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Uncertain dynamical systems and nonlinear adaptive control

Sangwin, Christopher James

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Uncertain dynamical systems and nonlinear adaptive control

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Christopher James Sangwin

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Signature of Author ....

Christopher James Sangwin

A Description of the admirable table of logarithmes*

To the Most Noble and Hopefull Prince Charles: Onely sonne of the high and mightie James by the grace of God, King of great Brittaine, France, and Ireland: Prince of Wales: Duke of Yorke and Rothesay: Great Steward of Scotland: and Lord of the Islands.

Most Noble Prince,

Seeing there is neither study, nor any kinde of learning that doth more acuate and stirre up generous and heroicall wits to excellent and eminent affaires: and contrariwise that doth more deiect and keepe downe sottish and dull mindes, then the Mathematikes. It is no marvell that learned and magnanimous Princes in all former ages have taken great delight in them, and that unskillfull and slothfull men have always pursued them with most cruell hatred, as utter enemies to their ignorance and sluggishnesse. Why then may not this my new invention (seeing it abhorreth blunt and base natures) seeke and flye unto your Highnesse most noble disposition and patronage? and especially seeing this new course of Logarithmes *doth cleane take away all the difficultie that heretofore hath beene in mathematicall calculations, (which otherwise might have been distastefull to your worthy towardnesse) and is so fitted to help the weakness of memory, that by meanes thereof it is easie to resolve moe Mathematical questions in one houres space, then otherwise by that wonted and commonly recived manner of Sines, Tangents and Secants, can bee done even in a whole day. And therefore this invention (I hope) will bee so much more acceptable to your Highnesse, as it yeeldeth a more easie and speedy way of accompt. For what can bee more delightfull and more excellent in any kinde of learning then to dispatch honourable and profound matters, exactly, readily, and without loss of either time or labour. I course therefore (most gracious Prince) that you would (according to your gentlenesse except this gift) though small and farr beneath the height of your deservings, and worth) as a pledge and token of my humble service: which if I understand you doe, you shall (even in this regard onely) encourage me that I am almost spent with sicknesse, shortly to attempt to other matters, perhaps greater then these, and more worthy so great a Prince. In the meane while, the supreame King of Kings, and Lord of Lords long defend and preserve to us the great light of great Brittaine, your renowned parents, and your selfe the noble branch of so noble a stemme, and the hope of our future tranquilitie: to him be given all honour and glory.*

Your Highnesse most devoted servant,

John Nepair

*Translated into English and published by Henry Briggs, (1618)

Thanks to my family.

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Summary

In this thesis we address the problem of adaptive feedback control design for various classes of dynamical systems modelled by nonlinear functional differential equations. In the formulation of the problem, we show that a diverse range of phenomena are included within these classes of systems, including, for example, retarded systems, diffusion processes and hysteresis.

The proposed controllers are high-gain, non-identifier based, adaptive feedback strategies. Some of the feedback strategies are discontinuous, and we develop and utilize a framework of non-smooth analysis and functional differential inclusions. This development includes results on existence of solutions to functional differential equations and inclusions, and on asymptotic behaviour of such solutions.

More specifically, we examine problems of feedback control for three principal systems: (a) single-input single-output systems; (b) planar systems; and (c) multi-input multi-output systems. In each case we model the system with a nonlinear functional differential equation. Furthermore, various of control objectives, such as asymptotic stability and output tracking of a reference signal to within a pre-specified accuracy quantified by $\lambda \geq 0$, are considered.

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Notation

$\mathbb{N}$	The natural numbers.
$\mathbb{R}$	The real numbers.
$\mathbb{R}_+$	$:= [0, \infty)$.
$\mathbb{C}$	The complex numbers.
$\mathbb{C}_-$ ($\mathbb{C}_+$)	The open left (right) complex half plane.
$\mathbb{B}_\delta(\bar{x})$	$:= \{x \in \mathbb{R}^n \mid \|x - \bar{x}\| \leq \delta\}$
$\lvert \cdot \rvert$	The modulus of a complex number (absolute value of a real number).
$\|\cdot\|$	The Euclidean norm, ie for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\|x\| := \sqrt{\sum_{i=1}^n x_i^2}$.
$\langle \cdot, \cdot \rangle$	The Euclidean inner-product, ie for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$. We view $x \in \mathbb{R}^n$ as both an n-tuple and as a vector with respect to a chosen basis. If a basis is chosen and $x \in \mathbb{R}^n$ is a vector we denote the *transpose* of x by x'.
spec(A)	The spectrum (ie the set of eigenvalues) of the matrix $A \in \mathbb{R}^{n \times n}$.
A'	The transpose of the matrix $A \in \mathbb{R}^{n \times n}$.
$\overline{B}$	Denotes the closure of the set B.
$\Omega(x)$	Denotes the ω-limit set of x. See Definition 36, pg 63.
$\boldsymbol{\Omega}(x)$	See (4.1), pg 64.

For $I \subset \mathbb{R}$ an interval we denote the following spaces of functions $f : I \to \mathbb{R}^Q$ by;

$C(I; \mathbb{R}^Q)$	continuous functions,
$AC(I; \mathbb{R}^Q)$	absolutely continuous functions,
$L^p(I; \mathbb{R}^Q)$	measurable p-integrable functions $p \geq 1$
$L^p_{\text{loc}}(I; \mathbb{R}^Q)$	measurable locally p-integrable functions $p \geq 1$
$L^\infty(I; \mathbb{R}^Q)$	measurable essentially bounded functions with norm $\|f\|_\infty :=$ ess $\sup_{t \in I} \|f(t)\|$
$L^\infty_{\text{loc}}(I; \mathbb{R}^Q)$	measurable locally essentially bounded functions,

We will also use the following spaces of functions;

$(\mathbf{X}, \|\cdot\|_\infty)$ the Banach space $C([-h,0];\mathbb{R}^N)$ with the usual sup norm, $\|x\|_\infty := \sup_{\in[-h,0]} \|x(t)\|$. See Section 4.2.2, pg 64.

$\mathcal{R}$ denotes the Sobolev space $W^{1,\infty}(\mathbb{R};\mathbb{R}^M)$, that is to say the space of bounded absolutely continuous functions $\mathbb{R} \to \mathbb{R}^M$ with essentially bounded derivatives, equipped with the norm

$$\|r\|_{1,\infty} = \sup_{t\in\mathbb{R}} \|r(t)\| + \text{ess-sup}_{t\in\mathbb{R}}\|\dot{r}(t)\|.$$

$\mathcal{I}$ $:= \{\gamma \in C(\mathbb{R}_+;\mathbb{R}_+) \mid \forall\, \delta \in \mathbb{R}_+ \ \exists\, \Delta \in \mathbb{R}_+ : \ \alpha(\delta\tau) \leq \Delta\alpha(\tau) \ \forall\, \tau \in \mathbb{R}_+\}$.

$\mathcal{K}$ $:= \{\gamma \in C(\mathbb{R}_+;\mathbb{R}_+) \mid \gamma(0) = 0, \gamma$ is strictly increasing$\}$.

$\mathcal{K}_\infty$ denotes the subclass of $\mathcal{K}$ for which $\gamma(t) \to \infty$ as $t \to \infty$.

$\mathcal{J}$ $:= \mathcal{I} \cap \mathcal{K}$. Examples and properties of class $\mathcal{J}$ functions are assembled in Section A.3, pg 132.

$\mathcal{J}_\infty$ $:= \mathcal{J} \cap \mathcal{K}_\infty$,

$\mathcal{KL}$ denotes the space of continuous functions $\gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for each fixed $s \in \mathbb{R}_+$ the map $\gamma(\cdot, s) \in \mathcal{K}$ and the map $\gamma(s,\cdot)$ is decreasing to zero.

For $x : I \to \mathbb{R}^N$, the restriction of x to $J \subset I$ is denoted by $x|_J$, an extension of x to $\mathbb{R}$ is denoted by x^e.

Given a function $f : X \to Y$ we define

$$\text{graph}(f) := \{(x, f(x)) \,|\, x \in X\} \subset X \times Y.$$

For a set-valued map F from X to the non-empty subsets of Y

$$\text{graph}(F) := \{(x,y)|y \in F(x)\,,\, x \in X\} \subset X \times Y.$$

Other classes of mappings and operators

For reference, we list here other classes of operators and indicate the whereabouts of their definitions;

$\mathcal{C}_i(\psi)$	Subclasses of $\mathcal{T}_h^{N,M}$.	Definition 16, pg 29.
$\mathcal{D}^{N,M}$	A class of functions.	Definition 26, pg 40.
$\mathcal{F}$	The class of upper-semicontinuous set-valued maps.	Definition 31, pg 57.
S_t	Shift operators on functions.	Definition 10, pg 25.
$\mathcal{S}(\psi)$	A subclass of $\mathcal{T}_h^{N,M}$.	Definition 17, pg 30.
$\mathcal{T}_h^{N,M}$	A class of nonlinear operators.	Definition 1, pg 20.
$\bar{\mathcal{T}}_h^{N,M}$	A class of nonlinear operators.	Definition 3, pg 21.

Chapter 1

Introduction

1.1 Control of dynamical systems

In control theory we consider a mathematical model of a system with inputs and outputs. By choosing the inputs (controls) we influence the behaviour of these outputs. In particular, we often seek inputs which give corresponding outputs that achieve a desired *control objective.* Such objectives are typically stabilization of an equilibrium state of the system, or tracking by the output of some given reference signal. The mathematical model is usually a dynamical system and in this thesis we consider dynamical systems generated by initial-value problems for controlled functional differential equations.

Interesting questions arise when only incomplete system information is known. In particular we can ask the question *"if the known information only approximates the process, can we construct a control that still achieves the desired control objective?"* Put another way, *"is a given control strategy* robust *to errors or uncertainties in the mathematical model?"* To achieve control when confronted by incomplete system information one approach is to use *adaptive control* wherein the control structure changes over time in an attempt to tune itself to the particular system. These ideas lead naturally to a *universal control*; that is a single control strategy that achieves the desired control objective for every member of an underlying class of systems.

This thesis formulates and investigates classes of control systems and studies the design and analysis of universal control strategies. We restrict our control objectives to be *positional*, which is to say we require that the system output approaches asymptotically some set (possibly time varying) as contrasted with an *optimal* control objective where one seeks to minimize some cost functional.

1.1.1 Model reference *vs* non-identifier based adaptive control

One classical approach to controller design is to make basic assumptions about the system one is attempting to control and use these to build a mathematical *model* which forms part of the controller itself. The control compares the measured output $y(\cdot)$ with the output predicted by the model. In the adaptive case one can attempt to tune the model, the controller, or of course, both. The quality of the resulting controller will crucially depend on the quality of the mathematical model. We do not consider this approach here. Instead the controllers considered in this thesis invoke no internal model, neither is there explicit estimation of plant parameters. Instead we take a more direct *non-identifier based* approach where we invoke only structural assumptions. For more information on the model reference approach see, for example, [57].

The following sections examine the development of high-gain non-identifier based adaptive controllers. These examples provide the prototypes for systems considered in this thesis.

1.2 Scalar single-input single-output linear systems

Perhaps the simplest control systems, and therefore the natural choice as a starting point, are the linear, scalar, single-input (u), single-output (y), systems given by

$$\left.\begin{array}{ll} \dot{x}(t) = ax(t) + bu(t), & x(0) = x^0 \in \mathbb{R} \\ y(t) = cx(t) & a, b, c \in \mathbb{R} \text{ with } cb \neq 0. \end{array}\right\} \tag{1.1}$$

The assumption $cb \neq 0$ is reasonable since $cb = 0$ implies either the control will have no effect, or there will only be a zero output. Our control objective is to choose $u(t)$ in such a way as to ensure $x(t) \to 0$ as $t \to \infty$. We will utilize *output feedback* in which we define $u(t)$ to be some function of the output $y(t)$, and possibly other variables internal to the control itself. These systems have been extensively studied in [27, 28] and in particular [30]. If we choose $u(t) := -ky(t)$ for some constant $k \in \mathbb{R}$ then system (1.1) becomes

$$\dot{x}(t) = (a - kbc)x(t), \quad x(0) = x^0.$$

Clearly the solution to this is $x(t) = x^0 e^{(a-kbc)t}$ so that if $a/|cb| < |k|$ and $\operatorname{sgn}(k) = \operatorname{sgn}(bc)$ then the system is exponentially stable. In particular $x(t) \to 0$ as $t \to \infty$.

If a, b and c are unknown, then how would one choose such a k? Let us first consider the case where a, b, c are unknown but $bc > 0$. To ensure that the solution exists for all $t > 0$ and that $x(t) \to 0$ as $t \to \infty$ we will introduce the following time-varying feedback:

$$u(t) = -k(t)y(t) \tag{1.2}$$

for some function k. We refer to a control system together with a specific feedback control strategy, eg (1.1) together with (1.2), as a *closed-loop system.* The idea behind such a strategy is simple: by gradually increasing $k(t)$ we will eventually achieve a gain sufficiently large so that $(a - k(t)bc) < 0$. This should stabilize the system. Furthermore, when $k(t)$ is large enough to achieve stability we should stop adapting. Thus the adaption law for k should take some account of the state $x(t)$.

It was shown in [86], as a corollary to more general result, that the adaptive law

$$\dot{k}(t) = y^2(t), \quad k(0) = k^0 \in \mathbb{R} \tag{1.3}$$

is sufficient in the sense that the following control objective is achieved:

(i) the solution exists for all $t > 0$;

(ii) $x(t) \to 0$ as $t \to \infty$; and

(iii) $k(t)$ remains bounded on $[0, \infty)$.

Note that (iii) implies convergence of $k(\cdot)$ by monotonicity. Notice further that whilst the above controller is adaptive, there is no attempt on the part of the controller to explicitly model the system being regulated. That is to say we make no attempt to estimate the system parameters a, b and c.

In fact, in this very simple case, the final value $k_\infty := \lim_{t\to\infty} k(t)$ can be calculated, [31], and more particularly [30], as

$$k_\infty = \frac{a}{cb} + \sqrt{\left(\frac{a}{cb} - k(0)\right)^2 + \frac{y(0)^2}{2}}\,.$$

1.2.1 Sign of $cb \neq 0$ unknown

In the above problem we assumed that the sign of cb was positive. We may simplify the problem by a change of co-ordinates, and consider the control problem for the equation

$$\dot{y}(t) = \bar{a}y(t) + \bar{b}u(t), \quad y(0) = y^0 \in \mathbb{R},$$

where $\bar{b} \neq 0$, and the output $y(t)$ is available for control purposes. In [62] the following problem was posed. *Do there exist differentiable functions u, ψ with the property that* *for all* $\bar{b} \neq 0$ *and* $(y^0, k^0) \in \mathbb{R}^2$ *the solution* (y, k) *of*

$$\left.\begin{aligned} \dot{y}(t) &= \bar{a}y(t) + \bar{b}u(y(t), k(t)), & y(0) &= y^0 \\ \dot{k}(t) &= \psi(y(t), k(t)), & k(0) &= k^0 \end{aligned}\right\} \tag{1.4}$$

satisfies the following:

(i) the solution exists for all $t > 0$*;*

(ii) $y(t) \to 0$ *as* $t \to \infty$*; and*

(iii) $k(t)$ *remains bounded on* $[0, \infty)$*?*

The key point here is, of course, that $\bar{b}$ could be either positive or negative. An answer to this question was provided by [65]. Differentiable (but not rational) functions do exist that satisfy these requirements. In particular [65] gives

$$\begin{aligned} u &: (y, k) \mapsto (k^2 + 1)\cos(\tfrac{1}{2}\pi k)e^{k^2}y \\ \psi &: (y, k) \mapsto y(k^2 + 1). \end{aligned}$$

This control was simplified significantly in [63] wherein it is proved that the strategy

$$\begin{aligned} u &: (y, k) \mapsto k^2 \cos(k)y \\ \psi &: (y, k) \mapsto y^2 \end{aligned}$$

is sufficient. This is in turn, a special case of the control introduced by [86]:

$$\left.\begin{aligned} u &: (y, k) \mapsto \nu(k)x \\ \psi &: (y, k) \mapsto y^2 \end{aligned}\right\} \tag{1.5}$$

where $\nu : \mathbb{R} \to \mathbb{R}$ is bounded on compact sets and satisfies the conditions,

$$\text{(a)} \quad \limsup_{\eta \to \infty} \frac{1}{\eta} \int_0^\eta \nu = +\infty, \qquad \text{(b)} \quad \liminf_{\eta \to \infty} \frac{1}{\eta} \int_0^\eta \nu = -\infty \tag{1.6}$$

first introduced in [65]. Functions ν which satisfy (1.6) have subsequently become known as *Nussbaum functions.* Other examples [32, Examples 4.1.2] include $k \mapsto k \sin\sqrt{|k|}$ and the discontinuous

$$k \mapsto \begin{cases} k & \text{if} \quad n^2 \leq |k| < (n+1)^2, \quad n \text{ even} \\ -k & \text{if} \quad n^2 \leq |k| < (n+1)^2, \quad n \text{ odd.} \end{cases}$$

The quadratic nature of the right hand side of (1.3) could result in rapid growth of k.

Moreover, given this quadratic term, the question of existence of solutions on a half line $[0, \infty)$ is no longer clear: the possibility of finite time blow up must be excluded. In fact, [86, Theorem 1] avoids this question by assuming *a priori* that a solution exists for all $t > 0$ ([65, Theorem 2] does not).

Gain growth conditions

As we have already pointed out, the quadratic nature of the right hand side of (1.5) could lead to rapid growth of $k(\cdot)$ which is undesirable. This can be weakened [31], and in the case of system (1.4) stability can be achieved by using

$$\dot{k}(t) = |y(t)|^p, \quad k(0) \in \mathbb{R} \tag{1.7}$$

for arbitrary $p \geq 1$. We will examine such growth conditions in Chapter 7 in the context of tracking problems.

In Sections 1.3–1.7.1 below we examine some generalizations which provide motivating examples for the classes of systems studied later.

1.3 Nonlinear scalar systems

For some known $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous and increasing, consider the class of nonlinear, single-input (u), single-output (y) systems

$$\left.\begin{array}{ll} \dot{y}(t) = f(t, y(t)) + bu(t), & y(0) = y^0 \in \mathbb{R} \\ f \text{ a Carathéodory function}, & b \neq 0 \\ |f(t,y)| \leq \mu\phi(|y|)\, \forall (t,y) \in \mathbb{R}^2, & \text{for some } \mu \in \mathbb{R}_+. \end{array}\right\} \tag{1.8}$$

These assumptions are reasonable. In particular, the assumption that f is a Carathéodory function ensures existence of solutions; if $b = 0$ the control will have no effect. Clearly to achieve control we must assume some bounds on the growth of the function f. In fact the class (1.8) is very general. If $\phi : r \mapsto e^r$ then f could be *any* polynomial of arbitrary degree with bounded time varying coefficients: $f(t, y) := \sum_{i=0}^{N} p_i(t) y^i$ for $p_i \in L^\infty(\mathbb{R})$. If an upper bound, M say, for the degree of the polynomial is known, then $\phi : r \mapsto 1 + r^M$ suffices.

To achieve universal adaptive stabilization for this class the (formal) controller

$$\left.\begin{array}{l} u(t) = \nu(k(t))\phi(|y(t)|)\,\mathrm{sgn}(y(t)) \\ \dot{k}(t) = \phi(|y(t)|)|y(t)|, \quad k(0) = k^0 \in \mathbb{R}, \end{array}\right\} \tag{1.9}$$

where ν satisfies (1.6) and $\operatorname{sgn}(r) := r/|r|$ if $r \neq 0$.

Of course, there are technical difficulties here since the discontinuity in the control places the resulting closed-loop system outside the classical theory for existence of solutions. To provide a rigorous mathematical formulation we adopt non-smooth analysis and a framework of differential inclusions [5, 18]. Detailed discussion of these issues is deferred until Chapter 3.

The use of a discontinuous controller is in some cases inevitable [76]. Consider the input (u), output (y), control system

$$\dot{y}(t) = f(y(t), u(t)), \quad y(0) = y^0 \in \mathbb{R}, \; u(t) \in \mathbb{R}$$

such that the control appears nonlinearly in f. It has been is proved [77] that asymptotic controllability is equivalent to the algebraic constraint

$$(\forall y \neq 0)(\exists u) \; yf(y,u) < 0.$$

The problem of feedback control is then choosing a feedback function $u : \mathbb{R} \to \mathbb{R}$ so that

$$(\forall y \neq 0) \; yf(y, u(y)) < 0.$$

However, a continuous $u(\cdot)$ need not exist. For example if $f : (y,u) \mapsto y + |y|u$ (continuous) then the system fails to be locally stabilizable by continuous feedback but is stabilizable by a discontinuous feedback such as $u : y \mapsto -2\operatorname{sgn}(y)$, [68].

Further, if the state space is a proper subset of $\mathbb{R}^n$, (so we can think of there being obstacles in the state space) then discontinuous feedbacks cannot be avoided. For examples of this phenomenon see [75, Chapter 4]. There may also be geometrical reasons why a discontinuity is needed. See "Brockett's example" in [9, 75, 68].

1.4 Higher order finite-dimensional linear systems

To examine control questions for higher order systems (ie $x(t) \in \mathbb{R}^n$) we first need some definitions. The m-input (u), m-output (y) linear system

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0 \in \mathbb{R}^n \\ y(t) &= Cx(t) + Du(t), \end{aligned}\right\} \tag{1.10}$$

is said to be *minimum-phase* if

$$\det \begin{bmatrix} sI - A & B \\ C & D \end{bmatrix} \neq 0 \text{ for all } s \in \overline{\mathbb{C}}_+.$$

First we consider the class $\mathcal{L}$ of finite-dimensional, linear, minimum-phase, relative-degree one (ie $CB \neq 0$), single-input $u(\cdot)$, single-output $y(\cdot)$ systems (1.10), with $D = 0$, $x(t) \in \mathbb{R}^n$ and $u(t), y(t) \in \mathbb{R}$.

Under a suitable coordinate transformation (see, for example, [32, Lemma 2.1.3]), every system in $\mathcal{L}$ can be expressed in the form of two coupled subsystems

$$\left.\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + b u(t) \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t) \end{aligned}\right\} \tag{1.11}$$

with $y(t), u(t) \in \mathbb{R}$, $z(t) \in \mathbb{R}^{n-1}$, $b := CB \neq 0$ and with A_4 having eigenvalues in the open left half complex plane.

It is well known (see, for example, [65], [63] and [86]), that the following output feedback strategy is an $\mathcal{L}$-universal stabilizer (in the sense that for each member of $\mathcal{L}$, every solution of the feedback control system is such that $(y(t), z(t)) \to (0, 0)$ as $t \to \infty$ and $k(\cdot)$ is bounded):

$$\left.\begin{aligned} u(t) &= \nu(k(t)) y(t), \\ \dot{k}(t) &= y^2(t), \quad k(0) = k^0, \end{aligned}\right\} \tag{1.12}$$

where $\nu(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function with the properties (1.6). Notice that (1.12) is essentially the control (1.5).

1.4.1 Switching functions and unmixing sets

So far the systems examined have been single-input single-output. Next consider the class $\bar{\mathcal{L}}$ of finite-dimensional, linear, minimum-phase, m-input ($u(t)$), m-output ($y(t)$), relative-degree one (ie CB non-singular), systems. Under a suitable coordinate transformation (see, for example, [32, Proposition 2.1.2]), every system in $\bar{\mathcal{L}}$ can be expressed in the form of two coupled subsystems

$$\left.\begin{aligned} \Sigma_1 : \quad \dot{y}(t) &= A_1 y(t) + A_2 z(t) + CBu(t), \quad & y(0) &= y^0 \\ \Sigma_2 : \quad \dot{z}(t) &= A_3 y(t) + A_4 z(t), & z(0) &= z^0 \end{aligned}\right\} \tag{1.13}$$

where the real matrices $A_1, \ldots, A_4$ have appropriate formats, with $y(t), u(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^{n-m}$, and with A_4 having eigenvalues in the open left half complex plane.

If the eigenvalues of CB lie in $\mathbb{C}_+$, the open right half complex plane then, for some

$k \in \mathbb{R}_+$ large enough, a static feedback of $-ky(t)$ will render the closed-loop system asymptotically stable.

The condition "eigenvalues of CB in the open right half complex plane" is the counterpart of the condition $cb > 0$ in the system (1.1). Of course, this condition is rather restrictive. If however, we know some $K \in \mathbb{R}^{m\times m}$ so that the eigenvalues of CBK lie in $\mathbb{C}_+$ (we say that the matrix K *unmixes the spectrum* of CB) then we simply use the feedback $-kKy(t)$, for $k \in \mathbb{R}_+$ large enough. Furthermore we can implement an analogous adaptive control in which we use the feedback

$$\left.\begin{aligned} u(t) &= -Kk(t)y(t), \\ \dot{k}(t) &= \|y(t)\|^2, \quad k(0) = k^0, \end{aligned}\right\} \tag{1.14}$$

to assure that k is "large enough".

In the case $cb \neq 0$, for system (1.1), we were able to use a Nussbaum switching function to adaptively control the system. If we assume that CB is non-singular but that an unmixing matrix K is not known a priori, we may modify our adaptive strategy.

Crucial to this is the idea of an *unmixing set.* That is a set of matrices $\{K_i\} \subset \mathbb{R}^{m\times m}$, at least one of which is guaranteed to unmix the spectrum of CB. That is, there exists some $K^* \in \{K_i\}$ such that the spectrum of CBK^* lies in $\mathbb{C}_+$. The results of [64] prove the existence of a *finite* unmixing set for all $m \in \mathbb{N}$. This approach to adaptive control has been taken in, for example, [4].

1.5 Infinite-dimensional linear systems

Many linear control problems arising from delay or distributed parameter systems (that is to say, systems arising from partial differential equations) may be formulated in terms of infinite-dimensional linear systems theory. Such a formulation typically consists of finite-dimensional input and output spaces (such as $\mathbb{R}^n$) and an infinite-dimensional Hilbert space X as the state space, together with maps between them. Background material may be found in, for example, [16, 17] together with examples. Such systems will be considered in this thesis in Section 2.2.2. Adaptive control problems for infinite-dimensional systems have an extensive literature, see for example, [51, 34] and also [55].

1.6 Asymptotic tracking

Another possible control objective is that of tracking asymptotically a reference signal $r(\cdot)$. Often it is possible, via a change of coordinates, to reduce a tracking problem to that of stabilization. For example, consider the class (1.8) of nonlinear systems. Given a reference signal $r \in \mathcal{R}$ we attempt to design a universal controller which assures that for all systems of the class, the error

$$e(t) := x(t) - r(t) \to 0, \text{ as } t \to \infty.$$

We employ feedback control using, at a given time t, only knowledge of $r(t)$, $y(t)$, $\phi(\cdot)$ and the structural information implicit in the formulation of the class (1.8).

If ϕ, of (1.8), satisfies

$$\forall\, R > 0\ \exists\ \mu_R > 0 \text{ such that } \phi(|e + r|) \le \mu_R \phi(|e|) \text{ for all } (e, r) \in \mathbb{R} \times [-R, R], \quad (1.15)$$

then it suffices to replace every occurrence of $y(t)$ by $e(t)$ in the control strategy (1.9). Thus at the expense of stronger assumptions on the bounding function ϕ, used to construct the controller, we are able to achieve asymptotic tracking. Notice that (1.15) forces $\phi(0) > 0$ and as concrete examples both $\phi_1 : y \mapsto \exp(|y|)$ and $\phi_2 : y \mapsto 1 + |y|^N$ for some $N \in \mathbb{N}$ satisfy (1.15). We will use these ideas in this thesis.

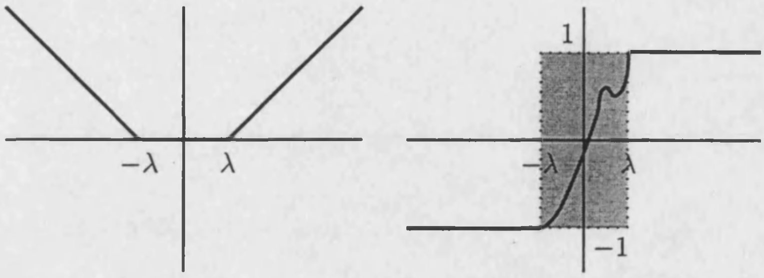
1.7 Robustness

One particular disadvantage of the controller (1.12), and those like it, is the lack of robustness to output noise. Assume, for example, that the output of the system (1.11) is corrupted by a noise term $\eta(\cdot)$ so that the measured output becomes $y(t) + \eta(t)$. In this case, the second equation governing the adapting gain in the control law (1.12) becomes

$$\dot{k}(t) = |y(t) + \eta(t)|^2.$$

Intuitively speaking, the consequence of such noise will be the continuing adaption of the controller gain, which may become unbounded. This phenomenon has been termed *parameter drift.* Arbitrarily small constant signals, for example, will have this effect. Hence, these controllers are not robust to noise in the output.

There are two methods of ameliorating this effect; the first, which has been termed "λ-tracking" involves the introduction of a dead-zone to prevent continued adaptation; the second, referred to as a "σ-modification" alters the dynamics of the controller gain k to stabilize it about some pre-specified set point.



(a) $d_\lambda(\cdot)$ (b) $s_\lambda(\cdot)$, $\lambda > 0$

Figure 1-1: Components of a λ-tracker.

1.7.1 λ-tracking

For pre-specified $\lambda > 0$, we define the distance function to the set $[-\lambda, \lambda]$ as

$$d_\lambda(x) := \max\{0, |x| - \lambda\}$$

and let $s_\lambda : \mathbb{R} \to \mathbb{R}$ be continuous and take the values $\text{sgn}(x)$ for $|x| > \lambda$ and $s_\lambda(x) \in [-1, 1]$ for $|x| \leq \lambda$. These generalize $|\cdot|$ and $\text{sgn}(\cdot)$ and are illustrated in Figure 1-1. It is well known (see, for example, [37]), that for systems of class $\bar{\mathcal{L}}$ with $\text{spec}(CB) \subset \mathbb{C}_+$, writing $e(t) := y(t) - r(t)$ (the tracking error), the output feedback strategy

$$\left.\begin{array}{rcl} u(t) & = & -k(t)e(t), \\ \dot{k}(t) & = & d_\lambda(\|e(t)\|)^2 \text{ and } k(0) = k^0 \end{array}\right\} \tag{1.16}$$

is a universal λ-servomechanism in the sense that, for each system of the class and every reference signal $r \in \mathcal{R}$, the strategy ensures:

(i) boundedness of the state;

(ii) convergence of the controller gain; and

(iii) output tracking with prescribed accuracy λ, ie $d_\lambda(\|e(t)\|) \to 0$ as $t \to \infty$.

In the context of nonlinear systems, given a reference signal $r \in \mathcal{R}$ and a system of class (1.8), implement a control of the form

$$\left.\begin{array}{rcl} u(t) & = & \nu(k(t))\phi(|e(t)|)\, s_\lambda(e(t)) \\ \dot{k}(t) & = & \phi(|e(t)|)d_\lambda(e(t)) \text{ and } k(0) = k^0 \end{array}\right\} \tag{1.17}$$

This control is a simple extension of (1.9). If the error satisfies $|e(t)| < \lambda$, then $\dot{k}(t) = 0$, ie if the error is smaller that the pre-specified λ then the adaptation stops.

Similarly, the control (1.17) achieves, for all systems (1.8) with ϕ satisfying (1.15):

(i) boundedness of the state;

(ii) convergence of the controller gain; and

(iii) output tracking with prescribed accuracy λ, ie $d_\lambda(\|e(t)\|) \to 0$ as $t \to \infty$.

This approach to control has been termed "*λ-tracking*" [37]. Compare (1.17) with the control (1.9): not only does the introduction of a dead-zone relieve the problem of noise, the control is now continuous.

In this thesis we construct λ-servomechanisms for various classes of control systems. Firstly we prove that relatively simple controllers may be successfully be applied to very general classes of systems modelled by functional differential equations. Secondly we examine the structure of such controls, in particular we consider in detail the gain adaption law.

1.7.2 σ-modification

Another method for compensating for the effects of disturbances in the output is the so called σ-modification, [40]. For example, consider the system (1.1) with the feedback law (1.2). In this case one changes the adaption law (1.3) to

$$\dot{k}(t) = -\sigma k(t) + y^2(t)$$

for some pre-specified $\sigma \in \mathbb{R}_+$. Intuitively speaking this modification should prevent $k(\cdot)$ from perpetually increasing in the presence of small noise terms. However, $k(\cdot)$ is no longer monotone. Not only does this complicate the stability analysis but now we have the possibility of "chaotic" dynamics in the closed-loop system, particularly if the controller uses a switching function. Despite this, one specific advantage of these controllers is that the gain $k(\cdot)$ decreases when stability of the system has been achieved. Thus, in some sense the gain does not remain excessively large. These issues are addressed in detail in [58, 39] and [41, Chapter 4].

1.8 Motivation for the system formulation

To motivate the approach we take here let us return to the class $\bar{\mathcal{L}}$ of linear systems given by (1.13). In this thesis we consider such systems to have the input-output block structure given by the schematic shown in Figure 1-2.

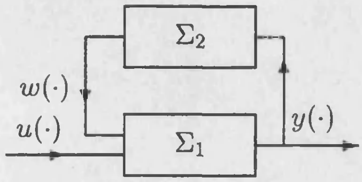


Figure 1-2: The generic system.

The dynamic block Σ_1, which can be influenced directly by the controller, is also driven by the output $w(\cdot)$ from the dynamic block Σ_2. Viewed abstractly, the block Σ_2 can be thought of as an operator which maps $y(\cdot)$ to $w(\cdot)$.

With respect to this operator theoretic viewpoint the dynamical behaviour of subsystem Σ_2 of Figure 1-2 is governed, in the case of class $\bar{\mathcal{L}}$ systems, by the second differential equation in (1.13) and hence $w(\cdot) = A_2 z(\cdot)$ is given by

$$w(t) = p(t) + (Ly)(t)$$

where the function p is given by $p(t) := A_2 \exp(A_4 t) z^0$ and the linear operator L is given by

$$(Ly)(t) := A_2 \int_0^t \exp(A_4(t-s)) A_3 y(s) ds \,. \tag{1.18}$$

System (1.13) can thus be interpreted as

$$\dot{y}(t) = A_1 y(t) + p(t) + (Ly)(t) + CBu(t), \quad y(0) = y^0. \tag{1.19}$$

In this thesis, we extend the above prototypical results to classes of nonlinear, infinite-dimensional systems. In particular, we consider various classes $\mathcal{N}_i$ (which will be made precise in due course) of nonlinear control systems having the same generic structure as in Figure 1-2, given by controlled nonlinear functional differential equations, typically of a form similar to

$$\left.\begin{aligned} \dot{y}(t) &= f(p(t), (\widehat{T}y)(t)) + g(p(t), (\widehat{T}y)(t), u(t)) \\ y|_{[-h,0]} &= y^0 \in C([-h,0]; \mathbb{R}^M) \end{aligned}\right\} \tag{1.20}$$

where, loosely speaking, $h \geq 0$ quantifies the "memory" of the system ($h = 0$ in the linear prototype), $p(\cdot)$ may be thought of as a (bounded) disturbance term (possibly dependent on initial data, as in the linear prototype) and $\widehat{T}$ is a nonlinear causal operator (replacing the operator L of the linear prototype).

We remark here that we assume the systems studied can be written in this form.

Co-ordinate changes, normal forms and nonlinear changes of variables to prove the equivalence of systems to this generic form are not considered.

1.8.1 The operators $\widehat{T}$

The operators $\widehat{T}$, introduced more fully in Chapter 2, play a central rôle in this thesis. In developing this theory there is a constant tension between, on the one hand, assumptions needed to prove the various results on existence and behaviour of solutions, and on the other hand, the need to make the minimum sufficient assumptions in order to include a wide range of interesting examples. Here we give the reader some pointers as to the choices made in this respect.

Firstly we introduce via Definition 1 the core space, $\mathcal{T} = \mathcal{T}_h^{N,M}$, of operators. Elements of this space, that is to say operators $\widehat{T} \in \mathcal{T}$, satisfy the minimum assumptions on their input-output behaviour, which may be thought of as representing the dynamics of the subsystem Σ_2 of Figure 1-2. Anticipating Definition 1, pg 20 we remark here that these assumptions, referred to in the text as (T1)-(T3), are respectively; a bounded-input locally bounded-output condition; a condition on causality; and a Lipschitz-like condition of local nature. All these requirements are reasonable for the input-output behaviour of an operator representing a dynamic block such as Σ_2. Furthermore, the parameters N, M denote the dimensions of the vector-valued inputs and outputs and the h represents the memory or delay in the system initially.

The class $\mathcal{T}$ as a whole is then shown to satisfy properties such as linearity:

$$T_1, T_2 \in \mathcal{T} \Rightarrow a_1 T_1 + a_2 T_2 \in \mathcal{T} \quad \forall\, a_1, a_2 \in \mathbb{R}.$$

This and other properties will allow operators to be built from simple prototypes in the control applications and examples in later chapters.

Other properties, such as (T3*), are introduced in Chapter 2. Not all subsequent results require these stronger assumptions and, furthermore, not all examples satisfy these extra conditions. Thus, to use them to define the core class would exclude examples unnecessarily.

Whilst a full description is postponed to later sections, we remark here that, from an input-output viewpoint, a diverse range of phenomena are incorporated within the framework developed here.

Operators are defined that capture the behaviour of simple delays (both point and distributed) of the form,

$$(\widehat{T}_p y)(t) := y^m(t-h), \quad (\widehat{T}_d y)(t) := \int_0^t y^m(s-h)ds.$$

Also included are hysteretic effects, the input-output behaviour of linear systems (as in the prototype) and the input-output behaviour of a class of driven nonlinear ordinary differential equations, specifically input-to-state stable systems. Readers seeking more concrete examples of systems that may be recast within such a framework should look ahead to Section 5.1, pg 72.

1.8.2 Existence of solutions to initial-value problems

The operators $\widehat{T}$ are functional operators which, in a certain sense contain "memory" and so their inclusion within a control system such as (1.20) places the resulting closed-loop control system outside classical theory of ordinary differential equations. Moreover, the control strategies will, as remarked before, often contain discontinuities. Hence, we develop and adopt a framework of functional differential inclusions. To study such control problems we present, in Chapter 3, a theory of existence of solutions to such functional differential inclusions which includes three proofs of existence of solutions. The first is a proof using minimal Carathéodory type conditions. In this case we get existence, but not uniqueness of solutions. Secondly assuming Lipschitz-like conditions we prove that solutions are unique. The third theorem concerns existence of solutions to a functional differential inclusion. We remark here that the functional nature of the initial value problem require an initial condition to be defined on an interval of the form $[-h, 0]$. These results, which we believe to be novel, are of independent interest to the control results of subsequent chapters, further justifying the formulation of the theory in terms of the class $\mathcal{T}$ and subsequent conditions.

1.8.3 Stability results for solutions to initial-value problems

In Chapter 4 we develop stability criteria which will underpin the stability analysis of control problems in subsequent chapters. These include applications of Barbălat's lemma (see Lemma 33, pg 61) to solutions of the initial value problems considered in Chapter 3. We further prove an integral invariance result in the context of autonomous systems. One may consider such systems to have as state space the Banach space of continuous functions $C([-h, 0]; \mathbb{R}^N)$ with suitable norm, denoted by $\mathbf{X}$. Systems of this nature have been well studied, however the invariance principle in this context is novel. Again, these results could be used without reference to the control problems of

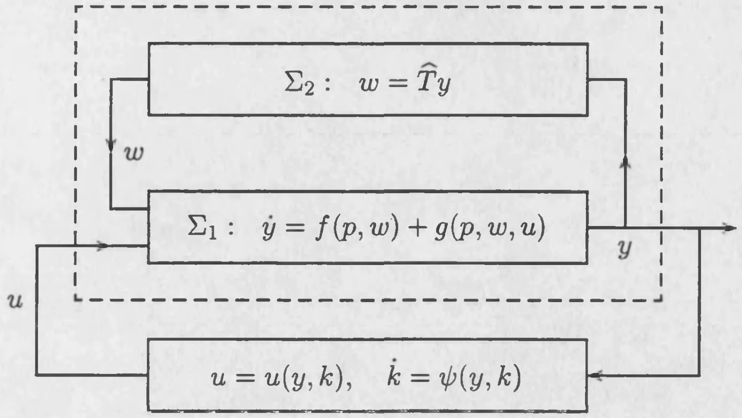


Figure 1-3: Typical adaptive control applied to systems of the form (1.20).

subsequent chapters.

1.8.4 Control systems

We solve various problems pertaining to the design of feedback controllers for systems such as (1.20). For example, as in the linear systems of Section 1.2.1, we seek functions $u(y,k)$ and $\psi(y,k)$ so that the feedback control $u(y(t),k(t))$, where

$$\dot{k}(t) = \psi(y(t),k(t)), \quad k(0) = k^0$$

achieves the control objective, such as, for example, stabilization. See Figure 1-3.

Chapter 5 examines single-input single-output control problems. Control objectives include attractivity of the system output to zero and λ-tracking ($\lambda \geq 0$). Examples are given and numerical simulations performed to illustrate the theory.

Chapter 6 constitutes a similar study in the context of a class of second order systems. It is also proved that under stronger assumptions on the operator, specifically in the context of autonomous systems, the integral invariance principle may be invoked to show that not only the system output $y(t)$ approaches zero asymptotically but the derivative, $\dot{y}(t)$, also approaches zero asymptotically.

Chapter 7 examines multi-input multi-output counterparts. Furthermore we introduce a class of controllers in Section 7.4 which themselves contain functional dependence. Whilst this increases the controller complexity to some extent it open other possibilities for controller design.

It should be noted that, whilst the results contained here incorporate wider classes of systems than previous work, the results of Chapter 7 themselves in the specialized context of the linear system (1.13), for example, constitute improvements on previous work.

1.8.5 Practical implementation of high-gain controllers

The high-gain approach to adaptive control attracts criticism on the grounds that the control strategies, particularly those utilizing switching function or dense searches, are not useful from a practical point of view. We believe that in the light of contemporary work such criticism is not founded. In recent research the high-gain style of adaptive λ-tracking controller has been implemented successfully in physical systems [2, 3, 38, 35, 36] and a modified asymptotic stabilizer incorporating Nussbaum type gains in [60].

Chapter 2

Classes of operators

This Chapter has two parts. First we introduce classes of operators, describe their properties, and establish their basic attributes. Secondly we show that commonly encountered systems and devices can, from an input-output viewpoint, be recast as such operators. The four examples we introduce here - linear systems, input-to-state stable systems, delays and hysteretic effects - have all been well studied however, one contribution of this thesis is to combine them within the following unified framework.

2.1 Classes of operators

Definition 1
For $h \geq 0$ and $N, M \in \mathbb{N}$, let $\mathcal{T}_h^{N,M}$ denote the space of operators $T : C([-h,\infty);\mathbb{R}^N) \to L^\infty_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^M)$ with the following properties.

(T1) *For every $r > 0$ and every bounded interval $I \subset \mathbb{R}_+$, there exists $R > 0$ such that, for all $x \in C([-h,\infty);\mathbb{R}^N)$,*

$$\sup_{t\in[-h,\infty)} \|x(t)\| < r \quad \Longrightarrow \quad \|(Tx)(t)\| < R \quad \text{for a.a. } t \in I.$$

(T2) *For all $t \geq 0$ and all $x, \xi \in C([-h,\infty);\mathbb{R}^N)$,*

$$x(s) = \xi(s) \text{ for all } s \in [-h,t] \quad \Longrightarrow \quad (Tx)(s) = (T\xi)(s) \text{ for a.a. } s \in [0,t].$$

(T3) *For all $t \geq 0$ and continuous $\zeta : [-h,t] \to \mathbb{R}^N$ there exist $r > 0$, $\tau > 0$ and $c > 0$ such that, for all $x, \xi \in C([-h,\infty);\mathbb{R}^N)$ with $x|_{[-h,t]} = \zeta = \xi|_{[-h,t]}$ and*

$x(s), \xi(s) \in \mathbb{B}_r(\zeta(t))$ *for all* $s \in [t, t+\tau]$,

$$\text{ess-sup}_{s\in[t,t+\tau]}\|(Tx)(s) - (T\xi)(s)\| \le c \sup_{s\in[t,t+\tau]} \|x(s) - \xi(s)\| . \tag{2.1}$$

Remarks 2

(i) The essence of (T2) *is that every* $T \in \mathcal{T}_h^{N,M}$ *is causal.*

(ii) Let $t \ge 0$ *and* $x \in C([-h,t);\mathbb{R}^N)$. *Let* x^e *be any class* $C([-h,\infty);\mathbb{R}^N)$ *extension of* x. *The function* $Tx^e|_{[0,t)}$ *is uniquely determined by* x *in the sense that, by virtue of* (T2), *the former is independent of the extension* x^e *chosen for the latter. On this basis, we will adopt the following notational convention: for* $s \in [0,t)$, *we simply write* $(Tx)(s)$ *in place of the notationally cumbersome* $(Tx^e)(s)$ *(where* x^e *is any class* $C([-h,\infty);\mathbb{R}^N)$ *extension of* x*).*

Furthermore, given $T \in \mathcal{T}_h^{N,M}$, $s \in \mathbb{R}_+$ *and* $x \in C((-\sigma,\infty);\mathbb{R}^N)$, *with* $h < \sigma \le \infty$ *we write* $(Tx)(s)$ *in place of the notationally cumbersome* $(Tx|_{[-h,\infty)})(s)$.

(iii) For notational convenience we write $\mathcal{T}_h^N$ *for* $\mathcal{T}_h^{N,N}$.

(iv) The assumption (T3)*may be thought of as a local Lipschitz-like condition.*

Definition 3

Let $\bar{\mathcal{T}}_h^{N,M} \subset \mathcal{T}_h^{N,M}$ *denote the space of operators which satisfy*

(T3*) *For all* $t \ge 0$, $\tau > 0$ *and continuous* $\zeta : [-h,t] \to \mathbb{R}^N$ *there exist* $r > 0$ *and* $c > 0$ *such that, for all* $x, \xi \in C([-h,\infty);\mathbb{R}^N)$ *with* $x|_{[-h,t]} = \zeta = \xi|_{[-h,t]}$ *and* $x(s), \xi(s) \in \mathbb{B}_r(\zeta(t))$ *for all* $s \in [t, t+\tau]$,

$$\text{ess-sup}_{s\in[t,t+\tau]}\|(Tx)(s) - (T\xi)(s)\| \le c\,\text{ess-sup}_{s\in[t,t+\tau]}\|x(s) - \xi(s)\| .$$

Notice the difference between (T3) and (T3*). Specifically that in (T3*) we may choose the $\tau > 0$ that governs the length of the interval over which the estimate (2.1) holds. Properties (T1), (T2) and (T3) are sufficient to prove existence of solutions to the various initial-value differential problems studied in Chapter 3. However, for the invariance principles of Chapter 4 we need the stronger Property (T3*).

We wish to prove that the classes $\bar{\mathcal{T}}_h^{N,M}$ and $\mathcal{T}_h^{N,M}$ satisfy various general and desirable properties. We do this by proving the results for the class $\mathcal{T}_h^{N,M}$ and providing details of how the proofs can be adapted for class $\bar{\mathcal{T}}_h^{N,M}$.

