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Branching Diffusion Processes

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Summary

In this thesis we study aspects of three branching diffusion models: a branching Brownian motion with an absorbing barrier; a typed branching Brownian motion, in which each particle has a 'type' that controls its spatial variance and breeding rate; and branching Brownian motion in a quadratic breeding potential. Within the context of these processes, we present results on a range of themes from the literature on branching diffusions. The major topics of interest are: links with differential equations, particularly travelling wave solutions; questions on exponential growth or eventual extinction in different regions of the domain of a process; and the rate of spatial spread of a process.

Additive martingales are fundamental to the study of branching diffusions, both for their convergence properties and their use in changes of measure, and consequently such martingales are used heavily throughout this thesis. Particularly important for us is the use of additive martingales in changes of measure known as spine constructions; in recent years this technique has been used to great effect in providing intuitive proofs of many important classical results, as well new ones, in the theory of branching processes.

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Contents

Chapter 1

Introduction

In this introductory chapter we discuss the themes in the branching-diffusion literature that are the focus of this thesis, and describe some of the most relevant work that has been done in these areas. Our intention here is to introduce some important concepts in the theory of branching processes, and also to provide motivation for the problems considered later on; the results that represent the original work of the thesis will be given chapter by chapter.

1.1 Branching Brownian motion

We begin with a description of the simplest branching diffusion — branching Brownian motion (BBM) on $\mathbb{R}$. It is a compound of three stochastic elements.

(i) The spatial motion. As the name suggests, during its lifetime each individual particle in the population diffuses as a Brownian motion on $\mathbb{R}$, independently of all the other individuals.

(ii) The branching rate, $\beta > 0$. Each individual has a lifetime which is distributed exponentially with parameter β. An individual's lifetime is independent of its spatial motion and the lifetimes of the other particles.

(iii) The offspring distribution, $\{p_k\}_{k\geq 0}$. When an individual dies, it is replaced (at the point where it died) by a random number of offspring. The number of offspring has distribution $\{p_k\}_{k\geq 0}$, and we will write $F(s) := \sum_{k=0}^{\infty} p_s s^k$ for the generating function. Conditional on their birth place and birth time, the offspring evolve independently of each other and repeat stochastically the behaviour of their parent.

There are many ways in which this basic model can be developed: the spatial motion can be an arbitrary (possibly inhomogeneous) diffusion; the branching rate can depend on the spatial motion; the offspring distribution can depend on the spatial motion; the domain can be a subset of $\mathbb{R}$ (or $\mathbb{R}^n$ for $n \geq 2$); or the particles can have a *type*, which can also evolve randomly and affect all three elements of the behaviour described above. In this thesis we will see examples of each of these modifications. For simplicity, however, beyond Chapter 1 we only consider dyadic branching, that is $p_2 = 1$, $F(s) = s^2$. Although this said, most of our results can be generalised to more complicated offspring distributions.

1.2 Branching diffusions and differential equations

The relationship between branching diffusion processes and non-linear differential equations has been a subject of much research since the landmark 1976 paper of McKean [73]. The probabilistic study of differential equations is actually much older than this — for example, it has been known since Kakutani [55] that the solution to the Dirichlet problem on a smooth bounded domain $D \subset \mathbb{R}^n$

$$\begin{aligned} \Delta v &= 0 \text{ in } D, \\ v &= f \text{ on } \partial D, \end{aligned}$$

can be expressed as

$$v(x) = \mathbb{E}^x f(B_\tau).$$

Here B is a Brownian motion started at the point $x \in D$, and τ is the first exit time of the Brownian motion from D.

Using branching diffusions rather than a single-particle Brownian motion we will shortly generalise this type of representation to reaction-diffusion equations, but we first note another important piece of classical theory — the Feynman-Kac formula. This states that for bounded $g(t,x) \in C([0,\infty) \times \mathbb{R}^n)$ and $f(x) \in C(\mathbb{R}^n)$,

$$v(t,x) := \mathbb{E}^x\Big(f(B_t) \exp\Big(\int_0^t g(t-s, B_s)\,\mathrm{d}s\Big)\Big)$$

is in $C^{1,2}((0,\infty)\times\mathbb{R}^n)$ and satisfies

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + gv \text{ in } (0,\infty)\times\mathbb{R}^n, \\ v(0,x) &= f(x) \text{ in } \mathbb{R}^n.\end{aligned} \tag{1.1}$$

(The stipulation that g and f be bounded is not the most general condition under which this result holds, and the formula also generalises to other diffusions.) Observe that if we set $g \equiv 0$ then (1.1) reduces to the heat equation, and as a solution we have $v(t,x) = \mathbb{E}^x(f(B_t))$. A variety of other parabolic and elliptic differential equations have probabilistic representations for their solutions; Durrett [25] contains a good discussion of some of these.

In a similar spirit, McKean [73, 74] gave a representation for the solution of a non-linear reaction-diffusion equation in terms of an expectation with respect to a branching Brownian motion. For the branching Brownian motion of Section 1.1, let N_t be the set of particles alive at time t, and write $X_u(t)$ for the spatial position of particle $u \in N_t$.

Theorem 1.2.1 (McKean representation). *For $f : \mathbb{R} \to [0,1]$, define*

$$u(t,x) := E^x\Big(\prod_{u\in N_t} f(X_u(t))\Big),$$

where E^x is the expectation with respect to P^x, the law of the branching Brownian motion started from a single particle at $x \in \mathbb{R}$. Then $u \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and satisfies

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \beta(F(u) - u) \\ u(0,x) &= f(x).\end{aligned} \tag{1.2}$$

This result is actually implicit in the work of Skorohod [83], but is generally credited to McKean. If we set $\beta = 0$, i.e. we stop any breeding in the BBM, then the McKean representation collapses to the solution of the heat equation. Taking $F(s) = s^2$ makes (1.2) the famous Fisher-Kolmogorov-Petrovski-Piscounov (FKPP) equation of population genetics. McKean used the representation of Theorem 1.2.1 to study particular solutions for the FKPP equation known as *travelling waves*. We will look at this in more detail in the following sections.

1.2.1 KPP equations and travelling waves

The Kolmogorov-Petrovski-Piscounov (FKPP) equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial t^2} + F(u), \tag{1.3}$$

for $u \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and $F \in C(\mathbb{R}^+)$, has a long history of study involving both analytic and probabilistic methods. This equation arose in biological models for the spread of advantageous genes through a population (see Fisher [36]), and so it is not surprising that wave-like solutions to the equation are of fundamental importance. In their 1937 paper Kolmogorov, Petrovski, and Piscounov [63] gave the first important results in this direction, two of which we summarise below.

Theorem 1.2.2 (Kolmogorov, Petrovski, Piscounov 1937). *(i) Suppose F defined on $[0,1]$ satisfies the conditions*

$$\begin{aligned} &F(0) = F(1) = 0; & &F(u) > 0, \quad 0 < u < 1; \\ &F'(0) = a > 0; & &F'(u) < a, \quad 0 < u \leq 1. \end{aligned}$$

Then solutions to the partial differential equation (1.3), with any initial condition satisfying $0 \leq u(0,x) \leq 1$, remain bounded in $[0,1]$ for all time: $0 \leq u(t,x) \leq 1$ for all $t \geq 0$.

(ii) As $t \to \infty$, solutions to (1.3) with Heaviside initial conditions will propagate on $\mathbb{R}$ at a constant speed c and assume a limiting shape. By this we mean $u(t, x+ct) \to v(x)$ as $t \to \infty$, where $v \in C^2(\mathbb{R})$ satisfies

$$c\frac{\mathrm{d}v}{\mathrm{d}x} = k\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + F(v) \tag{1.4}$$

with $v(x) \to 0$ as $x \to -\infty$ and $v(x) \to 1$ as $x \to +\infty$. This is known as the travelling wave solution of speed c, *and we refer to (1.4) as the* travelling-wave equation *for the reaction-diffusion equation (1.3).*

There is a vast literature on the analytic study of reaction-diffusion equations and wave-like solutions. Comprehensive summaries can be found in Britton [17] or Murray [75].

1.2.2 Branching Brownian motion and the FKPP equation

The probabilistic interest in FKPP-type equations dates from McKean's work [73, 74], which expressed for the first time the solution of an FKPP equation and the associated travelling-wave equation probabilistically. McKean showed that, for certain bounded initial conditions, solutions of FKPP equations can be expressed in terms of an expectation taken over all the particles in a branching Brownian motion (Theorem 1.2.1); and also that travelling wave solutions, when they exist, can be written as the Laplace transform of a martingale limit. Since McKean's work there has been much probabilistic analysis of the FKPP and related equations, for example Bramson [15, 16], Uchiyama [84], Neveu [76], Freidlin [37], Chauvin and Rouault [20, 21], Harris [46], and Kyprianou [66] to name just a few.

The starting point for our work in Chapter 2 is the FKPP equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \beta u(u-1), \tag{1.5}$$

where $\beta > 0$ and $u = u(t,x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$. The term $\beta(u^2 - u)$ belongs to the class of functions F in equation (1.3) studied by Kolmogorov *et al.* [63], and this particular reaction-diffusion equation has received much attention from both analysts and probabilists. We will now describe briefly some of the most important probabilistic work.

Travelling wave solutions of this equation have the form

$$u(t,x) = f(x - \rho t),$$

where $f \in C^2(\mathbb{R})$ and satisfies

$$\begin{aligned} \frac{1}{2}f'' + \rho f' + \beta f(f-1) &= 0 \text{ on } \mathbb{R}, \\ f(-\infty) &= 0, \\ f(\infty) &= 1. \end{aligned} \tag{1.6}$$

In this thesis we only consider monotone travelling waves, although non-monotonic travelling waves also exist and are of great interest to mathematical biologists — see Murray [75] or Britton [17] for example. It is very well known that monotonic solutions of the system (1.6) exist for all speeds $\rho \geq \sqrt{2\beta}$, and that these solutions are also unique up to translation. For $-\infty < \rho < \sqrt{2\beta}$ there are no monotone travelling wave solutions of speed ρ.

With the representation of Theorem 1.2.1 McKean was able to provide simplified proofs of the results of Kolmogorov *et al.* [63], and also extend some of them — in particular McKean showed that so long as the initial conditions have a certain asymptotic, the solution will converge to the travelling wave.

