# Wiener-Hopf factorization of symmetric processes and stochastic networks

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# Wiener-Hopf Factorization of Symmetric Processes and Stochastic Networks

submitted by

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**To my dear friend,**

the Revd Ted Parkinson

whose love, humour and understanding
over a bottle of wine
I can never replace.

Faith is the assurance of things hoped for,
the conviction of things not seen.

# Summary

In Part I we study certain random processes in the phase-plane $(t, x) \in \mathbb{R}^2$. In particular, we consider their windings around the origin, and the processes formed by setting $X^+(s)$ (respectively, $X^-(s)$) to be the position when the process first moves to the right of $t = s$ (the left of $t = -s$). The mathematical description of the link between these $X$ processes and the phase-plane process involves a type of Wiener-Hopf Factorization.

An account of the analysis for a particular well-studied case in Chapter 3 reveals an unexpected isomorphism between seemingly esoteric but naturally occurring spaces and the more common Sobolev Spaces.

A detailed exposition of the analysis, which previous work has avoided, shows how delicate the structures of some of the operators are. We focus here on the strange but crucial result that these isomorphisms are built from operators which are contractions, many with the same upper bound for their norms of $1/\sqrt{2}$.

More surprising results, this time of an algebraic nature, are found in the form of theorems discovered via numerical calculation. These arise when the process on the lower half plane is a Markov Chain. Again, the contraction result is proved, this time by means of an identity which provides a link between the $X$ processes and two other Markov Chains, which is itself of great interest.

In the shorter Part II, we look at a computational method for dealing with randomly grown networks and give the results of the programs. In an attempt to model a fairly wide class of biological processes, we transfer the mechanics of those processes to a more general setting.

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# Wiener-Hopf Theory

# Notation for Part I

| | | |
|---|---|---|
| $A := B$ | $A$ is defined to be equal to $B$ | |
| $\langle f, g \rangle_\mu$ | Integral $\int f g d\mu$ over the support of $\mu$ | p. 41 |
| $C_b(X)$ | Space of bounded continuous functions on $X$ | p. 16 |
| $C_b^2(X)$ | Space of functions $f \in C_b(X)$ with bounded and continuous first and second derivatives | p. 23 |
| $\mathbb{E}_x$ | Expectation for the process started at $x$ | p. 13 |
| $E$, $E^+$, $E^-$ | State space $E$ is a disjoint union $E^+ \cup E^-$ | p. 13 |
| $\gamma$ | $\gamma = \sqrt{2\lambda}$ | p. 16 |
| $\ell(t,x)$ or $L(t,x)$ | Local time | p. 13 |
| $n$ | Poisson measure on excursions | p. 20 |
| $\tilde{n}$ | Poisson measure on $\tilde{U}^*$ | p. 21 |
| $n_t$,$n_\lambda$ | Excursion entrance measure and its Laplace transform | p. 21 |
| $\mathbb{P}_x$ | Probability measure for the process started at $x$ | p. 14 |
| $\Pi^{\pm\mp}$ | Half winding operators from $E^\pm$ to $E^\mp$ | p. 13 |
| ${}^-R_\lambda$ | Resolvent of killed Brownian Motion | p. 17 |
| $\overline{\mathbb{R}^+}$ | One point compactification of $[0,\infty)$ | p. 23 |
| $R_\lambda f(x)$ | Resolvent operator | p. 16 |
| $R_\lambda(x, dy)$ | Resolvent density | p. 17 |
| $\tau_t^+$, $\tau_t^-$ | Times at which $\phi$ first leaves $(-\infty, t)$ and $(-t, \infty)$ | p. 13 |
| $U$, $\tilde{U}$, $\tilde{U}^*$ | Space of excursions, of marked excursions and of the union of both marked and unmarked | p. 21 |
| $\phi_t$ | A fluctuating time change | p. 13 |
| $W^2(\mathbb{R}^+)$ | Sobolev space of functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f$, $f'$ and $f''$ all in $L^2(\mathbb{R}^+)$ | p. 42 |
| $W^\pm$ | $W^\pm = \Pi^\pm \Pi^\mp$, the full winding operators | p. 14 |
| $\mathbf{1}(x)$ | Constant function $\mathbf{1}(x) = 1$ | |
| | | |
| [KW] | Reference Kennedy & Williams (1990) | p. 69 |
| [LMRW] | Reference London, McKean, Rogers & Williams (1982a) | p. 32 |
| Vol. I | Reference Rogers & Williams (1994) | p. 23 |
| Vol. II | Reference Rogers & Williams (1987) | p. 48 |
| [W] | Reference Williams (1991) | p. 15 |

# Chapter 1

# Introduction

## 1.1 Springs and Time

In his 1963 paper *A winding problem for a resonator driven by a white noise*, Henry McKean defines the phase plane for a resonator—such as a forced or damped spring—which is perturbed by a standard Brownian Motion, $B$. The equation governing the system is of the form

$$\ddot{u} + c_1(u)\dot{u} + c_2(u) = \dot{B}. \tag{1.1}$$

He then considers various questions concerning windings around the origin in the phase plane for the simplest case of $c_1 = c_2 = 0$. Thus $\dot{u}$ is a Brownian Motion, $u$ is the integral of that process and the phase plane is the graph of $(u, \dot{u})$. If we write $\phi_t$ for the additive functional $u$, the phase plane is the graph of $B_t$ against $\phi_t$.

This is very close to the familiar time-change using a perfect continuous homogeneous additive functional, with one alteration—the value of the functional may go down as well as up. The many and intriguing abstractions, simplifications and generalisations prompted by this investigation in the intervening thirty-five years have brought to light unexpected and often baffling half-answers. Thus the questions have become problems involving many areas of algebra, analysis and probability.

We move from McKean's scenario of a phase plane and think of the process as a generalised time change in which time can flow in either direction.

## 1.2 Definitions and Notation

The following definitions apply throughout Part I. A state space $E$ is decomposed into $E^+$ and its complement, $E^-$. The process $X(t,\omega) \in E$ is defined by the operator $Q$. Depending on the nature of $E$, this will either be an infinitesimal generator of a continuous stochastic process or a Q-matrix of a Markov Chain. (In the former case, we will denote the generator by $\mathcal{A}$.) The function $V : E \to \mathbb{R}$ is non-negative on $E^+$ and non-positive on $E^-$ and defines the additive functional $\phi$ and its inverses $\tau^{\pm}$ by

$$\phi_t := \int_0^t V(X_s)ds, \tag{1.2}$$

$$\tau_u^+ := \inf_{t>0}\{t : \phi_t > u\} \quad \text{and} \quad \tau_u^- := \inf_{t>0}\{t : -\phi_t > u\}. \tag{1.3}$$

In the case considered by McKean, $E$ is the real line decomposed into $\mathbb{R}^+$ and $\mathbb{R}^-$, $X$ is a Brownian Motion, and $V(x) = x$. Note that when we decompose the real line, or some part of it, into $E^+$ and $E^-$, the origin, 0, will *always* become part of $E^-$.

The function $V$ tells us how fast, and in which direction, time is running. We wish to generalise this concept. Certainly if $V$ is bounded, we can, by the occupational density formula, rewrite (1.2) as

$$\phi_t = \int_0^t V(X_s)ds = \int_E \ell(t,x)V(x)dx \tag{1.4}$$

where $\ell$ is the local time normalised such that

$$\int_0^t f(X_s)ds = \int_E \ell(t,x)f(x)dx.$$

Let us use local time to extend the notation of (1.4) and write, for a measure $\nu$,

$$\int_0^t f(X_s)d\nu_s(X) := \int_E \ell(t,x)f(x)d\nu(x). \tag{1.5}$$

Provided the process $(X_t, \phi_t)$ visits the origin at positive times with zero probability, we can talk about its windings around the origin. We define the half-winding operators probabilistically as

$$\Pi^{+-}f(x) = \mathbb{E}_x\, f(X(\tau_0^-)) \quad \text{and} \quad \Pi^{-+}g(y) = \mathbb{E}_y\, g(X(\tau_0^+)) \tag{1.6}$$

where $x \in E^+$, $y \in E^-$ and $\mathbb{E}_x$ is the expectation taken with respect to the probability measure when the process $X$ is started at $x$. Thus the half winding operators give the expected values of a function evaluated at the (random) point on the $\phi = 0$ axis after the

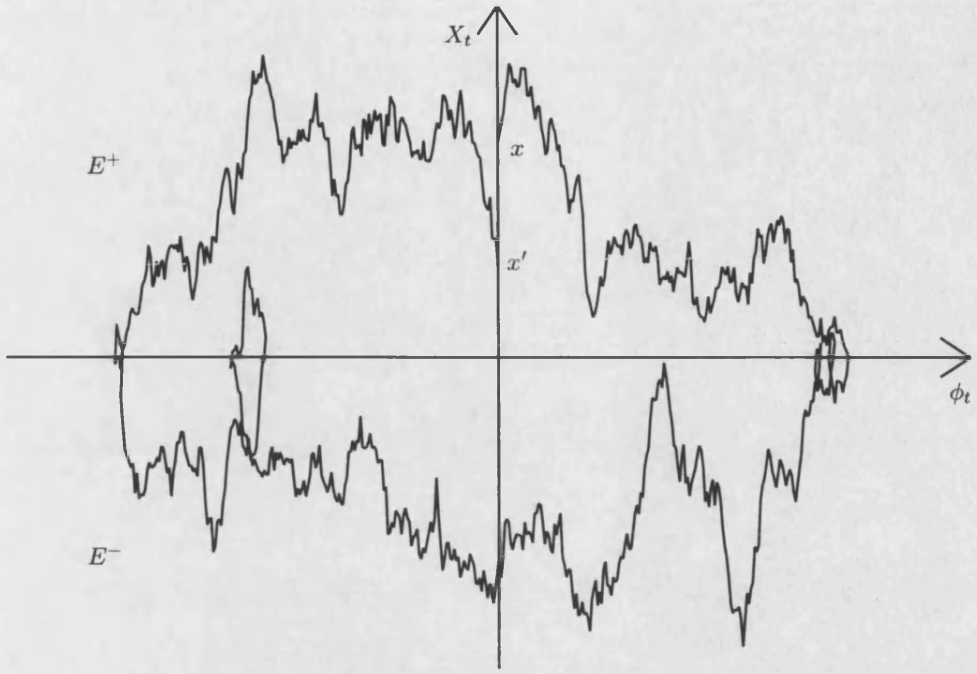



Figure 1-1: A full winding from $x$ to $x'$

process has wound round from $E^{\pm}$ to $E^{\mp}$. Let the full winding operators be defined by

$$W^+ f(x) = \Pi^{+-}\Pi^{-+} f(x) \tag{1.7}$$

and similarly $W^- f(y) = \Pi^{-+}\Pi^{+-} f(y)$. Methods (rather than a serious of brilliant but *ad hoc* techniques) for calculating these operators have always proved frustratingly elusive. In Chapter 3 we offer a specific case of a transform method first suggested by Ivan Graham. However, beyond the question of finding the kernels defining these operators is the largely ignored question of identifying their domains and ranges. We look in some detail at the answer to this, and at the relationships between the spaces as suggested by what Williams (1991) calls the Wiener-Hopf Factorization.

The first steps at pinning down connections between these winding processes and another equation (from Kennedy and Williams (1990)) which may also be called a Wiener-Hopf Factorization are taken in Chapters 4 and 5.

In cases where one or both of $E^+$ and $E^-$ are discrete, some of these operators are matrices, and we will refer to the entries of these matrices by notation such as $W^-(y, y')$. This example has the simple probabilistic meaning of $\mathbb{P}_y\{$first winding of the process around the origin is from $y$ to $y'\}$. We will also refer, where appropriate, to such quantities as $\Pi^{+-}(x, dy)$, being the probability density of half winding (with apologies for the neologism) from $x$ to $y$.

# Chapter 2

# A First Example

This chapter aims to establish the main existence results of Williams (1991) (to which we shall refer as [W]) for a specific continuous state-space Markovian process, rather than for a discrete Markov chain. This involves careful consideration of some analytical aspects that were unnecessary in that paper. It also provides a relatively explicit introduction to both the discrete and the continuous results of varying degrees of generality given later.

Sections 2.2 and 2.3 of this chapter deal with Markov processes on $[0, \infty)$ that behave like Brownian Motion away from zero. These are called Feller Brownian Motions and are discussed further in the appendix, Section A.1. Standard results mentioned there will be used without further comment.

The two sections on Feller Brownian Motions do not directly deal with winding problems, but much of our notation will anticipate those uses of the processes.

Many of our operators, in particular the half-winding operators, resolvents and transformed entrance laws of excursion theory, will adhere to the following notation. The notation is intended to be entirely standard and obvious and is defined only with the intention of avoiding, not creating, confusion.

(2.1) **Notation:** *Let $W$ be some subset of the real line or complex plane and let $\mathcal{W}$ be the sigma-algebra of measurable sets of $W$. For a function $A : W \times \mathcal{W} \to \mathbb{C}$, write*

$$Af(w) := \int_W A(w, dz) f(z).$$

*If the measure $A(w, \cdot)$ has a mass at $z$, then write*

$$A(w, z) := A(w, \{z\})$$

*and if it is absolutely continuous with respect to Lebesgue measure, write*

$$a(w,z) := A(w,dz)/dz.$$

*To denote, say, the operator $A$ acting on the function $z \mapsto \exp(\gamma z)$ evaluated at the point $w$, write*

$$A(e^{\gamma \cdot})(w)$$

*with the dot standing for the "dummy" variable of integration.*

The structure of the chapter is as follows. We discuss resolvents in general and the resolvent of killed Brownian Motion on the space of continuous bounded functions in particular. We then find the resolvents and generators for two Feller Brownian Motions. In Section 2.4 we define a time change for a Brownian Motion which gives rise to a Wiener-Hopf process around the origin. For this process, we find the full and half winding operators and see that the processes $X$ and $X^+$ are given by our two Feller Brownian Motions. We then prove the central result of this theory, the Wiener-Hopf factorization and finish the chapter by exploring the structure that can be seen when the operators are defined for $L^2$ spaces.

## 2.1 Resolvents and Killed Brownian Motion

For $\lambda > 0$, let $\gamma := \sqrt{2\lambda} > 0$. We will use this definition of $\gamma$ both here and in subsequent sections.

A family of bounded operators $\{R_\lambda : \lambda > 0\}$ on a Banach Space $B$ is a contraction resolvent if

- $\|\lambda R_\lambda\| \leq 1$ for $\lambda > 0$ and
- the resolvent equation holds for $\lambda$, $\mu > 0$,

$$R_\lambda - R_\mu + (\lambda - \mu) R_\lambda R_\mu = 0.$$

A contraction resolvent is an SCCR (Strongly Continuous Contraction Resolvent) if it also satisfies, for all $f \in B$,

- $\|\lambda R_\lambda f - f\| \to 0 \quad$ as $\lambda \to \infty$.

For an SCCR, the range $R_\lambda B$ is a dense subspace of $B$ and independent of $\lambda$.

In these sections, we use the Banach Space $B := C_b(\overline{\mathbb{R}^+})$ with the uniform norm.

A space of continuous, bounded functions with the uniform norm is the natural space for probability. In probabilistic terms, $R_\lambda$ is given by $\lambda R_\lambda f(x) = \mathbb{E}_x f(X_T)$ where $T$ is independent of the process and has exponential distribution with rate $\lambda$. From this definition it is clear that $\lambda R_\lambda$ is a contraction since $\mathbb{E}_x f(X_T) \leq \|f\|_\infty$ almost surely. In contrast to $L^2$ spaces, functions are well defined at every point and so there is no problem if $\mathbb{P}\{X_T = b\} > 0$ for some point $b$—we know that we can evaluate functions there.

The natural space to use is therefore $C_b(\mathbb{R}^+)$. However, for the Hille-Yosida theorem to apply, we require a dual space of measures and the Riesz Representation Theorem. Many texts use the space of functions which vanish at infinity, a space for which the Riesz Representation Theorem applies. Another strategy, and one which provides a space containing the constant functions, is to consider continuous bounded functions on $\overline{\mathbb{R}^+}$, the one point compactification of $\mathbb{R}^+$.

Towards the end of this chapter, we shall consider resolvents defined on $L^2$ spaces.

For $\lambda > 0$, the equality $R_\lambda = (\lambda - \mathcal{A})^{-1}$ holds between the SCCR and $\mathcal{A}$, the infinitesimal generator of the process. In particular, the domain of the generator $\mathcal{A}$ is the image of the whole space under $R_\lambda$. We will use the standard notation that for measurable sets $\Gamma$, $\lambda R_\lambda(x, \Gamma) = \mathbb{P}_x\{X_T \in \Gamma\}$. Then, following (2.1), $R_\lambda f(x) = \int R_\lambda(x, dy) f(y)$.

The resolvent of killed Brownian Motion is given by

$$ {}^{-}R_\lambda f(x) = \gamma^{-1} \int_0^\infty \left( e^{-\gamma|x-y|} - e^{-\gamma|x+y|} \right) f(y) dy. $$

We define killed Brownian Motion on $\mathbb{R}^-$ and denote its resolvent, which is essentially the same object, by ${}^{-}R_\lambda$ also.

Note that the resolvent of killed Brownian Motion is *not* strongly continuous on $C(\overline{\mathbb{R}^+})$. It maps $C(\overline{\mathbb{R}^+})$ into the space $C^2(\overline{\mathbb{R}^+}) \cap \{f : f(0) = 0\}$, which is not a dense subspace for the uniform norm. Since $\lambda\, {}^{-}R_\lambda f(0) = 0$ for all $\lambda$, $\lambda\, {}^{-}R_\lambda f$ does not in general tend to $f$ pointwise and hence certainly not in the uniform norm. Of course we have pointwise convergence if and only if $f(0) = 0$, and we then have uniform convergence.

The resolvent is only an SCCR on the smaller space $C(\overline{\mathbb{R}^+}) \cap \{f : f(0) = 0\}$.

The lack of strong continuity arises from the fact that the killed process jumps from 0+ to the coffin state, and hence 0 in effect acts as a branch point to a state at which the function is zero. Let us consider the resolvent of Brownian Motion *absorbed* at 0, which is given by

$$ (2.2)\quad {}^{-}R_\lambda f(x) + \lambda^{-1} e^{-\gamma x} f(0) = \gamma^{-1} \int_0^\infty \left[ e^{-\gamma|x-y|} - e^{-\gamma|x+y|} \right] f(y) dy + \lambda^{-1} e^{-\gamma x} f(0). $$

This maps $C(\overline{\mathbb{R}^+})$ to $C^2(\overline{\mathbb{R}^+})$, which *is* a dense subspace. Thus absorbed Brownian Motion, being a continuous process, has a strongly continuous resolvent on $C(\overline{\mathbb{R}^+})$.

This is a nuisance for the theory of Feller Brownian Motions since these processes have a clear relationship with a killed Brownian Motion. A Feller Brownian Motion and a killed Brownian Motion behave in identical fashions away from zero and then, when the killed Brownian Motion dies at 0, the Feller Brownian Motion does *something* and then behaves like another killed Brownian Motion until it next hits zero. Thus we would wish to express the resolvents of Feller Brownian Motions as at (2.2) where this decomposition is clear. However this is not necessarily the correct path since, for example, the strongly continuous domain (by which I mean the largest subspace of the domain on which the resolvent is strongly continuous) for the Feller Brownian Motion which is absorbed at zero is larger than the strongly continuous domain of killed Brownian Motion.

We finish this section with three lemmata.

(2.3) **Lemma:** *For continuous bounded $f$, the function ${}^{-}R_\lambda f$ is twice continuously differentiable with*

$$\frac{d}{dx}\,{}^{-}R_\lambda f(x) = \int_0^\infty \left(-\mathrm{sgn}(x-y)e^{-\gamma|x-y|} + e^{-\gamma|x+y|}\right) f(y)dy \tag{2.4}$$

*and*

$$\frac{d^2}{dx^2}\,{}^{-}R_\lambda f(x) = -2f(x) + 2\lambda\,{}^{-}R_\lambda f(x). \tag{2.5}$$

*Proof:* This can be shown by calculating the derivatives explicitly. □

The following lemma sets out the logic we will use on occasion to prove that a certain subspace is the image of a resolvent, and that the resolvent is injective.

(2.6) **Lemma:** *Let $R_\lambda$ be a resolvent on a Banach Space $B$ and suppose that there is an operator $\mathcal{A}$ whose domain, $D$, includes $R_\lambda B$ and that for any $f \in B$, $(\lambda - \mathcal{A})R_\lambda f = f$. Then if there exists a subspace $C$ such that $R_\lambda B \subseteq C \subseteq D$ and for $f \in C$, $(\lambda - \mathcal{A})f = 0$ implies that $f = 0$, then $R_\lambda B = C$ and $R_\lambda$ is a bijection from $B$ to $C$.*

*Also, if $R_\lambda$ is an SCCR, then $\mathcal{A}$ restricted to $C$ is the infinitesimal generator given by the resolvent $R_\lambda$.*

*Proof:* For $f \in C$, suppose that $R_\lambda(\lambda - \mathcal{A})f = g$. Therefore $g \in R_\lambda B \subseteq C$. By applying $(\lambda - \mathcal{A})$ to the equation, we have $(\lambda - \mathcal{A})(f - g) = 0$ and so $f - g = 0$. Hence $f \in R_\lambda B$ and $(\lambda - \mathcal{A})$ is a two-sided inverse for $R_\lambda$ which is therefore a bijection. □

(2.7) **Lemma:** *Let $\{R_\lambda f(0) : \lambda > 0\}$ be a family of linear functionals on $C(\overline{\mathbb{R}^+})$ satisfying $0 \le f \le 1 \Rightarrow 0 \le \lambda R_\lambda f(0) \le 1$ for all $\lambda > 0$. Define $R_\lambda f(x) := {}^{-}R_\lambda f(x) +$*

$\exp(-\gamma x)R_\lambda f(0)$. *Then if the image of* $C(\overline{\mathbb{R}^+})$ *under* $R_\lambda$ *is independent of* $\lambda$ *then* $R_\lambda$ *is a contraction resolvent.*

*Proof:* Since $\lambda\ {}^-R_\lambda 1(x)$ is just the probability that Brownian Motion killed at 0 is alive at an exponential time, it is equal to $1 - \exp(-\gamma x)$. Therefore,

$$|\lambda R_\lambda f(x)| \leq |\lambda\ {}^-R_\lambda f(x)| + e^{-\gamma x}|\lambda R_\lambda f(0)| \leq (1 - e^{-\gamma x})\|f\|_\infty + e^{-\gamma x}\|f\|_\infty = \|f\|_\infty$$

and so $\|\lambda R_\lambda\| \leq 1$.

We know that

$$\left(\lambda - \tfrac{1}{2}\frac{d^2}{dx^2}\right)\ {}^-R_\lambda f(x) = f(x) \quad \text{and} \quad \left(\lambda - \tfrac{1}{2}\frac{d^2}{dx^2}\right)e^{-\gamma x} = 0 \tag{2.8}$$

and so $(\lambda - \frac{1}{2}\frac{d^2}{dx^2})R_\lambda f = f$. Hence $R_\lambda$, possessing a left inverse, is injective and we have a bijection between $C(\overline{\mathbb{R}^+})$ and its image (assumed to be independent of $\lambda$) under $R_\lambda$ with $(\lambda - \frac{1}{2}\frac{d^2}{dx^2})$ being the two sided inverse.

For $f \in C(\overline{\mathbb{R}^+})$ let $g = R_\mu f$ so $(\mu - \frac{1}{2}\frac{d^2}{dx^2})g = f$. Then $R_\lambda f = g - (\lambda - \mu)R_\lambda g$ and so

$$(R_\lambda - R_\mu + (\lambda - \mu)R_\lambda R_\mu)f = g - (\lambda - \mu)R_\lambda g - g + (\lambda - \mu)R_\lambda g = 0,$$

and therefore $R_\lambda$ satisfies the resolvent equation and hence is a contraction resolvent. □

## 2.2 Sticky Brownian Motion

### 2.2.1 Definition

The first of our two Feller Brownian Motions is referred to as sticky Brownian Motion (or more precisely, reflected sticky Brownian Motion). This is a Brownian Motion on $(0, \infty)$ but it lingers a little too long at the origin—it actually spends positive real time there and the local time at zero is defined accordingly as

$$\ell_t := m^{-1}\int_0^t I_{\{X(s)=0\}}ds \tag{2.9}$$

for some mass $m > 0$. We have $m^{-1}$ rather than $m$ in the expression because for a small mass $m$, we want to build up local time more quickly. Then an excursion performed at a given local time will occur at a smaller real time. Thus the smaller the mass, the less time the process spends at 0.

Throughout we will take the normalization of local time for Brownian Motion to be as

given in Trotter's theorem so that for a bounded measurable function $f$ the local time $L$ is such that

$$\int_0^t f(B_s)ds = \int_{\mathbb{R}} f(x)L(t,x)dx.$$

As we mentioned briefly in the introduction, we use the following notation:

(2.10) **Notation:** *If a signed measure $\nu$ on $\mathbb{R}^+$ is absolutely continuous with respect to Lebesgue measure,* Leb, *there exists a function $f$ such that $d\nu(x) = f(x)d\text{Leb}(x) = f(x)dx$. For a diffusion $X$ on $\mathbb{R}^+$, define $d\nu_s(X)$ by*

$$d\nu_s(X) = f(X_s)ds = \frac{d\nu}{d\text{Leb}}(X_s)d\text{Leb}(s)$$

*and let us generalise this in the obvious way to other measures for which $f$ may be a generalised function. When a local time $L$ exists, then it can be used to define $d\nu_s(X)$ via*

$$\int_0^t f(X_s)d\nu_s(X) = \int_0^\infty f(x)L(x,t)d\nu(x).$$

We define $X$, the sticky Brownian Motion, in a way deliberately reminiscent of Chapter 1. Let the measure $\nu$ on $\mathbb{R}^+$ be Lebesgue measure together with a mass $m$ at the origin. We use the measure $\nu$ to define the time change

$$(2.11) \qquad \psi_t := \int_0^t d\nu_s(B) = \int_{0+}^\infty I_{\{B_s>0\}}ds + mL(t,0), \quad \sigma_u := \inf_{t>0}\{t : \psi_t > u\}$$

for a Brownian Motion $B_t$. Then the sticky Brownian Motion is given by $X(t) := B(\sigma(t))$. Compare this with (1.2) and (1.3).

We are using the notation that $L$ is the local time of the Brownian Motion and $\ell$ the local time of the sticky Brownian Motion.

An alternative description is via excursion theory—see Appendix A.2 for standard notation, definitions and results. If $U$ is the set of all Brownian excursions, and $L_t$ the local time of the Brownian Motion at zero, then there is a Poisson measure $n$ on $U$ such that for $\Xi$ and $\Gamma$ measurable subsets of $U$ and $\mathbb{R}^+$, the number of excursions in $\Xi$ occurring whilst the local time is in $\Gamma$ has a Poisson distribution with parameter $n(\Xi)\text{Leb}(\Gamma) \in [0,\infty]$. If we transform the Brownian Motion to sticky Brownian Motion via (2.11), then each excursion from 0 remains intact. However, instead of occurring as a Poisson Process on $U \times \mathbb{R}^+$, the space of excursions against local time $L_t$, the excursion points occur on the space of excursions against the *weighted real time* at zero, $\ell_t$. Each excursion is shifted forward in (real) time by an amount proportional to the local time when it occurs.

### 2.2.2 The Resolvent

We use excursion theory to find the resolvent of sticky Brownian Motion.

(2.12) **Theorem:** *The resolvent of sticky Brownian Motion, $R_\lambda$, is*

$$R_\lambda f(x) = {}^{-}R_\lambda f(x) + e^{-\gamma x} R_\lambda f(0) \tag{2.13}$$

*acting on $C(\overline{\mathbb{R}^+})$, where ${}^{-}R_\lambda$ is the resolvent of Brownian Motion killed at 0 and*

$$R_\lambda f(0) := \frac{1}{\frac{1}{2}\gamma(\gamma m + 1)} \left( m f(0) + \int_0^\infty e^{-\gamma x} f(x) dx \right). \tag{2.14}$$

*Proof:* Recall that ${}^{-}R_\lambda f(x)$, the resolvent of Brownian Motion killed at zero, is given by

$${}^{-}R_\lambda f(x) = \gamma^{-1} \int_0^\infty \left( e^{-\gamma|x-y|} - e^{-\gamma|x+y|} \right) f(y) dy.$$

We write $n_t$ for the entrance measure, so $n_t f$ is the expected value of $f$ evaluated at the point $\xi_t$ where $\xi$ is an excursion chosen according to $n$. Then since the excursions of sticky Brownian Motion are Brownian, the Laplace transform of the excursion entrance measure is given by

$$n_\lambda(dx) = e^{-\gamma x} dx \qquad x > 0 \tag{2.15}$$

where $\gamma = \sqrt{2\lambda}$. That is, the probability that an excursion is alive and in the set $\Gamma$ at exponential time $T$ is $n_\lambda(\Gamma) = \int_\Gamma n_\lambda(dx)$. We write $n_\lambda f$ for $\int n_\lambda(dx) f(x)$.

The set of all excursions is denoted by $U$. We now "mark" the process $X$ (or, equivalently, we mark each of its excursions independently) at rate $\lambda$ on the real time axis. Some of the excursions now contain a mark, and we will write $U^*$ for the set of excursions containing a mark. The set of all excursions, marked and unmarked, is now the disjoint union $\tilde{U}^* := U \cup U^*$ and let us write $\tilde{n}$ for the excursion measure on this larger set. See Appendix A.2 for further details. We also define the null marked excursion $\tilde{\partial}$, denoting the event of a mark occurring while the process is at zero. This has non-zero probability as the process spends positive time at the origin.

Let $T$ be the real time of the first mark. Thus $T$ has exponential distribution with rate $\lambda$ and $\ell_T$ has exponential distribution with rate equal to the rate of marked excursions, $\tilde{n}(\tilde{U}^* \cup \{\tilde{\partial}\})$. Since the rate of arrival of these excursions is the reciprocal of the mean inter-arrival time (measured in local time),

$$\begin{aligned} \tilde{n}(\tilde{U}^* \cup \{\tilde{\partial}\})^{-1} &= \mathbb{E}_0[\ell_T] \\ &= \mathbb{E}_0 \int_0^\infty e^{-\lambda t} d\ell(t) \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_0 \int_0^\infty m^{-1} e^{-\lambda t} I_{\{X_t=0\}} dt \\
&= (m\lambda)^{-1} \mathbb{P}_0\{X_T = 0\} = \\
\mathbb{E}_0[\ell_T] &= m^{-1} R_\lambda(0, \{0\}). \qquad (2.16)
\end{aligned}
$$

We obtain the *last-exit decomposition* (2.18), which expresses $\mathbb{P}_0\{X_t \in \Gamma\}$ as an integral over the last exit time of the process from 0, although for us the more important but less intuitive result is in fact its Laplace transform (2.17). We use result (A.4) and the two methods for marking a process to deduce that for $\Gamma \subseteq \mathbb{R}^+ \setminus \{0\}$,

$$
\begin{aligned}
R_\lambda(0,\Gamma) &= \lambda^{-1} \mathbb{P}_0\{X_T \in \Gamma\} \\
&= \lambda^{-1} \mathbb{P}\{\text{1st marked excursion} \in \Gamma \text{ at the mark}\} \\
&= \lambda^{-1} \mathbb{P}\{\xi_T \in \Gamma | \xi \text{ is marked}\} \\
&= \lambda^{-1} \frac{n(\{\xi_T \in \Gamma\})}{\tilde{n}(\tilde{U}^* \cup \{\tilde{\partial}\})} = \\
R_\lambda(0,\Gamma) &= n_\lambda(\Gamma) \mathbb{E}_0 \int_0^\infty e^{-\lambda t} d\ell(t) \qquad (2.17)
\end{aligned}
$$

(where $\xi$ denotes an excursion chosen randomly according to the measure $\tilde{n}$). Inverting the Laplace transforms gives us the decomposition

$$
\mathbb{P}_0\{X_t \in \Gamma\} = \mathbb{E}_0 \int_0^t n_{t-s}(\Gamma) d\ell(s). \qquad (2.18)
$$

From (2.16), we may write (2.17) as

$$
m^{-1} R_\lambda(0, \{0\}) n_\lambda(\Gamma) = R_\lambda(0, \Gamma) \qquad (2.19)
$$

and set $\Gamma = \mathbb{R}^+ \setminus \{0\}$ to obtain

$$
\begin{aligned}
m^{-1} \lambda R_\lambda(0, \{0\}) n_\lambda(\mathbb{R}^+ \setminus \{0\}) &= \lambda R_\lambda(0, \mathbb{R}^+ \setminus \{0\}) \\
&= \mathbb{P}_0\{X_T \neq 0\} \\
&= 1 - \mathbb{P}_0\{X_T = 0\} \\
&= 1 - \lambda R_\lambda(0, \{0\}).
\end{aligned}
$$

Hence, since $n_\lambda(\{0\}) = 0$,

$$
R_\lambda(0, \{0\}) = (\lambda + m^{-1} \lambda n_\lambda 1)^{-1} \qquad (2.20)
$$

where $1(x) := 1$ for all $x$. Note that from (2.15) $n_\lambda \mathbf{1} = \gamma^{-1}$ and so $\lambda + \lambda m^{-1} n_\lambda \mathbf{1} =$

$\gamma(m\gamma+1)/2m$. By (2.19) and for $0 \notin \Gamma$,

$$R_\lambda(0,\Gamma) = \frac{m^{-1}n_\lambda(\Gamma)}{\lambda + m^{-1}\lambda n_\lambda 1} = \frac{2n_\lambda(\Gamma)}{\gamma(m\gamma+1)} \tag{2.21}$$

and so

$$\begin{aligned} R_\lambda f(0) &= R_\lambda(0,\{0\})f(0) + \int_{0+}^{\infty} R_\lambda(0,dx)f(x) \\ &= \frac{f(0) + m^{-1}n_\lambda f}{\lambda + \lambda^{-1}n_\lambda 1} = \frac{mf(0) + n_\lambda f}{\frac{1}{2}\gamma(\gamma m+1)}. \end{aligned}$$

Finally, by (A.5),

$$R_\lambda f(x) = {}^{-}R_\lambda f(x) + e^{-\gamma x}R_\lambda f(0). \tag{2.22}$$

□

As with absorbed Brownian Motion, sticky Brownian Motion is a continuous process and the image of $C(\overline{\mathbb{R}^+})$ under the resolvent is a dense subspace. Its resolvent is strongly continuous on the whole of $C(\overline{\mathbb{R}^+})$.

By the Hille-Yosida theorem, there exists a unique strongly continuous contraction semigroup with $R_\lambda$ as its resolvent. From this we can construct what Rogers & Williams (1994) (to which we shall refer as Vol. I) calls a Feller-Dynkin process. By inverting the resolvent we can find the infinitesimal generator of this process. This is the subject of the next subsection.

By a Feller-Dynkin process on $\mathbb{R}^+$ we mean one generated by a Strongly Continuous Contraction Semigroup $P_t$ on $C(\overline{\mathbb{R}^+})$ with the property that $P_0 = I$, the identity.

The Hille-Yosida theorem relies on the generator having a dense subspace as its domain. When this condition fails, as it does for the Feller Brownian Motion of the next section, we have to consider the more general class of Ray processes.

### 2.2.3 The Generator

(2.23) **Theorem:** *For the space of continuous bounded functions with limits at infinity, the generator of sticky Brownian Motion is given by*

$$\mathcal{A}f = \tfrac{1}{2}f'' \tag{2.24}$$

*with domain*

$$C^2(\overline{\mathbb{R}^+}) \cap \{f : f'(0) = mf''(0)\} \tag{2.25}$$

*where $C^2(\overline{\mathbb{R}^+})$ is the space of functions $f : \mathbb{R}^+ \mapsto \mathbb{R}$ with a limit at $\infty$ and $f$, $f'$ and $f''$ all bounded and continuous.*

*Proof:* Since we have found the resolvent, we use the result that the domain of the generator is the range of the resolvent and

$$(\lambda - \mathcal{A})R_\lambda f = f. \tag{2.26}$$

By Lemma 2.3, for continuous bounded $f$, the function ${}^{-}R_\lambda f$ and hence $R_\lambda f$ is twice continuously differentiable. Thus $R_\lambda f$ is in $C^2(\overline{\mathbb{R}^+})$. Now by (2.26)

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dx^2}R_\lambda f(x) &= \frac{1}{2}\frac{d^2}{dx^2}\left[\,{}^{-}R_\lambda f(x) + e^{-\gamma x}R_\lambda f(0)\right] \\
&= -f(x) + \lambda\,{}^{-}R_\lambda f(x) + \lambda R_\lambda f(0) \\
&= (\lambda R_\lambda - I)f(x) \\
&= \mathcal{A}R_\lambda f(x)
\end{aligned}$$

and so $\mathcal{A}$ is half the second derivative. Also, by (2.4)

$$\begin{aligned}
(R_\lambda f)'(0) &= 2n_\lambda f - \gamma R_\lambda f(0) \\
&= 2n_\lambda f - \frac{mf(0) + n_\lambda f}{\frac{1}{2}(\gamma m + 1)} \\
&= \left[\frac{-mf(0) + \gamma m(n_\lambda f)}{\frac{1}{2}(\gamma m + 1)}\right] \\
&= m\left[-2f(0) + \gamma^2\left(\frac{mf(0) + n_\lambda f}{\frac{1}{2}\gamma(\gamma m + 1)}\right)\right] \\
&= m\left[-2f(0) + \gamma^2 R_\lambda f(0)\right] = \\
(R_\lambda f)'(0) &= m(R_\lambda f)''(0) \\
\Rightarrow \quad \frac{1}{2m}g'(0) &= \mathcal{A}g(0)
\end{aligned} \tag{2.27}$$

for $g$ in the domain of $\mathcal{A}$. Now for a function $h \in C^2(\overline{\mathbb{R}^+})$ satisfying this condition, $\lambda h - \frac{1}{2}h'' = 0$ implies that $h = 0$. Therefore, by Lemma 2.6, $\mathcal{A}$ has domain $C^2(\overline{\mathbb{R}^+}) \cap \{f : f'(0) = mf''(0)\}$. □

Lemma 2.7 now confirms that $R_\lambda$ is indeed a contraction resolvent.

This is the usual, and natural, definition of $\mathcal{A}$ for probabilistic purposes as was noted earlier. Let us expand the domain of $\mathcal{A}$, losing continuity of the functions in its range. We let $\mathcal{A}$ act on the whole of $C^2(\overline{\mathbb{R}^+})$ and let

$$\mathcal{A}f(x) = \begin{cases} \frac{1}{2}f''(x) & x > 0 \\ \frac{1}{2m}f'(0) & x = 0 \end{cases} \tag{2.28}$$

which maps into bounded measurable functions.

Note that this definition tallies with Dynkin's Formula that

$$\mathcal{A}f(0) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}_0 f(X(T_\varepsilon)) - f(0)}{\mathbb{E}_0 T_\varepsilon}$$

where $T_\varepsilon$ is the hitting time of $x = \varepsilon$. Excursions hitting level $\varepsilon$ occur at rate $2\varepsilon$ for reflected Brownian Motion and the length of time such an excursion takes to hit $\varepsilon$ is of order $\varepsilon^2$ (see Lemma 3.20). Therefore $\mathbb{E}_0 T_\varepsilon = m\mathbb{E}_0 \ell(T_\varepsilon) + o(\varepsilon^2)$ and

$$\mathcal{A}f(0) = \lim_{\varepsilon \downarrow 0} \frac{f(\varepsilon) - f(0)}{2m\varepsilon + o(\varepsilon^2)} = \frac{1}{2m} f'(0).$$

## 2.3 A Process with Non-local Behaviour

### 2.3.1 Definition

The second Feller Brownian Motion, denoted $X^+$, is a process that upon hitting 0 immediately re-enters according to the measure $\Pi^{-+}(0, dx) := m^{-1} \exp(-x/m)dx$. This is a finite measure and so, in contrast to Brownian Motion or sticky Brownian Motion, the first excursion (or the next excursion) is identifiable. The points $t$ for which $\lim_{s \uparrow t} X^+(s)$ is zero form a discrete set on the real time axis with only finitely many in any finite interval. The origin, although never entered by the process, is included in the state space as a branch point. We will follow the notation (2.1) and write $\Pi^{-+} f(0)$ for $\int_{\mathbb{R}^+} \Pi^{-+}(0, dx) f(x)$ and $\pi^{-+}(0, x)$ for $m^{-1} \exp(-x/m)$.

It is probabilistically intuitive that the process $X^+$ is well-defined by the description above. We know how to construct a Brownian excursion beginning at $x > 0$ and can easily imagine how we might perform the necessary formalities required to concatenate a finite number of them using the exponential distribution as given above. However, to deal with the analytical problems of Wiener-Hopf theory, we must take more care.

The process $X^+$ is not a Feller-Dynkin process—a fact which complicates our analysis. If we start the process at 0, it jumps away instantly and so the semigroup $P_t$ for the process started at position 0 and at time 0 has the density $P_0(0, dx) = m^{-1} \exp(-x/m)dx$ which is not the identity measure. We have, more importantly, also lost strong continuity of our semigroups. The closer we start to the origin, the sooner we are likely to jump and so we do not have uniform convergence of $\lambda R_\lambda f$ to $f$ as $\lambda \to \infty$ or (equivalently) $P_t f$ to $f$ as $t \downarrow 0$. Both of these pointwise convergences fail to be uniform near 0 and $P_t$ does not map $C(\overline{\mathbb{R}^+})$ to $C(\overline{\mathbb{R}^+})$ as it creates a discontinuity at 0.

