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Some topics in mathematical finance

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Some Topics in Mathematical Finance

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Summary

The first part of the thesis deals with the pricing of three very different types of option.

Firstly we look at Parisian options. These knock in or out when a barrier condition is met for a continuously occurring period of time. We derive expressions for the Laplace transforms in maturity of the prices of Parisian down-and-out put and call options. Prices are obtained by numerical inversion of the Laplace transform. We show that this method is accurate and extremely fast.

Secondly we consider continuous arithmetic-average Asian options. We give an expression for the double Laplace transform in strike and maturity of the option price and an efficient method for calculating this. Prices are obtained by numerical inversion of the Laplace transform and compared with results from the literature. We show that this approach is fast and accurate enough to be of practical use.

Thirdly we take an ab-initio approach to the valuation of options on multiple assets, in particular the 'min-put' option. We characterize the behaviour of the minimum process, given by the smallest of the log-asset-prices. We price European min-put options exactly and use a trinomial tree based method to find a fast lower bound for the price of American/Bermudan options.

The second part of the thesis develops the study of two-sector growth models of the form introduced by Arrow and Kurz (1970).

Being purely deterministic, their original model was unable to distinguish between open-loop and closed-loop control of the economy; by allowing stochastic terms into the model, we are able to resolve this difficulty. Moreover, we find that in some cases the model can be solved explicitly in closed form, and we can write down expressions for tax rates and interest rates. This leads to new one-factor interest-rate models, with related taxation policies; numerical examples show very reasonable behaviour.

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Chapter 1

Introduction to the thesis

1.1 Outline

The thesis is split into two unrelated parts. In Part I we attempt to find prices for three very different types of financial option. In Part II we construct a two-sector stochastic growth model, which we use to model the behaviour of tax and interest rates.

The emphasis throughout Part I of the thesis is on practical solutions to pricing problems. In particular, for a pricing method to be of use in the market a trader needs to be able to select the parameters of the model and obtain a corresponding price within a matter of seconds. With this in mind we aim for the informal goal of computing prices in at most a second of CPU time, sacrificing a small amount of accuracy if necessary to achieve this. We check the accuracy of the prices we obtain from these computations by comparing them with both our own benchmark prices and prices from the literature.

In Chapter 2 we look at Parisian options. These knock in or out when a barrier condition is met for a continuously occurring period of time. The work in this chapter is based on that of Chesney, Jeanblanc-Picqué, and Yor (1997) who first defined Parisian options, although we improve on many of their results or obtain them by alternate simpler means. We also give new *explicit* formulae for the Laplace transforms in maturity of the prices of Parisian down-and-out call and put options and show that these Laplace transforms can be inverted extremely rapidly and accurately numerically. The same techniques can be used to find the price of *any* vanilla Parisian option. This material is also presented in Hartley (2002b).

In Chapter 3 we consider continuous arithmetic-average Asian options. The material in this chapter is motivated by the work of Fu, Madan, and Wang (1998) who first considered the double Laplace transform of the Asian option price. However, the expression they gave for this Laplace transform was incorrect in several respects and they consequently failed to proceed any further with it. We give a correct expression for the double Laplace transform and then explain why this expression is very hard to calculate. We avoid this difficulty by providing an alternative method for calculating the Laplace transform numerically using a finite-difference scheme. Finally we obtain prices by numerical inversion of the Laplace transform that are substantially better than any prices previously obtained by Laplace transform methods and that compare well with prices obtained by other means. The material in this chapter is also presented in Hartley (2002a).

In Chapter 4 we take an ab-initio approach to the valuation of options on multiple assets, in particular the 'min-put' option, that is a put option on the minimum of several assets. We characterize the behaviour of the minimum process, given by the smallest of the assets, by finding the law of its distribution at a chosen time conditional on its position at an earlier time. This allows us to price European options exactly and we use a trinomial tree based method to find a fast lower bound for the price of American/Bermudan options. This chapter contains entirely new material some of which appeared in Hartley (2001).

Part II of the thesis develops the study of two-sector growth models of the form introduced by Arrow and Kurz (1970). This part of the thesis is considerably more theoretical than the first, although numerical methods do again feature strongly. Apart from a short section at the end of Appendix B.4 the whole of Part II consists of original material, some of which can also be found in Hartley and Rogers (2003).

Chapter 5 forms the theoretical foundation for this study. Firstly we set out the model and solve the *central planning problem* in which a communist style government, with total control of the economy, wishes to maximize a functional depending on consumption, levels of public services and leisure time. The resulting optimal choices are functions of the level of capital in the economy, i.e. open-loop control. Being purely deterministic, the original model of Arrow & Kurz was unable to distinguish between open-loop and closed-loop control of the economy; by allowing stochastic terms into the model, we are able to resolve this difficulty of interpretation. We then consider the more usual situation where the government controls the economy only through choice of tax rates and the issuing of debt at a particular rate of interest. Can it induce the private sector, consisting of individual optimizing households, to follow the government's original optimal policies? We give conditions on the tax and interest rates for this to be the case. The final section of Chapter 5 deals with the question of what price the private sector would be prepared to pay for a zero-coupon bond.

In Chapter 6 we consider explicit solutions to the problems of the previous chapter. We find that in some important cases the model can be solved explicitly in closed form, to the extent that we can write down expressions for tax rates and interest rates. This leads to new one-factor interest-rate models, with related taxation policies.

In Chapter 7 we demonstrate methods for finding numerical solutions to the central planning problem, given a full specification of the economy. As with the explicit solutions we can then find taxation and interest-rate policies. Unlike the explicit case we are also able to find the stationary distribution for the level of capital in the economy and use finite difference methods to solve a partial differential equation for the price of a zero-coupon bond. The corresponding bond yield curves strongly resemble those found in real markets, with increasing, decreasing and humped curves all present.

In Chapter 8 we give graphs showing explicit and numerical solutions for particular examples and conclude our discussion.

1.2 Notation and abbreviations

We will use the following mathematical notation:

$\sim$	Is distributed as, e.g. $X \sim N(\mu, \sigma^2)$
$\approx$	Is approximately equal to
$\equiv$	Is equivalent to; is defined to be equivalent to
$\dot{f}$	The derivative of f with respect to time
$\hat{f}$	The Laplace transform of the function f
$\binom{m}{k}$	The binomial coefficent $m!/k!(m-k)!$
$\mathbb{1}_{\{A\}}$	The indicator function of A
$\emptyset$	The empty set
$(X)^+$	The positive part of X, i.e. $\max(0, X)$
$X \wedge Y$	The minimum of X and Y
$\langle X, Y \rangle$	The quadratic-covariation process of X and Y
$\mathrm{Cov}[X, Y]$	The covariance of X and Y

$\mathbb{E}[X]$	The expected value of X
$\mathbb{E}[X \mid A]$	The expected value of X conditional on the event A having occured
$\mathbb{E}[X \,;\, A]$	The expected value of X when event A occurs too, i.e. $\mathbb{E}[X \, \mathbb{1}_{\{A\}}]$
$(\mathcal{F}_t)_{t\geq 0}$	The working filtration; all processes are adapted with respect to $\mathcal{F}_t$
$N(\mu, \sigma^2)$	A normal random variable with mean μ and variance σ^2
$\mathrm{O}\left(f(\Delta t)\right)$	A (possibly unknown) function $g(\Delta t)$ such that, for all sufficiently small Δt, $\lvert g(\Delta t)\rvert \leq c\, f(\Delta t)$ for some constant c. We also use the same notation with, for example, Δt replaced by some large value N. The definition stands as above but with Δt replaced by N and the word 'small' replaced by the word 'large'. It is always clear from the context which case applies.
$\mathbb{P}(A)$	The probability of event A
$\mathrm{Re}(X)$	The real part of X
$\mathrm{Var}[X]$	The variance of X

We also use the following abbreviations:

CRRA	Constant-relative-risk-aversion
HJB	Hamilton-Jacobi-Bellman
ODE	Ordinary differential equation
PDE	Partial differential equation
SDE	Stochastic differential equation

In Part I of the thesis the three chapters are each self-contained so any notation developed in one chapter does not carry through to the next. In particular, in Chapter 2 the functions C and P denote the prices of vanilla call and put options respectively; in Chapter 3 the same function names are used to denote the prices of *Asian* calls and puts. The measure $\mathbb{P}$ also varies between chapters: in Chapter 2 it is the measure under which the log-asset price has zero drift, in Chapters 3 and 4 it is the risk-neutral measure and in Part II of the thesis it is the 'real world' measure.

In Part II of the thesis the notation does carry over between chapters. Unfortunately a stochastic growth and taxation model is a complex beast with a corresponding volume of notation. Although some of this is standard growth theory notation a lot of it isn't. In an attempt to preserve the sanity of the reader there is a separate summary in Appendix B.5 of the notation specific to Part II of the thesis.

Part I

Three Option Pricing Problems

Chapter 2

Pricing Parisian options by Laplace inversion

Abstract

This chapter is concerned with Parisian options. These knock in or out when a barrier condition is met for a continuously occurring period of time. We derive expressions for the Laplace transform in maturity of the prices of Parisian down-and-out put and call options. Prices are then obtained by numerical inversion of the Laplace transform. We show that this method is accurate and extremely fast.

2.1 Introduction

Parisian options knock in or out when a barrier condition is met for a continuously occurring period of time. In particular we will consider the Parisian down-and-out option, which becomes worthless if the underlying asset remains under a specified barrier level for a fixed duration of time, but otherwise has the same payoff as a standard European option.

Such options are now widely traded in the over-the-counter market, particularly on exchange rates. In practice the monitoring of the asset price will be discrete so that, for example, the option will knock out if the asset price is below the barrier for five consecutive observation times occurring daily at midday. We focus only on the continuous time results, corresponding to a small time between successive asset price observations.

We follow closely the account of Chesney, Jeanblanc-Picqué, and Yor (1997), from now on referred to as CJY, by considering the Laplace transform in maturity of the Parisian option. We show that the Laplace transform can be evaluated explicitly and inverted extremely rapidly using the Euler method described by Abate and Whitt (1995). Chesney, Cornwall, Jeanblanc-Picqué, Kentwell, and Yor (1997) require a double numerical integration to give accurate results for a Parisian down-and-out put.

Alternatively the problem can be formulated as a partial differential equation in three variables: time, the price of the asset and the duration of time it has spent below the barrier. Vetzal and Forsyth (1999) and Haber, Schönbucher, and Wilmott (1999) use finite-difference methods to numerically obtain Parisian option prices. Zhu and Stokes (1999) use Galerkin finite-element methods. Avellaneda and Wu (1999) solve a PDE numerically using trinomial lattice methods. One further approach is that of Kwok and Lau (2001) who use a forward shooting grid method; this is a variant on the trinomial lattice method where an auxiliary state vector is used at each node on the lattice to capture the path-dependent feature of the option contract; in this case the auxiliary vector is used to characterize the duration of time the asset price has spent below the barrier. All these approaches have one major benefit - they are flexible enough to be easily modified to price more general options. For example Haber, Schönbucher, and Wilmott and Zhu and Stokes consider a 'Parasian' or 'cumulative Parisian' option[1], where the recorded duration is cumulative rather than continuous. The Laplace transform approach is very specific to the problem, but we shall see that what it lacks in flexibility it more than makes up for in accuracy and speed of computation.

This chapter is organized as follows. In Section 2.2 we define the Parisian down-and-out option and establish some parity identities relating it to various other types of Parisian option. In Section 2.3 we show how to price options where the initial asset price is above or below the barrier level in terms of the option price at the barrier. In Section 2.4 we assume that the initial asset price lies on the barrier and obtain explicit expressions for the Laplace transform of the Parisian down-and-out density and for the Laplace transform of the corresponding option prices. In Section 2.5 we give some option prices obtained by numerically inverting these Laplace transforms. We conclude in Section 2.6. Appendix A covers the method used for numerical inversion, an alternative derivation of the Parisian down-and-out density and details of how to perform some of the integrations that appear in the chapter.

[1]Again there is a discrete version of this option where, for example, the option knocks out when the asset price has been below the barrier on any five separate observation dates. In the currency markets this is known as a 'baseball' option as after a certain number of 'strikes' the option is knocked out.

2.2 Parisian options

Let us assume that under the risk-neutral measure $\tilde{\mathbb{P}}$ the asset price S_t obeys the stochastic differential equation

$$dS_t = S_t((r - \delta)dt + \sigma d\tilde{W}_t) \tag{2.1}$$

where r is the interest rate, σ the asset volatility and $\tilde{W}_t$ a standard Brownian motion under $\tilde{\mathbb{P}}$. The symbol δ denotes the continuous dividend rate in the case where S_t models a stock price; if S_t is a currency exchange rate then δ will be the interest rate of the second country. We can solve the SDE (2.1) explicitly and write

$$\begin{aligned} S_t &= S_0 e^{(r-\delta-\frac{1}{2}\sigma^2)t+\sigma\tilde{W}_t} \\ &\equiv e^{\sigma X_t} \end{aligned}$$

where

$$X_t \equiv x_0 + \tilde{W}_t + mt$$

is a Brownian motion with drift $m \equiv \frac{1}{\sigma}(r - \delta - \frac{\sigma^2}{2})$ and starting from $x_0 \equiv \frac{1}{\sigma}\log S_0$ at time 0. Under Black and Scholes (1973) assumptions a vanilla European call option on this asset, with strike price K and maturity T will have time-0 price[2]

$$C(T, x_0; K) \equiv \tilde{\mathbb{E}}^{x_0}[e^{-rT}(e^{\sigma X_T} - K)^+].$$

Here we write $\tilde{\mathbb{E}}^x$ to denote expectation under $\tilde{\mathbb{P}}$ with the process $(X_t)_{0\le t\le T}$ starting from $X_0 = x$. The payoff $(S_T - K)^+$ of the call depends only on the terminal asset price; for a barrier option the payoff will also depend on whether the asset price has reach a barrier level during the lifetime of the option. For example, a down-and-out call option has the same payoff as the vanilla call at time T, provided that the option has not been 'knocked out' by the asset price hitting a level $B < S_0$. If we write[3] $\tau_b \equiv \inf\{t : X_t = b\}$ where $b \equiv \frac{1}{\sigma}\log B$, then this option has time-0 value

$$\begin{aligned} DOC(T, x_0; K, b) &\equiv \tilde{\mathbb{E}}^{x_0}[e^{-rT}\left(e^{\sigma X_T} - K\right)^+ \mathbb{1}_{\{T<\tau_b\}}] \\ &\equiv \tilde{\mathbb{E}}^{x_0}[e^{-rT}\left(e^{\sigma X_T} - K\right)^+; T < \tau_b]. \end{aligned}$$

[2]We use the obvious notation to distinguish between the prices of these various types of option, so for example 'C' stands for 'call', 'PDOC' for 'Parisian down-and-out call' and so on. We think of these prices as functions of the maturity T and initial value x_0; the other (constant) parameters will appear in the function only for clarity or when necessary to make a point.

[3]We follow the usual convention that $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = +\infty$, where $\emptyset$ denotes the empty set.

For $K > B$ it is well known that this can be evaluated in terms of vanilla call prices as

$$DOC(T, x_0; K, b) = C(T, x_0; K) - e^{2m(b-x_0)}C(T, 2b - x_0; K). \tag{2.2}$$

The knock-out feature can substantially reduce the initial cost of a call option. However as the asset price only needs to touch the barrier for the option to be terminated this leaves the holder vulnerable to brief 'spikes' in the asset price or deliberate manipulation of the asset price by the writer of the option. Parisian options avoid this problem by allowing the buyer to specify a minimum amount of time that the asset price must be above/below the barrier in order to knock in/out.

We consider a Parisian down-and-out option which knocks out if the asset price process S_t spends a continuous duration D of time below a barrier level B. We will require that $0 < D < T$. If $D = 0$ then we have a standard down-and-out barrier option. If $D \geq T$ then we are effectively pricing a vanilla European option. If we define $g_{b,t}^- \equiv \sup\{0 \leq u \leq t : X_u > b\}$, the last time before t that the asset price was above the barrier level B, and hence $\tau_b^- \equiv \inf\{t : t - g_{b,t}^- > D\}$, the first time that the asset price spends a continuous time greater than D under the barrier, then a Parisian down-and-out call option has initial value

$$PDOC(T, x_0) = \tilde{\mathbb{E}}^{x_0}[e^{-rT}\left(e^{\sigma X_T} - K\right)^+; T < \tau_b^-].$$

If we now make a change of measure so that under the new measure $\mathbb{P}$, $\tilde{W}_t + mt$ is a standard Brownian motion[4] W_t and hence $X_t = x_0 + W_t$, then

$$\begin{aligned}
PDOC(T, x_0) &\equiv \mathbb{E}^{x_0}[e^{-rT}e^{m(X_T - x_0) - \frac{1}{2}m^2T}\left(e^{\sigma X_T} - K\right)^+; T < \tau_b^-] \\
&= e^{-(r+\frac{1}{2}m^2)T - mx_0}\ \mathbb{E}^{x_0}[\left(e^{\sigma X_T} - K\right)^+ e^{mX_T}; T < \tau_b^-] \\
&\equiv e^{-(r+\frac{1}{2}m^2)T - mx_0}\ \mathbb{E}^{x_0}[f(X_T); T < \tau_b^-] \\
&\equiv e^{-(r+\frac{1}{2}m^2)T - mx_0}\ {}^*PDOC(T, x_0)
\end{aligned}$$

where $f(z) \equiv (e^{\sigma z} - K)^+ e^{mz}$ and ${}^*PDOC(T, x) \equiv \mathbb{E}^x[f(X_T); T < \tau_b^-]$. In Sections 2.3 and 2.4 we will show how to determine the Laplace transform of ${}^*PDOC(T, x)$ and hence evaluate the price of the option numerically. A wider class of options can then be valued by use of various parity relations.

Firstly if a buyer holds both an 'out' option and an 'in' option (which will only pay out if the asset price has stayed under the barrier for duration D) then this is equivalent

[4] This is the Cameron-Martin-Girsanov Theorem; see, for example, Øksendal (1998) for an account.

to holding the vanilla option, so that for example

$$PDOC(T, x_0; K, b, D) + PDIC(T, x_0; K, b, D) = C(T, x_0; K) \tag{2.3}$$

where $PDIC$ denotes the price of a Parisian down-and-in call option. Similar identities hold for put options and up options (when the asset price has to stay *above* a barrier level B to knock in or out).

Secondly consider a Parisian up-and-out put. Following the methodology above this has initial value

$$PUOP(T, x_0) = e^{-(r+\frac{1}{2}m^2)T-mx_0}\,\mathbb{E}^{x_0}[\left(K - e^{\sigma X_T}\right)^+ e^{mX_T}; T < \tau_b^+]$$

where $\tau_b^+ \equiv \inf\{t : t - g_{b,t}^+ > D\}$ and $g_{b,t}^+ \equiv \sup\{0 \leq u \leq t : X_u < b\}$. Using the reflection principle we can replace X_T by $-X_T$ in the above and start the process instead at $-x_0$, so that

$$\begin{aligned}PUOP(T, x_0) &= e^{-(r+\frac{1}{2}m^2)T-mx_0}\,\mathbb{E}^{-x_0}[\left(K - e^{-\sigma X_T}\right)^+ e^{-mX_T}; T < \tau_{-b}^-]\\ &= Ke^{\sigma x_0}e^{-(r+\frac{1}{2}m^2)T+(m+\sigma)(-x_0)}\,\mathbb{E}^{-x_0}[\left(e^{\sigma X_T} - \frac{1}{K}\right)^+ e^{-(m+\sigma)X_T}; T < \tau_{-b}^-].\end{aligned}$$

It can be easily confirmed that

$$\begin{aligned}-(m+\sigma) &= \frac{1}{\sigma}(\delta - r - \frac{\sigma^2}{2}),\\ r + \frac{1}{2}m^2 &= \delta + \frac{1}{2}(m+\sigma)^2\end{aligned}$$

and hence

$$PUOP(T, x_0; K, b, D; \sigma, r, \delta) = Ke^{\sigma x_0}PDOC(T, -x_0; \frac{1}{K}, -b, D; \sigma, \delta, r),$$

or equivalently, abusing our notation slightly:

$$PUOP(T, S_0; K, B, D; \sigma, r, \delta) = K\,S_0\,PDOC(T, \frac{1}{S_0}; \frac{1}{K}, \frac{1}{B}, D; \sigma, \delta, r). \tag{2.4}$$

Similar identities hold for the other types of option.

We can represent the relationships described above diagrammatically:

$$\begin{array}{ccc} \text{PDOC} & \longleftrightarrow & \text{PDIC} \\ \updownarrow & & \updownarrow \\ \text{PUOP} & \longleftrightarrow & \text{PUIP} \end{array}$$

Similarly we have that:

$$\begin{array}{ccc} \text{PDOP} & \longleftrightarrow & \text{PDIP} \\ \updownarrow & & \updownarrow \\ \text{PUOC} & \longleftrightarrow & \text{PUIC} \end{array}$$

If we can value both types of Parisian down-and-out option (calls and puts) then we can find the price of *any* standard Parisian option.

2.3 Reduction to the case $x = b$

We are interested in Parisian down-and-out option prices of the form

$$^*PDO(T, x) \equiv \mathbb{E}^x[f(X_T); T < \tau_b^-]$$

where $f(z) \equiv (e^{\sigma z} - K)^+ e^{mz}$ for a call or $f(z) \equiv (K - e^{\sigma z})^+ e^{mz}$ for a put. We could equally well determine the Laplace transform of this and then use numerical inversion to find the option price. In Section 2.4 we will show how to do this when $x = b$, i.e. when the asset price process S_t starts at the barrier B. In this section we will show how to reduce the cases where the asset price starts above or below the barrier to the $x = b$ case. Firstly, a remark about Laplace transforms.

2.3.1 The Laplace transform : an example

Suppose that we have a function of the form

$$F(T, x) = \mathbb{E}^x[f(X_T); \tau_1 < T < \tau_2] \tag{2.5}$$

where $T > 0$, $(X_t)_{t \geq 0}$ is some process and τ_1 and τ_2 are $\mathcal{F}_t^X$ adapted stopping times where $(\mathcal{F}_t^X)_{t \geq 0}$ denotes the natural filtration of the process X. Laplace transforming

(2.5) in T yields for λ in the right complex half-plane

$$\begin{aligned}\hat{F}(\lambda,x) &= \int_0^\infty e^{-\lambda T}\,\mathbb{E}^x[f(X_T);\tau_1<T<\tau_2]\,dT\\ &= \frac{1}{\lambda}\mathbb{E}^x\left[\int_0^\infty \lambda e^{-\lambda T} f(X_T)\mathbb{1}_{\{\tau_1<T<\tau_2\}}\,dT\right]\\ &= \frac{1}{\lambda}\mathbb{E}^x[f(X_\xi);\tau_1<\xi<\tau_2]\end{aligned}$$

where ξ is an exponentially distributed random variable[5] with parameter λ. Crucially ξ is independent of $\mathcal{F}_t^X$ and hence independent of the process X and the stopping times τ_1 and τ_2.

2.3.2 Case 1 : $x < b$

Here the Brownian motion X starts below the barrier. Condition on the random time $\tau_b \equiv \inf\{t : X_t = b\}$, the first moment that X hits b. This either occurs before time D and thus before τ_b^-, or after time D in which case the option knocks out and expires worthless as we assume that $D < T$. Thus

$$\begin{aligned}\mathbb{E}^x[f(X_T);T<\tau_b^-] &= \mathbb{E}^x[f(X_T);T<\tau_b^-,D\le\tau_b] + \mathbb{E}^x[f(X_T);T<\tau_b^-,\tau_b<D]\\ &= \mathbb{E}^x[f(X_T);\tau_b<D,T<\tau_b^-]. \qquad (2.6)\end{aligned}$$

If we now condition on the exact time before D that X hits b we have that

$$\mathbb{E}^x[f(X_T);T<\tau_b^-] = \int_0^D n_t(b-x)\,\mathbb{E}^x[f(X_T);T<\tau_b^-|X_t=b]\,dt, \qquad (2.7)$$

where

$$n_t(y) \equiv \frac{|y|}{\sqrt{2\pi t^3}}e^{-\frac{1}{2t}y^2} \qquad (2.8)$$

denotes the entrance law of Brownian motion. This gives the distribution of the first hitting time of level y by a Brownian motion starting from 0. When the process X reaches b we can apply the strong Markov property and start the system again from b,

[5]It's not obvious that this is a well defined concept for complex λ, but the intuition we have about exponential random variables proves to be very helpful in motivating the results that follow. All the calculations could equally well be performed explicitly using integrals involving $\lambda e^{-\lambda T}$ rather than expectations involving ξ.

so that equation (2.7) gives us

$$^*PDO(T,x) = \int_0^D {}^*PDO(T-t,b)\, n_t(b-x)\, dt.$$

A similar identity appears in CJY. We have reduced valuation of the option starting from $x < b$ to an integral over options with $x = b$ but *different maturities.* We can do much better however. We are interested in finding

$$^*PDO(T,x) = \mathbb{E}^x[f(X_T); \tau_b < D, T < \tau_b^-].$$

We will define the function

$$F(T,x) \equiv \mathbb{E}^x[f(X_T); \tau_b < D, \tau_b \wedge D < T < \tau_b^-].$$

This function is identical to $^*PDO(T,x)$ on the set $\{D < T\}$ and so we can compute prices using $F(T,x)$ in the place of $^*PDO(T,x)$ as we always know that $D < T$. Following along the lines of the example outlined in Section 2.3.1 the Laplace transform in maturity of this expression is

$$\begin{aligned}
\hat{F}(\lambda,x) &= \frac{1}{\lambda}\mathbb{E}^x[f(X_\xi); \tau_b < D, \tau_b \wedge D < \xi < \tau_b^-] \\
&= \frac{1}{\lambda}\int_0^D e^{-\lambda t} n_t(b-x)\, \mathbb{E}^x[f(X_\xi); \xi < \tau_b^- | X_t = b, \xi > t]\, dt \\
&= {}^*\widehat{PDO}(\lambda,b) \int_0^D e^{-\lambda t} n_t(b-x)\, dt. \\
&= {}^*\widehat{PDO}(\lambda,b) \left[e^{-\theta(b-x)}\Phi\left(\theta\sqrt{D} - \frac{b-x}{\sqrt{D}}\right) + e^{\theta(b-x)}\Phi\left(-\theta\sqrt{D} - \frac{b-x}{\sqrt{D}}\right)\right] \qquad (2.9)
\end{aligned}$$

where $\theta \equiv \sqrt{2\lambda}$ and $\Phi(x) \equiv \int_{-\infty}^x \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$ is the standard cumulative normal distribution function. Note the considerable improvement! We condition on the time that the process hits b and *that* ξ *has not yet occurred* (hence the factor of $e^{-\lambda t}$). We can now use the lack-of-memory property of the exponential distribution and the strong Markov property to treat the entire system as having started again from b. Finally we can evaluate the integral on the second from last line explicitly. See Appendix A.3 for the details.

Suppose now that we would like to price an option that has already spent a duration of time $D_0 < D$ below the barrier. This is a case considered by Schröder (2002) who goes to some effort to modify the expressions given by CJY for the Laplace transform of the down-and-in density to allow for this. However, the modification needed to our approach is essentially trivial - we simply replace D by $D - D_0$ in equation (2.9) above.

2.3.3 Case 2 : $x > b$

Again we can condition on the first hitting time of b. If $T \leq \tau_b$ then clearly also $T < \tau_b^-$, so

$$\mathbb{E}^x[f(X_T); T < \tau_b^-] = \mathbb{E}^x[f(X_T); T \leq \tau_b] + \mathbb{E}^x[f(X_T); \tau_b < T < \tau_b^-].$$

The first term corresponds to a standard (non-Parisian) down-and-out option so we can use the explicit formula (2.2) to price this component. The second term can again best be treated by considering its Laplace transform

$$\begin{aligned}
\frac{1}{\lambda}\mathbb{E}^x[f(X_\xi); \tau_b < \xi < \tau_b^-] &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda t} n_t(x-b)\,\mathbb{E}^x[f(X_\xi); \xi < \tau_b^- | X_t = b, \xi > t]\,dt \\
&= {}^*\widehat{PDO}(\lambda, b)\int_0^\infty e^{-\lambda t} n_t(x-b)\,dt \\
&= {}^*\widehat{PDO}(\lambda, b)\,e^{-\theta(x-b)}. \qquad (2.10)
\end{aligned}$$

As in the $x < b$ case we use the lack-of-memory property of the exponential distribution and the strong Markov property to treat the whole system as starting again from b if the exponential random time ξ has not occurred before the process X hits b.

2.4 Valuation when $x = b$

We wish to find

$$\begin{aligned}
{}^*\widehat{PDO}(\lambda, b) &= \int_{-\infty}^\infty \int_0^\infty e^{-\lambda T} f(y)\,dT\,\mathbb{P}^b(X_T \in dy; T < \tau_b^-) \\
&\equiv \int_{-\infty}^\infty \int_0^\infty f(y)\,e^{-\lambda T} g_b(T, y)\,dT\,dy
\end{aligned}$$

where $g_b(T, y)$ is the density function of X_T on the event that $T < \tau_b^-$ starting from $X_0 = b$ and f is the relevant option payoff function. Writing $\hat{g}_b(\lambda, y)$ for the Laplace transform in maturity T of $g_b(T, y)$ we have that

$$^*\widehat{PDO}(\lambda, b) = \int_{-\infty}^\infty f(y)\,\hat{g}_b(\lambda, y)\,dy.$$

We wish to determine $\hat{g}_b(\lambda, y)$; note that we only need consider the case where $b = 0$ as then for general b we find by translation that

$$\hat{g}_b(\lambda, y) = \hat{g}_0(\lambda, y - b). \tag{2.11}$$

This agreed, we wish to determine

$$\lambda\, \hat{g}_0(\lambda, y)\, dy = \mathbb{P}^0(X_\xi \in dy; \xi < \tau_b^-)$$

where X_t is now a standard Brownian motion and ξ is an exponential distributed random variable with parameter $\lambda \equiv \frac{1}{2}\theta^2$. We will use excursion theory to do this - see Rogers (1989) or Rogers and Williams (2000) for a full account. We think of the path of the Brownian motion X_t as being decomposed into disjoint excursions away from zero, and we then *mark* these excursions with an independent Poisson process of rate λ. The time of the first mark thus corresponds to the value of the random variable ξ. We follow the notation of Rogers (1989) and write n for the excursion measure on the space U of excursions and for $A \subset \mathbb{R}/\{0\}$ we denote the Laplace transform of the entrance law (2.8) by

$$\begin{aligned} n_\lambda(A) &= \int_A \int_0^\infty e^{-\lambda t} n_t(y)\, dt\, dy \\ &= \int_A e^{-\theta|y|}\, dy. \end{aligned}$$

The rate of positive excursions with first mark when the excursion is in dy is

$$\begin{aligned} p_+(y)\, dy &= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \lambda n_\lambda([y - \varepsilon, y + \varepsilon])\, dy \\ &= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \lambda \left(\frac{e^{-\theta(y-\varepsilon)}}{\theta} - \frac{e^{-\theta(y+\varepsilon)}}{\theta} \right) dy \\ &= \lim_{\varepsilon \to 0} \frac{1}{2} \lambda \left(e^{-\theta(y-\varepsilon)} + e^{-\theta(y+\varepsilon)} \right) dy \\ &= \lambda e^{-\theta y}\, dy. \end{aligned}$$

The rate of negative excursions with first mark in dy before time D is

$$
\begin{aligned}
p_-(y)\,dy &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^D\int_{y-\varepsilon}^{y+\varepsilon}\lambda e^{-\lambda t}n_t(x)\,dx\,dt\,dy\\
&= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\lambda\int_0^D\frac{e^{-\lambda t}}{\sqrt{2\pi t}}\left(e^{-\frac{1}{2t}(y+\varepsilon)^2}-e^{-\frac{1}{2t}(y-\varepsilon)^2}\right)dt\,dy\\
&= \lim_{\varepsilon\to 0}\frac{1}{2}\lambda\int_0^D\frac{e^{-\lambda t}}{\sqrt{2\pi t}}\left(-\frac{y+\varepsilon}{t}e^{-\frac{1}{2t}(y+\varepsilon)^2}-\frac{y-\varepsilon}{t}e^{-\frac{1}{2t}(y-\varepsilon)^2}\right)dt\,dy\\
&= \lambda\int_0^D\frac{|y|e^{-\lambda t}}{\sqrt{2\pi t^3}}e^{-\frac{1}{2t}y^2}\,dt\,dy\\
&= \lambda\left[e^{-\theta y}\Phi\left(-\theta\sqrt{D}+\frac{y}{\sqrt{D}}\right)+e^{\theta y}\Phi\left(\theta\sqrt{D}+\frac{y}{\sqrt{D}}\right)\right]dy.
\end{aligned}
$$

See Appendix A.3 for details of the final step.

Finally, the rate of all negative excursions of duration greater than D and with no mark before D is

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\left[n(\{f\in U:\inf_{s>0}f(s)<-\varepsilon\})\,\mathbb{P}^{-\varepsilon}(\inf\{s>0:X_s=0\}>D)\,\mathbb{P}(\xi>D)\right]\\
&= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_D^\infty n_t(\varepsilon)\,dt\,e^{-\lambda D}\\
&= \frac{1}{2}e^{-\lambda D}\int_D^\infty\frac{1}{\sqrt{2\pi t^3}}dt\\
&= \frac{e^{-\lambda D}}{\sqrt{2\pi D}}.
\end{aligned}
$$

Here we have used Proposition 2 from Rogers (1989) which states that

$$
n(\{f\in U:\sup_{t>0}f(t)>a\})=(2a)^{-1}
$$

for each $a>0$.

We know also that

$$
\int_0^\infty p_+(y)\,dy=\frac{1}{2}\theta
$$

and

$$
\int_{-\infty}^0 p_-(y)\,dy=\theta\left[\Phi(\theta\sqrt{D})-\frac{1}{2}\right]
$$

by equation (A.8) of Appendix A.3. Assembling the various pieces:

$$\lambda\,\hat{g}_0(\lambda,y)\,dy = \mathbb{P}^0(X_\xi \in dy; \xi < \tau_b^-) = \frac{p_+(y)\mathbb{1}_{\{y>0\}} + p_-(y)\mathbb{1}_{\{y<0\}}}{\int_0^\infty p_+(y)\,dy + \int_{-\infty}^0 p_-(y)\,dy + \frac{e^{-\lambda D}}{\sqrt{2\pi D}}}\,dy$$

and substituting in the expressions derived above and rearranging we find that

$$\hat{g}_0(\lambda,y) = \frac{\sqrt{2\pi D}e^{\lambda D}}{\Psi(\theta\sqrt{D})}\Big(e^{-\theta y}\mathbb{1}_{\{y>0\}} + \left[e^{-\theta y}\Phi(-\theta\sqrt{D}+\frac{y}{\sqrt{D}}) + e^{\theta y}\Phi(\theta\sqrt{D}+\frac{y}{\sqrt{D}})\right]\mathbb{1}_{\{y<0\}}\Big) \tag{2.12}$$

where

$$\Psi(z) \equiv 1 + z\sqrt{2\pi}e^{\frac{1}{2}z^2}\Phi(z).$$

This expression is consistent with that obtained by CJY for the Laplace transform of the down-and-in density - see Appendix A.4.

2.4.1 Example 1 : Call option with $K > B$

We wish to find

$$^*\widehat{PDOC}(\lambda,b) = \int_{-\infty}^\infty f(y)\,\hat{g}_b(\lambda,y)\,dy.$$

where $f(z) \equiv (e^{\sigma z} - K)^+\,e^{mz}$ and $\hat{g}_b(\lambda,y)$ is as defined in equations (2.11) and (2.12) above. We are interested only in the case where $y > b$ as the call option we are considering only has value if the asset price exceeds $K > B$, so the density we need is given by

$$\hat{g}_b(\lambda,y) = \frac{\sqrt{2\pi D}e^{\lambda D}}{\Psi(\theta\sqrt{D})}e^{-\theta(y-b)}.$$

Setting $k \equiv \frac{1}{\sigma}\log K$ the Laplace transform of $^*PDOC(T,b)$ is then given explicitly by

$$\begin{aligned} ^*\widehat{PDOC}(\lambda,b) &= \frac{\sqrt{2\pi D}e^{\lambda D}\int_k^\infty (e^{\sigma y}-K)\,e^{my}e^{-\theta(y-b)}\,dy}{\Psi(\theta\sqrt{D})} \\ &= \frac{\sqrt{2\pi D}e^{\lambda D - k(\theta-m-\sigma)+\theta b}}{\Psi(\theta\sqrt{D})}\left(\frac{1}{\theta-m-\sigma} - \frac{1}{\theta-m}\right) \end{aligned} \tag{2.13}$$

under the assumption that

$$\mathrm{Re}(\theta - m - \sigma) > 0. \tag{2.14}$$

This assumption does not pose a problem as we shall see in Section 2.5.

2.4.2 Example 2 : Put option with $K > B$

The only difference from the previous example is that the put option will be algebraically much more messy, as we will have to integrate over both the $y > b$ and $y < b$ parts of the density. As before

$$^*\widehat{PDOP}(\lambda, b) = \int_{-\infty}^{\infty} f(y)\, \hat{g}_b(\lambda, y)\, dy$$

where this time we have $f(z) \equiv (K - e^{\sigma z})^+ e^{mz}$. Using equations (2.11) and (2.12) we find that

$$\begin{aligned}
^*\widehat{PDOP}(\lambda, b) &= \int_{-\infty}^{k} f(y)\, \hat{g}_b(\lambda, y)\, dy \\
&= \int_{-\infty}^{k-b} f(u+b)\, \hat{g}_0(\lambda, u)\, du \\
&= \int_{-\infty}^{k-b} \left(e^{\sigma k + m(u+b)} - e^{\sigma(u+b)+m(u+b)}\right) \hat{g}_0(\lambda, u)\, du \\
&= \frac{\sqrt{2\pi D} e^{\lambda D + mb}}{\Psi(\theta\sqrt{D})} \left[\int_0^{k-b} \left(e^{\sigma k + mu} - e^{\sigma b + (\sigma+m)u}\right) e^{-\theta u}\, du \right. \\
&\quad \left. + \int_{-\infty}^{0} \left(e^{\sigma k + mu} - e^{\sigma b + (\sigma+m)u}\right) \left(e^{-\theta u}\Phi\left(-\theta\sqrt{D} + \frac{u}{\sqrt{D}}\right) + e^{\theta u}\Phi\left(\theta\sqrt{D} + \frac{u}{\sqrt{D}}\right)\right) du \right] \\
&= \frac{\sqrt{2\pi D} e^{\lambda D + mb}}{\Psi(\theta\sqrt{D})} \left[\frac{e^{\sigma k}}{m-\theta}\left(e^{(m-\theta)(k-b)} - 1\right) \right. \\
&\quad - \frac{e^{\sigma b}}{\sigma + m - \theta}\left(e^{(\sigma+m-\theta)(k-b)} - 1\right) + e^{\sigma k}\left(I(-\theta\sqrt{D}, m-\theta) + I(\theta\sqrt{D}, m+\theta)\right) \\
&\quad \left. - e^{\sigma b}\left(I(-\theta\sqrt{D}, \sigma+m-\theta) + I(\theta\sqrt{D}, \sigma+m+\theta)\right) \right] \qquad (2.15)
\end{aligned}$$

where

$$I(\alpha, \beta) = \frac{1}{\beta}\left[\Phi(\alpha) - e^{\frac{1}{2}\beta^2 D - \alpha\beta\sqrt{D}}\Phi(\alpha - \beta\sqrt{D})\right].$$

See Appendix A.3 for more details on the integration. The other piece of information we need for valuation is the price of the conventional down-and-out put option which can be expressed as

$$\begin{aligned}
DOP(T, x_0; K) &= P(T, x_0; K) - P(T, x_0; B) - (K - B)P^D(T, x_0; B) \\
&\quad - e^{2m(b - x_0)}\left[P(T, 2b - x_0; K) - P(T, 2b - x_0; B) - (K - B)P^D(T, 2b - x_0; B)\right]
\end{aligned}$$

where $P(T, x_0; K)$ is the price of a vanilla put option with strike K, maturity T and initial asset price $S_0 = e^{\sigma x_0}$; $P^D(T, x_0; K)$ denotes the value of the digital *cash-or-nothing put* which pays 1 if $S_T < K$ and 0 otherwise.

2.5 Numerical examples

We employed the Euler method of Abate and Whitt (1995) as described in Appendix A.1 with parameters $A = 13.8$, $m = 30$ and $n = 40$. $A = 13.8$ was chosen so that error in the Abate-Whitt approximation was at most one part in a million relative to the price of the corresponding European option. The other two parameters were chosen to ensure convergence to around 5 significant figures for the range of option prices we consider. See Table 2.4 at the end of this section for some examples of how prices vary depending on the choice of m and n. Using the Euler scheme assumption (2.14) can be rewritten as a requirement that

$$A > \frac{T}{\sigma^2}\left(r - \delta + \frac{1}{2}\sigma^2\right)^2$$

which holds comfortably when $A = 13.8$ for all the cases we consider.

We implemented the above scheme in Matlab, using the algorithm for the error function described in Press, Flannery, Teukolsky, and Vetterling (1993) to calculate $\Phi(z)$ for complex values of z. Table 2.1 gives a selection of different call prices; each line of the table took around 0.03 seconds to calculate on a 600 MHz PC. Figure 2.1 shows how if the duration is small the Parisian down-and-out call option behaves like a conventional knock-out option whereas for long durations its behaviour is like that of a standard European call option.

There are no Parisian down-and-out call prices in the literature for us to check our results against. However Avellaneda and Wu (1999) use a trinomial lattice to calculate the price of a particular Parisian up-and-out put which we can calculate via the parity relation (2.4). For an up-and-out put with parameters $T = 1, S_0 = 100, K = 100, B = 120, D = 0.1, \sigma = 0.3, r = 0.1, q = 0$ we obtain a price of 7.0428 which compares well with their answer of 7.0392.

We also replicated the results of Chesney, Cornwall, Jeanblanc-Picqué, Kentwell, and Yor (1997) who consider an option on the Australian dollar/US dollar exchange rate with a face value of A\$1 million and priced in US dollars. Figure 2.2 shows the price of

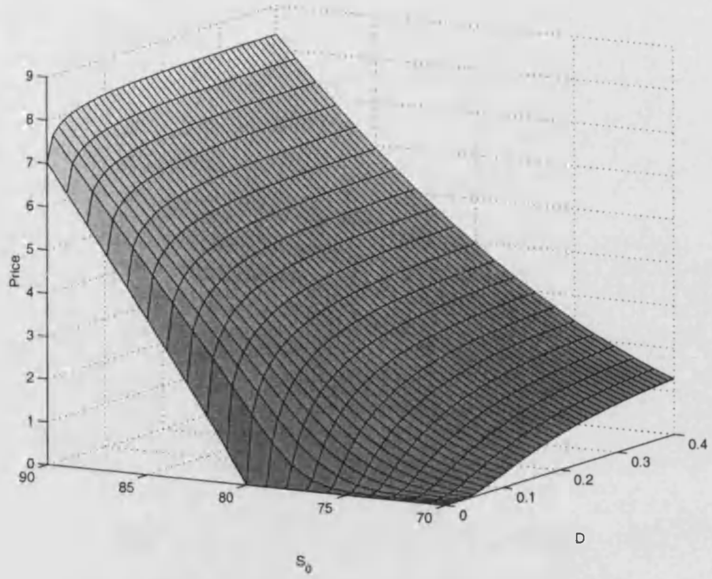

Figure 2.1: Price of Parisian down-and-out call, against initial asset price S_0 and duration D. $T = 1$, $K = 100$, $B = 80$, $r = 0.09$, $\delta = 0$, $\sigma = 0.25$.

S_0	D	C	$PDOC$	DOC
70.0000	0.0100	1.4575	0.0000	0
70.0000	0.0500	1.4575	0.0510	0
70.0000	0.1000	1.4575	0.3124	0
70.0000	0.2000	1.4575	0.8217	0
70.0000	0.4000	1.4575	1.2911	0
80.0000	0.0100	4.0011	1.5948	0
80.0000	0.0500	4.0011	2.7981	0
80.0000	0.1000	4.0011	3.3248	0
80.0000	0.2000	4.0011	3.7330	0
80.0000	0.4000	4.0011	3.9517	0
90.0000	0.0100	8.3426	7.6072	7.0130
90.0000	0.0500	8.3426	8.0137	7.0130
90.0000	0.1000	8.3426	8.1742	7.0130
90.0000	0.2000	8.3426	8.2856	7.0130
90.0000	0.4000	8.3426	8.3352	7.0130

Table 2.1: Prices of vanilla call, Parisian down-and-out call, and the ordinary down-and-out call for varying initial asset price S_0 and duration D. $T = 1$, $K = 100$, $B = 80$, $r = 0.09$, $\delta = 0$, $\sigma = 0.25$.

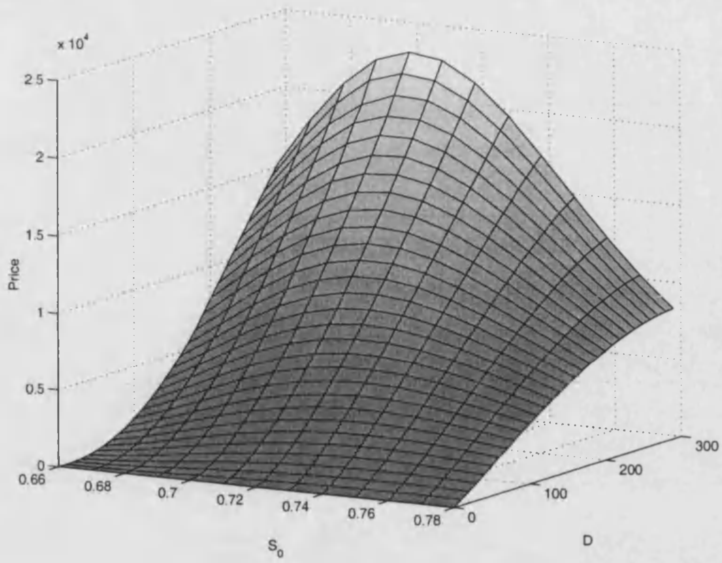

Figure 2.2: Price of Parisian down-and-out put against spot price S_0 and duration D in days. $T = 1$ year, $K = 0.7$, $B = 0.69$, $r = 0.05$, $\delta = 0.08$, $\sigma = 0.1$.

a one year down-and-out Parisian put option as a function of spot rate and duration. This figure is essentially the same as the one in Chesney, Cornwall et al. As one would expect the calculations for this figure took longer than for the call option, with each price taking around 0.20 seconds to calculate. Table 2.2 gives a sample of the prices used in the construction of this graph along with the corresponding values for the vanilla put and the standard down-and-out and up-and-in put options. For small values of the duration D the Parisian down-and-out put behaves like a conventional down-and-out put option. For D close to T the behaviour depends on the initial spot rate. If $S_0 > B$ then the Parisian option has a similar price to the vanilla put option; on the other hand if $S_0 < B$ then the Parisian option is basically an up-and-in put option for D close to T.

