theory.

1.6 Overview of the thesis

It is convenient to begin by mentioning a connection with classical partial differential equation theory. Setting $m = 1$ in the PME yields the classical heat equation (HE):

$$u_t = \Delta u \ \text{ in } \ \mathbb{R}^N \times \mathbb{R}_+, \tag{1.8}$$

which is the canonical second order partial differential equation. In Chapter 3 we set out some preliminary results concerning the HE (1.8) and its associated linear self-adjoint operator $\mathbf{B} \equiv \Delta - \frac{1}{2}y \cdot \nabla + \frac{N}{2}I$, which will be used in the remaining chapters. This operator plays a crucial role in the asymptotic theory for the HE, and its eigenfunctions describe all possible asymptotic patterns. We refer to this property as evolution completeness. It is known that the subset of eigenfunctions of $\mathbf{B}$ in a weighted L^2-space corresponding to the discrete spectrum $\sigma(\mathbf{B})$ is complete and closed and hence evolutionary complete.

We then consider the porous medium equation (PME) (1.1) and show that the set of its nonlinear eigenfunctions is evolutionary complete. The notion of evolution completeness for nonlinear equations and operators is introduced in order to cover all possible types of asymptotics for arbitrary initial data $\hat{u}$. Some of this work appears in "On evolution completeness of nonlinear eigenfunctions for the porous medium equation in the whole space" (Galaktionov, Harwin), to appear in Advances in Differential Equations [41]. It should be noted that the key new development of Sturmian Theory for solutions that change sign presented in Section 3.6 is my own work except for Lemma 3.11. Lemma 3.11 and the rest of the Chapter should be considered as joint work with Prof. Galaktionov.

In Chapter 4 we extend our previous investigations and look at the PME with absorption:

$$u_t = \Delta |u|^{m-1}u - |u|^{p-1}u, \text{ with } m > 1, \ p > m, \tag{1.9}$$

showing that there exist a countable sequence of critical exponents $\{p_k\}$. We show that, at each $p = p_k$, the generic asymptotics of solutions can change dramatically, and the time scaling factors for $t \gg 1$ can include extra $\ln t$-scaling in addition to the standard asymptotics attached to nonlinear eigenfunctions of the PME without absorption (4.26). We then extend these results to the fully nonlinear dual porous medium equation (DPME) with absorption $u_t = |\Delta u|^{m-1}\Delta u - |u|^{p-1}u$. Unlike the PME, the DPME does not have a conservation law for $\int u(x,t)\,\mathrm{d}x$, so finding its eigenvalues explicitly is more difficult. Some of this work has been published in "Spectra of critical exponents in nonlinear heat equations with absorption" (Galaktionov, Harwin), which has been published in Advances in Differential Equations [42]. It should be noted that subsection 4.2.2 concerning centre manifold behaviour is the work of Prof. Galaktionov; Propositions 4.2 and 4.5 are my own work as is Section 4.5. The rest of the Chapter should be considered as joint work with Prof. Galaktionov.

In Chapter 5 we consider the higher order semilinear parabolic equation

$$u_t = -(-\Delta)^m u + |u|^{p-1}u \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \ m \geq 2, \ p > 1, \tag{1.10}$$

with initial data $\hat{u} \in L^q(\mathbb{R}^N)$. We study other asymptotic aspects of such equations when $t \to 0^+$, and show that non-uniqueness occurs if $p > p_0 = 1 + \frac{2mq}{N}$.

Our analysis is based on the construction of self-similar solutions. For $m = 1$, when (1.10) is the classical semilinear heat equation from combustion theory

$$u_t = \Delta u + |u|^{p-1}u,$$

the non-uniqueness and similarity results are well known and were proved by Weissler and Haraux in the 1980s. We show that similar conclusions apply to higher order equations, though our techniques are different. Some of this work appears in "Non-uniqueness and global similarity solutions for a higher-order semilinear parabolic equation" (Galaktionov, Harwin), to appear in Nonlinearity [40]. It should be noted that all of the numerical calculation of the similarity profiles and the bifurcation diagrams is my own work. This includes the discovery of a non-standard boundary layer as $p \to \infty$, which seems to be the first one known in the theory of second and higher order ordinary differential equations. The rest of the Chapter should be considered as joint work with Prof. Galaktionov.

Chapter 2

Notation and Preliminaries

2.1 Notation

What follows are some key definitions, theorems and tools that will be used without comment throughout this thesis.

$\mathcal{L}(X, Y)$	the set of linear operators from X to Y
$C_0(X)$	the set of continuous compactly supported functions on X
H	a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$

2.2 Preliminaries

Definition 2.1 An operator L with dense domain D in H is said to be *symmetric* if for all $f, g \in D$ we have

$$\langle Lf, g \rangle = \langle f, Lg \rangle.$$

Definition 2.2 For an operator $L \in \mathcal{L}(H, H)$ the *adjoint operator* L^* is determined by the condition that

$$\langle Lf, g \rangle = \langle f, L^* g \rangle$$

for all $f, g \in H$.

Definition 2.3 An operator $L \in \mathcal{L}(H, H)$ is called *self-adjoint* if

$$\langle Lf, g \rangle = \langle f, Lg \rangle$$

for each $f, g \in H$. We say that L is *essentially self-adjoint* if it is symmetric and its closure is self-adjoint.

The Friedrichs extension

Every non-negative (or semi-bounded) symmetric operator **B** has at least one non-negative self-adjoint extension. If **B** is not essentially self-adjoint then this extension, called the Friedrichs extension, is one of infinitely many possible self-adjoint extensions. We use the Friedrich's extension as the minimal (extremal) extension of symmetric operators through quadratic forms.

Definition 2.4 An operator $L \in \mathcal{L}(X, Y)$ is called *Fredholm* if (a) the range of L is closed in Y, and (b) the numbers

$$n(L) = \dim(\mathrm{Ker} L) \quad and \quad d(L) = \dim(Y \setminus \mathrm{Im} A)$$

are finite.

Definition 2.5 Let X and Y be Banach spaces. A mapping $f : X \to Y$ is *Fréchet differentiable at x_0* if there exists $g \in \mathcal{L}(X, Y)$ such that in a neighbourhood U of x_0

$$\|f(x) - f(x_0) - g(x - x_0)\| = o(\|x - x_0\|).$$

In this case we write $g = f'(x_0)$, and $f'(x_0)$ is called the Fréchet derivative of f at x_0.

Definition 2.6 An operator $A : H \to H$ is said to be *potential* if there exists a C^1 functional $\Phi : H \to \mathbb{R}$, called the potential of A, such that $A = \Phi'$, where $'$ denotes the Fréchet derivative of Φ.

Theorem 2.7 *An operator $A : H \to H$ is potential if it has a self-adjoint Fréchet derivative $A'(u)$. If this is so then Φ in Definition 2.6 is given as*

$$\Phi(u) = \int_0^1 \langle A(\rho u), u \rangle \, \mathrm{d}\rho.$$

Definition 2.8 A functional Φ on a normed space E is called *coercive* if

$$\frac{\Phi(x)}{\|x\|^2} \to \infty \quad as \quad \|x\| \to \infty.$$

Definition 2.9 The *deficiency indices* of a symmetric ordinary differential operator L are defined to be the dimensions (possibly infinite) of the deficiency spaces

$$\begin{aligned} M^{\pm} :&= \{f \in \mathrm{Dom}(L^*) : L^* f = \pm \mathrm{i} f\} && (2.1)\\ &= \{f \in H : \langle Lh, f\rangle = \pm\mathrm{i}\langle h, f\rangle \text{ for all } h \in \mathrm{Dom}(L)\}. && (2.2)\end{aligned}$$

Definition 2.10 A partial differential equation of mth order is called
(i) *quasilinear* if it it linear with respect to the highest order deriviative; and
(ii) *fully nonlinear* if this is not true.

Definition 2.11 A *semigroup* is defined by a set and a binary operator from the set itself in which the multiplication operation is associative.

Definition 2.12 One space X is *embedded* in another space Y when the properties of Y restricted to X are the same as the properties of X.

Definition 2.13 Let X and Y be normed spaces. $L \in \mathcal{L}(X, Y)$ is *compact* if, for any bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ in Y contains a convergent subsequence.

Definition 2.14 A function f is in $L^p_\rho(\mathbb{R}^N)$, with a given positive weight ρ, if $\int_{\mathbb{R}^N} \rho|f|^p < \infty$.

Lemma 2.15 *If* $1 \le p < \infty$ *and* $\rho = \mathrm{e}^{|y|^\alpha}$, $\alpha > 0$, *then* $L^p_\rho(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$.

Definition 2.16 A function $u = u(x, t)$ is called a *weak solution* of the Cauchy problem

$$\begin{aligned} u_t &= \Delta u^m \text{ in } \mathbb{R}^N \times \mathbb{R}_+ && (2.3)\\ u(x, 0) &= \hat{u}(x) \text{ in } \mathbb{R}^N,\ \hat{u} \in L^1(\mathbb{R}^N) && (2.4)\end{aligned}$$

if
(i) $u, \nabla u^m \in L^2_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}_+)$; and
(ii) u satisfies the identity

$$\int_0^\infty \int_{\mathbb{R}^N} \nabla u^m \cdot \nabla\phi - u\phi_t \,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^N} \hat{u}(x)\phi(x, 0)\,\mathrm{d}x$$

for any test function $\phi(x, t) \in C^1_0(\mathbb{R}^N; \mathbb{R}_+)$.

The Lusternik–Schnirel'man theory for potential operators

The essence of the theory is as follows: Any positive even uniformly differentiable functional Φ in H has at least a countable subset of critical points on the unit sphere. These are eigenfunctions of Φ' and satisfy $\Phi'(u_k) = \lambda_k u_k$. Clark [20] and Rabinowitz [76] extended this theory to include non-positive functionals. Krasnosel'skii's genus theory [65] gives an insight into the actual structure of the nodal sets of eigenfunctions $u_k(x)$. (The complexity of these increases with k.)

Chapter 3

Evolution completeness of nonlinear eigenfunctions for the porous medium equation in the whole space

God exists since mathematics is consistent, and the Devil exists since we cannot prove it. - Andre Weil

In this chapter we ask if there exists any kind of *completeness property* for the nonlinear eigenfunctions of the porous medium equation (PME). This is a natural question to ask since the completeness and closure of countable subsets of eigenfunctions is common to classes of linear self-adjoint operators in Hilbert spaces; see Birman and Solomjak's book [13]. Thus, we will try to extend this notion to some nonlinear partial differential equations.

We consider the PME

$$u_t = \Delta(|u|^{m-1}u) \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \ m > 1,$$

with continuous, compactly supported initial data $\hat{u}$. (We consider only the one-dimensional PME or the PME in radially symmetric geometry.) The PME admits various similarity solutions of the form

$$u_k(x,t) = t^{-\alpha_k}\psi_k(x/t^{\beta_k}), \ k = 0, 1, 2, \ldots,$$

where each $\psi_k \in C_0(\mathbb{R}^N)$ satisfies a quasilinear elliptic equation in $\mathbb{R}^N$ and the exponents $\{\alpha_k, \beta_k\}$ are determined from the solubility of the resulting nonlinear eigenvalue problem. The nonlinear eigenfunction subset $\Phi = \{\psi_k\}$, which consists of a countable number of continuous families, is rather complicated but is known in one dimension. In radial geometry in $\mathbb{R}^N$, $N \geq 2$, only the first two eigenfunctions are known.

We show that the eigenfunction subset Φ is *evolutionary complete*, i.e. describes the asymptotics of arbitrary global C_0-solutions of the PME. We prove that this evolution completeness holds in one dimension and in radial geometry in $\mathbb{R}^N$. The analysis uses Sturm's Theorem on zero sets for parabolic equations, scaling techniques and theory of gradient dynamical systems. For $m = 1$, i.e. for the linear heat equation, the evolution completeness is a direct consequence of the fact that eigenfunction subset for the linear self-adjoint operator $\Delta + \frac{1}{2}y \cdot \nabla + \frac{N}{2}I$ in a weighted L^2-space, is complete and closed. These linear eigenfunctions are used as branching points of nonlinear ones for $m \approx 1^+$.

The work on evolution completeness is an extension of the work [38] (see references therein for previous work in this direction) in which the separable solutions of the PME in a bounded domain are shown to be evolutionary complete.

3.1 Introduction: nonlinear eigenfunctions and evolution completeness

Completeness and closure of countable orthonormal subsets $\Phi = \{\psi_\beta\}$ of eigenfunctions of classes of linear differential self-adjoint operators $\mathbf{A}$ in a Hilbert space H play an important role in the theory of evolution linear partial differential equations (PDEs)

$$u_t = \mathbf{A}u \text{ for } t > 0, \;\; u(0) = \hat{u} \in H. \tag{3.1}$$

Given initial data $\hat{u} = \sum c_\beta \psi_\beta$, the solution is prescribed by the eigenfunction expansion

$$u(t) = \sum c_j e^{\lambda_j t} \psi_j, \tag{3.2}$$

where $\sigma(\mathbf{A}) = \{\lambda_k, \, |\beta| = k = 0, 1, 2, \ldots\}$ is a monotone decreasing sequence of eigenvalues of $\mathbf{A}$. (3.2) represents the general solution of the equation and determines its asymptotic behaviour as $t \to \infty$. In this case the solution $u(t) \not\equiv 0$

approaches, for large times, the separable solution

$$u_k(t) = \sum_{|\beta|=k} c_\beta e^{\lambda_k t} \psi_\beta \tag{3.3}$$

where the finite $k = k(\hat{u})$ in the expansion (3.2) is such that $c_\beta = 0$ for all $|\beta| < k$ and there exists a $c_\beta \neq 0$ for some $|\beta| = k$. Then

$$u(t) = u_k(t) + O(e^{\lambda_{k+1} t}) \text{ as } t \to \infty. \tag{3.4}$$

For nonlinear evolution PDEs eigenfunction expansions do not apply. However, various nonlinear PDEs are known to admit countable or continuous subsets Φ of particular self-similar or other group invariant solutions, which are not separable as in (3.3) but are obtained from some lower order PDEs or ODEs. Our main goal is to consider the well known quasilinear porous medium equation (PME) and to explain how such a subset Φ of particular solutions (associated with *nonlinear eigenfunctions* of some operators) can be related to the asymptotics of the general solution in the sense of the *evolution completeness* of Φ introduced in [38].

So we consider the Cauchy problem for the classical PME

$$u_t = \Delta(|u|^{m-1}u) \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \ m > 1, \tag{3.5}$$

with continuous compactly supported initial data

$$u(x, 0) = \hat{u}(x) \in C_0(\mathbb{R}^N). \tag{3.6}$$

Problem (3.5), (3.6) has been studied since the 1950s and it is well known that there exists a unique global weak continuous bounded solution $u = u(x,t)$ decaying to zero as $t \to \infty$; see books by Lions [67] and DiBenedetto [27] and Kalashnikov's survey [58].

3.1.1 A countable subset of similarity patterns for the PME

We study the asymptotic behaviour of the solution as $t \to \infty$ and begin with different asymptotic patterns that can occur in the Cauchy problem for arbitrary initial functions $\hat{u} \in C_0$. Some particular asymptotic behaviour results for the

PME in $\mathbb{R}^N$ have been well known for many years. For instance, for nonnegative $\hat{u}$, the first general rigorous result is due to Friedman and Kamin [34] establishing that as $t \to \infty$, $u(x,t)$ approaches the *Zel'dovich-Kompaneetz-Barenblatt* similarity solution (known from the beginning of 1950s, [90, 6]) denoted here by $u_0(x,t)$ and given by

$$u_0(x,t) = t^{-N/[N(m-1)+2]}\psi_0(y), \quad y = x/t^{1/[N(m-1)+2]}, \quad \text{where} \tag{3.7}$$

$$\psi_0(y) = [B_0(b^2 - |y|^2)_+]^{1/(m-1)}, \quad B_0 = \tfrac{m-1}{2m(m+1)} \,. \tag{3.8}$$

Here $b > 0$ is an arbitrary parameter. The result of [34] says that if $u_0(x,t)$ has the same mass $M_0 = \int \hat{u} > 0$ (preserved in time) as $u(x,t)$, then

$$u(x,t) = u_0(x,t) + o(t^{-N/[N(m-1)+2]}) \quad \text{as } t \to \infty. \tag{3.9}$$

The PME in one dimension admits another explicit solution: the *Barenblatt-Zel'dovich* dipole solution [8]

$$u_1(x,t) = t^{-1/m}\psi_1(y), \quad y = x/t^{1/2m}, \quad \text{where} \tag{3.10}$$

$$\psi_1(y) = |y|^{1/m}[B_0(b^{(m+1)/m} - |y|^{(m+1)/m})_+]^{1/(m-1)}\operatorname{sign} y, \quad b > 0. \tag{3.11}$$

The stability hypotheses of $u_1(x,t)$ (see references in [46]) are as follows: if $M_0 = \int \hat{u} = 0$ and $M_1 = \int x\hat{u} \neq 0$, then $u(x,t)$ converges to the dipole solution with the same first momentum M_1,

$$u(x,t) = u_1(x,t) + o(t^{-1/m}) \quad \text{as } t \to \infty. \tag{3.12}$$

The solutions (3.7) and (3.10) of the PME are special since they can be represented explicitly whilst many other solutions cannot. It turns out that the PME in one dimension or in the radial geometry in $\mathbb{R}^N$ admits a countable subset of different similarity solutions (see [56] and further details in Section 3.3)

$$u_k(x,t) = t^{-\alpha_k}\psi_k(x/t^{\beta_k}), \tag{3.13}$$

where $k = 0, 1, 2, \ldots$ in the one-dimensional case and $k = 0, 2, 4, \ldots$ in the radial $\mathbb{R}^N$ case. Substituting (3.13) into the PME yields that $\psi_k \in C_0(\mathbb{R}^N)$ satisfies a quasilinear elliptic equation in $\mathbb{R}^N$. The exponents $\{\alpha_k, \beta_k\}$ are determined by whether such a nonlinear eigenvalue problem can be solved, as explained in Section 3.3. In view of the scaling invariance of the PME, the nonlinear eigenfunction subset

$$\Phi = \{\psi_k\},$$

consisting of a countable number of continuous families, is rather complicated. For $N \geq 2$ the non-radial eigenfunctions are not known except the second dipole-like one [57].

3.1.2 Main results: evolution completeness, connection with the linear theory and extensions

We show that Φ is *evolutionary complete*, i.e., describes the asymptotics of arbitrary global solutions of the PME with any $\hat{u} \in C_0$. In Section 3.4 we introduce our definition of evolution completeness and next, in Sections 3.5 and 3.6, present a proof for the one-dimensional and radial settings, where a key ingredient of the analysis is based on Sturm's Theorem on the zero set for parabolic equations.

It is worth mentioning a link between evolution completeness and classical results in linear operator theory. In the linear case $m = 1$, where the PME (3.5) becomes the canonical *heat equation*

$$u_t = \Delta u, \tag{3.14}$$

the evolution completeness follows from the completeness and closure of the eigenfunction subset for the linear self-adjoint operator

$$\mathbf{B}_1 = \Delta + \tfrac{1}{2} y \cdot \nabla + \tfrac{N}{2} I \tag{3.15}$$

in a weighted L^2-space. For convenience, we present a collection of related "linear" results in Section 3.2. Furthermore in Section 3.7, we show that eigenfunctions of $\mathbf{B}_1$ can be used in a branching analysis of nonlinear eigenfunctions for the PME occurring at $m = 1^+$.

The results on the evolution completeness can be extended to the quasilinear *p-Laplacian equation*

$$u_t = \nabla \cdot (|\nabla u|^{m-1} \nabla u), \tag{3.16}$$

and to the fully nonlinear *dual PME*

$$u_t = |\Delta u|^{m-1} \Delta u, \tag{3.17}$$

which admit known subsets of similarity solutions (recall that (3.17) reduces to the PME by the change $\Delta u \mapsto u$).

The PME in a bounded domain. It has been known since the 1970s that the PME in a *bounded* domain $\Omega \subset \mathbb{R}^N$ with the Dirichlet boundary condition $u = 0$ on the smooth boundary $\partial\Omega$ admits separable solutions that are simpler than (3.13),

$$u_k(x,t) = t^{-1/(m-1)}\psi_k(x), \quad k = 0, 1, 2, \ldots,$$

where each $\psi_k \neq 0$ satisfies a nonlinear elliptic equation

$$\Delta(|\psi_k|^{m-1}\psi_k) + \tfrac{1}{m-1}\psi_k = 0 \quad \text{in } \Omega, \qquad \psi_k = 0 \text{ on } \partial\Omega. \tag{3.18}$$

The existence of a countable subset $\Phi = \{\psi_k\}$ of these nonlinear eigenfunctions follows from the Lusternik-Schnirel'man theory of calculus of variations. The first similarity pattern $t^{-1/(m-1)}\psi_0(x)$, where $\psi_0 > 0$ in Ω, is known to be asymptotically stable as $t \to \infty$ and attracts all nontrivial solutions with integrable initial data $\hat{u} \geq 0$ [4]. The eigenfunction subset Φ given by (3.18) for the PME in Ω is much simpler than that of (3.13) for $\mathbb{R}^N$ (in fact, problem (3.18) fixes the single eigenvalue $\lambda_0 = -\frac{1}{m-1}$ of infinite multiplicity). As a consequence, if Φ is discrete then it is evolutionary complete [38], although proving that Φ is discrete for almost all smooth domains Ω is a challenging problem.

3.2 Discrete spectrum of a linear self-adjoint operator

The evolution completeness analysis leads to a complicated nonlinear eigenvalue problem. On the other hand, the linear counterpart for $m = 1$ (the heat equation (3.14)) deals with the standard theory of self-adjoint operators, with which it is convenient to begin. The spectral analysis of the classical singular linear Sturm-Liouville eigenvalue problem given below shows what spatial "shapes" of nonlinear eigenfunctions are expected to exist for $m > 1$, and allows us to observe what kind of branching is possible as $m \to 1^+$. These questions will be studied in Section 3.7.

For future comparison with properties of nonlinear parabolic equations, we briefly describe some well known facts concerning the linear diffusion operator with $m = 1$. Consider the heat equation (3.14) in $\mathbb{R}^N \times \mathbb{R}_+$ with initial data

$u(x,0) = \hat{u}(x) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. Let $b(x,t)$ denote the fundamental solution of the operator $\partial/\partial t - \Delta$:

$$b(x,t) = t^{-N/2} f(y), \;\; y = x/\sqrt{t}, \;\; \text{where } f(y) = (4\pi)^{-N/2} \mathrm{e}^{-|y|^2/4}. \tag{3.19}$$

Then we have that

$$u(x,t) = b(t) * \hat{u}(x) = t^{-N/2} \int_{\mathbb{R}^N} f((x-z)t^{-1/2}) \hat{u}(z) \, \mathrm{d}z. \tag{3.20}$$

Here $f(y)$ satisfies $\int f = 1$ and is the unique radial solution of the elliptic equation $\mathbf{B}_1 f = 0$ with the operator $\mathbf{B}_1$ given in (3.15). It admits the symmetric representation

$$\mathbf{B}_1 \equiv \tfrac{1}{\rho} \nabla \cdot (\rho \nabla) + \tfrac{N}{2} I \;\text{ with weight }\; \rho = \mathrm{e}^{|y|^2/4}$$

and $\mathbf{B}_1 : H^2_\rho(\mathbb{R}^N) \to L^2_\rho(\mathbb{R}^N)$ is a bounded self-adjoint operator with compact resolvent and discrete spectrum $\sigma(\mathbf{B}_1)$, [13] .

In order to classify the asymptotic behaviour of solutions as $t \to \infty$, we introduce the rescaled variables corresponding to the fundamental solution (3.19),

$$u(x,t) = t^{-N/2} w(y,\tau), \;\; y = x/\sqrt{t}, \;\; \tau = \ln t : \mathbb{R}_+ \to \mathbb{R}.$$

Then the rescaled solution w satisfies the evolution equation

$$w_\tau = \mathbf{B}_1 w, \tag{3.21}$$

where $w(y,\tau)$ is a solution of the Cauchy problem for (3.21) in $\mathbb{R}^N \times \mathbb{R}_+$ with initial data given at $\tau = 0$ (hence, at $t = 1$)

$$w_0(y) = u(y,1) \equiv b(1) * \hat{u} = f * \hat{u}. \tag{3.22}$$

The linear operator $\partial/\partial\tau - \mathbf{B}_1$ is the rescaled version of the parabolic operator $\partial/\partial t - \Delta$ and the corresponding semigroup $\mathrm{e}^{\mathbf{B}_1 \tau}$ is obtained from (3.20) by rescaling. This gives the eigenfunction expansion of the solution

$$w(y,\tau) = \sum \mathrm{e}^{\lambda_\beta \tau} M_\beta(\hat{u}) \psi_\beta(y),$$

where $\lambda_\beta = -\frac{|\beta|}{2}$ and $\psi_\beta(y)$ are the eigenvalues and eigenfunctions of $\mathbf{B}_1$,

$$\mathbf{B}_1\psi_\beta = \lambda_\beta\psi_\beta \text{ in } \mathbb{R}^N, \;\; \psi_\beta \in H^2_\rho(\mathbb{R}^N), \tag{3.23}$$

and $M_\beta(\hat{u}) = \int z^\beta \hat{u}(z)\,\mathrm{d}z$ are the corresponding momenta of the initial datum w_0 (recall the relation (3.22) between w_0 and $\hat{u}$). We next derive an equivalent representation of the semigroup by using another rescaling

$$u = (1+t)^{-N/2}w, \;\; y = x/(1+t)^{1/2}, \;\; \tau = \ln(1+t) : \mathbb{R}_+ \to \mathbb{R}_+.$$

Then $w(y,\tau)$ solves the Cauchy problem for equation (3.21) with initial data $w_0(y) \equiv \hat{u}(y)$. Rescaling the convolution (3.20) yields

$$w(y,\tau) = (1-\mathrm{e}^{-\tau})^{-N/2}\int_{\mathbb{R}^N} f((y-z\mathrm{e}^{-\tau/2})(1-\mathrm{e}^{-\tau})^{-1/2})w_0(z)\,\mathrm{d}z \equiv \mathrm{e}^{\mathbf{B}_1\tau}w_0. \tag{3.24}$$

The explicit representation of the resolvent of $\mathbf{B}_1$ is then constructed by the classical descent method [29]. Let $\lambda \in \mathbb{C}$, and consider the auxiliary equation

$$w_\tau = \mathbf{B}_1 w - \mathrm{e}^{\lambda\tau}g \text{ for } \tau > 0, \;\; w(0) = 0,$$

where $g \in L^2_\rho(\mathbb{R}^N)$. Setting $w = \mathrm{e}^{\lambda\tau}v$, we obtain the equation

$$v_\tau = (\mathbf{B}_1 - \lambda I)v - g,$$

and hence

$$v(\tau) = -\int_0^\tau \mathrm{e}^{(\mathbf{B}_1-\lambda I)(\tau-s)}g\,\mathrm{d}s.$$

Setting $\tau - s = \eta$ and passing to the limit $\tau \to \infty$ yields that the limit $v(\infty) = -\int_0^\infty \mathrm{e}^{(\mathbf{B}_1-\lambda I)\eta}g\,\mathrm{d}\eta \equiv (\mathbf{B}_1 - \lambda I)^{-1}g$ exists provided that the integral converges. Using the semigroup representation (3.24) and performing the change of variable $\mathrm{e}^{-\eta} = z \in (0,1)$ yields the integral operator

$$(\mathbf{B}_1 - \lambda I)^{-1}g = \int_{\mathbb{R}^N} K(y,\zeta)g(\zeta)\,\mathrm{d}\zeta, \text{ with the kernel} \tag{3.25}$$

$$K(y,\zeta) = -\int_0^1 z^{\lambda-1}(1-z)^{-N/2}f((y-\zeta z^{1/2})(1-z)^{-1/2})\,\mathrm{d}z. \tag{3.26}$$

This is the integral representation of the resolvent of $\mathbf{B}_1$ which is known to be a compact operator in $L^2_\rho(\mathbb{R}^N)$ for all $\lambda \in \mathbb{C} \setminus \sigma(\mathbf{B}_1)$. We summarise the main spectral properties of $\mathbf{B}_1$ as follows; see [13].

Lemma 3.1 *The spectrum of* $\mathbf{B}_1$ *in* $L^2_\rho(\mathbb{R}^N)$ *consists of real eigenvalues*

$$\sigma(\mathbf{B}) = \{\lambda_\beta = -\tfrac{|\beta|}{2},\ |\beta| = 0, 1, 2, \ldots\}. \tag{3.27}$$

The eigenvalues λ_β *have finite multiplicity with eigenfunctions*

$$\psi_\beta(y) = c_\beta D^\beta f(y) \equiv c_\beta H_\beta(y) f(y),\ \ c_\beta = (2^{|\beta|}\beta!)^{-1/2}, \tag{3.28}$$

where $H_\beta(y)$ *are Hermite polynomials and* $f(y)$ *is the fundamental solution. The orthonormal subset of eigenfunctions* $\Phi = \{\psi_\beta\}$ *is complete and closed in* $L^2_\rho(\mathbb{R}^N)$.

For clarity we present a proof of this classical result.
Proof. Let $l = |\beta|$. The existence of such eigenvalues and eigenfunctions follows by applying differentiation D^β to the elliptic equation (4.6)

$$D^\beta \mathbf{B}_1 f \equiv \mathbf{B}_1 D^\beta f + \frac{|\beta|}{2m} D^\beta f = 0. \tag{3.29}$$

It follows from the asymptotic analysis of the expansion (3.23) as $\tau \to \infty$ that no other eigenfunctions exist, all eigenvalues are real and are given in (3.28).
(ii) Completeness. Let us show that the system of the eigenfunctions $\{D^\beta f\}$ is complete in $L^2(\mathbb{R}^N)$. By the Riesz-Fischer theorem, we have to show that, given a function $g \in L^2(\mathbb{R}^N)$, the equalities

$$\int D^\beta f(x) g(x) dx = 0 \ \text{ for any } \ \beta \tag{3.30}$$

imply that $g = 0$. Let $F(\xi)$ and $G(\xi)$ be the Fourier transforms of f and g. Then

$$\int \xi^\beta F(\xi) G(-\xi) d\xi = 0 \ \text{ for any } \ \beta.$$

Applying the Fourier transform to equation (4.6) yields

$$|\xi|^2 F + \frac{1}{2}\xi \cdot \nabla F = 0,$$

and hence $F(\xi) = \mathrm{e}^{-|\xi|^2}$. Therefore,

$$\int \xi^\beta \mathrm{e}^{-|\xi|^2} G(-\xi) d\xi = 0 \ \text{ for any } \ \beta. \tag{3.31}$$

The function

$$M(z) = \int e^{-|\xi|^2} G(-\xi) e^{iz\xi} d\xi$$

is entire analytic in $\mathbb{C}^N$ (since $|e^{iz\xi}| \le e^{|\mathrm{Im}\, z||\xi|}$). Equality (3.31) means that $D^\beta M(0) = 0$ for any β. Therefore, $M(z) \equiv 0$. Thus, $G(\xi) = 0$ almost everywhere and $g = 0$. □

Evolution completeness of eigenfunctions. The eigenfunction expansion of $w(y, \tau)$, which is the solution of equation (3.21) with initial data $w_0 \in D(\mathbf{B}_1) = H^2_\rho(\mathbb{R}^N)$, takes the form

$$w(y,\tau) = \sum a_\beta e^{\lambda_\beta \tau} \psi_\beta(y), \quad a_\beta = \langle \hat{u}, \psi_\beta \rangle_\rho, \tag{3.32}$$

where $\langle \cdot, \cdot \rangle_\rho$ is the inner product in $L^2_\rho(\mathbb{R}^N)$. Since Φ is complete and closed in $L^2_\rho(\mathbb{R}^N)$ we have that, for any initial data $w_0 \in H^2_\rho(\mathbb{R}^N)$, $w_0 \neq 0$, there exists a finite integer $k = k(w_0) \geq 0$ such that, as $\tau \to \infty$,

$$w(y, \tau) = e^{-k\tau/2} [\psi_k(y) + o(1)], \tag{3.33}$$

where ψ_k is an eigenfunction of $\mathbf{B}_1$ with eigenvalue $-\frac{k}{2}$, i.e., $\psi_k = \sum_{|\beta|=k} b_\beta \psi_\beta$, where $\sum b_\beta^2 \neq 0$. Known spectral properties of $\mathbf{B}_1$ make it possible to give a complete description of the asymptotic patterns that can occur in the linear evolution equation (3.21). For the original heat equation (3.14) with initial data $\hat{u} \in H^2_\rho(\mathbb{R}^N)$, this gives a discrete subset of asymptotic patterns

$$\begin{aligned} u(x,t) &= t^{\lambda_k - N/2} \psi_k(y)(1 + o(1)), \\ y &= x/\sqrt{t}, \ \ \lambda_k = -k/2, \ \ k = 0, 1, 2, \dots . \end{aligned} \tag{3.34}$$

By completeness and closure, the eigenfunction subset Φ is also evolutionary complete in the sense that any nontrivial solution $u(\cdot, t) \in H^2_\rho(\mathbb{R}^N)$ has, for $t \gg 1$, the asymptotic behaviour (3.34) with a finite $k \geq 0$ that depends on initial data. Thus for such linear self-adjoint operators, the evolution completeness is a direct consequence of the standard completeness-closure of the eigenfunction subsets.

3.3 Nonlinear eigenfunctions and eigenvalues for the PME in one dimension and radial $\mathbb{R}^N$

We now return to the PME (3.5) and describe its nonlinear eigenfunctions. The PME is invariant under a group of scaling transformations and admits self-similar solutions that for convenience will be written in the following form:

$$u(x,t) = t^{\lambda-\mu_0}\psi(y), \;\; y = x/t^{\tilde{\beta}}, \tag{3.35}$$

where $\mu_0 = \frac{N}{N(m-1)+2}$ and $\tilde{\beta} = \frac{1+(m-1)(\lambda-\mu_0)}{2}$. In the linear case $m = 1$ we have $\tilde{\beta} = \frac{1}{2}$, $\mu_0 = \frac{N}{2}$ and these similarity solutions reduce to those given in (3.34). Substituting (3.35) into (3.5) yields that $\psi = \psi(y)$ is a weak solution of the following *nonlinear eigenvalue problem* in radially symmetric geometry:

$$\mathbf{B}_m(\psi) \equiv \Delta(|\psi|^{m-1}\psi) + \mathbf{C}_0\psi = \lambda\mathbf{C}\psi \;\text{ in }\; \mathbb{R}^N, \;\; \psi \in C_0(\mathbb{R}^N), \;\; \psi \neq 0, \tag{3.36}$$

where $\beta_0 = \frac{1}{N(m-1)+2}$ and $\mathbf{C}_0$, $\mathbf{C}$ are the linear first-order operators

$$\mathbf{C}_0 = \beta_0 y \cdot \nabla + \mu_0 I, \;\; \mathbf{C} = -\tfrac{1}{2}(m-1)y \cdot \nabla + I. \tag{3.37}$$

In most cases $\psi(y)$ is a typical example of similarity solutions of the *second kind* (a term introduced by Ya.B. Zel'dovich [88]), where suitable values $\lambda \in \mathbb{R}$ are obtained from whether the elliptic equation can be solved in the prescribed functional class; see details in [7]. For $m = 1$, where $\tilde{\beta} = 1/2$ and $\mathbf{C} = I$, (3.36) becomes the Sturm-Liouville eigenvalue problem (3.23) for the linear bounded self-adjoint operator $\mathbf{B}_1 : H^2_\rho(\mathbb{R}^N) \to L^2_\rho(\mathbb{R}^N)$ with real discrete spectrum (3.27). For $m > 1$, we have that $\tilde{\beta} = [1 + (m-1)(\lambda - \mu_0)]/2$ depends on the eigenvalues λ so that (3.36) is a nonlinear eigenvalue problem for a pencil of two operators, where $\mathbf{B}_m$ is nonlinear and $\mathbf{C}$ is a linear. Dealing with the nonlinear operators, we continue to denote the real "point" spectrum (real eigenvalues $\lambda = \lambda_k$) of operator (3.36) by $\sigma(\mathbf{B}_m)$, and then ψ are eigenfunctions (a standard terminology in the theory of nonlinear operators [65]).

3.3.1 Eigenfunctions in one dimension

The following Lemma (essentially Theorem 1.1 in [56]) gives all possible compactly supported similarity solutions (3.35) in the one-dimensional case.

Lemma 3.2 *Let $m > 1$ and $N = 1$. Then the eigenvalue problem* (3.36) *has a strictly decreasing sequence of eigenvalues*

$$\sigma(\mathbf{B}_m) = \{\lambda_k\} \downarrow -\tfrac{2}{m^2-1}, \quad \textit{where } \lambda_0 = 0 \textit{ and } \lambda_1 = -\tfrac{1}{m(m+1)}, \tag{3.38}$$

in the sense that (3.36) *has a compactly supported solution if, and only if, $\lambda = \lambda_k$ for some integer $k \geq 0$, and we normalise those such that*

$$\operatorname{supp} \psi_k = [-1, 1] \textit{ for any } k = 0, 1, 2, \dots . \tag{3.39}$$

Moreover, k equals exactly the number of sign changes of such $\psi_k(y)$, and $\psi_k(y)$ is symmetric (anti-symmetric) if k is even (odd).

One can see from the scaling (3.40) below that, for any function $\psi_k(y)$, there exists a b such that (3.39) holds. It follows from (3.38) that the exponents $\tilde{\beta} = \beta_k$ in (3.35) are strictly positive for all $k \geq 0$. When $N = 1$ equation (3.36) reduces to a first order ODE with the phase-plane studied by Barenblatt in the 1950s [6]. In this case the first eigenfunction $\psi_0 \geq 0$ was proved to exist for more general equations including gradient-dependent diffusion. The proof of Lemma 3.2 [56] is based on further delicate analysis of the phase-plane for different values of λ. In a particular representation, the phase-plane of (3.36) is known to admit limit cycles (see [17, 44, 83]) that generate a non-compactly supported profile $\psi_\infty(y)$. This profile has an infinite number of isolated zeros which are obtained as a result of an infinite number of rotations of the vector field. On this phase-plane the eigenfunctions $\{\psi_k\}$ with $\lambda = \lambda_k$ correspond to exactly k rotations around the origin; see details in [17].

In view of the scaling symmetry of equation (3.36), each ψ_k defines a one-parameter family of eigenfunctions

$$\psi_k(y; b) = b\psi_k(y/|b|^{(m-1)/2}) \text{ for any } b \neq 0. \tag{3.40}$$

Setting $m = 1$ in (3.38) yields precisely the spectrum (3.27) of the linear operator $\mathbf{B}_1$ in one dimension, and then scaling (3.40) reduces to multiplication by a constant b. It follows from (3.38) that in the linear case $m = 1$ the spectrum $\{\lambda_k\}$ is unbounded from below (a standard property of spectra of self-adjoint operators in Hilbert spaces with compact resolvents, [13]).