Claim 4 $\mathcal{T}_h^{N,M}$ *is a linear space:*

$$T_1, T_2 \in \mathcal{T}_h^{N,M} \implies (a_1T_1 + a_2T_2) \in \mathcal{T}_h^{N,M} \quad \forall\ a_1, a_2 \in \mathbb{R}.$$

Proof. Let $T_1, T_2 \in \mathcal{T}_h^{N,M}$ and $a_1, a_2 \in \mathbb{R}$. Define $T := (a_1 T_1 + a_2 T_2)$. Clearly $T : C([-h,\infty);\mathbb{R}^N) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^M)$.

Let $r > 0$ then there exist, by (T1), constants $R_1 > 0$ and $R_2 > 0$ such that for all $x \in C([-h,\infty);\mathbb{R}^N)$,

$$\sup_{t\in[-h,\infty)} \|x(t)\| < r \quad \Longrightarrow \quad \|(T_1x)(t)\| < R_1 \text{ and } \|(T_1x)(t)\| < R_2 \quad \text{for a.a. } t \in I\,.$$

Thus, by the triangle inequality,

$$\sup_{t\in[-h,\infty)} \|x(t)\| < r \quad \Longrightarrow \quad \|(Tx)(t)\| < R \quad \text{for a.a. } t \in I$$

where $R := |a_1|R_1 + |a_2|R_2$.

Clearly (T2) holds for T.

Let $t \geq 0$ and $\zeta \in C([-h,t],\mathbb{R}^N)$. By (T3), for $i = 1,2$ there exist constants r_i, c_i and $\tau_i > 0$ corresponding to the operators T_1 and T_2. Let $r := \min\{r_1, r_2\} > 0$, $\tau := \min\{\tau_1,\tau_2\} > 0$ and $c := |a_1|c_1 + |a_2|c_2 > 0$ then for all $x, \xi \in C([-h,\infty);\mathbb{R}^N)$ with $x|_{[-h,t]} = \zeta = \xi|_{[-h,t]}$ and $x(s), \xi(s) \in \mathbb{B}_r(\zeta(t))$ for all $s \in [t, t+\tau]$,

$$\begin{aligned}
&\text{ess-sup}_{s\in[t,t+\tau]}\|(Tx)(s) - (T\xi)(s)\| \\
&\leq |a_1|\,\text{ess-sup}_{s\in[t,t+\tau]}\|(T_1x)(s) - (T_1\xi)(s)\| + |a_2|\,\text{ess-sup}_{s\in[t,t+\tau]}\|(T_2x)(s) - (T_2\xi)(s)\| \\
&\qquad \leq |a_1|c_1 \sup_{s\in[t,t+\tau]}\|x(s) - \xi(s)\| + |a_2|c_2 \sup_{s\in[t,t+\tau]}\|x(s) - \xi(s)\| \\
&\qquad\qquad = c\,\text{ess-sup}_{s\in[t,t+\tau]}\|x(s) - \xi(s)\|\,.
\end{aligned}$$

This proves (T3) holds so that $T \in \mathcal{T}_h^{N,M}$.

Remark 5

The above proof can be easily adapted to show that the class $\bar{\mathcal{T}}_h^{N,M}$ is also a linear space.

Claim 6 *If $h_1 < h_2$, then $\mathcal{T}_{h_1}^{N,M} \subset \mathcal{T}_{h_2}^{N,M}$ in the sense that*

$$T \in \mathcal{T}_{h_1}^{N,M} \implies T^* \in \mathcal{T}_{h_2}^{N,M}$$

where $(T^*x)(t) := (Tx|_{[-h_1,\infty)})(t)$ for all $t \in \mathbb{R}_+$ and $x \in C([-h_2,\infty);\mathbb{R}^N)$.

Proof. Let $h_1 < h_2$ and $T \in \mathcal{T}_{h_1}^{N,M}$. For $x \in C([-h_2,\infty);\mathbb{R}^N)$ if we simply write $(Tx)(t)$ instead of $(Tx|_{[-h_1,\infty)})(t)$ for $t \geq 0$, then $T : C([-h_2,\infty);\mathbb{R}^N) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^M)$.

If

$$\sup_{t\in[-h_2,\infty)} \|x(t)\| < r \implies \sup_{t\in[-h_1,\infty)} \|x(t)\| < r$$
$$\implies \|(Tx)(t)\| < R \quad \text{for a.a. } t \in I.$$

proving that T satisfies (T1) of class $\mathcal{T}_{h_2}^{N,M}$.
Let $t \geq 0$ and $x, \xi \in C([-h_1,\infty);\mathbb{R}^N)$. Then

$$x(s) = \xi(s) \text{ for a.a. } s \in [-h_2, t] \implies x(s) = \xi(s) \text{ for a.a. } s \in [-h_1, t]$$
$$\implies (Tx)(s) = (T\xi)(s) \text{ for a.a. } s \in [0, t].$$

Finally, let $t \geq 0$ and continuous $\zeta : [-h_2, t] \to \mathbb{R}^N$. Then by (T3) there exist $r > 0$, $c > 0$ and $\tau > 0$ such that, for all $x, \xi \in C([-h_2,\infty);\mathbb{R}^N)$ with $x|_{[-h_1,t]} = \zeta|_{[-h_1,t]} = \xi|_{[-h_1,t]}$ and $x(s), \xi(s) \in \mathbb{B}_r(\zeta(t))$ for all $s \in [t, t+\tau]$,

$$\text{ess-sup}_{s\in[t,t+\tau]}\|(Tx)(s) - (T\xi)(s)\| \leq c\sup_{s\in[t,t+\tau]}\|x(s) - \xi(s)\|.$$

Proving that $T \in \mathcal{T}_{h_2}^{N,M}$.

Remark 7
Claim 6 also holds for the class $\bar{\mathcal{T}}_h^{N,M}$: if $h_1 < h_2$ then $\bar{\mathcal{T}}_{h_1}^{N,M} \subset \bar{\mathcal{T}}_{h_2}^{N,M}$. This can be seen as follows.
Let $t \geq 0$, $\tau > 0$ and continuous $\zeta : [-h_2, t] \to \mathbb{R}^N$. Then by (T3) there exist $r > 0$ and $c > 0$ such that, for all $x, \xi \in C([-h_2,\infty);\mathbb{R}^N)$ with $x|_{[-h_1,t]} = \zeta|_{[-h_1,t]} = \xi|_{[-h_1,t]}$ and $x(s), \xi(s) \in \mathbb{B}_r(\zeta(t))$ for all $s \in [t, t+\tau]$,*

$$\text{ess-sup}_{s\in[t,t+\tau]}\|(Tx)(s) - (T\xi)(s)\| \leq c\sup_{s\in[t,t+\tau]}\|x(s) - \xi(s)\|.$$

Claim 8
For all $r \in \mathcal{R} = W^{1,\infty}(\mathbb{R};\mathbb{R}^N)$, (the space of bounded absolutely continuous functions from $\mathbb{R}$ to $\mathbb{R}^N$ with essentially bounded derivatives).

$$T \in \mathcal{T}_h^{N,M} \implies T_r \in \mathcal{T}_h^{N,M}.$$

where T_r is defined by

$$(T_r y)(t) := (T(y+r))(t) \quad \forall t > 0, \quad y \in C([-h,\infty);\mathbb{R}^N). \tag{2.2}$$

Proof. Let $T \in \mathcal{T}_h^{N,M}$ and $r \in \mathcal{R}$, be fixed throughout. Define T_r by (2.2). Clearly

$$T_r : C([-h,\infty);\mathbb{R}^N) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^M).$$

Let I be a bounded interval and $\delta > 0$. Define $r^* := \sup_{s\in[-h,\infty)} \|r(s)\|$. By (T1) there exists some constant $\Delta > 0$ such that

$$\sup_{t\in[-h,\infty)} \|\hat{x}(t)\| < \delta + r^* \quad \Longrightarrow \quad \|(T\hat{x})(t)\| < \Delta \quad \text{for a.a. } t \in I\,.$$

Hence for all $x \in C([-h,\infty);\mathbb{R}^N)$, define $\hat{x}(t) := x(t) + r(t)$ then

$$\begin{aligned}\sup_{t\in[-h,\infty)} \|x(t)\| < \delta \quad &\Longrightarrow \quad \sup_{t\in[-h,\infty)} \|\hat{x}(t)\| < \delta + r^* \\ &\Longrightarrow \quad \|(T_r x)(t)\| < \Delta \quad \text{for a.a. } t \in I\,.\end{aligned}$$

Clearly T_r satisfies (T2).

Let $t > 0$ and $\zeta \in C([-h,t];\mathbb{R}^N)$. Define $\hat{\zeta}(s) := \zeta(s) + r(s)$ for all $s \in [-h,t]$, then by (T3) there exist $\hat{\delta} > 0$, $\hat{\tau} > 0$ and $c > 0$ such that, for all $\hat{x}, \hat{\xi} \in C([-h,\infty);\mathbb{R}^N)$ with $\hat{x}|_{[-h,t]} = \hat{\zeta} = \hat{\xi}|_{[-h,t]}$ and $\hat{x}(s), \hat{\xi}(s) \in \mathbb{B}_{\hat{\delta}}(\hat{\zeta}(t))$ for all $s \in [t, t+\hat{\tau}]$,

$$\text{ess-sup}_{s\in[t,t+\hat{\tau}]} \|(T\hat{x})(s) - (T\hat{\xi})(s)\| \le c \sup_{s\in[t,t+\hat{\tau}]} \|\hat{x}(s) - \hat{\xi}(s)\|\,. \tag{2.3}$$

Let $\delta := \hat{\delta}/2$ then by the essential boundedness of $\dot{r}$ there exists some $0 < \tau < \hat{\tau}$ such that $\|r(s) - r(t)\| < \delta$ for all $s \in [t, t+\tau]$.

Let $x, \xi \in C([-h,\infty);\mathbb{R}^N)$ with $x|_{[-h,t]} = \zeta = \xi|_{[-h,t]}$ and $x(s), \xi(s) \in \mathbb{B}_\delta(\zeta(t))$ for all $s \in [t, t+\tau]$. Define $\hat{x}(s) := x(s) + r(s)$ and $\hat{\xi}(s) := \xi(s) + r(s)$ so that $\hat{x}|_{[-h,t]} = \hat{\zeta} = \hat{\xi}|_{[-h,t]}$ and $\hat{x}(s), \hat{\xi}(s) \in \mathbb{B}_{\hat{\delta}}(\hat{\zeta}(t))$ for all $s \in [t, t+\hat{\tau}]$. Thus

$$\begin{aligned}\text{ess-sup}_{s\in[t,t+\tau]} \|(T_r x)(s) - (T_r \xi)(s)\| &= \text{ess-sup}_{s\in[t,t+\tau]} \|(T\hat{x})(s) - (T\hat{\xi})(s)\| \\ &\le c \sup_{s\in[t,t+\tau]} \|\hat{x}(s) - \hat{\xi}(s)\| \\ &= c \sup_{s\in[t,t+\tau]} \|x(s) - \xi(s)\|\,.\end{aligned}$$

Remark 9 *Let $N, M_1, M_2 \in \mathbb{N}$ and $T_1 \in \mathcal{T}_h^{N,M_1}$ and $T_2 \in \mathcal{T}_h^{N,M_2}$ then T defined by*

$$Ty := \begin{pmatrix} T_1 y \\ T_2 y \end{pmatrix}$$

is of class $\mathcal{T}_h^{N,M_1+M_2}$.

Definition 10 (Shift operator)
For $I \subseteq \mathbb{R}$ be an interval and $s \in \mathbb{R}$, let S_s denote the shift operator on functions $x : I \to \mathbb{R}^N$ given by $(S_s x)(t) := x(t+s)$ for all $t \in \{\tau - s \,|\, \tau \in I\}$.

Claim 11 *For all $s \in \mathbb{R}_+$*

$$T \in \mathcal{T}_h^{N,M} \quad \Longrightarrow \quad TS_{-s} \in \mathcal{T}_{h+s}^{N,M}. \tag{2.4}$$

Proof. Let $T \in \mathcal{T}_h^{N,M}$ and $s \in \mathbb{R}_+$. Assume that $r > 0$ and $I \subset \mathbb{R}_+$ is a bounded interval. By (T1) there exists $R > 0$ such that, for all $x \in C([-h,\infty);\mathbb{R}^N)$,

$$\sup_{t\in[-h,\infty)} \|x(t)\| < r \quad \Longrightarrow \quad \|(Tx)(t)\| < R \quad \text{for a.a. } t \in I.$$

Note that $x \in C([-(h+s),\infty);\mathbb{R}^N)$ if and only if $S_{-s}x \in C([-h,\infty);\mathbb{R}^M)$ and for such an x, $\sup_{t\in[-(h+s),\infty)} \|x(t)\| < r$ if and only if $\sup_{t\in[-h,\infty)} \|S_{-s}x(t)\| < r$. Hence $TS_{-s} : C([-(h+s),\infty);\mathbb{R}^N) \to L^\infty_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^M)$ and

$$\begin{aligned}\sup_{t\in[-(h+s),\infty)} \|x(t)\| < r \Longrightarrow \sup_{t\in[-h,\infty)} \|(S_{-s}x)(t)\| < r \\ \Longrightarrow \|(TS_{-s}x)(t)\| < R \quad \text{for a.a. } t \in I\end{aligned}$$

which proves that TS_{-s} satisfies (T1).
Let $t \geq 0$ and let $\hat{x}, \hat{\xi} \in C([-(h+s),\infty);\mathbb{R}^N)$. Then $\hat{x}(\tau) = \hat{\xi}(\tau)$ for all $\tau \in [-(h+s),t]$ if and only if

$$x(\tau) := (S_{-s}\hat{x})(\tau) = \hat{x}(\tau - s) = \hat{\xi}(\tau - s) = (S_{-s}\hat{\xi})(\tau) =: \xi(\tau) \quad \forall \tau \in [-h, t+s].$$

Hence, by (T2) of $T \in \mathcal{T}_h$

$$(Tx)(\tau) = (T\xi)(\tau) \quad \text{a.a. } \tau \in [0, t+s].$$

and so

$$(TS_{-s}\hat{x})(\tau) = (TS_{-s}\hat{\xi})(\tau) \quad \text{a.a. } \tau \in [0,t].$$

which proves that TS_{-s} satisfies (T2).
Let $t \geq 0$ and $\hat{\zeta} \in C([-(h+s),t];\mathbb{R}^N)$. Define $\zeta(\tau) := S_{-s}\hat{\zeta}(\tau)$ for all $\tau \in [-h, t+s]$. By (T3) there exist $c > 0$, $r > 0$ and $\gamma > 0$ such that, for all $x, \xi \in C([-h,\infty);\mathbb{R}^N)$ with

$x|_{[-h,t+s]} = \zeta|_{[-h,t+s]} = \xi|_{[-h,t+s]}$ and $x(\tau), \xi(\tau) \in \mathbb{B}_r(\zeta(t+s))$ for all $\tau \in [t+s, t+s+\gamma]$

$$\text{ess-sup}_{\tau\in[t+s,t+s+\gamma]}\|(Tx)(\tau) - (T\xi)(\tau)\| \le c \sup_{\tau\in[t+s,t+s+\gamma]} \|x(\tau) - \xi(\tau)\|.$$

Let $\hat{x}, \hat{\xi} \in C([-(h+s), \infty); \mathbb{R}^N)$ satisfy $\hat{x}(\tau) = \hat{\zeta}(\tau) = \hat{\xi}(\tau)$ for all $\tau \in [-(h+s), t]$ and $\hat{x}(\tau), \hat{\xi}(\tau) \in \mathbb{B}_r(\hat{\zeta}(t))$ for all $\tau \in [t, t+\gamma]$. Then, noting that for all $\tau \in [t, t+s]$,

$$x(\tau) := (S_{-s}\hat{x})(\tau) = \zeta(\tau) = (S_{-s}\hat{\xi})(\tau) =: \xi(\tau) \quad \forall \tau \in [-h, t+s]$$

it follows that $(Tx)(\tau) = (T\xi)(\tau)$ for almost all $\tau \in [-h, t+s]$ and

$$\begin{aligned}
\text{ess-sup}_{\tau\in[t,t+\gamma]}\|(TS_{-s}\hat{x})(\tau) - (TS_{-s}\hat{\xi})(\tau)\| &\le \text{ess-sup}_{\tau\in[t,t+s+\gamma]}\|(Tx)(\tau) - (T\xi)(\tau)\| \\
&= \text{ess-sup}_{\tau\in[t+s,t+s+\gamma]}\|(Tx)(\tau) - (T\xi)(\tau)\| \\
&\le c \sup_{\tau\in[t+s,t+s+\gamma]} \|x(\tau) - \xi(\tau)\| \\
&= c \sup_{\tau\in[t,t+\gamma]} \|\hat{x}(\tau) - \hat{\xi}(\tau)\|
\end{aligned}$$

which proves that (T3) holds. Hence $TS_{-s} \in \mathcal{T}_{h+s}$.

Claim 12 *For all $s \in \mathbb{R}_+$,*

$$T \in \mathcal{T}_h^{N,M} \quad \Longrightarrow \quad S_sTS_{-s} \in \mathcal{T}_{h+s}^{N,M}. \tag{2.5}$$

Proof. Let $T \in \mathcal{T}_h^{N,M}$ and $s \ge 0$ be fixed throughout. Let $\hat{T} := S_sTS_{-s}$.
First note that $TS_{-s}x \in L^\infty_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^N)$. Hence $S_sTS_{-s} : C([-(h+s), \infty); \mathbb{R}^N) \to L^\infty_{\text{loc}}([-s, \infty); \mathbb{R}^N)$. Thus we may, restrict the domain of $S_sTS_{-s}x$ to $[0, \infty)$ and regard S_sTS_{-s} as an operator $S_sTS_{-s} : C([-(h+s), \infty); \mathbb{R}^N) \to L^\infty_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^N)$.
Next assume that $r > 0$ and $I \subset \mathbb{R}_+$ is a bounded interval. Without loss of generalization, we may assume I closed, and so $I = [\alpha, \beta]$. (Of course, I need not be closed. The argument in the other cases is identical.) By (T1) there exists $R > 0$ such that, for all $x \in C([-(h+s), \infty); \mathbb{R}^N)$,

$$\sup_{t\in[-(h+s),\infty)} \|x(t)\| < r \quad \Longrightarrow \quad \|(Tx)(t)\| < R \quad \text{for a.a. } t \in [\alpha+s, \beta+s].$$

Thus

$$\sup_{t\in[-(h+s),\infty)} \|x(t)\| < r \quad \Longrightarrow \quad \|(Tx)(t+s)\| < R \quad \text{for a.a. } t \in [\alpha, \beta].$$

Hence

$$\sup_{t\in[-(h+s),\infty)} \|x(t)\| < r \quad \Longrightarrow \quad \|(S_sTS_{-s}x)(t)\| < R \quad \text{for a.a. } t \in I\,.$$

Which proves that S_sTS_{-s} satisfies (T1).

Let $\hat{t} \geq 0$ and define $t := \hat{t} + s$. Assume $\hat{x}, \hat{\xi} \in C([-(h+s),\infty);\mathbb{R}^N)$, with $\hat{x}(s) = \hat{\xi}(s)$ for all $s \in [-(h+s),\hat{t}]$. Let $x(\tau) := S_{-s}\hat{x}(\tau) = \hat{x}(\tau - s)$ and $\xi(\tau) := S_{-s}\hat{\xi}(\tau)$ for all $\tau \in [-h,t]$. Thus

$$x(\tau + s) = \hat{x}(\tau) = \hat{\xi}(\tau) = \xi(\tau + s) \quad \forall \tau \in [-(s+h),\hat{t}].$$

Then by (T2)

$$(Tx)(\tau) = (T\xi)(\tau) \quad \text{a.a. } \tau \in [0,t]$$

so that

$$(Tx)(\tau + s) = (T\xi)(\tau + s) \quad \text{a.a. } \tau \in [-s,\hat{t}].$$

Thus

$$(S_sTx)(\tau) = (S_sT\xi)(\tau) \quad \text{a.a. } \tau \in [-s,\hat{t}].$$

Hence

$$(S_sTS_{-s}\hat{x})(\tau) = (S_sTS_{-s}\hat{\xi})(\tau) \quad \text{a.a. } \tau \in [-s,\hat{t}].$$

This proves that

$$\hat{x}(\tau) = \hat{\xi}(\tau) \text{ for all } \tau \in [-(h+s),\hat{t}] \quad \Longrightarrow \quad (\hat{T}\hat{x})(\tau) = (\hat{T}\hat{\xi})(\tau) \text{ for a.a. } \tau \in [0,\hat{t}]$$

as required.

Let $\hat{t} \geq 0$ and define $t := \hat{t} + s$. Let $\hat{\zeta} \in C([-(h+s),t];\mathbb{R}^N)$, and define $\zeta(\tau) := \hat{\zeta}(\tau - s) = (S_{-s}\hat{\zeta})(\tau)$ for all $\tau \in [-h,t]$.

By (T3) there exist $\gamma > 0$, $r > 0$ and $c > 0$ such that, for all $x, \xi \in L^\infty_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ with $x|_{[-h,t]} = \zeta = \xi|_{[-h,t]}$ and $x(\tau), \xi(\tau) \in \mathbb{B}_r(\zeta(\tau))$ for all $\tau \in [t, t+\gamma]$,

$$\text{ess-sup}_{\tau\in[t,t+\gamma]}\|(Tx)(\tau) - (T\xi)(\tau)\| \leq c\sup_{\tau\in[t,t+\gamma]}\|x(\tau) - \xi(\tau)\|\,.$$

Let $\hat{x}, \hat{\xi} \in C([-(h+s),\infty);R^N)$ with $\hat{x}|_{[-(h+s),\hat{t}]} = \hat{\zeta} = \hat{\xi}|_{[-(h+s),\hat{t}]}$ and $\hat{x}(\tau), \hat{\xi}(\tau) \in \mathbb{B}_r(\hat{\zeta}(\hat{t}))$ for all $\tau \in [\hat{t},\hat{t}+\gamma]$. Define $x(\tau) := S_{-s}\hat{x}(\tau)$ and $\xi(\tau) := S_{-s}\hat{\xi}(\tau)$ for all $\tau \in [-h,t]$.

First note that $x(\tau) = \hat{x}(\tau - s) = \hat{\zeta}(\tau - s) = \zeta(\tau)$ for all $\tau \in [-h,t]$. Similarly $\xi(\tau) = \zeta(\tau)$ for all $\tau \in [-h,t]$.

Next note that $x(\tau) = \hat{x}(\tau - s) \in \mathbb{B}_r(\hat{\zeta}(\hat{t})) = \mathbb{B}_r(\zeta(t))$ for all $\tau \in [-h, t]$. Similarly $\xi(\tau) \in \mathbb{B}_r(\zeta(t))$ for all $\tau \in [-h, t]$.

Thus

$$\begin{aligned}
&\text{ess-sup}_{\tau\in[\hat{t},\hat{t}+\gamma]}\|(S_sTS_{-s}\hat{x})(\tau) - (S_sTS_{-s}\hat{\xi})(\tau)\| \\
&\qquad = \text{ess-sup}_{\tau\in[\hat{t},\hat{t}+\gamma]}\|(S_sTx)(\tau) - (S_sT\xi)(\tau)\| \\
&\qquad = \text{ess-sup}_{\tau\in[\hat{t},\hat{t}+\gamma]}\|(Tx)(\tau+s) - (T\xi)(\tau+s)\| \\
&\qquad = \text{ess-sup}_{\tau\in[t,t+\gamma]}\|(Tx)(\tau) - (T\xi)(\tau)\| \\
&\qquad \leq c\sup_{\tau\in[t,t+\gamma]}\|x(\tau) - \xi(\tau)\| \\
&\qquad = c\sup_{\tau\in[\hat{t},\hat{t}+\gamma]}\|\hat{x}(\tau) - \hat{\xi}(\tau)\|\,.
\end{aligned}$$

Thus,

$$\text{ess-sup}_{\tau\in[\hat{t},\hat{t}+\gamma]}\|(\hat{T}\hat{x})(\tau) - (\hat{T}\hat{\xi})(\tau)\| \leq c\sup_{\tau\in[\hat{t},\hat{t}+\gamma]}\|\hat{x}(\tau) - \hat{\xi}(\tau)\|\,.$$

This completes the proof.

Definition 13 (Bounded-input Bounded-output stable)
$T : C([-h,\infty);\mathbb{R}^N) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^M)$ is said to be bounded-input, bounded-output stable if

$$\forall r > 0\ \exists R > 0 :\ \forall x \in C([-h,\infty);\mathbb{R}^N),\quad \sup_{t\in[-h,\infty)} \|x(t)\| < r \implies \\ \|(Tx)(t)\| < R \text{ for a.a. } t \in \mathbb{R}_+\,. \tag{2.6}$$

If T is linear, T is bounded-input bounded-output stable if and only if T is a bounded operator.

Definition 14 (Right shift invariance)
$T \in \mathcal{T}_h^{N,M}$ is said to be right shift invariant *if for all $x \in C([-h,\infty);\mathbb{R}^N)$ and for all $s \in \mathbb{R}_+$*

$$(S_sTx)(t) = (TS_sx)(t) \quad \text{for a.a.} \quad t \in \mathbb{R}_+ \tag{2.7}$$

(where the right hand side is interpreted as $(T(S_sx)|_{[-h,\infty]})(t)$).

Claim 15
If T is a right shift invariant operator then for all $x, \xi \in C([-h,\infty);\mathbb{R}^N)$ and $t, \omega \geq 0$,

$$x(\tau) = \xi(\tau) \text{ for all } \tau \in [t-h, t+\omega] \quad \Longrightarrow \quad (Tx)(\tau) = (T\xi)(\tau) \text{ for a.a. } \tau \in [t, t+\omega].$$

Proof. Let $t, \omega \geq 0$ and $x, \xi \in C([-h,\infty);\mathbb{R}^N)$ satisfy $x(\tau) = \xi(\tau)$ for all $\tau \in$

$[t-h, t+\omega]$. Define $\hat{x}(\tau) := (S_t x)(\tau)$ and $\hat{\xi}(\tau) := (S_t\xi)(\tau)$ for all $\tau \in [-h, \omega]$. So that $\hat{x}|_{[-h,\omega]} = \hat{\xi}|_{[-h,\omega]}$. By (T2)

$$(T\hat{x})|_{[0,\omega]} = (T\hat{\xi})|_{[0,\omega]} \implies (TS_t x)|_{[0,\omega]} = (TS_t\xi)|_{[0,\omega]}$$

but T is right shift invariant and hence

$$(S_t Tx)|_{[0,\omega]} = (S_t T\xi)|_{[0,\omega]} \implies (Tx)(\tau) = (T\xi)(\tau) \quad \forall \tau \in [t, t+\omega]$$

as required.

For the purposes of constructing control strategies *a priori* estimates on the input-output behaviour of these operators will be needed.

Definition 16 (Control estimates)
For $h \geq 0$, $N, M \in \mathbb{N}$ and continuous $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ we consider the following subclasses of $\mathcal{T}_h^{N,M}$.

Let $\mathcal{C}_{1,h}^{N,M}(\psi) = \mathcal{C}_1(\psi)$ denote the subclass of $\mathcal{T}_h^{N,M}$ such that for some constant $\mu \geq 0$, the following holds: for each $x \in C([-h,\infty); \mathbb{R}^N)$, there exists a constant $c \geq 0$ such that

$$\int_0^t \|x(s)\| \, \|(Tx)(s)\| \, ds \leq c + \mu \int_0^t \psi(\|x(s)\|)\|x(s)\| \, ds \quad \forall\, t \geq 0. \tag{2.8}$$

Let $\mathcal{C}_{2,h}^{N,M}(\psi) = \mathcal{C}_2(\psi)$ denote the subclass of $\mathcal{T}_h^{N,M}$ such that for some constant $\mu \geq 0$, the following holds: for each $x \in C([-h,\infty); \mathbb{R}^N)$, there exists a constant $c \geq 0$ such that

$$\int_0^t \|(Tx)(s)\|^2 \, ds \leq c + \mu \int_0^t \psi(\|x(s)\|)\|x(s)\| \, ds \quad \forall\, t \geq 0. \tag{2.9}$$

Let $\mathcal{C}_{3,h}^{N,M}(\psi) = \mathcal{C}_3(\psi)$ denote the subclass of $\mathcal{T}_h^{N,M}$ such that for some constant $\mu \geq 0$ such that for each $x \in C([-h,\infty); \mathbb{R}^N)$,

$$\|(Tx)(t)\| \leq \mu(1 + \max_{s\in[0,t]} \psi(\|x(s)\|)) \quad \forall\, t \in \mathbb{R}_+ . \tag{2.10}$$

The differences between these classes will be illustrated with examples below. The estimate (2.8) will be used in Chapter 5, estimate (2.9) on the work on second order systems in Chapter 6, and estimate (2.10) in the work on multi-input multi-output systems in Chapter 7. We remark that any operator of class $\mathcal{C}_3$ will be bounded-input bounded-output stable in the sense of Definition 13.

Definition 17 (Class $\mathcal{S}$)
For $h \geq 0$, $N, M \in \mathbb{N}$, let $\mathcal{S}_h^{N,M} = \mathcal{S}$ denote the subclass of $\mathcal{T}_h^{N,M}$ for which there exists constants $\mu \geq 0$ and $\delta \geq 0$ such that for all $x, \xi \in C([-h,\infty);\mathbb{R}^N)$

$$\|T(x+\xi)(t)\| \leq \mu \left[\|(T\delta x)(t)\| + \|(T\delta\xi)(t)\|\right] \quad \forall t \in \mathbb{R}. \tag{2.11}$$

Remark 18 *If $T \in \mathcal{T}$ is linear then $T \in \mathcal{S}$.*

2.2 Examples

2.2.1 Input-to-state stable systems

Let $Z : \mathbb{R}^P \times \mathbb{R}^N \to \mathbb{R}^P$ be locally Lipschitz with $Z(0,0) = 0$. For $x \in L^\infty_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^N)$, let $z(\cdot, z^0, x)$ denote the unique maximal solution of the initial-value problem

$$\dot{z}(t) = Z(z(t), x(t)), \quad z(0) = z^0 \in \mathbb{R}^P. \tag{2.12}$$

Definition 19 (Input-to-state stability)
The system (2.12) is said to be input-to-state stable (ISS) if there exist functions $\theta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for all $(z^0, x) \in \mathbb{R}^P \times L^\infty_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^N)$,

$$\|z(t, z^0, x)\| \leq \theta(\|z^0\|, t) + \text{ess-sup}_{s\in[0,t]}\gamma(\|x(s)\|) \quad \forall\, t \geq 0. \tag{2.13}$$

There is an extensive literature concerning ISS systems. See [72], and for example, [78, 73, 80, 79, 46]. Also [75, Exercise 7.3.11]. In particular [74, Theorem 1];

Theorem 20 (Sontag)
System (2.12) is ISS if and only if there exist $\rho, \gamma, \kappa \in \mathcal{K}_\infty$ so that the following estimate holds for all initial states $z^0 \in \mathbb{R}^P$ and all inputs $x(\cdot)$:

$$\int_0^t \rho(\|z(s, z^0, x)\|)ds \leq \kappa(\|z^0\|) + \hat{\mu}\int_0^t \gamma(\|x(s)\|)ds \quad \forall\, t \geq 0. \tag{2.14}$$

Note that there is a related, but separate, notion termed Integral Input-to-State Stability [81]. In particular the system (2.12) is said to be Integral Input-to-state stable (IISS) if and only if there exist functions $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ such that for all $(z^0, x) \in \mathbb{R}^P \times L^\infty_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^N)$,

$$\|z(t, z^0, x)\| \leq \beta(\|z^0\|, t) + \gamma_1 \int_0^t \gamma_2(\|x(s)\|)ds \quad \forall\, t \geq 0.$$

The two concepts are not equivalent. In particular [81] proves that the system

$$\dot{z}(t) = -\arctan(z(t)) + x(t), \quad z(0) = z^0 \in \mathbb{R}$$

is IISS but with $z(0) = 1$ and $x(t) \equiv \pi/3$ the trajectory is unbounded and so the system is not ISS.

Assume that the system (2.12) is input-to-state stable (ISS). Let $W : \mathbb{R}^P \to \mathbb{R}^M$ be locally Lipschitz and such that there exists $L > 0$ such that $\|W(z)\| \leq L\|z\|$ for all z. Assume system (2.12) has output w given by

$$w(t) = W(z(t, z^0, x)).$$

Fix $z^0 \in \mathbb{R}^P$ arbitrarily. Define the operator $T : C(\mathbb{R}_+; \mathbb{R}^N) \to L^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R}^M)$ by

$$(Tx)(t) := W(z(t, z^0, x)). \tag{2.15}$$

In effect we define a family of operators T_{z^0}, parameterized by the initial condition z^0.

Membership of $\mathcal{T}_h^{N,M}$

Claim 21 *The operator (2.15) is of class $\mathcal{T}_0^{N,M}$.*

Proof. In view of (2.13) and properties of W, there exists $c > 0$ such that, for all $x \in C(\mathbb{R}_+; \mathbb{R}^N)$,

$$\|(Tx)(s)\| \leq c\left[1 + \sup_{\tau\in[0,s]}\gamma(\|x(\tau)\|)\right] \quad \forall\, s \geq 0. \tag{2.16}$$

(T1) and (T2) evidently hold. To establish that (T3*) also holds we proceed as follows. Let $t \geq 0$, $\rho > 0$ and $\zeta \in C([0,t]; \mathbb{R}^N)$. Let $r > 0$ and define $R := \sup_{\tau\in[0,t]} \|\zeta(\tau)\| + r$. By input-to-state stability there exists a compact set $K \subset \mathbb{R}^P$ such that, for all x with $\sup_{\tau\in\mathbb{R}_+} \|x(\tau)\| \leq R$, we have $z(s, z^0, x) \in K$ for all $s \geq 0$. Let $\lambda > 0$ be a Lipschitz constant for $Z(\cdot,\cdot)$ on the set $K \times \mathbb{B}_R(0)$. For all $x, y \in C(\mathbb{R}_+; \mathbb{R}^N)$ with $x|_{[0,t]} = \zeta = y|_{[0,t]}$ and $\|x(s)\|, \|y(s)\| \leq R$ for almost all $s \in [0, t+\rho]$,

$$\begin{aligned} \|z(s,z^0,x) - z(s,z^0,y)\| &\leq \int_0^s \|Z(z(\tau,z^0,x),x(\tau)) - Z(z(\tau,z^0,y),y(\tau))\|d\tau \\ &\leq \lambda \int_t^s [\|z(\tau,z^0,x) - z(\tau,z^0,y)\| + \|x(\tau) - y(\tau)\|]d\tau \end{aligned}$$

for all $s \in [t, t+\rho]$. By a version of Gronwall's Lemma, (see Lemma 82, pg 134), it follows that,

$$\|z(s,z^0,x) - z(s,z^0,y)\| \le \lambda \int_t^s \exp(\lambda(s-\tau))\|x(\tau) - y(\tau)\|ds \quad \forall\ s \in [t, t+\rho]. \quad (2.17)$$

We may now conclude that there exists a constant $c_R > 0$ such that, for all $x, y \in C(\mathbb{R}_+; \mathbb{R}^N)$ with $x|_{[0,t]} = \zeta = y|_{[0,t]}$ and $\|x(s)\|, \|y(s)\| \le R$ for almost all $s \in [t, t+\rho]$,

$$\|(Tx)(s) - (Ty)(s)\| \le c_R \mathrm{sup}_{s\in[t,t+\rho]}\|x(s) - y(s)\| \quad \forall\ s \in [t, t+\rho]. \quad (2.18)$$

Therefore, (T3*) holds.

Remark 22 *Note that (2.16) immediately implies that T is bounded-input bounded-output in the sense of Definition 13.*

Control estimates

Let $w(t) := W(z(t,z^0,x)) \in \mathbb{R}$ be the output of a single-input, single-output initial-value system (2.12) where W is locally Lipschitz and is such that $|W(z)| \le L\|z\|$ for all $z \in \mathbb{R}^P$. Let $\alpha, \beta, \kappa \in \mathcal{K}_\infty$. Define $\rho, \eta, \gamma \in \mathcal{K}_\infty$ to be the indefinite integrals of α, its inverse α^{-1}, and β, respectively and assume that

$$\rho(r) := \int_0^r \alpha, \quad \eta(r) := \int_0^r \alpha^{-1} \le r\alpha^{-1}(r), \quad \gamma(r) := \int_0^r \beta \le r\beta(r)\,.$$

(for example, for fixed $s > 0$ the $\mathcal{K}_\infty$ function $\phi : \tau \mapsto \tau^s$ satisfies $\int_0^r \phi \le r\phi(r)$ for all $r \in \mathbb{R}_+$.) Assume there exists some constant $\hat{\mu}$ such that, for all $(z^0, x) \in \mathbb{R}^P \times L^\infty_{\mathrm{loc}}(\mathbb{R})$,

$$\int_0^t \rho(\|z(s,z^0,x)\|)ds \le \kappa(\|z^0\|) + \hat{\mu}\int_0^t \gamma(|x(s)|)ds \quad \forall\ t \ge 0\,,$$

and so the system under consideration is input-to-state stable by Theorem 20.
Let $z^0 \in \mathbb{R}^P$ be arbitrary. Similar to (2.15), define $\widehat{T}$ by

$$(\widehat{T}x)(t) := w(t), \quad t \ge 0.$$

By Claim 21 this operator is of class $\mathcal{T}_0^1$. Invoking Young's inequality (Theorem 79, pg 133),

$$\begin{aligned} x(s)(\widehat{T}x)(s) &\leq L|x(s)|\|z(s,z^0,x)\| \\ &\leq L\int_0^s \alpha(\|z(\tau,z^0,x)\|) + \alpha^{-1}(|x(\tau)|)\, d\tau \\ &\leq L\left[\rho(\|z(s,z^0,x)\|) + \eta(|x(s)|)\right] \end{aligned}$$

whence, for $\mu := \hat{\mu} + 1$

$$\int_0^t x(s)(\widehat{T}x)(s)ds \leq L\left[\kappa(\|z^0\|) + \mu\int_0^t [\gamma(|x(s)|) + \eta(|x(s)|)]ds\right] \quad \forall\, t \geq 0.$$

Recalling that $\gamma(r) + \eta(r) \leq r[\alpha^{-1}(r) + \beta(r)]$, it follows that $\widehat{T} \in \mathcal{C}_1(\psi)$ provided that $\psi(r) \geq \alpha^{-1}(r) + \beta(r)$ for all $r \in \mathbb{R}_+$.

Next, if one knows *a priori* the function γ, in (2.16), we see that the operator is of class $\mathcal{C}_3(\gamma)$.

Control estimates - a specific example

Let $w(t) := W(z(t,z^0,x)) \in \mathbb{R}$ be the output of a single-input, single-output initial-value problem of the form

$$\left.\begin{aligned} \dot{z}(t) &= f(t,z(t)) + g(t,z(t),x(t)), \\ z(0) &= z^0 \in \mathbb{R}, \\ w(t) &= W(z(t)) \end{aligned}\right\} \tag{2.19}$$

where:

1. f is locally Lipschitz and there exists some $\epsilon > 0$ and $\alpha > 2$ such that

$$zf(t,z) \leq -\epsilon z^2(1 + |z|^{\alpha-2}) \text{ for all } t,z \in \mathbb{R},$$

2. g is locally Lipschitz and there exists $\tilde{\mu} \in \mathbb{R}_+$ and $n,m \in \mathbb{N}$ with $n < \alpha$ such that for all $t,z,x \in \mathbb{R}$, $|g(t,z,x)| \leq \tilde{\mu}(1 + |z|^{n-1})(|x| + |x|^m)$,

3. W is such that $|W(z)| \leq L\|z\|$ for some constant L and all $z \in \mathbb{R}$.

Remark 23

As a specific example of a function g that satisfies part 2, let $n,m \in \mathbb{N}$ be fixed with $n < \alpha$ and let $p_{i,j} \in L^\infty(\mathbb{R})$ with $\|p_{i,j}\|_\infty \leq p^*$ for all $0 \leq i < n$ and $0 < j \leq m$. Define

$g : \mathbb{R}^3 \to \mathbb{R}$ as

$$g(t,z,x) := \sum_{j=1}^{m}\sum_{i=0}^{n-1} p_{i,j}(t) z^i x^j.$$

Then we estimate

$$|g(t,z,x)| \le p^* \sum_{j=1}^{m}\sum_{i=0}^{n-1} |z|^i |x|^j \le p^* mn(1+|z|^{n-1})(|x|+|x|^m).$$

Claim 24 *Let $z^0 \in \mathbb{R}$ be arbitrary. Define $\widehat{T} \in \mathcal{T}_0^1$ by*

$$(\widehat{T}x)(t) := w(t)$$

then the system (2.19) is ISS, $\widehat{T} \in \mathcal{T}_0^1$ and there exist constants c and μ such that for all $x \in C(\mathbb{R}_+;\mathbb{R})$,

$$\int_0^t x(s)(\widehat{T}x)(s)\, ds \le cz^2(0) + \mu \int_0^t \psi(|x(s)|)|x(s)|\, ds \quad \forall\, t \ge 0 \tag{2.20}$$

where $\psi : x \mapsto 1 + x^{\bar{m}}$ and $\bar{m} \ge \max\{2m-1, \frac{\alpha m}{\alpha - n} - 1\}$.

Proof. The properties of f and g give

$$z(t)\dot{z}(t) \le -\epsilon z^2(t) - \epsilon |z|^\alpha(t) + \tilde{\mu}(|z(t)| + |z(t)|^n)(|x(t)| + |x(t)|^m)$$

for all $t \ge 0$. Estimating, by repeated use of Example 80, pg 133,

$$\left.\begin{array}{lcl} \tilde{\mu} z x^m & \le & \frac{\epsilon}{4} z^2 + \hat{\mu} x^{2m} \\ \tilde{\mu} z^n x^m & \le & \frac{\epsilon}{4} z^\alpha + \hat{\mu} x^{\frac{\alpha m}{\alpha - n}} \\ \tilde{\mu} z x & \le & \frac{\epsilon}{4} z^2 + \hat{\mu} x^2 \\ \tilde{\mu} z^n x & \le & \frac{\epsilon}{4} z^\alpha + \hat{\mu} x^{\frac{\alpha}{\alpha - n}} \end{array}\right\} \quad \forall z, x \in \mathbb{R}_+$$

for some constant $\hat{\mu}$ we have

$$\begin{aligned} z(t)\dot{z}(t) &\le -\frac{\epsilon}{2} z^2(t) - \frac{\epsilon}{2} z^\alpha(t) + \hat{\mu}\left(x^2(t) + x^{2m}(t) + |x(t)|^{\frac{\alpha m}{\alpha-n}} + |x(t)|^{\frac{\alpha}{\alpha-n}}\right) \\ &\le -\frac{\epsilon}{4} z^2(t) + 4\hat{\mu}\left(|x(t)| + |x(t)|^{\bar{m}+1}\right) \end{aligned}$$

where $\bar{m} + 1 \ge \max\{2m, \frac{\alpha m}{\alpha - n}\}$. Thus, for some constants c and d,

$$\int_0^t z^2(s)\, ds \le cz^2(0) + d \int_0^t |x(s)| + |x(s)|^{\bar{m}+1}\, ds$$

which proves that (2.19) is ISS. Further,

$$\begin{aligned}\int_0^t x(s)(\widehat{T}x)(s)\,ds &\le L\int_0^t |x(s)||z(s,z^0,x)|\,ds\\ &\le \frac{L}{2}\int_0^t z^2(s)\,ds + \frac{L}{2}\int_0^t x^2(s)\,ds\\ &\le cz^2(0) + \mu\int_0^t \psi(|x(s)|)|x(s)|\,ds\end{aligned}$$

for some constant μ.

Example 25 *As a specific example consider the system, [73],*

$$\dot{z}(t) = -a_0 z(t) - a_1 z^3(t) + a_2(1 + z^2(t))x(t) \tag{2.21}$$

with $a_0, a_1 > 0$. This is evidently of the form (2.19) with $\alpha = 4$, $n = 3$ and $m = 1$. Thus, taking $\bar{m} = 3$ is sufficient and we let $\psi(r) = 1 + r^3$. That system (2.21) is ISS and of class $C_1(\psi)$ follows from Claim 24.

2.2.2 Regular linear systems with bounded observation operator

Let $\mathbf{G}$ be the transfer function of a regular (in the sense of [85]), linear system with state space X (a Hilbert space), with generating operators (A, B, C, D) and with $\mathbb{R}^N$-valued input and $\mathbb{R}^M$-valued output. This means, in particular, that (i) A generates a strongly continuous semigroup $\mathbf{S} = (\mathbf{S}_t)_{t\ge 0}$ of bounded linear operators on X, (ii) the control operator B is a bounded linear operator from $\mathbb{R}^N$ to X_{-1}, (iii) the observation operator C is a bounded linear operator from X_1 to $\mathbb{R}^M$, and (iv) the feed through operator D is a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^M$. Here X_1 denotes the space dom(A) (the domain of A) endowed with the graph norm and X_{-1} denotes the completion of X with respect to the norm $\|z\|_{-1} = \|(s_0 I - A)^{-1} z\|$, where s_0 is any fixed element of the resolvent set of A and $\|\cdot\|$ denotes the norm on X. As a regular linear system, the transfer function $\mathbf{G}$ is holomorphic and bounded on every half-plane $\mathbb{C}_\alpha$ with $\alpha > \omega(\mathbf{S}) := \lim_{t\to\infty} t^{-1}\ln\|\mathbf{S}_t\|$. Moreover,

$$\lim_{s\to\infty,\ s\in\mathbb{R}} \mathbf{G}(s) = D\,.$$

The system is said to be exponentially stable if the semigroup $\mathbf{S}$ is exponentially stable, that is, if $\omega(\mathbf{S}) < 0$.

We assume that the observation operator C can be extended to a bounded linear operator from X to $\mathbb{R}^M$: this extended operator is again denoted by C and is the bounded observation operator referred to in the subsection title.