The significant result of this is that the image of $C(\overline{\mathbb{R}^+})$ under the resolvent is not a dense subspace. This means extra work in defining the semigroup or generator and proving uniqueness. Fortunately, the process is a Ray process and this extra work is done for us by Ray's Theorem.

### 2.3.2 The Resolvent

Let $R_\lambda$ be a contraction resolvent on $C(\overline{\mathbb{R}^+})$ with $R_\lambda \mathbf{1} = \mathbf{1}$.

(2.29) **Definition:** *For $\alpha > 0$, a function $f \in C(\overline{\mathbb{R}^+})$ is called an $\alpha$-super-median function if*

$$0 \leq \lambda R_{\lambda+\alpha} f \leq f \quad \forall \lambda > 0.$$

We can construct Ray processes from Ray resolvents. The theorem which does this relies on the idea that if the family of super-median functions is sufficiently rich then we can define the semigroup from these functions alone. This is in contrast to Feller-Dynkin processes for which the image of $R_\lambda$ is dense, which allows the semigroup to be defined more directly.

(2.30) **Definition:** *The resolvent $R_\lambda$ is a Ray resolvent on $C(\overline{\mathbb{R}^+})$ if the union over $\alpha > 0$ of all $\alpha$-super-median functions separates points in $\mathbb{R}^+$.*

Such resolvents uniquely define a semigroup, a generator and a Ray process on $\mathbb{R}^+$.

(2.31) **Theorem:** *The resolvent for the Feller Brownian Motion $X^+$ defined above is given by*

(2.32) $$R^+_\lambda f(x) = {}^-R_\lambda f(x) + e^{-\gamma x} R^+_\lambda f(0)$$

*where*

(2.33) $$R^+_\lambda f(0) = \frac{1}{\frac{1}{2}\gamma(\gamma m - 1)} \int_0^\infty \left(e^{-x/m} - e^{-\gamma x}\right) f(x) dx$$

*on the space $C(\overline{\mathbb{R}^+})$. As before, ${}^-R_\lambda$ is the resolvent for Brownian Motion killed at 0.*

*Proof:* We again use Itô Excursion Theory. Since we have only finitely many approaches to the origin in a finite interval, local time counts the number of excursions from zero. Note that if $X(0) = 0$ then $\ell(0+) = 1$ otherwise $\ell(0+) = 0$ and so we define

$$\ell_t := \left|\left\{u \in (0,t] : \lim_{s \uparrow u} X_s = 0\right\}\right| + I_{\{X(0)=0\}}.$$

The excursion measure $n(\Xi)$ (or $\tilde{n}(\Xi)$ for the measure of marked and unmarked excursions) is the probability that the next excursion lies in $\Xi \subset U$ (or $\Xi \subset \tilde{U}^*$). The local time when an excursion in some set $\Xi$ arrives is geometrically distributed and we still have that

$\mathbb{E}\,\ell = [n(\Xi)]^{-1}$. The Laplace Transform of the entrance measure is given by

$$n_\lambda(\Gamma) = \int_0^\infty \Pi^{-+}(0,dx)\ {}^-R_\lambda(x,\Gamma) \tag{2.34}$$

being the jump measure $\Pi^{-+}$ followed by killed Brownian Motion. We obtain a version of (2.16),

$$\mathbb{E}_0[\ell_T] = \mathbb{E}_0 \int_0^\infty e^{-\lambda t} d\ell(t) = \tilde{n}(\tilde{U}^*)^{-1} \tag{2.35}$$

(the marked null excursion $\tilde{\delta}$ is absent, since the process does not spend positive time at the origin) and the calculation leading up to (2.17) again yields

$$R_\lambda^+(0,\Gamma) = n_\lambda(\Gamma)\mathbb{E}_0 \int_0^\infty e^{-\lambda t} d\ell(t) \tag{2.36}$$

and so

$$R_\lambda^+ f(0) = (n_\lambda f)\,\mathbb{E}_0 \int_0^\infty e^{-\lambda t} d\ell(t). \tag{2.37}$$

The process is honest and so $\lambda^{-1} = R_\lambda^+(0,\mathbb{R}^+) = R_\lambda^+ \mathbf{1}(0)$ where $\mathbf{1}(x) = 1$ for all $x$, and so

$$R_\lambda^+ f(0) = \frac{R_\lambda^+ f(0)}{\lambda R_\lambda^+ \mathbf{1}(0)} = \frac{n_\lambda f}{\lambda n_\lambda \mathbf{1}}, \tag{2.38}$$

as required—the integral in (2.37) is a normalising constant. The resolvent is completed as before and as described in the appendix to obtain (2.32). The final step, determining $n_\lambda f$, is performed using

$$n_\lambda f = \int_0^\infty m^{-1} e^{-x/m}\ {}^-R_\lambda f(x)dx \tag{2.39}$$

and a straight calculation and use of Fubini's theorem yields

$$n_\lambda(dx) = \frac{2m}{(\gamma m)^2 - 1}\left(e^{-x/m} - e^{-\gamma x}\right)dx \tag{2.40}$$

and

$$n_\lambda f = \int_0^\infty n_\lambda(dx) f(x)$$

and therefore

$$n_\lambda \mathbf{1} = \frac{2m}{(\gamma m)^2 - 1}\left(m - \gamma^{-1}\right) = \frac{m}{\frac{1}{2}\gamma(\gamma m + 1)}. \tag{2.41}$$

□

(2.42) **Theorem:** *The resolvent $R_\lambda^+$ is a Ray resolvent.*

We must first prove the following lemma.

(2.43) **Lemma:** *The space* $R_\lambda^+ C(\overline{\mathbb{R}^+})$ *is equal to* $C^2(\overline{\mathbb{R}^+}) \cap C^{\Pi}(\overline{\mathbb{R}^+})$ *where*

$$C^{\Pi}(\overline{\mathbb{R}^+}) := C(\overline{\mathbb{R}^+}) \cap \{f : \Pi^{-+} f(0) = f(0)\}.$$

*Proof:* We know from Lemma 2.3 that functions $R_\lambda^+ f$ are twice continuously differentiable, and that $(\lambda - \frac{1}{2}\frac{d^2}{dx^2})R_\lambda^+ f = f$. Since $\Pi^{-+}(e^{-\gamma \cdot})(0) = m^{-1}(\gamma + m^{-1})^{-1}$,

$$\begin{aligned} \Pi^{-+} R_\lambda^+ f(0) &= \Pi^{-+} {}^{-}R_\lambda f(0) + R_\lambda^+ f(0) \Pi^{-+}(e^{-\gamma \cdot})(0) \\ &= n_\lambda f + \frac{R_\lambda^+ f(0)}{\gamma m + 1} \\ &= R_\lambda^+ f(0)\,(\lambda n_\lambda 1) + \frac{R_\lambda^+ f(0)}{\gamma m + 1} \\ &= R_\lambda^+ f(0) \qquad (2.44) \end{aligned}$$

and so the image is contained in $C^{\Pi}(\overline{\mathbb{R}^+})$ as well as in $C^2(\overline{\mathbb{R}^+})$.

Note that we used the identity $n_\lambda f = \Pi^{-+}({}^{-}R_\lambda f)(0)$ as at (2.34). This is probabilistically clear. Excursions of this process (which determine the left hand side of the equation) start according to the distribution $\Pi^{-+}$ and then behave like killed Brownian Motion, which is a description of the right hand side.

For a function $f$ in $C^2(\overline{\mathbb{R}^+}) \cap C^{\Pi}(\overline{\mathbb{R}^+})$, $(\lambda - \frac{1}{2}\frac{d^2}{dx^2})f = 0$ implies that $f = 0$. Thus by Lemma 2.6, this result is proved. □

*Proof: (Of Theorem 2.42)* For any positive function $f \in C(\overline{\mathbb{R}^+})$, $R_\alpha f$ is an $\alpha$-super-median function, a fact easily proved from the resolvent equation,

$$0 \le \lambda R_{\lambda+\alpha}^+ R_\alpha^+ f = R_\alpha^+ f - R_{\lambda+\alpha}^+ f \le R_\alpha^+ f.$$

Hence amongst super-median functions we have all twice differentiable functions in $C^{\Pi}(\overline{\mathbb{R}^+})$ and so they certainly separate points. As a concrete example to prove that $C^{\Pi}(\overline{\mathbb{R}^+})$ is not trivial, take $f(x) = \beta(m\alpha + 1)e^{-\alpha x} - \alpha(m\beta + 1)e^{-\beta x}$ for any $0 < \alpha < \beta$. Thus $R_\lambda^+$ is a Ray resolvent. □

### 2.3.3 The Generator

We define the generator $G^+$ of $X^+$ to be the operator whose domain is the image under $R_\lambda$ of the largest subspace of $C(\overline{\mathbb{R}^+})$ on which $R_\lambda$ is strongly continuous and is such that $(\lambda - G^+)^{-1} = R_\lambda$.

(2.45) **Lemma:** *The space* $C^{\Pi}(\overline{\mathbb{R}^+})$ *is not dense in* $C(\overline{\mathbb{R}^+})$.

*Proof:* The linear functional $f \mapsto (\Pi^{-+}f)(0) - f(0)$ is a bounded operator, since

$$|(\Pi^{-+}f)(0) - f(0)| \leq \|f\|_\infty + |f(0)| \leq 2\|f\|_\infty.$$

It is non-trivial since, for $\alpha > 0$, the function $f(x) := (m\alpha + 1)\exp(-\alpha x)$ is mapped to $-m\alpha$. The result is then immediate as the kernel of a bounded linear functional, here equal to $C^{\Pi}(\overline{\mathbb{R}^+})$, cannot be a dense subspace. □

Thus the resolvent is not strongly continuous on $C(\overline{\mathbb{R}^+})$, that is, $\|\lambda R_\lambda f - f\|_\infty \not\to 0$ as $\lambda \to \infty$. We can see this because $\lambda R_\lambda f(0) \to \Pi^{-+}f(0)$ as $\lambda \to \infty$ and so for $f$ as defined in the previous lemma's proof, $(\lambda R_\lambda - I)f(0)$ is bounded away from zero as $\lambda \to 0$. In fact, we knew that the resolvent could not be a strongly continuous contraction resolvent on $C(\overline{\mathbb{R}^+})$ because those conditions would be sufficient for it to have a strongly continuous contraction semigroup and the branch point at 0 tells us that this could not be so.

We can see that the resolvent is strongly continuous on $C^{\Pi}(\overline{\mathbb{R}^+})$ and so the generator has $R_\lambda C^{\Pi}(\overline{\mathbb{R}^+})$ for its domain.

(2.46) **Theorem:** *The process $X^+$ has generator $G^+$ given by $G^+f = \frac{1}{2}f''$ with domain comprising all functions $f \in C^2(\overline{\mathbb{R}^+})$ satisfying*

$$\Pi^{-+}f(0) = f(0) \quad \textit{and} \quad \Pi^{-+}G^+f(0) = G^+f(0). \tag{2.47}$$

*Proof:* This is proved by Lemmas 2.43 and 2.6. □

For a function $g_+$ satisfying $\Pi^{-+}g_+(0) = g_+(0)$, integration by parts tells us that

$$\Pi^{-+}G^+g_+(0) = \frac{1}{2m}\int_0^\infty e^{-x/m}g_+''(x)dx = -\frac{1}{2m}g_+'(0) \tag{2.48}$$

and so if we have $g_+$ satisfying (2.47), then

$$-g_+'(0) = mg_+''(0) \tag{2.49}$$

which should be compared with the condition on sticky Brownian Motion at (2.27). Note that this holds only because we restricted $R_\lambda$ to that part of its domain for which strong continuity holds.

As mentioned in the previous section, for a Strongly Continuous Contraction Semigroup on a Banach Space $B$ defining a process $X$, the generator $\mathcal{A}$ is given by

$$\mathcal{A}f(x) := \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}_x f(X_\varepsilon) - f(x)}{\varepsilon} \tag{2.50}$$

which is a nicely intuitive understanding of the generator. Of course, we expect this to be

true for $X^+$ only on the space where strong continuity holds. If we start $X^+$ at zero, it immediately jumps into $(0,\infty)$ according to $\Pi^{-+}$ and so the formula at (2.50) acting on a function $g_+$ in the domain of $G^+$ gives

$$\lim_{\varepsilon\downarrow 0}\frac{\mathbb{E}_0 g_+(X^+_\varepsilon)-g_+(0)}{\varepsilon}=\lim_{\varepsilon\downarrow 0}\frac{\Pi^{-+}(\mathbb{E}_\cdot[g_+(X^+_\varepsilon)])(0)-g_+(0)}{\varepsilon}=\Pi^{-+}G^+g_+(0)$$

which is not equal to $G^+g_+(0)$ unless $g_+=R^+_\lambda f_+$ for some $f_+\in C^\Pi(\overline{\mathbb{R}^+})$.

The Wiener-Hopf factorization proved in [W] is for finite dimensional Markov Chains, with Q-matrices rather than infinitesimal generators. For finite chains with state space $S$, the "domain" of the Q-matrix is trivial—if we keep everything real, then it is $\mathbb{R}^S$. For infinitesimal generators, it is not obvious what the domain should be, or whether the factorization holds for any given domain. The use of resolvents helps us to identify the domains that will give the results analogous to those in [W].

## 2.4 A Generalisation of $V$

We have given analytical descriptions of the two Feller Brownian Motions $X$ and $X^+$. We now turn to the connection provided by Wiener-Hopf techniques between these two processes.

Suppose that the standard one-dimensional Brownian Motion $B$, started at the origin, is time changed via

$$\phi_t=\int_0^t I_{\{B(s)>0\}}ds-m\ell_t=\int_0^t \operatorname{sgn}(B_s)d\nu_s(B) \tag{2.51}$$

where $\ell_t$ is the local time at the origin. The density $V(B_s)ds$ at (1.2) has been replaced with $\operatorname{sgn}(B_s)d\nu_s(B)$ where $\nu$ consists of Lebesgue measure on $\mathbb{R}^+=E^+$ and a mass $m$ at the origin.

The function sgn, to be consistent with our convention that $0\in E^-$, is equal to $+1$ on $(0,\infty)$ and $-1$ on $\{0\}$. Once we have performed the time change of (2.51) and (1.3), the half-line $(-\infty,0)$ plays no further rôle in this chapter.

## 2.5 Some Full and Half Winding Results

If the process is started at zero, the local time (which grows roughly like $\sqrt{t}$) will dominate for small $t$, and $\phi_t$ will thus be negative. For large $t$, $\phi_t$ will drift to infinity. That is, the

process started at 0 immediately moves to the left and we have $\tau_0^+$ almost surely strictly positive.

(2.52) **Theorem:** *The half-winding densities for the time-change at (2.51) are given by*

$$\Pi^{+-}(x,0) = e^{-x/m} \tag{2.53}$$

*and*

$$\Pi^{-+}(0,dx) = m^{-1}e^{-x/m}dx. \tag{2.54}$$

*Proof:* Let $\theta = m^{-1}$ and consider $N_t^+ := \exp(-\frac{1}{2}\theta^2\phi_t)f_+(B_t)$ where $f_+$ is given by

$$f_+(x) := I_{\{x>0\}}e^{-\theta x} + I_{\{x\leq 0\}}$$

and $I$ is the indicator function. The (weak) second derivative of $f_+$, $f_+''(dx)$, is equal to the measure

$$f_+''(dx) = \theta^2 f_+(x)I_{\{x>0\}}dx - \theta f_+(0)\delta_{\{x=0\}}dx$$

(where $\delta$ is the usual Dirac delta function) and so $\theta^2 f_+(B_t)d\phi_t = df_+''(B_t)$. Although $f_+$ is not $C^2$ at 0, we can apply Itô's formula to $N_t^+$ by using a reflected Brownian Motion and restricting $f_+$ to $\mathbb{R}^+ \cup \{0\}$. For a Brownian Motion, $\tilde{B}$, with local time at zero $\tilde{\ell}$, let

$$\beta_t := \int_0^t \operatorname{sgn}(\tilde{B}_t)d\tilde{B}_s, \qquad X_t := \beta_t + \tfrac{1}{2}\tilde{\ell}_t \tag{2.55}$$

and

$$\phi_t^X := t - m\tilde{\ell}_t.$$

Thus $dX_t = d\beta_t + \frac{1}{2}d\tilde{\ell}_t$, $d[X]_t = dt$ and $d\phi_t^X = dt - md\tilde{\ell}_t$. We still use the same normalization of local time, and thus Tanaka's formula implies the presence of a half in the definition of $X$ in (2.55).

Now we can redefine $N^+$ using $X$. This has removed the times when $B_t < 0$ and the value of $N^+$ is constant. Let $N_t^+ = \exp(-\frac{1}{2}\theta^2\phi_t^X)f_+(X_t)$. The reflected Brownian Motion $X$ is a semi-martingale, $\beta$ is a local martingale and Itô's formula for $N$ gives

$$\begin{aligned} dN_t^+ &= e^{-\frac{1}{2}\theta^2\phi_t}\Big[-\tfrac{1}{2}\theta^2 f_+(X_t)dt + \tfrac{1}{2}\theta^2 m f_+(X_t)d\tilde{\ell}_t \\ &\qquad + f_+'(X_t)d\beta_t + \tfrac{1}{2}f'(X_t)d\tilde{\ell}_t + \tfrac{1}{2}f_+''(X_t)dt\Big] \\ &= \theta e^{-\frac{1}{2}\theta^2\phi_t - \theta x}d\beta_t. \end{aligned}$$

Therefore $N_t^+$ is a local martingale.

When $B_0 > 0$, the quantity $N_t^+$ is bounded over the time interval $t \in (0, \tau_0^-)$, since $\phi$ is

positive until that time. Therefore $N_t^+$ is a true martingale on $[0, \tau_0^-)$.

By the optional stopping theorem, $\mathbb{E}_x N^+(\tau_0^-) = N^+(0)$ and we deduce that

$$\mathbb{E}_x N^+(\tau_0^-) = \mathbb{P}_x\{\text{process half winds round to } 0\} = \Pi^{+-}(x,0) = e^{-x/m} \tag{2.56}$$

thus obtaining one of the two half winding operators.

We repeat this procedure exactly on $N_t^- := \exp(\frac{1}{2}\gamma^2\phi_t) f_-(B_t)$ for any $\gamma > 0$ and for the function

$$f_-(x) = [(m\gamma^2)^{-1}\cos\gamma x + \gamma^{-1}\sin\gamma x] I_{\{x>0\}} + (m\gamma^2)^{-1} I_{\{x\le 0\}} \tag{2.57}$$

for which $\gamma^2 f_-(B_t) d\phi(B_t) = -df_-''(B_t)$. As with $N^+$, $N^-$ is a local martingale and since it is bounded on the interval $t \in (0, \tau_0^+)$ it is therefore a true martingale up to time $\tau_0^+$. We start the process at 0 and, by the optional stopping theorem, $N^-(0) = \mathbb{E}_0 N^-(\tau_0^+)$, and so

$$\begin{aligned} f_-(0) \;=\; (m\gamma^2)^{-1} \;&=\; \Pi^{-+} f_-(0) \\ &=\; \int_0^\infty \Pi^{-+}(0,dx)\,[(m\gamma^2)^{-1}\cos\gamma x + \gamma^{-1}\sin\gamma x] \end{aligned} \tag{2.58}$$

holds for all strictly positive values of $\gamma$. The key uniqueness theorem of Section 3 in the paper by London, McKean, Rogers & Williams (1982a) (to which we will refer as [LMRW1]) tells us that if for some measure $\Pi^{-+}(x,\cdot)$,

$$\mathbb{E}_x f_-(X(\tau_0^+)) = \int_0^\infty \Pi^{-+}(x,dy) f_-(y) \tag{2.59}$$

holds for all functions $f_-$ of the form (2.57), then $\Pi^{-+}(x,dy)$ *is* the half winding density. Now,

$$(m\gamma^2)^{-1} = m^{-1}\int_0^\infty e^{-x/m}\left[(m\gamma^2)^{-1}\cos\gamma x + \gamma^{-1}\sin\gamma x\right] dx \tag{2.60}$$

and so

$$\Pi^{-+}(0,dx) = m^{-1}e^{-x/m}dx \tag{2.61}$$

is the half winding density. □

Write $V(x) := \operatorname{sgn}(x)$. On $[0,\infty)$, $\mathcal{A}f_+(x) = \frac{1}{2}\theta^2 V(x) f_+(x)$. Note that $V^{-1}\mathcal{A}f_+$ is continuous at 0 since $-m^{-1}f_+'(0+) = f_+''(0+)$. Thus we can say that $f_+$ is an eigenfunction for $V^{-1}\mathcal{A}$ with eigenvalue $\frac{1}{2}\theta^2$. Similarly, $f_-$ is an eigenfunction for $V^{-1}\mathcal{A}$ with eigenvalue $-\frac{1}{2}\gamma^2$.

The spectrum of $V^{-1}\mathcal{A}$ is $\mathbb{R}^- \cup \{(2m^2)^{-1}\}$ and, as we shall see, the Wiener-Hopf factorization decomposes the operator $V^{-1}\mathcal{A}$ according to its positive and negative eigenvalues.

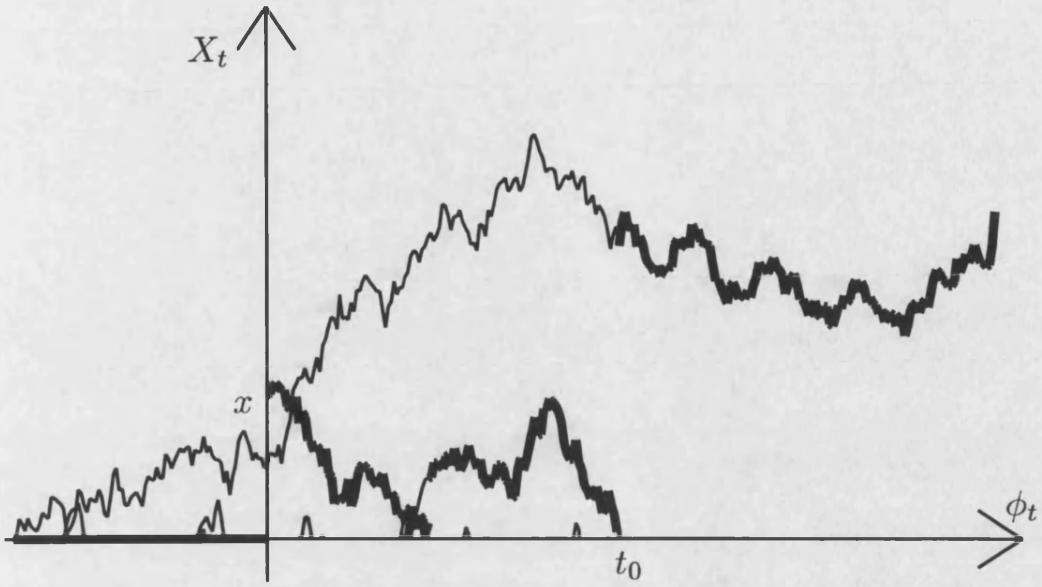



Figure 2-1: The processes $X_t^+$ and $X_t^-$

(2.62) **Theorem:** *The full winding operator $W^-$ is*

$$W^-(0,0) := \int_0^\infty \Pi^{-+}(0,dx)\Pi^{+-}(x,0) = \frac{1}{2}. \tag{2.63}$$

*Proof:* This follows from the definitions of $\Pi^{-+}$ and $\Pi^{+-}$. □

We see, as we would expect from scaling properties of Brownian Motion, that $W^-(0,0)$ is independent of $m$. Also, since the process started at $x = 0$ only has a probability of one half of returning to $\phi = 0$, $\liminf_{t\to\infty} \phi_t$ is strictly positive and so for large times the process drifts to $\phi = +\infty$.

The norms of $\Pi^{-+}$ and $\Pi^{+-}$ are both 1 in the uniform norm, since $\Pi^{-+}\mathbf{1}(0) = 1$ and $\Pi^{+-}\mathbf{1}(0) = 1$.

## 2.6 The Two Subprocesses, $X^+$ and $X^-$

We have the process $X$, sticky Brownian Motion defined on $[0,\infty)$, as shown plotted against $\phi$ in Figure 2-1. Its generator is given by $\mathcal{A}$, defined at (2.28), on the domain $C^2(\overline{\mathbb{R}^+}) \cap \{mf''(0) = f'(0)\}$. We also know, from equations (2.53) and (2.54), how the time changed process half winds around the origin.

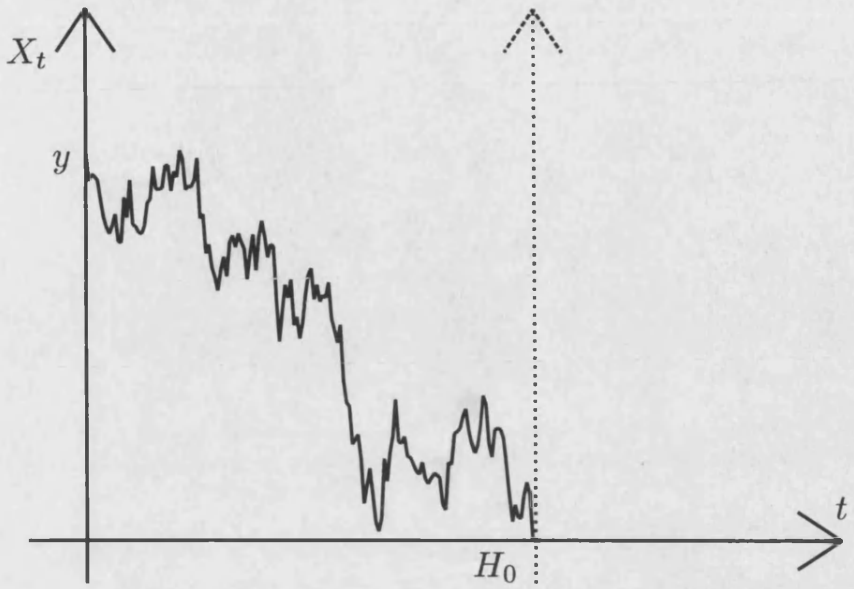



Figure 2-2: Translating the Time Axis

The subprocesses $X^+$ and $X^-$ are defined by

$$X_t^+ := X(\tau_t^+), \qquad X_t^- := X(\tau_t^-) \tag{2.64}$$

each on the appropriate half space $E^+$ or $E^-$. They are shown in bold in Figure 2-1, $X^+$ to the right of $\phi_t = 0$ and $X^-$ to the left. Let us call the generators of these processes $G^+$ and $G^-$ and, when we have found the domains, let us denote the functions on which they act by adding subscripts; $f_+$ and $f_-$.

It is clear that it should be possible to decompose the process $X$ into the two subprocesses $X^+$ and $X^-$. It also seems plausible that, given $X^+$ and $X^-$, together with the half winding operators, we could reconstruct the law of the entire process $X$. We shall do this and express the result using an isomorphism between the domains of the three generators.

Analysis of the process $X^-$, with its solitary state 0, in which it stays until the process dies, will not detain us for long. We need only determine the rate at which the process dies.

(2.65) **Theorem:** *The process $X^-$ consists of a solitary state 0 which the process leaves (to enter a coffin state) at rate $(2m^2)^{-1}$.*

*Proof:* As $X^-$ is a Markov process, it lives for an exponential time. Say that it dies at rate $\alpha$.

The probability that the winding process, started at $y > 0$, will half wind round to 0 is $\Pi^{+-}(y, 0) = \exp(-y/m)$. This is also the probability that the process $X^-$ will live for

time $H_0$ where $H_0$ is the hitting time of $x = 0$ for the winding process started at $y$. To see that this is the case, we start the winding process at height $y$ and wait for it to hit zero, which it does at time $H_0$ as in Figure 2-2. Now we consider this to be the starting point of a process $\tilde{X}^-$ identical in law to the process $X^-$ and defined using the winding process, but with the time axis shifted to the right by time $H_0$. For the winding process to complete a half-winding, $\tilde{X}^-$ must live for time $H_0$. Therefore, since the winding process, up to the time it hits zero, behaves like Brownian Motion, we use a standard result to say that

$$\Pi^{+-}(y,0) = e^{-y/m} = \int_0^\infty \mathbb{P}_y\{H_0 \in dt\}e^{-\alpha t} = \mathbb{E}_x[e^{-\alpha H_0}] = e^{-y\sqrt{2\alpha}}$$

and so $\alpha = (2m^2)^{-1}$. □

This result can also be seen from the proof of Theorem 2.52. The generator of the process $X^-$ corresponds to positive eigenvalues of $V^{-1}\mathcal{A}$—if $\frac{1}{2}\theta^2$ is an eigenvalue of $V^{-1}\mathcal{A}$ then $-\frac{1}{2}\theta^2$ is an eigenvalue of $G^-$. Hence $G^- = (2m^2)^{-1}$.

The process $X^+$ is more interesting. It is the Feller Brownian Motion of Section 2.3, the analysis of which we have already studied. This is immediate now that we have calculated the half winding operator $\Pi^{-+}(0, dx)$.

## 2.7 The Isomorphism

For functions $f_+$ in the domain of $G^+$ and functions $f_-$ in the domain of $G^-$, we consider the mapping

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \mapsto \begin{pmatrix} f^+ \\ f^- \end{pmatrix} := \begin{pmatrix} I^+ & \Pi^{+-} \\ \Pi^{-+} & I^- \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix}. \tag{2.66}$$

Given $(f^+, f^-)$ in $C_b(\overline{\mathbb{R}^+}) \times \mathbb{R}$, let

$$\begin{aligned} f_-(0) &= 2(f^-(0) - \Pi^{-+}f^+(0)) \\ f_+(x) &= f^+(x) - e^{-x/m}f_-(0) \end{aligned}$$

then $(f_+, f_-)$ is in $C_b(\overline{\mathbb{R}^+}) \times \mathbb{R}$ and $(f_+, f_-)$ maps to $(f^+, f^-)$ under (2.66). Therefore the mapping is surjective. Since it is linear, to prove injectivity only requires showing that $(f^+, f^-) = 0$ implies that $(f_+, f_-) = 0$. So, suppose that $(f^+, f^-) = 0$. Then $f_+(0) = -\exp(-x/m)f_-(0)$ and so $f^-(0) = \frac{1}{2}f_-(0) = 0$. Therefore the map is an automorphism on $C_b(\overline{\mathbb{R}^+}) \times \mathbb{R}$. We also have the following theorem.

(2.67) **Theorem:** *The mapping at (2.66) takes* $\{(f_+, f_-) : f_+$ in domain of $G^+$, $f_-$ in domain of $G^-\}$ *isomorphically to the domain of* $\operatorname{sgn}^{-1}\mathcal{A}$.

*Proof:* We define $\mathcal{A}$ as at (2.28) and then take as its domain the space of functions $f = (f^+, f^-)$ for which $\text{sgn}^{-1}(x)\mathcal{A}f$ is continuous and bounded. Since the multiplication by $\text{sgn}^{-1}$ changes the sign of the function at 0, we require

$$f \in C_b^2(\overline{\mathbb{R}^+}) \quad \text{and} \quad mf''(0) = -f'(0). \tag{2.68}$$

We know that the mapping is bijective between its domain and range.

Suppose that $(f_+, f_-)$ and $(f^+, f^-)$ are functions in $C_b^2(\overline{\mathbb{R}^+}) \times \mathbb{R}$ related via (2.66). Then $f_+$ is in the domain of $G^+$ if and only if $\Pi^{-+}f_+(0) = f_+(0)$ and $\Pi^{-+}f_+''(0) = f_+''(0)$. The function $(f^+, f^-)$ is in the domain of $\mathcal{A}$ if and only if $f^+(0) = f^-(0)$ (for continuity) and $f^+$ satisfies $-mf^{+''}(0) = f^{+'}(0)$.

Since $f^+(0) = f_+(0) + f_-(0)$ and $f^-(0) = \Pi^{-+}f_+(0) + f_-(0)$, the first of each of the pairs of conditions are equivalent. (Note that this implies that there is an isomorphic relationship between the extended domains where strong continuity does not hold. We now go on to prove that the smaller domains of strong continuity are also isomorphic.)

Integration by parts gives

$$\begin{aligned}
\Pi^{-+}f_+''(0) &= \int_0^\infty m^{-1}e^{-x/m}f_+''(x)dx \\
&= -m^{-1}f_+'(0) - m^{-2}f_+(0) + m^{-2}\Pi^{-+}f_+(0) \\
&= -m^{-1}f_+'(0)
\end{aligned}$$

and thus the second two conditions are also equivalent. □

This result shows the largely ignored difficulty of pinning down the correct domains. The domain of $\mathcal{A}$ as given at (2.25) is the natural (and dense) domain for sticky Brownian Motion. It suggests the domain given in (2.68) for the domain of $\text{sgn}^{-1}(x)\mathcal{A}$. This in turn is isomorphic to the domain of Strong Continuity of $G^+$ given at (2.47), which is not a dense subspace, and neither is it the natural domain for $G^+$.

To end the section, let us interpret the isomorphism (2.66) in terms of the winding process. It tells us that to evaluate $f^+(x)$ we must start the process at $x$, and allow it to half wind round to $E^-$ where it hits some point $y$, say. Then $f^+(x)$ is the expected value of $f_+(x) + f_-(y)$. We evaluate $f^-$ in a similar way.

If we formally invert the mapping then we obtain

$$\begin{aligned}
f_+(x) &= (I - \Pi^{+-}\Pi^{-+})^{-1}(f^+(x) - \Pi^{+-}f^-(x)) \\
f_-(x) &= (I - \Pi^{-+}\Pi^{+-})^{-1}(f^-(x) - \Pi^{-+}f^+(x))
\end{aligned}$$

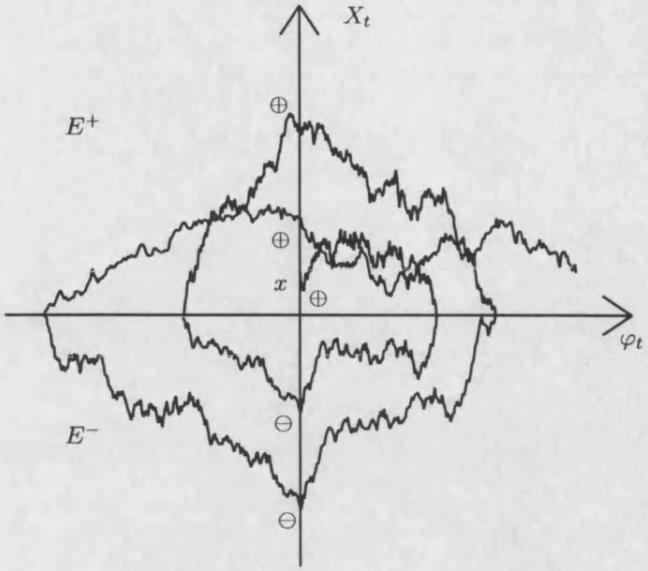



Figure 2-3: Inverting the Isomorphism

which we can expand as a power series and then also interpret probabilistically. To evaluate $f_+(x)$, start the process at $x$ and let it wind around the origin. Then $f_+(x)$ is the expectation of the sum of $f^+(X_t)$ for every time the process crosses $\phi(t) = 0$ whilst in $E^+$, minus the sum of $f^-(X_t)$ for every time the process crosses $\phi(t) = 0$ whilst in $E^-$. See Figure 2-3 where to calculate $f_+(x)$, for each of the points along $\phi_t = 0$, we evaluate and sum $f^+(X_t)$ (for points marked $\oplus$) or $-f^-(X_t)$ (for points marked $\ominus$). Again, $f^-$ has a similar interpretation and so we may write

$$f_+(x) = \mathbb{E}_x\left[\sum_{\substack{\phi(t)=0 \\ X_t \in E^+}} f^+(X_t) - \sum_{\substack{\phi(t)=0 \\ X_t \in E^-}} f^-(X_t)\right] \tag{2.69}$$

$$f_-(x) = \mathbb{E}_x\left[\sum_{\substack{\phi(t)=0 \\ X_t \in E^-}} f^-(X_t) - \sum_{\substack{\phi(t)=0 \\ X_t \in E^+}} f^+(X_t)\right]. \tag{2.70}$$

For the process we are considering, we have the result that $W^-(0,0)$ is equal to one half and so eventually $\phi(t)$ drifts to infinity. If we write $f_+(x) = \sum_0^\infty (W^+)^n f_+(x) - \sum_0^\infty (W^-)^n \Pi^{-+} f_+(x)$ and note that $(W^+)^{n+1} = \Pi^{+-}(W^-)^n \Pi^{-+}$ then we see that the expectations defining $f_+$ and $f_-$ are bounded for bounded functions.

## 2.8 The Wiener-Hopf Factorization

We have now completed the main work of the chapter. Before seeing the advantages of placing the results in a Hilbert Space setting, we prove the central result—the Wiener-Hopf factorization as expressed in Theorem 2.71. The subtleties of this result, with which we have already dealt, are those aspects concerning the domains and isomorphisms. All that remains is to verify a number of identities.

(2.71) **Theorem:** *The following equality holds on functions* $(f_+, f_-)$ *as in Section 2.7—*

$$\begin{pmatrix} I^+ & \Pi^{+-} \\ \Pi^{-+} & I^- \end{pmatrix}^{-1} \operatorname{sgn}(x)^{-1}\mathcal{A} \begin{pmatrix} I^+ & \Pi^{+-} \\ \Pi^{-+} & I^- \end{pmatrix} = \begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix}. \tag{2.72}$$

*Proof:* Take a pair $(f_+, f_-)$ with $f_+$ in the domain of $G^+$ and $f_-$ in the domain of $G^-$. Then,

$$\begin{aligned} \operatorname{sgn}(x)^{-1}\mathcal{A} \begin{pmatrix} I^+ & \Pi^{+-} \\ \Pi^{-+} & I^- \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix} &= \operatorname{sgn}(x)^{-1}\mathcal{A} \begin{pmatrix} f_+(x) + f_-(0)e^{-x/m} \\ f_+(0) + f_-(0) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}f_+'' + \frac{1}{2m^2}e^{-x/m}f_-(0) \\ -\frac{1}{2m}f_+'(0) + \frac{1}{2m^2}f_-(0) \end{pmatrix}. \end{aligned} \tag{2.73}$$

Also

$$\begin{pmatrix} I^+ & \Pi^{+-} \\ \Pi^{-+} & I^- \end{pmatrix} \begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} I^+ & \Pi^{+-} \\ \Pi^{-+} & I^- \end{pmatrix} \begin{pmatrix} \frac{1}{2}f_+'' \\ -G^-f_-(0) \end{pmatrix} \tag{2.74}$$

where $-G^-f_-(0) = (2m^2)^{-1}f_-(0)$ and so, using (2.48) this equals

$$\begin{pmatrix} \frac{1}{2}f_+'' + \frac{1}{2m^2}e^{-x/m}f_-(0) \\ \frac{1}{2}\Pi^{-+}f_+''(0) + \frac{1}{2m^2}f_-(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}f_+'' + \frac{1}{2m^2}e^{-x/m}f_-(0) \\ -\frac{1}{2m}f_+'(0) + \frac{1}{2m^2}f_-(0) \end{pmatrix}. \tag{2.75}$$

Therefore (2.75) and (2.73) are equal and

$$\begin{pmatrix} I^+ & \Pi^{+-} \\ \Pi^{-+} & I^- \end{pmatrix}^{-1} \operatorname{sgn}(x)^{-1}\mathcal{A} \begin{pmatrix} I^+ & \Pi^{+-} \\ \Pi^{-+} & I^- \end{pmatrix} = \begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix}. \tag{2.76}$$

□

This constitutes a factorization of $\operatorname{sgn}(x)^{-1}\mathcal{A}$ into its negative eigenvalues (the eigenvalues of $+G^+$) and positive eigenvalues (the eigenvalues of $-G^-$).

## 2.9 Exploiting Symmetry

We consider for a moment the Markov Chain situation. In [W], the Markov Chain $X$ on the finite state space $E$ has Q-matrix $Q$ which is symmetric with respect to a measure $m$ on $E$. The inner product

$$\langle f, g \rangle := \sum f_i m_i g_i$$

is defined and since $Q$ is $m$-symmetric, there are results to tell us that the chain is uniquely described by the Dirichlet Form $\mathcal{E}(f,g) := -\langle f, Qg \rangle$. (See Fukushima (1980) for the general Hilbert space setting. Applying this to Markov Chains is a great simplification of the theory given there.)

The theorem in [W] states that there exist inner products $\langle \cdot, \cdot \rangle_+$ for functions on $E^+$ and $\langle \cdot, \cdot \rangle_-$ for functions on $E^-$ such that

$$\begin{aligned} \langle f, Vg \rangle &= \langle f_+, g_+ \rangle_+ - \langle f_-, g_- \rangle_- \\ \mathcal{E}(f,g) &= \mathcal{E}_+(f_+, g_+) + \mathcal{E}_-(f_-, g_-) \end{aligned} \tag{2.77}$$

where the mapping between $f$ and $(f_+, f_-)$ is given by (2.66) as usual and the quadratic forms are given by $\mathcal{E}_+(f_+, g_+) = -\langle f_+, G^+ g_+ \rangle_+$ and $\mathcal{E}_-(f_-, g_-) = -\langle f_-, G^- g_- \rangle_-$. The operators $G^+$ and $G^-$ are symmetric with respect to these non-standard inner products.

The analytic definition of a Dirichlet Form is as follows. A symmetric, non-positive definite bilinear form $\mathcal{E}$ defined on a dense subspace of a Hilbert space $H$ is a Dirichlet Form if it is Markovian and closed. Let $\mathcal{D}(\mathcal{E})$ denote the domain of the Dirichlet Form and write $\mathcal{E}_\lambda(f,g)$ for $\mathcal{E}(f,g) + \lambda \langle f, g \rangle$.