In Lemma 3.2 the first two eigenvalues λ_0 and λ_1 and the corresponding eigenfunctions are obtained explicitly by using two known conservation laws for the PME. Namely, $\lambda_0 = 0$ ($\beta_0 = 1/(m+1)$) corresponds to the ZKB solution (3.7) with $N = 1$, [90, 6] satisfying the mass conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int u(x,t)\ \mathrm{d}x = 0. \tag{3.41}$$

The ODE (3.36) is integrated twice leading to the first eigenfunction (3.8) with $b = 1$. For $\lambda_1 = -1/m(m+1)$, $\beta_1 = 1/2m$, the similarity solution is Barenblatt-Zel'dovich dipole solution (3.10) with $N = 1$ [8], which corresponds to the momentum conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int xu(x,t)\ \mathrm{d}x = 0. \tag{3.42}$$

Integrating the ODE (3.36) leads to the odd eigenfunction (3.11) with $b = 1$. Figure 3.1 shows the third and fourth eigenfunctions of the PME plotted alongside its first two eigenfunctions. These profiles were produced using the Matlab boundary value problem solver bvp4c (see Appendix B for details).

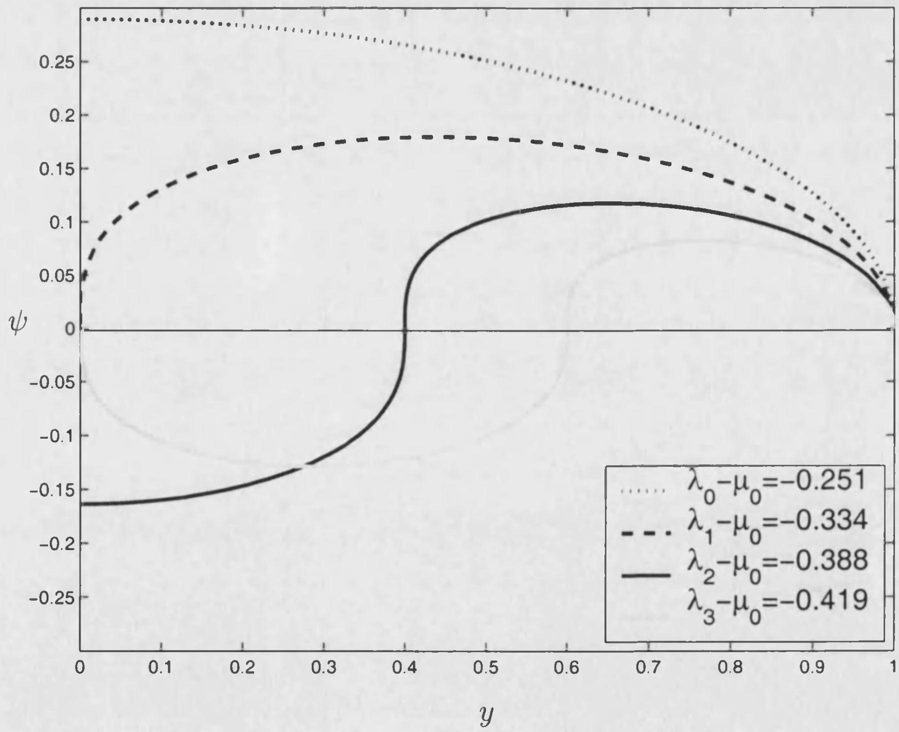


Figure 3.1: Four eigenvalues and their associated eigenfunctions of the PME with $m = 3$.

3.3.2 Eigenfunctions in radial geometry in $\mathbb{R}^N$

We now describe similarity patterns in $\mathbb{R}^N$, where the results can be formulated for radially symmetric solutions. The following Lemma is essentially Theorem 5.1 in [56].

Lemma 3.3 *Let $m > 1$ and $N > 1$. Then in the radial geometry, there exists a strictly monotone decreasing sequence of eigenvalues of the eigenvalue problem* (3.36)

$$\sigma(\mathbf{B}_m) = \{\lambda_k,\ k = 0, 2, 4, \ldots\} \downarrow -\tfrac{2}{(m-1)[N(m-1)+2]}, \quad \textit{where } \lambda_0 = 0, \tag{3.43}$$

so (3.36) *has radially symmetric, compactly supported solutions if, and only if, $\lambda = \lambda_k$ for some even integer $k \geq 0$, which is exactly the number of sign changes of $\psi_k(|y|)$ in $\mathbb{R}_+$. The normalisation condition is*

$$\operatorname{supp} \psi_k = \{|y| \leq 1\} \ \textit{for all}\ k = 0, 2, 4, \ldots. \tag{3.44}$$

Note that $\beta_k > 0$ for $k \geq 0$. Each profile ψ_k generates a one-parameter family (3.40) of solutions. As for $N = 1$, the first eigenvalue $\lambda_0 = 0$ corresponds to the ZKB solution (3.7) and ψ_0 is given by (3.8) with $b = 1$. Little is known for other non-radial eigenfunctions of the PME satisfying the elliptic equation (3.36). A multi-dimensional analogy of dipole pattern $\psi_1(y)$ [57] seems to be the only known non-radial nonlinear eigenfunction existing for all $m > 1$. (In Section 3.7 we present a branching analysis of nonlinear eigenfunctions applied for $m \approx 1^+$.)

According to the group of scalings (3.40), we specify the whole subset of nonlinear eigenfunctions of operator (3.36) as follows:

$$\Phi = \{\psi_k(y; b), k \geq 0, b \in \mathbb{R} \setminus \{0\}\}. \tag{3.45}$$

Thus, Φ consists of a countable subset of continuous one-parameter families of functions.

3.4 Notion of evolution completeness for the radial PME in $\mathbb{R}^N$

We consider the PME (3.5) in the radial setting with radial initial data $\hat{u} \in C_0$. A similar analysis applies to the equation in one dimension with arbitrary $\hat{u} \in C_0$, as explained below. We need an extra technical assumption on initial data

$$\hat{u} = \hat{u}(|x|) \text{ has a finite number of sign changes.} \tag{3.46}$$

Since the number of sign changes of $u(|x|, t)$ does not increase with time (Sturm's Theorem, see references in [39]), one can see that $u(0,t)$ can change sign only a finite number of times for all $t > 0$. We may then assume that $u(0,t) > 0$ for all $t \gg 1$. Otherwise, we replace $\hat{u} \mapsto -\hat{u}$ and hence $u \mapsto -u$. We cannot get rid of an assumption like (3.46) since a classification of local structures of zeros for the PME is still unknown or not rigorously justified in general. We let C_0^S denote the space of radial compactly supported continuous functions in $\mathbb{R}^N$ satisfying the Sturmian assumption (3.46).

It is convenient to rescale $u(x,t)$ according to (3.35) by setting

$$u(x,t) = (1+t)^{-\mu_0} v(y,\tau), \;\; y = x/(1+t)^{\beta_0}, \;\; \tau = \ln(1+t), \tag{3.47}$$

with $\mu_0 = \frac{N}{N(m-1)+2}$ and $\beta_0 = \frac{\mu_0}{N}$. Then $v(y,\tau)$ is a global solution of the rescaled equation

$$v_\tau = \mathbf{B}_m(v) \;\text{ for }\; \tau > 0, \;\; v(y,0) = v_0(y) \equiv \hat{u}(y), \tag{3.48}$$

where $\mathbf{B}_m$ is operator (3.36), so the profile (3.8) is stationary for this operator, $\mathbf{B}_m(\psi_0) = 0$. Recall that, under the above assumptions,

$$\phi(\tau) \equiv v(0,\tau) > 0 \;\text{ for all }\; \tau \gg 1. \tag{3.49}$$

For this rescaled nonlinear problem, we define evolution completeness as follows.

Definition 3.4 *The subset (3.45) of nonlinear eigenfunctions of problem (3.36) is evolutionary complete, if, for any initial data $\hat{u} \in C_0^S$, there exists a finite $k \geq 0$ and a constant $b \neq 0$ such that, as $t \to \infty$,*

$$w(z,\tau) \equiv \tfrac{1}{\phi(\tau)} v(z\phi^{(m-1)/2}(\tau), \tau) \to \psi_k(z;b) \;\; \textit{uniformly in } \mathbb{R}^N. \tag{3.50}$$

According to Lemma 3.3, for any fixed $k \geq 0$, we introduce the functional subsets

$$\mathcal{W}_k = \{v_0 \in C_0^S : \; \exists \; b \neq 0 \text{ such that } w(z,\tau) \to \psi_k(z;b) \text{ as } t \to \infty\}. \tag{3.51}$$

Then the evolution completeness assumes, in particular, that

$$\cup_{k \geq 0} \mathcal{W}_k = C_0^S \setminus \{0\}. \tag{3.52}$$

This implies that any $v_0 \neq 0$ belongs to $\mathcal{W}_k$ with some finite $k = k(v_0)$, and

$$\mathcal{W}_\infty = \{0\}, \tag{3.53}$$

where $\mathcal{W}_\infty$ is the set of initial data for which solutions $v(\cdot, \tau)$ have a "super-exponential" decay in the sense that, uniformly in $\mathbb{R}^N$,

$$v(y, \tau) = o(\mathrm{e}^{-K\tau}) \text{ as } \tau \to \infty \text{ for any constant } \quad K \gg 1. \tag{3.54}$$

The evolution completeness analysis consists of two parts.

3.5 First half: direct sum decomposition of C_0^S

Note that if $\hat{u}$ has nonzero mass, i.e., $M_0 = \int \hat{u} \neq 0$, then by the asymptotic stability of the ZKB similarity-solution, $v(y, \tau)$ is known to converge as $\tau \to \infty$ to the similarity profiles (3.8), (3.40) (or their reflection in the y axis if $M_0 < 0$) with the same mass (see further comments below). For general nonnegative data $\hat{u} \in L^1(\mathbb{R}^N)$, this is proved in [34]. For $\hat{u}$ changing sign with $M_0 > 0$, the "eventual positivity" for $t \gg 1$ for compactly supported solutions in radial geometry follows from intersection comparison techniques (see [77] and an "eventual monotonicity" approach in [48]).

We now consider general initial data with *zero mass* such that

$$v(\cdot, \tau) \to 0 \text{ as } \tau \to \infty \text{ uniformly.} \tag{3.55}$$

In order to clarify a possible asymptotic behaviour of the rescaled solution, we perform the extra rescaling for $\tau \gg 1$ given in (3.50), where the new rescaled function $w(z, \tau)$ satisfies the following perturbed parabolic equation:

$$w_\tau = \mathbf{B}_m(w) - g(\tau)\mathbf{C}w, \text{ with } \quad g(\tau) = \tfrac{\phi'(\tau)}{\phi(\tau)} \tag{3.56}$$

and $\mathbf{C}$ being the linear operator (3.37). Note that, by scaling (3.50), we have

$$w(0, \tau) \equiv 1 \text{ for } \tau \gg 1. \tag{3.57}$$

We now explain the main ingredients of the asymptotic analysis.

(i) **A bound on the rescaled orbits.** Without loss of generality we may assume that $\phi(\tau) \sim \max_y |v(y,\tau)|$ for $\tau \gg 1$, so that the orbit defined in (3.50) is uniformly bounded, and hence, by the standard parabolic theory of PME-type equations [27, 26], is compact in $C_{\rm loc}(\mathbb{R}^N)$. (By the Bernstein method, a uniform estimate $|(|u|^{m-1}u)_x| \le C$ is valid for $\tau \gg 1$.)

Alternatively, one can use another scaling:

$$\phi(\tau) = \max_y |v(y,\tau)|$$

(then $|w(z,\tau) \le 1$ for $\tau \gg 1$). This causes only minor changes in the analysis: Firstly, under hypothesis (3.46) $\phi'(\tau)$ exists for $\tau \gg 1$. Secondly, passing to the limit as in (3.60) we obtain another normalisation condition in (3.61):

$$\max_y |h(y,s)| \equiv 1 \ \text{ for } \ s \ge 0.$$

It is known that the intersection comparison argument guarantees that $h(\cdot, s)$ must be stationary (this is associated with the non-existence of inflection intersection points; see [50, p. 74]).

(ii) **First limit.** We claim for that there exists a finite limit along a sequence

$$g(\tau) \to \lambda \ \text{ as } \ \tau = \tau_j \to \infty. \tag{3.58}$$

(The proof of this claim is given by parts (iv) and (v) below. See particularly the remark after Proposition 3.5.) Then we claim that (see the oscillation analysis below)

$$g(\tau_j + s) \to \lambda \ \text{ uniformly on bounded intervals in } s. \tag{3.59}$$

In this case, setting $\tau = \tau_j + s$ and passing to the limit in equation (3.56) by using the standard regularity PME theory ((3.57) provides us with the crucial L^∞-estimate which makes it possible to pass to the limit in such PME-type equations via the general compactness result, [26]), we obtain that

$$w(\tau_j + s) \to h(s) \ \text{ in } \ L^\infty_{\rm loc}(\mathbb{R}_+; C_0), \tag{3.60}$$

where $h(s)$ solves the autonomous time-independent equation

$$h_s = \mathbf{B}_m(h) - \lambda \mathbf{C} h \ \text{ for } \ s > 0, \ \ h(0,s) \equiv 1. \tag{3.61}$$

By construction, $h(y, s)$ is a uniformly bounded weak C_0-solution.

The quasilinear parabolic equation (3.61) with a potential ordinary differential operator on the right-hand side is known to be a gradient system. The existence of a suitable integral Lyapunov function is proved by the general approach [91] with necessary modifications related to the degeneracy of the PME operator, [35, 19]. Furthermore, Sturm's Theorem prescribing the number of sign changes of solutions as a "discrete" Lyapunov function, guarantees that, in the gradient system (3.61), the non-empty ω-limit set $\omega(w_0)$ of any bounded orbit consists of stationary solutions only. Indeed, the boundary condition $h(0, s) \equiv 1$ implies by the strong Maximum Principle that $h(\cdot, s)$ must be a stationary solution. Further examples and references can be found in [39]. A different application of the Sturmian argument for solutions of changing sign will be presented below.

(iii) **$\boldsymbol{\lambda}$ is an eigenvalue.** The radial ODE

$$\mathbf{B}(f) - \lambda \mathbf{C} f = 0 \text{ in } \mathbb{R}^N, \;\; f \in C_0, \;\; f(0) = 1, \tag{3.62}$$

which coincides with the nonlinear eigenvalue problem (3.36) (note that the normalisation condition is different from (3.44)) must admit a solution. By Lemma 3.3 this means that the constant λ in (3.58) must coincide with one of the nonlinear eigenvalues of (3.36),

$$\lambda = \lambda_k \text{ for some } k \geq 0. \tag{3.63}$$

(iv) **Non-oscillatory property.** We now return to the condition (3.59). Assuming that it is not valid, we obtain a contradiction by the Sturmian intersection approach. A more detailed description of various aspects of the intersection comparison will be presented in later sections, where it is used for refined asymptotic estimates.

We may assume that $g(\tau)$ is oscillating around $\lambda = \lambda_k$ for $\tau \gg 1$ and that the oscillations are not small (in the sense that they have some minimum amplitude ε). We now compare two families of solutions. The first is the rescaled solution $v(y, \tau) = e^{\lambda_k \tau} \tilde{w}(z, \tau)$, with $z = y e^{-(m-1)\lambda_k \tau/2}$; the second is the rescaled (according to (3.47)) self-similar solutions (3.35), (3.40) which are denoted by $\bar{v}(y, \tau) = e^{\lambda_k \tau} \psi_k(z; b)$. The main idea of such a comparison is to show that each intersection of $g(\tau)$ with λ_k at some $\tau \gg 1$ would mean losing at least one intersection of $\tilde{w}(y, \tau)$ with the corresponding similarity solution $\psi_k(z; b_j)$, where we choose a particular scaling parameter $b = b_j$. To do this we must now prove that

$\tilde{w}(0, \tau)$ oscillates.

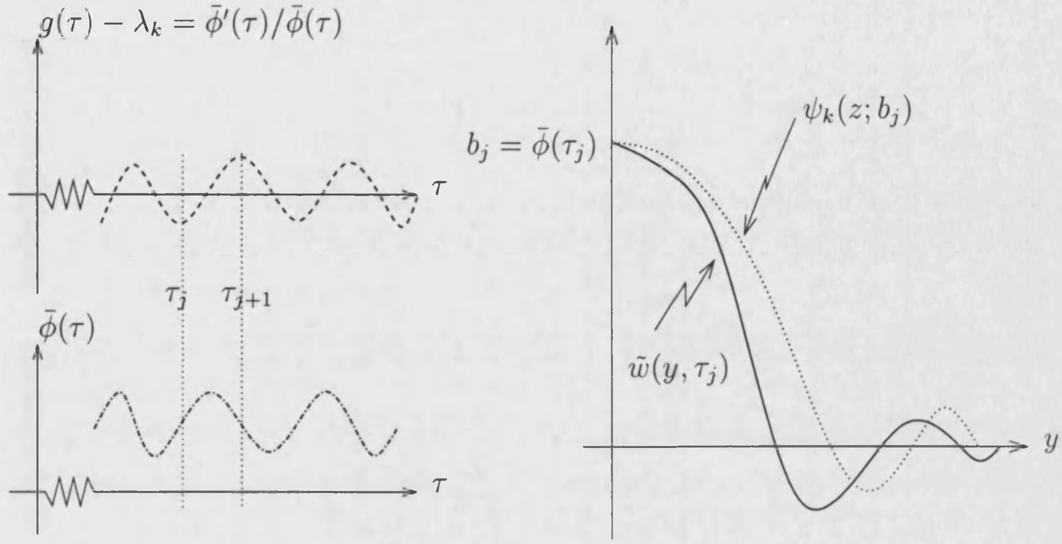


Figure 3.2: Setup and scaling.

We have assumed that $g(\tau)$ oscillates about some $\lambda_k < 0$. We now make the transformation

$$\phi(\tau) = \mathrm{e}^{\lambda_k \tau} \bar{\phi}, \tag{3.64}$$

so that $\bar{g}(\tau) = g(\tau) - \lambda_k = \bar{\phi}'(\tau)/\bar{\phi}(\tau)$ oscillates about zero. This is shown in Figure 3.2. For $\tau \gg 1$ we have $\phi > 0$ and hence $\bar{\phi} > 0$. The oscillations of $\bar{\phi}'(\tau)/\bar{\phi}(\tau)$ about zero now imply that $\bar{\phi}'(\tau)$ oscillates. Hence $\bar{\phi}(\tau) = \tilde{w}(0, \tau)$ oscillates as desired.

Since $\bar{g}(\tau)$ oscillates, we may form a sequence $\{\tau_j\} \to \infty$ such that, for all $j = 2, 4, 6, \ldots$,

$$\bar{g}'(\tau_j) = \bar{g}'(\tau_{j+1}) = 0 \quad \text{and} \quad \bar{g}(\tau_j)\bar{g}(\tau_{j+1}) < 0$$

(see Figure 3.2). We now chose b_j in (3.40) so that $\tilde{w}(0, \tau_j) = \bar{v}(0, \tau_j)$. This guarantees that at least one intersection between $\tilde{w}(y, \tau)$ and $\psi_k(z; b_j)$ disappears at $\tau = \tau_j$ (since $\bar{\phi}(\tau)$ has nonzero derivative here). By $\{b_j\}$ we denote the corresponding sequence of scaling parameters in $\psi_k(z; b_j)$.

First we consider the basic case where $\{b_j\}$ is uniformly bounded and is uniformly bounded away from zero. Passing to the limit $j \to \infty$ and using the compactness of the bounded family $\{\psi_k(z; b_j)\}$ of continuous functions, we find a profile $\psi_k(z; \bar{b})$ that has an infinite number of intersections with $\tilde{w}(y, \tau)$ for all $\tau \gg 1$. Since all the oscillations of $\bar{\phi}$ have an amplitude of at least $\varepsilon > 0$ there exists a family of self-similar solutions $\{\psi_k(z; b),\, b \in (\bar{b} - \varepsilon, \bar{b} + \varepsilon)\}$ such that each

$\psi_k(z;b)$ has an infinite number of intersections with $\tilde{w}$. This is impossible; see Proposition 7.1 in [37] which exactly that this cannot happen.

The analysis of the case where $b_j \to \infty$ is similar but we must perform an extra rescaling by using the group of transformations (3.40) with $b = b_j$ (leaving the rescaled equation invariant) for $\tau \approx \tau_j$ to get a bounded sequence and to repeat the above argument. We again assume that, after this extra rescaling, the oscillations are not small (otherwise we are done). Actually, in this intersection approach, the asymptotics of $\{b_j\}$ are not of principal importance since only the oscillatory property of $\tilde{w}(z,\tau)$ plays a key role. The case $b_j \to 0$ is similar with the same b-rescaling according to (3.40).

It now follows that $g(\tau)$ can only have a finite number of oscillations around λ_k. If $g(\tau)$ oscillates around another constant $\lambda \approx \lambda_k$, then the same argument applies where we have to use the similarity profiles satisfying equation (3.36) with the given λ. Then for $\lambda \notin \{\lambda_k\}$, ψ is not a solution in $\mathbb{R}^N$ and $\psi(y)$ is unbounded as $y \to \infty$. This simplifies the intersection comparison analysis with the bounded rescaled solution $\tilde{w}(y,\tau)$. We then compare the solutions on a bounded interval in y such that the necessary comparison is valid on the lateral boundary, where the difference does not change sign. In both cases, the assumption (3.58) implies (3.59) which makes it possible to pass to the limit $\tau_j + s \to \infty$.

Recalling that (3.63) holds for any partial limit (3.58), in view of discreteness of the spectrum $\{\lambda_k\}$, if there exists a partial limit (3.58), (3.63), then the function has the same limit,

$$\tfrac{\phi'(\tau)}{\phi(\tau)} \to \lambda_k \text{ as } \tau \to \infty. \tag{3.65}$$

Indeed, since $g(\tau)$ is continuous, the existence of two different partial limits $\lambda_* < \lambda^*$ would mean that any $\lambda \in (\lambda_*, \lambda^*)$ would correspond to a partial limit, i.e., problem (3.62) would have a nontrivial solution for a continuous interval of eigenvalues (meaning that $\omega(w_0)$ is connected) contradicting Lemma 3.3.

(v) **$\boldsymbol{\lambda = -\infty}$ is not possible.** We now need to rule out the possibility $\lambda = -\infty$ in (3.58). Actually, this shows that the super-exponential decay rate where

$$\phi(\tau) = o(\mathrm{e}^{-K\tau}) \text{ for } \tau \gg 1 \text{ with any } K \gg 1 \tag{3.66}$$

is possible for the trivial solution only, $v \equiv 0$.

It follows from the analysis presented above that we need to consider a function $\phi(\tau)$ such that

$$g(\tau) \to -\infty \text{ as } \tau \to \infty, \tag{3.67}$$

so that no finite partial limits (3.58) exist. Consider equation (3.56), where for convenience we replace $g(\tau) \mapsto -g(\tau)$ so that $g(\tau) > 0$ for $\tau \gg 1$. We perform the third scaling by setting

$$z = \zeta/\sqrt{g(\tau)}, \text{ where } s = -\ln \phi(\tau) \to \infty \text{ as } \tau \to \infty, \tag{3.68}$$

and then $w = w(\zeta, s)$ solves the perturbed equation

$$w_s = \mathbf{B}_\infty(w) + \rho(s)\zeta \cdot \nabla w + \tfrac{1}{g(\tau)}\,(\beta_0\zeta \cdot \nabla w + \mu_0 w), \;\; \rho(s) = -\tfrac{g'(\tau)}{2g^2(\tau)}, \tag{3.69}$$

$$\mathbf{B}_\infty(w) = \Delta|w|^{m-1}w - \tfrac{1}{2}(m-1)\zeta \cdot \nabla w + w. \tag{3.70}$$

It is important that if $g(\tau)$ has a super-exponential decay, then

$$\rho(s) \to 0 \text{ as } s \to \infty. \tag{3.71}$$

Therefore, passing to the limit along a subsequence $s = s_j \to \infty$, we have to have that $\omega(w_0) = \{f\}$ is non-empty and consists of nontrivial stationary solutions

$$\mathbf{B}_\infty(f) = 0 \text{ in } \mathbb{R}^N, \; f \in C_0, \; f(0) = 1. \tag{3.72}$$

Therefore we arrive at a contradiction in view of the following nonexistence result.

Proposition 3.5 *Problem* (3.72), (3.70) *does not have a solution.*

Proof. This follows from Lemma 3.3 describing all possible radial equations (3.36) ((3.72) belongs to the same type) admitting compactly supported solutions in $\mathbb{R}^N$.
□

Thus, (3.67) cannot happen. A similar argument to this holds to prove that $g(\tau) \not\to \infty$ and hence the claim (3.58) that $g(\tau)$ has a partial limit is proved true.

We have now established the first half of the evolution completeness theory for the radial PME in $\mathbb{R}^N$:

Theorem 3.6 *Let* (3.46) *hold. Then* (3.52) *is valid.*

It is important to note here that C_0^S cannot be replaced by L^1; see Appendix A.

3.6 Second half: uniqueness of the limit

Finally, we need to establish that, after the necessary rescaling in (3.50), (3.65), $\omega(w_0)$ consists of a *unique* similarity profile $\psi_k(\cdot; b)$. Recall that each nonlinear eigenfunction ψ_k generates by (3.40) a one-parameter family of stationary solutions, so that the unique choice of the parameter $b = b(\hat{u}) \neq 0$ is of principal importance. Such results based on using Sturm's Theorem on zero sets are well known in the asymptotic theory of PME-type equations but only for nonnegative solutions; see [1, 39, 48] and references therein. Below we present a detailed description of the modifications of the Sturmian analysis that are necessary to cover the case of solutions of changing sign. The regularity properties of solutions of changing sign and their interfaces are well known (though not in as much detail as for nonnegative solutions); see [12, 77] and references therein.

We will prove the following result finishing the evolution completeness analysis.

Theorem 3.7 *Let* (3.65) *hold with some finite* $k \geq 0$. *Then there exists a unique* $b \neq 0$ *such that*

$$\omega(w_0) = \{\psi_k(\cdot; b)\}. \tag{3.73}$$

We begin with the following auxiliary properties of the continuous branches of nonlinear eigenfunctions generated by scaling (3.40).

Proposition 3.8 *For any* $b_1, b_2 \in \mathbb{R}_+$, $b_1 \neq b_2$, *the profiles* $\psi_k(y; b_1)$ *and* $\psi_k(y; b_2)$ *have exactly* k *intersections.*

Proof. We first note that $\psi_k(y)$ is continuous and has exactly k sign changes [56]. Without loss of generality we study the one-dimensional problem and consider two cases.

Case 1: $\psi_k(y)$ is symmetric, k is even. Let us label the $\frac{k}{2}$ zeros of the profile $\psi_k(y; b_1)$ in the range $y > 0$ as $y_1, y_2, \ldots, y_{k/2}$, where $y_i < y_{i+1}$ for all i. Now label the zeros of the profile $\psi_k(y; b_2)$ in the same way as $\hat{y}_i$. Since $b_1 \neq b_2$ we may assume without loss of generality that $y_i < \hat{y}_i$ for all i; see the scaling (3.40). We have two sub-cases to consider:

(1a) b_1 is sufficiently close to b_2 to ensure that $\hat{y}_i \in (y_i, y_{i+1})$ for all i,

(1b) b_1 and b_2 are such that sub-case (1a) is not true.

Sub-case (1a). Note that the sign of the derivative with respect to y of $\psi_k(y; b_1)$ at y_i is the same as the sign of the derivative with respect to y of $\psi_k(y; b_2)$ at $\hat{y}_i$, since these zeros are simply scalings of one another. Due to the fact that there is no zero of $\psi_k(y; b_1)$ in $(y_{i+1}, \hat{y}_{i+1})$, we have that the sign of $\psi_k(y; b_1)$ remains the same for all $y \in (y_{i+1}, \hat{y}_{i+1})$. For definiteness let's say $\psi_k(y; b_1) < 0$ for $y \in (y_i, \hat{y}_i)$. Thus at

$y = \hat{y}_i$, $\psi_k(y; b_2) > \psi_k(y; b_1)$ since $\psi_k(\hat{y}_i; b_2) = 0$. Now at $y = \hat{y}_{i+1}$, $\psi_k(y; b_1) > 0$ so $\psi_k(y; b_2) < \psi_k(y; b_1)$. Thus, due to continuity, $\psi_k(y; b_2) - \psi_k(y; b_1)$ has at least one sign change in $(y_i, \hat{y}_{i+1})$ and so the profiles have at least one intersection in this interval. It remains to prove that they can have at most one intersection in this interval. This is done by means of the Maximum Principle.

Consider only the interval $(y_i, \hat{y}_{i+1})$ and assume that $\psi_k(y; b_1) < 0$ for $y \in (y_i, y_{i+1})$, so $\psi_k(y; b_2) < 0$ for $y \in (\hat{y}_i, \hat{y}_{i+1})$ (the proof is similar for the opposite sign). Now by the Maximum Principle, neither profile can have a local maximum in the intervals where they are negative. Neither can they have a point of inflection since this contradicts the Maximum Principle for their derivative with respect to y. Thus, both functions are convex in the region where they are negative and may only intersect each other once as a result of this and the ordering of their zeros. In the region where one function is positive, there can be no intersections since either $\psi_k(y; b_1) > \psi_k(y; b_2)$ or vice versa. Hence, they may intersect only once in the region $(y_i, \hat{y}_{i+1})$. This proves that there are $\frac{k}{2} - 1$ intersections in the range $(y_1, \hat{y}_{k/2})$.

The profiles must also intersect once in the interval $(\hat{y}_{k/2}, b_2^{(m-1)/2})$, i.e., between the last zero of $\psi(y; b_2)$ and its interface, since $\hat{y}_{k/2} < b_1^{(m-1)/2} < b_2^{(m-1)/2}$. Thus, the profiles have exactly $\frac{k}{2}$ intersections for $y \in (0, b^{(m-1)/2})$ and exactly k intersections overall.

Sub-case (1b). This is done by means of an evolution argument. Fix $b_1 = 1$. We pick b_2 such that we are in sub-case (1a) and then let b_2 vary to show that evolution in b_2 does not destroy or create intersections.

We first note that by the known regularity for ODE (3.36), all the intersections between the two profiles are transversal in terms of the variable $|\psi_k|^{m-1}\psi_k$, i.e., at any point of intersection,

$$\tfrac{d}{dy}[|\psi_k(y; b_2)|^{m-1}\psi_k(y; b_2) - |\psi_k(y; b_1)|^{m-1}\psi_k(y; b_1)] \neq 0. \qquad (3.74)$$

We have already showed that this difference cannot have a *point of inflection*, so the only possible type of intersection is transversal. These intersections cannot be lost without creating a situation in which one profile is tangent to the other. Then we have two solutions of the same ODE, which violates uniqueness, so this cannot occur; see Figure 3.3. No intersections may be gained for the same reason. Hence, the k transversal intersections found in sub-case (1a) remain for all values of b_2.

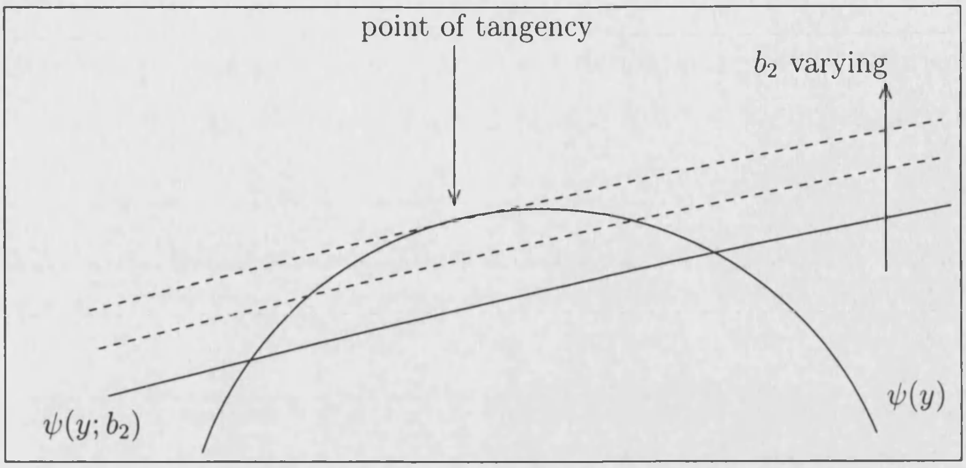


Figure 3.3: Two neighbouring intersections cannot be lost.

Case 2: $\psi_k(y)$ is anti-symmetric. This case differs from the symmetric case in that $\psi_k(y)$ passes though $(0,0)$ and thus has a fixed point under scaling (3.40). However, this does not create any problems since for $b_1 < b_2$ we have that $\psi_k(y; b_1) < \psi_k(y; b_2)$ for all $y \in (0, \hat{y}_1)$. The proof is very similar to the symmetric case and we find that there are exactly k intersections: $\lfloor \frac{k}{2} \rfloor$ in the domains $y > 0$ and $y < 0$ and one at $y = 0$. (Here $\lfloor \cdot \rfloor$ denotes the floor function for real numbers: $\lfloor s \rfloor$ is the largest integer j such that $j \leq s$.) □

Unperturbed equation. At this moment, assuming that (3.60) holds and, for convenience, replacing $h(s)$ again by $w(\tau)$, we study the ω-limit set for the rescaled equation (3.56) with $g(\tau) \equiv \lambda_k$ according to (3.65),

$$w_\tau = \mathbf{B}_m(w) - \lambda_k \mathbf{C} w \quad \text{for} \quad \tau > 0, \quad w(0) = f \in \omega(w_0). \tag{3.75}$$

By the regularity results for the PME [12, 27, 58, 77], we will use the fact that the rescaled solution (3.50) satisfies $|w|^{m-1}w \in C^1$ at least for all $\tau \gg 1$ (cf. typical results in [77] establishing by Sturm's Theorem that the solutions and interfaces attain extra regularity eventually in time). For ease of notation, we let $w^m(y,t) = |w(y,t)|^{m-1}w(y,t)$ and prove the following result.

Theorem 3.9 *The ω-limit set of the orbit of* (3.75) *consists of a single profile, i.e., there exists a unique constant $b \neq 0$ such that $\omega(f) = \{\psi_k(\cdot; b)\}$.*

Our study consists of two parts. Firstly, we extend Proposition 3.8 to intersection comparison with the rescaled solution $w(y, \tau)$ of (3.75).

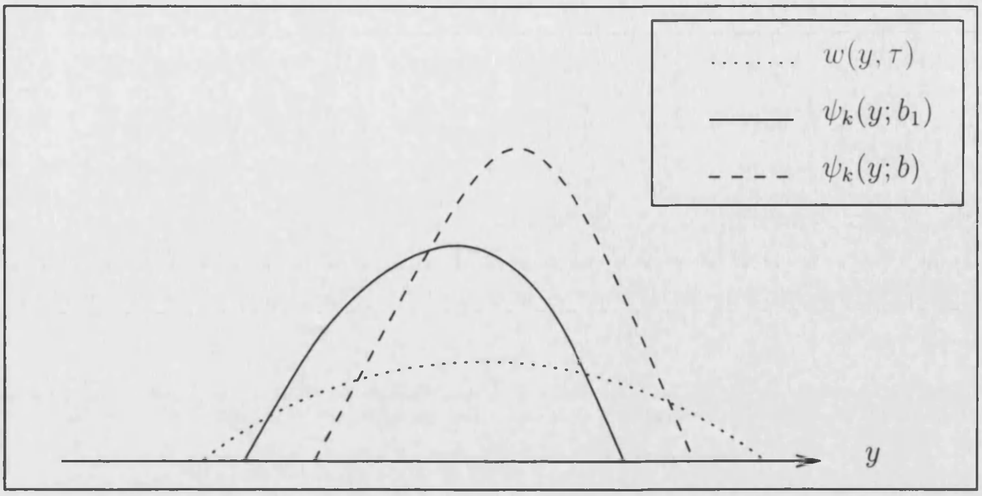


Figure 3.4: This situation can arise if $|b_1 - b| < \varepsilon_2$.

Lemma 3.10 *Let $\psi_k(\cdot;b_1) \in \omega(w_0)$ for some $b_1 \neq 0$, i.e., $w^m(y,\tau) \to \psi_k^m(y;b_1)$ uniformly along the sequence $\{\tau_j\} \to \infty$. Then there exist sufficiently small positive ε_1 and $\varepsilon_2 = \varepsilon_2(\varepsilon_1)$ such that when $\|w^m(y,\tau) - \psi_k^m(y;b_1)\|_{C^1} < \varepsilon_1$, $w(y,\tau)$ and $\psi_k(y;b)$ have exactly k intersections for all b such that $|b_1 - b| > \varepsilon_2$.*

Proof. We know that $w(y,\tau)$ can only intersect $\psi_k(y;b)$ in a set of neighbourhoods of the intersections between $\psi_k(y;b_1)$ and $\psi_k(y;b)$ defined by $N = \{y : \|\psi_k^m(y;b_1) - \psi_k^m(y;b)\|_C < \varepsilon_1\}$. We must now prove that $w(y,\tau)$ can only intersect $\psi_k(y;b)$ once in each of these neighbourhoods. The condition on b is needed to avoid

$$\|\psi_k^m(y;b_1) - \psi_k^m(y;b)\|_{C^1} < \varepsilon_1,$$

as this means that $w(y,\tau)$ could intersect $\psi_k(y;b)$ many times. This difficulty is shown in Figure 3.4: $\psi_k(y;b_1)$ and $\psi_k(y;b)$ intersect each other once on this range but $w(y,\tau)$ intersects both of them twice.

We label all the intersections between $\psi_k^m(y,b_1)$ and $\psi_k^m(y;b)$ by $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_k$. Let us consider a single, but arbitrary, intersection $\tilde{y}_i$. We now consider the subset $\tilde{N} \subset N$ that is a neighbourhood of $\tilde{y}_i$. Since all intersections between $\psi_k^m(y,b_1)$ and $\psi_k^m(y;b)$ are transversal, there is a positive angle between the gradients of the two curves at the point of intersection. Without loss of generality, assume that the gradient of $\psi_k^m(y,b_1)$ is strictly less than the gradient of $\psi_k^m(y;b)$ throughout $\tilde{N}$. Now choose ε_1 small enough to ensure that the gradient of $w^m(y,\tau)$ is also strictly less than the gradient of $\psi_k^m(y;b)$ for all $y \in \tilde{N}$. Thus, $w^m(y,\tau)$ may only intersect $\psi_k^m(y;b)$ once in $\tilde{N}$ provided ε_1 is small enough. □

Secondly, we need an extended version of the "tail lemma" (cf. [48, Lemma 8.2]) so we can apply it to solutions changing sign. This is needed to prove that the undesirable situation in Figure 3.5 may not occur for large τ. Suppose that, as shown in Figure 3.5, $w(y,\tau)$ is C^1-close to $\psi_k(y;b_2)$ but has a small negative tail. Now $w(y,\tau)$ can move freely between $\psi_k(y;b_2)$ and $\psi_k(y;b_1)$ without losing an intersection with any intermediate profile $\psi_k(y;b_*)$. This can be seen by recalling that each compactly supported solution is defined in the whole space: $\psi_k(y;b_*) \equiv 0$ for $y \notin \operatorname{supp}\psi_k(y;b_*)$, so $w(y,\tau)$ intersects $\psi_k(y;b_2)$ even when $y \notin \operatorname{supp}\psi_k(y;b_2)$.