In terms of the generating operators (A, B, C, D), the transfer function $\mathbf{G}$ is given by

$$\mathbf{G}(s) = C(sI - A)^{-1}B + D.$$

For any $z^0 \in X$ and input $x \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^N)$, the state z and the output w of the regular system (with bounded observation operator) satisfy the equations

$$\dot{z}(t) = Az(t) + Bx(t), \quad z(0) = z^0, \tag{2.22}$$

$$w(t) = Cz(t) + Dx(t) \tag{2.23}$$

for almost all $t \geq 0$. The derivative on the left-hand side of (2.22) has, of course, to be understood in X_{-1}. In other words, if we consider the initial-value problem (2.22) in the space X_{-1}, then for any $z^0 \in X$ and $x \in L^2_{\text{loc}}(\mathbb{R}; \mathbb{R}^N)$, (2.22) has unique strong solution given by the variation of parameters formula (see Theorem 2.9 (Chapter 4) in [67])

$$z(t) = \mathbf{S}_t z^0 + \int_0^t \mathbf{S}_{t-s} Bx(s)\, ds. \tag{2.24}$$

Moreover, for every $\tau > 0$, there exists a constant $a_\tau > 0$ such that, for all $z^0 \in X$ and all $x \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^N)$,

$$\|z(t)\| \leq a_\tau \left[\|z^0\| + \left(\int_0^t \|x(s)\|^2\, ds \right)^{\frac{1}{2}} \right] \quad \forall\, t \in [0, \tau]. \tag{2.25}$$

If the semigroup $\mathbf{S}$ is exponentially stable, then there exist constants $a, c_1, c_2 > 0$ such that

$$\|z(t)\| \leq a \left[\|z^0\| + \left(\int_0^t \|x(s)\|^2\, ds \right)^{\frac{1}{2}} \right], \quad \forall\, (t, z^0, x) \in \mathbb{R}_+ \times X \times L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^N). \tag{2.26}$$

$$\|z\|_{L^2(\mathbb{R}_+;X)} \leq c_1 \left[\|z^0\| + \|x\|_{L^2(\mathbb{R}_+;\mathbb{R}^N)} \right], \quad \forall\, (z^0, x) \in X \times L^2(\mathbb{R}_+; \mathbb{R}^N). \tag{2.27}$$

$$\|z\|_{L^\infty(\mathbb{R}_+;X)} \leq c_2 \left[\|z^0\| + \|x\|_{L^\infty(\mathbb{R}_+;\mathbb{R}^N)} \right], \quad \forall\, (z^0, x) \in X \times L^\infty(\mathbb{R}_+; \mathbb{R}^N). \tag{2.28}$$

Define the operator $T : C(\mathbb{R}_+; \mathbb{R}^N) \to L^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R}^M)$ by

$$(Tx)(t) := C \int_0^t \mathbf{S}_{t-s} Bx(s)\; ds + Dx(t), \quad t \geq 0 \tag{2.29}$$

and so $w(t) = C\mathbf{S}_t z^0 + (Tx)(t)$ for all $t \geq 0$. In general when solving a control problem, the contribution $C\mathbf{S}_t z^0$, from the initial state of the linear system, will be absorbed as a bounded perturbation, as in the finite-dimensional prototype (1.18). Note that, in view of (2.25) and boundedness of C, for each $\tau > 0$ there exists a_τ such that, for all $x \in L^\infty_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$,

$$\|(Tx)(t)\| \leq a_\tau \|C\| \left(\int_0^t \|x(s)\|^2 ds\right)^{\frac{1}{2}} + \|D\|\|x(t)\| \quad \text{for a.a. } t \in [0,\tau]$$
$$\text{and } \|(Tx)(t)\| = 0 \quad \forall t < 0. \tag{2.30}$$

Membership of $\bar{\mathcal{T}}_h^{N,M}$

In view of (2.30), (T1) evidently holds; setting $h = 0$, we see that (T2) also holds and (T3*) is a consequence of linearity of T and (2.30). Therefore, the operator T is of class $\bar{\mathcal{T}}_0^{N,M}$.

If the generator of the semigroup is exponentially stable then (2.28) gives bounded-input bounded-output stability.

Control estimates

Let ψ be such that $\psi(r) \geq r$ for all $r \in \mathbb{R}_+$. Let (A, B, C, D) be the generating operators of a single-input, single-output, regular linear system with bounded observation operator C. Assume that the semigroup $\mathbf{S}$ generated by A is exponentially stable. Similar to (2.29), define $\widehat{T}$ by

$$(\widehat{T}x)(t) := C\int_0^t \mathbf{S}_{t-s}Bx(s)\, ds + Dx(s), \quad t \geq 0.$$

We argue that $\widehat{T} \in \mathcal{C}$ by first noting that by (2.24) and (2.27) (with $z^0 = 0$)

$$\int_0^t \left(\int_0^\tau \mathbf{S}_{\tau-s}Bx(s)\, ds\right)^2 d\tau \leq c_1 \int_0^t |x(s)|^2 ds.$$

and hence for all $t > 0$,

$$\int_0^t |x(s)|\, |(Tx)(s)|\, ds \leq \int_0^t (1 + 2|D|^2 + 2c_1\|C\|^2)|x(s)|^2 \leq \mu \int_0^t \psi(|x(s)|)|x(s)|\, ds$$

for some constant μ and hence $\widehat{T} \in \mathcal{C}_1(\psi)$.

Note that (2.28) and causality imply the operator is of class $\mathcal{C}_3(\alpha_T)$ with the $\mathcal{J}$-function $\alpha_T(s) = s$.

Further by linearity $\widehat{T} \in \mathcal{S}$.

A specific example: controlled diffusion process

Consider a diffusion process, [15], on the one-dimensional spacial domain $\Omega = [0,1]$, with diffusion coefficient $\alpha > 0$, scalar pointwise input v applied at an interior point x_0 of $\Omega = [0,1]$

$$z_t(t,x) = \alpha z_{xx}(t,x) + \delta(x - x_0)v(t), \quad z(t,0) = 0 = z(t,1) \quad \forall t > 0$$

with zero initial conditions. With output given by the bounded delayed scalar observation generated by spatial averaging

$$(Tv)(t) := \frac{1}{2\epsilon}\int_{x_1-\epsilon}^{x_1+\epsilon} z(t-h,x)\, dx, \quad x_1 \in (\epsilon, 1-\epsilon)$$

this qualifies as a regular linear system with bounded observation operator and has been exploited as an example of a regular system in, for example, [54, 52, 53, 50, 49].

A specific example: controlled wave equation

Let $0 < x_1 < x_0 < 1$, $\epsilon > 0$ and $h > 0$ with $\min\{x_1, x_0 - x_1\} > \epsilon > 0$ be fixed. For a given input function $v(\cdot) \in C(\mathbb{R}_+;\mathbb{R})$, let $z(t,x) : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ be the solution to the point-forced wave equation with zero initial conditions and Dirichlet boundary conditions:

$$\left.\begin{array}{l} z_{tt}(t,x) = \alpha z_{xx}(t,x) - \beta z_t(t,x) \\ \lim_{\xi\uparrow x_0} z_x(t,\xi) - \lim_{\xi\downarrow x_0} z_x(t,\xi) = \gamma v(t) \\ z(t,0) = z(t,1) = 0 \text{ for all } t \geq 0 \\ z(t,x) = z_t(t,x) = 0 \text{ for all } x \in [0,1],\ t \in [-h,0] \end{array}\right\} \tag{2.31}$$

where $\alpha, \beta > 0$.

Define the scalar delay observation (output) operator by

$$(Tv)(t) := \frac{1}{2\epsilon}\int_{x_1-\epsilon}^{x_1+\epsilon} z(t-h,x)\, dx\,. \tag{2.32}$$

This is a bounded operator and for small $\epsilon > 0$ it approximates a point observation operator of the form $(Tv)(t) = z(t-h, x_1)$ which is unbounded.

Let $\mathcal{L}$ denote the Laplace transform then the following formal calculation shows that the transfer function $G(s)$ of the input/output system defined by (2.31) and (2.32) is given by

$$G(s) = \frac{e^{-hs}\gamma \sinh((1-x_0)\eta(s))[\cosh((x_1-\epsilon)\eta(s)) - \cosh((x_1+\epsilon)\eta(s))]}{2\epsilon(s^2+\beta s)\sinh(\eta(s))}$$

where $\eta(s) := \sqrt{s(s+\beta)/\alpha}$.

For each $x \in [0,1]$ let $\hat{z}(s,x) := \mathcal{L}z(\cdot,x)(s)$. This is a function of the transformed variable s, parametrized by x. Also let $\hat{v}(s) := \mathcal{L}v(s)$ denote the Laplace transform of v. Taking the Laplace transform of (2.31) and substituting the initial conditions gives

$$\left.\begin{array}{l} s^2\hat{z}(s,x) = \alpha\hat{z}_{xx}(s,x) - \beta s\hat{z}(s,x) \\ \lim_{\xi\uparrow x_0}\hat{z}_x(s,\xi) - \lim_{\xi\downarrow x_0}\hat{z}_x(s,\xi) = \gamma\hat{v}(s). \end{array}\right\}$$

For each fixed value of s this is a second order ordinary differential equation with constant coefficients and has a solution composed of two the pieces

$$\hat{z}(s,x) = \left\{\begin{array}{ll} a_1e^{\eta(s)x} + a_2e^{-\eta(s)x} & \text{for } x \in [0,x_0) \\ a_3e^{\eta(s)x} + a_4e^{-\eta(s)x} & \text{for } x \in (x_0,1] \end{array}\right.$$

for $\eta(s) := \sqrt{s(s+\beta)/\alpha}$ and appropriate constants a_1, a_2, a_3 and a_4 which are still to be determined. The boundary condition $z(\cdot,0) \equiv 0$ gives $\hat{z}(s,0) \equiv 0$. Thus $a_1 + a_2 = 0$. Any solution for $\hat{z}(\cdot;s)$ must be continuous at x_0. Thus $a_1e^{\eta(s)x_0} + a_2e^{-\eta(s)x_0} = a_3e^{\eta(s)x_0} + a_4e^{-\eta(s)x_0}$. Left and right derivatives of the two pieces differ by $\frac{\gamma}{\alpha}\hat{v}(s)$ at x_0 so that

$$a_1\eta(s)e^{\eta(s)x_0} - a_2\eta(s)e^{-\eta(s)x_0} = a_3\eta(s)e^{\eta(s)x_0} - a_4\eta(s)e^{-\eta(s)x_0} + \frac{\gamma}{\alpha}\hat{v}(s).$$

Lastly the boundary condition $z(\cdot,1) \equiv 0$ gives $a_3e^{\eta(s)} + a_4e^{\eta(s)} = 0$. This yields four equations in the four unknowns a_1, a_2, a_3 and a_4. Solving for a_1 and a_2 only (since $x_1 < x_0$) gives

$$a_1 = \frac{\gamma(e^{\eta(s)x_0} - e^{(2-x_0)\eta(s)})\hat{v}(s)}{2\alpha\eta(s)(1-e^{2\eta(s)})}$$

and

$$a_2 = \frac{-\gamma(e^{2\eta(s)x_0} - e^{2\eta(s)})\hat{v}(s)}{2\alpha\eta(s)e^{2\eta(s)x_0}(1-e^{2\eta(s)})}.$$

Substituting these and simplifying gives

$$\hat{z}(s,x) = \frac{\gamma\sinh((1-x_0)\eta(s))\sinh(\eta(s)x)\hat{v}(s)}{\alpha\eta(s)\sinh(\eta(s))}$$

for $x \in [0,x_0)$. Next taking the Laplace transform of $(Tv)(t)$ and applying both the

Fubini theorem and the shifting theorem gives

$$\begin{aligned}\mathcal{L}(Tv)(s) &= \frac{e^{-hs}}{2\epsilon}\int_{x_1-\epsilon}^{x_1+\epsilon} \hat{z}(s,x)\,dx \\ &= \frac{e^{-hs}\gamma\sinh((1-x_0)\eta(s))\hat{v}(s)}{\alpha\eta(s)\sinh(\eta(s))}\int_{x_1-\epsilon}^{x_1+\epsilon}\sinh(\eta(s)x)\,dx \\ &= \frac{e^{-hs}\gamma\sinh((1-x_0)\eta(s))[\cosh((x_1-\epsilon)\eta(s))-\cosh((x_1+\epsilon)\eta(s))]\hat{v}(s)}{2\epsilon(s^2+\beta s)\sinh(\eta(s))}.\end{aligned}$$

Hence

$$G(s) = \frac{\mathcal{L}(Tv)(s)}{\mathcal{L}v(s)} = \frac{e^{-hs}\gamma\sinh((1-x_0)\eta(s))[\cosh((x_1-\epsilon)\eta(s))-\cosh((x_1+\epsilon)\eta(s))]}{2\epsilon(s^2+\beta s)\sinh(\eta(s))}.$$

Thus, the input/output system defined by (2.31) and (2.32) qualifies as an exponentially stable regular linear system.

In the case $h = 0$, (with state space $X = H_0^1(\Omega) \times L^2(\Omega)$) the generator A of the exponentially stable semigroup S is given by

$$A = \begin{bmatrix} 0 & I \\ -Q & -\beta I \end{bmatrix}$$

where $Qz = -\alpha z_{xx}$ for all $z \in \mathrm{dom}(Q) = H_0^2(\Omega)$. The control, observation and feedthrough operators are given by

$$Bv = \begin{bmatrix} 0 \\ \gamma\delta(\cdot - x_0)v \end{bmatrix}, \quad C\zeta = C\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \frac{1}{2\epsilon}\int_{x_1-\epsilon}^{x_1+\epsilon}\zeta_1(\xi)\,d\xi\,, \quad D = 0$$

giving a regular linear system with

$$\zeta(t) = \begin{bmatrix} z(t,\cdot) \\ z_t(t,\cdot) \end{bmatrix}$$

taking the place of z (2.22)-(2.23).

2.2.3 Nonlinear delay elements

Definition 26

Let $\mathcal{D}^{N,M}$ denote the class of functions $(t,x) \mapsto \Psi(t,x)$, $\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^M$ that are measurable in t and locally Lipschitz in x uniformly with respect to t in compact sets, *precisely: for every compact $I \times K \subset \mathbb{R} \times \mathbb{R}^N$ there exists a constant L such that*

$$\text{for a.a } t \in I, \quad \|\Psi(t,x) - \Psi(t,y)\| \leq L\|x-y\| \quad \forall\, x,y \in K\,.$$

For $i = 0,...,n$, let $\Psi_i \in \mathcal{D}^{N,M}$ and $h_i \in \mathbb{R}_+$. Define $h := \max_i h_i$. For $x(\cdot) \in C([-h,\infty);\mathbb{R}^N)$, let

$$(Tx)(t) := \int_{-h_0}^{0} \Psi_0(s, x(t+s))\, ds + \sum_{i=1}^{n} \Psi_i(t, x(t-h_i))\,. \tag{2.33}$$

Membership of $\bar{\mathcal{T}}_h^{N,M}$

We argue that the operator $T : C([-h,\infty);\mathbb{R}^N) \to L^\infty_{\text{loc}}(\mathbb{R};\mathbb{R}^M)$, defined by (2.33), is of class $\bar{\mathcal{T}}_h^{N,M}$ by considering separately the cases of point and distributed delays. Claim 4 and Claim 6 will then suffice to show the operator (2.33) is of class $\bar{\mathcal{T}}_h^{N,M}$.

(a) Point delay. For $\Psi \in \mathcal{D}^{N,M}$, define $\widehat{T}$ by

$$(\widehat{T}x)(t) := \Psi(t, x(t-h)). \tag{2.34}$$

Then for $t \in [0,\infty)$, $(\widehat{T}x)(t)$ is well defined and the map $t \mapsto (\widehat{T}x)(t)$ is of class $C(\mathbb{R};\mathbb{R}^M)$. Properties of class $\mathcal{D}^{N,M}$ functions suffice to prove $\widehat{T}$ satisfies (T1) and (T3*). (T2) evidently holds.

(b) Distributed delay. For $\Psi \in \mathcal{D}^{N,M}$, define $\widehat{T}$ by

$$(\widehat{T}x)(t) := \int_{-h}^{0} \Psi(s, x(t+s))\, ds\,. \tag{2.35}$$

Again (T2) evidently holds.

Let $r > 0$, I be a bounded interval. Let $t^* := \sup\{t, t \in I\}$ then the properties of $\mathcal{D}^{N,M}$ functions ensure the existence of a constant k such that

$$\text{for a.a. } t \in [-h, t^*] \quad \|\Psi(t,x)\| \leq k \quad \forall\, \|x\| < r.$$

Then for all $t \in I$ and $x \in C([-h,\infty);\mathbb{R}^N)$ with $\sup_{t\in[-h,\infty)} \|x(t)\| < r$,

$$\|(\widehat{T}x)(t)\| \leq \int_{-h}^{0} \|\Psi(s, x(t+s))\|\, ds \leq h \text{ ess-sup}_{s\in[-h,0]} \|\Psi(s, x(t+s))\| \leq hk.$$

Let $t \geq 0$, $\tau > 0$, $r > 0$ and $\zeta : [-h,t] \to \mathbb{R}^N$ be continuous. Let $x, \xi \in C([-h,\infty);\mathbb{R}^N)$ with $x|_{[-h,t]} = \zeta = \xi|_{[-h,t]}$ and $x(s), \xi(s) \in \mathbb{B}_r(\zeta(t))$ for all $s \in [t, t+\tau]$, Then there exists some constant $k > 0$ such that

$$\text{for a.a. } s, \rho \in [-h, t+\tau], \quad \|\Psi(s, x(\rho)) - \Psi(s, \xi(\rho))\| \leq k\|x(\rho) - \xi(\rho)\|.$$

Then for $s \in [t, t+\tau]$,

$$\begin{aligned}
\|(\widehat{T}x)(s) - (\widehat{T}\xi)(s)\| &\leq \int_{-h}^{0} \|\Psi(\tau, x(s+\tau)) - \Psi(\tau, \xi(s+\tau))\| \, d\tau \\
&\leq h \text{ ess-sup}_{\tau\in[-h,0]}\|\Psi(\tau, x(s+\tau)) - \Psi(\tau, \xi(s+\tau))\| \\
&\qquad \leq hk \text{ sup}_{s\in[-h,0]}\|x(s+\tau) - \xi(s+\tau)\|
\end{aligned}$$

and hence

$$\begin{aligned}
\text{ess-sup}_{s\in[t,t+\tau]}\|(\widehat{T}x)(s) - (\widehat{T}\xi)(s)\| &\leq hk \text{ sup}_{s\in[t-h,t+\tau]}\|x(s) - \xi(s)\| \\
&= hk \text{ sup}_{s\in[t,t+\tau]}\|x(s) - \xi(s)\|
\end{aligned}$$

Thus (T3*) holds.

In addition, if for each fixed x, $\Psi_i(\cdot, x)$ is measurable and for every compact $K \subset \mathbb{R}^N$ there exists a constant k such that

$$\forall\, x \in K, \quad \|\Psi(t, x)\| \leq k \quad \text{for a.a. } t \in \mathbb{R}$$

then the operator (2.33) is bounded-input bounded-output stable.

Control estimates

In this section we derive control estimates for nonlinear delay elements. These derivations highlight the essential differences between the classes $\mathcal{C}_1(\psi)$ and $\mathcal{C}_2(\psi)$ of Definition 16, pg 29. As a motivating example, if for fixed $m \in \mathbb{N}$ the nonlinear delay element is of the form

$$(Ty)(t) := y^m(t-h), \tag{2.36}$$

then the argument below shows that such an operator is of class $\mathcal{C}_1(\psi_1)$ for $\psi_1 : r \mapsto r^m$ (simply taking $\alpha := (m+1)/m > 1$ in the analysis below). However, this delay element is of class $\mathcal{C}_2(\psi_2)$ where $\psi_2 : r \mapsto 1 + r^{2m}$. Thus, construction of universal control strategies using knowledge of ψ, that achieve the desired control objective for the operator (2.36) requires greater controller complexity in the context of the class $\mathcal{C}_2$.
Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and such that, for some $\alpha > 1$, $\psi^{\alpha-1}(r) \geq r$ for all

$r \in \mathbb{R}_+$. Let $\Psi \in \mathcal{D}^1$ (recall Definition 26) be such that, for some constant $\hat{\mu} > 0$,

$$|\Psi(t,x)|^\alpha \leq \hat{\mu}\psi(|x|)|x| \quad \text{for a.a. } t \text{ and all } x\,. \tag{2.37}$$

We will consider cases of point and distributed delays.

(a) Point delay. Define $\widehat{T} \in \bar{\mathcal{T}}_h^1$ by

$$(\widehat{T}x)(t) := \Psi(t, x(t-h)).$$

Clearly $\widehat{T}$ is bounded-input bounded-output stable. Moreover,

$$\begin{aligned}\int_0^t |(\widehat{T}x)(s)|^\alpha ds &= \int_{-h}^{t-h} |\Psi(s+h, x(s))|^\alpha ds \\ &\leq \hat{\mu}\int_{-h}^0 \psi(|x(s)|)|x(s)|\,ds + \hat{\mu}\int_0^t \psi(|x(s)|)|x(s)|\,ds\,.\end{aligned}$$

(b) Distributed delay. Define $\widehat{T} \in \bar{\mathcal{T}}_h^1$ by

$$(\widehat{T}x)(t) := \int_{-h}^0 \Psi(s, x(t+s))\,ds\,.$$

Again, $\widehat{T}$ is bounded-input bounded-output stable. Moreover,

$$\begin{aligned}\int_0^t |(\widehat{T}x)(s)|^\alpha ds &= \int_0^t \left|\int_{-h}^0 \Psi(\tau, x(s+\tau))d\tau\right|^\alpha ds \\ &\leq h^{\alpha-1}\int_0^t\int_{-h}^0 |\Psi(\tau, x(s+\tau))|^\alpha d\tau\, ds \text{ (by Hölder's inequality)} \\ &\leq h^{\alpha-1}\hat{\mu}\int_0^t\int_{s-h}^s \psi(|x(\sigma)|)|x(\sigma)|d\sigma\, ds \\ &= h^{\alpha-1}\hat{\mu}\int_{-h}^t\int_{\max\{0,\sigma\}}^{\min\{t,\sigma+h\}} \psi(|x(\sigma)|)|x(\sigma)|ds\, d\sigma \\ &\leq h^\alpha\hat{\mu}\left(\int_{-h}^0 \psi(|x(\sigma)|)|x(\sigma)|\,d\sigma + \int_0^t \psi(|x(\sigma)|)|x(\sigma)|\,d\sigma\right)\,.\end{aligned}$$

Thus in each case (point and distributed delay) and letting β denote the conjugate exponent of α (that is, $\beta^{-1}+\alpha^{-1}=1$), there exists a constant μ such that

$$\begin{aligned}\int_0^t x(s)(\widehat{T}x)(s)\,ds &\leq \frac{1}{\beta}\int_0^t |x(s)|^\beta ds + \frac{1}{\alpha}\int_0^t |(\widehat{T}x)(s)|^\alpha ds \text{ (by Young's inequality)}\\ &\leq \frac{\mu}{2}\left(\int_{-h}^0 \psi(|x(s)|)|x(s)|ds + \int_0^t \left(|x(s)|^{\frac{1}{\alpha-1}} + \psi(|x(s)|)\right)|x(s)|ds\right)\\ &\leq \text{constant} + \mu\int_0^t \psi(|x(s)|)|x(s)|ds \quad \forall\, t\geq 0\end{aligned}$$

and so $\widehat{T}\in\mathcal{C}_1(\psi)$.

Assume that $\alpha=2$ and that (2.37) holds then identical arguments show that both point and distributed delays are of class $\mathcal{C}_2(\psi)$.

Next, let $\alpha\in\mathcal{J}$, $h\in\mathbb{R}_+$ and $\Psi\in\mathcal{D}^{N,M}$ satisfy

$$\|\Psi(t,x)\| < \mu(1+\alpha(\|x\|)) \quad \forall (t,x)\in\mathbb{R}_+\times\mathbb{R}^N$$

for some $\mu\in\mathbb{R}_+$. Evidently for all $x\in C([-h,\infty);\mathbb{R}^M)$, both the point delay

$$(Ty)(t) = \Psi(t, x(t-h))\,.$$

and the distributed delay

$$(Ty)(t) = \int_{-h}^0 \Psi(s, x(t+s))\,ds$$

are of class $\mathcal{C}_3(\psi)$ with the $\mathcal{J}$-function $\psi=\alpha$.

A specific example

As a specific example, let $\psi(r)=1+r^3$. Let $h_1,h_2,h_3>0$ and $q_1,q_2,q_3\in L^\infty(\mathbb{R})$. Then the operator $\widehat{T}\in\mathcal{T}_h^1$, with $h:=\max\{h_1,h_2,h_3\}$, given by

$$(\widehat{T}x)(t) = q_1(t)x(t-h_1) + q_2(t)x^2(t-h_2) + \int_{-h_3}^0 q_3(s)x^3(t+s)ds$$

is bounded-input bounded-output stable in the sense of (2.6) and is of class $\mathcal{C}_1(\psi)$.

Membership of class $\mathcal{S}$

Let $\Psi\in\mathcal{D}^{N,M}$ satisfy

$$\exists\mu,\delta\geq 0: \quad \forall x,y\in\mathbb{R}^N, \quad \|\Psi(t,x+y)\| \leq \mu[\|\Psi(t,\delta x)\| + \|\Psi(t,\delta y)\|]. \tag{2.38}$$

Then it is clear that, for such a Ψ, both the point delays (2.34) and distributed delays (2.35) are of class $\mathcal{S}$.

As a concrete example recall that for each $n \in \mathbb{N}$ there exists a $\mu > 0$ so that

$$|x+y|^n \le \mu(|x|^n + |y|^n).$$

Thus the functions $\Psi \in \mathcal{D}^1$ of the form $\Psi : (t,y) \mapsto y^n$ for $n \in \mathbb{N}$ generate operators of class $\mathcal{S}$.

Right shift invariance

For $i = 0, ..., n$, let $\Psi_i \in \mathcal{D}^{N,M}$ and $h_i \in \mathbb{R}_+$. Define $h := \max_i h_i$. For $x(\cdot) \in C([-h.\infty); \mathbb{R}^N)$, let

$$(Tx)(t) := \int_{-h_0}^{0} \Psi_0(\tau, x(t+\tau))\, d\tau + \sum_{i=1}^{n} \Psi_i(0, x(t-h_i)). \tag{2.39}$$

Then T is shift invariant as will be seen as follows. Let $s \in \mathbb{R}_+$ and $x \in C([-h.\infty); \mathbb{R}^N)$.

$$\begin{aligned} (S_s T x)(t) &= S_s \left(\int_{-h_0}^{0} \Psi_0(\tau, x(t+\tau))\, d\tau + \sum_{i=1}^{n} \Psi_i(0, x(t-h_i)) \right) \\ &= \int_{-h_0}^{0} \Psi_0(\tau, S_s x(t+\tau))\, d\tau + \sum_{i=1}^{n} \Psi_i(0, S_s x(t-h_i)) \\ &= (T S_s x)(t). \end{aligned}$$

Notice that (2.39) differs from (2.33) in that the point delays have no explicit time dependence.

2.2.4 Hysteresis

In its most general sense any causal and rate-independent operator between spaces of scalar functions is said to be a hysteresis operator. For our purposes here we only consider the class of nonlinear operators $C(\mathbb{R}_+; \mathbb{R}) \to C(\mathbb{R}_+; \mathbb{R})$, which includes many physically motivated hysteretic effects, introduced in [50, Section 3]. These operators satisfy (T1)-(T3) however they do not satisfy (T3*).

Examples of such operators, including relay hysteresis, backlash hysteresis, elastic-plastic hysteresis and Preisach operators, are given in [50, Section 5]. By way of illustration, we describe in more detail the first two of these examples.

Relay hysteresis (see also [56, 44]). Let $a_1 < a_2$ and let $\rho_1 : [a_1, \infty) \to \mathbb{R}$, ρ_2 :

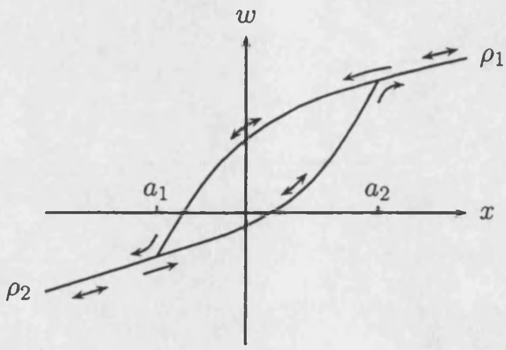
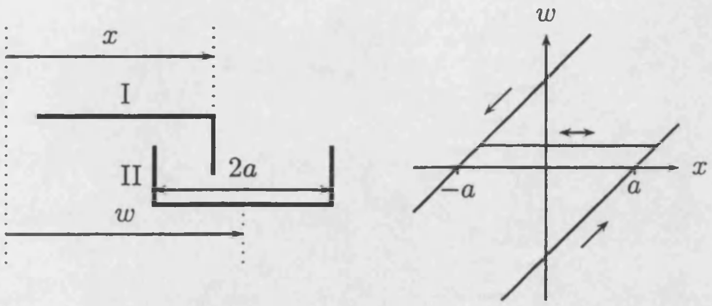


Figure 2-1: Relay hysteresis.



(2-2a) (2-2b)

Figure 2-2: Backlash hysteresis.

$(-\infty, a_2] \to \mathbb{R}$ be continuous, non-decreasing, globally Lipschitz and satisfy $\rho_1(a_1) = \rho_2(a_1)$ and $\rho_1(a_2) = \rho_2(a_2)$. For a given input $x \in C(\mathbb{R}_+;\mathbb{R})$ to the hysteresis element, the output w is such that $(x(t), w(t)) \in \text{graph}(\rho_1) \cup \text{graph}(\rho_2)$ for all $t \in \mathbb{R}_+$: the value $w(t)$ of the output at $t \in \mathbb{R}_+$ is either $\rho_1(x(t))$ or $\rho_2(x(t))$, depending on which of the threshold values a_2 or a_1 was "last" attained by the input x. This situation is illustrated by Figure 2-1. When suitably initialized, such a hysteresis element has the property that, to each input $x \in C(\mathbb{R}_+;\mathbb{R})$, there corresponds a unique output $w = Tx \in C(\mathbb{R}_+;\mathbb{R})$: the operator T, so defined, is of class $\mathcal{T}_0^1$. Full details may be found in [50, Section 5].

Backlash hysteresis. Next consider a one-dimensional mechanical link consisting of the two solid parts I and II, as shown in Figure 2-2.a, the displacements of which (with respect to some fixed datum) at time $t \geq 0$ are given by $x(t)$ and $w(t)$ with $|x(t) - w(t)| \leq a$ for all t, and $w(0) := x(0) + \xi$ for some pre-specified $-a \leq \xi \leq a$. Within the link there is *mechanical play*: that is to say the position $w(t)$ of II remains constant as long as the position $x(t)$ of I remains within the interior of II. Thus, assuming continuity of x, we have $\dot{w}(t) = 0$ whenever $|x(t) - w(t)| < a$. Given a continuous input $x \in C(\mathbb{R}_+;\mathbb{R})$, describing the evolution of the position of I, denote the corresponding position of II by $w = Tx$. The operator T, (in effect we define a family T_ξ of operators parameterized by the initial offset ξ) so defined, is known as *backlash* or *play* and is of class $\mathcal{T}_0^1$: full details may be found in [50, Section 5].

Control estimates

Assumption (N8) of [50] implies that the operator is of class $\mathcal{C}_3(\psi)$ with $\mathcal{J}$-function $\psi : s \mapsto s$.

Chapter 3

Existence of solutions for functional differential equations and inclusions

In this Chapter we present theorems on the existence and uniqueness of solutions to various initial-value problems for functional differential equations and inclusions. Such existence results form the basis of the stability theory of Chapter 4 and analysis of behaviour of the classes of control systems in the subsequent Chapters.

Existence results for such initial-value problems have an extensive literature. However, we present results here which are suited to our particular purposes and which differ in detail from those already in the literature. Many works on functional differential equations, [26] for example, study problems of the form

$$\left.\begin{array}{l} \dot{x}(t) = f(t, S_t x|_{[-h,0]}) \\ x|_{[-h,0]} = x^0 \in C\left([-h,0];\mathbb{R}^N\right) \end{array}\right\}$$

and assume $f : \mathbb{R} \times C([-h,0];\mathbb{R}^n) \to \mathbb{R}^n$ is continuous. This Chapter is a study of problems of the form

$$\left.\begin{array}{l} \dot{x}(t) = f(t, (Tx)(t)) \\ x|_{[-h,0]} = x^0 \in C\left([-h,0];\mathbb{R}^N\right) \end{array}\right\} \tag{3.1}$$

where $T \in \mathcal{T}_h^{N,K}$ and f is a Carathéodory function. Moreover, our local existence result is tailored to provide approximate solutions which play a key rôle in the proof of existence of solutions for an associated differential inclusion.

Differential inclusions with memory have been studied, [5, Chapter 4, §7], but without proofs of existence. We not only provide full proofs but present the theory in a form tailored for use with the control problems of the subsequent chapters.

3.1 Functional differential equations

By a *solution* of (3.1) we mean a function $x \in C([-h,\omega);\mathbb{R}^N)$ for some $\omega > 0$, such that $x|_{[-h,0]} = x^0$, and $x|_{[0,\omega)} \in AC([0,\omega);\mathbb{R}^N)$ with $\dot{x}(t) = f(t,(Tx)(t))$ for almost all $t \in [0,\omega)$ (recall our notational convention: for $t \in [0,\omega)$, $(Tx)(t)$ should be interpreted as $(Tx^e)(t)$ where x^e is any class $C([-h,\infty);\mathbb{R}^N)$ extension of x). The function $\xi : [-h,\rho) \to \mathbb{R}^N$ is a *right extension* of x if $\rho > \omega$ and $\xi|_{[-h,\omega)} = x$. A solution is said to be *maximal* if it does not have a right extension which is also a solution.

The following result plays a crucial rôle in the subsequent proof of existence of solutions of the initial-value problem (3.8) below.

Theorem 27
Let $N, K \in \mathbb{N}$, $T \in \mathcal{T}_h^{N,K}$ and $x^0 \in C([-h,0];\mathbb{R}^N)$. Assume $f : [-h,\infty) \times \mathbb{R}^K \to \mathbb{R}^N$ is a Carathéodory function. Then

(i) there exists, for some $\omega > 0$, a solution $x : [-h,\omega) \to \mathbb{R}^N$ of the initial-value problem (3.1);

(ii) every solution can be extended to a maximal solution;

(iii) if $f \in L^\infty_{\mathrm{loc}}([-h,\infty) \times \mathbb{R}^K;\mathbb{R}^N)$ and $x : [-h,\omega) \to \mathbb{R}^N$ is a bounded maximal solution, then $\omega = \infty$.

Proof. (i) By (T3) there exist $\tau > 0$, $\delta > 0$ and $c > 0$ such that, for all $x, \xi \in C([-h,\infty);\mathbb{R}^N)$ with $x|_{[-h,0]} = x^0 = \xi|_{[-h,0]}$ and $x(t), \xi(t) \in \mathbb{B}_\delta(x^0(0))$ for all $t \in [0,\tau]$,

$$\text{ess-sup}_{t\in[0,\tau]} \|(Tx)(t) - (T\xi)(t)\| \le c \sup_{t\in[0,\tau]} \|x(t) - \xi(t)\|.$$

By (T1) of T, there exists $\Delta > 0$ such that for all $x \in C([-h,\infty);\mathbb{R}^N)$,

$$\sup_{t\in[-h,\infty)} \|x(t)\| < \delta^* := \delta + \|x^0\|_\infty \quad \Longrightarrow \quad \|(Tx)(t)\| < \Delta \quad \text{for almost all} \quad t \in [0,\tau].$$

Since f is a Carathéodory function there exists integrable $\gamma : [0,\tau] \to \mathbb{R}_+$ such that

$$\|f(t,\xi)\| \le \gamma(t) \quad \text{for all} \quad (t,\xi) \in [0,\tau] \times \mathbb{B}_\Delta(0).$$

Define Γ by

$$\Gamma(t) := \begin{cases} 0, & t \in [-h, 0) \\ \int_0^t \gamma(s)ds, & t \in [0, \tau] \end{cases}$$

Thus, Γ is (absolutely) continuous and non-decreasing, with $\Gamma(0) = 0$. Let $0 < \beta < \tau$ be such that, $\Gamma(\beta) < \delta$.

Next, we construct a sequence $(x_n)_{n\in\mathbb{N}}$ of continuous functions $[-h, \beta] \to \mathbb{R}^N$ as follows. Let $n \in \mathbb{N}$. For $i = 1, ..., n$, define $x_n^i : [-h, i\beta/n] \to \mathbb{R}^M$ by the recursive formula:

$$i = 1: \quad x_n^1(t) := \begin{cases} x^0(t), & t \in [-h, 0] \\ x^0(0), & t \in (0, \beta/n] \end{cases}$$

$$i > 1: \quad x_n^i(t) := \begin{cases} x_n^{i-1}(t), & t \in [-h, (i-1)\beta/n] \\ x^0(0) + \int_0^{t-(\beta/n)} f(s, (Tx_n^{i-1})(s))ds, & t \in ((i-1)\beta/n, i\beta/n]\,. \end{cases}$$

Observe that, if $i \in \{1, ..., n-1\}$ and $\|x_n^i(t)\| < \delta^*$ for all $t \in [-h, (i\beta)/n]$, then (a) $\|x_n^{i+1}(t)\| < \delta^*$ for all $t \in [-h, (i\beta)/n]$ and (b) $\|(Tx_n^i)(t)\| < \Delta$ for all $t \in [0, (i\beta)/n]$ which, in turn, implies

$$\begin{aligned} \|x_n^{i+1}(t) - x^0(0)\| &\leq \int_0^{t-\beta/n} \|f(s, (Tx_n^i)(s))\|ds \leq \int_0^{t-\beta/n} \gamma(s)ds \\ &= \Gamma(t - \beta/n) < \delta \quad \forall\, t \in (i\beta/n, (i+1)\beta/n]\,. \end{aligned}$$

Noting that $\|x_n^1(t)\| \leq \|x^0\|_\infty < \delta^*$ for all $t \in [-h, \beta/n]$, we may now infer (by induction on i) that

$$\|x_n^i(t)\| < \delta^* \quad \forall\, i \in \{1, ..., n\} \quad \forall\, t \in [-h, i\beta/n]\,.$$

For notational convenience, we write $x_n := x_n^n$. By causality of T, the sequence $(x_n)_{n\in\mathbb{N}}$ so constructed has the property that, for each $n \in \mathbb{N}$,

$$x_n(t) = \begin{cases} x^0(t), & t \in [-h, 0] \\ x^0(0), & t \in (0, \beta/n] \\ x^0(0) + \int_0^{t-(\beta/n)} f(s, (Tx_n)(s))ds, & t \in (\beta/n, \beta]\,. \end{cases}$$

Moreover, for all $n \in \mathbb{N}$, $\|x_n(t)\| < \delta^*$ for all $t \in [-h, \beta]$ and so the sequence $(x_n)_{n\in\mathbb{N}}$ is uniformly bounded.

Next, we prove that the sequence $(x_n)_{n\in\mathbb{N}}$ is equicontinuous. Let $\epsilon > 0$. On the closed interval $[0,\beta]$, Γ is uniformly continuous and so there exists some $\bar{\delta} > 0$ such that

$$t, s \in [0,\beta] \quad \text{with} \quad |t-s| < \bar{\delta} \implies |\Gamma(t) - \Gamma(s)| < \epsilon. \tag{3.2}$$

Let $n \in \mathbb{N}$, $s, t \in [0,\beta]$ with $|t-s| < \bar{\delta}$. Without loss of generality, we assume that $s \leq t$. We consider three exhaustive cases.

First, if $0 \leq s \leq t \leq \beta/n$, then $\|x_n(t) - x_n(s)\| = 0$. Secondly, if $0 < s \leq \beta/n \leq t \leq \beta$, then $t - \beta/n < \bar{\delta}$ and so

$$\|x_n(t) - x_n(s)\| = \|x_n(t) - x^0(0)\| \leq \Gamma(t - \beta/n) < \epsilon .$$

Thirdly, if $\beta/n \leq s \leq t \leq \beta$, then

$$\|x_n(t) - x_n(s)\| \leq |\Gamma(t-\beta/n) - \Gamma(s-\beta/n)| < \epsilon.$$

Recalling that $x_n|_{[-h,0]} = x^0$ for all n, we conclude that the sequence $(x_n)_{n\in\mathbb{N}}$ is equicontinuous. By Theorem 75, pg 131, and extracting a subsequence if necessary, we may assume that the sequence $(x_n)_{n\in\mathbb{N}}$ converges uniformly on $[-h,\beta]$ to a continuous limit which we denote by x. Clearly $x|_{[-h,0]} = x^0$.

By (T3), $\lim_{n\to\infty}(Tx_n)(t) = (Tx)(t)$ for almost all $t \in [0,\beta]$ and so, by the continuity of the function $f(t,\cdot)$,

$$\lim_{n\to\infty} f(t, (Tx_n)(t)) = f(t, (Tx)(t)) \quad \text{for a.a.} \quad t \in [0,\beta].$$

Hence, by the Lebesgue dominated convergence theorem [84, §5.2]

$$\lim_{n\to\infty} \int_0^t f(s, (Tx_n)(s))ds = \int_0^t f(s, (Tx)(s))ds.$$

But

$$x_n(t) = x^0(0) + \int_0^t f(s, (Tx_n)(s))ds - \int_{t-\beta/n}^t f(s, (Tx_n)(s))ds$$

and

$$\int_{t-\beta/n}^t f(s, (Tx_n)(s))ds \ \to \ 0 \quad \text{as} \quad n \to \infty.$$

Hence,

$$x(t) = \begin{cases} x^0(t), & t \in [-h,0] \\ x^0(0) + \int_0^t f(s, (Tx)(s))ds, & t \in (0,\beta]. \end{cases}$$

and so x is a solution of the initial-value problem.

(ii) Let $x : [-h, \omega) \to \mathbb{R}^N$ be a solution of (3.1). Define

$$\mathcal{A} := \{(\rho, \xi) |\ \omega \le \rho \le \infty,\ \xi : [-h, \rho) \to \mathbb{R}^N \text{ is a solution of (3.1) with } \xi|_{[-h,\omega)} = x\}$$

On this non-empty set define a partial order $\preceq$ by

$$(\rho_1, \xi_1) \preceq (\rho_2, \xi_2) \iff \rho_1 \le \rho_2 \text{ and } \xi_1(t) = \xi_2(t) \text{ for all } t \in [-h, \rho_1).$$

Let $\mathcal{O}$ be a totally ordered subset of $\mathcal{A}$. Let $P := \sup\{\rho | (\rho, \xi) \in \mathcal{O}\}$ and let $\Xi : [-h, P) \to \mathbb{R}^M$ be defined by the property that, for every $(\rho, \xi) \in \mathcal{O}$, $\Xi|_{[0,\rho)} = \xi$. Then (P, Ξ) is in $\mathcal{A}$ and is an upper bound for $\mathcal{O}$. By Zorn's lemma, (Lemma 74), it follows that $\mathcal{A}$ contains at least one maximal element.

(iii) Assume that $x \in C([-h, \omega); \mathbb{R}^N)$ is a bounded maximal solution of (3.1) and that $f \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R}^K; \mathbb{R}^N)$. Seeking a contradiction, suppose $\omega < \infty$. By boundedness of x, together with (T1) of T it follows that $\dot{x}(\cdot)$ is essentially bounded. Therefore, x is uniformly continuous and so extends to a continuous function $x : [-h, \omega] \to \mathbb{R}^N$. x solves (3.1) if, and only if, $S_\omega x : [-(h + \omega), 0] \to \mathbb{R}^N$ solves

$$\dot{v}(t) = S_\omega f\big(t, (TS_{-\omega} v)(t)\big), \qquad v|_{[-(h+\omega),0]} = S_\omega x\,. \tag{3.3}$$

By (2.5) and the above existence result, the initial-value problem (3.3) has a solution $\tilde{v} : [-(h+\omega), \tau) \to \mathbb{R}^N$, $\tau > 0$. It follows that $\tilde{v}$ is a solution of the original initial-value problem (3.1) and is a right extension of the solution x. This contradicts maximality of x. Therefore, $\omega = \infty$.

Remark 28

Part (i) of the proof is inspired by an argument of [14, Theorem 1.1, pg 43]. Part (ii) is taken from [71, Theorem 8] and Part (iii) adapted from [71, Theorem 7].

Remark 29

We remark here that we have chosen $t^0 = 0$ throughout, even when considering non-autonomous problems. Hence we have, in some sense, given $t^0 = 0$ a distinguished status.

Under stronger hypotheses we can also obtain uniqueness of solutions. This result is proved via a fixed point argument akin to that used in [51].

Theorem 30

Let $h \ge 0$, $N, K \in \mathbb{N}$ and $T \in \mathcal{T}_h^{N,K}$. Let $f : \mathbb{R} \times \mathbb{R}^K \to \mathbb{R}^N$ be continuous and locally

Lipschitz uniformly in its second argument:

$$\text{for all compact } \Omega \subset \mathbb{R}^K, \quad \exists L > 0 : \\ \|f(t,u) - f(t,v)\| \le L\|u - v\| \quad \forall\, u, v \in \Omega, t \in \mathbb{R} \tag{3.4}$$

Let $x^0 \in C([-h,0];\mathbb{R}^N)$.

(i) *There exists a unique maximal solution* $x : [-h,\omega) \to \mathbb{R}^N$ *of the initial-value problem (3.1);*

(ii) *if* x *is bounded, then* $\omega = \infty$.

Proof. Let $\mathcal{A}$ denote the set of all pairs $(\rho,\xi) \in (0,\infty] \times C([-h,\rho);\mathbb{R}^N)$ such that ξ is a solution of the initial-value problem (3.1):

$$\mathcal{A} := \{(\rho,\xi)|\ 0 < \rho \le \infty,\ \xi \in C([-h,\rho);\mathbb{R}^N) \text{ is a solution of (3.1)}\}.$$

Our first objective is to show that $\mathcal{A} \neq \emptyset$ (implying the existence of at least one solution of (3.1)) and, for all $(\rho,\xi),(\sigma,\eta) \in \mathcal{A}$, if $\rho \le \sigma$ then $\eta|_{[-h,\rho)} = \xi$ (that is, any two solutions of (3.1) must coincide on the intersection of their domains). These two facts immediately lead to the first assertion of the theorem (namely, the existence of a unique maximal solution $x : [-h,\omega) \to \mathbb{R}^N$), via the following construction: let $\omega := \sup\{\rho|\ (\rho,\xi) \in \mathcal{A}\}$ and define $x \in C([-h,\omega);\mathbb{R}^N)$ by the property $x|_{[-h,\rho)} = \xi$ for all $(\rho,\xi) \in \mathcal{A}$.

By (T3) of T, there exists $\tau > 0$, $r \in (0,1)$ and $c > 0$ such that for all $x, \xi \in L^\infty_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$ with $x|_{[-h,0]} = x^0 = \xi|_{[-h,0]}$ and $x(t), \xi(t) \in \mathbb{B}_r(x^0(0))$ for almost all $t \in [0,\tau]$, the following holds

$$\|(Tx)(t) - (T\xi)(t)\| \le c\,\text{ess-sup}_{s\in[0,\tau]}\|x(s) - \xi(s)\|.$$

Define $R := \|x^0(0)\| + r$. By (T1) of T there exists $M > 0$ such that

$$\|x\|_{L^\infty(\mathbb{R};\mathbb{R}^N)} < R \quad \Longrightarrow \quad \|(Tx)(t)\| < M \quad \text{for a.a. } t \in [0,\tau).$$

By the properties of f, there exists a constant $L > 0$ such that

$$\|f(t,u) - f(t,v)\| \le L\|u - v\| \quad \forall\, u, v \in \mathbb{B}_M(0) \text{ and all } t \in \mathbb{R}.$$

For $\epsilon > 0$, define

$$C_\epsilon := \{x \in C([-h,\epsilon),\mathbb{R}^N) |\ x|_{[-h,0]} = x^0,\ \|x(t) - x^0(0)\| \le r/2\ \forall\ t \in (0,\epsilon)\}$$

which, when equipped with the metric

$$(x,\xi) \mapsto \Delta_\epsilon(x,\xi) := \sup_{-h\le t<\epsilon} \|x(t) - \xi(t)\|$$

is a complete metric space. For each $\epsilon > 0$, define the operator Γ_ϵ on C_ϵ by

$$(\Gamma_\epsilon x)(t) := \begin{cases} x^0(t), & t \in [-h,0] \\ x^0(0) + \int_0^t f(s,(Tx)(s))\,ds, & t \in (0,\epsilon). \end{cases}$$

(Again, recall our notational convention: for $t \in [0,\epsilon)$, $(Tx)(t)$ should be interpreted as $(Tx^e)(t)$ where x^e is any class $C([-h,\infty);\mathbb{R}^N)$ extension of x.) Clearly, any fixed point of Γ_ϵ is a solution of (3.1). Next, we show that Γ_ϵ is a contraction for all sufficiently small $\epsilon > 0$.