Zhu and Stokes (1999) give figures calculated by solving a PDE numerically and compare these with values obtained directly from Dr. Glenn Kentwell. Table 2.3 gives prices of a down-and-out put option for two different maturities along with calculation times where available. Notice firstly that our results are exactly the same as those of Kentwell, who uses a similar Laplace inversion method but requires a *double* integration. Secondly our times are several orders of magnitude better than those obtained by Zhu and Stokes (1999) using PDE methods.

S_0	D (days)	$P/10^4$	$PDOP/10^4$	$DOP/10^4$	$UIP/10^4$
0.68	1	4.9220	0.0000	0	3.3480
0.68	50	4.9220	0.0862	0	3.3480
0.68	130	4.9220	0.5148	0	3.3480
0.68	210	4.9220	1.2233	0	3.3480
0.68	290	4.9220	2.0842	0	3.3480
0.72	1	2.7142	0.0041	0.0011	0
0.72	50	2.7142	0.2099	0.0011	0
0.72	130	2.7142	0.7648	0.0011	0
0.72	210	2.7142	1.4307	0.0011	0
0.72	290	2.7142	2.1696	0.0011	0
0.74	1	1.9148	0.0059	0.0017	0
0.74	50	1.9148	0.2299	0.0017	0
0.74	130	1.9148	0.7349	0.0017	0
0.74	210	1.9148	1.2630	0.0017	0
0.74	290	1.9148	1.7457	0.0017	0

Table 2.2: Prices of vanilla put, Parisian down-and-out put, and the ordinary down-and-out and up-and-in puts for varying spot price S_0 and duration D. $T = 1$ year, $K = 0.7$, $B = 0.69$, $r = 0.05$, $\delta = 0.08$, $\sigma = 0.1$.

Time T	Laplace Inversion		Zhu & Stokes	
(years)	Hartley	Kentwell	Crank-Nicolson	Implicit
0.25	0.3028 (0.21s)	0.3028	0.302217 (12.19s)	0.299510 (10 s)
1.00	0.2748 (0.20s)	0.2748	0.274975 (47.91s)	0.265175 (41 s)

Table 2.3: Prices (and calculation times) for two Parisian down-and-out puts of different maturities. We give our Laplace inversion results, those of Kentwell and those from the two PDE methods of Zhu and Stokes. $S_0 = K = 10$, $B = 8$, $D = 0.1$, $r = 0.08$, $\delta = 0$, $\sigma = 0.2$.

$S_0 = 0.72$, $D = 130$ days			
m	n	$PDOP/10^4$	Time/s
3	5	0.73291407879699	0.030
7	10	0.73469987948380	0.053
11	15	0.73487530485423	0.078
15	20	0.73487855310620	0.104
20	30	0.73487861482601	0.145
30	40	0.73487861605727	0.202
40	50	0.73487861608157	0.260
60	80	0.73487861608258	0.403

$S_0 = 0.68$, $D = 290$ days			
m	n	$PDOP/10^4$	Time/s
3	5	1.97314259531518	0.035
7	10	2.09701008258086	0.061
11	15	2.08781642122231	0.090
15	20	2.08073735678438	0.122
20	30	2.08420985948142	0.171
30	40	2.08422147058169	0.238
40	50	2.08419619067505	0.306
60	80	2.08420406313746	0.475

Table 2.4: Prices of Parisian down-and-out put and calculation times as m and n vary for the Cornwall et al. example. $A = 13.8$; $T = 1$ year, $K = 0.7$, $B = 0.69$, $r = 0.05$, $\delta = 0.08$, $\sigma = 0.1$.

Finally Table 2.4 gives an idea of how the prices vary depending on the choices of the Abate-Whitt parameters m and n. In the first 'easy' example we get extremely rapid convergence to a stable price, so we could speed up calculations by using smaller values of m and n. However the second 'hard' example, with an initial spot rate below the barrier and a duration close to D shows why we have chosen the values $m = 30$ and $n = 40$ for all the other calculations in this section - convergence is much slower in this case. Note that the value these figures are converging to is not the true price but a price that is within the bound given by equation (A.2). In this second example the price is bounded by the price of an up-and-in put as given in Table 2.2 and so with $A = 13.8$ the values in the lower part of Table 2.4 will converge to a price within US\$3.4 of the true price (on a contract with face value of A\$1 million), or around 0.02%.

2.6 Summary and conclusions

In this chapter we defined a Parisian option, and gave parity relations to show that if we could evaluate Parisian down-and-out calls and puts then we could find the price of *any* standard Parisian option.

We showed that if we considered the Laplace transform of the price of any down-and-out Parisian option we could write this down *explicitly* in terms of the Laplace transform of the price of an identical option, but with the initial asset price S_0 being equal to the barrier level B. We also noted that we could do exactly the same thing for the Laplace transform of the price of an option where the asset had already spent a period of time $D_0 < D$ below the barrier level. These relations are substantially simpler than the identities given by Chesney, Jeanblanc-Picqué, and Yor (1997) or Schröder (2002).

We gave an excursion theory based derivation of the Laplace transform of the Parisian down-and-out density which is consistent with the expression obtained by Chesney, Jeanblanc-Picqué, and Yor for the Laplace transform of the Parisian down-and-in density obtained by other methods. We used this to obtain *explicit* expressions for the Laplace transform of the price of Parisian down-and-out call and put options in the case where $S_0 = B$ and $K > B$. Exactly the same techniques can be applied in the $K < B$ case.

Combining the claims of the previous two paragraphs, we thus showed how to find explicit expressions for the Laplace transform of any standard Parisian down-and-out option. We showed that we could invert these Laplace transforms numerically using the Euler method of Abate and Whitt (1995) and hence find the price of any standard Parisian option. Furthermore the prices so obtained were both extremely accurate and very quick to compute - properties that are not simultaneously present in any other method described in the literature.

Chapter 3

Pricing continuous Asian options by double Laplace inversion

Abstract

This chapter studies continuous arithmetic-average Asian options. We give a new expression for the double Laplace transform in strike and maturity of the price of such an option, and give an efficient method for calculating this Laplace transform numerically using a finite-difference scheme. Prices are then obtained by numerical inversion of the Laplace transform and these are compared with other results from the literature. We show that this approach is fast and accurate enough to be of practical use.

3.1 Introduction

Asian options are those whose payoff depends on the average of an asset price over a given period of time. This feature makes the option less sensitive to manipulation of the asset price near maturity, as well as providing a convenient way for the option buyer to hedge against adverse asset movements over long time periods.

The option can be either a continuous average, or a discrete average where the asset price is sampled at regularly spaced intervals of time. The standard method of averaging is to take the arithmetic average; a geometric average can also be taken and this leads to closed form solutions for vanilla Asian options, as under standard Black and Scholes (1973) assumptions the geometric average will itself be a log-normally distributed random variable. We will concentrate on the continuous, arithmetic average case with European style exercise and fixed strike. There is no closed form solution for the price of such an option. Pricing methods proposed in the literature fall roughly into four classes.

The first is Laplace transform methods. Geman and Yor (1993) derive an analytical expression for the Laplace transform in maturity of the price of a continuous Asian call option. Eydeland and Geman (1995) attempt to invert the Geman and Yor expression numerically using a fast Fourier transform - the resulting numbers are not very accurate. Fu, Madan, and Wang (1998) and Craddock, Heath, and Platen (2000) use numerical Laplace transform inversion methods on the same expression, although both have problems for short maturities and small volatilities.

Secondly there are Monte Carlo methods where thousands of asset price processes are simulated. Kemna and Vorst (1990) use the geometric Asian call option price as a control variate, an approach refined by Fu, Madan, and Wang (1998). In contrast to the Laplace inversion methods Monte Carlo techniques are particularly effective for short maturities and small volatilities since the variation from the sampling process is lower in these ranges. However they are in general slow to give results, although Lapeyre and Temam (2001) shows that the Monte Carlo approach can be competitive with other methods when high precision is required.

Thirdly there are various analytical approximations. Turnbull and Wakeman (1991) approximate the arithmetic average by a log-normal variable with matched moments. Curran (1992) conditions on another (highly correlated) random variable and then uses moment matching techniques. Rogers and Shi (1995) and Thompson (2000) use similar techniques to derive lower and upper bounds for the option price. Reiner, Davydov, and Kumanduri (2001) use density perturbation techniques for their approximation and Ju (2002) develops a Taylor expansion around zero volatility. Some of these approximations are extremely accurate.

Finally there are PDE methods. Rogers and Shi (1995) derive a one-dimensional PDE but this proves hard to solve numerically. Zvan, Forsyth, and Vetzal (1997/98) use flux-limiting techniques from computational fluid dynamics to solve the same PDE more accurately. Večeř (2001a), (2001b) derives a similar PDE that is more easily solved numerically. Zhang (2001) finds an analytical approximate formula with the error in this formula solving a PDE with smooth coefficients that can be evaluated accurately using numerical methods. The PDE category also includes tree methods such as that of Klassen (2001) which seems to give extremely fast and accurate results.

The remainder of this chapter is organised as follows. In Section 3.2 we define the Asian put option and then follow closely Fu et al. in deriving an ODE for the double Laplace transform (in strike and maturity) of the price of this option. In Section 3.3 we derive an analytic solution to this ODE and show that it is hard to evaluate accurately. We also explain why the solution that Fu et al. give in their paper is incorrect. Section 3.4 gives a much faster alternative way of computing the solution using finite-difference methods. We then utilize the Euler method described by Abate and Whitt (1995) to invert the double Laplace transform numerically and hence recover the original put price. Results of these computations are given in Section 3.5. We conclude in Section 3.6. Appendix A.1 summarizes the Euler method of Abate and Whitt and Appendix A.2 describes the accurate approximation method of Curran (1992) which we use as a benchmark.

3.2 The Asian put option

We wish to value the continuous arithmetic-average Asian put option, with value at time 0 given by

$$\begin{aligned} P(K,T;S_0) &\equiv \mathbb{E}\left[e^{-rT}\left(K-\frac{1}{T}\int_0^T S_t\,dt\right)^+\right] \\ &= \frac{S_0e^{-rT}}{T}\mathbb{E}\left[\left(\frac{KT}{S_0}-\int_0^T \frac{S_t}{S_0}\,dt\right)^+\right] \\ &\equiv \frac{S_0e^{-rT}}{T}p\left(\frac{KT}{S_0},T\right) \end{aligned} \tag{3.1}$$

where T is the exercise time, K is the strike price, r is the interest rate and under the risk-neutral measure $\mathbb{P}$ the asset price process S_t is given by

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}$$

with W_t a standard one-dimensional Brownian motion. From the form (3.1) of the price of the Asian put we can see that valuation simplifies to computing expressions of the form

$$p(k,T) \equiv \mathbb{E}\left[(k-A_T)^+\right]$$

under the condition $S_0 = 1$ and where

$$A_T \equiv \int_0^T S_u \, du.$$

As with vanilla options we have a put-call parity relation to allow us to obtain call prices from put prices. We are interested in finding the simplified call price expression $c(k,T) \equiv \mathbb{E}\left[(A_T - k)^+\right]$. At time T

$$(A_T - k)^+ - (k - A_T)^+ = A_T - k,$$

hence

$$\begin{aligned}
c(k,T) &= p(k,T) + \mathbb{E}[A_T - k] \\
&= p(k,T) + \int_0^T \mathbb{E}[S_t] \, dt - k \\
&= p(k,T) + \frac{e^{rT} - 1}{r} - k
\end{aligned}$$

and so the price of an Asian call option is

$$\begin{aligned}
C(K,T;S_0) &= \frac{S_0 e^{-rT}}{T} c\left(\frac{KT}{S_0}, T\right) \\
&= P(K,T;S_0) + \frac{S_0(1 - e^{-rT})}{rT} - Ke^{-rT}. \qquad (3.2)
\end{aligned}$$

One might also wish to price Asian options at times other than 0. Suppose that at time $t < T$ the asset price is S_t, and that over the time up to t a total of $A_t = \int_0^t S_u du$ has accrued towards A_T. The price at time t of an Asian put option with strike K and maturity T is then

$$\begin{aligned}
P(K,T;S_t,t,A_t) &\equiv \mathbb{E}\left[e^{-r(T-t)} \left(K - \frac{1}{T}\left(A_t + \int_t^T S_u du\right)\right)^+ \middle| \mathcal{F}_t\right] \\
&= \frac{T-t}{T}\mathbb{E}\left[e^{-r(T-t)} \left(\frac{KT - A_t}{T-t} - \frac{1}{T-t}\int_t^T S_u du\right)^+ \middle| \mathcal{F}_t\right] \\
&= \frac{T-t}{T} P\left(\frac{KT - A_t}{T-t}, T-t; S_t, 0, 0\right)
\end{aligned}$$

and similarly the time-t value of an Asian call is

$$C(K,T;S_t,t,A_t) = \frac{T-t}{T} C\left(\frac{A_t - KT}{T-t}, T-t; S_t, 0, 0\right).$$

Note that if $A_t > KT$ then the put option is worthless, and the corresponding call option is certain to be exercised.

3.2.1 A double Laplace transform

If we Laplace transform the reduced put price $p(k,T) \equiv \mathbb{E}\left[(k - A_T)^+\right]$ in k, we obtain for α in the right complex half-plane

$$\begin{aligned}
\hat{p}(\alpha, T) &= \mathbb{E}\left[\int_0^\infty e^{-\alpha k}\,(k - A_T)^+\,dk\right] \\
&= \mathbb{E}\left[e^{-\alpha A_T}\int_0^\infty u e^{-\alpha u}\,du\right] \\
&= \frac{1}{\alpha^2}\mathbb{E}\left[e^{-\alpha A_T}\right].
\end{aligned}$$

A second Laplace transform in the maturity T gives

$$\begin{aligned}
\hat{\hat{p}}(\alpha, \lambda) &= \frac{1}{\alpha^2}\mathbb{E}\left[\int_0^\infty e^{-\lambda T} e^{-\alpha A_T}\,dT\right] \\
&= \frac{1}{\alpha^2 \lambda} f(1),
\end{aligned}$$

where λ is in the right complex half-plane and

$$f(x) \equiv \mathbb{E}\left[\int_0^\infty \lambda e^{-\lambda T} e^{-\alpha A_T}\,dT \middle| S_0 = x\right]$$

is a decreasing function of x. Conditioning the expression inside the expectation on events up to time t we see that

$$M_t \equiv \int_0^t \lambda e^{-\lambda T} e^{-\alpha A_T} dT + e^{-\lambda t} e^{-\alpha A_t} f(S_t)$$

is a martingale. Applying Itô's Lemma

$$dM_t = e^{-\lambda t} e^{-\alpha A_t}\left[\lambda dt - (\lambda + \alpha S_t) f(S_t) dt + f'(S_t) dS_t + \frac{1}{2}\sigma^2 S_t^2 f''(S_t)\right]$$

and so $f(x)$ solves

$$\frac{1}{2}\sigma^2 x^2 f''(x) + r x f'(x) - (\lambda + \alpha x) f(x) = -\lambda \tag{3.3}$$

with boundary conditions $f(0) = 1$ and $f(x)$ decreasing to zero as $x \to \infty$.

3.3 An analytic solution

Firstly, we wish to solve

$$\frac{1}{2}\sigma^2 x^2 f''(x) + rxf'(x) - (\lambda + \alpha x)f(x) = 0, \tag{3.4}$$

the homogeneous version of the ODE (3.3). If we make the change of variables $u = \sqrt{x}$ so that

$$\frac{df}{dx} = \frac{1}{2u}\frac{df}{du} \qquad\qquad \frac{d^2 f}{dx^2} = \frac{1}{4u^2}\left(\frac{d^2 f}{du^2} - \frac{1}{u}\frac{df}{du}\right)$$

then (3.4) can be written as

$$u^2 f''(u) + \left(\frac{4r}{\sigma^2} - 1\right) uf'(u) - \frac{8}{\sigma^2}(\lambda + \alpha u^2)f(u) = 0. \tag{3.5}$$

Let's try a solution of the form

$$f(u) = u^\gamma g(u)$$

so that the ODE (3.5) becomes

$$u^2 g''(u) + \left(\frac{4r}{\sigma^2} - 1 + 2\gamma\right) ug'(u) - \left(\frac{8\alpha u^2}{\sigma^2} + \frac{8\lambda}{\sigma^2} - \gamma\left(\frac{4r}{\sigma^2} - 1\right) - \gamma(\gamma - 1)\right) g(u) = 0.$$

If we now set

$$\gamma = \frac{\sigma^2 - 2r}{\sigma^2}$$

this is

$$u^2 g''(u) + ug'(u) - \left(\frac{8\alpha u^2}{\sigma^2} + \nu^2\right) g(u) = 0 \tag{3.6}$$

where

$$\nu \equiv \frac{\sqrt{(\sigma^2 - 2r)^2 + 8\sigma^2\lambda}}{\sigma^2}.$$

This equation has a solution in terms of modified Bessel functions (see,for example, Jeffrey (1995)) and hence the solution to the homogeneous ODE (3.4) is

$$f(x) = x^{(\sigma^2 - 2r)/2\sigma^2}(B_0 I_\nu\left(\frac{\sqrt{8\alpha x}}{\sigma}\right) + B_1 K_\nu\left(\frac{\sqrt{8\alpha x}}{\sigma}\right)),$$

where B_0 and B_1 are constants and $I_\nu(x)$ and $K_\nu(x)$ denote modified Bessel functions of the first and second kind respectively.

Secondly, the inhomogeneous ODE (3.3) has a series solution of the form

$$f_1(x) \equiv \sum_{n\geq 0} a_n x^n$$

with the coefficients $(a_n)_{n\geq 0}$ satisfying $a_0 = 1$ and for $n \geq 1$ the recursion relation

$$(\frac{1}{2}\sigma^2 n(n-1) + rn - \lambda)a_n = \alpha a_{n-1},$$

or equivalently

$$(n-\rho_+)(n+\rho_-)a_n = \frac{2\alpha}{\sigma^2}a_{n-1},$$

where

$$\rho_+ = \frac{\nu+\gamma}{2}, \qquad \rho_- = \frac{\nu-\gamma}{2}.$$

Hence, we have for all n that

$$a_n = \frac{\Gamma(1+\rho_-)\Gamma(1-\rho_+)}{\Gamma(1+n+\rho_-)\Gamma(1+n-\rho_+)}\left(\frac{2\alpha}{\sigma^2}\right)^n,$$

and we can write $f_1(x)$ in the form

$$\begin{aligned} f_1(x) &= \sum_{n\geq 0} \frac{\Gamma(1+\rho_-)\Gamma(1-\rho_+)\Gamma(1+n)}{\Gamma(1+n+\rho_-)\Gamma(1+n-\rho_+)}\frac{1}{n!}\left(\frac{2\alpha x}{\sigma^2}\right)^n \\ &\equiv {}_1\mathrm{F}_2(1; 1+\rho_-, 1-\rho_+; \frac{2\alpha x}{\sigma^2}) \end{aligned} \tag{3.7}$$

where ${}_1\mathrm{F}_2$ is a generalized hypergeometric function.

With the relevant boundary conditions the general solution to (3.3) is then

$$f(x) = f_1(x) + B_0 f_0(x)$$

where

$$f_0(x) = x^{(\sigma^2-2r)/2\sigma^2} I_\nu\left(\frac{\sqrt{8\alpha x}}{\sigma}\right) \tag{3.8}$$

and B_0 is a constant. Fu, Madan, and Wang (1998) find a similar series solution[1], and then claim that as it has the right values for both $f(0)$ and $f'(0)$ it must be the

[1] Given by equation (38) in their paper, which is incorrectly derived from the steps above it. The factor of $\sigma^2/2(r-\nu)$ should not be present in the summation, or alternatively the factor of $\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)$ should be replaced by $\Gamma(2-\alpha_1)\Gamma(2-\alpha_2)$.

required solution to the ODE (3.3). However, note that for $\mathrm{Re}(\nu) > 1$ (which will hold in all the cases we consider),

$$I_\nu(0) = 0, \qquad\qquad \left.\frac{dI_\nu(x)}{dx}\right|_{x=0} = 0,$$

and hence the solution given in the appendix of Fu et al. is not in fact the required solution.

We can obtain B_0 explicitly by considering the asymptotics of f_0 and f_1. For $|\arg(z)| < \frac{\pi}{2}$ we have as $|z| \to \infty$ that[2],

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}\left(1 + \mathrm{O}\left(\frac{1}{z}\right)\right)$$

and for $|\arg(z)| < \pi$ similarly[3],

$${}_1\mathrm{F}_2(a_1; b_1, b_2; z) = \frac{\Gamma(b_1)\Gamma(b_2)}{2\sqrt{\pi}\Gamma(a_1)} z^{\frac{1}{2}(a_1 - b_1 - b_2 + \frac{1}{2})} e^{2\sqrt{z}}\left(1 + \mathrm{O}\left(\frac{1}{\sqrt{z}}\right)\right)$$

as $|z| \to \infty$. If we take square roots using the principal branch of the logarithm then the argument conditions needed above will be met and so for large x

$$\begin{aligned} f_0(x) &= \frac{1}{\sqrt{2\pi}} x^{\gamma/2} \left(\frac{\sqrt{8\alpha x}}{\sigma}\right)^{-\frac{1}{2}} e^{\sqrt{8\alpha x}/\sigma}\left(1 + \mathrm{O}\left(\frac{\sigma}{\sqrt{8\alpha x}}\right)\right), \\ f_1(x) &= \frac{\Gamma(1+\rho_-)\Gamma(1-\rho_+)}{2\sqrt{\pi}} \left(\frac{2\alpha x}{\sigma^2}\right)^{\frac{1}{2}(\gamma - \frac{1}{2})} e^{2\sqrt{2\alpha x}/\sigma}\left(1 + \mathrm{O}\left(\frac{\sigma}{\sqrt{2\alpha x}}\right)\right). \end{aligned}$$

We know that $f(x) = f_1(x) + B_0 f_0(x) \to 0$ as $x \to \infty$ and so, as the powers of x and $e^{\sqrt{x}}$ match in the two formulae above, we deduce that we must have

$$B_0 = -\Gamma(1+\rho_-)\Gamma(1-\rho_+)\left(\frac{2\alpha}{\sigma^2}\right)^{\gamma/2} \tag{3.9}$$

and so the required solution to the ODE (3.3) is

$$f(x) = {}_1\mathrm{F}_2(1; 1+\rho_-, 1-\rho_+; \frac{2\alpha x}{\sigma^2}) - \Gamma(1+\rho_-)\Gamma(1-\rho_+)\left(\frac{2\alpha x}{\sigma^2}\right)^{\gamma/2} I_\nu\left(\frac{\sqrt{8\alpha x}}{\sigma}\right). \tag{3.10}$$

From the analytic solution given by (3.10) we can see that an important factor in the

[2]http://functions.wolfram.com/03.02.06.0006.01
[3]http://functions.wolfram.com/07.22.06.0005.01

solution is α/σ^2. The real part of α/σ^2, which will stay constant as we invert $\hat{\hat{p}}$ (see Appendix A.1 for details of the inversion method and the constant A), is

$$\begin{aligned} c &\equiv \operatorname{Re}(\alpha/\sigma^2) \\ &= \frac{A}{2k\sigma^2} \\ &= \frac{A}{2}\frac{S_0}{KT\sigma^2}. \end{aligned}$$

c is inversely proportional to the parameter $q \equiv \frac{\sigma^2 KT}{4S_0}$ used by both Fu, Madan, and Wang (1998) and by Craddock, Heath, and Platen (2000). The latter find that when q is small (corresponding to c large) their Abate-Whitt based method, which also requires the evaluation of hypergeometric functions, is slow to converge. The problem we have with computing $f(1)$ from equation (3.10) is that when c is large the various components (3.7), (3.8) and (3.9) of the analytic solution at $x = 1$ can be numbers of very large magnitude. As we are seeking an accurate solution for $f(1)$, which we know to lie between 0 and 1, we would have to evaluate these components to a correspondingly large number of decimal places. For example, suppose we wish to find $f(1)$ from (3.10) for the (fairly typical) parameter values $r = 0.02$, $\sigma = 0.1$, $\alpha = 6.9 + 10\pi i$ and $\lambda = 6.9$. Using Maple we find that (to 3 significant figures)

$$\begin{aligned} {}_1F_2(1; 1+\rho_-, 1-\rho_+; \frac{2\alpha}{\sigma^2}) &= 0.363 \times 10^{48} - 0.102 \times 10^{49}\, i \\ \Gamma(1+\rho_-)\Gamma(1-\rho_+)\left(\frac{2\alpha}{\sigma^2}\right)^{\gamma/2} &= 6.10 + 12.28\, i \\ I_\nu\left(\frac{\sqrt{8\alpha}}{\sigma}\right) &= -0.546 \times 10^{47} - 0.567 \times 10^{47}\, i \end{aligned}$$

so it looks like we are going to need to evaluate these numbers to at least 50 decimal places to get an accurate answer. This turns out to be the case; Table 3.1 shows how varying the number of decimal places used for evaluation in Maple affects the answer we get. Unfortunately, at the required level of accuracy, Maple takes over a second to compute $f(1)$. We will need to perform several thousand such calculations for different α and λ in order to get a good answer when we invert the Laplace transform, so clearly a more efficient way to compute $f(1)$ is needed.

No. of decimal places	Re($f(1)$)
20	-0.310×10^{31}
30	-0.459×10^{21}
40	0.327×10^{11}
50	0.150
60	0.08091250643
70	0.08091250645
80	0.08091250645

Table 3.1: Value of Re($f(1)$) obtained against the number of decimal places used to evaluate equation (3.10) using Maple.

3.4 A fast method to compute $f(1)$

Another approach is to solve the ODE (3.3) by finite-difference methods. Construct a uniform mesh

$$x_i = i.\Delta x$$

for $i = 0, \ldots, N$, where we choose N and x_N and hence $\Delta x = x_N/N$. Writing $f_i \equiv f(x_i)$ a second-order-accurate finite-difference scheme for the ODE (3.3) is given by

$$\frac{1}{2}\sigma^2 x_i^2 \left(\frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2} \right) + r x_i \left(\frac{f_{i+1} - f_{i-1}}{2\Delta x} \right) - (\lambda + \alpha x_i) f_i = -\lambda$$

for $1 \leq i \leq N-1$ with boundary conditions $f_0 = 1$ and $f_N = 0$. Solving for f_i can then be done in the usual way by solving the corresponding tridiagonal system of equations. As one would expect the smaller the mesh size Δx and the larger the value chosen for x_N the more accurate the result.

Bearing in mind that we are interested only in the value of $f(1)$, we will employ a more sophisticated version of this scheme using two meshes. Firstly we pick a value of x_N considerably greater than 1 and solve the scheme above. We then pick points x_- and x_+ on the original mesh such that $x_- < 1 < x_+$ and solve the same scheme on a finer mesh

$$x_i = x_- + i.\Delta x$$

for $i = 0, \ldots, M$ where we pick M and set $\Delta x = (x_+ - x_-)/M$. The boundary conditions at x_- and x_+ will be taken to be the values of $f(x_-)$ and $f(x_+)$ from the first solution. In practice this scheme exhibits a much better tradeoff between accuracy and speed than a scheme based on a single uniform mesh.

Scheme	x_N	N	x_-	x_+	M	m_α	n_α	m_λ	n_λ
'Accurate'	8	400	0.5	1.5	400	20	30	20	30
'Fast'	3	36	0.75	1.25	36	10	14	9	12

Table 3.2: 'Accurate' and 'fast' scheme parameters. $A = 13.8$ was used for both schemes.

3.4.1 Choice of parameters

For the numerical inversion we employed the Euler method of Abate and Whitt (1995) as described in Appendix A.1 with $A = 13.8$. Two parameter schemes were used, an 'accurate' scheme and a 'fast' scheme. See Table 3.2 for the values used. The parameters m_α, n_α and m_λ, n_λ respectively denote the Abate-Whitt parameters m, n used in the inversion of $\hat{p}(\alpha, T)$ with respect to α and $\hat{\hat{p}}(\alpha, \lambda)$ with respect to λ. The values of x_N and N have been chosen so that x_- and x_+ are mesh points for the first scheme and similarly M has been chosen so $x = 1$ is always a mesh point so as to avoid interpolation issues.

The time taken for the numerical inversion can be halved by making use of the following result. Denoting the complex conjugate of $a \equiv a_1 + ia_2$ by $a^* = a_1 - ia_2$ we have that

$$\begin{aligned}
\hat{\hat{p}}(\alpha^*, \lambda^*) &= \frac{1}{(\alpha^*)^2}\mathbb{E}\left[\int_0^\infty e^{-\lambda^* T} e^{-\alpha^* A_T}\, dT\right] \\
&= \frac{\alpha^2}{(\alpha^*)^2\alpha^2}\mathbb{E}\left[\int_0^\infty \left(e^{-\lambda T}\right)^* \left(e^{-\alpha A_T}\right)^*\, dT\right] \\
&= \frac{((\alpha\alpha)^*)^*}{(\alpha^*)^2\alpha^2}\mathbb{E}\left[\int_0^\infty \left(e^{-\lambda T} e^{-\alpha A_T}\right)^*\, dT\right] \\
&= \frac{((\alpha^*)^2)^*}{(\alpha^*)^2\alpha^2}\left(\mathbb{E}\left[\int_0^\infty e^{-\lambda T} e^{-\alpha A_T}\, dT\right]\right)^* \\
&= \left(\frac{(\alpha^*)^2\mathbb{E}\left[\int_0^\infty e^{-\lambda T} e^{-\alpha A_T}\, dT\right]}{(\alpha^*)^2\alpha^2}\right)^* \\
&= \hat{\hat{p}}(\alpha, \lambda)^*
\end{aligned}$$

and hence

$$\operatorname{Re}\left(\hat{\hat{p}}(\alpha^*, \lambda^*)\right) = \operatorname{Re}\left(\hat{\hat{p}}(\alpha, \lambda)\right).$$

3.5 Numerical examples

Our discussion so far has focused on Asian put options but the put-call parity relation (3.2) means that we can equivalently price call options. We shall price Asian call options on 25 parameter combinations, consisting of the 19 cases considered by Craddock, Heath, and Platen (2000) and 9 cases from Rogers and Shi (1995). 3 of the cases are common to both papers and the first 7 cases are also covered for various methods in Table 4 of Fu, Madan, and Wang (1998). The parameter values for the 25 cases are shown in Table 3.3 along with the corresponding values of c (with $A = 13.8$). We will

Case	S_0	K	T	r	σ	c
1	1.9	2	1	0.05	0.50	26.22
2	2	2	1	0.05	0.50	27.60
3	2.1	2	1	0.05	0.50	28.98
4	2	2	1	0.02	0.10	690.00
5	2	2	1	0.18	0.30	76.67
6	2	2	2	0.0125	0.25	55.20
7	2	2	2	0.05	0.50	13.80
8	17	16	2.5	0.06	0.30	32.58
9	17	17	2.5	0.06	0.30	30.67
10	17	18	2.5	0.06	0.30	28.96
11	53	51	1.5	0.07	0.40	29.88
12	53	53	1.5	0.07	0.40	28.75
13	53	55	1.5	0.07	0.40	27.70
14	29	27	0.5	0.11	0.15	658.77
15	29	29	0.5	0.11	0.15	613.33
16	29	31	0.5	0.11	0.15	573.76
17	100	95	1	0.09	0.10	726.32
18	100	100	1	0.09	0.10	690.00
19	100	105	1	0.09	0.10	657.14
20	100	90	1	0.09	0.30	85.19
21	100	100	1	0.09	0.30	76.67
22	100	110	1	0.09	0.30	69.70
23	100	90	1	0.09	0.50	30.67
24	100	100	1	0.09	0.50	27.60
25	100	110	1	0.09	0.50	25.09

Table 3.3: The 25 cases we wish to price and the corresponding values of c.

compare our figures with those from the Curran (precise) method, which Reiner, Davydov, and Kumanduri (2001) show to be an extremely accurate method of computing Asian option prices. Details of the Curran method are given in Appendix A.2.

Case	CHP	RS	Accurate	Curran	Fast	CPU
1	0.194		0.1932	0.1932	0.1932	0.97
2	0.248		0.2464	0.2464	0.2464	0.95
3	0.308		0.3062	0.3062	0.3062	0.92
4	0.055		0.0560	0.0560	0.0559	0.96
5	0.222		0.2184	0.2184	0.2184	0.93
6	0.172		0.1723	0.1723	0.1724	0.94
7	0.340		0.3501	0.3501	0.3500	0.93
8	2.808		2.8159	2.8158	2.8158	0.91
9	2.305		2.3109	2.3109	2.3111	0.91
10	1.875		1.8791	1.8790	1.8789	0.94
11	7.903		7.8959	7.8957	7.8961	0.91
12	6.942		6.9356	6.9354	6.9360	0.96
13	6.077		6.0711	6.0710	6.0711	0.94
14	2.808		2.6979	2.6979	2.6984	0.93
15	1.129		1.1348	1.1347	1.1333	0.92
16	0.278		0.2854	0.2853	0.2863	0.92
17		8.91	8.9120	8.9118	8.9148	0.91
18		4.92	4.9151	4.9151	4.9075	0.93
19		2.07	2.0702	2.0701	2.0697	0.92
20	15.056	14.98	14.9841	14.9839	14.9857	0.98
21	8.964	8.83	8.8289	8.8287	8.8332	0.97
22	4.700	4.70	4.6969	4.6967	4.6941	0.99
23		18.18	18.1890	18.1887	18.1859	0.94
24		13.02	13.0284	13.0281	13.0288	0.97
25		9.18	9.1246	9.1244	9.1220	0.92

Table 3.4: Asian call option prices, and CPU time in seconds taken to calculate the 'fast' results. 'CHP' denotes the figures from Craddock, Heath, and Platen (2000), and 'RS' those from Rogers and Shi (1995).

Table 3.4 shows the Asian option prices obtained using the 'accurate' method, along with the results from the benchmark Curran method, from the Craddock et al. Abate-Whitt method and from Rogers and Shi's lower bound method where relevant. The prices given by the Laplace transform inversion method agree extremely well with those from our benchmark method and (apart from case 25) with the lower bound given by Rogers and Shi. Reasonable agreement is also obtained with the results of Craddock et al. although comparison is not entirely fair as their results were (mostly) calculated in a few seconds. Note that good results are obtained regardless of the value of the parameter c. These 'accurate' prices took an average of 800 seconds each to compute, but very considerable speed improvements can be obtained without too much loss of accuracy, as can be seen from the last two columns of Table 3.4 which show the prices obtained using the 'fast' parameter scheme to calculate the Laplace transform values and perform the inversion. Also shown are the CPU times in seconds taken to perform each of these Laplace transform inversion calculations - an average time per calculation of 0.94 seconds (all calculations were done in Matlab on a 600 MHz PC), with each calculation taking less than a second.

3.6 Summary and conclusions

In this chapter we defined the Asian put option, gave a put-call parity relation linking it to the Asian call option and showed how to price Asian options at times other than the start of the accrual period.

Following Fu, Madan, and Wang (1998) we derived an ODE for the double Laplace transform (in strike and maturity) of the Asian put price. We then derived a new analytic solution to this ODE in terms of a generalized hypergeometric function, gamma functions and a Bessel function of the second kind. However this analytic solution proved difficult to evaluate numerically so we proposed a method for solving the ODE numerically using finite-difference methods. This gave accurate answers extremely quickly.

The difficulties in pricing Asian options by inversion of a double Laplace transform come mostly from the evaluation of the Laplace transform itself rather than from the inversion procedure. We used the Euler method of Abate and Whitt (1995) to invert the Laplace transform twice numerically and the resulting prices compared extremely well with those from the method of Curran (1992). Unlike Craddock, Heath, and Platen (2000) we did not encounter problems for any particular ranges of parameter values. Moreover, we were able to fine tune our method to give accurate prices in under a second, showing that Laplace inversion methods could be used in practice for the pricing of Asian options.

Chapter 4

Pricing a multi-asset American option

Abstract

In this chapter we take an ab-initio approach to the valuation of options on multiple assets, in particular the 'min-put' option, that is a put option on the minimum of several assets. We characterize the behaviour of the minimum process, given by the smallest of the assets, by finding the law of its distribution at a chosen time conditional on its position at an earlier time. This allows us to price European options exactly and we use a trinomial tree based method to find a fast lower bound for the price of American/Bermudan options.

4.1 Introduction

Multi-asset options are increasingly being used for risk management, and traded in the financial markets. Options on the maximum or minimum of several assets have been studied for some time. Johnson (1987) derives an explicit formula for the price of a European call option on the maximum of n assets (a 'max-call' option) in terms of n-variate cumulative normal functions. Broadie and Detemple (1997) give valuation formulae in non-explicit form for American max-call options and characterize the corresponding optimal exercise regions. In this chapter we consider the closely related problem of pricing options on the *minimum* of several assets. We will then apply the same methods to valuation of the max-call option.

The valuation of an American option is a maximization problem over choice of exercise policy, and for a given exercise policy there is a corresponding lower bound to the price of the option. We will restrict ourselves to an exercise rule based solely on the level of the minimum asset, which means that we can model just the minimum process itself rather than all n assets. The main advantage of this method is that it scales extremely well as the number of assets increases and we are able to calculate option prices extremely rapidly. The resulting lower bounds are found to be close to the true price of the option.

Standard tree, lattice and finite-difference methods are not feasible for finding the true price of multi-asset options as the computing time required increases exponentially with the number of assets. Hence the literature on pricing such options is entirely based on Monte Carlo techniques where multiple asset price processes are simulated. The main difficulty in applying Monte Carlo to American options is obtaining the optimal exercise strategy. In order to reduce the computational effort required, researchers often concentrate on pricing Bermudan options, where exercise is only allowed at a fixed number of equally spaced dates.

Raymar and Zwecher (1997) use Monte Carlo simulation and reduce the n simulated asset prices at a particular time by assigning them to a particular 'bucket' region. Probabilities of moving between different buckets at different times are determined by repeated simulation and then a dynamic programming approach is used to iterate backwards determining optimal exercise. This is a similar approach to ours in that the choice of the 'buckets' determines a simplified exercise rule - for example they reduce the n simulated asset prices to just the maximum, or to the maximum and second maximum asset, with the values in each case represented by 200 discrete regions - the 'buckets'. Broadie and Glasserman (1997) develop a stochastic mesh method that allows them to generate both lower and upper bounds for an option price, with the property that these bounds converge to the true price as the numbers of paths simulated is increased. Rogers (2002) and Haugh and Kogan (2001) both independently propose a dual method for computing upper bounds using Monte Carlo by representing the American option price as the solution of a dual minimization problem. Haugh and Kogan simulate the sub-optimal exercise strategy implied by an approximate option price (i.e. the lower bound) as the basis for their calculations of the upper bound. Andersen and Broadie (2001) build on this dual approach and use an approximate exercise policy instead of an approximate option price to compute upper and lower bounds. This leads to tighter upper bounds but at a computational cost, particularly for a large number of allowed exercise periods.

The layout of the rest of the chapter is as follows. Section 4.2 describes the type of options we wish to value. In Section 4.3 we derive a law for the position of the minimum process at time $t + \Delta t$ given its position at time t, and give a similar law for the maximum process. Section 4.4 explains the numerical methods we use for pricing the options. The resulting prices are given in Section 4.5. We compare these with values from the literature and show that we can obtain prices extremely quickly using our scheme. We conclude in Section 4.6.

4.2 Options on the minimum of n assets

We consider a frictionless market with constant interest rate r and n risky assets with prices $S_t^1, \ldots, S_t^n$ at time t. We will assume that under the risk-neutral measure $\mathbb{P}$ these asset prices obey the stochastic differential equations

$$dS_t^i = S_t^i((r - \delta_i)dt + \sigma_i dW_t^i) \qquad i = 1, \ldots, n \tag{4.1}$$

where σ_i is the volatility of asset i, δ_i the continuous dividend rate paid by asset i and $W^1, \ldots, W^n$ are standard Brownian motions with covariance structure given by

$$\langle W^i, W^j \rangle_t = \mathbb{1}_{\{i=j\}} t.$$

We could in principle deal with correlated assets, so that $\langle W^i, W^j \rangle_t = \rho_{ij} t$, but we would require an efficient way to calculate n-variate cumulative normal functions. At time t the price of asset i is

$$\begin{aligned} S_t^i &= S_0^i e^{(r-\delta_i-\frac{1}{2}\sigma_i^2)t+\sigma_i W_t^i} \\ &\equiv e^{X_t^i} \end{aligned}$$

where

$$X_t^i \equiv a_i + \mu_i t + \sigma_i W_t^i \qquad i = 1, \ldots, n$$

are the log-asset-price processes. Each X_t^i is a Brownian motion starting at $a_i \equiv \log S_0^i$ with drift $\mu_i \equiv r - \delta_i - \frac{1}{2}\sigma_i^2$ and volatility σ_i.

We are interested in pricing an option with payoff $f(Z_t)$ depending on the minimum process

$$Z_t \equiv \min_{i=1,\ldots,n} X_t^i .$$

In particular we will consider the 'min-put' option, a put option on the minimum of the n assets; this will have a payoff given by $f(Z_t) = (K - e^{Z_t})^+$ at time t. We will assume that the option expires at time T and that exercise of the option is allowed at times in the set $R \subseteq [0, T]$ which we will take to be

$$R = \begin{cases} \{t_i = iT/d : i = 0, 1, \ldots, d\} & \text{when } d > 0, \\ \{T\} & \text{when } d = 0, \end{cases}$$

for d a non-negative integer. This covers European ($d = 0$) and Bermudan options and provides a good approximation to American-style exercise as d grows large. Under Black and Scholes (1973) assumptions such an option will have time-0 price

$$V(t, x^1, \ldots, x^n) \equiv \sup_{\tau \in \mathcal{T}_{R \cap [t,T]}} \mathbb{E}\left[e^{-r(\tau - t)} f(Z_\tau) \middle| X_t^1 = x^1, \ldots, X_t^n = x^n \right].$$

Here we write $\mathcal{T}_A$ to denote the set of all $\mathcal{F}_t$-adapted stopping times τ which satisfy $\tau \in A$ almost surely. The conventional approach to pricing this option would be to use an n-dimensional scheme to model the evolution of the n log-asset-price processes $X_t^1, \ldots, X_t^n$. The time taken to evaluate the price would then grow exponentially as n increases. We will avoid this problem by considering the law of the minimum process Z_t directly, so that we only need to use a one-dimensional method and the value $V(t, z)$ of the option at time t will depend only on the position $Z_t = z$ of the minimum process at that time. In doing this we will lose information about the position of the individual X^is, which may cause us to exercise at a sub-optimal time in the non-European cases. However, in practice, we only slightly under price.

4.3 The minimum process Z_t

We wish to derive the law

$$F(x; t, \Delta t, z) \equiv \mathbb{P}(Z_{t+\Delta t} < x | Z_t = z) \tag{4.2}$$

for $\Delta t \leq T - t$ which we will do by finding $\overline{F}(x; t, \Delta t, z) \equiv 1 - F(x; t, \Delta t, z)$, the probability that $Z_{t+\Delta t}$ is *above* a level x given $Z_t = z$. Firstly we will calculate the probability that a given process i is the minimal one at time t. We know the starting values and drifts of all the individual processes $X_t^i \equiv a_i + \mu_i t + \sigma_i W_t^i$ and so the probability that the minimum is above a level x at time t is

$$\mathbb{P}(Z_t > x) = \prod_{i=1}^{n} \overline{\Phi}\left(\frac{x - \mu_i t - a_i}{\sigma_i \sqrt{t}}\right) \tag{4.3}$$

where $\overline{\Phi}(x) = 1 - \Phi(x)$ and $\Phi(x)$ is the standard cumulative normal distribution function. Differentiating and changing the sign of equation (4.3) yields the corresponding density function

$$\begin{aligned} g(x;t) &\equiv \sum_{i=1}^{n} \frac{1}{\sigma_i\sqrt{2\pi t}} \exp\left(-\frac{(x - \mu_i t - a_i)^2}{2\sigma_i^2 t}\right) \prod_{j\neq i} \overline{\Phi}\left(\frac{x - \mu_j t - a_j}{\sigma_j\sqrt{t}}\right) \\ &\equiv \sum_{i=1}^{n} \tilde{p}_i(t,x) \end{aligned} \tag{4.4}$$

with the appropriate identifications. This density consists of a sum of contributions from the individual density of each X_t^i multiplied by the probability that the other log-asset processes are above it. The probability that a particular process is the minimum is thus

$$\begin{aligned} p_i(t,z) &\equiv \mathbb{P}(Z_t = X_t^i | Z_t = z) \\ &= \frac{\tilde{p}_i(t,z)}{\sum_{j=1}^{n} \tilde{p}_j(t,z)}. \end{aligned}$$

We are now ready to look at the law of $Z_{t+\Delta t}$ conditional on $Z_t = z$,

$$\begin{aligned} \overline{F}(x;t,\Delta t,z) &\equiv \mathbb{P}(Z_{t+\Delta t} > x | Z_t = z) \\ &= \sum_{i=1}^{n} p_i(t,z)\, \mathbb{P}(Z_{t+\Delta t} > x | Z_t = X_t^i = z) \\ &= \sum_{i=1}^{n} p_i(t,z)\, \overline{\Phi}\left(\frac{x - (z + \mu_i \Delta t)}{\sigma_i\sqrt{\Delta t}}\right) \prod_{j\neq i} \overline{F}_j(x;t,\Delta t,z) \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \overline{F}_j(x;t,\Delta t,z) &\equiv \mathbb{P}(X_{t+\Delta t}^j > x | Z_t = z < X_t^j) \\ &= \mathbb{P}(X_{t+\Delta t}^j > x, X_t^j > z | Z_t = z < X_t^j) \\ &= \mathbb{P}(X_t^j + \Delta_j > x, X_t^j > z | Z_t = z < X_t^j) \end{aligned}$$

and $\Delta_j \sim N(\mu_j \Delta t, \sigma_j^2 \Delta t)$ is independent of X_t^j. X_t^j has a $N(\mu_j t + a_j, \sigma_j^2 t)$ distribution and so using conditional probability we find that

$$\overline{F}_j(x; t, \Delta t, z) = \frac{\mathbb{P}\left(X_t^j + \Delta_j > x, X_t^j > z\right)}{\overline{\Phi}\left(\frac{z-\mu_j t - a_j}{\sigma_j \sqrt{t}}\right)}$$
$$= \frac{\mathbb{P}\left(X > \frac{x-\mu_j(t+\Delta t)-a_j}{\sigma_j\sqrt{t+\Delta t}}, Y > \frac{z-\mu_j t-a_j}{\sigma_j\sqrt{t}}\right)}{\overline{\Phi}\left(\frac{z-\mu_j t-a_j}{\sigma_j\sqrt{t}}\right)}$$

where X,Y are standard normal variables with correlation

$$\rho = \frac{\mathbb{E}[XY]}{\sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}} = \frac{t}{\sqrt{t}\sqrt{t+\Delta t}} = \frac{1}{\sqrt{1+\frac{\Delta t}{t}}}. \tag{4.6}$$

The earliest time we wish to evaluate this at is when $t = \Delta t$ giving $\rho^2 = \frac{1}{2}$ and hence $\rho^2 > \frac{1}{2}$ for times $t > \Delta t$, so the standard procedure of using the tetrachoric series (an expansion in powers of ρ) is not well suited to our purpose. A good numerical method for computing this bivariate normal is that of Vasicek (1998) which involves an expansion in terms of powers of $(1-\rho^2)$ and so converges rapidly for the values of ρ we consider. Vasicek describes a method for computing the bivariate cumulative normal distribution function

$$N_2(x, y, \rho) = \int_{-\infty}^{x} \int_{-\infty}^{y} n_2(u, v, \rho)\, du\, dv \tag{4.7}$$

where the bivariate normal density is given by

$$n_2(u, v, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right).$$

Substituting $u' = -u$,$v' = -v$ into (4.7) we get

$$N_2(-x, -y, \rho) = \int_{x}^{\infty} \int_{y}^{\infty} n_2(u', v', \rho)\, du'\, dv',$$

and hence

$$\overline{F}_j(x; t, \Delta t, z) = \frac{N_2\left(-\frac{x-\mu_j(t+\Delta t)-a_j}{\sigma_j\sqrt{t+\Delta t}}, -\frac{z-\mu_j t-a_j}{\sigma_j\sqrt{t}}, \rho\right)}{\overline{\Phi}\left(\frac{z-\mu_j t-a_j}{\sigma_j\sqrt{t}}\right)} \tag{4.8}$$

with ρ as given in (4.6). We are now able to evaluate (4.5).