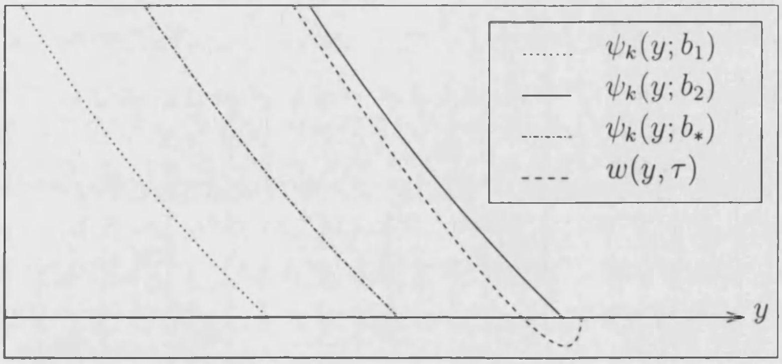


Figure 3.5: Our solution can move freely between the profiles $\psi_k(y;b_2)$ and $\psi_k(y;b_1)$ without losing an intersection with any intermediate profile $\psi_k(y;b_*)$.

As above, we assume that $\psi_k(\cdot;b) \in \omega(w_0)$, and, for definiteness, we also assume that $\psi_k(y;b) > 0$ for $y \in (y_1, y_0)$, where $y_1 < y_0 = b^{(m-1)/2}$ and y_1 is the largest zero of $\psi_k(y;b)$.

Lemma 3.11 *Let $\varepsilon > 0$ be small enough and $\|w^m(y,\tau) - \psi_k^m(y;b)\|_{C^1} < \varepsilon$ for some time $\tau = \tau_j \gg 1$. There exists positive $\bar{\varepsilon}(\varepsilon) \sim \varepsilon$ as $\varepsilon \to 0$ such that if* $\operatorname{supp} w(y,\tau_j) > \operatorname{supp}\psi_k(y;b) + \bar{\varepsilon}$, *then there exist $s > 0$ and small $\delta > 0$ both independent of ε such that $w(y,\tau_j+s) = 0$ for all $y \geq y_0 + \delta$ and $w(y,\tau_j+s) \geq 0$ for $y \geq \frac{1}{2}(y_0+y_1)$.*

Proof. We look for a weak super-solution in the form of a travelling wave. Set

$$\bar{w}(y,\tau) = h(\eta) \geq 0, \quad \eta = y + \lambda\tau \text{ with } \lambda > 0, \text{ so that}$$
$$\mathbf{D}(h) \equiv (h^m)'' + (\beta_k y - \lambda)h' + \alpha_k h \leq 0$$

in the positivity domain plus typical regularity at the interface (see below). Actually, we need this super-solution for a local comparison in a neighbourhood of

$y = y_0$, namely for $y > \frac{1}{2}(y_0 + y_1)$. Assuming that $h' \leq 0$, we have that

$$\mathbf{D}(h) \leq \bar{\mathbf{D}}(h) \equiv (h^m)'' + (\beta_k y_1 - \lambda)h' + \alpha_k h. \tag{3.76}$$

Here $\bar{\mathbf{D}}$ is an operator with constant coefficients so it admits a standard weak super-solution $h(\eta) = A(\eta_0 - \eta)_+^{1/(m-1)}$ for some constant $A > 0$ and arbitrary $\eta_0 > y_0$. Then (3.76) for $\tau = 0$ ($\eta = y$) reads

$$\tfrac{m}{m-1}A^{m-1} + (\lambda - \beta_k y_1) + (m-1)\alpha_k(\eta_0 - y) \leq 0,$$

where, for comparison from below, we need only to check this inequality in a sufficiently small neighbourhood of $y = \eta_0^-$. For instance, we can take

$$\lambda = \tfrac{1}{2}\beta_k y_1 \text{ and } A^{m-1} \leq \tfrac{\beta_k y_k}{4m(m-1)}, \tag{3.77}$$

and this provides us with the required super-solution. This super-solution moves to the left and destroys the positive part of the tail that is far from the interface of $\psi_k(y; b)$ by the usual comparison, and the tail is destroyed up to a certain small right-hand δ-neighbourhood of point $y = y_0$. This process is shown on Figures 3.6(a), 3.6(b) and 3.6(c).

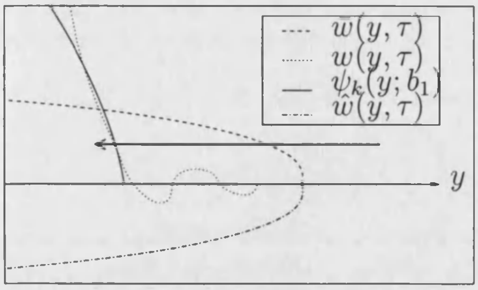
Concerning the negative part of the tail, the same comparison idea can be used to destroy it completely by means of a sub-solution $\hat{w} = -\bar{w}$. Then, since $w(y, \tau) > 0$ on $[y_1 + \delta, y_0 - \delta]$, obviously, the negative part of the small tail will be destroyed up to $y = y_1 + \delta$. See Figures 3.6(c) and 3.6(d). □

The rest of the analysis uses the same intersection comparison ideas as for nonnegative solutions; cf. [39, 48] and [1].

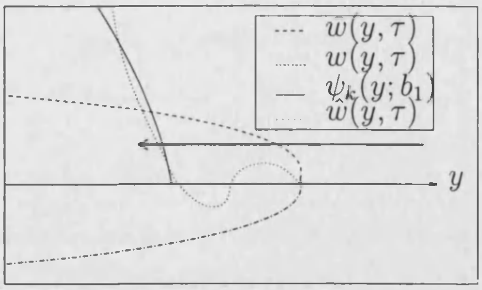
Lemma 3.12 *Let $b_1 < b_2$ be fixed. Assume that there exists a small $\varepsilon > 0$ such that at some time $\tau_j \gg 1$, $w(y, \tau)$ is such that $\|w^m(y, \tau_j) - \psi_k^m(y; b_1)\|_{C^1} < \varepsilon$, and for some time $\bar{\tau}_j \gg \tau_j$, we have $\|w^m(y, \bar{\tau}_j) - \psi_k^m(y; b_2)\|_{C^1} < \varepsilon$.*

Then $w(y, \bar{\tau}_j)$ has lost at least one intersection with all profiles $\psi_k(y; b_*)$ with b_* satisfying $|b_1 - b_*| > \delta(\varepsilon) > 0$ and $|b_2 - b_*| > \delta(\varepsilon) > 0$.

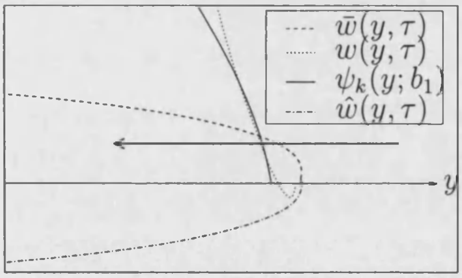
Proof. Firstly, note that if we have an ordered system $\{\psi_k(\cdot; b), b \geq 0\}$, i.e., none of the profiles $\psi_k(y; b)$ intersect each other (this happens for $k = 0$ only), then it is impossible for $w(y, \tau)$ to move from being close to $\psi_k(y; b_1)$ to being close to $\psi_k(y; b_2)$ by the usual comparison. Thus, there must be at least one intersection between all profiles for this movement to occur and this is where the principles



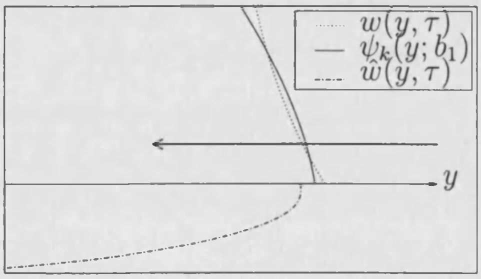
(a) Initial situation.



(b) $\bar{w}$ begins to destroy the tail.



(c) $\bar{w}$ has destroyed the positive tail far from the interface and now comparison with $\bar{w}$ begins to fail. We won't need $\bar{w}$ any further.



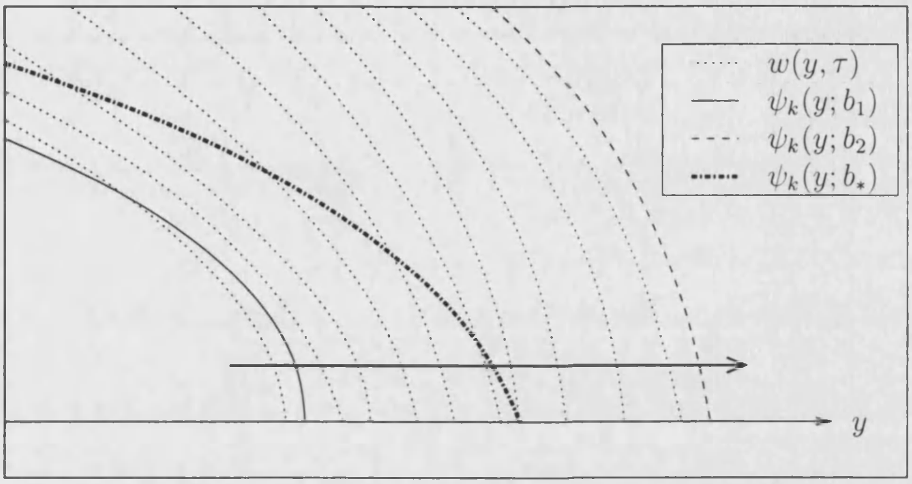
(d) Now $\hat{w}$ has completely destroyed the negative tail.

Figure 3.6: The travelling wave $\bar{w}$ destroys the tail as it moves in the direction of the arrow.

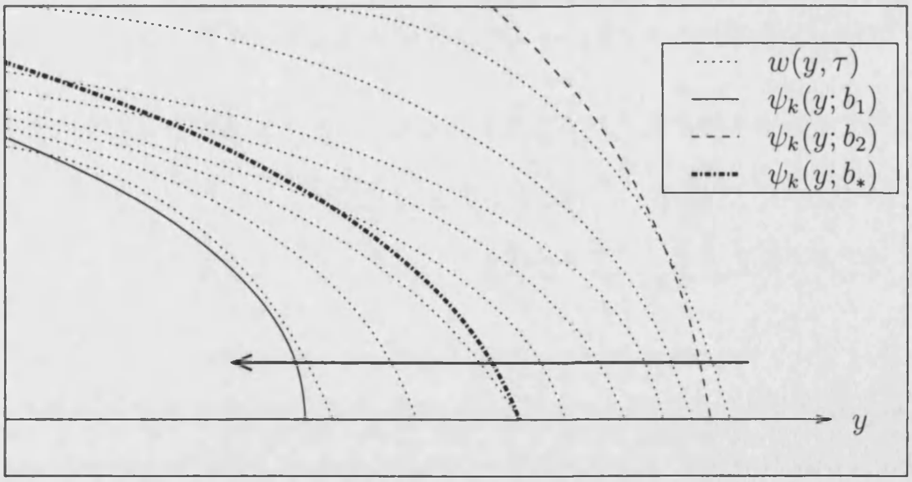
of intersection comparison apply. We now show that one of these intersections will be lost during this movement.

We only sketch the idea for the remainder of the proof. Details can be found in [48, Lemma 8.2]. Consider only the interfaces of the profiles $\psi_k(y;b_1)$ and $\psi_k(y;b_2)$. We have shown by Lemma 3.11 that if $\|w^m(y,\tau) - \psi_k^m(y,b_2)\|_{C^1} < \varepsilon$ for some small positive ε, then in a region of their interfaces $w(y,\tau)$ and $\psi_k(y;b_1)$ have the same sign. Also, if $w^m(y,\tau)$ is C^1-close to $\psi_k^m(y;b_1)$, then Lemma 3.10 implies that $w(y,\tau)$ must have precisely k zeros with all other profiles $\psi_k(y;b_*)$ with intermediate values b_*. Figure 3.7 shows how $w(y,\tau)$ can move from being C^1-close to $\psi_k(y;b_1)$ to being C^1-close to $\psi_k(y;b_2)$. Note that an intersection is lost with any intermediate profile $\psi_k(y;b_*)$ at its interface. This means that at least one intersection must be lost with each intermediate profile during this transition time. $\square$

Proof of Theorem 3.9. Assume that there are two distinct values $0 < b_1 < b_2$



(a) Passage from $\psi_k(y;b_1)$ to $\psi_k(y;b_2)$.



(b) Passage from $\psi_k(y;b_2)$ to $\psi_k(y;b_1)$.

Figure 3.7: An intersection is lost with any intermediate profile on its interface. The interface of $w(y,\tau)$ moves in the direction of the arrow as time increases.

such that $\psi_k(\cdot; b_{1,2}) \in \omega(f)$. In this case there must be sequences $\{\tau_j\} \to \infty$ and $\{\bar{\tau}_j\} \to \infty$ such that

$$w(\tau_j) \to \psi_k(\cdot, b_1) \text{ as } j \to \infty, \text{ but } w(\bar{\tau}_j) \to \psi_k(\cdot, b_2) \text{ as } j \to \infty.$$

We may now arrange for an infinite number of profiles to lie between $\psi_k(y; b_1)$ and $\psi_k(y; b_2)$ close to their interfaces by picking values of b_* in between b_1 and b_2. Taking ε_1, and hence ε_2, small enough in Lemma 3.10 we may pick b_*, $b_1 < b_* < b_2$, such that $|b_* - b_1| > \varepsilon_2$ and $|b_* - b_2| > \varepsilon_2$, so if $w(y, \tau)$ is close enough to either $\psi_k(y; b_1)$ or $\psi_k(y; b_2)$ then it has exactly k intersections with each of the profiles $\psi_k(y; b_*)$.

Now at some time $\tau_j \gg 1$, $\|w(y, \tau_j) - \psi_k(y; b_1)\| < \varepsilon_1$ and thus $w(y, \tau_j)$ has exactly k intersections with some $\psi_k(y; b_*)$. Using Lemma 3.12, we now see that if $w(y, \tau)$ were to converge to $\psi_k(y; b_2)$ it would lose an intersection with this intermediate profile. Since we know by Sturm's Theorem that intersections cannot be gained, we have a contradiction to Lemma 3.10, which states that if $w(y, \tau)$ is close to $\psi_k(y; b_2)$ then it has exactly k intersections with our $\psi_k(y; b_*)$. $\square$

Perturbed equation. Returning back to the full rescaled equation (3.56) and passing to the limit (3.60), we have that the constant b in Theorem 3.9 does not depend on the sequence $\{\tau_j\} \to \infty$ in view of equality (3.57), which itself selects the unique limit profile.

Completeness in one dimension: the end of the proof for odd k. For $N = 1$, Lemma 3.2 gives the complete description of all the nonlinear eigenfunctions. The asymptotic completeness analysis remains the same if k is even. However, for odd values of k, the scaling function (3.49) is not suitable since for the exact similarity solutions, $v(0, \tau) \equiv 0$. In this case one needs to pick another scaling function, e.g.

$$\phi(\tau) = \sup_\xi v(\xi, \tau). \tag{3.78}$$

Then, in order to ensure that $\phi(\tau)$ is C^1-smooth for $\tau \gg 1$, we need to impose an extra condition on initial data (cf. (3.46))

$$\hat{u}(x) \text{ has a finite number of extrema,} \tag{3.79}$$

which guarantees that for $\tau \gg 1$, $v(y, \tau)$ has isolated extrema so that $\phi(\tau)$ is smooth. The proof is similar to the property of eventual monotonicity, [48].

Thus, scaling (3.50) will provide us with a uniformly bounded rescaled orbit and rescaled equation (3.56). The rest of the analysis including the uniqueness conclusion contains no novelties.

3.7 The branching of nonlinear eigenfunctions at $m = 1$

In non-radial geometry, (3.36) is a difficult open nonlinear eigenvalue problem. The operators involved are not potential and the problem does not admit a variational setting. Therefore, the classical Lusternik-Schnirel'man category theory [65, Chapter 8] does not apply. Nevertheless, we expect (3.36) to admit at least a countable subset of nonlinear eigenfunctions (different up to the scaling (3.40)), which is a typical feature of potential operators with uniformly differentiable even functionals, [65, Theorem 57.2].

3.7.1 Derivation of the branching equation

We will apply the classical perturbation, branching approach to problem (3.36) using the known eigenfunctions of the linear eigenvalue problem (3.23) corresponding to $m = 1$. Bifurcations of non-radial eigenfunctions from known radial ones are not expected to occur at a sequence of critical exponents $\{m = m_k > 1\}$. Such an approach is fruitful for other types of nonlinear operators corresponding to finite time blow-up (focusing) self-similar phenomena in reaction-diffusion problems; see bifurcation scenarios in [17] and [2], where countable sequences of bifurcation exponents actually occur.

Thus we set

$$m = 1 + \varepsilon, \quad \text{where} \quad 0 < \varepsilon \ll 1, \tag{3.80}$$

and fix an eigenvalue $\lambda_\beta = -\frac{k}{2}$ with an arbitrary $k \geq 1$ from the discrete spectrum (3.27) possessing the eigenspace

$$\Phi_k = \operatorname{Span}\{\psi_\beta, \ |\beta| = k\} \tag{3.81}$$

of finite dimension K (the number of distinct multi-indices β of the fixed length k).

Consider the nonlinear eigenvalue problem (3.36), where we estimate the coefficients in the operators given in (3.37)

$$\beta_0 = \tfrac{1}{2} - \tfrac{1}{4}N\varepsilon + O(\varepsilon^2), \;\; \mu_0 = \tfrac{N}{2} - \tfrac{1}{4}N^2\varepsilon + O(\varepsilon^2) \tag{3.82}$$

and set

$$\lambda = -\tfrac{k}{2} + \mu, \tag{3.83}$$

where $\mu = \mu(\varepsilon)$ is a new unknown parameter. We use the representation

$$\Delta(|\psi|^{m-1}\psi) = \Delta\psi + \Delta(|\psi|^{m-1}\psi) - \Delta\psi \equiv \Delta\psi + \Delta G_\varepsilon(\psi),$$

where in the nonlinear perturbation

$$G_\varepsilon(\psi) = \psi(|\psi|^\varepsilon - 1), \tag{3.84}$$

so, as $\varepsilon \to 0^+$,

$$G_\varepsilon(s) = \varepsilon g(s) + O(\varepsilon^2), \;\; \text{where} \;\; g(s) = s \ln |s|, \tag{3.85}$$

uniformly on any compact subset bounded away from zero. We have that $G_\varepsilon(\psi)$ is continuously differentiable in the variables ψ and ε at any point including $\{\psi = 0, \, \varepsilon = 0\}$.

We then arrive at the following perturbed problem:

$$(\mathbf{B}_1 + \tfrac{k}{2} I)\psi = \mu\psi - \Delta G_\varepsilon(\psi) + \varepsilon(\mathcal{L}_1\psi - \tfrac{\mu}{2}\, y \cdot \nabla\psi) + O(\varepsilon^2)\mathcal{L}_2\psi, \tag{3.86}$$

where $\mathcal{L}_1$ is the first order linear operator

$$\mathcal{L}_1 = \tfrac{N+k}{4}\, y \cdot \nabla + \tfrac{N^2}{4}\, I, \tag{3.87}$$

and $\mathcal{L}_2$ is another similar first order differential operator which will play no role in the local branching analysis. It follows from the construction that the linear operator on the left-hand side has the kernel of dimension K,

$$\ker\,(\mathbf{B}_1 + \tfrac{k}{2}\, I) = \Phi_k.$$

In view of the completeness of the eigenfunction subset Φ in L^2_ρ, the range of $\mathbf{B}_1 + \frac{k}{2} I$ has the same codimension K. Then $(\mathbf{B}_1 + \frac{k}{2} I)^{-1}$ is a bounded Fredholm operator with the deficiency index K. Hence we can use the classical Lyapunov–

Schmidt method to construct asymptotic expansions of solutions. As standard practice, the method applies to the equivalent integral equation with bounded and compact operators. To derive this we take the negative invertible operator $\mathbf{B}_1 - I$ and apply the compact operator $(\mathbf{B}_1 - I)^{-1}$ to both parts of equation (3.86). This gives compact linear integral operators in the linear terms. Concerning the nonlinear term, we assume temporarily that the nonlinearity $\psi(|\psi|^\varepsilon - 1)$ is approximated by a uniformly Lipschitz continuous function $h_L(\psi)$ with a parameter $L \gg 1$ such that $h_L(\psi) \le C(1 + |\psi|)$, $h_L(\psi) \equiv \psi(|\psi|^\varepsilon - 1)$ for $|\psi| \le L$, and $h_L(\psi) \to \psi(|\psi|^\varepsilon - 1)$ as $L \to \infty$ uniformly on compact subsets. Fixing an $L \gg 1$, we replace $\psi(|\psi|^\varepsilon - 1)$ by $h_L(\psi)$. Since we are looking for uniformly bounded solutions ψ, such a truncation of the equation does not affect the main results of the analysis, though we will need to check that the perturbation techniques yield uniformly bounded solutions. Bearing in mind this approximation and continuing to use the original notation for the nonlinearity, we obtain a compact Hammerstein operator, [64, Chapter 5].

Thus, we consider a nonlinear integral equation

$$\psi = \mathbf{A}(\psi, \rho), \quad \text{with a parameter } \rho = (\varepsilon, \mu) \in \mathbb{R}^2, \tag{3.88}$$

where $\mathbf{A}$ is compact in L^2_ρ. The unperturbed (linear) problem has a K-dimensional subspace of solutions, i.e.,

$$\psi_\beta = \mathbf{A}(\psi_\beta, 0) \text{ for any } |\beta| = k, \text{ where} \tag{3.89}$$

$$\mathbf{A}'(0,0) = -(1 + \tfrac{k}{2})(\mathbf{B}_1 - I)^{-1}. \tag{3.90}$$

Since $\sigma((\mathbf{B}_1 - I)^{-1}) = \{-(1 + \frac{j}{2})^{-1}, j \ge 0\}$, 1 is the eigenvalue of $\mathbf{A}'(0,0)$ corresponding to $j = k$. Note again that, returning to the differential problem (3.86), the operator on the left-hand side is self-adjoint in $L^2_\rho(\mathbb{R}^N)$, but the rest of the linear and nonlinear operators on the right-hand side are not self-adjoint or potential in this space.

Thus we use the Lyapunov-Schmidt method of asymptotic expansions; see [65, Section 54.3] and [82, Section 23] for equations with compact operators. According to the general branching theory, for $\rho \approx 0$ we are looking for a solution of the form

$$\psi = u + v, \quad \text{where } u \in \Phi_k \text{ and } v \in \Phi_k^\perp, \tag{3.91}$$

i.e., we take

$$u = \sum_{|\beta|=k} C_\beta \psi_\beta, \text{ assuming that } \|C\| \neq 0. \tag{3.92}$$

According to (3.89), u is a nontrivial solution of the linear equation with $\rho = 0$. Projecting equation (3.88) onto $\Phi_k^\perp$ yields an equation for v which, under the given assumptions, is uniquely solved locally to give $v = v(u, \rho)$ [82, p. 326] and substituting this v into the projection of (3.88) onto Φ_k gives the *Lyapunov–Schmidt branching equation* for unknowns $\{C_\beta, \mu\}$. Then (3.91) establishes a one-to-one correspondence between the solutions of (3.36) (close to Φ_k) and the solutions of the branching equation [82, p. 329]. For convenience and without loss of rigor, we derive this branching equation using the differential problem (3.86) instead of the equivalent integral equation (3.88). In view of the specific non-analytic structure of the nonlinearity (3.84) for small $\varepsilon > 0$, we will use the branching theory in the case of finite regularity, [82, Section 27]. Notice that compactly supported solutions with $m > 1$ are well-suited to the L^2_ρ-setting of the integral equation (3.88).

By (3.90), 1 is the eigenvalue of the linearised operator. As a natural step, in view of the linear dependence on ε in the main nonlinear perturbation in (3.85) and in (3.82), we consider a similar expansion of the parameter μ in (3.83) and v in (3.91) by setting

$$\mu = \varepsilon\nu + o(\varepsilon) \text{ and } v = \varepsilon\phi + o(\varepsilon). \tag{3.93}$$

Here ϕ is an unknown function and ν is an unknown parameter to be determined from the final branching equation. Substituting (3.93) into the differential equation (3.86), using (3.85) and taking into account the terms of order $O(\varepsilon)$ yields the following equation for ϕ:

$$(\mathbf{B}_1 + \tfrac{k}{2} I)\phi = \nu u - \Delta g(u) + \mathcal{L}_1 u, \quad \phi \in L^2_\rho. \tag{3.94}$$

In view of kernel (3.81), by Fredholm's Theorem, the criterion for solubility consists of the K orthogonality conditions obtained via multiplying by any ψ_γ, $|\gamma| = k$, in L^2_ρ. This gives K algebraic equations in the unknown coefficients $\{C_\beta\}$ and ν

$$[\nu + \tfrac{k(k+2N)}{4}] C_\gamma = \langle g(\textstyle\sum C_\beta \psi_\beta), \Delta\psi_\gamma^* \rangle \text{ for any } |\gamma| = k, \tag{3.95}$$

where, for convenience, we use the scalar product $\langle \cdot, \cdot \rangle$ in $L^2(\mathbb{R}^N)$. Then the adjoint eigenfunctions $\psi_\gamma^* = \rho\psi_\gamma$ become the orthonormal Hermite polynomials

$c_\gamma H_\gamma$. To get the left-hand side in (3.95), we observed from (3.28) and (3.87) that, by the orthonormality,

$$\langle \mathcal{L}_1 \sum C_\beta \psi_\beta, \psi_\gamma^* \rangle = \langle \sum C_\beta \psi_\beta, \mathcal{L}_1^* \psi_\gamma^* \rangle = -\tfrac{k(k+2N)}{4} C_\gamma,$$

with the adjoint operator $\mathcal{L}_1^* = -\frac{N+k}{4} y \cdot \nabla - \frac{NK}{4} I$.

Thus, we need to study the solubility of the system (3.95) of K equations for K unknowns $\{C_\beta,\ |\beta| = k\}$ plus ν with a normalisation condition saying that $u \neq 0$. For instance, one can impose the normalisation condition

$$\|C\| = 1, \tag{3.96}$$

though others would do, for instance, $\langle |g(\sum C_\beta \psi_\beta)|, 1 \rangle = 1$.

3.7.2 Preliminary properties of the branching equation

We begin with some preliminary properties of this algebraic branching system.
(i) System (3.95), (3.96) is not gradient and the right-hand side in (3.95) is not the gradient of a function $\mathbb{R}^K \to \mathbb{R}$. One can see that a gradient system can occur if $\psi_\gamma \Delta \psi_\beta^* = \psi_\beta \Delta \psi_\gamma^*$ for any $|\beta| = |\gamma| = k$, which is not true if $k \geq 2$. Therefore, we cannot rely on the critical point variational theory that is known to simplify the bifurcation and branching analysis. Indeed, if the problem were gradient governed by a sufficiently smooth even functional in $\mathbb{R}^K$, this would mean that the algebraic problem would have at least K different branches of solutions, similar to the linear eigenvalue problem with $m = 1$ admitting precisely K linearly independent orthonormal solutions. This is a result we would like to expect for the nonlinear problem; see below.
(ii) The following "linear" property of the nonlinear system (3.95) holds:

$$\text{if } C \text{ is a solution of (3.95), then } \alpha C \text{ is a solution for any } \alpha \in \mathbb{R}. \tag{3.97}$$

Indeed substituting αC into (3.95) and using the orthogonality property of the Hermite polynomials, $\langle \psi_\beta, \psi_\gamma^* \rangle = \delta_{\beta\gamma}$, yields that, for any $|\beta| = |\gamma| = k$,

$$\langle \psi_\beta, \Delta \psi_\gamma^* \rangle = \langle \Delta \psi_\beta, \psi_\gamma^* \rangle = 0$$

since $\Delta \psi_\beta \sim \psi_{\bar\beta}$ with $|\bar\beta| = k + 2$.
(iii) Studying the algebraic system (3.95) with the extra normalisation equation

(3.96), we should take into account various orthogonal transformations (e.g. rotations) in $\mathbb{R}^N$ under which the nonlinear equation (3.36) is invariant. Namely, any transformation

$$y \mapsto ay \text{ with } \|a\| = 1 \tag{3.98}$$

leaves (3.36) invariant. Obviously, this invariant property affects the dimension of the manifold of solutions of the corresponding algebraic system (3.95), (3.96).

(iv) The parameter value

$$\nu_* = -\tfrac{k(k+2N)}{4} \tag{3.99}$$

plays a special role in the analysis. It follows from (3.95) that ν_* is the only possible choice for non-radial solutions. Indeed, in this case, if $\nu \neq \nu_*$, then in view of the existence of a multi-dimensional Lie group of invariant transformations, the system (3.95) becomes overdetermined. On the other hand, in the radial case, where (3.95) reduces to a single equation, in general, $\nu \neq \nu_*$ (see examples below).

(v) As we know from Lemma 3.3, there must exist bifurcations of the radial solutions from radial linear eigenfunctions for all even $k = 0, 2, 4, \ldots$ [For odd values of k, the eigenfunctions (3.28) are not even in y and cannot generate even nonlinear eigenfunctions.] Obviously, those branches of even nonlinear eigenfunctions correspond to the following choice of the unknowns $\{C_\beta\}$:

$$C_\beta = \begin{cases} \frac{1}{\sqrt{N}} & \text{if } \beta = (0, \ldots, 0, k, 0, \ldots, 0) \equiv \bar{\beta}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.100}$$

It is easy to check that the corresponding algebraic system (3.95) reduces to a simple single algebraic equation admitting such a solution.

3.7.3 Existence of nonlinear eigenfunctions for $m \approx 1$

The existence and multiplicity of distinct nonlinear eigenfunctions is associated with the existence of different solutions of the branching equation (3.95), (3.96). It turns out that the general solubility analysis, and hence existence of various non-radial nonlinear eigenfunctions (at least for all $m \approx 1^+$) is a difficult algebraic problem. We illustrate some typical difficulties by studying the following simple example.

3.7.4 $k = 2$ on the plane

In $\mathbb{R}^2$, according to (3.28), there exist three linearly independent eigenfunctions denoted now by

$$\psi_1 = c_*(\tfrac{1}{2}\,y_1^2 - 1)e^{-|y|^2/4}, \;\; \psi_2 = c_*(\tfrac{1}{2}\,y_2^2 - 1)e^{-|y|^2/4}, \;\; \psi_3 = c_*\tfrac{1}{2}y_1y_2e^{-|y|^2/4}, \tag{3.101}$$

where $c_* = \frac{1}{2\sqrt{2}}$ is the normalisation constant. The general representation of $u \in \ker(\mathbf{B}_1 + \frac{k}{2}\,I)$ will be written in the form

$$u = C_1\psi_1 + C_2\psi_2 + C_3\psi_3. \tag{3.102}$$

The radial eigenfunction. We begin with the radial case, where

$$C_1 = C_2 = \tfrac{1}{\sqrt{2}} \;\text{ and }\; C_3 = 0.$$

Calculating $\Delta\psi_1^* = \Delta\psi_2^* = c_*$ and substituting into (3.95) yields the single equation

$$\tfrac{\nu+3}{\sqrt{2}} = \tfrac{1}{2\sqrt{2}} \int_{\mathbb{R}^2} g(\tfrac{1}{\sqrt{2}}(\psi_1 + \psi_2))\,dy \equiv \tfrac{1}{2\sqrt{2}} \int_{\mathbb{R}^2} g(\tfrac{1}{4}(\tfrac{1}{2}\,|y|^2 - 2)e^{-|y|^2/4})\,dy.$$

By the radial change of variable $s = \frac{|y|^2}{4} \geq 0$, the last integral denoted by μ_r reduces to

$$\mu_r = 2\pi \int_0^\infty (s-1)e^{-s} \ln|(s-1)e^{-s}|\,ds. \tag{3.103}$$

It seems that this integral cannot be calculated explicitly and its numerical value is approximately

$$\mu_r = -4.380. \tag{3.104}$$

This gives the unique value of the parameter $\nu = \nu_{rad}$ in (3.93)

$$\nu_{\rm rad} = -3 + \tfrac{1}{2}\,\mu_r = -5.190\,. \tag{3.105}$$

As we know from Lemma 3.3, this bifurcation at $m = 1^+$ leads to the solution branch existing for all $m > 1$.

Non-radial solutions. For general solutions (3.102), we obtain from (3.95) a system of three algebraic equations

$$(\nu+3)C_1 = c_* \textstyle\int g(u)\,dy, \;\; (\nu+3)C_2 = c_* \int g(u)\,dy, \;\; (\nu+3)C_3 = 0.$$

The first two equations mean $(\nu+3)C_1 = (\nu+3)C_2$ so that either $C_1 = C_2$, which leads to the above radial case, or

$$\nu = \nu_* = -3. \tag{3.106}$$

The solution (3.102) takes the form

$$u = \tfrac{1}{2}c_*[\mathbf{Q}(y) - 2(C_1 + C_2)]\mathrm{e}^{-(y_1^2+y_2^2)/4}, \tag{3.107}$$

where $\mathbf{Q}$ denotes the quadratic form

$$\mathbf{Q}(y) = C_1 y_1^2 + C_2 y_2^2 + C_3 y_1 y_2. \tag{3.108}$$

We need to consider three cases depending on the index of $\mathbf{Q}$.

(i) **Index zero.** Let $C_1C_2 \neq 0$. By an orthogonal transformation on the $\{y_1, y_2\}$-plane, one can always reduce the quadratic form in (3.107) to a diagonal form while the positive definite form $|y|^2$ remains unchanged. Therefore in this case, we may set $C_3 = 0$. Then we obtain a system of two equations

$$\begin{cases} \int g(C_1\psi_1 + C_2\psi_2)\,\mathrm{d}y = 0, \\ C_1^2 + C_2^2 = 1. \end{cases} \tag{3.109}$$

Since the function $g(s)$ is odd, this system admits the obvious non-symmetric solution

$$C_1 = -C_2 = \tfrac{1}{\sqrt{2}} \implies u = c_* \tfrac{1}{8}\,(y_1^2 - y_2^2)\mathrm{e}^{-|y|^2/4}, \tag{3.110}$$

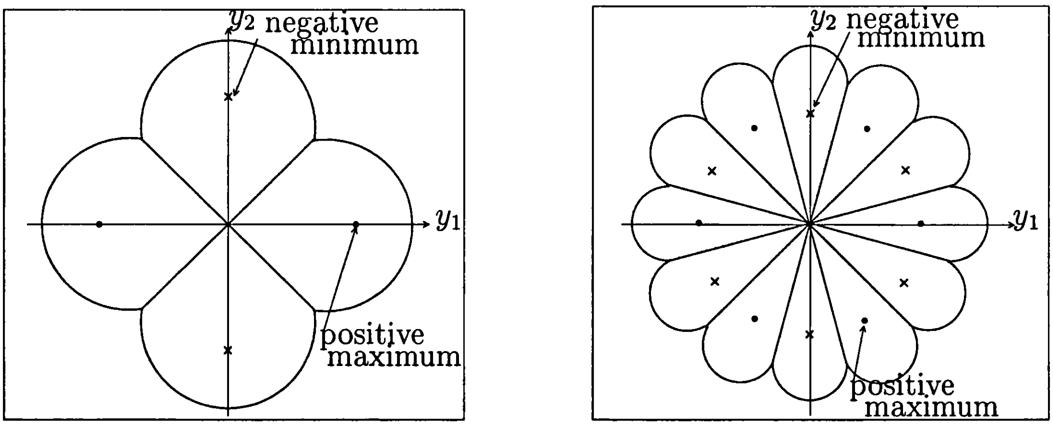
for which the index of (3.108) is equal 0. In the polar coordinates

$$\begin{cases} y_1 = r\cos\sigma, \\ y_2 = r\sin\sigma, \end{cases} \tag{3.111}$$

this solution is angular π-periodic,

$$u = c_* \tfrac{1}{8}\, r^2 \mathrm{e}^{-r^2/4} \cos 2\sigma. \tag{3.112}$$

By the branching theory [82], we conclude that a nonlinear eigenfunction bifurcates at $m = 1^+$ from u with $\nu = -3$ in (3.93). As usual, due to the finite propagation for $m > 1$, the exponentially decaying linear eigenfunction (3.110) for $m = 1$ will generate a nonlinear one with bounded support, which inherits the symmetries admitted by the Laplacian. Figure 3.8(a) shows a plausible schematic star-shaped support of this nonlinear eigenfunction. This solution of (3.36) is anti-symmetric relative to the axes $y_1 \pm y_2 = 0$ and can be obtained in the quarter plane $\{y_1 > 0,\ -y_1 < y_2 < y_1\}$ with the appropriate conditions at $y_2 = \pm y_1$ admitting reflections with the sign change $\psi \mapsto -\psi$.



(a) A plausible support when $k = 2$

(b) A plausible support when $k = 4$

Figure 3.8: Two plausible supports of nonlinear eigenfunctions in $\mathbb{R}^2$.

(ii) **Index two.** We continue to study the solubility of system (3.109). Looking for a positive solution

$$C_1 \in (0,1) \text{ and } C_2 = \sqrt{1 - C_1^2} > 0, \tag{3.113}$$

for which index of (3.108) with $C_3 = 0$ by diagonalisation is equal to 2, yields the equation

$$H(C_1) \equiv \int g(C_1\psi_1 + \sqrt{1 - C_1^2}\psi_2)\, \mathrm{d}y = 0. \tag{3.114}$$

Numerically we have that

$$H(1) = \sqrt{\tfrac{\pi}{2}} \int_{-\infty}^{\infty} (\tfrac{1}{2}\, y^2 - 1) \mathrm{e}^{-y^2/4} [\ln |\tfrac{1}{2} y^2 - 1| - \tfrac{1}{4}\, y^2 - \tfrac{1}{2}]\, \mathrm{d}y = 1.70\,.$$

On the other hand, from the calculations in the radial case we have by (3.104) that

$$H(\tfrac{1}{\sqrt{2}}) = \mu_r < 0.$$

Therefore since $H(1)H(\frac{1}{\sqrt{2}}) < 0$, (3.114) has a new solution $C_1 \in (0, \frac{1}{\sqrt{2}})$ by continuity.

(iii) **Index one.** Finally, consider the last case where the index of (3.108) is equal to 1, e.g. $C_2 = C_3 = 0$. Then (3.106) holds and $C_1 \neq 0$ must satisfy the condition $\int g(C_1\psi_1) = 0$, or, equivalently,

$$\int_{\mathbb{R}^2} (\tfrac{1}{2}y_1^2 - 1)\mathrm{e}^{-|y|^2/4} \ln |(\tfrac{1}{2}y_1^2 - 1)\mathrm{e}^{-|y|^2/4}| \, \mathrm{d}y_1 \mathrm{d}y_2 = 0.$$

This is equivalent to the equality

$$\mu_1 \equiv \int_{-\infty}^{\infty} (\tfrac{1}{2}y_1^2 - 1)\mathrm{e}^{-y_1^2/4} \ln |\tfrac{1}{2}y_1^2 - 1| \, \mathrm{d}y_1 = 12,$$

which is not true (numerically, $\mu_1 \approx 1.9512$). So the linear eigenfunction $u = C_1\psi_1(y)$ at $m = 1$ cannot generate a nonlinear one for $m \approx 1^+$.