The McKean representation has a particularly elegant form if we take the Heaviside initial condition $u(0, x) = \mathbf{1}_{\{x>0\}}$, in which case

$$\begin{aligned} u(t,x) &= E^x\Big(\prod_{u\in N_t} \mathbf{1}_{\{X_u(t)>0\}}\Big) = P^x(X_u(t) > 0, \forall u \in N_t) \\ &= P^0(X_u(t) > -x, \forall u \in N_t) = P^0(L_t > -x) = P^0(R_t < x), \qquad (1.7) \end{aligned}$$

where $L_t := \inf_{u\in N_t} X_u(t)$ and $R_t := \sup_{u\in N_t} X_u(t)$ are, respectively, the left- and right-most particles in the branching Brownian motion. The behaviour of the right-most particle plays a leading role in BBM studies of the FKPP equation — see Bramson [15], and Chauvin and Rouault [20, 21]. In particular we note that the spatial spread of BBM is asymptotically linear with speed $\sqrt{2\beta}$, by which we mean that

$$\lim_{t\to\infty} \frac{R_t}{t} = \sqrt{2\beta}$$

almost surely. We also note that the speed of the right-most particle corresponds to the threshold speed for existence of travelling waves.

Associated with the BBM are the additive martingales

$$Z_\lambda(t) := \sum_{u\in N_t} e^{\lambda X_u(t) - (\frac{1}{2}\lambda^2+\beta)t},$$

which are defined for all $\lambda \in \mathbb{R}$. These martingales are uniformly integrable if and only if $|\lambda| < \sqrt{2\beta}$, in which case $Z_\lambda(\infty) > 0$ almost surely; and if $|\lambda| \geq \sqrt{2\beta}$ then $Z_\lambda(\infty) = 0$ with probability one (Neveu [76]). These martingales can be used to establish both existence and uniqueness for solutions of the system (1.6), and for $\rho > \sqrt{2\beta}$ the unique (up to translation) travelling wave is

$$f(x) = E^x \exp(-Z_\lambda(\infty)).$$

We also observe that, by symmetry, there exist monotone travelling waves of speeds $\rho < -\sqrt{2\beta}$ which *decrease* from 1 to 0. The case $\rho = \sqrt{2\beta}$ requires a more delicate analysis using the 'derivative' martingale

$$Z'_\lambda(t) := \frac{\partial}{\partial\lambda} Z_\lambda(t) = \sum_{u\in N_t} (X_u(t) - \lambda t) e^{\lambda X_u(t) - (\frac{1}{2}\lambda^2+\beta)t},$$

full details of which can be found in Harris [46].

Thus far we have discussed the 'classical' travelling-wave problem, which has been studied since McKean's initial paper [73]; there have been numerous extensions of the theory to related problems, one of which forms Chapter 2 of this thesis, but before moving on to this we highlight two recent studies of the FKPP equation (1.5) that are of particular relevance to Chapter 2. The first *entirely probabilistic* proofs of the results described in this section so far were given in Harris [46], the key innovation being a probabilistic derivation of a large-x asymptotic for travelling waves (when they exist), from which uniqueness modulo translation follows. Following this Kyprianou [66] gave an alternative probabilistic analysis of the same problem using spine constructions, where the martingales Z_λ and Z'_λ were used to change measure on the probability space of the BBM. Spine techniques are central to many of the proofs in this thesis, and a detailed introduction to these ideas is given in Section 1.3.

1.2.3 The one-sided FKPP system

Since 1976 much effort has gone into the extension of McKean's ideas to other partial and ordinary differential equations. We devote Chapter 2 to a probabilistic analysis of the modification of the system (1.6) given below, which we shall refer to as the *one-sided FKPP system.*

$$\begin{aligned} \frac{1}{2}f'' - \rho f' + \beta f(f-1) &= 0 \text{ on } (0,\infty), \\ f(0+) &= 0, \\ f(\infty) &= 1. \end{aligned} \tag{1.8}$$

Both ρ and β are strictly positive constants. If $f : (0,\infty) \to \mathbb{R}$ is a solution of the above system, then $f(x+\rho t)$ solves the FKPP equation (1.5). We note, however, that $f(x+\rho t)$ is not a travelling wave in the sense of Theorem 1.2.2: it is not the limiting shape of any solutions of (1.5).

We study the existence and uniqueness for solutions to the system (1.8) by considering a branching Brownian motion with absorption. This is a dyadic branching

Brownian motion in which the spatial motion of the particles is a Brownian motion with drift $-\rho < 0$, the breeding rate is $\beta > 0$, and particles are killed (removed from the process) on hitting the origin. Clearly there is a chance that this process can become extinct, and we will denote by ζ the extinction time, so that $\{\zeta = \infty\}$ is the event that the BBM survives forever. It is the study of the extinction probabilities $P^x(\zeta < \infty)$ that yields existence and uniqueness for (1.8); and we prove a large-x asymptotic for the one-sided travelling wave, when it exists, using martingales for a single-particle Brownian motion. We are also able to prove the following asymptotic result for the right-most particle in the killed BBM.

Lemma 1.2.3. *Suppose $\rho < \sqrt{2\beta}$. For all $x > 0$ we have*

$$\lim_{t\to\infty} \frac{R_t}{t} = \sqrt{2\beta} - \rho,$$

P^x-almost everywhere on $\{\zeta = \infty\}$.

This is telling us that, to order t, on $\{\zeta = \infty\}$ the asymptotic speed of the right-most particle in the killed BBM is the same as it would be if we did not have the absorbing barrier at the origin.

With the asymptotic speed of the right-most particle in the killed BBM established, one naturally wonders if anything be said about the rate of growth in the numbers of particles travelling at slower speeds, or if we can find the rate of decay of the probability that the right-most particle travels at a faster speed. These questions motivate the work in Chapter 3, in which we consider particles with spatial positions above the spatial ray of gradient $\lambda > 0$. For the 'super-critical' region $0 < \lambda < \sqrt{2\beta} - \rho$, we find the almost-sure asymptotic rate of exponential growth in the number of particles situated above the spatial ray of gradient λ; and in the 'sub-critical' region $\lambda > \sqrt{2\beta} - \rho$ we find the asymptotic rate of decay of the probabilities $P^x(R_t > \lambda t + \theta)$, where $\theta \geq 0$ is some constant. Our study of the probabilities $P^x(R_t > \lambda t + \theta)$ draws on work of Chauvin and Rouault [20] for standard branching Brownian motion — their work is discussed in some detail in Section 1.3.2 below.

The work in Chapters 2 and 3 is joint with S. C. Harris and A. E. Kyprianou, and has appeared in Harris *et al.* [45].

A brief note on the more general theory

This introductory section has barely scratched the surface of the vast body of work on the probabilistic analysis of ordinary and partial differential equations. Comprehensive surveys of the relationships between branching diffusions and differential equations can be found in, to cite just a few examples, Freidlin [37], Dynkin [28], and Ikeda *et al.* [50, 51, 52]. Another area of great importance in the probabilistic literature is the use of superprocesses in the analysis of differential equations. Similar issues, such as the representation of solutions probabilistically, arise there; but superprocesses are not the object of study of this thesis and so for an account of this subject we refer the reader to Dynkin [27, 29], Le Gall [69], Etheridge [33], and references therein.

1.3 Introduction to the spine approach

During the last decade or so *spine constructions* have become an important tool in the study of branching processes — for Galton-Watson processes, branching random walks and superprocesses, as well as branching diffusions. In this section we will give a few examples of spine constructions, and describe some applications. We give particular attention to the results of Chauvin and Rouault [20], which motivated many of our results in Chapters 3 and 4 on BBM with absorption at the origin.

1.3.1 The spine construction for the Galton-Watson process

A spine construction for a branching process is a change of measure that only alters the behaviour of the single original individual: all subtrees branching off from the 'spine' behave as they did under the original measure. This idea was first formalised in the context of branching Brownian motion by Chauvin and Rouault [20, 21].

Although a few earlier papers — and several later ones, e.g. Chauvin *et al.* [22], and Waymire and Williams [86, 87] — used size-biased tree constructions, the spine ideas were not fully exploited until the relatively recent series of papers Lyons *et al.* [72], Lyons [71], and Kurtz *et al.* [65]. These papers all considered discrete-time processes (Galton-Watson processes and the branching random walk), and as a first example of a spine construction we give the spine change of measure for a standard Galton-Watson process, taken from Lyons *et al.* [72].

We start with a single individual in generation 0. At each subsequent time-step every individual alive divides independently into a random number of offspring, X, with the number of offspring distributed according to $\mathbb{P}(X = k) = p_k$, for $k \in \{0, 1, 2, \ldots\}$. We define $m := \mathbb{E}X < \infty$, and denote the law of this process by $\mathbf{GW}$.

Lyons *et al.* [72] defined a change of measure on the probability space of the Galton-

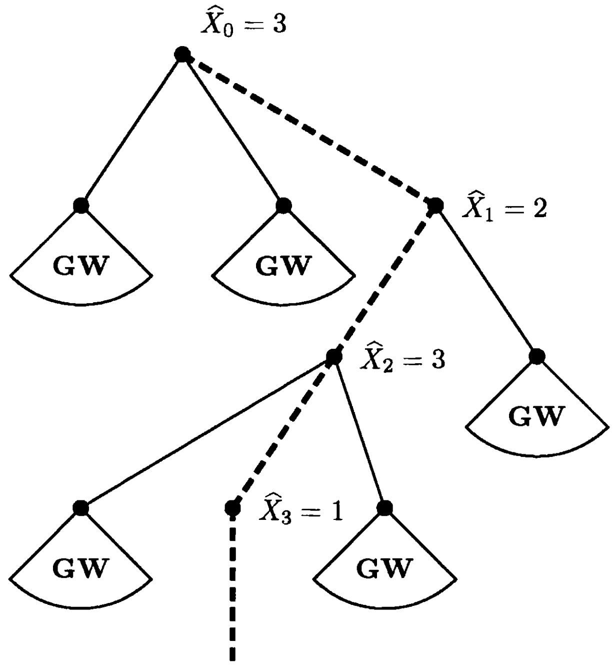
Watson process via

$$\left.\frac{\mathrm{d}\widehat{\mathbf{GW}}}{\mathrm{d}\mathbf{GW}}\right|_{\mathcal{F}_n} = \frac{Z_n}{m^n} =: M_n. \tag{1.9}$$

Here $(\mathcal{F}_n)_{n\geq 0}$ is the natural filtration. Under the measure $\widehat{\mathbf{GW}}$, the initial particle gives birth to a random number of offspring according to the *size-biased* distribution $\widehat{X}$, where

$$\mathbb{P}(\widehat{X} = k) := \widehat{p}_k = kp_k/m$$

and $k \in \{0, 1, 2, \dots\}$. One of these offspring is selected uniformly at random, and this chosen offspring repeats stochastically the behaviour of its parent; all other offspring behave as they did under the measure $\mathbf{GW}$. Notice that $\mathbb{P}(\widehat{X} = 0) = 0$, so the spine is immortal under the measure $\widehat{\mathbf{GW}}$. Writing $\widehat{X}_n$ for the number of offspring the spine gives birth to in generation $n \in \{0, 1, 2, \dots\}$, this construction of the process under the law $\widehat{\mathbf{GW}}$ can be easily understood with the aid of the graphical representation below.