First note that f, as a continuous function, is bounded on the compact set $[0,\tau]\times\mathbb{B}_M(0)$ by some constant $\bar{f}$ say. ie

$$\|f(s,v)\| \le \bar{f} \quad \forall\ s \in [0,\tau],\ v \in \mathbb{B}_M(0).$$

Thus fix $\epsilon^* \in (0,\tau)$ sufficiently small so that $\epsilon^* cL < 1$ and $2\epsilon^*\bar{f} < r$. Let $\epsilon \in (0,\epsilon^*)$ and $x \in C_\epsilon$. By definition, $(\Gamma_\epsilon x)|_{[-h,0]} = x^0$ and

$$\begin{aligned} \|(\Gamma_\epsilon x)(t) - x^0(0)\| &= \left\| \int_0^t f(s,(Tx)(s))\ ds \right\| \\ &\le \int_0^\epsilon \|f(s,(Tx)(s))\| ds \\ &\le \epsilon\bar{f} \\ &< r/2 \quad \forall\ t \in (0,\epsilon]. \end{aligned}$$

Therefore, $(\Gamma_\epsilon x)(\cdot) \in C_\epsilon$ and so $\Gamma_\epsilon : C_\epsilon \to C_\epsilon$. Furthermore, for all $x, \xi \in C_\epsilon$,

$$\begin{aligned}\Delta_\epsilon(\Gamma_\epsilon x, \Gamma_\epsilon \xi) &= \sup_{t\in(0,\epsilon)} \left\| \int_0^t f(s,(Tx)(s)) - f(s,(T\xi)(s))\, ds \right\| \\ &\le \int_0^\epsilon \|f(s,(Tx)(s)) - f(s,(T\xi)(s))\|\, ds \\ &\le \epsilon L \,\text{ess-sup}_{s\in[0,\epsilon)} \|Tx(s) - T\xi(s)\| \\ &\le \epsilon c L \Delta_\epsilon(x,\xi)\end{aligned}$$

and, since $\epsilon c L < 1$, it follows that Γ_ϵ is a contraction on C_ϵ.

We have now shown that, for all $\epsilon \in (0,\epsilon^*)$, $\Gamma_\epsilon : C_\epsilon \to C_\epsilon$ is a contraction. By the contraction mapping theorem, [45, §5.1–2], for each $\epsilon \in (0,\epsilon^*)$, Γ_ϵ has a unique fixed point. Therefore, (3.1) has a solution, equivalently, $\mathcal{A} \neq \emptyset$. Let $(\rho,\xi),(\sigma,\eta) \in \mathcal{A}$. By continuity of ξ and η, $\xi|_{[-h,\epsilon)}, \eta|_{[-h,\epsilon)} \in C_\epsilon$ for all sufficiently small $\epsilon \in (0,\epsilon^*)$ and, since these restrictions are fixed points of the contraction Γ_ϵ, they must coincide.

The above argument has established the following facts, assembled for later referral.

$$\left.\begin{array}{l}\text{For every initial-value, (3.1) has a solution and all}\\ \text{solutions must coincide on some interval } [-h,\epsilon) \text{ with } \epsilon > 0\,.\end{array}\right\} \tag{3.5}$$

To show that any two solutions must coincide on the intersection of their domains, we argue as follows. Let $(\rho,\xi),(\sigma,\eta) \in \mathcal{A}$ and assume, without loss of generality, that $\rho \le \sigma$. Seeking a contradiction, suppose $\eta|_{[-h,\rho)} \neq \xi$. Then there exists $t \in (0,\rho)$ such that $\xi(t) \neq \eta(t)$. Let $t^* := \inf\{t \in (0,\rho) \mid \xi(t) \neq \eta(t)\}$. In view of (3.5), we have $t^* > 0$. Define $z^0 \in C([-(h+t^*),0];\mathbb{R}^N)$ by $z^0(t) := \xi(t+t^*) = \eta(t+t^*)$ for all $t \in [-(h+t^*),0]$, $\hat{f}(t,u) := f(t+t^*,u)$ and consider the initial-value problem

$$\left.\begin{array}{l}\dot{z}(t) = \hat{f}(t,(S_{t^*}TS_{-t^*}z)(t))\\ z|_{[-(h+t^*),0]} = z^0\,.\end{array}\right\} \tag{3.6}$$

In view of (2.5), together with (3.5) interpreted in the context of the initial-value problem (3.6), we conclude that, for some $\epsilon \in (0,\rho - t^*)$, (3.6) has precisely one solution z with domain $[-(h+t^*),\epsilon)$.

Now define $\xi^*, \eta^* \in C([-h,\epsilon);\mathbb{R}^N)$ by

$$\xi^*(t) = \xi(t+t^*), \quad \eta^*(t) := \eta(t+t^*) \quad \forall\, t \in [-(h+t^*),\epsilon)\,.$$

For almost all $t \in [0, \epsilon)$,

$$\dot{\xi}^*(t) = \dot{\xi}(t + t^*) = f(t + t^*, (T\xi)(t + t^*)) = \hat{f}(t, (S_{t^*}TS_{-t^*}\xi^*)(t))$$

and so $\xi^* = z$ (the unique solution of (3.6) with domain $[-(h + t^*), \epsilon)$). Similarly, we have $\eta^* = z$ and so

$$\xi(t) = \eta(t) \quad \forall\, t \in [t^*, t^* + \epsilon)$$

which contradicts the definition of t^*. Therefore, $\eta|_{[-h,\rho)} = \xi$. The existence of a unique maximal solution of (3.1) follows.

Assume that the unique maximal solution $x \in C([-h, \omega); \mathbb{R}^N)$ of (3.1) is bounded. Seeking a contradiction, suppose $\omega < \infty$. By boundedness of x, together with (3.4) and (T1) of T, there exists $M > 0$ such that $\|\dot{x}(t)\| \leq M$ for almost all $t \in [0, \omega)$. Therefore, x is uniformly continuous and so extends to a continuous function $\xi : [-h, \omega] \to \mathbb{R}^N$. Define $z^0 \in C([-(h + \omega), 0]; \mathbb{R}^N)$ by

$$z^0(t) := \xi(\omega + t) \quad \forall\, t \in [-(h + \omega), 0]\,.$$

For $\hat{f}(t, u) := f(t + t^*, u)$ consider the initial-value problem

$$\left.\begin{aligned} \dot{z}(t) &= \hat{f}(t, (S_\omega TS_{-\omega}z)(t)) \\ z|_{[-(h+\omega),0]} &= z^0\,. \end{aligned}\right\} \tag{3.7}$$

By (2.4) and the above existence result, there exists a solution $z : [-(h+\omega), \omega_z) \to \mathbb{R}^N$ ($\omega_z > 0$) of (3.7). Now define $x^* : [-h, \omega + \omega_z) \to \mathbb{R}^N$ as follows

$$x^*(t) := \begin{cases} x(t), & t \in [-h, \omega) \\ z(t - \omega), & t \in [\omega, \omega + \omega_z) \end{cases}$$

Noting that, for almost all $t \in [\omega, \omega + \omega_z)$,

$$\dot{x}^*(t) = \dot{z}(t - \omega) = \hat{f}(t - \omega, (S_\omega TS_{-\omega}z)(t)) = f(t, (Tx^*)(t)),$$

it follows that x^* is a solution of the original initial-value problem (3.1) and is a right extension of the solution x. This contradicts maximality of x. Therefore, $\omega = \infty$.

3.2 Differential inclusions

The classical theory for existence of solutions for a differential equation such as (1.8) requires continuity of $f(t,x)$ in x, but as outlined in Section 1.3, discontinuities may be an unavoidable feature of a particular control strategy. To overcome these technical difficulties we develop a framework of *differential inclusions* and set-valued analysis. This area is well developed: see [21, 22, 66, 70] and also [5, 18, 12, 13] to name but a few.

Definition 31
For $N \in \mathbb{N}$, let $\mathcal{F}^{N,M}$ denote the space of set-valued maps $x \mapsto F(x) \subset \mathbb{R}^M$ that (i) are upper semicontinuous on $\mathbb{R}^N$, and (ii) take non-empty, convex, compact values.

We now focus on the question of existence of a solution of the initial-value problem

$$\left.\begin{aligned} &\dot{x}(t) \in F(t,(Tx)(t)) \\ &x|_{[-h,0]} = x^0 \in C\left([-h,0];\mathbb{R}^N\right) \end{aligned}\right\} \tag{3.8}$$

with $T \in \mathcal{T}_h^{N,M}$ and $F \in \mathcal{F}^{M+1,N}$.
By a *solution* of (3.8) we mean a function $x \in C([-h,\omega);\mathbb{R}^N)$ for some $\omega > 0$, such that $x|_{[-h,0]} = x^0$, and $x|_{[0,\omega)} \in AC([0,\omega);\mathbb{R}^N)$ with $\dot{x}(t) \in F(t,(Tx)(t))$ for almost all $t \in [0,\omega)$. A solution is said to be *maximal* if it does not have a right extension which is also a solution.

Theorem 32
Let $h \geq 0$, $N, M \in \mathbb{N}$, $F \in \mathcal{F}^{M+1,N}$, $T \in \mathcal{T}_h^{N,M}$ and $x^0 \in C([-h,0];\mathbb{R}^N)$. Then:

(i) There exists a solution to the initial-value problem (3.8) on some interval $[-h,\omega)$ with $\omega > 0$;

(ii) every solution can be maximally extended;

(iii) if a maximal solution $x : [-h,\omega) \to \mathbb{R}^N$ is bounded, then $\omega = \infty$.

Proof. (i) Let $(\epsilon_n)_{n\in\mathbb{N}} \subset (0,1)$ be a monotone sequence with $\epsilon_n \downarrow 0$ as $n \to \infty$. By Proposition 88, for each n, there exists a locally Lipschitz function f_n with

$$\operatorname{graph}(f_n) \subset \operatorname{graph}(F) + \mathbb{B}_{\epsilon_n}(0). \tag{3.9}$$

By Theorem 30, for each $n \in \mathbb{N}$, the initial-value problem

$$\left.\begin{aligned} \dot{x}(t) &= f_n(t,(Tx)(t)) \\ x|_{[-h,0]} &= x^0 \in C([-h,0];\mathbb{R}^N) \end{aligned}\right\} \tag{3.10}$$

has a maximal solution which we denote by $\xi_n : [-h,\omega_n) \to \mathbb{R}^N$. Let

$$\tau_n := \inf\{t \in (0,\omega_n) |\ \|\xi_n(t) - x^0(0)\| = 1/2\} \quad \forall\, n \in \mathbb{N},$$

with the convention that $\inf \emptyset := \infty$.

Evidently, for each $n \in \mathbb{N}$, $\xi_n(t) \in \mathbb{B}_1(x^0(0))$ for all $t \in [0,\tau_n)$ and so

$$\|\xi_n(t)\| < R := \max_{s\in[-h,0]} \|x^0(s)\| + 1 \quad \forall\, t \in [-h,\tau_n)\,.$$

By (T1) of T, there exists $K > 0$ such that, for all $n \in \mathbb{N}$,

$$\|(T\xi_n)(t)\| < K \quad \text{for a.a. } t \in [0,\tau_n) \cap [0,1)\,.$$

By Proposition 89, pg 137 there exists a compact set $B \subset \mathbb{R}^N$ such that

$$F([0,1] \times \mathbb{B}_K(0)) \subset B.$$

Let $R := 1 + \sup\{\|v\|, v \in B\}$ then, for each $n \in \mathbb{N}$,

$$\|f_n(t,x)\| < R \quad \forall\, (t,x) \in [0,1] \times \mathbb{B}_K(0). \tag{3.11}$$

We may conclude that, for all $n \in \mathbb{N}$,

$$\|\xi_n(t) - \xi_n(s)\| \le \int_t^s \|\dot{\xi}_n(\sigma)\| d\sigma \le \int_t^s \|f_n(\sigma,(T\xi_n)(\sigma))\| d\sigma < R|t-s| \tag{3.12}$$

for all $s,t \in [0,\tau_n) \cap [0,1)$, $t \le s$. Next, we define $\omega^* := \inf_{n\in\mathbb{N}} \tau_n \ge 0$ ($\omega^* = \infty$ is possible) and claim that $\omega^* \ne 0$. Suppose otherwise, then (τ_n) has a subsequence (which we do not relabel) with $\tau_n < 1$ for all n and $\tau_n \downarrow 0$ as $n \to \infty$, which, together with (3.12), yields the contradiction:

$$0 < 1/2 = \|\xi_n(\tau_n) - x^0(0)\| < R\tau_n \to 0 \text{ as } n \to \infty\,.$$

Therefore, $\omega^* > 0$.

Fix $\rho \in \mathbb{R}$ such that $0 < \rho < \min\{1,\omega^*\}$. For each n, define $x_n := \xi_n|_{[0,\rho)} \in$

$AC([0,\rho);\mathbb{R}^N)$ and let $x_n^* \in C([-h,\rho);\mathbb{R}^N)$ denote the concatenation of x_n and x^0 given by

$$x_n^*(t) := \begin{cases} x^0(t), & t \in [-h,0] \\ x_n(t), & t \in [0,\rho) \end{cases}$$

For all $t \in [0,\rho)$, $(x_n(t))_{n\in\mathbb{N}} \subset \mathbb{B}_{1/2}(x^0(0))$ and, by (3.11), for all $n \in \mathbb{N}$,

$$\|\dot{x}_n(t)\| < R \quad \text{for a.a. } t \in [0,\rho)\,. \tag{3.13}$$

By Theorem 90, pg 137 and extracting a subsequence if necessary, we may assume that (x_n) converges uniformly to a function $x \in AC([0,\rho);\mathbb{R}^N)$, and $(\dot{x}_n)$ converges weakly in $L^1([0,\rho);\mathbb{R}^N)$ to $\dot{x}$. Let $x^* \in C([-h,\rho);\mathbb{R}^N)$ denote the concatenation of x and x^0. By (T3) of T, there exists $\omega \in (0,\rho)$ and $c > 0$ such that, for all n sufficiently large,

$$\|(Tx_n^*)(t) - (Tx^*)(t)\| \le c \sup_{s\in[0,\omega)} \|x_n^*(s) - x^*(s)\| \quad \text{for a.a. } t \in [0,\omega)\,. \tag{3.14}$$

Define $I := [0,\omega)$. We will show that $x^*|_I$ is a solution of the initial-value problem (3.8). By construction, $x^*|_{[-h,0]} = x^0$ and $x^*|_{[0,\omega)} = x$. Thus, it remains only to show that $\dot{x}(t) \in F(t,(Tx)(t))$ for almost all $t \in I$.

For each $n \in \mathbb{N}$, define $y_n : I \to \mathbb{R}^M$ by $y_n(t) := (Tx_n^*)(t)$ for all $t \in I$ and let $y : I \to \mathbb{R}^M$ be given by $y(t) = (Tx^*)(t)$ for all $t \in I$. By (3.14) and the uniform convergence of x_n^* to x^*, y_n tends to y almost everywhere on I. Further by (3.14),

$$\int_0^\omega \|y_n(t) - y(t)\|dt \le c\omega \sup_{s\in[0,\omega)} \|x_n^*(s) - x^*(s)\| \;\to\; 0 \quad \text{as } n \to \infty\,.$$

Therefore, (y_n) converges (strongly) in $L^1(I;\mathbb{R}^N)$ to $y \in L^1(I;\mathbb{R}^N)$.

Let z_n denote the sequence of functions z_n given by $z_n(t) = f_n(t,y_n(t))$ for all $t \in I$. Then, for all n, $z_n(t) = \dot{x}_n(t)$ for almost all $t \in I$. By weak convergence of $(\dot{x}_n)$ it follows that (z_n) converges weakly in $L^1(I;\mathbb{R}^N)$ to $z \in L^1(I;\mathbb{R}^N)$.

Let $\epsilon > 0$ be arbitrary. By (3.9) and since $\epsilon_n \downarrow 0$ as $n \to \infty$, there exists n_ϵ such that, for all $t \in I$,

$$((t,y_n(t)),z_n(t)) \in \operatorname{graph}(F) + \mathbb{B}_\epsilon(0) \quad \forall\, n > n_\epsilon\,.$$

By Theorem 91, pg 138 we conclude that

$$((t,y(t)),z(t)) \in \operatorname{graph}(F) \quad \text{for a.a. } t \in I\,,$$

equivalently,

$$z(t) = \dot{x}(t) \in F(t,(Tx)(t)) \quad \text{for a.a. } t \in I.$$

This establishes assertion (i).

(ii) Let $x : [-h, \omega) \to \mathbb{R}^N$ be a solution of (3.8). Define

$$\mathcal{A} := \{(\rho, \xi) | \omega \leq \rho \leq \infty,\ \xi : [-h, \rho) \to \mathbb{R}^N \text{ is a solution of (3.8) with } \xi|_{[-h,\omega)} = x\}$$

On this non-empty set define a partial order $\preceq$ by

$$(\rho_1, \xi_1) \preceq (\rho_2, \xi_2) \quad \Longleftrightarrow \quad \rho_1 \leq \rho_2 \text{ and } \xi_1(t) = \xi_2(t) \text{ for all } t \in [-h, \rho_1).$$

Let $\mathcal{T}$ be a totally ordered subset of $\mathcal{A}$. Let $P := \sup\{\rho | (\rho, \xi) \in \mathcal{T}\}$ and let $\Xi : [-h, P) \to \mathbb{R}^N$ be defined by the property that, for every $(\rho, \xi) \in \mathcal{T}$, $\Xi|_{[0,\rho)} = \xi$. Then (P, Ξ) is in $\mathcal{A}$ and is an upper bound for $\mathcal{T}$. By Zorn's lemma, (Lemma 74), it follows that $\mathcal{A}$ contains at least one maximal element.

(iii) Assume that $x \in C([-h, \omega); \mathbb{R}^N)$ is a bounded maximal solution of (3.8). Seeking a contradiction, suppose $\omega < \infty$. By boundedness of x, together with Proposition 89 and (T1) of T, there exists $M > 0$ such that $\|\dot{x}(t)\| \leq M$ for almost all $t \in [0, \omega)$. Therefore, x is uniformly continuous and so extends to a continuous function $\xi : [-h, \omega] \to \mathbb{R}^N$. Define $z^0 \in C([-(h+\omega), 0]; \mathbb{R}^N)$ by

$$z^0(t) := \xi(\omega + t) \quad \forall\, t \in [-(h+\omega), 0]$$

and $\hat{F} : (t, x) \mapsto F(t + \omega, x)$. Consider the initial-value problem

$$\left.\begin{aligned} &\dot{z}(t) \in \hat{F}(t, (S_\omega T S_{-\omega} z)(t)) \\ &z|_{[-(h+\omega),0]} = z^0 . \end{aligned}\right\} \tag{3.15}$$

By (2.5) and the above existence result, there exists a solution $z : [-(h+\omega), \omega_z) \to \mathbb{R}^N$ $(\omega_z > 0)$ of (3.15). Now define $x^* : [-h, \omega + \omega_z) \to \mathbb{R}^N$ as follows

$$x^*(t) := \begin{cases} x(t), & t \in [-h, \omega) \\ z(t - \omega), & t \in [\omega, \omega + \omega_z) \end{cases}$$

Noting that, for almost all $t \in [\omega, \omega + \omega_z)$,

$$\dot{x}^*(t) = \dot{z}(t - \omega) \in \hat{F}(t - \omega, (S_\omega T S_{-\omega} z)(t - \omega)) = F(t, (Tx^*)(t)),$$

it follows that x^* is a solution of the original initial-value problem (3.1) and is a right extension of the solution x. This contradicts maximality of x. Therefore, $\omega = \infty$.

Chapter 4

Stability theory

4.1 Barbălat's lemma and its consequences

Let $(X, \|\cdot\|_X)$ be a metric space and $I \subset \mathbb{R}$ be an interval. A function $x : I \to X$ is *uniformly continuous* if for all $\epsilon > 0$ there exists some $\delta > 0$ such that if $s, t \in I$ with $|s - t| < \delta$ then $\|x(s) - x(t)\|_X < \epsilon$. The following facts are stated here for later reference:

(i) if I is compact and $x : I \to X$ is continuous then x is uniformly continuous on I;

(ii) if $x : \mathbb{R}_+ \to \mathbb{R}$ is absolutely continuous with essentially bounded derivative then x is uniformly continuous.

Many of the results in this section are consequences of the following elementary result known as Barbălat's lemma [6]. The proof which can also be found in, for example, [42] is included for completeness.

Lemma 33 (Barbălat)
If $x : \mathbb{R}_+ \to \mathbb{R}_+$ is a uniformly continuous function and

$$\lim_{t\to\infty} \int_0^t x(s)\, ds < \infty$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose for a contradiction that $x(t)$ does not tend to 0 as $t \to \infty$. Then there exists a positive $\epsilon > 0$ and $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $t_{n+1} > t_n + 1$ and $x(t_n) > \epsilon$. By the uniform continuity of x there exists $1 > k > 0$ such that $|x(t) - x(t + \tau)| \le \epsilon/2$

for all $t \geq 0$ and all $0 \leq \tau \leq k$. Hence

$$\begin{aligned} x(s) &= x(t_n) - (x(t_n) - x(s)) \\ &\geq x(t_n) - |x(t_n) - x(s)| \\ &> \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2} \end{aligned}$$

for all $s \in [t_n, t_n + k]$. Thus

$$\int_{t_n}^{t_n+k} x(s)\, ds > \frac{\epsilon k}{2} \quad \forall n \in \mathbb{N}.$$

Which contradicts the hypothesis that the integral $\int_0^t x(s)\, ds$ converges to a finite limit as $t \to \infty$.

Corollary 34 *Let $\xi \in AC \cap L^1(\mathbb{R}_+;\mathbb{R})$. If $\dot{\xi} \in L^\infty(\mathbb{R}_+;\mathbb{R})$ then $\xi(t) \to 0$ as $t \to \infty$.*

Proof. The result follows from Barbălat's Lemma on noting that absolute continuity of $\xi(\cdot)$ implies almost everywhere differentiability and that essential boundedness of the derivative implies uniform continuity.

Corollary 35
Let $F \in \mathcal{F}^{N,N}$ and $x : \mathbb{R}_+ \to \mathbb{R}^N$ be a bounded solution to the differential inclusion

$$\dot{x}(t) \in F(x(t)), \text{ almost all } t > 0, \quad x(0) = x^0.$$

If $x \in L^p(\mathbb{R}_+;\mathbb{R}^N)$ for some $p \geq 1$ then $x(t) \to 0$ as $t \to \infty$.

Proof. Since x is a bounded solution there exists some $R > 0$ such that $\|x(s)\| < R$ for all $t > 0$. Then (i) the map $\xi : s \mapsto \|x(s)\|^p$ is absolutely continuous, (ii) by Proposition 89, pg 137, $\dot{\xi}(t)$ is essentially bounded, and (iii) by hypothesis $\int_0^\infty \xi(s) ds < \infty$. Thus, by Corollary 34, $\|x(t)\| \to 0$ as $t \to \infty$.

Stability results, such as Corollary 35, using either integrals of the state or integral criteria based on observations of the state (such as a function $s \mapsto l(x(s))$) have their origins in the work of [83] and [25]. More recent work such as [19] generalizes these results, which differ fundamentally from the classical Lyapunov function approach wherein properties of the derivative of the observation $s \mapsto V(x(s))$ are utilized.

In the remainder of this Chapter we examine stability in the context of differential inclusions such as (3.8). These results will underpin the analysis and design of the control strategies of subsequent chapters.

4.2 ω-limit sets

Let (X, ρ) be a metric space. Given a non-empty set $A \subset X$, we say that A is *relatively compact* if $\bar{A}$ is compact. The distance to set map $\mathrm{d}_A : X \to \mathbb{R}_+$ defined by $\mathrm{d}_A(x) := \inf\{\rho(x, a) \mid a \in A\}$ is globally Lipschitz with constant 1: $|\mathrm{d}_A(x) - \mathrm{d}_A(y)| \leq \rho(x, y)$ for all $x, y \in X$.

A function $x : \mathbb{R}_+ \to X$ is said to *approach* A if $\mathrm{d}_A(x(t)) \to 0$ as $t \to \infty$.

Definition 36 (ω-limit set)
The ω-limit set of a function $x : \mathbb{R}_+ \to X$ is the set

$$\Omega(x) := \{\bar{x} \in X \mid \exists (t_i) \subset \mathbb{R}_+, t_i \to \infty, x(t_i) \to \bar{x} \text{ as } i \to \infty\}.$$

If $\Omega(x) \neq \emptyset$, an element of $\Omega(x)$ is said to be an ω-limit point of x. The set $\Omega(x)$ is always closed. Following [19, §4] we will use the concept of a robust ω-limit set.

Definition 37 (Robust ω-limit sets)
Let $x : \mathbb{R}_+ \to X$ be measurable with non-empty ω-limit set $\Omega(x)$. An ω-limit point $\bar{x} \in \Omega(x)$ is said to be robust if, for all sequences $(s_n)_{n \in \mathbb{N}}$, $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ with (i) $\lim_{n \to \infty} s_n \to \infty$, (ii) $t_n \geq s_n$ and (iii) $\lim_{n \to \infty}(t_n - s_n) = 0$,

$$\lim_{n \to \infty} x(s_n) = \bar{x} \implies \lim_{n \to \infty} x(t_n) = \bar{x}.$$

If every $\bar{x} \in \Omega(x)$ is robust then the ω-limit set $\Omega(x)$ is said to be robust.

For further discussion of this concept see [19], however it will be sufficient for our purposes to know that if $x : \mathbb{R}_+ \to X$ is uniformly continuous with non-empty ω-limit set then $\Omega(x)$ is robust.

Lemma 38
Let $x : \mathbb{R}_+ \to X$. If the range of x is relatively compact then $\Omega(x)$ is non-empty, compact and approached by x.

Proof. Let $(t_n)_{n \in \mathbb{N}} \to \infty$ be any sequence in $\mathbb{R}_+$. Since the range of x is relatively compact the sequence (x_n), given by $x_n := x(t_n)$, contains a convergent subsequence, also denoted by (x_n). The limit of this subsequence is an element of $\Omega(x)$.

Since X is a metric space it will be sufficient to prove that $\Omega(x)$ is sequentially compact. Thus we are required to prove that any sequence in $\Omega(x)$ has a convergent subsequence. Let $(\bar{x}_n)_{n=1}^{\infty} \subset \Omega(x)$ so that for each $n \in \mathbb{N}$ there exists a sequence $(\tau_{n,i})_{i=1}^{\infty} \subset \mathbb{R}_+$ such that (i) $\tau_{n,i} \to \infty$ and (ii) $\rho(x(\tau_{n,i}), \bar{x}_n) \to 0$ as $i \to \infty$. So for each $n \in \mathbb{N}$ there exists an $m(n)$ such that $\rho(x(\tau_{n,m(n)}), \bar{x}_n) < 1/2^n$ and $|\tau_{n,m(n)}| > 2^n$.

Let $z_n \in X$ be defined by $z_n := x(\tau_{n,m(n)})$. Since the range of x is relatively compact the sequence (z_n) contains a subsequence, (z_{n_j}), converging to x^*.

By construction $x^* \in \Omega(x)$. Moreover for all $j \in \mathbb{N}$, $\rho(z_{n_j}, \bar{x}_{n_j}) \leq 1/2^{n_j} \leq 1/2^j$. Relabel (z_{n_j}) as (z_j) and $(\bar{x}_{n_j})$ as $(\bar{x}_j)$ then given any $\epsilon > 0$ there exists some $M \in \mathbb{N}$ such that $\rho(\bar{x}_j, z_j) < \frac{\epsilon}{2}$ and $\rho(z_j, x^*) < \frac{\epsilon}{2}$ for all $j > M$. Then

$$\rho(\bar{x}_j, x^*) \leq \rho(\bar{x}_j, z_j) + \rho(z_j, x^*) < \epsilon.$$

Thus $\bar{x}_j \to x^*$ and we have proved that a subsequence converges to a limit in $\Omega(x)$.

To complete the proof of the Lemma suppose that $\Omega(x)$ is not approached by x. Then there exists an $\epsilon > 0$ and an increasing sequence (t_n) with $t_n \to \infty$ such that $x_n := x(t_n)$ converges to $\bar{x}$, say, and $\mathrm{d}_{\Omega(x)}(x_n) > \epsilon$ for all $n \in \mathbb{N}$. Clearly $\bar{x} \in \Omega(x)$ so that $\mathrm{d}_{\Omega(x)}(\bar{x}) = 0$. This is a contradiction to the continuity of $\mathrm{d}_{\Omega(x)}(\cdot)$.

4.2.1 Real Euclidean space $\mathbb{R}^N$

For example, given the Euclidean space $\mathbb{R}^N$ and a bounded continuous function $x : \mathbb{R}_+ \to \mathbb{R}^N$ the closure of the image of x is a compact set. Hence, for a bounded continuous function $x : \mathbb{R}_+ \to \mathbb{R}^N$, $\Omega(x)$ is non-empty, compact and approached by x.

4.2.2 The space of continuous $\mathbb{R}^N$-valued functions

In the remainder of this chapter we let $N \in \mathbb{N}$ and $h > 0$ be fixed and define $I := [-h, 0]$. Define $\mathbf{X}$ to be the Banach space $C(I; \mathbb{R}^N)$ with the norm $\|x\|_\infty := \sup_{t \in I} \|x(t)\|$.

Given a continuous function $x : [-h, \infty) \to \mathbb{R}^N$ we consider the map from $\mathbb{R}_+$ to $\mathbf{X}$ given by $s \mapsto (S_s x)|_I$. If $x : [-h, \infty) \to \mathbb{R}^N$ is a bounded uniformly continuous function then the family $\Phi := \bigcup\{(S_s x)|_I \mid s \in \mathbb{R}_+\} \subset \mathbf{X}$ is both uniformly bounded and equicontinuous. Hence by Arzelà-Ascoli theorem, (Theorem 75), is relatively compact. Thus, by Lemma 38, the ω-limit set is again non-empty, bounded and approached by $s \mapsto (S_s x)|_I$ as $s \to \infty$.

Given a continuous function $x : [-h, \infty) \to \mathbb{R}^N$ we denote the ω-limit set of the map $s \mapsto (S_s x)|_I$ by $\mathbf{\Omega}(x)$. Thus

$$\mathbf{\Omega}(x) = \{\bar{x} \in \mathbf{X} | \exists (t_i) \subset \mathbb{R}, t_i \to \infty, \|x(t_i + \cdot)|_I - \bar{x}(\cdot)\|_\infty \to 0 \text{ as } i \to \infty\}. \tag{4.1}$$

4.2.3 ω-limit sets and dynamical systems

Note that Definition 36 and Lemma 38 are statements about functions, whereas discussion of ω-limit sets more often (for example [23, 24, 26, 47]) takes place in the context of a local semi-flow generated by a local semi-dynamical system such as that generated by the differential equation

$$\dot{x}(t) = f(x(t)) \text{ for a.a. } t > 0 \text{ with } x(0) = x^0 \in \mathbb{R}^N \tag{4.2}$$

for a locally Lipschitz f. Let $\varphi : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ denote the local semi-flow generated by (4.2). Thus for all $x^0 \in \mathbb{R}^N$, $\varphi(0, x^0) = x^0$ and almost all $t > 0$

$$\frac{\partial \varphi}{\partial t} = f(\varphi(t, x^0)).$$

Assume, instead of (4.2), the dynamical system is modelled by the differential inclusion

$$\dot{x}(t) \in F(x(t)) \quad \text{for a.a. } t > 0 \text{ with } x(0) = x^0 \in \mathbb{R}^N \tag{4.3}$$

where $F \in \mathcal{F}^N$. Now we have possibly non-unique solutions for a given initial condition. The semi-flow approach presupposes unique forward trajectories which do not necessarily occur in the context of differential inclusions such as (4.3) or indeed (3.8).

Returning to the semi-flow generated by (4.2), if $x := \varphi(\cdot, x^0)$ is bounded then $\Omega(x) \neq \emptyset$ is an *invariant set* with respect to the semi-flow generated by (4.2). That is to say for all $\bar{x} \in \Omega(x)$, $\varphi(t, \bar{x}) \in \Omega(x)$ for all $t > 0$. ie the solution of (4.2) with initial condition $\bar{x}$ remains in $\Omega(x)$.

In the case of (4.3), where we have possibly non-unique solutions, we only obtain *weak-invariance* of the ω-limit set [70]. That is to say for all $\bar{x} \in \Omega(x)$ there exists a solution of (4.3) with initial condition $\bar{x}$ that remains in $\Omega(x)$ for all $t > 0$.

In Section 4.5 below we prove counterparts of these results for solutions of functional differential inclusions such as (3.8).

4.3 Asymptotic behaviour of solutions

In this section we focus on behaviour of solutions of the initial-value problem

$$\left.\begin{aligned} &\dot{x}(t) \in F(t, (Tx)(t)) \\ &x|_{[-h,0]} = x^0 \in C\left([-h, 0]; \mathbb{R}^N\right) \end{aligned}\right\} \tag{4.4}$$

with $T \in \mathcal{T}_h^{N,M}$ bounded-input bounded-output stable, and $F \in \mathcal{F}^{M+1,N}$ uniformly bounded in t on compact sets. That is to say for all $K \subset \mathbb{R}^M$ compact, there exists some $k \geq 0$ such that

$$\sup\{\|x\| \mid x \in F(t,w), t \in \mathbb{R},\ w \in K\} \leq k. \tag{4.5}$$

Next, we present two results on asymptotic behaviour of solutions of (4.4). The first requires a continuous observation function l and the second only lower semicontinuous l. These results plays a rôle in the analysis of the adaptive control strategies developed in the following chapters.

Lemma 39

Let $h \geq 0$, $N, M \in \mathbb{N}$, $F \in \mathcal{F}^{M+1,N}$ satisfy (4.5) and $T \in \mathcal{T}_h^{N,M}$ be bounded-input bounded-output stable. Let $l : \mathbb{R}^N \to \mathbb{R}_+$ be continuous. If $x : [-h, \infty) \to \mathbb{R}^N$ is a bounded solution of (4.4) and

$$\int_0^\infty l(x(t))\ dt < \infty\,, \tag{4.6}$$

then $x(t)$ tends, as $t \to \infty$, to the zero level set $l^{-1}(0)$ of l.

Proof. Since x is bounded there exists a compact set $K \subset \mathbb{R}^N$ such that $x(t) \in K$ for all $t \geq 0$. Further, x is a solution of (4.4). From the bounded input bounded output stability of T and properties of F it follows that $\dot{x}$ is essentially bounded. Thus x is uniformly continuous.

Since l is continuous it is uniformly continuous on the compact set K. Further, the composition $l(x(\cdot))$ of the uniformly continuous x (with range K) and a function that is uniformly continuous on K is also uniformly continuous. Apply Barbălat's Lemma to $l(x(\cdot))$ and conclude that $l(x(t)) \to 0$ as $t \to \infty$ and so $x(t) \to l^{-1}(0)$.

The following is essentially Lemma 9(ii) of [19] in the context of solutions of the system (4.4) with a direct proof.

Lemma 40

Let $h \geq 0$, $N, M \in \mathbb{N}$ and $F \in \mathcal{F}^{M+1,N}$ satisfy (4.5). Let $T \in \mathcal{T}_h^{N,M}$ be bounded-input, bounded-output stable in the sense of Definition 13. Let $l : \mathbb{R}^N \to \mathbb{R}_+$ be lower semicontinuous (Recall Definition 77, pg 132). If $x : [-h, \infty) \to \mathbb{R}^N$ is a bounded solution of (4.4) and, for some $\tau > 0$,

$$\lim_{t\to\infty} \int_t^{t+\tau} l(x(s))\ ds = 0\,, \tag{4.7}$$

then $x(t)$ tends, as $t \to \infty$, to the zero level set $l^{-1}(0)$ of l.

Proof. By boundedness of x and properties of F and T, it follows from (4.4) that $\dot{x}$ is essentially bounded and so x is uniformly continuous. By boundedness, x has non-empty ω-limit set $\Omega(x) \subset \mathbb{R}^N$. Since x approaches $\Omega(x)$, it suffices to show that $\Omega(x) \subset l^{-1}(0)$. Seeking a contradiction, suppose that there exists $z \in \Omega(x)$ with $z \notin l^{-1}(0)$. Define $\epsilon := l(z)/2 > 0$. Since $z \in \Omega(x)$, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to \infty$ such that $x(t_n) \to z$ as $n \to \infty$. By (4.7), and passing to a subsequence if necessary, we may assume

$$t_{n+1} - t_n > \frac{\tau}{n} \quad \text{and} \quad \int_{t_n}^{t_n+\frac{\tau}{n}} l(x(s))ds < \frac{\epsilon\tau}{n} \quad \forall\, n.$$

Therefore, for each n, there exists $s_n \in (t_n, t_n + \tau/n)$ such that $l(x(s_n)) < \epsilon$. Since $s_n - t_n \downarrow 0$ as $n \to \infty$, it follows by uniform continuity of x that $\|x(s_n) - x(t_n)\| \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} x(s_n) = \lim_{n\to\infty} x(t_n) = z\,.$$

Invoking lower semicontinuity of l, we arrive at a contradiction:

$$\epsilon \geq \liminf_{n\to\infty} l(x(s_n)) \geq l(z) = 2\epsilon > 0\,.$$

Remarks 41

(i) Note that the conclusion of Lemma 40 does not assert the convergence of the solution to an invariant subset of $l^{-1}(0)$ as in [10, 69]. Results on weak invariance are given in Section 4.4 below.

(ii) The requirement that F satisfy (4.5), or some similar bound on the growth of F, seems to be necessary. A non-autonomous differential equation is constructed in [25], the right hand ride of which fails to satisfy (4.5), with solution $x : [t_0, \infty) \to \mathbb{R}$, for which $l(|x(\cdot)|) \in L^1([t_0, \infty); \mathbb{R})$ for some continuous strictly monotone increasing function $l(\cdot)$ satisfying $l(0) = 0$ but for which $\limsup_{t\to\infty} \|x(t)\| > 0$.

4.4 Autonomous systems

Definition 42 *For $M, N \in \mathbb{N}$, $F \in \mathcal{F}^{M+1,N}$ is said to be* autonomous *if*

$$F(t,x) = F(s,x) \quad \forall x \in \mathbb{R}^M \text{ and } t,s \in \mathbb{R}. \tag{4.8}$$

Definition 43

An inclusion (3.8) is called autonomous *if T is of class $\mathcal{T}_h^{N,M}$, bounded input bounded* *output stable in the sense of (2.6) and is right shift invariant, and F is of class $\mathcal{F}^{M+1,N}$ and autonomous. Otherwise (3.8) is said to be* non-autonomous.

Remark 44

We write an autonomous system as

$$\left.\begin{array}{l}\dot{x}(t) \in F((Tx)(t)),\ \textit{for a.a. } 0 \le t < \omega \\ x|_I \in \mathbf{X}\end{array}\right\} \tag{4.9}$$

Furthermore, assume that $x : [-h, \infty) \to \mathbb{R}^N$ is a solution of (4.9) and let $t^*, t, \omega \ge 0$. Define $z(s) := S_{t^*}x(s)$ for all $s \in [t-h, t+\omega]$ then z satisfies

$$\left.\begin{array}{l}\dot{z}(s) \in F((Tz)(s)),\ \textit{for a.a. } t \le s \le t+\omega \\ z|_{[t-h,t]} \in C\left([t-h,t];\mathbb{R}^N\right)\end{array}\right\}.$$

Remark 45

The lack of time dependence in $F(\cdot)$ is not a restriction in some circumstances: $F(\cdot)$ is often constructed, in applications of interest, from a function $(t,x) \mapsto f(t,x) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, which is bounded in the first variable, by setting $F(x) := \overline{\{f(t,x) : t \in \mathbb{R}\}}$. Given a time dependent set valued map $F^*(\cdot,\cdot)$ which is bounded in the first variable, a time independent set valued map $F(\cdot)$ can be constructed by setting $F(x) := \overline{\bigcup_{t\in\mathbb{R}} F^*(t,x)}$. In applications where this construction is used it will be possible to relate conclusions about the behaviour of solutions of systems generated by differential inclusions with right hand side $F(\cdot)$, back to give conclusions about the behaviour of solutions of the original systems since any solution, $x(\cdot)$ of

$$\dot{x}(t) \in F^*(t, x(t)) \textit{ with } x(0) = x^0 \in \mathbb{R}^N$$

or

$$\dot{x}(t) \in f(t, x(t)) \textit{ with } x(0) = x^0 \in \mathbb{R}^N$$

will be a solution to

$$\dot{x}(t) \in F(x(t)) \textit{ with } x(0) = x^0 \in \mathbb{R}^N.$$

Remark 46

If x is a bounded maximal solution of (4.9) then it exists on $[-h,\infty)$ and moreover, since $\|(Tx)(\cdot)\|$ is bounded and F is upper semicontinuous, $\dot{x}(\cdot)$ is bounded by Proposition 89 and so x is uniformly continuous. Thus $\Omega(x) \ne \emptyset$. (Recall (4.1)).

In the remainder of this Chapter we investigate the properties of solutions of autonomous inclusions.

4.5 An invariance principle

Lemma 47 (Compactness of trajectories)
Let $N \in \mathbb{N}$, $t, \omega \geq 0$ and assume $(x_j)_{j\in\mathbb{N}} \subset C([t-h, t+\omega]; \mathbb{R}^N)$ is a uniformly bounded sequence of solutions of the autonomous inclusion (recall Definition 43)

$$\dot{x}_j(s) \in F((Tx_j)(s)), \text{ for a.a. } t \leq s \leq t+\omega$$

with initial condition $x_j|_{[t-h,t]}$. If $(x_j|_{[t-h,t]})$ is equicontinuous then there exists a subsequence which converges uniformly to a function $x^* : [t-h, t+\omega] \to \mathbb{R}^N$ which satisfies

$$\left.\begin{array}{l} \dot{x}^*(s) \in F((Tx^*)(s)), \text{ for a.a. } t \leq s \leq t+\omega \\ x^*|_{[t-h,t]} \in C([t-h, t]; \mathbb{R}^N) \end{array}\right\} \tag{4.10}$$

Proof. Let $J := [t-h, t]$. By the uniform boundedness and equicontinuity of $x_j|_J$ and Theorem 75, pg 131 there exists a subsequence, denoted also by $(x_j(\cdot))$, which converges uniformly on J.

Since (x_j) is uniformly bounded it follows from (2.6) and Proposition 89 that $(\dot{x}_j)$ is uniformly essentially bounded on $[t, t+\omega]$. Apply Theorem 90 to conclude that there exists a subsequence (again denoted by) (x_j) which converges uniformly to an absolutely continuous function $x^* : [t, t+\omega] \to \mathbb{R}^N$ and $\dot{x}_n$ converges weakly to $\dot{x}^*$ in $L^1([t, t+\omega]; \mathbb{R}^N)$.

Let $z_j(s) := (Tx_j)(s)$ for all $s \in [t, t+\omega]$ and $j \in \mathbb{N}$. Then

$$\dot{x}_j(s) \in F(z_j(s)), \text{ for a.a. } t \leq s \leq t+\omega$$

which is to say that $(z_j(s), \dot{x}_j(s)) \in \text{graph}(F)$. By (T3*), $(Tx_j)(\cdot)$ converges to $(Tx^*)(\cdot)$ almost everywhere. Thus (z_j) converges to $z^*(\cdot) := (Tx^*)(\cdot)$. Apply the Theorem 91 to conclude that $(z^*(s), \dot{x}^*(s)) \in \text{graph}(F)$. That is to say that $\dot{x}^*(s) \in F((Tx^*)(s))$ for almost all $s \in [t, t+\omega]$ as required.

Definition 48 (Weak invariance)
$\emptyset \neq \Gamma \subset \mathbf{X}$ is weakly invariant with respect to solutions of the autonomous inclusion (4.9) if for every $\bar{x} \in \Gamma$ there is a solution $x^* : [-h, \infty) \to \mathbb{R}^N$ of (4.9) with initial condition $x^*|_I = \bar{x}$ and for all $s \geq 0$, $(S_s x^*)|_I \in \Gamma$.

Lemma 49 (Weak invariance of $\Omega(x)$)
If x is a bounded maximal solution of the autonomous inclusion (4.9) then $\Omega(x)$ is weakly invariant.

Proof. Since x is a bounded maximal solution of (4.9) it is defined on $[-h, \infty)$, is uniformly continuous and $\mathbf{\Omega}(x) \neq \emptyset$. Let $\bar{x} \in \mathbf{\Omega}(x)$.

The proof is by induction on m with the following induction hypothesis: there exists an $x^* : [-h, m] \to \mathbb{R}^N$ which satisfies;

$$
\begin{aligned}
&\dot{x}^*(s) \in F((Tx^*)(s)) \quad \text{for all } s \in [0, m],\\
&x^*|_I = \bar{x},\\
&(S_s x^*)|_I \in \mathbf{\Omega}(x) \quad \forall s \in [0, m].
\end{aligned}
$$

Since $\bar{x} \in \mathbf{\Omega}(x)$ there exists an unbounded sequence of times $(t_n)_{n=1}^{\infty} \subset \mathbb{R}_+$ and a corresponding sequence $x_n : s \mapsto x(t_n + s)$ for all $s \in I$ such that $\|\bar{x} - x_n\|_\infty \to 0$ as $n \to \infty$.

Let x_n be extended to functions on the interval $[-h, 1]$ by letting $x_n(s) := x(t_n + s)$ for all $s \in [-h, 1]$. Then by Remarks 44 x_n is a solution of (4.9) on $[-h, 1]$ with initial condition $x_n|_I$.

The family $(x_n)_{n \in \mathbb{N}}$ is uniformly bounded by the bound on x and equicontinuous by the uniform continuity of x so by Lemma 47 there exists a subsequence, also denoted by (x_n), which converges uniformly to a function $x^* : [-h, 1] \to \mathbb{R}^N$ which is a solution of (4.9) with initial condition $\bar{x}$. Moreover, for each $0 \le s \le 1$, $[s - h, s] \subset [-h, 1]$ and $x_n(s) = x(t_n + s)$ so that $(S_s x^*)\,|_I \in \mathbf{\Omega}(x)$.

Next assume that there exists a solution $x^* : [-h, m] \to \mathbb{R}^N$ of (4.9) with initial condition $\bar{x}$ such that for all $0 \le s \le m$, $(S_s x^*)|_I \in \mathbf{\Omega}(x)$. Then, in particular, there exists an increasing and unbounded sequence of times (τ_n) such that $\|x(\tau_n + \cdot) - x^*(m + \cdot)\|_\infty \to 0$. Define the functions $x_n^* : [m - h, m + 1] \to \mathbb{R}^N$ by $x_n^*(t) := x(\tau_n + t)$. Again by Lemma 47 there exists a subsequence, which is also denoted by (x_n^*), such that $x_n^*|_{[m-h,m+1]}$ converges uniformly to some function $\tilde{x} : [m - h, m + 1] \to \mathbb{R}^N$ and $\tilde{x}$ is a solution of (4.9) with initial condition $\tilde{x}|_{[m-h,m]} = x^*|_{[m-h,m]}$. Let $x^*(\cdot)$ be extended to $[-h + m, m + 1]$ by

$$
x^*(t) := \begin{cases} x^*(t) & t \in [-h, m] \\ \tilde{x}(t) & t \in (m, m+1] \end{cases}
$$

then x^* is a solution of (4.9) on $[-h, m+1]$ with initial condition $\bar{x}(\cdot)$ and by construction for every $s \in [0, m + 1]$, $(S_s x^*)\,|_I \in \mathbf{\Omega}(x)$.