Thus we have detected at least three different types of nonlinear eigenfunctions existing for $m \approx 1^+$:

(i) the radially symmetric one corresponding to the value (3.105),

(ii) the angular-symmetric one (3.112) with the parameter (3.106), and

(iii) the eigenfunction corresponding to coefficients (3.113) with no obvious symmetry.

Notice that the total number *three* of distinct nonlinear eigenvalues for $m > 1$ coincides with, or at is least not less than, the dimension of the corresponding eigenspace for $m = 1$ (though of course any linear properties of operators and envelopes are no longer valid for $m > 1$).

3.7.5 Nonlinear eigenfunctions generated by periodic harmonic polynomials

As a final step towards constructing non-radial nonlinear eigenfunctions, we consider the general case of arbitrary even $k \geq 2$ in $\mathbb{R}^N$ and will describe angular periodic eigenfunctions of the linear eigenvalue problem (3.23) with $|\beta| = k$ coinciding with (3.112) for $k = 2$. In polar coordinates $y = (r, \sigma)$ in $\mathbb{R}^N$, the Laplacian takes the form

$$\Delta = \Delta_r + \tfrac{1}{r^2}\Delta_\sigma, \;\; \Delta_r = \tfrac{\mathrm{d}^2}{\mathrm{d}r^2} + \tfrac{N-1}{r}\tfrac{\mathrm{d}}{\mathrm{d}r}, \tag{3.115}$$

where Δ_σ is the Laplace-Beltrami operator on the unit sphere S^{N-1} in $\mathbb{R}^N$, which is a regular operator with discrete spectrum in $L^2(S^{N-1})$ (each eigenvalue repeated as many times as its multiplicity)

$$\sigma(-\Delta_\sigma) = \{c_j = j(j + N - 2), \; j \geq 0\}, \tag{3.116}$$

and an orthonormal, complete, closed subset $\{f_k(\sigma)\}$ of eigenfunctions being kth order homogeneous harmonic polynomials restricted to S^{N-1}.

Performing the separation of variables in (3.23), we look for eigenfunctions in the form

$$\psi_\beta(y) = \phi_\beta(r) f_j(\sigma), \tag{3.117}$$

where f_j is an eigenfunction of Δ_σ,

$$-\Delta_\sigma f_j = c_j f_j. \tag{3.118}$$

Substituting (3.117) into (3.23) yields an ODE problem for the radial function $\phi = \phi_\beta$

$$\mathbf{D}_1 \phi \equiv \Delta_r \phi + \tfrac{1}{2} r \phi' + \left(\tfrac{N+k}{2} - \tfrac{c_j}{r^2}\right)\phi = 0. \tag{3.119}$$

The spectral properties of such operators are well known and, in particular, occur in the study of the Cauchy problem for the heat equation with inverse-square potentials; see references in [84]. We need to compute the radial parts $\phi(r)$ of eigenfunctions. Setting $\phi = \mathrm{e}^{-r^2/4}\phi^*$ yields the eigenvalue problem for the adjoint operator

$$\mathbf{D}_1^* \phi^* \equiv \Delta_r \phi^* - \tfrac{1}{2} r (\phi^*)' + \left(\tfrac{k}{2} - \tfrac{c_j}{r^2}\right)\phi^* = 0. \tag{3.120}$$

Let us show that, for any even $j \leq k$ such that

$$k - j = 2l \ \text{ with an } \ l \geq 0, \tag{3.121}$$

it admits a polynomial eigenfunction. Looking for a solution of (3.120) in the form of Kummer's series

$$\phi^*(r) = \textstyle\sum_{i=0}^{\infty} C_i r^{j+2i}, \tag{3.122}$$

where the extra exponent j in the terms r^{j+2i} is due to the singular inverse-square potential $\frac{c_j}{r^2}$ in (3.120), it is easy to derive the recurrent relation for the expansion coefficients

$$C_{i+1} = \tfrac{j+2i-k}{2[(j+2i+2)(j+2i+N)-c_j]}\, C_i \ \text{ for } \ i \geq 0. \tag{3.123}$$

It follows from (3.123) that, in the case (3.121), the coefficients C_{i+1} vanish for all $i \geq l$ so that there exists an eigenfunction $\phi_l^*(r)$ that is a kth order polynomial. In particular, for $j = k$, this eigenfunction is

$$\phi_0^*(r) = r^k. \tag{3.124}$$

Hence using (3.117), we obtain the eigenfunctions

$$\psi_\beta(y) = \phi_l^*(|y|)\mathrm{e}^{-|y|^2/4} f_j(\sigma), \;\; j = k - 2l, \;\; l = 0, 1, \ldots, \tfrac{k}{2}. \tag{3.125}$$

For $j = k$ (i.e., $l = 0$) we obtain the following special eigenfunction:

$$\psi_\beta(y) = |y|^k \mathrm{e}^{-|y|^2/4} f_k(\sigma). \tag{3.126}$$

Returning to the branching equation (3.95) with ν given by (3.99), we recall that each $\Delta\psi_\gamma^*$ is a polynomial of the even order $2l-2$ and is an eigenfunction with the eigenvalue $-\frac{k}{2}+1$. By the orthogonality of eigenfunctions corresponding to mutually distinct eigenvalues, the periodic eigenfunctions (3.125) are orthogonal to each of these polynomials. The validity of the branching equation (3.95), (3.99) for general eigenfunctions (3.125) is not straightforward. But due to the "maximal" symmetry and changing sign properties of the eigenfunction (3.126), the nonlinear orthogonality (branching) condition holds automatically for any lower-order polynomials $\Delta\psi_\gamma^*$, i.e.,

$$\langle g(\psi_\beta), \Delta\psi_\gamma^* \rangle = 0, \;\; |\gamma| = k. \tag{3.127}$$

Hence branching always occurs from the linear eigenfunction (3.126) at $m = 1^+$.

Notice also that the first radial eigenfunction for $j = 0$ must also satisfy conditions (3.127) (from the ODE theory we know that branching occurs for $j = 0$). Consider briefly another simple example.

3.7.6 $k = 4$ on the plane

We take into account three eigenfunctions given by (3.28), where we omit the normalisation constants,

$$\psi_1 = (\tfrac{1}{8}y_1^4 - \tfrac{3}{2}y_1^2 + \tfrac{3}{2})\mathrm{e}^{-|y|^2/4}, \;\; \psi_2 = (\tfrac{1}{8}y_2^4 - \tfrac{3}{2}y_2^2 + \tfrac{3}{2})\mathrm{e}^{-|y|^2/4},$$

$$\psi_3 = (\tfrac{1}{4}y_1^2 y_2^2 - \tfrac{1}{2}|y|^2 + 1)\mathrm{e}^{-|y|^2/4}.$$

The radially symmetric eigenfunction is obtained from the linear combination

$$\psi_1 + \psi_2 + \psi_3 = (\tfrac{1}{8}|y|^4 - 2|y|^2 + 4)\mathrm{e}^{-|y|^2/4}. \tag{3.128}$$

In the polar coordinates (3.111) we next consider the linear combination

$$\psi_1 + \psi_2 - \psi_3 = \tfrac{1}{16} r^4 e^{-|y|^2/4} \cos 4\sigma + \tfrac{1}{2}\left(\tfrac{1}{8} r^4 - 2r^2 + 4\right) e^{-|y|^2/4}. \tag{3.129}$$

The first term on the right-hand side corresponds to the adjoint eigenfunction (3.124) with $k = 4$ and (3.129) is the Fourier expansion of the eigenfunction that consists of two terms $\cos j\sigma$ with $j = 4$ and $j = 0$. Obviously, the linear eigenfunction

$$\psi_4 = |y|^4 e^{-|y|^2/4} \cos 4\sigma$$

satisfies the branching equation. The corresponding nonlinear eigenfunction possesses a star-shaped support that has twelve vertices, as Figure 3.8(b) shows.

Despite the above perturbation results that are local in $m \approx 1^+$, a complete description of all the non-radial nonlinear eigenfunctions of the PME in $\mathbb{R}^N$ and their evolution completeness remain a challenging open problem where new ideas and techniques are necessary.

Chapter 4

Spectra of critical exponents in nonlinear heat equations with absorption

> *The startling truth finally became apparent, and it was this: Numbers written on restaurant checks within the confines of restaurants do not follow the same mathematical laws as numbers written on any other pieces of paper in any other parts of the Universe. This single statement took the scientific world by storm. So many mathematical conferences got held in such good restaurants that many of the finest minds of a generation died of obesity and heart failure, and the science of mathematics was put back by years.* - Douglas Adams. Taken from "Life, the Universe and Everything."

In this chapter we extend the PME by considering an absorption term. It has been known since the 1980's that global L^1-solutions of the classical PME with absorption $u_t = \Delta u^m - u^p$ in $\mathbb{R}^N \times \mathbb{R}_+$, with $m, p > 1$, change their large-time behaviour at the critical absorption exponent $p_0 = m + 2/N$ (also known as the Fujita critical exponent). Work has been done in analysing such behaviour; see the surveys and references in the book [50].

We extend these results by showing that, provided the solution $u(x,t)$ is allowed to change sign, there exists an infinite sequence $\{p_k,\ k \geq 0\}$ of critical exponents generating a countable subset of different non-self-similar asymptotic patterns. These results are extended to the fully nonlinear dual porous medium equation with absorption where only the first critical exponent for the DPME with absorption in $\mathbb{R} \times \mathbb{R}_+$ was known [57].

4.1 Introduction: critical absorption exponents

We describe new types of asymptotic behaviour of global compactly supported solutions of the PME with absorption

$$u_t = \Delta(|u|^{m-1}u) - |u|^{p-1}u \text{ in } \mathbb{R}^N \times \mathbb{R}_+ \text{ with exponents } m > 1, \ p > 1, \quad (4.1)$$

which was studied extensively in the theory of nonlinear degenerate parabolic partial differential equations from the beginning of the 1970's. The PME and related degenerate parabolic PDEs were the main models in the theory of free boundaries, [32]. Our aim is to introduce a sequence of *critical* exponents $\{p = p_k, \ k \geq 0\}$ for (4.1) corresponding to non-scaling invariant asymptotic behaviour as $t \to \infty$. Such critical asymptotic phenomena are not exceptional and are a common feature for other equations with power nonlinearities. As the second example, we extend the results to the fully nonlinear dual PME with absorption

$$u_t = |\Delta u|^{m-1}\Delta u - |u|^{p-1}u \text{ in } \mathbb{R}^N \times \mathbb{R}_+ \ (m > 1, \ p > 1). \quad (4.2)$$

We consider the Cauchy problem with bounded, integrable, compactly supported initial data $\hat{u}$. It is known that such nonlinear heat equations admit solutions that are unique, global in time, and vanish as $t \to \infty$ with rates depending on the exponents m, p and the space dimension N. We refer to Kalashnikov's survey [58], DiBenedetto's book [27] and [60], [63], [10] for fully nonlinear equations.

Concerning the precise asymptotic behaviour of global solutions, a complete classification was achieved in the 1980-90's for nonnegative solutions of the PME with absorption (4.1). It was proved that $p_0 = m + 2/N$ is the critical exponent in the sense that (see key references in [16], [47], [61] and Chapt. 2 in [78]):

(I) In the subcritical range $p \in (1, p_0)$ the asymptotic behaviour of $u(x,t) \geq 0$ as $t \to \infty$ is governed by the unique *very singular* self-similar solution;

(II) In the supercritical range $p > p_0$ the solution converges as $t \to \infty$ to the self-similar Zel'dovich - Kompaneetz - Barenblatt (ZKB) solution of the PME: $u_t = \Delta u^m$; and

(III) In the critical case $p = p_0$ the asymptotic behaviour is given by a unique ZKB solution with an extra $\ln t$ scaling in u and x (see [47] and earlier references therein).

The main goal of this chapter is to show that the equations (4.1) and (4.2) admit a *monotone sequence of critical exponents* $\{p_k\}$, of which p_0 is the first. These critical exponents will be shown to generate special asymptotic patterns in the Cauchy problem with initial data $\hat{u}$ that changes sign. Therefore, the critical behaviour at $p = p_k$ for any $k \geq 1$ cannot be observed in the class of nonnegative solutions. It is worth mentioning that critical phenomena are also of crucial importance for reaction-diffusion equations with source terms $+|u|^{p-1}u$, where p_0, known as the *Fujita* critical exponent, stays the same and affects various blow-up and stability properties; see references in Chapt. 4 of [78].

Concerning free-boundaries, we will show that our patterns for (4.1) at $p = p_k$ exhibit the following spectrum of asymptotic behaviour as $t \to \infty$:

$$|x_k(t)| = Ct^{1/2}(t \ln t)^{-(m-1)\delta_k/2}(1 + o(1)) \text{ for } k = 0, 1, 2, \dots, \tag{4.3}$$

with a positive sequence of exponents $\{\delta_k = \mu_0 - \lambda_k\}$, where $\{\lambda_k\}$ are the corresponding "nonlinear" eigenvalues and $\mu_0 = N/[N(m-1)+2]$

Similar logarithmically perturbed solutions occur for the semilinear heat equation ($m = 1$)

$$u_t = \Delta u - |u|^{p-1}u \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \;\; p > 1 \tag{4.4}$$

(obviously, the free-boundary phenomena are not exhibited). In this case, the sequence of critical exponents can be calculated explicitly (see (4.9) below):

$$p_k = 1 + 2/(k + N), \;\; k = 0, 1, 2, \dots, \tag{4.5}$$

and is connected with the discrete spectrum of the linear differential operator (associated with the heat equation) in a weighted L^2-space

$$\mathbf{B} = \Delta + \frac{1}{2} y \cdot \nabla + \frac{N}{2} I. \tag{4.6}$$

Moreover, similar effects are observed for higher order semilinear equations

$$u_t = -(-\Delta)^l u - |u|^{p-1}u \text{ with any integer } l \geq 2,$$

where $p_0 = 1 + 2l/N$. For nonlinear second order diffusion operators, such a connection with spectral theory of linear operators is not available to us, though we show how a discrete subset of asymptotic patterns ("nonlinear eigenfunctions") of the purely diffusive equations generate the corresponding sequence of critical exponents in reaction-absorption equations (4.1) and (4.2). It is convenient to begin with the semilinear equation (4.4) and well established spectral properties of the linear operator (4.6). We next extend these ideas to quasilinear and fully nonlinear operators using different mathematical tools.

4.2 Critical exponents in the semilinear heat equation with absorption

4.2.1 Rescaled equations and a sequence of critical exponents

We show how the discrete spectrum of the linear operator $\mathbf{B}$ in (4.6) is associated with critical phenomena for the semilinear equation (4.4). Bearing in mind the typical asymptotic behaviour in the rescaled heat equation (see (3.33), where $k = 0, 1, \dots$) we perform the following rescaling in (4.4):

$$u(x,t) = t^{-(k+N)/2} v(y,\tau), \quad y = x/t^{1/2}, \quad \tau = \ln t : (1,\infty) \to \mathbb{R}_+. \tag{4.7}$$

The rescaled solution $v(y,\tau)$ satisfies the perturbed equation

$$v_\tau = (\mathbf{B} + \frac{k}{2} I)v - e^{-\gamma_k \tau} g(v), \quad g(v) \equiv |v|^{p-1} v, \quad \gamma_k = \frac{1}{2}(p-1)(k+N) - 1. \tag{4.8}$$

Setting $\gamma_k = 0$ gives us a sequence $\{p_k\}$ of critical exponents:

$$\gamma_k = 0 \Longrightarrow p = p_k = 1 + 2/(k+N). \tag{4.9}$$

In these critical cases we arrive at the autonomous parabolic equation

$$\begin{gathered} v_\tau = \mathbf{B}_k v - g(v), \quad \text{where } \mathbf{B}_k = \mathbf{B} + \frac{k}{2} I, \\ \text{with spectrum } \sigma(\mathbf{B}_k) = \{\lambda_\beta^{(k)} = \frac{1}{2}(k - |\beta|)\} \end{gathered} \tag{4.10}$$

We consider sufficiently small initial data with exponential decay at infinity: $|v_0(y)| \le c e^{-a|y|^2}$ in $\mathbb{R}^N$ with c, a positive and $v_0 \in H^2_\rho(\mathbb{R}^N)$. Let $O^+(v) = \{v(\tau),\, \tau > 0\}$ be the corresponding global forward orbit given by the rescaled equation (4.10). The linear operator $\mathbf{B}$ with the discrete spectrum and a single simple eigenvalue λ_0 with $\operatorname{Re}\lambda_0 = 0$, is sectorial in $L^2_\rho(\mathbb{R}^N)$ and generates a strong continuous analytic semigroup $\{e^{\mathbf{B}\tau}, \tau \geq 0\}$. The asymptotic behaviour with a finite-dimensional centre manifold is covered by the invariant manifold theory in [68], Chapt. 9.

By regularity theory for parabolic partial differential equations [29, 33], $v(y,\tau)$ is sufficiently smooth for $\tau \geq 0$. In view of the completeness and orthonormality of eigenfunctions, we use the convergent eigenfunction expansion of the solution,

$$v(y,\tau) = \sum a_\beta(\tau)\psi_\beta(y), \tag{4.11}$$

where the expansion coefficients satisfy the DS

$$\dot{a}_\beta = \lambda_\beta^{(k)} a_\beta - \langle g(v), \psi_\beta \rangle_\rho \text{ for any } \beta. \tag{4.12}$$

4.2.2 Centre manifold behaviour and a generating algebraic system

We now look for a solution $v(\cdot,\tau)$ with the behaviour, for $\tau \gg 1$ on the centre manifold known to be tangent to the centre subspace of the linearised operator $\mathbf{B}_k = \mathbf{B} + \frac{k}{2}I$. Such a centre subspace asymptotic dominance assumes that in the eigenfunction expansion of $v(y,\tau)$ of the form (3.32), the leading term as $\tau \to \infty$ is given by

$$v(\tau) = \sum_{|\beta|=k} a_\beta(\tau)\psi_\beta + \ldots, \tag{4.13}$$

where we omit higher order terms. Then the expansion coefficients satisfy a perturbed finite-order DS

$$\dot{a}_\beta = -\langle g(\textstyle\sum a_\beta \psi_\beta), \psi_\beta \rangle_\rho + \ldots, \quad |\beta| = k. \tag{4.14}$$

We are interested in looking for asymptotic solutions of (4.14) of the form

$$v(\tau) = a_k(\tau)\phi_k + o(a_k(\tau)) \text{ with an eigenfunction } \phi_k = \sum_{|\gamma|=k} b_\gamma \psi_\gamma, \tag{4.15}$$

where the leading term consists of functions of the same order of decay as $\tau \to \infty$. Here $a_k(\tau)$ is a single unknown function to be determined together with the coefficients $\{b_\beta\}$ which are not arbitrary (unlike the linear expansion (3.33)). Substituting (4.15) into the equation (4.10) and multiplying by ψ_β in $L^2_\rho(\mathbb{R}^N)$, we have that the coefficients $\{b_\beta\}$ satisfy the following *generating algebraic system* (GAS):

$$\langle g(\sum b_\gamma \psi_\gamma), \psi_\beta \rangle_\rho = b_\beta \text{ for all } |\beta| = k, \tag{4.16}$$

which characterises a subset of centre manifold patterns of the form (4.15). Then $a_k(\tau)$ is determined from the DS

$$\dot{a}_k = -g(a_k)[1 + o(1)] \text{ for } \tau \gg 1. \tag{4.17}$$

Integrating (4.17) (as a standard ordinary differential equation) yields decaying solutions with the behaviour

$$a_k(\tau) = \pm[(p-1)\tau]^{-1/(p-1)}(1 + o(1)) \to 0 \text{ as } \tau \to \infty. \tag{4.18}$$

In terms of the original (x, t, u)-variables, this behaviour takes the form of a logarithmically perturbed linearised pattern as $t \to \infty$

$$u(x,t) = \pm C_k (t \ln t)^{-(k+N)/2}[\phi_k(x/t^{1/2}) + o(1)] \tag{4.19}$$

with $C_k = [2/(k+N)]^{-(k+N)/2}$. Let us return to the solubility of the GAS (4.16). In general, for arbitrary $k \geq 1$ and in sufficiently large dimensions $N > 1$, it is a complex nonlinear algebraic system of ν_k equations with ν_k unknowns $\{\beta, |\beta| = k\}$ (ν_k being the number of distinct multi-indices of the length k). A complete description of a (countable) subset of possible distinct solutions is unknown. We only consider important particular examples.

The GAS is easily solved for the first critical exponent corresponding to $k = 0$ where the centre subspace $E^c = \text{Span}\{f\}$ (where $f = \psi_0$) of the operator $\mathbf{B}_0 = \mathbf{B}$ is one-dimensional. Then $\phi_0 = b_0 f$, where the constant $b_0 \neq 0$ is obtained from the equation (4.16):

$$b_0 = \langle g(b_0 f), f \rangle_\rho \equiv |b_0|^{p-1} b_0 \langle |f|^{p+1}, 1 \rangle_\rho \Longrightarrow b_0 = \pm \langle |f|^{p+1}, 1 \rangle_\rho^{-1/(p-1)}. \tag{4.20}$$

This gives a unique stable asymptotic pattern on the centre manifold (4.19). Such a stable generic asymptotic behaviour has been known about for a long time; see [45], [52] and Chapter 2 in [78]. In [15], [16] such asymptotic behaviour was established by using the perturbation theory of linear self-adjoint operators.

Consider the more delicate case $k > 0$ that corresponds to the higher order critical exponents $p = p_k$, where $\mathbf{B}_k$ has nontrivial unstable subspace $E^u =$

Span$\{\psi_\beta,\ |\beta| < k\}$ and hence the asymptotic behaviour on the centre manifold is not stable (though exists).

One-dimensional geometry. If $N = 1$, then for any $k = 1, 2, \ldots$, the centre subspace $E^c = \text{Span}\{\psi_k\}$ is one-dimensional and the GAS (4.16) always gives a suitable solution

$$\phi_k = b_k\psi_k, \ \text{ with } \ b_k = \pm\langle|\psi_k|^{p+1}, 1\rangle_\rho^{-1/(p-1)}. \tag{4.21}$$

Multi-dimensional geometry. For $N > 1$, we first restrict our attention to the radially symmetric case and fix a unique radial eigenfunction in (4.15), $\phi_k(r) = b_k\psi_k(r)$ with $r = |y|$. It exists for any even $k > 0$ and is an eigenfunction of the ordinary differential operator

$$\frac{1}{r^{N-1}}(r^{N-1}\psi')' + \frac{1}{2}\psi' r + \frac{N}{2}\psi = -\frac{k}{2}\psi; \tag{4.22}$$

see [81]. In the radial setting $E^c = \text{Span}\{\psi_k\}$ is one-dimensional, the GAS (4.16) reduces to a single equation and, similar to (4.20), (4.21), we arrive at a unique asymptotic pattern with the constant b_k:

$$\phi_k(r) = b_k\psi_k(r) \Longrightarrow b_k = \pm\langle|\psi_k|^{p+1}, 1\rangle_\rho^{-1/(p-1)}, \ \ k = 2, 4, \ldots. \tag{4.23}$$

Let us show that there exist non-symmetric patterns in $\mathbb{R}^N$. Let $k = 1$. For convenience set $\phi_1 = 2^{-1/2}\sum_{j=1}^N b_j f_j$, where $f_j \equiv \partial f/\partial y_j$ where $f = \psi_0$ is the rescaled Gaussian kernel. We now arrive at the GAS

$$2^{-(p+1)/2}\langle g(\sum b_j f_j), f_i\rangle_\rho = b_i, \ \ i = 1, 2, \ldots, N. \tag{4.24}$$

As a first solution, we choose equal coefficients, $b_j = b_0$ for all j. Then (4.24) reduces to a single equation for b_0,

$$2^{-(p+1)/2}|b_0|^{p-1}\langle g(\sum f_j), f_i\rangle_\rho = 1,$$

where i is arbitrary by symmetry. On the other hand there exists another solution $\{b_j\} = \{b_1, 0, \ldots, 0\}$. Indeed, the system (4.24) reduces to the first equation for b_1,$2^{-(p+1)/2}|b_1|^{p-1}\int \rho|f_1|^{p+1} = 1$. It is important that in all cases the leading term of the asymptotic behaviour of the critical asymptotic patterns (4.19) does not depend on the initial data.

Remark: a countable subset of exponentially decaying patterns on the

stable manifold. Equation (4.8) can admit orbits on the infinite-dimensional stable manifold of the origin, which can be seen by the eigenfunction expansion of solutions. The linear diagonal structure of the system (4.12) shows that if the nonlinear term g forms an exponentially decaying perturbation as $\tau \to \infty$ (unlike the centre manifold behaviour studied above), then there exist patterns with the following exponential decay as $\tau \to \infty$:

$$v_\beta(y,\tau) = Ce^{\lambda_\beta \tau}(\psi_\beta(y) + o(1)), \;\; C = C(\hat{u}) \neq 0, \tag{4.25}$$

where ψ_β is a suitable eigenfunction with $\lambda_\beta < 0$ for $|\beta| > 0$. Such results are well known in linear perturbation theory; see [33], p. 226 and [23].

4.3 Discrete spectra and eigenfunctions for nonlinear operators

We now return to the nonlinear parabolic equations (4.1) and (4.2). Following the same lines as in the semilinear case, we first study the asymptotic behaviour for the purely diffusive equations: the PME

$$u_t = \Delta(|u|^{m-1}u) \;\text{ in }\; \mathbb{R}^N \times \mathbb{R}_+, \;\; m > 1, \tag{4.26}$$

and the dual PME

$$v_t = |\Delta v|^{m-1}\Delta v \;\text{ in }\; \mathbb{R}^N \times \mathbb{R}_+, \;\; m > 1. \tag{4.27}$$

We restrict our attention to a class of bounded compactly supported initial data.

4.3.1 Discrete spectrum and similarity patterns for the PME

The PME (4.26) is invariant under various groups of scaling transformations and admits self-similar solutions that will be written in the following form:

$$u(x,t) = t^{\lambda-\mu_0}\psi(y), \;\; y = x/t^{\tilde{\beta}}, \tag{4.28}$$

where $\mu_0 = N/[N(m-1)+2]$ and $\tilde{\beta} = [1+(m-1)(\lambda-\mu_0)]/2$. In the linear case $m = 1$ we have $\tilde{\beta} = 1/2$, $\mu_0 = N/2$ and these similarity solutions reduce to (3.34) with $\lambda = \lambda_k$ being the spectrum (3.27). Substituting (4.28) into (4.26) yields that $\psi = \psi(y)$ is a weak solution of the following *nonlinear eigenvalue problem* for the quasilinear elliptic equation

$$\begin{gathered} \mathbf{B}(\psi) \equiv \Delta(|\psi|^{m-1}\psi) + \tilde{\beta}\nabla\psi \cdot y + \mu_0\psi = \lambda\psi \ \text{ in } \mathbb{R}^N, \\ \psi \in C_0(\mathbb{R}^N), \ \ \psi(y) \not\equiv 0. \end{gathered} \tag{4.29}$$

Using the results of the phase-plane analysis of the ODE (4.29), we present two asymptotics of eigenfunctions at singular points. Let $y_0 = \sup \operatorname{supp} \psi_k$. Then

$$\begin{gathered} \psi_k(y) = \pm A_k(y_0 - y)^{1/(m-1)}(1 + o(1)) \\ \text{as } y \to y_0^-, \ \ A_k = [(m-1)\beta_k/m]^{1/(m-1)}. \end{gathered} \tag{4.30}$$

If $\psi_k(y)$ vanishes at an internal point y_1 of $\operatorname{supp}\psi_k$, then the asymptotics behaviour is

$$\psi_k(y) = B|y_1 - y|^{\frac{1}{m}-1}(y_1 - y)(1 + o(1)) \ \text{ as } \ y \to y_1, \tag{4.31}$$

where $B \neq 0$ is a constant. Actually, (4.30) is the limit case of (4.31) with $B = 0$.

Local asymptotic properties of radial eigenfunctions $\psi_k(\bar{y})$ with $\bar{y} = |y| \geq 0$ are the same as for $N = 1$ and (4.30) and (4.31) hold in the new notation.

4.3.2 Discrete spectrum and eigenfunctions for the dual PME

We begin with the one-dimensional case $N = 1$ where the transformation of (4.26) into the dual PME (4.27) is straightforward. Performing one integration yields the p-Laplacian equation in radially symmetric geometry

$$w(x,t) = \int_{-\infty}^{x} u(s,t)\,\mathrm{d}s \Longrightarrow w_t = (|w_x|^{m-1}w_x)_x, \tag{4.32}$$

and a second integration leads to the dual PME in one dimension

$$v(x,t) = \int_{-\infty}^{x} w(s,t)\,\mathrm{d}s \Longrightarrow v_t = |v_{xx}|^{m-1}v_{xx}. \tag{4.33}$$

We then translate the subset of similarity patterns for the PME from Lemma 3.2 to the dual PME (cf. [11]) and obtain a unique family of nonnegative compactly supported similarity solutions of (4.33) for any $k = 0, 1, 2, \dots$

$$v(x,t) = t^{\Lambda_k - \mu_0} \Psi_k(y), \; y = x/t^{\beta_k}, \tag{4.34}$$

$$\Lambda_k = \lambda_{k+2} + 2\beta_{k+2} \equiv -(m-1)\mu_0 + 1 + m\lambda_{k+2}, \tag{4.35}$$

$$\Psi_k(y) = \int_{-\infty}^{y} \int_{-\infty}^{\xi} \psi_{k+2}(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}\xi, \tag{4.36}$$

where eigenfunctions $\{\Psi_k\}$ satisfy the nonlinear eigenvalue problem

$$\mathbf{B}(\Psi) \equiv |\Psi''|^{m-1}\Psi'' + \tilde{\beta}\Psi' y + \mu_0 \Psi = \Lambda \Psi \text{ in } \mathbb{R}, \quad \Psi \in C_0^2(\mathbb{R}), \quad \Psi \not\equiv 0, \tag{4.37}$$

with $\tilde{\beta}$ as in (4.28). The transformation (4.36) on first two similarity PME profiles ψ_0 and ψ_1 yields unbounded and non-compactly supported functions respectively, which do not belong to our functional class. We next classify the spectrum of similarity solutions (4.34). It follows that $\beta_k > 0$ for $k \geq 0$.

Lemma 4.1 *Let $m > 1$ and $N = 1$. Then the spectrum $\sigma(\mathbf{B})$ of operator* (4.37) *consists of a strictly decreasing sequence of negative eigenvalues*

$$\Lambda_k = 2/(m+1) + m\lambda_{k+2} \downarrow -2/(m^2-1), \quad k = 0, 1, 2, \dots, \tag{4.38}$$

so that problem (4.37) *has a compactly supported C^2 solution if, and only if, $\Lambda = \Lambda_k$ for some integer $k \geq 0$, which is the number of sign changes of $\Psi_k(y)$.*

It follows from (4.36) that $\Psi_k'' = \psi_{k+2}$ has precisely $k+2$ sign changes in $\mathbb{R}$. The standard Sturmian property of eigenfunctions is proved below.

Proposition 4.2 *Ψ_k has k sign changes.*

Proof. Recall that Ψ_k has $k+2$ isolated inflection points. We prove the result in three steps.

Step 1: For $\xi > 0$, Ψ_k has exactly one sign change between any two neighbouring inflection points. Suppose not. Then there may exist two neighbouring inflection points, say $\Psi_k(a)$, $\Psi_k(b)$, such that either $\Psi_k(\xi) > 0$ or $\Psi_k(\xi) < 0$ for all $\xi \in (a, b)$. (We note that an inflection point can only occur when $\Psi_k' < 0$ (respectively $\Psi_k' > 0$) if $\Psi_k > 0$ (respectively $\Psi_k < 0$). This follows from the ODE for Ψ_k given by $\mathbf{B}_k(\Psi_k) \equiv \mathbf{B}(\Psi_k) - \Lambda_k \Psi_k = 0$.) When $\Psi_k'' = 0$ we have that

$$(\Lambda_k - \mu_0)\Psi_k(\xi) = \beta_k \xi \Psi_k'(\xi), \quad \text{where } \Lambda_k - \mu_0 < 0, \; \beta_k > 0. \tag{4.39}$$

We assume that $\Psi_k, \Psi_k'' > 0$ on (a,b) and thus that Ψ_k' is an increasing function on this interval (all other combinations of signs of Ψ_k and Ψ_k'' require only slight variations to the proof below). For some $\varepsilon > 0$, there exists intervals $(a-\varepsilon, a)$, $(b, b+\varepsilon)$ such that $\Psi_k'' < 0$, and thus Ψ_k' is a decreasing function on these intervals. Since $\Psi_k' < 0$ at any inflection point we have that $\Psi_k'(\xi) < 0$ for $\xi \in (a-\varepsilon, b+\varepsilon)$. Thus at b, Ψ_k' achieves a local maximum whilst it is still negative which is a violation of the Maximum Principle. Thus, Ψ_k must have at least one sign change between two neighbouring inflection points. The proof that it has at most one sign change is a direct consequence of the fact that Ψ_k is a convex function on (a,b). A similar proof yields that Ψ_k must have exactly one sign change between any two neighbouring inflection points in the domain $\xi < 0$.

Step 2: If Ψ_k is a symmetric function then it has exactly k sign changes. Since Ψ_k is symmetric and k is even, we know by symmetry that Ψ_k has $k/2+1$ inflection points in the domain $\xi > 0$. By Step 1, Ψ_k has exactly $k/2$ sign changes in the domain $\xi > 0$, and thus k sign changes overall.

Step 3: If Ψ_k is an anti-symmetric function then it has exactly k sign changes. Since Ψ_k is anti-symmetric, k is odd and we know that Ψ_k has $\lfloor k/2+1 \rfloor$ inflection points in the domain $\xi > 0$ and one inflection point at $\xi = 0$. (Here $\lfloor \cdot \rfloor$ denotes the floor function for real numbers: $\lfloor k \rfloor$ is the largest integer j such that $j \leq k$.) Thus it must have $\lfloor k/2 \rfloor$ sign changes in the domain $\xi > 0$ by step 1 and thus k sign changes overall. $\square$

The one-parameter families of eigenfunctions take the form

$$\Psi_k(y; b) = b\Psi_k(y/b^{(m-1)/2m}) \text{ for any } b > 0. \tag{4.40}$$

Instead of (4.30) and (4.31) we obtain the following asymptotics of eigenfunctions:

$$\Psi_k(y) = \pm A_k(y_0 - y)^\delta(1 + o(1)) \text{ as } y \to y_0^-, \ \delta = \tfrac{2m-1}{m-1} > 2, \tag{4.41}$$

$$\Psi_k(y) = B|y_1 - y|^{(m+1)/m}(1 + o(1)) \text{ as } y \to y_1, \tag{4.42}$$

where $A_k = [\beta_k y_0 \delta^{1-m}(\delta - 1)^{-m}]^{1/(m-1)}$ and $B \neq 0$ is an arbitrary constant. These expansions show that $\Psi_k \in C^{2+\alpha}$ are smooth, classical solutions of the ODE (4.37).

Radial eigenfunctions in $\mathbb{R}^N$. In the multi-dimensional case $N > 1$ the dual PME (4.27) is related to (4.26) by the change $u = -\Delta v$. Let Γ denote the fundamental solution of Laplace's operator in $\mathbb{R}^N$:

$$\Gamma(x) = \frac{1}{N(2-N)\omega_N}|x|^{2-N} \text{ for } N \geq 3, \;\; \Gamma(x) = \frac{1}{2\pi}\ln|x| \text{ for } N = 2,$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$. Then we have $v = -\Gamma * u$, and Lemma 3.3 gives a unique family of similarity solutions (4.34) of the dual PME corresponding to radially symmetric profiles

$$\Psi_k(y) = -\int_{\mathbb{R}^N} \Gamma(y-\xi)\psi_{k+2}(\xi)\,\mathrm{d}\xi, \;\; k = 0, 2, 4, \ldots, \tag{4.43}$$

where each function Ψ_k is a classical radial C^2 solution of the elliptic eigenvalue problem

$$\begin{gathered}\mathbf{B}(\Psi) \equiv |\Delta\Psi|^{m-1}\Delta\Psi + \tilde{\beta}\nabla\Psi\cdot y + \mu_0\Psi = \Lambda\Psi \;\text{ in }\; \mathbb{R}^N, \\ \Psi \in C_0^2(\mathbb{R}^N), \;\; \Psi \not\equiv 0.\end{gathered} \tag{4.44}$$

Similar to the case $N = 1$, for $N \geq 3$ this transformation of the first radial profile ψ_0 yields a bounded but not compactly supported pattern. For $N = 2$ the transformation of similarity solutions (4.28) takes the form

$$v(x,t) = -t^{\lambda_k + 2\beta_k - \mu_0}\left[\frac{\beta_k \ln t}{2\pi}\int_{\mathbb{R}^2}\psi_k(\xi)\,\mathrm{d}\xi + \int_{\mathbb{R}^2}\Gamma(y-\xi)\psi_k(\xi)\,\mathrm{d}\xi\right], \tag{4.45}$$

and therefore for $k = 0$, where $\int \psi_0 > 0$, it does not belong to our functional class. From Lemma 3.3 we obtain the following subset of similarity solutions and again discover that $\beta_k > 0$ for $k \geq 0$.

Lemma 4.3 *Let $m > 1$ and $N > 1$. Then in radial geometry, operator* (4.44) *has a strictly decreasing sequence of negative eigenvalues*

$$\Lambda_k = \frac{2}{N(m-1)+2} + m\lambda_{k+2} \downarrow \frac{-2}{(m-1)[N(m-1)+2]}, \;\; k = 0, 2, 4, \ldots, \tag{4.46}$$

so that problem (4.44) *has a compactly supported, radially symmetric solution if, and only if, $\Lambda = \Lambda_k$, and then Ψ_k has exactly k sign changes in $\mathbb{R}_+$.*

The proof that these eigenfunctions satisfy the Sturmian zero property is similar to that in Proposition 4.2. The scalings (4.40) determine one-parameter families of nonlinear radial eigenfunctions and the asymptotics (4.41) and (4.42) remain valid in terms of the radial variable $\bar{y} = |y|$.