A form is *closed* if whenever some sequence $f_n$ in $\mathcal{D}[\mathcal{E}]$ satisfies $\mathcal{E}_1(f_n - f_m, f_n - f_m) \to 0$ as $n, m \to \infty$ then there exists some $f \in \mathcal{D}[\mathcal{E}]$ such that $\mathcal{E}_1(f_n - f, f_n - f) \to 0$ as $n \to \infty$.

There exists a one to one correspondence between closed symmetric, non-positive definite bilinear forms on $H$ and non-positive definite self-adjoint operators on $H$.

A form is *Markovian* if whenever $f \in \mathcal{D}[\mathcal{E}]$ then $g := 0 \vee f \wedge 1$ is also in $\mathcal{D}[\mathcal{E}]$ and $\mathcal{E}(g,g) \leq \mathcal{E}(f,f)$.

A Dirichlet Form is *regular* if the space $\mathcal{D}(\mathcal{E}) \cap \{f : f \in C_b(\mathbb{R}^+), f \text{ has compact support}\}$ is dense in $\mathcal{D}(\mathcal{E})$ with respect to the norm $\mathcal{E}_1$ and dense in $\{f : f \in C_b(\mathbb{R}^+), f \text{ has compact support}\}$ with respect to the uniform norm.

If a regular Dirichlet Form is defined on a Hilbert space carrying a standard inner product, that is if $H = L^2(E, m)$ for some measure $m$, then it defines a unique symmetric Markovian

process. In fact, this process is a Hunt process; it is strong Markov, right continuous and quasi-left continuous.

If the inner product is not standard, then the correspondence between Dirichlet Forms and Hunt processes breaks down. Even in the simplest case of finite state Markov Chains, a non-standard inner product can yield Dirichlet Forms corresponding to a symmetric, non-negative definite $Q$ which is not a Q-matrix. Given a process $X$ wih Q-matrix $Q$ and a non-standard inner product $\langle \cdot, \cdot \rangle_+$, we will call $\mathcal{E}_+(f,g) := -\langle f, Qg \rangle_+$ the quadratic form related to $X$, or just the related quadratic form, and reserve the name Dirichlet Form for a standard inner product where the relationship between the form and the Hunt process is well established.

Neither of the inner products $\langle \cdot, \cdot \rangle_+$ and $\langle \cdot, \cdot \rangle_-$ in [W] is standard and so there is no general theory that says that the quadratic forms $\mathcal{E}_+$ and $\mathcal{E}_-$ define the processes $X^+$ and $X^-$. Nevertheless, (2.77) does exhibit the structure of the analysis underlying the Wiener-Hopf factorization. The Dirichlet Form $\mathcal{E}$ is defined with respect to a standard inner product and so does define the Markov Chain $X$. Future work establishing the exact relationship between quadratic forms and stochastic processes for non-standard inner products would tidy up this area of the theory.

Most of the above comments also apply to our continuous state space situation although we no longer have the simple situation that $C_b(E) = L^2(E,m)$. Therefore to show rigorously that the Dirichlet Form, or the associated resolvent or generator on the Hilbert space, defined the process, we would have to show that the objects on the Hilbert space defined the corresponding objects on the spaces of continuous functions. This is what Fukushima does for symmetric processes and standard inner products and provides the motivation for the definition of regularity of a Dirichlet Form. Since the constant functions are not in the closure (under the uniform norm) of the space of continuous $L^2(\overline{\mathbb{R}^+}, \nu)$ functions, to make the transition from a Dirichlet Form on $L^2(\overline{\mathbb{R}^+}, \nu)$ to a resolvent on $C(\overline{\mathbb{R}^+})$ requires the imposition of the result that $R_\lambda^+ \mathbf{1} = \lambda^{-1}$.

We now return to the case where $X$ is sticky Brownian Motion. If we consider the resolvent of this process and define $R_\lambda(x,y)$ as in (2.1) with respect to $\nu$, so that $R_\lambda f(x) = \int R_\lambda(x,y) f(y) \nu(dy)$, then for $x$, $y > 0$,

$$R_\lambda(x,y) = \gamma^{-1}(e^{-\gamma|x-y|} - e^{-\gamma|x+y|}) + \frac{e^{-\gamma|x+y|}}{\frac{1}{2}\gamma(\gamma m + 1)} = R_\lambda(y,x)$$

and when $x = 0$, $R_\lambda(0,y) = R_\lambda(y,0)$. Thus sticky Brownian Motion is a symmetric process with respect to the measure $\nu$ and can be defined by a Dirichlet Form on the Hilbert space $H = L^2(\mathbb{R}^+, \nu)$.

Define the generator $\mathcal{A}$ as at (2.28) on $W^2(\mathbb{R}^+,\nu)$, a subspace of $H := L^2(\mathbb{R}^+,\nu)$. (The measure $\nu$ is Lebesgue measure on $\mathbb{R}^+$ with a mass $m$ at the origin.) The inner product on $H$ is defined by

$$\langle f,g\rangle_\nu = mf(0)g(0) + \int_0^\infty f(x)g(x)dx$$

for $L^2(\mathbb{R}^+,\nu)$ functions $f$ and $g$ and the Dirichlet Form for $X$ is given by

$$\mathcal{E}_\nu(f,g) = -\langle f,\mathcal{A}g\rangle_\nu = \tfrac{1}{2}\int_0^\infty f'(x)g'(x)dx. \tag{2.78}$$

This is defined initially for functions $f$ and $g$ in the domain of $\mathcal{A}$. To obtain the domain $\mathcal{D}(\mathcal{E}_\nu)$, we take the closure of the domain of $\mathcal{A}$ with respect to the Dirichlet Form. Therefore $\mathcal{D}(\mathcal{E}_\nu)$ is equal to $W^1(\mathbb{R}^+,\nu)$. The latter equality in (2.78) holds by integration by parts (and since $\mathcal{A}f(0) := (2m)^{-1}f'(0)$). Note that since $\nu$ has a mass at zero, $L^2(\mathbb{R}^+,\nu)$ functions are well defined at zero.

In the same way, we define the generator $G^+$ by $G^+f_+ = \frac{1}{2}f''$ on the domain $W^2(\mathbb{R}^+,\text{Leb})\cap$ $\{f : \Pi^{-+}f_+(0) = f_+(0)\}$ which is dense in $L^2(\mathbb{R}^+,\text{Leb})$.

We have the useful result that the half winding operators are adjoints of one another, $(\Pi^{-+})^* = \Pi^{+-}$, in that the kernels for the operators with respect to the measure $\nu$ satisfy $\Pi^{-+}(0,x) = \Pi^{+-}(x,0)$. Therefore $\langle \Pi^{-+}f_+, g_-\rangle_{\nu^-} = \langle f_+, \Pi^{+-}g_-\rangle_{\nu^+}$. Furthermore, these operators are both contractions. Unlike most other examples, here the norms of $\Pi^{+-}$ and $\Pi^{-+}$ are both easily calculated. We find that

$$\|\Pi^{+-}\|_2 = \|\Pi^{-+}\|_2 = 2^{-\frac{1}{2}}. \tag{2.79}$$

The value of the square root of one half for the norm is by no means restricted to this simple case. We shall see that it occurs either as the norm or as an upper bound for the norm in many situations.

If we write the mapping (2.66) (now acting on the space $L^2(\mathbb{R}^+,\nu)$) as $(I+T)(f_+,f_-)$ where $T(f_+,f_-) = (\Pi^{-+}f_-, \Pi^{-+}f_+)$, then from (2.79),

$$\begin{aligned} \|T(f_+,f_-)\|_2^2 &= \|\Pi^{+-}f_-\|_2^2 + \|\Pi^{-+}f_+\|_2^2 \\ &\le \tfrac{1}{2}(\|f_-\|_2^2 + \|f_+\|_2^2) \\ &= \tfrac{1}{2}\|(f_+,f_-)\|_2^2 \end{aligned} \tag{2.80}$$

and we see that $T$ is a contraction with norm no greater than $2^{-\frac{1}{2}}$. The mapping at (2.66) is therefore invertible and hence an automorphism on $L^2(\mathbb{R}^+,\nu)$. It takes functions $(f^+,f^-)$ in the domain of $\mathcal{A}$ to the space of functions $(f_+,f_-)$ for $f_+$ in the domain of $G^+$ and $f_-$ in the domain of $G^-$. Assuming that $(f^+,f^-)$ is in the domain of $\mathcal{A}$, then it is immediate that $f_+$ is in $W^2(\mathbb{R}^+,\text{Leb})$. The continuity of $(f^+,f^-)$ guarantees that $\Pi^{-+}f_+(0) = f_+(0)$.

We have already observed that the operators we used to describe killed Brownian Motion are not strongly continuous. This arises from the fact that functions for which $f(0)$ is zero are not dense in the space $C_b(\overline{\mathbb{R}^+})$ with respect to the uniform norm and thus the image of the resolvent is not a dense subspace. We show here that the resolvent ${}^{-}R_\lambda$ of killed Brownian Motion *is* an SCCR on the space $H = L^2(\mathbb{R}^+, \text{Leb})$. The image of its domain in $H$ also contains only functions for which $f(0) = 0$, but is nevertheless dense in $H$ with respect to the inner product.

As well as showing it to be strongly continuous, we must show that the resolvent has norm satisfying $\|\lambda\ {}^{-}R_\lambda\|_2 \leq 1$. We use the following lemma to prove this.

(2.81) **Lemma:** *Suppose that $\sigma$ is a non-negative function on $\mathbb{R}^+ \times \mathbb{R}^+$ for which there exist finite constants $c_x$ and $c_y$ such that*

$$\int_0^\infty |\sigma(x,y)|dx < c_x \quad \forall y > 0 \quad \textit{and} \quad \int_0^\infty |\sigma(x,y)|dy < c_y \quad \forall x > 0.$$

*Define $\Sigma f(x) := \int_{\mathbb{R}^+} \sigma(x,y)f(y)dy$ on $L^2(\mathbb{R}^+)$. Then $\Sigma$ is a bounded operator on $L^2(\mathbb{R}^+)$ with norm less than $\sqrt{c_x c_y}$.*

*Proof:* For $h \in L^1(\mathbb{R}^+)$,

$$\int_0^\infty |\Sigma h(x)|dx \leq \int_0^\infty \int_0^\infty |\sigma(x,y)|dx\ |h(y)|dy$$

by Fubini and this is bounded above by $c_x\|h\|_1$. Thus $\Sigma h$ is in $L^1(\mathbb{R}^+)$.

Now take $f \in L^2(\mathbb{R}^+)$. By Cauchy-Schwartz,

$$|\Sigma f(x)| \leq \int_0^\infty \left[|\sigma(x,y)|^{1/2}\right]\left[|\sigma(x,y)|^{1/2}|f(y)|\right] dy \leq \sqrt{c_y}(\Sigma(f^2)(x))^{1/2}$$

and since $f^2$ is an $L^1(\mathbb{R}^+)$ function, $[\Sigma(f^2)(x)]^{1/2}$ is an $L^2(\mathbb{R}^+)$ function and so

$$\|\Sigma f(x)\|_2 \leq \sqrt{c_y c_x}\sqrt{\|f^2\|_1} = \sqrt{c_x c_y}\|f\|_2.$$

Therefore $\|\Sigma\|_2 \leq \sqrt{c_x c_y}$. □

(2.82) **Lemma:** *The operator ${}^{-}R_\lambda$ is an SCCR on $L^2(\mathbb{R}^+)$.*

*Proof:* With $\sigma(x,y) = \gamma^{-1}(\exp(-\gamma|x-y|) - \gamma^{-1}\exp(-\gamma|x+y|))$ in Lemma 2.81, we see that the resolvent ${}^{-}R_\lambda$ is a bounded operator on $L^2(\mathbb{R}^+)$. We can take $c_x = c_y = \lambda^{-1}$ and thus $\|\lambda\ {}^{-}R_\lambda\|_2 \leq 1$.

The image of $H$ under ${}^{-}R_\lambda$ is the dense subspace $\{f \in W^2(\mathbb{R}^+) : f(0) = 0\}$ and therefore the resolvent is strongly continuous. □

This result gives us in turn the result that our other operators based on the resolvent for killed Brownian Motion are contractions and hence are well defined resolvents.

## 2.10 Inner Products and Dirichlet Forms

We enrich the structure of the isomorphism (2.66) of Section 2.7 by finding inner products $\langle \cdot, \cdot \rangle_+$ and $\langle \cdot, \cdot \rangle_-$ on $L^2(\mathbb{R}^+, \nu^+)$ and $L^2(\{0\}, \nu^-)$ which satisfy

$$\langle f, \mathrm{sgn}(x) g \rangle_\nu = \langle f^+, g^+ \rangle_{\nu^+} - \langle f^-, g^- \rangle_{\nu^-} = \langle f_+, g_+ \rangle_+ - \langle f_-, g_- \rangle_- . \tag{2.83}$$

As usual, we use the notation

$$\begin{pmatrix} f^+ \\ f^- \end{pmatrix} = \begin{pmatrix} f_+ & + & \Pi^{+-} f_- \\ \Pi^{-+} f_+ & + & f_- \end{pmatrix}, \qquad \begin{pmatrix} g^+ \\ g^- \end{pmatrix} = \begin{pmatrix} g_+ & + & \Pi^{+-} g_- \\ \Pi^{-+} g_+ & + & g_- \end{pmatrix}. \tag{2.84}$$

We now reformulate the Wiener-Hopf factorization (2.72) in terms of Dirichlet Forms.

(2.85) **Theorem:** *There exist unique inner products $\langle \cdot, \cdot \rangle_+$ on $(0, \infty)$ and $\langle \cdot, \cdot \rangle_-$ on $\{0\}$ and quadratic forms $\mathcal{E}_+$ and $\mathcal{E}_-$ related to the processes $X^+$ and $X^-$ such that*

$$\langle f, \mathrm{sgn}(x) g \rangle_\nu = \langle f_+, g_+ \rangle_+ - \langle f_-, g_- \rangle_- \tag{2.86}$$

*and, for functions $f$ and $g$ in the domain of $\mathcal{A}$, and their images under (2.66),*

$$\mathcal{E}_\nu(f, g) = \mathcal{E}_+(f_+, g_+) + \mathcal{E}_-(f_-, g_-). \tag{2.87}$$

*Proof:* Using the result $(\Pi^{-+})^* = \Pi^{+-}$ (so $\langle \Pi^{+-} f_-, g_+ \rangle_{\nu^+}$ is equal to $\langle f_-, \Pi^{-+} g_+ \rangle_{\nu^-}$),

$$\begin{aligned} \langle f, \mathrm{sgn}(x) g \rangle_\nu &= \langle f_+ + \Pi^{+-} f_-, g_+ + \Pi^{+-} g_- \rangle_{\nu^+} - \\ &\qquad - \langle \Pi^{-+} f_+ + f_-, \Pi^{-+} g_+ + g_- \rangle_{\nu^-} \\ &= \langle f_+, g_+ \rangle_{\nu^+} + \langle \Pi^{+-} f_-, \Pi^{+-} g_- \rangle_{\nu^+} - \\ &\qquad - \langle \Pi^{-+} f_+, \Pi^{-+} g_+ \rangle_{\nu^-} - \langle f_-, g_- \rangle_{\nu^-} \\ &= \langle f_+, g_+ \rangle_+ - \langle f_-, g_- \rangle_- \end{aligned} \tag{2.88}$$

gives the expressions for the inner products as

$$
\begin{aligned}
\langle f_+, g_+\rangle_+ &= \langle f_+, g_+\rangle_{\nu+} - \langle \Pi^{-+} f_+, \Pi^{-+} g_+\rangle_{\nu-} \\
&= \int_0^\infty f_+ g_+ - m\Pi^{-+} f_+(0)\Pi^{-+} g_+(0) \\
\langle f_-, g_-\rangle_- &= \langle f_-, g_-\rangle_{\nu-} - \langle \Pi^{+-} f_-, \Pi^{+-} g_-\rangle_{\nu+} \\
&= \tfrac{1}{2} m f_-(0) g_-(0).
\end{aligned}
\tag{2.89}
$$

The inner product $\langle \cdot, \cdot \rangle_+$ is a true inner product, because $\Pi^{-+}$ has norm less than $1/\sqrt{2}$.

These inner products (and hence the quadratic forms) must be unique, since we have obtained them constructively. The related quadratic forms for the $X^+$ and $X^-$ processes are given by

$$
\begin{aligned}
\mathcal{E}_+(f_+, g_+) &= -\langle f_+, G^+ g_+\rangle_+ &&= \tfrac{1}{2}\int_0^\infty f_+' g_+' \\
\mathcal{E}_-(f_-, g_-) &= -\langle f_-, G^- g_-\rangle_- &&= \tfrac{1}{4m} f_-(0) g_-(0)
\end{aligned}
\tag{2.90}
$$

for functions in the domains of $G^+$ and $G^-$. (Recall that $\Pi^{-+} f_+(0) = f_+(0)$ and that $g_+'(0) = -m\Pi^{-+} g_+''(0)$.)

Using the definition of $\mathcal{A}$ at (2.25), we can write

$$
\begin{aligned}
\mathcal{E}_\nu(f, g) &= -\langle f, \mathcal{A} g\rangle_\nu \\
&= -\tfrac{1}{2}\langle f_+ + \Pi^{+-} f_-, g_+'' + (\Pi^{+-} g_-)''\rangle_{\nu+} - \\
&\qquad -\tfrac{1}{2m}\langle \Pi^{-+} f_+ + f_-, g_+'(0) + (\Pi^{+-} g_-)'(0)\rangle_{\nu-}
\end{aligned}
\tag{2.91}
$$

and see that this equals $\mathcal{E}_+(f_+, g_+) + \mathcal{E}_-(f_-, g_-)$ as below.

Integration by parts gives

$$
\Pi^{-+} G^+ g_+(0) = -G^+ g_+(0) \tag{2.92}
$$

and hence the terms in (2.91) involving $f_+$ and $g_+$ are

$$
\begin{aligned}
-\tfrac{1}{2}\langle f_+, g_+''\rangle_{\nu+} &- \tfrac{1}{2m}\langle \Pi^{-+} f_+, g_+'(0)\rangle_{\nu-} = \\
-\langle f_+, G^+ g_+\rangle_{\nu+} &+ \langle \Pi^{-+} f_+, G^+ g_+(0)\rangle_{\nu-} = \\
-\langle f_+, G^+ g_+\rangle_{\nu+} &- \langle \Pi^{-+} f_+, \Pi^{-+} G^+ g_+(0)\rangle_{\nu-} = \mathcal{E}_+(f_+, g_+).
\end{aligned}
\tag{2.93}
$$

By direct calculation, the $f_-$ and $g_-$ terms are

$$
-\tfrac{1}{2}\langle \Pi^{+-} f_-, (\Pi^{+-} g_-)''\rangle_{\nu+} - \tfrac{1}{2m}\langle f_-, (\Pi^{+-} g_-)'(0)\rangle_{\nu-} = \mathcal{E}_-(f_-, g_-). \tag{2.94}
$$

By (2.54)

$$
\langle \Pi^{+-} f_-, g_+''\rangle_{\nu+} = \langle f_-, \Pi^{-+} g_+''\rangle_{\nu-} = \frac{1}{m}\langle f_-, g_+'(0)\rangle_{\nu-} \tag{2.95}
$$

and so the $f_-$ and $g_+$ cross terms cancel. Finally, either by the symmetry of the Dirichlet Form, or by one more calculation, the $f_+$ and $g_-$ cross terms are found to cancel.

Therefore we have inner products and quadratic forms satisfying

$$\begin{aligned} \langle f, \mathrm{sgn}(x) g \rangle_\nu &= \langle f_+, g_+ \rangle_+ - \langle f_-, g_- \rangle_- \\ \mathcal{E}_\nu(f, g) &= \mathcal{E}_+(f_+, g_+) + \mathcal{E}_-(f_-, g_-). \end{aligned} \tag{2.96}$$

□

# Chapter 3

# The Canonical Continuous Case

## 3.1 Definitions

We now consider the important case of Brownian Motion on $\mathbb{R}$ time changed as at (1.2) with $V(x) = \text{sgn}(x)$, so $E^+ = (0, \infty)$ and $E^- = (-\infty, 0]$. An approximation of this is shown in Figure 1-1 on page 14. The expressions for the operators can be found in [W], but the analysis is wholly absent there and also, *a fortiori*, proofs concerning domains, ranges and isomorphisms.

This case is worthy of careful study as it can give results for more complicated cases via transformations, as in McGill (1989b).

We will follow the same general scheme as Chapter 2. Thus although we begin with a process on $\mathbb{R}$ (Brownian Motion), we will later decompose it into the $X^+$ and $X^-$ processes on $\mathbb{R}^+$ and $\mathbb{R}^-$ respectively. For this reason it is helpful to adopt the superscript and subscript notation from Section 2.10 and, anticipating later definitions, let

$$f = \begin{pmatrix} f^+ \\ f^- \end{pmatrix} = \begin{pmatrix} f_+ & + & \Pi^{+-} f_- \\ \Pi^{-+} f_+ & + & f_- \end{pmatrix} \tag{3.1}$$

and similarly for other functions. Again, once we have defined the operators related to these processes, we will use $f = (f^+, f^-)$ for functions in the domain of $\text{sgn}^{-1}\mathcal{A}$ and $f_+$ and $f_-$ for functions in the domains of $G^+$ and $G^-$.

Following the notation of Chapter 2, we define the operator

$$\mathcal{A}f(x) = \begin{cases} \frac{1}{2}(f^+)''(x) & x > 0 \\ \frac{1}{2}(f^-)''(x) & x \leq 0 \end{cases} \tag{3.2}$$

on the domain

$$(3.3) \quad \{f : f^+ \in W^2(\mathbb{R}^+), f^- \in W^2(\mathbb{R}^-), f^+(0+) = f^-(0), (f^+)'(0+) = (f^-)'(0)\}$$

where $W^2$ denotes a Sobolev space, $W^2(\mathbb{R}^\pm) := \{f : f, f', f'' \in L^2(\mathbb{R}^\pm)\}$.

In fact, this domain is simply $W^2(\mathbb{R})$. The reason for expressing the domain as at (3.3) is for comparison with the domains of $G^+$ and $G^-$. As before, we shall find an operator which maps $(f^+, f^-)$ in the domain of $\mathcal{A}$ to $(f_+, f_-)$ in the domains of $G^+$ and $G^-$ respectively. These pairs of functions will both lie in $W^2(\mathbb{R}^+) \times W^2(\mathbb{R}^-)$ and both will have two further conditions imposed upon them. Those two conditions for this domain are $f^+(0+) = f^-(0)$ and $(f^+)'(0+) = (f^-)'(0)$. Two conditions for $f_+$ and $f_-$ will emerge later, although we note here that the conditions on $f_+$ must be independent of $f_-$ and *vice versa.* The interdependence of the two functions is factorized out by the transformation.

Note that the processes will be described by operators on $L^2$ spaces from the beginning. In the previous section, we were dealing with a transient process and so the inverse of the isomorphism (2.66) could be calculated probabilistically for bounded continuous functions, even though the half winding operators were not contractions in the uniform norm. The situation was simpler because although the half winding operators were not contractions, the full winding operators were. Here we have a balanced process which is not transient. Properties of Brownian Motion tell us that "the Brownian Motion will eventually perform a large enough excursion to bring $\phi_t$ back to zero." (The probability of $\phi_t$ drifting to infinity must be either one or zero, since it is an event depending on the tail $\sigma$-algebra. However, it must be equal to the probability of drifting to $-\infty$ and hence both probabilities must be zero.) Therefore the full winding operators are not $L^\infty$ contractions and if, for example, we tried to find the inverse of the function $\mathbf{1}(x)$ using (2.69), we would be faced with the difference of two unbounded sums. Since the process is recurrent, the full winding operators are not contractions on the continuous bounded spaces either. We also see that the mapping (2.66) is not invertible on $C_b(\overline{\mathbb{R}^+}) \times C_b(\overline{\mathbb{R}^-})$, since $\Pi^{-+}\mathbf{1}(x) = 1$ and so $\operatorname{sgn}(x)$ is mapped to zero.

Also define

$$(3.4) \quad \mathcal{A}_c f(x) = \tfrac{1}{2} f''(x)$$

(where $f''(0)$ is defined to be $f''(0-)$) on $f^+ \in C_b^2(\overline{\mathbb{R}^+})$, $f^- \in C_b^2(\overline{\mathbb{R}^-})$, $f^+(0+) = f^-(0)$ and $(f^+)'(0+) = (f^-)'(0)$. Note that this maps into the space $C_b(0, \infty] \times C_b[-\infty, 0]$.

## 3.2 Calculation of $\Pi^{-+}$

The first result pertaining to this case can be found in [W] and other papers. However, we prove the result here using Mellin Transforms in a way which tantalisingly suggests a more generally applicable method and also yields the new result on the norm of the half-winding operators.

(3.5) **Theorem:** *For $x < 0$, the half-winding density is given by*

$$\Pi^{-+}f^{+}(x) = \int_0^\infty \frac{\sqrt{-2xy}}{\pi(x^2+y^2)} f^{+}(y)dy \tag{3.6}$$

*which, as an operator on $L^2(\mathbb{R}^+)$, has norm $2^{-\frac{1}{2}}$.*

*Proof:* As in Section 2.5, we use the optional stopping theorem on a martingale, this time of the form $N_t^- := \exp(\frac{1}{2}\lambda^2\phi_t)f(B_t)$ for a bounded continuous function $f$. For this to be a local martingale, we require $\mathcal{A}_c f(x) = -\frac{1}{2}\lambda^2 \mathrm{sgn}(x) f(x)$, which is solved by

$$f(x) = \begin{cases} \sqrt{2}\sin(\lambda x + \pi/4) & x > 0 \\ \exp(-\lambda|x|) & x \leq 0 \end{cases} \tag{3.7}$$

for any $\lambda > 0$. Note that there is no non-trivial solution of $\mathcal{A}f(x) = -\frac{1}{2}\lambda^2 \mathrm{sgn}(x) f(x)$ (that is, for $f \in W^2(\mathbb{R})$) and so we are forced to work with the generator $\mathcal{A}_c$ in the continuous bounded space $C_b(\overline{\mathbb{R}^+}) \times C_b(\overline{\mathbb{R}^-})$ (so we allow, as we must, discontinuities in $\mathcal{A}f$ at the origin).

Itô's formula is valid here, as $f$ is $C^1$ everywhere and has a bounded and measurable (weak) second derivative, in that there exists a bounded, measurable function $f''$ such that

$$f'(x) - f'(y) = \int_y^x f''(w)dw.$$

For this result, see Rogers & Williams (1987) (Vol. II), Section IV.45 although the proof there can be simplified by decomposing the second derivative into its positive and negative parts. Then the function is a difference of two convex functions and a previous result then applies.

As in Chapter 2, the local martingale $N^-$ is bounded on $[0, \tau_0^+]$ and we deduce via the Optional Stopping Theorem that for $x < 0$,

$$f^{-}(x) = \mathbb{E}_x N^{-}(\tau_0^+) = \Pi^{-+}f^{+}(x) = \int_0^\infty \Pi^{-+}(x, dy) f^{+}(y) \tag{3.8}$$

for some measure $\Pi^{-+}(x, dy)$. Again, by the theorem in [LMRW1], if we can find $\Pi^{-+}$ that solves (3.8) for all functions of the form (3.7), then we have found the transition density

$\Pi^{-+}(x, dy)$. So now we need not worry about domains at all. We are finding the half winding density, and to do this we only need to find a measure $\Pi^{-+}(x, dy)$ satisfying (3.8).

To find this density, we take the Mellin Transforms of $f^+$ and $f^-$, to be denoted $\widehat{f^+}$ and $\widehat{f^-}$, so

$$\widehat{f^+}(s) := \int_{\mathbb{R}^+} x^{s-1} f^+(x) dx \qquad \widehat{f^-}(s) := \int_{\mathbb{R}^+} x^{s-1} f^-(-x) dx.$$

If we look at $\mathrm{Re}(s) = \frac{1}{2}$, this is a Fourier Transform by another name. For an $L^2(\mathbb{R}^+)$ function $f$, we can define $g(x) := \exp(x/2) f(\exp x)$ which is an $L^2(\mathbb{R})$ function with

$$\|f\|_2 = \|g\|_2 \quad \text{and} \quad \widehat{f}(\tfrac{1}{2} + iu) = \widetilde{g}(u),$$

where $\widetilde{g}$ is the Fourier Transform. Therefore as a function on the line Re $z = \frac{1}{2}$, $\|\widehat{f}\|_2 = \|f\|_2$.

For our functions $f^+$ and $f^-$ from (3.7),

$$\begin{aligned} \widehat{f^+}(s) &= \lambda^{-s}\Gamma(s)\sqrt{2}\sin(\tfrac{\pi s}{2} + \tfrac{\pi}{4}) \\ \widehat{f^-}(s) &= \lambda^{-s}\Gamma(s) \end{aligned} \tag{3.9}$$

on $0 <$ Re $s < 1$ and so if we set

$$\widehat{\kappa}(s) := \frac{1}{\sqrt{2}} \mathrm{cosec}\left(\frac{\pi s}{2} + \frac{\pi}{4}\right) \tag{3.10}$$

then $\widehat{f^-}(s) = \widehat{\Pi^{-+} f^+}(s) = \widehat{\kappa}(s)\widehat{f^+}(s)$ and so the operator $\Pi^{-+}$ is a multiplier in the transform space. The inverse Mellin Transform of $\widehat{\kappa}$ is

$$\kappa(x) := \frac{\sqrt{2x}}{\pi(1 + x^2)} \tag{3.11}$$

and Mellin Transform theory results tell us that for $x \in \mathbb{R}^-$

$$\Pi^{-+} f^+(x) = \int_0^\infty y^{-1} \kappa(-x/y) f^+(y) dy = \int_0^\infty \frac{\sqrt{-2xy}}{\pi(x^2 + y^2)} f^+(y) dy \tag{3.12}$$

and so

$$\Pi^{-+}(x, dy) = \frac{\sqrt{-2xy}}{\pi(x^2 + y^2)}\, dy. \tag{3.13}$$

(Note that $-2xy > 0$ in this integral.) Now viewing $\widehat{\Pi^{-+}}$ as a multiplication operator on $L^2(\{z : \mathrm{Re}\ z = \frac{1}{2}\})$ and since $\|\widehat{f}\|_2 = \|f\|_2$, we deduce the result that $\|\widehat{\Pi^{-+}}\|_2 = \|\Pi^{-+}\|_2$ and so,

$$\|\Pi^{-+}\|_2 = \sup_{\{\mathrm{Re}\ s = \frac{1}{2}\}} |\widehat{\kappa}(s)| = \sup_{y \in \mathbb{R}} \frac{1}{\sqrt{2}\cosh y} = \frac{1}{\sqrt{2}}. \tag{3.14}$$

□

See Chapter 6 for another proof that this operator has norm $1/\sqrt{2}$.

By symmetry, $\Pi^{+-}$ is defined by

$$\Pi^{+-}f^{-}(y) = \int_{-\infty}^{0} \frac{\sqrt{-2xy}}{\pi(x^2+y^2)} f^{-}(x)dx \qquad y > 0 \tag{3.15}$$

and is also a contraction. The mapping defined at (2.66) is also an isomorphism for the canonical case definitions.

We now see that we do indeed have the result

$$\Pi^{-+}\mathbf{1}(x) = \int_{0}^{\infty} \frac{\sqrt{-2xy}}{\pi(x^2+y^2)} dy = 1 \tag{3.16}$$

where $\mathbf{1}(x) = 1$ for all $x < 0$ (see Gradshteyn and Rhyzik (1980) p.369). This confirms that the time changed process, started at $x < 0$ will almost surely do a half winding and so $\mathbb{P}_x\{\tau_0^+ < \infty\} = 1$.

## 3.3 The $X^+$ and $X^-$ processes

We will consider the process $X^+$ and its generator $G^+$ only. Symmetric results will also hold for $X^-$. The process $X^+$ behaves like a Brownian Motion away from zero (it is a Feller Brownian Motion), and so $G^+$ is half the second derivative away from zero. We will calculate $G^+f_+(0)$ to find the domain of the generator.

The behaviour of $X^+$, or of the time changed process $(\phi_t, X_t)$, at the origin is not as simple as in the local time case of Chapter 2. There, we had $\tau_0^+ > 0$ almost surely—we could identify the "next" excursion. Here, $\tau_0^+ = 0$ almost surely and, just as there is no first excursion from zero of a Brownian Motion, so there is no first jump into $(0, \infty)$ of $X^+$.

It is easy to understand the behaviour of $X^+$ if we start the winding process at $x < 0$. It will jump into $(0, \infty)$ at time 0 according to the measure $\Pi^{-+}(x, dy)$. However, it is not true to say that if we start at $x = 0$ it re-enters according to $\Pi^{-+}(0, dy)$ as this measure is zero away from 0 and is singular at 0. If we consider the behaviour of Brownian Motion (and see Figure 1-1), we realise that $X^+$ *must* make some jumps from zero to $y > 0$.

As is proved in, for example, McGill (1989b), the process $X^+$ can be defined via the Laplace transform of its excursion measure, which is given by

$$n_\lambda(dx) = \int_0^\infty y^{-3/2}\, {}^{-}R_\lambda(y, dx)dy. \tag{3.17}$$

Informally, this says that $X^+$ jumps into $(0,\infty)$ according to the (infinite) measure $x^{-3/2}$. Formally, the (local time) rate of arrival of excursions starting in $\Gamma \subseteq \mathbb{R}^+$ is given by some constant multiple of the possibly infinite integral

$$\int_\Gamma x^{-3/2} dx.$$

We will prove that (3.17) is a valid definition later.

### 3.3.1 Calculation of $G^+f(0)$

#### Via Excursion Theory

We use excursion theory to determine the value of $G^+f(0)$ and the domain of the operator in the canonical case.

(3.18) **Theorem:** *For the canonical case,* $G^+f(0) = \frac{1}{2}f''(0)$ *with domain*

(3.19) $$C_b^2(\overline{\mathbb{R}^+}) \cap \left\{ f : \int_0^\infty x^{-3/2} \left[ f(x) - f(0) \right] dx = 0 \right\}.$$

(3.20) **Lemma:** *Let* $H_y$ *denote the hitting time of level* $y$ *for a Brownian Motion. Then*

$$\begin{aligned}
&i) && \mathbb{P}_x\{H_0 < H_\varepsilon\} &&= \frac{\varepsilon - x}{\varepsilon} \\
&ii) && \mathbb{P}_x\{H_\varepsilon < H_0\} &&= \frac{x}{\varepsilon} \\
&iii) && \mathbb{E}_x[H_0 | H_0 < H_\varepsilon] &&= \frac{x}{3}(2\varepsilon - x) \\
&iv) && \mathbb{E}_x[H_\varepsilon | H_\varepsilon < H_0] &&= \frac{\varepsilon^2 - x^2}{3}.
\end{aligned}$$

*Proof:* The first two are standard results, and the last two can be deduced via differentiation from

(3.21) $$\mathbb{E}_x\left[ e^{-\alpha H_\varepsilon} | H_\varepsilon < H_0 \right] = \frac{\varepsilon \sinh(x\sqrt{2\alpha})}{x \sinh(\varepsilon\sqrt{2\alpha})}.$$

□

Let $T_\varepsilon$ be the crossing time of $\varepsilon$ for the process $X^+$, that is, $T_\varepsilon := \inf\{t : X_t^+ > \varepsilon\}$.

(3.22) **Theorem:**

(3.23) $$\mathbb{E}\, T_\varepsilon = \frac{\varepsilon^2}{3}.$$

*Proof:* Let us categorise each excursion from 0 as follows

- **Type 1($x$)**: Excursions starting at $x > \varepsilon$.

$$\text{Rate} = x^{-3/2}dx =: \rho_1(dx)$$

- **Type 2($x$)**: Excursions starting at $x < \varepsilon$ and hitting $\varepsilon$.

$$\text{Rate} = \varepsilon^{-1}x^{-1/2}dx =: \rho_2(dx)$$

- **Type 3($x$)**: Excursions starting at $x < \varepsilon$ and not hitting $\varepsilon$.

$$\text{Rate} = \varepsilon^{-1}(\varepsilon - x)x^{-3/2}dx =: \rho_3(dx)$$

  - **Type 3$^\delta$($x$)**: Excursions starting at $\delta < x < \varepsilon$ and not hitting $\varepsilon$.

$$\text{Rate} = \rho_3(dx)$$

Only **type 3** are of infinite rate.

Let us stop the process after exponential (rate $\alpha$) local time. Given two Poisson processes of rates $\beta$ and $\gamma$, the expected number of $\beta$-events by the time of the first $\gamma$-event is $\gamma^{-1}\beta$. Thus the expected number of **type 3$^\delta$** excursions is $\alpha^{-1}\beta_\delta$ where

$$\beta_\delta = \int_\delta^\varepsilon \rho_3(dx) = \text{Total rate of } \textbf{type 3}^\delta \text{ excursions.} \tag{3.24}$$

The probability that a **type 3$^\delta$** excursion starts at $x$ is $\beta_\delta^{-1}\rho_3(dx)/dx$ and so

$$\mathbb{E}\left[\text{length of a } \textbf{type 3}^\delta\right] = \beta_\delta^{-1}\int_\delta^\varepsilon \rho_3(dx)\frac{x}{3}(2\varepsilon - x) \tag{3.25}$$

$$= \frac{1}{3\varepsilon\beta_\delta}\int_\delta^\varepsilon x^{-1/2}(\varepsilon - x)(2\varepsilon - x)dx \tag{3.26}$$

$$= \frac{1}{3\varepsilon\beta_\delta}\left[4\varepsilon^2 x^{1/2} - 2\varepsilon x^{3/2} + \frac{2}{5}x^{5/2}\right]_\delta^\varepsilon. \tag{3.27}$$

Therefore the expected length of real time elapsed by exponential rate $\alpha$ local time due to **type 3$^\delta$** excursions is $\mathbb{E}$ [length] $\times$ $\mathbb{E}$ [number] =

$$\frac{1}{3\varepsilon\alpha}\left[4\varepsilon^2 x^{1/2} - 2\varepsilon x^{3/2} + \frac{2}{5}x^{5/2}\right]_\delta^\varepsilon \to \frac{4\varepsilon^{3/2}}{5\alpha} \qquad \text{as } \delta \downarrow 0. \tag{3.28}$$

The total rate of **type 2** excursions is given by

$$\int_0^\varepsilon \rho_2(dx) = \int_0^\varepsilon x^{-1/2}dx = \frac{2}{\sqrt{\varepsilon}}. \tag{3.29}$$

The total rate of **type 1** excursions is given by

$$\int_\varepsilon^\infty \rho_1(dx) = \int_\varepsilon^\infty x^{-3/2}dx = \frac{2}{\sqrt{\varepsilon}}. \tag{3.30}$$

Thus excursions exceeding $\varepsilon$ occur at rate $4\varepsilon^{-1/2}$ and **types 1** and **2** occur with equal probability. That is, given that the process has exceeded $\varepsilon$, there is probability one half that it jumped above $\varepsilon$ and probability one half that it jumped to a point below $\varepsilon$ and then crossed it continuously.

By these results, the time elapsed due to **type 3** excursions by $T_\varepsilon$, the time at which $X^+$ exceeds $\varepsilon$, is

$$\frac{4\varepsilon^{3/2}}{5}.\frac{\sqrt{\varepsilon}}{4} = \frac{\varepsilon^2}{5} \tag{3.31}$$

The starting distribution for **type 2** excursions is

$$\frac{\rho_2(dx)}{\rho([0,\varepsilon])} = \frac{dx}{\varepsilon\sqrt{x}}\frac{\sqrt{\varepsilon}}{2} = \frac{dx}{2\sqrt{x\varepsilon}} \tag{3.32}$$

and thus the expected time for a **type 2** excursion to hit $\varepsilon$ is given by

$$\int_0^\varepsilon \frac{dx}{2\sqrt{x\varepsilon}}\frac{\varepsilon^2 - x^2}{3} = \frac{1}{6\sqrt{\varepsilon}}\left[2\varepsilon^2 x^{1/2} - \frac{2}{5}x^{5/2}\right]_0^\varepsilon = \frac{4\varepsilon^2}{15}. \tag{3.33}$$

The expected time to exceed $\varepsilon$ is therefore

$$\mathbb{E}\,T_\varepsilon = \underset{\textbf{type 3}}{\frac{\varepsilon^2}{5}} + \underset{\textbf{type 1}}{\frac{1}{2}.0} + \underset{\textbf{type 2}}{\frac{1}{2}.\frac{4\varepsilon^2}{15}} = \frac{\varepsilon^2}{3} \tag{3.34}$$

□

*Proof:* of Theorem 3.18

We find $G^+f(0)$ using Dynkin's formula that for $f \in C_b^2(\overline{\mathbb{R}^+})$

$$G^+f(0) = \lim_{\varepsilon\downarrow 0}\frac{\mathbb{E}_0 f(X^+(T_\varepsilon)) - f(0)}{\mathbb{E}\,T_\varepsilon}. \tag{3.35}$$

To be finite, we require the numerator to tend to zero as $\varepsilon$ tends to zero.

Using $\int_\varepsilon^\infty x^{-3/2}dx = 2/\sqrt{\varepsilon}$ and the fact that this is the total rate of **type 1** excursions,

$$\begin{aligned} \mathbb{E}_0 f(X^+(T_\varepsilon)) &= \tfrac{1}{2}f(\varepsilon) + \tfrac{1}{2}.\frac{\sqrt{\varepsilon}}{2}\int_\varepsilon^\infty f(x)\rho_1(dx) \\ &= \tfrac{1}{2}\left[f(\varepsilon) + f(0) + \frac{\sqrt{\varepsilon}}{2}\int_\varepsilon^\infty x^{-3/2}(f(x)-f(0))dx\right]. \end{aligned} \tag{3.36}$$

Since $f$ is bounded and $C^1$ near 0, $\int_0^\infty x^{-3/2}(f(x)-f(0))dx$ is finite and so the limit of $\mathbb{E}_0 f(X^+(T_\varepsilon))$ as $\varepsilon$ tends to zero is $f(0)$ as required.