The line of descent marked with the dashed line produces offspring according to the size-biased distribution $\widehat{X}$. This is the spine.

Lyons *et al.* [72] used this spine construction to give elegant and intuitive proofs of some classical results of Kesten and Stigum [60], Heathcote *et al.* [49], and Kesten *et al.* [59] for discrete-time branching processes. Another observation from their paper that is of relevance to our later work is the following. Let $m \leq 1$, so that the process will $\mathbf{GW}$-almost surely become extinct in finite time, and then for $s \in \mathbb{Z}^+$ and $A \in \mathcal{F}_s$ we have

$$\lim_{n\to\infty} \mathbf{GW}(A|Z_{s+n} > 0) = \widehat{\mathbf{GW}}(A). \tag{1.10}$$

So we can think of $\widehat{\mathbf{GW}}$ as the law of a sub-critical Galton-Watson process 'conditioned to survive forever'. The interpretation of a spine construction as the limit obtained when conditioning on an event that has probability tending to zero appears in Chapters 3 and 4.

1.3.2 A first spine construction for branching Brownian motion

Spine constructions for branching diffusions arise from changing measure with an additive martingale. To give an example we consider changing measure with the martingale Z_λ, which gives rise to a spine construction for branching Brownian motion. Using the notation from Section 1.2.2 we define the change of measure

$$\left.\frac{\mathrm{d}\pi_\lambda^x}{\mathrm{d}P^x}\right|_{\mathcal{F}_t} := \frac{Z_\lambda(t)}{Z_\lambda(0)}.$$

Here $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration; by this we mean $\mathcal{F}_t$ is the σ-field that 'encodes information up to time t'. Under the measure π_λ^x the BBM may be reconstructed in law as follows:

- starting from position x, the initial ancestor diffuses according to a Brownian motion with drift λ;
- at rate 2β the particle undergoes fission producing two particles;
- one of these particles is selected at random, each with probability one half;
- this chosen particle repeats stochastically the behaviour of their parent;
- the other particle initiates from its birth position an independent copy of a standard BBM with law $P^\cdot$, and so on.

This is a special case the construction of Chauvin and Rouault [20], which allowed random numbers of offspring. If we allow random family sizes then the martingale Z_λ has a very slightly modified form, and under the new measure the offspring distribution for the spine is size-biased — as was the case with the Galton-Watson process. To properly justify the behaviour under the measure π_λ^x requires significantly more notation than we have developed so far, and a careful construction of several different filtrations on the probability space of the BBM; full details of this are given in Chapter 3.

We recall from Section 1.2.2 that, for any $x \in \mathbb{R}$, under the measure P^x the right-most particle in the BBM has the following asymptotic:

$$\lim_{t\to\infty} \frac{R_t}{t} = \sqrt{2\beta} \qquad P^x\text{-almost surely.} \tag{1.11}$$

This can be proved using the Z_λ martingales, or alternatively by using the relationship between the position of the right-most particle and the FKPP equation given at (1.7) (see, for example, Chauvin and Rouault [20]). In order to describe the results of Chauvin and Rouault [20] we require the counting function

$$N_t(a,b) := \sum_{u\in N_t} \mathbf{1}_{\{X_u(t)\in(a,b)\}},$$

for $a, b \in [-\infty, +\infty]$. Chauvin and Rouault proved the following result on the probability of finding particles that have travelled at speeds greater than $\sqrt{2\beta}$: for $\lambda > \sqrt{2\beta}$ and $\theta \in \mathbb{R}$,

$$P^x(N_t(\lambda t+\theta, +\infty) > 0) \underset{t\to+\infty}{\sim} C\lambda E^x N_t(\lambda t + \theta, +\infty) > 0, \tag{1.12}$$

where $C \in \mathbb{R}$ is a constant that does not depend on x. Now

$$\{N_t(\lambda t+\theta, +\infty) > 0\} = \{R_t > \lambda t + \theta\},$$

and the McKean representation can be used to show that

$$u(t,x) := 1 - P^x(N_t(\lambda t + \theta, +\infty) > 0)$$

is a solution of the FKPP equation (1.5) with initial condition $u(0,x) := \mathbf{1}_{\{x<\theta\}}$. The Feynman-Kac formula allows us to express the right-hand side of (1.12) as a solution of the associated *linear* partial differential equation

$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2} + \beta w, \qquad w(t,x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}),$$

so that (1.12) may also be viewed as a statement about certain solutions of non-linear partial differential equations being asymptotically equal to a constant multiple of a solution of an associated linear problem.

Since $\lambda > \sqrt{2\beta}$ we know that

$$P^x(N_t(\lambda t + \theta, +\infty) > 0) \xrightarrow[t\to\infty]{} 0,$$

and so it is natural to consider whether or not a Yaglom-type result holds. Chauvin and Rouault [20] proved that there exists a probability distribution $(\Pi_i)_{i\geq 0}$ on $\mathbb{N}$ such that

$$\lim_{t\to\infty} P^x\Big(N_t(\lambda t + \theta, +\infty) = i \Big| N_t(\lambda t + \theta, +\infty) > 0\Big) = \Pi_i,$$

and that this distribution has finite expectation $1/C\lambda$.

Finally, the spine construction appears as the limit of the law of the process conditioned on non-extinction above spatial rays of gradient $\lambda > \sqrt{2\beta}$; that is for $s > 0$, and any $A \in \mathcal{F}_s$,

$$\lim_{t\to\infty} P^x\Big(A \Big| N_{t+s}(\lambda(t+s), +\infty) > 0\Big) = \pi^x_\lambda(A).$$

Compare this with (1.10): as we mentioned above, conditioning on a null event can often be interpreted using a spine change of measure. We shall return to this idea in Chapters 3 and 4.

Much of Chapter 3 is devoted to proving results for the branching Brownian motion with absorption that are analogous to the Chauvin and Rouault [20] results for standard BBM. Although some of the proofs for standard branching Brownian motion adapt easily to the killed process, significant refinements are necessary in places. In particular, Z_λ is not a martingale for the killed branching Brownian motion, and so we introduce new families of additive martingales for this process. Naturally, the close connection to standard BBM means that we can (and do) make use of some existing techniques, but, nevertheless, adding killing at the origin yields some significant new problems.

The idea that a spine change of measure, suitably interpreted, can be seen as a conditioning of a branching process on a null event motivates the results in Chapter 4. We will see in Chapter 2 that the extinction time for the branching Brownian motion with absorption is almost-surely finite, that is $P^x(\zeta < \infty) = 1$, if $\rho \geq \sqrt{2\beta}$; and in this case we say that the process is sub-critical. Noting that survival of the process until time t is equivalent to the right-most particle, R_t, being strictly positive, one might then ask what can be said about the conditioned law $P^x(A|R_{s+t} > 0)$ for $A \in \mathcal{F}_s$, as $t \to \infty$. The first step in Chauvin and Rouault [20], and also in Chapter 3, is to prove the asymptotic for the probability $P^x(R_t > \lambda t + \theta)$. Following this strategy, the main result in Chapter 4 is an asymptotic result for the survival probability $P^x(R_t > 0)$ in the case $\rho > \sqrt{2\beta}$. Although this looks a very similar result to the asymptotic for $P^x(R_t > \lambda t + \theta)$, a significantly different approach is needed in the proof. Once we have this asymptotic result, the sub-critical process 'conditioned to survive forever' can be interpreted as a spine construction in which the spine behaves like a Bessel-3 process.

The work in Chapter 4 appears in Harris and Harris [44].

1.3.3 More applications of spines

Thus far we have discussed spines primarily in the context of null-conditioning for branching processes. However the appeal of the spine approach is due mainly to the way it has yielded intuitively simple, elegant proofs of both new and old results in the theory of branching processes. We now mention briefly some other applications of spine techniques that appear in this thesis.

Martingale convergence theorems

The fundamental role of additive martingales in the theory of branching processes means that there is much interest in results on the convergence in $\mathcal{L}^p$-spaces of these martingales, for $p \geq 1$. The Lyons *et al.* papers [65, 71, 72] used a dichotomy between the behaviour of the limit of the change-of-measure martingale under the original measure as compared to the spine measure to give short proofs of results from Kesten and Stigum [60, 61], and of Biggins [5]. Kyprianou [66] used similar ideas in the context of branching Brownian motion to give new derivations of the $\mathcal{L}^1$ convergence properties of the Z_λ martingales. Hardy and Harris [42] refined the method yet further to tackle $\mathcal{L}^p$ convergence in the case $p > 1$.

These authors combine the aforementioned measure-theoretic dichotomy with another powerful spine technique — the decomposition of an additive martingale by taking a suitable conditional expectation. Introduced in Lyons [71], this has the effect of reducing calculations involving all particles in a branching process to one-particle calculations involving the spine only. Kyprianou and Rahimzadeh Sani [68], Kuhlbusch [64], Olofsson [77], Biggins and Kyprianou [11], and Athreya [3] provide further applications of related spine techniques.

We use a modification of the approach of Hardy and Harris [42] in Chapter 4, when studying the convergence of additive martingales for the branching Brownian motion with absorption.

Path-wise large deviations

Hardy and Harris [43] used spine techniques to give a 'conceptually simple' proof of a large-deviations result of Lee [70]. Loosely speaking, this result gives the rate of decay (to exponential order) of the probability that a single particle in a branching Brownian motion follows a 'deviant' path — i.e. a generalisation to BBM of Schilder's theorem for a single Brownian motion. The main innovation of Hardy and Harris was the use of a spine change of measure to prove a difficult lower bound for the probability that deviant paths are followed.