4.4 Critical asymptotic behaviour for the PME with absorption

Using the above "nonlinear spectral analysis" of operator (4.29), similar to the semilinear case in Section 4.2, we study the critical asymptotic behaviour for the PME with absorption (4.1). According to (4.28), we introduce the rescaled variables (cf. (4.7))

$$u(x,t) = t^{\lambda_k-\mu_0} v(y,\tau), \;\; y = x/t^{\beta_k}, \;\; \tau = \ln t, \tag{4.47}$$

to get the following rescaled equation:

$$v_\tau = \mathbf{B}_k(v) - \mathrm{e}^{-\gamma_k \tau} g(v), \;\; \gamma_k = p(\mu_0 - \lambda_k) - (1 + \mu_0 - \lambda_k), \tag{4.48}$$

where $g(v) = |v|^{p-1}v$, $\mathbf{B}_k = \mathbf{B} - \lambda_k I$ and $\mathbf{B}$ is as in (4.29). Similar to (4.9) we obtain a sequence of critical exponents generated by the nonlinear spectrum $\sigma(\mathbf{B})$

$$\gamma_k = 0 \Longrightarrow p = p_k = 1 + 1/(\mu_0 - \lambda_k), \tag{4.49}$$

where $k = 0, 1, 2, \ldots$ for $N = 1$ (Lemma 3.2) and $k = 0, 2, 4, \ldots$ for $N > 1$ (Lemma 3.3). It follows from (3.43) that

$$p_k \downarrow 1 + 1/(m-1) \;\; \text{as} \;\; k \to \infty. \tag{4.50}$$

Hence, $p_\infty > 1$ if $m > 1$ unlike the linear case (4.5), where $p_\infty = 1$ due to the fact that the spectrum (3.27) of the linear self-adjoint operator is unbounded.

In the critical case $p = p_k$ we arrive at the autonomous rescaled equation (cf. (4.10))

$$v_\tau = \mathbf{B}_k(v) - g(v), \;\; \tau \gg 1. \tag{4.51}$$

We will describe a special asymptotic behaviour admitted by equation (4.51), where

$$v(\cdot, \tau) \to 0, \;\; \text{not exponentially fast}, \;\; \text{as} \;\; \tau \to \infty. \tag{4.52}$$

By the construction of the nonlinear eigenfunctions, $\mathbf{B}_k(\psi_k) = 0$, and therefore, similar to the linear expansion case (4.13), using the scaling invariance (3.40) we will study the asymptotic behaviour close to the one-dimensional manifold $W^{(k)} = \{\pm\psi_k(y; b), \, b > 0\}$. The existence of a free parameter b implies the possibility of the orbits moving along the curve (formally $b = b(\tau)$ for $\tau \gg 1$). In order to describe such "slow motion" of these orbits close to $W^{(k)}$, given a solution $v(y, \tau)$ of equation (4.51) and an eigenfunction $\psi_k(y)$, we perform an extra rescaling (as suggested by (3.40))

$$v(y,\tau) = b(\tau)w(\xi,\tau), \;\; \xi = y/b^{(m-1)/2}(\tau), \;\; \text{with a positive function } b(\tau), \tag{4.53}$$

$$b(\tau) \to 0, \;\; \text{not exponentially fast,} \;\; \text{as } \tau \to \infty, \tag{4.54}$$

to be determined in such a way that the new rescaled orbit $\{w(\cdot,\tau)\}$ stabilises as $\tau \to \infty$ to a non-trivial limit equilibrium. The assumption (4.54) is essential (we will show that there exist many "stable manifold patterns" with $b(\tau)$ decaying exponentially).

In view of the scaling invariance (3.40), w satisfies an equation with a non-autonomous perturbation

$$w_\tau = \mathbf{B}_k(w) + q(\tau)\mathbf{C}w - b^{p-1}(\tau)g(w), \;\; \mathbf{C} = \nu_*\xi\cdot\nabla - I, \;\; q(\tau) = \frac{\dot b(\tau)}{b(\tau)}, \tag{4.55}$$

where $\nu_* = (m-1)/2$. If (4.54) holds and $q(\tau) \to 0$ as $\tau \to \infty$, then (4.55) is an asymptotically small perturbation of the autonomous equation

$$w_\tau = \mathbf{B}_k(w) \tag{4.56}$$

admitting a one-parameter family (3.40) of equilibria. We will show that for a particular choice of the scaling function $b(\tau)$ there exists a unique eigenfunction ψ_k such that

$$w(\cdot,\tau) \to \psi_k \;\; \text{as} \;\; \tau \to \infty. \tag{4.57}$$

In one dimension or in radial geometry in $\mathbb{R}^N$, the asymptotic analysis of the perturbed equation (4.55) for arbitrary k is based on the classical theory of singular ordinary differential operators [70].

4.4.1 One-dimensional case

We consider a class of symmetric (anti-symmetric) solutions $u(x,t)$ for even (odd) k and choose

$$b^{(m-1)/2}(\tau) = \sup \operatorname{supp} v(\cdot,\tau), \;\; \tau \gg 1. \tag{4.58}$$

Regularity of interfaces for the PME with reaction-absorption terms are well known, see [32], [58], and [12], [77] for solutions changing sign. Hence, we may assume that $b(\tau)$ is sufficiently smooth for $\tau \gg 1$. Note that equation (4.55) is understood in the weak sense so that actually at this stage we do not need C^1-regularity of the interfaces. Then (4.58) implies that

$$w(\pm 1, \tau) = 0 \text{ and } \operatorname{supp} w(\cdot, \tau) \subseteq [-1, 1] \text{ for } \tau \gg 1. \tag{4.59}$$

Using (3.40), we choose the eigenfunction $\psi_k(\xi)$ such that

$$\operatorname{supp} \psi_k = [-1, 1]. \tag{4.60}$$

We first treat two particular cases.

Example 1: mass conservation for $\mathbf{k = 0}$. The analysis becomes easier for $k = 0$ when the operator $\mathbf{B}_0 = \mathbf{B}$ admits the mass conservation, $\int \mathbf{B}_0(w) d\xi \equiv 0$. Following [45] and [47], integrating equation (4.55) over $\mathbb{R}^N$ we obtain that the mass $M_0(\tau) = \int w$ satisfies

$$M' = q(\tau) \int \mathbf{C} w - b^{p-1} \int g(w). \tag{4.61}$$

Assuming that (4.57) holds for the unique ψ_0 satisfying (4.60), we have that

$$M' = -\varepsilon_1 \frac{\dot{b}}{b}(1 + o(1)) - \varepsilon_2 b^{p-1}(1 + o(1)) \text{ for } \tau \gg 1, \tag{4.62}$$

where $\varepsilon_1 = -\int \mathbf{C}\psi_0 > 0$ and $\varepsilon_2 = \int g(\psi_0) > 0$. Since the mass $M(\tau)$ of a compact orbit is uniformly bounded and nonzero and the first perturbation $\dot{b}/b$ is not integrable on $(1, \infty)$ by (4.54), we have to have that $b(\tau)$ satisfies the "ODE" that corresponds to neglecting M' (see proof in [47])

$$\dot{b} = -b^p(\varepsilon_2/\varepsilon_1)(1 + o(1)) \Longrightarrow b(\tau) = C\tau^{-1/(p-1)}(1 + o(1)) \text{ for } \tau \gg 1, \tag{4.63}$$

where $C = [(p-1)\varepsilon_2/\varepsilon_1]^{-1/(p-1)}$.

Example 2: momentum conservation for $\boldsymbol{k = 1}$. We have the momentum conservation $\int \xi \mathbf{B}_1(w) \equiv 0$ when $k = 1$. Therefore, for the momentum $M(\tau) = \int \xi w$ we obtain equation (4.62), where $\varepsilon_1 = \int \xi \mathbf{C}\psi_1$ and $\varepsilon_2 = \int \xi g(\psi_1)$ and finally the asymptotic behaviour (4.63), which is generic for such anti-symmetric solutions of zero mass; see proof in [46].

Arbitrary $k \geq 2$. No explicit conservation laws for the rescaled operators $\mathbf{B}_k$ apply and it seems that an independent asymptotic ODE for the scaling function $b(\tau)$ in (4.53) cannot be derived. Therefore, we have to study the behaviour of $b(\tau)$ as $\tau \to \infty$ together with the rate of convergence in (4.57).

We begin with the "linearisation" by setting

$$w(\xi,\tau) = \psi_k(\xi) + Y(\xi,\tau) \text{ for } \tau \gg 1, \ \operatorname{supp} Y(\cdot,\tau) \subseteq [-1,1], \tag{4.64}$$

leading to the perturbed equation

$$\begin{gathered} Y_\tau = \mathbf{A}_k Y + q(\tau)P_1 - b^{p-1}(\tau)P_2 + q(\tau)\mathbf{C}Y + \mathbf{D}(Y) - b^{p-1}(\tau)\mathbf{E}(Y), \\ \text{with } \mathbf{A}_k = \mathbf{B}_k'(\psi_k), \ P_1(\xi) = \mathbf{C}\psi_k(\xi), \ P_2(\xi) = g(\psi_k(\xi)) \end{gathered} \tag{4.65}$$

and the nonlinear perturbations

$$\mathbf{D}(Y) = \mathbf{B}_k(\psi_k + Y) - \mathbf{B}_k'(\psi_k)Y, \ \mathbf{E}(Y) = g(\psi_k + Y) - g(\psi_k), \tag{4.66}$$

which are quadratic in Y as $Y \to 0$ in the corresponding metrics. We need to describe special asymptotic behaviour of global uniformly bounded solutions of (4.65) satisfying

$$Y(\cdot,\tau) \to 0 \text{ not exponentially fast as } \tau \to \infty. \tag{4.67}$$

We now write down the Fréchet derivative $\mathbf{A}_k$ in Sturm-Liouville form

$$\mathbf{A}_k Y = m(|\psi_k|^{m-1}Y)'' + \beta_k Y'\xi + (\mu_0 - \lambda_k)Y \equiv \frac{1}{\rho}[(pY')' - qY], \tag{4.68}$$

$$\text{with } p(\xi) = |\psi_k(\xi)|^{2(m-1)} \exp\left\{\frac{\beta_k}{m}\int |\psi_k(\xi)|^{1-m}\xi \, d\xi\right\}, \tag{4.69}$$

$$\rho(\xi) = \frac{p(\xi)}{m|\psi_k(\xi)|^{m-1}} \text{ and } q(\xi) = \rho(\xi)[\lambda_k - \mu_0 - m(|\psi_k(\xi)|^{m-1})''].$$

We next need spectral properties of this symmetric singular ordinary differential operator on the bounded interval $(-1,1)$. It is symmetric relative to the inner product $\langle\cdot,\cdot\rangle_\rho$ in the weighted space $L^2_\rho(I)$ with the induced norm denoted by $\|\cdot\|_\rho$. It follows from (4.31) that $1/p(\xi)$ is locally integrable, $1/p \in L^1_\rho$, in a neighbourhood of any internal point $\xi_1 \in (-1,1)$, where $p(\xi)$ vanishes. Hence, $\mathbf{A}_k$ has only two singular points $\xi = \pm 1$; see [70], Chapter V. It suffices to consider

$\xi = 1$. Let $s = 1 - \xi > 0$. Then the operator $\mathbf{A}_k$ takes the following form

$$\mathbf{A}_k Y \sim \tfrac{1}{s^{\kappa-1}}[(s^\kappa Y')' - s^{\kappa-1} Y].$$

We use the asymptotics (4.30) and classical asymptotic ordinary differential equation theory to get that

$$p(s) \sim s^\kappa \;\text{ and }\; \rho(s), q(s) \sim s^{\kappa-1}, \tag{4.70}$$

where $\kappa = (2m-3)/(m-1)$. In order to get the deficiency indices of $\mathbf{A}_k$, we fix $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and, using (4.70), solve equation the $\mathbf{A}_k Y = \lambda Y$ by standard asymptotic ordinary differential equation techniques to get two linearly independent solutions as $s \to 0$ (the term λY is negligible)

$$\begin{aligned} &Y_1 = 1 + O(s) \;\; (Y,\, Y' \text{ are bounded}) \;\text{ and} \\ &Y_2 \sim s^{1-\kappa},\; m \neq 2;\;\; Y_2 \sim \ln s,\; m = 2. \end{aligned} \tag{4.71}$$

Since $Y_2 \in L^2_\rho$ and $Y_1 \in L^2_\rho$ if, and only if, $\kappa > 0$, i.e., $m > 3/2$ we have that the deficiency indices are $(2,2)$ if $m > 3/2$ ($\xi = 1$ is in the limit-circle case of singular end-point) and are $(1,1)$ if $m \in (1, 3/2]$ (the limit-point case). Therefore, for $m > 3/2$ any real self-adjoint extension of $\mathbf{A}_k$ has a discrete spectrum, [70], p. 84. For $m \in (1, 3/2]$ a similar result follows from the L^2_ρ-integrability of the kernel of the inverse operator; see examples in [70], Section 23. One can see, by a direct construction via the variation of constants formula, that the inverse operator $(\mathbf{A}_k + \varepsilon I)^{-1} : L^2_\rho \to L^2_\rho$ for some $\varepsilon < 0$ has a Hilbert-Schmidt kernel; see [70], p. 88. We summarise these results in the following Proposition by using Naimark's theory of self-adjoint extensions [70, §18 and §19]

Proposition 4.4 *(i) If $m \in (1, 3/2]$, then operator* (4.68) *in $L^2_\rho(I)$ has a discrete spectrum $\sigma(\mathbf{A}_k) = \{\lambda_j^{(k)},\, j = 0, 1, 2, \ldots\}$ which is a strictly monotone decreasing sequence of eigenvalues. The corresponding eigenfunctions $\Phi^{(k)} = \{\psi_j^{(k)}\}$ form an orthonormal basis in $L^2_\rho(I)$.*
(ii) If $m \in (3/2, 2)$, then the same is true in the class of functions satisfying

$$Y(\pm 1) = 0. \tag{4.72}$$

(iii) If $m \geq 2$, then the same is true with the condition

$$Y(\xi) \text{ is bounded as } \xi \to \pm 1. \tag{4.73}$$

(iv) For $m \in (1, 2]$ the centre subspace E^c of such self-adjoint extensions $\mathbf{A}_k$ is one-dimensional and is spanned by the eigenfunction

$$\psi_k^{(k)}(\xi) = c_k \frac{\mathrm{d}}{\mathrm{d}b} \psi_k(\xi; b)\Big|_{b=1} \equiv -c_k \mathbf{C} \psi_k(\xi) \quad \text{with } \lambda_k^{(k)} = 0, \tag{4.74}$$

where $c_k > 0$ is a normalisation constant.

In cases (i) - (iii) of Proposition 4.4, suitable choices of real self-adjoint extensions are governed by choosing the corresponding symmetric, unitary 2×2 matrix $[u]$ that describes the singularity behaviour of admitted solutions as $\xi \to \pm 1$; [70], pp. 74-81. Indeed, in all the cases we deal with the unique, extremal Friedrichs self-adjoint extension of $\mathbf{A}_k$ [13] obtained by using the positive quadratic form $\langle \mathbf{C}w, w \rangle_\rho$ with $\mathbf{C} = -\mathbf{A}_k + cI$, $c \gg 1$, in completing $C_0^\infty((-1,1))$ $(m < 2)$ via the induced norm. For $m \geq 2$, we take space $C_N^\infty((-1,1))$ of functions which are constant for $\xi \approx \pm 1$. This corresponds to the Neumann-type boundary condition (4.73). The eigenfunction (4.74) is obtained by differentiating equation (4.29) with respect to b and using (3.40). By (4.68) this yields $\mathbf{A}_k \psi_k^{(k)} = 0$ where $\psi_k^{(k)}$ satisfies necessary growth conditions at singular endpoints, e.g. (4.72) or (4.73).

We now prove that $\psi_k^{(k)}$ has precisely k zeros on I so that it is kth eigenfunction by Sturm's Theorem.

Proposition 4.5 *$\psi_k^{(k)}$ has exactly k zeros.*

Proof. We know that $\psi_k(\xi)$ has exactly k zeros (Lemma 3.2). Take an interval $[c, d]$ such that $\psi_k' > 0$ on (c, d), $\psi_k'(c) = \psi_k'(d) = 0$ and $\psi_k(\xi) = 0$ for exactly one $\xi \in (c, d)$. We prove that $\psi_k^{(k)}$ must also have exactly one zero in (c, d). (On an interval in which $\psi_k' < 0$ put $\psi_k = -\psi_k$ in the proof below to gain a similar contradiction.)

When $\psi_k' = 0$ we have, by (4.74), that $\operatorname{sign}(\psi_k^{(k)}) = \operatorname{sign}(\psi_k)$. This shows that $\psi_k^{(k)}$ has at least one zero in $[c, d]$ since $\psi_k(c) < 0$ and $\psi_k(d) > 0$ by the Maximum Principle. Hence, it remains to prove that $\psi_k^{(k)}$ can have at most one zero in (c, d). We do this by examining the sign of $\left(\psi_k^{(k)}\right)'$ at a zero of $\psi_k^{(k)}$. Now, for $\xi \in (c,d)$, substituting for ψ_k'' using $\mathbf{B}_k(\psi_k) = 0$ yields

$$\left(\psi_k^{(k)}\right)' = -c_k\left(\frac{(m-3)}{2}\psi_k' + \frac{m-1}{2}\xi\psi_k''\right)$$
$$= -c_k\left\{\frac{(m-3)}{2}\psi_k' + \frac{m-1}{2}\xi\left[\frac{(\lambda_k-\mu_0)\psi - \beta_k\xi\psi'}{m|\psi|^{m-1}} - (m-1)\frac{\psi}{|\psi|^2}(\psi')^2\right]\right\}. \tag{4.75}$$

At any zero ξ_0 of $\psi_k^{(k)}$ we have that

$$(m-1)\psi_k'(\xi_0)\xi_0 = 2\psi_k(\xi_0). \tag{4.76}$$

We now have two distinct cases to consider.

Case (i): $\xi_0 \neq 0$. Substituting for ψ_k' in (4.75) using (4.76) yields

$$\left(\psi_k^{(k)}\right)'(\xi_0) = -c_k\left[\frac{(m-3)a}{(m-1)\xi_0} + \frac{(m-1)\xi_0(\lambda_k-\mu_0)a}{2m|a|^{m-1}} - \frac{a\beta_k\xi_0}{m|a|^{m-1}} - \frac{2a}{\xi_0}\right], \tag{4.77}$$

where $a = \psi(\xi_0)$. Noting that $\lambda_k < 0$, $\beta_k > 0$ by Lemma 3.2 and that at a zero of $\psi_k^{(k)}$, $\operatorname{sign}(a) = \operatorname{sign}(\xi_0)$ by equation (4.76) we have that $(m-3)a/(m-1)\xi_0 < 2a/\xi_0$, $(m-1)\xi_0(\lambda_k-\mu_0)a/2m|a|^{m-1} < 0$ and $a\beta_k\xi_0/m|a|^{m-1} > 0$ if $m > 1$. Thus, at any zero $\xi_0 \in (c,d)$ of $\psi_k^{(k)}$ we know that $\left(\psi_k^{(k)}\right)'$ is positive. Hence, there can be only one. See Figure 4.1.

Case (ii): $\xi_0 = 0$. We have shown that at any other zero ξ_1 of $\psi_k^{(k)}$ the gradient must be positive so a second zero clearly cannot exist. □

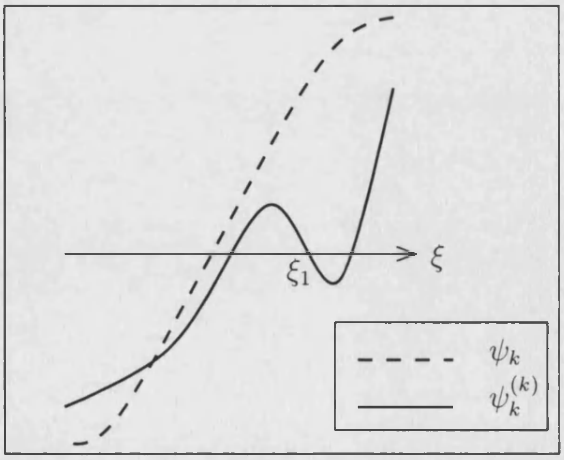


Figure 4.1: The zero marked ξ_1 cannot occur since the gradient of $\psi_k^{(k)}$ at this point is negative.

Asymptotic patterns. It follows from the completeness of the subset $\Phi^{(k)}$ of eigenfunctions in $L^2_\rho(I)$ that

$$L^2_\rho(I) = E^u \oplus E^c \oplus E^s, \quad \text{where} \tag{4.78}$$
$$E^u = \mathrm{Span}\{\psi_1^{(k)}, \dots, \psi_{k-1}^{(k)}\}, \ \ E^c = \mathrm{Span}\{\psi_k^{(k)}\}, \ \ E^s = \mathrm{Span}\{\psi_{k+1}^{(k)}, \psi_{k+2}^{(k)}, \dots\}.$$

Using Proposition 4.4 and (4.59) we get the convergent eigenfunction expansion of the bounded, continuous, weak solution

$$Y(\xi, \tau) = \sum a_j(\tau) \psi_j^{(k)}(\xi). \tag{4.79}$$

The expansion coefficients $\{a_j(\tau)\}$ satisfy the infinite-dimensional dynamical system obtained by substituting (4.79) into (4.65) and multiplying by $\psi_j^{(k)}$ in $L^2_\rho(I)$:

$$\begin{aligned} \dot a_j = \lambda_j^{(k)} a_j + q(\tau)\langle P_1, \psi_j^{(k)}\rangle_\rho - b^{p-1}(\tau)\langle P_2, \psi_j^{(k)}\rangle_\rho + q(\tau)\langle \mathbf{C}Y, \psi_j^{(k)}\rangle_\rho \\ +\langle \mathbf{D}(Y), \psi_j^{(k)}\rangle_\rho - b^{p-1}(\tau)\langle \mathbf{E}(Y), \psi_j^{(k)}\rangle_\rho, \ \ j = 0, 1, 2, \dots . \end{aligned} \tag{4.80}$$

By the PME regularity theory [58], [27], we may assume that (4.67) holds in $L^2_\rho(I)$ and in $H^2_\rho(I)$ and the expansion coefficients are uniformly small:

$$\|Y(\cdot, \tau)\|^2_\rho = \sum a_j^2(\tau) \to 0 \ \text{ as } \ \tau \to \infty. \tag{4.81}$$

We are interested in the critical asymptotic behaviour corresponding to the evolution close to the centre subspace of $\mathbf{A}_k$, so that we exclude both the unstable and stable exponentially decaying patterns. Unlike the semilinear equation in Section 4.2, in the case of the quasilinear parabolic equation (4.65) with the degenerate singular linear operator $\mathbf{A}_k$, we do not know that a centre manifold exists and we cannot use a standard invariant manifold theory; cf. [68] and [74]. Moreover, to our knowledge, for the case of singular operators like (4.68) with degenerate non-constant coefficients, the only known rigorous result is Angenent's analysis [71] of the rate of convergence to the ZKB-profiles for the PME (i.e., for equation like (4.55) without non-autonomous perturbations), where $\ln t$-perturbations were shown to exist (but actual convergence of the asymptotic series was not achieved).

Therefore we perform further asymptotic analysis under the assumption of centre subspace dominance, assuming that the behaviour for $\tau \gg 1$ of the kth coefficient is dominant in the sense that

$$Y(\tau) = a_k(\tau)\psi_k^{(k)} + o(a_k(\tau)) \text{ as } \tau \to \infty \tag{4.82}$$

uniformly on compact subsets and in H^2_ρ. Under this assumption, performing necessary expansions on the right-hand side of (4.4.1), we obtain the following system:

$$\begin{aligned} \dot{a}_j = \lambda_j^{(k)} a_j - \varepsilon_{1,j}\frac{\dot{b}}{b} - \varepsilon_{2,j} b^{p-1} + A_j a_k^2 + B_j a_k^3 + C_j a_k \frac{\dot{b}}{b} \\ + D_j a_k b^{p-1} + O(a_k^2(b^{p-1} + a_k^2)), \end{aligned} \tag{4.83}$$

$$\varepsilon_{1,j} = -\langle \mathbf{C}\psi_k, \psi_j^{(k)}\rangle_\rho, \quad \varepsilon_{2,j} = \langle g(\psi_k), \psi_j^{(k)}\rangle_\rho,$$

$$A_j = \frac{1}{2}m(m-1)\langle [|\psi_k|^{m-3}\psi_k(\psi_k^{(k)})^2]'', \psi_j^{(k)}\rangle_\rho, \quad C_j = \langle \mathbf{C}\psi_k^{(k)}, \psi_j^{(k)}\rangle_\rho,$$

$$B_j = \frac{1}{6}m(m-1)(m-2)\langle [|\psi_k|^{m-3}(\psi_k^{(k)})^3]'', \psi_j^{(k)}\rangle_\rho, \quad D_j = \langle g'(\psi_k), \psi_j^{(k)}\rangle_\rho > 0.$$

Using asymptotics (4.30) one can see that all the expansion coefficients above are finite if $m \in (1, 3/2)$. Roughly speaking, this means that the expansion techniques apply if the PME operator is not "very nonlinear". Such a restriction is natural when dealing with weak solutions, where, in general, expansion methods are hard to apply.

Setting $j = k$ in (4.83) with $\lambda_k^{(k)} = 0$ yields the ODE for a_k describing the behaviour close to the centre subspace

$$\begin{aligned} \dot{a}_k = -\varepsilon_{1,k}\frac{\dot{b}}{b} - \varepsilon_{2,k} b^{p-1} + A_k a_k^2 + B_k a_k^3 + C_k a_k \frac{\dot{b}}{b} \\ + D_k a_k b^{p-1} + O(a_k^2(b^{p-1} + a_k^2)), \end{aligned} \tag{4.84}$$

$$\varepsilon_{1,k} = -\langle P_1, \psi_k^{(k)}\rangle_\rho = c_k \|\mathbf{C}\psi_k\|_\rho^2 > 0, \tag{4.85}$$

$$\varepsilon_{2,k} = \langle P_2, \psi_k^{(k)}\rangle_\rho = c_k\langle g(\psi_k), \psi_k\rangle_\rho - \frac{1}{2}(m-1)c_k\langle g(\psi_k), \psi_k'\xi\rangle_\rho. \tag{4.86}$$

Observe that $A_k = B_k = 0$ for $m = 1$ so that these quadratic and cubic terms on the right-hand side of (4.84) do not occur in the linear case (cf. (4.14)), which makes the centre manifold analysis in Section 4.2 essentially easier.

Consider the dynamical system (4.83), (4.84). It is of crucial importance that under assumption (4.54), $\dot{b}/b \notin L^1(\mathbb{R}_+)$. Therefore, in all the equations this term cannot be balanced by the derivatives $\dot{a}_j(\tau)$ which are integrable by (4.81). We then observe a typical "centre manifold" case where all nonintegrable terms on the right-hand sides must cancel each other.

Proposition 4.6 *Let $m \in (1, 3/2)$ and (4.54) hold, $\varepsilon_{2,k} > 0$, and let the rescaled orbit $\{Y(\tau)\}$ approach the centre subspace in the sense of (4.82) with the rate of convergence satisfying*

$$a_k^2(\tau) = o(b^{p-1}(\tau)) \quad as \ \tau \to \infty. \tag{4.87}$$

Then the scaling function $b(\tau)$ given in (4.53) satisfies

$$b(\tau) = C_k \tau^{-1/(p-1)}(1 + o(1)) \quad as \ \tau \to \infty, \quad C_k = [(p-1)\varepsilon_{2,k}/\varepsilon_{1,k}]^{-1/(p-1)}. \tag{4.88}$$

In view of the existence of infinitely many other patterns on the "stable manifold" as well as infinitely many other "centre manifold" patterns (for different k's), we claim that a sufficiently constructive condition that guarantees the behaviour (4.87) cannot be easily achieved.

Proof of Proposition 4.6. Under the assumption (4.87) the last five terms depending on the rate of convergence $a_k(\tau)$ on the right-hand side of (4.84) are negligible in comparison with the first two terms which thus form nonintegrable perturbations. Therefore, $b(\tau)$ can be determined from the asymptotic ODE

$$-\varepsilon_{1,k}[\dot{b}(\tau)/b(\tau)] - \varepsilon_{2,k} b^{p-1}(\tau) + \nu(\tau) = 0 \ \text{ for } \ \tau \gg 1, \tag{4.89}$$

where $\nu(\tau) \in L^1((1,\infty))$ is an integrable function. Writing down (4.89) in the form

$$(\phi(\tau)b)' = -\frac{\varepsilon_{2,k}}{\varepsilon_{1,k}} b^p \phi(\tau), \ \text{ where } \ \phi(\tau) = \exp\left\{-\frac{1}{\varepsilon_{1,k}}\int_0^\tau \nu(s)\,\mathrm{d}s\right\},$$

integrating it in terms of the new function $B = b\phi(\tau)$ and using that, by the assumption on $\nu(\tau)$ the limit $\phi(\infty)$ exists, we obtain (4.88). Actually, this means that $b(\tau)$ can be obtained from the following equivalent "autonomous" ODE:

$$\dot{b} = -[\varepsilon_{2,k}/\varepsilon_{1,k}] \ b^p(1 + o(1)), \quad \tau \gg 1, \tag{4.90}$$

which was observed earlier in particular examples. $\square$

It is important that (4.88) establishes the same rate of decay of such asymptotic patterns as those in (4.63) already known for two particular cases. The assumption on the centre subspace dominance and nonexponential decay rate as

$\tau \to \infty$ are essential. By Proposition 4.4, equation (4.68) is expected to admit a countable subset of other exponentially decaying patterns corresponding to the behaviour close to the infinite-dimensional stable subspaces themselves corresponding to negative eigenvalues $\lambda_j^{(k)} < 0$ for any $j > k$.

Finally, bearing in mind the critical exponents (4.49) and going back to the original variables, by using the two scalings (4.47) and (4.53) and equality (4.88), we obtain the following subset of critical asymptotic patterns as $t \to \infty$:

$$\begin{aligned} u(x,t) &= C_k (t \ln t)^{-\delta_k} [\psi_k(\eta) + o(1)], \\ \eta &= \frac{x}{t^{1/2}} (t \ln t)^{(m-1)\delta_k/2}, \quad \delta_k = \mu_0 - \lambda_k > 0. \end{aligned} \tag{4.91}$$

It follows from the rescaled variable η in (4.91) that the interface (free boundary) has the behaviour given in (4.3).

4.4.2 Radial multi-dimensional case

In radial geometry with the single spatial variables $\bar{y} = |y| \geq 0$ and $\bar{\xi} = |\xi|$, $k \geq 0$ is always even and we keep the same scalings and transformations as for $N = 1$. Example 1 is true [47]. The Fréchet derivative $\mathbf{A}_k$ in (4.65) is a singular ordinary differential operator on $(0,1)$

$$\mathbf{A}_k Y = m \bar{\xi}^{1-N} (\bar{\xi}^{N-1} (|\psi_k|^{m-1} Y)')' + \beta_k Y' \bar{\xi} + (\mu_0 - \lambda_k) Y, \tag{4.92}$$

which admits a symmetric Sturm-Liouville representation similar to (4.68), where the coefficients include the Jacobian $\bar{\xi}^{N-1}$, (4.69) for p reads

$$p(\bar{\xi}) = \bar{\xi}^{N-1} |\psi_k(\bar{\xi})|^{2(m-1)} \exp \left\{ \frac{\beta_k}{m} \int |\psi_k(\bar{\xi})|^{1-m} \bar{\xi} \, \mathrm{d}\bar{\xi} \right\}$$

and the formula for the weight ρ stays the same. The singular end-point $\bar{\xi} = 1$ has the same properties and deficiency indices as for $N = 1$. In view of the radial Laplacian in (4.92), the origin $\bar{\xi} = 0$ is a singular point (in the limit circle or point case) and the symmetry Neumann condition

$$Y'(0) = 0 \ \ (\text{or } Y(\bar{\xi}) \text{ is bounded as } \bar{\xi} \to 0) \tag{4.93}$$

does not change the spectral properties of necessary self-adjoint extensions formulated in Proposition 4.4. As for the radial Laplacian, the Friedrichs extension is obtained by completing via the positive quadratic form of the space $C_0^\infty([0,1))$, where functions are constant near $\bar{\xi}=0$. The rest of the critical asymptotic construction is the same and finally we arrive at asymptotics (4.91) for even $k \geq 0$ provided that $\varepsilon_{2,k} > 0$.

4.5 Critical asymptotic behaviour for the dual PME with absorption

Similar to Section 4.4 we study the critical asymptotic behaviour of the dual PME with absorption (4.2). Some of the results rely on the transformation (4.32), (4.33) but there are some essential differences between the analysis in this Section and that of Section 4.4 including the linearisations and the spectral analysis. According to (4.34) we introduce the rescaled variables (cf.(4.47))

$$u(x,t) = t^{\Lambda_k - \mu_0} v(y,\tau), \;\; y = x/t^{\beta_k}, \;\; \tau = \ln t, \tag{4.94}$$

to get the following rescaled equation:

$$v_\tau = \mathbf{B}_k(v) - \mathrm{e}^{-\Gamma_k \tau} g(v), \;\; \Gamma_k = p(\mu_0 - \Lambda_k) - (1 + \mu_0 - \Lambda_k), \tag{4.95}$$

where $g(v) = |v|^{p-1}v$ and $\mathbf{B}_k(v) = |\Delta v|^{m-1}\Delta v + \beta_k \nabla v \cdot y + (\mu_0 - \Lambda_k)v$. Similar to (4.49) we obtain a sequence of critical exponents

$$\Gamma_k = 0 \Longrightarrow p = p_k = 1 + 1/(\mu_0 - \Lambda_k), \tag{4.96}$$

where $k = 0,1,2,\ldots$ for $N=1$ (Lemma 4.1) and $k = 0,2,4,\ldots$ for $N>1$ (Lemma 4.3). It follows from (4.46) that (4.50) holds. In the critical case $p = p_k$, the rescaled equation is exactly (4.51) and is autonomous.

We study exponentially decaying asymptotic patterns of equation (4.51) satisfying (4.52). By construction, $\mathbf{B}_k(\Psi_k) = 0$, and by scaling invariance (4.40) we describe the asymptotic behaviour close to the one-dimensional manifold $W^{(k)} = \{\pm\Psi_k(y;b), b>0\}$. To this end we perform an extra rescaling (cf. (4.53))

$$v(y,\tau) = b(\tau) w(\xi,\tau), \;\; \xi = y/b^{(m-1)/2m}(\tau), \tag{4.97}$$

with a positive function $b(\tau)$ satisfying (4.54) such that the rescaled orbit $\{w(\cdot,\tau)\}$ stabilises as $\tau \to \infty$ to a non-trivial limit equilibrium. In view of the scaling invariance (4.40), w satisfies the perturbed equation (4.55) with $\nu_* = (m-1)/2m$, which is an asymptotically small perturbation of the autonomous equation (4.56) admitting equilibria (4.40). We will show that for a particular choice of the scaling function $b(\tau)$ there exists a unique eigenfunction Ψ_k such that

$$w(\cdot, \tau) \to \Psi_k \text{ as } \tau \to \infty.$$

The operator $\mathbf{B}_k$ for any $k \geq 0$ does not admit any conservation laws (unlike the corresponding operator for the PME) and thus the only simplification occurs in the one-dimensional case.

4.5.1 One-dimensional case

In the class of symmetric (anti-symmetric) solutions $u(x,t)$ for even (odd) k we set (cf. (4.58))

$$b^{(m-1)/2m}(\tau) = \sup \operatorname{supp} v(\cdot, \tau), \quad \tau \gg 1. \tag{4.98}$$

In view of transformation (4.33), we may assume that $b(\tau)$ is sufficiently smooth for $\tau \gg 1$ as follows from the PME theory. Then (4.59) holds and by (4.40), $\operatorname{supp} \Psi_k = [-1, 1]$. Since we do not have any explicit conservation laws we study the behaviour of $b(\tau)$ as $\tau \to \infty$ together with the rate of convergence. The linearisation (4.64) gives the perturbed equation (4.65) with the coefficients and perturbation terms given by:

$$\mathbf{A}_k = \mathbf{B}_k'(\Psi_k), \; P_1(\xi) = \mathbf{C}\Psi_k(\xi), \; P_2(\xi) = g(\Psi_k(\xi)), \tag{4.99}$$

$$\mathbf{D}(Y) = \mathbf{B}_k(\Psi_k + Y) - \mathbf{B}_k'(\Psi_k)Y, \; \mathbf{E}(Y) = g(\Psi_k + Y) - g(\Psi_k). \tag{4.100}$$

We study global uniformly bounded solutions of (4.65) satisfying

$$Y(\cdot, \tau) \to 0 \text{ as } \tau \to \infty \text{ not exponentially fast.} \tag{4.101}$$

In Sturm-Liouville form

$$\mathbf{A}_k Y = m|\Psi_k''|^{(m-1)} Y'' + \beta_k \xi Y' + (\mu_0 - \Lambda_k)Y \equiv \frac{1}{\rho}[(pY')' - qY], \tag{4.102}$$

$$\text{with } p(\xi) = \exp\left\{\frac{\beta_k}{m} \int |\Psi_k''(\xi)|^{1-m} \xi \, d\xi\right\}, \; \rho(\xi) = \frac{p(\xi)}{m|\Psi_k''|^{m-1}}$$

and $q(\xi) = \rho(\xi)(\Lambda_k - \mu_0)$. We now need spectral properties of this singular ordinary differential operator in $L^2_\rho(I)$ having two singular points $\xi = \pm 1$. Consider $\xi = 1$. Let $s = 1 - \xi$, we use the asymptotics (4.41) to get that (4.70) holds with $\kappa = -1/(m-1)$. Calculating the deficiency indices of $\mathbf{A}_k$, for any fixed $\lambda \in \mathbb{C}\backslash\mathbb{R}$ we obtain solutions (4.71) without the extra case for $m = 2$, and hence it follows, since $\kappa < 0$, that the deficiency indices are always $(1, 1)$. It is not difficult to see that the inverse operator $(\mathbf{A}_k + \varepsilon I)^{-1}$ with a large constant $\varepsilon < 0$, has L^2_ρ-kernel, so we have a discrete spectrum of the Friedrichs extension.

Proposition 4.7 *(i) The operator* (4.102) *in* $L^2_\rho(I)$ *has a discrete spectrum* $\sigma(\mathbf{A}_k) = \{\Lambda_j^{(k)},\ j = 0, 1, 2, \dots\}$ *which is a strictly monotone decreasing sequence of eigenvalues. The corresponding eigenfunctions* $\Phi^{(k)} = \{\Psi_j^{(k)}\}$ *form an orthonormal basis in* $L^2_\rho(I)$.
(ii) The centre subspace E^c *of such self-adjoint extensions* $\mathbf{A}_k$ *is one-dimensional and is spanned by the eigenfunction*

$$\Psi_k^{(k)}(\xi) = c_k \frac{\mathrm{d}}{\mathrm{d}b} \Psi_k(\xi; b)\Big|_{b=1} \equiv -c_k \mathbf{C} \Psi_k(\xi), \text{ i.e., } \Lambda_k^{(k)} = 0, \tag{4.103}$$

where $c_k > 0$ *is a normalisation constant and* $\mathbf{C}$ *is as in equation (4.55) with* $\nu_* = (m-1)/2m$.

The eigenfunction (4.103) is obtained by differentiating equation (4.44) with respect to b and using (4.40). By (4.102) this yields $\mathbf{A}_k \Psi_k^{(k)} = 0$. We now prove that $\Psi_k^{(k)}$ has precisely k zeros.