Claim 50 *If $x : [-h, \infty) \to \mathbb{R}^N$ is uniformly continuous then the map $s \mapsto (S_s x)|_I$ from $\mathbb{R}_+$ to $\mathbf{X}$ is also uniformly continuous.*

Proof. Let $\epsilon > 0$ then by the uniform continuity of x there exists a $\delta > 0$ such that

$$t, s > -h, |t-s| < \delta \quad \Rightarrow \quad \|x(t) - x(s)\| < \epsilon.$$

Hence for all $t, s > 0$ with $|t-s| < \delta$, if $\tau \in [-h, 0]$, then $\|x(t+\tau) - x(s+\tau)\| < \epsilon$. Thus $\|(S_t x)|_I - (S_s x)|_I\|_\infty < \epsilon$ which is to say that $s \mapsto (S_s x)|_I$ is uniformly continuous.

Theorem 51 (An integral invariance principle)
Let $l : \mathbf{X} \to \mathbb{R}_+$ be lower semicontinuous. If x is a bounded solution of the autonomous inclusion (4.9) with

$$\int_0^\infty l((S_s x)|_I)\, ds < \infty$$

then $(S_s x)|_I$ approaches the largest weakly invariant set in $\Sigma := l^{-1}(0)$.

Proof. By Theorem 32, x has maximal interval of existence $[-h, \infty)$ and $\Omega(x) \neq \emptyset$ by Remark 46. By Claim 50 $s \mapsto (S_s x)|_I$ is uniformly continuous and so by [19, pg 8, Remarks 2] $\Omega(x)$ is robust (recall Definition 37). Thus by [19, Lemma 9(ii)] $\Omega(x) \subset \Sigma$. By Lemma 38 $s \mapsto (S_s x)|_I$ approaches $\Omega(x)$ which is weakly invariant by Lemma 49. Therefore $(S_s x)|_I$ approaches the largest weakly-invariant set in Σ.

Chapter 5

Single-input single-output first order systems

5.1 The generic SISO system

In this chapter we consider the problem of adaptive feedback control of nonlinear, single-input $u(\cdot)$, single-output $y(\cdot)$ systems $(b, g, p, \widehat{T})$, given by a controlled nonlinear functional differential equation of the form

$$\dot{y}(t) = g(p(t), y(t)) + (\widehat{T}y)(t) + b(t, y(t))u(t). \tag{5.1}$$

The nature of the control objective, nonlinearities g, b and operator $\widehat{T}$ will be made precise in due course. The function p (assumed to be essentially bounded) models general perturbations, disturbances and time-varying parameters. By way of illustration, the following three systems can be recast within the framework of (5.1):

$$\text{Example 1:} \quad \left\{ \begin{array}{l} \dot{y}(t) = g(p(t), y(t)) + q_1(t)y(t-h_1) + q_2(t)y^2(t-h_2) \\ \qquad + \int_{-h_3}^{0} q_3(s)y^3(t+s)ds + bu(t) \end{array} \right. \tag{5.2}$$

$$\text{Example 2:} \quad \left\{ \begin{array}{l} \dot{y}(t) = g(p(t), y(t)) + \frac{c_1}{2\epsilon} \int_{\xi_o-\epsilon}^{\xi_o+\epsilon} z(t,\xi)\, d\xi + bu(t) \\ z_{tt}(t,\xi) = a_0 z_{\xi\xi}(t,\xi) - a_1 z_t(t,\xi) + c_0\delta(\xi - \xi_i)y(t), \\ z(t,0) = 0 = z(t,1) \end{array} \right. \tag{5.3}$$

$$\text{Example 3:} \quad \left\{ \begin{array}{l} \dot{y}(t) = g(p(t), y(t)) + W(z(t)) + bu(t) \\ \dot{z}(t) = -a_0 z(t) - a_1 z^3(t) + a_2(1 + z^2(t))y(t) \end{array} \right. \tag{5.4}$$

with q_1, q_2, q_3 essentially bounded, $b \neq 0$ is an unknown constant, $a_0, a_1 > 0$ and where the second of equations (5.3) has spatial domain $\Omega = [0,1]$ and is subject to Dirichlet boundary conditions.

5.2 Systems of class $\mathcal{N}_1$

Let $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions, available for control purposes, with the property that

$$\Phi(r) := \max\{\phi(r), \psi(r)\} > 0 \text{ for all } r \neq 0. \tag{5.5}$$

We first consider the stabilization problem of feedback control to ensure $y(t) \to 0$ as $t \to \infty$ for a class $\mathcal{N}_1$ ($= \mathcal{N}_1(\phi, \psi)$) of nonlinear, single-input $u(\cdot)$, single-output $y(\cdot)$ systems $(b, g, p, \widehat{T})$, of the form (5.1) that satisfy Assumption 52 below.

Assumption 52

1. *$b \neq 0$ is constant.*

2. *$g : \mathbb{R}^P \times \mathbb{R} \to \mathbb{R}$ is continuous and, for each compact $K \subset \mathbb{R}^P$ there exists $\mu_K \geq 0$ such that*

$$|g(p, y)| \leq \mu_K \phi(|y|) \quad \forall\ (p, y) \in K \times \mathbb{R}. \tag{5.6}$$

3. *$p \in L^\infty(\mathbb{R}; \mathbb{R}^P)$.*

4. *For some $h \geq 0$, $\widehat{T} \in \mathcal{T}_h^1 \cap \mathcal{C}_1(\psi)$ (recall Definition 16, pg 29) and is bounded-input, bounded-output stable in the sense of (2.6).*

If $\phi(y) := \exp(|y|)$, then all polynomial function of arbitrary degree and with time varying coefficients $p(\cdot) = (p_1(\cdot), \ldots, p_P(\cdot)) \in L^\infty(\mathbb{R}; \mathbb{R}^P)$ of the form

$$g(p(t), y(t)) = \sum_{i=0}^{P} p_i(t) y^i(t)$$

satisfy Assumption 52 (2) and (3). If an upper bound S is known for the degree, P, of the polynomial then $\exp(|y|)$ can be replaced by $\phi(y) := 1 + |y|^S$.

The problem to be addressed is that of control design to ensure that, for all $(b, g, p, \widehat{T}) \in \mathcal{N}_1 = \mathcal{N}_1(\phi, \psi)$ and all initial data $y|_{[-h,0]} = y^0 \in C([-h, 0]; \mathbb{R})$, every solution $y(\cdot)$ of (5.1) approaches zero.

5.2.1 $\mathcal{N}_1$-universal stabilizer

Writing

$$\Phi : \mathbb{R}_+ \to \mathbb{R}_+, \quad r \mapsto \max\{\phi(r), \psi(r)\}$$

then the adaptive feedback strategy (appropriately initialized and with the discontinuity interpreted in the set-valued sense)

$$\left.\begin{array}{rcl} u(t) & = & \nu(k(t))\Phi(|y(t)|)\,\mathrm{sgn}(y(t)), \\ \dot{k}(t) & = & \Phi(|y(t)|)|y(t)| \end{array}\right\} \tag{5.7}$$

where $\nu : \mathbb{R} \to \mathbb{R}$ is a continuous function with properties (1.6) will be shown to be a $\mathcal{N}_1$-universal stabilizer assuring that, for all $(b, g, p, \widehat{T}) \in \mathcal{N}_1$ and all y^0, $y(t) \to 0$ as $t \to \infty$ whilst maintaining boundedness of the controller function k.

Embedding the closed-loop system into a differential inclusion

In view of the discontinuous nature of the feedback, the first of equations (5.7) is interpreted in the set-valued sense

$$u(t) \in \nu(k(t))\Phi(|y(t)|)\sigma(y(t)), \tag{5.8}$$

with $y \mapsto \sigma(y) \subset \mathbb{R}$ given by

$$\sigma(y) := \begin{cases} \{\mathrm{sgn}(y)\}, & y \neq 0 \\ [-1, 1], & y = 0. \end{cases} \tag{5.9}$$

Let $(b, g, p, \widehat{T}) \in \mathcal{N}_1$. By properties of g and essential boundedness of p, there exists $\mu \in \mathbb{R}_+$ such that, for all $y \in \mathbb{R}$ and almost all t, $|g(p(t), y)| \leq \mu\phi(|y|)$. Define $x \mapsto F_1(x) \subset \mathbb{R}^2$ by

$$F_1(x) = F_1(y, k) := \{v + bu|\ |v| \leq \mu\phi(|y|), u \in \nu(k)\Phi(|y|)\sigma(y)\} \times \{\Phi(|y|)|y|\}.$$

$F_1(\cdot)$ is upper semicontinuous with non-empty, compact and convex values and so $F_1 \in \mathcal{F}^2$. Define $T \in \mathcal{T}_h^2$ by

$$(Tx)(t) = (T(y, k))(t) := (\widehat{T}y, 0)(t) \quad \forall\, t \in \mathbb{R}_+, \tag{5.10}$$

for all $x \in C([-h, \infty); \mathbb{R}^2)$. We now embed the feedback-controlled system in a functional differential inclusion:

$$\dot{x}(t) - (Tx)(t) \in F_1(x(t)), \quad x|_{[-h,0]} = (y^0, k^0) \in C\left([-h,0];\mathbb{R}^2\right). \tag{5.11}$$

Since $F_1 \in \mathcal{F}^2$ and $T \in \mathcal{T}_h^2$ it follows, by Theorem 32, that, for all x^0, (5.11) has a solution and every solution can be maximally extended.

Note that any absolutely continuous function $s \mapsto (e(s), k(s))$ which satisfies the combined equations (5.1) and (5.7) will be a solution to the inclusion (5.11). We say that the closed-loop system comprising (5.1) and (5.7) is *embedded* into (5.11).

Theorem 53

Let $x = (y,k) : [-h,\omega) \to \mathbb{R}^2$ be a maximal solution of the initial-value problem (5.11). Then:

(i) $\omega = \infty$*;*

(ii) $\lim_{t\to\infty} k(t)$ *exists and is finite;*

(iii) $y(t) \to 0$ *as* $t \to \infty$.

Proof. For almost all $t \in [0,\omega)$, we have

$$\begin{aligned} y(t)\dot{y}(t) &\le y(t)(\widehat{T}y)(t) + [\mu + b\nu(k(t))]\Phi(|y(t)|)|y(t)| \\ &= y(t)(\widehat{T}y)(t) + [\mu + b\nu(k(t))]\dot{k}(t), \end{aligned} \tag{5.12}$$

which, on integration and invoking properties of $C_1(\psi)$-class operators, yields

$$\begin{aligned} 0 &\le y^2(t) \le y^2(\tau) + c^* + 2\int_\tau^t [\mu^* + b\nu(k(s))]\,\dot{k}(s)\,ds \\ &= y^2(\tau) + c^* + 2\mu^*[k(t) - k(\tau)] + 2b\int_{k(\tau)}^{k(t)} \nu(s)ds \quad \forall\, \tau, t \in [0,\omega),\ \tau \le t \end{aligned} \tag{5.13}$$

for some constants c^* and μ^*. Seeking a contradiction, suppose k is unbounded and so, by monotonicity, $k(t) \to \infty$ as $t \uparrow \omega$. Fix $\tau \in [0,\omega)$ such that $k(\tau) \ge 1$. Dividing by $k(t) \ge k(\tau) \ge 1$ in (5.13) gives

$$0 \le y^2(\tau) + c^* + 2\mu^* + \frac{2b}{k(t)}\int_{k(\tau)}^{k(t)} \nu(s)ds \quad \forall\, t \in [\tau,\omega). \tag{5.14}$$

By the properties (1.6) of the ν there exist two monotonic increasing sequences $(\eta_n)_{n\in\mathbb{N}}$

and $(\hat{\eta}_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\frac{1}{\eta_n}\int_{k(0)}^{\eta_n}\nu = +\infty \text{ and } \lim_{n\to\infty}\frac{1}{\hat{\eta}_n}\int_{k(0)}^{\hat{\eta}_n}\nu = -\infty.$$

By the continuity, monotonicity and supposed unboundedness of $k(\cdot)$ there exist strictly increasing sequences $(t_n)_{n\in\mathbb{N}}, (\hat{t}_n)_{n\in\mathbb{N}} \subset [0,\omega)$ such that $\eta_n = k(t_n)$ and $\hat{\eta}_n = k(\hat{t}_n)$ for all $n \in \mathbb{N}$. Although we do not know the sign of b in (5.14) if $b < 0$ then $\lim_{n\to\infty}\frac{2b}{k(t_n)}\int_{k(\tau)}^{k(t_n)}\nu = -\infty$ and if $b > 0$ then $\lim_{n\to\infty}\frac{2b}{k(\hat{t}_n)}\int_{k(\tau)}^{k(\hat{t}_n)}\nu = -\infty$. In either case we gain a contradiction to (5.14). Therefore, k is bounded.

Boundedness of y, and hence x, follows by (5.14). By boundedness of $x = (y,k)$, we conclude that $\omega = \infty$, which is assertion (i) of the theorem. By boundedness and monotonicity of k, assertion (ii) holds. Finally, by boundedness of k,

$$\infty > \int_0^\infty \dot{k}(t)dt = \int_0^\infty \Phi(|y(t)|)|y(t)|dt$$

and so, by Lemma 40, we conclude that $y(t) \to 0$ as $t \to \infty$. This completes the proof.

Remark 54

Let $\mathcal{N}_1^*$ be the subclass of $\mathcal{N}_1$ in which knowledge of $\mathrm{sgn}(b)$ *is available to the controller. Then we may avoid the use of a Nussbaum switching function and replace the control (5.7) with the formal control*

$$\left.\begin{aligned} u(t) &= -\,\mathrm{sgn}(b)k(t)\Phi(|y(t)|)\,\mathrm{sgn}(y(t)),\\ \dot{k}(t) &= \Phi(|y(t)|)|y(t)|. \end{aligned}\right\}$$

We embed the closed-loop system within a differential inclusion and the analysis proceeds as before yielding the estimate

$$y(t)\dot{y}(t) \le y(t)(\widehat{T}y)(t) + [\mu - |b|k(t)]\dot{k}(t)\,,$$

in place of (5.12), which, on integration, easily provides a contradiction in the case of unbounded k. Thus k, y and hence x are bounded. The rest of the proof remains unchanged.

5.2.2 Numerical examples

Consider the specific examples (5.2), (5.3) and (5.4) given Section 5.1 with $q_1, q_2, q_3 \in L^\infty(\mathbb{R})$, $a_0, a_1 > 0$ and where the second of equations (5.3) has spatial domain $\Omega = [0,1]$ and is subject to Dirichlet boundary conditions as in Section 2.2.2. Each of these three

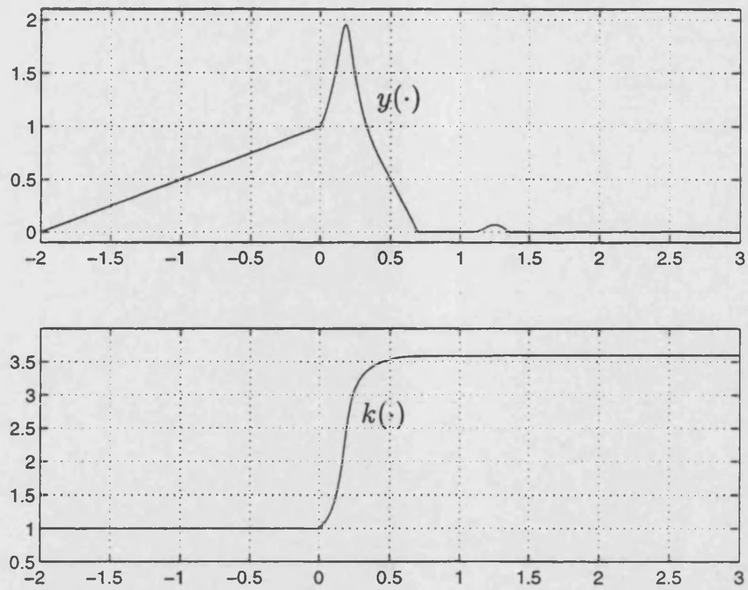


Figure 5-1: Typical behaviour of Example 1 under adaptive control.

systems may be expressed in the form (5.1), with the operator $\widehat{T}$ defined as in Sections 2.2.3, 2.2.2 and 2.2.1, respectively. We remark that, in the case of Example 2, the contribution of the initial state of the wave equation

$$t \mapsto \mathbf{S}_t \begin{bmatrix} z^0 \\ z^1 \end{bmatrix}$$

may, by exponential stability of the semigroup, be absorbed by the function g.
Defining

$$\phi = \psi : r \mapsto 1 + r^3$$

and assuming that $b \neq 0$ and $p \in L^\infty(\mathbb{R}; \mathbb{R}^P)$, then each of Examples 1, 2 and 3 is of class $\mathcal{N}_1 = \mathcal{N}_1(\phi, \psi)$ provided that g satisfies Assumption 52(2). Therefore, by Theorem 53, the following control (appropriately initialized and with the discontinuity interpreted in the set-valued sense)

$$\left.\begin{array}{rcl} u(t) & = & k^2(t)\cos(k(t))\phi(|y(t)|)\mathrm{sgn}(y(t)), \\ \dot{k}(t) & = & \phi(|y(t)|)|y(t)| \end{array}\right\}$$

stabilizes each of the three widely disparate uncertain systems: in all three cases, the function $y(t) \to 0$ as $t \to \infty$ and the monotone adapting parameter $k(\cdot)$ converges.

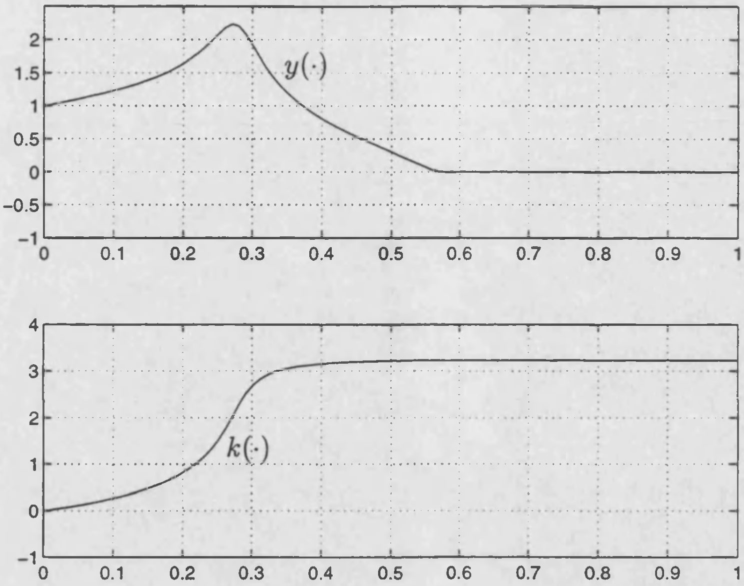


Figure 5-2: Typical behaviour of Example 2 under adaptive control.

Example 1

By way of illustration, consider (5.2) with $b = \frac{1}{2}$, $g : (p, y) \mapsto y^{\frac{1}{3}} + y^3 + p$, $p : t \mapsto \sin(7t)$, $h_1 = 1$, $h_2 = h_3 = 2$, $q_1(t) = q_2(t) = 1$ for all t, $q_3 : t \mapsto \sin(-\frac{1}{2}\pi t)$, and with initial conditions $y(t) = \frac{1}{2}t + 1$ for $-2 \leq t \leq 0$ and $k(0) = 1$. Figure 5-1 depicts the system behaviour (computed using an improved Euler method within MATLAB) under the adaptive control.

Example 2

Consider (5.3) with parameters $a_0 = 10$, $a_1 = 1$, $c_0 = 10$, $c_1 = 1$, $\xi_o = 1/3$, $\xi_i = 2/3$, $\epsilon = 0.01$, with Dirichlet boundary conditions ($v(t, 0) = v(t, 1) = 0$ for all $t \geq 0$) on the first of equations (5.3), with $b = 1/2$, $g : (r, y) \mapsto y^3 + r$, with initial conditions $z(0, \cdot) = 0 = z_t(0, \cdot)$, $y(t) = 1$ for all $t \in [-1, 0]$, $k(0) = 0$, and with $p(\cdot) = p_1(\cdot)$ where $p_1(\cdot)$ is the first coordinate of the solution of the initial-value problem for the Lorenz system

$$\left.\begin{aligned} \dot{p}_1(t) &= p_2(t) - p_1(t) \\ \dot{p}_2(t) &= 2.8p_1(t) - 0.1p_2(t) - p_1(t)p_3(t) \\ \dot{p}_3(t) &= p_1(t)p_2(t) - \tfrac{8}{30}p_3(t) \\ (p_1(0), p_2(0), p_3(0)) &= (1, 0, 3) \end{aligned}\right\} \qquad (5.15)$$

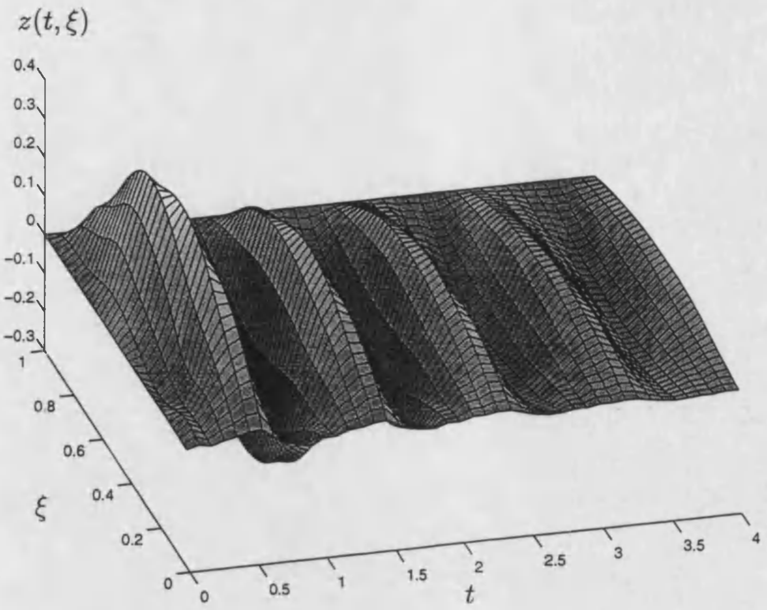


Figure 5-3: State evolution for the wave equation in Example 2.

(The solution is bounded on $\mathbb{R}_+$: see, for example, [82, Appendix C].) Figures 5-2 and 5-3 depict the system behaviour (computed - adopting a truncated eigenfunction expansion, of order 8, to model the wave equation - using SIMULINK within MATLAB) under the adaptive control; Figure 5-3 shows the state evolution of the second of equations (5.3).

Example 3

Consider (5.4) with $a_0 = 2$, $a_1 = 1$, $a_2 = 3$, $b = -\frac{1}{2}$, $p(\cdot) = (p_1(\cdot), p_2(\cdot), p_3(\cdot))$, $W : z \mapsto z$, and

$$g(p(t), y(t)) = p_1(t)y(t) + p_2(t)y^3(t) + p_3(t)\sqrt{|y(t)|}$$

where $(p_1(\cdot), p_2(\cdot), p_3(\cdot))$ is the solution of the initial-value problem (5.15). For initial conditions $y(0) = \frac{1}{2}$, $z(0) = -2$ and $k(0) = 1$, Figure 5-4 depicts the system behaviour (computed using SIMULINK within MATLAB) under the adaptive control.

5.3 Systems of class $\mathcal{N}_2$

We extend the class of the previous Section by considering a class of systems $\mathcal{N}_2 \supset \mathcal{N}_1$ of systems of the form (5.1). For such systems, we again address the stabilization

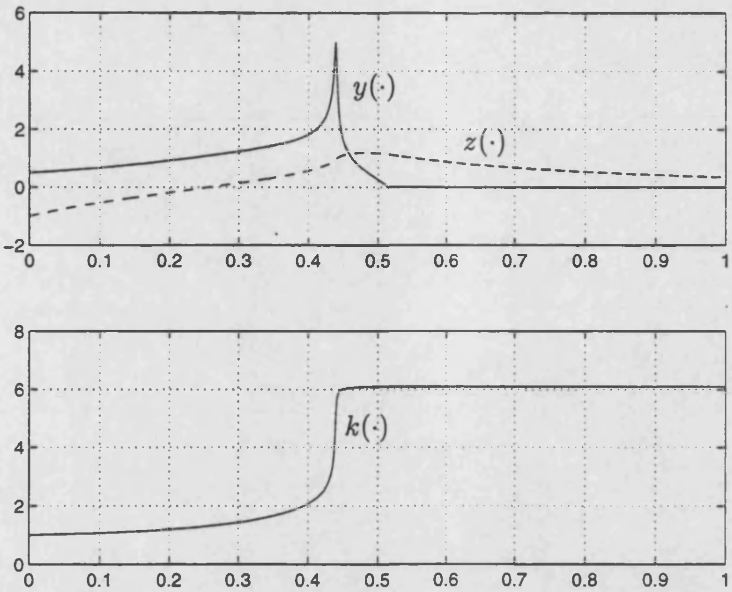


Figure 5-4: Typical behaviour of Example 3 under adaptive control.

problem of feedback control to ensure $y(t) \to 0$ as $t \to \infty$.

Let $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions, available for control purposes, which satisfy (5.5). The class $\mathcal{N}_2$ $(= \mathcal{N}_2(\phi, \psi))$ is defined to be the family of all those systems $(b, g, p, \widehat{T})$ of the form (5.1) that satisfy Assumption 55 below.

Assumption 55

The function $b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist constants $0 < \gamma_1 < \gamma_2$ such that

$$\gamma_1 < |b(t, y)| < \gamma_2 \quad \forall \ (t, y) \in \mathbb{R} \times \mathbb{R}. \tag{5.16}$$

The nonlinearities g, p and $\widehat{T}$ satisfy Assumption 52 parts 2-4.

Implementation of an $\mathcal{N}_2$-universal strategy will use a *scaling-invariant Nussbaum function:*

Definition 56

The function $\nu : \mathbb{R} \to \mathbb{R}$ is a scaling-invariant Nussbaum function *if for all α, β with* $\mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta)$ *the function*

$$t \mapsto \tilde{\nu}(t) := \begin{cases} \alpha\nu(t) & \text{if} \quad \nu(t) \geq 0 \\ \beta\nu(t) & \text{if} \quad \nu(t) < 0 \end{cases}$$

satisfies the properties (1.6) (ie $\tilde{\nu}(\cdot)$ is itself a Nussbaum function).

This definition originates in [51] and, for example (see [51, §4.2]), $k \mapsto \cos(\frac{\pi}{2}k)e^{k^2}$ is a scaling-invariant Nussbaum function. In fact, [51] require the constants α, β in Definition 56 to both be positive. Since ν is a Nussbaum function if and only if $-\nu$ is a Nussbaum function, our definition is equivalent.

5.3.1 $\mathcal{N}_2$-universal stabilizer

Let $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous. Writing

$$\Phi : \mathbb{R}_+ \to \mathbb{R}_+, \quad r \mapsto \max\{\phi(r), \psi(r)\}$$

as before, then the following adaptive feedback strategy (appropriately interpreted) will be shown to be an $\mathcal{N}_2$-universal stabilizer assuring that, for all $(b, g, p, \widehat{T}) \in \mathcal{N}_2$ and all initial data y^0, $y(t) \to 0$ as $t \to \infty$ whilst maintaining boundedness of the controller function $k(\cdot)$.

$$\left.\begin{array}{l} u(t) = \nu(k(t))\Phi(|y(t)|)\operatorname{sgn}(y(t)) \\ \dot{k}(t) = \Phi(|y(t)|)|y(t)| \\ k|_{[-h,0]} = k^0 \in C([-h,0];\mathbb{R}) \end{array}\right\} \tag{5.17}$$

where $\nu(\cdot) : \mathbb{R} \to \mathbb{R}$ is a *scaling-invariant Nussbaum function*.

Notice that only knowledge of ϕ, ψ is used in building this controller. In particular the values of γ_1, γ_2 and $\operatorname{sgn}(b)$ are not available.

Embedding the closed-loop system into a differential inclusion

In view of the discontinuous nature of the feedback, the first of equations (5.17) is interpreted in the set-valued sense

$$u(t) \in \nu(k(t))\Phi(|y(t)|)\sigma(y(t)), \tag{5.18}$$

with $y \mapsto \sigma(y) \subset \mathbb{R}$ given by (5.9).

Let $(b, g, p, \widehat{T}) \in \mathcal{N}_2$. By properties of b, g and essential boundedness of p, there exists $\mu \in \mathbb{R}_+$ and $0 < \gamma_1 < \gamma_2$ such that, for all $y \in \mathbb{R}$ and almost all t, $|g(p(t), y)| \le \mu\phi(|y|)$ and $\gamma_1 < |b(t, y)| < \gamma_2$. Moreover since b is continuous and strictly bounded away from zero $(t, y) \mapsto b(t, y)$ is of constant sign $\beta := \operatorname{sgn}(b) = \pm 1$. Define $x \mapsto F_2(x) \subset \mathbb{R}^2$ by

$$\begin{aligned} F_2(x) &= F_2(y, k) \\ &:= \{v + \beta w\nu(k)u \mid |v| \le \mu\phi(|y|), \gamma_1 < w < \gamma_2,\ u \in \Phi(|y|)\sigma(y)\} \times \{\Phi(|y|)|y|\}. \end{aligned}$$

F_2 is upper semicontinuous with non-empty, compact and convex values and so $F_2 \in \mathcal{F}^2$. Define $T \in \mathcal{T}_h^2$ by (5.10). We now embed the feedback-controlled system in a functional differential inclusion:

$$\dot{x}(t) - (Tx)(t) \in F_2(x(t)), \quad x|_{[-h,0]} = x^0 := (y^0, k^0) \in C\left([-h,0];\mathbb{R}^2\right). \tag{5.19}$$

Since $F_2 \in \mathcal{F}^2$ and $T \in \mathcal{T}_h^2$ it follows, by Theorem 32, that, for all x^0, (5.19) has a solution on some interval $[-h,\omega)$ with $\omega > 0$, (ii) every solution can be maximally extended, (iii) if a maximal solution $x : [-h,\omega) \to \mathbb{R}^2$ is bounded, then $\omega = \infty$.

Theorem 57

Let $x(\cdot) = (y(\cdot), k(\cdot)) : [-h,\omega) \to \mathbb{R}^2$ be a maximal solution of the initial-value problem (5.19). Then:

(i) $\omega = \infty$;

(ii) $\lim_{t\to\infty} k(t)$ *exists and is finite;*

(iii) $y(t) \to 0$ *as* $t \to \infty$.

Proof. First note that $\{yu|\ u \in \sigma(y)\} = \{|y|\}$ for all $y \in \mathbb{R}$. Next define

$$\tilde{\nu}(k) := \begin{cases} \gamma_2\nu(k), & \text{if } \nu(k) \geq 0,\ \beta = 1 \\ \gamma_1\nu(k), & \text{if } \nu(k) < 0,\ \beta = 1 \\ -\gamma_1\nu(k), & \text{if } \nu(k) \geq 0,\ \beta = -1 \\ -\gamma_2\nu(k), & \text{if } \nu(k) < 0,\ \beta = -1 \end{cases}$$

so that $\tilde{\nu}$ is a Nussbaum function for both $\beta = \pm 1$. Moreover, if $\gamma_1 < w < \gamma_2$ then $\beta w \nu(k) \leq \tilde{\nu}(k)$.

Then for almost all $t \in [0,\omega)$, we have

$$\begin{aligned} y(t)\dot{y}(t) &\leq y(t)(\widehat{T}y)(t) + [\mu + \tilde{\nu}(k(t))]\Phi(|y(t)|)|y(t)| \\ &= y(t)(\widehat{T}y)(t) + [\mu + \tilde{\nu}(k(t))]\dot{k}(t), \end{aligned}$$

which, on integration and invoking properties of $C_1(\psi)$-class operators, yields

$$\begin{aligned} 0 \leq y^2(t) &\leq y^2(\tau) + c^* + 2\int_\tau^t [\mu^* + \tilde{\nu}(k(s))]\,\dot{k}(s)\,ds \\ &= c^* + 2\mu^*[k(t) - k(\tau)] + 2\int_{k(\tau)}^{k(t)} \tilde{\nu}(s)ds \quad \forall\ \tau, t \in [0,\omega),\ \tau \leq t \end{aligned} \tag{5.20}$$

for some constants c^* and μ^*. Seeking a contradiction, suppose k is unbounded and so, by monotonicity, $k(t) \to \infty$ as $t \uparrow \omega$. Fix $\tau \in [0, \omega)$ such that $k(\tau) \geq 1$. Dividing by $k(t) \geq k(\tau) \geq 1$ in (5.20) gives

$$0 \leq y^2(\tau) + c^* + 2\mu^* + \frac{2}{k(t)} \int_{k(\tau)}^{k(t)} \tilde{\nu}(s)ds \quad \forall\, t \in [\tau, \omega). \tag{5.21}$$

Recalling that $\tilde{\nu}$ is a Nussbaum function we arrive at a contradiction to property (b) of (1.6). Therefore, k is bounded. Boundedness of y follows by (5.21). By boundedness of $x = (y, k)$, we conclude that $\omega = \infty$, which is assertion (i) of the theorem. By boundedness and monotonicity of k, assertion (ii) holds. Finally, by boundedness of k,

$$\infty > \int_0^\infty \dot{k}(t)dt \geq \int_0^\infty \Phi(|y(t)|)|y(t)|dt$$

and so, by Lemma 40, we conclude that $y(t) \to 0$ as $t \to \infty$. This completes the proof.

5.4 Tracking control of single-input single-output systems

The problem addressed in this section is of feedback control of (5.1) to ensure tracking, by the output, of an arbitrary reference signal $r \in \mathcal{R}$ to within a pre-specified error $\lambda \geq 0$. That is to say that the error $e(t) := y(t) - r(t) \to [-\lambda, \lambda]$ as $t \to \infty$.

For $\lambda \geq 0$ we define the distance function to the set $[-\lambda, \lambda]$ as

$$d_\lambda(x) := \max\{0, |x| - \lambda\}.$$

Let $s_\lambda(\cdot) : \mathbb{R} \to \mathbb{R}$ be defined as follows; for $\lambda > 0$ let $s_\lambda(\cdot)$ take the value $\operatorname{sgn}(x)$ for $|x| > \lambda$ and $s_\lambda(x) \in [-1, 1]$ for $|x| \leq \lambda$; for $\lambda = 0$ let $s_\lambda(x) := \operatorname{sgn}(x)$ if $x \neq 0$ and $s_0(0) \in [-1, 1]$.

5.5 Systems of class $\mathcal{N}_3$

Let $\psi \in \mathcal{I}$ (for example, $\forall n \in \mathbb{N}$, the function $y \mapsto 1 + |y|^n$ if of class $\mathcal{I}$) and ϕ satisfy

$$\left.\begin{array}{l} \phi \in C(\mathbb{R}; \mathbb{R}_+), \quad \text{non-decreasing} \\ \forall\, R > 0\ \exists\ \mu_R^* > 0 \text{ such that} \\ \phi(|e + r|) \leq \mu_R^* \phi(|e|) \text{ for all } (e, r) \in \mathbb{R} \times [-R, R]. \end{array}\right\} \tag{5.22}$$

Notice this forces $\phi(0) > 0$ and as concrete examples both $\phi_1 : y \mapsto \exp(|y|)$ and $\phi_2 : y \mapsto 1 + |y|^n$ for some $n \in \mathbb{N}$ satisfy (5.22).

In this section we consider the class $\mathcal{N}_3 = \mathcal{N}_3(\phi, \psi)$ which is defined to be the family of all those systems $(b, g, p, \widehat{T})$ of the form (5.1) that satisfy Assumption 58 below.

Assumption 58

1. *The system* $(b, g, p, \widehat{T}) \in \mathcal{N}_1(\phi, \psi)$ *(recall Assumption 52, pg 73.)*
2. *The operator* $\widehat{T} = \sum_{i=1}^{m} \widehat{T}_i$ *where, for all* $1 \le i \le m$,
   (a) $\widehat{T}_i \in S$ *(recall Definition 17, pg 30),*
   (b) $\widehat{T}_i \in C_1(\psi)$,
   (c) *each* $\widehat{T}_i$ *satisfies (2.6).*

5.5.1 $\mathcal{N}_3$-universal stabilizer

Define the map

$$\Phi : e \mapsto \psi(d_\lambda(e)) + \phi(|e|)$$

so that $\Phi(e) > 0$ for all $e \in \mathbb{R}$. Consider the control strategy

$$\left.\begin{array}{rcl} u(t) & = & \nu(k(t))\Phi(|e(t)|)\, s_\lambda(e(t)) \\ \dot{k}(t) & = & \Phi(|e(t)|)d_\lambda(e(t)) \text{ and } k(0) = k^0 \end{array}\right\} \tag{5.23}$$

It will be the objective of this section to prove that the closed-loop system (5.1) and (5.23) achieves the following: for every reference signal $r \in \mathcal{R}$ and every system in the class $\mathcal{N}_3$ the solution of the closed-loop system exists for all $t > 0$, the adapting parameter k remains bounded and so, by monotonicity, converges, and the error $e(t)$ between the the system output at time t and the reference signal approaches the set $[-\lambda, \lambda]$ as $t \to \infty$. The phrase λ-*tracking* has been used, [1, 7, 32, 37, 34, 33, 61], to describe such a control objective for the case $\lambda > 0$.

The control will be discontinuous for $\lambda = 0$ (and possibly for $\lambda > 0$) and so the analysis of the closed-loop control system will take place in the context of a differential inclusion.

This tracking result for the class $\mathcal{N}_3$ differs from and, in some aspects, extends the work of [7, 34]. Both [7] and [34] consider the multi-input multi-output case with the "eigenvalues of CB" in the left complex half plane and assume $\lambda > 0$. We consider the single-input single-output case with $b \neq 0$ and take $\lambda \geq 0$. In [7] subsystem Σ_1 from Figure 1-2 is modelled by a nonlinear functional operator and Σ_2 is a finite-dimensional linear system. In the present context Σ_1 is a nonlinear function and Σ_2 is taken to be an operator. Further, and in contrast to the results here, in [7] the interconnections between the two subsystems are assumed to satisfy some bound [7, Equation (22)]. We extend the results of [34] by modelling the dynamics of Σ_1 by a nonlinear function, and Σ_2 by a nonlinear operator. Furthermore we provide a unified analysis for the cases $\lambda > 0$ and $\lambda = 0$.

The argument below is set out in the following steps; first we recast the system (5.1) in terms of the error $e(\cdot)$; then applying the feedback control strategy defined in (5.23) we embed the closed-loop system into a differential inclusion (see (5.27) below); estimates on the derivative will allow us to conclude that k is bounded, then an application of Lemma 40 proves that solutions of the inclusion have the requisite properties.

Recasting in terms of the system error

Define the continuous function $f : \mathbb{R}^{P+2} \times \mathbb{R} \to \mathbb{R}$ by

$$f((p,r,s),e) := g(p, e+r) - s \text{ for all } p \in \mathbb{R}^P \text{ and } e,r,s \in \mathbb{R}.$$

Define $\tilde{p}$ as $t \mapsto (p(t), r(t), \dot{r}(t))$. By the properties of $p(\cdot)$ and $r(\cdot)$, $\tilde{p}(\cdot)$ takes values in some compact $\tilde{K} \subset \mathbb{R}^{P+2}$. Invoking the estimates (5.6) for g and (5.22) for ϕ there exists some constant $c_1 > 0$ such that

$$|f(\tilde{p}(t), e)| \leq c_1 \phi(|e|) \text{ for all } t \in \mathbb{R} \text{ and } e \in \mathbb{R}.$$

Let $\widehat{T}_r$ be defined as $(\widehat{T}_r x)(\cdot) := (\widehat{T}(x+r))(\cdot)$ for all $x \in C([-h,0];\mathbb{R})$. Expressed in terms of the error the dynamics have the form

$$\left.\begin{aligned} \dot{e}(t) - (\widehat{T}_r e)(t) &= f(\tilde{p}(t), e(t)) + bu(t) \\ e|_{[-h,0]} &= (y^0 - r)(\cdot) \in C([-h,0];\mathbb{R}) \end{aligned}\right\} \tag{5.24}$$

for $h \geq 0$.

Embedding the closed-loop system into a differential inclusion

Embed s_λ into the set-valued map $e \mapsto \sigma_\lambda(e)$ defined by

$$\sigma_\lambda(e) := \begin{cases} \{\mathrm{sgn}(e)\}, & |e| > \lambda \\ [-1,1], & |e| \leq \lambda \end{cases} \tag{5.25}$$

(so that s_λ is a selection from σ_λ). Define a set-valued map F_3 by

$$\begin{aligned} F_3(x) &= F_3(e,k) \\ &:= \{v + bu \mid |v| \leq c_1\phi(|e|),\ u \in \nu(k)\Phi(e)\sigma_\lambda(e)\} \times \{\Phi(e) d_\lambda(e)\}. \end{aligned}$$

Note that F_3 is upper semicontinuous with non-empty compact convex values in $\mathbb{R}^2$. Extend $k(s)$ for $s < 0$ by defining $k(s) = k^0$ for all $s < 0$ and let $x(s) := (e(s), k(s))$. Define $T : C(\mathbb{R}; \mathbb{R}^2) \to L^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R}^2)$ by

$$T(e,k)(t) := ((\widehat{T}(e+r))(t), 0), \quad \forall t \in \mathbb{R}, \tag{5.26}$$

and consider the following differential inclusion

$$\left.\begin{aligned} \dot{x}(t) - (Tx)(t) \in F_3(x(t)), \text{ for a.a. } t > 0 \\ x|_{[-h,0]} = x^0 := ((y^0 - r)(\cdot), k^0) \in C\left([-h,0]; \mathbb{R}^2\right). \end{aligned}\right\} \tag{5.27}$$

Since F_3 is upper semicontinuous with non-empty convex compact values and, by Claim 8 and Remark 9, T is of class $\mathcal{T}_h^2$ it follows, by Theorem 32, that (i) there exists a solution to the initial-value problem (5.27) on some interval $[-h, \omega)$ with $\omega > 0$, (ii) every solution can be maximally extended, (iii) if a maximal solution $x : [-h, \omega) \to \mathbb{R}^2$ is bounded, then $\omega = \infty$.

Theorem 59

Let $x = (e, k) : [-h, \omega) \to \mathbb{R}^2$ be a maximal solution to (5.27). Then:

(i) $\omega = \infty$;

(ii) $\lim_{t\to\infty} k(t)$ *exists and is finite; and*

(iii) $d_\lambda(e(t)) \to 0$ *as* $t \to \infty$.

Proof. Composing the locally Lipschitz d_λ with the absolutely continuous e gives the absolutely continuous function $d_\lambda(e(\cdot))$. Thus $d_\lambda(e(\cdot))$ is differentiable almost everywhere and if $t \in (0, \omega)$, $|e(t)| \neq \lambda$ and $\dot{e}(t)$ exists then,

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2} d_\lambda(e(t))^2\right) = d_\lambda(e(t)) \operatorname{sgn}(e(t)) \dot{e}(t).$$

For all $t \in [-h, \omega)$ define $\zeta(t) := d_\lambda(e(t)) \operatorname{sgn}(e(t))$ and note that $t \mapsto e(t) - \zeta(t)$ is bounded. By Property 2 of Assumption 58 the operator $\widehat{T} = \sum_{i=1}^m \widehat{T}_i$ where $\widehat{T}_i \in \mathcal{S}$. Hence there exist constants δ_i and μ_i such that each $\widehat{T}_i$ satisfies (2.11). Further, since each $\widehat{T}_i$ is bounded-input bounded-output stable there exists some $c_2 > 0$ such that

$$\sum_{i=1}^m |(\widehat{T}_i(\delta_i(e - \zeta + r)))(t)| \leq c_2 \phi(0) \leq c_2 \phi(|e(t)|)$$

for almost all $t > 0$ (since ϕ is increasing). Thus

$$\sum_{i=1}^{m} |(\widehat{T}_i(\zeta + e - \zeta + r))(s)| + c_1\phi(|e(s)|)$$
$$\leq \sum_{i=1}^{m} \mu_i \left[|(\widehat{T}_i\delta_i\zeta)(s)| + |\widehat{T}_i(\delta_i(e - \zeta + r))(s)|\right] + c_1\phi(|e(s)|)$$
$$\leq \mu \sum_{i=1}^{m} |(\widehat{T}_i\delta_i\zeta)(s)| + (c_1 + \mu c_2))\phi(|e(s)|)$$

where $\mu := \max\{\mu_i | i = 1, \cdots, m\}$. Further, for $1 \leq i \leq m$

$$\begin{aligned}\int_0^t \frac{1}{\delta_i} \left|\widehat{T}_i(\delta_i\zeta)(s)\right| \delta_i\zeta(s)\; ds &\leq \frac{\hat{c}_i}{\delta_i} + \hat{\mu}\int_0^t \psi(\delta_i d_\lambda(s))d_\lambda(s)\; ds \\ &\leq \frac{\hat{c}_i}{\delta_i} + \Delta_i\hat{\mu}\int_0^t \psi(d_\lambda(s))d_\lambda(s)\; ds. \quad (5.28)\end{aligned}$$

Noting that for all $u \in \nu(k(s))\Phi(e(s))\sigma_\lambda(e(s))$

$$b\zeta(s)u = b\nu(k(s))\dot{k}(s),$$

assume that $0 < t < \omega$ and integrate $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}d_\lambda(e(t))^2\right)$ from 0 to t so that

$$\begin{aligned} d_\lambda(e(t))^2 &= d_\lambda(e(0))^2 + 2\int_0^t \zeta(s)\dot{e}(s)\; ds \\ &\leq d_\lambda(e(0))^2 + 2\int_0^t \mu\sum_{i=1}^{m}\frac{1}{\delta_i}|(\widehat{T}_i\delta_i\zeta)(s)|\delta_i\zeta(s)\; ds \\ &\quad +2\int_0^t (c_1 + \mu c_2)\phi(|e(s)|)\zeta(s) + b\nu(k(s))\dot{k}(s)\; ds \\ &\leq d_\lambda(e(0))^2 + c + c_3\int_0^t \Phi(e(s))d_\lambda(e(s))\; ds + 2b\int_{k^0}^{k(t)} \nu(s)\; ds \\ &= d_\lambda(e(0))^2 + c + c_3(k(t) - k^0) + 2b\int_{k^0}^{k(t)} \nu(s)\; ds \end{aligned}$$

where $c_3 := 2\max\{(c_1 + \mu c_2), m\mu\hat{\mu}\Delta_i | 1 \leq i \leq m\}$ and $c := 2m\mu\max\{\hat{c}_i/\delta_i | 1 \leq i \leq m\}$. Seeking a contradiction, suppose k is unbounded and so, by monotonicity, $k(t) \to \infty$ as $t \uparrow \omega$. For t sufficiently large divide by $k(t) > 0$ and we have

$$0 \leq \text{const} + \frac{2b}{k(t)}\int_{k^0}^{k(t)} \nu(s)\; ds. \quad (5.29)$$

Recalling that $b \neq 0$ and taking limits as $t \uparrow \omega$ in (5.29) we contradict one or the other of properties (1.6) of ν. Therefore, k is bounded.