The generalised Many-to-One lemma

In the calculation of expectations involving sums over the particles in a branching diffusion, the use of 'Many-to-One' ideas is at least as old as Sawyer [81]. For branching Brownian motion the Many-to-One lemma states that, for measurable $g : \mathbb{R} \to \mathbb{R}$,

$$P^x\bigg(\sum_{u\in N_t} g(X_u(t))\bigg) = e^{\beta t}\mathbb{P}^x(g(Y_t)),$$

where Y is a single Brownian motion under $\mathbb{P}$. This is the formalisation of the intuitive notion that expectations of sums over the branching particles are equal to the expectation for a single particle, scaled up by the expected population size.

A significant generalisation has been given recently in Hardy and Harris [41]. Hardy and Harris showed that we can actually allow f to be a function of the paths $\{Y_u(s)\}_{0\leq s\leq t}$, rather than merely the time t position of the particles. Their result also holds for typed branching diffusions, general offspring distributions, and position-dependent breeding rates.

The Hardy and Harris proof of the general Many-to-One lemma relies on their improved formulation of the underlying tree-space for spine techniques. This has several advantages over the constructions of Lyons *et al.* [65, 71, 72] and Kyprianou [66]. We will look at the differences in Section 3.2, when we introduce the full notation for spines.

1.4 Some other branching diffusion models

In this section we briefly mention some other directions in which the basic BBM model has been developed, and introduce the models that are studied in Chapters 5 and 6.

1.4.1 Typed branching diffusions

A multi-type branching process is a process in which particles can be of different *types*, and the type of the particle affects its offspring distribution and (in spatial processes) its spatial movement. The multi-type Galton-Watson process and multi-type branching random walk are well-studied random processes — in addition to their theoretical importance, multi-type models in biological applications can capture variation within a population.

In contrast there has been comparatively little work done on typed branching diffusions, perhaps because even the simplest multi-type behaviour can add significantly to the difficulty of the analysis. Asmussen and Hering [2] studied branching diffusions on bounded domains with a general set of types, but in this thesis we will concentrate on a particular family of typed branching diffusions for which the space and type motions are both diffusions on $\mathbb{R}$.

To give an example of a multi-type branching diffusion we will describe the model of Champneys *et al.* [18]. Let $a_1, a_2, r_1, r_2, q_1, q_2$ be fixed positive constants. Let θ be a positive parameter, which we can think of as the 'temperature' of the system, and define the matrices

$$A := \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \qquad R := \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \qquad Q := \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}.$$

Retaining the notation of Section 1.2.2 let N_t be the set of particles alive at time at time t, and let $X_u(t)$ denote the spatial position at time t of particle $u \in N_t$. In addition each particle alive at time t now has a type $Y_u(t) \in I := \{1, 2\}$, which evolves as autonomous Markov chain on I with Q-matrix θQ. Whilst a particle is of type $y \in I$, its spatial position moves as a driftless Brownian motion on $\mathbb{R}$ with constant variance coefficient a_y, and it undergoes dyadic branching at rate r_y. When a new particle is born, it inherits its parent's spatial position and type. Once born, particles are immortal and behave independently of one another.

Using this process Champneys *et al.* undertook a probabilistic analysis of travelling wave solutions for the coupled system of FKPP equations

$$\frac{\partial u}{\partial t} = \frac{1}{2} A \frac{\partial^2 u}{\partial x^2} + Ru(u-1) + \theta Q u, \tag{1.13}$$

where $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ so that, for example,

$$\frac{\partial u_1(t,x)}{\partial t} = \frac{1}{2} a_1 \frac{\partial^2 u_1(t,x)}{\partial x^2} + r_1 u_1(t,x)(u_1(t,x) - 1) - \theta q_1 u_1(t,x) + \theta q_1 u_2(t,x).$$

Many of the results for the classical FKPP equation have analogues here: for example, solutions of (1.13) that are bounded in $[0,1]$ can be shown to have a McKean representation, and there exists a threshold speed for the existence of monotone travelling waves connecting 0 to 1. A travelling wave (of speed c) here is a solution of the form $u(t,x) = w(x - ct)$, where $w : \mathbb{R} \to \mathbb{R}^2$ is monotone increasing and satisfies

$$\frac{1}{2} A w'' + cw' + Rw(w-1) + \theta Q w = 0 \tag{1.14}$$

with $w(-\infty) = (0,0)$ and $w(\infty) = (1,1)$. Champneys *et al.* [18] showed that such solutions exist for all sufficiently large speeds. As was the case for standard BBM, the probabilistic interpretation of the critical speed for existence of travelling waves is that it corresponds to the almost-sure asymptotic speed of the right-most particle in the branching diffusion.

Hardy [40] extended the methods of Champneys *et al.* [18] to study a coupled system of N reaction-diffusion equations, and then the particles in the related branching diffusion have a type that evolves as a Markov chain on the enlarged (but still finite) state space $I = \{1, 2, \ldots, N\}$.

In Chapter 5 we study a typed branching diffusion for which the type-space I is now $\mathbb{R}$, and the type evolves as an ergodic Ornstein-Uhlenbeck process rather than a Markov chain. The branching mechanism is also modified from the typed models described above: we still have dyadic branching, but this branching now occurs at rate that is a quadratic function of the particle's type position $y \in \mathbb{R}$. In the main results of Chapter 5 we find the almost-sure asymptotic 'shape' (in $\mathbb{R}^2$) of this branching diffusion, and also the almost-sure exponential growth rate of the number of particles at a given space-type location within this region. Additive martingales and spine constructions are at the heart of this chapter, and, in a manner related to the techniques of Hardy and Harris [43], we use a spine change of measure to find a lower bound for the probability that a single particle in the branching diffusion follows a 'difficult' path.

The work in Chapter 5 appears in Git *et al.* [39].

1.4.2 BBM with quadratic breeding rate

In Chapter 6 we study BBM with a quadratic breeding potential, i.e. the instantaneous rate of dyadic fission of a particle $u \in N_t$ is $\beta Y_u(t)^2$, where $\beta > 0$ and $Y_u(t)$ is the particle's spatial position. BBM with this quadratic breeding rate exhibits the interesting feature (noted in Itô and McKean [53, pp 200–211]) that the expected total number of particles blows up in finite time, but the total number of particles remains finite almost surely for all time. This behaviour makes it a difficult model to study.

The main result of Chapter 6 is a lower bound on the asymptotic position of the right-most particle. In contrast to the linear spatial spread of standard BBM, with quadratic breeding the right-most particle has a displacement from the origin that grows (at least) exponentially in t. We currently do not have an upper bound on the position of the right-most particle, but we conclude this thesis with a conjecture and some ideas for the further study of this process.

Chapter 2

The one-sided FKPP equation

In this chapter, a study of branching Brownian motion with absorption allows us to prove existence and uniqueness for solutions of the one-sided FKPP equation, and to represent these solutions as extinction probabilities for the killed BBM. In the spirit of the probabilistic studies of the classical travelling wave problem for the standard FKPP equation, we shall see that the behaviour of the right-most particle in the killed BBM is central to the argument. Our analysis includes an application of a spine decomposition which yields, as a by-product of the KPP study, the asymptotic speed of the right-most particle in the killed BBM (on the event that the process survives for all time).

The work in this chapter, joint with S. C. Harris and A. E. Kyprianou, has appeared in Harris *et al.* [45].

2.1 Introduction and summary of results

In this chapter we restrict our attention to the particular FKPP-type equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \beta(u^2 - u), \tag{2.1}$$

where $u \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and we have some given initial condition $u(0,x) := f(x)$. As we mentioned before, the nonlinear term u^2 leads to study of BBM with dyadic branching, but the probabilistic analysis extends readily to the more general equations considered in, for example, Chauvin and Rouault [20].

Searching for travelling wave solutions of speed ρ, i.e. FKPP solutions of the form

$u(t,x) = f(x+\rho t)$, for $f \in C^2(\mathbb{R})$, leads to the FKPP travelling-wave equation

$$\begin{aligned} \frac{1}{2}f'' - \rho f' + \beta(f^2 - f) &= 0 \text{ on } \mathbb{R} \\ f(-\infty) &= 1, \\ f(\infty) &= 0. \end{aligned} \tag{2.2}$$

It is well known that monotone travelling waves exist and are unique (up to translation) for all speeds $\rho \geq \sqrt{2\beta}$. For $-\infty < \rho < \sqrt{2\beta}$, there exist no monotone travelling wave solutions of speed ρ.

One of the probabilistic methods for studying equations (2.1) and (2.2) is via their links with branching Brownian motion. We consider a BBM with drift $-\rho$, where $\rho \in \mathbb{R}$, and dyadic branching rate β; that is to say a branching process where particles diffuse according to a Brownian motion with drift $-\rho$, and after an exponentially distributed (rate β, independent of the spatial motion) length of time divide in two. From their birth positions these particles repeat stochastically the behaviour of their parents. All particles are independent of each other. We shall refer to this process as a $(-\rho, \beta; \mathbb{R})$-BBM, with probabilities $\{P^x : x \in \mathbb{R}\}$ where P^x is the law of the process initiated from a single particle positioned at x. The configuration of space at time t is then given by the point process $\mathcal{X}_t^{-\rho}$, with points $\{\mathcal{Y}_u(t) : u \in \mathcal{N}_t^{-\rho}\}$, where $\mathcal{N}_t^{-\rho}$ is the set of individuals alive at time t.

Associated with the $(-\rho, \beta; \mathbb{R})$-BBM are the positive martingales

$$Z_\lambda(t) := \sum_{u \in \mathcal{N}_t^{-\rho}} e^{(\lambda+\rho)\mathcal{Y}_u(t) - (\frac{1}{2}(\lambda^2 - \rho^2) + \beta)t}, \tag{2.3}$$

which are defined for each $\lambda \in \mathbb{R}$. It is well known that such martingales are uniformly integrable with strictly positive limit precisely when $|\lambda + \rho| < \sqrt{2\beta}$, otherwise they have an almost-sure zero limit. These martingales can be used to establish both the existence and uniqueness of the travelling wave solutions to the system (2.2) where, in particular, $f(x) = E^x \exp(-Z_\lambda(\infty))$ gives a representation for the travelling wave of speed ρ when λ satisfies $\frac{1}{2}(\lambda+\rho)^2 - \rho(\lambda+\rho) + \beta = 0$. See McKean [73, 74] and Neveu [76]; and we remind the reader that Kyprianou [66] and Harris [46] give complete probabilistic expositions that are particularly relevant for the techniques used in this chapter. These latter references also include derivations of the asymptotic behaviour of the travelling wave solution. We note that the martingale results for a constant drift of $\rho \in \mathbb{R}$ follow trivially from the $\rho = 0$ case found in these, and subsequent, references.