We will be particularly interested in the first two moments of the distribution of the minimum process at time $t + \Delta t$ given that $Z_t = z$. We define the drift $m(t, \Delta t, z)$ of Z_t by the equation

$$m(t, \Delta t, z)\, \Delta t = \mathbb{E}[Z_{t+\Delta t} - Z_t | Z_t = z] \\ = \int_{-\infty}^{\infty} x\, F'(x; t, \Delta t, z)\, dx - z \tag{4.9}$$

where F' denotes the derivative of F with respect to x. Similarly we define the volatility $s(t, \Delta t, z)$ of Z_t via

$$\begin{aligned} s(t, \Delta t, z)^2\, \Delta t &= \mathbb{E}\left[(Z_{t+\Delta t} - \mathbb{E}[Z_{t+\Delta t} | Z_t = z])^2 \,| Z_t = z \right] \\ &= \mathbb{E}\left[(Z_{t+\Delta t} - z)^2 \,| Z_t = z \right] - m(t, \Delta t, z)^2\, \Delta t^2 \\ &= \mathbb{E}\left[Z_{t+\Delta t}^2 | Z_t = z \right] - \mathbb{E}[Z_{t+\Delta t} | Z_t = z]^2 \\ &= \int_{-\infty}^{\infty} x^2\, F'(x; t, \Delta t, z)\, dx - \left(\int_{-\infty}^{\infty} x\, F'(x; t, \Delta t, z)\, dx \right)^2 . \end{aligned} \tag{4.10}$$

4.3.1 The maximum process

The above calculations can be easily modified to give the equivalent distribution for the maximum process

$$Z_t^{max} \equiv \max_{i=1,\dots,n} X_t^i . \tag{4.11}$$

Instead of calculating the probability that all the processes are above a certain minimum level we now work out the probability that all the processes are *below* a given maximum level. We find that the corresponding cumulative probability distribution function is given by

$$\begin{aligned} F^{max}(x; t, \Delta t, z) &\equiv \mathbb{P}(Z_{t+\Delta t}^{max} < x | Z_t^{max} = z) \\ &= \sum_{i=1}^{n} p_i^{max}(t, z)\, \Phi\left(\frac{x - (z + \mu_i \Delta t)}{\sigma_i \sqrt{\Delta t}} \right) \prod_{j \neq i} F_j^{max}(x; t, \Delta t, z). \end{aligned}$$

where

$$F_j^{max}(x; t, \Delta t, z) = \frac{N_2 \left(\frac{x - \mu_j (t+\Delta t) - a_j}{\sigma_j \sqrt{t+\Delta t}}, \frac{z - \mu_j t - a_j}{\sigma_j \sqrt{t}}, \frac{1}{\sqrt{1 + \frac{\Delta t}{t}}} \right)}{\Phi\left(\frac{z - \mu_j t - a_j}{\sigma_j \sqrt{t}} \right)}$$

and

$$\begin{aligned} p_i^{max}(t,z) &\equiv \mathbb{P}(Z_t^{max} = X_t^i | Z_t^{max} = z) \\ &= \frac{\tilde{p}_i^{max}(t,z)}{\sum_{j=1}^n \tilde{p}_j^{max}(t,z)}, \end{aligned}$$

with

$$\tilde{p}_i^{max}(t,z) = \frac{1}{\sigma_i\sqrt{2\pi t}} \exp\left(-\frac{(z-\mu_i t - a_i)^2}{2\sigma_i^2 t}\right) \prod_{j\neq i} \Phi\left(\frac{z-\mu_j t - a_j}{\sigma_j\sqrt{t}}\right).$$

4.4 Calculation of option prices

4.4.1 European options

In the European option case, where exercise is allowed only at the expiry time T, we can price the option price *exactly* from the law of the minimum process. If the option pays $f(Z_T)$ at time T then, integrating this with respect to the time-T minimum process density $g(x;T)$ given by equation (4.4), gives the European option price

$$e^{-rT}\mathbb{E}[f(Z_T)] = e^{-rT}\int_{-\infty}^{\infty} f(x)\, g(x;T)\, dx.$$

We can calculate this numerically using, for example, the 'quad' function of Matlab which performs quadrature to a requested level of precision. Answers to within an accuracy of 10^{-10} can be computed in around a tenth of a second.

4.4.2 A trinomial tree scheme

We price American/Bermudan options using a recombinant trinomial tree. We will evaluate the option price at the $N+1$ times

$$t_k = k.\Delta t \qquad k = 0,\ldots,N$$

where $\Delta t \equiv T/N$ is fixed. If we are valuing a Bermudan option N must be chosen carefully so that all the allowed exercise times lie on the lattice. We choose also the spacial points

$$z_k^i = c_k + i.\Delta z \qquad i = -k,\ldots,k$$

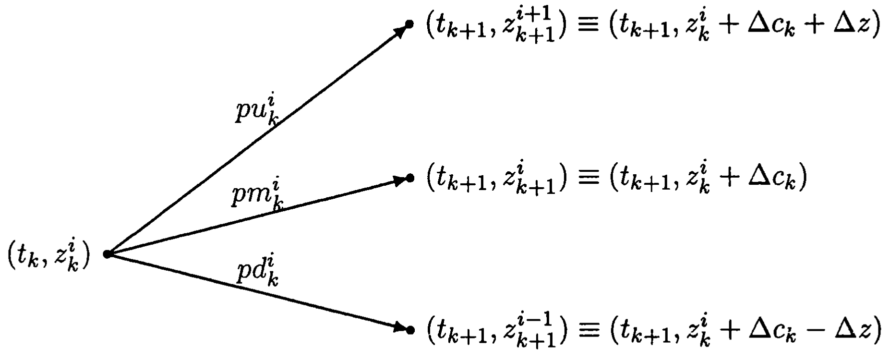

Figure 4.1: A section of the trinomial tree showing branches from (t_k, z_k^i).

where c_k denotes the spacial point at the center of the tree at time t_k and $\Delta z > 0$ is a fixed spatial discretization parameter. We will use the notation $\Delta c_k \equiv c_{k+1} - c_k$ for the amount that the centre of the tree changes by between times t_k and t_{k+1}. We write $V_k^i \equiv V(t_k, z_k^i)$ for the value of the option at time t_k and with the minimum process at z_k^i. Similarly we will abbreviate the drift (4.9) and volatility (4.10) of the minimum process by writing $m_k^i \equiv m(t_k, \Delta t, z_k^i)$ and $s_k^i \equiv s(t_k, \Delta t, z_k^i)$.

Figure 4.1 shows one section of the trinomial tree. From the point (t_k, z_k^i) the tree branches to the points $(t_{k+1}, z_k^i + \Delta c_k + \Delta z)$, $(t_{k+1}, z_k^i + \Delta c_k)$ and $(t_{k+1}, z_k^i + \Delta c_k - \Delta z)$ with probabilities pu_k^i, pm_k^i and pd_k^i respectively. Equating the first and second moments of the random variable described by this one section of the tree with m_k^i and s_k^i and noting that the probabilities sum to unity, yields the equations

$$\begin{aligned} m_k^i \Delta t &= \Delta c_k + (pu_k^i - pd_k^i)\Delta z \,, \\ (m_k^i)^2 \Delta t^2 + (s_k^i)^2 \Delta t &= (\Delta c_k)^2 + 2(pu_k^i - pd_k^i)\Delta c_k \, \Delta z + (pu_k^i + pd_k^i)\Delta z^2 \,, \end{aligned}$$

and so the probabilities we need are

$$pu_k^i = \frac{(s_k^i)^2 \Delta t + (m_k^i \Delta t - \Delta c_k)^2}{2\,\Delta z^2} + \frac{m_k^i \Delta t - \Delta c_k}{2\,\Delta z}\,, \tag{4.12}$$

$$pm_k^i = 1 - \frac{(s_k^i)^2 \Delta t + (m_k^i \Delta t - \Delta c_k)^2}{\Delta z^2}\,, \tag{4.13}$$

$$pd_k^i = \frac{(s_k^i)^2 \Delta t + (m_k^i \Delta t - \Delta c_k)^2}{2\,\Delta z^2} - \frac{m_k^i \Delta t - \Delta c_k}{2\,\Delta z}\,. \tag{4.14}$$

In order for these values to be genuine probabilities lying in $[0,1]$ we will require that

$$\frac{(s_k^i)^2\Delta t + (m_k^i\Delta t - \Delta c_k)^2}{\Delta z^2} \leq 1\,, \tag{4.15}$$

and

$$\Delta z|m_k^i\Delta t - \Delta c_k| \leq (s_k^i)^2\Delta t + (m_k^i\Delta t - \Delta c_k)^2\,. \tag{4.16}$$

Typically we will choose Δc_k to be $\mathrm{O}(\Delta t)$ so these constraints are approximately

$$(s_k^i)^2 \leq \frac{\Delta z^2}{\Delta t}\,, \tag{4.17}$$

$$\Delta z \leq \frac{(s_k^i)^2}{|m_k^i - \Delta c_k/\Delta t|}\,. \tag{4.18}$$

The first of these constraints is a serious restriction - if we wish to improve accuracy by halving Δz then we must quadruple the number of time steps N.

We calculate the value of the option as follows. At the terminal nodes of the tree we set the price to the amount we get by exercising the option

$$V_N^i = f(z_N^i) \qquad i = -N, \ldots, N$$

and then we work backwards in time through the tree calculating option values using the rule

$$V_k^i = \begin{cases} \max\left(e^{-r\Delta t}(pu_k^i V_{k+1}^{i+1} + pm_k^i V_{k+1}^i + pd_k^i V_{k+1}^{i-1}), f(z_k^i)\right) & \text{if } t_k \in R, \\ e^{-r\Delta t}(pu_k^i V_{k+1}^{i+1} + pm_k^i V_{k+1}^i + pd_k^i V_{k+1}^{i-1}) & \text{otherwise,} \end{cases} \tag{4.19}$$

so that the option price at time t_k is either the discounted expected price of the option at time t_{k+1} or the amount gained by immediate exercise if this is allowed and is more profitable.

4.4.3 Interpretation as a finite-difference scheme

Let us suppose that we can approximate the dynamics of the minimum process by

$$dZ_t = m(t, Z_t)\,dt + s(t, Z_t)\,dW_t \tag{4.20}$$

where W_t is a standard Brownian motion (independent of those previously considered) and $m(t,z)$ and $s(t,z)$ are suitably smooth functions. The time-t value of an option on the minimum process will now be given by

$$V(t, z) \equiv \sup_{\tau \in \mathcal{T}_{R \cap [t,T]}} \mathbb{E}\left[e^{-r(\tau - t)} f(Z_\tau) \middle| Z_t = z \right]$$

with the supremum being taken over stopping times adapted to the filtration of W. Suppose firstly that we wish to value a European option, so that $R \equiv \{T\}$. Taking the expectation of the expression $e^{-rT} f(Z_T)$ conditional on events up to time t we find that $e^{-rt} V(t, Z_t)$ is a martingale and hence $V(t, z)$ satisfies the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} s(t, z)^2 \frac{\partial^2 V}{\partial z^2} + m(t, z) \frac{\partial V}{\partial z} - rV = 0, \tag{4.21}$$

with boundary condition $V(T, z) = f(z)$. If the option we wish to value is not European then $V(t, z)$ will still obey the same PDE, but only on the region where it not optimal to immediately exercise.

In order to solve this PDE numerically we will need approximate values for $m(t, z)$ and $s(t, z)$ at various points (t, z). Integrating equation (4.20) between times t and $t + \Delta t$

$$Z_{t+\Delta t} - Z_t = \int_t^{t+\Delta t} m(u, Z_u)\, du + \int_t^{t+\Delta t} s(u, Z_u)\, dW_u \tag{4.22}$$

and so taking the expectation of this conditional on $Z_t = z$ we find that

$$\begin{aligned} \mathbb{E}[Z_{t+\Delta t} - Z_t | Z_t = z] &= \mathbb{E}\left[\int_t^{t+\Delta t} m(u, Z_u)\, du \middle| Z_t = z \right] \\ &\approx m(t, z)\, \Delta t \end{aligned}$$

for Δt small. Similarly, squaring equation (4.22) and taking expectations

$$\mathbb{E}\left[(Z_{t+\Delta t} - Z_t)^2 | Z_t = z \right] \approx m(t, z)^2\, \Delta t^2 + s(t, z)^2\, \Delta t.$$

Hence we can approximate $m(t, z)$ by the drift $m(t, \Delta t, z)$ and $s(t, z)$ by the volatility $s(t, \Delta t, z)$.

The trinomial tree scheme of section 4.4.2 can now be viewed as an explicit finite-difference scheme. We can set the centre of the tree at the constant level Z_0 so that $\Delta c_k \equiv 0$ for all k. From equation (4.19) and from the expressions (4.12)–(4.14) for the various probabilities it follows that if it is not optimal to exercise immediately V_k^i

satisfies the equation

$$\frac{V_{k+1}^i - V_k^i}{\Delta t} + \frac{1}{2}(s_k^i)^2 \left(\frac{V_{k+1}^{i+1} - 2V_{k+1}^i + V_{k+1}^{i-1}}{\Delta z^2} \right) + m_k^i \left(\frac{V_{k+1}^{i+1} - V_{k+1}^{i-1}}{2\,\Delta z} \right) - rV_k^i = \mathrm{O}(\Delta t).$$

Thus we have an explicit finite-difference scheme for solving the PDE (4.21). We solve the above equation (setting the right hand side equal to zero) for V_k^i and then replace V_k^i by the immediate exercise price if it is bigger. The constraints (4.15) and (4.16) that had to be satisfied for the probabilities to lie in $[0,1]$ are also the conditions necessary for this scheme to converge to a stable solution. See Wilmott (1998) for discussion of stability conditions for the explicit finite-difference scheme. In principal other types of finite-difference scheme could be used to solve the PDE (4.21).

4.4.4 Implementation issues

The trinomial tree scheme described above was programmed in Matlab with the moments m_k^i and s_k^i calculated by a compiled C subroutine called from Matlab, giving roughly a 100 times increase in speed over writing the code in Matlab alone. Two different routines for calculating $F(x;t,\Delta t,z)$ were used - one for the case where there are two assets with (possibly) differing initial values, volatilities and dividend rates and one for n identical assets, when the expression (4.5) for $\overline{F}(x;t,\Delta t,z)$ can be simplified considerably to

$$\overline{F}(x;t,\Delta t,z) = \overline{\Phi}\left(\frac{x-(z+\mu_1\Delta t)}{\sigma_1\sqrt{\Delta t}}\right)\overline{F}_1(x;t,\Delta t,z)^{n-1}.$$

The centre of the tree : The obvious choice for the centre of the tree would be the expected value of the minimum process so that

$$c_k = \mathbb{E}[Z_{t_k}].$$

However this choice leads to oscillatory behaviour for the price as N increases. This sort of problem is common with trees, see for example Klassen (2001), and the solution is to centre the tree so that the strike K is always in the centre of the tree at maturity. Hence we set $c_k \equiv Z_0 + k\,(\log(K) - Z_0)\,\Delta t/T$ so that $\Delta c_k = (\log(K) - Z_0)\,\Delta t/T$ for all k. This ensures that the calculated price of the option increases as we increase N and also means that Richardson extrapolation is viable as we shall see in Section 4.5.5.

Constraints : We require that the constraints (4.15) and (4.16) are satisfied everywhere on the tree. Looking first at (4.15) we require that

$$(s_k^i)^2\Delta t + (m_k^i\Delta t - \Delta c_k)^2 \leq \Delta z^2\,,$$

everywhere on the tree. For the minimum process the minimum upward[1] drift will occur when all the asset prices are at the minimum point - we can calculate this numerically. The maximum upward drift will occur when just one asset price is at the minimum and the others are well above this - this will have value $\max_{i=1,\ldots,n}\mu_i$. Hence we can find the maximum value of $(m_k^i\Delta t - \Delta c_k)^2$. Similarly the maximum value of the volatility will occur either when all the asset prices are together or when the asset with the highest individual volatility has a price well below that of the other assets. This gives us an upper bound B_0 say on $(s_k^i)^2\Delta t + (m_k^i\Delta t - \Delta c_k)^2$ and we initially choose $\Delta z \equiv \sqrt{B_0}$. If we do this we find that violations of the second constraint (4.16) occur at some points on the tree. We overcome this problem by choosing a smaller Δz as follows. Having picked $\Delta z \equiv \sqrt{B_0}$ we calculate the moments m_{N-1}^i and s_{N-1}^i for all $i = -(N-1),\ldots,N-1$ at the penultimate time step on the lattice. We can now find

$$B_1 \equiv \max_{i=1,\ldots,n}\left\{(s_{N-1}^i)^2\Delta t + (m_{N-1}^i\Delta t - \Delta c_{N-1})^2\right\}$$

then set $\Delta z \equiv \sqrt{B_1}$ and start the price computations from the beginning. Following this procedure no significant violations of either constraint were found for the examples we considered.

Numerical accuracy of distribution calculations : There are very fast and accurate ways to compute the cumulative normal distribution function $\Phi(x)$ for a given x - we do this using the method for calculating the error function described in Press, Flannery, Teukolsky, and Vetterling (1993). However calculations of the bivariate normals $N_2(x, y, \rho)$ are much slower. We sum the series as given by Vasicek (1998) and stop summing when we reach a term smaller than 10^{-6}; this provides a good balance between accuracy and speed of execution. When calculating $\overline{F}_j(x; t, \Delta t, z)$ we will have problems when z is significantly bigger than $(\mu_j t + a_j)$ so that both the numerator and denominator of the fraction (4.8) will be small. In order to avoid this sort of numerical error we will have to switch to some less accurate way of calculating the drift for large z. However

$$\mathbb{P}(Z_t > z) \leq \min_{i=1,\ldots,n}\mathbb{P}(X_t^i > z), \tag{4.23}$$

[1] Of course the drift is likely to be downward - we use the word upward only to indicate the sign of the drift.

and so the region in which the denominator of (4.8) is less than ε for some j is precisely the region which the minimum process has a probability less than ε of visiting. As we are pricing a put option this region also contributes least to the option price, so we can safely ignore it for a suitable choice of ε. We choose $\varepsilon = 10^{-5}$.

Quadrature : We approximate the integral $\int x\, F'(x; t, \Delta t, z)\, dx$ by an appropriate summation. We pick a positive integer M, set $\Delta x \equiv \min_{i=1,\dots,n} \sigma_i \sqrt{\Delta t}/M$ and write

$$x_i = z + \Delta t \min_{j=1,\dots,n} \mu_j + i\Delta x$$

for i an integer. We find i_L and i_U such that

$$i_L = \max\{i \in \mathbb{Z} : F(x_i; t, \Delta t, z) < \varepsilon\}, \qquad i_U = \min\{i \in \mathbb{Z} : F(x_i; t, \Delta t, z) > 1 - \varepsilon\}. \tag{4.24}$$

We then have that

$$\int x\, F'(x; t, \Delta t, z)\, dx \approx \sum_{i=i_L}^{i_U} \frac{1}{2}(x_i + x_{i-1})\, [F(x_i; t, \Delta t, z) - F(x_{i-1}; t, \Delta t, z)] \tag{4.25}$$

with a similar approximation holding for $\int x^2\, F'(x; t, \Delta t, z)\, dx$. Of course in practice the determination of the limits (4.24) is done during the calculation of (4.25), starting at a value between the limits and working outwards and the calculated values of F are reused for the calculation of $\int x^2\, F'(x; t, \Delta t, z)\, dx$. We also artificially double the number of F points using simple cubic interpolation. If we have values F_0, F_1, F_2 and F_3 at equally spaced x points x_0, x_1, x_2 and x_3 then fitting a cubic to these can yield a further point midway between x_1 and x_2 (i.e. at $\frac{1}{2}(x_1 + x_2)$) with value $(-F_0 + 9F_1 + 9F_2 - F_3)/16$. Repeating this for every group of 4 consecutive points doubles the number of values available for the quadrature. We choose $\varepsilon = 10^{-8}$ and $M = 100$ although in Section 4.5.5 we will lower M to obtain more rapid results.

Interpolation of drift and volatility : The main factor influencing the speed of the algorithm is the need to compute lots of (slow) bivariate normals. Some time can be saved by reducing the accuracy of these computations, but for substantial speed improvements it is best to do as few as possible. We can do this by reducing the number of computation points used in the quadrature as described above, but a more effective way is to interpolate values for the drift and volatility using Matlab's cubic spline interpolation routine. We compute m_k^i and s_k^i at a fixed time point t_k for a subset of the points $i = -k, \dots, k$ and then interpolate to find m_k^i and s_k^i at the other values of i. We will not do this initially, but in Section 4.5.5 we illustrate how considerable computation time can be saved using this technique, with little loss of accuracy.

4.5 Numerical examples

4.5.1 Example 1 : American min-put option on 2 assets

As a first example we will find the price of an American min-put on 2 assets. We use a 200 step lattice with exercise allowed at every time-step. As a benchmark we will compare these with 'true' prices computed via a 200×200 two-dimensional tree that models the movements of both asset processes. Table 4.1 and Table 4.2 give the resulting prices for options on assets with various different volatilities and initial values. The 'European' column is the price of the option with European-style exercise, calculated using the Matlab 'quad' function as described in Section 4.4.1. The 'Tree' column is the price calculated using a trinomial tree to model the behaviour of the minimum of the asset prices. The '2-D' column denotes the benchmark results calculated on a tree modelling the behaviour of both assets. The prices obtained using our method are all lower than the benchmark prices - this is to be expected as we are restricting ourselves to a simpler exercise rule based only on the minimum of the two asset prices rather than on the positions of them both. However the under pricing is not too large, with 'Tree' prices within approximately 1% of the benchmark values.

4.5.2 Example 2 : American min-put option on n identical assets

We will now price an American min-put option on the minimum of $i = 1, \ldots, n$ assets, all with the same initial value S_0^i, dividend rate δ_i and volatility σ_i. We compare our results with those of Rogers (2002) who uses a dual approach to compute an upper bound for the option price using Monte-Carlo methods. Table 4.3 gives the prices - as we would expect our lower bounds for the prices are generally below, and fairly close to, the Monte-Carlo upper bounds. This is not true for the 15 asset result, but it is still well within one standard error. As the number of assets increases the American price gets closer to the European price. Again this is to be expected as with a large number of assets the likelihood of improving on the current minimum at any moment is increased and hence the early exercise premium of the American option is small.

S_0^1	S_0^2	European	Tree	2-D
90	90	17.1232	19.925	20.137
90	100	14.8448	17.400	17.560
90	110	13.2854	15.827	15.951
100	100	12.2800	14.251	14.370
100	110	10.5091	12.237	12.321
110	110	8.5805	9.898	9.962

Table 4.1: Prices of American min-put on two assets with the same volatility. $K = 100$, $T = 3$, $r = 0.05$, $\delta_1 = \delta_2 = 0$, $\sigma_1 = \sigma_2 = 0.2$, $N = 200$.

S_0^1	S_0^2	European	Tree	2-D
90	90	27.0174	29.460	29.659
90	100	24.4921	26.569	26.851
90	110	22.3922	24.214	24.578
100	90	25.3413	27.833	27.873
100	100	22.6268	24.652	24.771
100	110	20.3599	22.076	22.235
110	90	24.1929	26.723	26.693
110	100	21.3440	23.383	23.393
110	110	18.9580	20.619	20.678

Table 4.2: Prices of American min-put on two assets with different volatilities. $K = 100$, $T = 3$, $r = 0.05$, $\delta_1 = \delta_2 = 0$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $N = 200$.

n	European	Tree	MC	SE
2	24.7703	24.864	25.16	0.057
3	31.2487	31.307	31.76	0.095
4	35.8092	35.858	36.28	0.081
5	39.1666	39.211	39.47	0.095
10	48.0214	48.064	48.33	0.100
15	52.1261	52.169	52.14	0.108

Table 4.3: Prices of American min-put on n assets. MC denotes Monte-Carlo upper bound of Rogers (2002) with standard error given in the SE column. $K = 100$, $T = 0.5$, $r = 0.06$, $N = 200$. $S_0^i = 100$, $\delta_i = 0$, $\sigma_i = 0.6$ the same for all assets.

4.5.3 Example 3 : Bermudan max-call option on n identical assets

We now price Bermudan max-call options. As the name suggests, the max-call option is a call option on the maximum of n dividend paying assets. As with the min-put we denote the strike price by K so that if the option is exercised at some time $\tau \in R$ it has payoff given by $f(Z_\tau^{max}) = (e^{Z_\tau^{max}} - K)^+$, and we take the set of allowed exercise times to be

$$R = \{t_i = iT/d : i = 0, 1, \dots, d\}.$$

We use exactly the same trinomial tree method as we did for the min-put option, differing only in that we use the distribution $F^{max}(x; t, \Delta t, z)$ of the maximum process as given in Section 4.3.1 in place of $F(x; t, \Delta t, z)$. We will be particularly interested in the cases where $d = 3, 6$ or 9 hence we choose $N = 180$ steps so that the corresponding Bermudan allowed exercise times occur on our tree. We also use $d = 180$ to approximate American-style exercise although this will be covered more thoroughly in Example 4.

One further difference from the min-put option is that we now see the reason for the inclusion of a dividend-yield-rate term δ_i in the definition (4.1) of the asset price processes. It is well known (see for example Musiela and Rutkowski (1998), Corollary 8.2.1) that if $r \geq 0$ an American call option on a non-dividend-paying asset is worth the same as the equivalent European call option; it is never optimal to exercise the option before the expiry date T. This is basically because under the risk-neutral measure the asset price S_t has positive drift so it is always better to wait and see if it increases rather than exercise immediately. If it isn't optimal to exercise a call option on any of n individual assets then it certainly won't be optimal to exercise a call option on the maximum of these assets. In all the examples we will be interested in we will assume that all the assets are identical and hence to get non-trivial answers we must choose a value for the dividend yield δ big enough that $r - \delta$ is negative.

There are several sources of good max-call prices to use as benchmarks in the literature. The dual approach of Rogers is used by Andersen and Broadie (2001) to obtain upper bounds on Bermudan max-call options with $d = 9$ and they use conventional (i.e. primal) Monte Carlo simulation to also obtain lower bounds. Table 4.4 gives their 95% confidence intervals for the price of such Bermudan options along with our lower bounds based again on the simplified exercise strategy of looking only at the minimum process. Also given for $n = 2$ and 3 are binomial values that Andersen and Broadie determine from the multidimensional BEG routine of Boyle, Evnine, and Gibbs (1989). Our results are again all within around 1% of the BEG values and the lower end of the 95% confidence interval of the benchmark results.

n	S_0^i	European	Tree	BEG	AB 95% CI
2	90	6.6551	8.065	8.075	[8.053,8.082]
2	100	11.1957	13.841	13.902	[13.892,13.924]
2	110	16.9286	21.183	21.345	[21.316,21.359]
3	90	9.5380	11.229	11.29	[11.265,11.308]
3	100	15.6807	18.529	18.69	[18.661,18.728]
3	110	23.1284	27.231	27.58	[27.512,27.663]
5	90	14.5856	16.492		[16.602,16.655]
5	100	23.0516	25.832		[26.109,26.292]
5	110	32.6852	36.271		[36.704,36.832]

Table 4.4: Bermudan max-call on n assets with $d = 9$ exercise opportunities at times iT/d where $i = 0, \dots, d$. Prices are from Andersen and Broadie (2001), their 95% confidence interval and also prices they have determined using the multidimensional BEG routine. $K = 100$, $T = 3$, $r = 0.05$, $N = 180$. S_0^i, $\delta_i = 0.1$, $\sigma_i = 0.2$ the same for all assets.

For the 5 asset case, Broadie and Glasserman (1997) compute a 90% confidence interval using a stochastic mesh method for Bermudan options with a range of exercise frequencies. Table 4.5 gives their results along with those from our approximate method. Broadie and Glasserman also give European prices which agree exactly with ours, although they don't say how these are obtained[2]. Our lower bound is again close to the benchmark prices. As the initial asset price increases or the number of exercise periods grows the gap between our lower bound and the prices of Broadie and Glasserman also increases, reflecting the value lost through our choice of simplified exercise strategy in these cases.

Table 4.6 gives Bermudan max-call prices on 10 assets with $d = 9$ and gives results from Raymar and Zwecher (1997) for comparison. Raymar and Zwecher use Monte Carlo simulation and reduce the simulated asset price vector at a particular time by assigning it to a particular 'bucket' region. Probabilities of moving between different buckets at different times are determined by repeated simulation and then a similar procedure to our trinomial tree is used to iterate backwards determining optimal exercise. Their '200 $\times$ 1' results have 200 'buckets' at each stage to represent the value of the maximum asset and so their results should be comparable to our method which is very similar in philosophy. This is indeed the case, although our prices are generally higher than those of Raymar and Zwecher for lower strikes and vice-versa for higher strikes. The

[2]Other authors give either Monte-Carlo European prices or inaccurate prices using an unspecified method, indicating that it perhaps isn't well known that these options can be priced exactly in the European case.

d	S_0^i	Tree	BG 90% CI
3	90	16.000	[15.995,16.016]
3	100	25.227	[25.267,25.302]
3	110	35.568	[35.679,35.710]
6	90	16.364	[16.438,16.505]
6	100	25.674	[25.889,25.948]
6	110	36.097	[36.466,36.527]
9	90	16.492	[16.602,16.710]
9	100	25.832	[26.101,26.211]
9	110	36.271	[36.719,36.842]
180	90	16.733	
180	100	26.127	
180	110	36.621	

Table 4.5: Prices of max-call on 5 assets with d exercise opportunities at times iT/d where $i = 0, \dots, d$. Benchmark prices are the 90% confidence interval computed by Broadie and Glasserman (1997). $K = 100$, $T = 3$, $r = 0.05$, $N = 180$. S_0^i, $\delta_i = 0.1$, $\sigma_i = 0.2$ the same for all assets. The European exercise prices are 14.586, 23.052 and 32.685 for $S_0^i = 90$, 100 and 110 respectively.

K	European	Tree	200×1	SE	50×4	SE
85	40.6463	40.966	40.931	0.045	41.216	0.043
100	26.4082	26.674	26.652	0.045	26.911	0.044
115	13.2807	13.482	13.486	0.041	13.695	0.039
130	4.8074	4.918	4.943	0.026	5.043	0.027
145	1.3658	1.403	1.426	0.014	1.466	0.015

Table 4.6: Prices of 10 asset Bermudan ($d = 9$) max-call option with exercise allowed at times iT/d where $i = 0, \dots, d$. 200×1 and 50×4 are Monte-Carlo results of Raymar and Zwecher (1997) with corresponding standard errors. $T = 1$, $r = 0.05$, $N = 180$. $S_0^i = 100$, $\delta_i = 0.1$, $\sigma_i = 0.2$ are the same for all assets.

'50×4' results are obtained using 50 'buckets' to represent the value of the maximum asset and another 4 'buckets' at each maximum 'bucket' to represent the value of the second largest asset, giving a total of 200 'buckets' again. These results are based on a more complex exercise rule than our own and so are all higher in value. Note that the Bermudan prices are not much higher than the value of the European option, reflecting the fact that with 10 assets, all with the potential to rise in value, it will usually be optimal not to exercise until maturity.

4.5.4 Example 4 : American max-call option on n identical assets

We can approximate American exercise by setting d large in the Bermudan example above. We compare our results with those of Haugh and Kogan (2001), who find prices of max-calls on 5 and 10 assets for $d = 99$, so that there are 100 exercise periods in total. We use a tree with $N = 198$ time steps and compute the same prices for $d = 99$ and $d = 198$. Table 4.7 and Table 4.8 show the results for 5 and 10 assets respectively. Doubling the number of exercise opportunities makes little difference to our prices which are below those of Haugh and Kogan, although only by at most 2%. We notice again that for 10 assets the early-exercise premium on the option is significantly lower than for 5 assets.

d	S_0^i	European	Tree	HK LB	SE	HK UB	SE
99	90	14.586	16.729	16.962	0.0056	17.030	0.0218
99	100	23.052	26.122	26.611	0.0066	26.666	0.0184
99	110	32.685	36.612	37.332	0.0075	37.442	0.0247
198	90	14.586	16.740				
198	100	23.052	26.135				
198	110	32.685	36.623				

Table 4.7: Prices of max-call on 5 assets with d exercise opportunities at times iT/d where $i = 0, \ldots, d$. Benchmark prices are the lower (HK LB) and upper (HK UB) bounds computed by Haugh and Kogan (2001) with the corresponding standard errors given in the SE column. $K = 100$, $T = 3$, $r = 0.05$, $N = 198$. S_0^i, $\delta_i = 0.1$, $\sigma_i = 0.2$ the same for all assets.

d	S_0^i	European	Tree	HK LB	SE	HK UB	SE
99	90	14.747	14.989	15.178	0.0039	15.283	0.0136
99	100	26.408	26.725	26.996	0.0045	27.070	0.0117
99	110	38.529	38.917	39.223	0.0050	39.306	0.0201
198	90	14.747	14.991				
198	100	26.408	26.728				
198	110	38.529	38.920				

Table 4.8: Prices of max-call on 10 assets with d exercise opportunities at times iT/d where $i = 0, \ldots, d$. Benchmark prices are the lower (HK LB) and upper (HK UB) bounds computed by Haugh and Kogan (2001) with the corresponding standard errors given in the SE column. $K = 100$, $T = 1$, $r = 0.05$, $N = 198$. S_0^i, $\delta_i = 0.1$, $\sigma_i = 0.2$ the same for all assets.

One advantage of modelling the maximum process directly is that the resulting exercise boundary is much simpler[3]; rather than having to consider the prices of all n assets when deciding whether to exercise, the holder of a max-call option need only consider the highest priced asset. The quickest way to calculate the optimal exercise boundary for a max-call option is to read it directly off the tree used for pricing, as at each node of the tree we know if the price of the option at that point is obtained by immediate exercise of the option. We thus obtain upper and lower bounds on the position of the optimal exercise boundary as shown in Figure 4.2 for a 5 asset American max-call - it's easy to see where the exercise boundary will lie. Thus we have a simple exercise rule based on the position of the maximum in contrast to the true optimal exercise boundary for the problem which will be a surface in $\mathbb{R}^n$.

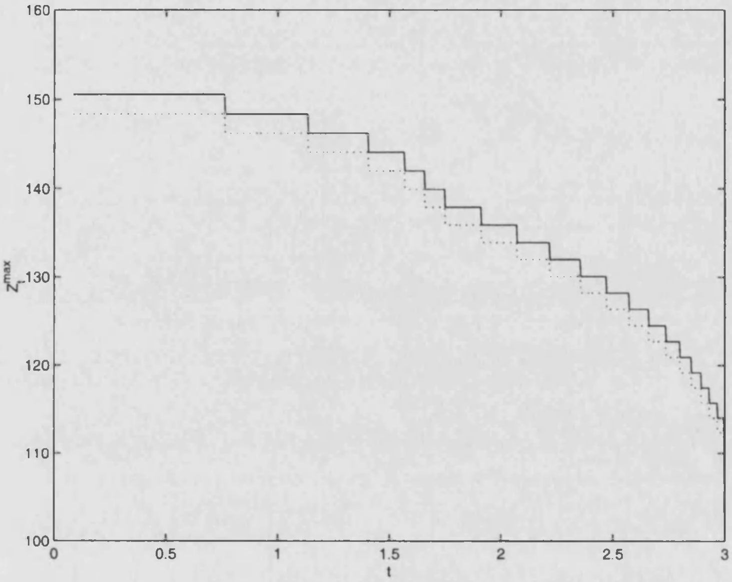

Figure 4.2: Bounds on the optimal exercise boundary for an American max-call option on 5 assets. The holder should certainly exercise if the maximum goes above the solid line, and not exercise if the maximum is below the dotted line. $K = 100$, $T = 3$, $r = 0.05$, $N = 1000$, $d = 1000$. $S_0^i = 100$, $\delta_i = 0.1$, $\sigma_i = 0.2$ the same for all assets.

[3] Look at Broadie and Detemple (1997), Figure 7.2 to see how complicated things can get.

4.5.5 Calculation times

One benefit of our approach is the speed of calculation. The computation time is linear in the number of assets n for non-identical assets, and independent of n for identical assets. The speed of computation is also completely independent of the number of allowed exercise periods d for a fixed N. In comparison the method of Haugh and Kogan has a computational effort that grows linearly with d while the method of Andersen and Broadie is quadratic in d.

The 'Tree' prices in Table 4.5 took around 4 minutes each to compute on a Sun UltraSPARC-III, but these times can be improved considerably. Firstly we can use cubic spline fitting of the drift and volatility curves as described in Section 4.4.4. Table 4.9 shows the effect on the prices of 5 asset max-call options. The prices obtained are all within 0.01 of their original values but take only 15% of the original CPU time to calculate.

d	S_0^i	No spline	CPU	Spline	CPU
3	90	16.000	221.03	16.004	32.94
3	100	25.227	219.81	25.232	32.85
3	110	35.568	218.67	35.575	32.66
6	90	16.364	221.03	16.367	32.94
6	100	25.674	219.86	25.679	32.85
6	110	36.097	218.73	36.104	32.67
9	90	16.492	221.05	16.496	32.94
9	100	25.832	219.86	25.838	32.84
9	110	36.271	218.73	36.279	32.65
180	90	16.733	221.03	16.736	33.00
180	100	26.127	219.88	26.133	32.88
180	110	36.621	218.77	36.627	32.70

Table 4.9: Prices of max-call on 5 assets with d exercise opportunities at times iT/d where $i = 0, \dots, d$. Prices computed with and without cubic spline fitting of the drift and volatility of the minimum process and the corresponding CPU times in seconds. $K = 100$, $T = 3$, $r = 0.05$, $N = 180$. S_0^i, $\delta_i = 0.1$, $\sigma_i = 0.2$ the same for all assets.

Next we consider the use of Richardson extrapolation as described in, for example, Klassen (2001). We have a good test case in the European-style max-call option - we can compute its value exactly as in Section 4.4.1, but we could also use a N time-step tree to compute it. The resulting error will give us an idea of the errors we will get from using the tree to price more complicated options. Figure 4.3 shows how the price of a

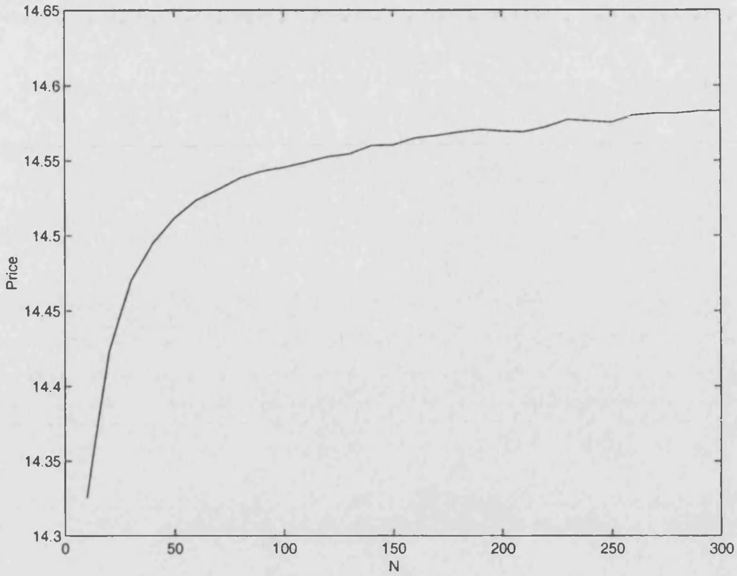

Figure 4.3: Price of European max-call option on 5 assets calculated on a tree with N time-steps. $K = 100$, $T = 3$, $r = 0.05$. $S_0^i = 90$, $\delta_i = 0.1$, $\sigma_i = 0.2$ the same for all assets.

European max-call computed on a tree varies as we increase the number of steps in the tree. We see that the price increases reasonably smoothly towards the true price of the option so that when $N = 300$ the computed price of 14.583 is just under the true price of 14.586. We can think of the trinomial tree scheme as an explicit finite-difference method (see Section 4.4.3) with a theoretical accuracy of $\mathrm{O}(\Delta t, \Delta z^2)$. The constraint (4.17) means that we must choose Δz^2 to be of order Δt so the tree scheme should have a highest order error term proportional to $1/N$. This turns out to be almost the case for our European max-call example when we restrict ourselves to the range of N which we will want to use for extrapolation; a plot (see Figure 4.4) of the logarithm of the error against $\log N$ produces a straight line with a slope of around -0.81. Given this, we will assume that the leading error term is of order $1/N$. If we compute option prices V_{N_1} and V_{N_2} on trees with N_1 and N_2 time-steps respectively and use Richardson extrapolation to get rid of the leading error term then we find that the option price should be

$$\frac{N_1 V_{N_1} - N_2 V_{N_2}}{N_1 - N_2}.$$

Taking $N_1 = 18$ and $N_2 = 36$ we get the results in Table 4.10. The extrapolated results compare well with the more accurate results that we obtained earlier in Table 4.5. The CPU times are now around a second, aided by a reduction in accuracy of

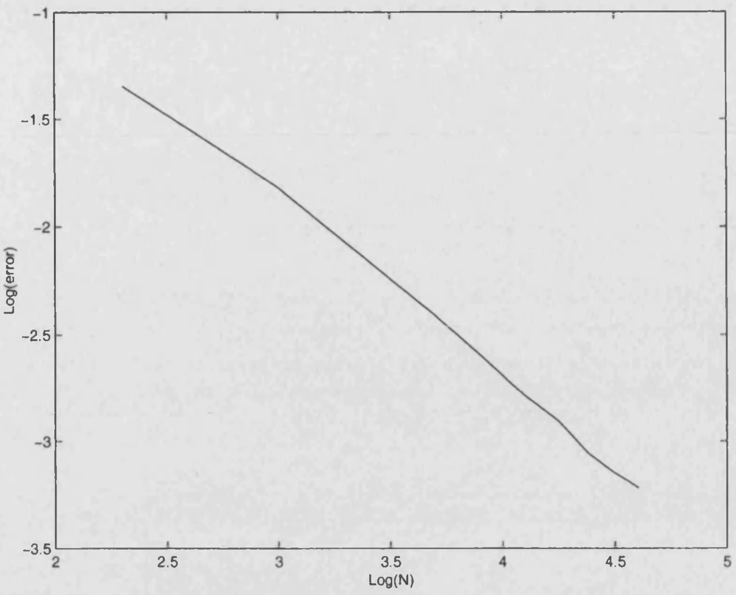

Figure 4.4: The logarithm of the difference between the true price of a European max-call option on 5 assets and the price calculated on a tree with N time-steps plotted against $\log N$. $K = 100$, $T = 3$, $r = 0.05$. $S_0^i = 90$, $\delta_i = 0.1$, $\sigma_i = 0.2$ the same for all assets.

d	S_0^i	$N = 18$	$N = 36$	RE	Accurate	CPU
0	90	14.4208	14.4994	14.5781	14.586	1.10
0	100	22.8884	22.9896	23.0908	23.052	1.07
0	110	32.5437	32.6114	32.6791	32.685	1.08
3	90	15.7466	15.8606	15.9746	16.000	1.09
3	100	24.8299	25.0367	25.2436	25.227	1.10
3	110	35.1280	35.3132	35.4983	35.568	1.08
6	90	16.0977	16.2292	16.3608	16.364	1.10
6	100	25.2542	25.4863	25.7185	25.674	1.09
6	110	35.5536	35.8353	36.1170	36.097	1.08
9	90	16.2274	16.3406	16.4538	16.492	1.10
9	100	25.4228	25.6384	25.8539	25.832	1.09
9	110	35.7717	36.0160	36.2604	36.271	1.09

Table 4.10: Prices of max-call on 5 assets with d exercise opportunities at times iT/d where $i = 0, \ldots, d$. Shown are the prices computed for 18 and 36 steps and the Richardson extrapolation price (RE). 'Accurate' denotes prices computed by quadrature in the case of the European options and prices from Table 4.5 computed using a tree with $N = 180$ steps in the other cases. The CPU column is the total time in seconds taken to obtain the Richardson extrapolation price. $K = 100$, $T = 3$, $r = 0.05$. S_0^i, $\delta_i = 0.1$, $\sigma_i = 0.2$ the same for all assets.

the quadrature used to compute moments as described in Section 4.4.4, by setting $M = 10$. This is extremely fast in comparison with Monte Carlo based methods which take several minutes to obtain results.