By an application of l'Hôpital's rule,

$$\begin{aligned} G^+f(0) &= \lim_{\varepsilon\downarrow 0}\left(\frac{1}{\varepsilon^2/3}\right)\left\{\tfrac{1}{2}f(\varepsilon) + \frac{\sqrt{\varepsilon}}{4}\int_\varepsilon^\infty x^{-3/2}f(x)dx - f(0)\right\} \\ &= \lim_{\varepsilon\downarrow 0}\left(\frac{3}{4\varepsilon}\right)\left\{f'(\varepsilon) + \frac{1}{4\sqrt{\varepsilon}}\int_\varepsilon^\infty x^{-3/2}f(x)dx - \frac{1}{2\varepsilon}f(\varepsilon)\right\} \end{aligned} \tag{3.37}$$

which, to be finite, requires

$$\lim_{\varepsilon\downarrow 0}\left[f'(\varepsilon) + \frac{1}{4\sqrt{\varepsilon}}\int_\varepsilon^\infty x^{-3/2}\left(f(x)-f(0)\right)dx - \frac{1}{2\varepsilon}\left(f(\varepsilon)-f(0)\right)\right] = 0. \tag{3.38}$$

This certainly requires that $\int_0^\infty x^{-3/2}(f(x)-f(0))dx = 0$, and when this condition holds,

$$\begin{aligned} \lim_{\varepsilon\downarrow 0}\ & \frac{1}{4\sqrt{\varepsilon}}\int_\varepsilon^\infty x^{-3/2}[f(x)-f(0)]dx &&= \\ \lim_{\varepsilon\downarrow 0}\ & \frac{1}{2\varepsilon^{-1/2}}\left(-\varepsilon^{-3/2}[f(\varepsilon)-f(0)]\right) &&= \\ \lim_{\varepsilon\downarrow 0}\ & \frac{1}{2\varepsilon}\left(-[f(\varepsilon)-f(0)]\right) &&= -\tfrac{1}{2}f'(0) \end{aligned} \tag{3.39}$$

and so (3.38) holds for functions $f$ for which

$$\int_0^\infty x^{-3/2}\left(f(x)-f(0)\right)dx = 0. \tag{3.40}$$

Therefore,

$$G^+f(0) = \lim_{\varepsilon\downarrow 0}\left(\frac{3}{4\varepsilon}\right)\left\{f'(\varepsilon) + \frac{1}{4\sqrt{\varepsilon}}\int_\varepsilon^\infty x^{-3/2}f(x)dx - \frac{1}{2\varepsilon}f(\varepsilon)\right\} \tag{3.41}$$

and multiplying top and bottom by $\sqrt{\varepsilon}$ gives

$$G^+f(0) = \lim_{\varepsilon\downarrow 0}\left(\frac{3}{4\varepsilon^{3/2}}\right)\left\{\sqrt{\varepsilon}f'(\varepsilon) + \frac{1}{4}\int_\varepsilon^\infty x^{-3/2}f(x)dx - \frac{1}{2\sqrt{\varepsilon}}f(\varepsilon)\right\}. \tag{3.42}$$

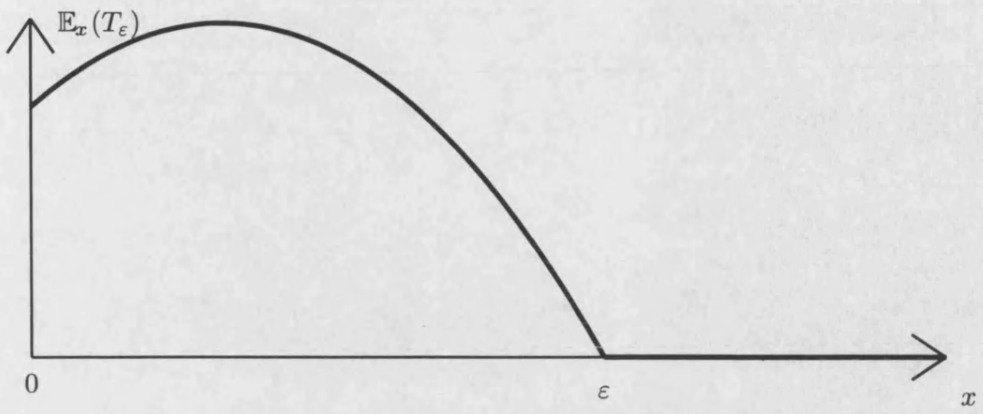



Figure 3-1: Expected time to exceed $\varepsilon$

Applying l'Hôpital's rule, we obtain

$$G^+f(0) = \lim_{\varepsilon\downarrow 0}\left(\frac{1}{2\varepsilon^{1/2}}\right)\left\{\sqrt{\varepsilon}f''(\varepsilon) + \tfrac{1}{2}\varepsilon^{-1/2}f'(\varepsilon) - \frac{1}{4}\varepsilon^{-3/2}f(\varepsilon)\right.$$

$$\left. - \frac{1}{2\sqrt{\varepsilon}}f'(\varepsilon) + \frac{1}{4}\varepsilon^{-3/2}f(\varepsilon)\right\} = \tfrac{1}{2}f''(0) \tag{3.43}$$

for $C_b^2(\overline{\mathbb{R}^+})$ functions satisfying (3.40). □

**Via Martingales**

We now derive the result of Theorem 3.22 assuming the domain of $G^+$ and using a martingale method.

(3.44) **Theorem:** *Assume that the process $X^+$ is defined by the generator $G^+f = \frac{1}{2}f''$ on functions $f \in C_b^2(\overline{\mathbb{R}^+})$ satisfying (3.38). Then Theorem 3.22 holds.*

*Proof:* For a fixed $\varepsilon > 0$, define the function

$$f(x) = \begin{cases} -(x-\varepsilon)\,(x+\frac{\varepsilon}{3}) & x \le \varepsilon \\ 0 & x > \varepsilon \end{cases}$$

and let $M_t = f(X_t^+) + t$. Then $\int_{\mathbb{R}^+} x^{-3/2}[f(x) - f(0)]dx = 0$ and so $f$ satisfies (3.38). Now $f$ is in the closure of the domain of $G^+$, and if $f_\eta \to f$ as $\eta \downarrow 0$ then

$$\lim_{\eta\to 0} G^+f_\eta(x) = \begin{cases} \frac{1}{2}f''(x) = -1 & 0 \le x < \varepsilon \\ \frac{4\varepsilon}{3}\delta_\varepsilon & x = \varepsilon \\ 0 & x > \varepsilon. \end{cases}$$

Thus Itô's formula says that

$$dM_t \doteq \lim_{\eta \to 0} G^+ f_\eta(X_t^+)dt + dt$$

and since this is zero on $[0, T_\varepsilon)$ whenever $X_0^+ < \varepsilon$, then $M_t$ is a true martingale up to that time.

The process $M$ jumps when $X$ hits $\varepsilon$ and increases with the local time at that point. Since we stop $M$ at $T_\varepsilon$, the local time is zero and we have no contribution from the $\delta_\varepsilon$ function.

Therefore, by the optional stopping theorem, for $x < \varepsilon$,

$$f(x) = \mathbb{E}_x(f(X(T_\varepsilon)) + T_\varepsilon) = \mathbb{E}_x T_\varepsilon = -(x-\varepsilon)(x+\varepsilon/3)$$

and in particular, (3.23) holds. □

We can also deduce, via martingales, the result that $\mathbb{P}_0\{X^+(T_\varepsilon) = \varepsilon\} = \frac{1}{2}$ by considering

$$N_t := g(X_t^+) := \frac{1}{2\varepsilon}(X_t^+ + \varepsilon) I_{\{x \le \varepsilon\}}.$$

Since $g$ satisfies (3.38), we can approximate it by continuous piecewise linear functions $g^\eta$ which are zero on $(\varepsilon+\eta, \infty)$ and then approximate the $g^\eta$ by functions $g_n^\eta$ in the domain of $G^+$ with $G^+ g_n^\eta(X_t)$ zero on $[0, T_\varepsilon]$. Then $N_t^\eta$ is a martingale with $N^\eta(T_\varepsilon)$ equal to zero if $X^+(T_\varepsilon) > \varepsilon + \eta$ and so, letting $\eta \to 0$, if $X^+(T_\varepsilon) \neq \varepsilon$ then $N(T_\varepsilon) = 0$. If $X^+(T_\varepsilon) = \varepsilon$ then $N(T_\varepsilon) = 1$. Therefore, by the optional stopping theorem, $g(0) = \mathbb{P}_0\{X^+(T_\varepsilon) = \varepsilon\} = \frac{1}{2}$.

### 3.3.2 The resolvent of $G^+$

For $\lambda \notin \mathbb{R}$, define $\gamma$ and $\tilde{\gamma}$ to have positive real parts such that $\gamma^2 = 2\lambda$ and $\tilde{\gamma}^2 = -2\lambda$. We will adopt this notation for the remainder of the chapter.

(3.45) **Theorem:** *The resolvent $R_\lambda^+$ of $X^+$ on $C_b(\overline{\mathbb{R}^+})$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}^-$ is given by*

$$R_\lambda^+ f(x) = {}^-R_\lambda f(x) + \frac{n_\lambda^+ f}{\lambda n_\lambda^+ 1} e^{-\gamma x} \tag{3.46}$$

*where*

$$n_\lambda^+ f = \int_0^\infty x^{-3/2}\, {}^-R_\lambda f(x) dx. \tag{3.47}$$

*Proof:* We first take $\lambda > 0$ and define $X^+$ via its entrance measure $n_\lambda^+$ as at (3.17). Then using excursion theory as in Chapter 2, this is immediate provided that the functions in the image space of this mapping are well defined. Since $f$ is bounded, $n_\lambda^+ f$ will be finite

if and only if $n_\lambda^+ \mathbf{1}$ is finite. That this is the case can be shown by evaluating

$$\gamma^{-1} \int_{x=0}^{\infty} x^{-3/2} \int_{y=0}^{\infty} \left( e^{-\gamma|x-y|} - e^{-\gamma|x+y|} \right) dy dx = 2\gamma^{-2} \int_{x=0}^{\infty} x^{-3/2} \left( 1 - e^{-\gamma x} \right) dx.$$

If $\gamma$ has a positive real part then this is a finite integral and so $n_\lambda^+ \mathbf{1} < \infty$.

We extend $R_\lambda^+ f$ by analytic continuation to all values of $\lambda$ away from $\mathbb{R}^-$ (where $\gamma$ does not have positive real part). We thus see that the expression gives the resolvent for the process for all values $\lambda \in \mathbb{C} \setminus \mathbb{R}^-$. □

The above expression for $R_\lambda^+$ also holds for $L^2(\mathbb{R}^+)$ functions. The resolvent of killed Brownian Motion, ${}^-R_\lambda$, maps $L^2(\mathbb{R}^+)$ into $W^2(\mathbb{R}^+)$ and has the property that ${}^-R_\lambda f(0) = 0$. Therefore, $|n_\lambda^+ f| < \infty$ for $f \in L^2(\mathbb{R}^+)$ since $L^2(\mathbb{R}^+)$ functions which are zero and differentiable at the origin give a finite result when integrated against $x^{-3/2}$.

Similarly, we have that the resolvent $R_\lambda^- := (\lambda + G^-)^{-1}$ is given by

$$R_\lambda^- f(x) = - {}^-R_{-\lambda} f(x) + \frac{n_\lambda^- f}{\lambda n_\lambda^- \mathbf{1}} e^{\tilde{\gamma} x} \tag{3.48}$$

where

$$n_\lambda^- f = \int_{-\infty}^{0} |x|^{-3/2} \, {}^-R_\lambda f(x) dx \tag{3.49}$$

and $\tilde{\gamma}^2 = -2\lambda$ with the real part strictly positive. This holds for $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$.

By Fubini's theorem, we may write $n_\lambda^+ f$ as

$$\int_0^\infty {}^-R_\lambda(\cdot^{-3/2})(x) f(x) dx$$

which, as we have seen, is finite for any $f \in L^2(\mathbb{R}^+)$. Therefore, ${}^-R_\lambda(\cdot^{-3/2})(x)$ is also in $L^2(\mathbb{R}^+)$.

We can derive the resolvent of $V^{-1}\mathcal{A}$, which we will denote $S_\lambda$, heuristically using excursion theory ideas. The Laplace Transform of the entrance law for reflected Brownian Motion is $\int_{\mathbb{R}^+} e^{-\gamma x} f(x) dx$. Since $(\lambda - V^{-1}(x)\mathcal{A}) = -((-\lambda) - V^{-1}(-x)\mathcal{A})$ we expect the Laplace Transform of the entrance law for "Reflected Brownian Motion on $\mathbb{R}^-$ time changed to be travelling backwards in time" to be $-\int_{\mathbb{R}^-} e^{\tilde{\gamma} x} f(x) dx$. Then the transformed entrance law for the wound process, modulo an irrelevant constant, would be a combination of these two. This is proved in Theorem 3.51 below.

We must be careful in our terminology. There is a perfectly standard winding process $t \mapsto (\phi(t), X_t)$ on the phase plane $\mathbb{R}^2$. This is not the process we are considering. Our more unorthodox process has a one dimensional state space, with time capable of flowing

in both directions. Rather than being a two dimensional process that *does wind* about the origin, it is the result of taking a one dimensional process which *has been wound* around the origin according to some function $V$. Thus when we speak of the generator of the wound process, we mean $V^{-1}(x)\mathcal{A}$ where $\mathcal{A}$ is the generator of the process that is being wound.

(3.50) **Lemma:** *We have*

$$\frac{1}{\lambda n_\lambda^+ 1} = \frac{1}{2\sqrt{\gamma\pi}}.$$

*Proof:* Since $\lambda\ {}^-R_\lambda 1(x)$ is just the probability that a Brownian Motion started at $x$ has not hit zero by an exponential time, it is equal to $1 - e^{-\gamma x}$. Therefore

$$\begin{aligned} \lambda n_\lambda^+ 1 &= \lambda \int_0^\infty x^{-3/2}\ {}^-R_\lambda 1(x)dx \\ &= \int_0^\infty x^{-3/2}(1 - e^{-\gamma x})dx = 2\sqrt{\gamma}\,\Gamma(-\tfrac{1}{2}) = 2\sqrt{\gamma\pi}. \end{aligned}$$

□

## 3.4 The Wound Process

We must be aware which operators have a clear probabilistic meaning and which are analytical objects only. The usual interpretation of a resolvent, that is, the expected value of a function evaluated at $X_T$ for an exponential random variable $T$, does not apply to the resolvent of the wound process. However, we must calculate this resolvent in order to show the relationship between it and the resolvents that do relate directly to standard stochastic processes and do have standard stochastic meanings. The resolvent for the wound process is valid only for $\lambda \notin \mathbb{R}$ and so it is not meaningful to talk about stopping a process at a random exponential time of rate $\lambda$.

(3.51) **Theorem:** *The resolvent of the wound process (that is, the resolvent of $V^{-1}\mathcal{A}$) is given by*

$$S_\lambda f(x) = \begin{cases} {}^-R_\lambda f^+(x) + \dfrac{n_\lambda f}{\lambda n_\lambda 1} e^{-\gamma x} & x > 0 \\ \\ -\ {}^-R_{-\lambda} f^-(x) + \dfrac{n_\lambda f}{\lambda n_\lambda 1} e^{\tilde{\gamma} x} & x \le 0 \end{cases} \tag{3.52}$$

*where $\lambda \notin \mathbb{R}$,*

$$n_\lambda f := \int_0^\infty e^{-\gamma x} f^+(x)dx - \int_{-\infty}^0 e^{\tilde{\gamma} x} f^-(x)dx \tag{3.53}$$

*and $f^+$ and $f^-$ are the restrictions of $f$ to $\mathbb{R}^+$ and $\mathbb{R}^-$.*

*Proof:* We have

$$\frac{1}{\lambda n_\lambda \mathbf{1}} = (\gamma^{-1} + \tilde{\gamma}^{-1}).$$

From the form of (3.52) we know that $S_\lambda f$ lies in $W^2(\mathbb{R}^+) \times W^2(\mathbb{R}^-)$. Suppose that it is $C^1$ at the origin and is therefore in the domain of $\mathcal{A}$. Then, by the definition of ${}^-R_\lambda$, $(\lambda - V^{-1}(x)\mathcal{A})S_\lambda f(x) = f(x)$ and so our proof that $S_\lambda$ is the resolvent of $V^{-1}\mathcal{A}$ would be complete.

Therefore we must prove that $S_\lambda f$ and its first derivative are continuous at the origin.

Since ${}^-R_\lambda$ is the resolvent of Brownian Motion killed at zero, both ${}^-R_\lambda f^+(0)$ and ${}^-R_{-\lambda} f^-(0)$ are zero and $S_\lambda f$ is therefore continuous at the origin.

Next we look at the first derivative. Since $|x-y| = (x-y)\mathrm{sgn}(x-y)$,

$$\begin{aligned}
\frac{d}{dx}\, {}^-R_\lambda f^+(x)\Big|_{x=0} &= \int_0^\infty \left[-\mathrm{sgn}(x-y)e^{-\gamma|x-y|} + e^{-\gamma|x+y|}\right] f^+(y)dy\Big|_{x=0} \\
&= 2\int_0^\infty e^{-\gamma y} f^+(y)dy.
\end{aligned}$$

Similarly

$$\frac{d}{dx}\, {}^-R_{-\lambda} f^-(x)\Big|_{x=0} = -2\int_{-\infty}^0 e^{\tilde{\gamma} y} f^-(y)dy.$$

Thus, using $\gamma\tilde{\gamma}^{-1} = -\tilde{\gamma}\gamma^{-1}$,

$$\begin{aligned}
(S_\lambda f)'(0+) &= 2\int_0^\infty e^{-\gamma y} f^+(y)dy - \gamma(\gamma^{-1} + \tilde{\gamma}^{-1}) n_\lambda f \\
&= (1 - \gamma\tilde{\gamma}^{-1})\int_0^\infty e^{-\gamma y} f^+(y)dy + (1 + \gamma\tilde{\gamma}^{-1})\int_{-\infty}^0 e^{\tilde{\gamma} y} f^-(y)dy \\
&= 2\int_{-\infty}^0 e^{\tilde{\gamma} y} f^-(y)dy + \\
&\qquad (1 + \tilde{\gamma}\gamma^{-1})\left\{\int_0^\infty e^{-\gamma y} f^+(y)dy - \int_{-\infty}^0 e^{\tilde{\gamma} y} f^-(y)dy\right\} \\
&= 2\int_{-\infty}^0 e^{\tilde{\gamma} y} f^-(y)dy + \tilde{\gamma}(\tilde{\gamma}^{-1} + \gamma^{-1}) n_\lambda f \\
&= (S_\lambda f)'(0-)
\end{aligned}$$

and the proof is complete. □

## 3.5 The Wiener-Hopf Factorization

We now come to the central equation of the theory, the factorization (3.55). For the Canonical Case, most of our work has been concerned with finding the right domains, and this is the main obstacle to proving the factorization. It is quite simple to show that the equation, if it makes sense at all, is true. It is harder to show that it is true, and sensible, for exactly those functions for which we require it to be true. The result that the isomorphism (2.66) defined by the $\Pi^{\pm\mp}$ operators takes the product of two spaces, each with an unusual side condition and maps them exactly to the Sobolev space $W^2(\mathbb{R})$ is certainly an unexpected one.

(3.54) **Theorem:** *The expression*

$$\text{(3.55)} \qquad \begin{pmatrix} I & \Pi^{+-} \\ \Pi^{-+} & I \end{pmatrix}^{-1} V^{-1}\mathcal{A} \begin{pmatrix} I & \Pi^{+-} \\ \Pi^{-+} & I \end{pmatrix} = \begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix}$$

*holds for functions $(f_+, f_-)$ in the domains of $G^+$ and $G^-$.*

We prove the following lemma at the end of this section.

(3.56) **Lemma:** *The functions*

$$\begin{aligned} &i) && {}^{-}R_\lambda(\cdot^{-3/2})(x) \\ &ii) && {}^{-}R_\lambda(\Pi(\cdot, y))(x) \\ &iii) && \Pi^{+-}(e^{\tilde{\gamma}\cdot})(x) \\ &iv) && {}^{-}R_\lambda(\Pi(x, \cdot))(y) \end{aligned}$$

*are all $C^2(\overline{\mathbb{R}^+})$ in $x$ and, under the operator $\lambda - \frac{1}{2}\frac{d^2}{dx^2}$, map to*

$$x^{-3/2}, \quad \Pi(x,y), \quad \frac{1}{2}\sqrt{\frac{\tilde{\gamma}}{2\pi}}x^{-3/2}, \quad \textit{and} \quad -\Pi(x,y) + \frac{\sqrt{2}}{8\pi}x^{-3/2}\,{}^{-}R_\lambda(\cdot^{-3/2})(y)$$

*respectively.*

From the result of the lemma we see that

$$\left(\lambda - \frac{d^2}{dx^2}\right)\left[\frac{1}{2}\sqrt{\frac{\tilde{\gamma}}{2\pi}}\,{}^{-}R_\lambda(\cdot^{-3/2})(x) - \Pi^{+-}(e^{\tilde{\gamma}\cdot})(x) + e^{-\gamma x}\right]$$

is well-defined and zero and so the bracketed expression, being zero at 0 and at infinity, must be identically zero and hence

$${}^{-}R_\lambda(\cdot^{-3/2})(x) = 2\sqrt{\frac{2\pi}{\tilde{\gamma}}}(\Pi^{+-}(e^{\tilde{\gamma}\cdot}) - e^{-\gamma\cdot})(x).$$

*Proof:* (of Theorem 3.54) We prove the result that for $x > 0$ and for $\lambda \notin \mathbb{R}$,

$$S_\lambda f(x) = R^+_\lambda f_+(x) + \Pi^{+-} R^-_{-\lambda} f_-(x), \tag{3.57}$$

where

$$f = \begin{pmatrix} f^+ \\ f^- \end{pmatrix} = \begin{pmatrix} I^+ & \Pi^{+-} \\ \Pi^{-+} & I^- \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$$

first for $f_- = 0$, then for $f_+ = 0$. Then the result for general $(f_+, f_-)$ holds by linearity and for all $x \in \mathbb{R}$ by symmetry. The generator version (3.55) and the resolvent version (3.57) together with its counterpart on $\mathbb{R}^-$ are equivalent and thus the theorem will be proved.

Choose $x > 0$ and let $f_- = 0$. Define $f^+(y) := f_+(y)$ for $y > 0$ and $f^-(y) := \Pi^{-+} f_+(y)$ for $y \leq 0$. We must prove that $S_\lambda f(x) = R^+_\lambda f_+(x)$. That is,

$$^-R_\lambda f_+(x) + \frac{n_\lambda f}{\lambda n_\lambda 1} e^{-\gamma x} = {}^-R_\lambda f_+(x) + \frac{n^+_\lambda f_+}{\lambda n^+_\lambda 1} e^{-\gamma x} \tag{3.58}$$

which reduces after cancellation to

$$\frac{n_\lambda f}{\lambda n_\lambda 1} = \frac{n^+_\lambda f}{\lambda n^+_\lambda 1}. \tag{3.59}$$

A glance back at Lemma 3.50 and equations (3.47) and (3.53) tells us that this, when written out in full, becomes

$$\int_0^\infty \left\{ (\gamma^{-1} + \tilde{\gamma}^{-1}) e^{-\gamma x} - (\gamma^{-1} + \tilde{\gamma}^{-1}) \Pi^{+-}(e^{\tilde{\gamma}\cdot})(x) - \frac{1}{2\sqrt{\gamma\pi}} {}^-R_\lambda(\cdot^{-3/2})(x) \right\} f_+(x) dx = 0 \tag{3.60}$$

and so we require the bracketed expression to be identically zero.

Note that the square of $(\gamma^{-1} + \tilde{\gamma}^{-1})$ is $2\gamma^{-1}\tilde{\gamma}^{-1}$ and so

$$\left[ (\gamma^{-1} + \tilde{\gamma}^{-1}) \times \frac{1}{2} \sqrt{\frac{\tilde{\gamma}}{2\pi}} \right]^2 = \frac{1}{4\pi} \gamma^{-1}. \tag{3.61}$$

By the results in Lemma 3.56, the bracketed expression in (3.60) is a combination of the exponentials $\exp(-\gamma x)$ and $\exp(\gamma x)$. Since it is zero at $x = 0$ and $x = \infty$, it is identically zero.

Still with $x > 0$, we let $f_+ = 0$. Define $f^+(y) := \Pi^{+-} f_-(y)$ for $y > 0$ and $f^-(y) := f_-(y)$

for $y \leq 0$. We must prove that $S_\lambda f(x) = \Pi^{+-} R^-_{-\lambda} f_-(x)$. That is,

$$(3.62) \qquad {}^-R_\lambda \Pi^{+-} f_-(x) + \frac{n_\lambda f}{\lambda n_\lambda \mathbf{1}} e^{-\gamma x} = -\Pi^{+-}({}^-R_{-\lambda} f_-)(x) + \frac{n^-_\lambda f_-}{\lambda n^-_\lambda \mathbf{1}} \Pi^{+-}(e^{\tilde{\gamma}\cdot})(x).$$

Writing this out in a similar way to (3.60) as an integral of some function against $f_-$ we require the following expression to be zero,

$$\begin{aligned} &{}^-R_\lambda(\Pi(\cdot, y))(x) + \frac{1}{\lambda n_\lambda \mathbf{1}} e^{-\gamma x} \left\{ \Pi^{+-}(e^{\gamma\cdot})(y) - e^{-\tilde{\gamma} y} \right\} \\ &\qquad + {}^-R_{-\lambda}(\Pi(x, \cdot))(y) - \frac{1}{\lambda n^-_\lambda \mathbf{1}} \Pi^{+-}(e^{\tilde{\gamma}\cdot})(x)\ {}^-R_{-\lambda}(\cdot^{-3/2})(y). \end{aligned} \tag{3.63}$$

Applying the operator $(\lambda - \frac{1}{2}\frac{d^2}{dx^2})$ and using the results of Lemma 3.56 yields zero, and hence (3.63) is again a combination of the exponentials $\exp(\gamma x)$ and $\exp(-\gamma x)$. As $x$ tends to infinity, the expression tends to zero for all $y$ and so we need only prove that it is zero at $x = 0$. That is, that

$$(3.64) \qquad (\gamma^{-1} + \tilde{\gamma}^{-1}) \left\{ \Pi^{+-}(e^{\gamma\cdot})(y) - e^{-\tilde{\gamma} y} \right\} - \frac{1}{2\sqrt{\pi\tilde{\gamma}}}\ {}^-R_{-\lambda}(\cdot^{-3/2})(y) = 0.$$

This expression is zero at $y = 0$ and as $y \to \infty$ and so applying $(-\lambda - \frac{1}{2}\frac{d^2}{dy^2})$ we obtain zero once more and see that (3.63) is indeed identically zero.

This completes the proof of the theorem. □

*Proof:* (Of Lemma 3.56) Parts *i*) and *ii*). We know that the functions in *i*) and *ii*) are both well defined and, by inverting ${}^-R_\lambda$, we have the result given.

We use the three facts that $\Pi^{+-}\mathbf{1} = \mathbf{1}$,

$$\left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) \Pi(x, y) = -\frac{\sqrt{2}}{4\pi} (xy)^{-3/2}$$

(let us use the notation $\Pi_x(x, y)$ for the derivative with respect to $x$) and that $x\Pi_x(x, y) \to 0$ as $x \to 0$.

Part *iii*). By integration by parts,

$$\int_0^\infty \Pi_{yy}(x, y)[e^{-\tilde{\gamma} y} - 1] dy = -2\lambda \Pi^{+-}(e^{\tilde{\gamma}\cdot})(x)$$

and

$$\begin{aligned} \tfrac{1}{2}\tfrac{d^2}{dx^2} \Pi^{+-}(e^{\tilde{\gamma}\cdot})(x) &= \textstyle\int_0^\infty \Pi_{xx}(x, y)[e^{-\tilde{\gamma} y} - 1] dy \\ &= -\textstyle\int_0^\infty (\Pi_{yy}(x, y) + \frac{\sqrt{2}}{4\pi}(xy)^{-3/2})[e^{-\tilde{\gamma} y} - 1] dy \end{aligned}$$

and therefore

$$(\lambda - \tfrac{1}{2}\frac{d^2}{dx^2})\Pi^{+-}(e^{\tilde{\gamma}\cdot})(x) = \int_0^\infty \frac{\sqrt{2}}{4\pi}(xy)^{-3/2}[e^{-\tilde{\gamma}y} - 1]dy = \frac{1}{2}\sqrt{\frac{\tilde{\gamma}}{2\pi}}x^{-3/2}.$$

Part *iv*). Note that we are differentiating with respect to $x$, not $y$, and so ${}^{-}R_\lambda(\Pi(x,\cdot))(y)$ does *not* map to $\Pi(x,y)$.

Again by integration by parts (in which all the boundary terms are zero)

$$\begin{aligned}
\gamma^{-1}\int_0^\infty [e^{-\gamma|w-y|} - e^{-\gamma|w+y|}]\Pi_{ww}(x,w)dw = & \\
&= -\int_0^\infty [\mathrm{sgn}(y-w)e^{-\gamma|w-y|} + e^{-\gamma|w+y|}]\Pi_w(x,w)dw \\
&= \gamma\int_0^\infty [e^{-\gamma|w-y|} - 2\delta(y-w)e^{-\gamma|w-y|} - e^{-\gamma|w+y|}]\Pi(x,w)dw \\
&= -2\Pi(x,y) + \gamma^2\, {}^{-}R_\lambda(\Pi(x,\cdot))(y).
\end{aligned}$$

So

$$\begin{aligned}
\frac{d^2}{dx^2}\,{}^{-}R_\lambda(\Pi(x,\cdot))(y) &= \gamma^{-1}\int_0^\infty [e^{-\gamma|w-y|} - e^{-\gamma|w+y|}]\Pi_{xx}(x,w)dw \\
&= -\gamma^{-1}\int_0^\infty [e^{-\gamma|w-y|} - e^{-\gamma|w+y|}]\left(\Pi_{ww}(x,w) + \frac{\sqrt{2}}{4\pi}(xw)^{-3/2}\right)dw \\
&= 2\Pi(x,y) - \gamma^2\,{}^{-}R_\lambda(\Pi(x,\cdot))(y) - \tfrac{\sqrt{2}}{4\pi}x^{-3/2}\,{}^{-}R_\lambda(\cdot^{-3/2})(y)
\end{aligned}$$

and

$$\left(\lambda - \frac{1}{2}\frac{d^2}{dx^2}\right)\,{}^{-}R_\lambda(\Pi(x,\cdot))(y) = -\Pi(x,y) + \frac{\sqrt{2}}{8\pi}x^{-3/2}\,{}^{-}R_\lambda(\cdot^{-3/2})(y)$$

as required. □

## 3.6 Dirichlet Forms

We finish these two chapters concentrating on the analysis of Wiener-Hopf techniques by finding, as before, inner products and related quadratic forms on the spaces satisfying the equations (3.66) and (3.67) below. This provides us with another way of viewing the Wiener-Hopf factorization and the structure behind it. We also prove that $G^+$ is self-adjoint with respect to $\langle\cdot,\cdot\rangle_+$ and similarly for $G^-$ and $\langle\cdot,\cdot\rangle_-$.

(3.65) **Theorem:** *There exist inner products $\langle\cdot,\cdot\rangle_+$ on $L^2(\mathbb{R}^+)$ and $\langle\cdot,\cdot\rangle_-$ on $L^2(\mathbb{R}^-)$ such that the Dirichlet Form $\mathcal{E}$ and the quadratic forms $\mathcal{E}_+$ and $\mathcal{E}_-$ associated with the processes satisfy*

$$\langle f, \mathrm{sgn}(x) g\rangle_\nu = \langle f_+, g_+\rangle_+ - \langle f_-, g_-\rangle_- \tag{3.66}$$

*and*

$$\mathcal{E}_\nu(f,g) = \mathcal{E}_+(f_+,g_+) + \mathcal{E}_-(f_-,g_-), \tag{3.67}$$

*for functions $f$ and $g$ in the domain of $\mathcal{A}$ and their images $f_+$ and $f_-$ under the isomorphism.*

*Proof:* This is essentially the same theorem and proof as in Theorem 2.85. We define the inner products as before with

$$\begin{aligned}\langle f_+, g_+\rangle_+ &:= \langle f_+, g_+\rangle_{\nu^+} - \langle \Pi^{-+} f_+, \Pi^{-+} g_+\rangle_{\nu^-} \\ \langle f_-, g_-\rangle_- &:= \langle f_-, g_-\rangle_{\nu^-} - \langle \Pi^{+-} f_-, \Pi^{+-} g_-\rangle_{\nu^+}\end{aligned}$$

where the measures $\nu^+$ and $\nu^-$ are Lebesgue measure on $\mathbb{R}^+$ and $\mathbb{R}^-$. By the isomorphism between the domains and the same calculation as at equation (2.88), we see that (3.66) holds. By the definition of a Dirichlet Form associated with a process, we have that

$$\mathcal{E}(f,g) = -\langle f, \mathcal{A} g\rangle_\nu \tag{3.68}$$

and the quadratic forms $\mathcal{E}_+$ and $\mathcal{E}_-$ are defined similarly. Writing $\mathcal{E}(f,g)$ in terms of $f_+$ and $f_-$ using the Wiener-Hopf factorization gives us

$$-\langle f_+ + \Pi^{+-} f_-, G^+ g_+ - \Pi^{+-} G^- g_-\rangle_{\nu^+} + \langle \Pi^{-+} f_+ + f_-, \Pi^{-+} G^+ g_+ - G^- g_-\rangle_{\nu^-}. \tag{3.69}$$

Multiplying this out shows that the terms involving $f_+$ and $g_+$ and $f_-$ and $g_-$ are equal to $\mathcal{E}_+(f_+,g_+)$ and $\mathcal{E}_-(f_-,g_-)$ and, using the adjointness of $\Pi^{+-}$ and $\Pi^{-+}$, the cross terms cancel so that

$$\mathcal{E}(f,g) = \mathcal{E}_+(f_+,g_+) + \mathcal{E}_-(f_-,g_-). \tag{3.70}$$

□

We finish our work on the Canonical Case with the following result.

(3.71) **Lemma:** *$G^+$ is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_+$.*

*Proof:* A function $g_+$ is in the domain of $G^{+*}$ if there exists an $h$ (and this $h$ will be unique) such that for all $f_+$ in the domain of $G^+$,

$$\langle G^+ f_+, g_+\rangle_+ = \langle f_+, h\rangle_+$$

and then $G^{+*}g_+$ is defined to be equal to $h$.

The symmetry of the operator $G^+$ follows from the self-adjointness of $\mathcal{A}$ as follows. For a function $f_+$ in the domain of $G^+$, let $f$ equal $f_+$ on $\mathbb{R}^+$ and $\Pi^{-+}f_+$ on $\mathbb{R}^-$ and similarly for functions $g_+$ and $g$. Then

$$\langle f_+, G^+ g_+\rangle_+ = -\mathcal{E}_+(f_+, g_+) = -\mathcal{E}(f, g) = -\mathcal{E}(\overline{g}, \overline{f}) = \langle \overline{g}_+, G^+\overline{f_+}\rangle_+ = \langle G^+ f_+, g_+\rangle_+.$$

Then for $f_+$ and $g_+$ in the domain of $G^+$,

$$\langle G^+ f_+, g_+\rangle_+ = \langle f_+, G^+ g_+\rangle_+$$

and so $g_+$ is in the domain of $G^{+*}$ and $G^{+*}g_+ = G^+ g_+$.

We must prove that $f_+$ in the domain of $G^{+*}$ also lies in the domain of $G^+$.

For $\lambda$ not in the spectrum of $G^+$ and $f_+$ in the domain of $G^{+*}$, we define $g_+$ to be $R^+_\lambda(\lambda - G^{+*})f_+$. Then $g_+$ is in the domain of $G^+$ and hence of $G^{+*}$. Thus

$$\begin{aligned}(\lambda - G^{+*})f_+ &= (\lambda - G^+)g_+ \\ &= (\lambda - G^{+*})g_+\end{aligned}$$

and so $(\lambda - G^{+*})(f_+ - g_+) = 0$.

Then for $h \in L^2(\mathbb{R}^+)$,

$$0 = \langle(\lambda - G^{+*})(f_+ - g_+), R^+_\lambda h\rangle_+ = \langle(f_+ - g_+), (\lambda - G^+)R^+_\lambda h\rangle_+ = \langle f_+ - g_+, h\rangle_+$$

and hence $f_+ = g_+$ and $f_+$ is in the domain of $G^+$. Therefore $G^+$ and $G^{+*}$ have identical domains and hence are equal. □

# Chapter 4

# Atomic Measures I – Mass at Zero

In this and the following chapter we consider the Wiener-Hopf decomposition of a Brownian Motion time changed via measures consisting of a finite number of atoms on $\mathbb{R}^-$ and Lebesgue measure on $\mathbb{R}^+$. One motive for studying such a case is the possibility of performing numerical calculations. Approximate values for measures and operators on the atoms and for probabilities can be computed and these results suggest the (occasionally surprising) theorems of later sections.

The atomic measures allow us to work with finite dimensional function spaces and the analysis is greatly simplified. Chapters 4 and 5 investigate instead a certain algebraic relation buried inside the structure underlying these processes. This is not a total departure from the earlier chapters. Proving that the half winding operators are contractions is still the primary aim, if not the primary result.

## 4.1 The Theory

We begin with strictly positive masses $m_0, m_1, \ldots, m_n$ distributed over the negative real line. Let $R_i$ be the position of the $i^{th}$ mass, such that $R_0 = 0$ and define the inter-mass distances $d_i := R_i - R_{i+1} > 0$ for $i = 0, 1, \ldots, n-1$. Let $S := \{R_0, R_1, \ldots, R_n\}$ and let the measure $\nu$ be as shown in Figure 4-1 with Lebesgue measure on $\mathbb{R}^+$ and masses $m_i$ at $R_i \in \mathbb{R}^-$. The measure restricted to $\mathbb{R}^-$ is denoted $\nu^-$.

As stated in (2.1), we will write $\Pi^{+-}(x, R_i)$ for $\Pi^{+-}(x, \{R_i\})$, the (strictly positive) probability that the usual winding process given by Brownian Motion time changed by $\nu$ and started at $x > 0$ half winds round to mass $i$.

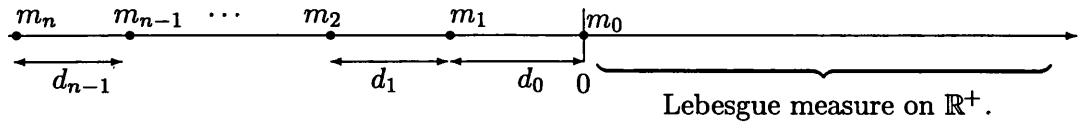



Figure 4-1: A Measure with Atoms

We time change Brownian Motion as previously using the measure $\nu$ (no reversal of time or signed measures yet—this is straight time change with an additive functional). The resulting process, $X$, is now a Brownian Motion on $(0,\infty)$ and a Markov Chain on $S\backslash\{R_0\}$. The behaviour at 0 is, as always, rather more subtle. From the time when the process $X$ enters the state $R_i$ for $i \in \{1,2,\ldots,n\}$, we can represent the Brownian Motion as a series of excursions away from $R_i$. The process $X$ leaves the state $R_i$ when the Brownian Motion performs a positive excursion of height at least $d_{i-1}$ or a negative excursion of height at least $d_i$ (take $d_n = \infty$). The time at which $X$ leaves $R_i$ is $m_i$ times the local time accumulated by the Brownian Motion at $R_i$ when such an excursion occurs. A positive excursion hitting $R_{i-1}$ arrives at exponential local time at rate $d_{i-1}^{-1}$ and a negative excursion hitting $R_i$ arrives at exponential local time at rate $d_i^{-1}$. Thus the process $X$ jumps from state $R_i$ to $R_{i-1}$ at rate $(d_{i-1}m_i)^{-1}$ and to $R_{i+1}$ at rate $(d_i m_i)^{-1}$.

When the process is at $R_0 = 0$ it jumps to $R_1$ after an exponential length local time at 0 at rate $(d_0 m_0)^{-1}$. Whilst at zero, it performs positive excursions according to the Brownian Motion excursion measure $n$.

For a function $f$, let us write $f = (f, \mathbf{f})$ where $\mathbf{f}$ is the vector on $S$ $(f(R_0)\ f(R_1)\ \cdots\ f(R_n))^T$. Let $[\mathbf{f}]_k$ denote the $k^{th}$ element $f(R_k)$. The generator of the process, $X$, is given by

$$\mathcal{A}f := \frac{1}{2}\frac{d}{d\nu}\frac{d}{dx}f = \begin{cases} \frac{1}{2}f''(x) & \text{on } \mathbb{R}^+ \\ \frac{1}{2}\left([Q\mathbf{f}]_k + m_0^{-1}f'(0+)I_{\{k=0\}}\right) & \text{at } x = R_k \end{cases} \tag{4.1}$$

(see, for example, Vol. II Section V.47) where

$$Q := M^{-1}\begin{pmatrix} -c_0 & c_0 & 0 & . & 0 & 0 & 0 \\ c_0 & -(c_0+c_1) & c_1 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & c_{n-2} & -(c_{n-2}+c_{n-1}) & c_{n-1} \\ 0 & 0 & 0 & . & 0 & c_{n-1} & -c_{n-1} \end{pmatrix}, \tag{4.2}$$

and $c_i := d_i^{-1}$. The value of $\mathcal{A}f$ at $R_i$ is half the change in the derivative divided by the mass. The left hand derivative at $R_n$ must be zero, else the function would not be bounded. We also define the $S \times S$ matrix $V$, whose only non-zero entry is $V_{00} = m_0^{-1}$.