The system that is the object of study in this chapter is the FKPP travelling wave equation on a modified domain: we shall consider solutions to the FKPP travelling wave equation defined on $\mathbb{R}^+$ that satisfy $f : \mathbb{R}^+ \to [0,1]$, $f \in C^2(0,\infty)$ and

$$\begin{aligned} \frac{1}{2}f'' - \rho f' + \beta(f^2 - f) &= 0 \text{ on } (0,\infty), \\ f(0+) &= 1, \\ f(\infty) &= 0. \end{aligned} \tag{2.4}$$

We refer to this as the one-sided FKPP equation, but remind ourselves that solutions to this system are not travelling waves in the sense of being the 'limiting shape' of solutions of (2.1).

Note that without the boundary conditions we always have that the constant functions 0 and 1 are solutions to (2.4). Interestingly, solutions to the one-sided FKPP equation occur precisely at wave speeds for which there are no (monotone) solutions to the FKPP travelling wave equation on $\mathbb{R}$.

Theorem 2.1.1. *The system* (2.4) *has a unique solution if and only if* $-\infty < \rho < \sqrt{2\beta}$, *in which case*

$$\lim_{x\uparrow\infty} e^{-(\rho-\sqrt{\rho^2+2\beta})x} f(x) = k$$

for some constant $k \in (0,\infty)$. *Further, if* $\rho \geq \sqrt{2\beta}$, *there is no solution to* (2.4).

Existence, uniqueness, and a weaker asymptotic result were established analytically in Pinsky [78], who himself cites Aronson and Weinberger [1]. The stronger asymptotic result given here can be extracted from Kametaka [57], who also uses analytic methods — namely classical phase-plane techniques (as described in Coddington and Levinson [23]). Some care is required, though, as Kametaka's paper is predominantly concerned with the case $\rho \geq \sqrt{2\beta}$.

In the spirit of Harris [46] and Kyprianou [66], we shall provide a new proof of Theorem 2.1.1 using probabilistic means alone which, for the most part, means that we appeal either to martingale arguments, spine decompositions, or fundamental properties of both branching and single-particle Brownian motion.

In contrast to the probabilistic study of travelling waves on $\mathbb{R}$, our analysis involves a branching Brownian motion with drift $-\rho$ where particles are *killed* at the origin. For the purpose of the forthcoming analysis, we will construct this killed BBM, $X^{-\rho}$, from the part of the BBM $\mathcal{X}^{-\rho}$ that survives killing at the origin. Considering $X^{-\rho}$ as a subprocess of $\mathcal{X}^{-\rho}$, we shall again work with the probabilities $\{P^x : x > 0\}$. We shall denote the configuration of particles alive at time t by $\{Y_u(t) : u \in N_t^{-\rho}\}$, where $N_t^{-\rho}$ is the set of surviving particles. In keeping with previous notation, we shall refer to this killed BBM process as a $(-\rho, \beta; \mathbb{R}^+)$-BBM. We define $\zeta := \inf\{t > 0 : N_t^{-\rho} = \varnothing\}$ to be the extinction time of the $(-\rho, \beta; \mathbb{R}^+)$-BBM, and then $\{\zeta = \infty\}$ is the event the process survives forever.

In Section 2.2, we shall briefly discuss the details of a spine construction for the $(-\rho, \beta; \mathbb{R})$-BBM. In particular, if we change the measure of a $(-\rho, \beta; \mathbb{R})$-BBM using the Z_λ additive martingale, the process under the new measure can be constructed by first laying down the motion of a single particle, the spine, as a Brownian motion with modified drift λ, which gives birth at an accelerated rate 2β to independent $(-\rho, \beta; \mathbb{R})$-BBMs. These changes of measure and their associated spine constructions prove a key tool in our later analysis.

In Section 2.3 we look at some important properties of the drifting branching Brownian motion with killing at the origin. In particular, we look at the behaviour of the right-most particle, R_t, the relationship with the survival set, and survival probabilities.

In Sections 2.4–2.6, we prove Theorem 2.1.1 via a sequence of smaller results. These are: non-existence of solutions to the system (2.2) for $\rho \geq \sqrt{2\beta}$, as a consequence of $P^x(\zeta < \infty) \equiv 1$ in the $(-\rho, \beta; \mathbb{R}^+)$-BBM; existence of a solution for $-\infty < \rho < \sqrt{2\beta}$ in the form of the (non-trivial) extinction probability $P^x(\zeta < \infty)$; uniqueness of travelling waves for $-\infty < \rho < \sqrt{2\beta}$; and finally the asymptotic result.

In Section 2.7, we show how our intuitive spine approach to killed BBM in Section 2.3 also allows us to deduce the following asymptotic result for the right-most particle in the $(-\rho, \beta; \mathbb{R}^+)$-BBM.

Lemma 2.1.2. *For all $x > 0$ we have*

$$\lim_{t \uparrow \infty} \frac{R_t}{t} = \sqrt{2\beta} - \rho$$

on $\{\zeta = \infty\}$, P^x-almost surely.

We note that Sevast′yanov [82] and Watanabe [85] have used extinction probabilities for BBM with absorption on *bounded* domains to study diffusion-type equations on those domains.

2.2 Spine constructions for BBM

In this section, we briefly recall a change of measure and its associated spine construction. This will be a key tool in our proofs in Section 2.3.

When $|\lambda + \rho| < \sqrt{2\beta}$ one can define an equivalent change of measure on the probability space of the $(-\rho, \beta; \mathbb{R})$-BBM as

$$\left.\frac{d\pi_\lambda^x}{dP^x}\right|_{\mathcal{F}_t} = \frac{Z_\lambda(t)}{Z_\lambda(0)} = e^{-(\lambda+\rho)x} Z_\lambda(t) \, .$$

Under π_λ^x the tree of the $(-\rho, \beta; \mathbb{R})$-BBM can be reconstructed in law in the following way:

- starting from position x, the initial ancestor diffuses according to a Brownian motion with drift λ;
- at rate 2β the particle undergoes fission producing two particles;
- one of these particles is selected at random with probability one half;
- this chosen particle repeats stochastically the behaviour of their parent;
- the other particle initiates from its birth position an independent copy of a $(-\rho, \beta; \mathbb{R})$-BBM with law $P^{\cdot}$, and so on.

The selected line of descent is referred to as the spine. The spine moves as a Brownian motion with drift λ, giving birth at an accelerated rate 2β along its path to independent $(-\rho, \beta; \mathbb{R})$-BBMs. This change of measure has been used before by Kyprianou [66], and also by Chauvin and Rouault [20]. We give the detailed set-up and notation for such changes of measure in Chapter 3, when it becomes necessary to introduce new families of martingales for the $(-\rho, \beta; \mathbb{R}^+)$-BBM and describe their associated spine constructions.

2.3 Killed branching Brownian motion

It will turn out that the existence and uniqueness result in Theorem 2.1.1 can be proved probabilistically by analysing the $(-\rho, \beta; \mathbb{R}^+)$-BBM, and in particular the behaviour of the position of its right-most particle, defined by

$$R_t = \sup\{Y_u(t) : u \in N_t^{-\rho}\}$$

on $\{\zeta > t\}$, and zero otherwise.

Theorem 2.3.1. *We have for all $x > 0$ and $\rho \in \mathbb{R}$,*

$$\limsup_{t\uparrow\infty} R_t = \infty$$

on $\{\zeta = \infty\}$, P^x-almost surely.

Proof. Let $Y = \{Y(t) : t \geq 0\}$ be a Brownian motion with drift $-\rho$ and probabilities $\{\mathbb{P}^x_{-\rho} : x \in \mathbb{R}\}$, and let $\tau_0 = \inf\{t \geq 0 : Y(t) = 0\}$. Note that

$$P^x(\zeta < \infty|\mathcal{F}_t) \geq \prod_{u\in N_t^{-\rho}} \mathbb{P}^{Y_u(t)}_{-\rho}(\tau_0 < \mathbf{e}_\beta) = \prod_{u\in N_t^{-\rho}} \mathbb{E}^{Y_u(t)}_{-\rho}(e^{-\beta\tau_0}),$$

where $\mathbf{e}_\beta$ is an exponential variable (independent of Y) with rate β. The above inequality holds on account of the fact that extinction would occur if each of the individuals alive at time t hit the origin before splitting. Setting $\alpha = \rho - \sqrt{\rho^2 + 2\beta} < 0$, standard expressions for the one-sided exit problem for Brownian motion — see, for example, Borodin and Salminen [14] — imply that for all $x > 0$

$$P^x(\zeta < \infty|\mathcal{F}_t) \geq \prod_{u\in N_t^{-\rho}} e^{\alpha Y_u(t)} = \exp\left(\alpha \sum_{u\in N_t^{-\rho}} Y_u(t)\right).$$

Now $\mathbf{1}_{\{\zeta<\infty\}}$ is a $\mathcal{F}_\infty$-measurable random variable, and so $P^x(\zeta < \infty|\mathcal{F}_t)$ is a uniformly integrable martingale with limit $\mathbf{1}_{\{\zeta<\infty\}}$. Thus, P^x-almost everywhere on $\{\zeta = \infty\}$, it is clear that $P^x(\zeta < \infty|\mathcal{F}_t)$ converges to zero, and hence for all $x > 0$

$$\lim_{t\uparrow\infty} \sum_{u\in N_t^{-\rho}} Y_u(t) = \infty \text{ on } \{\zeta = \infty\},\ P^x\text{-a.s.}$$

Now let Γ_z be the event that the $(-\rho, \beta; \mathbb{R}^+)$-BBM is contained entirely in the strip $(0, z)$. For the process Y define the stopping time $\tau_z = \inf\{t \geq 0 : Y(t) = z\}$. We have, for $0 < x < z$,

$$P^x(\Gamma_z|\mathcal{F}_t) \leq \prod_{u\in N_t^{-\rho}} \mathbb{P}^{Y_u(t)}_{-\rho}(\tau_0 < \tau_z) \tag{2.5}$$

on $Y_u(t) \in (0, z)$, for $u \in N_t^{-\rho}$. This inequality follows from the fact that Γ_z implies that the spatial path of each of the lines of descent emanating from the configuration at time t must hit the origin before hitting z. First consider the case that $-\infty < \rho < 0$.