4.6 Summary and conclusions

In this chapter we looked at options on the minimum or maximum of several assets, in particular the min-put option. We gave an explicit formula for the distribution of the minimum process at time $t + \Delta t$, given its position at an earlier time t, in terms of cumulative normal and bivariate normal functions.

We described how to evaluate this formula rapidly using the expansion of Vasicek (1998) and gave a quadrature method for the exact pricing of European options and a trinomial tree method for finding a lower bound for the price of American/Bermudan options. We gave numerical examples for both min-put and max-call prices which compared well with benchmark prices and those from the literature. In contrast to the standard Monte-Carlo approach we showed that our method could give good answers in seconds rather than minutes. Another benefit of our approach is that the exercise boundary can be found in a much simpler form.

One possibility for future research would be to investigate the extent that a more complicated exercise rule leads to better lower bounds for the price of multiple asset options, for example exercise based on the value of the second lowest priced asset as well as the lowest priced asset. The evolution of the joint law of these two processes could be modelled on a suitable two-dimensional tree. Other possibilities would be to investigate the use of other finite-difference schemes or to consider the case where the assets are correlated.

Part II

A Two Sector Stochastic Growth Model

Chapter 5

A two-sector stochastic growth model

Abstract

This part of the thesis develops the study of two-sector growth models of the form introduced by Arrow and Kurz (1970).

Being purely deterministic, the original model of Arrow and Kurz was unable to distinguish between open-loop and closed-loop control of the economy; by allowing stochastic terms into the model, we are able to resolve this difficulty of interpretation. Moreover, we find that in some important cases the model can be solved explicitly in closed form, to the extent that we can write down expressions for tax rates and interest rates. This leads to new one-factor interest-rate models, with related taxation policies; numerical examples, including calculation of zero-coupon bond yield curves, show very reasonable behaviour.

5.1 Introduction

The history of growth models is long and illustrious, stretching back at least to Ramsey (1928). Throughout this development, much attention has been devoted to single-sector models, where there is just one type of capital or good, which is produced at a rate depending on current capital levels, labour force and technology levels, and is then either consumed or reinvested into capital. One analogy is a farm producing corn which can either be eaten or used to produce more corn. There are two basic types of continuous-time single-sector growth model appearing in the economic literature. Firstly the Solow model as developed by Solow (1956) and Swan (1956). This is a growth model with an exogenously given savings rate which determines the proportion of capital reinvested (and hence also the proportion consumed). Denison (1961) showed that this model was able to explain trends in empirical growth data for the United States. Secondly there is the Ramsey model. This was originally conceived by Ramsey (1928) but the term is now used to refer to the version as refined by Cass (1965) and Koopmans (1965). This is a growth model with consumer optimization - households choose their rates of consumption over time to maximize a utility functional. See, for example, the books of Romer (2001) and Barro and Sala-I-Martin (1995) for a more complete description of these models and their variants. Ramsey's original model was actually more subtle than that of Cass/Koopmans in some respects, for example it included a disutility function to reflect the amount of labour supplied (i.e. the longer the hours worked the less the utility). We will adopt a similar approach.

The first two-sector model was developed by Uzawa (1961), (1963) who considered an economy with two produced goods, a consumer good and an investment good, produced by investment capital and labour. Again using the farm analogy, this is using labour and tractors to make corn and tractors. Uzawa (1965) then refined this model to one where the two goods are physical capital and human capital, both of which are required for production of further physical capital (by manufacturing) and human capital (by education). Arrow and Kurz (1970) chose public capital rather than human capital and our work in this chapter develops this model.

Arrow and Kurz proposed a deterministic model where there were two types of capital, government capital and private capital, which were both needed in the production of the single consumption good. They first set about solving the government's optimization problem, where the government's objective was to maximize the integrated discounted felicity from *per capita* consumption, where the felicity also depends on the *per capita* level of government capital. This feature of the model recognises that the felicity of the population is improved if the provision of education, health care, transportation, *etc.* is improved, and that such infrastructure is provided largely (if not exclusively) by government capital. Since Arrow and Kurz assume that private and government capital can be freely switched at any time, it is clear that the state of the optimally-controlled system at any time is completely described by the total amount of capital, the split between government and private sectors being dictated by optimality.

The problem gets more interesting when it comes to the behaviour of the private sector, which is viewed as very large collection of identical non-collaborating small households, each individually optimizing its common objective, which is again an integrated discounted felicity of *per capita* consumption and government capital, but not of course the same as the government's objective. History and fashion have overwhelmed the centrally-planned economy, so we suppose that the government's control of the economy is exercised only through levying various proportional taxes, and issuing and retiring riskless debt from time to time. The central question studied by Arrow and Kurz is: *can the government manipulate taxes and debt in such as way as to induce the private sector to follow the government's optimal policy?*

The analysis of Arrow and Kurz is quite involved, but they are able to conclude that, under certain conditions, various combinations of taxes and debt can steer the economy onto the government's desired trajectory. However, the solutions they find are in terms of deterministic trajectories for the various tax rates for all future times, and this leaves undecided the interpretation of the solution: *is this open-loop or closed-loop control?* That is, do we think of the solution for the income tax rate (which will be an explicit function of time) as something that the government commits to at time 0, or do we rather think of the income tax rate as being a function of the underlying state variable (the total amount of capital)? The former interpretation seems viable only if we assume that the world really is deterministic, and that the government can predict with perfect foresight for all time. Casual observation suggests that this is very unlikely to be the case, so we would prefer to have a solution where tax rates would be expressed in terms of the current state of the economy, rather than being set according to a centuries-old plan. In the deterministic model of Arrow and Kurz, these two cannot be distinguished. See Christiaans (2001) for further discussion on this point - he concludes that open-loop solutions of dynamic optimization problems are unstable and therefore provide no reasonable basis for a positive theory of economic growth.

Another feature of Arrow and Kurz's solution is that we have little insight into the properties of the tax regimes the government would need to follow: in particular, are the tax rates always between 0 and 1? If not, are the suggested values actually credible?

To address these issues, we plan in this chapter to take the model of Arrow and Kurz, and modify it in two respects:

(i) introduce random fluctuations in output and population size;

(ii) allow the population to choose their level of effort.

The first modification allows us to distinguish between solutions which are functions of the underlying state of the economy, and solutions which are pre-determined processes. Without the second modification, we find that the effects of income tax are unrealistic. Once again, it turns out that the optimal solution of the government's problem can be expressed in terms of a single underlying state variable, the technology-adjusted *per capita* capital in the economy, which now follows a stochastic differential equation, and is thus a diffusion. We are then able to solve the private sector's problem, and deduce relations which must be satisfied by the various tax rates and by government debt in order to induce the private sector to follow the trajectory desired by government. In particular, we look for (and find) solutions for the tax rates which are functions only of the state process.

As yet, these expressions for tax rates are still quite opaque, so we are no better placed to decide whether they will always be between 0 and 1, for example. Our response to that is twofold. Firstly we find explicit examples which can be *solved in closed form*, and where it is possible to find the range of any of the tax rates, as these are expressed now as explicit functions of the state variable. A collection of such examples helps us to build up a library of possible behaviours, may lead us to other interesting questions, and allows us to check further analytical and numerical work. The approach we use is simply to take the inverse problem; write down the solution we would like, and then see whether we can find a model to which that is the solution! So we obtain a simple solution to a possibly slightly complicated model, rather than no solution to a simple model. This approach applies even to the basic one-sector model, and we show in an appendix some of the solutions which can be obtained for that. Our consideration of explicit examples is similar to the so called "inverse optimal" problem first studied by Kurz (1968) of constructing the class of objective functions that could give rise to given specified consumption-investment functions. Chang (1988) solves a similar stochastic inverse optimal problem.

Secondly we use numerical methods to find tax rates for an explicitly specified model. For the deterministic model of Arrow and Kurz we can use standard differential equation solving techniques. In the deterministic case there is an equilibrium point which the system tends towards and we can solve backwards in time from this. Similar methods for the Ramsey model are discussed in Judd (1998). For the full stochastic model no such equilibrium point exists and we must use more sophisticated techniques. We are interested in finding policies rules that depend only on the total capital and not on time, so we approximate our continuous model by a choosing a discrete set of points for the allowed values of total capital and calculate the probabilities of transitions (up or down) between these discrete levels, given a fixed policy, and the expected time these transitions will take. We then use dynamic programming to find the optimal policy for this approximate model.

A good review of the literature on dynamic programming applied to economic problems is given in the introduction to Chapter 14 of Amman, Kendrick, and Rust (1996). Previous applications of dynamic programming to growth models seem to have been on discrete time models (although usually with continuous state spaces for the levels of capital, consumption etc.) such as the model of Brock and Mirman (1972) which is a single-sector stochastic model with the volume of production at each time depending on a random variable as well as the levels of capital and labour. Taylor and Uhlig (1990) contains the results of a 'horse race' in which a number of alternative methods (from papers in the 1990 *Journal of Business and Economic Statistics* appearing immediately after the article of Taylor and Uhlig) compete in their ability to solve a more general version of the Brock and Mirman model. No conclusive winner was found.

Shortly after the work of Arrow and Kurz growth theory fell out of favour, not making a return until the mid-1980s. Lucas (1988) extended the work of Denison (1961) by showing that a two-sector model can explain not only the trends in growth data, but also diversity between countries in the data. Consequently much of the recent growth literature deals with economies with two capital goods, the first usually being physical capital and examples of the second including human capital, public capital, financial capital, quality of products and embodied and disembodied knowledge (Mulligan and Sala-I-Martin 1993).

Models considering directly the effects of public investment come in two formulations. The first considers how the *rate* of government expenditure on public services effects the productivity of the economy; see Aschauer (1988) for a discrete example or Barro (1990) for a continuous time model. The second type of formulation considers the total *stock* of public capital, invested in such things as roads and hospitals, as the key input to the production rate. This was the problem first studied by Arrow and Kurz, with the stock of government capital appearing in the utility function as well as the production function. This second approach is arguably more realistic but has not been widely adopted, although Futagami, Morita, and Shibata (1993) have extended the model of Barro to include government capital, and Fisher and Turnovsky (1998) have adopted a Ramsey style framework, although in both these models the public capital only appears in the production function and not also the utility. This is true for most other two-sector models too - usually the utility function is restricted to being a function of consumption and not of levels of capital or rates of investment. However Baxter and King (1993) do consider a discrete time model very similar to that of Arrow and Kurz.

Use of continuous time stochastic calculus in economic growth models first appeared in the papers of Bourguignon (1974), Merton (1975) and Bismut (1975). These extend the Solow growth model to a random setting by addition of a Brownian element to the labour supply (Bourguignon, Merton) or to the production process (Bourguignon, Bismut). Merton also considers a stochastic version of the Ramsey problem, again with Brownian motion appearing in the dynamics of the labour supply. Chapter 3 of Malliaris and Brock (1982) contains a good overview of these and similar models. More recent contributions building on Merton's 'Stochastic Ramsey Model' include Foldes (1978), (2001) who adds Brownian motions to further parameters of the model, and Amilon and Bermin (2001) who allow the government to control the expected growth rate of the labour supply. We have been unable to find any continuous time stochastic two-sector (private sector and government capital) models anywhere in the literature.

One of the possible uses of a stochastic growth model is to study interest-rate dynamics. Merton (1975) does this for the stochastic Solow model using a Cobb-Douglas production function and a constant savings ratio. Amilon and Bermin use a stochastic Ramsey model and generate a variety of interest-rate processes by considering different production and utility functions. Cox, Ingersoll, and Ross (1985a), (1985b) develop a simple stochastic model of capital growth which they use to determine the behaviour of asset prices including the term structure of interest rates. Sundaresan (1984) builds on this work and that of Merton by considering multiple consumption goods with a Cobb-Douglas production function.

We will proceed as follows. In Section 5.2 we describe our model and consider the *central-planning problem* where the government has total control over the economy and wishes to maximize its own utility functional. We give conditions that the government's optimal choices must obey. Section 5.3 introduces taxation and a private sector independently optimizing its own utility functional subject to taxation constraints. We find expressions that the tax rates must satisfy in order to force the private sector to follow the *government's* optimal choices. In Chapter 6 we look at ways to find explicit solutions to these problems and give an example. In Chapter 7 we show how to calculate numerical solutions to more generally specified problems. We give plots illustrating our explicit and numerical solutions and conclude in Chapter 8. Appendices B.1 and B.2 gives proofs that our explicit example satisfies various technical conditions. Appendix

B.3 is a (very technical) discussion of the behaviour of the level of government debt. Appendix B.4 shows how our results simplify to the one-sector Ramsey model. Finally Appendix B.5 contains a useful summary of the notation used in this part of the thesis.

5.2 The government's problem

The dynamics of the total capital K_t in the economy at time t evolves according to the equation

$$dK(t) = K(t)dZ_t^0 + \Big[F(K_p(t), K_g(t), T(t)L(t)\pi(t)) - \delta K(t) - C(t) \Big] dt, \tag{5.1}$$

where Z^0 is some multiple of a standard Brownian motion, $L(t)$ is the size of the population at time t, $\pi(t) \in [0,1]$ is the proportion of the population's effort devoted to production, and $K_p(t)$ is the amount of private capital in existence at time t, $K_g(t) \equiv K(t) - K_p(t)$ the amount of government capital at time t. The parameter δ is the rate of depreciation, a positive constant, the process C is the aggregate consumption rate, and the process T is the labour-augmenting effect of improvements in technology. We shall assume always that K, K_g, K_p and C are non-negative. As a notational convenience we will use subscript and argument notations $K_t \equiv K(t)$ interchangeably throughout, and will omit appearance of the time argument where there is no risk of confusion. Concerning the production function F, we shall make the usual assumption of homogeneity of degree 1, which is to say that

$$F(\lambda K_p, \lambda K_g, \lambda L) = \lambda F(K_p, K_g, L) \tag{5.2}$$

for any $\lambda > 0$. We shall also suppose that

$$\begin{aligned} dL_t &= L_t(dZ_t^L + \mu_L dt), \quad &(5.3)\\ dT_t &= \mu_T T_t dt, \qquad T_0 = 1, \quad &(5.4) \end{aligned}$$

where μ_L and $\mu_T \geq 0$ are constants, Z^L is again a multiple of a standard Brownian motion and we specify the covariance structure of the Brownian processes by

$$\langle Z^i, Z^j \rangle_t = v_{ij} t, \qquad i,j \in \{0, L\}. \tag{5.5}$$

The objective of the government is to maximise

$$\mathbb{E}\int_0^\infty e^{-\rho_g t} L_t\, U\bigg(\frac{C_t}{L_t}, \frac{K_g(t)}{L_t}, \pi_t \bigg)\, dt, \tag{5.6}$$

where the *felicity*[1] U is strictly concave and increasing in the first two arguments, decreasing in the last and $\rho_g > 0$ is constant. The objective (5.6) depends on *per capita* consumption and *per capita* government capital, and the felicity is weighted according to the current population size. In order to have the prospect of a time-homogeneous solution, we require that U is also homogeneous of degree $(1 - R_g)$ in the first two arguments for some $R_g > 0$ different from 1; this means that U can be represented as

$$U(C, K_g, \pi) = K_g^{1-R_g} h(\xi, \pi), \qquad \xi \equiv C/K_g \tag{5.7}$$

for some C^2 function h. Our main results will be proved only subject to the assumption that h is either non-negative or non-positive. This restriction seems to be satisfied in many interesting cases, and is probably not really necessary; we require it to save us from over-clumsy statements of results.

As a consequence of the assumptions so far, it turns out to be advantageous to work with *per capita* technology-adjusted variables, rather than their aggregated equivalents. We define

$$\eta_t \equiv L_t T_t = L_0 \exp\bigg\{ Z_t^L + (\mu_L - \frac{1}{2}v_{LL} + \mu_T)t \bigg\}, \tag{5.8}$$

and then define

$$k_t \equiv K_t/\eta_t, \quad k_g(t) \equiv K_g(t)/\eta_t, \quad k_p(t) \equiv K_p(t)/\eta_t, \quad c_t \equiv C_t/\eta_t, \tag{5.9}$$

and so forth. Applying Itô's Lemma to the definition (5.8) of η_t we find that the dynamics of η_t^{-1} are given by

$$d\eta_t^{-1} = \eta_t^{-1} \left(-dZ_t^L + \mu_0 dt \right) \tag{5.10}$$

where $\mu_0 \equiv v_{LL} - \mu_L - \mu_T$. The dynamics of $k_t \equiv \eta_t^{-1} K_t$ follow from this and the dynamics (5.1) of K and are given by

$$dk_t = k_t(dZ_t^0 - dZ_t^L) + \bigg[F(k_p(t), k_g(t), \pi_t) - \gamma k_t - c_t \bigg] dt, \tag{5.11}$$

[1] It would be nice to imagine that this expression was chosen for its dictionary definition 'a cause of happiness'. However it's just economics jargon for 'flow of utility'.

where

$$\gamma \equiv \delta + v_{0L} - \mu_0$$
$$= \delta + \mu_L + \mu_T + v_{0L} - v_{LL}.$$

It is now necessary to also re-express the government objective (5.6) in terms of *per capita* technology-adjusted variables, and here the assumption that U is homogeneous of degree $(1-R_g)$ enters in an essential way. We find that the objective of the government can be expressed as

$$\begin{aligned}
\mathbb{E}\int_0^\infty & e^{-\rho_g t} L_t\, U\left(\frac{C_t}{L_t}, \frac{K_g(t)}{L_t}, \pi_t\right) dt \\
&= \mathbb{E}\int_0^\infty e^{-\rho_g t} L_t\, U(c_t T_t, k_g(t) T_t, \pi_t)\, dt \\
&= \mathbb{E}\int_0^\infty e^{-\rho_g t} L_t T_t^{1-R_g} U(c_t, k_g(t), \pi_t)\, dt \\
&= L_0 \mathbb{E}\int_0^\infty e^{-\rho_g t} e^{(\mu_L - \frac{1}{2}v_{LL})t + Z_t^L} e^{\mu_T(1-R_g)t}\, U(c_t, k_g(t), \pi_t)\, dt \\
&= L_0 \mathbb{E}_g \int_0^\infty e^{-\lambda_g t} U(c_t, k_g(t), \pi_t)\, dt \qquad (5.12)
\end{aligned}$$

where

$$\lambda_g \equiv \rho_g - (1-R_g)\mu_T - \mu_L,$$

and the final expectation is with respect to the measure $\mathbb{P}_g$ which is absolutely continuous with respect to $\mathbb{P}$ on every $\mathcal{F}_t$ and has density

$$\left.\frac{d\mathbb{P}_g}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp(Z_t^L - \frac{1}{2}v_{LL}t).$$

The effect of changing measure from $\mathbb{P}$ to $\mathbb{P}_g$ is to introduce additional drift into the Brownian motions Z^0 and Z^L; precisely, we have

$$\begin{aligned}
Z_t^0 &= z_t^0 + v_{0L}t, \\
Z_t^L &= z_t^L + v_{LL}t,
\end{aligned}$$

where (z^0, z^L) are two $\mathbb{P}_g$-martingales possessing the same covariance structure as (Z^0, Z^L). This is the Cameron-Martin-Girsanov Theorem; see, for example, Øksendal

(1998) for an account. The dynamics (5.11) of k_t under $\mathbb{P}$ become under $\mathbb{P}_g$

$$dk_t = k_t(dz_t^0 - dz_t^L) + \Bigg[F(k_p(t), k_g(t), \pi_t) - \gamma_g k_t - c_t \Bigg] dt, \tag{5.13}$$

where the constant γ_g is given by

$$\gamma_g \equiv \gamma - v_{0L} + v_{LL} = \delta + \mu_L + \mu_T.$$

In order to maximise (5.12) with the dynamics (5.13), we can proceed to find the Hamilton-Jacobi-Bellman equation for the value function

$$V(k) \equiv \sup_{c,k_g,0\leq\pi\leq1} \mathbb{E}_g \left[\int_0^\infty e^{-\lambda_g t} U(c_t, k_g(t), \pi_t)\, dt \;\middle|\; k_0 = k \right]. \tag{5.14}$$

Conditioning the expression inside the expectation on events up to time t we see that

$$\int_0^t e^{-\lambda_g s} U(c_s, k_g(s), \pi_s)\, ds + e^{-\lambda_g t} V(k_t)$$

is a supermartingale, and a martingale under optimal control. Hence the HJB equation satisfied by V is

$$\sup_{c,k_g,0\leq\pi\leq1} U(c, k_g, \pi) - \lambda_g V(k) + \frac{1}{2}\sigma^2 k^2 V''(k) + [F(k - k_g, k_g, \pi) - \gamma_g k - c] V'(k) = 0, \tag{5.15}$$

where

$$\sigma^2 \equiv v_{00} - 2v_{0L} + v_{LL}.$$

From this, we deduce the necessary first-order conditions for optimality:

$$\begin{aligned} U_c(c, k_g, \pi) &= V'(k) && (5.16)\\ U_g(c, k_g, \pi) &= V'(k)(F_p(k_p, k_g, \pi) - F_g(k_p, k_g, \pi)) && (5.17)\\ U_\pi(c, k_g, \pi) &= -V'(k) F_\pi(k_p, k_g, \pi), && (5.18) \end{aligned}$$

where we use subscripts to denote differentiation, as in the abbreviations:

$$U_c \equiv \frac{\partial U}{\partial c}, \quad U_g \equiv \frac{\partial U}{\partial k_g}, \quad U_\pi \equiv \frac{\partial U}{\partial \pi}, \quad F_p \equiv \frac{\partial F}{\partial k_p}, \quad F_g \equiv \frac{\partial F}{\partial k_g}, \quad F_\pi \equiv \frac{\partial F}{\partial \pi}.$$

The conditions (5.16), (5.17) and (5.18) arise from considering the optimization problem

$$\sup_{c,k_g,0\leq\pi\leq1} U(c, k_g, \pi) + p[\, F(k - k_g, k_g, \pi) - c\,]; \tag{5.19}$$

implicit in the statements (5.16), (5.17) and (5.18) is the following assumption:

$$\textit{For every } p, k > 0, \textit{ the problem } (5.19) \textit{ has an interior solution which depends in a } C^1 \textit{ fashion on } (p, k) \tag{5.20}$$

This assumption does not always hold, but we shall make it for the sake of the simplifications in the statements and proofs of results; no doubt similar conclusions can be reached without it, but we leave that as an issue for further research.

The observation that the optimizing values (c, k_g, π) are uniquely determined as functions of (p, k) reduces the HJB equation (5.15) to a non-linear differential equation for V; once the solution is found, we are able to express the optimal values of (c, k_g, π) as functions of $(V(k), k)$, or, more simply put, functions of k. We shall henceforth use the notation c^*,k_g^* and π^* for these optimal functions of the underlying state variable k, along with the notation k_p^*, with the obvious interpretation $k_p^*(k) = k - k_g^*(k)$. We shall also introduce the notation

$$\Phi(k) = F(k_p^*(k), k_g^*(k), \pi^*(k)) - \gamma_g k - c^*(k) \tag{5.21}$$

for the drift in the dynamics (5.13), which therefore are more compactly expressed as

$$\begin{aligned} dk_t &= k_t(dz_t^0 - dz_t^L) + \Phi(k_t)dt \\ &= \sigma k_t dw_t + \Phi(k_t)dt, \end{aligned} \tag{5.22}$$

where the $\mathbb{P}_g$-Brownian motion w is defined by $w \equiv (z^0 - z^L)/\sigma$. Under the original measure $\mathbb{P}$ the dynamics (5.11) can be written as

$$\begin{aligned} dk_t &= k_t(dZ_t^0 - dZ_t^L) + \tilde{\Phi}(k_t)dt, \\ &= \sigma k_t dW_t + \tilde{\Phi}(k_t)dt, \end{aligned} \tag{5.23}$$

with the identifications $\tilde{\Phi}(k) \equiv \Phi(k) + (\gamma_g - \gamma)k$, and $W \equiv (Z^0 - Z^L)/\sigma$. Under mild conditions on Φ, for example global Lipschitz (Rogers and Williams 2000, Theorem V.11.2.), the SDE (5.22) has a pathwise-unique strong solution, and the value function V will satisfy the equation

$$U(c^*(k), k_g^*(k), \pi^*(k)) - \lambda_g V(k) + \frac{1}{2}\sigma^2 k^2 V''(k) + \Phi(k)V'(k) = 0. \tag{5.24}$$

Although there may be some issues concerning smoothness of the (c, k_g, π) optimizing in (5.20), the following result is the starting point of our investigations.

Theorem 1 *(i) Assuming that the value function* (5.14) *is finite valued and* C^3*, and that assumption* (5.20) *holds then there exist differentiable functions* Φ*,* c^**,* k_g^**,* π^* *and twice-differentiable* $\Psi \equiv V'$ *such that the equalities*

$$0 = \Psi(F_p - \gamma_g - \lambda_g) + \Psi'(\Phi + \sigma^2 k) + \frac{1}{2}\sigma^2 k^2 \Psi'' \tag{G1}$$

$$U_c = \Psi \tag{G2}$$

$$U_\pi = -F_\pi \Psi \tag{G3}$$

$$U_g = (F_p - F_g)\Psi \tag{G4}$$

$$\Phi = F - \gamma_g k - c \tag{G5}$$

hold along the path given by $(c^*(k), k_g^*(k), \pi^*(k))$[2]*.*

(ii) Conversely suppose that there exist differentiable functions Φ*,* c^**,* k_g^* *and* π^* *and twice-differentiable* Ψ *such that the equalities* (G1)–(G5) *hold along the path given by* (c^*, k_g^*, π^*)*. If* k^* *is the solution to the SDE* (5.22) *then provided the transversality condition*

$$\sup_t e^{-\lambda_g t} k_t^* \Psi(k_t^*) \in L^1, \qquad \lim_{t\to\infty} e^{-\lambda_g t} k_t^* \Psi(k_t^*) = 0 \tag{GT}$$

holds, the policy given by (c^*, k_g^*, π^*) *is optimal for the government, the optimally-controlled economy follows the dynamics* (5.22) *and there is a value function* $V(k)$ *given by*

$$V(k) \equiv \int_1^k \Psi(y)\, dy + V_1$$

which satisfies the HJB equation

$$0 = -\lambda_g V + V'\Phi + \frac{1}{2}\sigma^2 k^2 V'' + U \tag{G6}$$

along the optimal path, where V_1 *is a constant that can be determined explicitly.*

Proof of Theorem 1 : (i) follows from the discussion above; (G1) is obtained by differentiating the HJB equation (5.15) with respect to k and then making use of conditions (5.16)–(5.18).

(ii) Suppose that the process k_t has dynamics given by (5.22) for some consumption process c_t and some choice $k_g(t)/k(t)$ of the proportion of capital held by the government. We introduce a (Lagrangian) semimartingale $e^{-\lambda_g t}\Psi_t \equiv e^{-\lambda_g t}\Psi(k_t^*)$ where k^* is

[2]This means, for example, that $U_g(c^*(k), k_g^*(k), \pi^*(k)) = (F_p - F_g)(k_p^*(k), k_g^*(k), \pi^*(k))\Psi(k)$ in the case of (G4).

the conjectured optimal process, satisfying (5.22), and where

$$d\Psi_t \equiv \Psi_t(a_t dw + b_t dt).$$

We now consider the integral

$$\int_0^\tau e^{-\lambda_g t}\Psi_t dk_t$$

where τ is some stopping time. Firstly, from the dynamics of k_t we know that

$$\int_0^\tau e^{-\lambda_g t}\Psi_t dk_t = \int_0^\tau e^{-\lambda_g t}\Psi_t \left(\sigma k_t dw_t + [F(k_p(t), k_g(t), \pi_t) - \gamma_g k_t - c_t]dt\right).$$

Secondly, integrating by parts, we find that

$$\int_0^\tau e^{-\lambda_g t}\Psi_t dk_t = e^{-\lambda_g \tau}k_\tau \Psi_\tau - k_0\Psi_0 - \int_0^\tau e^{-\lambda_g t}\Psi_t \left(a_t k_t dw_t + (\sigma a_t + b_t - \lambda_g)k_t dt\right).$$

These two expressions must be equal, hence we have for any stopping time τ that (omitting explicit appearance of t in most places)

$$\int_0^\tau e^{-\lambda_g t}U(c, k_g, \pi)dt = \int_0^\tau e^{-\lambda_g t}\Big[U(c, k_g, \pi) + \Psi(F(k_p, k_g, \pi) - \gamma_g k - c) + k\Psi(b - \lambda_g) + \sigma a k \Psi\Big]dt + k_0\Psi_0 - e^{-\lambda_g \tau}k_\tau\Psi_\tau + M_\tau \quad (5.25)$$

for some $\mathbb{P}_g$-local martingale M starting at 0. We now consider the maximisation over k, c, k_g and π of the integrand on the right-hand side of (5.25): the first-order conditions we obtain will be

$$\begin{aligned}
\Psi(k^*)(F_p(k_p, k_g, \pi) - \gamma_g) &= (\lambda_g - b - a\sigma)\Psi(k^*)\\
U_c(c, k_g, \pi) &= \Psi(k^*)\\
U_g(c, k_g, \pi) &= \Psi(k^*)(F_p - F_g)\\
U_\pi(c, k_g, \pi) &= -\Psi(k^*)F_\pi.
\end{aligned}$$

The last three of these are satisfied at $c = c^*(k^*)$, $k_g = k_g^*(k^*)$, $\pi = \pi^*(k^*)$ in view of (G2), (G3) and (G4). The first is satisfied due to (G1), since from the Itô expansion of $\Psi(k^*)$ we must have that

$$\begin{aligned}
a &= \frac{\sigma k^* \Psi'(k^*)}{\Psi(k^*)}\\
b &= \frac{\Phi(k^*)\Psi'(k^*) + \frac{1}{2}\sigma^* k^{*2}\Psi''(k^*)}{\Psi(k^*)}.
\end{aligned}$$

To summarise then: the integrand on the right-hand side of (5.25) is maximised at $c = c^*(k^*)$, $k_p = k_p^*(k^*)$, $k_g = k_g^*(k^*)$, $\pi = \pi^*(k^*)$. Reversing the integration-by-parts argument by which we arrived at (5.25) we have for any stopping time τ that

$$\begin{aligned}\int_0^\tau e^{-\lambda_g t} U(c, k_g, \pi)\, dt &\leq \int_0^\tau e^{-\lambda_g t} U(c^*(k_t^*), k_g^*(k_t^*), \pi^*(k_t^*))\, dt \\ &\qquad + e^{-\lambda_g \tau}(k_\tau^* - k_\tau)\Psi(k_\tau^*) + M_\tau - \tilde{M}_\tau \\ &\leq \int_0^\tau e^{-\lambda_g t} U(c^*(k_t^*), k_g^*(k_t^*), \pi^*(k_t^*))\, dt \\ &\qquad + e^{-\lambda_g \tau} k_\tau^* \Psi(k_\tau^*) + M_\tau - \tilde{M}_\tau\end{aligned}$$

where $\tilde{M}$ is another continuous $\mathbb{P}_g$-local martingale starting at 0. Take a sequence of stopping times $\tau_n \uparrow \infty$ which reduce both M and $\tilde{M}$ strongly. For example if S_n is a localizing sequence for M and T_n is a localizing sequence for $\tilde{M}$ then $\tau_n \equiv S_n \wedge T_n$ is a localizing sequence for both M and $\tilde{M}$. We can now take expectations to obtain

$$\begin{aligned}\mathbb{E}_g \int_0^{\tau_n} e^{-\lambda_g t} U(c, k_g, \pi)\, dt \leq \mathbb{E}_g \int_0^{\tau_n} e^{-\lambda_g t} U(c^*(k_t^*), k_g^*(k_t^*), \pi^*(k_t^*))\, dt \\ + \mathbb{E}_g \left[e^{-\lambda_g \tau_n} k_{\tau_n}^* \Psi(k_{\tau_n}^*) \right]\end{aligned}$$

then let the reducing time τ_n tend to infinity, and appeal to the transversality condition (GT) to give us the required optimality result.

Finally, suppose that we take $V(k)$ given by

$$V(k) \equiv \int_1^k \Psi(y)\, dy + V_1$$

where

$$V_1 \equiv \frac{1}{\lambda_g} \left[\Psi(1)\Phi(1) + \frac{1}{2}\sigma^2 \Psi'(1) + U(c^*(1), k_g^*(1), \pi^*(1)) \right].$$

If we differentiate

$$-\lambda_g V(k) + V'(k)\Phi(k) + \frac{1}{2}\sigma^2 k^2 V''(k) + U(c(k), k_g(k), \pi(k)) \tag{5.26}$$

with respect to k, using the fact that $V'(k) = \Psi(k)$ we obtain

$$\begin{aligned}&\Psi(-\lambda_g + (1 - k_g')F_p + k_g' F_g + \pi' F_\pi - \gamma_g - c') \\ &\quad + \Psi'(\Phi + \sigma^2 k) + \frac{1}{2}\sigma^2 k^2 \Psi'' + c' U_c + k_g' U_g + \pi' U_\pi = 0\end{aligned}$$

by (G1)–(G4). Hence expression (5.26) is constant and this constant is 0 by the construction of V_1. $\square$

Theorem 1 characterizes the optimal solution to the government's problem, but what can we do with it? Are there examples where the solution can be expressed in closed form? In view of the complicated way in which the optimizing values c^*, k_g^* and π^* were defined, it appears at first sight unlikely, but we shall see in Chapter 6 that it *is* possible to exhibit explicit solutions by considering the *inverse problem*, where we postulate a form for the solution and seek a problem whose solution is as postulated. A contrasting approach is to explicitly specify the production and government felicity functions along with the coefficients λ_g, γ_g and σ. We can then use numerical methods to determine the government's optimal path (c^*, k_g^*, π^*) as a function of k along with the corresponding value function $V(k)$. We deal with this in Chapter 7.

5.3 Government borrowing and taxation

The government's optimal policy has been determined, but the issue now is how to implement that policy when the government cannot directly control the private sector, but can only shape its choices through taxation and the issuing of government debt. Since the optimal policy of the previous section was Markovian, in the sense that the total technology-adjusted *per capita* capital k was a Markov process (even a diffusion), we shall now seek Markovian taxation policies, which are defined by the property that the rates of tax are functions only of k.

Before we can understand the effects of government fiscal policy, we have to understand the behaviour of the private sector on which it acts, and we turn to that now. We think of the private sector as made up of a very large number L_0 of identical non-collaborating households; if one of these households receives a cash-flow process of $(\tilde{C}_t)_{t\geq 0}$ while working for a proportion $(\tilde{\pi}_t)_{t\geq 0}$ of time, then it values this cash flow as

$$\mathbb{E}\int_0^\infty e^{-\rho_p t} u\left(\frac{L_0\tilde{C}_t}{L_t}, \frac{K_g(t)}{L_t}, \tilde{\pi}_t\right) dt, \tag{5.27}$$

and it wishes to maximise this. Here u is strictly concave and increasing in its first two arguments, and decreasing in the third, and $\rho_p > 0$ is constant. The felicity u depends on the *per capita* level of government capital, and on the *per capita* rate of consumption for the household, which is assumed to be subject to the same size fluctuations as the entire population; it also varies inversely with the proportion of effort devoted to production. As with the government objective, we assume that u is homogeneous of degree $(1 - R_p)$ in the first two arguments, where $R_p > 0$ is different from 1, and typically different from R_g.

We suppose that the objectives of the government and private sector are different, and that the government aims to set taxes and to borrow in such a way as to induce the private sector to follow the government's desired path. We need now to decompose the dynamics (5.1) of the economy so as to understand the effects of the taxes. Homogeneity of order 1 of F implies[3] that we may express the output as the sum of three terms,

$$\begin{aligned} F(K_p, K_g, \pi LT) &= K_p F_p(K_p, K_g, \pi LT) + K_g F_g(K_p, K_g, \pi LT) + \pi LT F_\pi(K_p, K_g, \pi LT) \\ &= K_p F_p(k_p, k_g, \pi) + K_g F_g(k_p, k_g, \pi) + \pi LT F_\pi(k_p, k_g, \pi) \end{aligned} \tag{5.28}$$

which are interpreted as the return on private capital, the return on government capital, and the return on labour, respectively. Including the random effects term (dZ^0) then, the returns on private capital, government capital and labour are (respectively)

$$K_p dZ_t^0 + K_p F_p dt, \quad K_g dZ_t^0 + K_g F_g dt, \quad \pi LT F_\pi dt. \tag{5.29}$$

We shall suppose that the government is able to appropriate some fixed proportion $1 - \theta_p - \theta_L$ of the returns to its capital by direct charging for services such as toll roads, university tuition fees, subsidized rail fares, and some health-care costs, but it is in the nature of government expenditure that much of the return on government capital cannot be directly appropriated, so in practice this proportion may be near to zero. A proportion θ_p of the returns to government capital are included in the returns to private capital, and the remaining proportion θ_L is included in returns to labour, so that from an accounting point of view we suppose that the returns on private capital and labour are (respectively)

$$K_p dZ_t^0 + K_p F_p dt + \theta_p(K_g dZ_t^0 + K_g F_g dt), \quad \theta_L(K_g dZ_t^0 + K_g F_g dt) + \pi LT F_\pi dt, \tag{5.30}$$

with the remaining $(1 - \theta_p - \theta_L)(K_g dZ_t^0 + K_g F_g dt)$ going directly to government.

The evolution of the levels of private and government capital are determined by the

[3]Differentiate the identity (5.2) with respect to λ to obtain the first line. For the second line

$$\begin{aligned} F_p(k_p, k_g, \pi) &\equiv \frac{\partial}{\partial k_p} F(k_p, k_g, \pi) \\ &= \frac{\partial}{\partial k_p} \frac{F(LTk_p, K_g, \pi LT)}{LT} \\ &= F_p(K_p, K_g, \pi LT) \end{aligned}$$

and similarly for F_g and F_π.

equations

$$\begin{aligned} dK_p &= dI_p - \delta K_p dt, & (5.31) \\ dK_g &= dI_g - \delta K_g dt, & (5.32) \end{aligned}$$

where $I_p(t)$ is the total amount invested in private capital by time t.

The government will issue debt and levy taxes; returns on private capital will be taxed at rate $1 - \beta_k$, wages (return on labour) at rate $1 - \beta_w$, consumption at rate $1 - \beta_c$, and interest on government debt at rate $1 - \beta_r$, so that the private sector's aggregate budget equation is therefore

$$\begin{aligned} dI_p + dD + \beta_c^{-1} C dt = \beta_k \big[K_p(dZ_t^0 + F_p dt) + \theta_p(K_g dZ_t^0 + K_g F_g dt) \big] + r\beta_r D dt \\ + \beta_w \big[\theta_L(K_g dZ_t^0 + K_g F_g dt) + \pi L T F_\pi dt \big] \qquad (5.33) \end{aligned}$$

where D_t denotes the amount of government debt at time t. The interpretation of the left-hand side is that this is the total outgoings of the private sector: the investment in private capital, the investment in government debt, and the cost of consumption[4]. The right-hand side (5.33) is the after-tax income of the private sector: return on private capital plus interest on government debt plus wage income. Arrow and Kurz have also a tax on savings, which alters the term $dI_p + dD$ in equation (5.33) to $\beta_s^{-1}(dI_p + dD)$. Since this could be absorbed into our formulation simply by reinterpreting the other $\beta_{\cdot}$, we lose no generality by studying the equations as given.

The relation (5.31) can be used to eliminate dI_p so that we can rewrite the private-sector budget equation as

$$\begin{aligned} dK_p + dD &= K_p \big[\beta_k dZ_t^0 + (\beta_k F_p - \delta) dt \big] + r\beta_r D dt - \beta_c^{-1} C dt \\ &\quad + \beta_w \pi \eta F_\pi dt + (\beta_k \theta_p + \beta_w \theta_L)(K_g dZ_t^0 + K_g F_g dt). \qquad (5.34) \end{aligned}$$

This bears the simple interpretation that the change in private-sector wealth is accounted for by the return on private capital (adjusted for depreciation) plus the return on government debt, less consumption plus the wage income and a share of the returns on government capital.

[4]If the private sector attempts to consume C it will actually consume only $\beta_c C$ after taxation, hence it must allow an amount $\beta_c^{-1} C$ in order to consume the desired amount after taxation. This is in contrast to a conventional consumption tax, such as VAT, which adds a charge onto the original desired level of consumption. It is easy to move between the two types of tax specification so we lose no generality by assuming the form we do.

Recall that we seek tax rates as functions of k which will cause the private sector to follow the government's optimal trajectory. So we shall suppose that such tax rates have been set, the economy as a whole is following the government's optimal policy as discussed in Section 5.2, and shall consider the optimization problem faced by a single household. *If any deviation from the government's optimal path is suboptimal for the individual household, then we have an equilibrium in which all households follow the government's optimal path;* we shall suppose that this is what happens, and deduce the implications for the tax rates and borrowing policy. These are summarized in the following result.

Theorem 2 *Suppose that the government sets proportional taxes $1-\beta_c$ on consumption, $1-\beta_w$ on income, $1-\beta_k$ on returns on private capital, and $1-\beta_r$ on returns on government debt, all functions only of the total technology-adjusted* per capita *capital k in the economy at the time. If there exists a C^2 function ψ, and a function r such that the equations*

$$\begin{aligned} 0 &= \psi(\beta_k F_p - \gamma - \lambda_p + v_{0L}(1-\beta_k)) && \text{(PS1)} \\ &\qquad + \psi'(\tilde{\Phi} + \beta_k \sigma^2 k + (1-\beta_k)(\gamma_g - \gamma)k) + \frac{1}{2}\sigma^2 k^2 \psi'' \\ u_c &= \beta_c^{-1}\psi && \text{(PS2)} \\ u_\pi &= -\beta_w F_\pi \psi && \text{(PS3)} \\ 0 &= \psi(r\beta_r + \mu_0 - \lambda_p) + \psi'(\tilde{\Phi} + (\gamma_g - \gamma)k) + \frac{1}{2}\sigma^2 k^2 \psi'' && \text{(PS4)} \end{aligned}$$

all hold along the government's optimal path[5]*, where $\lambda_p = \rho_p - (1-R_p)\mu_T$, then the private sector faced with these tax rates will choose to follow the government's optimal path, provided the transversality condition*

$$\sup_t e^{-\lambda_p t}|x_t|\psi(k_t^*) \in L^1, \qquad \lim_{t\to\infty} e^{-\lambda_p t} x_t \psi(k_t^*) = 0 \qquad \text{(PST)}$$

is satisfied, where $x \equiv k_p + \Delta_p$ is the total technology-adjusted per capita wealth of the private sector, split between private capital k_p and government debt $\Delta_p \equiv D/\eta$.

PROOF OF THEOREM 2 : The strategy is firstly to discover the dynamics faced by a single household optimizing in an economy which is following the government's optimal path. Next we rework the private household's objective, expressing it in intensive variables. We then use the Lagrangian method to characterize the private household's

[5] For example, in full (PS3) says $u_\pi(c^*(k), k_g^*(k), \pi^*(k)) = -\beta_w(k)F_\pi(k_p^*(k), k_g^*(k), \pi^*(k))\psi(k)$.

optimal path.

We first derive a *household* budget equation analogous to the aggregate budget equation (5.34). Households are small relative to the size of the population and do not collaborate with each other and so the coefficients $F_\bullet \equiv F_\bullet(K_p, K_g, \pi LT) = F_\bullet(k_p, k_g, \pi)$ which are functions of the *total* private sector capital, and *total* number of technology-adjusted man-hours worked will *not* be significantly affected by the choices of one household. Similarly the tax coefficients $\beta_\bullet$ are fixed as functions of k by the government. We will consider an individual household with $\tilde{K}_p$ invested in capital, $\tilde{D}$ in government debt, consuming at rate $\tilde{C}$ and devoting a proportion $\tilde{\pi}$ of its effort to production; this household will thus have a budget equation given by

$$d\tilde{K}_p + d\tilde{D} = \tilde{K}_p\big[\beta_k dZ_t^0 + (\beta_k F_p - \delta)dt\big] + r\beta_r\tilde{D}dt - \beta_c^{-1}\tilde{C}dt \\ + \beta_w\tilde{\pi}\frac{\eta}{L_0}F_\pi dt + \frac{K_g}{L_0}(\beta_k\theta_p + \beta_w\theta_L)(dZ_t^0 + F_g dt). \qquad (5.35)$$

This agreed, the problem facing the typical private sector household is to optimize the objective (5.27) with the dynamics given by (5.35), where the tax coefficients $\beta_\bullet$ and the interest rate r are fixed functions of k and the partial derivatives $F_\bullet$ and government capital K_g are all evaluated along the government's optimal path. As with the government's problem, we first reduce to technology-adjusted *per capita* variables, expressing the objective as

$$\mathbb{E}\int_0^\infty e^{-\rho_p t}u\left(\frac{L_0\tilde{C}_t}{L_t}, \frac{K_g(t)}{L_t}, \tilde{\pi}_t\right)dt, \quad = \quad \mathbb{E}\int_0^\infty e^{-\rho_p t}T_t^{1-R_p}u(\tilde{c}_t, k_g^*(t), \tilde{\pi}_t)\,dt \\ = \quad \mathbb{E}\int_0^\infty e^{-\lambda_p t}u(\tilde{c}_t, k_g^*(t), \tilde{\pi}_t)\,dt, \qquad (5.36)$$

where $\lambda_p \equiv \rho_p - (1-R_p)\mu_T$. We are reserving starred variables (k_p^*, k_g^*) for the government's optimal values, and using the notation

$$\tilde{k}_p \equiv \tilde{K}_p L_0/\eta, \quad \tilde{\Delta}_p \equiv \tilde{D}L_0/\eta, \quad \tilde{c}_t \equiv \tilde{C}_t L_0/\eta_t.$$

The typical household will have capital $\tilde{K}_p = K_p/L_0$, so as long as the household stays on the government's optimal path we will have that $\tilde{k}_p = k_p$. Of course it will turn out in the end that the private sector will choose $\tilde{c}_t = c^*(k_t^*)$, $\tilde{k}_p(t) = k_p^*(k_t^*)$ and $\tilde{\pi} = \pi^*(k_t^*)$. The dynamics (5.35) along with the dynamics (5.10) of η^{-1} imply the following dynamics under $\mathbb{P}$ for the (technology-adjusted *per capita*) private-sector wealth process $\tilde{x} \equiv \tilde{k}_p + \tilde{\Delta}_p$:

$$\begin{aligned}
d\tilde{x} &= \tilde{x} d\eta^{-1} + L_0\eta^{-1}(d\tilde{K}_p + d\tilde{D}) + d\langle \eta^{-1}, L_0(\tilde{K}_p + \tilde{D})\rangle_t \\
&= (\tilde{k}_p + \tilde{\Delta}_p)\big[-dZ^L + \mu_0 dt\big] + \tilde{k}_p\big[\beta_k dZ^0 + (\beta_k F_p - \delta)dt\big] + r\beta_r\tilde{\Delta}_p dt - \beta_c^{-1}\tilde{c}\,dt \\
&\quad + \beta_w\tilde{\pi}F_\pi dt + k_g^*(\beta_k\theta_p + \beta_w\theta_L)(dZ^0 + F_g dt) - \big[\tilde{k}_p\beta_k + k_g^*(\beta_k\theta_p + \beta_w\theta_L)\big]v_{0L}dt \\
&= \tilde{k}_p\big[\beta_k dZ^0 - dZ^L + (\beta_k F_p - \gamma + v_{0L}(1-\beta_k))dt\big] + \beta_w\tilde{\pi}F_\pi dt \\
&\quad + \tilde{\Delta}_p\big[-dZ^L + (\mu_0 + r\beta_r)dt\big] - \beta_c^{-1}\tilde{c}dt + AdZ^0 + Bdt, \qquad (5.37)
\end{aligned}$$

where we have used the abbreviations $A = (\beta_k\theta_p + \beta_w\theta_L)k_g^*$ and $B = k_g^*(\beta_k\theta_p + \beta_w\theta_L)(F_g - v_{0L})$ and made use of the identity $\gamma = \delta + v_{0L} - \mu_0$.