We now reverse time when the Brownian Motion is below zero, so $\phi$ is defined as at (1.2) with the function there called $V(x)$ equal to $\mathrm{sgn}(x)$ and the winding process $X$ is defined as usual. We find a local martingale of the form $N_t := \exp(-\frac{1}{2}\lambda^2\phi(t))f(X_t)$ by solving

$$(\tfrac{1}{2}\lambda^2 - \mathrm{sgn}^{-1}(x)\mathcal{A})f(x) = 0 \tag{4.3}$$

for bounded continuous functions. Thus $f$ is the exponential function $\exp(-\lambda x)$ on $\mathbb{R}^+$ and is piecewise linear on $\mathbb{R}^-$. We can, and do, normalise the eigen-functions so that $f(0) = 1$.

Since $f'(0+) = -\lambda$ and we require $[\mathbf{f}]_0 = 1$ by continuity, we deduce that $[V\mathbf{f}]_k = m_0^{-1} I_{\{k=0\}}$. Therefore to find the eigen-function on $S$, we must solve

$$\left(Q - \lambda V + \lambda^2 I\right)\mathbf{f} = 0 \tag{4.4}$$

subject to $[\mathbf{f}]_0 = 1$. This is a polynomial in $\lambda$ of degree $2|S|$. We assume (and later prove) that it has exactly $|S|$ strictly positive roots and label these $\lambda_0 < \lambda_1 < \cdots < \lambda_n$. We write $\mathbf{e}_j$ for the solution corresponding to the root $\lambda_j$. Let $f_j = (f_j, \mathbf{e}_j)$ where $\mathbf{e}_j$ satisfies (4.4) with $\lambda = \lambda_j$ and $f_j(x) := \exp(-\lambda_j x)$ on $x > 0$. The optional stopping theorem applied to $N_t$ with $X_0 = x$ gives

$$N_0 = e^{-\lambda_j x} = \mathbb{E}\, N_{\tau^-(0)} = \sum_k \Pi^{+-}(x, R_k)[\mathbf{e}_j]_k. \tag{4.5}$$

By considering all $|S|$ solutions, we see that there must exist some matrix $\Pi$ such that $\Pi^{+-}(x, R_j) = [\boldsymbol{\epsilon}_x \Pi]_j$ where

$$\boldsymbol{\epsilon}_x = \begin{pmatrix} e^{-\lambda_0 x} & e^{-\lambda_1 x} & \cdots & e^{-\lambda_n x} \end{pmatrix}. \tag{4.6}$$

Therefore, defining $K$ by $K_{ij} := [\mathbf{e}_j]_i$, we have that for all $x > 0$,

$$\boldsymbol{\epsilon}_x = \boldsymbol{\epsilon}_x \Pi K \quad \text{and hence} \quad \Pi = K^{-1} \text{ and } \Pi^{+-}(x, R_j) = [\boldsymbol{\epsilon}_x \Pi]_j. \tag{4.7}$$

Since our present aim is to find plausible results, which we later prove, we *assume* the $\Pi$-duality result: $\Pi^{-+}(R_j, dx) = [M^{-1}\Pi^T \boldsymbol{\epsilon}_x^T]_j dx$. The full winding operator $W^-$ is then given by

$$W^-_{ij} = \int_0^\infty \Pi^{-+}(R_i, dx)\Pi^{+-}(x, R_j) = \left[\int_0^\infty M^{-1}\Pi^T \boldsymbol{\epsilon}_x^T \boldsymbol{\epsilon}_x \Pi dx\right]_{ij}. \tag{4.8}$$

Let

$$\Theta_{ij} = \int_0^\infty \left[\boldsymbol{\epsilon}_x^T \boldsymbol{\epsilon}_x\right]_{ij} dx = \int_0^\infty e^{-(\lambda_i + \lambda_j)x} dx = \frac{1}{\lambda_i + \lambda_j} \tag{4.9}$$

and so

(4.10) $$W^- = M^{-1}\Pi^T\Theta\Pi.$$

By following this procedure, the matrix $W^-$ can be computed numerically.

## 4.2 The Theorems

The program's results (see Appendix C) suggest the theorem that $\|W^-\| = \frac{1}{2}$ for any combination of masses, provided $m_0$ at the origin is positive.

(4.11) **Theorem:** *For Lebesgue measure on $\mathbb{R}^+$ and a finite number of atoms on $\mathbb{R}^-$, including one atom at 0, the full winding operator $W^-$ has norm one half.*

We define a matrix $\Gamma_-$ by taking the positive eigenvalues $\lambda_0, \ldots, \lambda_n$ and their related eigenvectors $\mathbf{e}_j$ for the quadratic equation (4.4), and setting

(4.12) $$\Gamma_-\mathbf{e}_j = -\lambda_j\mathbf{e}_j$$

for each positive eigenvalue $\lambda_j$. We define $\Gamma_+$ similarly, with

(4.13) $$\Gamma_+\mathbf{f}_j^+ = \lambda_j^+\mathbf{f}_j^+$$

for each negative eigenvalue, $\lambda_j^+ \leq 0$ and eigen-vector $\mathbf{f}_j^+$.

A theorem also suggested by the results is that $\Gamma_+ = (I - 2W)\Gamma_-$ which, as we shall see, almost immediately implies the first.

Note that since $Q$ is a strict Q-matrix, it has eigen-value 0 and hence $\lambda = 0$ is a solution to (4.4).

(4.14) **Theorem:** *For the matrices defined above,*

(4.15) $$\Gamma_+ = (I - 2W)\Gamma_-.$$

## 4.3 Proof of Theorem 4.11

In Subsection 4.3.1 we recall facts from Kennedy & Williams (1990) (= [KW]). By using the methods of [KW], *we enrich the structure*: for example, the $\Gamma_-$ operator does not feature in the [LMRW1] story. However, it is the case that $G^- = -\frac{1}{2}\Gamma_-^2$. The relevance of Subsection 4.3.1 to Theorem 4.11 will to some extent already be apparent.

### 4.3.1 Wiener-Hopf factorization of a quadratic

With notation as before and for an $S \times S$ matrix $G$, let $G^*$ denote the adjoint of $G$ with respect to $L^2(S, \nu^-)$, so that, for column vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^S$,

$$< \mathbf{u}, G^*\mathbf{v} >:= \mathbf{u}^T M G^* \mathbf{v} =< G\mathbf{u}, \mathbf{v} >= \mathbf{u}^T G^T M \mathbf{v}, \text{ so that } G^* = M^{-1} G^T M. \tag{4.16}$$

Let $\mathcal{Q}$ denote the set of real $S \times S$ matrices with non-negative off-diagonal elements and with non-positive row sums; and let $\mathcal{Q}_0$ consist of those elements of $\mathcal{Q}$ for which all row sums are exactly 0. That is, $\mathcal{Q}$ is the set of $Q$-matrices and $\mathcal{Q}_0$ the set of strict $Q$-matrices. Let $Q \in \mathcal{Q}_0$, and assume that $Q$ is $\nu^-$-*symmetrizable* in that

$$Q = Q^*; \quad \text{equivalently, } MQ \text{ is symmetric.} \tag{4.17}$$

Let $V$ be any real diagonal $S \times S$ matrix.

In [KW] it is shown that there exists a unique pair $(\Gamma_-, \Gamma_+)$ of elements of $\mathcal{Q}$ such that

$$t^2 I - tV + Q = (tI - \Gamma_+^*)(tI + \Gamma_-) = (tI + \Gamma_-^*)(tI - \Gamma_+), \tag{4.18}$$

$I$ denoting the identity $S \times S$ matrix. As consequences, we have

$$\Gamma_+^* - \Gamma_- = V, \tag{4.19}$$

and

$$\Gamma_+^* \Gamma_- = -Q = \Gamma_-^* \Gamma_+. \tag{4.20}$$

It can immediately be verified that

$$\Gamma_+^2 - V\Gamma_+ + Q = 0, \qquad \Gamma_-^2 + V\Gamma_- + Q = 0. \tag{4.21}$$

We shall be interested only in the case when $Q$ is *irreducible* (in that for any disjoint partition $S = S_1 \cup S_2$ of $S$ into non-empty sets, we can find $i \in S_1$ and $j \in S_2$ with $Q(i,j) > 0$) and $V$ has non-negative diagonal entries which are not all zero. We now assume that we are working with that case. Then $\Gamma_+ \in \mathcal{Q}_0$, and $\Gamma_- \in \mathcal{Q} \setminus \mathcal{Q}_0$. We can find a basis $(\mathbf{e}_j : j \in S)$ of $\mathbb{R}^S$ and strictly positive numbers which we write $(\lambda_j : j \in S)$ such that

$$(\lambda_j^2 I - \lambda_j V + Q)\mathbf{e}_j = \mathbf{0}; \quad \Gamma_- \mathbf{e}_j = -\lambda_j \mathbf{e}_j. \tag{4.22}$$

Note that the $Q$ of Section 4.1 is irreducible.

### 4.3.2 An algebraic identity

We continue with the notation of the last subsection. The key step will be to apply the following lemma to the case when $W$ is the $W^-$ of (4.10).

(4.23) **Lemma:** *Suppose that the eigenvalues $(-\lambda_j : j \in S)$ of $\Gamma_-$ are distinct. Suppose also that an $S \times S$ matrix $W$ satisfies*

$$\langle W\mathbf{e}_k, \mathbf{e}_j\rangle = \frac{\langle V\mathbf{e}_j, \mathbf{e}_k\rangle}{\lambda_j + \lambda_k} \qquad (j,k \in S). \tag{4.24}$$

*Then*

$$(I - 2W)\Gamma_- = \Gamma_+. \tag{4.25}$$

*Proof:* From (4.22) and (4.20), we have

$$\lambda_j^2\langle \mathbf{e}_j, \mathbf{e}_k\rangle - \lambda_j\langle V\mathbf{e}_j, \mathbf{e}_k\rangle = -\langle Q\mathbf{e}_j, \mathbf{e}_k\rangle = \langle \Gamma_-\mathbf{e}_j, \Gamma_+\mathbf{e}_k\rangle, \tag{4.26}$$

and since $\lambda_j \neq 0$ and $\Gamma_-\mathbf{e}_j = -\lambda_j\mathbf{e}_j$, we have

$$\lambda_j\langle \mathbf{e}_j, \mathbf{e}_k\rangle - \langle V\mathbf{e}_j, \mathbf{e}_k\rangle = -\langle \mathbf{e}_j, \Gamma_+\mathbf{e}_k\rangle. \tag{4.27}$$

Next, from (4.26) and the fact that $Q = Q^*$ and $V = V^*$, we have

$$\lambda_j^2\langle \mathbf{e}_j, \mathbf{e}_k\rangle - \lambda_j\langle V\mathbf{e}_j, \mathbf{e}_k\rangle = \lambda_k^2\langle \mathbf{e}_j, \mathbf{e}_k\rangle - \lambda_k\langle V\mathbf{e}_j, \mathbf{e}_k\rangle,$$

so that

$$\langle \mathbf{e}_j, \mathbf{e}_k\rangle = \frac{\langle V\mathbf{e}_j, \mathbf{e}_k\rangle}{\lambda_j + \lambda_k} \quad (j \neq k). \tag{4.28}$$

By combining (4.27) with (4.28), we have

$$\langle \Gamma_+\mathbf{e}_k, \mathbf{e}_j\rangle = \frac{\lambda_k}{\lambda_j + \lambda_k}\langle V\mathbf{e}_j, \mathbf{e}_k\rangle = \lambda_k\langle \mathbf{e}_k, \mathbf{e}_j\rangle \quad (j \neq k). \tag{4.29}$$

From (4.27) with $k = j$, we have

$$\langle \Gamma_+\mathbf{e}_j, \mathbf{e}_j\rangle = -\lambda_j\langle \mathbf{e}_j, \mathbf{e}_j\rangle + \langle V\mathbf{e}_j, \mathbf{e}_j\rangle. \tag{4.30}$$

Next, from (4.24),

$$\langle (I - 2W)\Gamma_-\mathbf{e}_k, \mathbf{e}_j\rangle = -\lambda_k\langle (I - 2W)\mathbf{e}_k, \mathbf{e}_j\rangle = -\lambda_k\langle \mathbf{e}_k, \mathbf{e}_j\rangle + \frac{2\lambda_k}{\lambda_j + \lambda_k}\langle V\mathbf{e}_j, \mathbf{e}_k\rangle.$$

Hence, using (4.28) and (4.29),

$$\langle (I - 2W)\Gamma_- \mathbf{e}_k, \mathbf{e}_j \rangle = \lambda_k \langle \mathbf{e}_k, \mathbf{e}_j \rangle = \langle \Gamma_+ \mathbf{e}_k, \mathbf{e}_j \rangle \quad (j \neq k),$$

and, from (4.24), and (4.30),

$$\langle (I - 2W)\Gamma_- \mathbf{e}_j, \mathbf{e}_j \rangle = -\lambda_j \langle \mathbf{e}_j, \mathbf{e}_j \rangle + \langle V\mathbf{e}_j, \mathbf{e}_j \rangle = \langle \Gamma_+ \mathbf{e}_j, \mathbf{e}_j \rangle.$$

Hence, the lemma is proved. □

We suppose that the eigenvectors $\mathbf{e}_j$ of $\Gamma_-$ are all normalized (see Note below) so that

(4.31) $$\mathbf{e}_j(0) = 1.$$

Then, since $V(0,0) = m(0)^{-1}$, we have

(4.32) $$\langle V\mathbf{e}_j, \mathbf{e}_k \rangle = 1 = \langle \mathbf{e}_j, V\mathbf{1} \rangle,$$

$\mathbf{1}$ here denoting the vector $(1, 1, \ldots, 1)^T$ on $S$.

*Note.* Suppose that $\mathbf{e}_j$ is an eigenvector of $\Gamma_-$ such that $\mathbf{e}_j(0) = 0$. Then $V\mathbf{e}_j = \mathbf{0}$, and

$$\mathbf{0} = (\lambda_j^2 I - \lambda_j V + Q)\mathbf{e}_j = (\lambda_j^2 I + Q)\mathbf{e}_j.$$

But the tri-diagonal nature of $Q$ would now imply that $\mathbf{e}_j = \mathbf{0}$. Hence the normalization (4.31) is possible. □

### 4.3.3 The key to duality

We now turn to the operator $\Pi^{-+}$ and use the notation that $\Pi^{-+} f(R_j) = \int_0^\infty \pi(j,x) f(x) dx$. To find $\pi(\cdot,\cdot)$ requires the solution of

(4.33) $$\mathcal{A} f_\lambda^-(x) = -\tfrac{1}{2}\lambda^2 \mathrm{sgn}(x) f_\lambda^-(x),$$

and we choose the normalization $(f_\lambda^-)'(0+) = 1$. This has the solution on $x > 0$ of

(4.34) $$f_\lambda^-(x) = f_\lambda^-(0)\cos\lambda x + \lambda^{-1}\sin\lambda x,$$

and on $x \leq 0$,

(4.35) $$(\lambda^2 I - Q)\mathbf{f}_\lambda^- = V\mathbf{1} \qquad (\lambda \neq 0).$$

These eigen-functions are easier to find since the function space is two dimensional and so we can specify the function's value at zero independently of its derivative.

Since $Q = Q^*$ and $\Gamma_-^2 + V\Gamma_- + Q = 0$, we have from (4.35) and (4.32), for $\lambda \neq 0$, and with $\langle \cdot, \cdot \rangle$ as the $\nu^-$ scalar product,

$$\begin{aligned} 1 = \langle \mathbf{e}_j, V\mathbf{1} \rangle &= \langle \mathbf{e}_j, (\lambda_j^2 I - Q)\mathbf{f}_\lambda^- \rangle \\ &= \langle (\lambda_j^2 I - Q)\mathbf{e}_j, \mathbf{f}_\lambda^- \rangle = \langle (\lambda_j^2 I + \Gamma_-^2 + V\Gamma_-)\mathbf{e}_j, \mathbf{f}_\lambda^- \rangle \\ &= (\lambda_j^2 + \lambda_k^2)\langle \mathbf{e}_j, \mathbf{f}_\lambda^- \rangle - \lambda_j \langle V\mathbf{e}_j, \mathbf{f}_\lambda^- \rangle = (\lambda^2 + \lambda_j^2)\langle \mathbf{e}_j, \mathbf{f}_\lambda^- \rangle - \lambda_j f_\lambda^-(0), \end{aligned}$$

whence

$$\begin{aligned} \langle \mathbf{e}_j, \mathbf{f}_\lambda^- \rangle &= \frac{\lambda_j}{\lambda^2 + \lambda_j^2} f_j^-(0) + \frac{1}{\lambda^2 + \lambda_j^2} \\ &= \int_0^\infty \mathrm{e}^{-\lambda_j y} \left\{ f_\lambda^-(0) \cos \lambda y + \lambda^{-1} \sin \lambda y \right\} \, \mathrm{d}y \\ &= \int_0^\infty \mathrm{e}^{-\lambda_j y} f_\lambda^-(y) \, \mathrm{d}y. \end{aligned}$$

But

$$\mathbf{f}^-(\cdot) = \int_0^\infty \pi(\cdot, y) f_\theta(y) \, \mathrm{d}y,$$

and we have the following theorem.

(4.36) **Theorem:** *The map $\pi(\cdot, y)$ on $S$ is characterized by*

$$\langle \mathbf{e}_j, \pi(\cdot, y) \rangle = \mathrm{e}^{-\lambda_j y},$$

*so that if $\varepsilon_j$ is the function in $(0, \infty)$ with*

$$\varepsilon_j(y) = \mathrm{e}^{-\lambda_j y},$$

*then with $(\Pi^{-+})^* : L^2(S, \nu^-) \to L^2(0, \infty)$ the Hilbert-space adjoint of $\Pi^{-+}$, we have*

$$(\Pi^{-+})^* \mathbf{e}_j = \varepsilon_j.$$

This analytic expression of the 'duality principle' really states that the *other* half winding operator $\Pi^{+-}$ is just $(\Pi^{-+})^*$. We have

$$\mathrm{sgn}(\cdot)^{-1} \mathcal{A} \varepsilon_j = +\tfrac{1}{2} \lambda_j^2 \varepsilon_j \quad \text{on } (0, \infty) \cup S,$$

if we regard $\mathbf{e}_j$ as the extension to $S$ of $\varepsilon_j$.

### 4.3.4 Completion of proof of Theorem 4.11

Suppose that $-\lambda_j$ is an eigenvalue of $\Gamma_-$ and that $\Gamma_-\mathbf{e} = -\lambda_j\mathbf{e}$. Then, by equation (4.18),

$$(\lambda_j^2 I + Q)\mathbf{e} = \lambda_j V\mathbf{e}.$$

The tri-diagonal nature of $Q$ now implies that the eigenspace of $\Gamma_-$ corresponding to eigenvalue $-\lambda_j$ is one dimensional, and so all eigenvalues of $\Gamma_-$ are distinct.

Let $W^-$ denote the full winding operator

$$W^- := \Pi^{-+}(\Pi^{-+})^*,$$

so that $W^- = (W^-)^*$ as an $S \times S$ matrix. We need only show that

$$\|W^-\|_2 = \tfrac{1}{2}.$$

Now we have, using Theorem 4.36 and equation (4.32),

$$\begin{aligned}\langle \mathbf{e}_k, W^-\mathbf{e}_j\rangle &= \langle \mathbf{e}_k, \Pi^{-+}(\Pi^{-+})^*\mathbf{e}_j\rangle_{\nu^-} = \langle (\Pi^{-+})^*\mathbf{e}_k, (\Pi^{-+})^*\mathbf{e}_j\rangle_{\nu^+} \\ &= \langle \varepsilon_k, \varepsilon_j\rangle_{\nu^+} = \int_0^\infty \mathrm{e}^{-\lambda_j y}\mathrm{e}^{-\lambda_k y}\,\mathrm{d}y = \frac{1}{\lambda_j+\lambda_k} = \frac{\langle V\mathbf{e}_j, \mathbf{e}_k\rangle}{\lambda_j+\lambda_k}.\end{aligned}$$

Since we have proved that $\Gamma_-$ has distinct eigenvalues, we can now apply Lemma 4.23 to deduce that

$$(I - 2W^-)\Gamma_- = \Gamma_+.$$

Now $\Gamma_+ \in \mathcal{Q}_0$ and so $\Gamma_+\mathbf{1} = 0$, so that the Perron-Frobenius eigenvalue of $\Gamma_+$ is 0. Thus there exists a positive row vector $\mathbf{p}$ on $S$ with $\mathbf{p}\Gamma_+ = \mathbf{0}$. Since $\Gamma_-$ is invertible, we have $\mathbf{p}(I - 2W^-) = \mathbf{0}$, and therefore

$$\mathbf{p}W^- = \tfrac{1}{2}\mathbf{p}.$$

But $W^-$ is a positive matrix, and hence its Perron-Frobenius eigenvalue is $\frac{1}{2}$. The proof of Theorem 4.11 is complete. □

### 4.3.5 A Note on a Related Pair of Processes

In [KW], the operators $\Gamma_-$ and $\Gamma_+$ occur as the Q-matrices of two different processes. In our case, those processes may be thought of as being constructed as follows. Take a Markov Chain $Y$ with Q-matrix $Q$ on the set of masses $S$. Now let time evolve as a Brownian Motion, but when the process is in state 0, the Brownian Motion has added to it

a positive drift of magnitude $m_0^{-1}$. The two processes we consider, call them $Y^+$ and $Y^-$, are constructed by setting $Y_t^+$ to be in the same state as the Markov Chain $Y$ when the Brownian Motion (plus drift when in state 0) first exceeds $t$. Similarly $Y_t^-$ is constructed by taking the state of the Chain $Y$ when the Brownian Motion first drops below $-t$. Then the Q-matrices for $Y^+$ and $Y^-$ are $\Gamma_+$ and $\Gamma_-$.

Clearly, the equation (4.25), containing as it does the Q-matrices of these two chains, must express some fact about the process. Also, we would expect the full and half winding operators for the Wiener-Hopf theory to have some interpretation in terms of these processes. This is in fact the case. These relations are not obvious, although they are fairly easily proved if correctly guessed.

Further investigation of these relations would certainly prove interesting and could also uncover more structure of these Wiener-Hopf operators.

# Chapter 5

# Atomic Measures II – No Mass at Zero

We have shown that if there is a non-zero mass at the origin then the full winding operator has norm exactly one half. A natural extension would seek the more general result for a completely arbitrary set of masses. Extensive numerical examples (together with a strong prior belief) suggest that $\|W^-\| < \frac{1}{2}$ for cases with no mass at zero. Naturally, if a mass is removed, a full winding becomes less likely and so we would expect $\|W^-\|$ to decrease. However, beyond this Wiener-Hopf question, the equation (4.25) suggests that $W^-$ must have some other probabilistic interpretation. The Kennedy-Williams paper tells us how to construct the Markov Chains with Q-matrices $\Gamma_+$ and $\Gamma_-$ and we have proved the algebraic relationship (4.25) between them.

We prove that $\|W^-\| < \frac{1}{2}$ and also find an equation analogous to (4.25) although it should be noted that the [KW] results do not apply here. In particular, $\Gamma_+$ is not necessarily a Q-matrix.

## 5.1 Mass at the Origin

To recapitulate last chapter's notation, we are given masses $m_1, m_2, \ldots, m_n$, distances $d_0, d_1, \ldots, d_{n-1}$ and $\varepsilon > 0$. Place a mass $\varepsilon$ at the origin. Define $M$, the diagonal matrix of masses with entries $\varepsilon, m_1, m_2, \ldots, m_n$ and the Q-matrix $Q := M^{-1}C$, where $C$ is the

symmetric matrix

$$C := \begin{pmatrix} -c_0 & c_0 & 0 & . & 0 & 0 & 0 \\ c_0 & -(c_0+c_1) & c_1 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & c_{n-2} & -(c_{n-2}+c_{n-1}) & c_{n-1} \\ 0 & 0 & 0 & . & 0 & c_{n-1} & -c_{n-1} \end{pmatrix} \tag{5.1}$$

and $c_k := d_k^{-1}$ $(1 \le k \le n-1)$. Let $V$ be the $n+1$ by $n+1$ matrix with

$$V_{00} := -1/\varepsilon, \quad V_{ij} := 0 \text{ for all other pairs } (i,j). \tag{5.2}$$

We proceed as before, with the single change of notation that $m_0 = \varepsilon$ and solve the equation

$$(Q+\lambda^2)\begin{pmatrix} 1 \\ \mathbf{e}_\lambda \end{pmatrix} = \begin{pmatrix} \lambda/\varepsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{5.3}$$

where $\mathbf{e}_\lambda$ is a vector of size $n$, and obtain $n+1$ positive and $n+1$ non-positive values for $\lambda$ and the corresponding vectors $\mathbf{e}_\lambda$. Let us order the solutions $\lambda_{2n+1} < \ldots < \lambda_{n+1} = 0 < \lambda_0 < \ldots < \lambda_n$. It is clear from the differential equation leading to (5.3) that if $\lambda(\varepsilon) \to \lambda < \infty$ as $\varepsilon$ tends to zero then the first component of $\mathbf{e}_\lambda$ tends to $1 + \lambda d_0$.

We use these vectors to define the matrices $\Gamma_+$ and $\Gamma_-$, which we know to be Q-matrices;

$$\lambda \le 0 \quad \Rightarrow \quad \Gamma_+ \begin{pmatrix} 1 \\ \mathbf{e}_\lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \mathbf{e}_\lambda \end{pmatrix} \tag{5.4}$$

$$\lambda > 0 \quad \Rightarrow \quad \Gamma_- \begin{pmatrix} 1 \\ \mathbf{e}_\lambda \end{pmatrix} = -\lambda \begin{pmatrix} 1 \\ \mathbf{e}_\lambda \end{pmatrix}. \tag{5.5}$$

Now construct the $n+1$ by $n+1$ matrices $K$ and $J$ whose columns consist of the vectors corresponding to the positive and the non-positive solutions $\lambda$ respectively. Let $\Pi := K^{-1}$. The probability distribution on the masses after a half winding when the process is started at a point $x > 0$ is given by $(\exp(-\lambda_0 x)\ \exp(-\lambda_1 x)\ \ldots\ \exp(-\lambda_n x))\Pi$.

Apart from the definition of $J$, this is all as defined in the previous chapter.

## 5.2 No Mass at the Origin

We define corresponding matrices in $n$ dimensions and use a tilde to denote that we are in the case with no mass at the origin. Thus $\widetilde{M}$ is a diagonal matrix with entries $m_1, m_2, \ldots, m_n$, and $\widetilde{Q}$ is an $n$ by $n$ matrix defined using $d_1, d_2, \ldots, d_{n-1}$.

The problem in the case of no mass at the origin leads us to require the solution of

$$(\widetilde{Q} + \lambda^2)\mathbf{e}_\lambda = \begin{pmatrix} \lambda/m_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{5.6}$$

subject to $\mathbf{e}_\lambda$ having first element $1 + \lambda d_0$ and $\lambda \neq 0$. This condition on the first element of the eigen-vector is the only place where $d_0$ occurs.

## 5.3 Matrix Limits

### 5.3.1 Finite solutions as $\varepsilon \to 0$

Suppose that we have a solution to (5.3) given by $\lambda(\varepsilon)$ and $\mathbf{e}_\lambda(\varepsilon)$ which tend to finite limits $\lambda$ and $\mathbf{e}_\lambda$ as $\varepsilon \to 0$. Then we can easily show that $\lambda$ and $\mathbf{e}_\lambda$ also give a solution to (5.6). We use these solutions to form the matrices $\widetilde{K}$ and $\widetilde{J}$ which are related to $K$ and $J$ via

$$K = \left(\begin{array}{c|c} 1 \cdots 1 & 1 \\ \hline \widetilde{K} & w \end{array}\right) \quad J = \left(\begin{array}{c|c} 1 & 1 \cdots 1 \\ \hline 1 & \\ \vdots & \widetilde{J} \\ 1 & \end{array}\right), \tag{5.7}$$

for some vector $w$ and define $\widetilde{\Pi} := \widetilde{K}^{-1}$. Now define $\widetilde{\Gamma}_-$ to be the matrix with eigen-vectors $\mathbf{e}_\lambda$ corresponding to eigen-values $-\lambda < 0$ and similarly for $\widetilde{\Gamma}_+$ with $\lambda \leq 0$.

### 5.3.2 The Unbounded Solution

Using the equation obtained from (5.3), we find that there is a solution $\lambda$ which is equal to $1/\varepsilon + 1/d_0 - \varepsilon/d_0^2 + O(\varepsilon^2)$. From this we can deduce that the corresponding eigen-vector

is equal to

$$\mathbf{e}_\lambda = \begin{pmatrix} 1 \\ -\varepsilon^2/m_1 d_0 + O(\varepsilon^3) \\ \varepsilon^4/m_1 m_2 d_0 d_1 + O(\varepsilon^5) \\ \vdots \end{pmatrix} \tag{5.8}$$

which in turn allows us to deduce that, for some vector $u$ independent of $\varepsilon$, the structure of $\Gamma_-$ is

$$\Gamma_- = \left(\begin{array}{c|c} -1 & u^T \\ \hline 0 & \\ 0 & 0 \\ \vdots & \\ 0 & \end{array}\right)\varepsilon^{-1} + \left(\begin{array}{c|c} -d_0^{-1} & O(1) \\ \hline 0 & \\ 0 & \widetilde{\Gamma}_- \\ \vdots & \\ 0 & \end{array}\right) + \left(\begin{array}{c|c} O(\varepsilon) & 0 \cdots 0 \\ \hline O(\varepsilon) & \\ O(\varepsilon^2) & O(\varepsilon) \\ \vdots & \\ O(\varepsilon^2) & \end{array}\right). \tag{5.9}$$

For all solutions bounded as $\varepsilon \to 0$, we therefore require that $u^T \mathbf{e}_\lambda = 1$ and so

$$u^T = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \widetilde{\Pi}, \quad u_j = \mathbb{P}_{0+}\{\text{half wind to } j^{th} \text{ mass}\} \tag{5.10}$$

where $\mathbb{P}_{0+}$ indicates that we start the process at $x > 0$ and take the limit as $x$ tends to zero. There is a discontinuity of the behaviour at 0. For all $x > 0$, $\phi_t$ is positive at small times, and for $x = 0$, $\phi_t$ is negative at small times.

Using the relations $\Gamma_+ - \Gamma_-^* = V$ and $\Gamma_-^* \Gamma_+ = -Q$ and the structure already obtained, we find that

$$\Gamma_- = \left(\begin{array}{c|c} -1 & u^T \\ \hline 0 & \\ 0 & 0 \\ \vdots & \\ 0 & \end{array}\right)\varepsilon^{-1} + \left(\begin{array}{c|c} -d_0^{-1} & v^T \\ \hline 0 & \\ 0 & \widetilde{\Gamma}_- \\ \vdots & \\ 0 & \end{array}\right) +$$

$$+ \left(\begin{array}{c|c} k_1 & 0 \cdots 0 \\ \hline 1/m_1 d_0 & \\ 0 & O(1) \\ \vdots & \\ 0 & \end{array}\right)\varepsilon + O(\varepsilon^2) \tag{5.11}$$

where $k_1 = -(m_1 d_0)^{-1}(1\ 0 \cdots 0)u + 1/d_0^2$ and $v$ is independent of $\varepsilon$. We also find that $u^T(\widetilde{\Gamma}_- - d_0^{-1}) = (-d_0^{-1}\ 0 \cdots 0)$.

From this representation of $\Gamma_-$, we can prove the following lemma.

(5.12) **Lemma:** *The determinant of* $\widetilde{\Gamma}_-$ *is non-zero.*

*Proof:* Since the matrix $\Gamma_-$ is symmetric, its determinant is given by the product of its eigen-values. As $\varepsilon \to 0$, all but one of these eigen-values are bounded away from 0 and from $\infty$ and the remaining one is of order $\varepsilon^{-1}$. Therefore the determinant of $\Gamma_-$ is of order $\varepsilon^{-1}$.

Computing the order of the determinant again using (5.9), we see that the only way in which a term of order $\varepsilon^{-1}$ could arise is by the multiplication of $-\varepsilon^{-1}$ and det $\widetilde{\Gamma}_-$. Therefore det $\widetilde{\Gamma}_-$ must be non-zero. □

We can immediately write down the result that

$$\Gamma_+ = \Gamma_-^* + V = \left(\begin{array}{c|c} -d_0^{-1} & d_0^{-1}\,0\,\cdots\,0 \\ \hline \widetilde{M}^{-1}u & \widetilde{\Gamma}_-^* \end{array}\right) + O(\varepsilon) \tag{5.13}$$

and hence for $\lambda < 0$,

$$\Gamma_+ \begin{pmatrix} 1 \\ \mathbf{e}_\lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ \widetilde{\Gamma}_+ \mathbf{e}_\lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ \widetilde{M}^{-1}u + \widetilde{\Gamma}_-^* \mathbf{e}_\lambda \end{pmatrix} + O(\varepsilon) \tag{5.14}$$

and, using $(1\ \cdots\ 1)\widetilde{J}^{-1}\mathbf{e}_\lambda = (1)$, $\widetilde{\Gamma}_+\mathbf{e}_\lambda = \widetilde{\Gamma}_-^*\mathbf{e}_\lambda + \widetilde{M}^{-1}u = (\widetilde{\Gamma}_-^* + \widetilde{M}^{-1}u(1\ \cdots\ 1)\widetilde{J}^{-1})\mathbf{e}_\lambda$ and so

$$\widetilde{\Gamma}_+ = \widetilde{\Gamma}_-^* + \widetilde{M}^{-1}u(1\ \cdots\ 1)\widetilde{J}^{-1}. \tag{5.15}$$

## 5.4 The Full Winding Matrix

We know that $K$ has the structure

$$K = \left(\begin{array}{c|c} 1\ \cdots\ 1 & 1 \\ \hline & O(\varepsilon^2) \\ \widetilde{K} & O(\varepsilon^4) \\ & \vdots \end{array}\right) \tag{5.16}$$

and thus we can deduce that its inverse $\Pi$ has the structure

$$\Pi = \left(\begin{array}{c|c} 0 & \\ \vdots & \widetilde{\Pi} \\ 0 & \\ \hline 1 & -u^T \end{array}\right) + O(\varepsilon). \tag{5.17}$$

This makes probabilistic sense. It says that for a very small mass at the origin, the process started at 0 will half wind round to the mass at the origin with probability approaching one and to the other masses with probability approaching zero. Also, when the process is started away from 0, the probability of half winding to the mass at the origin is almost zero and the probabilities of hitting the other masses are close to the probabilities in the case with no mass at the origin. The process is only affected by the mass at the origin when it is at the origin. Starting the process at the origin here is shorthand for "start the process at $x$ and take the result as $x \to 0$." The origin is in $E^-$ and not in $E^+$.

Knowing the structure of $\Pi$ and the value of the unbounded $\lambda$ allows us to write down the structure of the full winding matrix $W^-$ as

$$W = \left(\begin{array}{c|c} 1/2 & u^T/2 \\ \hline \frac{1}{2}\varepsilon\widetilde{M}^{-1}u & \widetilde{W}^- \end{array}\right) + \text{higher orders} \tag{5.18}$$

which again makes probabilistic sense. Given a single mass at zero, the probability of completing a winding from zero to zero is one half. As the mass is reduced, the size of the excursion from the origin will also reduce. Therefore, for a small mass at the origin, the presence of the other masses will have little significance and we would expect $W^-_{00} \to \frac{1}{2}$. Given that it does not perform an almost immediate winding back to zero (which occurs with probability one half), the process restarts close to zero, where the chance of hitting the other masses is given by the vector $u$. Hence the top row of $W^-$. The left hand column is completed from the fact that $W^-$ is self-adjoint with respect to $M$. Probabilities of windings between other masses will tend to the values they take when the mass at the origin is absent.

*Note:* That the size of the winding when one mass is present tends to zero as $\varepsilon$ tends to zero is strongly implied by the form of $\Pi^{-+}$ in Chapter 2. With a small number of implicit assumptions, we can show that the expected time of the first winding (when it occurs) tends to zero. We can scale the reflected Brownian Motion so that $\tilde{B}_t := c^2 B(t/c)$ and place a mass $\tilde{\varepsilon} := c^2\varepsilon$ at the origin. Then the local time at 0 for the scaled Brownian Motion is given by $\tilde{\ell}(t) = c^{-1}\ell(t/c)$. Then if $T$ and $\tilde{T}$ are the random times of the first winding (possibly infinite) then

$$\begin{aligned} T &= \inf_{t>0}\{t - \varepsilon\ell_t = 0\} = \inf_{t>0}\left\{t - c^{-1}\tilde{\varepsilon}\tilde{\ell}(ct) = 0\right\} \\ &= \inf_{t>0}\left\{ct - \tilde{\varepsilon}\tilde{\ell}(ct) = 0\right\} = c^{-1}\tilde{T}. \end{aligned}$$

Therefore if we let $f(x) := \mathbb{E}\,[T|T < \infty]$ when $\varepsilon = x$, then $cf(x) = f(c^2x)$. Therefore $f(0+) = \lim_{n\to\infty}(2^{-2n}x) = \lim_{n\to\infty} 2^{-n}f(x) = 0$. This assumes that $f$ is finite some-

where. If we assume continuity of $f$ then we have that $f(x) = f(1)\sqrt{x}$.

It is now clear that the following theorem holds.

(5.19) **Theorem:** *For any finite arrangement of masses on* $\mathbb{R}^- \setminus \{0\}$,

$$\|W^-\| < \tfrac{1}{2}.$$

*Proof:* The result follows from the fact that eigen-values are continuous functions of the entries of a matrix. As $\varepsilon$ tends to zero, the matrix $W^-$ tends to

$$W_0 := \lim_{\varepsilon \to 0} W^- = \left(\begin{array}{c|c} 1/2 & u^T/2 \\ \hline 0 & \widetilde{W}^- \end{array}\right).$$

Let us write $\theta_0, \theta_1, \ldots, \theta_n$ for the eigen-values of this matrix. The eigen-value with the largest absolute value is $\theta_0 = \frac{1}{2}$ and the remaining eigen-values $\theta_1, \theta_2, \ldots, \theta_n$ are the eigen-values of $\widetilde{W}^-$. These all have modulus less than or equal to one half by continuity. We now show that for $j > 0$, $|\theta_j|$ is strictly less than one half.

Suppose that $\widetilde{W}^-$ has eigen-value $\theta = \frac{1}{2}$. Then it has at least one eigen-vector satisfying $\widetilde{W}^-\mathbf{v} = \frac{1}{2}\mathbf{v}$. We use the result of Theorem 5.20, the proof of which does not depend on this result, to say that for this vector $\mathbf{v}$, $\widetilde{\Gamma}_+(I - 2\widetilde{W}^-)\mathbf{v} = 0$ and hence $\widetilde{\Gamma}_-\mathbf{v} = 0$ and the determinant of $\widetilde{\Gamma}_-$ is therefore zero. This is a contradiction of Lemma 5.12. Therefore $\frac{1}{2}$ is not an eigen-vector of $\widetilde{W}^-$.

Furthermore, $\widetilde{W}^-$ is a strictly positive matrix and so has a Perron-Frobenius eigen-value, that is, a positive eigen-value which is strictly greater than the absolute value of all the other eigen-values (see Seneta (1981)). Since this eigen-value must be in the range $(0, \frac{1}{2})$ we have $|\theta_j| < \frac{1}{2}$ for $j > 0$. □

If we normalize the first eigen-vector such that $[\mathbf{v}_0]_0 = 1$ we have that $\frac{1}{2}u^T + \tilde{\mathbf{v}}_0^T\widetilde{W}^- = \frac{1}{2}\tilde{\mathbf{v}}_0^T$, or that $u^T = \tilde{\mathbf{v}}_0^T(I - 2\widetilde{W}^-)$. Inverting the ubiquitous $(I - 2\widetilde{W}^-)$ as a power series, the equation $u^T(I - 2\widetilde{W}^-)^{-1} = \tilde{\mathbf{v}}_0^T$, says that the $i$th entry $[\tilde{\mathbf{v}}_0]_i$ is the sum

$$\sum_{k=0}^{\infty} 2^k \mathbb{P}_0\{\text{Pass through } i \text{ after } k \text{ windings and a half}\}.$$

We finish the work on atomic measures by proving this analogue to (4.25).

(5.20) **Theorem:** *In contrast to the case for mass at the origin,*

$$\tilde{\Gamma}_+(I - 2\widetilde{W}^-) = \tilde{\Gamma}_-. \tag{5.21}$$

*Proof:* First note that $(1\ 0\ \cdots\ 0)J = (1\ \cdots\ 1)$ and $(1\ 0\ \cdots\ 0)(I - 2W^-) = (0\ \ -u^T)$ to order $\varepsilon$. From this we can deduce that

$$(1\ \cdots\ 1)\tilde{J}^{-1}(I - 2\widetilde{W}^-) = -u^T. \tag{5.22}$$

We know, in the case with mass at the origin, that

$$(I - 2W^-)\Gamma_- = \Gamma_+ \tag{5.23}$$

or, equivalently, that

$$\Gamma_+(I - 2W^-) = \Gamma_- + 2V(I - W^-). \tag{5.24}$$

The left hand side of this is equal, to order $\varepsilon$, to

$$\text{LHS} = \left(\begin{array}{c|c} 0 & d_0^{-1}u^T + \begin{pmatrix} d_0^{-1} & 0 & \cdots & 0 \end{pmatrix}(I - 2\widetilde{W}^-) \\ \hline 0 \\ \vdots & -\widetilde{M}^{-1}uu^T + \tilde{\Gamma}_-^*(I - 2\widetilde{W}^-) \\ 0 \end{array}\right) = \Gamma_- + 2V(I - W^-)$$

and so $-\widetilde{M}^{-1}uu^T + \tilde{\Gamma}_-^*(I - 2\widetilde{W}^-) = \tilde{\Gamma}_-$ or, from (5.22) and (5.15),

$$\left(\widetilde{M}^{-1}u(1\ \cdots\ 1)\tilde{J}^{-1} + \tilde{\Gamma}_-^*\right)(I - 2\widetilde{W}^-) = \tilde{\Gamma}_+(I - 2\widetilde{W}^-) = \tilde{\Gamma}_-. \tag{5.25}$$

□

This also tells us that the $v^T$ in (5.11) is equal to $d_0^{-1}(u^T + (1\ 0\ \cdots\ 0)(I - 2\widetilde{W}^-))$.