Proposition 4.8 $\Psi_k^{(k)}$ *has* k *zeros.*

Proof. We know that Ψ_k has exactly k sign changes (Proposition 4.2). Take an interval $[c, d]$ such that $\Psi_k' > 0$ on (c, d), $\Psi_k'(c) = \Psi_k'(d) = 0$ and $\Psi_k(\xi) = 0$ for exactly one $\xi \in (c, d)$. We prove that $\Psi_k^{(k)}$ must also have exactly one zero in (c, d). (On an interval in which $\Psi_k' < 0$ put $\Psi_k = -\Psi_k$ in the proof below to gain a similar contradiction.)

When $\Psi_k' = 0$ we have, by (4.103), that $\operatorname{sign}(\Psi_k^{(k)}) = \operatorname{sign}(\Psi_k)$ (This shows that $\Psi_k^{(k)}$ has at least one zero in $[c, d]$ since $\Psi_k(c) < 0$ and $\Psi_k(d) > 0$ by the Maximum Principle.) so it remains to prove that $\Psi_k^{(k)}$ can have at most one zero in (c, d). We do this by examining the sign of $(\Psi_k^{(k)})'$ at a zero of $\Psi_k^{(k)}$. Now, for $\xi \in (c, d)$,

$$(\psi_k^{(k)}(\xi))' = -c_k \left[\frac{(m-1)\xi}{2m} \Psi_k''(\xi) - \frac{m+1}{2m} \Psi_k'(\xi) \right]. \tag{4.104}$$

Substituting for Ψ_k'' using $\mathbf{B}_k(\Psi_k) = 0$ yields

$$\left(\Psi_k^{(k)}\right)' = -c_k\left[\frac{(m-1)\xi}{2m}|\alpha_k\Psi_k - \beta_k\xi\Psi_k'|^{\frac{1}{m}-1}(\alpha_k\Psi_k - \beta_k\xi\Psi_k') - \frac{m+1}{2m}\Psi_k'\right]. \tag{4.105}$$

At any zero ξ_0 of $\Psi_k^{(k)}$ we have that

$$\frac{m-1}{2m}\Psi_k'(\xi_0)\xi_0 = \Psi_k(\xi_0). \tag{4.106}$$

We now have two distinct cases to consider.

Case (i) $\xi_0 \neq 0$. Substituting (4.106) into (4.105) yields

$$\begin{aligned}\left(\Psi_k^{(k)}\right)'(\xi_0) = c_k\Bigg\{&\frac{(m+1)a}{(m-1)\xi_0} \qquad (4.107)\\ &-\frac{(m-1)\xi_0}{2m}\left|(\lambda_k-\mu_0)a - \frac{\beta_k m a}{m-1}\right|^{\frac{1}{m}-1}\left[(\lambda_k-\mu_0)a - \frac{\beta_k m a}{m-1}\right]\Bigg\}\end{aligned}$$

where $a = \Psi(\xi_0)$. Noting that at a zero of $\Psi_k^{(k)}$, sign (a) = sign (ξ_0) by equation (4.106), we have that $(\Psi_k^{(k)})'(\xi_0) > 0$ since both terms in equation (4.107) are positive.

Thus at any zero $\xi_0 \in (c,d)$ of $\Psi_k^{(k)}$ we know that the gradient of $\psi_k^{(k)}$ is positive and so there can be only one.

Case (ii) $\xi_0 = 0$. We have shown that at any other zero ξ_1 of $\Psi_k^{(k)}$ the gradient must be negative so a second zero clearly cannot exist. □

Asymptotic patterns. By the completeness in $L^2_\rho(I)$ of the subset $\{\Phi^{(k)}\}$ of eigenfunctions, using Proposition 4.7, equation (4.65) with (4.99) and equation (4.100), we get the uniformly convergent eigenfunction expansion of the bounded, smooth solution $Y(\xi,\tau) = \sum a_j(\tau)\Psi_j^{(k)}(\xi)$. The expansion coefficients $\{a_j(\tau)\}$ satisfy the dynamical system (4.4.1) with $\lambda_j^{(k)}$ replaced by $\Lambda_j^{(k)}$, subject to (4.99), (4.100). We may assume that the expansion coefficients are uniformly small. Being interested in the critical asymptotic behaviour corresponding to the evolution close to the centre subspace of $\mathbf{A}_k$, we exclude both the stable and unstable exponentially decaying patterns. We perform further formal asymptotic analysis under the assumption of centre subspace dominance and assume that the behaviour for $\tau \gg 1$ of the kth coefficient is dominant in the sense that (cf. (4.82))

$$Y(\tau) = a_k(\tau)\Psi_k^{(k)} + o(a_k(\tau)) \;\; \text{as} \;\; \tau \to \infty \tag{4.108}$$

uniformly on compact subsets and in H^2_ρ. Under this assumption, performing necessary expansions on the right-hand side of (4.4.1), with (4.99),(4.100), we obtain the dynamical system (4.83) with coefficients

$$\varepsilon_{1,j} = -\langle \mathbf{C}\Psi_k, \Psi_j^{(k)}\rangle_\rho, \quad \varepsilon_{2,j} = \langle g(\Psi_k), \Psi_j^{(k)}\rangle_\rho,$$
$$A_j = \frac{1}{2}m(m-1)\langle |\Psi_k''|^{m-3}\Psi_k[(\Psi_k^{(k)})'']^2, \Psi_j^{(k)}\rangle_\rho, \quad D_j = \langle g'(\Psi_k), \Psi_j^{(k)}\rangle_\rho > 0,$$
$$B_j = \frac{1}{6}m(m-1)(m-2)\langle |\Psi_k''|^{m-3}[(\Psi_k^{(k)})'']^3, \Psi_j^{(k)}\rangle_\rho, \quad C_j = \langle \mathbf{C}\Psi_k^{(k)}, \Psi_j^{(k)}\rangle_\rho.$$

By asymptotics (4.41) one can see that all the expansion coefficients above are finite if $m \in [1,2)$. For $j = k$ with $\Lambda_k^{(k)} = 0$ we obtain the key ODE (4.83) for $a_k(\tau)$ with the coefficients

$$\varepsilon_{1,j} = -\langle P_1, \Psi_k^{(k)}\rangle_\rho, = c_k\|\mathbf{C}\Psi_k\|_\rho^2 > 0$$
$$\varepsilon_{2,j} = \langle P_2, \Psi_k^{(k)}\rangle_\rho = c_k\langle g(\Psi_k), \Psi_k\rangle_\rho - \tfrac{c_k}{2m}(m-1)\langle g(\Psi_k), \Psi_k'\xi\rangle_\rho.$$

Obviously, $\mathbf{A}_k = \mathbf{B}_k = 0$ for $m = 1$.

The asymptotic analysis of this dynamical system is the same as in Proposition 4.6 (see also the remark afterwards). Finally, in the original variables we obtain the critical asymptotic patterns:

$$u(x,t) = C_k(t\ln t)^{-\delta_k}[\Psi_k(\eta) + o(1)], \quad \eta = \frac{x}{t^{1/2m}}(t\ln t)^{(m-1)\delta_k/2m}, \tag{4.109}$$

where $\delta_k = \mu_0 - \Lambda_k > 0$ and C_k is as given in (4.88).

4.5.2 Radial geometry.

The analysis is quite similar; see the end of Section 4.4. $\mathbf{A}_k$ is a singular ordinary differential operator on $(0,1)$

$$\mathbf{A}_k Y = m|\Delta_{\bar\xi}\Psi_k|^{m-1}\Delta_{\bar\xi}Y + \beta_k\bar\xi Y' + (\mu_0 - \Lambda_k)Y \tag{4.110}$$

with the symmetric representation (4.102), where

$$p(\bar\xi) = \bar\xi^{N-1}\exp\left\{\frac{\beta_k}{m}\int \bar\xi|\Delta_{\bar\xi}\Psi_k|^{1-m}\mathrm{d}\bar\xi\right\}, \quad \rho(\bar\xi) = \frac{p(\bar\xi)}{m|\Delta_{\bar\xi}\Psi_k|^{m-1}}.$$

One can see that the end-point $\bar\xi = 1$ has the same deficiency indices as for $N = 1$. At the origin $\bar\xi = 0$ a symmetry condition (4.93) is imposed. Finally, we obtain critical asymptotics (4.109) for any $k = 0, 2, 4, \ldots$.

Chapter 5

Non-uniqueness and global similarity solutions for a higher order semilinear parabolic equation

According to the Nobel Prize-winning physicist Richard Feynman, any theorem, no matter how difficult to prove in the first place, is viewed as "trivial" by mathematicians once it has been proven. Therefore, there are exactly two types of mathematical objects: trivial ones, and those which have not yet been proven.

In this chapter we will extend some of the ideas and techniques used in the previous two chapters to a higher order parabolic equation. Most techniques fail since they are based on the maximum principle or on comparison theorems. Nonetheless, our aim is to show that some techniques apply in a generalised form.

We consider the $2m$th order ($m \geq 2$) semilinear parabolic equation of reaction-diffusion type

$$u_t = -(-\Delta)^m u + |u|^{p-1}u \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \text{ with exponent } p > 1$$

and initial data $\hat{u} \in L^q(\mathbb{R}^N)$, $q \geq 1$. This is a higher order extension of the classical semilinear heat equation for $m = 1$ from Combustion Theory. It is well known from Weissler's results (1979, 1980) that, for $p < p_0 = 1 + \frac{2mq}{N}$, there exists a unique local in time solution as a continuous curve $u(t) : [0, T] \to L^q(\mathbb{R}^N)$ for sufficiently small $T > 0$. For $m = 1$, it was proved that local nonexistence can happen for $p > p_0$. In the case $m = 1$, which was studied in greater detail in the 1980s, the precise range $p \leq 1 + \frac{2q}{N}$ for uniqueness of such solutions has been established. For $p > 1 + \frac{2q}{N}$, the non-uniqueness was proved by Haraux and Weissler (1982) by constructing similarity solutions.

Our goal is to show that non-uniqueness takes place for the higher order parabolic equations if $p > p_0$. To this end, we describe a discrete subset of similarity solutions

$$u_*(x,t) = t^{-1/(p-1)} V(y), \quad y = x/t^{1/2m},$$

where each V is a radial, exponentially decaying solution of the elliptic equation

$$-(-\Delta)^m V + \tfrac{1}{2m} \nabla V \cdot y + \tfrac{1}{p-1} V + |V|^{p-1} V = 0 \text{ in } \mathbb{R}^N.$$

By perturbation techniques, we establish the existence of radially symmetric similarity profiles V_l for p close to critical bifurcation exponents $p_l = 1 + \frac{2m}{N+l}$, $l = 0, 2, \dots$, and prove that all the p-bifurcation branches remain in the subcritical Sobolev range $p < p_S = \frac{N+2m}{(N-2m)_+}$. By using analytic, asymptotic and numerical methods we justify some global properties of the bifurcation diagram. We also demonstrate that the similarity profiles satisfy Sturm's zero property in a certain "approximate" sense.

5.1 Introduction: non-uniqueness and similarity solutions

5.1.1 A higher order semilinear parabolic equation admitting blow-up

We consider the Cauchy problem for the $2m$th order semilinear parabolic equation

$$u_t = -(-\Delta)^m u + u^p \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \quad u(x,0) = \hat{u}(x) \in L^q(\mathbb{R}^N), \ q \geq 1, \tag{5.1}$$

where Δ denotes the Laplacian in $\mathbb{R}^N$, and, for convenience, we use the notation

$$u^p := |u|^{p-1} u, \quad \text{with a fixed exponent } p > 1.$$

The second order case $m = 1$ corresponds to the classical *semilinear heat equation* from Combustion Theory [89]

$$u_t = \Delta u + u^p. \tag{5.2}$$

From the 1960s this equation became a crucial nonlinear model describing various classes of global and blow-up solutions in reaction-diffusion theory. A number of general techniques and results on *blow-up*, i.e., *global* (in time) *nonexistence* of solutions, starting from Fujita's analysis (1966), were first developed for (5.2) and for the *Frank-Kamenetskii equation* with exponential nonlinearity

$$u_t = \Delta u + e^u,$$

also known as the *solid fuel model*, [89]. There is a large amount of mathematical literature on this subject; see the survey on blow-up problems for nonlinear parabolic equations [49]. The blow-up behaviour is typically studied for sufficiently smooth initial data (e.g., bounded and integrable) that guarantee the equation can be solved locally in time in the classical sense. Then the solutions can blow-up in finite time, as $t \to T^-$, meaning the formation of an *evolution singularity* of a special space-time structure which was carefully studied in the last twenty years for $m = 1$; see the above survey and books [78, Chapter 4], [50, Chapters 9,10]. For $m \geq 2$, blow-up singularity formation phenomena have been studied less. Self-similar and approximate self-similar blow-up patterns for $2m$th order equations like (5.1) are described in [18, 36].

5.1.2 Local nonexistence and continuous dependence

The problems of *local existence* and *uniqueness* of solutions of (5.1) for general initial data $\hat{u} \in L^q(\mathbb{R}^N)$ are no less important for general PDE theory. These are key questions of the classical theory of parabolic equations; see the books by Eidel'man [29], Friedman [33] and Henry [55], which treated wide classes of nonlinearities and initial data.

Given general data $\hat{u} \in L^q(\mathbb{R}^N)$ ($\hat{u} \notin L^\infty$ so that the unbounded nonlinearity u^p in (5.1) can play the crucial role at the initial moment of time $t = 0^+$), by using the analogy with blow-up, the Cauchy problem can be treated as the analysis of formation (collapse) of the *initial singularity* posed at $t = 0$. The first question one must answer is whether we have local existence and uniqueness of solutions, which are understood, as standard practice, as proper continuous curves, e.g. $u : [0,T] \to L^q(\mathbb{R}^N)$, satisfying the integral equation

$$u(t) = \mathrm{e}^{-(-\Delta)^m t}\hat{u} + \int_0^t \mathrm{e}^{-(-\Delta)^m (t-s)} u^p(s)\,\mathrm{d}s \quad \text{for } t > 0, \tag{5.3}$$

where $\{\mathrm{e}^{-(-\Delta)^m t}\}$ is the semigroup with the infinitesimal generator $-(-\Delta)^m$. For data $\hat{u} \in L^q(\mathbb{R}^N)$ with $q > 1$, these questions were first systematically studied by Weissler [85, 86]. In particular, he showed that a unique solution of (5.3) that is local in time always exists if

$$p < p_0(m,q) = 1 + \tfrac{2mq}{N}; \tag{5.4}$$

see the results in [86, pp. 87-90], for $2m$th order equations like (5.1). For $m = 1$, the end point $p = p_0(1,q)$ was shown to be included into the local existence-uniqueness range. More recent results on local and global existence for higher order parabolic equations like (5.1) can be found in [5, 22, 28].

It is important that the local existence-uniqueness range (5.4) for $m = 1$ is optimal in the sense that, for

$$p > p_0(1,q) = 1 + \tfrac{2q}{N}, \tag{5.5}$$

we have that

(i) there exist initial data $\hat{u} \in L^q$ such that no solution as a curve $u \in C([0,T]; L^q) \cap C^1((0,T]; L^q)$ exists for arbitrarily small $T > 0$; see [86, Theorem 1]; and

(ii) if in addition to (5.5),

$$p < p_S(1) = \tfrac{N+2}{(N-2)_+} \quad \text{(the critical Sobolev exponent for } m = 1\text{)}, \tag{5.6}$$

then (5.3) (or (5.1)) admits a nontrivial *nonnegative* solution with $\hat{u} = 0$ in $L^q(\mathbb{R}^N)$, i.e., uniqueness fails, at least, for zero initial data. This *non-uniqueness* result was proved in [54] by constructing self-similar solutions of the semilinear heat equation under consideration; see more detailed comments below.

5.1.3 Main results on non-uniqueness via similarity solutions

Note that counterexamples for the uniqueness theory are not known for $2m$th order equations and require a delicate analysis of the similarity solutions of (5.1) to be studied in this chapter.

For the higher order equation (5.1) with $m \geq 2$, we study the existence and multiplicity of similarity solutions of the form

$$u_*(x,t) = t^{-1/(p-1)}V(y), \;\; y = x/t^{1/2m}, \tag{5.7}$$

derived by noting that for any $\lambda > 0$

$$u_\lambda(x,t) = \lambda^{-1/(p-1)}u(x/\lambda^{1/2m}, t/\lambda) \tag{5.8}$$

is also a solution with compactly supported initial data; and then setting $\lambda = t$. Substituting (5.7) into the PDE yields that V is a non-trivial solution of the elliptic equation

$$\mathbf{B}_1 V + V^p \equiv -(-\Delta)^m V + \tfrac{1}{2m}\nabla V \cdot y + \tfrac{1}{p-1}V + V^p = 0 \;\text{ in }\; \mathbb{R}^N, \tag{5.9}$$

$$V(y) \;\text{ decays exponentially fast as }\; |y| \to \infty. \tag{5.10}$$

Results for $m = 1$. This second order equation with $m = 1$ is well known from the 1980s. It was studied in detail by using phase-plane analysis in the radial case [54, 85] and by the variational approach in the elliptic setting [87] based on the earlier pioneer work [66] on variational methods in weighted Sobolev spaces. For $m = 1$, the linear part $\mathbf{B}_1$ in (5.9) is symmetric and the equation can be written in the form

$$\tfrac{1}{\rho}\nabla \cdot (\rho \nabla V) + \tfrac{1}{p-1}V + V^p = 0, \;\text{ with exponential weight }\; \rho(y) = \mathrm{e}^{|y|^2/4}. \tag{5.11}$$

This determines a potential operator in the weighted space $L^2_\rho(\mathbb{R}^N)$, so the solutions of (5.11) are obtained as the critical points of the functional

$$F(V) = -\tfrac{1}{2}\int \rho|DV|^2 + \tfrac{1}{2(p-1)}\int \rho V^2 + \tfrac{1}{p+1}\int \rho|V|^{p+1}. \tag{5.12}$$

The even smooth functional F is not positive but the well known versions [20, 76] that extend the classical Lusternik–Schnirel'man category theory [69] to non-positive functionals apply. In the subcritical Sobolev range (5.6) (necessary for compact embedding of the functional spaces H^1_ρ, L^2_ρ and l^{p+1}_ρ that are involved in (5.12)), this gives a countable subset of different similarity profiles $\{V_l, l = 0,1,2,\ldots\}$, where, for $p > 1+\frac{2}{N}$, the first profile $V_0(y)$ is strictly positive in $\mathbb{R}^N$ by the variational setting, see [87]. The same result is derived by the phase-plane analysis (a shooting-type argument) of the second order radial ODE (5.11), [85]. In both approaches typical features of the Maximum Principle for the second order elliptic and parabolic PDEs are used.

$\boldsymbol{m \geq 2}$ For equation (5.9) with $m \geq 2$, most known techniques fail because of principal difficulties occurring for such higher order operators,

(i) the linear operator $\mathbf{B}_1$ is not symmetric in L^2_ρ-spaces, hence

(ii) the nonlinear operator $\mathbf{B}_1 V + V^p$ cannot be written in a potential form in any weighted L^2 space (a proof is available in [51, Section 7]),

(iii) the shooting argument on the phase-plane becomes extremely difficult even for $m = 2$ where two shooting parameters occur, and the dimension of the parameter space increases dramatically if m gets larger,

(iv) the semigroup corresponding to the PDE (5.1) is not order-preserving (this means that all ODE or PDE arguments connected with the Maximum Principle are no longer valid).

Our main goal is to justify that, nevertheless, several properties of the subset of similarity solutions for $m > 1$ remain the same as for $m = 1$. For the higher order problem (5.9), (5.10), we apply the approach based on bifurcation analysis with respect to the exponent p as a bifurcation parameter. We prove that, under suitable assumptions, there exists a continuous p-branch of similarity profiles $V_0(y)$ defined in a small right-hand neighbourhood of the first critical Fujita exponent

$$p_0 = 1 + \tfrac{2m}{N} \ \ (= p_0(m,1)). \tag{5.13}$$

Continuing this branch for $p > p_0$, the critical Sobolev exponent

$$p_S(m) = \tfrac{N+2m}{(N-2m)_+} \tag{5.14}$$

is shown to play a role and this branch does not enter the range $p \geq p_S$. Further global properties of this bifurcation branch are described by asymptotic, analytic and numerical methods.

The similarity solutions (5.7) determine typical properties of local solutions of (5.1) and the actual range of non-uniqueness. Indeed, one can see that

$$\|u_*(\cdot,t)\|_q = t^\gamma \|V\|_q, \text{ with } \gamma = -\tfrac{1}{p-1} + \tfrac{N}{2mq} \text{ and } \|V\|_q < \infty. \tag{5.15}$$

Therefore $\gamma > 0$ in the range

$$p > 1 + \tfrac{2mq}{N}. \tag{5.16}$$

Hence, if (5.1) admits a nontrivial similarity solution (5.7), then $u_*(\cdot,0) = 0$ in L^q so that the uniqueness for (5.1) fails, at least, for zero initial data.

We also study other branches of similarity solutions associated with a countable number of the critical bifurcation points (see Proposition 5.4)

$$p_l = 1 + \tfrac{2m}{N+l}, \quad l = 0, 1, 2, \ldots, \tag{5.17}$$

and show that, in general, problem (5.9), (5.10) can admit various solutions. Actually, in view of the infinite number of bifurcation points (5.17) concentrated at $p = 1^+$ for $l \gg 1$, we expect that, for any $p \in (1, p_S)$, there exists an infinite countable subset of different similarity profiles $\{V_l, l = 0, 1, 2, \ldots\}$, which can be obtained via p-parameter continuation from the bifurcation points (5.17). Note that in the $2m$th order case, existence of such a countable family cannot be rigorously connected with the Lusternik–Schnirel'man critical point theory (applied to the potential case $m = 1$ only), though we show that this theory can be used "asymptotically". The first solution V_0 bifurcating at $p = p_0^+$ is not strictly positive in $\mathbb{R}^N$ for $m > 1$, unlike the case $m = 1$, where the fact that V_0 is positive follows from the variational statement associated with the Maximum Principle.

It is important that the same critical exponents (5.17) occur in the similarity analysis of *very singular solutions* of semilinear equation with absorption

$$u_t = -(-\Delta)^m u - u^p \tag{5.18}$$

studied in [51], though the local and global bifurcation diagram are quite different (in particular, the direction of p-branches is opposite) to say nothing of the evolution properties of the solutions. Indeed, (5.18) has the strictly monotone and coercive operator in $L^2(\mathbb{R}^N)$ and no blow-up occurs (at least, for $p \in (1, p_S)$), while in (5.1) the operator is neither monotone nor coercive. A list of references concerning equation (5.18) is available in [51].

The plan of this chapter is as follows. In Section 5.2 we briefly describe properties of the fundamental solution of the $2m$th order linear parabolic operator in

(5.1) and explain spectra and eigenfunctions of linear non-self-adjoint operators associated with $\mathbf{B}_1$ given in (5.9). Section 5.3 is devoted to the asymptotic analysis of the radial ODE (5.9) for similarity profiles. The local bifurcation problem for (5.9), (5.10) is studied in Section 5.4 together with a stability analysis, while in Section 5.5 we present some results on global continuation of the p-bifurcation branches and study some structural properties of the similarity profiles. In particular, we observe that Sturm's property of zeros, which is true for $m = 1$ only, can be applied to higher order equations in an "approximate" sense to be properly introduced. We also present an asymptotic argument concerning the "approximate" application of the Lusternik–Schnirel'man critical point theory showing why equation (5.9), with the non-potential operator (for $m \geq 2$), can admit a countable subset of large solutions in the subcritical range $p \in (1, p_S)$.

Finally, in Section 5.6 we study the limit $p \to \infty$ and derive *limit* linear inhomogeneous "eigenvalue" problems describing the limits of similarity profiles $\{V_l\}$. In particular, we describe the corresponding *boundary layer* phenomenon occurring in the ODE (5.9) as $p \to \infty$ in dimension $N < 2m$.

5.2 Fundamental solution and spectral properties of linear operators

5.2.1 Estimate of the fundamental solution

The fundamental solution $b(x,t)$ of the linear parabolic equation

$$u_t = -(-\Delta)^m u \tag{5.19}$$

takes the standard similarity form

$$b(x,t) = t^{-N/2m} f(y), \ \ y = x/t^{1/2m}. \tag{5.20}$$

The rescaled kernel f is the unique radial solution of the elliptic equation

$$\mathbf{B} f \equiv -(-\Delta)^m f + \tfrac{1}{2m} y \cdot \nabla f + \tfrac{N}{2m} f = 0 \ \text{ in } \mathbb{R}^N, \ \text{ with } \int f = 1. \tag{5.21}$$

Then $f(|y|)$ is known to be oscillatory as $|y| \to \infty$ and satisfies the estimate [29]

$$|f(y)| < D\mathrm{e}^{-d|y|^\alpha} \ \text{in } \mathbb{R}^N, \ \text{ where } \alpha = \tfrac{2m}{2m-1} \in (1,2), \tag{5.22}$$

for some positive constants D and d depending on m and N. Such estimates are sufficient [28] to establish global existence of small solutions for $p > p_0$ for equations like (5.1) with the lower-order term replaced by $\pm|u|^p$ or $\pm|u|^{p-1}u$. For the reaction-diffusion equation

$$u_t = -(-\Delta)^m u + |u|^p, \tag{5.23}$$

(5.13) is the first critical Fujita exponent; see [28], [36]. The linear operator $\mathbf{B}_1$ in equation (5.9) is connected with operator (5.21) for the rescaled kernel f in (5.20) by the formula

$$\mathbf{B}_1 = \mathbf{B} + c_1 I, \quad \text{where } c_1 = \tfrac{N(p_0-p)}{2m(p-1)}. \tag{5.24}$$

5.2.2 Point spectrum of the non self-adjoint operator B

In view of (5.24), in order to study the similarity solutions, we need the spectral properties of $\mathbf{B}$ and of the corresponding adjoint operator $\mathbf{B}^*$. Both are considered in weighted L^2-spaces with the weight functions induced by the exponential estimate of the rescaled kernel (5.22).

For $m \geq 2$, we consider $\mathbf{B}$ in the weighted space $L^2_\rho(\mathbb{R}^N)$ with the exponentially growing weight function

$$\rho(y) = \mathrm{e}^{a|y|^\alpha} > 0 \text{ in } \mathbb{R}^N, \tag{5.25}$$

where $a \in (0, 2d)$ is a sufficiently small constant. We ascribe to $\mathbf{B}$ the domain $H^{2m}_\rho(\mathbb{R}^N)$ being a Hilbert space with the norm $\|v\|^2 = \int \rho(y) \sum_{k=0}^{2m} |D^k v(y)|^2 \, \mathrm{d}y$, induced by the corresponding inner product. Then $H^{2m}_\rho \subset L^2_\rho \subset L^2$. The spectral properties of $\mathbf{B}$ are as follows [28].

Lemma 5.1 *(i)* $\mathbf{B} : H^{2m}_\rho \to L^2_\rho$ *is a bounded linear operator with the real point spectrum*

$$\sigma(\mathbf{B}) = \{\lambda_l = -\tfrac{l}{2m}, \ l = 0, 1, 2, \ldots\}. \tag{5.26}$$

The eigenvalues λ_l *have finite multiplicity with eigenfunctions*

$$\psi_\beta(y) = \tfrac{(-1)^{|\beta|}}{\sqrt{\beta!}} D^\beta f(y), \ \ \textit{for any } |\beta| = l. \tag{5.27}$$

(ii) The subset $\Phi = \{\psi_\beta, \ |\beta| = 0, 1, 2, \ldots\}$ *is complete and the resolvent* $(\mathbf{B} - \lambda I)^{-1}$ *is compact in* L^2_ρ.

In the classical case $m = 1$, $f(y) = (4\pi)^{-N/2}e^{-|y|^2/4}$ is the rescaled positive Gaussian kernel and the eigenfunctions are $\psi_\beta(y) = e^{-|y|^2/4}H_\beta(y)$, where H_β are Hermite polynomials in $\mathbb{R}^N$ [13]. The operator $\mathbf{B}$, with the domain H^2_ρ, $\rho = e^{|y|^2/4}$, is self-adjoint and the eigenfunctions form an orthonormal basis in L^2_ρ.

By Lemma 5.1, the centre and stable subspaces of $\mathbf{B}$ are given by $E^c = \text{Span}\{\psi_0 = f\}$, $E^s = \text{Span}\{\psi_\beta,\ |\beta| > 0\}$.

5.2.3 Polynomial eigenfunctions and the spectrum of the adjoint operator B^*

Consider the adjoint operator

$$\mathbf{B}^* = -(-\Delta)^m - \tfrac{1}{2m} y \cdot \nabla. \tag{5.28}$$

For $m = 1$, $\mathbf{B}^* \equiv \frac{1}{\rho^*}\nabla \cdot (\rho^* \nabla)$, $\mathcal{D}(\mathbf{B}^*) = H^2_{\rho^*}$, with weight $\rho^*(y) = e^{-|y|^2/4}$, is self-adjoint in $L^2_{\rho^*}$ and has a discrete spectrum. The eigenfunctions form an orthonormal basis in $L^2_{\rho^*}$. For $m > 1$, we consider $\mathbf{B}^*$ in $L^2_{\rho^*}$ with the exponentially decaying weight function $\rho^*(y) = \frac{1}{\rho(y)} \equiv e^{-a|y|^\alpha} > 0$.

Lemma 5.2 *(i)* $\mathbf{B}^* : H^{2m}_{\rho^*} \to L^2_{\rho^*}$ *is a bounded linear operator with the same spectrum* (5.26) *as* $\mathbf{B}$. *The eigenfunctions* $\psi^*_\beta(y)$ *with* $|\beta| = l$ *are* l*th order polynomials*

$$\psi^*_\beta(y) = \tfrac{1}{\sqrt{\beta!}}\big[y^\beta + \textstyle\sum_{j=1}^{[|\beta|/2m]} \frac{1}{j!}(-\Delta)^{mj} y^\beta\big]. \tag{5.29}$$

(ii) The subset $\Phi^* = \{\psi^*_\beta\}$ *is complete and the resolvent* $(\mathbf{B}^* - \lambda I)^{-1}$ *is compact in* $L^2_{\rho^*}$.

It follows that the orthonormality condition holds

$$\langle \psi_\beta, \psi^*_\gamma \rangle = \delta_{\beta\gamma}, \tag{5.30}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard $L^2(\mathbb{R}^N)$ inner product. For $m = 1$, these are well known properties of Hermite polynomials generated by the corresponding self-adjoint Sturm-Liouville problem [13].

Using (5.30), we introduce the subspaces of eigenfunction expansions and begin with the operator $\mathbf{B}$. We denote by $\tilde{L}^2_\rho$ the subspace of eigenfunction expansions $v = \sum c_\beta \psi_\beta$ with coefficients $c_\beta = \langle v, \psi^* \rangle$ defined as the closure of the finite sums $\{\sum_{|\beta| \le M} c_\beta \psi_\beta\}$ in the norm of L^2_ρ. Similarly, for the adjoint operator

$\mathbf{B}^*$, we define the subspace $\tilde{L}^2_{\rho^*} \subseteq L^2_{\rho^*}$. Note that since the operators are not self-adjoint and the eigenfunction subsets are not orthonormal, in general, these subspaces can be different from L^2_ρ and $L^2_{\rho^*}$, and the equality is guaranteed in the self-adjoint case $m = 1$, $a = \frac{1}{4}$ only.

5.2.4 First application: the stability of zero in the rescaled equation

Following (5.7), we use the similarity scaling

$$u = (1+t)^{-1/(p-1)}v, \quad y = x/(1+t)^{1/2m}, \quad \tau = \ln(1+t) : \mathbb{R}_+ \to \mathbb{R}_+ . \tag{5.31}$$

Then the rescaled solution $v = v(y, \tau)$ solves the autonomous equation

$$v_\tau = \mathbf{B}_1 v + v^p \text{ for } \tau > 0, \quad v(y,0) = v_0(y) \equiv \hat{u}(y), \tag{5.32}$$

and similarity profiles satisfying (5.9), (5.10) are its stationary solutions. We show that, at $p = p_0$, the trivial stationary solution $v \equiv 0$ changes its stability, which is a crucial characterisation of this first critical exponent. As is well known in the general stability and bifurcation theory [23, 65], often this means that $p = p_0$ is a bifurcation point of equilibria, as will be proved in Section 5.4.

Proposition 5.3 *The trivial solution $v \equiv 0$ of equation* (5.32) *is unstable for $p \in (1, p_0)$, and is stable for $p > p_0$.*

Proof. It follows from (5.24), (5.26) that the operator $\mathbf{B}_1$ in (5.25) that is the linearisation about $v = 0$ of the nonlinear operator in (5.32), has the discrete spectrum

$$\sigma(\mathbf{B}_1) = \{\nu_l = c_1 - \tfrac{l}{2m}, \ l = 0, 1, 2, \ldots\}, \tag{5.33}$$

so that $\nu_0 > 0$ for $p \in (1, p_0)$ (since $c_1 > 0$) and $\nu_0 < 0$ for $p > p_0$ (when $c_1 < 0$). In view of the known spectral properties of $\mathbf{B}$ (see Lemma 5.1 and [28]), this stability/instability result follows from the principle of linearised stability, see [68, Chapter 9]. □

In view of the blow-up results in [28] establishing that, for any $p \in (1, p_0]$, there are blowing-up solutions with arbitrarily small initial data, we have that zero is unstable also in the critical case $p = p_0$. For $p > p_0$, there exist global sufficiently small solutions of (5.1) decaying as $t \to \infty$ at least as $O(t^{-N/2m})$. Actually, in this case, (5.23) admits a countable subset of various global asymptotic patterns [28].

5.3 Exponential bundle as $y \to \infty$ in the ODE

It is important to know the actual dimension of the subset of exponentially decaying solutions of (5.9). We describe the asymptotics of its small solutions satisfying $V(y) \to 0$ as $y \to \infty$, where y now denotes the radial variable $|y| \geq 0$. The linearisation of (5.9) about $V = 0$ gives

$$\mathbf{B}_1 V = 0 \ \text{ for } \ y > 0. \tag{5.34}$$

For the decaying solutions, (5.9) is an asymptotically small perturbation of the linear equation (5.34). The asymptotic analysis of perturbed higher order ODEs such as (5.34) is standard in classical ODE theory; see [21, Chapters III-V], and general asymptotic methods in [30].

According to [21], we first derive the leading differential operator with constant coefficients, which defines the asymptotic behaviour. Starting with the ODE (5.34),

$$(-1)^{m+1}[V^{(2m)} + \tfrac{m(N-1)}{y} V^{(2m-1)} + \ldots] + \tfrac{1}{2m} V' y + \tfrac{1}{p-1} V = 0, \tag{5.35}$$

we set $z = y^\alpha$ to obtain the following equation:

$$V^{(2m)} - a_1 V' - \tfrac{1}{z} a_2 V + z^{-1} \mathbf{C}(z) V = 0, \tag{5.36}$$

where $a_1 = \frac{1}{2m}(-1)^m \alpha^{1-2m}$, $a_2 = \frac{1}{p-1}(-1)^m \alpha^{-2m}$ and

$$\mathbf{C}(z)V = \textstyle\sum_{j=1}^{2m-1} \gamma_j z^{j+1-2m} V^{(j)}$$

is a linear operator with bounded coefficients as $z \to \infty$. In this sum the coefficient of the highest derivative $V^{(2m-1)}$ is $\gamma_{2m-1} = \frac{1}{2m}(-1)^{m+1}[1 + m(N-1)(2m-1)]$ and the coefficient of the first derivative V' is of order $O(z^{2-2m}) = o(z^{-1})$ as $z \to \infty$. By perturbation theory for higher order linear ODEs, the leading terms of exponentially decaying solutions are described via those for the operator in

(5.36) with constant coefficients,

$$V^{(2m)} - a_1 V' = 0. \tag{5.37}$$

Setting $V = e^{\mu z}$, $\mu \neq 0$, gives the characteristic equation $\mu^{2m} - a_1\mu = 0$, whence

$$\mu^{2m-1} = a_1 = \tfrac{1}{2m}(-1)^m \alpha^{1-2m} \equiv \rho_0^{2m-1}(-1)^m, \ \text{where } \rho_0 > 0. \tag{5.38}$$

For any $m \geq 1$, there exist $2m-1$ roots $\{\mu_0, \mu_1, \dots, \mu_{2m-2}\}$ given by

$$\mu_k = \begin{cases} \rho_0 e^{i(2k+1)\pi/(2m-1)}, & m = 2l+1, \\ \rho_0 e^{i2\pi k/(2m-1)}, & m = 2l, \end{cases} \tag{5.39}$$

where the m roots have negative real parts, $\operatorname{Re}\mu_k < 0$. These correspond to $k = l, l+1, \dots, 3l$ for $m = 2l+1$ and $k = l, l+1, \dots, 3l-1$ for $m = 2l$. Bearing in mind that, for odd m's, the root for $k = m$ is real, $\mu_m = -\rho_0$, and for any complex root the corresponding subspace of solutions is two-dimensional, we have an

$$m\text{-dimensional bundle of exponentially decaying solutions as } y \to \infty. \tag{5.40}$$

For the second order case $m = 1$, the bundle is simply one-dimensional, which made it possible to use a phase-plane analysis or apply a monotone parabolic method via simple super- and sub-solutions of (5.2) for $p > p_0 = 1 + \frac{2}{N}$; see [78, Chapter 4].

On the other hand, (5.36) also admits solutions with algebraic decay as $z \to \infty$ (corresponding to the characteristic root $\mu = 0$) described by the first order operator

$$-a_1 V' - \tfrac{1}{z} a_2 V = 0 \implies V(z) = C\, z^{-(2m-1)/(p-1)}.$$

For the linearised equation (5.34), we obtain the algebraic asymptotic behaviour

$$V(y) = C|y|^{-2m/(p-1)}(1 + o(1)) \ \text{as } y \to \infty, \ \text{with any } C \neq 0. \tag{5.41}$$

Such solutions do not satisfy condition (5.10) and represent another family of asymptotic similarity patterns for the PDE (5.1).

Thus, if V solves the problem (5.9), (5.10) with the asymptotics from the exponentially decaying bundle, then

$$|V(y)| \le D_1 e^{-d_1|y|^\alpha} \text{ in } \mathbb{R}^N, \text{ with some constants } D_1,\, d_1 > 0. \tag{5.42}$$

Passing to the limit $t \to 0^+$ in (5.7), it follows that such similarity solutions satisfy

$$u_*(x,t) \to 0 \text{ for } x \neq 0, \text{ and } |u_*(x,t)|^r \to \text{const.}\,\delta(x), \;\; r = \tfrac{(p-1)N}{2m} < 1, \tag{5.43}$$

in the sense of bounded measures in $\mathbb{R}^N$, where $\delta(x)$ denotes Dirac's mass. Solutions with algebraic decay (5.41) form the following initial data with the uniform convergence in $|x|$ on $[\varepsilon, \infty)$, $\varepsilon > 0$ as $t \to 0^+$:

$$u_*(x,0^+) = C|x|^{-2m/(p-1)}. \tag{5.44}$$

5.4 Existence of similarity profiles close to bifurcation points

Consider the ODE problem (5.9), (5.10). Using the linear analysis of Section 5.2, we formulate the bifurcation problems, which guarantee the existence of a similarity solution in a neighbourhood of bifurcation points.