Let us now combine the objective (5.36) with the dynamics (5.37) using a Lagrangian process $e^{-\lambda_p t}\psi_t^* \equiv e^{-\lambda_p t}\psi(k_t^*)$, where by Itô's Lemma

$$d\psi^* = \psi^*\big[a^*(dZ^0 - dZ^L) + b^* dt\big], \qquad (5.38)$$

using the notation $a_t^* = a(k_t^*)$, $b_t^* = b(k_t^*)$, and where

$$a(k) = k\psi'(k)/\psi(k) \qquad (5.39)$$

$$b(k) = \frac{\frac{1}{2}\sigma^2 k^2\psi''(k) + \psi'(k)\tilde{\Phi}(k)}{\psi(k)}. \qquad (5.40)$$

Again omitting superfluous appearances of the time variable and evaluating the integral $\int_0^\tau e^{-\lambda_p t}\psi^* d\tilde{x}$ in two different ways (see proof of Theorem 1) gives us for any stopping time τ that

$$\begin{aligned}
\int_0^\tau e^{-\lambda_p t}u(\tilde{c}, k_g^*, \tilde{\pi})\,dt = \int_0^\tau e^{-\lambda_p t}\Big[&u(\tilde{c}, k_g^*, \tilde{\pi}) + \tilde{x}\psi^*(b^* - \lambda_p) + \psi^*\{\beta_w\tilde{\pi}F_\pi - \beta_c^{-1}\tilde{c} \\
&+ \tilde{k}_p(\beta_k F_p - \gamma + v_{0L}(1-\beta_k)) + \tilde{\Delta}_p(r\beta_r + \mu_0) + B\} \\
&+ a^*\psi^*\{(\beta_k\tilde{k}_p + A)(v_{00} - v_{0L}) + \tilde{x}(v_{LL} - v_{0L})\}\Big]\,dt \\
&+ \tilde{x}_0\psi_0^* - \tilde{x}_\tau e^{-\lambda_p\tau}\psi_\tau^* + M_\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\tau e^{-\lambda_p t}\Big[u(\tilde{c}, k_g^*, \tilde{\pi}) - \beta_c^{-1}\psi^*\tilde{c} + \beta_w F_\pi \psi^* \tilde{\pi} + \psi^*(B + Aa^*(v_{00} - v_{0L})) \\
&+ \psi^* \tilde{k}_p\{\beta_k F_p - \gamma - \lambda_p + v_{0L}(1-\beta_k) + b^* + a^*\beta_k(v_{00} - v_{0L}) + a^*(v_{LL} - v_{0L})\} \\
&+ \psi^* \tilde{\Delta}_p\{r\beta_r + \mu_0 - \lambda_p + b^* + a^*(v_{LL} - v_{0L})\}\Big]\, dt + \tilde{x}_0\psi_0^* - \tilde{x}_\tau e^{-\lambda_p \tau}\psi_\tau^* + M_\tau \\
&= \int_0^\tau e^{-\lambda_p t}\Big[u(\tilde{c}, k_g^*, \tilde{\pi}) - \beta_c^{-1}\psi^*\tilde{c} + \beta_w F_\pi \psi^* \tilde{\pi} + \psi^*(B + Aa^*(v_{00} - v_{0L})) \\
&+ \tilde{k}_p\{\psi^*(\beta_k F_p - \gamma - \lambda_p + v_{0L}(1-\beta_k)) + \frac{1}{2}\sigma^2 k^2 \psi''(k^*) \\
&\qquad\qquad + \psi'(k^*)(\tilde{\Phi} + (\beta_k(v_{00} - v_{0L}) + v_{LL} - v_{0L})k)\} \\
&+ \tilde{\Delta}_p\{\psi^*(r\beta_r + \mu_0 - \lambda_p) + \frac{1}{2}\sigma^2 k^2 \psi''(k^*) + \psi'(k^*)(\tilde{\Phi} + (v_{LL} - v_{0L})k)\}\Big]\, dt \\
&\qquad\qquad + \tilde{x}_0\psi_0^* - \tilde{x}_\tau e^{-\lambda_p \tau}\psi_\tau^* + M_\tau
\end{aligned}
$$

where M is some continuous local martingale. We now consider the maximization of the integrand on the right-hand side of this equation over the household's choices $\tilde{c}$, $\tilde{\pi}$, $\tilde{k}_p$ and $\tilde{\Delta}_p$. For the maximization with respect to $\tilde{k}_p$ and $\tilde{\Delta}_p$ to be non-trivial (and hence to lead to an interesting economic solution!) we require that

$$
\begin{aligned}
0 &= \psi^*(\beta_k F_p - \gamma - \lambda_p + v_{0L}(1-\beta_k)) + \frac{1}{2}\sigma^2 k^2 \psi''(k^*) \\
&\qquad + \psi'(k^*)(\tilde{\Phi} + (\beta_k(v_{00} - v_{0L}) + v_{LL} - v_{0L})k) \\
0 &= \psi^*(r\beta_r + \mu_0 - \lambda_p) + \frac{1}{2}\sigma^2 k^2 \psi''(k^*) + \psi'(k^*)(\tilde{\Phi} + (v_{LL} - v_{0L})k).
\end{aligned}
$$

These two equations hold due to conditions (G1) and (G4) of Theorem 2 and the identities $v_{LL} - v_{0L} = \gamma_g - \gamma$ and $v_{00} - v_{0L} = \sigma^2 - (\gamma_g - \gamma)$. Writing $\tilde{c}^*$ and $\tilde{\pi}^*$ for the maximizing values of $\tilde{c}$ and $\tilde{\pi}$ respectively, we require also that

$$
\begin{aligned}
u_c(\tilde{c}^*, k_g^*, \tilde{\pi}^*) &= \beta_c^{-1}\psi^* \\
u_\pi(\tilde{c}^*, k_g^*, \tilde{\pi}^*) &= -\beta_w F_\pi \psi^*
\end{aligned}
$$

where we shall assume that the maximum is attained uniquely. Now due to conditions (G2) and (G3) of Theorem 2 and the uniqueness of the maximum, we see that the maximum occurs when the household's maximizing choices of $\tilde{c}^*$ and $\tilde{\pi}^*$, are equal to the government's choices c^* and π^*. Hence we deduce that

$$\begin{aligned}\int_0^{\tau} e^{-\lambda_p t} u(\tilde{c}, k_g^* \tilde{\pi})\, dt &\leq \int_0^{\tau} e^{-\lambda_p t}\big[u(c^*, k_g^*, \pi^*) - \beta_c^{-1}\psi^* c^* + \beta_w F_\pi \psi^* \pi^* \\ &\quad + \psi^*(B + Aa^*(v_{00} - v_{0L}))\big]\, dt + \tilde{x}_0 \psi_0^* - \tilde{x}_\tau e^{-\lambda_p \tau}\psi_\tau^* + M_\tau \\ &= \int_0^{\tau} e^{-\lambda_p t} u(c^*, k_g^*, \pi^*)\, dt + (x_\tau^* - \tilde{x}_\tau) e^{-\lambda_p \tau}\psi_\tau^* + \tilde{M}_\tau \\ &\leq \int_0^{\tau} e^{-\lambda_p t} u(c^*, k_g^*, \pi^*)\, dt + x_\tau^* e^{-\lambda_p \tau}\psi_\tau^* + \tilde{M}_\tau\end{aligned}$$

for some other[6] local martingale $\tilde{M}$. Here we obtained the second line by reversing the integration-by-parts method used on the Lagrangian form. Choosing a τ that reduces $\tilde{M}$ strongly and taking expectations gives us that

$$\mathbb{E}\int_0^{\tau} e^{-\lambda_p t} u(\tilde{c}, k_g^*, \tilde{\pi})\, dt \leq \mathbb{E}\int_0^{\tau} e^{-\lambda_p t} u(c^*, k_g^*, \pi^*)\, dt + \mathbb{E}x_\tau^* e^{-\lambda_p \tau}\psi_\tau^*,$$

and the transversality condition (PST) allows us to let $\tau \to \infty$ to conclude that

$$\mathbb{E}\int_0^{\infty} e^{-\lambda_p t} u(\tilde{c}, k_g^*, \tilde{\pi})\, dt \leq \mathbb{E}\int_0^{\infty} e^{-\lambda_p t} u(c^*, k_g^*, \pi^*)\, dt$$

as required. □

REMARKS. (i) Of course, the way we plan to use Theorem 2 is to enable us to *find* the tax regimes which will persuade the private sector to follow the government's optimal path. So if we suppose that the government's optimal path has been determined, as in Section 5.2, we want now to know whether it is possible to have the conditions (PS1), (PS2), (PS3) and (PS4) all holding at the same time. But this is in fact quite easy: for example, if we *choose* the functional form of β_c and β_r, then (PS2) determines the function ψ and then β_k, β_w and r are determined from (PS1), (PS3) and (PS4) respectively.

(ii) Note the similarities between conditions (PS1), (PS2) and (PS3) and the corresponding conditions (G1), (G2) and (G3) of Theorem 1. If we set the tax rates to zero (so $\beta_k = 1$ etc.) then these conditions of Theorem 2 are identical in form to those of Theorem 1; however they depend on the private sector parameters λ_p and γ and on the private sector felicity function u rather than the corresponding government quantities. Only if the private sector and government share identical values $\lambda_p = \lambda_g$, $\gamma = \gamma_g$ and $u \equiv U$ will the private sector follow the government's optimal path under a no-tax

[6] $\tilde{M}$ corresponds to the term $M - \tilde{M}$ in the proof of Theorem 1.

regime.

(iii) If we subtract equation (PS4) from (PS1) then, after some rearrangement, we obtain

$$\beta_k \left(F_p - v_{0L} + \frac{\psi'}{\psi}(v_{00} - v_{0L})k \right) = r\beta_r + \delta. \tag{5.41}$$

Thus the net return on private capital $\beta_k F_p$ is equal to the net return on debt $r\beta_r$ plus depreciation δ and some 'price of risk' terms. In the formulation used by Arrow and Kurz (no uncertainty or depreciation and one tax rate on investment income so that $\beta_k = \beta_r$ in our notation) this equation tells us that $F_p = r$ so that rates of return are equal on both capital and debt. Note that this equation has *arisen naturally* from our consideration; the private sector maximize over choices of both private capital and debt. On the other hand Arrow and Kurz *assume* that $F_p = r$ so that the private sector is concerned only with its total amount of material assets, i.e. capital and debt together. The debt is then just used as a further instrument to ensure correct levels of private sector capital - the private sector is assumed to hold exactly the amount of debt issued by the government. In our treatment the role of debt is more subtle, and arguably more realistic.

(iv) We do not claim (nor is it true in general) that the solution is fully Markovian, because the process Δ_p may fail to be a function only of k^*. However, under certain conditions we can characterize the long-term behaviour of the debt, writing it in the form

$$\begin{aligned}\Delta_p(t) &= G_1(k_t)k_t^{-a/\sigma} \\ &\quad + \int_{-\infty}^{t} \left(\frac{k_u}{k_t}\right)^{a/\sigma} e^{-b(W'_t - W'_u) - \frac{1}{2}b^2(t-u) + \int_u^t G_0(k_v)dv} \left\{G_2(k_u)dW'_u + G_3(k_u)du\right\},\end{aligned}$$

where G_0, G_1, G_2, G_3 are given functions of k and W' is a Brownian motion that is *completely independent* of the Brownian motion $W \equiv (Z^0 - Z^L)/\sigma$ that drives the dynamics of k. See Appendix B.3 for the details. This is the kind of condition we might wish for - the debt is partly a function of k_t and partly a buffer for the fluctuations in the economy unrelated to changes in the level of capital. However this is still a far from satisfactory solution.

In a model with tax rates as proposed above we are unlikely to be able to find a solution with a Markovian debt process. If we require that Δ_p is a function of k then $x \equiv \Delta_p + k_p$

must also be a function of k and hence

$$dx = x'(\tilde{\Phi}dt + k(dZ^0 - dZ^L)) + \frac{1}{2}\sigma^2 k^2 x'' dt. \tag{5.42}$$

Equating the dZ^L term in the above with that in the other expression we have for the private sector dynamics (5.37) we find that

$$x'k = x$$

and hence $x \equiv \Gamma k$ for some constant Γ. With this identification we can now equate the dZ^0 and dt terms in (5.42) and (5.37) giving (after some rearrangement)

$$(\Gamma - \beta_k)k = \{\beta_w \theta_L - (1 - \theta_p)\beta_k\} k_g \tag{5.43}$$

and

$$\Gamma\tilde{\Phi} + \beta_c^{-1} c - \beta_w \pi F_\pi = k_p \left[\beta_k (F_p - F_g) - \delta - r\beta_r\right] + \Gamma k \left[\mu_0 - v_{0L} + r\beta_r + F_g\right]. \tag{5.44}$$

We now effectively have six equations (PS1)–(PS4), (5.43), (5.44) in five unknowns ψ, β_c, β_k, $r\beta_r$ and β_w so we are unlikely to be able to find a consistent solution to these, and even if we can the solution is likely to be highly dependent on exact choice of parameters.

One way round this might be to require that the proportions θ_p and θ_L are also functions of capital which the government is free to choose. However this is not very sound economically so we do not investigate this idea any further. It is interesting to note however that θ_p and θ_L *do not appear at all* in conditions (PS1)–(PS4) of Theorem 2, and affect only the amount of debt held by the private sector.

5.4 State-price densities, consumption rates of interest and bond prices

So far we have established the government's optimal policy and shown that through appropriate choice of tax and interest rates the private sector can be persuaded to follow this optimal policy. Assuming that this is the case, what price would the government or the private sector be prepared to pay for a new claim introduced into the model? In this section we will informally derive the government and private sector state-price density processes, along with the corresponding (consumption) rates of interest. See Breeden

(1986) for more detail. We will then use these to find the price of a zero-coupon bond.

Suppose that at time t the government is offered an $\mathcal{F}_T$-measurable (random) amount B_T at time T by some outside source, for example the government of another country, in exchange for a payment of B_t at time t. We assume that both B_t and B_T are extremely small, so that they have a negligible effect on the economy, and that at time t the total capital in the economy is given by $k_t = k$. We can find the maximum price B_t that the government would be prepared to pay by considering the change that this contract will make to the government's overall expected future *value* at time t, $V(k_t)$, where as before,

$$V(k) \equiv \mathbb{E}_g\left[\int_0^\infty e^{-\lambda_g t} U(c^*(k_t), k_g^*(k_t), \pi^*(k_t))\, dt \,\middle|\, k_0 = k \right].$$

Similarly, if the government was taking the other side of the agreement and delivering B_T at time T in exchange for B_t, we can find the minimum price B_t that the government would accept. We will locate the *indifference price*, at which the government is indifferent between buying or selling the claim, or indeed doing neither. In time-t-discounted notation the amount B_T becomes $\eta_T^{-1} B_T / \eta_t^{-1}$ and so indifference valuation of the claim gives that

$$\begin{aligned}
0 &= V(k_t + B_t) + e^{-\lambda_g (T-t)} \mathbb{E}_g[V(k_T - \frac{\eta_T^{-1}}{\eta_t^{-1}} B_T) - V(k_T) | \mathcal{F}_t] - V(k_t) \\
&= B_t V'(k_t) - e^{-\lambda_g (T-t)} \mathbb{E}_g[\frac{\eta_T^{-1}}{\eta_t^{-1}} B_T V'(k_T) | \mathcal{F}_t] + \mathrm{O}(B_t^2) + \mathrm{O}(B_T^2) \\
&= B_t \Psi(k) - \frac{\mathbb{E}_g[e^{-\lambda_g T} \eta_T^{-1} B_T \Psi(k_T) | \mathcal{F}_t]}{e^{-\lambda_g t} \eta_t^{-1}} + \mathrm{O}(B_t^2) + \mathrm{O}(B_T^2).
\end{aligned}$$

It follows that, to leading order, the time-t price the government will assign to a claim paying B_T at time T is given by

$$\begin{aligned}
B_t &= \frac{\mathbb{E}_g[e^{-\lambda_g T} \eta_T^{-1} \Psi(k_T) B_T | \mathcal{F}_t]}{e^{-\lambda_g t} \eta_t^{-1} \Psi(k_t)}. \\
&= \frac{\mathbb{E}_g[e^{-\lambda_g T} \eta_T^{-1} U_c(k_T) B_T | \mathcal{F}_t]}{e^{-\lambda_g t} \eta_t^{-1} U_c(k_t)}.
\end{aligned}$$

using equation (G2) and writing $U_c(k) \equiv U_c(c^*(k), k_g^*(k), \pi^*(k))$. We have thus identified the government's state-price density process

$$\zeta_t^g \equiv e^{-\lambda_g t} \eta_t^{-1} \Psi(k_t) = e^{-\lambda_g t} \eta_t^{-1} U_c(k_t). \tag{5.45}$$

We can now proceed to find what Arrow and Kurz refer to as the government's *consumption rate of interest*, the rate of interest that the government would have to receive in order to exchange consumption now for consumption later, or vice-versa; we shall denote this by $r_g(k_t)$. To find r_g we use the identity

$$d\zeta_t^g \equiv \zeta_t^g(-r_g dt + d\tilde{M}_t),$$

where $\tilde{M}_t$ is some $\mathbb{P}_g$-martingale. We can thus decompose the government's state-price density process ζ_t^g into a discount factor and a change of measure, writing

$$\zeta_t^g \equiv \exp\left(-\int_0^t r_g(s)\,ds\right)\tilde{\zeta}_t^g, \tag{5.46}$$

where

$$d\tilde{\zeta}_t^g = \tilde{\zeta}_t^g d\tilde{M}_t.$$

Under $\mathbb{P}_g$ the process η_t^{-1} has dynamics

$$d\eta_t^{-1} = \eta_t^{-1}\left(-dz_t^L + (\mu_0 - v_{LL})dt\right)$$

and applying Itô's Lemma to equation (5.45) we find that

$$\frac{d\zeta_t^g}{\zeta_t^g} = -\lambda_g dt + \frac{d\eta_t^{-1}}{\eta_t^{-1}} + \frac{1}{\Psi}\left(\Psi'(\Phi dt + k(dz_t^0 - dz_t^L)) + \frac{1}{2}\sigma^2 k^2 \Psi'' + k\Psi'(v_{LL} - v_{0L})\right)$$

so that r_g is indeed a function of k as stated, and

$$\begin{aligned}
r_g &= \lambda_g - \mu_0 + v_{LL} - \frac{1}{\Psi}\left(\Psi'(\Phi + k(v_{LL} - v_{0L})) + \frac{1}{2}\sigma^2 k^2 \Psi''\right) \\
&= \lambda_g - \mu_0 + v_{LL} - \frac{1}{\Psi}\left(k\Psi'(v_{LL} - v_{0L} - \sigma^2) - \Psi(F_p - \gamma_g - \lambda_g)\right) \\
&= F_p - \gamma_g - \mu_0 + v_{LL} - k\frac{\Psi'}{\Psi}(v_{LL} - v_{0L} - \sigma^2) \\
&= F_p - \delta + k\frac{\Psi'}{\Psi}(v_{00} - v_{0L}),
\end{aligned}$$

where we have made use of condition (G1) of Theorem 1 in moving from the first to the second line above. The consumption rate of interest is equal to the rate of return on private capital, minus depreciation and some 'price of risk' terms. In the Arrow-Kurz model this equation is simply $r_g = F_p$ so that the government is happy with the same rate of return it has arranged for the population.

The analogous state-price density process for the private sector will be given by

$$\zeta_t^p \equiv e^{-\lambda_p t}\eta_t^{-1}\psi(k_t) = e^{-\lambda_p t}\eta_t^{-1}\beta_c(k_t)u_c(k_t) \tag{5.47}$$

under $\mathbb{P}$. The private sector's time-t indifference price for a claim B_T delivered at time T will be given by

$$\frac{\mathbb{E}[\zeta_T^p B_T|\mathcal{F}_t]}{\zeta_t^p}.$$

Notice that when valuing future claims the private sector is concerned about whether tax rates on consumption will change during the interval, as well as the other factors common with the government. The claim B_T no longer needs to be thought of as coming from outside the economy; we view the private sector as a large number of independent households, each free to buy or sell claims to or from each other or the government without affecting the economy as a whole with their decisions.

We can calculate the private sector's consumption rate of interest r_p and find that

$$\begin{aligned} r_p &= \lambda_p - \mu_0 - \frac{1}{\psi}\left(\psi'(\tilde{\Phi} + k(v_{LL} - v_{0L})) + \frac{1}{2}\sigma^2 k^2 \psi''\right) \\ &= \lambda_p - \mu_0 + \frac{1}{\psi}\psi(r\beta_r + \mu_0 - \lambda_p) \\ &= r\beta_r \end{aligned}$$

where we have made use of equation (PS4) and $v_{LL} - v_{0L} = \gamma_g - \gamma$. The $\mathbb{P}$-dynamics of ζ_t^p are given by

$$\frac{d\zeta_t^p}{\zeta_t^p} = -r\beta_r dt - dZ_t^L + \frac{\psi'}{\psi}\sigma k dW_t. \tag{5.48}$$

The private sector's consumption rate of interest is exactly the same as the net rate they are receiving from government debt. If this was not the case then the private sector would not be investing in it!

REMARKS (i) We can now easily derive a PDE for the price of a zero-coupon bond. We shall write $B(t,k;T)$ for the time-t price of a zero-coupon bond paying one unit of capital at time T, where $k_t = k$ at time $t \leq T$. The price the private sector will be prepared to pay for such a bond is

$$B(t,k;T) \equiv \frac{\mathbb{E}\left[\zeta_T^p.1 \,|\, k_t = k\right]}{\zeta_t^p}.$$

Conditioning ζ_T^p on events up to time t we see that

$$M_t \equiv \zeta_t^p B(t, k_t; T)$$

is a martingale. Using Itô's Lemma we find that

$$\begin{aligned} dM_t &= \zeta_t^p \left(dB + \frac{B d\zeta_t^p}{\zeta_t^p} + \frac{d\langle \zeta_t^p, B \rangle}{\zeta_t^p} \right) \\ &= \zeta_t^p \left(\frac{\partial B}{\partial t} dt + \frac{\partial B}{\partial k} dk + \frac{1}{2}\sigma^2 k^2 \frac{\partial^2 B}{\partial k^2} dt + \frac{B d\zeta_t^p}{\zeta_t^p} + \frac{\partial B}{\partial k} k \left(\frac{\psi'}{\psi}\sigma^2 k + v_{LL} - v_{0L} \right) \right) \end{aligned}$$

and hence the zero-coupon bond price $B(t, k; T)$ satisfies the partial differential equation

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 k^2 \frac{\partial^2 B}{\partial k^2} + \frac{\partial B}{\partial k} \left(\tilde{\Phi} + \sigma^2 k^2 \frac{\psi'}{\psi} + (v_{LL} - v_{0L})k \right) - r\beta_r B = 0 \tag{5.49}$$

with terminal condition $B(T, k; T) = 1$. In Chapter 7 we will show how to calculate the price of zero-coupon bonds numerically using this equation. We could equally well price more complicated assets using this method.

(ii) Decomposing the private sector's state-price density in an analogous way to equation (5.46) we see that the price of a zero-coupon bond can also be written as

$$B(t, k; T) = \tilde{\mathbb{E}} \left[\exp\left(- \int_t^T r_s \beta_r(s)\, ds \right) \middle| \mathcal{F}_t \right] \tag{5.50}$$

where the expectation is with respect to a measure $\tilde{\mathbb{P}}$ which is absolutely continuous with respect to $\mathbb{P}$ on every $\mathcal{F}_t$ and has density

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \tilde{\zeta}_t^p$$

where

$$d\tilde{\zeta}_t^p = \tilde{\zeta}_t^p \left(-dZ_t^L + \frac{\psi'}{\psi}\sigma k dW_t \right).$$

If we set $\beta_r \equiv 1$ then equation (5.50) gives the conventional representation of a bond price given a short rate process r_t and a pricing measure $\tilde{\mathbb{P}}$. See, for example, Musiela and Rutkowski (1998, Chapter 12).

(iii) Our definition of the private sector's consumption rate of interest r_p differs from that of Arrow and Kurz in that our definition is a consumption rate *in the presence of taxation* whereas Arrow and Kurz use the rate that the private sector would use if there were no taxation, which we shall denote by r_p^{AK}. Under the conditions of their model we then have that

$$\begin{aligned} r_p^{AK} &\equiv \lambda_p + \gamma - \frac{d}{dt}\log u_c \\ &= \lambda_p - \mu_0 - \frac{\dot{u}_c}{u_c}, \end{aligned}$$

which is related to our rate r_p by the identity

$$\begin{aligned} r_p^{AK} &= r_p + \frac{\dot{\beta}_c}{\beta_c} \\ &= \beta_r F_p + \frac{\dot{\beta}_c}{\beta_c}. \end{aligned} \tag{5.51}$$

If we now introduce taxation at rate $1 - \beta_s$ on savings as in the Arrow-Kurz model by replacing β_c^{-1} by $v \equiv \beta_s \beta_c^{-1}$ and β_r by $\beta_s \beta_r$, then equation (5.51) can be written as

$$r_p^{AK} = \beta_s \beta_r F_p - \frac{\dot{v}}{v},$$

which is condition (a) of Proposition 1 on page 182 of Arrow and Kurz. This equation describes how the tax rates are related to the consumption interest rate (without taxation) of the private sector. Other identities from Arrow and Kurz can be written down from our results if so desired.

Chapter 6

Explicit solutions

Theorem 1 tells us that provided there exist functions Ψ, Φ, c^*, k_g^* and π^* satisfying the equations (G1)–(G5) and the transversality condition (GT), then we have a solution to the government's original problem. In general, it will be hard to find explicit solutions for a given problem; nonetheless, we shall show in this chapter that explicit solutions abound, and can be manufactured readily by considering the *inverse problem*, where we postulate a form for the solution and seek a problem whose solution is as postulated.

We begin in Section 6.1 by assuming that instead of specifying a production rate function $F(k_p, k_g, \pi)$ we instead choose the government's value function $V(k)$ and the desired optimal trajectory $k_g^*(k)$ for the level of government capital. As in the original problem the government's felicity function $U(c, k_g, \pi)$ will also be specified. We give a general characterization of the properties these chosen functions must have in order that we can construct a valid solution (and in particular a well behaved production function F) for the government's problem.

In Section 6.1.1 we show that choosing $V(k)$ to be CRRA leads to a considerable simplification of this characterization. We specialize further in Section 6.1.2 by choosing an explicit form for the government's felicity function U and give a lemma listing the properties we now require from a certain function of $k_g^*(k)$ in order that we obtain a valid solution to the original problem. In Section 6.1.3 we give an example function that satisfies these properties.

Having shown how to obtain explicit solutions to the government's problem we move on to considering the implications for tax and interest rates. In Section 6.2 we propose two possible approaches to taxation given explicit solutions as proposed in Section 6.1.2.

In the first approach a clever choice for the consumption and wage taxes leads to a nice form for the interest rate r, which in a special case can be compared with an example previously studied by Merton (1975). In the second approach we are able to choose *constant* consumption and wage taxes.

6.1 An explicit solution to the government's problem

We will assume that we have chosen a form for the government's felicity function $U(c, k_g, \pi)$ which is decreasing in the last argument and strictly concave, increasing and homogeneous of degree $1 - R_g$ in the first two arguments. This allows us to write

$$U(c, k_g, \pi) \equiv k_g^{1-R_g} h(\xi, \pi) \tag{6.1}$$

where $h(x, \pi) \equiv U(x, 1, \pi)$, and $\xi \equiv c/k_g$. For U to have the properties listed above we need h to be strictly concave and increasing in its first argument, decreasing in its second and to satisfy the conditions

$$\begin{aligned} \xi h_\xi - (1 - R_g)h &< 0, \\ \xi^2 h_{\xi\xi} + 2R_g \xi h_\xi - R_g(1 - R_g)h &< 0, \\ R_g h_\xi^2 + (1 - R_g)hh_{\xi\xi} &< 0. \end{aligned} \tag{6.2}$$

Differentiation of equation (6.1) gives

$$U_c(c, k_g, \pi) = k_g^{-R_g} h_\xi(\xi, \pi), \tag{6.3}$$

$$U_g(c, k_g, \pi) = k_g^{-R_g} \left[(1 - R_g)h(\xi, \pi) - \xi h_\xi(\xi, \pi) \right], \tag{6.4}$$

$$U_\pi(c, k_g, \pi) = k_g^{1-R_g} h_\pi(\xi, \pi). \tag{6.5}$$

To find an explicit solution[1], we first *make a choice of the functions Ψ (equivalently V), U (equivalently, h and R_g), and k_g*. Not all such choices will result in soluble problems; for example, we will have to have that V is concave. Moreover, we shall require of our proposed solution that

$$0 \leq k_g \leq k \tag{6.6}$$

to avoid the possibility that either of k_p, k_g should be negative[2]. However, we can from

[1] Please excuse us if we do not use superscript asterisks in this discussion.

[2] Depending on the form of the production and felicity functions, negative values may be mathematically possible, but we shall restrict attention to more realistic situations where this doesn't happen.

these choices deduce what the solution (if it exists) must be, by solving the equations (G1)–(G5) and (6.1)–(6.5). To see how this is done, first note that as F is required to be homogeneous of order 1, we have a consistency condition

$$F = k_p F_p + k_g F_g + \pi F_\pi \,, \tag{6.7}$$

hence

$$\begin{aligned} \Phi + \gamma_g k + c &= k_p F_p + k_g F_g + \pi F_\pi \\ &= k F_p - (F_p - F_g) k_g + \pi F_\pi \,, \end{aligned} \tag{6.8}$$

by (G5). Since (G1) can be written as[3]

$$0 = V'(-\Phi' + F_p - \gamma_g) - U' \tag{6.9}$$

where U' denotes the derivative with respect to k of the function

$$U(k) \equiv U(c(k), k_g(k), \pi(k)), \tag{6.10}$$

we can use conditions (G1), (G3) and (G4) to rewrite (6.8) as

$$V'(\Phi + \gamma_g k + c) = k(U' + V'\gamma_g + V'\Phi') - k_g U_g - \pi U_\pi .$$

Now as U is homogeneous of order $(1 - R_g)$ and from (G2) we know that $(1 - R_g)U = cU_c + k_g U_g = cV' + k_g U_g$ it follows that

$$V'\Phi = k(U' + V'\Phi') - (1 - R_g)U - \pi U_\pi$$

and so

$$(V' + kV'')\Phi = kU' + k(V'\Phi)' - (1 - R_g)U - \pi U_\pi .$$

We can use equation (G6) to get an expression for Φ and $V'\Phi$, giving

$$\left(1 + \frac{kV''}{V'}\right)(\lambda_g V - U - \frac{1}{2}\sigma^2 k^2 V'') = kU' + k(\lambda_g V - U - \frac{1}{2}\sigma^2 k^2 V'')' - (1 - R_g)U - \pi U_\pi ,$$

[3]To see this, note that if we differentiate equation (G6) with respect to k we find that

$$0 = U' + \frac{1}{2}\sigma^2 k^2 V''' + V''(\Phi + \sigma^2 k) + V'(\Phi' - \lambda_g)$$

and subtracting this from equation (G1) gives the desired result.

which can be rearranged to give

$$
\begin{aligned}
\left(\frac{\lambda_g V - \frac{1}{2}\sigma^2 k^2 V''}{kV'}\right)' &= \frac{1}{k^2 V'}\left[\pi U_\pi - \left(R_g + \frac{kV''}{V'}\right)U\right] \\
&= \frac{k_g^{1-R_g}}{k^2 V'}\left[\pi h_\pi - \left(R_g + \frac{kV''}{V'}\right)h\right]. \qquad (6.11)
\end{aligned}
$$

(G2) gives us that

$$V' = U_c = k_g^{-R_g} h_\xi(\xi, \pi). \qquad (6.12)$$

We will require that this equation along with (6.11) *allows us to determine* $\xi(k)$ *and* $0 \le \pi(k) \le 1$ *for all* k. As we originally chose a specific $k_g(k)$ we now also know $c(k) \equiv k_g(k)\xi(k)$, and hence the form of the function $U(k) \equiv U(c(k), k_g(k), \pi(k))$. We can recover $\Phi(k)$ from (G6) and we can express F_p, evaluated along the path $(k_p(k), k_g(k), \pi(k))$, explicitly using (6.9). Similarly, combining (6.4) with (G4) gives the relation

$$F_p - F_g = \frac{(1 - R_g)h(\xi, \pi) - \xi h_\xi(\xi, \pi)}{k_g^{R_g} V'(k)}, \qquad (6.13)$$

expressing the difference $F_p - F_g$ as a known function of k, and combined with our knowledge of F_p we get F_g as a function of k. Finally we obtain F_π from (6.5) combined with (G3). How near are we to a solution? Equations (G1)–(G4), (6.1), (6.3) and (6.4) hold along the trajectory by construction; equation (G5) could be used to define the value of F along the trajectory as a function of k, but is this consistent with the forms of F_p, F_g and F_π which we have just found? We have to check that if F_p, F_g and F_π are obtained as above then

$$
\begin{aligned}
(\Phi + \gamma_g k + c)' &= \frac{d}{dk} F(k_p(k), k_g(k), \pi(k)) \\
&= F_p - (F_p - F_g)k_g' + F_\pi \pi'.
\end{aligned}
$$

Multiplying throughout by V', what we have to show is

$$
\begin{aligned}
V'\Phi' + V'c' &= V'(F_p - \gamma_g) - k_g' U_g - \pi' U_\pi \\
&= U' + V'\Phi' - k_g' U_g - \pi' U_\pi
\end{aligned}
$$

which is equivalent to proving

$$U' = c' U_c + k_g' U_g + \pi' U_\pi,$$

and this is immediate.

By this inverse approach we have constructed a trajectory $(c(k), k_g(k), \pi(k))_{k\geq 0}$, and have found the values of the production function F along this path. What we still need to check is that the function F can be extended off the path where it known to some concave function $\tilde{F}(K_p, K_g, L)$ increasing in all its arguments, homogeneous of degree 1, that agrees with F along the path $(k_p(k), k_g(k), \pi(k))_{k\geq 0}$. Let us abbreviate $F(k_p(k), k_g(k), \pi(k))$ to $F(k)$, with similar interpretations of $F_p(k)$, $F_g(k)$ and $F_\pi(k)$. Clearly, if there *is* such a concave increasing function F, we shall have to have at very least the conditions

$$F_p(k) \geq 0, \quad F_g(k) \geq 0, \quad F_\pi(k) \geq 0 \qquad \forall k \geq 0, \tag{6.14}$$

along with the homogeneity condition (6.7), which holds by construction, and the 'tangent inequality'

$$F(k) \leq \Lambda(k_p(k), k_g(k), \pi(k); w), \quad \forall k, w \geq 0 \tag{6.15}$$

where

$$\begin{aligned}\Lambda(x, y, z; w) &\equiv F(w) + (x - k_p(w))F_p(w) + (y - k_g(w))F_g(w) + (z - \pi(w))F_\pi(w)\\ &= xF_p(w) + yF_g(w) + zF_\pi(w)\end{aligned}$$

is the equation of the tangent plane to F at $(k_p(w), k_g(w), \pi(w))$. However, these three conditions (6.14), (6.7) and (6.15) are already almost enough. Defining

$$\tilde{F}(x, y, z) \equiv \inf_{w\geq 0} \Lambda(x, y, z; w), \tag{6.16}$$

it is clear that $\tilde{F}$ is concave and increasing in all its arguments. If we assume also

$$\text{the infimum in (6.16) is attained uniquely,} \tag{6.17}$$

then for a general (x, y, z) there exists a unique $w_0 = w_0(x, y, z)$ such that

$$\begin{aligned}\tilde{F}(x, y, z) &= \Lambda(x, y, z; w_0),\\ \tilde{F}_p(x, y, z) &= F_p(w_0),\\ \tilde{F}_g(x, y, z) &= F_g(w_0),\\ \tilde{F}_\pi(x, y, z) &= F_\pi(w_0)\end{aligned}$$

and so

$$\begin{aligned}
&\tilde{F}(x,y,z) - x\tilde{F}_p(x,y,z) - y\tilde{F}_g(x,y,z) - z\tilde{F}_\pi(x,y,z) \\
&= \Lambda(x,y,z;w_0) - xF_p(w_0) - yF_g(w_0) - zF_\pi(w_0) = 0
\end{aligned}$$

as required. The non-negativity of $\tilde{F}$ still needs to be checked, but because of the non-negativity of F_p, F_g and F_π, it is immediate that $\Lambda(x,y,z;w)$ is non-negative for any $x,y,z \geq 0$, and so $\tilde{F}$ is non-negative.

Thus we see that in general if we propose k_g, concave U homogeneous of degree $1-R_g$, and concave V, we can construct a candidate solution: provided we can check (6.14), (6.15), (6.17), and (GT), then we have a solution. It may well be, of course, that the production function defined by (6.16) cannot be expressed more simply; in this sense, then, we will have built an explicit solution to a problem whose statement is somewhat implicit, which is arguably more use than an implicit solution to an explicit problem.

Taking the right-hand side of the tangent inequality (6.15) less the left-hand side and differentiating with respect to k gives us

$$k_p'(k)\,[F_p(w) - F_p(k)] + k_g'(k)\,[F_g(w) - F_g(k)] + \pi'(k)\,[F_\pi(w) - F_\pi(k)] \tag{6.18}$$

as $F' = k_p'F_p + k_g'F_g + \pi'F_\pi$. If we can show that this expression is non-negative for $k \geq w$ and non-positive for $k \leq w$ then the tangent inequality (6.15) follows. We want a solution where k_p and k_g are increasing functions of k and π decreases with k so that means that a sufficient condition for the tangent equality to hold is that F_p and F_g are decreasing functions of k and F_π is an increasing function of k. In practice the following reworking will prove more useful. Using the abbreviation

$$[F_p]_k^w = F_p(w) - F_p(k)$$

and similar, (6.18) above can be written as

$$\begin{aligned}
&k_p'(k)\,[F_p]_k^w + k_g'(k)\,[F_g]_k^w + \pi'(k)\,[F_\pi]_k^w \\
&\quad = \left[k_p'F_p + k_g'F_g + \pi'F_\pi\right]_k^w - F_p(w)\left[k_p'\right]_k^w - F_g(w)\left[k_g'\right]_k^w - F_\pi(w)\left[\pi'\right]_k^w \\
&\quad = \left[F'\right]_k^w + (F_p(w) - F_g(w))\left[k_g'\right]_k^w + F_\pi(w)\left[-\pi'\right]_k^w . \qquad (6.19)
\end{aligned}$$

6.1.1 Specializing: V is CRRA

If we now suppose that

$$V(k) = \frac{A_g k^{1-S}}{(1-S)}$$

for some $S > 0$ different from 1, and A_g a positive constant, it turns out that the form of the candidate solution simplifies considerably. (G6) is now

$$\begin{aligned} U &= V' \left[\left(\frac{\lambda_g}{1-S} + \frac{1}{2}\sigma^2 S \right) k - \Phi \right] \\ &\equiv V'[Qk - \Phi], \end{aligned}$$

where $Q \equiv \lambda_g/(1-S) + \frac{1}{2}\sigma^2 S$ and (6.9) gives

$$F_p = \gamma_g + Q - \frac{S}{k}(Qk - \Phi). \tag{6.20}$$

With this form for V the left hand side of (6.11) is identically zero, hence we require simply that

$$\pi h_\pi = (R_g - S)h. \tag{6.21}$$

Equation (6.12) becomes

$$A_g k^{-S} = k_g^{-R_g} h_\xi. \tag{6.22}$$

With these simplifications it is now possible to follow the steps of Section 6 and obtain the following relations:

$$F_p = \gamma_g + Q - \frac{S}{k}\frac{U}{V'} \tag{6.23}$$

$$F_p - F_g = \frac{(1-R_g)}{k_g}\frac{U}{V'} - \xi \tag{6.24}$$

$$F_\pi = -\frac{(R_g - S)}{\pi}\frac{U}{V'} \tag{6.25}$$

$$F = (\gamma_g + Q)k + c - \frac{U}{V'} \tag{6.26}$$

$$\Phi = Qk - \frac{U}{V'} \tag{6.27}$$

where

$$\frac{U}{V'} = k_g \frac{h}{h_\xi} = c\frac{h}{\xi h_\xi}.$$

As V' is positive we will find it easier to ensure $F_p \geq 0$ if we consider $U < 0$. We have also

$$F_g = \gamma_g + Q + \xi - \frac{U}{V'}\left(\frac{S}{k} - \frac{(R_g - 1)}{k_g}\right);$$

we will require this to be non-negative too.

Suppose that we now assume a form for $h(\xi, \pi)$. Equations (6.21) and (6.22) will then be used to determine k_g, π and ξ as functions of k (we would have to assume the form of k_g and possibly also π depending on the form of h). The consumption rate c is given by $c = \xi k_g$ and equations (6.23)–(6.27) above determine the other quantities we require. We need to show that the conditions (6.14) and (6.17) are satisfied along with the transversality condition (GT). All that then remains is to check the tangent inequality holds which we will do by considering (6.18) or (6.19).

6.1.2 Specializing further: $h(\xi, \pi) \equiv h_1(\xi)h_2(\pi)$

We will assume that h is of product form, so that

$$h(\xi, \pi) \equiv h_1(\xi)h_2(\pi)$$

where we assume that we know the form of the (monotone) functions h_1 and h_2 and also of k_g. Equations (6.21) and (6.22) become

$$\pi h_2'(\pi) = (R_g - S)h_2(\pi) \tag{6.28}$$

$$A_g k^{-S} = k_g^{-R_g} h_1'(\xi)h_2(\pi). \tag{6.29}$$

The first of these determines π as a *constant value for all* k (there may be ambiguity if h_2 solves the ODE on an interval). The second equation then determines $\xi(k)$ and hence c. Finally

$$\frac{U}{V'} = c\frac{h_1(\xi)}{\xi h_1'(\xi)}. \tag{6.30}$$

Let's assume that we know k_g and

$$h_1(\xi) = -\frac{\xi^{-\nu}}{\nu}, \qquad\qquad h_2(\pi) = (1 - \pi)^{-\kappa},$$

so that

$$
\begin{aligned}
U(c, k_g, \pi) &= -\frac{k_g^{-(R_g-1-\nu)} c^{-\nu} h_2(\pi)}{\nu} \\
&\equiv -\frac{k_g^{-\omega} c^{-\nu} (1-\pi)^{-\kappa}}{\nu}
\end{aligned}
\tag{6.31}
$$

where $\omega \equiv R_g - 1 - \nu$. We will assume that

$$\nu > 0, \quad \omega > 0, \quad \kappa > 0, \quad R_g > S > 1,$$

so that U has all the required properties (concave and increasing in k_g and c, decreasing in π and the conditions (6.2) are satisfied). U is negative so the derived value function V must also be negative, therefore $S > 1$, and condition (6.28) then means that $R_g > S$. From (6.28) the optimal π is given by

$$\pi = \frac{R_g - S}{\kappa + R_g - S} \in (0, 1)$$

and the value of h_2 at the optimal π is thus

$$\Theta = \left(\frac{\kappa}{\kappa + R_g - S} \right)^{-\kappa}.$$

Note that the specific choice of the function $h_2(\pi)$ doesn't really matter; the solution depends only on the parameter Θ which we can choose arbitrarily, e.g. by multiplying the choice of h_2 above by a constant.

From (6.29) we obtain

$$\xi = \left(A_g \Theta^{-1} k^{-S} k_g^{R_g} \right)^{-1/(1+\nu)} \tag{6.32}$$

and hence

$$c = \left(A_g \Theta^{-1} k^{-S} k_g^{\omega} \right)^{-1/(1+\nu)}. \tag{6.33}$$

We also find that

$$\frac{U}{V'} = -\frac{c}{\nu}$$

and so from equations (6.23)–(6.27) we obtain

$$
\begin{aligned}
F_p &= \gamma_g + Q + \frac{S}{\nu}\frac{c}{k} && (6.34)\\
F_p - F_g &= \frac{\omega}{\nu}\xi && (6.35)\\
F_\pi &= \frac{(R_g - S)}{\nu}\frac{c}{\pi} && (6.36)\\
F &= (\gamma_g + Q)k + (1 + \frac{1}{\nu})c && (6.37)\\
\Phi &= Qk + \frac{c}{\nu}. && (6.38)
\end{aligned}
$$

As we will choose k_g to be non-negative, c is also non-negative and thus F_π, F_p and F are also non-negative for suitably large $\gamma_g + Q$.