In this case we can write from (2.5) that, on the event that $Y_u(t) \in (0,z)$ for each $u \in N_t^{-\rho}$,

$$P^x(\Gamma_z|\mathcal{F}_t) \le \prod_{u\in N_t^{-\rho}} e^{-|\rho|Y_u(t)} \frac{\sinh|\rho|(z - Y_u(t))}{\sinh|\rho|\, z} \le \exp\Big(-|\rho| \sum_{u\in N_t^{-\rho}} Y_u(t)\Big) \to 0$$

on the event $\{\zeta = \infty\}$ as t tends to infinity. Now consider the case that $\rho > 0$. It follows, again by using classical results for the two-sided exit problem, that on the event that $Y_u(t) \in (0,z)$ for each $u \in N_t^{-\rho}$

$$\begin{aligned} P^x(\Gamma_z|\mathcal{F}_t) &\le \prod_{u\in N_t^{-\rho}} \Big(1 - \frac{e^{-\rho z}}{\sinh \rho z} e^{\rho Y_u(t)} \sinh \rho Y_u(t)\Big) \\ &\le \exp\Big(-\frac{e^{-\rho z}}{\sinh \rho z}\rho \sum_{u\in N_t^{-\rho}} Y_u(t)\Big), \end{aligned} \tag{2.6}$$

where we have used the inequalities $e^{-x}\sinh x \le x$ and $1 - x \le e^{-x}$. The exponential at (2.6) tends to zero on the event $\{\zeta = \infty\}$ as $t \to \infty$. Finally, for the case that $\rho = 0$, on $Y_u(t) \in (0,z)$ for $u \in N_t^{-\rho}$

$$P^x(\Gamma_z|\mathcal{F}_t) \le \prod_{u\in N_t^{-\rho}} \Big(1 - \frac{Y_u(t)}{z}\Big) \le \exp\Big(-\frac{1}{z} \sum_{u\in N_t^{-\rho}} Y_u(t)\Big) \to 0$$

on the event $\{\zeta = \infty\}$ as $t \to \infty$. In conclusion, for any $z > 0$,

$$P^x\Big(R_t > z \text{ infinitely often}\Big|\zeta = \infty\Big) = 1$$

and the statement of the theorem holds. □

Theorem 2.3.2. *If $\rho \ge \sqrt{2\beta}$ then $P^x(\zeta < \infty) = 1$ for all $x > 0$.*

Proof. Suppose that $\mathcal{R}_t$ is the position of the right most particle in a $(-\rho, \beta; \mathbb{R})$-BBM, and consider the 'critical' martingale $Z_{\sqrt{2\beta}-\rho}(t)$. We have

$$Z_{\sqrt{2\beta}-\rho}(t) \ge e^{\sqrt{2\beta}(\mathcal{R}_t - (\sqrt{2\beta}-\rho)t)} \ge 0,$$

and it is well known (see Neveu [76], Harris [46], or Kyprianou [66]) that $Z_{\sqrt{2\beta}-\rho}(t) \to 0$

almost surely, from which we may deduce that

$$\lim_{t\uparrow\infty}\left(\mathcal{R}_t-(\sqrt{2\beta}-\rho)t\right)=-\infty \qquad \text{a.s.}$$

From our construction of the $(-\rho,\beta;\mathbb{R}^+)$-BBM, extinction of this process is guaranteed when the right most particle in the $(-\rho,\beta;\mathbb{R})$-BBM drifts to $-\infty$. Thus, when $\rho\geq\sqrt{2\beta}$, this happens with probability one. $\square$

Theorem 2.3.3. *If $-\infty<\rho<\sqrt{2\beta}$, then for each $x>0$ and $\lambda\in(0,\sqrt{2\beta}-\rho)$*

(i) $E^x(Z_\lambda(\infty);\liminf_{t\uparrow\infty}R_t/t\geq\lambda)=\pi^x_\lambda(\liminf_{t\uparrow\infty}R_t/t\geq\lambda)\geq 1-e^{-2\lambda x}$*;*

(ii) $P^x(\zeta<\infty)\in(0,1)$*;*

(iii) $\lim_{x\downarrow 0}P^x(\zeta<\infty)=1$*; and*

(iv) $\lim_{x\uparrow\infty}P^x(\zeta=\infty)=\lim_{x\uparrow\infty}P^x(\liminf_{t\uparrow\infty}R_t/t\geq\lambda;\zeta=\infty)=1.$

Proof. (i) Recall that when $|\lambda+\rho|<\sqrt{2\beta}$, and in particular when $\lambda\in(0,\sqrt{2\beta}-\rho)$, under the measure π^x_λ (defined in Section 2.2) the $(-\rho,\beta;\mathbb{R})$-BBM has one line of descent, the spine, which has an exceptional drift λ. The probability that this spine never meets the origin is the probability that a Brownian motion started from $x>0$ and with drift λ has an all time infimum that is strictly positive — this is well known to be $1-\exp(-2\lambda x)$. If we write $\xi=\{\xi_t:t\geq 0\}$ for the spatial path of any surviving line of descent in $X^{-\rho}$, then we have established that

$$E^x\left(Z_\lambda(\infty);\zeta=\infty \text{ and } \exists\xi \text{ in } X^{-\rho} \text{ such that } \lim_{t\uparrow\infty}\frac{\xi_t}{t}=\lambda\right)\geq 1-e^{-2\lambda x}.$$

Now note that

$$\left\{\zeta=\infty \text{ and } \exists\xi \text{ in } X^{-\rho} \text{ such that } \lim_{t\uparrow\infty}\frac{\xi_t}{t}=\lambda\right\}\subseteq\{\liminf_{t\uparrow\infty}R_t/t\geq\lambda\},$$

and the statement of part (i) follows.

(ii) To prove that $P^x(\zeta<\infty)>0$, note that there is a strictly positive probability that the initial ancestor in the process $X^{-\rho}$ hits the origin before reproducing, thus resulting in extinction. To prove that $P^x(\zeta<\infty)<1$, or equivalently that $P^x(\zeta=\infty)>0$, recall from part (i) that under π^x_λ the probability that the (λ-drifting) spine in the branching Brownian motion does not meet the origin is strictly positive. This implies that $E^x(Z_\lambda(\infty); \zeta = \infty) > 0$; and since $P^x(Z_\lambda(\infty) > 0) = 1$, it follows that $P^x(\zeta = \infty) > 0$.

(iii) Since extinction in a finite time is guaranteed if the original ancestor is killed before reproducing,

$$P^x(\zeta < \infty) \geq \mathbb{P}^x_{-\rho}(\tau_0 < \mathbf{e}_\beta) = e^{-(\sqrt{\rho^2+2\beta}-\rho)x} \uparrow 1$$

as $x \to 0$. Recall that $\tau_0 = \inf\{t \geq 0 : Y(t) = 0\}$, and also that $\mathbf{e}_\beta$ is exponentially distributed with parameter β and independent of the Brownian motion $(Y, \mathbb{P}^x_{-\rho})$.

(iv) Note that $P^x(\liminf_{t\uparrow\infty} R_t/t \geq \lambda; \zeta = \infty)$ is an increasing function of x and therefore has a limit. Suppose this limit is not equal to one. Then since it was shown in part (i) of the proof that

$$\lim_{x\uparrow\infty} E^x\left(Z_\lambda(\infty); \liminf_{t\uparrow\infty} R_t/t \geq \lambda\right) = 1,$$

there is a contradiction, since for all $x > 0$

$$P^x(Z_\lambda(\infty) > 0) = 1 \text{ and } E^x(Z_\lambda(\infty)) = 1.$$

Finally, noting that $P^x(\zeta = \infty) \geq P^x(\liminf_{t\uparrow\infty} R_t/t \geq \lambda; \zeta = \infty)$, the proof is complete. □

2.4 Non-existence for $\rho \geq \sqrt{2\beta}$

Theorem 2.4.1. *No travelling wave solutions to (2.4) exist for* $\rho \geq \sqrt{2\beta}$.

Proof. Suppose that f is a solution to (2.4). It follows that, for all $x > 0$,

$$\prod_{u \in N_t^{-\rho}} f(Y_u(t))$$

is a martingale which converges almost surely and in $\mathcal{L}^1(P^x)$. We have seen in Theorem 2.3.2 that if $\rho \geq \sqrt{2\beta}$, then $P^x(\zeta < \infty) = 1$ for all $x > 0$ and hence

$$\lim_{t\uparrow\infty} \prod_{u \in N_t^{-\rho}} f(Y_u(t)) = 1$$

almost surely, implying that $f \equiv 1$; that is to say there is no non-trivial solution. □

2.5 Existence and uniqueness for $-\infty < \rho < \sqrt{2\beta}$

Theorem 2.5.1. *Solutions of the system (2.4) exist and are unique for $-\infty < \rho < \sqrt{2\beta}$. Further, the unique solution can be represented by the extinction probability for the $(-\rho, \beta; \mathbb{R}^+)$-BBM, that is*

$$f(x) = P^x(\zeta < \infty). \tag{2.7}$$

Remark 2.5.2. The representation (2.7) trivially shows that the unique solution to (2.4) is strictly monotone decreasing, although this wasn't an initial restriction. Indeed, one could assume monotonicity instead of $f(\infty) = 0$ and again reach the same conclusions. Also note that one might naively try to extended this solution to produce a travelling wave of speed $\rho < \sqrt{2\beta}$ on the whole of $\mathbb{R}$, but such a solution would clearly fail to satisfy equation (2.2) at a single point (due to a discontinuity in the first derivative at the origin).

Proof. For $x \geq 0$ define $p(x) := P^x(\zeta < \infty)$. From Theorem 2.3.3 we have $p(x) \in (0, 1)$ for each $x > 0$, $\lim_{x\uparrow\infty} p(x) = 0$, $\lim_{x\downarrow 0} p(x) = 1$, and, in addition, $p(0) = 1$ because of instantaneous killing.