5.4.1 Countable subset of bifurcation points $\{p_l\}$

Taking p close to the critical values, as defined in (5.17), we look for small solutions of (5.9). At $p = p_l$, the linear operator $\mathbf{B}_1$ has a nontrivial kernel, hence, the following result.

Proposition 5.4 *If for an integer $l \ge 0$, the eigenvalue $\lambda_l = -\frac{l}{2m}$ of operator* (5.21) *is of odd multiplicity, then the critical exponent* (5.17) *is a bifurcation point for the problem* (5.9), (5.10).

Proof. Given an $n \gg 1$, we denote by $(V^p)_n$ a suitable uniformly Lipschitz continuous truncation of the nonlinearity V^p such that $(V^p)_n \equiv V^p$ for $|V| \le n$ so

$$(V^p)_n \to V^p \text{ as } n \to \infty \text{ uniformly on compact subsets.}$$

Consider in L^2_ρ the truncated equation

$$\hat{\mathbf{B}}V = -(1+c_1)V - (V^p)_n, \ \text{where } \hat{\mathbf{B}} = \mathbf{B}_1 - (1+c_1)I \equiv \mathbf{B} - I. \quad (5.45)$$

It follows from (5.33) that the spectrum $\sigma(\hat{\mathbf{B}}) = \{-1 - \frac{l}{2m}\}$ consists of strictly negative eigenvalues. The inverse operator $\hat{\mathbf{B}}^{-1}$ is known to be compact, [28, Proposition 2.4]. Therefore, in the corresponding integral equation

$$V = \mathbf{A}(V) \equiv -(1+c_1)\hat{\mathbf{B}}^{-1}V - \hat{\mathbf{B}}^{-1}(V^p)_n, \quad (5.46)$$

the right-hand side is a compact Hammerstein operator; see [64, Chapter V] or [18, 51]. In view of the known spectral properties of $\hat{\mathbf{B}}^{-1}$ (Section 5.2), bifurcations in the problem (5.46) occur if the derivative $\mathbf{A}'(0) = -(1+c_1)\hat{\mathbf{B}}^{-1}$ has the eigenvalue 1 of odd multiplicity, [65, 64]. Since $\sigma(\mathbf{A}'(0)) = \{(1+c_1)/(1+\frac{l}{2m})\}$, we obtain the critical values (5.17). By construction, the solutions of (5.46) for $p \approx p_l$ are small in L^2_ρ and, as can be seen from the properties of the inverse operator, V is small in the domain H^{2m}_ρ of $\mathbf{B}$. Since the weight (5.25) is a monotone growing function as $|y| \to \infty$, using the known asymptotic properties of solutions of the ODE (5.9) in Section 5.3, $V \in H^{2m}_\rho$ is a uniformly bounded, continuous function (for $N < 2m$ this directly follows from Sobolev's embedding theorem). Note that, for even m's, solutions of (5.9) may blow-up at finite $y = y_0$ (a striking contrast to second order ODEs) forming singularities $\sim |y-y_0|^{-2m/(p-1)} \notin L^2_\rho$ locally. Therefore, for $p \approx p_l$, we only have bounded, small solutions. Hence the same bifurcations occur in the original non-truncated equation (5.46) corresponding to $n = \infty$. □

Thus $l = 0$ is always a bifurcation point since $\lambda_0 = 0$ is simple. In general, for $l = 1, 2, \ldots$ the odd multiplicity occurs depending on the dimension N. For instance, for $l = 1$, the multiplicity is N, and, for $l = 2$, it is $\frac{N(N+1)}{2}$. In the case of even multiplicity of λ_l, extra analysis is necessary to guarantee that a bifurcation occurs [65].We do not perform this study here and note that the non-degeneracy of this vector field is not straightforward. It is crucial that, for the main applications ($N = 1$ and the radial setting in $\mathbb{R}^N$), the eigenvalues (5.26) are simple and (5.17) are always bifurcation points. The nonlinear perturbation term in the integral equation (5.46) is an odd smooth operator. This implies the following result describing the local behaviour of bifurcation branches occurring in the main applications; see [64] and [65, Chapter 8].

Proposition 5.5 *Let λ_l be a simple eigenvalue of* $\mathbf{B}$ *with eigenfunction ψ_l, and let*

$$\kappa_l = \langle \psi_l^p, \psi_l^* \rangle. \tag{5.47}$$

Then: (i) if $\kappa_l > 0$, then problem (5.9), (5.10) *has precisely two small solutions for $p \approx p_l^+$ and no solutions for $p \approx p_l^-$, and (ii) if $\kappa_l < 0$, then it has precisely two small solutions for $p \approx p_l^-$ and no solutions for $p \approx p_l^+$.*

We next describe the behaviour of solutions for $p \approx p_l$ and apply the classical Lyapunov-Schmidt method, [65, Chapter 8], to equation (5.46) with the operator $\mathbf{A}$ that is differentiable at 0. Since, under the assumptions of Proposition 5.5, the kernel $E_0 = \ker \mathbf{A}'(0) = \operatorname{Span}\{\psi_l\}$ is one-dimensional, we set $V = V_0 + V_1$, where $V_0 = \varepsilon_l \psi_l \in E_0$ and $V_1 = \sum_{k \neq l} \varepsilon_k \psi_k \in E_1$, here E_1 is the complementary (orthogonal to ψ_l^*) invariant subspace. Let P_0 and P_1, $P_0 + P_1 = I$, be projections onto E_0 and E_1 respectively. Projecting (5.46) (with $n = \infty$) onto E_0 yields

$$\gamma_l \varepsilon_l = -\langle \hat{\mathbf{B}}^{-1}(V^p), \psi_l^* \rangle, \;\; \gamma_l = 1 - \tfrac{1+c_1}{1+l/2m} = \tfrac{(N+l)s}{(p-1)(2m+l)}, \tag{5.48}$$

where we denote $s = p - p_l$. By the bifurcation theory (see [65, p. 355] or [25, p. 383], note that operator $\mathbf{A}'(0)$ is Fredholm of index zero), $V_1 = o(\varepsilon_l)$ as $\varepsilon_l \to 0$, so that ε_l is calculated from (5.48) as follows

$$\gamma_l \varepsilon_l = -\varepsilon_l^p \langle \hat{\mathbf{B}}^{-1} \psi_l^p, \psi_l^* \rangle + o(\varepsilon_l^p) \implies |\varepsilon_l|^{p-1} = \hat{c}_l (p - p_l)[1 + o(1)], \;\; \hat{c}_l = \tfrac{(N+l)^2}{4m^2 \kappa_l}.$$

We have used the fact that $\langle \hat{\mathbf{B}}^{-1} \psi_l^p, \psi_l^* \rangle = \langle \psi_l^p, (\hat{\mathbf{B}}^*)^{-1} \psi_l^* \rangle = -\kappa_l / (1 + \frac{l}{2m})$ (recall the identity $(\hat{\mathbf{B}}^{-1})^* = (\hat{\mathbf{B}}^*)^{-1}$).

From the numerical calculation of the bifurcation diagrams performed in Subsection 5.5.2 we expect that $\kappa_l > 0$. Indeed, in view of the orthonormality property (5.30), for $p = 1$, we have $\kappa_l = 1$, so that, by continuity with respect to p,

$$\kappa_l > 0 \;\text{ for all }\; p \approx 1^+, \tag{5.49}$$

where the eigenfunctions $\{\psi_l\}$ and the adjoint polynomials $\{\psi_l^*\}$ are given in (5.27) and (5.29) respectively. However, deducing that the scalar product (5.47) is positive for arbitrary $p > 1$ is not straightforward, and we shall rely on numerical evidence; see below.

By assumption (5.49), we obtain a countable sequence of bifurcation points (5.17) satisfying $p_l \to 1^+$ as $l \to \infty$, with typical pitch-fork bifurcation branches in right-hand neighbourhoods for $p > p_l$. The behaviour of solutions in H^{2m}_ρ takes the form

$$V_l(y) = \pm \left[\hat{c}_l(p-p_l)\right]^{1/(p-1)} (\psi_l(y) + o(1)) \text{ as } p \to p_l^+. \tag{5.50}$$

For convenience, we now fix the main result concerning the local (in p) existence and instability of the similarity profile $V_0(y)$ corresponding to the first bifurcation point, $p = p_0$. If $\kappa_0 > 0$, then two bifurcation branches exist for $p > p_0$.

Theorem 5.6 *For $p \approx p_0^+$, problem* (5.9), (5.10) *admits a solution $V_0(y)$ provided that $\frac{2m}{N}$ is small enough, and it is an unstable stationary solution of the rescaled equation* (5.32).

Proof. Recall that by Proposition 5.3, in the parameter range $p > p_0$, the trivial solution $V = 0$ is asymptotically stable. As we have seen, two continuous branches bifurcating at $p = p_0^+$ exist if

$$\kappa_0 = \langle \psi_0^{p_0}, \psi_0^* \rangle \equiv \int |f|^{2m/N} f > 0 \;\; (\psi_0^* \equiv 1). \tag{5.51}$$

Since the rescaled fundamental solution f satisfies $\int f = 1$, (5.51) holds by continuity provided that $\frac{2m}{N} \ll 1$. Hence, there exists a solution (5.50) with $l = 0$ satisfying, for small $s = p - p_0 > 0$,

$$V_0(y) = (\hat{c}_0 s)^{1/(p-1)}[f(y) + o(1)], \text{ where } \hat{c}_0 = \tfrac{N^2}{4m^2\kappa_0}. \tag{5.52}$$

Let us next estimate the spectrum of the linearised operator of equation (5.32)

$$\mathbf{D}_0 = \mathbf{B}_1 + p|V_0|^{p-1}I. \tag{5.53}$$

Some of the eigenvalues of (5.53) follow from symmetries of the original PDE (5.1). Namely, the stable eigenspace with $\hat{\lambda} = -1$ and $\hat{\psi} = \frac{1}{p-1}V_0 + \frac{1}{2m}\nabla V_0 \cdot y \in L^2_\rho$ follows from the time-translational invariance of the PDE. For $N = 1$, translations in x yield another pair $\hat{\lambda} = -\frac{1}{2m}$, $\hat{\psi} = V_{0y} \in L^2_\rho$. For $N > 1$, in the non-radial setting, this $\hat{\lambda}$ has multiplicity N with eigenfunctions V_{0y_i}. This is not the first pair with the maximal $\operatorname{Re}\hat{\lambda}$.

Bearing in mind that the spectrum (5.26) of the unperturbed operator $\mathbf{B}$ has the unique, non-hyperbolic eigenvalue $\lambda_0 = 0$, we use (5.52) to obtain

$$\mathbf{D}_0 = \mathbf{B} - s(1 + o(1))\mathbf{C}, \tag{5.54}$$

where, as follows from (5.51) and (5.52) at $p = p_0$, the perturbation has the form

$$\mathbf{C} = \tfrac{N^2}{4m^2}\big(1 - \tfrac{p_0}{\kappa_0}|f|^{2m/N}\big)I. \tag{5.55}$$

Therefore, we consider the spectrum of the perturbed operator

$$\tilde{\mathbf{D}}_0 = \mathbf{B} - s\mathbf{C}. \tag{5.56}$$

Since $(\mathbf{B} - I)^{-1}\mathbf{C}$ is bounded,

$$(\tilde{\mathbf{D}}_0 - I)^{-1} = (I - s(\mathbf{B} - I)^{-1}\mathbf{C})^{-1}(\mathbf{B} - I)^{-1}$$

is compact for small $|s|$ as the product of a compact and bounded operators. Hence, the spectrum of $\tilde{\mathbf{D}}_0$ is discrete. By the classical perturbation theory of linear operators [53], the eigenvalues and eigenvectors of $\tilde{\mathbf{D}}_0$ can be constructed as a perturbation of the discrete spectrum $\sigma(\mathbf{B})$ consisting of eigenvalues of finite multiplicity. We are interested in the perturbation of the first simple eigenvalue $\lambda_0 = 0$ where the computations are simplest. Setting

$$\tilde{\lambda}_0 = s\mu_0 + o(s), \;\; \tilde{\psi}_0 = \psi_0 + s\varphi_0 + o(s) \quad \text{as} \quad s \to 0$$

and putting these expansions into the eigenvalue equation $\tilde{\mathbf{D}}_0\tilde{\psi}_0 = \tilde{\lambda}_0\tilde{\psi}_0$ yields

$$\mathbf{B}\varphi_0 = (\mathbf{C} + \mu_0 I)\psi_0. \tag{5.57}$$

We then obtain the solubility (orthogonality) condition

$$\langle(\mathbf{C} + \mu_0 I)\psi_0, \psi_0^*\rangle = 0 \implies \mu_0 = -\langle \mathbf{C}f, 1\rangle.$$

Using (5.55) yields $\mu_0 = \frac{N}{2m} > 0$. Therefore, $\operatorname{Re}\tilde{\lambda}_0 > \frac{sN}{4m} > 0$ for all $p \approx p_0^+$ and hence $V_0(y)$ is unstable in H_ρ^{2m}. □

Inequality (5.51) is expected to be valid for any m and N, and then the whole branch of similarity profiles $V_0(y)$ bifurcating from $p = p_0$ remains unstable for all $p > p_0$, though the proof would require establishing that the eigenvalue of $\sigma(\mathbf{D}_0)$ with maximal real part does not touch the imaginary axis. In particular, this open problem means that a new saddle-node bifurcation never occurs on this p_0-branch, i.e., it does not have turning points in p. Actually, this means that the first bifurcation branch starting at $p = p_0$ describes similarity solutions which are border ones between the stable zero solutions and the blow-up solutions, which form a generic class for the PDE (5.1). Therefore, V_0 cannot be stable for the rescaled evolution equation (5.32).

Obviously, the other bifurcation branches are "more" unstable than the first one. Taking any $l \geq 1$, instead of (5.54) we now have

$$\mathbf{D}_l = \mathbf{B}_1 + p|V_l|^{p-1}I \equiv \mathbf{B} + \left[c_1 + sp_l\hat{c}_l(|\psi_l|^{p-1} + o(1))\right] I, \quad s = p - p_l.$$

From the definition of $\mathbf{B}_1$, (5.24), $c_1 > 0$ for all $p \approx p_l$, thus V_l is unstable for any l.

5.5 Global p-bifurcation diagram and similarity profiles

In this section we describe some global and asymptotic properties of the bifurcation diagram and similarity profiles $\{V_l\}$.

5.5.1 Nonexistence of similarity solutions for $p \geq p_S$

We begin with a first estimate establishing that, in the supercritical Sobolev p-range, similarity solutions do not exist. Namely, we prove that the Sobolev exponent (5.14) is the critical one for existence.

Theorem 5.7 *For $p \geq p_S$, problem* (5.9), (5.10) *does not have a solution* $V(y) \not\equiv 0$.

Proof. We follow the lines of a similar analysis for $m = 1$; see [54, Section 6] and [78, pp. 228-232], where the quasilinear PME diffusion operator was studied. Assume for contradiction that there exists a nontrivial similarity profile $V(y)$ and let $u_*(x,t)$ given by (5.7) be the corresponding similarity solution of the PDE, where, for convenience, we replace $t \mapsto 1+t$. Equation (5.1) is a gradient system and the potential

$$E(t) = \tfrac{1}{2}\int |D^m u|^2 - \tfrac{1}{p+1}\int |u|^{p+1} \qquad (5.58)$$

gives a Lyapunov function that is non-increasing on evolution orbits,

$$E'[u](t) = -\int (u_t)^2 \le 0. \qquad (5.59)$$

The potential is well defined on $u = u_*(\cdot, t) \in H^m \cap L^{p+1}$ for $t > 0$ and substituting $u_*(x, 1+t)$ from (5.7) yields

$$E(t) = (1+t)^\gamma E(0), \;\; \text{where} \;\; \gamma = \tfrac{(N-2m)(p-p_S)}{2m(p-1)}. \qquad (5.60)$$

Then the constant $E(0)$ is bounded by (5.10). In the critical case $p = p_S$, this immediately leads to a contradiction since $\gamma = 0$ in (5.60), hence $E(t) \equiv$ constant contradicting (5.59), where $(u_*)_t \ne 0$.

If $p > p_S$, then $\gamma > 0$, and in order to have the potential to be strictly decreasing according to (5.59), we need $E(0) < 0$ and then

$$E(t) < 0 \;\; \text{for all} \; t > 0. \qquad (5.61)$$

Let us show that any solution satisfying (5.61) cannot be global and blows up in finite time. Together with (5.59), we will use another identity obtained by multiplying (5.1) by u in L^2, which by (5.61), (5.58) gives the following estimate for $G(t) = \int u^2(t)$:

$$G' = 2\int u u_t = -2\int |D^m u|^2 + 2\int |u|^{p+1} > -2(p+1)E. \qquad (5.62)$$

Next, by the Cauchy-Buniakovskii-Schwarz inequality and (5.62),

$$-G(t)E'(t) = \int u^2 \int (u_t)^2 \ge \left(\int u u_t\right)^2 = \tfrac{1}{4} G'G' \ge -\mu G'E, \;\; \mu = \tfrac{1}{2}(p+1).$$

Therefore,

$$GE' - \mu G'E \le 0 \implies \left(\tfrac{G^\mu}{E}\right)' \ge 0. \qquad (5.63)$$

Integrating over $(0,t)$ yields $G^\mu(t) \le c_0 E(t)$ with $c_0 = \frac{G^\mu(0)}{E(0)} < 0$. Using (5.62), we have, for any $t > 0$ that $G^\mu \le |c_0||E| \le c_1 G'$ with $c_1 = \frac{|c_0|}{2(p+1)} > 0$, which is a standard ordinary differential inequality $G' \ge c_2 G^\mu$ for $t > 0$, where $c_2 = \frac{1}{c_1} > 0$. Since $\mu > 1$, this means that $G = \int u^2$ blows up in finite time contradicting the fact that $u = u_*$ is a global in time solution. $\square$

5.5.2 The p-bifurcation diagram and similarity profiles $\{V_l\}$

On Figure 5.1 we present the results of the numerical simulation of the problem (5.9), (5.10) performed by using Matlab boundary value problem solver bvp4c. Details on how this was achieved are in Appendix C. We observe from Figure 5.1(a) that along the bifurcation branches the parameter p is strictly increasing. Figures 5.1(b) and 5.1(c) show the six similarity profiles V_0, V_1, V_2 V_3, V_4, and V_5 for $m = 2$, $p = 6$ in one dimension. These profiles $V(y)$ have asymptotics as $y \to \infty$ from the exponential bundle described in Section 5.3. Note that all the profiles are oscillatory as $y \to \infty$, which is a property inherited from the oscillatory rescaled kernel $f(y)$ of the fundamental solution (5.20). Sturm's zero property that $V_l(y)$ has precisely l zeros cannot be true for $m > 1$, and is valid for $m = 1$ only.

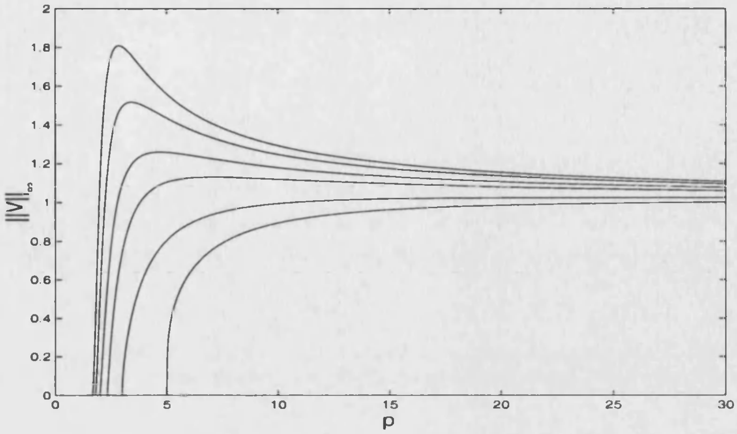
The first profile $V_0(y)$ has the simplest spatial shape with the absolute maximum at the origin $y = 0$. As we mentioned, this profile changes sign and is oscillatory as $y \to \infty$. Using the asymptotic analysis in Section 5.3, from formula (5.39) we have that, for $m = 2$, each $V_l(y)$ has the asymptotics

$$V_l(y) = C_l e^{-ay^{4/3}}[\cos(by^{4/3} - c_l) + o(1)], \quad a = \tfrac{\sqrt{3}}{2}\rho_0, \quad b = \tfrac{1}{2}\rho_0, \quad \rho_0 = \tfrac{3}{4}4^{-1/3}, \tag{5.64}$$

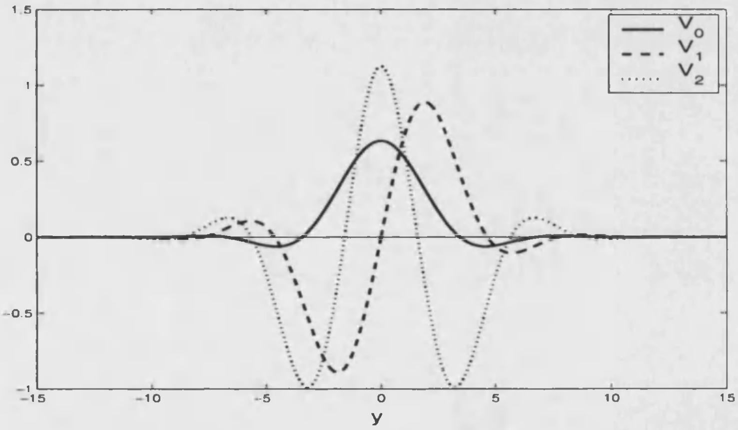
where $C_l \neq 0$ and c_l are some constants depending on l. The profile $V_1(y)$ belongs to the second p_1-bifurcation branch and is non-monotone with two essential, dominant extremum points. As we know, V_1 is more unstable than V_0. The third profile $V_2(y)$ has three essential extrema, and so on.

One can observe from Figure 5.1(b) that the number of "essential" zeros, that are not related to the exponentially small tail oscillations given by (5.64), increases with l according to the standard Sturmian property, i.e., V_0 has no essential zeros, while V_1 has one and V_2 has two, etc. Furthermore, concerning the number of essential extrema, we also observe Sturm's property: V_0 has a single essential maximum, V_1 has two extrema, V_3 has three, etc. In this sense, Sturm's zero and extrema properties remain valid in a certain "approximate" sense, where one needs to detect and distinguish essential zeros and extrema of V_l from an infinite number of the non-essential ones in the oscillating exponential tail with the behaviour (5.64). Other similarity profiles also exhibit approximate Sturmian property.

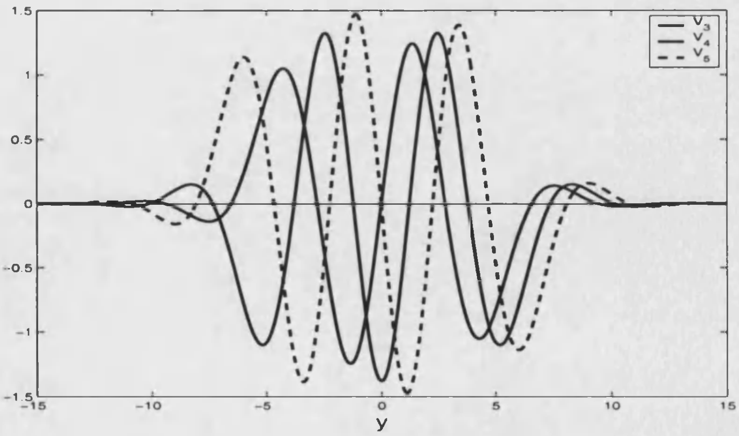
Such an "approximate" Sturm's theory for higher order nonlinear ODEs such



(a) The bifurcation diagram.



(b) The similarity profiles V_0, V_1 and V_2.



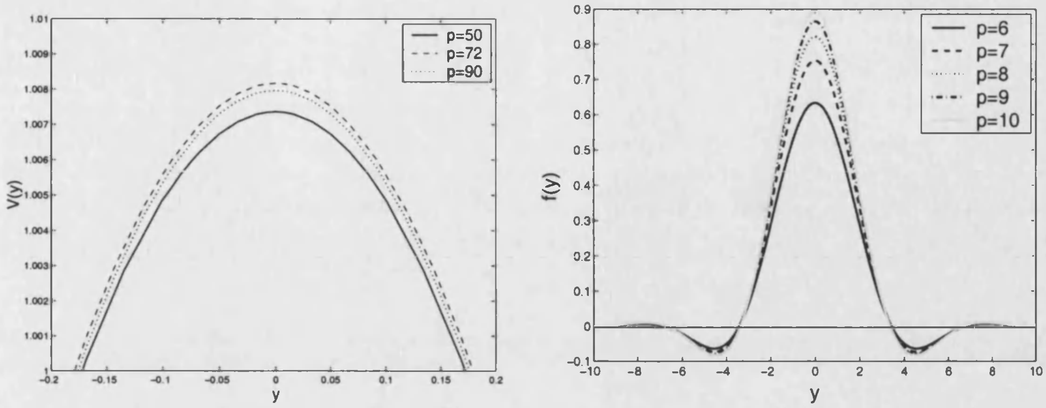
(c) The similarity profiles V_3, V_4 and V_5.

Figure 5.1: The p_0, p_1, p_2, p_3, p_4 and p_5 branches of the bifurcation diagram when $m = 2$, $N = 1$ and the corresponding similarity profiles when $p = 6$.

as (5.9) is not expected to admit an easy rigorous statement though below we present some "asymptotic evidence" related to known Sturmian properties for higher order, self-adjoint, positive ordinary differential operators and Krasnosel'skii's genus theory of critical points. An effective approximation and extension of this classical Sturm Theorem for $m = 1$ to higher order nonlinear operators is a challenging open problem. On the other hand, for practical reasons, a straightforward, naïve approach can be used. Namely, a hint for distinguishing the essential zeros from the exponential tail can be formulated by using the sharp asymptotics (5.64). We say that the infinite set of ordered positive zeros $\{y_1, y_2, \ldots\}$ consists of non-essential tail zeros if, with a sufficient relative accuracy (with respect to the corresponding differences $y_2 - y_1$, $y_3 - y_2, \ldots$), this sequence obeys the expansion (5.64), i.e., there exists a constant $c_l \in \mathbb{R}$ and an integer m such that $by_k^{4/3} - c_l \approx \frac{\pi}{2} + \pi(k+m)$ for all $k = 1, 2, \ldots$. To increase the accuracy of such a representation of zeros from the exponential tail, one can improve the quality of the asymptotic expansion (5.64) by including extra higher order exponential terms.

Let us return to the most important first bifurcation branch. Figure 5.1(a) shows the p_0-bifurcation branch when $m = 2$ and $N = 1$ alongside the p_1, p_2, p_3, p_4 and p_5 branches. (Note that $p_S = \infty$ here since $N < 2m$.) The p_0-branch has several properties of note. Although it is monotone in the bifurcation parameter p we note that it is not monotone in $\|V\|_\infty$. In fact, although we have strong numerical evidence that its limit as $p \to \infty$ is $\|V\|_\infty = 1$ (see also a boundary layer theory in Section 5.6), the branch passes through $\|V\|_\infty = 1$ at approximately $p = 29$, has a maximum at roughly $p = 71.5$ and then approaches $\|V\|_\infty = 1$ from above. This turning point can be seen in Figure 5.2(a) whilst Figure 5.2(b) shows the dependence of the similarity profiles $V(y)$ on the exponent p (see 5.6 for a boundary layer analysis for $p \gg 1$).

The p_1, p_2, p_3, p_4 and p_5 branches also pass through $\|V\|_\infty = 1$ before approaching it from above as $p \to \infty$. The following table gives a rough summary of the critical points.



(a) The turning point in the p_0-bifurcation branch.

(b) The dependence of $V_0(y)$ on $6 \le p \le 10$.

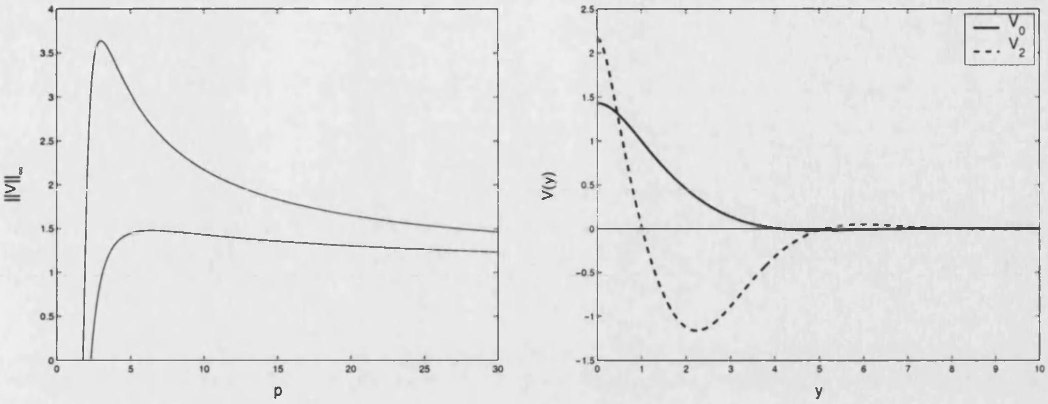
Figure 5.2: The dependence on p of the similarity profiles.

Bifurcation branch	Value at which $\|V\|_\infty = 1$	Turning point
p_0	$p = 28.832$	$p = 71.572$
p_1	$p = 10.587$	$p = 25.501$
p_2	$p = 3.795$	$p = 7.051$
p_3	$p = 2.973$	$p = 4.702$
p_4	$p = 2.227$	$p = 3.413$
p_5	$p = 1.964$	$p = 2.846$

The picture remains essentially the same in higher dimensions. On Figure 5.3(b) we present the radially symmetric profiles $V_0(|y|)$ and $V_2(|y|)$ for $m = 2$ and $N = 3$, where, with y as the radial variable,$\Delta^2 V(y) = V^{(4)}(y) + \frac{4}{y} V'''(y)$. The p_0 and p_2 branches of the bifurcation diagram for this case are shown in Figure 5.3(a). We do not expect any great distinction in the general shape of the p_l-branches, emanating from the corresponding bifurcation points (5.17), for the different values of N that keep $p < p_s$. Further, the solutions and bifurcation diagrams are qualitatively similar for the sixth order equation; see Figure 5.4.

5.5.3 The maximum points of the p-bifurcation branches

Figures 5.1(a) and 5.4(a) and the related analysis demonstrate that the p_l-branches have maximum points, thus posing a natural asymptotic problem to



(a) The bifurcation diagram.

(b) The first two similarity profiles for $p = 10$.

Figure 5.3: The p_0 and p_2 branches of the bifurcation diagram and their similarity profiles when $m = 2$, $N = 3$.

detect those by an analytic argument. It is remarkable that the functions

$$h_l(p) = (p - p_l)^{1/(p-1)} \tag{5.65}$$

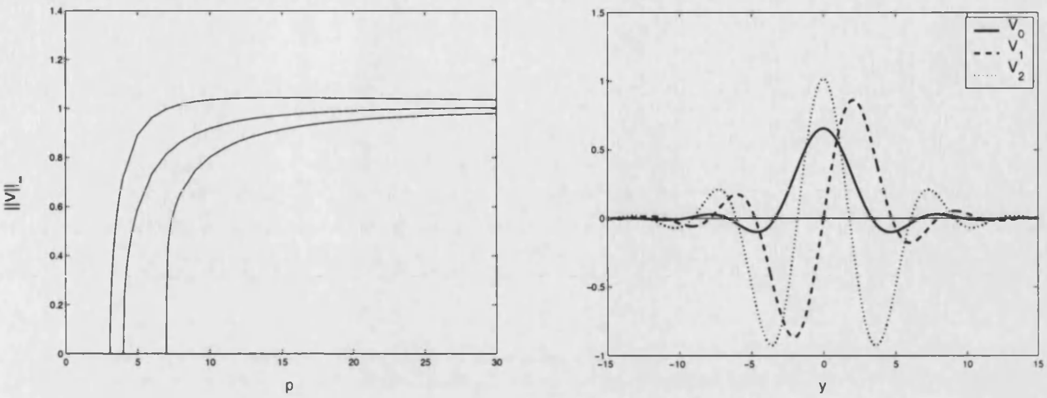
obtained in (5.50) by the linearised bifurcation theory (and hence applied for $p \approx p_l^+$ only) have a single maximum and correctly describe the single maximum shape of the bifurcation branches. Moreover $h_l(p)$ exhibits the following asymptotic behaviour:

$$h_l(p) = 1 + \tfrac{\ln p}{p} + \dots \quad \text{for} \quad p \gg 1. \tag{5.66}$$

We next show that this is a reasonable estimate of the behaviour of the bifurcation branches as $p \to \infty$.

5.6 A limit linear inhomogeneous problem as $p \to \infty$ for $N < 2m$

In this section we present another asymptotic analysis, which can be used for proving the existence of similarity profiles for sufficiently large values of the exponent p. We consider the case $N < 2m$, where the critical Sobolev exponent is infinite, $p_S = \infty$, and the p-bifurcation branches are expected to be well defined for all $p \gg 1$ (this actually happens in the second order case $m = 1$, [85, 87]).



(a) The bifurcation diagram.

(b) The first three similarity profiles for $m = 3$.

Figure 5.4: The p_0, p_1 and p_2 branches of the bifurcation diagram and their similarity profiles when $m = 3$, $N = 1$.

Then we obtain an interesting new problem concerning the asymptotic behaviour of branches and of profiles $V_l(y)$ as $p \to \infty$.

As above, we concentrate on the one-dimensional case with $m = 2$, where the ODE is

$$-V^{(4)} + \tfrac{1}{4} V'y + \tfrac{1}{p-1} V + V^p = 0 \text{ for } y > 0, \quad V'(0) = V'''(0) = 0, \tag{5.67}$$

and $V(y)$ decays exponentially as $y \to \infty$. For this case, Figure 5.1(a) clearly shows that the branches approach the limit

$$\|V\|_\infty \to 1 \text{ as } p \to \infty. \tag{5.68}$$

It follows from the ODE (5.67) that a boundary layer near the origin $y = 0$ occurs as $p \to \infty$ and we will study its asymptotic structure. We begin with a detailed study of the boundary layer occurring for the most important, first p_0-branch where $V_0(y)$ has the unique dominant maximum at $y = 0$ and therefore, in view of (5.68), we use the fact that

$$\|V\|_\infty = V(0) \to 1 \text{ as } p \to \infty. \tag{5.69}$$

5.6.1 Limit linear problem for $\bar{V}_0$

We start with an easy observation that in the outer region, where $|V| \le \delta < 1$ with an arbitrary constant $\delta \in (0,1)$, the ODE (5.9) for $p \gg 1$ is a regular small perturbation of the *limit linear ODE* containing the first two differential terms only

$$\mathbf{B}_* \bar{V} \equiv -\bar{V}^{(4)} + \tfrac{1}{4}\bar{V}'y = 0, \tag{5.70}$$

with the condition of exponential decay (5.10). Furthermore, it follows from the standard estimates of the ODE theory that, in $\{|\bar{V}| \le \delta\}$, the subset of solutions $\{V, \, p \gg 1\}$ is uniformly bounded and equi-continuous and hence compact in $C_{\rm loc}(\mathbb{R})$. Therefore there exists a finite limit

$$V(y) \to \bar{V}(y) \;\; \text{as} \;\; p \to \infty, \tag{5.71}$$

possibly along a subsequence $\{p = p_j\}$. The convergence is uniform on any bounded interval. Passing to the limit as $p = p_j \to \infty$ in the equation (5.9) yields that $\bar{V}$ satisfies the linear ODE (5.70) and the condition (5.10). In view of (5.68) and the symmetry condition at the origin posed for $V(y)$, we impose the following conditions at the origin for the limit function:

$$\bar{V}(0) = 1, \;\; \bar{V}'(0) = 0. \tag{5.72}$$

Obviously, these conditions can apply provided that $\bar{V}(y) < 1$ for small $y > 0$, which we prove next.

Proposition 5.8 *The problem* (5.70), (5.10), (5.72) *has the unique solution* $\bar{V}_0(y)$ *that satisfies*

$$\bar{V}_0''(0) = -2\alpha < 0. \tag{5.73}$$

Proof. By the analysis of the linearised equation in Section 5.3, we have from (5.40) that the ODE (5.70) has a two-parameter family of exponentially solutions, from which we choose two linearly independent ones denoted below by $Y_{1,2}(y)$. Then taking the general solution of (5.70) $\bar{V}(y) = C_1Y_1(y) + C_2Y_2(y)$, we have that it satisfies the conditions (5.72) provided that the constants C_1, C_2 solve the linear system

$$\begin{cases} C_1Y_1(0) + C_2Y_2(0) = 1, \\ C_1Y_1'(0) + C_2Y_2'(0) = 0. \end{cases}$$

Since the Wronskian

$$W(Y_1, Y_2)(0) = \begin{vmatrix} Y_1(0) & Y_2(0) \\ Y_1'(0) & Y_2'(0) \end{vmatrix}$$

is not singular, this system has a unique solution $\{C_1, C_2\}$ which determines the limit profile $\bar{V}(y)$.

Let us prove (5.73). First, multiplying equation (5.70) by $\bar{V}$ and integrating over $\mathbb{R}_+$ yields

$$\bar{V}'''(0) = \textstyle\int [(V'')^2 + \frac{1}{8}V^2] > 0. \tag{5.74}$$

It is clear from the ODE (5.70) that if $\bar{V}''(0) \geq 0$, then $\bar{V}(y)$ is strictly increasing. We show this by deriving its analytic expansion. Differentiating the ODE $\bar{V}^{(4)} = \frac{1}{4}\bar{V}'y$, we have that, for any $k \geq 0$, $\bar{V}^{(k+4)} = \frac{k}{4}\bar{V}^{(k)} + \frac{1}{4}\bar{V}^{(k+1)y}$, so at the origin

$$\bar{V}^{(k+4)}(0) = \tfrac{k}{4}\bar{V}^{(k)}(0). \tag{5.75}$$

Therefore, for any $k \geq 1$,

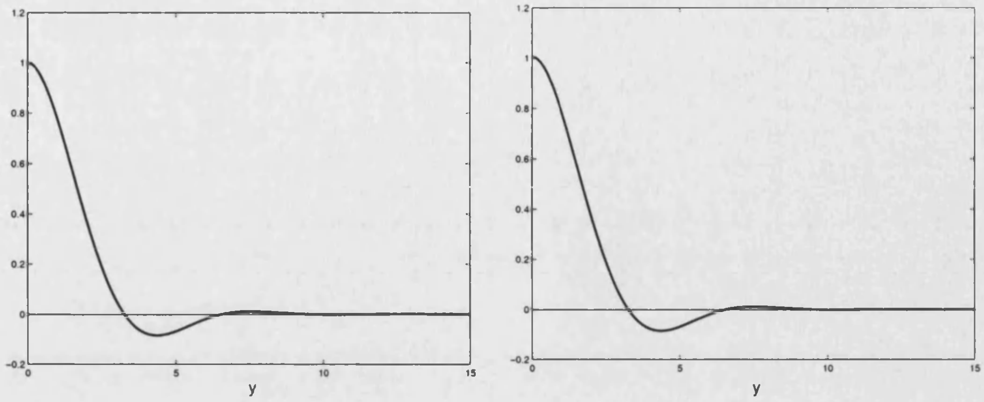
$$|\bar{V}^{(k)}(0)| \leq \text{const.}\tfrac{k(k-4)(k-8)\ldots\gamma_k}{4^{(k-4)/4}}, \tag{5.76}$$

where $\gamma_k = 1, 2$ or 3 depending on $k = 4l + \gamma_k$. It follows that the Taylor series of the solution,

$$\bar{V}(y) = \textstyle\sum_{k=0}^{\infty} \frac{1}{k!}\bar{V}^{(k)}(0)y^k, \tag{5.77}$$

converges uniformly on any bounded interval (obviously $\bar{V}(y)$ must be analytic as a solution of equation (5.70) with analytic coefficients). Now arguing by contradiction and assuming that $\bar{V}''(0) \geq 0$, we obtain from (5.75) and (5.74) that $\bar{V}^{(k)}(0) \geq 0$ for all $k \geq 0$ and then (5.77) determines an analytic function that is increasing in y. This contradicts the exponential decay condition (5.10) since $\bar{V}(0) = 1$. □

By uniqueness of $\bar{V}_0$, the convergence (5.71) holds along any subsequence $\{p_k\} \to \infty$. Figure 5.5 shows the graph of $\bar{V}_0(y)$ calculated numerically alongside the profile V_0 when $p = 240$. The two graphs are presented on separate figures as they would have been indistinguishable otherwise.