# Chapter 6

# Other Winding Operator Results

In this chapter we present results for other cases where the norms of the half winding operators can be calculated explicitly. Proofs and derivations of the half winding operators can be found in Rogers & Williams (1984).

We begin with a theorem from Toland and Williams (1998).

(6.1) **Theorem:** *Consider Brownian Motion on* $\mathbb{R}$ *time changed in the usual way with the measure*

$$\nu^+ = \text{Leb on } (0,\infty); \quad \nu^- = c^2 \times \text{Leb on } [-1,0], \quad \nu^- = 0 \text{ on } (-\infty,-1)$$

*and wound about the origin via* $V(x) = \text{sgn}(x)$. *Then*

$$\|\Pi^{-+}\|_2 = \begin{cases} \dfrac{\sqrt{2c}}{1+c} & \text{if } c \leq 1, \\ 2^{-\frac{1}{2}} & \text{if } c \geq 1. \end{cases} \tag{6.2}$$

The winding process is given by Brownian Motion reflected at $-1$, with time below the origin scaled by $c$ and travelling in the negative direction. The maximum value of the norm occurs when $c \geq 1$. Thus we have the upper bound of $1/\sqrt{2}$ for Brownian Motion with a reflecting barrier at $-1$.

Note that we can use Brownian scaling to deduce that the operators have norms given by (6.2) when $\nu^- = c^2 \times$Leb on *any* finite interval $[-k,0]$. We can do this by scaling the Brownian Motion and using the idea that for a fixed Brownian path, the value of $\Pi^{-+}f(x)$ is unaffected for any dilation of time. We scale both time and space to transform reflecting Brownian Motion on $[-1,\infty)$ into reflecting Brownian Motion on $[-k,\infty)$ by defining

$\tilde{B}_t = kB(k^{-2}t)$. Using tildes to denote objects relating to the scaled Brownian Motion, we have $\tilde{\phi}_t = k^2\phi(k^{-2}t)$ and $\tilde{\tau}_0^+ = k^2\tau_0^+$. Therefore if $\tilde{f}(kx) = f(x)$, then

$$\begin{aligned}\tilde{\Pi}^{-+}\tilde{f}(kx) &= \widetilde{\mathbb{E}}_{kx}\tilde{f}(\tilde{B}(\tilde{\tau}_o^+)) = \widetilde{\mathbb{E}}_{kx}\tilde{f}(\tilde{B}(k^2\tau_o^+)) \\ &= \mathbb{E}_x\tilde{f}(kB(\tau_0^+)) = \mathbb{E}_x f(B(\tau_0^+)) = \Pi^{-+}f(x).\end{aligned}$$

Then, from the fact that $\|\tilde{f}\|_2 = \sqrt{k}\|f\|_2$, we have $\|\Pi^{-+}\|_2 = \|\tilde{\Pi}^{-+}\|_2$.

(6.3) **Proposition:** *For the Canonical Case of Brownian Motion wound about the origin via $V(x) = \mathrm{sgn}(x)$, the half winding operators, given in Chapter 3, have norm $1/\sqrt{2}$.*

*Proof:* We use a specific case of the result in Theorem 319 from Hardy, Littlewood and Pólya (1934). Suppose that $K(x,y)$ is non-negative and homogeneous of degree $-1$ so that $K(x,y) = aK(ax,ay)$ for $a \in \mathbb{R}$ and $K(x,y) = K(y,x)$. Then

$$\int_0^\infty K(x,1)x^{-\frac{1}{2}}dx = \int_0^\infty K(1,x^{-1})x^{-\frac{1}{2}}\frac{dx}{x} = \int_0^\infty K(1,y)y^{\frac{1}{2}}\frac{dy}{y} = \int_0^\infty K(1,y)y^{-\frac{1}{2}}dy$$

and writing $K(r) := K(r,1)$, the H-L-P result says that the map

$$f(x) \mapsto \int_0^\infty K(r)f(rx)dr \tag{6.4}$$

on $L^2(\mathbb{R}^+)$ has norm

$$\int_0^\infty K(r)r^{-1/2}dr.$$

We let

$$K(r) := \Pi^{+-}(r,1) = \frac{\sqrt{2r}}{\pi(1+r^2)}$$

so that the operator $\Pi^{+-}$ is given by the mapping (6.4). Then we can calculate the norm to be

$$\|\Pi^{+-}\|_2 = \int_0^\infty K(r)r^{-1/2}dr = \frac{\sqrt{2}}{\pi}\left[\tan^{-1}r\right]_0^\infty = 2^{-1/2}$$

as required. □

(6.5) **Proposition:** *For the case above, but with $\nu^- = c^2\times\mathrm{Leb}$ on $\mathbb{R}^-$ for some $c > 0$,*

$$\|\Pi^{+-}\|_2 = \frac{\sqrt{2c}}{c+1}.$$

This case, which reduces to the previous on setting $c = 1$, has half winding density given by the kernel

$$\Pi^{+-}(x,y) := \frac{2c^\alpha x^\alpha |y|^{1-\alpha}\sin(\pi\alpha/2)}{\pi(c^2x^2+y^2)}$$

for $x > 0$ and $y \le 0$ and where $c^{-1} = \tan(\pi\alpha/2)$.

This defines a mapping from $L^2(\mathbb{R}^+,\text{Leb})$ to $L^2(\mathbb{R}^-, c^2\text{Leb})$ and so to find the norm we can set $K(r) = \Pi^{+-}(1,r)$ as previously and then evaluate $\int_0^\infty cK(r)r^{-1/2}dr$. Thus, using the notation and result of Lemma 6.7,

$$\begin{aligned}
\|\Pi^{+-}\|_2 &= \int_0^\infty cK(r)r^{-1/2}dr \\
&= 2\pi^{-1}c^{1+\alpha}\sin\frac{\pi\alpha}{2}\int_0^\infty \frac{r^{\frac{1}{2}-\alpha}}{(c^2+r^2)}dr \\
&= 2\pi^{-1}c^{1+\alpha}\mathcal{I}(2,\tfrac{1}{2}-\alpha,c)\sin\frac{\pi\alpha}{2} \\
&= \left(\frac{\sqrt{c}}{\cos\frac{\pi}{2}(\frac{1}{2}-\alpha)}\right)\sin\frac{\pi\alpha}{2} \\
&= \frac{\sqrt{2c}}{\cot\frac{\pi\alpha}{2}+1} = \frac{\sqrt{2c}}{c+1}.
\end{aligned}$$

□

Note that duality of the half winding operators implies that $\|\Pi^{-+}\|_2 = \|\Pi^{+-}\|_2$. For the norms of these operators, interchanging $\mathbb{R}^+$ and $\mathbb{R}^-$ is equivalent to performing the mapping $c \mapsto c^{-1}$. In each case, the measure on $\mathbb{R}^-$ has density equal to $c^{-1}$ times the density of the measure on $\mathbb{R}^+$. For this reason, the norm of the operator is invariant under $c \mapsto c^{-1}$.

Before giving one last case and proving the contour integration result, we compare our results.

| $\nu^-$ | | $\nu^+$ | norm |
|---|---|---|---|
| $c^2\times$Leb on $[-k,0]$ | $0 < c \le 1$ | Leb on $(0,\infty)$ | $\sqrt{2c}/(1+c)$ |
| $c^2\times$Leb on $[-k,0]$ | $c \ge 1$ | Leb on $(0,\infty)$ | $1/\sqrt{2}$ |
| $c^2\times$Leb on $(-\infty,0]$ | $c > 0$ | Leb on $(0,\infty)$ | $\sqrt{2c}/(1+c)$ |
| Finite number of atoms including one at zero. | | Leb on $(0,\infty)$ | $1/\sqrt{2}$ |

We see that the norms do not behave continuously as limits are taken. If we attempted to use the results of Chapter 4 to approximate Lebesgue measure on $[-k,0]$ then the limit of the matrix norm gives the $L^2$ norm only when $c \ge 1$. Similarly, as the interval $[-k,0]$ increases towards $(-\infty,0]$, the limit of the norm is the norm of the limit only when $c \le 1$.

(6.6) **Proposition:** *For the measure*

$$\nu(dx) := \begin{cases} x^\beta dx & x > 0 \\ c^{2+\beta}x^\beta dx & x \le 0 \end{cases}$$

*where $\beta > -1$, the norm of the half winding operator is given by*

$$\|\Pi^{-+}\|_2 = \|\Pi^{+-}\|_2 = \frac{2\sqrt{c}\sin\frac{\delta}{2}}{1+c}$$

*where $\delta = \pi/(2+\beta)$, $\alpha \in (0,1)$, $\sin([1-\alpha]\delta) = c\sin\alpha\delta$ and $C\delta = c^\alpha \sin\alpha\delta$.*

*Proof:* The operator $\Pi^{-+}$ is a map from $L^2(\mathbb{R}^+, y^\beta dy)$ to $L^2(\mathbb{R}^-, c^{2+\beta}x^\beta dx)$. Define the operator $H$ by

$$Hf = c^{\frac{1}{2}\beta+1}x^{\frac{1}{2}\beta}\Pi^{-+}(x^{-\frac{1}{2}\beta}f)$$

as a map from $L^2(\mathbb{R}^+, dy)$ to $L^2(\mathbb{R}^-, dx)$. Since the maps $f \mapsto x^{-\beta/2}f$ from $L^2(\mathbb{R}^-, dx)$ to $L^2(\mathbb{R}^+, y^\beta dy)$ and $f \mapsto c^{\beta/2+1}x^{\beta/2}f$ from $L^2(\mathbb{R}^-, c^{2+\beta}x^\beta dx)$ to $L^2(\mathbb{R}^-, dx)$ are isometries, the norm of $\Pi^{+-}$ and the norm of $H$ must be equal.

The kernel for the operator $\Pi^{-+}$ is given by

$$\pi(x,y) = \frac{Cx^\alpha y^{1-\alpha+\beta}}{(cx)^{2+\beta} + y^{2+\beta}}$$

and so

$$\begin{aligned} Hf(x) &= c^{\frac{1}{2}\beta+1}x^{\frac{1}{2}\beta}\int_0^\infty \pi(x,y)y^{-\frac{1}{2}\beta}f(y)dy \\ &= Cc^{\frac{1}{2}\beta+1}\int_0^\infty \frac{r^{\frac{1}{2}\beta+1-\alpha}}{c^{2+\beta}+r^{2+\beta}}f(rx)dr = \int_0^\infty K(r)f(rx)dr \end{aligned}$$

where

$$K(r) := \frac{Cc^{\frac{1}{2}\beta+1}r^{1-\alpha+\frac{1}{2}\beta}}{c^{2+\beta}+r^{2+\beta}}.$$

Therefore by the H-L-P result, the norm is equal to

$$\|\Pi^{-+}\|_2 = \|H\|_2 Cc^{\frac{1}{2}\beta+1}\int_0^\infty \frac{r^{\frac{1}{2}-\alpha+\frac{1}{2}\beta}}{c^{2+\beta}+r^{2+\beta}}dr = Cc^{\frac{1}{2}\beta+1}\mathcal{I}(2, \tfrac{1}{2}-\alpha\tfrac{1}{2}\beta, c).$$

Then since $\mathcal{I}(2, \frac{1}{2}-\alpha\frac{1}{2}\beta, c) = \delta c^{-h-\frac{\beta}{2}-\alpha}\mathrm{cosec}(\delta(\frac{3}{2}-\alpha+\frac{\beta}{2}))$ and $(\beta+2)\delta = \pi$, the norm is equal to

$$\frac{\sqrt{c}\sin\alpha\delta}{\sin(\frac{\beta+2}{2}+\frac{1}{2}-\alpha)\delta} = \frac{\sqrt{c}\sin\alpha\delta}{\cos(\alpha-\frac{1}{2})\delta}.$$

Since $c\sin\alpha\delta = \sin(\delta-\alpha\delta)$, trigonometric identities give us $c = 2\sin\frac{\delta}{2}\cos\frac{\delta}{2}\cot\alpha\delta - \cos^2\frac{\delta}{2} + \sin^2\frac{\delta}{2}$ and so $c+1 = (2\sin\frac{\delta}{2})\cos\frac{\delta}{2}\cot\alpha\delta + \sin\frac{\delta}{2}$. Then we use this result with the fact that $\cos(\alpha-\frac{1}{2})\delta = \sin\alpha\delta(\cot\alpha\delta\cos\frac{\delta}{2} + \sin\frac{\delta}{2})$ to finish the proof:

$$\|H\|_2 = \frac{\sqrt{c}\sin\alpha\delta}{\cos(\alpha-\frac{1}{2})\delta}$$

$$= \frac{\sqrt{c}}{\cot \alpha\delta \cos \frac{\delta}{2} + \sin \frac{\delta}{2}}$$
$$= \frac{2\sqrt{c} \sin \frac{\delta}{2}}{c+1}.$$

□

(6.7) **Lemma:** *For $c > 0$ and $\alpha > \beta + 1 > 0$,*

$$\mathcal{I}(\alpha, \beta, c) := \int_0^\infty \frac{r^\beta}{c^\alpha + r^\alpha} dr = \frac{\pi c^{\beta+1-\alpha}}{\alpha \sin \frac{\pi}{\alpha}(\beta+1)}.$$

*Proof:* First, to find $\mathcal{I}(2, \beta, c)$, define $f(r)$ continuously in the complex plane $\mathbb{C}$ minus a cut from 0 into the lower half plane by

$$f(r) = \frac{r^\beta}{c^2 + r^2}.$$

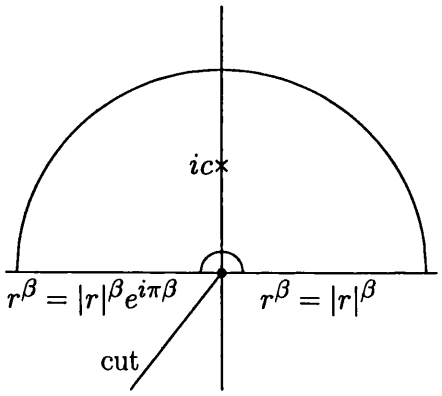



Figure 6-1: Contour Integration

The chosen branch of $r^\beta$ is shown in Figure 6-1. This function has poles at $r = \pm ic$ and the residual value at $ic$ is given by

$$\lim_{r \to ic} \frac{(r - ic) r^\beta}{c^2 + r^2} = \lim_{r \to ic} \frac{(\beta+1) r^\beta - \beta i c r^{\beta-1}}{2r}$$
$$= \frac{(ic)^\beta}{2ic}$$
$$= \frac{1}{2i} c^{\beta-1} e^{i\beta\pi/2}.$$

Therefore using contour integration over the semi-circle in Figure 6-1 we obtain $(1 + e^{i\pi\beta})\, \mathcal{I}(2, \beta, c) = \pi c^{\beta-1} e^{i\beta\pi/2}$ and hence

$$\mathcal{I}(2, \beta, c) = \frac{\pi c^{\beta-1}}{2 \cos \frac{\pi\beta}{2}}.$$

Now for general $\alpha$, transform $r$ and $c$ to $s$ and $d$ via $r^\alpha = s^2$ and $c^\alpha = d^2$. Then using the previous result, we prove the lemma,

$$\begin{aligned}
\mathcal{I}(\alpha, \beta, c) &= \frac{2}{\alpha} \mathcal{I}\left(2, \frac{2}{\alpha}(\beta+1) - 1, d\right) \\
&= \frac{2\pi}{2\alpha} \cdot \frac{d^{\frac{2}{\alpha}(\beta+1-\alpha)}}{\cos\frac{\pi}{2}[\frac{2}{\alpha}(\beta+1)-1]} \\
&= \frac{\pi c^{\beta+1-\alpha}}{\alpha \sin\frac{\pi}{\alpha}(\beta+1)}.
\end{aligned}$$

□

# Chapter 7

# A Connection with Other Work

In the closing chapter of Part I we look briefly at a probabilistic factorization that arises from consideration of a random walk. It is this that is most commonly referred to as a Wiener-Hopf factorization and is closely linked to our wound processes. In this way, we connect our results to a larger body of work, further details of which can be found in Kennedy (1994) and (1997), Asmussen (1998) and Feller (1971).

Let $Y_1, Y_2, \ldots$ be independent random variables, each with distribution $F(dy)$ on $\mathbb{R}$ having some finite mean $\mu$. Assume $0 < F(0) < 1$. Then define $S_0 := 0$ and $S_n := \sum_{i=1}^{n} Y_i$ for $n = 1, 2, \ldots$. This is the random walk associated with the distribution $F$ as shown in Figure 7-1.

Consider the stopping times $\tau_+$ and $\tau_-$, known as the first strict ascending and first weak descending ladder times, given by

$$\tau_+ := \inf\{n > 0 : S_n > 0\} \quad \text{and} \quad \tau_- := \inf\{n > 0 : S_n \leq 0\}.$$

That the descending ladder times are weak is equivalent to our convention that $0 \in E^-$. Denote the distributions of the random values $S(\tau_+)$ and $S(\tau_-)$ by $F_+$ and $F_-$. Then the following Wiener-Hopf factorization holds—

$$(\delta_0 - F) = (\delta_0 - F_+) * (\delta_0 - F_-) \tag{7.1}$$

where $\delta_0$ denotes a unit mass at the origin and $*$ is the convolution of the two distributions.

It is clear that there is some relationship between this and the processes and stopping times we have so far considered. The ladder times look very much like the half winding times when the process is started at 0. In both cases we consider the first time at which

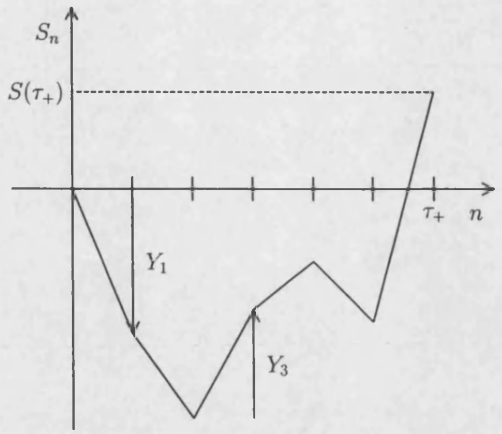



Figure 7-1: The Random Walk $S_n$ with Jump Distribution $F$

a fluctuating additive functional first becomes either positive or negative and find the distribution of the process at that time. We now make this connection precise.

Define a right continuous Markov process $X$ starting at 0 which, while away from the origin drifts back towards it at rate 1 and upon approaching the origin jumps away according to the distribution $F$.

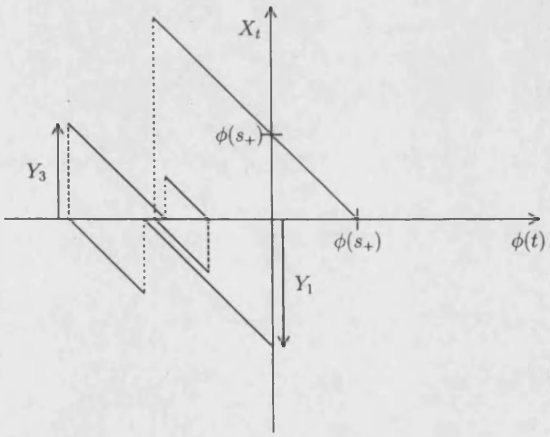



Figure 7-2: A Simple Winding Process

Let $Y$ be a random variable with distribution $F$. The process we have defined then has generator $\mathcal{A}$ given by

$$\mathcal{A}f(x) = \begin{cases} -f'(x) & x > 0 \\ \mathbb{E}\,\mathrm{sgn}(Y) f'(Y-) & x = 0 \\ +f'(x) & x < 0 \end{cases} \tag{7.2}$$

on the domain $C^1(\mathbb{R}^+) \times \mathbb{C}^1(\mathbb{R}^-)$ subject to the two conditions $f(0+) = f(0-)$ and $\mathbb{E}\, f(Y) = f(0)$. We see again the two conditions on the domain, which will correspond

via the usual kind of isomorphism to two different conditions, one each on the domains of $G^+$ and $G^-$.

Now change time as usual using the definitions $E^+ := (0, \infty)$, $E^- := (-\infty, 0]$ and $V(x) := \text{sgn}(x)$. Start the wound process at the origin, let $s_0 = 0$ and then let

$$s_{i+1} := \inf_{u > s_i} \left\{ \lim_{u' \uparrow u} X_{u'} = 0 \right\} \qquad \text{for } i = 0, 1, 2, \ldots .$$

If we now build the random walk $S$ and the wound process $(X, \phi)$ using the same sequence $Y_1, Y_2, \ldots$ for each, then we have

$$S_n = \phi(s_n)$$

for $n = 0, 1, 2, \ldots$. This is illustrated in Figures 7-1 and 7-2.

*Remark:* A problem arises if the density of the $Y_i$s has non-zero mass at the origin. The random walk and the wound process are no longer almost surely equivalent—we cannot see in the wound process jumps of size zero in the random walk. If $Y_1 = S_1 = 0$ then $\tau_- = 1$ and $S(\tau_-) = 0$, but $s_- \neq 0$ and $X(s_-)$ is not necessarily equal to 0. We therefore assume that $\mathbb{P}\{Y_i = 0\} = 0$.

The first strict ascending ladder time corresponds to the first time $X$ approaches zero when $\phi > 0$ and occurs at time $s_n$ for some random $n$. Let us call this time $s_+$. Then $\phi(s_+) = S(\tau_+)$. However, since the process drifts at rate one, the value of $\phi(s_+)$ is equal to the value of $X$ when $\phi$ was last equal to 0. Thus, the value of $S$ at the first ladder time is the same value of $X$ at the first half winding time. The same is true for the descending ladder time, where $s_-$ is the first $s_n > 0$ for which $\phi(s_n) \leq 0$.

We therefore have, with our usual notation, that the distribution $\Pi^{-+}(0, dx)$ is equal to $F_+(dx)$ and the distribution $\Pi^{+-}(0+, dx)$ is equal to $F_-(dx)$.

If we were given the two distributions $F_+(dx)$ and $F_-(dx)$, then we could immediately describe the $X^+$ and $X^-$ processes. They drift towards the origin at rate one and jump away from it according to the distributions $F_+$ and $F_-$. Therefore the domains of the generators $G^+$ and $G^-$ consist of functions $f_+$ and $f_-$ where $f_+ \in C^1(\mathbb{R}^+)$ and $f_- \in C^1(\mathbb{R}^-)$ subject to the conditions that $\mathbb{E}\, f_+(Y_+) = f_+(0)$ and $\mathbb{E}\, f_-(Y_-) = f_-(0)$ where $Y_+$ and $Y_-$ have distributions $F_+$ and $F_-$.

The Wiener-Hopf factorization given at (7.1) is a factorization of random variables. It therefore decomposes a function, namely, the distribution. The factorizations we have been considering are decompositions of random processes, which involve factorizing generators and their domains.

# Part II

# Simulating Network Growth

# Chapter 8

# Introduction

In this shorter second part, we present the development and results of computer simulations of randomly generated networks.

The work was prompted and to some degree guided by biologically motivated ideas of Dr. Alan Rayner (University of Bath). Much work has been done to attempt to use Brownian Motion and related processes to model the growth of blood vessels near a cancerous tumour.

Chaplain & Anderson (1996) model angiogenesis (the growth of blood vessels) by looking in some detail at the mechanism behind such growth. They achieve this by using a system of three coupled partial differential equations with terms representing the concentrations of three substances in the blood which in turn affect growth probabilities. Results given in the paper of simulations on a grid show networks exhibiting anastomosis (joining of two growths) and the "brush border", a dense growth on the edge of the network nearest the tumour. Both of these features are observed in angiogenesis. The structures presented in the paper are qualitative and allow general deductions to be made, such as how soon anastomosis occurs after initial growth or whether the proliferation of sprouts near the tumour is sufficient to cause the formation of a brush border.

Stokes & Lauffenburger (1991a) concentrate on one of the three substances—a type of cell—considered by Chaplain and Anderson. They model the movement of individual cells, which form vessels behind them as they migrate through the surrounding tissue. The trajectory of each of the cells is modelled by an integrated Ornstein-Uhlenbeck process (see Stokes & Lauffenburger (1991b)). The results in the paper give the fine structure of the vasculature systems they produce. They attempt to set the parameters for the simulation using experimentally obtained values.

The kinds of patterns that are exhibited by blood vessels in angiogenesis can also be found in other biological settings. Similar networks are formed, for example, in the vein system of a healthy leaf or in the paths that certain fungi take during different phases of growth. Attempts to use random processes to generate pictures that look qualitatively like real life often succeed in demonstrating one or other particular feature of the development of such networks, but it was felt that it may be possible to find more universally applicable models. We present here an attempt to use a computer to model the growth of a general stochastic network.

A problem inherent in computer modelled simulations is a tendency to begin with the results and then to engineer a program to reproduce them or to make discoveries, not about the original problem, but purely about the implementation of the model. For example, one may develop a program to model these structures which includes a term controlling how strongly the direction of growth is attracted towards the tumour. To discover that the experimentally observed level of anastomosis does not occur when this parameter is set too high is not to have investigated the model or the mechanism of the original structures, but to have realised that vessels growing in parallel do not meet. Similarly, if great proliferation occurs near the tumour, then part of the model must allow this to be reproduced. However, there is possibly only a slight difference between a program which shows a natural way in which proliferation can occur and a program which has been specifically designed to reproduce known results.

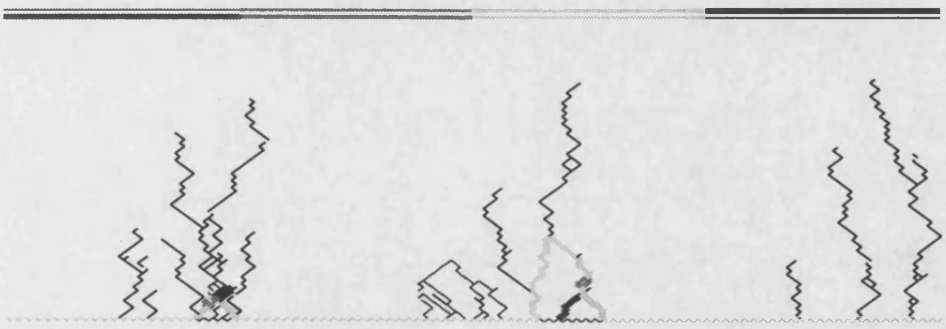


Figure 8-1: A Random Network

The primary purpose of these simulations might be said to be geometrical. We would like to discover how the shape of the network might affect the growth of the network. When parts of a network start to grow, they behave independently of one another up until the time at which they meet. It is the joining, or fusing, of two separate strands of the network to form a loop that determines the geometry. In contrast to other simulations, we put most emphasis on this fusing and the change in flow around the network that it causes, rather than on the behaviour of the tips of the branches.

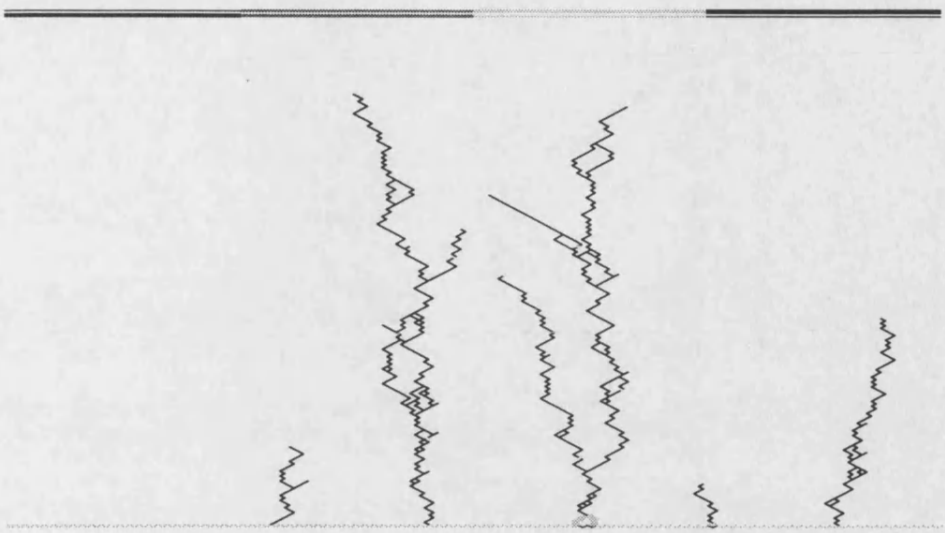


Figure 8-2: Shortly Before Anastomosis

We shall look in some detail at a method for dealing with an entire network's geometry that balances the two demands of storage and processing time made by any computer program. In view of the problem inherent in modelling noted above, we have aimed to keep the model simple in order to extract more structure from the programs than we put into them. Results will be presented which seem to produce networks and network development with a rich structure, but some of these results are unstable in the sense that they occurred for very specific parameters and even then only for a few "lucky" runs of the program.

In Figure 8-1, the thickness of the lines represents their resistance to the flow—thicker lines representing lower resistances—and where loops have formed, the colour indicates the amount of flow, as shown in the bar along the top, from smallest flow to largest.

In Figures 8-2 and 8-3 we see an example of two branches joining and causing a large flow diversion away from the original vessel and up towards the tumour, supposed to be located at the top of the figures.

## 8.1 A Biological Setting

We briefly and simplistically give the principal factors believed to be responsible for the development of blood vessel networks which have grown as a response to the presence of a tumour.

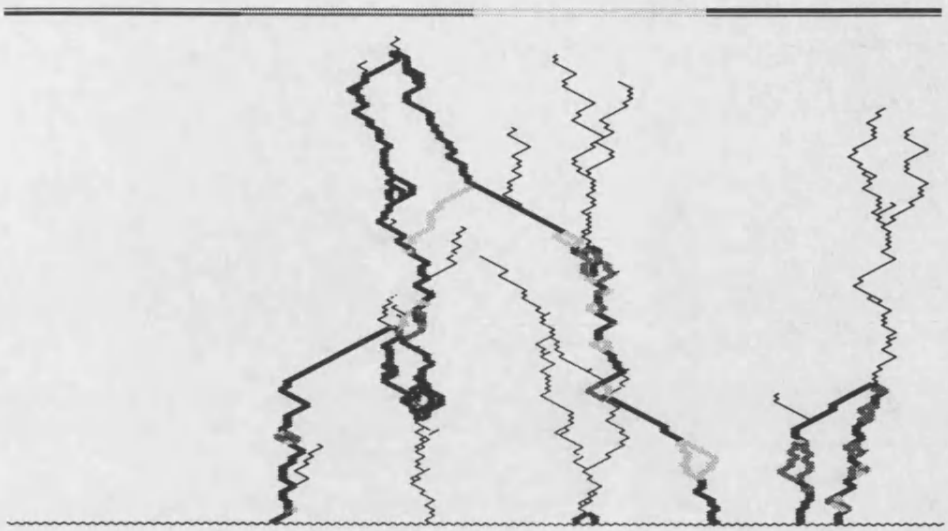


Figure 8-3: Shortly After Anastomosis

When a tumour develops, it produces chemicals which promote the growth of new blood vessels. We refer to this growth of blood vessels as angiogenesis. They sprout from some existing blood vessel and grow towards the tumour. When two collide they fuse, or anastomose. This puts these new vessels in a parallel formation with the original one.

- New blood vessels are formed by the presence of microvessel endothelial cells (MEC) in the blood stream.
- The tumour produces chemicals that cause the MEC concentration to rise.
- The wall of an existing vessel degrades and the MEC migrate into the surrounding tissue forming sprouts of new blood vessels.
- When two of these sprouts meet, they anastomose, blood flows in the new vessel, and the process repeats.

We reduce these ideas to the abstract notion of a network being driven from below by the flow of the network (by the MEC in the blood) and attracted from above by some attractor (by the chemicals from the tumour). We take the attractor to be uniform along some line parallel to the original vessel. We therefore assume that the vessel is far from the tumour, or the tumour is fairly large, to justify the uniformity of the attractor.

In this way, we distance ourselves from this specific biological case and can consider a stochastic network *per se*.

# Chapter 9

# The Structure as an Electrical Network

To generalise the setting, we consider the growth of an electrical network of wires and fix the voltage along one wire. In the case of blood vessel growth, this represents the original vessel. This voltage is kept fixed under the assumption that flow at a distance from the tumour is unaffected by the new growth. By storing the voltages at the points of the grid and the resistance of the wires, we can calculate the electrical current along them.

We first see how this is achieved, then see what probabilities are used to obtain the results described.

## 9.1 The Electrical Network

The program randomly constructs an electrical network on a grid, beginning with a single line along one of the edges of the square. We define "grid" and "network" as this sentence suggests. The grid is a set of points and the network is composed of the subset of the grid joined by wires together with those wires. The ends of the initial wire are kept at constant voltages 0 and 1. Wires grow diagonally upwards from this base edge. For example, a simple growth is shown in Figure 9-1, with the voltages of the initial wire labelled.

No growth occurs from the ends of the initial wire. Neither do we allow growth from points with 0, 3 or 4 adjacent points. Adjacent means connected by a single wire segment. In Figure 9-1, we allow growth from A or D, as they have 2 and 1 adjacent points, but not from B or C, which each have 3 points adjacent.

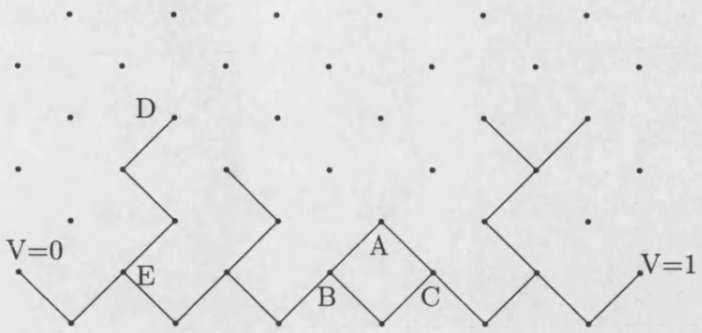



Figure 9-1: A simple network

We would like to use a fine grid with the wires fairly widely spaced so that the model approximates to some continuous limit. To write a program capable of generating sufficiently complex structures, we must store the network in a way that is compact in terms of memory and that also allows quick calculation of voltages and of resistance between points.

### 9.1.1 Points, Vertices and Nodes

We distinguish between three types of location.

- A *Point* refers to a position on the grid, which may or may not be part of the network.

The other two types of location are more topological than geometrical. The distance between them, as represented on the grid, may be zero. For example, we assume that "in reality" two wires do not hit head on, but that one reached the point of intersection first and the second wire hit it some way down from the tip. So, what looks like $\wedge$ on the grid is in fact stored as a $\lambda$ or its mirror image. This occurs at Point A in Figure 9-1.

- A *Vertex* exists at any Point in the network which is not in the middle of a wire. That is, it is a junction where a wire has hit another or it is the loose end of a wire. The Vertices are numbered and their present positions stored in arrays. As mentioned above, two Vertices may coexist at a single Point. In fact two vertices will exist anywhere on the grid (except the base line) where a $\wedge$ or $\times$ occurs.

In the example above, the network contains 23 Points and 14 Vertices, two at position A. In fact, we also introduce a Vertex 0 attached to our original two Vertices 1 and 2.

We now divide the network into "Clusters". Cluster 0 consists of all *Vertices* with flow through them. From this we may explain the addition of Vertex 0. We imagine an electrical cell placed at 0 and define Cluster 0 to be all those Vertices through which this cell causes current to flow. (Note that in Figure 9-1, of the two Vertices at position A, only one has flow through it.) In the same way, we label all such self-contained circuits in the network as being Clusters. The labelling of the Clusters is quite intuitive; it is the "lowest" Vertex in the Cluster and the position of the imaginary cell. However, although we can easily see which Vertex is the lowest by looking at the network, lowest does not equate to smallest numbered.

In each Cluster, there are important and unimportant Vertices. Consider Cluster 0 in Figure 9-1 and suppose that everything not a part of this Cluster has been removed. We have remaining the wire from Point 0 to Point 1 and the wires up to Point A. Vertices B and C are important; they are at a junction of Cluster 0 where the flow branches. Vertex E is unimportant. Once we have stripped away everything not in Cluster 0, (which includes the wire DE), Vertex E is just a point in the middle of a wire. The single Vertex remaining at position A is similarly unimportant. The only important Vertices in Cluster 0 are the original two Vertices together with B and C. We also include Vertex 0.

- A *Node of Cluster c* is an important Vertex which is in Cluster *c*. Vertex *c* is always important.

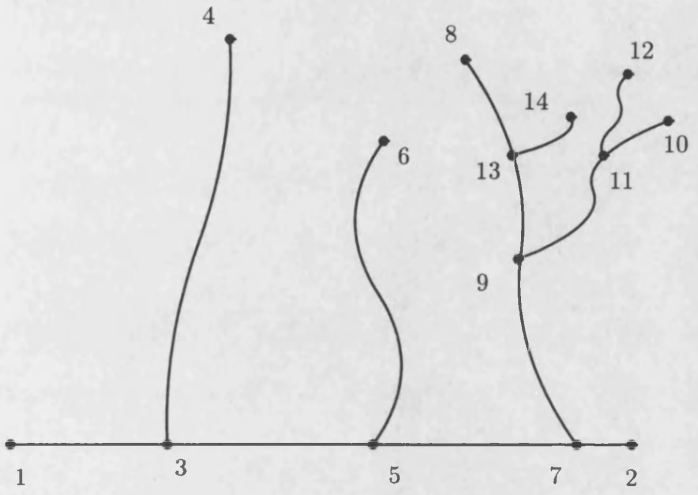
The idea is that we want to reduce the network to its simplest form before we calculate the voltages. Calculating voltages involves the inversion of a matrix and so we want the matrix to be as small as possible. We do this by first disregarding everything not in Cluster 0, since only Cluster 0 has flow in it. Then we discard all the remaining Vertices which are not Nodes, because by now non-Node Vertices lie in the middle of wires and are unimportant. However, we must keep track of the other Clusters because these may later be part of Cluster 0. If two wires join to allow current to flow through some Vertex, then current will flow through all Vertices in the same Cluster. This is one advantage of tracking the development of the network in this way.

In the example, 14 Points have flow through them. However, only 8 are Vertices and only 4 are Nodes, namely, B, C and the two original Nodes. Thus we need only invert a $4 \times 4$ matrix, rather than a $14 \times 14$ one.

By storing the grid in this way, we can quickly find the voltages of the network.

### 9.1.2 The Tree Structures and Network Structure

We make great use of the fact that the network is very nearly a tree structure. A tree structure is easy to store. At each Vertex, we only need to know the unique Vertex below it, and we can navigate around the tree. In our case, until two wires hit, we have a tree structure which we store in the array `exit1`.




| i | node | exit1 | cluster |
|---|---|---|---|
| 0 | 0 | – | – |
| 1 | 0 | 0 | 0 |
| 2 | 0 | 7 | 0 |
| 3 | – | 1 | 0 |
| 4 | – | 3 | 3 |
| 5 | – | 3 | 0 |
| 6 | – | 5 | 5 |
| 7 | – | 5 | 0 |
| 8 | – | 13 | 13 |
| 9 | – | 7 | 7 |
| 10 | – | 11 | 11 |
| 11 | – | 9 | 9 |
| 12 | – | 11 | 11 |
| 13 | – | 9 | 9 |
| 14 | – | 13 | 13 |

Figure 9-2: A tree structure

In Figure 9-2, the diagram of the tree is equivalent to the columns headed `i` and `exit1` in that either the diagram or the columns can be reproduced from the other. (A null value is represented by '–'.) The array entry `node[vx]` is given the value $c$ if `vx` is an important Point of Cluster $c$. Otherwise it is null.

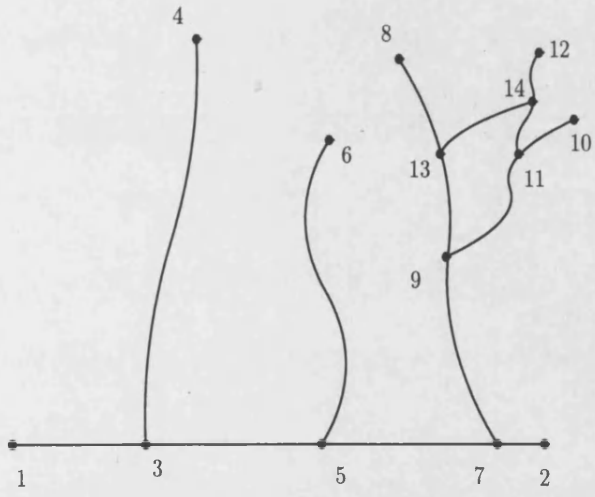
The array `exit1` gives us the full tree structure. The array `cluster` also gives a tree structure, but instead of tracking through a Cluster one Vertex at a time, when a Vertex in a Cluster is reached, the next move is directly to the base Vertex of the Cluster. In Figure 9-2, the path down the tree from Vertex 9 is 9-7-5-3-1-0 (look at `exit1`). However, as soon as we reach Vertex 7, we are in Cluster 0, so the path down the tree given by the array `cluster` is 9-7-0.