Hence $d_\lambda(e(\cdot))$ is bounded which implies that e is bounded. Thus x is bounded and so $\omega = \infty$. Define $l : \mathbb{R}^2 \to \mathbb{R}_+$ by $l(\cdot,\cdot) : (e,k) \mapsto \Phi(e)d_\lambda(e)$. By the boundedness of k

$$\infty > \int_0^\infty \dot{k}(s)\,ds = \int_0^\infty l(e(s),k(s))\,ds.$$

Apply Lemma 40 to conclude that $d_\lambda(e(t)) \to 0$ as $t \to \infty$.

Remark 60 (Noise corrupted output)
If the output is corrupted by an unknown noise term $\eta \in \mathcal{R}$ so that the term $\hat{y}(t) := y(t) + \eta(t)$ is available to the controller, then defining $\hat{r}(t) := r(t) - \eta(t)$ we see that $\lim_{t\to\infty} d_\lambda(y(t) - \hat{r}(t)) = 0$. ie the corrupted output measurement $\hat{y}(t)$ tends, as $t \to \infty$, towards the λ-neighbourhood of the reference signal r.

Remark 61 (Constant b)
As in Remark 54 let $\mathcal{N}_3^* \subset \mathcal{N}_3$ in which knowledge of $\operatorname{sgn}(b)$ *is available to the controller. Then we may replace the control (5.23) with*

$$\left.\begin{array}{rcl} u(t) &=& -\operatorname{sgn}(b)k(t)\Phi(|e(t)|)\,s_\lambda(e(t)) \\ \dot{k}(t) &=& \Phi(|e(t)|)d_\lambda(e(t)) \text{ and } k(0) = k^0 \end{array}\right\} \tag{5.30}$$

The analysis proceeds as before.

5.5.2 Numerical example

Again, by way of illustration consider Example 1 of Section 5.1 with $b = \frac{1}{2}$, $g : (p,y) \mapsto y^{\frac{1}{3}} + y^3 + p$, $p : t \mapsto \sin(7t)$, $h_1 = 1$, $h_2 = h_3 = 2$, $q_1(t) = q_2(t) = 1$ for all t, $q_3 : t \mapsto \sin(-\frac{1}{2}\pi t)$, and with initial conditions $y(t) = t + 1$ for $-1 \leq t \leq 0$, $y(t) = 0$, $t < -1$ and $k(0) = 0$.

We define the functions

$$\phi = \psi : r \mapsto 1 + r^3$$

and assuming that $\operatorname{sgn}(b)$ is known, we illustrate Remark 61 and implement a control of the form (5.30).

Taking $r : t \mapsto -1 + \frac{1}{2}\sin(t) + \frac{1}{4}\sin(2t)$, Figures 5-5 and 5-6 depict the system behaviour (computed using an improved Euler method within MATLAB) under the adaptive control with, respectively, $\lambda = 1/5$ and $\lambda = 1/50$.

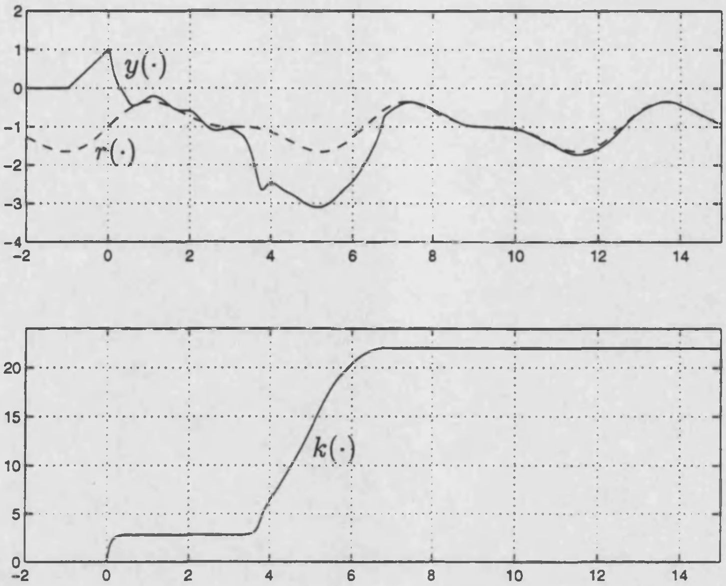


Figure 5-5: Typical behaviour of Example 1 under adaptive tracking control.

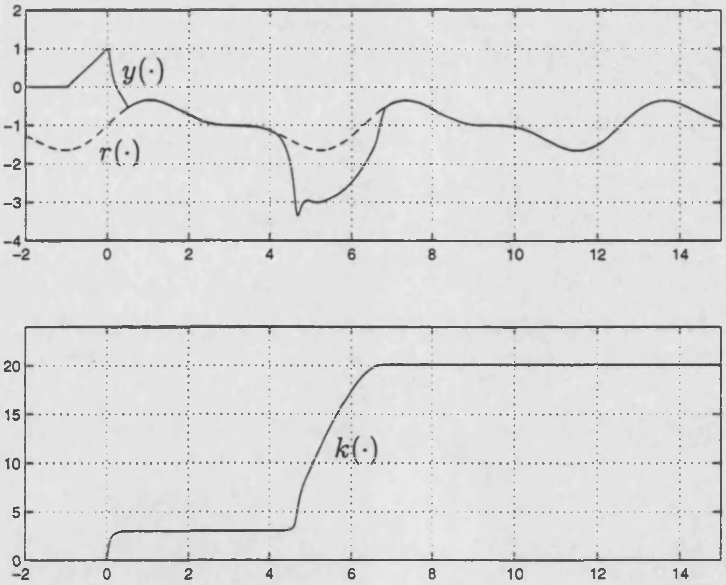


Figure 5-6: Typical behaviour of Example 1 under adaptive tracking control.

5.6 Static feedback

If more information about the system is available for control purposes it is possible to control of such systems via non-adaptive feedback strategies. For example, consider the class $\mathcal{N}_1$ of Section 5.2 with the control objective of stabilization, as before.

Let $(b, g, p, \widehat{T}) \in \mathcal{N}_1^*(\phi, \psi)$ so that in addition to knowledge of ϕ and ψ, we have available $\operatorname{sgn}(b)$.

We implement a static feedback control of the form

$$u(t) = -k^* \operatorname{sgn}(b)\Phi(|y(t)|) \operatorname{sgn}(y(t))$$

where Φ is defined by (5.5) and $k^* \in \mathbb{R}_+$.

By properties of g and essential boundedness of p, there exists $\mu \in \mathbb{R}_+$ such that, for all $y \in \mathbb{R}$ and almost all t, $|g(p(t), y)| \le \mu\phi(|y|)$. As before, we adopt a framework of differential inclusions: define $x \mapsto F_1^*(x) \subset \mathbb{R}^2$ by

$$F_1^*(y) := \{v - |b|k^*u \;|v| \le \mu\phi(|y|), u \in \Phi(|y|)\sigma(y)\}\,.$$

$F_1^*(\cdot)$ is upper semicontinuous with non-empty, compact and convex values and so $F_1^* \in \mathcal{F}^1$. Embed the feedback-controlled system in a functional differential inclusion:

$$\dot{y}(t) - (\widehat{T}y)(t) \in F_1^*(y(t)), \quad y^0 \in C\left([-h, 0]; \mathbb{R}^1\right). \tag{5.31}$$

Since $F_1^* \in \mathcal{F}^1$ and $\widehat{T} \in \mathcal{T}_h^2$ it follows, by Theorem 32, that, for all y^0, (5.31) has a solution and every solution can be maximally extended.

Lemma 62 *Let $y : [-h, \omega) \to \mathbb{R}$ be a maximal solution of the initial-value problem (5.31). Then for some $k^* \in \mathbb{R}_+$ large enough, $\omega = \infty$ and $y(t) \to 0$ as $t \to \infty$.*

Proof. For almost all $t \in [0, \omega)$, we have

$$y(t)\dot{y}(t) \le y(t)(\widehat{T}y)(t) + [\mu - |b|k^*]\Phi(|y(t)|)|y(t)|$$

which, on integration and invoking properties of $C_1(\psi)$-class operators, yields, for all $t \in [0, \omega)$

$$0 \le y^2(t) \le y^2(0) + c^* + 2\int_0^t [\mu^* - |b|k^*]\,\Phi(|y(s)|)|y(s)|\;ds$$

for some constants c^* and μ^*. Then for $k^* > \mu^*/|b|$, we have $y^2(t) \le y^2(0) + c^*$ for all $t \in [0, \omega)$. Hence $\omega = \infty$.

Furthermore,

$$y^2(0) + c^* \geq 2 \int_0^t [|b|k^* - \mu^*]\, \Phi(|y(s)|)|y(s)|\; ds$$

and so by Lemma 40, we conclude that $y(t) \to 0$ as $t \to \infty$. This completes the proof.

Remarks 63

(a) Of course, to implement such a control one would need a lower bound for k^*. This could be calculated from knowledge of the constant μ from (5.6) and the constants in the estimate (2.8) for $\widehat{T}$.

(b) One could also prove that analogous non-adaptive controllers achieve tracking.

Chapter 6

Single-input single-output second order systems

6.1 The generic second order system

In this chapter we consider the problem of adaptive feedback control of a class of second order single-input $u(\cdot)$, single-output $y(\cdot)$ systems $(b, f, d, p, \widehat{T})$ given by a controlled nonlinear functional differential equation of the form

$$\ddot{y}(t) = d(t)\dot{y}(t) + (\widehat{T}y)(t) + f(p(t), y(t)) + bu(t), \text{ for a.a. } t > 0 \tag{6.1}$$

where $f(\cdot,\cdot)$ is a nonlinear function, $d(\cdot)$ and $p(\cdot)$ are bounded disturbances, $\widehat{T}$ is a causal operator and $u(\cdot)$ is the control term. The output $y(t)$ but not its derivative $\dot{y}(t)$ is available for control purposes. Study of such systems extends the work of [69] by the addition of causal operators $\widehat{T}$ in (6.1).

Many physically motivated examples, particularly mechanical or electrical systems, may be recast in the form (6.1).

Initially in this chapter we employ an adaptive control that achieves the following control objective: for every system of the class, solutions to the closed-loop initial-value problem exist on $\mathbb{R}_+$, the adapting parameter remains bounded and $y(t) \to 0$ as $t \to \infty$. Under stronger hypotheses on $\widehat{T}$ it also follows that $\dot{y}(t) \to 0$.

Next we consider a problem of tracking control to ensure, for an arbitrary reference signal of a given class, that for every system of the class solutions of the corresponding closed-loop initial-value problem exists on $\mathbb{R}_+$, the adapting parameter remains bounded and $e(t) := y(t) - r(t) \to 0$ as $t \to \infty$.

In each case the control incorporates a discontinuous feedback control action and to overcome this lack of smoothness we again consider an analytical framework of functional differential inclusions in which the non-smooth closed-loop system comprising (6.1) and the control takes the form

$$\dot{x}(t) \in F(t, (Tx)(t)), \quad \text{for a.a. } t \geq 0,$$

with prescribed initial data of the form $x|_{[-h,0]} = x^0 \in C\left([-h, 0]; \mathbb{R}^N\right)$.
That $\dot{y}(t) \to 0$, as $t \to \infty$ follows from an integral invariance principle which invokes properties of the ω-limit sets of functional differential inclusions discussed in Chapter 4. This approach uses functional counterparts of the concepts and results of [70].

6.2 Systems of class $\mathcal{N}_4$

Consider the single-input, single-output, second order control system (6.1) where the parameters $b \in \mathbb{R}$, $P \in \mathbb{N}$, the functions d, f, p and the operator $\widehat{T}$ are unknown. The instantaneous value of the state $y(t)$, but *not* its derivative $\dot{y}(t)$, is available for feedback. We identify the system (6.1) with $(b, d, f, p, \widehat{T})$.
For $\delta > 0$ and $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous non-decreasing functions, available for control purposes, satisfying (5.5), we denote by $\mathcal{N}_4 = \mathcal{N}_4(\phi, \psi, \delta)$ the class of all systems (6.1) satisfying the following.

Assumption 64

1. *$b \neq 0$ is constant,*

2. *$d \in C(\mathbb{R}; \mathbb{R})$ with $D \leq d(t) \leq -(\delta + 3\epsilon)$ for some $\epsilon > 0$, $D < -(\delta + 3\epsilon)$ and all $t \in \mathbb{R}$,*

3. *$f(\cdot, \cdot) : \mathbb{R}^P \times \mathbb{R} \to \mathbb{R}$ is continuous and continuously differentiable in its first argument. Both f and $\partial f / \partial p$ are bounded in the following way: for each compact set $K \subset \mathbb{R}^P$ there is a constant μ_K such that*

$$|f(\rho, y)| + \|\frac{\partial}{\partial \rho} f(\rho, y)\| \leq \mu_K \phi(|y|) \text{ for all } (\rho, y) \in K \times \mathbb{R}, \tag{6.2}$$

4. *$p \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^P)$,*

5. *the operator $\widehat{T} \in \mathcal{T}_h^1 \cap C_2(\psi)$ (recall Definition 16, pg 29).*

It will be proved below that for any $(b, d, f, p, \widehat{T}) \in \mathcal{N}_4$ knowledge of δ, ϕ and ψ are sufficient for the construction of an adaptive control that stabilizes (6.1) in the sense that $y(t) \to 0$ as $t \to \infty$ whilst assuring convergence of all internal states. In addition, if $\widehat{T} \in \mathcal{T}_h^1$ is shift invariant (recall Definition 14, pg 28) then $\dot{y}(t) \to 0$ as $t \to \infty$.

6.2.1 $\mathcal{N}_4$-universal stabilizer

Define $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\gamma(\xi) := \xi + \phi(\xi) + \psi(\xi)$$

and let Γ denote its indefinite integral

$$\Gamma(\xi) := \int_0^\xi \gamma(s)\, ds.$$

It will be proved below that the following control strategy is a $\mathcal{N}_4$-universal stabilizer;

$$\left.\begin{array}{rcl} u(t) & = & \nu(\eta(t))\gamma(|y(t)|)\,\mathrm{sgn}(y(t)) \\ \eta(t) & = & \delta k(t) + \Gamma(|y(t)|) \\ \dot{k}(t) & = & \gamma(|y(t)|)|y(t)| \text{ with } k(0) = k^0 \in \mathbb{R} \end{array}\right\} \tag{6.3}$$

where $\nu : \mathbb{R} \to \mathbb{R}$ is a continuous function with the properties (1.6). The controller is discontinuous and is interpreted in the set-valued sense

$$\left.\begin{array}{rcl} u(t) & \in & \nu(\eta(t))\gamma(|y(t)|)\sigma(y(t)) \\ \eta(t) & = & \delta k(t) + \Gamma(|y(t)|) \\ \dot{k}(t) & = & \gamma(|y(t)|)|y(t)| \text{ with } k(0) = k^0 \in \mathbb{R} \end{array}\right\} \tag{6.4}$$

where the map σ is defined by (5.9).

A coordinate change

By Assumption 64 (3-4) there exists a constant μ such that

$$|f(p(t), y)| + \|\frac{\partial}{\partial p} f(p(t), y)\| \le \mu\phi(|y|) \text{ for almost all } t \in \mathbb{R}, \text{ and all } y \in \mathbb{R}.$$

Using the coordinate transformation

$$z(t) = \dot{y}(t) + \delta y(t),$$

extending k by setting $k(t) = k^0$ for all $t \in I := [-h, 0]$ and by writing $x(\cdot) = (y(\cdot), z(\cdot), k(\cdot))$, we may express the open-loop control system (6.1) as

$$\left.\begin{array}{rcl} \dot{y}(t) &=& -\delta y(t) + z(t) \\ \dot{z}(t) &=& (\delta + d(t))(z(t) - \delta y(t)) + (\widehat{T}y)(t) + f(p(t), y(t)) + bu(t) \\ \dot{k}(t) &=& \gamma(|y(t)|)|y(t)| \end{array}\right\} \tag{6.5}$$

for almost all $t > 0$ with $x|_I = (y^0, z^0, k^0 \in C\left(I; \mathbb{R}^3\right)$.

Embed into a differential inclusion

The closed-loop system, comprising (6.5) with controller (6.4), is embedded into the (non-autonomous) differential inclusion

$$\dot{x}(t) \in F_4(t, (Tx)(t)) \tag{6.6}$$

where $(Tx)(t) := (y(t), z(t), k(t), (\widehat{T}y)(t))$ and the set-valued map F_4 is given by

$$F_4(t, (y, z, k, w)) = F_a(x) \times F_b(t, (y, z, k, w)) \times F_c(x)$$

where

$$\begin{array}{rcl} F_a(x) &:=& \{-\delta y + z\}, \\ F_b(t, (y, z, k, w)) &:=& \{w + (\delta + d(t))(z - \delta y) + f(p(t), y) \\ && +bu|\ u \in \nu(\delta k + \Gamma(|y|))\gamma(|y|)\sigma(y)\}, \\ F_c(x) &:=& \{\gamma(|y|)|y|\}. \end{array}$$

F_4 is upper semicontinuous with non-empty, convex, compact values in $\mathbb{R}^3$. Therefore, by Theorem 32, for each $x \in C\left([-h, 0]; \mathbb{R}^3\right)$ the functional differential inclusion (6.6) has a solution and every solution can be extended to a maximal solution.

The following argument is a modification of [70, Lemma 3.4].

Theorem 65

Let $x : [-h, \omega) \to \mathbb{R}^3$ be a maximal solution of (6.6). Then:

(i) $\omega = \infty$;

(ii) $\lim_{t\to\infty} k(t)$ *exists and is finite;*

(iii) $y(t) \to 0$ *as* $t \to \infty$; *and*

(iv) if $\widehat{T}$ is right shift invariant (recall Definition 14, pg 28) then $\dot{y}(t) \to 0$ as $t \to \infty$.

Proof. Let $x = (y, z, k)$ be a maximal solution to (6.6) on the interval $[-h, \omega)$. On $\mathbb{R}$ define the locally Lipschitz function $\Phi : r \mapsto \Gamma(|r|)$, with directional derivative at r in the direction s given by

$$\Phi^{\dagger}(r; s) := \lim_{h \downarrow 0} \frac{\Phi(r + hs) - \Phi(r)}{h} = \begin{cases} \gamma(|r|)\,\mathrm{sgn}(r)s, & r \neq 0 \\ \gamma(0)|s|, & r = 0 \end{cases}$$

Let $G_1 \in AC([0, \omega); \mathbb{R})$ denote the composition $\Phi \circ y$. Then the derivative of G_1 exists almost everywhere and $\dot{G}_1(t) = \Phi^{\dagger}(y(t); \dot{y}(t))$. Moreover

$$\dot{\eta}(t) = \delta\gamma(|y(t)|)|y(t)| + \Phi^{\dagger}(y(t); \dot{y}(t)). \tag{6.7}$$

By the properties of f and p, the function

$$G_2 : t \mapsto \int_0^{y(t)} f(p(t), \xi)\, d\xi$$

is of class $AC([0, \omega); \mathbb{R})$ and hence is differentiable almost everywhere with

$$\dot{G}_2(t) = \int_0^{y(t)} < \frac{\partial}{\partial p} f(p(t), \xi), \dot{p}(t) > \; d\xi + f(p(t), y(t))\dot{y}(t).$$

Recall that $\dot{y}(t) = z(t) - \delta y(t)$ and the property (6.2) of f implies the existence of some constant $\bar{\mu} > 0$ so that

$$\max\left\{ |f(p(t), \xi)|, \|\frac{\partial}{\partial p} f(p(t), \xi)\| \|\dot{p}(t)\| \right\} \leq \bar{\mu}\phi(|\xi|) \text{ for all } (t, \xi) \in \mathbb{R}^2.$$

It follows that

$$\dot{G}_2(t) \geq -\mu\phi(|y(t)|)|y(t)| + z(t) f(p(t), y(t))$$

for almost all $t \in [0, \omega)$ where $\mu := \bar{\mu}(1 + \delta)$.

Let $\mathcal{O}_0$ be the set (of measure zero) of points $t \in \mathbb{R}_+$ at which at least one of the derivatives of G_1, G_2 or y fail to exist.

Claim: For almost all $t \in [0, \omega)$

$$uz(t) = \dot{\eta}(t) \quad \forall u \in \gamma(|y(t)|)\sigma(y(t)). \tag{6.8}$$

Let $t \in Y := [0, \omega) \backslash \mathcal{O}_0$ then there are three exhaustive cases;

(a) $y(t) \neq 0$,
(b) $(y(t), \dot{y}(t)) = (0, 0)$,
(c) $y(t) = 0$ and $\dot{y}(t) \neq 0$.

We first consider case (a). If $y(t) \neq 0$, then $\gamma(|y(t)|)\sigma(y(t)) = \{\gamma(|y(t)|)\operatorname{sgn}(y(t))\}$. Further by (6.7)

$$\dot{\eta}(t) = \delta\gamma(|y(t)|)|y(t)| + \gamma(|y(t)|)\operatorname{sgn}(y(t))\dot{y}(t) = z(t)\gamma(|y(t)|)\operatorname{sgn}(y(t)) = uz(t).$$

Next, (b), if $(y(t), \dot{y}(t)) = (0, 0)$ then $z(t) = 0$. Further, since $\Phi^{\dagger}(0; 0) = 0$ and $y(t) = 0$ (6.7) shows that $\dot{\eta}(t) = 0$. Thus (6.8) holds.

Finally, (c), since $\dot{y}(\cdot)$ is continuous, every $t \in Y$ at which $y(t) = 0$ and $\dot{y}(\cdot) \neq 0$ is isolated. Thus the set $\mathcal{O}_1 := \{t \in Y | y(t) = 0, \dot{y}(t) \neq 0\}$ is of measure zero.

Define $\mathcal{O} := \mathcal{O}_0 \cup \mathcal{O}_1$. Then for all $t \in [0, \omega)\backslash\mathcal{O}$, (6.8) holds and moreover

$$uz(t) = \nu(\eta(t))\dot{\eta}(t) \quad \forall u \in \nu(\eta(t))\gamma(|y(t)|)\sigma(y(t)). \tag{6.9}$$

Define $G_c \in AC([0, \omega); \mathbb{R})$, parameterized by $c > 0$, as $G_c : t \mapsto cG_1(t) - G_2(t)$. Then for all c sufficiently large,

$$G_c(t) \geq \frac{cy^2(t)}{2} \tag{6.10}$$

and G_c is differentiable on $[0, \omega)\backslash\mathcal{O}$ with (recalling (6.7))

$$\begin{aligned}\dot{G}_c(t) = c\dot{G}_1(t) - \dot{G}_2(t) &= c\,(\dot{\eta}(t) - \delta\gamma(|y(t)|)|y(t)|) - \dot{G}_2(t)\\ &\leq c\dot{\eta}(t) + (\mu - c\delta)\gamma(|y(t)|)|y(t)| - z(t)f(p(t), y(t)).\end{aligned} \tag{6.11}$$

Define the map

$$V_c(t, y, z) = G_c(t, y) + \frac{1}{2}z^2.$$

Anticipating our result, the function V_c is a Lyapunov-like candidate combining a quadratic term in z and the term $G_c(t, y)$ that we have shown in the estimate (6.10) to dominate a quadratic term in y for sufficiently large constants $c > 0$. We now proceed to derive an estimate on the rate of change of the absolutely continuous - and hence differentiable almost everywhere - map $t \mapsto V_c(t, y(t), z(t))$. Thus for almost all $t \in \mathbb{R}_+$

$$\begin{aligned}
\dot{V}_c(t, y(t), z(t)) &= \dot{G}_c(t, y(t)) + z(t)\dot{z}(t) \\
&= \dot{G}_c(t, y(t)) + (\delta + d(t))z^2(t) - \delta(\delta + d(t))z(t)y(t) \\
&\quad + z(t)(\widehat{T}y)(t) + z(t)f(p(t), y(t)) + b\nu(\eta(t))\dot{\eta}(t) \\
&\leq \dot{G}_c(t, y(t)) - 2\epsilon z^2(t) - \delta(\delta + d(t))z(t)y(t) + \frac{1}{4\epsilon}(\widehat{T}y)^2(t) \\
&\quad + z(t)f(p(t), y(t)) + b\nu(\eta(t))\dot{\eta}(t) \qquad (6.12) \\
&\leq (\mu - c\delta + \frac{\Delta^2}{4\epsilon})\gamma(|y(t)|)|y(t)| \\
&\quad + \frac{1}{4\epsilon}(\widehat{T}y)^2(t) - \epsilon z^2(t) + c\dot{\eta}(t) + b\nu(\eta(t))\dot{\eta}(t).
\end{aligned}$$

The first inequality follows by Assumption 64 (2) and the estimate (A.3). The second inequality follows using the estimate for $\dot{G}_c$ given in (6.9) and the following inequality

$$-\delta(\delta + d(t))y(t)z(t) \leq \Delta|y(t)z(t)| \leq \epsilon z^2(t) + \frac{\Delta^2}{4\epsilon}y^2(t) \leq \epsilon z^2(t) + \frac{\Delta^2}{4\epsilon}\gamma(|y|)|y(t)|$$

with $\Delta := (\|d(\cdot)\|_{L^\infty} + \delta)\delta$.

Next take c large enough so that both estimate (6.10) and $\mu - c\delta + \frac{\Delta^2}{4\epsilon} \leq 0$ hold. Then

$$V_c(t, y(t), z(t)) \geq \frac{1}{2}[cy^2(t) + z^2(t)]$$

and

$$\dot{V}_c(t, y(t), z(t)) \leq \frac{1}{4\epsilon}(\widehat{T}y)^2(t) + c\dot{\eta}(t) + b\nu(\eta(t))\dot{\eta}(t)$$

for almost all t. On integration, writing V_c^0 in place of $V_c(0, y(0), z(0))$,

$$\begin{aligned}
\frac{1}{2}[cy^2(t) + z^2(t)] &\leq V_c(t, y(t), z(t)) = V_c^0 + \int_0^t \dot{V}_c(s, y(s), z(s))\, ds \\
&\leq V_c^0 + \int_0^t \frac{1}{4\epsilon}(\widehat{T}y)^2(s) + c\dot{\eta}(s) + b\nu(\eta(s))\dot{\eta}(s)\, ds \\
&\leq V_c^0 + a_1 + \int_0^t \frac{a_2}{4\epsilon}\psi(|y(s)|)|y(s)| + c\dot{\eta}(s) + b\nu(\eta(s))\dot{\eta}(s)\, ds \\
&\leq V_c^0 + a_1 + \int_0^t \frac{a_2}{4\epsilon}\dot{k}(s) + c\dot{\eta}(s) + b\nu(\eta(s))\dot{\eta}(s)\, ds \\
&\leq V_c^0 + a_1 + (\frac{a_2}{4\epsilon\delta} + c)[\eta(t) - \eta(0)] + b\int_{\eta(0)}^{\eta(t)} \nu(\xi)\, d\xi
\end{aligned}$$

for all $t \in [0, \omega)$.

Thus

$$0 \leq \frac{1}{2}[cy^2(t) + z^2(t)] \leq \text{ constant } + (\frac{a_2}{4\epsilon\delta} + c)\eta(t) + b\int_{\eta(0)}^{\eta(t)} \nu(\xi)\, d\xi \tag{6.13}$$

for all $t \in [0, \omega)$.

We first show that the function η (and hence k) is bounded. By the properties of ν, there exist two unbounded monotonic increasing sequences $(\tilde{\eta}_n)_{n\in\mathbb{N}}$ and $(\hat{\eta}_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{1}{\tilde{\eta}_n} \int_{\eta(0)}^{\tilde{\eta}_n} \nu = +\infty \text{ and } \lim_{n\to\infty} \frac{1}{\hat{\eta}_n} \int_{\eta(0)}^{\hat{\eta}_n} \nu = -\infty.$$

Without loss of generality we assume that $\tilde{\eta}_1 > 1$ and $\hat{\eta}_1 > 1$. Assume that η is unbounded above, then there exist two increasing sequences $(\tilde{t}_n) \subset [0, \omega)$ and $(\hat{t}_n) \subset [0, \omega)$ such that for all $n \in \mathbb{N}$, $\eta(\tilde{t}_n) = \tilde{\eta}_n$ and $\eta(\hat{t}_n) = \hat{\eta}_n$. There are two exhaustive cases.

Case 1: $b > 0$. Divide (6.13) by $b\hat{\eta}_n \geq b\hat{\eta}_1 \geq b > 0$ and gain the contradiction

$$0 \leq \text{constant} + \frac{1}{\hat{\eta}_n} \int_{\hat{\eta}_1}^{\hat{\eta}_n} \nu(\xi)\, d\xi \to -\infty \text{ as } n \to \infty.$$

Case 2: $b < 0$. Again, divide (6.13) by $|b|\tilde{\eta}_n \geq |b|\tilde{\eta}_1 \geq |b| > 0$ and gain the contradiction

$$0 \leq \text{constant} - \frac{1}{\tilde{\eta}_n} \int_{\tilde{\eta}_1}^{\tilde{\eta}_n} \nu(\xi)\, d\xi \to -\infty \text{ as } n \to \infty.$$

Therefore η and hence k remains bounded. The boundedness of η with (6.13) imply the boundedness of y and z. Thus x is bounded and hence $\omega = \infty$.

Since k is a bounded monotone increasing function $\lim_{t\to\infty} k(t)$ exists and, moreover

$$\int_0^\infty \gamma(|y(s)|)|y(s)|\, ds = \int_0^\infty \dot{k}(s)\, ds < \infty.$$

Apply Lemma 39 to conclude that $\gamma(|y(s)|)|y(s)| \to 0$ as $s \to \infty$. Hence $y(t) \to 0$.

It remains to prove that if $\widehat{T} \in \mathcal{T}_h^1$ is shift invariant then $\dot{y}(t)$ tends to zero as $t \to \infty$. By the boundedness of d, p, x and $\widehat{T}y$ there exists some $\rho > 0$ such that $F_b(t, (z(t), y(t), k(t), (\widehat{T}y)(t))) \subset \mathbb{B}_\rho$ for all $t \geq 0$ and so x is a solution to the autonomous differential inclusion

$$\dot{x}(t) \in \{-\delta y(t) + z(t)\} \times \mathbb{B}_\rho \times \{\gamma(|y(t)|)|y(t)|\}, \quad x|_I = x^0. \tag{6.14}$$

For $\bar{x}(\cdot) = (\bar{y}, \bar{z}, \bar{k}) \in C(I; \mathbb{R}^3)$ define

$$l(\bar{y}, \bar{z}, \bar{k}) := \gamma(|\bar{y}(0)|)|\bar{y}(0)| \geq 0.$$

If x is a bounded solution to (6.6) then

$$\int_0^\infty l((S_s x)|_I)\, ds = \int_0^\infty \gamma(|y(s)|)|y(s)|\, ds < \infty.$$

so that, by the Integral Invariance Principle in Theorem 51, $(S_s x)|_I$ approaches the largest weakly invariant set (relative to (6.14)) in $\Sigma := \{\bar{x} \in C(I; \mathbb{R}^3) \mid \bar{y}(0) = 0\}$, denoted by Σ^*.

By the weak invariance of Σ^*, if $\bar{x} \in \Sigma^*$ then there exists a solution $x^* = (x_1^*, x_2^*, x_3^*) : [-h, \infty) \to \mathbb{R}^3$ of

$$\dot{x}^*(t) = (\dot{x}_1^*(t), \dot{x}_2^*(t), \dot{x}_3^*(t)) \in \{-\delta x_1^*(t) + x_2^*(t)\} \times \mathbb{B}_\rho \times \{\gamma(|x_1^*(t)|)|x_1^*(t)|\},$$

with initial condition $x^*|_I = \bar{x} \in \Sigma^*$ and for all $s \geq 0$, $(S_s x^*)|_I \in \Sigma^* \subseteq \Sigma$. That is to say for all $s \in \mathbb{R}_+$, $x_1^*(s) = 0$ on $[0, \infty)$. Since $\dot{x}_1^*(t) = -\delta x_1^*(t) + x_2^*(t)$ and $x_1^*(t) = 0$ for all $t \in [0, \infty)$ it follows that $x_2^*(t) = 0$ for all $t \in [0, \infty)$ also.

The largest weakly invariant subset of Σ is contained in the set $\{(y, z, k \in C(I; \mathbb{R}^3) : y(0) = z(0) = 0\}$.

Now $(S_s x)|_I$ approaches the largest weakly invariant set in $\Sigma := l^{-1}(0)$ and so $(S_s x)|_I$ approaches the set $\{(y, z, k) \in C(I; \mathbb{R}^3) : y(0) = z(0) = 0\}$. Hence

$$\lim_{s\to\infty} (S_s y)(0) = 0 = \lim_{s\to\infty} (S_s z)(0).$$

Thus $\lim_{s\to\infty}(y(s), \dot{y}(s)) = 0$.

6.2.2 Numerical example

A numerical simulation of

$$\ddot{y}(t) = d\dot{y}(t) + a_1 y(t) + a_2 y^3(t) + a_3 \cos(\omega t) + (\widehat{T}y)(t) + bu(t), \tag{6.15}$$

the nonlinear Duffing equation with disturbances and delays was performed. This is an adaptation of the original Duffing oscillator [20] which uses a cubic term to model a stiffening spring effect. The shift-invariant operator $\widehat{T}$ was taken to be

$$(\widehat{T}y)(t) := \frac{1}{4}y^2(t - \tfrac{1}{2}) - \frac{3}{4}y(t - 3/4).$$

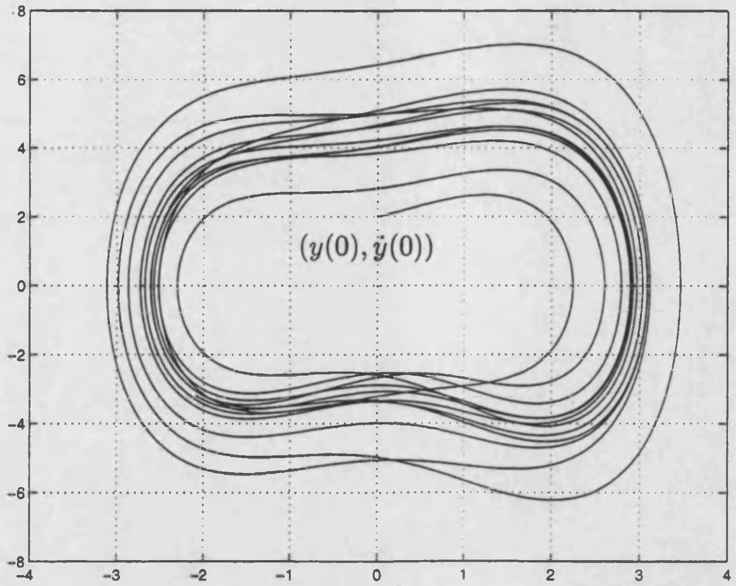


Figure 6-1: A plot of $\dot{y}(t)$ against $y(t)$ showing the behaviour of (6.15) in the absence of control.

The simulation used $d = -0.4$, $a_1 = 3/2$, $a_2 = -1$, $a_3 = 2$ and $\omega = 1.8$. The initial conditions were taken as $(y, \dot{y}, k)|_{[-\frac{3}{4},0]} = (0, 2, 0)$ and in the absence of control (ie $u \equiv 0$) complex behaviour is observed. This is illustrated in Figure 6-1.

Taking the unknown $b = 1$ and a controller with $\delta = \frac{1}{5}$, $\phi(\xi) = \psi(\xi) = 1 + \xi^3$ and $\nu : k \mapsto k^2 \cos(k)$, a numerical simulation was performed using the SIMULINK package within MATLAB, the results of which are shown in Figure 6-2.

A phase plane portrait of the controlled system showing y against $\dot{y}$ is given in Figure 6-3. The control action $u(t)$ is plotted in Figure 6-4. This highlights the discontinuous nature of the controller and illustrates so-called *'chatter'*, the major drawback that control strategies such as (6.3) exhibit.

6.3 Tracking control of second order systems

In this section we consider the feedback control problem for systems of the form (6.1), of asymptotically tracking, by the system output, of an arbitrary reference signal $r : \mathbb{R} \to \mathbb{R}$, which is bounded, continuously differentiable with bounded first derivative $\dot{r}$, and essentially bounded second derivative $\ddot{r}$.

Essentially, we replace every occurrence of y in the previous argument by the error $e(t) := y(t) - r(t)$ and then recast the problem in terms of the system error. We then apply Theorem 65 to prove stability.

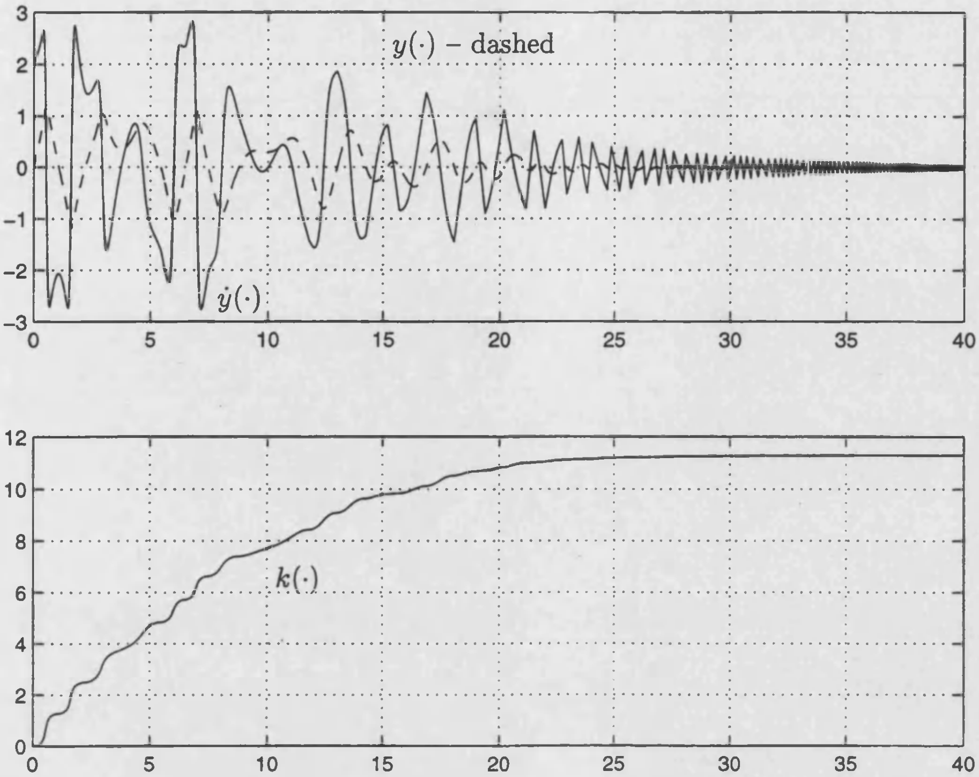


Figure 6-2: A numerical simulation of a second order system under adaptive control.

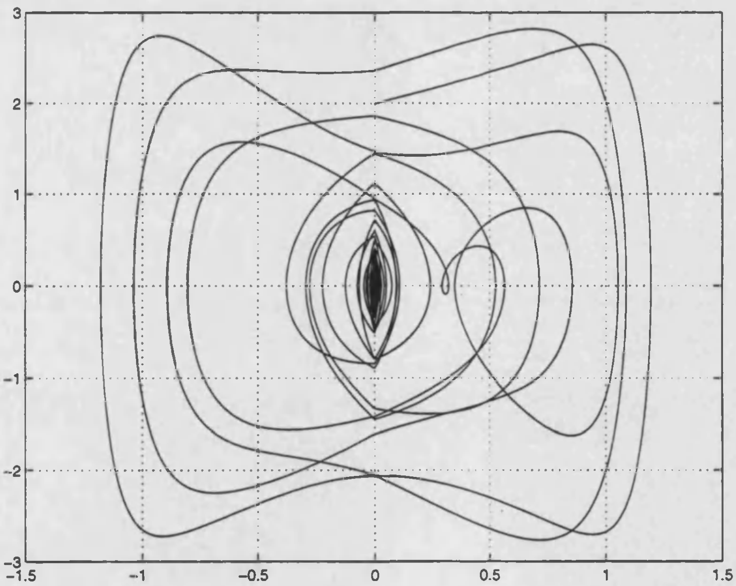


Figure 6-3: A phase plane plot showing $\dot{y}(t)$ against $y(t)$ for a Duffing Oscillator with control.

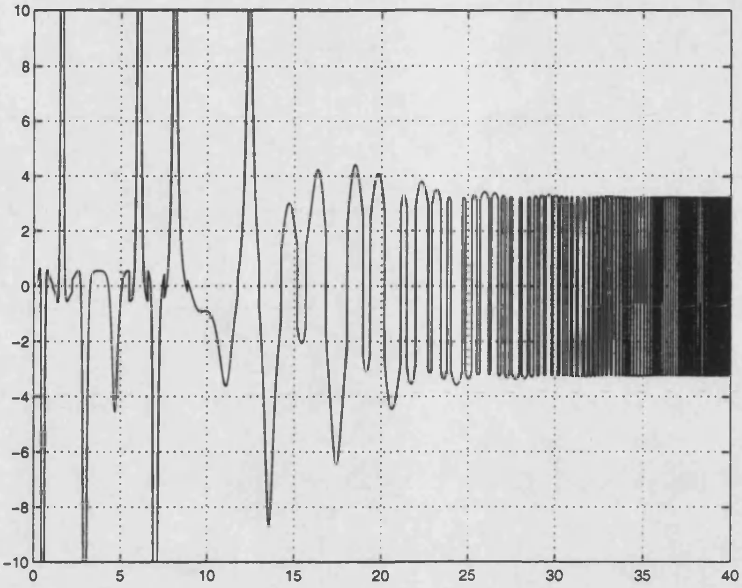


Figure 6-4: The control action $u(t)$ for the Duffing Oscillator illustrating 'chatter'.

6.4 Systems of class $\mathcal{N}_5$

For $\delta > 0$, $\psi \in C(\mathbb{R}; \mathbb{R}_+)$ and ϕ satisfying (5.22) available for control purposes, we denote by $\mathcal{N}_5 = \mathcal{N}_5(\delta, \phi, \psi)$ the class of systems

$$\ddot{y}(t) = d(t)\dot{y}(t) + (\widehat{T}y)(t) + g(\tilde{p}(t), y(t)) + bu(t) \tag{6.16}$$

satisfying Assumption 66.

Assumption 66

1. *The system $(b, d, g, \tilde{p}, \widehat{T}) \in \mathcal{N}_4(\delta, \phi, \psi)$ (recall Assumption 64, pg 93.)*

2. *The operator $\widehat{T}$ is linear.*

Define $\widehat{T}_r$ by (5.26) and,

$$p : \mathbb{R} \to \mathbb{R}^{P+5}, \quad t \mapsto (\tilde{p}(t), d(t), r(t), \dot{r}(t), \ddot{r}(t), (\widehat{T}r)(t))$$

and $f(q, e) : \mathbb{R}^{P+5} \times \mathbb{R} \to \mathbb{R}$,

$$f((\tilde{p}, q_1, q_2, q_3, q_4, q_5), e) := g(\tilde{p}, e + q_2) + q_1 q_3 - q_4 + q_5.$$

Then we see that (6.16) may be written in the form (6.1) with every occurrence of y, $\dot{y}$ and $\ddot{y}$ replaced by e, $\dot{e}$ and $\ddot{e}$ respectively. Moreover, it is evident that this system is of class $\mathcal{N}_4(\delta, \phi, \psi)$. Next, define the discontinuous control by (6.3) replacing every occurrence of $y(t)$ by $e(t)$. Embed the closed-loop system in a differential inclusion of the form (6.4) where $x(t) := (e(t), z(t), k(t))$ then a solution exists on $[-h, \omega)$ and we may apply Theorem 65 to conclude that:

(i) $\omega = \infty$;

(ii) $\lim_{t\to\infty} k(t)$ exists and is finite; and

(iii) $e(t) \to 0$ as $t \to \infty$.

Remarks 67

(a) If r is non-constant then $\widehat{T}_r$ is not shift invariant. Hence the conditions under which Theorem 65 give convergence of $\dot{e}$ to 0, ie linear $\widehat{T}$ and constant r, are rather restrictive.

(b) The assumption that $\widehat{T}$ is linear is perhaps disappointingly strong. One would like to weaken this by imposing, for example, conditions similar to those in Assumption 58(2). If we do this we are unable to absorb the term $(\widehat{T}r)(\cdot)$ as an additional perturbation in the function f, and instead must cancel a constant term (or term such as $\phi(|e(t)|)$ if we impose (5.22)) in the counterpart of (6.12) by noting that there exists some constant c_1 such that

$$\begin{aligned}\left(\sum_{i=1}^{m}(\widehat{T}_i e + r)(t)\right)^2 &\leq 2\mu^2\left(\sum_{i=1}^{m}(\widehat{T}_i\delta_i e)(t)\right)^2 + 2\mu^2\left(\sum_{i=1}^{m}(\widehat{T}_i\delta_i r)(t)\right)^2 \\ &\leq c_1\sum_{i=1}^{m}(\widehat{T}_i\delta_i e)(t)^2 + c_1\phi(0) \leq c_1\sum_{i=1}^{m}(\widehat{T}_i\delta_i e)(t)^2 + c_1\phi(|e(t)|)\end{aligned}$$

where $\mu := \max\{\mu_i | i = 1, \cdots, m\}$ and estimating

$$\begin{aligned}z(t)(\widehat{T}_r e)(t) &\leq \epsilon z^2(t) + \frac{1}{4\epsilon}\left(\sum_{i=1}^{m}(\widehat{T}_i e + r)(t)\right)^2 \\ &\leq \epsilon z^2(t) + c_2\sum_{i=1}^{m}(\widehat{T}_i\delta_i e)(t)^2 + c_2\phi(|e(t)|) \qquad (6.17)\end{aligned}$$

for some constant $c_2 > 0$.

This term neither cancels nor can be absorbed into or estimated by $\dot{k}$ or $\dot{\eta}$. Thus it is not clear how this particular form of analysis *could be adapted to prove a tracking*

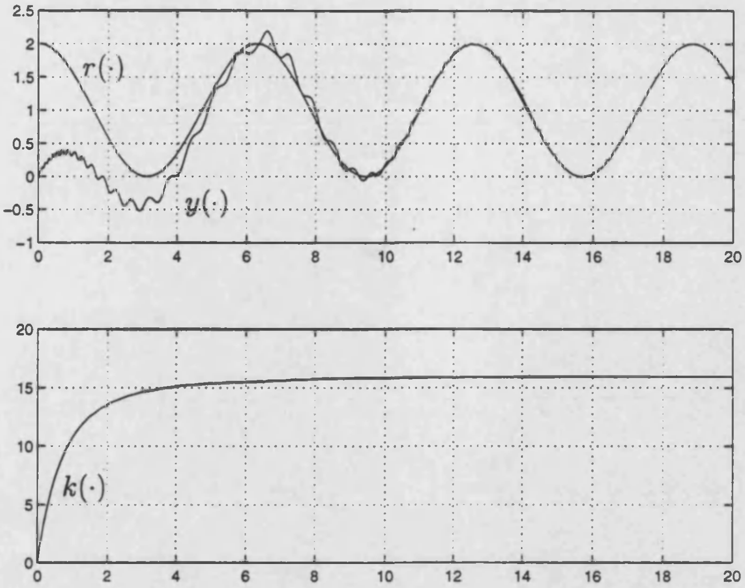


Figure 6-5: A Duffing Oscillator under adaptive tracking control.

result under weaker hypotheses.