The remaining problem is to make a good choice of k_g. We may think of the problem as one of choosing a non-negative function

$$
\varphi(k) \equiv \left(\frac{k_g(k)}{k}\right)^{-\omega/(1+\nu)} \qquad (6.39)
$$

in such a way as to guarantee non-negativity of F_g together with the tangent inequality and the inequality (6.6): $0 \le k_g \le k$. The final inequality (6.6) can be equivalently expressed by saying that we need to have $\varphi(k) \ge 1$. In terms of φ, we have more simply

$$
\begin{aligned}
c &= Ak^B\varphi && (6.40)\\
\xi &= Ak^{B-1}\varphi^{R_g/\omega} && (6.41)\\
F_g &= \gamma_g + Q + Ak^{B-1}(S\varphi - \omega\varphi^{R_g/\omega})/\nu && (6.42)\\
\Phi &= Qk + Ak^B\varphi/\nu && (6.43)
\end{aligned}
$$

where the parameters A and B are related to the other parameters by

$$
\Theta A^{-(1+\nu)} = A_g, \qquad (6.44)
$$

$$
(1+\nu)B = S - \omega. \qquad (6.45)
$$

Non-negativity of F_p will be guaranteed by

$$
\gamma_g + Q \equiv \gamma_g + \frac{\lambda_g}{1-S} + \frac{1}{2}\sigma^2 S \ge 0. \qquad (6.46)
$$

Non-negativity of F_g needs to be checked case by case. Note that equation (6.45) implies that

$$(1-B)(1+\nu) = R_g - S$$

and hence $B < 1$. If we consider the limit as $k \downarrow 0$ of (6.42), we see that we must have

$$S\varphi(0) - \omega\varphi(0)^{R_g/\omega} \geq 0,$$

and since $\varphi \geq 1$ we conclude from this that a necessary condition to be satisfied is

$$S \geq \omega,$$

from which it follows from (6.45) that $B > 0$.

The derivative (6.19) of the tangent inequality (bearing in mind we have $\pi'(k) = 0$) is

$$\left(1 + \frac{1}{\nu}\right)(c'(w) - c'(k)) + \frac{\omega}{\nu}\xi(w)(k_g'(w) - k_g'(k))$$

so if $c'(k)$ and $k_g'(k)$ are decreasing functions of k then the tangent inequality will hold.

Alternatively observe from (6.33) that

$$k_g' = \frac{S}{\omega}\frac{k_g}{k} - \frac{(1+\nu)}{\omega}\frac{c'}{\xi}$$

and so the derivative (6.19) can be written as

$$\left(1 + \frac{1}{\nu}\right)c'(k)\left(\frac{\xi(w)}{\xi(k)} - 1\right) + \frac{S}{\nu}\xi(w)\left(\frac{k_g(w)}{w} - \frac{k_g(k)}{k}\right)$$

so that if c is increasing and k_g/k and ξ are decreasing functions then the tangent inequality will be satisfied. For the example we consider below this turns out to be a *more restrictive condition* on the range of parameters we can use than the previous condition.

We also need to check condition (6.17), that $\Lambda(x, y, z; w)$ has a unique infimum over w for any fixed (x, y, z). In this case

$$\begin{aligned}\Lambda(x, y, z; w) &= (x+y)F_p(w) - y(F_p(w) - F_g(w)) + zF_\pi(w)\\ &= (x+y)(\gamma_g + Q + \frac{S}{\nu}\frac{c(w)}{w}) - y(\frac{\omega}{\nu}\xi(w)) + z(\frac{(R_g - S)}{\nu}\frac{c(w)}{\pi}).\end{aligned}$$

We summarize the discussion above in the following lemma.

Lemma 1 *Suppose that we have suitable positive constants σ, A, ν, λ_g, κ and further constants satisfying the relations*

$$R_g > S > 1, \qquad S > \omega = R_g - 1 - \nu > 0,$$
$$B = \frac{S-\omega}{1+\nu}, \qquad \gamma_g + Q = \gamma_g + \frac{\lambda_g}{1-S} + \frac{1}{2}\sigma^2 S \geq 0.$$

Take a function $\varphi(k) \geq 1$ satisfying

$$\varphi(0) \leq \left(\frac{S}{\omega}\right)^{\omega/(1+\nu)},$$

define

$$c(k) = Ak^B\varphi(k), \qquad \xi(k) = Ak^{B-1}\varphi(k)^{R_g/\omega}, \qquad k_g(k) = c(k)/\xi(k),$$

and k^* solving

$$\begin{aligned} dk^* &= \sigma k^* dw + \Phi(k^*)dt \\ &= \sigma k^* dw + (Qk^* + \varphi(k_t^*){k^*}^B A/\nu)dt. \end{aligned} \tag{6.47}$$

Assume also that the following following four conditions hold:

(L1) $F_g = \gamma_g + Q + Ak^{B-1}(S\varphi - \omega\varphi(k)^{R_g/\omega})/\nu \geq 0$ *for all k.*

(L2) Either

$$c'' < 0, \qquad k_g'' < 0$$

or

$$c' > 0, \qquad \left(\frac{k_g}{k}\right)' < 0, \qquad \xi' < 0.$$

(L3) The expression

$$(x + Sy)\,\frac{c}{k} - \omega y\,\xi + z\,c \tag{6.48}$$

attains its infimum over $k \geq 0$ uniquely for all non-negative x, y, z.

(L4) k^* satisfies the transversality condition (GT)

$$\sup_t e^{-\lambda_g t}{k_t^*}^{1-S} \in L^1, \qquad \lim_{t\to\infty} e^{-\lambda_g t}{k_t^*}^{1-S} = 0.$$

Then we have constructed an explicit solution to the problem of Theorem 1 with

$$V(k) = \frac{\Theta A^{-(1+\nu)} k^{1-S}}{(1-S)}, \qquad U(c, k_g, \pi) = -\frac{k_g^{-\omega} c^{-\nu}(1-\pi)^{-\kappa}}{\nu},$$

$$F(x,y,z) = \inf_{w \geq 0} \left\{ (x+y)(\gamma_g + Q + \frac{S}{\nu}\frac{c(w)}{w}) - \frac{\omega y}{\nu}\xi(w) + \frac{z(R_g - S)}{\nu}\frac{c(w)}{\pi^*} \right\}$$

and constants π^* and Θ given by

$$\pi^* = \frac{R_g - S}{\kappa + R_g - S}, \qquad \Theta = \left(\frac{\kappa}{\kappa + R_g - S} \right)^{-\kappa}.$$

6.1.3 An example

We now consider choices of φ of the form

$$\varphi(k) = \varphi_0 (1 + ak)^{\varepsilon}, \tag{6.49}$$

and check the conditions of Lemma 1. We will take $\varepsilon \geq 0$ and $a \geq 0$ (so that $\varphi \geq 1$), and $\varphi_0 \geq 1$ will be chosen small enough so that

$$S\varphi_0 > \omega \varphi_0^{R_g/\omega}; \tag{6.50}$$

this *can* always be done, since $S > \omega$. We will also choose ε so that

$$\varepsilon \leq \frac{\omega(1-B)}{R_g + S}. \tag{6.51}$$

The function φ is increasing and hence k_g/k is decreasing. In order that we have $F_g \geq 0$ we demand that

$$\xi = A\varphi_0^{R_g/\omega} k^{B-1}(1+ak)^{\varepsilon R_g/\omega}$$

be decreasing (and for A to be small enough in relation to $\gamma_g + Q$). This will be true provided

$$\varepsilon \leq \frac{\omega(1-B)}{R_g},$$

which follows from equation (6.51).

For the tangent inequality to hold we need $c''(k)$ and $k_g''(k)$ negative. In order for $c''(k)$ to be negative, it is sufficient that $\varepsilon \leq 1 - B$ which is guaranteed by inequality (6.51).

This condition also ensures that c/k is decreasing and hence, from equation (6.34), F_p is decreasing. For $k_g''(k)$ to be negative, it is sufficient that

$$\varepsilon \leq \frac{\omega}{1+\nu} = \frac{\omega(1-B)}{R_g - S}$$

which again follows from equation (6.51).

Finally, again under the condition (6.51), we can show that (L3) and (L4) hold - see Appendices B.1 and B.2 for proofs.

One particular case included in this example is obtained by setting either a or ε equal to zero so that $\varphi(k) \equiv \varphi_0$ is constant. The ratio of government to total capital k_g/k will then also be constant and the consumption rate will be given simply by $c(k) = A\varphi_0 k^B$.

6.2 Introducing taxes

The government's choice of taxes will depend on the private sector's preferences, which we will assume here are of the form

$$u(c, k_g, \pi) = -\frac{k_g{}^{-\omega_p} c^{-\nu_p} (1-\pi)^{-\kappa_p}}{\nu_p}, \tag{6.52}$$

where $\nu_p > 0$, $\omega_p \equiv R_p - 1 - \nu_p > 0$ and $\kappa_p > 0$. We modify the notation of the previous subsections by writing ω_g in place of ω, ν_g in place of ν and so on, to emphasize the distinction between government and private-sector parameters in what is an otherwise similar specification. We shall assume that condition (PST) of Theorem 2 holds. With the private sector's felicity function specified as above conditions (PS2) and (PS3) from Theorem 2 combined with the very similar conditions (G2) and (G3) from Theorem 1 tell us that

$$\begin{aligned} \beta_c \beta_w &= \frac{u_\pi U_c}{u_c U_\pi} \\ &= \frac{\kappa_p \nu_g}{\kappa_g \nu_p}. \end{aligned} \tag{6.53}$$

Combining the expressions (6.39) for k_g and (6.40) for c in terms of k and φ, and using condition (PS2) of Theorem 2, we have

$$\begin{aligned}\beta_c^{-1}\psi &= A^{-(1+\nu_p)}\Theta_p k^{-R_p+(1-B)(1+\nu_p)}\varphi^{-(1+\nu_p)+\omega_p(1+\nu_g)/\omega_g} \\ &\equiv A^{-(1+\nu_p)}\Theta_p k^{-S_p}\varphi^{-\alpha} \end{aligned} \tag{6.54}$$

where $\Theta_p \equiv (1-\pi^*)^{-\kappa_p}$, $S_p \equiv R_p - (1-B)(1+\nu_p)$ is defined in an analogous manner to S_g (6.45) and $\alpha \equiv 1+\nu_p - \omega_p(1+\nu_g)/\omega_g$. *We shall assume the inequality*

$$\frac{R_p}{1+\nu_p} \leq \frac{R_g}{1+\nu_g}, \tag{6.55}$$

which is easily seen to be equivalent to

$$\alpha \equiv 1+\nu_p - \omega_p(1+\nu_g)/\omega_g \geq 0.$$

There seem to be two approaches we can take to taxation, depending on whether we take equation (6.53) or (6.54) as our starting point.

Approach 1 : Given the form of (6.54) and looking back at our choice of government $\Psi \equiv V'$ of

$$\Psi = A_g k^{-S_g} \equiv A_g k^{-R_g+(1-B)(1+\nu_g)},$$

it seems natural to pick an analogous function for the private sector's function $\psi(k)$, i.e.

$$\psi \equiv A_p k^{-R_p+(1-B)(1+\nu_p)} \equiv A_p k^{-S_p}, \tag{6.56}$$

where A_p is a constant which we shall choose as follows. We will pick some $\beta_w(0) \in [0,1]$ and then equation (6.53) determines $\beta_c(0)$. The consumption tax is then

$$\beta_c = \beta_c(0)\left(\frac{\varphi}{\varphi_0}\right)^{\alpha},$$

and hence $A_p \equiv \beta_c(0)\varphi_0^{-\alpha}A^{-(1+\nu_p)}\Theta_p$. Similarly

$$\beta_w = \beta_w(0)\left(\frac{\varphi}{\varphi_0}\right)^{-\alpha}.$$

As $\alpha > 0$ we have automatically ensured the desirable property $0 \leq \beta_w \leq 1$ where the tax rate on wages $1-\beta_w$ increases as k increases. We have also constructed a consumption tax $1-\beta_c$ that decreases as k increases, eventually becoming a subsidy at high k (if it wasn't already a subsidy to begin with). This is intuitively correct - in a poor economy the population should be encouraged to work more and consume less, whilst in a very rich economy the population should be taxed highly on their income to pay for better public services and should also be consuming more of their capital.

The main benefit of this approach comes when we wish to find β_k and $r\beta_r$ from equations (PS1) and (PS4) respectively. We obtain relatively simple expressions due to the simple form (6.56) of the expression for ψ. Solving (PS4) for $r\beta_r$ gives us

$$r\beta_r = A\frac{S_p}{\nu_g}k^{B-1}\varphi(k) + A_r, \tag{6.57}$$

where

$$A_r = \lambda_p - \mu_0 + S_p(Q + 2(\gamma_g - \gamma) - \frac{1}{2}\sigma^2(1 + S_p)).$$

For large enough values of ρ_p the constant A_r will be non-negative, and thus $r\beta_r$ will be non-negative and decreasing. A more enlightening way to express equation (6.57) is in the form

$$r\beta_r = \frac{S_p}{\nu_g}\frac{c}{k} + A_r,$$

so that the (taxed) rate of return on investment in government debt is equal to a constant rate A_r plus a component proportional to c/k, the ratio between rate of consumption and total capital in the economy. When this ratio is high then interest rates must also be high to attract investment in preference to consumption.

Similarly we can obtain the tax coefficient β_k from condition (PS1) and find that

$$\beta_k(F_p - S_p v_{00} + (S_p - 1)v_{0L}) = r\beta_r + \delta. \tag{6.58}$$

Thus the return on private capital $\beta_k F_p$ is equal to the return on debt $r\beta_r$ plus depreciation δ and some 'price of risk' terms. Substituting in the expression (6.34) for F_p we have that

$$\begin{aligned}\beta_k &= \frac{\delta + r\beta_r}{\gamma_g + Q - S_p v_{00} + (S_p - 1)v_{0L} + A\nu_g^{-1}S_g k^{B-1}\varphi(k)} \\ &= \frac{\delta + A_r + A\nu_g^{-1}S_p k^{B-1}\varphi(k)}{A_k + A\nu_g^{-1}S_g k^{B-1}\varphi(k)}\end{aligned} \tag{6.59}$$

where $A_k \equiv \gamma_g + Q - S_p v_{00} + (S_p - 1)v_{0L}$ will be positive for large enough γ_g.

If we make the plausible assumption that $\beta_k = \beta_r$, so that the tax rates on investment in capital and bonds are the same, then there is an explicit expression for the interest-rate process r:

$$r = \left\{ A_k + A\nu_g^{-1} S_g k^{B-1} \varphi(k) \right\} \frac{A_r + A\nu_g^{-1} S_p k^{B-1} \varphi(k)}{\delta + A_r + A\nu_g^{-1} S_p k^{B-1} \varphi(k)}. \tag{6.60}$$

The interest rate r is thus expressed as a function of the diffusion process k which solves the SDE

$$dk = \sigma k dW + \left[(Q + \gamma_g - \gamma)k + Ak^B \varphi(k)/\nu_g\right] dt.$$

Specializing further by assuming that φ is constant, this can be reduced to *linear* form by considering instead the variable $\zeta \equiv k^{1-B}$, which solves

$$d\zeta = (1-B)\left[\sigma\zeta dW + (Q + \gamma_g - \gamma - \frac{1}{2}B\sigma^2)\zeta dt\right] + A(1-B)\varphi_0 dt/\nu_g.$$

Merton (1975) finds structurally similar interest rate processes in a study of a single-sector growth model, and Kloeden and Platen (1992) present this under the name of the stochastic Verhulst equation. We do not necessarily require that φ be constant so we have a more general form than both of these.

Approach 2 : Equation (6.53) tells us that $\beta_c\beta_w$ is constant so we choose both β_c and β_w to be constant for all k. Equation (6.54) gives

$$\psi = \beta_c \equiv A^{-(1+\nu_p)} \Theta_p k^{-S_p} \varphi^{-\alpha} \tag{6.61}$$

and then, as before, (PS1) and (PS4) can be used to determine β_k and $r\beta r$ (and thus r assuming $\beta_r \equiv \beta_k$). These can again be given in explicit form, although due to the more complicated form of the expression (6.61) for ψ the expressions for β_k and r are considerably less enlightening than those from the first approach (and hence we omit them). This is the price that we pay for the convenience of constant consumption and wage taxes. However we will see in Chapter 8 that we can get very reasonable values for the capital income tax and interest rates from this approach. This will also be the method we will tend to use when obtaining results by numerical means, as we shall see in the next chapter.

Chapter 7

Numerical solutions

In this chapter we will show how to solve the government's problem numerically and also describe how to then determine tax and interest rates. In Section 7.1.1 we look at the deterministic model of Arrow and Kurz (which is a special case of our model) and find solutions to the government's problem using numerical differential equation techniques. In Section 7.1.2 we will find optimal trajectories for the full stochastic model as described in Section 5.2 using policy improvement methods. One convenient result of the way we do this is that we can easily find the stationary distribution of the level of capital k and we show how to do this in Section 7.2. In Section 7.3 we look at the (considerably easier!) problem of finding the tax and interest rates specified by Theorem 2. Finally in Section 7.4 we describe how to calculate zero-coupon bond prices and the corresponding yields.

7.1 The government's problem

Suppose that we are given a suitable production function $F(k_p, k_g, \pi)$, a felicity function

$$U(c, k_g, \pi) \equiv k_g^{(1-R_g)} h\left(\frac{c}{k_g}, \pi\right)$$

and various constants as specified in Section 5.2; we can't in general obtain explicit expressions for the government's value function $V(k)$ and the functions $c^*(k)$, $k_g^*(k)$ and $\pi^*(k)$ that give the optimal consumption rate, government capital and population effort for a given total capital k. However we *can* obtain these functions numerically.

7.1.1 The Arrow-Kurz Model

We can recover the (deterministic) model studied in Chapter IV of Arrow and Kurz by removing the Brownian motion terms (i.e. setting $v_{00} = v_{0L} = v_{LL} = 0$) and stipulating that the workforce dedicate maximum effort to production so that $\pi \equiv 1$. In the deterministic case $\gamma_g = \gamma$ so we will drop the subscript. Writing $f(k_p, k_g) \equiv F(k_p, k_g, 1)$ and dropping also the superfluous π from the government's felicity function, the HJB equation (G6) and the optimality conditions (G2) and (G4) of Theorem 1 are now given by

$$0 = -\lambda_g V(k) + V'(k)(f(k-k_g, k_g) - \gamma k - c) + U(c, k_g) \tag{7.1}$$

$$U_c(c, k_g) = \Psi(k) \tag{7.2}$$

$$U_g(c, k_g) = \Psi(k)(f_p(k-k_g, k_g) - f_g(k-k_g, k_g)) \tag{7.3}$$

where again $\Psi(k) \equiv V'(k)$. The dynamics of the system are given by

$$\dot{k} = f(k-k_g^*, k_g^*) - \gamma k - c^* \tag{7.4}$$

$$\dot{\Psi} = \Psi(\lambda_g + \gamma - f_p(k-k_g^*, k_g^*)), \tag{7.5}$$

where c^* and k_g^* are obtained by solving (7.2) and (7.3) given k and Ψ. Equation (7.4) is just equation (G5) and equation (7.5) follows from equation (G1) which, in full, states that

$$\Psi(f_p(k-k_g^*, k_g^*) - \lambda_g - \gamma) + \Psi'(f(k-k_g^*, k_g^*) - \gamma k - c^*) = 0. \tag{7.6}$$

To solve the system we can use an ODE solver such as Scilab's 'DASSL' command to solve (7.6), and then use (7.1) to find $V(k)$. DASSL requires a starting point $(k^\infty, \Psi(k^\infty))$ and for accurate results an initial value for $\Psi'(k^\infty)$. Fortunately in the deterministic case we have an equilibrium point $\dot{k} = \dot{\Psi} = 0$ which we can locate. This is the point that the system will head towards eventually (as $t \to \infty$), hence the notation. Suppose that we have Cobb-Douglas style production and felicity functions, e.g.

$$\begin{aligned} f(k_p, k_g) &= Ak_p^\alpha k_g^\beta \\ U(c, k_g) &= \frac{(c^a k_g^{1-a})^{1-R_g}}{1-R_g}, \end{aligned}$$

where we require that $0 < \alpha, \beta, \alpha+\beta, a < 1$ and $R_g > 0$, $R_g \neq 1$ for the various homogeneity conditions to be satisfied. We can now find expressions in closed form for the equilibrium point :

$$\begin{aligned}
k_g^\infty &= \left(\frac{(\lambda_g+\gamma)\theta^{1-\alpha}}{\alpha A}\right)^{-\frac{1}{1-\alpha-\beta}} \\
k_p^\infty &= \theta k_g^\infty \\
k^\infty &= k_g^\infty + k_p^\infty \\
c^\infty &= \frac{(\lambda_g+\gamma)k_p^\infty}{\alpha} - \gamma k^\infty \\
\Psi^\infty &= U_c(c^\infty, k_g^\infty),
\end{aligned}$$

where we write $c^\infty \equiv c(k^\infty)$ etc. and

$$\theta \equiv \frac{k_p^\infty}{k_g^\infty} = \frac{\alpha(a\lambda_g+\gamma)}{a(\lambda_g+\gamma)\beta + (1-a)(\lambda_g+\gamma-\alpha\gamma)}.$$

To solve for $\Psi'(k^\infty)$ we can differentiate (7.6) and the optimality conditions (7.2) and (7.3) with respect to k at the equilibrium point yielding (dropping the k^∞ dependence for convenience) :

$$\Psi'((1-k_g')f_p + k_g' f_g - \gamma - c') + \Psi((1-k_g')f_{pp} + k_g' f_{pg}) = 0 \tag{7.7}$$

and

$$\begin{aligned}
c'U_{cc} + k_g' U_{cg} &= \Psi' \\
c'U_{cg} + k_g'(U_{gg} + \Psi(f_{pp} - 2f_{pg} + f_{gg})) &= \Psi'(f_p - f_g) + \Psi(f_{pp} - f_{pg}).
\end{aligned}$$

These two equations can be solved simultaneously to determine $c'(k^\infty)$ and $k_g'(k^\infty)$ in terms of $\Psi'(k^\infty)$ and then substituted into (7.7) which then becomes a quadratic in $\Psi'(k^\infty)$. We take the negative of the two roots as we know that $V(k)$ is a concave function of k and hence $V''(k^\infty) \equiv \Psi'(k^\infty) < 0$. This knowledge now allows us to use DASSL to solve away from the equilibrium point (i.e. backwards in time) in each direction to find $\Psi(k)$ for a much wider range of k, and hence $V(k)$ from equation (7.1).

7.1.2 The stochastic two-sector model

Numerical solution is more complicated for the full stochastic model as described in Section 5.2. We have a second order ODE given by (G1) to solve, and at each point (k, Ψ) we need to solve the three equations (G2)–(G4) in order to find the optimal c, k_g and π. This adds a level of complexity, but the main reason we cannot use a differential equations approach similar to that we used for the previous two models is that there is no equilibrium point in this stochastic model. We have boundary conditions of sorts - the form of the felicity function may tell us that e.g. $V(0) = 0$ or $V(0) = -\infty$ but we need to know also the first and second derivatives of V at zero and these will usually either be infinite or impossible to determine. We will use instead a policy improvement method to calculate the government's value function $V(k)$. As we wish to find optimal policies (and later optimal tax and interest rates) that are functions of k we will discretize the capital and then try to find the optimal policy at each different capital level.

We will change the main variable we consider from k to $x \equiv \log k$. Firstly this will aid calculation, and secondly this fits the structure of the problem, as for small k the value function V changes much more rapidly for a fixed injection of capital than it does for larger k. From the dynamics (5.22) of k and using Itô's Lemma we find that the $\mathbb{P}_g$-dynamics of x are

$$\begin{aligned} dx_t &= \frac{1}{k}\,dk_t - \frac{1}{2k^2}\sigma^2 k^2 dt \\ &= \left(\frac{F(k-k_g, k_g, \pi) - c}{k} - \gamma_g - \frac{1}{2}\sigma^2\right) dt + \sigma dw_t \\ &\equiv \mu(x; k_g, c, \pi)\, dt + \sigma dw_t, \end{aligned} \tag{7.8}$$

with the appropriate identification. We wish to find V and the corresponding optimal c, k_g and π at the points

$$x_i = x_0 + i\Delta x \qquad 0 \leq i \leq N$$

for some suitable choice of x_0 and Δx. We will use the notation $k_i \equiv e^{x_i}$, $V_i \equiv V(k_i)$, $k_g^i \equiv k_g(k_i)$ etc. We proceed with the policy improvement as follows:

Step 0 : Firstly we pick an initial value function. We will use the value function corresponding to the policy of disposing of all the capital in some suitable[1] way. The homogeneity of F (providing it has some dependence on capital and isn't just a function of labour) then means that the economy will have zero capital in perpetuity, and the population will live in misery. Hence the initial value function is

$$V_i = \int_0^\infty e^{-\lambda_g t} U(0,0,0)\, dt = \frac{U(0,0,0)}{\lambda_g} \quad \forall i$$

[1] Dumping it into the sea, building a Millennium Dome with it, etc.

which will usually be zero for U a non-negative felicity function, and $-\infty$ for U a non-positive felicity function. For the purpose of numerical calculations we will substitute a negative number of large magnitude for $-\infty$.

Step 1 : Given the value function $(V_i)_{0 \le i \le N}$ we now wish to find a better policy to follow. To find the new optimal policy (k_g^i, c_i, π_i) to follow at point x_i we proceed as follows. We will assume that the diffusion x_t starts at value x_i at time 0 and continues until the process x_t hits either the level above (i.e. x_{i+1}) or below (x_{i-1}) at the random time

$$\tau \equiv \inf \{ t > 0 : x_t \in \{x_{i-1}, x_{i+1}\} \} .$$

We will approximate x_t in the neighbourhood of x_i by the process $\tilde{x}_t$ which also starts at level x_i at time 0, but obeys the stochastic differential equation

$$d\tilde{x}_t = \tilde{\mu}_i dt + \sigma dw_t$$

where $\tilde{\mu}_i$ is now a constant with value

$$\tilde{\mu}_i \equiv \mu(x_i; k_g^i, c_i, \pi_i).$$

We define also the stopping time

$$\tilde{\tau}_i \equiv \inf \{ t > 0 : \tilde{x}_t \in \{x_{i-1}, x_{i+1}\} \} ,$$

denote by $\tilde{p}_i^+$ the probability that $\tilde{x}_{\tilde{\tau}_i} = x_{i+1}$ and define the corresponding probability of hitting the level x_{i-1} before x_{i+1} by $\tilde{p}_i^- \equiv 1 - \tilde{p}_i^+$. We know the values V_{i-1} and V_{i+1} and wish to find

$$\begin{aligned}
&\max_{k_g, c, \pi} \mathbb{E}_g^{x_i} \left[\int_0^\infty e^{-\lambda_g t} U(c_t, k_g(t), \pi_t)\, dt \right] \\
&= \max_{k_g, c, \pi} \mathbb{E}_g^{x_i} \left[\int_0^\tau e^{-\lambda_g t} U(c_t, k_g(t), \pi_t)\, dt + e^{-\lambda_g \tau} V(x_\tau) \right] \\
&\approx \max_{k_g^i, c_i, \pi_i} U(c_i, k_g(i), \pi_i) \mathbb{E}_g^{x_i} \left[\int_0^{\tilde{\tau}_i} e^{-\lambda_g t}\, dt \right] + \mathbb{E}_g^{x_i} \left[e^{-\lambda_g \tilde{\tau}_i} \right] (\tilde{p}_i^- V_{i-1} + \tilde{p}_i^+ V_{i+1}) \\
&= \max_{k_g^i, c_i, \pi_i} \frac{U(c_i, k_g(i), \pi_i)}{\lambda_g} \left(1 - \mathbb{E}_g^{x_i} \left[e^{-\lambda_g \tilde{\tau}_i} \right] \right) + \mathbb{E}_g^{x_i} \left[e^{-\lambda_g \tilde{\tau}_i} \right] (\tilde{p}_i^- V_{i-1} + \tilde{p}_i^+ V_{i+1}). \qquad (7.9)
\end{aligned}$$

This maximization is not as straightforward as it may at first appear as $\tilde{\tau}_i$, $\tilde{p}_i^+$ and $\tilde{p}_i^-$ are all functions of $\tilde{\mu}_i$ and thus of k_g^i, c_i and π_i. However the whole expression to be maximized above can be written down explicitly using the following calculations.

First note that a scale function for the process $\tilde{x}_t$ is given by $s(y) = -e^{-2\tilde{\alpha}_i y}$ where $\tilde{\alpha}_i \equiv \tilde{\mu}_i/\sigma^2$ and so the probability of the process $\tilde{x}_t$ hitting the level x_{i+1} before the level x_{i-1} is given by

$$\tilde{p}_i^+ = \frac{s(x_i) - s(x_{i-1})}{s(x_{i+1}) - s(x_{i-1})} = \frac{e^{2\tilde{\alpha}_i \Delta x} - 1}{e^{2\tilde{\alpha}_i \Delta x} - e^{-2\tilde{\alpha}_i \Delta x}}.$$

Secondly Proposition 1 of Rogers and Stapleton (1998) tells us that

$$\mathbb{E}\left[e^{-\lambda_g \tilde{\tau}_i}\right] = \frac{\cosh(\tilde{\alpha}_i \Delta x)}{\cosh(\tilde{\beta}_i \Delta x)} \tag{7.10}$$

where $\tilde{\beta}_i \equiv \sqrt{\tilde{\mu}_i^2 + 2\lambda_g \sigma^2}/\sigma^2 > \tilde{\alpha}_i$.

Computationally the maximization of (7.9) can be reduced to a maximization over just one variable by use of Theorem 1. For a small enough discretization the optimal solution to the policy improvement problem should be very close to the solution to the full problem given by Theorem 1 and so eliminating Ψ from equations (G2)–(G4) tells us that

$$U_\pi(c, k_g, \pi) = -F_\pi(k - k_g, k_g, \pi) U_c(c, k_g, \pi) \tag{7.11}$$

$$U_g(c, k_g, \pi) = (F_p(k - k_g, k_g, \pi) - F_g(k - k_g, k_g, \pi)) U_c(c, k_g, \pi) \tag{7.12}$$

and so, for example, if we have chosen a value for k_g^i we can determine the corresponding optimal values of c_i and π_i from these two equations. The main example which we will consider is as follows. We will assume a Cobb-Douglas style production function of the form

$$F(K_p, K_g, L) = A K_p^\alpha K_g^\beta L^{1-\alpha-\beta} \tag{7.13}$$

where $0 < A$ and $0 < \alpha, \beta, \alpha + \beta < 1$. We will also assume that the felicity function is CRRA of the form

$$\begin{aligned} U(c, k_g, \pi) &\equiv k_g^{1-R_g} h(\xi, \pi) \\ &= -k_g^{1-R_g} \frac{\xi^{-\nu}(1-\pi)^{-\kappa}}{\nu} \\ &= -\frac{k_g^{-\omega} c^{-\nu} (1-\pi)^{-\kappa}}{\nu} \end{aligned} \tag{7.14}$$

where $\xi \equiv \frac{c}{k_g}$, $\omega \equiv R_g - 1 - \nu$ and either

$$R_g > 1, \quad \nu > 0, \quad \omega > 0, \quad \kappa > 0$$

or

$$1 > R_g > 0, \quad -1 < \nu < 0, \quad -1 < \omega < 0, \quad \kappa < 0.$$

With these forms for F and U then for a given k and k_g (and hence also $k_p = k - k_g$) equations (7.11) and (7.12) explicitly give π via

$$\pi\left(\frac{\alpha k_g - \beta k_p}{(1-\alpha-\beta)k_p} + \frac{\omega}{\kappa}\right) = \frac{\omega}{\kappa} \tag{7.15}$$

and then

$$c = \frac{A\nu}{\omega} k_p^{\alpha-1} k_g^{\beta} \pi^{1-\alpha-\beta}(\alpha k_g - \beta k_p). \tag{7.16}$$

Thus (7.9) is reduced to a maximization over k_g^i alone which we solve numerically using Matlab, in the range

$$\frac{\beta k_i}{\alpha+\beta} \leq k_g^i \leq k_i.$$

The left hand limit is what the optimal k_g would be if k_g didn't appear in the felicity function, and so the optimal rate of production for a given π is obtained by finding k_g such that $F_p(k - k_g, k_g, \pi) = F_g(k - k_g, k_g, \pi)$. Clearly picking k_g less than this would be suboptimal as both the rate of production of capital and the felicity could be increased by increasing k_g. Note also that this choice of lower limit for k_g ensures that the expressions for the optimal π and c given by equations (7.15) and (7.16) are always non-negative, and so also π is always in the interval $[0, 1]$.

The final issue we need to consider is boundary conditions. We will take

$$V_{-1} = \frac{U(0,0,0)}{\lambda_g}, \qquad V_{N+1} = V_N$$

so that if the level $x_0 - \Delta x$ is hit all the capital is thrown away, and if the level $x_N + \Delta x$ above is hit enough capital is thrown away that the government has an amount k_N of capital again.

Step 2 : Having computed a new optimal policy given the value function, we now compute a new value function given this optimal policy. This is simply a matter of solving the relevant simultaneous equations, which follow from equation (7.9). Given a policy (c_i, k_g^i, π_i) and the corresponding quantities $\tilde{p}_i^+$, $\tilde{p}_i^-$ and $\mathbb{E}\left[e^{-\lambda_g \tilde{\tau}_i}\right]$ as computed above, we find the new value function $(\tilde{V}_i)_{0 \leq i \leq N}$ by solving the *linear* system

$$\tilde{V}_i = \frac{U(c_i, k_g(i), \pi_i)}{\lambda_g}\left(1 - \mathbb{E}_g^{x_i}\left[e^{-\lambda_g \tilde{\tau}_i}\right]\right) + \mathbb{E}_g^{x_i}\left[e^{-\lambda_g \tilde{\tau}_i}\right](\tilde{p}_i^- \tilde{V}_{i-1} + \tilde{p}_i^+ \tilde{V}_{i+1})$$

for $0 \leq i \leq N$ and with the same boundary conditions as previously, i.e.

$$\tilde{V}_{-1} = \frac{U(0,0,0)}{\lambda_g}, \qquad\qquad \tilde{V}_{N+1} = \tilde{V}_N.$$

Step 3: We now check for convergence. If

$$\max_i |\tilde{V}_i - V_i| < \varepsilon$$

for a suitable choice of ε then we halt, and take the current $\tilde{V}$ as the correct value function and the corresponding policy as the optimal policy. Otherwise we set

$$V_i := \tilde{V}_i \qquad \forall i$$

and return to Step 1. In practice with $\varepsilon = 10^{-8}$ only around 5–15 iterations are needed!

7.2 The stationary distribution of k

Having calculated the government's optimal policy and value function using the policy improvement method of Section 7.1.2 it is now easy to find the stationary distribution of k, or equivalently $x \equiv \log k$, under this optimal policy. We can approximate the process x_t by a continuous time Markov Chain on the discrete state space

$$x_i = x_0 + i\Delta x \qquad 0 \leq i \leq N,$$

with the non-zero entries in the Q-matrix given by

$$\begin{aligned} q_{i,i+1} &= \frac{\tilde{p}_i^+}{\mathbb{E}[\tilde{\tau}_i]} && 0 \leq i < N \\ q_{i,i-1} &= \frac{\tilde{p}_i^-}{\mathbb{E}[\tilde{\tau}_i]} && 0 < i \leq N \\ q_{i,i} &= -q_{i,i+1} - q_{i,i-1} && 0 \leq i \leq N \end{aligned}$$

with all quantities calculated as in Section 7.1.2 under the optimal policy. The quantity $\mathbb{E}[\tilde{\tau}_i]$ can be obtained from equation (7.10) by differentiating both sides with respect to λ_g and then setting $\lambda_g = 0$, yielding

$$\mathbb{E}[\tilde{\tau}_i] = \frac{\Delta x}{\tilde{\mu}_i} \tanh(\tilde{\alpha}_i \Delta x). \tag{7.17}$$

Having constructed the Q-matrix we can find the stationary distribution

$$g \equiv [g_0 \quad g_1 \quad \cdots \quad g_{N-1} \quad g_N]$$

by solving the system $g\,Q = 0$ subject to the normalising condition $\sum_{i=0}^{N} g_i = 1$. Thus we have obtained the probability g_i of the chain being in state x_i in equilibrium, for all states.

7.3 The taxation problem

Tax and interest rates for the two-sector stochastic model can now be calculated from Theorem 2. If we *choose* the form of the consumption tax $1 - \beta_c(k)$ then equation (PS2) gives $\psi(k)$, and equations (PS1), (PS3) and (PS4) then allow us to determine β_k, β_w and $r\beta_r$ respectively. Alternatively we could follow a similar procedure having first assumed the form of the wage tax $1 - \beta_w$. A more detailed example follows.

We will assume that the government felicity function $U(c, k_g, \pi)$ is as given by equation (7.14), but we will modify the notation by writing ν_g in place of ν, ω_g in place of ω etc. We now define similarly the private sector's felicity function by

$$u(c, k_g, \pi) = -\frac{k_g^{-\omega_p} c^{-\nu_p} (1 - \pi)^{-\kappa_p}}{\nu_p} \tag{7.18}$$

where $\omega_p \equiv R_p - 1 - \nu_p$ and with restrictions on the parameters analogous to those for the government felicity function. Conditions (PS2) and (PS3) from Theorem 2 combined with the very similar conditions (G2) and (G3) from Theorem 1 give

$$\begin{aligned} \beta_c \beta_w &= \frac{u_\pi U_c}{u_c U_\pi} \\ &= \frac{\kappa_p \nu_g}{\kappa_g \nu_p}, \end{aligned} \tag{7.19}$$

so that $\beta_c \beta_w$ is a constant. We could choose β_c constant so that β_w is also constant, but we could equally well choose a more variable tax regime where, for example, the consumption tax falls as the amount of capital available to consume increases, with the wage tax then increasing to balance this. Assuming the form of β_c, equation (PS2) gives us $\psi = \beta_c u_c(c^*, k_g^*, \pi^*)$, with β_w determined by (7.19) above. We can now proceed to finding β_k. We again change variables to $x = \log k$ so that

$$\frac{d\psi}{dk} = \frac{1}{k}\frac{d\psi}{dx}, \qquad\qquad \frac{d^2\psi}{dk^2} = \frac{1}{k^2}\left(\frac{d^2\psi}{dx^2} - \frac{d\psi}{dx}\right) \tag{7.20}$$

and so we can determine β_k from equation (PS1), on the mesh $x_i = x_0 + i\Delta x$ that we solved the government's problem on in Section 7.1.2, by use of the following standard second-order-accurate finite-difference approximations:

$$\begin{aligned}
\psi'(x_0) &= \frac{-\psi(x_2) + 4\psi(x_1) - 3\psi(x_0)}{2\Delta x} \\
\psi'(x_i) &= \frac{\psi(x_{i+1}) - \psi(x_{i-1})}{2\Delta x} \\
\psi'(x_N) &= \frac{\psi(x_{N-2}) - 4\psi(x_{N-1}) + 3\psi(x_N)}{2\Delta x} \\
\psi''(x_0) &= \frac{-\psi(x_3) + 4\psi(x_2) - 5\psi(x_1) + 2\psi(x_0)}{\Delta x^2} \\
\psi''(x_i) &= \frac{\psi(x_{i+1}) - 2\psi(x_i) + \psi(x_{i-1})}{\Delta x^2} \\
\psi''(x_N) &= \frac{-\psi(x_{N-3}) + 4\psi(x_{N-2}) - 5\psi(x_{N-1}) + 2\psi(x_N)}{\Delta x^2},
\end{aligned}$$

with $1 \leq i \leq (N-1)$ in the second and fifth of these equations. We can similarly find $r\beta_r$ from equation (PS4). If we assume that the tax rates on income from capital and debt will be the same so that $\beta_r \equiv \beta_k$, then this enables us to find r.

7.4 Bond prices

Now that we have the (short-term) interest rate as a function of capital we can attempt to find bond prices. In Section 5.4 we derived a PDE (5.49) for $B(t, k; T)$, the time-t price of a zero-coupon bond paying one unit of capital at time T, where $k_t = k$ at time $t \leq T$. We need to change variables from k to $x = \log k$ in this PDE. We can write the $\mathbb{P}$-dynamics of x as

$$dx_t \equiv \mu_x(x_t)dt + \sigma dW_t$$

where $\mu_x(x_t) \equiv \tilde{\Phi}(e^{x_t})e^{-x_t} - \frac{1}{2}\sigma^2$. Now using identities similar to those in (7.20) we find that the zero-coupon bond price $B(t, x; T)$, written in terms of x, satisfies the partial differential equation

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 B}{\partial x^2} + \frac{\partial B}{\partial x}\left(\mu_x + \sigma^2\frac{1}{\psi}\frac{d\psi}{dx} + v_{LL} - v_{0L}\right) - r\beta_r B = 0 \tag{7.21}$$

with terminal condition $B(T, x; T) = 1$ and μ_x as defined above. We can solve this PDE using the Crank-Nicolson finite-difference scheme over the range $[x_0, x_N] \times [0, T]$ with the same spatial discretization as previously and a suitable choice for the number of time steps (e.g. 100). See, for example, Mitchell and Griffiths (1980) or Wilmott (1998) for a description of the Crank-Nicolson scheme.

Having obtained the prices of a bond paying out at time T_{max} for a range of x and all times $0 \leq t \leq T_{max}$ the corresponding yields are given by

$$Y(t, x; T_{max}) = -\frac{1}{T_{max} - t} \log B(t, x; T_{max})$$

for $0 \leq t < T_{max}$. Time-homogeneity then implies that the yield of a bond of maturity T is given at time-0 by

$$Y(0, x; T) = Y(T_{max} - T, x; T_{max})$$

for $0 < T \leq T_{max}$. This allows us to plot yield curves for varying k.

Chapter 8

Examples and conclusions

In this chapter we will illustrate the explicit and numerical solutions of the previous two chapters graphically. The aim of this chapter is to give a broad idea of the sort of features and behaviour exhibited by the model rather than an attempt at anything more specific. This model could be used as the basis for investigations into a wide range of economic topics - obvious examples include allocation of capital or human capital resources, rates of time preference and risk aversion levels, fiscal policy and taxation issues and modelling of interest rates.

The remainder of this chapter is laid out as follows. We can use almost exactly the same basic model specification for both explicit and numerical solutions - we give this in Section 8.1. This approach will also allow us to compare the numerical and explicit results. In Section 8.2 we exhibit solutions to the government's problem. In Section 8.3 we look at taxation policy and give plots of the policies arising from our numerical and explicit examples, along with bond yield curves. We conclude our examination of the stochastic two-sector model in Section 8.4.

8.1 Model specification

We will take the growth rates of population and technology and the rate of depreciation of capital to be

$$\mu_L = 0.01, \qquad \mu_T = 0.05, \qquad \delta = 0.10,$$

respectively. For the covariances between the two driving Brownian motions we will choose

$$v_{LL} = 0.01, \qquad v_{00} = 0.02, \qquad v_{0L} = 0.005,$$

and for the discount rates of the government and the private sector we will take

$$\rho_g = 0.10, \qquad \rho_p = 0.15.$$

We now turn to the specification of the felicity functions. In both our explicit and numerical examples these are given by

$$U(c, k_g, \pi) = -\frac{k_g^{-\omega_g} c^{-\nu_g} (1-\pi)^{-\kappa_g}}{\nu_g}$$

for the government, where $\omega_g = R_g - 1 - \nu_g$, and

$$u(c, k_g, \pi) = -\frac{k_g^{-\omega_p} c^{-\nu_p} (1-\pi)^{-\kappa_p}}{\nu_p}$$

for the private sector with a similar definition for ω_p. Notice that we can write

$$k_g^{-\omega_g} c^{-\nu_g} = (k_g^{1-\theta_g} c^{\theta_g})^{1-R_g}$$

where $\theta_g \equiv \nu_g/(R_g - 1)$ reflects the importance in the felicity function of consumption over government capital. We will take

$$\theta_g = 0.4, \qquad \theta_p = 0.7,$$

with the obvious definition of θ_p; this means that the private sector's felicity function places a greater importance on consumption, as opposed to general welfare levels, than the government's does. Once we have chosen R_g this along with θ_g defines ν_g and hence ω_g and similarly for R_p. Finally we will take

$$|\kappa_g| = 0.2, \qquad |\kappa_p| = 0.1$$

with κ_g positive if $R_g > 1$ and negative otherwise, and similarly for κ_p.

The choice of the parameters made above is fairly arbitrary, but the choices can be justified as plausible. It is less obvious what a plausible value for the coefficient of relative risk aversion R would be. Many studies have been conducted by economists in this area. Arrow (1971) summarizes a number of them, concluding that relative risk aversion with respect to wealth is almost constant and arguing on theoretical grounds that R should be 1. Mehra and Prescott (1985) summarize several studies all of which suggest values for R between 1 and 2. More recently Romer (2001) states that a value of 4 is 'towards the high end of values that are viewed as plausible', although he also shows that methods for calculating R based on equity premiums can give values as high as 240! Anderson and Dillon (1992) give a classification scale for R, where 0.5 corresponds to 'hardly risk averse at all', 1.0 'somewhat risk averse (normal)' and 2.0, 3.0 and 4.0 'rather', 'very' and 'extremely' risk averse respectively. We shall pick values based on this sort of range.

To complete the specification of the examples, for the numerical examples we will take the production function specified as in (7.13) by

$$F(K_p, K_g, L) = K_p^{0.4} K_g^{0.3} L^{0.3}.$$

In the case of the explicit example the production function is not known explicitly. Instead we specify the form of the government's value function, defined as in Lemma 1 by choosing values for A and B (and hence S) that satisfy the various relations. We also need to pick the constants φ_0, a and ε that define our example $\phi(k) = \varphi_0(1 + ak)^\varepsilon$ so that relations (6.50) and (6.51) are both satisfied.

Having specified the model we will use for the two types of examples, we can now move on to illustrate both the numerical (Section 8.2.1) and explicit solutions (Section 8.2.2) to the government's problem of choosing the optimal consumption rate c, work rate π and level of government capital k_g for any given level of total capital k.

8.2 Solutions to the government's problem

8.2.1 Numerical examples

For the government's problem we'll look at three different examples and solve them using the method of Section 7.1.2; in the first we take $R_g = 0.8$ (and hence $\nu_g = -0.08$, $\omega_g = -0.12$, $\kappa_g = -0.2$) so that the government is not particularly risk averse and has a positive felicity function. In the second example we will take double the coefficient of risk aversion, so that $R_g = 1.6$ ($\nu_g = 0.24$, $\omega_g = 0.36$, $\kappa_g = 0.2$) and the corresponding government felicity function is negative. The value functions calculated for these two examples are shown in Figure 8.1. Our final example will have a very risk-averse value

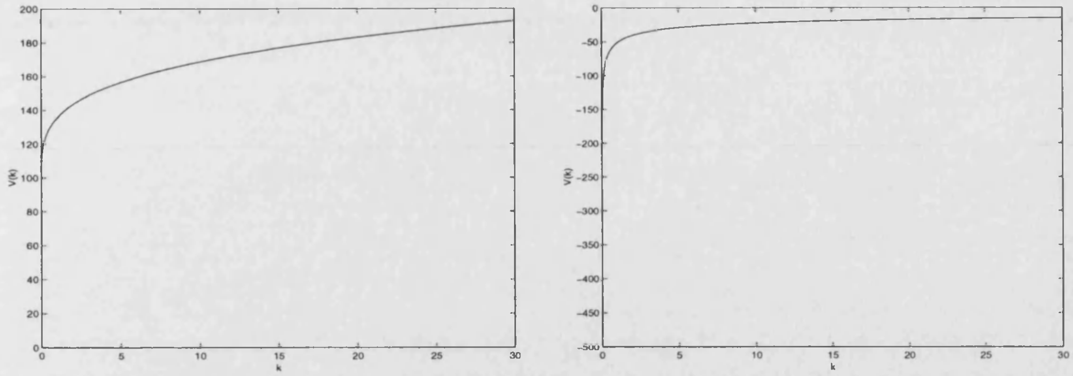

Figure 8.1: The government's value function $V(k)$. The left hand plot corresponds to the numerical example with $R_g = 0.8$ and the right hand plot has $R_g = 1.6$.

of $R_g = 3.2$ (with $\nu_g = 0.88$, $\omega_g = 1.32$ and $\kappa_g = 0.2$). The value function for this example is shown in Figure 8.6.