Since in our example, most of the Vertices are not in any Cluster (only Cluster 0 exists so far) the array `cluster` is almost a duplicate of `exit1`. We need to know, for each entry of `cluster`, whether it is duplicating `exit1` and sending us back one vertex, or whether it is sending us all the way to the base Vertex of a Cluster. In the example from Figure 9-2, the

answer is clear. Travelling from Vertex 9 to Vertex 7 must be stepping back one Vertex because there is no Cluster 7. From 7 to 0 must be travelling to a base vertex, because there is a Cluster 0. Since there is a Cluster $c$ if and only if Vertex $c$ is an important Vertex of Cluster $c$, i.e., if and only if `node[c]=c`, we know which rôle `cluster` is playing for each Vertex.

The array `cluster` stores the minimum amount of information necessary to determine which Vertices have flow through them. However, it forgets the internal structure of all the Clusters. The array `exit1` does not lose this information.

Of course, the network is *not* a tree. When two wires join, a Cluster grows or a new one is formed and a richer structure appears. This extra structure is stored in the array `exit2` in a similar way to `exit1`. The way in which the tree structure is lost is that a Vertex hits another wire and there are now two ways of travelling down the network from that location. For example, suppose Vertex 14 hits wire 11–12. See Figure 9-3. Now Vertex 12 is above Vertex 14 and Vertex 14 is above Vertex 13 (as before) *and* 11. We set `exit2[14]=11`.




| i | node | exit1 | exit2 | cluster |
|---|---|---|---|---|
| 0 | 0 | – | – | – |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 7 | – | 0 |
| 3 | – | 1 | 1 | 0 |
| 4 | – | 3 | – | 3 |
| 5 | – | 3 | 1 | 0 |
| 6 | – | 5 | – | 5 |
| 7 | – | 5 | 1 | 0 |
| 8 | – | 13 | – | 13 |
| 9 | 9 | 7 | 7 | 7 |
| 10 | – | 11 | – | 11 |
| 11 | – | 9 | 9 | 9 |
| 12 | – | 14 | – | 14 |
| 13 | – | 9 | 7 | 9 |
| 14 | – | 13 | 11 | 9 |

Figure 9-3: First loop formed

Since "loose" Vertices (the only Vertices that can anastomose) are always even-numbered, the odd-numbered entries in `exit2` appear redundant. We use them to record the wire from which the odd numbered Vertex grew. In the above diagram, Vertex 13 is on wire 7–8, as is Vertex 9. These are both labelled 7.

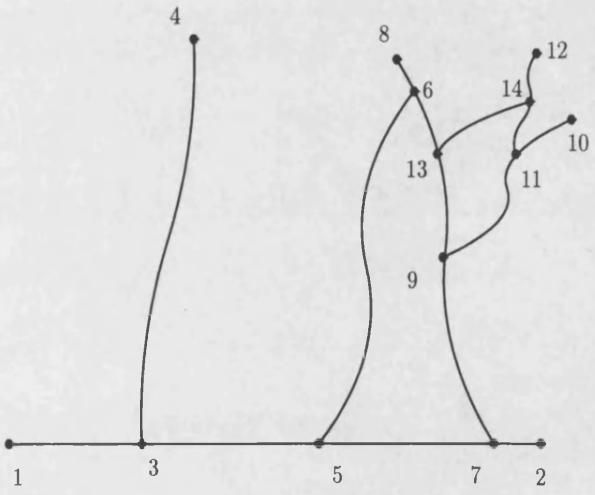
### 9.1.3 Updating the Structure

Now let us look in more detail at the joining of vertices.

Suppose Vertex 6 grows to join wire 13–8 as in Figure 9-4. First, we find which of 8 and 13 is higher (higher means further from Cluster 0 in the tree structure `Cluster`) and whether the existence of the wire is stored in array `exit1` or `exit2`. In this case, we see that 8 is higher and that `exit1[8]=13`.

If 8 and 13 were in the same Cluster then 6 would now be a junction of the Cluster and hence would become a *Node* of this Cluster. However, here 13 is in Cluster 9 and 8 is not in any Cluster.

Now the program looks down the network to find which Cluster will grow or whether a new one will form. (One or other of these will occur.) It looks down the network from Vertices 6 and 13. It is clear by observation that 6 joins Cluster 0 at Vertex 5 and 13 joins it at 7. Also, no higher numbered Cluster is encountered on both of the paths. Since 5 is lower than 7, meaning that it is closer to 0 in the tree structure given by `exit1`, let us describe this by saying that Vertices 13 and 6 meet at 5.




| i | node | exit1 | exit2 | cluster |
|---|---|---|---|---|
| 0 | 0 | – | – | – |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 7 | 0 | 0 |
| 3 | – | 1 | 1 | 0 |
| 4 | – | 3 | – | 3 |
| 5 | 0 | 3 | 1 | 0 |
| 6 | – | 5 | 13 | 0 |
| 7 | 0 | 5 | 1 | 0 |
| 8 | – | 6 | – | 6 |
| 9 | 0 | 7 | 7 | 0 |
| 10 | – | 11 | – | 11 |
| 11 | – | 9 | 9 | 0 |
| 12 | – | 14 | – | 14 |
| 13 | 0 | 9 | 7 | 0 |
| 14 | – | 13 | 11 | 0 |

Figure 9-4: Second loop formed

However, it is not so easy for the computer to follow the diagram. *We* find their meeting point by putting one finger on 6 and one on 13 and moving down through the vertices and Clusters until we find the first Cluster on both paths. We want the computer to do

the same. One way of doing this would be to work out both paths down and then cross check to find the first Cluster on both paths. This is obviously an inefficient method. We want to move pointers down each path in the sure and certain hope that if a Cluster lies on both paths, the pointers will arrive there simultaneously. A nice property that would allow this method would be if `cluster[c]<c`. Then we could move whichever pointer was at a higher number until their numbers matched. However, the network does not possess this property.

A property it does possess is that a wire cannot grow from a higher numbered wire. That is, for odd Vertices `vx`, `exit2[vx]<vx`. Also, two Vertices always meet at an odd-numbered Vertex. A bit of thought shows that we can use the tree structure given by `exit1` for even Vertices and `exit2` for odd Vertices and employ the same strategy to the one suggested in the previous paragraph of reducing the higher number.

In this example, we start with our pointers at $(6, 13)$. Since Vertices never meet at an even Vertex, we move away from Vertex 6 and set the pointers to $(5, 13)$. Vertex 5 is on wire 1 and Vertex 13 is on wire 7, so 13 must be higher than 5 and we move to $(5, 7)$. Now Vertices 5 and 7 are on the same wire (wire 1) and we need only determine which is the *lower* of the two (it is of course Vertex 5). Since Vertex 5 is in Cluster 0, the fusing of Vertex 6 will enlarge Cluster 0. If Vertex 5 had been a Vertex not in any Cluster then a new Cluster (labelled 5) would have formed.

We now go back to our original Vertices 6 and 13, armed with the information that they meet at a point in Cluster 0. We travel down from each of the two Vertices using the tree structure given by `cluster` and setting everything in our path to be a part of Cluster 0.

Some of the Vertices become Nodes of Cluster 0. All Nodes of the Clusters encountered on the way down become Nodes of Cluster 0. Here, only Vertex 9 falls into this category. Also, the first Vertex of any Cluster we encounter becomes a Node, in this case, Vertex 13 for Cluster 9 and Vertices 5 and 7 for Cluster 0.

Finally, the arrays `exit1` and `exit2` are updated.

### 9.1.4 Navigating the tree

Using this system, every edge between Vertices is labelled exactly once. We introduce arrays `rest1` and `rest2` to store the resistances of the wires along these edges, the entries in these arrays corresponding to the entries in `exit1` and `exit2`.

We wish to find all the paths between *Nodes* in Cluster 0. There are three types of these paths. If `n` is a Node of Cluster 0 and `m:=exit1[n]` is in Cluster 0 then we can

| | Grid | Tree | |
|---|---|---|---|
| 1 |  |  | One Loose Vertex |
| 2 |  |  | On a Wire |
| 3 |  |  | Two Vertices, One Loose |
| 4 |  |  | One Attached Vertex |
| 5 |  |  | Two Attached Vertices |
| 6 |  |  | One Attached Vertex |

Figure 9-5: Correspondence between tree and grid

follow the unique path in Cluster 0 from Vertex `m` to the next Node. The same applies if `m:=exit2[n]` is in Cluster 0 and `m` is even. The third kind of path occurs if there is an even numbered Vertex `n` which is not a Node but for which `exit1[n]` and `exit2[n]` are both in Cluster 0. We must follow the paths from both of these vertices until we reach Nodes. In our example, we have paths 5–3–1, 7–5, 9–7 and 13–9 of the first type, 2–7 of the second and 5–6–13 and 13–14–11–9 of the third type.

The ideas outlined above, together with the algorithm for finding `exit2[2n+1]`, form the basis of the part of the program which stores the network.

Further parts of the program relate this structure to the actual grid. It is important to realize the correspondence between the grid and the tree. Figure 9-5 sets out this relationship modulo symmetric patterns.

The possibility of new growth exists only in the situations given in 1, 2 or 3. Growing from a Point looking like 1 or 3 would cause a loose Vertex to move to a new position, while growing from a Point like 2 causes a new wire to grow. Likewise, if a wire grows, then it must either add a Point to the network, or join to an existing point looking like 1, 2 or 6.

### 9.1.5 Navigating the grid

One advantage of this way of storing the information has been mentioned, namely, that calculating the voltages is much easier and faster. The other main advantage is that we can run through all of the network in a systematic manner. We therefore need not waste time scanning empty sections of the grid when determining where growth occurs. We scan the base line separately as it does not have the same properties as the rest of the grid. We can then scan through from Vertex 3 to the current highest numbered Vertex and see if there is a wire either north-west or north-east of the vertex. If either (or both) of these wires are present, then we can follow the *unique* path upwards in that direction until we hit another Vertex. The only exception to this is that if two Vertices exist at the same Point, we only check upwards from one of them. This is easily done, because the tree structure still knows which is higher, and so we only go upwards from the higher Vertex at any point.

This method is used for scanning through the wires, through the Points and also for scanning through Cluster 0 when plotting the flow.

## 9.2 Algorithms for Calculating Probabilities

In this section we describe the way in which the program decides whether a wire will grow, die or change resistance.

### 9.2.1 Probability of Growth of a Wire

We consider each Point of the network in turn, calculate a probability for its growth and then use this and a uniform random variable to determine whether growth actually occurs.

The value of the probability of growth depends on three factors; the height of the Point, the distance (in terms of resistance) between that Point and the nearest flow, and the size of that flow. The higher the point, the closer it is to the attractor at the top and so the higher the probability of growth. Similarly, a higher rate of flow simulates growth and raises the probability.

The probability is given by the expression "1 in $F/f + R*r + H/h$", where $F$, $R$ and $H$ are variables set by the user and the lower case variables are the values of the distance to the nearest flow ($r$, as distance is measured in terms of resistance), the size of the flow at that point ($f$), and the height ($h$). In theory, this expression is not guaranteed to give a value

below 1, but in practice, all of our probabilities are in fact very small (to approximate to continuous time).

The probability of growth at a particular Point represents the probability of a loose wire at that Point growing up to the next level. If there is no loose wire at that Point, a new sprout must grow and we multiply the growth probability by a *fixed* amount. This is because the attractor and the flow are taken to stimulate growth only, not budding of new sprouts.

### 9.2.2 Changes in Resistance of Wires

We allow the resistance (or in the case of angiogenesis, we may think of the diameter) of the new connections to alter over time. In angiogenesis, it is observed that a small number of dominant paths carrying most of the flow develop. Therefore we seek to decrease the resistance of "good" paths and increase the resistance of less used paths. Altering the resistances also promotes the growth of a stable and simple final structure, rather than an intricate web of wires across the whole grid.

We employ a feedback effect so that wires carrying a heavy load have their resistances lowered and wires whose load is small have theirs increased. Other than a positivity constraint, there are no bounds to the values the resistances can take. If the resistance of a wire reaches a certain level, it is depicted as a dead wire.

### 9.2.3 Killing of Wires

We wish to kill wires which seem to be redundant, that is, carrying little or no flow. In fact, rather than removing wires, which complicates the network and destroys the structure on which the program relies, we give the wire a very large resistance.

The obvious idea is to look at the flow in a wire and then kill it with some probability related to the value of that flow. However, all newly created wires carry zero flow, and would have a high chance of being killed immediately. To give them a chance to participate in the network, we do not use the flow, $I(n)$, to determine whether to kill. Instead, we use a value $J(n)$ which is given some initial value $J(0) > 0$ and is then altered using the present current to $J(n) = \lambda I(n) + (1 - \lambda)J(n - 1)$, thus giving us a weighted average.

Wires carrying no flow do not have their resistances altered, although these wires may be killed.

### 9.2.4 Direction of Growth

Wires always grow towards the attractor (that is, upwards) but there is no preference between left and right. However, tips are attracted by the nearby presence of other wires. This is incorporated because of the result that in more than two dimensions, the wires would not collide at all if left to grow in random directions.

# Chapter 10

# Results

The first piece of output, shown in Figure 8-1, demonstrates the general appearance of the networks obtained. At first, sprouts appear in series with one another. Without anastomosis, the network would consist of these single sprouts and tree-like structures. Figure 10-1 shows a network before anastomosis has caused the flow to alter. The fusing of the sprouts causes the series network to become a parallel one, and current flows round the loops.

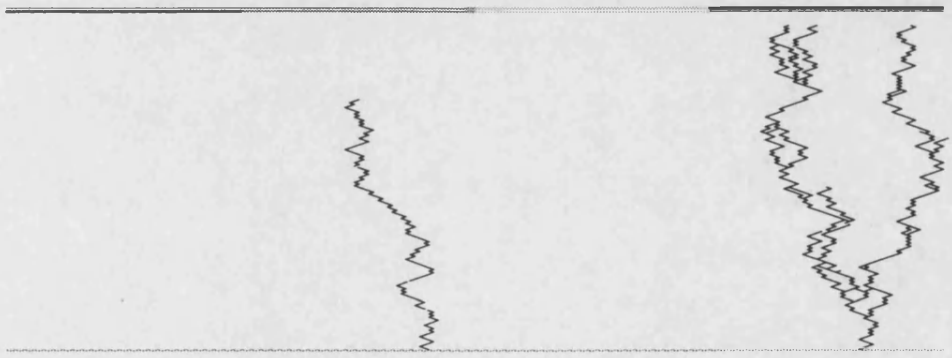


Figure 10-1: A Structure with no flow

These loops may or may not flourish, as seen in Figure 8-1, depending on whether or not sufficient flow is diverted through them. This means that if a number of paths are possible, the one with the lowest resistance is most likely to survive. Figures 8-2 and 8-3 show how the main flow can be diverted away from the original vessel and up towards the attractor to such an extent that this new path and not the original vessel carries most of the flow.

A succesful divertion of flow can cause the process to repeat, as in Figure 10-2 where a second loop has formed on top of the first.

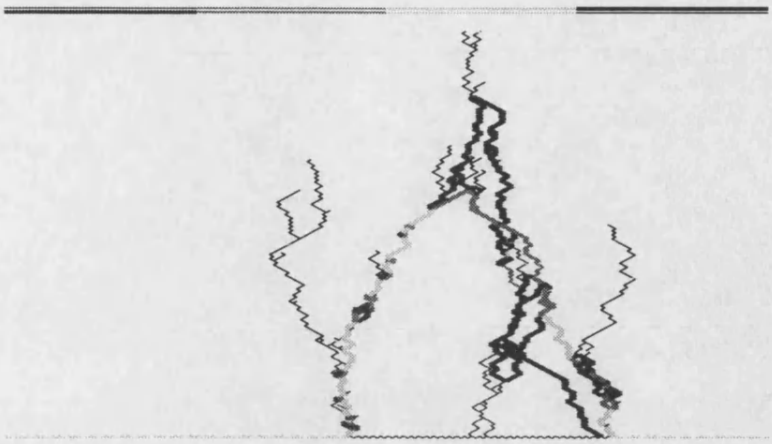


Figure 10-2: A Secondary Growth

It will be seen that the flow tends to choose one dominant path and that other paths inside this loop quickly die away. This agrees with observed networks in biological settings.

We now mention two results which were observed and whose occurrence is capable of explanation but which occurred maybe only once over a large number of random runs with the same parameters. They are included because they are features of network growth which may have some practical relevance and could possibly be stable results for slightly different models to the one used here.

Something similar to the brush border mentioned in the introduction was seen on occasion. Indeed, the effect of being close to the attractor is to stimulate growth, and if this effect is strong enough then something like a brush border will appear. See Figure 10-3 where this effect has been forced. More interesting is the occasional observation of a sudden change in the growth and budding rate, often at a single point where anastomosis has just occured. We would expect that the sudden introduction of flow to a point close to the attractor would immediately raise the growth probability and give rise to a sudden emergence of growth from that single point. Although this occasionally was observed in varying degrees, the difficulty in finding parameters for which it was a common feature leaves doubt as to whether it is truly a feature of this model. Emergence at a point is an observed feature of fungal movement.

A second feature of the networks which was difficult to reproduce for different runs of the program was the joint effect of killing the new sprouts and the speed of growth. It was seen, on a few occasions, that an increase in the rate of growth in a network did not necessarily produce a more healthy network. Since this model has the seemingly reasonable property that vessels are most likely to die if they have no flow through them, it is possible for a slower network to outgrow a faster one. In the latter case, a new sprout will develop and

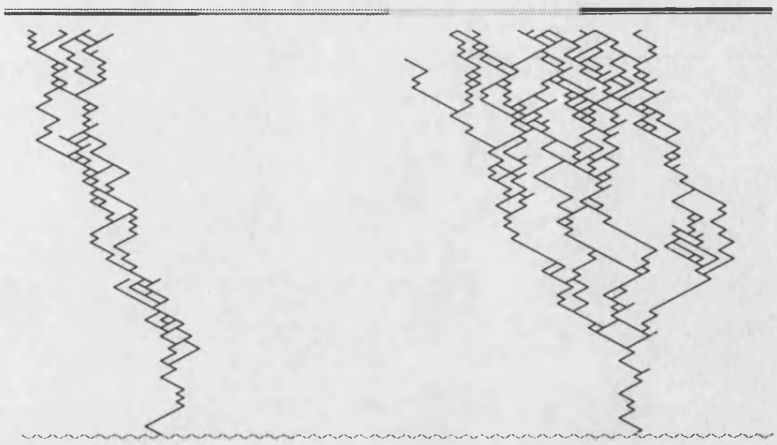


Figure 10-3: An example of a network with a denser structure at the top

will grow quickly before any other vessels grow near enough for anastomosis to occur. It therefore has no chance to carry any flow and, being very long, is susceptible to being killed. In this way, the network develops a sequence of dead, trailing vessels carrying no flow to the attractor. A slower growth results in small, less easily killed vessels which are more likely to join to other nearby growths being formed. These growths have had time to grow because the smaller wires last longer on average than the longer ones. In this way, the slower network gradually pushes the flow forwards and manages sustainable growth, while the faster network over-reaches itself and fails to advance the flow at all.

In our model, this feature, if it is there at all, is difficult to distinguish from background noise. However, it may be the case that there is, for the reason given, an optimal growth rate linked to the rate at which vessels die and that other implementations of these ideas may be able to confirm or refute this.

•

# Appendices

# Appendix A

# Standard Results

In this chapter we give standard results and references for the two areas of Feller Brownian Motions and Excursion Theory.

## A.1 Feller Brownian Motions

A Feller Brownian Motion is a Markov process on $[0, \infty)$ that behaves like a standard Brownian Motion away from 0. Details can be found in §5.7 of Itô & McKean (1974).

A reflected Brownian Motion is a familar example of a Feller Brownian Motion. It has generator $\mathcal{G}$ given by $\mathcal{G}f = \frac{1}{2}f''$ acting on the set of $C_b^2(\mathbb{R}^+)$ functions with $f'(0) = 0$. Killed Brownian Motion, another Feller Brownian Motion, has generator $\mathcal{G}f = \frac{1}{2}f''$ with domain $C_b^2(\mathbb{R}^+) \cap \{f : f(0) = 0\}$.

Feller showed that all such processes have generator $\mathcal{G}f = \frac{1}{2}f''$ with a domain that can be specified by a quadruple $(p_1, p_2, p_3, p_4)$, unique up to multiples, where $p_1, p_2, p_3 \geq 0$ and $p_4$ is a positive measure on $\mathbb{R}^+$ with $\int (x \wedge 1) p_4(dx)$ finite. The domain of $\mathcal{G}$ is the set of $C_b^2(\mathbb{R}^+)$ functions satisfying

$$p_1 f(0) + p_2 f'(0) + p_3 \mathcal{G}f(0) + \int_0^\infty (f(x) - f(0)) p_4(dx) = 0. \tag{A.1}$$

The numbers $p_1$ and $p_2$, as we saw above, can be thought of as specifying the rates of killing and reflection. The value $p_3$ is related to the stickiness at zero. See Section 2.2, where a reflecting Brownian Motion is delayed at the origin and so in the definition of the domain as given at (2.25) we have $p_1$ and $p_4$ both equal to zero and $p_2, p_3 \neq 0$. The measure $p_4$ is the discontinuous jump measure out of 0, as can be seen at (2.47).

If $\nu^+$ is Lebesgue measure on $\mathbb{R}^+$ and $\nu^-$ is some measure on $\mathbb{R}^-$, then the process $X^+$ formed via the now familiar procedure in Part I will be a Feller Brownian Motion. The process $X^+$ cannot be sticky at zero, since $0 \in E^-$. Therefore the resultant Feller Brownian Motion will have domain specified by $(p_1, p_2, 0, p_4)$.

In Section VI.50 of Vol. II the resolvents of Feller Brownian Motions are found using excursion theory and these results are used to find the resolvents of the two Feller Brownian Motions in Chapter 2.

## A.2 Itô Excursion Theory

### A.2.1 Excursion Theory Description of Brownian Motion

We briefly recall the basic results of Itô Excursion Theory as applied to reflected Brownian Motion, partly as the simplest way to define the notation of Part I.

Reflected Brownian Motion is decomposed into its excursions away from 0. An excursion is a continuous function $\xi : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{\partial\}$ such that $\xi(0) = 0$ and if $H(\xi) := \inf\{t > 0 : \xi_t = 0 \text{ or } \xi_{t-} = 0\}$ then $\xi(t) = \partial$ for all $t \geq H(\xi)$. Let $U$ denote the set of all such functions. The state $\partial$ represents a coffin state and we kill the excursions when they hit 0.

The path of a Brownian Motion can now be described by specifying each excursion and the local time at zero at which it occurs, that is, by describing a point process on $U \times \mathbb{R}^+$. Itô's central result was to show that this is a Poisson point process and there exists a measure $n$ on $U$ such that for $\Xi \subseteq U$ and $\Gamma \subseteq \mathbb{R}^+$, the number of points in $\Xi \times \Gamma$ is Poisson with parameter $n(\Xi)\text{Leb}\,(\Gamma)$. This parameter may be infinite. For example, reflected Brownian Motion performs infinitely many excursions of height less than $\varepsilon > 0$ by any positive time. We are ignoring measurability conditions.

We define the entrance law, $n_t$ to be the measure such that for $\Gamma \subseteq \mathbb{R}^+$,

$$n_t(\Gamma) := n(\{\xi : \xi_t \in \Gamma, H(\xi) > t\}).$$

This is the entrance law in the sense that $n_t P_s^- = n_{t+s}$ where $P^-$ is the transition semigroup for killed Brownian Motion.

We also define the Laplace transformed entrance law,

$$n_\lambda(dx) := \int_{t=0}^{\infty} e^{-\lambda t} n_t(dx)dt. \tag{A.2}$$

For all $t > 0$, we have $n_t(\{0\}) = 0$, since the process dies on hitting 0. Therefore, $n_\lambda(\{0\}) = 0$ also.

### A.2.2 Marking the Excursions

The great drawback of excursion theory is that it deals with local time, which can be difficult to relate to real time. To overcome this difficulty, and also to introduce a powerful technique, we mark the real time axis, independently of the process, with a Poisson process of rate $\lambda$. We now differentiate between marked and unmarked excursions. A marked excursion is one which contains a mark. Let $U^*$ denote marked excursions and $\tilde{U} := U \cup U^*$ denote all excursions, marked and unmarked.

The Poisson nature of the markings means that we can perform this procedure in two equivalent ways. We can mark the real time axis and decompose that into marked and unmarked excursions, or decompose the process into excursions and then mark each excursion individually at rate $\lambda$. Let $\tilde{n}$ denote the Poisson excursion measure on $\tilde{U}$.

We mark the process as above and let $T$ be the real time of the first mark. Then since $T$ occurs at an exponential time independent of the process and for a subset $\Gamma \subseteq \mathbb{R}^+$

$$\lambda R_\lambda(x, \Gamma) = \mathbb{P}_x\{X_T \in \Gamma\}. \tag{A.3}$$

Let $H_\Xi$ be the local time at which an excursion in the set $\Xi \subseteq U$ occurs. Then for $A$ and $B$ subsets of $U$ with $n(A) < \infty$, standard Poisson process results tell us that

$$\mathbb{P}\{H_A \leq H_B\} = \frac{n(A)}{n(A \cup B)}. \tag{A.4}$$

Let $X$ be a Feller Brownian Motion and let $^-X$ denote that process killed when it hits zero. Then we use the Strong Markov Property at $H_0$ the hitting time of 0 and the memoryless property of the exponential random variable $T$ to deduce that

$$\begin{aligned} R_\lambda f(x) &= \lambda^{-1}\mathbb{E}_x f(X_T) \\ &= \lambda^{-1}\mathbb{E}_x[f(X_T); H_0 \geq T] + \lambda^{-1}\mathbb{E}_x[f(X_T); H_0 < T] \\ &= \lambda^{-1}\mathbb{E}_x[f(^-X_T); H_0 \geq T] + \lambda^{-1}\mathbb{P}_x\{H_0 < T\}\mathbb{E}_0[f(X_T)]. \end{aligned}$$

Then since $\mathbb{P}_x\{T > H_0\} = \mathbb{E}_x \exp(-\lambda H_0)$, we have

$$R_\lambda f(x) = {}^-R_\lambda f(x) + \lambda^{-1}\mathbb{E}_x[e^{-\lambda H_0}]\mathbb{E}_0[f(X_T)]$$

$$= \quad {}^{-}R_\lambda f(x) + e^{-\gamma x} R_\lambda f(0). \tag{A.5}$$

Therefore to find the resolvent of a Feller Brownian Motion, it suffices to find $R_\lambda f(0)$.

# Appendix B

# Some Numerical Results

The following pages contain program results. The results relating to Chapter 4 for a mass $m_0$ at the origin are on pp. 118—121 and the results relating to Chapter 5 for $m_0 \to 0$ on pp. 122—127.

Mass 0: $m_0 = 1.00$
Mass 1: $m_1 = 1.00 \quad d_0 = 1.00$

Characteristic equation for $Q - V\theta + \theta^2$ is

$$\theta^4 - \theta^3 - 2.00\,\theta^2 + \theta = 0$$

which has roots

$$\theta_0^+ = 0.44504187 \quad \theta_0^- = 0.0$$
$$\theta_1^+ = 1.80193774 \quad \theta_1^- = -1.2469796$$

$$W^- = \begin{pmatrix} 0.400968868 & 0.178447934 \\ 0.178447934 & 0.178447934 \end{pmatrix}$$

Characteristic equation for $W^-$ is

$$x^2 - 0.57942\,x + 0.03971 = 0$$

which has roots

$$0.07941680 \qquad 0.5$$

$$\Gamma^+ = \begin{pmatrix} -1.445042 & 0.801938 \\ 0.445042 & -0.801938 \end{pmatrix} \quad \Gamma^- = \begin{pmatrix} -0.445042 & 0.445042 \\ 0.801938 & -0.801938 \end{pmatrix}$$

Mass 0: $m_0 = 1.00$

Mass 1: $m_1 = 100.00 \quad d_0 = 2.00$

Characteristic equation for $Q - V\theta + \theta^2$ is

$$\theta^4 - \theta^3 - 0.505\,\theta^2 + 0.49\,\theta = 0$$

which has roots

$$\theta_0^+ = 0.00971588 \quad \theta_0^- = 0.0$$
$$\theta_1^+ = 1.36679976 \quad \theta_1^- = -0.3765156$$

$$W^- = \begin{pmatrix} 0.368059626103473980 & 0.482365060468602447 \\ 0.004823650604686025 & 0.482365060467679241 \end{pmatrix}$$

Characteristic equation for $W^-$ is

$$x^2 - 0.85042\,x + 0.17521 = 0$$

which has roots

$$0.35042469 \qquad 0.5$$

$$\Gamma^+ = \begin{pmatrix} -1.363235975 & 1.3279660964 \\ 0.0036323598 & -0.013279661 \end{pmatrix} \quad \Gamma^- = \begin{pmatrix} -0.363235975 & 0.3632359755 \\ 0.0132796610 & -0.013279661 \end{pmatrix}$$

Mass 0: $m_0 = 1.00$
Mass 1: $m_1 = 3.00 \quad d_0 = 5.00$

Characteristic equation for $Q - V\theta + \theta^2$ is

$$\theta^4 - \theta^3 - 0.267\,\theta^2 + 0.067\,\theta = 0$$

which has roots

$$\theta_0^+ = 0.16487142 \quad \theta_0^- = 0.0$$
$$\theta_1^+ = 1.17829778 \quad \theta_1^- = -0.3431692$$

$$W^- = \begin{pmatrix} 0.444570807430946846 & 0.216950660409590795 \\ 0.072316886803196923 & 0.216950660409527152 \end{pmatrix}$$

Characteristic equation for $W^-$ is

$$x^2 - 0.66152\,x + 0.08076 = 0$$

which has roots

$$0.16152147 \qquad 0.5$$

$$\Gamma^+ = \begin{pmatrix} -1.148901568 & 0.5828028891 \\ 0.0496338561 & -0.194267630 \end{pmatrix} \quad \Gamma^- = \begin{pmatrix} -0.148901568 & 0.1489015683 \\ 0.1942676297 & -0.194267630 \end{pmatrix}$$

Mass 0: $m_0 = 1.00$
Mass 1: $m_1 = 3.00 \quad d_0 = 7.00$
Mass 2: $m_1 = 5.00 \quad d_0 = 9.00$

Characteristic equation for $Q - V\theta + \theta^2$ is

$$\theta^6 - \theta^5 - 0.250\,\theta^4 + 0.107\,\theta^3 + 0.95\,\theta^2 - 0.01\,\theta = 0$$

which has roots

$$\begin{array}{ll} \theta_0^+ = 0.06876172 & \theta_0^- = 0.0 \\ \theta_1^+ = 0.28170085 & \theta_1^- = -0.3395610 \\ \theta_2^+ = 1.13130970 & \theta_2^- = -0.1422113 \end{array}$$

$$W^- = \begin{pmatrix} 0.453989786622365166 & 0.160546074685967405 & 0.114744182433550099 \\ 0.053515358228655802 & 0.134535317542425981 & 0.164214290186336653 \\ 0.022948836486710020 & 0.098528574111802006 & 0.145301335111560487 \end{pmatrix}$$

Characteristic equation for $W^-$ is

$$x^3 - 0.73383\,x^2 + 0.11919\,x - 0.00114 = 0$$

which has roots

$$0.01016350 \qquad 0.22366294 \qquad 0.5$$

$$\Gamma^+ = \begin{pmatrix} -1.114209726 & 0.4329812284 & 0.2274283579 \\ 0.0351021895 & -0.248586894 & 0.1224832927 \\ 0.0017806313 & 0.0625558909 & -0.118975647 \end{pmatrix}$$

$$\Gamma^- = \begin{pmatrix} -0.114209726 & 0.1053065685 & 0.0089031576 \\ 0.1443270762 & -0.248586894 & 0.1042598181 \\ 0.0454856716 & 0.0734899756 & -0.118975647 \end{pmatrix}$$

## B.1 Small Mass at Zero

Let $m_\varepsilon$ denote Lebesgue measure on $\mathbb{R}^+$, a mass $\varepsilon$ at nought and unit masses at $-1$ and $-2$. Let us denote a function by $(f, \mathbf{e})$ where $f$ is a function on $\mathbb{R}^+$ and $\mathbf{e}$ is a vector giving the value of the function at the atoms. The function is linear between the masses, and constant below $-2$. We wish to solve

$$\frac{1}{2}\frac{d^2}{dm_\varepsilon dx}(f,\mathbf{e}) = \tfrac{1}{2}\lambda^2(f,-\mathbf{e}) \tag{B.1}$$

for bounded, continuous functions, twice differentiable away from the masses and with left and right first derivatives at the masses. For negative $\lambda$, this will enable us to construct certain martingales.

We first consider the two dimensional problem, when $\varepsilon = 0$. Let $f(x) := \exp(-\lambda x)$ and let

$$Q := \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \tag{B.2}$$

so that

$$\frac{1}{2}\frac{d^2}{dmdx}(f,\mathbf{e}) = \tfrac{1}{2}\left(\lambda^2 f, Q\mathbf{e} + \begin{pmatrix} -\lambda \\ 0 \end{pmatrix}\right). \tag{B.3}$$

By continuity, we require $\mathbf{e}_1 = 1 + \lambda$ and so we must solve

$$(Q + \lambda^2)\mathbf{e}_\lambda = \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \tag{B.4}$$

subject to this condition. The solutions, which are solutions to the equation $\lambda^4 + \lambda^3 - 3\lambda^2 - 2\lambda + 1 = 0$, are given below.

| approx $\lambda$ | exact $\lambda$ | $\mathbf{e}_\lambda$ |
|---|---|---|
| $-1.8794$ | $2\cos\frac{8\pi}{9}$ | $\begin{pmatrix} -.87939 \\ .34737 \end{pmatrix}$ |
| $-1$ | $2\cos\frac{6\pi}{9}$ | $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ |
| $.34730$ | $2\cos\frac{4\pi}{9}$ | $\begin{pmatrix} 1.3473 \\ 1.5321 \end{pmatrix}$ |
| $1.5321$ | $2\cos\frac{2\pi}{9}$ | $\begin{pmatrix} 2.5321 \\ -1.8794 \end{pmatrix}$ |
| | $\lambda$ | $\begin{pmatrix} 1+\lambda \\ (1-\lambda)^{-1} \end{pmatrix}$ |

Given these quadratic eigenvalues and eigenvectors, we construct $\Gamma_+$ and $\Gamma_-$ via $\Gamma_+\mathbf{e}_\lambda = -\lambda\mathbf{e}_\lambda$ for $\lambda > 0$ and $\Gamma_-\mathbf{e}_\lambda = \lambda\mathbf{e}_\lambda$ for $\lambda < 0$. Thus, letting $\eta := 2\cos\frac{2\pi}{9}$,

$$\begin{aligned} \Gamma_+ &= (\eta - 1)\begin{pmatrix} -2 & (1+\eta)(2-\eta) \\ 1 & -\eta \end{pmatrix} = \begin{pmatrix} -1.0642 & .63042 \\ .53209 & -.81521 \end{pmatrix} \\ \Gamma_- &= \begin{pmatrix} 2\cos\frac{8\pi}{9} & 0 \\ 2\cos\frac{4\pi}{9} & 2\cos\frac{6\pi}{9} \end{pmatrix} = \begin{pmatrix} -1.8794 & 0 \\ .34730 & -1 \end{pmatrix} \end{aligned} \tag{B.5}$$

The fact that $\Gamma_-$ is a Q-matrix here is a coincidence.