6.4.1 An example

To illustrate adaptive tracking control we consider again the system (6.15) with linear operator

$$(\widehat{T}y)(t) := 2y(t - \tfrac{1}{2}) - 2y(t - 3/4).$$

The simulation used $d = -0.8$, $a_1 = 3/2$, $a_2 = -1$, $a_3 = 2$ and $\omega = 1.8$. The initial conditions were taken as $(y, \dot{y}, k)|_{[-\frac{3}{4},0]} = (0, 2, 0)$.

Taking the unknown $b = 1$ and building a controller with $\delta = \frac{1}{2}$, $\phi(\xi) = \psi(\xi) = 1 + \xi^3$ and $\nu : k \mapsto k^2 \cos(k)$, to track the reference signal $r : t \mapsto \cos(t) + 1$, a numerical simulation was performed using the SIMULINK package within MATLAB, the results of which are shown in Figures 6-5 and 6-6.

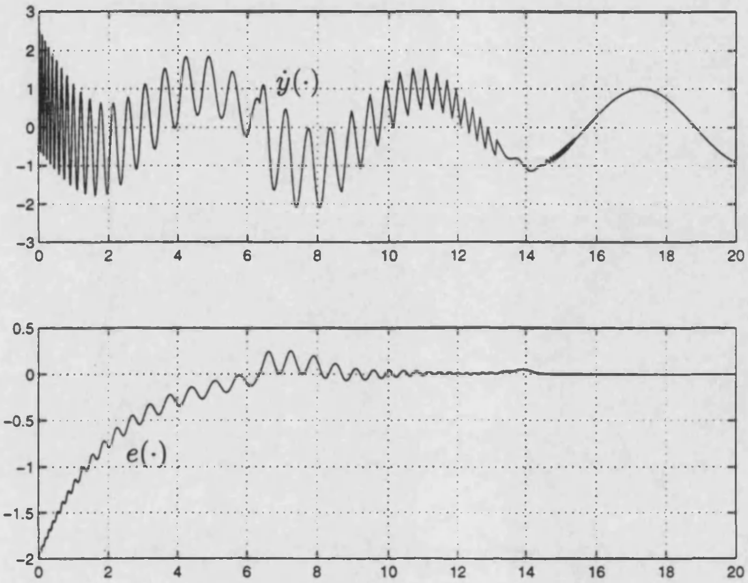


Figure 6-6: A Duffing Oscillator under adaptive tracking control.

Chapter 7

Adaptive control of multi-input multi-output systems

In this chapter we consider two classes of multi-input multi-output systems and prove counterparts of the results of previous chapters. We remark here that in addition to the differences in algebraic structure, (compare (7.1) and (7.8) below), the operator $\widehat{T}$ is assumed to satisfy different control estimates in each case. Specifically, for the class $\mathcal{N}_6$ we assume $\widehat{T}$ is of class $\mathcal{C}_1(\psi)$ and for the class $\mathcal{N}_7$ below, we assume $\widehat{T}$ is of class $\mathcal{C}_3(\psi)$. In the case of $\mathcal{N}_7$-class systems we consider three control objectives: stabilization by output feedback; asymptotic tracking by the output of an arbitrary reference signal of class $\mathcal{R}$; and construction of a λ-universal servomechanism for reference signals of class $\mathcal{R}$.

7.1 Systems of class $\mathcal{N}_6$

We first consider the stabilization problem of feedback control to ensure $y(t) \to 0$ as $t \to \infty$ for a class of appropriately initialized, nonlinear, multi-input $u(t) \in \mathbb{R}^N$, multi-output $y(t) \in \mathbb{R}^N$ systems of the form

$$\dot{y}(t) = f(p(t), y(t)) + (\widehat{T}y)(t) + Bu(t). \tag{7.1}$$

7.2 Adaptive control

Let $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions, available for control purposes, with the properties that $\phi(y) > 0, \psi(y) > 0$ for all $y \neq 0$. The class $\mathcal{N}_6 = \mathcal{N}_6(\phi, \psi, \hat{B})$ is defined to be the family of all those systems $(f, B, p, \widehat{T})$ of the form (7.1) that satisfy

Assumption 68 below.

Assumption 68

1. *$B \in \mathbb{R}^{N\times N}$ and there exists some known $\hat{B} \in \mathbb{R}^{N\times N}$ such that* $\operatorname{spec}(B\hat{B}) \subset \mathbb{C}_+$.
2. *$f : \mathbb{R}^P \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous, and for each compact $K \subset \mathbb{R}^P$ there exists $\mu_K \geq 0$ such that*
$$\|f(r,y)\| \leq \mu_K \phi(\|y\|) \quad \forall\ (r,y) \in K \times \mathbb{R}^N . \tag{7.2}$$
3. *$p \in L^\infty(\mathbb{R};\mathbb{R}^P)$.*
4. *For some $h \geq 0$, $\widehat{T} \in \mathcal{T}_h^N \cap \mathcal{C}_1(\psi)$ (recall Definition 16, pg 29) and is bounded-input, bounded-output stable in the sense of (2.6).*

Remark 69 *In the case $N = 1$ it is sufficient to let $\hat{B} \in \mathbb{R}$ with $\hat{B} \neq 0$ satisfy* $\operatorname{sgn}(\hat{B}) = \operatorname{sgn}(B)$. *In this case $\mathcal{N}_6 \subset \mathcal{N}_1^*$, recall Remark 54, pg 76. For $N > 1$, and in the language of Section 1.4.1, we say that $\hat{B}$ unmixes the spectrum of B.*

7.2.1 $\mathcal{N}_6$-universal stabilizer

The problem to be addressed is that of control design to ensure that, for all systems $(f,B,p,\widehat{T}) \in \mathcal{N}_6$ and all initial data $y|_{[-h,0]} = y^0 \in C([-h,0];\mathbb{R}^N)$, every solution $y(\cdot)$ of the closed-loop system approaches zero asymptotically.

Writing
$$\Phi : \mathbb{R}_+ \to \mathbb{R}_+, \quad r \mapsto \max\{\phi(r), \psi(r)\}$$

then $0 \notin \Phi^{-1}((0,\infty))$ and the following adaptive feedback strategy (appropriately interpreted) will be shown to be a $\mathcal{N}_6$-universal stabilizer assuring that, for all systems $(f,B,p,\widehat{T}) \in \mathcal{N}_6$ and all $y^0 \in C([-h,0];\mathbb{R}^N)$, $y(t) \to 0$ as $t \to \infty$ whilst maintaining boundedness of the controller function k:

$$\left.\begin{aligned} u(t) &= -\hat{B}k(t)\Phi(\|y(t)\|)y(t)\|y(t)\|^{-1}, \\ \dot{k}(t) &= \Phi(\|y(t)\|)\|y(t)\|, \\ k|_{[-h,0]} &= k^0 \in C([-h,0];\mathbb{R}) \text{ with } k^0(0) > 0. \end{aligned}\right\} \tag{7.3}$$

In view of the discontinuous nature of the feedback, the first of equations (7.3) is interpreted in the set-valued sense

$$u(t) \in -\hat{B}k(t)\Phi(\|y(t)\|)\bar{\sigma}(y(t)), \tag{7.4}$$

with $y \mapsto \bar{\sigma}(y) \subset \mathbb{R}^N$ given by

$$\bar{\sigma}(y) := \begin{cases} \{y\|y\|^{-1}\}, & y \neq 0 \\ \bar{\mathbb{B}}_1(0), & y = 0. \end{cases} \tag{7.5}$$

Let the system $(f, B, p, \widehat{T})$ be given. By properties of f and essential boundedness of p, there exists some $\mu \in \mathbb{R}_+$ such that for all $y \in \mathbb{R}^N$ and almost all t, $\|f(p(t), y)\| \leq \mu\phi(\|y\|)$. Define $x \mapsto F_6(x) \subset \mathbb{R}^{N+1}$ by

$$F_6(x) = F_6(y,k) := \left\{v - \hat{B}ku \mid \|v\| \leq \mu\phi(\|y\|),\ u \in \Phi(\|y\|)\bar{\sigma}(y)\right\} \times \{\Phi(\|y\|)\|y\|\}.$$

Then $F_6(\cdot)$ is upper semicontinuous with non-empty, compact and convex values and so $F_6 \in \mathcal{F}^{N+1}$.
Define $T \in \mathcal{T}_h^{N+1}$ by

$$(Tx)(t) = (T(y,k))(t) := (\widehat{T}y, 0)(t) \quad \forall\, t.$$

We now embed the feedback-controlled system in a functional differential inclusion:

$$\left.\begin{array}{l} \dot{x}(t) - (Tx)(t) \in F_6(x(t)), \\ x|_{[-h,0]} = x^0 = (y^0, k^0) \in C\left([-h,0]; \mathbb{R}^{N+1}\right). \end{array}\right\} \tag{7.6}$$

Since $F_6 \in \mathcal{F}^{N+1}$ and $T \in \mathcal{T}_h^{N+1}$ it follows, by Theorem 32, that, for all x^0, (7.6) has a solution and every solution can be maximally extended.

Theorem 70 *Let $x = (y,k) : [-h, \omega) \to \mathbb{R}^{N+1}$ be a maximal solution of the initial-value problem (7.6). Then:*

(i) $\omega = \infty$;

(ii) $\lim_{t\to\infty} k(t)$ exists and is finite;

(iii) $y(t) \to 0$ as $t \to \infty$.

Proof. Since $\mathrm{spec}(B\hat{B}) \subset \mathbb{C}_+$ there exists, by Theorem 84, pg 135 some $G \in \mathbb{R}^{N\times N}$ satisfying

$$G = G' > 0 \quad \text{and} \quad (B\hat{B})'G + GB\hat{B} = 2I.$$

Let $V : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$V(y) = \tfrac{1}{2}\langle y, Gy\rangle.$$

Then for almost all $t \in [0, \omega)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} V(y(t)) = \tfrac{1}{2}\langle \dot{y}(t), Gy(t)\rangle + \tfrac{1}{2}\langle y(t), G\dot{y}(t)\rangle = \langle y(t), G\dot{y}(t)\rangle.$$

First note that if $y \neq 0$ then $\bar{\sigma}(y) = \{y/\|y\|\}$ so that

$$\langle y, GB\hat{B}u\rangle = \tfrac{1}{2}\langle u, (B\hat{B})'Gy\rangle + \tfrac{1}{2}\langle y, GB\hat{B}u\rangle = \|y\|$$

for all $y \in \mathbb{R}^N$ and $u \in \bar{\sigma}(y)$. Hence

$$\begin{aligned}
&\langle y(t), -GB\hat{B}k(t)\Phi(\|y(t)\|)u\rangle \\
&\qquad = -k(t)\Phi(\|y(t)\|)\left[\langle y(t), GB\hat{B}u\rangle\right] \\
&\qquad\qquad = -k(t)\Phi(\|y(t)\|)\|y(t)\|
\end{aligned}$$

for all $t \in [0, \omega)$ and $u \in \bar{\sigma}(y(t))$.

Thus for almost all $t \in [0, \omega)$, $\dot{y}(t)$ exists and

$$\frac{\mathrm{d}}{\mathrm{d}t} V(y(t)) \leq \|G\|\|(\widehat{T}y)(t)\|\,\|y(t)\| + \mu\|G\|\phi(\|y(t)\|)\|y(t)\| - k(t)\dot{k}(t).$$

Integrating from 0 to t and invoking Assumption 68(4), yields

$$\begin{aligned}
0 \leq V(y(t)) &\leq V(y(0)) + c + \int_0^t \bar{\mu}\psi(\|y(s)\|)\|y(s)\| + (\mu\|G\| - k(s))\dot{k}(s)ds \\
&\leq V(y(0)) + c + \int_0^t (\bar{\mu} + \mu\|G\| - k(s))\dot{k}(s)ds \\
&\leq V(y(0)) + c + \int_0^t (\mu^* - k(s))\dot{k}(s)ds
\end{aligned}$$

for some constant $\mu^* \geq 0$. Hence, for almost all $t \in [0, \omega)$

$$0 \leq V(y(t)) \leq c^* + k(t)(\mu^* - k(t)/2) \tag{7.7}$$

where $c^* := V(y(0)) + c - k(0)(\mu^* - k(0)/2)$. Seeking a contradiction, suppose k is unbounded. By monotonicity $k(t) \to \infty$ as $t \uparrow \omega$ and we easily arrive at a contradiction to (7.7). Therefore, k is bounded. Boundedness of y also follows by (7.7). By boundedness of $x = (y, k)$, we conclude that $\omega = \infty$, which is assertion (i) of the theorem. By boundedness and monotonicity of k, assertion (ii) holds. Finally, by boundedness of k,

$$\infty > \int_0^\infty \dot{k}(t)dt = \int_0^\infty \Phi(\|y(t)\|)\|y(t)\|dt$$

and so, by Lemma 40, we conclude that $y(t) \to 0$ as $t \to \infty$. This completes the proof.

7.3 Systems of class $\mathcal{N}_7$

Next we consider a class $\mathcal{N}_7$ of nonlinear N-input (u), N-output (y), systems $(p, f, g, \widehat{T})$, having the same structure as in Figure 1-2, given by a controlled nonlinear functional differential equation of the form

$$\left.\begin{aligned} &\dot{y}(t) = f(p(t), (\widehat{T}y)(t)) + g(p(t), (\widehat{T}y)(t), u(t)), \\ &\dot{y}(t\, y|_{[-h,0]} = y^0 \in C([-h,0];\mathbb{R}^N), \\ &y|_{[-h,0]} = y^0 \in C([-h,0];\mathbb{R}^N), \end{aligned}\right\} \tag{7.8}$$

where $h \geq 0$, p may be thought of as a (bounded) disturbance term and $\widehat{T}$ is a nonlinear causal operator.

We will consider three control objectives. The first is to design a $\mathcal{N}_7$-universal stabilizer to ensure that (i) controller gains converge and (ii) $y(t) \to 0$ as $t \to \infty$. We show that such a stabilizer can be modified to prove the second control objective: universal tracking by the output of a reference signal of class $\mathcal{R}$.

The third control objective is to determine a $\mathcal{N}_7$-universal λ-servomechanism: specifically, to determine continuous functions $\phi : \mathbb{R}^N \to \mathbb{R}^N$ and $\psi_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ (parameterized by $\lambda > 0$) such that, for each system of class $\mathcal{N}_7$ and every reference signal $r \in \mathcal{R}$, the control

$$\left.\begin{aligned} &u(t) = -k(t)\phi(\|y(t) - r(t)\|), \\ &\dot{k}(t) = \psi_\lambda(\|y(t) - r(t)\|), \\ &k|_{[-h,0]} = k^0 \end{aligned}\right\} \tag{7.9}$$

ensures (i) convergence of the controller gain, and (ii) tracking of r with prescribed asymptotic accuracy quantified by $\lambda > 0$ in the sense that $d_\lambda(\|y(t) - r(t)\|) \to 0$ as $t \to \infty$.

The contribution of the theory of this λ-servomechanism is twofold: firstly, we develop universal servomechanisms for this class of nonlinear, infinite-dimensional systems; secondly, we determine relatively weak hypotheses on the righthand side ψ_λ of the gain adaptation equation in (7.9) under which the tracking objective is achievable. One particular consequence in the very specific context of linear systems (1.13), is that the righthand side of the differential equation in (1.14) may be replaced by any continuous function $\psi_\lambda : [0,\infty) \to [0,\infty)$ with the properties $\psi_\lambda^{-1}(0) = [0,\lambda]$ and $\liminf_{s\to\infty} \psi_\lambda(s) \neq 0$: in particular, ψ_λ may be chosen to be a bounded function, one such choice is given by $\psi_\lambda(s) = d_\lambda(s)/s$ for $s > 0$ with $\psi_\lambda(0) := 0$. This ensures that the gain k can exhibit linear growth at most, a feature with attendant advantages from a practical viewpoint.

Let $\alpha_f, \alpha_T \in \mathcal{J}$. The class $\mathcal{N}_7 = \mathcal{N}_7(\alpha_f, \alpha_T)$ is defined to be the family of all those systems $(p, f, g, \widehat{T})$ of the form (7.8) that satisfy Assumption 71 below (with $P, Q \in \mathbb{N}$ arbitrary):

Assumption 71

1. *$p \in C([-h,\infty); \mathbb{R}^P)$;*
2. *$f : \mathbb{R}^P \times \mathbb{R}^Q \to \mathbb{R}^N$, $Q \in \mathbb{N}$, is continuous and, for every compact set $K \subset \mathbb{R}^P$, there exists constant $c_f \geq 0$ such that*

$$\|f(p,w)\| \leq c_f\,[1 + \alpha_f(\|w\|)] \quad \forall\ (p,w) \in K \times \mathbb{R}^Q\,;$$

3. *$g : \mathbb{R}^P \times \mathbb{R}^Q \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and, for every compact set $K \subset \mathbb{R}^P$, there exists positive definite, symmetric $G \in \mathbb{R}^{N\times N}$ such that*

$$\langle Gu, g(p,w,u)\rangle \geq \|u\|^2 \quad \forall (p,w,u) \in K \times \mathbb{R}^Q \times \mathbb{R}^N\,;$$

4. *$\widehat{T} : C([-h,\infty); \mathbb{R}^N) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^Q)$ is of class $\mathcal{T}_h^{N,Q}$ and there exist $\alpha_T \in \mathcal{J}$ and constant $c_T \geq 0$ such that, for all $y \in C([-h,\infty); \mathbb{R}^N)$,*

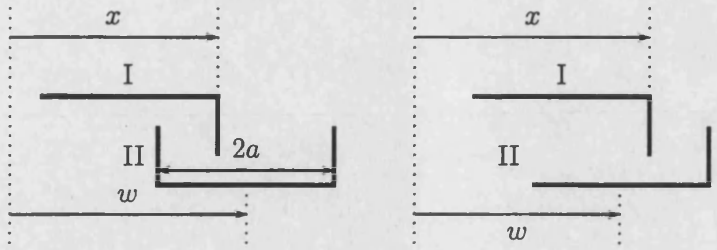
$$\|(\widehat{T}y)(t)\| \leq c_T \left[1 + \max_{s\in[0,t]} \alpha_T(\|y(s)\|)\right] \quad \forall\ t \in \mathbb{R}_+\,. \tag{7.10}$$

That is to say $\widehat{T} \in \mathcal{C}_3(\alpha_T)$.

For example, if $g(p,w,u) = B_1 u$ as in the linear prototype (1.13) then B_1 having eigenvalues in the open right half complex plane yields, by Theorem 84, pg 135, the existence of a positive definite $G \in \mathbb{R}^{N\times N}$ satisfying

$$GB_1 + B_1'G = 2I$$

whence Property 3, Assumption 71.



(7-1a) Backlash. (7-1b) The operator M.

Figure 7-1: The operator M.

7.4 A $\mathcal{N}_7$-universal stabilizer

In this section we show that by implementing a control of the form

$$\left.\begin{aligned} u(t) &= -k(t)\left[1+\max\nolimits_{s\in[0,t]}\alpha(\|y(s)\|)\right]y(t)\|y(t)\|^{-1},\\ \dot{k}(t) &= \left[1+\max\nolimits_{s\in[0,t]}\alpha(\|y(s)\|)\right]\|y(t))\|,\\ k(0) &= k^0>0, \end{aligned}\right\} \tag{7.11}$$

where $\alpha := \alpha_f \circ \alpha_T$, we achieve universal $\mathcal{N}_7$-stabilization assuring that, for all systems $(p, f, g, \widehat{T}) \in \mathcal{N}_7$ and all $y^0 \in C([-h, 0]; \mathbb{R}^N)$, $y(t) \to 0$ as $t \to \infty$ whilst maintaining boundedness of the controller function k.

This control is considerably more complex than those of previous sections. In particular one would need to keep track of the maximum value of $\alpha(\|y(\cdot)\|)$ obtained. Further note that the differential equation in (7.11) defining k is itself functional. To make sense of such a control we define the operator $M : C(\mathbb{R}_+; \mathbb{R}^N) \to C(\mathbb{R}_+; \mathbb{R}_+)$ by

$$(My)(t) := \max_{s\in[0,t]} \|y(s)\|.$$

Then M is clearly causal and BIBO stable. Further

$$|(Mx)(t) - (Mz)(t)| \leq \max_{s\in[0,t]} \|x(s) - z(s)\| = (M(x-z))(t)$$

by the reverse triangle inequality, (A.1), for the norms $\|x|_{[0,t]}\|_\infty$. Hence $M \in \mathcal{T}_0^{N,1}$. In fact, in the case $N = 1$, M can be thought of as a hysteresis operator, by comparing its behaviour with that of a "one-sided" backlash operator, as illustrated in Figure 7-1.

By Proposition 78(3), pg 132, the control may be expressed as

$$u(t) = -k(t)\left[1+\alpha((My)(t))\right]y(t)\|y(t)\|^{-1}.$$

As usual, this is interpreted in the set-valued sense

$$u(t) \in -k(t)\,[1+\alpha((My)(t))]\,\bar{\sigma}(y(t)), \tag{7.12}$$

with $y \mapsto \bar{\sigma}(y) \subset \mathbb{R}^N$ given by (7.5).

Let the system $(p, f, g, \widehat{T}) \in \mathcal{N}_7$ be given. By properties of f and essential boundedness of p, there exists $c_f \in \mathbb{R}_+$ such that for all $w \in \mathbb{R}^Q$ and almost all t, $\|f(p(t), w)\| \leq c_f \alpha_f(\|w\|)$. Define $x \mapsto F_7(x) \subset \mathbb{R}^{N+1}$ by

$$\begin{aligned} F_7(t,(w,m,y,k)) \;&:=\; \{v + g(p(t), w, u) \,|\, \|v\| \leq c_f[1+\alpha_f(\|w\|)], u \in [1+\alpha(m)]\,\bar{\sigma}(y)\} \\ &\quad\; \times \{[1+\alpha(m)]\,\|y\|\}\,. \end{aligned}$$

Then F_7 is upper semicontinuous with non-empty, compact and convex values and so $F_7 \in \mathcal{F}^{Q+N+3,N+1}$.

Define $T \in \mathcal{T}_h^{N+1,Q+N+2}$ by

$$(Tx)(t) = (T(y,k))(t) := \Big((\widehat{T}y)(t), (My)(t), y(t), k(t)\Big) \quad \forall\, t.$$

We now embed the feedback-controlled system in a functional differential inclusion:

$$\left.\begin{aligned} &\dot{x}(t) \in F_7(t, T(x(t))), \\ &x|_{[-h,0]} = (y^0, k^0) \in C\left([-h,0]; \mathbb{R}^{N+1}\right). \end{aligned}\right\} \tag{7.13}$$

Since $F_7 \in \mathcal{F}^{Q+N+3,N+1}$ and $T \in \mathcal{T}_h^{N+1,Q+N+2}$ it follows, by Theorem 32, that, for all x^0, (7.13) has a solution and every solution can be maximally extended.

Theorem 72 *Let $x = (y,k) : [-h, \omega) \to \mathbb{R}^{N+1}$ be a maximal solution of the initial-value problem (7.13). Then:*

(i) $\omega = \infty$*;*

(ii) $\lim_{t\to\infty} k(t)$ *exists and is finite;*

(iii) $y(t) \to 0$ *as* $t \to \infty$*.*

Proof. Similar to the proof of Theorem 70 define $V : \mathbb{R}^N \to \mathbb{R}$ by

$$V(y) := \tfrac{1}{2}\langle Gy, y\rangle$$

so that for almost all $t \in [0, \omega)$, $\dot{V}(y(t)) = \langle Gy(t), \dot{y}(t) \rangle$. Moreover,

$$\langle Gy(t), g(p(t), (\widehat{T}y)(t), u) \rangle \le -k(t)\,[1 + \alpha(My(t))]\,\|y(t)\|$$

for all $u \in -k(t)\,[1 + \alpha(My(t))]\,\bar{\sigma}(y(t))$.

Thus

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(y(t)) &\le c_f[1 + \alpha_f(\|\widehat{T}y(t)\|)]\ \|y(t)\| - k(t)\,[1 + \alpha(My(t))]\,\|y(t)\| \\
&\le c_f\left[1 + \alpha_f\left(c_T[1 + \max_{s \in [0,t]} \alpha_T(\|y(s)\|)]\right)\right] \|y(t)\| \\
&\quad -k(t)\,[1 + \alpha(My(t))]\,\|y(t)\| \\
&\le \mu[1 + \alpha(My(t))]\ \|y(t)\| - k(t)\dot{k}(t)
\end{aligned}$$

for some constant $\mu >_{0} \le V(y(t)) \le V(y(0)) + \int_0^t \mu\dot{k} - k(t)\dot{k}(t)dt$

$$0 \le V(y(t)) \le V(y(0)) + \int_0^t \mu\dot{k} - k(t)\dot{k}(t)dt$$

and so

$$0 \le V(y(t)) < c + (\mu - k(t))k(t)$$

for some constant $c > 0$. Boundedness of k and hence y follow by previous arguments: hence $\omega = \infty$. Next, as before, we conclude k is convergent by monotonicity. Finally we note that

$$\begin{aligned}
\infty > \int_0^\infty \dot{k}(s)\ ds = \int_0^\infty \left[1 + \max_{\tau \in [0,s]} \alpha(\|y(\tau)\|)\right] \|y(s))\|\ ds \\
\ge \int_0^\infty [1 + \alpha(\|y(s)\|)]\ \|y(s))\|\ ds
\end{aligned}$$

and so that by Lemma 40, we conclude that $y(t) \to 0$ as $t \to \infty$. This completes the proof.

7.5 A $\mathcal{N}_7$-universal asymptotic tracker

Next we consider the problem of asymptotic tracking for systems of class $\mathcal{N}_7$. Specifically, given $r \in \mathcal{R}$, the problem of controller design to ensure that the error between the system output and the reference signal tends to zero asymptotically: $e(t) := y(t) - r(t) \to 0$ as $t \to \infty$. Essentially we show that, under stronger assumptions on the functions α_f, α_T we may replace every occurrence of $y(t)$ in the control (7.11) by the error $e(t)$.

Let $r \in \mathcal{R}$, $(p, f, g, \widehat{T}) \in \mathcal{N}_7(\alpha_f, \alpha_T)$ where, comparing with (5.22), α_f and α_T satisfy

$$\left.\begin{array}{l} \forall\ R > 0\ \exists\ \mu_R^* > 0 \text{ such that} \\ \alpha(|e+r|) \leq \mu_R^* \alpha(|e|) \text{ for all } (e,r) \in \mathbb{R} \times [-R, R]. \end{array}\right\} \tag{7.14}$$

Define $\widehat{T}_r : C([-h,\infty);\mathbb{R}^N) \to L^\infty_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^Q)$ by

$$(\widehat{T}_r e) := (\widehat{T}e + r)(t),$$

$\tilde{p} := (p, \dot{r})$, and

$$\tilde{f}((p,s),w) := f(p,w) - s.$$

Then, replacing every occurrence of $y(t)$ and $\dot{y}(t)$ in (7.8) and (7.11) by $e(t)$ and $\dot{e}(t)$ respectively, the system $(\tilde{p}, \tilde{f}, g, \widehat{T}_r) \in \mathcal{N}_7$. We embed the closed-loop system in a differential inclusion, such as (7.13), and apply Theorem 72 to conclude that for all maximal solutions, $x = (e,k) : [-h, \omega) \to \mathbb{R}^{N+1}$, of the initial-value problem differential inclusion:

(i) $\omega = \infty$;

(ii) $\lim_{t\to\infty} k(t)$ exists and is finite;

(iii) $e(t) \to 0$ as $t \to \infty$.

7.6 A $\mathcal{N}_7$-universal λ-servomechanism

The control strategy of the previous section was complex and the control was discontinuous. In this section we construct an $\mathcal{N}_7$-universal λ-servomechanism. Specifically we choose functions ϕ and ψ_λ so that a control of the form (7.9) achieves, for all reference signals $r \in \mathcal{R}$, λ-tracking whilst maintaining boundedness of the internal state k. The advantage of this is that the control is of much reduced complexity: the max operator M of (7.11) is absent, and the control is now continuous.

For $\alpha_f, \alpha_T \in \mathcal{J}$ let $(p, f, g, \widehat{T}) \in \mathcal{N}_7(\alpha_f, \alpha_T)$. In fact p may, for the purposes of this section be of class $L^\infty([-h,\infty),\mathbb{R}^P)$ rather than the more restrictive $C([-h,\infty),\mathbb{R}^P)$. Choose $\alpha \in \mathcal{J}_\infty$ with the property

$$\liminf_{s\to\infty} \frac{\alpha(s)}{s + \alpha_f(\alpha_T(s))} \neq 0. \tag{7.15}$$

For example, the choice $\alpha : s \mapsto s + \alpha_f(\alpha_T(s))$ suffices. For $\lambda > 0$, choose $\psi_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$

to be a continuous function with the properties:

$$\begin{aligned} &\text{(i)} \quad \liminf_{s\to\infty} \tfrac{s\psi_\lambda(s)}{\alpha(s)} \neq 0, \\ &\text{(ii)} \quad \psi_\lambda^{-1}(0) := \{s \mid \psi_\lambda(s) = 0\} = [0,\lambda]\,. \end{aligned} \tag{7.16}$$

For example, the choice ψ_λ given by $\psi_\lambda(s) = d_{\alpha(\lambda)}(\alpha(s))/s$ for $s > 0$ with $\psi_\lambda(0) = 0$ suffices.

Define the *continuous* function $\phi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\phi(e) := \begin{cases} \alpha(\|e\|)\|e\|^{-1}e, & e \neq 0 \\ 0, & e = 0 \end{cases} \tag{7.17}$$

Our objective is to show that the following strategy

$$u(t) = -k(t)\phi(e(t)), \quad \dot{k}(t) = \psi_\lambda(\|e(t)\|), \quad e(t) := y(t) - r(t) \tag{7.18}$$

is a $\mathcal{N}_7$-universal λ-servomechanism for all reference signals $r \in \mathcal{R}$.

We emphasize that, in the construction of an $\mathcal{N}_7$-universal λ-servomechanism, the tracking error $e(t) = y(t) - r(t)$ only is assumed available for feedback and the only *a priori* structural information assumed is knowledge of the functions α_f, $\alpha_T \in \mathcal{J}$.

Theorem 73 *Let $\alpha_f, \alpha_T \in \mathcal{J}$. Choose $\alpha \in \mathcal{J}_\infty$ such that (7.15) holds and define the continuous $\phi : \mathbb{R}^N \to \mathbb{R}^N$ by (7.17). Let $\lambda > 0$ and let $\psi_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and non-decreasing with properties (7.16).*

Then the feedback strategy (7.18) is an $\mathcal{N}_7$-universal λ-servomechanism in the sense that for all $r \in \mathcal{R}$, $(p, f, g, \widehat{T}) \in \mathcal{N}_7(\alpha_f, \alpha_T)$ and $(y^0, k^0) \in C([-h,0];\mathbb{R}^{N+1})$, the feedback-controlled initial-value problem

$$\left.\begin{aligned} &\dot{y}(t) = f(p(t), (\widehat{T}y)(t)) + g(p(t), (\widehat{T}y)(t), -k(t)\phi(y(t) - r(t))), \\ &\dot{k}(t) = \psi_\lambda(\|y(t) - r(t)\|), \\ &(y,k)|_{[-h,0]} = (y^0, k^0) \in C([-h,0];\mathbb{R}^{N+1}), \end{aligned}\right\} \tag{7.19}$$

has a solution. Every solution can be extended to a maximal solution and every maximal solution $(y,k) : [0,\omega) \to \mathbb{R}^{N+1}$ has the following properties:

(i) (y,k) is bounded;

(ii) $\omega = \infty$;

(iii) $\lim_{t\to\infty} k(t)$ exists and is finite;

(iv) $\lim_{t\to\infty} d_\lambda(\|y(t)-r(t)\|) = 0$.

Proof.

Write $K := Q+N+1$ and define the (single-valued) function $\bar{f} : [-h,\infty)\times\mathbb{R}^K \to \mathbb{R}^{N+1}$ by

$$\bar{f} : (t,w,y,k) \mapsto (f(p(t),w) + g(p(t),w,-k\phi(\|y-r(t)\|)),\ \psi_\lambda(\|y-r(t)\|)), \tag{7.20}$$

and define $T : C([-h,\infty);\mathbb{R}^{N+1}) \to L^\infty_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^K)$ by

$$(Tx)(t) = (T(y,k))(t) := ((\widehat{T}y)(t), y(t), k(t)). \tag{7.21}$$

thus, the initial-value problem

$$\dot{x}(t) = \bar{f}(t,(Tx)(t)),$$

that is to say (7.19), is equivalent to (3.1). By continuity of f, g, ϕ, ψ_λ and (essential) boundedness of p, it follows that $\bar{f}$ is a Carathéodory function with the property that, for each $w\in\mathbb{R}^K$, $\bar{f}(\cdot,w)\in L^\infty_{\text{loc}}([-h,\infty);\mathbb{R}^{N+1})$. By assumption, $\widehat{T}\in\mathcal{T}_h^{N,Q}$ and so $T\in\mathcal{T}_h^{N+1,K}$. Therefore, by Theorem 27, (7.19) has a solution and every solution can be maximally extended: moreover, every *bounded* maximal solution has interval of existence $[-h,\infty)$.

Let (y,k) be a solution of (7.19) on its maximal interval of existence $[-h,\omega)$ where $\omega\in(0,\infty]$. Writing $e := y-r$, we have

$$\left.\begin{aligned} \dot{e}(t) &= f(p(t),(\widehat{T}(e+r))(t)) \\ &\qquad + g(p(t),(\widehat{T}(e+r))(t)),-k(t)\phi(e(t))) - \dot{r}(t) \\ \dot{k}(t) &= \psi_\lambda(\|e(t)\|) \end{aligned}\right\} \quad \text{for a.a. } t\in[0,\omega). \tag{7.22}$$

By (essential) boundedness of p and Property 3 (Definition 71) of g, there exists positive definite, symmetric G such that

$$\begin{aligned} &\langle Ge(t), g(p(t),(\widehat{T}(e+r))(t)),-k(t)\phi(e(t)))\rangle \\ &\qquad\qquad \le -k(t)\alpha(\|e(t)\|)\|e(t)\| \quad \text{for a.a. } t\in[0,\omega). \end{aligned} \tag{7.23}$$

Define $c_0 := \sqrt{2\|G^{-1}\|}$ and $c_1 := \sqrt{2/\|G\|}$. For notational convenience, we introduce functions $V,W\in AC([0,\omega);\mathbb{R}_+)$ given by

$$V(t) := \tfrac{1}{2}\langle Ge(t),e(t)\rangle \quad\text{and}\quad W(t) := \sqrt{V(t)}$$

with

$$c_0^{-1}\|e(t)\| \le W(t) \le c_1^{-1}\|e(t)\| \quad \text{for all } t \in [0,\omega). \tag{7.24}$$

By (7.22), (7.23) and properties of f, g and $\widehat{T}$, together with (essential) boundedness of p, r and $\dot{r}$, there exist constants $c_f, c_T > 0$ such that

$$\begin{aligned}\dot{V}(t) = \langle Ge(t), \dot{e}(t)\rangle \le c_f\|G\|[1+\alpha_f(c_T + c_T \max_{s\in[0,t]} \alpha_T(\|e(s)+r(s)\|))]\|e(t)\| \\ - k(t)\alpha(\|e(t)\|)\|e(t)\| + \|G\|\|r\|_{1,\infty}\|e(t)\| \quad \text{for a.a. } t \in [0,\omega).\end{aligned} \tag{7.25}$$

Invoking properties of $\mathcal{J}$ functions, we may conclude that, for some constant $c_2 > 0$,

$$\dot{V}(t) \le c_2\left[1 + \max_{s\in[0,t]} \alpha_f(\alpha_T(\|e(s)\|))\right]\|e(t)\| - k(t)\alpha(\|e(t)\|)\|e(t)\| \quad \text{a.a. } t \in [0,\omega). \tag{7.26}$$

By property (7.15) and the first of properties (7.16), there exist constants $\gamma > \|e(0)\|$, c_γ, $\tilde{c}_\gamma > 0$ such that

$$\alpha_f(\alpha_T(s)) \le c_\gamma\alpha(s) \quad \text{for all } s \ge \gamma \qquad \text{and} \qquad \psi_\lambda(s) \ge \frac{c_\gamma\alpha(s)}{\tilde{c}_\gamma s} \quad \text{for all } s \ge \gamma. \tag{7.27}$$

With a view to proving Assertion (i), we first show that e is bounded. Seeking a contradiction, suppose that e is unbounded; equivalently suppose, that W is unbounded. For each $n \in \mathbb{N}$, define

$$\begin{aligned}\tau_n &:= \inf\{t \in [0,\omega) \mid c_1W(t) = n+1+\gamma\},\\ \sigma_n &:= \sup\{t \in [0,\tau_n] \mid c_1W(t) = n+\gamma\}.\end{aligned}$$

Recalling that $\gamma > \|e(0)\| \ge c_1W(0)$, this construction yields a sequence of disjoint intervals (σ_n,τ_n) such that

$$\left.\begin{aligned}&\max\nolimits_{t\in[0,\tau_n]} c_1W(t) = c_1W(\tau_n) = n+1+\gamma\\ &c_1W(\sigma_n) = n+\gamma\\ &c_1W(t) \in (n+\gamma, n+1+\gamma) \ \text{ for all } t \in (\sigma_n,\tau_n)\end{aligned}\right\} \quad \text{for all } n \in \mathbb{N}.$$

Moreover, for all $n \in \mathbb{N}$,

$$\max_{s\in[0,t]} c_1W(s) = \max_{s\in[\sigma_n,t]} c_1W(s) \le n+1+\gamma < 2n+2\gamma \le 2c_1W(t) \quad \text{for all } t \in [\sigma_n,\tau_n]$$

which, together with (7.24) and properties of $\mathcal{J}$ functions, implies the existence of constants $c_3, c_4 > 0$ such that, for all $n \in \mathbb{N}$,

$$\max_{s\in[0,t]} \alpha(\|e(s)\|) \le \max_{s\in[0,t]} \alpha(c_0 W(s)) \le \alpha(2c_0 W(t)) \le \alpha(2c_0 c^{-1}\|e(t)\|)$$
$$\le c_3\alpha(\|e(t)\|) \le c_3\alpha(c_0 W(t)) \le c_4\alpha(c_1 W(t)) \quad \text{for all } t \in [\sigma_n, \tau_n]. \quad (7.28)$$

Noting that, for all $n \in \mathbb{N}$, $\alpha(\|e(t)\|) \ge \alpha(\gamma)$ for all $t \in [\sigma_n, \tau_n]$ and invoking (7.28) together with (7.24), (7.26) and (7.27), we may conclude the existence of constants $c_5, c_6 > 0$ such that, for all $n \in \mathbb{N}$,

$$\dot V(t) \le [c_5 - k(t)]\alpha(\|e(t)\|)\|e(t)\| \le c_6\alpha(c_1 W(t))W(t) \quad \text{for all } t \in [\sigma_n, \tau_n]. \quad (7.29)$$

Our immediate task is to show that a contradiction to the supposition of unboundedness of e arises in each of the two cases of (a) bounded k and (b) unbounded k.

Case (a): Assume that k is bounded, ie

$$\infty > \int_0^\omega \psi_\lambda(\|e(t)\|)dt. \quad (7.30)$$

Invoking (7.27), (7.29), and (7.24) yields

$$2\ln\left(\frac{n+1+\gamma}{1+\gamma}\right) = \ln V(\tau_n) - \ln V(\sigma_1) = \sum_{j=1}^{n}[\ln V(\tau_j) - \ln V(\sigma_j)]$$
$$= \sum_{j=1}^{n}\int_{\sigma_j}^{\tau_j} \frac{\dot V(t)}{V(t)}\,dt \le c_6\sum_{j=1}^{n}\int_{\sigma_j}^{\tau_j}\frac{\alpha(c_1 W(t))}{W(t)}\,dt \le c_6 c_0\sum_{j=1}^{n}\int_{\sigma_j}^{\tau_j}\frac{\alpha(\|e(t)\|)}{\|e(t)\|}\,dt. \quad (7.31)$$

By construction of (σ_n, τ_n) and since $c_0 \ge c_1$ by (7.24) we have

$$\gamma < \|e(t)\| \qquad \text{if } t \in (\sigma_j, \tau_j).$$

Hence substituting the second inequality of (7.27) into (7.31) yields,

$$2\ln\left(\frac{n+1+\gamma}{1+\gamma}\right) \le c_6 c_0 \frac{\tilde c_\gamma}{c_\gamma}\sum_{j=1}^{n}\int_{\sigma_j}^{\tau_j}\psi_\lambda(\|e(t)\|)\,dt,$$

contradicting (7.30).

Case (b): Now assume that k is unbounded. Then $k(t) \to \infty$ as $t \uparrow \omega$. Let $n^* \in \mathbb{N}$ be such that $k(\sigma_{n^*}) \ge 2c_5$. By (7.29),

$$\dot V(t) \le -c_5\alpha(\|e(t)\|)\|e(t)\| < 0 \quad \text{for a.a. } t \in [\sigma_{n^*}, \tau_{n^*}],$$

which contradicts the fact that $V(\tau_{n^*}) = W^2(\tau_{n^*}) > W^2(\sigma_{n^*}) = V(\sigma_{n^*})$.

We may now conclude that e is bounded. By continuity of ψ_λ, it follows that $\dot{k}$ is bounded and k is bounded on every compact subinterval of $[0, \omega)$. Therefor $\omega = \infty$.

Next, we prove boundedness of k. By boundedness of e and (7.26), there exists a constant $c_9 > 0$ such that

$$\dot{V}(t) \le c_9 - k(t)\beta(V(t)) \quad \text{a.a. } t \in [0, \infty),$$

where $\beta \in \mathcal{K}$ is given by $\beta(s) = \alpha\,(c_1\sqrt{s})\,c_1\sqrt{s}$. Seeking a contradiction, suppose k is unbounded. Then $k(t) \uparrow \infty$ as $t \to \infty$ and so, by Proposition 83, $V(t) \to 0$ as $t \to \infty$. Therefore, there exists $\tau \in [0, \infty)$ such that $\|e(t)\| < \lambda$ for all $t \in [\tau, \infty)$ and so $\dot{k}(t) = 0$ for all $t \in [\tau, \infty)$, which again contradicts the supposition of unboundedness of k.

We have now established Assertions (i) and (ii). Assertion (iii) follows by boundedness and monotonicity of k. By boundedness of e and continuity of ψ_λ, we see that $\psi_\lambda(\|e(\cdot)\|)$ is uniformly continuous. By boundedness of k, $\int_0^\infty \psi_\lambda(\|e(t)\|)dt < \infty$. By Barbălat's Lemma [6], we conclude that $\psi_\lambda(\|e(t)\|) \to 0$ as $t \to \infty$ whence, recalling that $\psi_\lambda^{-1}(0) = [0, \lambda]$, Assertion (iv).

7.7 Discussion

7.7.1 Noise corrupted output

Assume that the output measurement is corrupted by noise $\eta \in \mathcal{R}$ so that the control and gain adaptation become

$$\left.\begin{array}{l} u(t) = -k(t)\phi(y(t) - r(t) + \eta(t)), \\ \dot{k}(t) = \psi_\lambda(\|y(t) - r(t) + \eta(t)\|), \\ k|_{[-h,0]} = k^0. \end{array}\right\}$$

It follows from Theorem 73 that $\lim_{t\to\infty} d_\lambda(\|y(t) + \eta(t) - r(t)\|) = 0$, ie the corrupted output measurement $y(t) + \eta(t)$ tends, as $t \to \infty$, towards the λ-neighbourhood of the reference signal $r(t)$.

Therefore, if an *a priori* bound on the magnitude of the noise is available, then in practice one should choose λ commensurate with such a bound.

7.7.2 Linear systems

To encompass linear systems, such as the motivating class $\bar{\mathcal{L}}$ of finite-dimensional, linear, minimum-phase systems described in Section 1.4.1, with spec$(CB) \subset \mathbb{C}_+$, each of α_f and α_T can be taken to be the identity map $id : s \mapsto s$. In this context, $\alpha : s \mapsto s$ and $\psi_\lambda : s \mapsto d_\lambda^2(s)$ are allowable choices, in which case we recover (1.16). Note that the latter choice for ψ_λ, being quadratic in nature, implies that the controller gain $k(\cdot)$ can exhibit rapid growth whenever the tracking error is large. Such behaviour may be undesirable from a practical viewpoint.

As an alternative, and simple but admissible choice of ψ_λ we may set

$$\psi_\lambda(\|e\|) = \begin{cases} \frac{\|e\|-\lambda}{\|e\|}, & \|e\| \geq \lambda \\ 0, & \|e\| < \lambda \end{cases} \tag{7.32}$$

or, as an eventually constant function for some arbitrary $\bar{\psi}, \delta > 0$

$$\psi_\lambda(\|e\|) := \min\left\{\frac{\bar{\psi}}{\delta} d_\lambda(\|e\|), \bar{\psi}\right\}. \tag{7.33}$$

Both choices ensure that k can exhibit at most linear growth.

7.7.3 Numerical examples

Example 1: a scalar linear system

We consider the problem of feedback control in the specific case of a (suitably initialized) scalar linear time delay system

$$\dot{y}(t) = a_1 y(t) + a_2 y(t-h) + u(t). \tag{7.34}$$

As remarked before, we may choose both α_f and α_T to be the identity map $s \mapsto s$ and function ψ_λ in (7.18) to be bounded which, for this simulation, we choose to be (7.33) with $\bar{\psi} = \delta = 1$ giving $\psi_\lambda(s) := \min\{d_\lambda(s), 1\}$. In this case the control (7.18) reduces to

$$\left.\begin{aligned} u(t) &= -k(t)(y(t) - r(t)), \\ \dot{k}(t) &= \min\{d_\lambda(\|y(t) - r(t)\|), 1\} \\ k|_{[-h,0]} &= k^0 \end{aligned}\right\} \tag{7.35}$$

Theorem 73 proves that under such control all systems (7.34) λ-track the reference signal r.