An application of the branching Markov property (cf. Chauvin [19]) together with the tower property of conditional expectation gives

$$p(x) = E^x\Big(P^x(\zeta < \infty | \mathcal{F}_t)\Big) = E^x\Big(\prod_{u \in N_t^{-\rho}} p(Y_u(t))\Big). \tag{2.8}$$

As this equality holds for all $x, t > 0$, one can see that $\prod_{u\in N_t^{-\rho}} p(Y_u(t))$ is a martingale which converges almost surely and in $L^1(P^x)$. Note that on $\{\zeta < \infty\}$ it is clear that the martingale limit is equal to 1 — the empty product. Note however that this martingale cannot be identically equal to 1 because its mean, $p(x)$, is strictly less than 1. An application of Kolmogorov's backwards equations (cf. Champneys *et al.* [18] or Dynkin [27, Theorem II.3.1]) yields that p belongs to $C^2(0, \infty)$ and is a solution to the ODE in (2.4).

To show uniqueness, suppose that f is a solution to (2.4) when $-\infty < \rho < \sqrt{2\beta}$. Again we can construct a positive martingale $M_t := \prod_{u\in N_t^{-\rho}} f(Y_u(t))$ which is bounded, and hence uniformly integrable. Clearly $M_\infty = 1$ on $\{\zeta < \infty\}$. Theorem 2.3.1 gives $\limsup_{t\uparrow\infty} R_t = \infty$ on $\{\zeta = \infty\}$ almost surely, and moreover

$$\begin{aligned} M_\infty = \lim_{t\uparrow\infty} \prod_{u\in N_t^{-\rho}} f(Y_u(t)) &= \liminf_{t\uparrow\infty} \prod_{u\in N_t^{-\rho}} f(Y_u(t)) \\ &\leq \liminf_{t\uparrow\infty} f(R_t) \\ &\leq f(\limsup_{t\uparrow\infty} R_t). \end{aligned}$$

Since $f(+\infty) = 0$, we can identify the limit as $M_\infty = \mathbf{1}_{\{\zeta<\infty\}}$ almost surely. Finally,

$$f(x) = E^x(M_0) = E^x(M_\infty) = P^x(\zeta < \infty) = p(x)$$

and uniqueness follows. □

2.6 Asymptotic when $-\infty < \rho < \sqrt{2\beta}$

In this section we determine the asymptotic for the solution to the one-sided FKPP system (2.4). As a first step, the following lemma shows that the unique solution decays exponentially for sufficiently large y.

Lemma 2.6.1. *Let f be the unique solution of the system (2.4) when $-\infty < \rho < \sqrt{2\beta}$. Let $x_0 > 0$ and define $\mu := \sqrt{\rho^2 + 2\beta(1 - f(x_0))} - \rho > 0$. Then*

$$f(y) \leq \left(f(x_0)e^{\mu x_0}\right) e^{-\mu y}$$

for all $y > x_0$.

Proof. Recall that Y is a Brownian motion with drift $-\rho$ starting from $x > 0$ under $\mathbb{P}^x_{-\rho}$, and that for any $z \geq 0$, $\tau_z := \inf\{t : Y_t = z\}$. Itô's formula implies that

$$M_t := f(Y_{t\wedge\tau_0}) \exp\left(\beta \int_0^{t\wedge\tau_0} (f(Y_s) - 1)\,\mathrm{d}s\right) \tag{2.9}$$

is a $\mathbb{P}^x_{-\rho}$-local martingale, and, since $0 \leq f \leq 1$, it is actually a bounded martingale. Suppose that $y > x_0$. Since $\tau_{x_0} < \infty$ almost surely under $\mathbb{P}^y_{-\rho}$, the optional stopping theorem and the monotonicity of f (see remark after Theorem 2.5.1) yield

$$f(y) = \mathbb{E}^y_{-\rho}\left(f(x_0) \exp\left(\beta \int_0^{\tau_{x_0}} (f(Y_s) - 1)\,\mathrm{d}s\right)\right) \leq f(x_0)\mathbb{E}^y_{-\rho}\left(e^{\beta(f(x_0)-1)\tau_{x_0}}\right). \tag{2.10}$$

It can be shown that (see, for example, Borodin and Salminen [14])

$$\mathbb{E}^{y}_{-\rho}\left(e^{\beta(f(x_0)-1)\tau_{x_0}}\right) = e^{-\mu(y-x_0)},$$

where $\mu := \sqrt{\rho^2 + 2\beta(1-f(x_0))} - \rho > 0$. Inequality (2.10) then becomes

$$f(y) \leq f(x_0)e^{-\mu(y-x_0)}$$

as required. □

As a corollary, we gain a new and straightforward probabilistic proof of the weaker asymptotic result found in Pinsky [78] that first motivated this work.

Corollary 2.6.2. *When it exists, the solution to the system* (2.4) *satisfies*

$$\lim_{x\to\infty} \frac{1}{x} \ln f(x) = \rho - \sqrt{\rho^2 + 2\beta} < 0.$$

Proof. For any fixed $x_0 > 0$ and $y > x_0$, taking logarithms in Lemma 2.6.1 yields

$$\frac{\ln f(y)}{y} \leq \mu\frac{x_0}{y} + \frac{\ln f(x_0)}{y} - \mu$$

and hence

$$\limsup_{y\to\infty} \frac{\ln f(y)}{y} \leq -\mu.$$

As this is true for arbitrary $x_0 > 0$, and $f(x_0) \to 0$ as $x_0 \to \infty$, we find that

$$\limsup_{y\to\infty} \frac{\ln f(y)}{y} \leq \rho - \sqrt{\rho^2 + 2\beta}.$$

To prove the lower bound, recall that M_t defined at (2.9) is a uniformly integrable martingale and hence for $y > 0$, remembering that $f(x) \in [0,1]$ with $f(0) = 1$, we have

$$\begin{aligned} f(y) &= \mathbb{E}^{y}_{-\rho}\left(\exp\left(\beta\int_0^{\tau_0}(f(Y_s)-1)\,\mathrm{d}s\right)\right) \\ &\geq \mathbb{E}^{y}_{-\rho}(e^{-\beta\tau_0}) = e^{-(\sqrt{\rho^2+2\beta}-\rho)y}. \end{aligned} \tag{2.11}$$

Hence

$$\frac{\ln f(y)}{y} \geq \rho - \sqrt{\rho^2 + 2\beta},$$

and taking a $\liminf_y$ completes the proof. □

As another corollary to Lemma 2.6.1, we can find an exponentially decaying bound for f valid on the whole of $(0, \infty)$. This is of importance in the proof of the strong asymptotic of Theorem 2.1.1.

Corollary 2.6.3. *Let f be the unique solution of the system* (2.4) *when $-\infty < \rho < \sqrt{2\beta}$. Given any $K > 1$, there exists a $\kappa > 0$ such that*

$$f(y) \leq K e^{-\kappa y}$$

for all $y \geq 0$.

Proof. For $K > 1$, choose $x_0 > 0$ such that $K = e^{\mu x_0}$. Note that $f(x_0) \in (0,1)$ and then set $\kappa = \sqrt{\rho^2 + 2\beta(1 - f(x_0))} - \rho > 0$. Lemma 2.6.1 says that $f(y) \leq K e^{-\kappa y}$ for all $y \geq x_0$. Also, since $0 \leq f \leq 1$ and for $y < x_0$ we have $K e^{-\kappa y} > 1$, we trivially have $f(y) \leq K e^{-\kappa y}$ for all $y \leq x_0$. □

We now extend the analysis to prove the stronger asymptotic of Theorem 2.1.1. Crucial to this argument is the following proposition, which we shall prove at the end of this section.

Proposition 2.6.4. *With $\tilde{\rho} := \sqrt{\rho^2 + 2\beta}$, $x > 0$, and $f(x)$ the unique travelling wave at speed $-\infty < \rho < \sqrt{2\beta}$,*

$$\lim_{x \to \infty} \mathbb{E}^x_{-\tilde{\rho}} \exp\left(\beta \int_0^{\tau_0} f(Y_s)\, \mathrm{d}s\right) < +\infty. \tag{2.12}$$

Proof of Theorem 2.1.1 (asymptotics). Working with the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^x_{\lambda-\rho}}{\mathrm{d}\mathbb{P}^x_{-\rho}}\right|_{\mathcal{F}_t} = e^{\lambda(Y_t + \rho t - x) - \frac{1}{2}\lambda^2 t}$$

for $\lambda \in \mathbb{R}$ and $x > 0$, we have from the martingale at (2.9) that

$$e^{-\lambda x} f(x) = \mathbb{E}^x_{-\rho}\left(e^{-\lambda Y_{t\wedge\tau_0}} f(Y_{t\wedge\tau_0}) e^{\beta \int_0^{t\wedge\tau_0} f(Y_s)\, \mathrm{d}s} e^{\lambda(Y_{t\wedge\tau_0} - x) - \beta(t\wedge\tau_0)}\right).$$

Now choose $\lambda = \alpha := \rho - \sqrt{\rho^2 + 2\beta} < 0$, so that $\beta + \rho\lambda = \frac{1}{2}\lambda^2$. Defining $v(x) := e^{-\alpha x} f(x)$ and $\tilde{\rho} := \sqrt{\rho^2 + 2\beta} > 0$ yields

$$v(x) = \mathbb{E}^x_{-\tilde{\rho}}\left(v(Y_{t\wedge\tau_0}) \exp\left(\beta \int_0^{t\wedge\tau_0} f(Y_s)\, \mathrm{d}s\right)\right),$$

whence

$$v(Y_{t\wedge\tau_0})\exp\left(\beta\int_0^{t\wedge\tau_0} f(Y_s)\,\mathrm{d}s\right) \tag{2.13}$$

is a $\mathbb{P}_{-\tilde{\rho}}$-martingale which is positive and therefore converges almost surely. As $\tau_0 < \infty$ $\mathbb{P}_{-\tilde{\rho}}$-almost surely, we also have $v(Y_{t\wedge\tau_0}) \to 1$ almost surely under $\mathbb{P}^x_{-\tilde{\rho}}$; but v is not (yet) known to be a bounded function so we cannot immediately conclude that this martingale is uniformly integrable. However, using the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^x_{-\tilde{\rho}}}{\mathrm{d}\mathbb{P}^x_{-\rho}}\right|_{\mathcal{F}_{\tau_0}} = e^{(\sqrt{\rho^2+2\beta}-\rho)x-\beta\tau_0},$$

(which is possible because $\exp(\alpha(Y_{t\wedge\tau_0}-x)-\beta(t\wedge\tau_0))$ is a uniformly integrable martingale) we may transform (2.11) to

$$v(x) = \mathbb{E}^x_{-\tilde{\rho}}\left(\exp\left(\beta\int_0^{\tau_0} f(Y_s)\,\mathrm{d}s\right)\right), \tag{2.14}$$

and hence the $\mathbb{P}^x_{-\tilde{\rho}}$ martingale in (2.13) is uniformly integrable. Note that from (2.14) it is clear that v is monotone increasing in x, and hence its limit exists as $x \to \infty$.