We construct $\Gamma_+$ and $\Gamma_-$ in the same way for the three dimensional problem with $\epsilon > 0$. Continuity now gives the condition $\mathbf{e}_1 = 1$ and we solve

$$(Q + \lambda^2)\mathbf{e}_\lambda = \begin{pmatrix} \lambda/\epsilon \\ 0 \\ 0 \end{pmatrix} \quad \text{where} \quad Q := \begin{pmatrix} -\epsilon^{-1} & \epsilon^{-1} & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}. \tag{B.6}$$

| $\varepsilon$ | $\Gamma_+$ | eigenvectors$^T$ |
|---|---|---|
| 2 | $\begin{pmatrix} -.83811 & .36811 & .16188 \\ .51959 & -1.2431 & .51959 \\ .15664 & .50691 & -.84336 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1.1244 & 1.1882 \\ 1 & 0 & -1 \\ 1 & -3.0379 & 1.6286 \end{pmatrix}$ |
| 1 | $\begin{pmatrix} -1.4789 & .68251 & .28463 \\ .39074 & -1.20362 & .54620 \\ .088156 & .52111 & -.83083 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1.2036 & 1.3097 \\ 1 & .59436 & -.83084 \\ 1 & -.76352 & .28464 \end{pmatrix}$ |
| 1/2 | $\begin{pmatrix} -2.6238 & 1.2675 & .50111 \\ .27030 & -1.1638 & .57171 \\ .041576 & .53005 & -.82227 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1.2656 & 1.4054 \\ 1 & 1.3912 & -1.2093 \\ 1 & -.17080 & .024409 \end{pmatrix}$ |
| 1/4 | $\begin{pmatrix} -4.7525 & 2.3810 & .89909 \\ .17168 & -1.1290 & .59296 \\ .016452 & .53379 & -.81773 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1.3039 & 1.8533 \\ 1 & 1.9393 & -1.5257 \\ 1 & -.046329 & .0020473 \end{pmatrix}$ |
| 1/8 | $\begin{pmatrix} -8.8510 & 4.5483 & 1.6600 \\ .10080 & -1.1029 & .60837 \\ .0055723 & .53437 & -.81588 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1.3249 & 1.4973 \\ 1 & 2.2317 & -1.6993 \\ 1 & -.012925 & .00016490 \end{pmatrix}$ |
| 1/16 | $\begin{pmatrix} -16.916 & 8.8316 & 3.1529 \\ .055606 & -1.0858 & .61824 \\ .0016740 & .53380 & -.81530 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1.33591.5144 \\ 1 & 2.3809 & -1.7885 \\ 1 & -.0035061 & .000012250 \end{pmatrix}$ |
| 1/32 | $\begin{pmatrix} -32.955 & 17.361 & 6.1181 \\ .029393 & -1.0757 & .62398 \\ .00046403 & .53313 & -.81517 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1.3416 & 1.5232 \\ 1 & 2.4564 & -1.8339 \\ 1 & -.00092156 & .848 \times 10^{-6} \end{pmatrix}$ |
| 1/64 | $\begin{pmatrix} -64.977 & 34.397 & 12.036 \\ .015142 & -1.0701 & .62710 \\ .00012259 & .53266 & -.81516 \end{pmatrix}$ | – |
| 1/128 | $\begin{pmatrix} -128.99 & 68.455 & 23.865 \\ .0076892 & -1.0672 & .62873 \\ .00003153 & .53239 & -.81518 \end{pmatrix}$ | – |
| 1/256 | $\begin{pmatrix} -256.99 & 136.56 & 47.520 \\ .0038751 & -1.0657 & .62957 \\ .00000800 & .53224 & -.81519 \end{pmatrix}$ | – |
| $\to 0$ | $\begin{pmatrix} -(1+\varepsilon^{-1}) & .53209\varepsilon^{-1} & .18479\varepsilon^{-1} \\ \varepsilon & -1.0642 & .63041 \\ .53209\varepsilon^2 & .53209 & -.81521 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1.3473 & 1.5321 \\ 1 & 2.5321 & -1.8794 \\ 1 & -\varepsilon^2 & \varepsilon^4 \end{pmatrix}$ |
| 0 | $\begin{pmatrix} -1.0642 & .63042 \\ .53209 & -.81521 \end{pmatrix}$ | $\begin{pmatrix} 1.34730 & 1.5321 \\ 2.5321 & -1.8794 \end{pmatrix}$ |

| $\varepsilon$ | $\Gamma_-$ | eigenvectors |
|---|---|---|
| 2 | $\begin{pmatrix} -.33811 & .25979 & .078320 \\ .73621 & -1.2431 & .50691 \\ .32377 & .51959 & -.84336 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -.95497 & -2.3441 \\ 1 & -6.1316 & 3.5271 \end{pmatrix}$ |
| 1 | $\begin{pmatrix} -.4789 & .39074 & .08816 \\ .68251 & -1.2036 & .52111 \\ .28463 & .54620 & -.83083 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -.52111 & -1.6825 \\ 1 & -3.5133 & 1.9190 \end{pmatrix}$ |
| 1/2 | $\begin{pmatrix} -.62375 & .54060 & .083152 \\ .63373 & -1.1638 & .53005 \\ .25056 & .57171 & -.82227 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -.28299 & -1.3427 \\ 1 & -2.2030 & 1.1222 \end{pmatrix}$ |
| 1/4 | $\begin{pmatrix} .75253 & -.68673 & -.065808 \\ .59524 & -1.1290 & .53379 \\ .22477 & .59296 & -.81773 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -.15101 & -1.1705 \\ 1 & -1.5457 & .72951 \end{pmatrix}$ |
| 1/8 | $\begin{pmatrix} -.85101 & .80643 & .044578 \\ .56854 & -1.1029 & .53437 \\ .20750 & .60837 & -.81588 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -.078867 & -1.0846 \\ 1 & -1.2149 & .53648 \end{pmatrix}$ |
| 1/16 | $\begin{pmatrix} -.91648 & .88970 & .026783 \\ .55197 & -1.0858 & .53380 \\ .19705 & .61824 & -.81530 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -.040465 & -1.0420 \\ 1 & -1.0481 & .44128 \end{pmatrix}$ |
| 1/32 | $\begin{pmatrix} -.95544 & .94059 & .014849 \\ .54253 & -1.0757 & .53313 \\ .19119 & .62398 & -.81517 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -.020521 & -1.0209 \\ 1 & -.96406 & .39412 \end{pmatrix}$ |
| 1/64 | $\begin{pmatrix} -.97693 & .96908 & .0078458 \\ .53745 & -1.0701 & .53266 \\ .18807 & .62710 & -.81516 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -0.010337 & -1.0104 \\ 1 & -.92181 & .37066 \end{pmatrix}$ |
| 1/128 | $\begin{pmatrix} -.98825 & .98422 & .0040368 \\ .53481 & -1.0672 & .53239 \\ .18645 & .62873 & -.81518 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -.0051882 & -1.0052 \\ 1 & -.90062 & .35897 \end{pmatrix}$ |
| 1/256 | $\begin{pmatrix} -.99407 & .99202 & .0020480 \\ .53346 & -1.0657 & .53224 \\ .18563 & .62957 & -.81519 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -.0025991 & -1.0026 \\ 1 & -.89001 & .35313 \end{pmatrix}$ |
| $\to 0$ | $\begin{pmatrix} -1 & 1 & 0 \\ .53209 & -1.06418 & .53209 \\ .18479 & .63041 & -.81520 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -.87938 & .34729 \end{pmatrix}$ |
| 0 | $\begin{pmatrix} -1.8794 & 0 \\ .34730 & -1 \end{pmatrix}$ | $\begin{pmatrix} 0 & -1 \\ -.87939 & .34730 \end{pmatrix}$ |

The eigenvectors for $\Gamma_+$ are used to form a matrix, $K$, with $K_{ij} := \mathbf{e}(\lambda_j)_i$ and we define $\Pi := K^{-1}$.

| $\varepsilon$ | +ive Rts | -ive Rts | $\Pi$ |
|---|---|---|---|
| | .23188 | 0 | $\begin{pmatrix} .31634 & .27373 & .31634 \\ .56657 & -.045860 & -.43343 \\ .11708 & -.22787 & .11708 \end{pmatrix}$ |
| 2 | 1.0000 | −.76980 | |
| | 1.6927 | −1.6548 | |
| | .28463 | 0 | $\begin{pmatrix} .12972 & .31104 & .37864 \\ .37437 & .28584 & -.54852 \\ .49591 & -.59688 & .16989 \end{pmatrix}$ |
| 1 | 1.3097 | −.83083 | |
| | 1.9190 | −1.6825 | |
| | .31536 | 0 | $\begin{pmatrix} .043925 & .31398 & .39754 \\ .068954 & .35146 & -.36557 \\ .88712 & -.66544 & -.031962 \end{pmatrix}$ |
| 1/2 | 1.4664 | −.88838 | |
| | 2.8280 | −1.7214 | |
| | .33129 | 0 | $\begin{pmatrix} .013432 & .30759 & .39978 \\ .014199 & .29447 & -.27185 \\ .97237 & -.60206 & -.12793 \end{pmatrix}$ |
| 1/4 | 1.5070 | −.93327 | |
| | 4.8609 | −1.7660 | |
| | .33930 | 0 | $\begin{pmatrix} .0038330 & .30163 & .39840 \\ .0034737 & .26572 & -.23745 \\ .99269 & -.56735 & -.16096 \end{pmatrix}$ |
| 1/8 | 1.5210 | −.96296 | |
| | 8.9095 | −1.8068 | |
| | .34330 | 0 | $\begin{pmatrix} .0010391 & .29776 & .39696 \\ .00088664 & .25210 & -.22297 \\ .99807 & -.54986 & -.17399 \end{pmatrix}$ |
| 1/16 | 1.5268 | −.98039 | |
| | 16.947 | −1.8372 | |
| | .34530 | 0 | $\begin{pmatrix} .00027203 & .29555 & .39603 \\ .00022641 & .24548 & -.21636 \\ .99950 & -.54103 & -.17967 \end{pmatrix}$ |
| 1/32 | 1.5295 | −.98990 | |
| | 32.971 | −1.8563 | |
| | .34630 | 0 | $\begin{pmatrix} .00006971 & .29437 & .39550 \\ .00005739 & .24221 & -.21320 \\ .99987 & -.53658 & -.18230 \end{pmatrix}$ |
| 1/64 | 1.5308 | −.99487 | |
| | 64.985 | −1.8673 | |
| | .34680 | 0 | $\begin{pmatrix} .00001765 & .29376 & .39522 \\ .00001446 & .24058 & -.21166 \\ .99997 & -.53434 & -.18356 \end{pmatrix}$ |
| 1/128 | 1.5315 | −.99742 | |
| | 128.99 | −1.8732 | |
| | .34705 | 0 | $\begin{pmatrix} .00000444 & .29344 & .39508 \\ .00000363 & .23977 & -.21089 \\ .99999 & -.53321 & -.18418 \end{pmatrix}$ |
| 1/256 | 1.5318 | −.99870 | |
| | 257.00 | −1.8763 | |
| | .34730 | 0 | $\begin{pmatrix} 0 & .29313 & .39493 \\ 0 & .23896 & -.21014 \\ 1 & -.53211 & -.18478 \end{pmatrix}$ |
| $\to 0$ | 1.5329 | −1 | |
| | $\infty$ | −1.8794 | |
| 0 | .34730 | | $\begin{pmatrix} .29313 & .39493 \\ .23896 & -.21014 \end{pmatrix}$ |
| | 1.5329 | | |

We define $\Theta$ by $\Theta_{ij} := (\lambda_i + \lambda_j)^{-1}$ and $W^- := \Pi^* \Theta \Pi$.

| $\varepsilon$ | $W^-$ | E-vals | Left E-vecs |
|---|---|---|---|
| 2 | $\begin{pmatrix} .37954 & .10461 & .087762 \\ .20922 & .10052 & .094572 \\ .17552 & .094572 & .091958 \end{pmatrix}$ | .5<br>.071302<br>.00071799 | ( .61625 .20394 .17981 )<br>( −.61926 .45581 .54419 )<br>( .12477 −.96017 .87523 ) |
| 1 | $\begin{pmatrix} .38145 & .14045 & .10477 \\ .14045 & .13131 & .11935 \\ .10477 & .11935 & .11435 \end{pmatrix}$ | .5<br>.12536<br>.0017552 | ( .51175 .26668 .22157 )<br>( −.47398 .46541 .53459 )<br>( .09763 −.93707 .90237 ) |
| 1/2 | $\begin{pmatrix} .40100 & .17561 & .11233 \\ .087804 & .15872 & .13853 \\ .056165 & .13853 & .13055 \end{pmatrix}$ | .5<br>.18671<br>.0035552 | ( .42759 .32176 .25065 )<br>( −.33306 .48065 .51935 )<br>( .069388 −.90936 .93061 ) |
| 1/4 | $\begin{pmatrix} .42867 & .20640 & .11148 \\ .051601 & .18038 & .15135 \\ .027870 & .15135 & .14039 \end{pmatrix}$ | .5<br>.24352<br>.0059236 | ( .36812 .36439 .26749 )<br>( −.21451 .49921 .50079 )<br>( .044634 −.88167 .95537 ) |
| 1/8 | $\begin{pmatrix} .45461 & .23001 & .10648 \\ .028750 & .19560 & .15900 \\ .013310 & .15900 & .14564 \end{pmatrix}$ | .5<br>.28758<br>.0082830 | ( .33033 .39374 .27593 )<br>( −.12745 .51674 .48326 )<br>( .026283 −.85879 .97372 ) |
| 1/16 | $\begin{pmatrix} .47373 & .24579 & .10122 \\ .015362 & .20519 & .16322 \\ .0063260 & .16322 & .14823 \end{pmatrix}$ | .5<br>.31703<br>.010128 | ( .30825 .41192 .27982 )<br>( −.070931 .53002 .46998 )<br>( .014481 −.84286 .98552 ) |
| 1/32 | $\begin{pmatrix} .48573 & .25522 & .097388 \\ .0079756 & .21072 & .16543 \\ .0030434 & .16543 & .14947 \end{pmatrix}$ | .5<br>.33459<br>.011332 | ( .29614 .42229 .28158 )<br>( −.037710 .53856 .46144 )<br>( .0076406 −.83315 .99236 ) |
| 1/64 | $\begin{pmatrix} .49254 & .26043 & .095061 \\ .0040693 & .21371 & .16657 \\ .0014853 & .16657 & .15007 \end{pmatrix}$ | .5<br>.34429<br>.012030 | ( .28975 .42787 .28238 )<br>( −.019491 .54346 .45654 )<br>( .0039307 −.82772 .99607 ) |
| 1/128 | $\begin{pmatrix} .49618 & .26319 & .093775 \\ .0020561 & .21527 & .16714 \\ .0007326 & .16714 & .15036 \end{pmatrix}$ | .5<br>.34941<br>.012407 | ( .28647 .43078 .28276 )<br>( −.0099159 .54611 .45389 )<br>( .0019944 −.82484 .99801 ) |
| 1/256 | $\begin{pmatrix} .49807 & .26460 & .093097 \\ .0010336 & .21607 & .16743 \\ .0003636 & .16743 & .15050 \end{pmatrix}$ | .5<br>.35204<br>.012604 | ( .28480 .43226 .28294 )<br>( −.0050020 .54748 .45252 )<br>( .0010047 −.82336 .99900 ) |
| $\to 0$ | $\begin{pmatrix} .5 & .26604 & .092396 \\ 0 & .21688 & .16772 \\ 0 & .16772 & .15064 \end{pmatrix}$ | .5<br>.35472<br>.012806 | ( .28312 .43376 .28312 )<br>( 0 .54889 .45111 )<br>( 0 −.82184 1 ) |
| 0 | $\begin{pmatrix} .21688 & .16772 \\ .16772 & .15064 \end{pmatrix}$ | .35472<br>.012806 | ( .54889 .45111 )<br>( −.82184 1 ) |

For non-zero mass at nought, we have proved the relationship $(I - 2W^-)\Gamma_+ = \Gamma_-$. For zero mass at nought, we have $\Gamma_-(I - 2W^-) = \Gamma_+$.

# Appendix C

# Programs for Calculating Full Winding Operators

This appendix contains the C code programs used to find the norms of the half winding operators for Chapters 4 and 5.

The definitions in `NormN.h` (p. 130, column 1, lines 13–26) allow us to pass matrices and vectors between routines using pointers. We can also use the easily read notation of `Mx (A,i,j)` for the $(i,j)$th entry of an $N$ by $N$ matrix $A$ and `Vr (w,i)` for the $i$th entry of a vector $w$ of dimension $N$. The notation `DMx` and `DVr` refer to matrices and vectors with dimensions twice the size.

The main routine is in `NormN.c` (p. 131, 1, 20–37). The program requests the input of the masses and the distances between them and stores their reciprocals.

The routine `FindPoly` (p. 134,1,5) constructs the matrix $Q$ and then finds the co-efficients of $p(\lambda) := \det\,(Q - \lambda V + \lambda^2)$ by solving

$$\begin{pmatrix} p(1) \\ p(2) \\ \vdots \\ p(2|S|+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^{2|S|} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2|S|+1 & (2|S|+1)^2 & \cdots & (2|S|+1)^{2|S|} \end{pmatrix} \mathbf{p}$$

so that $p(\lambda) = (1\ \lambda\ \cdots\ \lambda^{2|S|})\mathbf{p}$ and $[\mathbf{p}]_k$ is the coefficient of $\lambda^k$. The program assumes that this polynomial has distinct roots and (in `FindRoots`, p. 135, 1, 7) repeatedly differentiates the polynomial. The root of the linear polynomial obtained by the final differentiation is

found and then, inductively, the roots of the $k$ degree polynomial are found using bisection on each of the $k$ intervals formed by dissecting the real line at the roots of the $k-1$ degree polynomial.

The routine `CalcW` (p. 135, 2, 13) finds the solutions to $(Q - \lambda V + \lambda^2) = 0$, a task easily performed due to the tri-diagonal structure of the matrix. Then $W^-$ is calculated using equation 4.10.

Procedures concerned with displaying the results are omitted.

Some results from these programs are given in Appendix B.

NormN.h

```
/*Finding the Norm of Pi-+Pi+-    David  */
/*for three masses.               Marles */
#include <stdio.h>
#include <stdlib.h>
#include <math.h>

#define N 2 /*FOR OTHER DIMENSIONS,
              CHANGE DEFN OF N*/
            /*THEN touch *.c and compile  */

#define EPS         1E-15
#define DEPS        1E-50
#define Mtrx        struct Matrix
#define Mx(A,i,j)   (*A).x[i][j]
#define Vctr        struct Vector
#define Vr(v,i)     (*v).x[i]
#define DMtrx       struct DoubleMatrix
#define DMx(A,i,j)  (*A).x[i][j]
#define DVctr       struct DoubleVector
#define DVr(v,i)    (*v).x[i]
#define DVctr       struct DoubleVector

Vctr  {double x[N];};
Mtrx  {double x[N][N];};
DVctr {double x[N+N];};
DMtrx {double x[N+N][N+N];};

/*OUTPUT*/
extern FILE  *fp,*fq;

/*ROOTS AND MASSES AND THEIR POSITIONS*/
extern Vctr thetaV,*theta,thetamV,
                        *thetam,mV,*m,dV,*d;

/*MATRICES*/
extern Mtrx PiM,*Pi,kM,*k,MM,*M,AM,*A,QM,*Q;
extern Mtrx EfnM,*Efn,EfnmM,*Efnm,GpM,*Gp,GmM,*Gm;

/*POLYNOMIAL CO-EFFS AND ROOTS*/
extern DMtrx pDM,*p,rootDM,*root;

/*EIGENVALUES AND EIGENVECTORS*/
extern Vctr evalV,*eval,wV,*w;

/*DISPLAY*/

void   line();
double ex (int n);
void   sig (FILE *ft,double x, int n);
int    length (double x);
void   pwr (FILE *ft,double x,int n);
void   disp (FILE *fp,double x, int n);
void    PrintMatrix(FILE *ft, Mtrx *A,int acc);
void   DPrintMatrix(DMtrx *A,int acc);
void   PrintMatrixMult(Mtrx *A, Mtrx *B,
                       Mtrx *C,int acc);
void   PrintRightVectorMult(Mtrx *A, Vctr *v,
                       Vctr *w,double theta,int acc);
void   PrintLeftVectorMult (Vctr *v, Mtrx *B,
                       Vctr *w,double theta,int acc);
void   PrintEqtn (int n,int acc);
void   PrintDeterminant (Mtrx *A,int n);

/*MATRIX*/

void    MatrixEquals( Mtrx *A,  Mtrx *B);
void   DMatrixEquals(DMtrx *A, DMtrx *B);
void    TransposeMatrix ( Mtrx *A);
void   DTransposeMatrix (DMtrx *A);
void   MatrixMult(Mtrx *A, Mtrx *B, Mtrx *C);
void    RightVectorMult( Mtrx *A,  Vctr *v,
                                    Vctr *w);
void   DRightVectorMult(DMtrx *A, DVctr *v,
                                   DVctr *w);
void   LeftVectorMult(Vctr *v, Mtrx *B, Vctr *w);
double  Cofactor( Mtrx *A,int n,int m);
double DCofactor(DMtrx *A,int n,int m);
double  Determinant ( Mtrx *A,int n);
double DDeterminant (DMtrx *A,int n);
void    InvertMatrix( Mtrx *A,  Mtrx *B);
void   DInvertMatrix(DMtrx *A, DMtrx *B);
void   FindRightEFn (Mtrx *A,Vctr *B,double theta);
void   FindLeftEFn (Vctr *A,Mtrx *B,double theta);

/*CALCULATIONS*/

void   FindPoly();
double q(double x,int n);
double RootBetween(int n,double lo, double hi);
void   FindRoots(int n);
void   CalcW();
void   CharEqtn(Mtrx *A);
void   CalcNorm();

/*CONTROL*/

void   Start();
void   Finish();
void   InputData();
void   SetQ();
void   warn(double x,double a,double b,int n);
void   EigenVectors(Mtrx *A, Vctr *v);
void   GammaPlus();
void   GammaMinus();
```

NormN.c

```
/*Finding the Norm of Pi-+Pi+-    David  */
/*for three masses.               Marles */
#include "NormN.h"

FILE  *fp,*fq;                    /*OUTPUT*/    (5)
    /*ROOTS AND MASSES AND THEIR POSITIONS*/
Vctr thetaV,*theta,mV,*m,dV,*d;
Vctr thetamV,*thetam;

Mtrx PiM,*Pi,kM,*k,MM,*M;       /* SINGLE */   (10)
Mtrx AM,*A,QM,*Q,EfnM,*Efn;     /*MATRICES*/
Mtrx GpM,*Gp,EfnmM,*Efnm,GmM,*Gm;

           /*POLYNOMIAL CO-EFFS AND ROOTS*/
DMtrx pDM,*p,rootDM,*root;                      (15)

          /*EIGENVALUES AND EIGENVECTORS*/
Vctr evalV,*eval,wV,*w;

main(){                                         (20)

  Start();
  InputData();
  FindPoly();
  line();                                       (25)
  fprintf (fp,"\nEquation formed from
                        these values is:\n");
  PrintEqtn(N+N-1,5);
  FindRoots(N+N-1);
  CalcPi();                                     (30)
  line();
  CalcNorm();
  line();
  EigenVectors(Pi,eval);
  GammaPlus();                                  (35)
  GammaMinus();
  Finish();

}
```

## Control.c

```
/*Finding the Norm of Pi-+Pi+-    David  */
/*for three masses.               Marles */
#include "NormN.h"

void Start(){

  theta=&thetaV;
  thetam=&thetamV;
  m=&mV;
  d=&dV;
  Pi=&PiM;
  k=&kM;
  M=&MM;
  A=&AM;
  Q=&QM;
  p=&pDM;
  root=&rootDM;
  eval=&evalV;
  w=&wV;
  Gp=&GpM;
  Gm=&GmM;
  Efn=&EfnM;
  Efnm=&EfnmM;
  fp=fopen("output.txt","w");
  fq=fopen("messages.txt","w");
  fprintf (fp,"Finding norm of Pi Pi for
                      Lebesgue Measure on R+");
  fprintf (fp," and %i masses in R-.\n",N);
  fprintf (fp,"000 denotes less than %4.1g",DEPS);
  fprintf (fp,"\nxEy denotes x * 10^y.
                        xey denotes x * 10^-y.");
  fprintf (fp,"\nExponent of ++ or --
                  denotes outside [-99,+99].\n");
  fprintf (fq,"Finding norm of Pi Pi for
                      Lebesgue Measure on R+");
  fprintf (fq," and %i masses in R-.\n\n",N);
  fprintf (fq,"Warnings and messages:\n\n");
  line();
}

void Finish(){

  fprintf (fp,"\n\f");
  fclose (fp);
  fclose (fq);
  system ("emacs --title=NormN_Messages --g
                    91x42+0+0 messages.txt &");
  system ("emacs --title=NormN_Output --g
                 91x42+150+150 output.txt &");
  printf ("\nCompleted\n");

}

void InputData(){
  int i;

  for (i=0;i<N;i++){
    Vr(m,i)=-1;
    Vr(d,i)=-1;
  }
  for (i=0;i<N;i++)
    while (Vr(m,i)<0){
      printf (" Enter mass %i : ",i);
      scanf ("%lf",&(Vr(m,i)));
      Vr(m,i)=1/Vr(m,i);
    }
  for (i=0;i<N-1;i++)
    while (Vr(d,i)<0){
      printf (" Enter distance from mass %i
                                  to mass %i : ",i,i+1);
      scanf ("%lf",&(Vr(d,i)));
      Vr(d,i)=1/Vr(d,i);
    }
  for (i=0;i<N-1;i++){
    fprintf (fp,"\nMass %i : ",i);
    disp (fp,1/Vr(m,i),8);
    fprintf (fp,"     Distance to mass %i : ",i+1);
    disp (fp,1/Vr(d,i),8);
  }
  fprintf (fp,"\nMass %i : ",N-1);
  disp (fp,1/(Vr(m,N-1)),8);
}

void SetQ(){
  int i,j;

  Mx(Q,0,0)=-(Vr(m,0))*(Vr(d,0));
  Mx(Q,0,1)=-Mx(Q,0,0);
  for (j=2;j<N;j++)
    Mx(Q,0,j)=0;

  for (i=1;i<N-1;i++){
    for (j=0;j<i-1;j++)
      Mx(Q,i,j)=0;
    Mx(Q,i,i-1)=(Vr(m,i))*(Vr(d,i-1));
    Mx(Q,i,i+1)=(Vr(m,i))*(Vr(d,i));
    Mx(Q,i,i)=-Mx(Q,i,i-1)-Mx(Q,i,i+1);
    for (j=i+2;j<N;j++)
      Mx(Q,i,j)=0;
  }

  for (j=0;j<N-2;j++)
    Mx(Q,N-1,j)=0;
  Mx(Q,N-1,N-2)=(Vr(m,N-1))*(Vr(d,N-2));
  Mx(Q,N-1,N-1)=-Mx(Q,N-1,N-2);
}

  void warn(double x,double a,double b,int n){
    int i;

    i=-1;
    while (x<1){
```

```
    x=x*10;
    i++;
  }
  fprintf (fq,"\n\n     1000 ITERATIONS
              COMPLETED WHILE FINDING ROOT");  (5)
  fprintf (fq,"\n     Accuracy=10^-%i
                              root between ",i);
  disp (fq,a,20);
  fprintf (fq," and");
  disp (fq,b,20);                              (10)
  fprintf (fq,"\n
                   polynomial at 1st value = ");
  disp (fq,q(a,n),20);
  fprintf (fq,"\n
                polynomial at 2nd value = "); (15)
  disp (fq,q(b,n),20);
  fprintf (fq,"\n\n");
}

void EigenVectors(Mtrx *A,Vctr *u){            (20)
  int i;
  Vctr vV,*v;

  v=&vV;
  fprintf (fp,"Right eigenvectors :\n");       (25)
  for (i=0;i<N;i++){
    FindRightEFn(A,v,Vr(u,i));
    RightVectorMult (A,v,w);
    PrintRightVectorMult(A,v,w,Vr(u,i),10);
  }                                            (30)
  line();
  fprintf (fp,"Left eigenvectors :\n");
  for (i=0;i<N;i++){
    FindLeftEFn(v,A,Vr(u,i));
    LeftVectorMult (v,A,w);                    (35)
    PrintLeftVectorMult(v,A,w,Vr(u,i),10);
  }
}
```

CalcN.c

```
/*Finding the Norm of Pi-+Pi+-    David  */
/*for three masses.               Marles */
#include "NormN.h"

void FindPoly(){
  DMtrx AM,*A,CM,*C;
  Mtrx BM,*B;
  DVctr wV,*w,uV,*u;
  int i,j,k;
  double x,y;

  u=&uV;
  C=&CM;
  w=&wV;
  A=&AM;
  B=&BM;
  SetQ();
  x=1.0;
  for (k=0;k<N+N;k++){
    MatrixEquals(Q,B);
    for (i=0;i<N;i++)
      Mx(B,i,i)=Mx(B,i,i)+x*x;
    Mx(B,0,0)=Mx(B,0,0)-(Vr(m,0)*x);
    Vr (w,k)=Determinant(B,N);
    Vr (w,k)=Vr (w,k)/(k+1);
    x=x+1.0;
  }
  x=1.0;
  for (i=0;i<N+N;i++){
    y=1.0;
    for (j=0;j<N+N;j++){
      Mx(A,i,j)=y;
      y=y*x;
    }
    x=x+1.0;
  }
  DInvertMatrix (A,C);
  DRightVectorMult(C,w,u);
  for (i=0;i<N+N;i++){
    Mx(p,N+N-1,i)=Vr(u,i);
  }
}

double q(double x,int n){
  int i;
  double val;

  val=0.0;
  for (i=0;i<=n;i++)
    val=val+(Mx(p,n,i))*pow(x,(double)i);
  return (val);
}

double RootBetween(int n,double lo, double hi){
  int midsgn,hisgn,i;
  double mid;

  i=0;
  if (q(lo,n)*q(hi,n)>0){
    fprintf (fq,"\n************");
    fprintf (fq,"\nAttempting to find root between");
    disp (fq,lo,12);
    fprintf (fq," and");
    disp (fq,hi,12);
    fprintf (fq,"\nPolynomial of degree %i",n);
    fprintf (fq," takes values");
    disp (fq,q(lo,n),12);
    fprintf (fq," and");
    disp (fq,q(hi,n),12);
    fprintf (fq," at these points");
    fprintf (fq,".\n************\n");
  }
  hisgn=-1;
  if (q(hi,n)>0)
    hisgn=1;
  while (hi-lo>DEPS){
    if (fabs(q(hi,n))<DEPS){
      fprintf (fq,"\n\n     HIT ROOT");
      disp (fq,hi,20);
      fprintf (fq," on ITERATION %i
                              where polynomial =",i);
      disp (fq,q(hi,n),20);
      return (hi);
    }
    if (fabs(q(lo,n))<DEPS){
      fprintf (fq,"\n\n     HIT ROOT");
      disp (fq,lo,20);
      fprintf (fq," on ITERATION %i
                              where polynomial =",i);
      disp (fq,q(lo,n),20);
      return (lo);
    }
    i++;
    if (i==1000){
      warn(hi-lo,lo,hi,n);
      mid=lo;
      if (fabs (q(hi,n))<fabs (q(lo,n)))
  mid=hi;
      fprintf (fq,"          Chosen root        ");
      disp (fq,mid,20);
      fprintf (fq,"\n");
      return (mid);
    }
    mid=(hi/2)+(lo/2);
    midsgn=-1;
    if (q(mid,n)>0)
      midsgn=1;
    if (midsgn*hisgn==1)
      hi=mid;
    else
      lo=mid;
    }
```

```
  fprintf (fq,"          Chosen root          ");
  disp (fq,mid,20);
  fprintf (fq,"\n");
  return (mid);
}

void FindRoots(int n){
  int i,j;
  double step,new,lo,hi;

  for(i=n-1;i>0;i--)
    for(j=i;j>=0;j--)
      Mx(p,i,j)=Mx(p,i+1,j+1)*((float)j+1.0);
  Mx(root,1,1)=-Mx(p,1,0)/Mx(p,1,1);
  for (i=2;i<=n;i++){
    step=1.0;
    lo=q(Mx(root,i-1,1),i);
    new=Mx(root,i-1,1)-1;
    new=new-step;
    step=step*2;
    while (q(new,i)*lo>0){
      new=lo-step;
      step=step*2;
    }
    fprintf (fq,"\n\nFINDING ROOTS OF
            POLYNOMIAL DEGREE %i with CO-EFFS",i);
    for (j=i;j>=0;j--){
      fprintf (fq,"\nCo-eff of x^%i ",j);
      disp (fq,Mx(p,i,j),20);
    }
    Mx(root,i,1)=RootBetween(i,new,Mx(root,i-1,1));
    step=1.0;
    hi=q(Mx(root,i-1,i-1),i);
    new=Mx(root,i-1,i-1)+1;
    step=step*2;
    while (q(new,i)*hi>0){
      new=new+step;
      step=step*2;
    }
    Mx(root,i,i)=
          RootBetween(i,Mx(root,i-1,i-1),new);
    if (i>2)
      for (j=2;j<i;j++)
       Mx(root,i,j)=RootBetween(i,Mx(root,i-1,j-1),
                                Mx(root,i-1,j));
  }
  fprintf (fp,"\nRoots of polynomial are:\n");
  for (j=1;j<=n;j++){
    fprintf (fp,"       ");
    disp (fp,Mx(root,n,j),10);
    fprintf (fp,"           ");
    fprintf (fp,"where polynomial=");
    disp (fp,q(Mx(root,n,j),n),10);
    fprintf (fp,"\n");
  }
  if (n==N+N-1){
    Mx(root,N+N-1,0)=0;
    for (j=0;j<N;j++){
      Vr(thetam,j)=Mx(root,N+N-1,j);
      Vr(theta,j)=Mx(root,N+N-1,j+N);
      fprintf (fp,"\ntheta+[%i]=",j);
      disp (fp,Vr(theta,j),10);
      fprintf (fp,"          theta-[%i]=",j);
      disp (fp,Vr(thetam,j),10);
    }
  }
}

void CalcW(){
  int i,j,i2,s,t;
  double val,x;
  Mtrx AM,*A;
  Vctr BV,*B;

  A=&AM;
  B=&BV;
  for (i=0;i<N;i++){
    x=Vr(theta,i);
    MatrixEquals(Q,A);
    for (i2=0;i2<N;i2++)
      Mx(A,i2,i2)=Mx(A,i2,i2)+x*x;
    Mx(A,0,0)=Mx(A,0,0)-(Vr(m,0)*x);
    Mx(Efn,i,0)=1;
    Mx(Efn,i,1)=-Mx(A,0,0)/Mx(A,0,1);
    for (j=1;j<N-1;j++){
      Mx(Efn,i,j+1)=-(Mx(A,j,j-1)*Mx(Efn,i,j-1))-
                          (Mx(A,j,j)*Mx(Efn,i,j));
      Mx(Efn,i,j+1)=Mx(Efn,i,j+1)/Mx(A,j,j+1);
    }
  }
  InvertMatrix (Efn,k);
  for (i=0;i<N;i++)
    for (j=i;j<N;j++){
      val=0.0;
      for (s=0;s<N;s++)
for (t=0;t<N;t++)
      val=val+(Mx(k,i,s)*(Mx(k,j,t)))/(Vr(theta,s)
                                        +Vr(theta,t));
      Mx(Pi,i,j)=val*(Vr(m,i));
      Mx(Pi,j,i)=val*(Vr(m,j));
    }
  line();
  fprintf (fp,"The operator Pi +- Pi -+ equals\n");
  PrintMatrix(fp,Pi,20);
}

void CharEqtn(Mtrx *A){
  int i,j;
  Mtrx BM,*B,CM,*C;
  Vctr wV,*w,uV,*u;
  double x,y;
```

```
  u=&uV;
  w=&wV;
  B=&BM;
  C=&CM;
  x=1.0;
  for (i=0;i<N;i++){
    MatrixEquals(A,C);
    for (j=0;j<N;j++)
      Mx(C,j,j)=Mx(C,j,j)-x;
    Vr (w,i)=Determinant(C,N);
    if (N%2==1)
      Vr (w,i)=-Vr (w,i);
    Vr (w,i)=Vr (w,i)-(pow(x,(double)N));
    x=x+1.0;
  }
  x=1.0;
  for (i=0;i<N;i++){
    y=1.0;
    for (j=0;j<N;j++){
      Mx(B,i,j)=y;
      y=y*x;
    }
    x=x+1.0;
  }
  InvertMatrix (B,C);
  RightVectorMult(C,w,u);
  for (i=0;i<N;i++){
    Mx(p,N,i)=Vr(u,i);
  }
  Mx(p,N,N)=1;
  for(i=N-1;i>0;i--){
    for(j=i;j>=0;j--){
      Mx(p,i,j)=Mx(p,i+1,j+1)*((float)j+1.0);
    }
  }
}

void CalcNorm(){
  int j;

  CharEqtn(Pi);
  fprintf (fq,"\n\nFor the matrix Pi Pi:");
  PrintDeterminant (Pi,8);
  fprintf (fp,"\nCharacteristic equation for
                              this matrix is:\n");
  PrintEqtn (N,7);
  FindRoots(N);
  for (j=0;j<N;j++){
    Vr(eval,j)=Mx(root,N,j+1);
  }
}

void GammaPlus(){
  Mtrx BM,*B,CM,*C;
  int i,j;

  TransposeMatrix(Efn);
  B=&BM;
  C=&CM;
  for (i=0;i<N;i++)
    for (j=0;j<N;j++)
      Mx(B,i,j)=-Mx(Efn,i,j)*Vr(theta,j);
  InvertMatrix(Efn,C);
  MatrixMult(B,C,Gp);
  line();
  fprintf (fp,"\nGamma plus equals :\n");
  PrintMatrix (fp,Gp,12);/*
  fprintf (fp,"\n\nMatrix of eigenvectors:\n");
  PrintMatrix (fp,Efn,12);*/
  fprintf (fp,"\n");
  for (i=0;i<N;i++)
    Vr(theta,i)=-Vr(theta,i);
  EigenVectors(Gp,theta);
}

void GammaMinus(){
  int i,j,i2,s,t;
  double val,x;
  Mtrx AM,*A,CM,*C;
  Vctr vV,*v,uV,*u;

  A=&AM;
  v=&vV;
  u=&uV;
  for (i=0;i<N;i++){
    x=Vr(thetam,i);
    MatrixEquals(Q,A);
    for (i2=0;i2<N;i2++)
      Mx(A,i2,i2)=Mx(A,i2,i2)+x*x;
    Mx(A,0,0)=Mx(A,0,0)-(Vr(m,0)*x);
    Mx(Efnm,0,i)=1;
    Mx(Efnm,1,i)=-Mx(A,0,0)/Mx(A,0,1);
    for (j=1;j<N-1;j++){
      Mx(Efnm,j+1,i)=-(Mx(A,j,j-1)*Mx(Efnm,j-1,i))-
                          (Mx(A,j,j)*Mx(Efnm,j,i));
      Mx(Efnm,j+1,i)=Mx(Efnm,j+1,i)/Mx(A,j,j+1);
    }
  }
  C=&CM;
  for (i=0;i<N;i++)
    for (j=0;j<N;j++)
      Mx(A,i,j)=Mx(Efnm,i,j)*Vr(thetam,j);
  InvertMatrix(Efnm,C);
  MatrixMult(A,C,Gm);
  line();
  fprintf (fp,"\nGamma minus equals :\n");
  PrintMatrix (fp,Gm,12);
  fprintf (fp,"\n");
  EigenVectors(Gm,thetam);   }
```

## MatrixN.c

```
/*Finding the Norm of Pi-+Pi+-    David  */
/*for three masses.               Marles */
#include "NormN.h"

void MatrixEquals(Mtrx *A, Mtrx *B){
  int i,j;

  for (i=0;i<N;i++)
    for (j=0;j<N;j++)
      Mx (B,i,j)=Mx(A,i,j);
}

void DMatrixEquals(DMtrx *A, DMtrx *B){
  int i,j;

  for (i=0;i<N+N;i++)
    for (j=0;j<N+N;j++)
      DMx (B,i,j)=DMx(A,i,j);
}

void TransposeMatrix (Mtrx *A){
  Mtrx BM,*B;
  int i,j;

  B=&BM;
  for (i=0;i<N;i++)
    for (j=0;j<N;j++)
      Mx(B,i,j)=Mx(A,j,i);
  MatrixEquals (B,A);
}

void DTransposeMatrix (DMtrx *A){
  DMtrx BM,*B;
  int i,j;

  B=&BM;
  for (i=0;i<N+N;i++)
    for (j=0;j<N+N;j++)
      DMx(B,i,j)=DMx(A,j,i);
  DMatrixEquals (B,A);
}

void MatrixMult(Mtrx *A, Mtrx *B, Mtrx *C){
  int i,j,k;
  double val;

  for (i=0;i<N;i++)
    for (j=0;j<N;j++){
      val=0.0;
      for (k=0;k<N;k++)
val=val+Mx(A,i,k)*Mx(B,k,j);
      Mx(C,i,j)=val;
    }
}

void RightVectorMult(Mtrx *A, Vctr *v, Vctr *w){
      int i,j;
      double val;

      for (i=0;i<N;i++){
 (5)    val=0.0;
        for (j=0;j<N;j++)
          val=val+((Mx(A,i,j))*(Vr(v,j)));
        Vr(w,i)=val;
      }
(10) }

    void DRightVectorMult(DMtrx *A,DVctr *v,DVctr *w){
      int i,j;
      double val;
(15)
      for (i=0;i<N+N;i++){
        val=0.0;
        for (j=0;j<N+N;j++)
          val=val+((DMx(A,i,j))*(DVr(v,j)));
(20)    DVr(w,i)=val;
      }
    }

    void LeftVectorMult(Vctr *v, Mtrx *B, Vctr *w){
(25)  int i,j;
      double val;

      for (j=0;j<N;j++){
        val=0.0;
(30)    for (i=0;i<N;i++)
          val=val+Vr(v,i)*Mx(B,i,j);
        Vr(w,j)=val;
      }
    }
(35)
    double Cofactor (Mtrx *A, int i, int j){
      Mtrx BM,*B;
      int ia,ib,ja,jb;

(40)  B=&BM;
      ib=0;
      for (ia=0;ia<N;ia++)
        if (ia!=i){
          jb=0;
(45)      for (ja=0;ja<N;ja++)
    if (ja!=j){
      Mx (B,ib,jb)=Mx(A,ia,ja);
      jb++;
    }
(50)      ib++;
        }
      return (Determinant (B,N-1));
    }

(55) double DCofactor (DMtrx *A, int i, int j){
      DMtrx BDM,*B;
```

```
  int ia,ib,ja,jb;
  double val;

  B=&BDM;
  ib=0;
  for (ia=0;ia<N+N;ia++)
    if (ia!=i){
      jb=0;
      for (ja=0;ja<N+N;ja++)
        if (ja!=j){
  val=DMx(A,ia,ja);
  DMx (B,ib,jb)=val;
  jb++;
}
      ib++;
    }
  val=DDeterminant (B,N+N-1);
  return (val);
}

double Determinant (Mtrx *A,int n){
  double det;
  int i,j,s;
  Mtrx BM,*B;

  if (n==1)
    return (Mx(A,0,0));
  B=&BM;
  det=0.0;
  s=1;
  for (i=1;i<n;i++)
    for (j=1;j<n;j++)
      Mx(B,i-1,j-1)=Mx(A,i,j);
  for (i=0;i<n;i++){
    det=det+s*Mx(A,0,i)*Determinant(B,n-1);
    s=-s;
    for (j=1;j<n;j++)
      Mx(B,j-1,i)=Mx(A,j,i);
  }
  return (det);
}

double DDeterminant (DMtrx *A,int n){
  double det;
  int i,j,s;
  DMtrx BDM,*B;

  if (n==1)
    return (DMx(A,0,0));
  B=&BDM;
  det=0.0;
  s=1;
  for (i=1;i<n;i++)
    for (j=1;j<n;j++)
      DMx(B,i-1,j-1)=DMx(A,i,j);
  for (i=0;i<n;i++){
      det=det+s*Mx(A,0,i)*DDeterminant(B,n-1);
      s=-s;
      for (j=1;j<n;j++)
        DMx(B,j-1,i)=DMx(A,j,i);
(5)   }
      return (det);
    }

    void InvertMatrix(Mtrx *A, Mtrx *B){
(10)  Mtrx TransAM,*TransA;
      int i,j,s,t;
      double det;

      TransA=&TransAM;
(15)  fprintf (fq,"\n\nFINDING INVERSE OF");
      PrintMatrix(fq,A,7);

      det=Determinant(A,N);
      PrintDeterminant (A,8);
(20)
      if (fabs(det)<0.0001){
        fprintf (fq,"\n********");
        fprintf (fq,"\nWARNING :
              SMALL DETERMINANT IN 'INVERTMATRIX'\n");
(25)    fprintf (fq,"determinant = %10.8f",det);
        fprintf (fq,"\n********\n\n");
      }

      det=1/det;
(30)
      if (fabs(det)<0.0001){
        fprintf (fq,"\n********");
        fprintf (fq,"\nWARNING :
              LARGE DETERMINANT IN 'INVERTMATRIX'\n");
(35)    fprintf (fq,"reciprocal of determinant =
                                          %10.8f",det);
        fprintf (fq,"\n********\n\n");
      }

(40)  MatrixEquals(A,TransA);
      TransposeMatrix (TransA);
      s=1;
      t=1;
      if (N%2==0)
(45)    t=-1;
      for (i=0;i<N;i++){
        for (j=0;j<N;j++){
          Mx(B,i,j)=det*Cofactor(TransA,i,j)*s;
          s=-s;
(50)    }
        s=s*t;
      }
    }

(55) void DInvertMatrix(DMtrx *A, DMtrx *B){
      DMtrx TransADM,*TransA;
```

```
  int i,j,s,t;
  double det,val;

  TransA=&TransADM;

  det=DDeterminant(A,N+N);
  det=1/det;

  DMatrixEquals(A,TransA);
  DTransposeMatrix (TransA);
  s=1;
  t=-1;
  for (i=0;i<N+N;i++){
    for (j=0;j<N+N;j++){
      val=DCofactor(TransA,i,j);
      DMx(B,i,j)=det*val*s;
      s=-s;
    }
    s=s*t;
  }
}

void FindRightEFn (Mtrx *A,Vctr *w,double theta){
  Mtrx CM,*C;
  Vctr vV,*v;
  int i,j,k;
  double x;

  C=&CM;
  v=&vV;
  MatrixEquals(A,C);
  fprintf (fq,"\nTa daaa\n");
  for (i=0;i<N;i++)
    Mx(C,i,i)=Mx(C,i,i)-theta;
  PrintMatrix (fq,C,10);
  for (i=0;i<N-1;i++)
    for (j=i+1;j<N;j++){
      x=-Mx(C,j,i)/Mx(C,i,i);
      Mx(C,j,i)=0.0;
      for (k=i+1;k<N;k++)
Mx(C,j,k)=Mx(C,j,k)+x*Mx(C,i,k);
    }
  for (i=0;i<N-1;i++)
    Vr(w,i)=0;
  Vr(w,N-1)=1;
  for (i=N-2;i>=0;i--){
    RightVectorMult(C,w,v);
    Vr(w,i)=-Vr(v,i)/(Mx(C,i,i));
  }
  x=0.0;
  for (i=0;i<N;i++)
    x=x+Vr(w,i);
  if (fabs(x)>0.001)
    for (i=0;i<N;i++)
      Vr(w,i)=Vr(w,i)/x;
}

void FindLeftEFn (Vctr *v,Mtrx *B,double theta){
  Mtrx CM,*C;

  C=&CM;
  MatrixEquals (B,C);
  TransposeMatrix (C);
  FindRightEFn (C,v,theta);
}
```

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