The government's optimization problem is a balance between two factors. The first is the growth rate $\Phi(k)$, which can be increased by a lower rate of consumption, higher effort and a higher proportion of capital in private hands. The second is felicity, which can be increased by a higher rate of consumption, lower effort and a higher proportion of capital controlled by the government. These two aims are in opposition to each other with the balance between them determined by how risk averse the government is. A government that is not particularly risk averse (in our examples $R_g = 0.8$) strongly favours growth over felicity at low capital levels, whereas a very risk averse ($R_g = 3.2$) government desires higher felicity levels and consequently slower growth. The different growth rate functions $\Phi(k)$ are shown in Figure 8.2 for our three examples and reflect the trends just discussed. The optimal values of π, k_g/k and c reflect the desired balance between felicity and growth, and are shown for the three examples in Figures 8.3, 8.4 and 8.5 respectively. Also shown is g, the stationary distribution of k, scaled so that its maximum is 1.

Looking at how the optimal path changes over time, the contrast between Figures 8.3 and 8.5 is noticeable. In the first with $R_g = 0.8$ the population's work rate drops dramatically as the capital in the economy increases and emphasis changes from growing capital to gaining felicity. Similarly the proportion of government capital increases as total capital increases for the same reason. The rate of consumption of capital is roughly linear with k. In contrast when $R_g = 3.2$ the government is so risk averse that the population actually works slightly harder as the total capital increases - in a poor

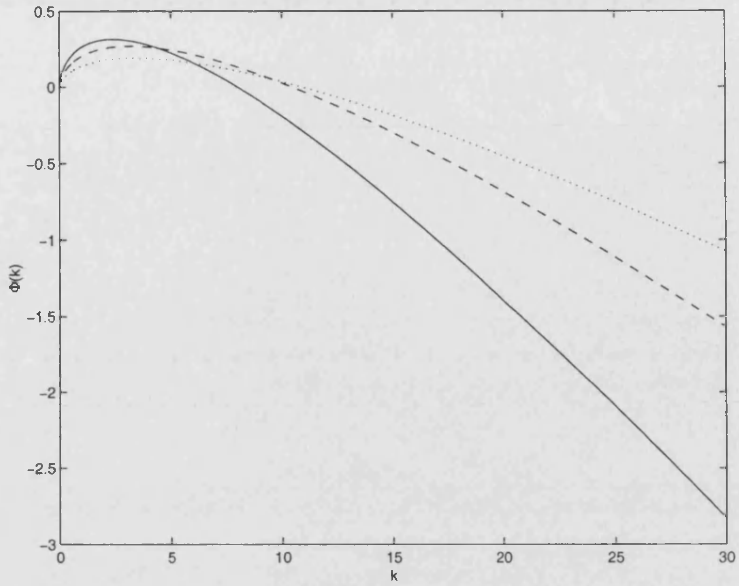

Figure 8.2: Plots of the optimal growth rate $\Phi(k)$ against k for the numerical example with coefficients of relative risk aversion $R_g = 0.8$ (solid line), $R_g = 1.6$ (dashed line) and $R_g = 3.2$ (dotted line).

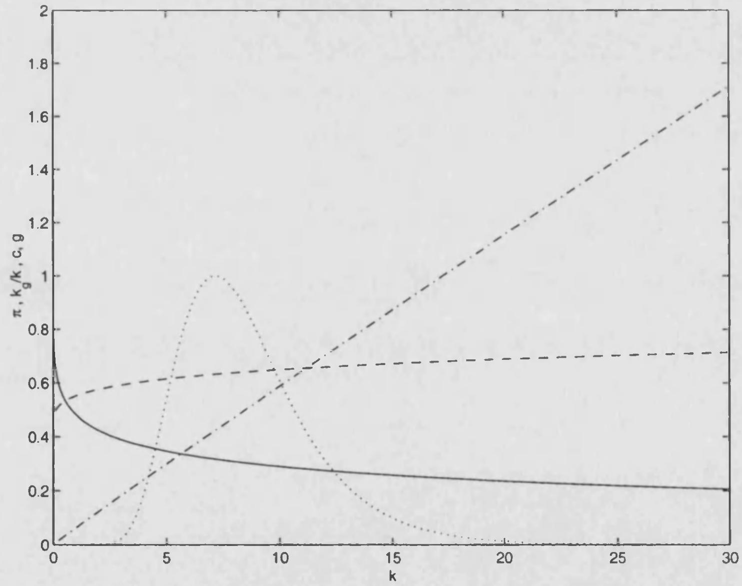

Figure 8.3: Plots of the optimal π (solid line), k_g/k (dashed line), c (dash-dot line) and scaled g (dotted line) against total capital k for the numerical example with $R_g = 0.8$.

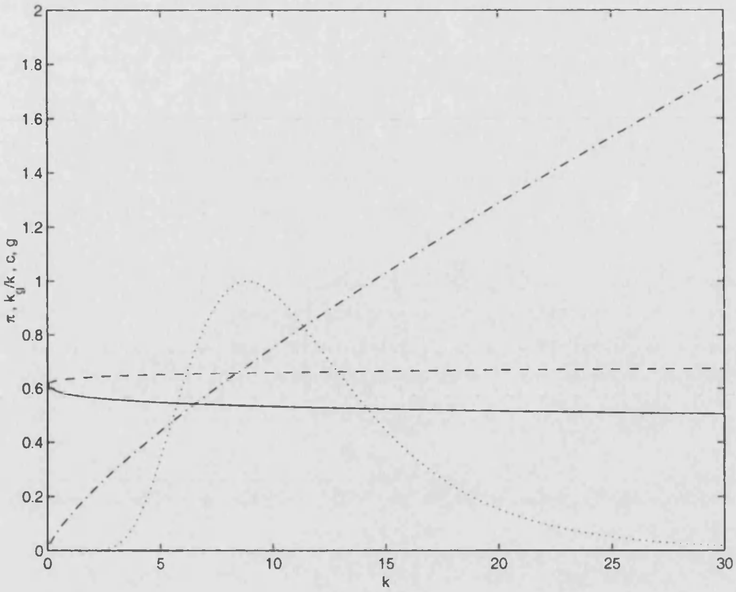

Figure 8.4: Plots of the optimal π (solid line), k_g/k (dashed line), c (dash-dot line) and scaled g (dotted line) against total capital k for the numerical example with $R_g = 1.6$.

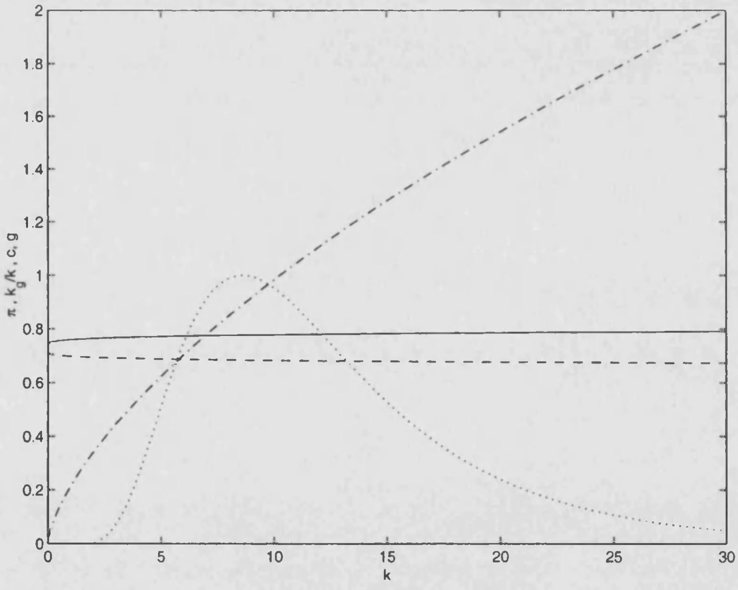

Figure 8.5: Plots of the optimal π (solid line), k_g/k (dashed line), c (dash-dot line) and scaled g (dotted line) against total capital k for the numerical example with $R_g = 3.2$.

economy one of the ways that a sufficiently high felicity can be obtained is through the population having greater leisure time. For the same reason the proportion of capital invested in government services is higher and the level of consumption in proportion to total capital is higher at lower capital levels. The stationary distribution of k is also wider for the more risk averse government, as for low capital levels growth rates are lower and at high capital levels a more risk averse government will not burn through its capital so rapidly in order to gain a higher felicity.

8.2.2 The explicit example

In this section we will give graphs to show typical optimal trajectories for the explicit example of Chapter 6 and we will compare these with the numerical example of the previous section. The explicit example is constructed so that k_g/k is a decreasing function of k so we may get similar results to the numerical example above by trying the case when $R_g = 3.2$. This turns out to be the case. We pick constants $A = 0.25$ and $B = 0.5$ (so that $S = 2.26$) along with $\varphi_0 = 1.2$, $a = 1$ and $\varepsilon = 0.02$ to specify $\varphi(k)$. All the relations of Lemma 1 and equations (6.50) and (6.51) are satisfied as required. Figure 8.6 shows the resulting value function $V(k)$ with the numerical example from the previous section for comparison, and Figure 8.7 shows the optimal π, k_g/k and c against total capital k for the explicit example - note that π is constant by construction. We see that this explicit example is very similar to the numerical example of Figure 8.5. This means that the explicit example is indeed useful to study as it shares many of the characteristics of the numerical example derived from a more conventional model, while being more tractable for analytic purposes. It also validates our numerical methods in that we get very similar numbers from our numerical optimization procedure as we do from the explicit model.

If we now lower R_g the range of valid parameter values gets smaller. Picking $R_g = 2.0$, for $\gamma_g + Q$ to be positive we need S to be at least 1.79 and hence B to be at least 0.85. We choose $B = 0.9$. Equation (6.51) now restricts ε to being less than around 0.0155 - we choose $\varepsilon = 0.01$ (in the previous example we could have chosen ε as high as 0.12 without violating equation (6.51)). We pick $a = 1$ and $\varphi_0 = 1.2$ as before, and then choose $A = 0.07$ so that the optimal choice of c is on a similar scale to that of previous examples. Figure 8.8 shows the optimal π, k_g/k and c for this example. For the model as specified in Section 8.1 it becomes impossible to find an explicit solution with parameters satisfying the requirements of Lemma 1 for choices of R_g below around 1.8.

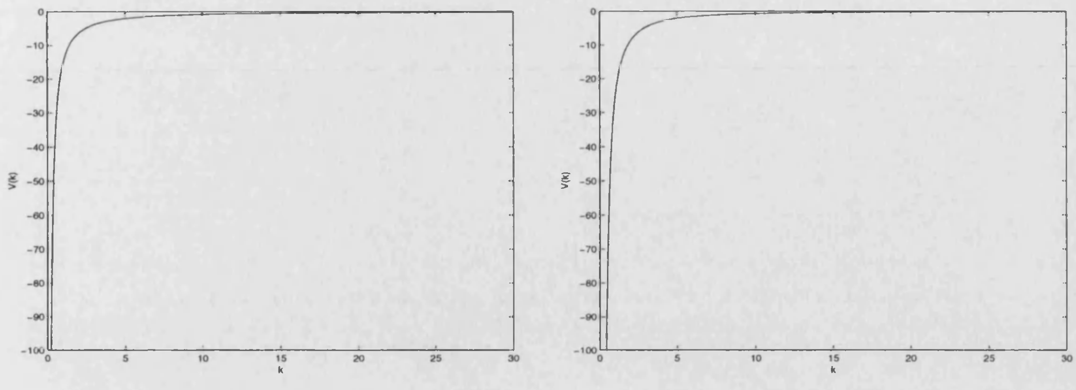

Figure 8.6: The government's value function $V(k)$ for the explicit (left hand plot) and numerical examples when $R_g = 3.2$.

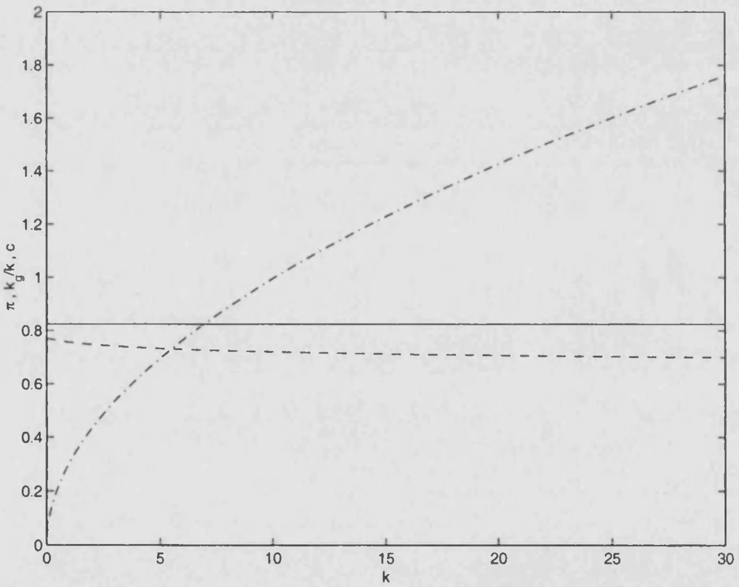

Figure 8.7: Plots of the optimal π (solid line), k_g/k (dashed line) and c (dash-dot line) against total capital k for the explicit example with $R_g = 3.2$.

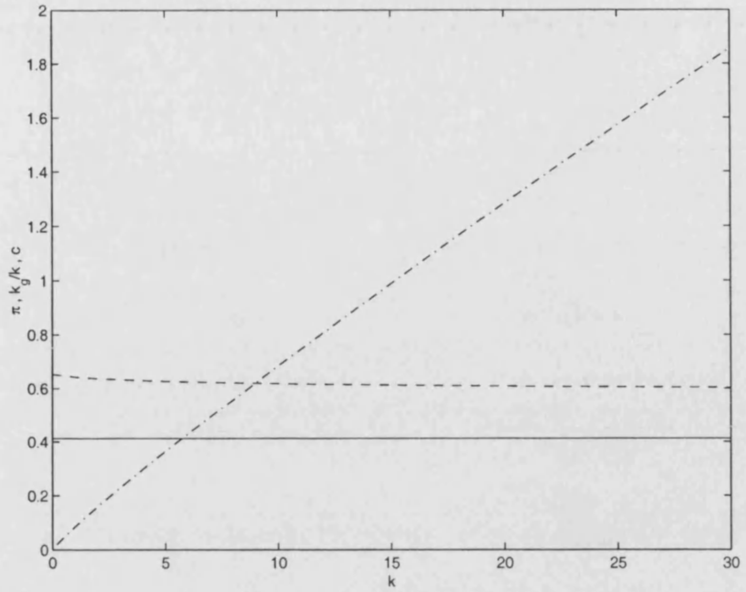

Figure 8.8: Plots of the optimal π (solid line), k_g/k (dashed line) and c (dash-dot line) against total capital k for the explicit example with $R_g = 2.0$.

8.3 Taxation policy

What would happen if the private sector was running the country instead of the government? The results of this would be given by solving the government's problem as in Section 5.2 but with all the government specific quantities and functions (ρ_g, U etc.) replaced by their corresponding private sector specific counterparts (ρ_p, u and so on). Figure 8.9 shows the resulting optimal c, k_g/k and π computed numerically for the private sector with $R_p = 0.8$, 1.6 and 3.2 (and ν_p, ω_p and κ_p then defined as in Section 8.1). The same general trends are visible here as in the government's choices as the coefficient of relative risk aversion increases. However, for a given level of risk-aversion, the consumption levels and proportion of time spent working are higher and the proportion of capital invested publicly is lower. The population is also poorer, as shown by a narrower stationary distribution for the capital located closer to zero.

These are the values which the private sector would choose *if it was allowed to allocate its capital as it pleased.* However the government will implement taxation policy to ensure this does not happen; the other plot in Figure 8.9 shows the values a government with $R_g = 1.6$ would like for consumption etc. The choice of taxation policy, as given in Theorem 2, will ensure that the private sector, whatever their preferences, ends up

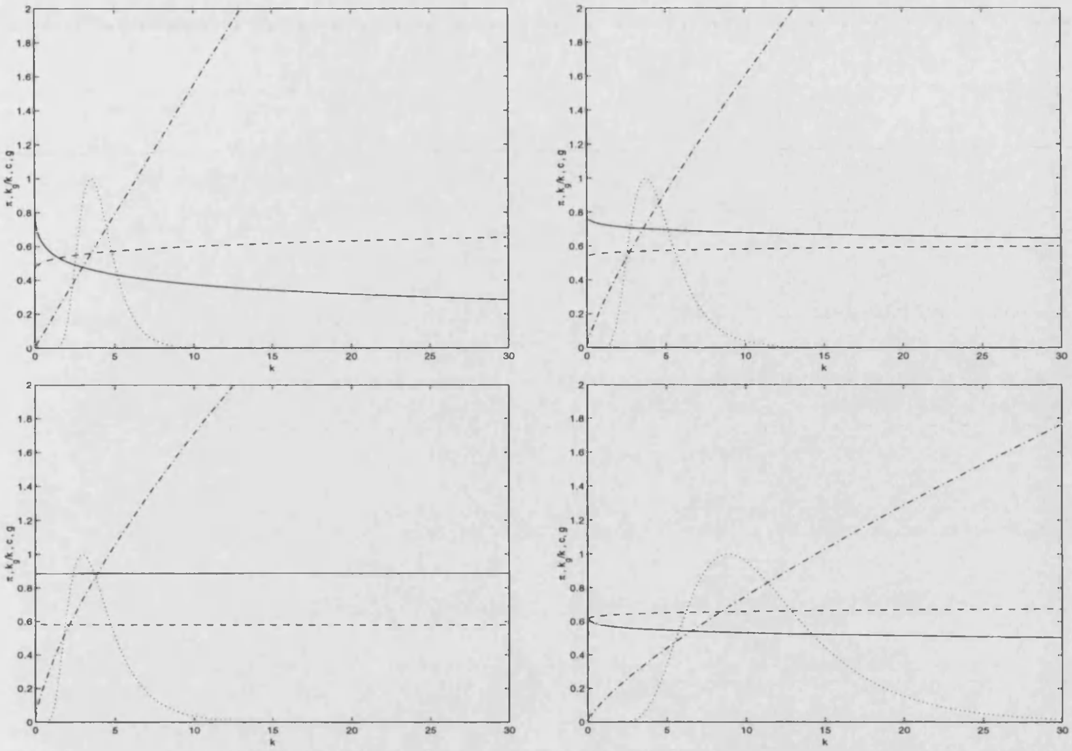

Figure 8.9: Plots of the optimal π (solid line), k_g/k (dashed line), c (dash-dot line) and scaled g (dotted line) against total capital k for the numerical example with the private sector having total control. $R_p = 0.8$ (top-left), $R_p = 1.6$ (top-right) and $R_p = 3.2$ (bottom-left). The bottom-right plot is Figure 8.4 again for comparison - the government's optimal policy when $R_g = 1.6$.

following the government's optimal policy rather than the one that they would prefer. We will see the taxation policy choices resulting from this in the next section.

8.3.1 Numerical solutions to the taxation question

We'll stick with the second example of Section 8.2.1, with $R_g = 1.6$ so that the government is fairly risk averse. The government's choice of taxes will depend on the private sector's preferences which we take to be as defined in Section 8.1 along with different choices of R_p which we will choose as follows.

Firstly we'll take $R_p = 0.8$ so that the private sector is less risk averse. We get $\beta_c\beta_w = 0.857$ and so we'll pick $\beta_c = 1$, $\beta_w = 0.857$, so that there is no consumption tax and the tax on wage income is around 14%. Figure 8.10 shows the capital income

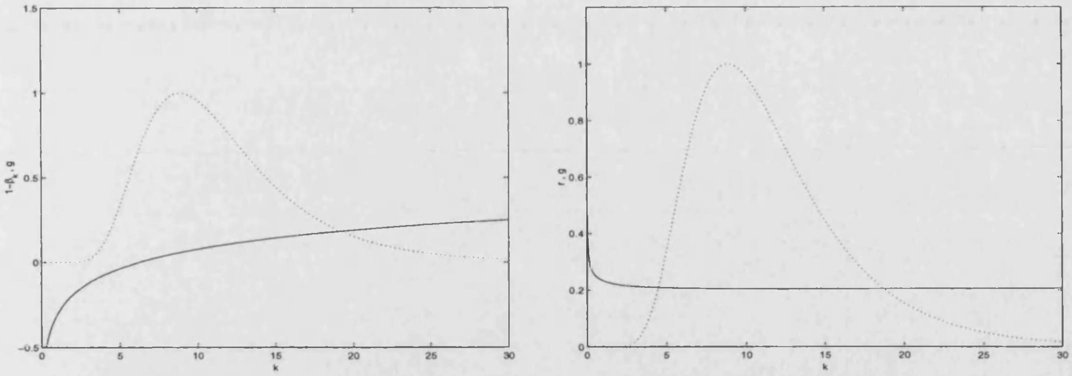

Figure 8.10: Plots of the optimal capital income tax $1-\beta_k$ (left hand plot) and interest rate r (right hand plot) and scaled g, the stationary distribution of k (dotted line in both plots), against total capital k when $R_g = 1.6$, $R_p = 0.8$.

tax rate $1 - \beta_k$ and interest rate r as total capital varies. Note that for small values of capital the income tax is effectively a subsidy. Looking at Figure 8.9 this corresponds to the region where the private sector wishes to consume *less* than the government and hence has to be given more income to persuade it to raise its consumption rate. The tax then rises as the capital in the economy grows and the capital flow to the private sector has to be reduced to lower its consumption rate. The interest-rate levels do the opposite - initially they are large, reflecting the similarly high returns available on capital, and then they fall as the economy becomes richer.

We will now take $R_p = 1.6$ so that the government and private sector share the same level of risk aversion (although they have different felicity functions still - the private sector's is weighted more towards consumption for example). We find that $\beta_c\beta_w = 0.286$ and so take $\beta_c = 0.65$, $\beta_w = 0.44$ corresponding to a wage tax of around 56% and a consumption tax[1] of around 54%. Figure 8.11 shows the capital tax rates and interest rates again. This time the capital tax rate does not become a subsidy for small k as the private sector always wishes to consume more than the government wants it to. The interest rate curve is steeper than before - the private sector needs a larger incentive to invest capital rather than consume it at low capital levels.

One final example. What if the private sector is more risk averse than the government? We take the government's coefficient of relative risk aversion as $R_g = 1.6$ again, but change the private sector's coefficient of risk aversion to $R_p = 3.2$. This gives $\beta_c\beta_w =$

[1] The consumption tax is given by $\beta_c^{-1} - 1$ as $\beta_c c$ is the amount actually consumed if the private sector tries to consume c, whereas conventional consumption taxes (e.g. VAT) add a charge onto the amount that the private sector actually consumes.

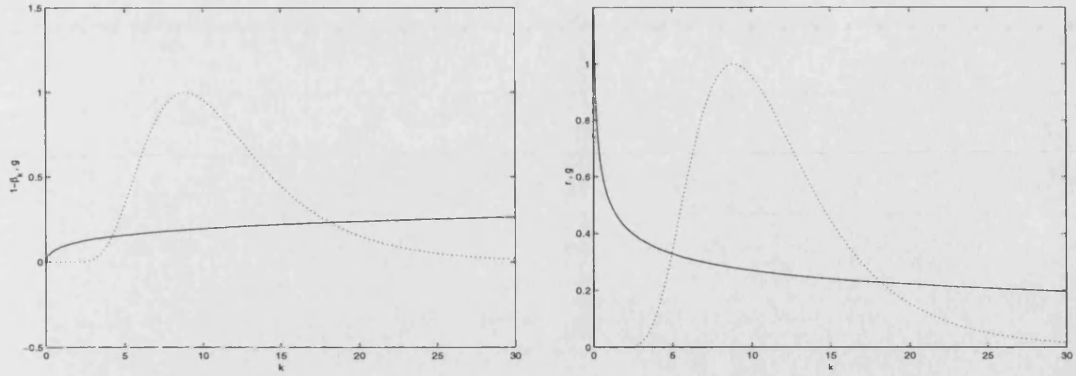

Figure 8.11: Plots of the optimal capital income tax $1-\beta_k$ (left hand plot) and interest rate r (right hand plot) and scaled g, the stationary distribution of k (dotted line in both plots), against total capital k when $R_g = 1.6$, $R_p = 1.6$.

0.0779 and we'll pick $\beta_c = 0.4$ (a 150% consumption tax!) and $\beta_w = 0.195$. Figure 8.12 shows the resulting tax rate $1 - \beta_k$ and interest rate. In this case the tax rate on capital income starts off high and falls as the capital in the economy increases, but the interest-rate curve is still of the same form as previously.

If we calculate bond prices for this example using the procedure of Section 7.3 then we get yields as shown in Figure 8.13. At high levels of capital the yield curve is a conventional increasing curve and at low levels of capital the yield curve is inverted. Figure 8.14 shows a selection of equally spaced (with respect to k) curves taken from the surface in Figure 8.13. We see that between the conventional and the inverted yield curves there is a (slightly!) humped yield curve. If we multiply all the covariance parameters by 4 in the specification of Section 8.1 (so that $v_{LL} = 0.04$, $v_{00} = 0.08$, $v_{0L} = 0.02$) and recalculate everything then the resulting yield curves are shown in Figure 8.15. The structure is much more visible in this diagram with three humped yield curves between the decreasing and increasing yield curves. These humped types of curve are occasionally spotted in the bond markets as the yield curve makes a transition from increasing to inverted or vice-versa, so it is good that they arise also in our model. One interesting experiment would be to try and choose the parameters of our economic model to fit yield data from the markets. We leave this for future research.

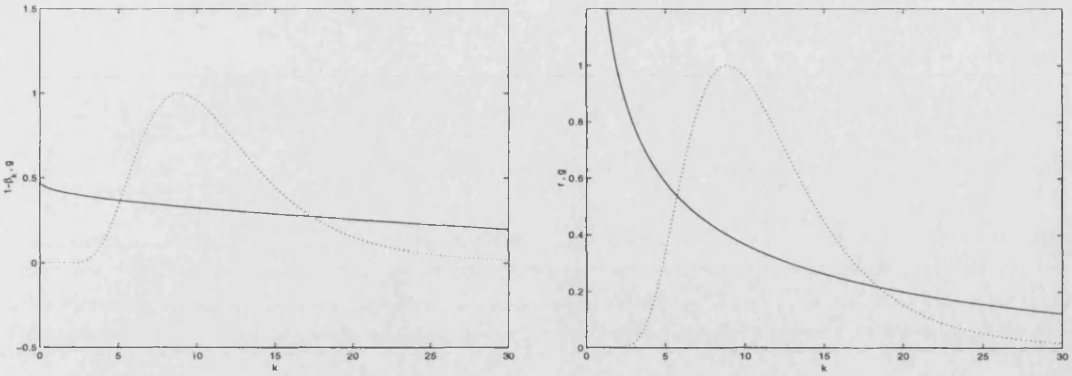

Figure 8.12: Plots of the optimal capital income tax $1-\beta_k$ (left hand plot) and interest rate r (right hand plot) and scaled g, the stationary distribution of k (dotted line in both plots), against total capital k when $R_g = 1.6$, $R_p = 3.2$.

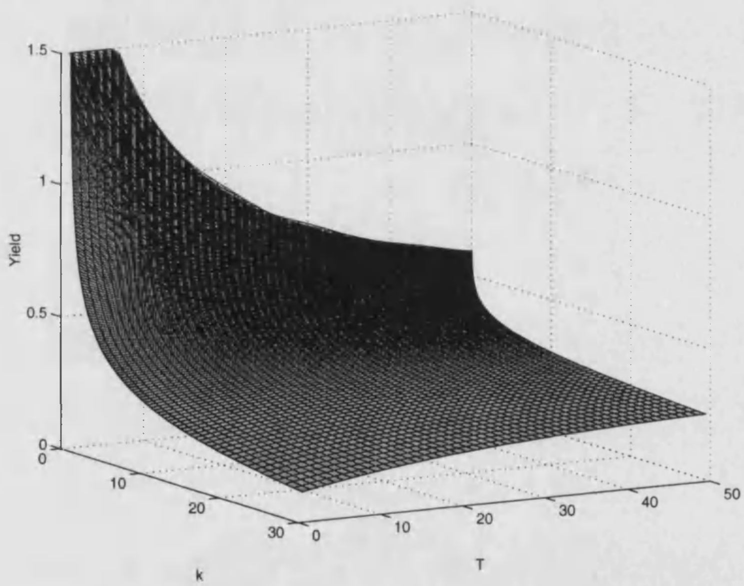

Figure 8.13: Yield at time-0 of a zero-coupon bond of maturity T, against T and time-0 capital level k. $R_g = 1.6$, $R_p = 3.2$.

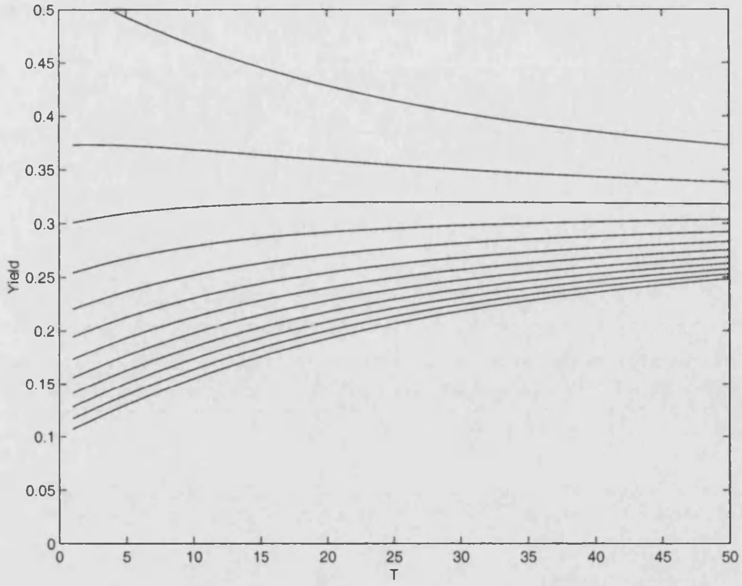

Figure 8.14: Yield at time-0 of a zero-coupon bond of maturity T, against T. Each line is for a different initial k with the line corresponding to the smallest k at the top of the picture. $R_g = 1.6$, $R_p = 3.2$.

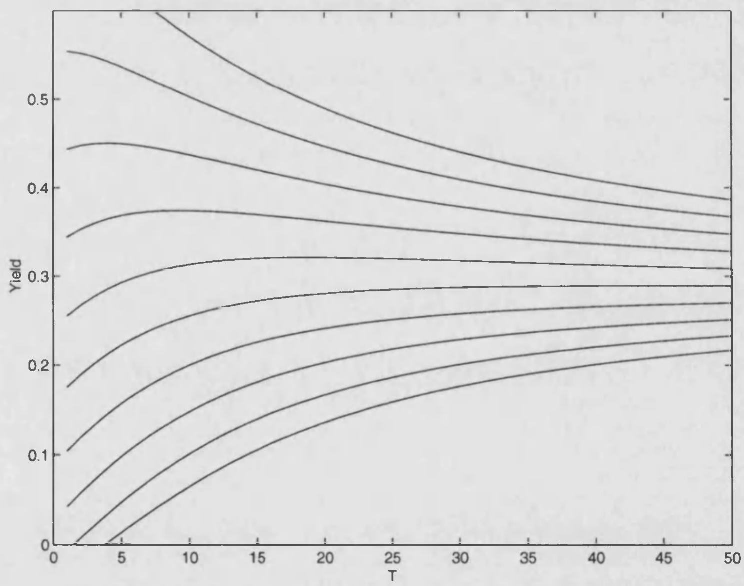

Figure 8.15: Yield at time-0 of a zero-coupon bond of maturity T, against T. Each line is for a different initial x with the line corresponding to the smallest x at the top of the picture. $R_g = 1.6$, $R_p = 3.2$. $v_{LL} = 0.04$, $v_{00} = 0.08$, $v_{0L} = 0.02$.

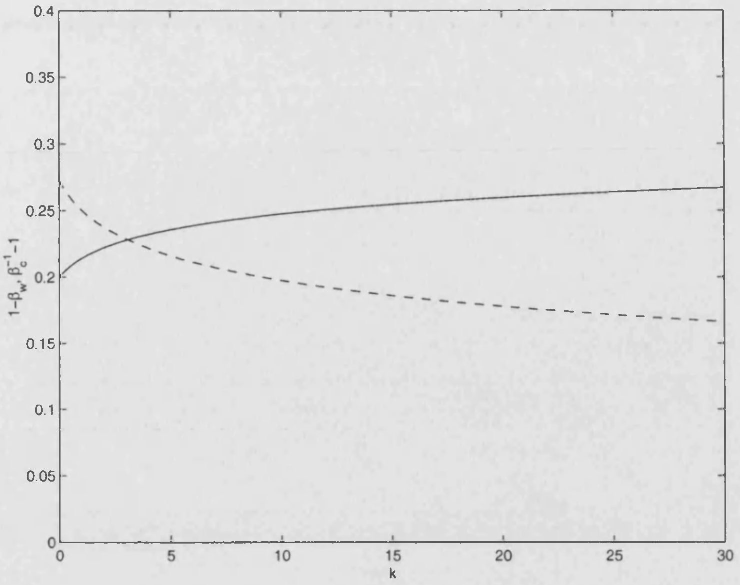

Figure 8.16: Plot of the wage tax rate (solid line) and the consumption tax rate (dashed line) against total capital k for the explicit example using Approach 1. $R_g = 3.2$, $R_p = 2.0$.

8.3.2 The explicit example

We now look at taxation for the explicit example with values as in Section 8.2.2 so that $R_g = 3.2$ and for the private sector's preferences we will pick $R_p = 2.0$ (and hence $\nu_p = 0.7$, $\omega_p = 0.3$ and $\kappa_p = 0.1$). We then find from equation (6.53) that $\beta_c \beta_w = 0.6286$. The taxation policies arising from the two approaches given in Section 6.2 are as follows.

Approach 1 : The consumption and wage taxes are given by

$$\beta_c = \beta_c(0)(1 + ak)^{\alpha\varepsilon},$$
$$\beta_w = \beta_w(0)(1 + ak)^{-\alpha\varepsilon},$$

where $\alpha \equiv 1 + \nu_p - \omega_p(1 + \nu_g)/\omega_g = 1.2727$ is a positive constant as required. We choose $\beta_w(0) = 0.8$, so that $\beta_c(0) = 0.7857$; the resulting wage and consumption tax rates $1 - \beta_w$ and $\beta_c^{-1} - 1$ are shown in Figure 8.16. These are very reasonable looking rates - the tax on consumption falls as more capital is available in the economy for consumption whereas the wage tax is larger for a richer economy.

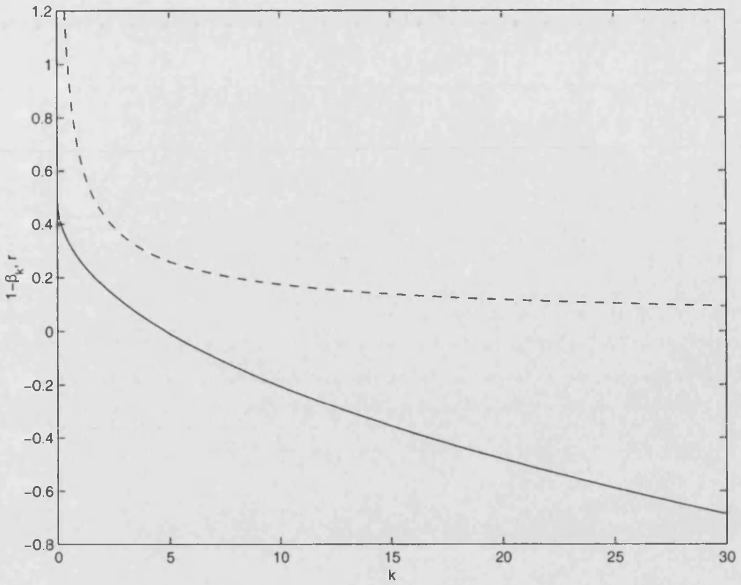

Figure 8.17: Plot of the capital income tax rate (solid line) and the interest rate r (dashed line) against total capital k for the explicit example using Approach 1. $R_g = 3.2$, $R_p = 2.0$.

The capital income tax rate $1 - \beta_k$, from equation (6.59), and interest rate r, from equation (6.60) are shown in Figure 8.17. The capital income tax rate is in fact a subsidy for all but very low values of k. This reflects the differences in risk aversion between the government and private sector - at higher capital levels the less risk averse private sector would prefer to consume more than the government would wish it to, and consequently invest less and the government thus has to subsidize the private sector's income from capital in order to achieve its own objectives.

Approach 2 : Here we choose both β_c and β_w constant for all k. We will take $\beta_c = 1$ and $\beta_w = 0.6286$ so that there is no consumption tax and the wage tax is just over 37%. Calculating the capital income tax rate and interest rate as described in Section 6.2 gives the functions shown in Figure 8.18. The values obtained for this explicit example are very reasonable - the capital income tax rate is a subsidy for small k to encourage investment, and then becomes an increasing conventional tax rate for higher k. Figure 8.19 shows $1 - \beta_k$ and r again, for the numerical example as described in Section 8.3.1 and with the same assumptions about R_g, R_p, β_c and β_w as for the explicit example. The capital income tax rate is very similar for the numerical example, and here we can see that the tax rate will be a subsidy for a fairly small proportion of time in the long run by looking at the stationary distribution of k. The interest rates in both the

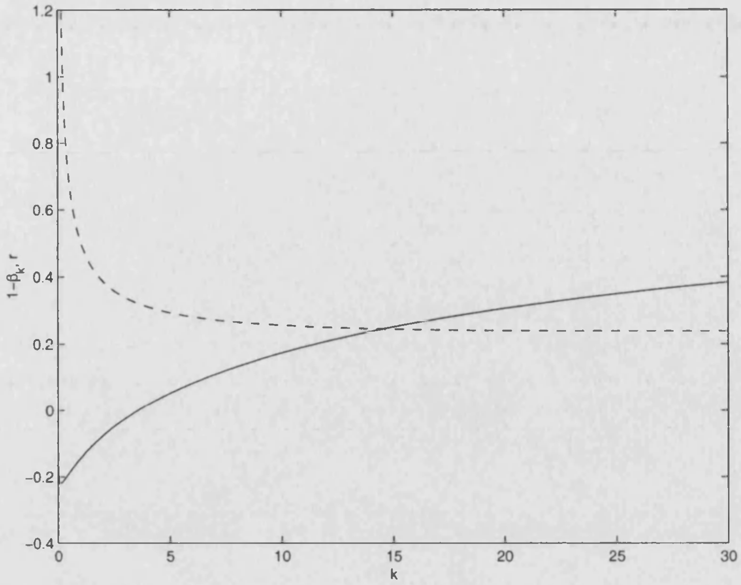

Figure 8.18: Plot of the capital income tax rate (solid line) and the interest rate r (dashed line) against total capital k for the explicit example with constant wage and consumption tax. $R_g = 3.2$, $R_p = 2.0$.

explicit and numerical example are also very similar again showing that this explicit example is worth investigating.

8.4 Summary, conclusions and suggestions for future research

We have introduced stochastic terms into the model of Arrow and Kurz (1970) and also added a factor to account for the proportion of work devoted to labour, as in the original model of Ramsey (1928). With these modifications we have then found optimality conditions for the government's central-planning problem, where the government has complete control over the economy and wishes to maximize a utility functional.

In the more realistic situation where the government's control of the economy is through taxation and debt policy, we have found sufficient conditions on the tax rates so that the private sector, in maximizing its own utility functional subject to these tax rates, chooses to follow the government's desired optimal trajectory. The resulting tax and interest rates are functions of per-capita capital, giving closed-loop control. Being

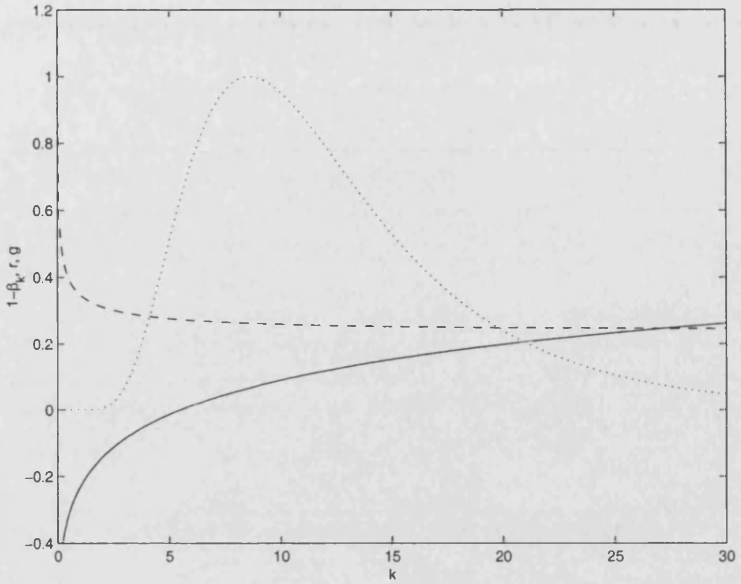

Figure 8.19: Plot of the capital income tax rate (solid line) and the interest rate r (dashed line) against total capital k for the numerical example with constant wage and consumption tax. Also shown is a scaled version of g, the stationary distribution of k (dotted line). $R_g = 3.2$, $R_p = 2.0$.

purely deterministic, the original model of Arrow and Kurz was unable to distinguish between open-loop and closed-loop control of the economy; by allowing stochastic terms into the model, we have resolved this difficulty of interpretation. We have also found the government and private sector's state-price density processes and the corresponding consumption rates of interest. We can thus price any asset - we considered a zero-coupon bond and gave a PDE obeyed by the price of such a bond.

We have shown how to find explicit solutions to the government's problem by considering the *inverse* problem, where the desired solution is chosen in advance and then the original problem that would produce such a solution is constructed. We have given an example of such a solution and shown that the consideration of such explicit solutions can lead to novel interest-rate models. These interest-rate models arise naturally from our model, in contrast to conventional models of the short rate which are generally chosen for convenience of calculation with little or no economic justification.

We described procedures for calculating the solution to the government's problem numerically for both deterministic and stochastic models. In the stochastic case the methods used allowed us to easily also calculate stationary distributions and bond prices as well as tax and interest rates. We gave illustrative example solutions to the government's problem exhibiting reasonable behaviour and showed also that the explicit example solution exhibited similar behaviour. We similarly gave examples of tax rates and saw that in many cases these were sensible (i.e. between 0 and 1 or small subsidies). Numerical computation of yield curves revealed that the interest-rate models arising from this stochastic two-sector model displayed behaviour found in the markets, with increasing, decreasing and humped yield curves all present depending on the level of capital in the economy.

Suggestions for future research

In a sense we have achieved very little here - we have built a model and shown it gives sensible answers but we have not actually used it for anything meaningful!

On the economic side, the model forms a good basis for investigations into a wide range of issues. It's a growth model, so it can be used for a studying growth. It's a two-sector model so issues of resource allocation can be considered - we have modelled public and private capital but the model could equally well be used to study, for example, physical and human capital. As a stochastic model, questions not even accessible to deterministic models can be asked. For example: how does the level of volatility of production affect the balance between consumption and investment?

The obvious application of the model is in the study of fiscal policy. The stochastic nature of the model has removed the difficulties of interpretation present in the original deterministic model of Arrow and Kurz and the study of how taxation rates vary with capital is interesting in itself. More complex questions could be asked. How does the private sector's level of risk aversion affect VAT levels? In what circumstances is government subsidy of investment in industry necessary? In a more volatile economy are income taxes generally higher or lower? What about other taxation schemes? The assumptions we have made about the instruments of taxation available could be changed leading to results analogous to those of Theorem 2. In particular, it would be useful if we could find a taxation scheme where we could say something more useful about the levels of debt. Perhaps introducing the concept of money into the model would help with this, although again the study of the role of money in an economy is interesting in its own right. Modelling money is notoriously tricky but may be possible within the stochastic framework of the model; in a deterministic model, with capital growing risklessly, money is a dominated asset.

On the financial side, we have a new selection of interest-rate models that could be interesting to study. How are spot rates affected by the values of underlying economic variables? How do bond prices from the model compare with those from other short-rate models? Can we calibrate the model to the market yield curves?

All these issues could be investigated either numerically, using the techniques of Chapter 7 or analytically using the examples, or general approach, given in Chapter 6. Either way this model seems to have a huge array of potential uses.

Appendix A

Appendix to Part I

A.1 The Abate-Whitt Euler method

For numerical inversion of Laplace transforms we employ the Euler method of Abate and Whitt (1995). This uses a version of the Poisson summation formula to give an approximation with an explicit error bound. Suppose we have $\hat{f}(\lambda)$, the Laplace transform of f, and we wish to find the value of f at a specific point $f(t)$. Then the Abate-Whitt inversion rule says that

$$f(t) \approx \frac{e^{A/2}}{2t} \sum_{k=-\infty}^{\infty} (-1)^k \operatorname{Re}(\hat{f}) \left(\frac{A + 2k\pi i}{2t} \right). \tag{A.1}$$

Note that the choice of line in the complex plane along which we sum is *determined by the particular* t for which we wish to find $f(t)$. The error is given by

$$e_d = \sum_{k=1}^{\infty} e^{-kA} f((2k+1)t)$$

so that the size of A controls the error made in the approximation. If $f(t)$ is bounded, e.g. if $0 \leq f \leq B$ for some fixed B then the error is bounded by

$$|e_d| \leq \frac{Be^{-A}}{1 - e^{-A}} \approx Be^{-A}. \tag{A.2}$$

For example, if f is the price of a put option it will be bounded by the strike price K; if f is the price of a Parisian option it will be bounded by the price of the equivalent European option. Abate and Whitt recommend $A = 18.4$ which gives a maximum error of $B \times 10^{-8}$; we will use $A = 13.8$ giving an error of at most $B \times 10^{-6}$.