By way of illustration we take $a_1 = -1$, $a_2 = 3/2$ and $h = 3$. In the absence of control we may apply a classical frequency domain approach, [59]. Taking Laplace transforms

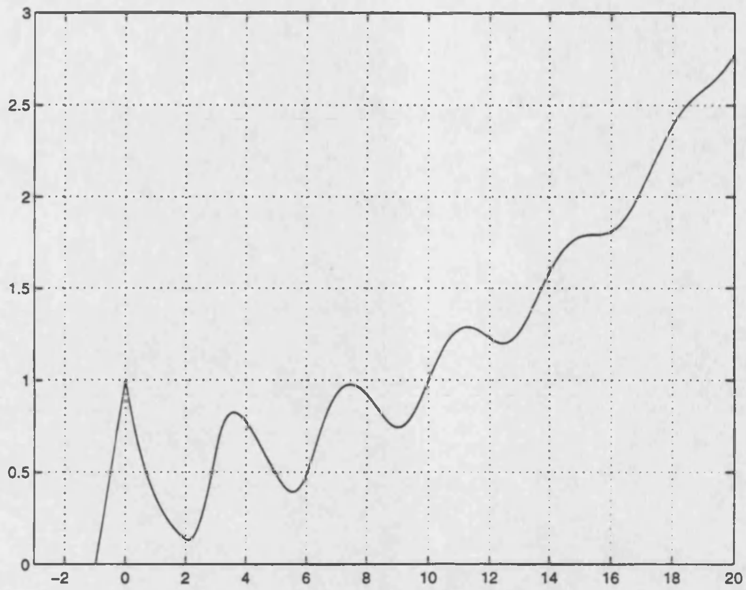


Figure 7-2: System (7.34) in the absence of control.

and solving we see that the stability of this linear system is governed by the signs of the real parts of the poles which are, in this case, the solutions to the equation

$$s - a_1 - a_2 e^{-sh} = 0.$$

In the case $h = 0$, $a_1 = -1$ and $a_2 = 3/2$ this equation has only one solution $s = 1/2$. Thus without delay the system would be unstable. For $h > 0$ we will, in general, have infinitely many solutions and cannot hope to solve this exactly. Thus we consider h as a parameter and seek only values on the imaginary axis at which the poles cross. In this case a routine calculation [59, §2.4] leads us to solve

$$W(\omega^2) = (i\omega - a_1)(-i\omega - a_1) - a_2^2 = 0$$

which in our case gives the imaginary axis crossing points for the poles as $\omega = \pm i\sqrt{5}/2$. We must also check that at such a value of h the pole crosses from the left half plane to the right as h increases - ie is destabilizing. Hence we examine $\operatorname{sgn} W'(\omega^2)$ which in our case is 1. Thus all poles cross to the right half plane as h increases. We conclude that for all $h \geq 0$ the system is unstable.

For the initial condition

$$y|_{[-3,0]} = \begin{cases} 0, & t \leq -1 \\ t+1, & -1 < t \leq 0 \end{cases}$$

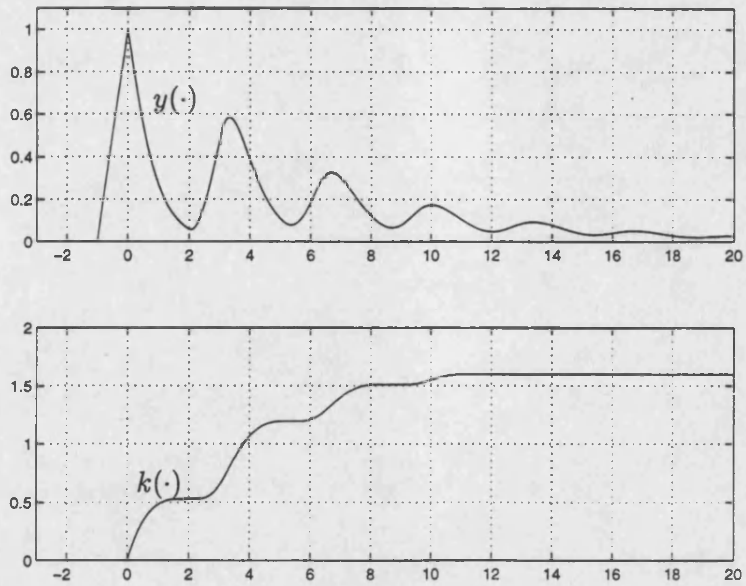


Figure 7-3: System (7.34) under feedback control.

and $k^0 = 0$, a numerical simulation for the system in the absence of control is shown in Figure 7-2.

Taking $\lambda = 1/10$ and reference signal $r(t) = 0$ for all $t \in \mathbb{R}$ (ie λ-stabilization), the behaviour of the closed-loop system under adaptive control (7.35) is in shown in Figure 7-3.

Taking $\lambda = 1/10$ and

$$r(t) := 1/5\sin(2t) + 1/20\sin(3t)$$

we see the behaviour of the closed-loop system under adaptive control in Figure 7-4. The gain k is slow to converge, however for $k^0 = 4.5$ (a value commensurate with the order of magnitude of other system parameters) tracking to within the specified λ is achieved and the gain is constant. A simulation of this is shown in Figure 7-5, taking zero initial conditions for y.

Example 2: a nonlinear system

As an example, consider the suitably initialized, nonlinear system

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = \begin{pmatrix} a_1 y_1(t) + a_2|y_2(t)|^{\frac{1}{2}} + a_3 y_1(t) y_1(t-h_1) \\ a_4 y_1(t) + a_5 y_2(t) y_1(t-h_2)^3 + p(t) \end{pmatrix} + Bu(t) \tag{7.36}$$

for constants $a_1, \ldots, a_5 \in \mathbb{R}$, $p \in L^\infty(\mathbb{R})$ and $B \in \mathbb{R}^{2\times 2}$ with $\mathrm{spec}(B) \subset \mathbb{C}_+$.

Let $h := \max\{h_1, h_2\}$, then by defining $y := (y_1, y_2)$, the operator $\widehat{T} : C([-h,\infty]; \mathbb{R}^2) \to$

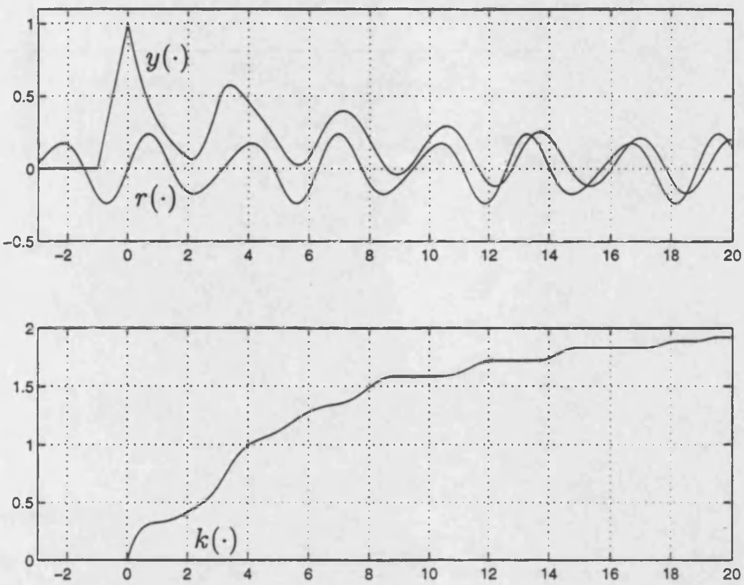


Figure 7-4: System (7.34) under tracking feedback control.

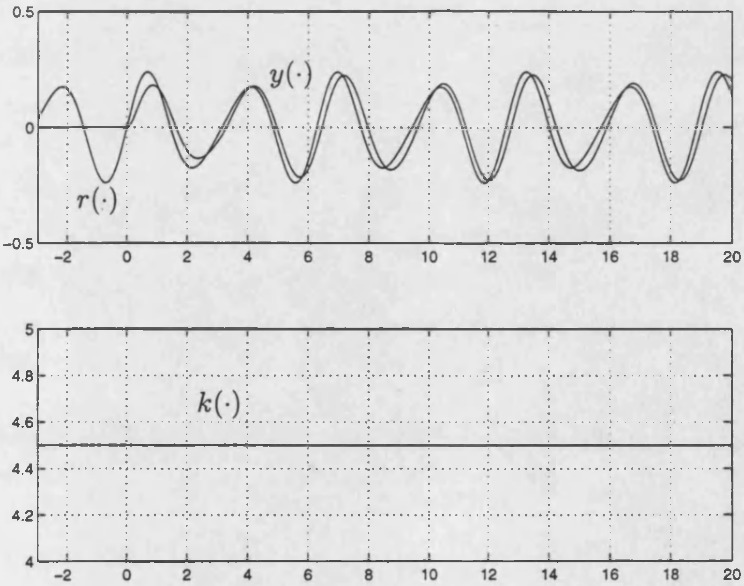


Figure 7-5: System (7.34) tracking for large k.

$L^\infty(\mathbb{R}_+;\mathbb{R}^4)$ by

$$(\widehat{T}y)(t) := (y_1(t), y_2(t), y_1(t-h_1), y_1(t-h_2))$$

and $f : \mathbb{R}^5 \to \mathbb{R}^2$ by

$$f(p, w_1, w_2, w_3, w_4) := \begin{pmatrix} a_1 w_1 + a_2 |w_2|^{\frac{1}{2}} + a_3 w_1 w_3 \\ a_4 w_1 + a_5 w_2 w_4^3 + p \end{pmatrix}$$

the system (7.36) is of class $\mathcal{N}_7$ with $\alpha_f(s) := s^4$, $\alpha_T(s) := s$. Thus taking

$$\alpha(s) := s + \alpha_f(\alpha_T(s)) = s + s^4,$$

and

$$\psi_\lambda := \frac{d_{\alpha(\lambda)}(\alpha(s))}{s + \bar{\epsilon}}$$

for some pre-specified $\lambda, \bar{\epsilon} > 0$, the control (7.18) is a λ-servomechanism for all reference signals $r \in \mathcal{R}$, for the system (7.36).

Numerical simulations were performed with $a_i = 1$, $i = 1, \cdots, 5$, $h_1 = 1$, $h_2 = 1/2$ and $p(t) = \cos(10t)$ and B the identity matrix. The reference signal $r = (r_1, r_2)$ was taken to be

$$r(t) := \begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\cos(t) \\ \sin(2t) + \frac{1}{2}\sin(4t) \end{pmatrix},$$

and $\lambda = 1/10$, $\bar{\epsilon} = 1/100$.

Taking the initial condition $(y_1(t), y_2(t), k(t)) \equiv (3/2, -2, 0)$ for all $t \leq 0$, the results of the simulation are shown in Figure 7-6. As in the case of the linear systems (7.34), the gain k is slow to converge. Taking the initial condition $(y_1(t), y_2(t), k(t)) \equiv (1/2, 0, 45)$ for all $t \leq 0$, the results of the simulation are shown in Figure 7-7, showing that for k large enough, tracking to within $\lambda = 1/10$ is achieved, and the gain k converges. All simulations were performed using SIMULINK, within MATLAB.

7.7.4 Gain adaptation $\psi_\lambda(\cdot)$ for nonlinear systems

For linear systems, linearly bounded gain adaptation as in (7.32) or (7.33) yields the control objective. By considering the perturbed linear system

$$\dot{y}(t) = |y(t)|^\epsilon y(t) + u(t) \tag{7.37}$$

where $\epsilon > 0$ is arbitrary, we show that stronger conditions are required on the gain adaption. Certainly, the nonlinearity must be compensated via the feedback so that

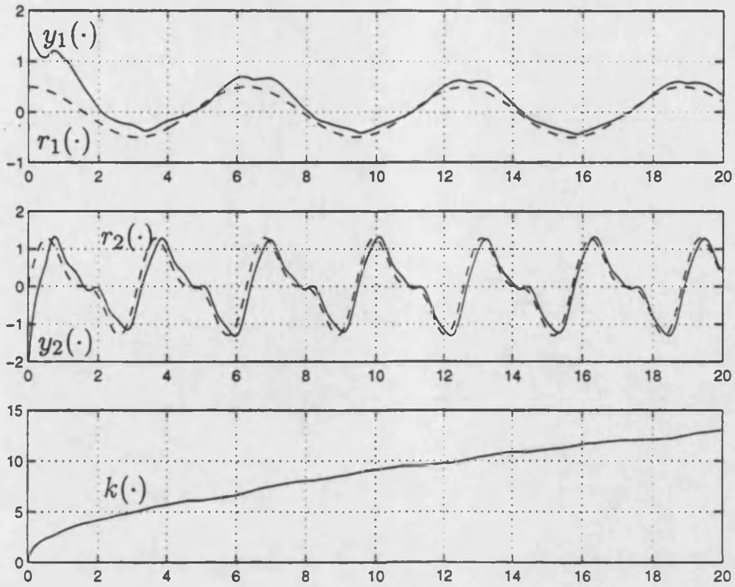


Figure 7-6: System (7.36) under tracking feedback control.

one may choose

$$u(t) = -k(t)|y(t)|^\varepsilon y(t) \tag{7.38}$$

and (7.15) is satisfied. Note that we may choose $\alpha_T(s) = s$. However, the closed-loop system comprising (7.37), (7.38) with the gain adaptation governed by

$$\dot{k}(t) = \psi_\lambda(\|e(t)\|)$$

may exhibit finite escape time where ψ_λ is given by (7.33), as we will show in the following.

Suppose $y(0) = y^0 > 0$ and $k(0) = 0$. Then the closed-loop system (7.37), (7.38) satisfies, as long as $k(t) < 1$,

$$\dot{y}(t) = [1 - k(t)]\, y(t)^{1+\varepsilon},$$

and separating variables and using $k(\tau) \leq \bar{\psi}\,\tau$ for all $\tau \geq 0$ leads to

$$y(t)^{-\varepsilon} \leq (y^0)^{-\varepsilon} + \varepsilon\bar{\psi}t^2/2 - \varepsilon t,$$

and thus the right hand side of the latter inequality becomes zero for

$$t' = \bar{\psi}^{-1}[1 - \sqrt{1 - 2\bar{\psi}(y^0)^{-\varepsilon}/\varepsilon}]. \tag{7.39}$$

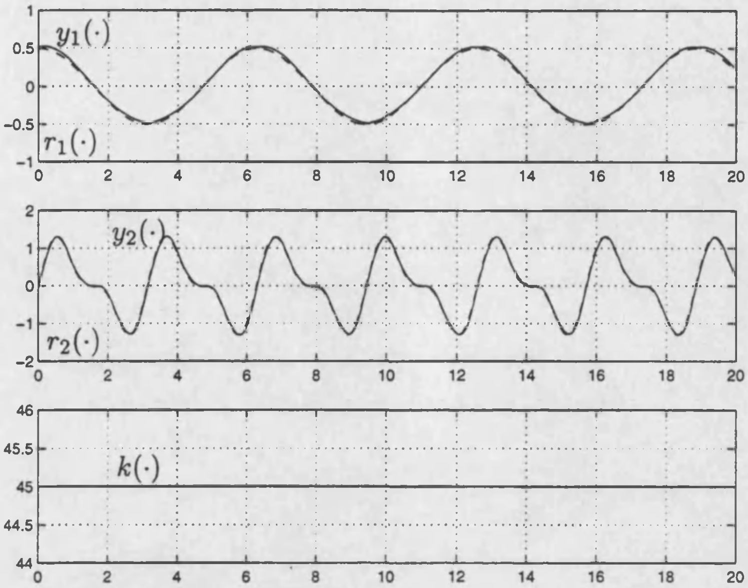


Figure 7-7: System (7.36) tracking for large k.

Therefore, if

$$2\bar{\psi} < \varepsilon(y^0)^{\varepsilon}, \tag{7.40}$$

then (7.37), (7.38) (7.33) exhibits finite escape time at t'. This justifies the stronger assumption on the increase of the gain, which for the above example would be

$$\lim \sup_{s\to\infty} \psi_\lambda(s)\, s^{-\varepsilon} > 0.$$

Taking $\bar{\psi} = 1$ (as in (7.35)) and $\epsilon = 1/10$, and evaluating (7.40) we see that if $y^0 > 1.024 \times 10^{13}$ then finite escape time is guaranteed.

Three numerical simulations were performed to test this. Firstly $y^0 = 10^{13}$ did not produce finite escape time. Secondly taking $y^0 = 1.1 \times 10^{13}$ and evaluating (7.39) at these values predicts $t' = 0.915538\cdots$. The numerical simulation demonstrated finite escape time at $t = 0.915537\cdots$. Lastly for $y^0 = 1.03 \times 10^{13}$ evaluating (7.39) predicts $t' = 0.975833\cdots$. The simulation demonstrated finite escape time at $t = 0.975831\cdots$. All simulations were performed using SIMULINK, within MATLAB with variable step-size numerical routines, set to a tolerance of 10^{-6}, giving excellent correspondence between theory and experiment.

Chapter 8

Conclusion

In this thesis we have addressed the problem of adaptive feedback control design for controlled functional differential equations, and we showed that such a framework allowed the inclusion of a diverse range of phenomena.

The control strategies developed in this thesis were, on occasion, discontinuous, and we developed and adopted a framework of non-smooth analysis and functional differential inclusions within which to perform the analysis.

An existence theory has been developed for solutions of the initial-value problem for a general class of functional differential equations and inclusions. This class includes those arising from the control problems and the behaviour of such solutions was studied. In particular stability criteria were developed for the solutions to such equations.

We considered, and designed controllers for, three principal feedback systems. The first was a class of single-input, single-output, scalar systems. We reiterate here however, that the presence of the nonlinear operator allows the inclusion within this class of, higher-order, relative-degree one, minimum-phase systems. Next we considered a class of planar systems and finally we considered multi-input, multi-output systems. These controllers were high-gain, non-identifier based, adaptive feedback strategies.

In all three cases we consider a variety of control objectives. In particular we required that:

(i) the solution to the closed-loop system should exist for all times $t \geq 0$;

(ii) all controller gains should converge; and

(iii) the system output should achieve the desired (positional) control objective.

Control objectives considered in this thesis were attractivity of the system output to zero, ie $y(t) \to 0$ as $t \to \infty$, and output tracking of a reference signal to within a pre-specified $\lambda \geq 0$. In the case $\lambda > 0$ a continuous controller could be adopted. Such a control strategy ameliorated the problem of 'chatter'.

In many cases the theory was illustrated with numerical experiments.

The following are areas which merit further investigation.

- The use of finite unmixing sets in the multi-input multi-output case to provide counterparts to SISO results which utilize switching functions could be investigated. Such an approach has been taken in [4] for systems of ordinary differential equations.

- Although the effect of the gain adaptation law has been studied in Section 7.6, we have not quantitatively evaluated the effects of implementing differing strategies ψ_λ and ϕ for such systems. Furthermore, the behaviour of system transients in this and other cases has not been addressed.

- In the tracking problem for systems of class $\mathcal{N}_6$ we assumed that the operator $\widehat{T}$ was linear. Whether it is possible to remove the assumption of linearity could be studied further. See Remark 67 (b).

- The use of feedback control strategies involving functional terms could be further investigated. This area has received some attention in this thesis in Section 7.4. However, the bounded-input bounded-output, class $\mathcal{T}_h^1$ operator

  $$(Dy)(t) := \frac{y(t) - y(t-h)}{h}$$

  approximates the value of $\dot{y}(t)$ for small $h > 0$. The use of operators such as D to build control strategies warrants further study.

Appendix A

Background material

A.1 Sets and spaces

In this thesis we adopt the Axiom of Choice in the form of Zorn's lemma:

Lemma 74 (Zorn's Lemma)
Let $\mathcal{A} \neq \emptyset$ be a partially ordered set. If every totally ordered subset $\mathcal{B} \subset \mathcal{A}$ has an upper bound, then $\mathcal{A}$ contains at least one maximal element.

Let $(x, \|\cdot\|_X)$ be a metric space. A set $A \subset X$ is *relatively compact* if the closure of A, ie $\bar{A}$, is compact.

A family $\Phi \subset C([a,b];\mathbb{R}^n)$ is said to be *uniformly bounded* if there exists some $k > 0$ such that

$$\|x(t)\| < k \text{ for all } t \in [a,b],\ x \in \Phi.$$

A family $\Phi \subset C([a,b];\mathbb{R}^n)$ is said to be *equicontinuous* if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $s,t \in [a,b]$

$$|s-t| < \delta \implies \|x(s)-x(t)\| < \epsilon \text{ for all } x \in \Phi.$$

Theorem 75 (Arzelà-Ascoli)
A family $\Phi \subset C([a,b];\mathbb{R}^n)$ is relatively compact if and only if it is uniformly bounded and equicontinuous.

See, for example, [43, §11, Theorem 4].

A.2 Real-valued functions

The following definitions are basic.

Definition 76 (Carathéodory function)
A function $f : I \times \mathbb{R}^K \to \mathbb{R}^N$, $I \subset \mathbb{R}$, *is a* Carathéodory function *if:*

(i) for each fixed $t \in \mathbb{R}$, $z \mapsto f(t,z)$ *is continuous;*

(ii) for each fixed $z \in \mathbb{R}^K$, $t \mapsto f(t,z)$ *is measurable;*

(iii) for each compact $K \subset I \times \mathbb{R}^K$ *there exists some* $\gamma \in L^1_{\text{loc}}(\mathbb{R})$ *such that*

$$\|f(t,x)\| \leq \gamma(t) \quad \textit{for all} \quad (t,x) \in K.$$

Definition 77 (Lower semicontinuity) *Given a normed space* $(X, \|\cdot\|)$, *a functional* $f : X \to \mathbb{R}$ *is said to be* lower semicontinuous *at* $x \in X$ *if for every sequence* $(x_n) \subset X$ *converging to* x *we have*

$$f(x) \leq \liminf_{n \to \infty} f(x_n).$$

If f *is lower semicontinuous at every* $x \in X$ *we say that* f *itself is lower semicontinuous.*

If $f : X \to \mathbb{R}$ is continuous then f is lower semicontinuous.

A.3 Class $\mathcal{J}$ functions

For example, (a) for each $s > 0$, the function $\tau \mapsto \tau^s$ is of class $\mathcal{J}_\infty$, (b) the function $\tau \mapsto \ln(1+\tau)$ is of class $\mathcal{J}_\infty$; its inverse $\tau \mapsto \exp(\tau) - 1$ is of class $\mathcal{K}_\infty$ but is *not* of class $\mathcal{J}$. The following properties of class $\mathcal{J}$ functions are assembled for reference.

Proposition 78

1. $\alpha, \beta \in \mathcal{J} \implies \alpha \circ \beta \in \mathcal{J}$ *and* $\alpha + \beta \in \mathcal{J}$;
2. $\alpha \in \mathcal{J} \implies \exists \Delta > 0 : \ \alpha(a+b) \leq \Delta[\alpha(a) + \alpha(b)] \quad \forall\, a, b \in \mathbb{R}_+$;
3. *Let* $t > 0$, $I = [0,t]$, $\xi \in C([0,t]; \mathbb{R}_+)$ *and* $\alpha, \beta \in \mathcal{K}$.

$$\alpha(\max_{s \in I} \beta(\xi(s))) = \max_{s \in I} \alpha(\beta(\xi(s))).$$

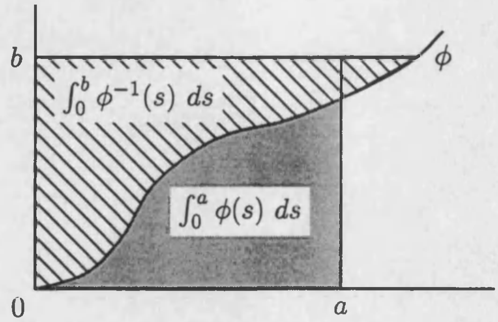


Figure A-1: Young's inequality.

Proof. Assertion 1 is an immediate consequence of the definition of $\mathcal{J}$.
Assertion 2 again follows from the definition of $\mathcal{J}$ on noting that, being of class $\mathcal{J}$, α is *a fortiori* a $\mathcal{K}$ function and so has the property that $\alpha(a+b) \leq \alpha(2a) + \alpha(2b)$.
Assertion 3 is a simple consequence of strict monotonicity of $\mathcal{K}$ functions.

A.4 Classical estimates, inequalities and results

We first recall the *reverse triangle inequality.* If $(X, \|\cdot\|)$ is a normed space then for all $x, y \in X$,

$$|\|x\| - \|y\|| \leq \|x - y\|. \tag{A.1}$$

Theorem 79 (Young's Inequality)
Let $\phi(\cdot) \in \mathcal{K}_\infty$ then for all $a, b \in \mathbb{R}_+$

$$ab \leq \int_0^a \phi(s)\ ds + \int_0^b \phi^{-1}(s)\ ds \tag{A.2}$$

and equality holds if and only if $b = \phi(a)$.

See, for example, [29, Theorem 13.2], but Figure A-1 illustrates the idea behind this Theorem.

Example 80 *Consider the $\mathcal{K}_\infty$ functions $s \mapsto s^r$ for fixed $r > 0$. Then for all $a, b \in \mathbb{R}_+$*

$$\begin{aligned} ab &\leq \int_0^a s^r\ ds + \int_0^b s^{\frac{1}{r}}\ ds \\ &= \frac{a^{r+1}}{r+1} + \frac{r}{r+1} b^{\frac{r+1}{r}} \end{aligned}$$

which, for $r = 1$ reduces to the familiar

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

for all $a, b \in \mathbb{R}_+$. Moreover for $p > 1$ and $\epsilon > 0$ applying Theorem 79 to the $\mathcal{K}_\infty$ function $s \mapsto \epsilon s^{p-1}$ gives the inequality

$$ab \leq \epsilon a^p + \frac{b^q}{q(p\epsilon)^{1/(p-1)}} \quad \forall a, b \in \mathbb{R}_+$$

where q is the conjugate exponent of p, ie $1/p + 1/q = 1$. For $p = q = 2$ and $\epsilon > 0$ this reduces to

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon} \quad \text{for all } a, b \in \mathbb{R}_+. \tag{A.3}$$

Lemma 81 (Hölder's Inequality)
Let $p > 1$ and $q > 0$ be the conjugate exponent satisfying $1/p + 1/q = 1$. Let $I \subset \mathbb{R}$ and $x, \xi \in L^\infty(I; \mathbb{R}^N)$ then

$$\int_I |x(s)\xi(s)|ds \leq \left(\int_I x^p(s)ds\right)^{\frac{1}{p}} \left(\int_I \xi^q(s)ds\right)^{\frac{1}{q}}.$$

Lemma 82 (Gronwall's Lemma)
Let $\phi \in L^1([t, t^*]; \mathbb{R})$, $\phi(\tau) \geq 0$, ψ be absolutely continuous on $[t, t^*]$. If $\xi \in L^\infty([t, t^*]; \mathbb{R})$ satisfies

$$\xi(s) \leq \psi(s) + \int_t^s \phi(\tau)\xi(\tau)d\tau, \quad \forall s \in [t, t^*]$$

then

$$\xi(s) \leq \psi(t) \exp\left(\int_t^s \phi(\tau)\right) + \int_t^s \psi'(\tau) \exp\left(\int_t^s \phi(\rho)d\rho\right) ds, \quad \forall \tau \in [t, t^*].$$

See, for example, [15, Lemma 8.1].

Lemma 83
Let $\xi \in AC(\mathbb{R}_+; \mathbb{R}_+)$, $k \in C(\mathbb{R}_+; \mathbb{R}_+)$, $\beta \in \mathcal{K}$ and $c \geq 0$. If k is monotonically non-decreasing and unbounded as $t \to \infty$ and

$$\dot{\xi}(t) \leq c - k(t)\beta(\xi(t)) \quad a.a.\ t \in \mathbb{R}_+,$$

then $\xi(t) \to 0$ as $t \to \infty$.

Proof. Assume $k(t) \uparrow \infty$ as $t \to \infty$ and, seeking a contradiction, suppose $\xi(t) \not\to 0$ as $t \to \infty$. Then there exists an $\epsilon > 0$ and sequence $(t_n)_{n \in \mathbb{N}}$ such that $\xi(t_n) \geq 2\epsilon$

for all $n \in \mathbb{N}$ and $t_n \to \infty$ as $n \to \infty$. Clearly, ξ cannot be bounded away from zero (otherwise, $\dot{\xi}(t) \leq -c < 0$ for all t sufficiently large, whence a contradiction) so there exists a sequence $(s_n)_{n\in\mathbb{N}}$ such that $\xi(s_n) \leq \epsilon$ for all $n \in \mathbb{N}$ and $s_n \to \infty$ as $n \to \infty$. By continuity of ξ, it follows that there exists a sequence of disjoint intervals $I_n = [\sigma_n, \tau_n]$ such that $\sigma_n \to \infty$ as $n \to \infty$ and, for all $n \in \mathbb{N}$, $\sigma_n < \tau_n$, $\xi(\sigma_n) = \epsilon$, $\xi(\tau_n) = 2\epsilon$ and $\xi(t) \in [\epsilon, 2\epsilon]$ for all $t \in I_n$. Choosing n^* sufficiently large such that $k(\sigma_{n^*}) \geq 2c/\beta(\epsilon)$, we arrive at a contradiction

$$0 < \epsilon = \xi(\tau_{n^*}) - \xi(\sigma_{n^*}) = \int_{I_{n^*}} \dot{\xi}(t)dt \leq \int_{I_{n^*}} c - \frac{2c}{\beta(\xi(t))}\beta(\xi(t))\, dt \leq -c\int_{I_{n^*}} dt < 0\,.$$

Therefore, $\xi(t) \to 0$ as $t \to \infty$.

Theorem 84 (Lyapunov's Equation)
The matrix $A \in \mathbb{R}^{n\times n}$ *has* $\mathrm{spec}(A) \subset \mathbb{C}_+$ *if and only if for all symmetric and positive definite* $M \in \mathbb{R}^{n\times n}$*, there exists a unique real positive definite symmetric matrix* P *which satisfies*

$$A'P + PA = M.$$

See, for example, [48, Theorem XI, pg 81], [8, pg 61], or [11, pg 186-189].

Appendix B

Set-valued analysis

Let X and Y be two non-empty sets. Then we may define a *set-valued map* F, between X and Y, to be a map which assigns to each element of X a non-empty subset of Y. That is to say $F(x) \subset Y$. The *values* of F are the sets $F(x)$ for $x \in X$. Such a map has *convex values* if $F(x)$ is convex for all $x \in X$, and *compact values* if $F(x)$ is compact for all $x \in X$.

We define the *graph* of F to be

$$\mathrm{graph}(F) := \{(x, y) \in X \times Y |\ y \in F(x)\}.$$

Definition 85 (Set-valued Upper Semicontinuity)
A set-valued map F from a metric space X to the non-empty subsets of metric space Y is said to be upper semicontinuous *at $\bar{x} \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $F(x) \subset F(\bar{x}) + \mathbb{B}_\epsilon$ for all x with $|x - \bar{x}| < \delta$.*

F is upper semicontinuous if it is upper semicontinuous at every $\bar{x} \in X$.

For a detailed discussion see [5, Chapter 1]. The following facts are well known.

Theorem 86 (A Closed Graph Theorem)
Let $x \mapsto F(x) \subset Y$ be a set-valued map on a space $X \neq \emptyset$. If $F(X) := \bigcup_{x \in X} F(x)$ has compact closure and the graph is closed then F is upper semicontinuous.

See [5, Corollary 1, page 41]. This result is applied to prove the upper semicontinuity of the maps σ, σ_λ and $\bar{\sigma}$, recall (5.9), (5.25) and (7.5). Let $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous then a direct application of Theorem 86 proves that the set-valued map $F(x) := \{f(x)\}$ is upper semicontinuous. The next example will be important in applications.

Claim 87 *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and increasing. Define the set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$ by*

$$F(x) := \{v \in \mathbb{R}^m \mid \|v\| \leq \phi(\|x\|)\}.$$

Then F is upper semicontinuous.

Proof. Let $\epsilon > 0$. By the continuity of the increasing function ϕ, there exists a $\delta > 0$ such that if $|a - a'| < \delta$ then $\phi(a') < \phi(a) + \epsilon$.
Thus if $\bar{x}, x \in \mathbb{R}^n$ with $\|\bar{x} - x\| < \delta$ then for all $y \in F(x)$,

$$\|y\| \leq \phi(\|x\|) < \phi(\|\bar{x}\|) + \epsilon.$$

Thus $y \in F(\bar{x}) + \mathbb{B}_\epsilon$. Hence if $\|\bar{x} - x\| < \delta$ then $F(x) \subset F(\bar{x}) + \mathbb{B}_\epsilon$, as required.

We note here that sums and compositions of upper semicontinous set-valued maps are also upper semicontinuous.

Proposition 88 (An Approximate Selection Theorem)
Let $F \in \mathcal{F}^N$. For each $\epsilon > 0$, there exists a locally Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}^N$ such that $\mathrm{graph}(f) \subset \mathrm{graph}(F) + \mathbb{B}_\epsilon(0)$. *(Here, $\mathbb{B}_\epsilon(0)$ is the open ball of radius ϵ centred at $0 \in \mathbb{R}^N \times \mathbb{R}^N$.)*

See [5, Proposition 3, page 42].

Proposition 89 *Let $F \in \mathcal{F}^{N,M}$. If $K \subset \mathbb{R}^N$ is compact, then $F(K) := \cup_{x \in K} F(x)$ is compact.*

See [5, Theorem 1, page 84]).

Theorem 90 (A Compactness Theorem)
Let $(x_n)_{n \in \mathbb{N}} \subset AC(I; \mathbb{R}^N)$ for some interval $I \subseteq \mathbb{R}$ satisfying

1. *For all $t \in I$, $(x_n(t))_{n \in \mathbb{N}}$ is bounded in $\mathbb{R}^N$*
2. *There exists a positive function $c(\cdot) \in L^1(I)$ such that, for almost all $t \in I$, $\|\dot{x}_n(t)\| \leq c(t)$*

Then there exists a subsequence (again denoted by) $x_n(\cdot)$ converging to an absolutely continuous function $x(\cdot)$ from I to X in the sense that

1. *$x_n(\cdot)$ converges uniformly to $x(\cdot)$ over compact subsets of I.*

2. $\dot{x}_n(\cdot)$ *converges weakly to* $\dot{x}(\cdot)$ *in* $L^1(I, \mathbb{R}^N)$.

For a proof see [5, Theorem 4, pg 13]

Theorem 91 (Convergence Theorem)
Let F *be an upper semicontinuous set-valued map from a Banach space* X *with non-empty closed and convex values in a Banach space* Y. *Let* I *be an interval in* $\mathbb{R}$ *and* $x_k(\cdot) : I \to X$ *and* $y_k(\cdot) : I \to Y$ *be measurable functions such that*

1. *for almost all* $t \in I$, *for every neighbourhood* $\mathcal{N}$ *of* 0 *in* $X \times Y$ *there exists* $k_0 := k_0(t, \mathcal{N})$ *such that*

$$(x_k(t), y_k(t)) \in \text{graph}(F) + \mathcal{N} \textit{ for all } k \geq k_0$$

2. $x_k(\cdot)$ *converges almost everywhere to a function* $x(\cdot) : I \to X$

3. $y_k(\cdot) \in L^1(I, Y)$ *and converges weakly to* $y(\cdot)$ *in* $L^1(I, Y)$

then for almost all $t \in I$,

$$(x(t), y(t)) \in \text{graph}(F) \tag{B.1}$$

ie $y(t) \in F(x(t))$.

For a proof see [5, Theorem 1, pg 60].

Bibliography

[1] F. Allgöwer, J. Ashman, and A. Ilchmann. High-gain adaptive λ-tracking for nonlinear systems. *Automatica*, 33:881–888, 1997.

[2] F. Allgöwer and A. Ilchmann. Multivariable adaptive λ-tracking for nonlinear chemical processes. *Proceedings of 3rd European Control Conference*, pages 1645–1651, September 1995.

[3] F. Allgöwer and A. Ilchmann. Process control applications of adaptive λ-tracking. *Proceedings of the American Control Conference*, pages 734–738, June 1997.

[4] J. A. Ashman. *Adaptive control of uncertain nonlinear systems.* PhD thesis, Department of Mathematical Sciences, University of Bath, 1994.

[5] J. P. Aubin and A. Cellina. *Differential inclusions.* Springer, 1984.

[6] I. Barbălat. Systèmes d'équations différentielles d'oscillations non linéaires. *Revue de Mathématiques Pures et Appliquées, Bucharest*, IV:267–270, 1959.

[7] F. Blanchini and E. P. Ryan. A Razumikhin-type lemma for functional differential equations with applications to adaptive control. *Automatica*, 35(5):809–818, 1998.

[8] R. W. Brockett. *Finite dimensional linear systems.* John Wiley & Sons, inc., 1970.

[9] R. W. Brockett. *Asymptotic stability and feedback stabilization.* Differential Geometric Control Theory. Birkhauser, Boston, 1983.

[10] C. I. Byrnes and C. F. Martin. An integral-invariance principle for nonlinear systems. *IEEE Transactions on Automatic Control*, 40(6):983–994, 1995.

[11] F. M. Callier and C. A. Desoer. *Linear systems theory.* Springer, 1991.

[12] F. H. Clarke. *Optimization and nonsmooth analysis.* Wiley, 1983.

[13] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth analysis and control theory*, volume 178 of *Graduate Texts in Mathematics.* Springer, 1998.

[14] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations.* McGraw-Hill, 1955.

[15] R. F. Curtain and A. J. Pritchard. *Functional analysis in modern applied mathematics*, volume 132 of *Mathematics in Science and Engineering.* Academic Press, 1977.

[16] R. F. Curtain and A. J. Pritchard. *Infinite dimensional linear systems theory*, volume 8 of *Lecture notes in control and information sciences.* Springer, 1978.

[17] R. F. Curtain and H. J. Zwart. *An introduction to infinite dimensional linear systems theory.* Springer, 1995.

[18] K. Deimling. *Multivalued differential equations.* Walter de Gruyter, 1992.

[19] W. Desch, H. Logemann, E. P. Ryan, and E. D. Sontag. Meagure functions and asymptotic behaviour of dynamical systems. *Nonlinear Analysis: Theory, Methods and Applications*, To appear.

[20] G. Duffing. *Erzwungene Schwingungen bei Veränderlicher Eigenfrequenz.* Braunschweig, 1918.

[21] A. F. Filippov. Differential equations with discontinuous right-hand sides. *American Mathematical Society Translations*, 43:199–232, 1964.

[22] A. F. Filippov. *Differential equations with discontinuous right-hand sides.* Kluwer, Dordrecht, 1988.

[23] P. Glendinning. *Stability, instability and chaos : an introduction to the theory of nonlinear differential equations.* Cambridge University Press, 1994.

[24] J. Guckenheimer and P. Holmes. *Nonlinear ocillations, dynamical systems, and bifurcations of vector fields.* Springer, 1983.

[25] W. Hahn. On a new type of stability. *Journal of Differential Equations*, 3:440–448, 1967.

[26] J. K. Hale and S. M. V. Lunel. *Introduction to functional differential equations.* Springer, 1993.

[27] U. Helmke and D. Prätzel-Wolters. Stability and robustness properties of universal adaptive controllers for 1st-order linear-systems. In C. I. Byrnes, C. F. Martin, and R. E. Saeks, editors, *Analysis and Control of Nonlinear Systems*, pages 17–26. North-Holland, 1988.

[28] U. Helmke and D. Prätzel-Wolters. Stability and robustness properties of universal adaptive controllers for 1st-order linear-systems. *International Journal of Control*, 48:1153–1182, 1988.

[29] E. Hewitt and K. Stromberg. *Real and abstract analysis.* Springer, 1965.

[30] A. C Hicks and S. Townley. On exact-solutions of differential-equations arising in universal adaptive stabilization. *Systems and Control Letters*, 20(2):117–129, 1993.

[31] A. Ilchmann. Non-identifier-based adaptive control of dynamical systems: a survey. *IMA Journal of Mathematical Control and Information*, 8:321–366, 1991.

[32] A. Ilchmann. Non-identifier-based high-gain adaptive control. *Lecture Notes in Control and Information Science*, 189, 1993.

[33] A. Ilchmann. Adaptive λ-tracking for polynomial minimal phase systems. *Dynamics and Stability of Systems*, 13(4):341–371, 1998.

[34] A. Ilchmann and H. Logemann. Adaptive λ-tracking for a class of infinite-dimensional systems. *Systems and Control Letters*, 34(1–2):11–21, 1998.

[35] A. Ilchmann, A. Mazur, and C. Schmidt. Adaptive λ-tracking for rigid manipulators. *Proceedings of IEEE International conference on robotics and Automation, Louvain-LaNeuve, Belgium*, 1998.

[36] A. Ilchmann and M. Pahl. Adaptive multivariable regulation of a biogas tower reactor. *European Journal of Control*, 4:116–131, 1998.

[37] A. Ilchmann and E. P. Ryan. Universal lambda-tracking for nonlinearly-perturbed systems in the presence of noise. *Automatica*, 30:337–346, 1994.

[38] A. Ilchmann and S. Townley. Adaptive regulation for bio-processes. *Proceedings of Fourth International Symposium on Models and Methods in Automation and Robotics*, pages 1203–1209, 26-29 August 1997.

[39] P. Ioannou and P. V. Kokotović. *Adaptive systems with reduced models.* Springer, 1983.

[40] P. Ioannou and P. V. Kokotović. Instability analysis and improvement for robustness of adaptive control. *Automatica*, 20:583–594, 1984.

[41] H. Kaufman, I. Bar-Kana, and K. Sobel. *Direct adaptive control algorithms.* Springer, 1994.

[42] H. K. Khalil. *Nonlinear systems.* Macmillan, 1992.

[43] A. N. Kolmorogorov and S. V. Fomin. *Introductory real analysis.* Dover, 1970.

[44] M. A. Krasnosel'skiĭ and A. V. Pokrovskiĭ. *Systems with hysteresis.* Springer, 1989.

[45] E. Kreyszig. *Introductory functional analysis with applications.* Wiley, 1978.

[46] M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and adaptive control design.* Wiley, New York, 1995.

[47] J. P. LaSalle. *The stability of dynamical systems.* Arrowsmith, 1976.

[48] J. P. LaSalle and S. Lefschetz. *Stability by Liapunov's direct method with applications.* Academic Press, 1961.

[49] H. Logemann and R. F. Curtain. Absolute stability results for infinite-dimensional well-posed systems with applications to low-gain control. *ESAIM: Control, Optimisation and Calculus of Variations*, 5, 2000.

[50] H. Logemann and A. Mawby. *Low-gain integral control of infinite-dimensional regular linear systems subject to input hysteresis.* In *Advances in mathematical systems theory.* Colonius, F. *ed.* Birkhäuser, 2001. 255–293.

[51] H. Logemann and D. H. Owens. Input-output theory of high-gain adaptive stabilization of infinite-dimensional systems with non-linearities. *International Journal of Adaptive Control and Signal Processing*, 2:193–216, 1988.

[52] H. Logemann and E. P. Ryan. Time-varying and adaptive discrete-time low-gain control of infinite-dimensional linear systems with input nonlinearities. *Mathematics of Control, Signals and Systems*, To appear.

[53] H. Logemann and E. P. Ryan. Time-varying and adaptive integral control of infinite-dimensional regular linear systems with input nonlinearities. *SIAM Journal of Control and Optimization*, 38:1120–1144, 2000.

[54] H. Logemann, E. P. Ryan, and S. Townley. Integral control of infinite-dimensional linear systems subject to input saturation. *SIAM Journal On Control And Optimization*, 36(6):1940–1961, 1998.

[55] H. Logemann and S. Townley. Adaptive stabilization without identification for gdistributed parameter systems: an overview. *IMA Journal of Math. Control and Information*, 14:175–206, 1997.

[56] J. W. Macki, P. Nistri, and P. Zecca. Mathematical models for hysteresis. *SIAM Review*, 35:94–123, 1993.

[57] I. Mareels and J. W. Polderman. *Adaptive systems an introduction.* Birkhäuser, 1996.

[58] I. Mareels, S. Van Gils, J. W. Polderman, and A. Ilchmann. Asymptotic dynamics in adaptive gain control. In Paul M. Frank, editor, *Advances in control: highlights of ECC'99*, pages 29–63. Springer, 1999.

[59] J. E. Marshall, H. Górecki, K. Walton, and A. Korytowski. *Time-delay systems stability and performance criteria with applicatons.* Ellis Horwood, 1992.

[60] D. Matko, N. Yajima, and M. Hinada. Adaptive stabilization of lateral motion of a balloon gondola. *Control Engineering Practice*, 5(6):763–769, 1997.

[61] D. E. Miller and E. J. Davison. An adaptive controller which provides an arbitrary good transient and steady-state response. *IEEE Trans. Autom. Control*, AC-36:68–81, 1991.

[62] A. S. Morse. Recent problems in parameter adaptive control. *Proc. CNRS Colloquim on Development and Utilization of Mathmatical Models in Automatic Control*, Sept. 1982.

[63] A. S. Morse. An adaptive control for globally stabilizing linear systems with unknown high-frequency gains. *Lecture Notes in Control and Information Science*, 62:58–68, 1984.

[64] B. Mårtensson. The unmixing problem. *IMA Journal of Mathematical Control and Information*, 8(4):367–377, 1991.

[65] R. D. Nussbaum. Some remarks on a conjecture in parameter adaptive control. *Systems and Control Letters*, 3:243–246, 1983.

[66] B. E. Paden and S. S. Sastry. A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators. *IEEE Transaction on Circits and Systems*, 34(1):73–82, 1987.

[67] A. Pazy. *Semigroups of linear operators and applications to partial differential equations.* Springer, 1983.

[68] E. P. Ryan. On Brocketts condition for smooth stabilizability and its necessity in a context of nonsmooth feedback. *SIAM Journal On Control Optimization*, 32:1597–1604, 1994.

[69] E. P. Ryan. A universal servomechanism for a class of nonlinear planar systems. *Applied Mathematics Computation*, 78:229–243, 1996.

[70] E. P. Ryan. An integral invariance principle for differential inclusions with applications in adaptive control. *SIAM Journal of Control and Optimization*, 36:960–980, 1998.

[71] E. P. Ryan and C. J. Sangwin. Functional differential inclusions and adaptive stabilization. *International Journal of Control*, To appear.

[72] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443, 1989.

[73] E. D. Sontag. On the input-to-state stability property. *European Journal of Control*, 1:24–36, 1995.

[74] E. D. Sontag. Comments on integral variants of iss. *Systems and Control Letters*, 34:93–100, 1998.

[75] E. D. Sontag. *Mathematical control theory, deterministic finite dimensional systems*. Springer, 2nd edition, 1998.

[76] E. D. Sontag. Stability and stabilization: Discontinuities and the effect of disturbances. In F.H. Clarke and R.J. Stern, editors, *Nonlinear Analysis, Differential Equations, and Control (Proc. NATO Advanced Study Institute, Montreal)*, pages 551–598. Kluwer, Dordrecht, Jul/Aug 1999.

[77] E. D. Sontag and H. J. Sussmann. Remarks on continuous feedback. *Proceedings IEEE Conference Decision and Control*, pages 916–921, December 1980.

[78] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems and Control Letters*, 24:351–359, 1995.

[79] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property with respect to compact sets. In *Proceedings of IFAC Non-Linear Control Systems Design Symposium (NOLCOS '95), Tahoe City, CA*, pages 226–231, June 1995.

[80] E. D. Sontag and Y. Wang. New characterizations of the input-to-state stability property. *IEEE Transations on automatic control*, 41:1283–1294, 1996.

[81] E. D. Sontag, Y. Wang, and D. Angeli. A characterization of integral input to state stability. *IEEE Transactions Automatic Control*, to appear.

[82] C. Sparrow. *The Lorenz equations: bifurcations, chaos and strange attractors*, volume 41 of *Applied Mathematical Sciences*. Springer, 1982.

[83] A. Strauss. Liapunov functions and l^p solutions of differential equations. *Transactions of the American Mathematical Society*, 119:37–50, 1965.

[84] A. J. Weir. *Lebesgue integration and measure.* Cambridge, 1973.

[85] G. Weiss. Transfer-functions of regular linear-systems, part 1: Characterization of regularity. *Transactions of American Mathematical Society*, 342:827–854, 1994.

[86] J. C. Willems and C. I. Byrnes. Global addaptive stabilization in the absence of information on the sign of the high frequency gain. *Lecture Notes in Control and Information Science*, 62:49–57, 1984.