(a) The limit profile $\bar{V}_0(y)$.

(b) The V_0 similarity profile when $p = 240$.

Figure 5.5: The limit profile $\bar{V}_0(y)$ and an approximation to it.

5.6.2 Limit linear "eigenvalue" problems for $\bar{V}_l$ with $l \geq 1$

The limit inhomogeneous problems for any $\bar{V}_l$ are similar and need some modifications.

The second, anti-symmetric "eigenfunction" $\bar{V}_1$. For solutions on the second p_1-branch composed of odd profiles $V_1(y)$ that are anti-symmetric in y, i.e., satisfying $V(0) = V''(0) = 0$, we have the convergence (5.71), where the limit function $\bar{V}_1(y)$ solves the following problem for the ODE (5.70) with anti-symmetry conditions (cf. (5.72)):

$$\bar{V}(0) = \bar{V}''(0) = 0, \text{ and } \tag{5.78}$$
$$\sup \bar{V}(y) = 1 \text{ is attained at some } y = y_1 > 0 \text{ with } \bar{V}(y_1) = 1, \; \bar{V}'(y_1) = 0,$$

where $y_1 > 0$ is an unknown parameter ("eigenvalue"). Note that, in general, $\bar{V}_1(y)$ does not satisfy the ODE (5.70) at $y = y_1$ and we consider the limit linear ODE (5.70) in two intervals $(0, y_1)$ and (y_1, ∞). The second, *outer* problem on (y_1, ∞) has a unique solution for any fixed parameter $y_1 > 0$. The proof repeats that of Proposition 5.8 via the Wronskian of the exponentially decaying solutions of (5.70).

We now check what kind of regularity can be imposed on $\bar{V}_1(y)$ at $y = y_1$. The outer problem determines the values $\bar{V}''(y_1^+)$ and $\bar{V}'''(y_1^+)$ but we cannot guarantee that $\bar{V}_1'''(y)$ is continuous at $y = y_1$. Instead, studying the *inner* problem on $(0, y_1)$, we use $\beta_1 = \bar{V}'''(y_1^-)$ as a parameter and consider the Cauchy problem for the limit equation (5.70) with the four conditions at $y = y_1$

$$\bar{V}(y_1) = 1, \;\; \bar{V}'(y_1) = 0, \;\; \bar{V}''(y_1^-) = \bar{V}''(y_1^+), \;\; \bar{V}'''(y_1^-) = \beta_1, \tag{5.79}$$

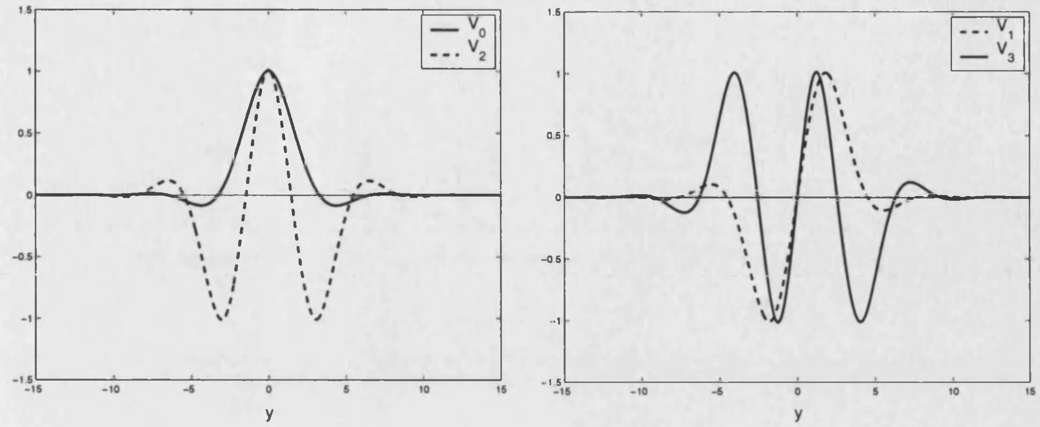
with two free parameters y_1 and β_1.

We next show how to use these two parameters for "shooting" from $y = y_1$ to ensure that two conditions (5.78) hold at the left-hand end point $y = 0$. In view of the fact that the operator (5.70) is analytic, conditions (5.78) provide us with two analytic algebraic equations admitting not more than a countable subset of "eigenvalues" y_1 and $\beta_1 = \beta_1(y_1)$. It is convenient to choose a branch $\{\bar{V}(y) = \bar{V}(y; y_1),\ y_1 > 0\}$ of solutions on $(0, y_1)$ satisfying the second condition $\bar{V}''(0) = 0$. It is easy to fix such a branch for $0 < y_1 \ll 1$, where, by continuity, as $y_1 \to 0^+$, the outer solution approaches the function $\bar{V}_0$ described by Proposition 5.8. This fixes the derivative $\bar{V}''(y_1^-) \approx -2\alpha < 0$. Since the interval $(0, y_1)$ is arbitrarily small, choosing the minimal β_1 to get $\bar{V}''(0) = 0$ determines the necessary branch and we extend it by continuity. One can see that then $\bar{V}(0) \to 1^-$ as $y_1 \to 0^+$, so such $\bar{V}(y)$ cannot satisfy the first condition (5.78) for small $y_1 > 0$. For $y_1 \gg 1$, we have to use the full general solution of (5.70) consisting of four terms $\bar{V}(y) = C_1Y_1(y) + C_2Y_2(y) + C_3Y_3(y) + C_4Y_4(y)$, where $Y_{1,2}$ are exponentially decaying as in (5.64), $Y_3 \equiv 1$ and $Y_4(y) \sim e^{\rho_0 y^{4/3}}$ is strictly monotone increasing for $y \gg 1$. It can be shown that the first two oscillatory terms are still dominant for $y < y_1$ (as they are for $y > y_1$ where $C_3 = C_4 = 0$), and then $\bar{V}(y; y_1)$ changes sign near $y = y_1^-$, and it is a routine calculation to conclude that, by continuity, there exists a $y_1 > 0$ such that $\bar{V}(0) = 0$. Indeed, on the given branch, $\bar{V}(0; y_1)$ then changes sign as $y_1 > 0$ increases and then the first zero ("eigenvalue") y_1 gives the second "eigenfunction" $\bar{V}_1(y) > 0$.

A numerical approximation to the function $\bar{V}_1$ is given on Figure 5.6(b) which shows the convergence to the unique limit function $\bar{V}_1$ corresponding to a unique value of $y_1 > 0$.

Limit problems for arbitrary profiles $\bar{V}_l$. Let us introduce the linear inhomogeneous problem occurring for an arbitrary p_{2l}-branch. (For odd, p_{2l+1}-branches, the construction is similar.) Then we have l unknown parameters $\{y_1, \ldots, y_l\}$ with $y_0 = 0$ and consider the ODE (5.70) on l disjoint intervals (y_k, y_{k+1}), $k = 0, 1, \ldots, l-1$, with conditions

$$\bar{V}(y_k) = (-1)^k, \;\; \bar{V}'(y_k) = 0, \;\; k = 0, 1, \ldots, l. \tag{5.80}$$



(a) The V_0 and V_2 similarity profiles when $p = 240$.

(b) The V_1 and V_3 similarity profiles when $p = 240$.

Figure 5.6: Large p approximations to the limit profiles.

Let us show that y_l can be considered as the actual single "eigenvalue". On the last unbounded interval (y_l, ∞), we have got a standard problem demanding an exponentially decaying solution, which exists and is unique by the Wronskian properties (the 2×2 Wronskian matrix of the corresponding linear system is non-singular). Next, taking the problem on the interval (y_{l-1}, y_l) with $y_{l-1} = y_{l-1}(y_l)$ to be determined, as above, we use the continuity of the second derivative,

$$\bar{V}''(y_l^-) = \bar{V}''(y_l^+), \tag{5.81}$$

and use the parameter $\beta_l = \bar{V}'''(y_l^-)$ to get (5.80) to hold at some $y = y_{l-1} > 0$, etc. Finally, we obtain the Cauchy problem on the last interval $(0, y_1)$, where $y_1 = y_1(y_l) > 0$ and use the last parameter $\beta_1 = \bar{V}'''(y_1^-)$ for shooting to get the corresponding conditions (5.72), which give two analytic equations on y_l and β_1 having not more than a countable subset of roots. For arbitrary l, a proof of existence of such a y_l is a more difficult problem. We have observed existence and uniqueness of y_l by using numerical methods. With a high level of confidence from the numerical calculations of $\bar{V}$, we have that V_1, V_2 and V_3 will be good approximations to their limit profiles when $p = 240$. As such, these profiles are presented in Figures 5.6(a) and 5.6(b).

We do not expect essential changes in the construction if the linear inhomogeneous problems in the general radial case $N < 2m$. On the other hand, for $N > 2m$, a singular boundary layer is expected to occur as $p \to p_S^-$, as Theorem

5.7 shows. The asymptotic structure of such a layer is more delicate.

The limit linear inhomogeneous problems for the ODE (5.70) can be used for establishing the existence of the "nonlinear" eigenfunctions $\{V_l\}$ for $p \gg 1$ on the basis of a perturbation technique. Then the boundary layer structure plays a key role and we will complete our analysis with this study.

5.6.3 The structure of the boundary layer as $p \to \infty$

We return to convergence (5.71) of the first profile $V_0(y)$ and describe the boundary layer, which occurs at the origin $y = 0$. Performing the linearisation by setting $V = \bar{V}_0 + Y$ gives two main perturbation terms in the equation

$$-Y^{(4)} + \tfrac{1}{4} Y'y + \tfrac{1}{p} \bar{V}_0 + V_0{}^p + \ldots = 0. \tag{5.82}$$

It follows from (5.74) (recall that $V_0'''(0) = 0$ by symmetry) that the derivative $V_0^{(4)}(0)$ cannot be bounded as $p \to \infty$. Indeed, otherwise, $V_0''' \to \bar{V}_0'''$ uniformly, hence $\bar{V}_0'''(0) = 0$ contradicting (5.74). In view of the main nonlinear perturbation V^p in (5.82), we then set

$$Y(y) = g(p)V_1(y)(1 + o(1)), \quad \text{where } g(p) \gg \tfrac{1}{p} \text{ as } p \to \infty.$$

Substituting this expansion into (5.82) and taking into account the leading terms of the order $O(g(p))$, we obtain the following expansion in the outer region:

$$V_0(y) = (1 + g(p))\bar{V}_0(y) + \ldots \tag{5.83}$$

with a yet unknown function $g(p) \to 0$ as $p \to \infty$. Note that by (5.73), as $y \to 0$,

$$\bar{V}_0(y) = 1 - \alpha y^2 + O(y^3), \quad \text{where } \alpha = -\tfrac{1}{2} \bar{V}_0''(0) > 0. \tag{5.84}$$

We next extend this expansion into the inner region near the origin. Using, as the first approximation, the outer expansion (5.83), (5.84) in the nonlinear term V^p in the original ODE (5.67) yields, for sufficiently small $y \geq 0$,

$$V_y^{(4)} \sim (1 + g(p))^p (1 - \alpha y^2 + \ldots)^p \sim e^{pg(p)} e^{-\alpha p y^2 + \cdots} + \ldots .$$

The last exponential factor determines the rescaled boundary layer variable $py^2 =$

z^2, i.e., $y = z/\sqrt{p}$, and then, on small compact subsets in z, one obtains

$$V_z^{(4)} \sim \tfrac{1}{p^2}\, e^{pg(p)} e^{-\alpha z^2+\dots} + \dots . \tag{5.85}$$

This gives an approximation for large p of the balance between these two leading terms of the ODE, $\frac{1}{p^2}e^{pg(p)} \sim 1$ and therefore $g(p) \sim \frac{\ln p}{p}(1+o(1))$. Then at the origin

$$V(0) = 1 + O(\tfrac{\ln p}{p}) \;\text{ for } p \gg 1; \tag{5.86}$$

cf. (5.66) obtained from the local bifurcation analysis. Such an asymptotic behaviour is confirmed by numerical experiments suggesting the refined asymptotic behaviour

$$V(0) = 1 + M\,\tfrac{\ln p}{p} + \dots \;\text{ for }\; 2000 \le p \le 10000,$$

where the correction factor M satisfies $0.30 \le M \le 0.35$, is strictly increasing with p and its rate of increase decreases with p. This shows that we have sufficient accuracy in the boundary layer estimate (5.86). Similar boundary layers (determining matching conditions like (5.81)) occur at the "eigenvalue" points $y = y_k$ for solutions $\bar{V}_l$ on other bifurcation branches.

Chapter 6

Conclusions and further work

This thesis has primarily described the large-time behaviour of partial differential equations by studying their similarity solutions. The foundation of this work has been to try to extend the known theory for the heat equation and the semilinear heat equation to more complicated models. Chapter 3 has shown that the subset of self-similar solutions of the porous medium equation can be evolutionary complete, in one dimension and in radial geometry in $\mathbb{R}^N$, in the sense that its large-time behaviour can be described in terms of its nonlinear eigenfunctions. This is an important result because it extends the known theory of evolution completeness for the heat equation (completeness and closure of eigenfunction subsets for linear self-adjoint operators) and mirrors it in many ways. We also managed to calculate the third and fourth nonlinear eigenvalue-eigenfunction pairs for the porous medium equation. As an extension, it may be possible to translate these into the first and second eigenvalue-eigenfunction pairs for the dual porous medium equation. Calculating the eigenvalue-eigenfunction pairs for the dual porous medium equation numerically may also be possible, but would require more expertise.

Little is known about similarity solutions of higher order quasilinear parabolic PDEs. For instance, the fourth order *thin film equation* (TFE)

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u), \quad n > 0,$$

admits finite-mass solutions of the ZKB type (see references in [31]), but other nonlinear eigenfunctions are difficult to detect even in one dimension, where a fourth order ODE occurs. (See also the paper [14] devoted to dipole-type solutions of the TFE.) For the $2l$th order p-Laplacian equation with the monotone coercive operator (these guarantee existence and uniqueness of a weak solution of the Cauchy problem)

$$u_t = (-1)^{l+1}\textstyle\sum_{|\sigma|=l} D^\sigma(|D^l u|^{m-1} D^\sigma u) \tag{6.1}$$

($|D^l u|$ is the length of the vector $\{D^\sigma u,\ |\sigma| = l\}$), substituting (3.13) leads to a higher order elliptic equation (an ODE in one dimension) with an unknown subset of nonlinear eigenfunctions. Furthermore, for any $l \geq 2$, the rescaled operators are not potential and hence the rescaled equations are not gradient systems. In view of the lack of the Maximum Principle, Sturm's Theorem is not valid. This creates an essential difficulty in asymptotic analysis and evolution completeness remains an open problem.

Chapter 4 extends the idea of countable sets of critical exponents from the quasilinear heat equation to the porous medium equation with absorption and then subsequently to the dual porous medium equation with absorption. This was an important extension because our theory allows for solutions of changing sign, a non physical but interesting property of solutions. We also expect that similar critical phenomena occur for a class of quasilinear higher order parabolic equations including the *thin film equation with absorption*

$$u_t = -(|u|^n u_{xxx})_x - |u|^{p-1}u$$

(the first critical exponent is $p_0 = n+5$ for $n \in (0,3)$), though the corresponding mathematical analysis becomes much more involved, as happens with several asymptotic results for the thin film equation, generating non-symmetric and non-potential rescaled operators.

Chapter 5 is concerned with extending the non-uniqueness results of Haraux and Weissler for the second order semilinear heat equation to the $2m$th order semilinear equation $u_t = -(-\Delta)^m u + |u|^{p-1}u$. This has been successful in the sense that similarity solutions have been used to show that non-uniqueness can occur in $L^q(\mathbb{R}^N)$ if $p > 1 + 2mq/N$, at least, for zero initial data. The global structure of such similarity solutions has been demonstrated via numerical calculation of the bifurcation diagram in one space dimension when m equals 2 and 3 and in radially geometry when $N = 3$ and $m = 2$.

We have not been able to demonstrate numerically that no nontrivial solutions exist for $p > p_s$, the Sobolev critical exponent. This is due to the fact that we require the dimension N to be at least five and this makes the numerical calculations difficult even in radially symmetric geometry. We expect blow-up in the sense of the supremum norm to occur at $p = p_s$ and it would be interesting to try test this theory by using more sophisticated numerical techniques. An analytic analysis of boundary layers is also a challenging problem.

Appendix A

C_0^S cannot be replaced by L^1

Oh bother. - Winnie the Pooh.

Let us show that in the completeness analysis, the space C_0^S cannot be replaced by the usual space $L^1(\mathbb{R}^N)$ that plays a key role in the general PME regularity theory; see [27] and [50, Chapter 2]. This is important for arbitrary initial data of changing sign. Obviously, for any integrable data $\hat{u} \geq 0$, the rescaled solution converges to the ZKB profile with the total mass $\hat{u}_0$. The proof is achieved by approximation via compactly supported data, [34]. On the other hand, note that the L^1-setting is not suitable for the asymptotic analysis of the nonnegative solutions of the PME with critical absorption where some delicate logarithmically perturbed patterns can occur [50, p. 98].

The following proposition demonstrates that we cannot replace C_0^S by L^1.

Proposition A.1 *There exist initial data $\hat{u} \in L^1(\mathbb{R}^N)$ for which the convergence* (3.50) *does not hold for any finite k.*

Proof. Recall that such data $\hat{u} \in L^1$ are not compactly supported. Our construction is as follows. We fix two strictly monotone positive sequences, $\{\kappa_n\}$, $\{\rho_n\}$ such that

$$\kappa_n \to \infty, \;\; \kappa_{n+1} - \kappa_n \to \infty \;\; \text{and} \;\; \rho_n \to 0, \;\; \text{all sufficiently fast.} \tag{A.1}$$

For instance we can take $\kappa_n = \mathrm{e}^{n^n}$ and $\rho_n = \exp\{-\mathrm{e}^{n^n}\}$, or take more exponential functions if necessary. We determine the following initial function:

$$\tilde{u}_0(x) = -\delta'(x) + \sum_{n=1}^{\infty} \rho_n[\delta(x-\kappa_{2n-1}) + \delta(x+\kappa_{2n-1}) - \delta(x-\kappa_{2n}) - \delta(x+\kappa_{2n})], \tag{A.2}$$

i.e., we put at the points $x = \pm\kappa_{2n-1}$ and $x = \pm\kappa_{2n}$ Dirac delta functions with the weights ρ_n alternating their signs. Next, for any $n \geq 1$ we define

$$\hat{u}^{\pm n} = \frac{1}{n^{3n}}[\delta(x \mp n^n) - \delta(x \mp 2n^n)].$$

Finally, we define the initial data by

$$\tilde{u}_0(x) = \sum_{n=0}^{\infty} \hat{u}^{\pm n}(x) \not\equiv 0.$$

One can see that, after regular compactly supported approximations to δ, $\tilde{u}_0 \in L^1$ and moreover $x\tilde{u}_0 \in L^1$ and hence there exists the corresponding global solution $\tilde{u}(x,t)$ that is a continuous function for $t > 0$.

Let us discuss its asymptotic behaviour for large times. We display the behaviour for $t \gg 1$ of the central pattern of this solution having a bounded connected (almost symmetric) support on the interval $x \in [\tilde{s}_-(t), \tilde{s}_+(t)]$. It follows that if, for some integer $n \gg 1$, $|\tilde{s}_\pm(t)| \in (\kappa_{2n}, \kappa_{2n+1})$ and, in addition, no interaction with the interfaces of the neighbour small solution parts has happened, then this part of the solution has in the support $[\tilde{s}_-(t), \tilde{s}_+(t)]$ the zero mass,

$$\textstyle\int \tilde{u}(x,t)\, \mathrm{d}x = 0, \;\text{ but }\; \int x\tilde{u}(x,t)\, \mathrm{d}x = 1.$$

The first moment 1 corresponds to $-\delta'$ in (A.2) since the rest of the δ-functions create the resulting zero momentum. Therefore, for sufficiently large $t \gg 1$, the solution after scaling (3.50) must take the form of the dipole profile (3.11) of the same momentum 1. By the construction of the sequences in (A.1), we can always guarantee that there exists a sufficient interval of time to gain this dipole shape approximately and moreover with increasing accuracy for $n \gg 1$.

Then, after this period of stabilisation to a dipole profile ψ_1, when, similarly, under the assumption $|\tilde{s}_\pm(t)| \in (\kappa_{2n+1}, \kappa_{2n+2})$ with no interaction with neighbouring parts, this part of the solution has nonzero mass since

$$\textstyle\int \tilde{u}(x,t)\, \mathrm{d}x = 2\rho_{n+1} \neq 0.$$

Therefore, after sufficiently large time, this part of the solution approximately takes the form of a ZKB-solution with profile (3.8) of the same mass. We again assume that there exists a sufficiently long period of time for such an approximate stabilisation. Continuing this asymptotic analysis with $n \to \infty$ yields that the central part of the solution takes with time an alternating sequence of the dipole and the ZKB-structures with the arbitrary accuracy depending on the choice of sequences (A.1). Since those similarity solutions (3.35) have different scaling factors and exponents, eigenvalues λ_1 and λ_0 and essentially different nonlinear eigenfunctions ψ_1 and ψ_0, we do not have any chance to observe convergence after scaling in (3.50). $\square$

Appendix B

The Matlab code used to calculate the nonlinear eigenvalues and eigenfunctions for the PME

On two occasions I have been asked [by members of Parliament], "Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?" I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question.
- Charles Babbage.

Below is my code to find the first three nonlinear eigenvalues and eigenfunctions of the PME. In it's current incantation it will calculate the third eigenvalue-eigenfunction pair when $m = 3$. Since no parameters can be passed into the initial guess function, the initial guess for the nonlinear eigenvalue α in (3.13) and the value of m must be hard coded into this function.

The code uses Matlab boundary value problem solver `bvp4c`. We wish to solve the ODE

$$\alpha\psi(\xi) - \beta\xi\psi'(\xi) = (|\psi(\xi)|^{m-1}\psi(\xi))''$$

(this is the ODE (3.36) for $N = 1$) on the closed interval $[0, 1]$ (the support of the nonlinear eigenfunction when $\xi > 0$) subject to the boundary conditions

$$\begin{aligned} &\psi'(0) = 0, \quad \psi(1) = 0 \quad (|\psi|^{m-1}\psi)'(1) = 0 \quad \text{for even solutions,} \\ &\psi(0) = 0, \quad \psi(1) = 0 \quad (|\psi|^{m-1}\psi)'(1) = 0 \quad \text{for odd solutions.} \end{aligned}$$

We cannot solve this ODE directly for $\psi(\xi)$ (since it has an infinite gradient at the end point $\xi = 1$) so we use the standard pressure variable:

$$v(\xi) = |\psi(\xi)|^{m-2}\psi(\xi) \Longrightarrow \psi(\xi) = |v(\xi)|^{(2-m)/(m-1)}v.$$

(Note that v is Lipschitz continuous.) This yields the modified ODE

$$\alpha|v(\xi)|^{(2-m)/(m-1)}v(\xi) - \frac{\beta\xi}{m-1}|v(\xi)|^{(2-m)/(m-1)}v'(\xi) = (|v(\xi)|^{1/(m-1)}v(\xi))''.$$

We must regularise $|v|$ in this equation so that Matlab can handle this problem. We set $|v| = (v^2+\varepsilon^2)^{1/2}$ and let $\varepsilon \to 0$. Now dividing by $\frac{m}{m-1}(v^2(\xi)+\varepsilon^2)^{1/(2(m-1))}$ yields

$$\frac{\alpha v(\xi)m-1}{m(v^2(\xi)+\varepsilon^2)^{1/2}} - \frac{\beta\xi v'(\xi)}{m(v^2(\xi)+\varepsilon^2)^{1/2}} - \frac{v(\xi)}{(m-1)(v^2(\xi)+\varepsilon^2)}(v'(\xi))^2 = v''(\xi).$$

This can then be written as a system of first order equations and solved using `bvp4c` on a closed interval $[0,1]$ subject to the boundary conditions

$$v'(0) = 0,\ \ v(1) = 0,\ \ v'(1) = -\frac{\beta(m-1)}{m} \quad \text{for even solutions,}$$
$$v(0) = 0,\ \ v(1) = 0,\ \ v'(1) = -\frac{\beta(m-1)}{m} \quad \text{for odd solutions.}$$

An initial guess for the nonlinear eigenvalue is passed as parameter to `bvp4c` and it finds a value for this parameter that solves our ODE and boundary conditions. The Matlab code:

```
function planerun

%In this vesion we use the pressure variable v:
%\psi=abs(v)^((2-m)/(m-1))v and we find v.

% planerun Solves my ODE given a starting guess
% for alpha (the nonlinear eigenvalue) and
%returns the actual value of alpha

a=input('Input a starting guess for alpha: ');
%a=-0.25 for ZKB, a=-0.33 for dipole,
%a=-0.4 for the third eigenvalue
```

```
m=input('Input a value for the parameter m: ');
% Epsilon is used in the regularisation of the ODE.
e=input('Input a value for the parameter epsilon: ');
f=input('Input the initial number of mesh points you require: ');
g=input('Input the maximum number of mesh points you allow: ');

opts=bvpset('Nmax',g,'RelTol',1e-3,'AbsTol',1e-6);

solinit=bvpinit(linspace(0,1,f),@planeinit,a);
sola=bvp4c(@plane,@planebc,solinit,opts,m,e);


fprintf('The value of alpha is: %7.3f.\n', sola.parameters);
v=sola.y(1,:);
psi=(abs(v).^(-1+1/(m-1))).*v;
plot(sola.x,psi,'-','LineWidth',2)
%hold on;
%To get a smoother plot.
%yy=spline(sola.x,sola.y(1,:),linspace(0,1,g));
%plot(linspace(0,1,g),yy,'r','LineWidth',2)

legend([repmat('\alpha = ',1,1) num2str(sola.parameters)])
xlabel('\xi','FontSize',16)
ylabel('\psi','Rotation',0,'FontSize',16)
str=strcat('m= ',num2str(m));
str=strcat(str,' ');
str=strcat(str,' \epsilon= ');
str=strcat(str,num2str(e));
title(str);

%-------------------------------------------------------
% Subfunctions
%-------------------------------------------------------

function yprime = plane(x,y,a,m,e)
% PLANE YPRIME=PLANE(X,Y,ALPHA) evaluates derivative
```

```
b=(a*(m-1)+1)/2;

yprime=[y(2);(a*(m-1)/m)*y(1)*(abs(e^2+y(1)^2))^(-1/2)...
            -(b/m)*y(2)*x*(abs(e^2+y(1)^2))^(-1/2)-...
             (y(1)/(m-1))*(y(2)^2)*(abs(e^2+y(1)^2))^(-1)];

%-------------------------------------------------------------
function res = planebc(ya,yb,a,m,e)
%PLANEBC evaluates residual

b=(a*(m-1)+1)/2;
res=[ya(2);yb(1);yb(2)+b*(m-1)/m]; %even eigenfunctions
%res=[ya(1);yb(1);yb(2)+b*(m-1)/m]; %dipole
%-------------------------------------------------------------
function yinit=planeinit(x)
% PLANEINIT evaluates the inital guess at x

% The following conditions are only needed for
% the third eigenvalue with m=3

m=3;
c=-0.4; %Third eigenvalue m=3
b=(c*(m-1)+1)/2; % for m=3
z=0.4; % for m=3
C=(2*c+1)/(6*(1-z^2)); % for m=3
%z=0.25 %4th  eigenvalue when m=3
%Select the correct initial guess...

%yinit=[sin(pi*x);pi*cos(pi*x)]; %dipole
%yinit=[cos(pi*x/2);(-pi/2)*sin(pi*x/2)]; %ZKB
yinit=[C*(x^2-z^2)*(1-x^2);...
                                 2*C*x*(1-2*x^2+z^2)];%Third eigenfunction
%yinit=[(1-x.^2).*(x.^2-z^2).*(x.^2-a^2);...
%        -6.*(x.^5)+4.*(x.^3).*(a^2)+4.*(x^3).*(z^2)...
%        -2.*x.*(z^2).*(a^2)+4.*(x^3)-2.*x.*(a^2)...
```

```
%           -2.*x.*(z^2)]; %4th eigenfunction
```

Appendix C

The Matlab code used to generate the similarity profiles and bifurcation diagrams for $u_t = -(-\Delta)^m u + |u|^{p-1}u$

To err is human, but to really foul things up requires a computer.
- Farmers' Almanac, 1978.

Matlab codes have been written to solve the following ODE on closed interval $[a, b]$:

$$-(-\Delta)^m V(y) + \frac{1}{2m}\nabla V \cdot y + \frac{1}{p-1}V(y) + |V(y)|^{p-1}V(y) = 0 \tag{C.1}$$

subject to one of the following two sets of boundary conditions:

$$\begin{aligned} &V(-b) = V'(b) = 0, \;\; V(b) = V'(b) = 0 \quad \text{for even solutions;} \\ &V(0) = V''(0) = 0, \quad V(b) = V'(b) = 0 \quad \text{for odd solutions;} \end{aligned}$$

using Matlab boundary value problem solver bvp4c. For sufficiently large b, these conditions are enough to approximate the solutions with exponential decay. The initial guesses are exponential functions multiplied by trigonometric functions. This is because we expect the solutions to mimic the behaviour of the exponential kernel observed in the linear case $m = 1$.

The codes also plot the p-bifurcation diagram by tracing out the desired bifurcation branch. This is done by providing the program with a good initial guess close to one of the bifurcation points, varying the parameter p and feeding in the last solution as the initial guess for the next iteration. This is successful in the most part but the branches become more unstable as the bifurcation points p_k become closer to 1. The result is that when you try to get onto a fairly unstable branch, even though you provide a fairly accurate initial guess, you immediately converge to one of the more stable branches. Thus, at most, I have only managed to trace out the first six branches: p_0,$p_1, \ldots p_5$.

For even values of k the p_kth bifurcation branch consist of even solutions with $k+1$ clear maxima. (They also have other maxima but these are inherited from their exponentially decaying tail.) The odd values of k yield branches from $y = p_k$ consisting of odd solutions with $k+1$ clear maxima. The distinction between the maxima and the exponential tail becomes more subtle for large values of k.

The programs are also set up to calculate when the bifurcation branches pass through $||V||_\infty = 1$ and when the have a maximum. This is done by means of a simple test.

In the radial setting when $m = 2$ we have

$$\Delta^2 V = V^{(4)} + \frac{2(N-1)}{y}V''' + \frac{(N-1)(N-3)}{y^2}V'' - \frac{(N-1)(N-3)}{y^3}V'. \quad (C.2)$$

Thus when $N = 3$ and $m = 2$ we have that, in the radial setting, equation (C.1) becomes

$$V^{(4)} + \frac{4}{y}V''' + \frac{1}{2m}V'y + \frac{1}{p-1}V + |V|^{p-1}V = 0, \quad (C.3)$$

where y now denotes the radial variable. We impose the boundary conditions

$$V(0) - v'(0) = 0 \;\; V(b) = V'(b) = 0.$$

My Matlab code to plot the bifurcation diagram and similarity profiles for $u_t = -\Delta^2 u + |u|^{p-1}u$ when $m = 2$ and $N = 1$:

```
function out=planerun(initial,pinit,pend,pstep)
%In this vesion we solve the ODE related to:
% u_t=-(-\Delta)^m u+|u|^{p-1} u
opts=bvpset('Nmax',1000,'RelTol',1e-5,'AbsTol',1e-8,...
'FJacobian',@fjac,'BCJacobian',@bjac,'Stats','on');
if(nargin<2)
pinit=input('Enter an initial value for p:');
pend=input('Enter a final value for p:');
```

```
pstep=input('Enter the step size:');
end
pvals=[pinit:pstep:pend];
ptp=0; %turning point indicator.
b=0; %birfucation branch >1 indicator.
for j=1:length(pvals)
    p=pvals(j) % pick the correct value for p
    if(j==1) % first time round
        if(nargin<1) % no user data supplied in initial
            solinit=bvpinit(linspace(-20,20,100),@planeinit);
        else % user supplied initial data
            solinit=initial;
        end
    else % otherwise we have the initial guess being
         % the old solution
        solinit=sol;
    end
sol=bvp4c(@plane,@planebc,solinit,opts,p);
    bd(j)=max(abs(sol.y(1,:)));
    if(bd(j)<1e-6) %Stop the calculation once
                  %you obtain the zero solution
        fprintf(1,'The zero solution has been found');
        plot(sol.x,sol.y(1,:),'-','LineWidth',2)
        %print out the last result
        xlabel('y','FontSize',16)
        ylabel('f(y)','FontSize',16)
        str=strcat('p=',num2str(p));
        title(str);
        break
    end
    if((bd(j)>1)&(b==0))
        b=1;
        fprintf('bd=%7.10f.\n',bd(j));
        %Give an approximation to
        %the crossing point
        fprintf('p=%7.3f.\n',pvals(j));
```

```
    end
    if((j>1)&((bd(j)<bd(j-1))&(ptp==0)))
        ptp=1;
        fprintf('ptp=%7.10f.\n',(bd(j)+bd(j-1))/2);
        %Give an approximation to
        %the turning point
        fprintf('p=%7.3f.\n',(pvals(j)+pvals(j-1))/2);
    end
    if(j==length(pvals))
        plot(sol.x,sol.y(1,:),'-','LineWidth',2)
        xlabel('y','FontSize',16)
        ylabel('f(y)','Rotation',0,'FontSize',16)
        str=strcat('p=',num2str(p));
        title(str);
    end
end
figure %Plot the bifurcation diagram
plot(pvals,bd,'-')
xlabel('p','Fontsize',16)
ylabel('||V||_\infty','Rotation',0,'FontSize',16)

out=sol; %To plot this type plot(out.x,out.y(1,:))

%-----------------------------------------------------
% Subfunctions
% ----------------------------------------------------

function yprime = plane(x,y,p)
% PLANE YPRIME=PLANE(X,Y,ALPHA) evaluates derivative
yprime=[y(2);y(3);y(4);...
        (1/4)*y(2)*x+(1/(p-1))*y(1)+y(1)*abs(y(1)).^(p-1)];

%----------------------------------------------------------

function fjac=fjac(x,y,p)
fjac = [0,1,0,0;...
```

```
       0,0,1,0;...
       0,0,0,1;...
   (1/(p-1))+p*abs(y(1)).^(p-1), x.*1/4,0,0];

%--------------------------------------------------------------
function res = planebc(ya,yb,p)
%PLANEBC evaluates residual
res=[ya(1);ya(2);yb(1);yb(2)]; %even eigenfunctions
%res=[ya(1);ya(3);yb(1);yb(2)]; %odd eigenfunctions

%---------------------------------------------------------------
function [bja,bjb]=bjac(ya,yb,p)
bja=[1,0,0,0;...
        0,1,0,0;...
        0,0,0,0;...
        0,0,0,0];
bjb=[0,0,0,0;...
        0,0,0,0;...
        1,0,0,0;...
        0,1,0,0];
%---------------------------------------------------------------
function yinit=planeinit(x)
% PLANEINIT evaluates the inital guess at x

%% Initial guess for the p_0 branch - start at p=6
yinit=[exp(-x^2/20)*cos(x/2);...
        (-1/10)*exp(-x^2/20)*(x*cos(x/2)+5*sin(x/2));...
        (1/100)*exp(-x^2/20)*(-35*cos(x/2)+(x^2)*cos(x/2)...
               +10*x*sin(x/2));...
        (-1/1000)*exp(-x^2/20)*(-105*x*cos(x/2)-275*sin(x/2)...
               +(x^3)*cos(x/2)+15*(x^2)*sin(x/2))];

%% Initial guess for the p_1 branch - start at p=4
%yinit=[exp(-(x^2)/20)*sin(x/2);...
%         (-1/10)*(exp(-(x^2)/20))*(x*sin(x/2)-5*cos(x/2));...
%         (1/100)*(exp(-(x^2)/20))*(-35*sin(x/2)+(x^2)*sin(x/2)...
```

```
%               -10*x*cos(x/2));...
%         (-1/1000)*(exp(-(x^2)/20))*(-105*x*sin(x/2)...
%               +275*cos(x/2)+(x^3)*sin(x/2)-15*(x^2)*cos(x/2))];

%% Initial guess for the p_2 branch - start at p=4
%yinit=[exp(-x^2/20)*cos(x);...
%         (-1/10)*exp(-x^2/20)*(x*cos(x)+10*sin(x));...
%         (1/100)*exp(-x^2/20)*(-110*cos(x)+(x^2)*cos(x)+...
%               20*x*sin(x));...
%         (-1/1000)*exp(-x^2/20)*(-330*x*cos(x)-1300*sin(x)...
%               +(x^3)*cos(x)+30*(x^2)*sin(x))];

%% Initial guess for the p_3 branch - start at p=2.5
%yinit=[exp(-(x^2)/20)*sin(x);...
%          (1/10)*exp(-(x^2)/20)*(x*sin(x)-10*cos(x));...
%          (1/100)*(exp(-(x^2)/20))*(-110*sin(x)+(x^2)*sin(x)...
%               -20*x*cos(x));...
%          (1/1000)*(exp(-(x^2)/20))*(-330*x*sin(x)+1300*cos(x)...
%               +(x^3)*sin(x)-30*(x^2)*cos(x))];
```

The same code is used, with suitable modifications to the ODE to be solved, the boundary conditions and the initial guess, to plot the bifurcation diagram and eigenfunctions when either $m = 3$ and $N = 1$ or $m = 2$ and $N = 3$.

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