Euler summation can be used to accelerate the convergence of this alternating series. Let $s_n(t)$ be the approximation of $f(t)$ in (A.1) with the infinite series truncated to $2n + 1$ terms, i.e.

$$s_n(t) = \frac{e^{A/2}}{2t} \sum_{k=-n}^{n} (-1)^k \operatorname{Re}(\hat{f}) \left(\frac{A + 2k\pi i}{2t} \right).$$

Then the Euler sum approximation to $f(t)$ is given by

$$f(t) \approx \sum_{k=0}^{m} \binom{m}{k} 2^{-m} s_{n+k}(t).$$

The parameters m and n are chosen to give the desired level of accuracy. For example, Abate and Whitt recommend $m = 11$, $n = 15$. Note that if the function we are inverting is symmetric in the sense that $\operatorname{Re}(\hat{f})(a + ib) = \operatorname{Re}(\hat{f})(a - ib)$ we need only compute $1 + n + m$ terms of (A.1) rather than $1 + 2(n + m)$, effectively halving the computation time.

A.2 Curran's method for calculating Asian option prices

Following Curran (1992) we consider the payoff of an Asian call option conditioned on the further random variable $G_T \equiv \int_0^T W_t \, dt$. This is highly correlated with A_T and is normally distributed, $G_T \sim N(0, \frac{T^3}{3})$.

$$\begin{aligned} c(k, T) &= \mathbb{E}\left[(A_T - k)^+\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[(A_T - k)^+ \,|G_T = z\right]\right] \\ &= \int_{-\infty}^{\infty} \mathbb{E}\left[(A_T - k)^+ \,|G_T = z\right] \mathbb{P}(G_T \in dz). \end{aligned} \tag{A.3}$$

Now approximate A_T conditional on $G_T = z$ by a log-normal distribution

$$\log A_T \sim N(\mu_z, \sigma_z^2)$$

where μ_z, σ_z are chosen to match the first two moments of A_T, i.e.

$$\begin{aligned}
\mathbb{E}[A_T|G_T = z] &= e^{\mu_z + \frac{1}{2}\sigma_z^2} \\
\mathrm{Var}\,[A_T|G_T = z] &= e^{2\mu_z + \sigma_z^2}\left(e^{\sigma_z^2} - 1\right) \\
&= \mathbb{E}[A_T|G_T = z]^2\left(e^{\sigma_z^2} - 1\right).
\end{aligned}$$

Given the conditional moments the above equations can be easily solved for μ_z and σ_z and (A.3) can then be found by numerical integration of a series of standard Black-Scholes terms of the form

$$e^{\frac{1}{2}\sigma_z^2 + \mu_z}\Phi\left(\frac{\mu_z + \sigma_z^2 - \log k}{\sigma_z}\right) - k\Phi\left(\frac{\mu_z - \log k}{\sigma_z}\right),$$

where $\Phi(x) \equiv \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy$ is the standard cumulative normal distribution function.

To evaluate the conditional moments we make use of the following results from Rogers and Shi (1995):

$$\begin{aligned}
\mathbb{E}[W_t|G_T = z] &= \tfrac{3t(2T-t)}{2T^3}\,z &&\equiv m_t z \\
\mathrm{Cov}\,[W_s, W_t|G_T = z] &= s \wedge t - \tfrac{3st(2T-s)(2T-t)}{4T^3} &&\equiv v_{s,t}
\end{aligned}$$

and hence also

$$\mathrm{Var}\,[W_t|G_T = z] \equiv v_{t,t} = t - m_t^2 v$$

where $v \equiv \mathrm{Var}(G_T) = \frac{T^3}{3}$. Then

$$\begin{aligned}
\mathbb{E}[A_T|G_T = z] &= \mathbb{E}\left[\int_0^T e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t}dt\,\middle|\,G_T = z\right] \\
&= \int_0^T e^{(r-\frac{1}{2}\sigma^2)t}\mathbb{E}\left[e^{\sigma W_t}|G_T = z\right]dt \\
&= \int_0^T e^{(r-\frac{1}{2}\sigma^2)t}e^{\sigma m_t z + \frac{1}{2}\sigma^2(t - m_t^2 v)}dt \\
&= \int_0^T e^{\sigma m_t z - \frac{1}{2}\sigma^2 m_t^2 v + rt}dt
\end{aligned}$$

and similarly

$$\mathrm{Var}\,[A_T|G_T = z] = \int_0^T\int_0^T e^{\sigma z(m_s + m_t) - \frac{1}{2}\sigma^2 v(m_s^2 + m_t^2) + r(s+t)}\left(e^{\sigma^2 v_{s,t}} - 1\right)dt\,ds.$$

Evaluation of (A.3) is thus effectively a triple numerical integration. However this can still be calculated in a reasonable amount of time - around a minute in Matlab on a 686 PC. If a small loss of accuracy is acceptable then this time can be reduced to a few seconds.

A.3 Some integrals

This section contains details of how to perform various integrals. Of particular interest is the integration of $K_{\lambda,D}(y)$ which was done incorrectly by Chesney, Jeanblanc-Picqué, and Yor (1997) in both the $y > 0$ and $y < 0$ cases. Throughout this section we assume that λ is in the right complex half-plane and the principal branch of the logarithm is used to define $\theta \equiv \sqrt{2\lambda}$.

Integral 1 The first hitting time of level y by a Brownian motion starting from 0 has density

$$n_t(y) \equiv \frac{|y|}{\sqrt{2\pi t^3}} e^{-\frac{1}{2t}y^2}.$$

We wish to evaluate

$$\begin{aligned}
\int_0^D n_t(y)e^{-\lambda t}\,dt &= \int_0^D \frac{|y|}{\sqrt{2\pi t^3}} e^{-\frac{1}{2}(\frac{y^2}{t}+\theta^2 t)}\,dt \\
&= \frac{1}{2}e^{-\theta|y|}\int_0^D \left(\frac{|y|}{\sqrt{2\pi t^3}} + \frac{\theta}{\sqrt{2\pi t}}\right) e^{-\frac{1}{2}(\frac{y^2}{t}-2\theta|y|+\theta^2 t)}\,dt \\
&\quad + \frac{1}{2}e^{\theta|y|}\int_0^D \left(\frac{|y|}{\sqrt{2\pi t^3}} - \frac{\theta}{\sqrt{2\pi t}}\right) e^{-\frac{1}{2}(\frac{y^2}{t}+2\theta|y|+\theta^2 t)}\,dt \\
&= e^{-\theta|y|}\int_{\frac{|y|}{\sqrt{D}}-\theta\sqrt{D}}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx + e^{\theta|y|}\int_{\frac{|y|}{\sqrt{D}}+\theta\sqrt{D}}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx \\
&= e^{-\theta|y|}\Phi\left(\theta\sqrt{D} - \frac{|y|}{\sqrt{D}}\right) + e^{\theta|y|}\Phi\left(-\theta\sqrt{D} - \frac{|y|}{\sqrt{D}}\right) \qquad \text{(A.4)}
\end{aligned}$$

using the substitutions $x = \frac{|y|}{\sqrt{t}} \mp \theta\sqrt{t}$ and the identity

$$1 - \Phi(x) = \Phi(-x). \qquad \text{(A.5)}$$

It follows immediately that

$$\int_0^\infty n_t(y)e^{-\lambda t}\,dt = e^{-\theta|y|}.$$

Integral 2 For any complex number z and for real numbers $0 \le x_1 < x_2 \le \infty$ we wish to find

$$\begin{aligned}\int_{x_1}^{x_2} xe^{-\frac{1}{2}x^2+zx}\,dx &= \int_{x_1}^{x_2} (x-z)e^{-\frac{1}{2}x^2+zx}\,dx + z\int_{x_1}^{x_2} e^{-\frac{1}{2}(x-z)^2+\frac{1}{2}z^2}\,dx \\ &= -\left[e^{-\frac{1}{2}x^2+zx}\right]_{x_1}^{x_2} + z\sqrt{2\pi}e^{\frac{1}{2}z^2}\int_{x_1-z}^{x_2-z}\frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}\,du \\ &= e^{-\frac{1}{2}x_1^2+zx_1} - e^{-\frac{1}{2}x_2^2+zx_2} + z\sqrt{2\pi}e^{\frac{1}{2}z^2}(\Phi(x_2-z)-\Phi(x_1-z)),\end{aligned}$$

using the substitution $u = x - z$. We can now put $x_1 = 0$, $x_2 = \infty$ and use (A.5) to confirm the calculation in CJY that

$$\Psi(z) \equiv \int_0^\infty xe^{-\frac{1}{2}x^2+zx}\,dx = 1 + z\sqrt{2\pi}e^{\frac{1}{2}z^2}\Phi(z). \tag{A.6}$$

Integral 3 We wish to compute the function

$$K_{\lambda,D}(y) \equiv \int_0^\infty xe^{-\frac{1}{2D}x^2-|y+x|\theta}\,dx$$

defined as in CJY. For $y > 0$, substituting $x = u\sqrt{D}$ and using Integral 2, we find that

$$\begin{aligned}K_{\lambda,D}(y) &= \int_0^\infty xe^{-\frac{1}{2D}x^2-\theta x}e^{-\theta y}\,dx \\ &= De^{-\theta y}\int_0^\infty ue^{-\frac{1}{2}u^2-\theta\sqrt{D}u}\,du \\ &= De^{-\theta y}\Psi(-\theta\sqrt{D}).\end{aligned}$$

Similarly if $y < 0$ then

$$\begin{aligned}K_{\lambda,D}(y) &= \int_0^{-y} xe^{-\frac{1}{2D}x^2+\theta x}e^{\theta y}\,dx + \int_{-y}^\infty xe^{-\frac{1}{2D}x^2-\theta x}e^{-\theta y}\,dx \\ &= De^{\theta y}\int_0^{-\frac{y}{\sqrt{D}}} ue^{-\frac{1}{2}u^2+\theta\sqrt{D}u}\,du + De^{-\theta y}\int_{-\frac{y}{\sqrt{D}}}^\infty ue^{-\frac{1}{2}u^2-\theta\sqrt{D}u}\,du \\ &= De^{\theta y}\left[1 - e^{-\frac{y^2}{2D}-\theta y} + \theta\sqrt{2\pi D}e^{\lambda D}(\Phi(-\frac{y}{\sqrt{D}}-\theta\sqrt{D}) - \Phi(-\theta\sqrt{D}))\right] \\ &\quad + De^{-\theta y}\left[e^{-\frac{y^2}{2D}+\theta y} - \theta\sqrt{2\pi D}e^{\lambda D}(1-\Phi(-\frac{y}{\sqrt{D}}+\theta\sqrt{D}))\right] \\ &= De^{\theta y}\Psi(-\theta\sqrt{D}) \\ &\quad - D\theta\sqrt{2\pi D}e^{\lambda D}\left[e^{-\theta y}\Phi(-\theta\sqrt{D}+\frac{y}{\sqrt{D}}) - e^{\theta y}\Phi(-\theta\sqrt{D}-\frac{y}{\sqrt{D}})\right].\end{aligned}$$

Here we have made use of Integral 2 and equation (A.5). Combining the expressions above we have that

$$
\begin{aligned}
K_{\lambda,D}(y) &= De^{-\theta|y|}\Psi(-\theta\sqrt{D}) \\
&- D\theta\sqrt{2\pi D}e^{\lambda D}\left[e^{-\theta y}\Phi(-\theta\sqrt{D}+\frac{y}{\sqrt{D}}) - e^{\theta y}\Phi(-\theta\sqrt{D}-\frac{y}{\sqrt{D}})\right]\mathbb{1}_{\{y<0\}}.
\end{aligned} \tag{A.7}
$$

Integral 4 In order to find explicitly the Laplace transform of various Parisian option prices we will need to evaluate for real $\alpha > 0$ integrals of the form

$$
\begin{aligned}
I_{y_1}^{y_2}(\alpha,\beta,\theta) &\equiv \int_{y_1}^{y_2} e^{\theta y}\Phi(\alpha y+\beta)\,dy \\
&= \left[\frac{1}{\theta}e^{\theta y}\Phi(\alpha y+\beta)\right]_{y_1}^{y_2} - \frac{\alpha}{\theta}\int_{y_1}^{y_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\alpha y+\beta)^2+\theta y}\,dy \\
&= \left[\frac{1}{\theta}e^{\theta y}\Phi(\alpha y+\beta)\right]_{y_1}^{y_2} - \frac{\alpha}{\theta}\int_{y_1}^{y_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\alpha y+\beta-\frac{\theta}{\alpha})^2+\frac{1}{2}(\frac{\theta^2}{\alpha^2}-\frac{2\beta\theta}{\alpha})}\,dy \\
&= \frac{1}{\theta}\left[e^{\theta y_2}\Phi(\alpha y_2+\beta) - e^{\theta y_1}\Phi(\alpha y_1+\beta)\right] \\
&\quad - \frac{1}{\theta}e^{\frac{\theta^2}{2\alpha^2}-\frac{\beta\theta}{\alpha}}\int_{\alpha y_1+\beta-\frac{\theta}{\alpha}}^{\alpha y_2+\beta-\frac{\theta}{\alpha}}\frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}\,du \\
&= \frac{1}{\theta}\left[e^{\theta y_2}\Phi(\alpha y_2+\beta) - e^{\theta y_1}\Phi(\alpha y_1+\beta)\right] \\
&\quad - \frac{1}{\theta}e^{\frac{\theta^2}{2\alpha^2}-\frac{\beta\theta}{\alpha}}\left[\Phi(\alpha y_2+\beta-\frac{\theta}{\alpha}) - \Phi(\alpha y_1+\beta-\frac{\theta}{\alpha})\right].
\end{aligned}
$$

Here we have used the substitution $u = \alpha y + \beta - \theta/\alpha$. For convenience we will denote the integral we will most commonly wish to compute by

$$
\begin{aligned}
I(\beta,\theta) &\equiv I_{-\infty}^{0}(1/\sqrt{D},\beta,\theta) \\
&= \frac{1}{\theta}\left[\Phi(\beta) - e^{\frac{1}{2}\theta^2 D-\beta\theta\sqrt{D}}\Phi(\beta-\theta\sqrt{D})\right].
\end{aligned} \tag{A.8}
$$

A.4 An alternative derivation of $\hat{g}_b(\lambda, y)$

In this section we determine the Laplace transform $\hat{g}_b(\lambda, y)$ of the Parisian down-and-out density when $x = b$ using the information given in Chesney, Jeanblanc-Picqué, and Yor (1997). CJY give the Laplace transform in maturity $\hat{h}_b(\lambda, y)$ of the density function $h_b(T, y)$ for a Parisian down-and-in option. We have a bit of translation between notations to do here. CJY have $b_{CJY} \equiv \frac{1}{\sigma}\log(B/S_0) = x - b$. Also their density refers to a Brownian motion starting at 0 rather that $x = b$, hence we must also put $y_{CJY} = y - b$. With these changes and writing $\theta \equiv \sqrt{2\lambda}$ we from CJY have that

$$\hat{h}_b(\lambda, y) = \frac{K_{\lambda,D}(y-b)}{D\theta\Psi(\theta\sqrt{D})}$$

where Ψ and $K_{\lambda,D}$ are as defined in CJY and our Section A.3. To obtain the Laplace transform of the Parisian down-and-out we make use of the in-out parity relation (2.3). The Laplace transform in maturity of the Black-Scholes density

$$BS(T, y) = \frac{e^{-(y-b)^2/2T}}{\sqrt{2\pi T}}$$

is given by

$$\widehat{BS}(\lambda, y) = \frac{e^{-\theta|y-b|}}{\theta}$$

so (doing the $b = 0$ case for notational simplicity)

$$\begin{aligned}
\hat{g}_0(\lambda, y) &= \widehat{BS}(\lambda, y) - \hat{h}_0(\lambda, y) \\
&= \frac{e^{-\theta|y|}\Psi(\theta\sqrt{D}) - \frac{1}{D}K_{\lambda,D}(y)}{\theta\Psi(\theta\sqrt{D})} \\
&= \frac{e^{-\theta|y|}\left[\Psi(\theta\sqrt{D}) - \Psi(-\theta\sqrt{D})\right]}{\theta\Psi(\theta\sqrt{D})} \\
&\quad + \frac{\theta\sqrt{2\pi D}e^{\lambda D}\left[e^{-\theta y}\Phi\left(-\theta\sqrt{D} + \frac{y}{\sqrt{D}}\right) - e^{\theta y}\Phi\left(-\theta\sqrt{D} - \frac{y}{\sqrt{D}}\right)\right]}{\theta\Psi(\theta\sqrt{D})}\mathbb{1}_{\{y<0\}} \\
&= \frac{\sqrt{2\pi D}e^{\lambda D}}{\Psi(\theta\sqrt{D})}\left(e^{-\theta|y|} + \left[e^{-\theta y}\Phi\left(-\theta\sqrt{D} + \frac{y}{\sqrt{D}}\right) - e^{\theta y}\Phi\left(-\theta\sqrt{D} - \frac{y}{\sqrt{D}}\right)\right]\mathbb{1}_{\{y<0\}}\right) \\
&= \frac{\sqrt{2\pi D}e^{\lambda D}}{\Psi(\theta\sqrt{D})}\Big(e^{-\theta y}\mathbb{1}_{\{y>0\}} \\
&\quad + \left[e^{-\theta y}\Phi\left(-\theta\sqrt{D} + \frac{y}{\sqrt{D}}\right) + e^{\theta y}\Phi\left(\theta\sqrt{D} + \frac{y}{\sqrt{D}}\right)\right]\mathbb{1}_{\{y<0\}}\Big)
\end{aligned} \tag{A.9}$$

where we use the correct evaluation (A.7) of $K_{\lambda,D}(y - b)$ along with definition (A.6) and equation (A.5). For general b we get

$$\hat{g}_b(\lambda, y) = \hat{g}_0(\lambda, y - b).$$

This is the same as the density in equation (2.12) we obtained using excursion theory.

Appendix B

Appendix to Part II

B.1 Proof that condition (L3) holds for $\varphi(k) = \varphi_0(1+ak)^\varepsilon$

We need to show that the expression

$$\begin{aligned}\Lambda_0(k) &\equiv \frac{1}{A}\left[(x+Sy)\,\frac{c}{k} - \omega y\,\xi + z\,c\right] \\ &= (x+Sy)k^{B-1}\varphi(k) - \omega y k^{B-1}\varphi(k)^{R_g/\omega} + zk^B\varphi(k)\end{aligned}$$

attains its infimum over $k \geq 0$ uniquely for all non-negative x, y, z. If $x = y = 0$ it will attain its infimum at $k = 0$ as c is increasing, and similarly if $y = z = 0$ the infimum will be attained at $k = \infty$ as c/k is decreasing. We will assume from now on that either y is non-zero or both x and z are non-zero.

Differentiating Λ_0 with respect to k, and using the fact that $R_g/\omega = 1 + (1+\nu)/\omega$, gives

$$
\begin{aligned}
\Lambda_0'(k) &= k^{B-2}\varphi'\Bigg[(x+Sy)\left(k+(B-1)\frac{\varphi}{\varphi'}\right)+z\left(k^2+Bk\frac{\varphi}{\varphi'}\right) \\
&\quad -\omega y\left((B-1)\frac{\varphi}{\varphi'}\varphi^{(1+\nu)/\omega}+k\frac{R_g}{\omega}\varphi^{(1+\nu)/\omega}\right)\Bigg] \\
&= \varphi_0 k^{B-2}(1+ak)^{\varepsilon-1}\Bigg[az(\varepsilon+B)k^2+(zB-a(x+Sy)(1-B-\varepsilon))\,k \\
&\quad -(1-B)(x+Sy)+\omega y\varphi_0^{(1+\nu)/\omega}(1+ak)^{\varepsilon(1+\nu)/\omega}\left((1-B)+a(1-B-\varepsilon\frac{R_g}{\omega})k\right)\Bigg] \\
&\equiv \varphi_0 k^{B-2}(1+ak)^{\varepsilon-1}\Bigg[az(\varepsilon+B)k^2+(zB-a(x+Sy)(1-B-\varepsilon))\,k \\
&\quad -(1-B)(x+Sy)+\omega y\varphi_0^{(1+\nu)/\omega}(1+ak)^{\varepsilon_0}(1-B+a_0k)\Bigg] \\
&\equiv \varphi_0 k^{B-2}(1+ak)^{\varepsilon-1}f(k)
\end{aligned}
$$

with the appropriate identifications. If we can show that the equation $f(k)=0$ holds for only one point k then the minimum of Λ_0 must be attained uniquely. Firstly observe that

$$f(0)=-(1-B)\left[x+y(S-\omega\varphi_0^{(1+\nu)/\omega})\right]$$

is negative due to equation (6.50). Secondly $a_0=a(1-B-\varepsilon R_g/\omega)$ is positive because of (6.51) and so $f(k)\to\infty$ as $k\to\infty$. Finally

$$
\begin{aligned}
f''(k) &= \omega y\varphi_0^{(1+\nu)/\omega}(1+ak)^{\varepsilon_0-2}\left[a_0a^2\varepsilon_0(\varepsilon_0+1)k+a\varepsilon_0(2a_0+(1-B)a(\varepsilon_0-1))\right] \\
&\quad +2az(\varepsilon+B)
\end{aligned}
$$

and so a sufficient condition for f to be convex is

$$a\varepsilon_0(2a_0+(1-B)a(\varepsilon_0-1))\geq 0$$

which is easily seen to be equivalent to

$$\varepsilon\leq\frac{\omega(1-B)}{R_g+S}$$

and this is assumption (6.51). Putting these three facts together we can conclude that $f(k)$ has only one root and so $\Lambda_0(k)$ does attain its infimum uniquely. □

B.2 Proof that condition (L4) holds for $\varphi(k) = \varphi_0(1+ak)^\varepsilon$

We wish to show the transversality condition (L4), given here by

$$\sup_t e^{-\lambda_g t} k_t^{*1-S} \in L^1, \qquad \lim_{t\to\infty} e^{-\lambda_g t} k_t^{*1-S} = 0, \tag{B.1}$$

where $(1-S)$ is negative. The dynamics of k^* are

$$\begin{aligned} dk^* &= \sigma k^* dw + (Qk^* + \varphi(k_t^*)k^{*B} A/\nu)dt \\ &= \sigma k^* dw + (Qk^* + \varphi_0(1+ak^*)^\varepsilon k^{*B} A/\nu)dt \end{aligned}$$

so if we now introduce the process $x \equiv \log(k^*)$, we see that x satisfies the SDE

$$dx = \sigma dw + (Q - \frac{1}{2}\sigma^2 + h(x))dt,$$

where

$$h(x) \equiv \frac{A\varphi_0}{\nu}(1+ae^x)^\varepsilon e^{(B-1)x}.$$

We are interested in

$$e^{-\lambda_g t} k_t^{*1-S} = e^{-(S-1)(x_t + \frac{\lambda_g t}{S-1})}$$

and our task is therefore to establish lower bounds on the process

$$\tilde{x}_t \equiv x_t + \frac{\lambda_g t}{S-1}.$$

Now the process $\tilde{x}$ itself is not a diffusion; however, because $\varepsilon < 1 - B$ from (6.51), it is readily seen that h is decreasing, and so $h(x_t) \geq h(\tilde{x}_t)$. Using this, we can apply the Yamada-Watanabe stochastic comparison theorem[1]; the process $\tilde{x}$ dominates the process y which starts at the same value, but solves instead the SDE

$$\begin{aligned} dy &= \sigma dw + (Q - \frac{1}{2}\sigma^2 + \frac{\lambda_g}{S-1} + h(y))dt \\ &= \sigma dw + (\frac{1}{2}\sigma^2(S-1) + h(y))dt \end{aligned}$$

[1]See, for example, Rogers and Williams (2000, V.43).

The process y is a diffusion, with scale function s which satisfies

$$\begin{aligned} s'(y) &= \exp\left(-2\int_0^y \frac{\frac{1}{2}\sigma^2(S-1)+h(v)}{\sigma^2}\,dv\right) \\ &= \exp\left(-(S-1)y + \int_y^0 \frac{2h(v)}{\sigma^2}\,dv\right). \end{aligned}$$

Without loss of generality we can set

$$s(y) = -\int_y^\infty s'(u)\,du$$

so that the scale function has the properties $s(-\infty) = -\infty$, $s(\infty) = 0$. If we now denote $Y \equiv \inf_t y_t$ we have for all $b < 0$ that

$$\begin{aligned} \mathbb{P}^0(Y < b) &= \mathbb{P}^0(\inf\{t > 0 : y_t = b\} < \infty) \\ &= \frac{s(0) - s(\infty)}{s(b) - s(\infty)} \\ &= \frac{s(0)}{s(b)} \\ &\leq \frac{-s(0)}{\int_b^0 s'(u)\,du}. \end{aligned}$$

Writing $\alpha \equiv 2\sigma^2 A\varphi_0/\nu > 0$ we find that

$$\begin{aligned} \int_b^0 s'(u)\,du &= \int_b^0 \exp\left(-(S-1)u + \alpha\int_u^0 (1+ae^v)^\varepsilon e^{-(1-B)v}\,dv\right)du \\ &\geq \int_b^0 \exp\left(-(S-1)u + \alpha\int_u^0 dv\right)du \\ &= \frac{\exp\left(-(\alpha+S-1)b\right) - 1}{\alpha+S-1} \end{aligned}$$

and so

$$\mathbb{P}^0(Y < b) \leq \frac{\beta}{e^{-(\alpha+S-1)b} - 1},$$

where $\beta \equiv -s(0)(\alpha + S - 1)$ is a positive constant. Hence, picking any $B > 0$, we have that

$$
\begin{aligned}
\mathbb{E}^0\left[e^{-(S-1)Y}\right] &= (S-1)\int_{-\infty}^{\infty} e^{-(S-1)b}\,\mathbb{P}^0(Y<b)\,db \\
&\leq (S-1)\int_{-B}^{\infty} e^{-(S-1)b}\,db + (S-1)\int_{-\infty}^{-B} \frac{\beta e^{-(S-1)b}\,db}{e^{-(\alpha+S-1)b}-1} \\
&\leq e^{(S-1)B} + (S-1)\int_{-\infty}^{-B} \frac{\beta e^{-(S-1)b}\,db}{e^{-(\alpha+S-1)b}-e^{-(\alpha+S-1)(b+B/2)}} \\
&= e^{(S-1)B} + (S-1)\int_{-\infty}^{-B} \frac{\beta e^{\alpha b}\,db}{1-e^{-(\alpha+S-1)B/2}} \\
&< \infty
\end{aligned}
$$

and (B.1) follows easily from this. □

B.3 The debt process Δ_p

We define a $\mathbb{P}$-Brownian motion W by $\sigma W \equiv Z^0 - Z^L$ so that $\sigma^2 \equiv v_{00} - 2v_{0L} + v_{LL}$ and the dynamics (5.23) of k are

$$dk = \sigma k dW + \tilde{\Phi}(k)dt. \tag{B.2}$$

From (5.37) and Itô applied to $k_p(k)$ the dynamics of Δ_p are given by

$$
\begin{aligned}
d\Delta_p = \Delta_p\big[-dZ^L + (\mu_0 + r\beta_r)dt\big] - k_p'(\sigma k dW + \tilde{\Phi}dt) - \frac{1}{2}\sigma^2 k^2 k_p'' dt + AdZ^0 + Bdt \\
+ k_p\big[\beta_k dZ^0 - dZ^L + (\beta_k F_p - \gamma + v_{0L}(1-\beta_k))dt\big] + \beta_w \tilde{\pi} F_\pi dt - \beta_c^{-1} c dt.
\end{aligned} \tag{B.3}
$$

We wish to express Z^0 and Z^L in terms of W so we write

$$dZ^L \equiv adW + bdW'$$

where W' is a $\mathbb{P}$-Brownian motion independent of W so that

$$a\sigma = v_{0L} - v_{LL} \qquad\qquad a^2 + b^2 = v_{LL}$$

and then dZ^0 is given by $dZ^0 = (a+\sigma)dW + bdW'$. Inserting these expressions into (B.3) and collecting Δ_p, dW, dW' and dt terms gives

$$\begin{aligned}
d\Delta_p &= \Delta_p\big[-adW - bdW' + (\mu_0 + r\beta_r)dt\big] \\
&\quad + \big[(A+\beta_k k_p)(a+\sigma) - ak_p - \sigma k k_p'\big]dW + \big[Ab + b\beta_k k_p - bk_p\big]dW' \\
&\quad + \big[B - k_p'\tilde{\Phi} - \frac{1}{2}\sigma^2 k^2 k_p'' + k_p(\beta_k F_p - \gamma + v_{0L}(1-\beta_k)) + \beta_w \tilde{\pi} F_\pi - \beta_c^{-1} c\big]dt \\
&\equiv \Delta_p\big[-adW - bdW' + (\mu_0 + r\beta_r)dt\big] + A_0(k)dW + A_1(k)dW' + \Gamma_0(k)dt \quad \text{(B.4)}
\end{aligned}$$

with the necessary identifications. To deal firstly with the Δ_p term we consider Z solving the homogeneous stochastic differential equation

$$dZ = Z\big[-adW - bdW' + (\mu_0 + r\beta_r)dt\big]. \tag{B.5}$$

The solution to this stochastic differential equation is given (up to a constant) by

$$Z_t = \exp\left(-aW_t - bW_t' - \frac{1}{2}(a^2+b^2)t + \int_0^t (\mu_0 + r(k_s)\beta_r(k_s))ds\right). \tag{B.6}$$

Observe that from (B.2)

$$\begin{aligned}
\sigma dW &= \frac{dk}{k} - \frac{\tilde{\Phi}(k)}{k}dt \\
&= d(\log k) + \left(\frac{1}{2}\sigma^2 - \frac{\tilde{\Phi}(k)}{k}\right)dt
\end{aligned}$$

and so we can write equation (B.6) as

$$Z_t = k_t^{-a/\sigma} \exp\left(-bW_t' - \frac{1}{2}b^2 t + \int_0^t G_0(k_s)ds\right), \tag{B.7}$$

where $G_0(k) \equiv \mu_0 + r(k)\beta_r(k) + \frac{1}{2}b^2 - \frac{1}{2}v_{0L} + a\tilde{\Phi}(k)/\sigma k$. Combining the dynamics for Δ_p (B.4) and Z (B.5) gives

$$\begin{aligned}
d\left(\frac{\Delta_p}{Z}\right) &= \frac{d\Delta_p}{Z} + \Delta_p d\left(\frac{1}{Z}\right) + d\left\langle \Delta_p, \frac{1}{Z}\right\rangle \\
&= \frac{d\Delta_p}{Z} + \frac{\Delta_p}{Z}\left(-\frac{dZ}{Z} + v_{LL}dt\right) + \frac{1}{Z}(aA_0(k) + bA_1(k) - \Delta_p v_{LL})dt \\
&= \frac{1}{Z}\left(A_0(k)dW + A_1(k)dW' + (\Gamma_0(k) + aA_0(k) + bA_1(k))dt\right) \\
&= \frac{1}{Z}\left(A_0(k)dW + A_1(k)dW' + \Gamma_1(k)dt\right)
\end{aligned}$$

where $\Gamma_1(k) \equiv \Gamma_0(k) + aA_0(k) + bA_1(k)$. Thus for $s < t$

$$\frac{\Delta_p(t)}{Z_t} = \frac{\Delta_p(s)}{Z_s} + \int_s^t Z_u^{-1} \left\{ A_0(k_u)dW_u + A_1(k_u)dW'_u + \Gamma_1(k_u)du \right\}. \tag{B.8}$$

We can re-express the dW part of this integral. Define $G_1(k)$ so that

$$\sigma k G'_1(k) = k^{a/\sigma} A_0(k)$$

and then we have that

$$\begin{aligned} dG_1(k) &= G'_1(k)(\sigma k dW + \tilde{\Phi}(k)dt) + \frac{1}{2}\sigma^2 k^2 G''_1(k)dt \\ &= k^{a/\sigma} A_0(k)dW + \mathcal{L}G_1(k)dt \end{aligned}$$

where $\mathcal{L}$ is the generator of the process k_t. We can now rewrite the dW term in expression (B.8) as follows

$$\begin{aligned} \int_s^t Z_u^{-1} A_0(k_u)dW_u &= \int_s^t e^{bW'_u + \frac{1}{2}b^2 u - \int_0^u G_0(k_v)dv} \left\{ dG_1(k_u) - \mathcal{L}G_1(k_u)du \right\} \\ &= \left[G_1(k_u) k_u^{-a/\sigma} Z_u^{-1} \right]_s^t - \int_s^t k_u^{-a/\sigma} Z_u^{-1} G_1(k_u) \left\{ bdW'_u + (b^2 - G_0(k_u))du \right\} \\ &\quad - \int_s^t k_u^{-a/\sigma} Z_u^{-1} \mathcal{L}G_1(k_u)du. \end{aligned} \tag{B.9}$$

Hence expression (B.8) can be written as

$$\begin{aligned} \frac{\Delta_p(t)}{Z_t} &= \frac{\Delta_p(s)}{Z_s} + \frac{G_1(k_t)k_t^{-a/\sigma}}{Z_t} - \frac{G_1(k_s)k_s^{-a/\sigma}}{Z_s} \\ &\quad + \int_s^t Z_u^{-1} \left\{ A_1(k_u) - bk_u^{-a/\sigma} G_1(k_u) \right\} dW'_u \\ &\quad + \int_s^t Z_u^{-1} \left\{ \Gamma_0(k_u) - k_u^{-a/\sigma}\mathcal{L}G_1(k_u) + k_u^{-a/\sigma}(G_0(k_u) - b^2) \right\} du \\ &\equiv \frac{\Delta_p(s)}{Z_s} + \frac{G_1(k_t)k_t^{-a/\sigma}}{Z_t} - \frac{G_1(k_s)k_s^{-a/\sigma}}{Z_s} + \int_s^t Z_u^{-1} \left\{ G_2(k_u)dW'_u + G_3(k_u)du \right\} \end{aligned} \tag{B.10}$$

and Δ_p is given by

$$\Delta_p(t) = G_1(k_t)k_t^{-a/\sigma} + (\Delta_p(s) - G_1(k_s)k_s^{-a/\sigma}) \left(\frac{k_s}{k_t}\right)^{a/\sigma} e^{-b(W'_t - W'_s) - \frac{1}{2}b^2(t-s) + \int_s^t G_0(k_u)du}$$
$$+ \int_s^t \left(\frac{k_u}{k_t}\right)^{a/\sigma} e^{-b(W'_t - W'_u) - \frac{1}{2}b^2(t-u) + \int_u^t G_0(k_v)dv} \left\{G_2(k_u)dW'_u + G_3(k_u)du\right\}. \tag{B.11}$$

Suppose that we start with zero debt so that $\Delta_p(s) = 0$ at some time s in the past. Can we hold t fixed, let $s \to -\infty$ and get some meaningful limit? We would like something like

$$\lim_{s\to\infty} G_1(k_s)e^{bW'_s - \frac{1}{2}b^2 s + \int_0^s G_0(k_u)du} = 0,$$
$$\mathbb{E}^\pi \int_0^\infty k_u^{2a/\sigma} G_2(k_u)^2 e^{b^2 u + 2\int_0^u G_0(k_s)ds} du < \infty,$$
$$\mathbb{E}^\pi \int_0^\infty \left|k_u^{a/\sigma} G_3(k_u)\right| e^{\int_0^u G_0(k_s)ds} du < \infty$$

where π is the invariant law of k. An example of the sort of simpler (sufficient) conditions needed for these to hold would be

$$G_1(k), G_2(k), G_3(k) \qquad \text{all bounded,}$$
$$\frac{\tilde{\Phi}(k)}{k} \to -\varepsilon < 0 \quad \text{as} \quad k \to \infty,$$
$$\sup_k G_0(k) < -\frac{1}{2}b^2.$$

The condition on $\tilde{\Phi}$ makes the tail of the invariant law of k like a Gaussian, so all moments exist, and the condition on $\sup G_0$ makes the exponential term decreasing, so then we do get convergence as $s \to -\infty$, with

$$\Delta_p(t) = G_1(k_t)k_t^{-a/\sigma} \tag{B.12}$$
$$+ \int_{-\infty}^t \left(\frac{k_u}{k_t}\right)^{a/\sigma} e^{-b(W'_t - W'_u) - \frac{1}{2}b^2(t-u) + \int_u^t G_0(k_v)dv} \left\{G_2(k_u)dW'_u + G_3(k_u)du\right\}.$$

B.4 The one-sector government problem

In a one-sector model there is no distinction between public and private capital, and we can follow a similar development; or we may alternatively deduce the one-sector results as special cases of the two-sector results. Either way, we will assume that the private sector works all the hours available to them ($\pi = 1$ in the previous notation) so that the rate of production, which is now a function only of labour and total capital, is given simply by $F(K, LT) \equiv LTf(k)$ where $f(k) \equiv F(k, 1)$. As before the government is concerned with maximising expected levels of per-capita consumption

$$\mathbb{E}\int_0^\infty e^{-\rho_g t} L_t U\left(\frac{C_t}{L_t}\right) dt = L_0 \mathbb{E}_g \int_0^\infty e^{-\lambda_g t} U(c_t) dt$$

where we use exactly the same notation as in the two-sector problem, and again assume that U is homogeneous of order $1 - R_g$. This is a stochastic extension of the model considered in Chapter III of Arrow and Kurz (1970). The deterministic version of this model was originally proposed by Ramsey (1928) although the term 'Ramsey model' generally refers to the version as refined by Cass (1965) and Koopmans (1965).

Returning to stochastic one-sector model just described the HJB equation for the value function $V(k)$ is now

$$-\lambda_g V(k) + V'(k)(f(k) - \gamma_g k - c) + \frac{1}{2}\sigma^2 k^2 V''(k) + U(c) = 0, \tag{B.13}$$

and setting $\Psi(k) = V'(k)$ the optimality equations corresponding to those of Theorem 1 are

$$0 = \Psi(f' - \gamma_g - \lambda_g) + \Psi'(\Phi + \sigma^2 k) + \frac{1}{2}\sigma^2 k^2 \Psi'' \tag{1SG1}$$

$$U' = \Psi \tag{1SG2}$$

$$\Phi = f - \gamma_g k - c. \tag{1SG5}$$

We have assumed that U is homogeneous of order $1 - R_g$ so (up to multiplication by a constant) it must have the CRRA form

$$U(c) = \frac{c^{1-R_g}}{1 - R_g},$$

with $R_g > 0$ and $R_g \neq 1$. Again it is possible to construct an explicit solution to the government's problem; choosing V, we find the optimal c from (1SG2), then deduce Φ from (B.13), and then deduce f from (1SG5). It remains only to check that the f so obtained is concave, increasing and non-negative.

As a simple example, if we pick a value function that is also CRRA

$$V(k) = \frac{A_V^{-R_g} k^{1-S}}{1-S},$$

with $A_V > 0$, $S > 0$ and $S \neq 1$ then (1SG2) gives us

$$c(k) = A_V k^{S/R_g}$$

and then (B.13) yields

$$\begin{aligned}
\Phi(k) &= \left(\frac{\lambda_g}{1-S} + \frac{1}{2}\sigma^2 S\right) k - \frac{A_V k^{S/R_g}}{1-R_g} \\
&\equiv Qk - \frac{A_V k^{S/R_g}}{1-R_g}.
\end{aligned}$$

Finally (1SG5) gives

$$\begin{aligned}
f(k) &= (\gamma_g + Q)k + \left(1 - \frac{1}{1-R_g}\right) c \\
&= (\gamma_g + Q)k + \frac{R_g A_V k^{S/R_g}}{R_g - 1}.
\end{aligned}$$

For $f(k)$ to be concave, increasing and non-negative we will require that

$$Q + \gamma_g \geq 0, \quad R_g > S > 1.$$

Alternatively we can consider numerical solutions - the methods described in Section 7.1.2 for solution of the two-sector stochastic problem can be easily adapted to the one-sector problem so we will consider only the deterministic case. From equation (1SG2) we find that $c(k) = I(\Psi(k))$ where $I(.)$ is the inverse function of $U'(c)$. The dynamics of k and Ψ are then given by equations (1SG5) and (1SG1) respectively as[2]

$$\begin{aligned}
\dot{k} &= f(k) - \gamma k - I(\Psi) \\
\dot{\Psi} &= \Psi(\lambda_g + \gamma - f'(k)).
\end{aligned}$$

[2]In the deterministic case $\gamma_g = \gamma$ hence we drop the subscript.

We can locate the equilibrium point $\dot{k} = \dot{\Psi} = 0$ by solving the equations

$$
\begin{aligned}
f'(k^\infty) &= \lambda_g + \gamma \\
c^\infty &= f(k^\infty) - \gamma k^\infty \\
\Psi^\infty &= U'(c^\infty).
\end{aligned}
$$

The ODE for $\Psi(k)$ we wish to solve numerically is

$$\Psi(k)(f'(k) - \lambda_g - \gamma) + \Psi'(k)(f(k) - \gamma k - I(\Psi)) = 0. \tag{B.14}$$

Differentiating this with respect to k and evaluating at the equilibrium point yields

$$\Psi^\infty f''(k^\infty) + \Psi'(k^\infty)(f'(k^\infty) - \gamma - \Psi'(k^\infty)I'(\Psi^\infty)) = 0,$$

which is a quadratic in $\Psi'(k^\infty)$ and hence

$$\Psi'(k^\infty) = \frac{f'(k^\infty) - \gamma + \sqrt{(f'(k^\infty) - \gamma)^2 + 4\Psi^\infty I'(\Psi^\infty) f''(k^\infty)}}{2I'(\Psi^\infty)}.$$

Note that we take the positive root to ensure that we get a negative value for $\Psi'(k^\infty)$, as $I'(\Psi^\infty) < 0$ due to $U(c)$ being a concave felicity function. We can now compute the optimal $\Psi(k)$ by solving the ODE (B.14) working outwards (so that time is reversed) from the equilibrium point k^∞ using, for example, Scilab's 'DASSL' routine with the initial conditions $k = k^\infty$, $\Psi = \Psi^\infty$ and $\Psi'(k) = \Psi(k^\infty)$. Such numerical methods for the Ramsey model are well known; see, for example, Judd (1998).

B.5 Summary of notation

A t argument/subscript denotes a quantity at time t. Other subscripts are used to denote partial differentiation in the case of functions of two or more variables (e.g $F_g \equiv \partial F/\partial k_g$).Time-$t$ price of a zero-coupon bond paying 1 at time T given $k_t = k$

$B(t, k; T)$ Consumption rate $= k$

C $\equiv \eta^{-1}C$

c Optimal value of c for a given k

$c^*(k)$ Level of government debt Optimal value of c for a given k

D Level of government debt

$\mathbb{E}, \mathbb{E}_g$	Expectation taken under $\mathbb{P}, \mathbb{P}_g$ respectively
$F(K_p, K_g, \pi LT)$	Production (rate) function
g	The stationary/equilibrium distribution of k
$h(\xi, \pi)$	$\equiv U(\xi, 1, \pi)$
I_g	Amount invested in government capital
I_p	Amount invested in private capital
K	Total capital
K_g	Government capital
K_p	Private sector capital
k, k_g, k_p	Technology-adjusted per capita capital levels $\eta^{-1}K$, $\eta^{-1}K_g$, $\eta^{-1}K_p$
$k_g^*(k)$	Optimal value of k_g for a given k
L	Labour force / population size
$\mathbb{P}$	Real world probability measure
$\mathbb{P}_g$	Government's valuation measure
R_g	U is homogeneous of order $1 - R_g$ in c,k_g
R_p	u is homogeneous of order $1 - R_p$ in c,k_g
r	Interest rate on government debt
r_g, r_p	Consumption rates of interest for the government and private sector respectively
T	Technology level
$U(c, k_g, \pi)$	Government felicity function
$u(c, k_g, \pi)$	Private sector felicity function
$V(k)$	Government value function
v_{ij}	Covariation (per unit time) of Z^i and Z^j, $i, j \in 0, L$
W	A $\mathbb{P}$-Brownian motion defined by $\sigma W \equiv Z^0 - Z^L$
w	A $\mathbb{P}_g$-Brownian motion defined by $\sigma w \equiv z^0 - z^L$
X	Total private sector wealth $K_p + D$
x	$\equiv \eta^{-1}X$
Z^0, Z^L	Multiples of standard Brownian motions

(z^0, z^L)	Two $\mathbb{P}_g$-Brownian motions with exactly the same covariance structure as (Z^0, Z^L)
$1 - \beta_c$	Tax rate on consumption; if a household wishes to consume c after tax then it must consume $\beta_c^{-1}c$ before tax
$1 - \beta_k$	Tax rate on returns on private capital
$1 - \beta_r$	Tax rate on returns on government debt
$1 - \beta_w$	Tax rate on wages
γ	$\equiv \delta + \mu_L + \mu_T + v_{0L} - v_{LL}$
γ_g	$\equiv \gamma - v_{0L} + v_{LL}$
Δ_p	$\equiv D/\eta$
δ	Rate of depreciation of capital
ζ_t^g, ζ_t^p	State price density processes of the government and private sector
η	$\equiv LT$
θ_p, θ_L	Proportion of return on government's capital included in returns to private sector capital and labour respectively
λ_g	$\equiv \rho_g - (1 - R_g)\mu_T - \mu_L$
λ_p	$\equiv \rho_p - (1 - R_p)\mu_T$
μ_0	$\equiv v_{LL} - \mu_L - \mu_T$. Exponential drift of η^{-1}
μ_L	Exponential drift term of labour
μ_T	Exponential growth rate of technology level
ξ	$\equiv C/K_g \equiv c/k_g$
π	Proportion of population's effort devoted to production
$\pi^*(k)$	Optimal values of π for a given k
ρ_g, ρ_p	Government and private sector utility time-discount factors
σ^2	$\equiv v_{00} - 2v_{0L} + v_{LL}$
$\Phi(k)$	$\equiv F(k_p^*(k), k_g^*(k), \pi^*(k)) - \gamma_g k - c^*(k)$. The drift in k along the optimal path under $\mathbb{P}_g$
$\tilde{\Phi}(k)$	$\equiv F(k_p^*(k), k_g^*(k), \pi^*(k)) - \gamma k - c^*(k)$. The drift in k along the optimal path under $\mathbb{P}$
Ψ	$\equiv V'(k)$. The Lagrange multiplier process corresponding to the government's optimization problem.

ψ The Lagrange multiplier process corresponding to the private sector's optimization problem.

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