All that remains is to prove that v converges to a finite limit as x tends to infinity, which is precisely Proposition 2.6.4. Thus

$$v(x) := f(x)e^{-\alpha x} \uparrow k \in (0,\infty) \text{ as } x \to \infty.$$

Hence $f(x)$ asymptotically looks like the decaying solution of

$$\frac{1}{2}f'' - \rho f' - \beta f = 0,$$

which is the linearisation of equation (2.2) about the origin. □

Proof of Proposition 2.6.4. Recall that $\tilde{\rho} = \sqrt{\rho^2+2\beta}$, and for $y > 0$

$$\mathbb{E}^y_{-\tilde{\rho}}\left(e^{\gamma\tau_0}\right) = e^{(\tilde{\rho}-\sqrt{\tilde{\rho}^2-2\gamma})y}$$

provided that $2\gamma < \tilde{\rho}^2$ (for later use we observe that, in particular, this holds for all $\gamma \le \beta$).

Note that for any $y > 0$, since $f \in [0,1]$ we have

$$\mathbb{E}^{y}_{-\tilde{\rho}} \exp\left(\beta \int_0^{\tau_0} f(Y_s)\,\mathrm{d}s\right) \le \mathbb{E}^{y}_{-\tilde{\rho}}(e^{\beta\tau_0}) = e^{(\sqrt{\rho^2-2\beta}-\rho)y} < \infty, \tag{2.15}$$

and, for any $y_0 > y_1 > 0$, the strong Markov property gives

$$\mathbb{E}^{y_0}_{-\tilde{\rho}}\left(e^{\beta\int_0^{\tau_0} f(Y_s)\,\mathrm{d}s}\right) = \mathbb{E}^{y_0}_{-\tilde{\rho}}\left(e^{\beta\int_0^{\tau_{y_1}} f(Y_s)\,\mathrm{d}s}\right)\mathbb{E}^{y_1}_{-\tilde{\rho}}\left(e^{\beta\int_0^{\tau_0} f(Y_s)\,\mathrm{d}s}\right). \tag{2.16}$$

Fix any $K > 1$, and recall from Corollary 2.6.3 that there then exists $\mu > 0$ such that

$$f(x) \le Ke^{-\mu x} \qquad \forall x \ge 0.$$

Now fix any $d > 0$. Choose a fixed $M \in \mathbb{N}$ sufficiently large such that $Ke^{-\mu y_1} < 1$ where $y_1 := Md$. Then, for any $N \in \mathbb{N}$ and $y_0 := (M+N)d$, and with $S_i := \tau_{(M+i-1)d} - \tau_{(M+i)d}$ so that the S_i are independent and identically distributed like the first hitting time of 0 by a Brownian motion started at d, we have (defining $\tau_{(M+N)d} := 0$)

$$\begin{aligned}
\mathbb{E}^{y_0}_{-\tilde{\rho}} \exp\left(\beta\int_0^{\tau_{y_1}} f(Y_s)\,\mathrm{d}s\right) &\le \mathbb{E}^{y_0}_{-\tilde{\rho}} \exp\left(\beta K \int_0^{\tau_{y_1}} e^{-\mu Y_s}\,\mathrm{d}s\right)\\
&= \mathbb{E}^{y_0}_{-\tilde{\rho}} \exp\left(\beta K \sum_{n=1}^{N} \int_{\tau_{(M+N-n+1)d}}^{\tau_{(M+N-n)d}} e^{-\mu Y_s}\,\mathrm{d}s\right)\\
&\le \mathbb{E}^{y_0}_{-\tilde{\rho}} \exp\left(\beta K \sum_{n=1}^{N} e^{-\mu(M+N-n)d} S_{N-n+1}\right)\\
&= \mathbb{E}^{y_0}_{-\tilde{\rho}} \exp\left(\beta K e^{-\mu y_1} \sum_{k=0}^{N-1} e^{-\mu k d} S_{k+1}\right)\\
&\le \mathbb{E}^{y_0}_{-\tilde{\rho}} \prod_{k=0}^{N-1} \exp(\beta e^{-\mu k d} S_{k+1})\\
&= \prod_{k=0}^{N-1} \mathbb{E}^{y_0}_{-\tilde{\rho}} \exp(\beta e^{-\mu k d} S_{k+1})\\
&= \exp\left(d \sum_{k=0}^{N-1} \left(\tilde{\rho} - \sqrt{\tilde{\rho}^2 - 2\beta e^{-\mu k d}}\right)\right).
\end{aligned} \tag{2.17}$$

Since

$$\tilde{\rho} - \sqrt{\tilde{\rho}^2 - 2\beta e^{-\mu k d}} = \sqrt{\rho^2 + 2\beta}\left(1 - \sqrt{1 - \left(\frac{2\beta}{\rho^2 + 2\beta}\right) e^{-\mu k d}}\right)$$
$$= \left(\frac{\beta}{\sqrt{\rho^2 + 2\beta}}\right) e^{-\mu k d} + o(e^{-\mu k d}),$$

the ratio test reveals the sum appearing in (2.17) is convergent when $N \to \infty$. Using this fact together with monotone convergence and equations (2.15) and (2.16) now gives the required result. □

2.7 Right-most particle asymptotic $-\infty < \rho < \sqrt{2\beta}$

The intention of Theorem 2.3.3 was to establish properties of the probability of extinction in order to justify it as a solution to the travelling-wave equation. However, in light of parts (i) and (iii) of this same theorem there is reason to believe that, like the $(-\rho, \beta; \mathbb{R})$-BBM, the $(-\rho, \beta; \mathbb{R}^+)$-BBM has a right-most particle with asymptotic drift $\sqrt{2\beta} - \rho$ (but now it is necessary to specify that this happens on the survival set). This is indeed the case. After considerable extra work Lemma 2.1.2 will actually follow from the stronger result given in Theorem 3.1.3, but we can already give a direct alternative proof which we include for now for interest.

Proof of Lemma 2.1.2. We shall prove this result by establishing separately that

$$\liminf_{t\uparrow\infty} \frac{R_t}{t} \geq \sqrt{2\beta} - \rho \text{ and } \limsup_{t\uparrow\infty} \frac{R_t}{t} \leq \sqrt{2\beta} - \rho \text{ on } \{\zeta = \infty\} \ P^x\text{-a.s.}$$

Theorem 2.3.1 shows that for each $x > 0$, on $\{\zeta = \infty\}$, $\limsup_{t\uparrow\infty} R_t = +\infty$ P^x-almost surely, and hence $\sigma_y := \inf\{t \geq 0 : X^{-\rho}(y, \infty) > 0\}$ is P^x-almost surely finite for each $y > 0$ on $\{\zeta = \infty\}$. This implies that for any $\lambda > 0$,

$$P^x\left(\liminf_{t\uparrow\infty} R_t/t \geq \lambda; \zeta = \infty\right) = P^x\left(\liminf_{t\uparrow\infty} R_t/t \geq \lambda; \sigma_y < \infty; \zeta = \infty\right).$$

Thus for any $y > x$

$$\begin{aligned}
&P^x\left(\liminf_{t\uparrow\infty} R_t/t \geq \lambda; \zeta = \infty\right) \\
&\qquad = P^x\left(\sigma_y < \infty\right) P^x\left(\liminf_{t\uparrow\infty} R_t/t \geq \lambda; \zeta = \infty \middle| \sigma_y < \infty\right) \\
&\qquad \geq P^x\left(\sigma_y < \infty\right) P^y\left(\liminf_{t\uparrow\infty} R_t/t \geq \lambda; \zeta = \infty\right), \qquad (2.18)
\end{aligned}$$

where the inequality follows from the fact that at time σ_y there is one particle positioned at y which, given $\mathcal{F}_{\sigma_y}$, gives rise to an branching tree independent of other particles alive at time σ_y and further whose right-most particle is bounded above by the right-most particle of $X^{-\rho}$. Recalling Theorem 2.3.1, now note that as $y \to \infty$,

$$P^x\left(\sigma_y < \infty\right) \uparrow P^x\left(\limsup_{t\uparrow\infty} R_t = \infty\right) = P^x\left(\zeta = \infty\right).$$

With the help of Theorem 2.3.3(iv), it follows from (2.18) that, when we further insist that $\lambda \in (0, \sqrt{2\beta} - \rho)$,

$$\begin{aligned}
P^x(\zeta = \infty) &\geq P^x\left(\liminf_{t\uparrow\infty} R_t/t \geq \lambda; \zeta = \infty\right) \\
&\geq \lim_{y\uparrow\infty} P^x(\sigma_y < \infty) P^y\left(\liminf_{t\uparrow\infty} R_t/t \geq \lambda; \zeta = \infty\right) \\
&= P^x(\zeta = \infty).
\end{aligned}$$

We thus deduce that for any $\varepsilon > 0$, P^x-almost everywhere on the event $\{\zeta = \infty\}$, we have

$$\liminf_{t\uparrow\infty} R_t/t \geq \sqrt{2\beta} - \rho - \varepsilon.$$

Additionally note that on $\{\zeta = \infty\}$, R_t is P^x-almost surely stochastically bounded above by the right-most particle $\mathcal{R}_t$ of the *unkilled* $(-\rho, \beta; \mathbb{R})$-BBM and recall, for example, $Z_\lambda(t) \geq \exp\left((\lambda + \rho)\mathcal{R}_t - \frac{1}{2}(\lambda^2 - \rho^2)t - \beta t\right)$, yielding

$$\limsup_{t\uparrow\infty} R_t/t \leq \sqrt{2\beta} - \rho$$

P^x-almost everywhere on the event $\{\zeta = \infty\}$. $\square$

Chapter 3

Further analysis of the BBM with absorption

In this Chapter we extend our analysis of the $(-\rho, \beta; \mathbb{R}^+)$-BBM. We know from Chapter 2 that, on non-extinction, the right-most particle travels asymptotically at speed $\sqrt{2\beta} - \rho > 0$. This divides the upper half-plane into two regions: the 'sub-critical' region above the spatial ray of gradient $\sqrt{2\beta} - \rho$; and the 'super-critical' region below.

We introduce an additive martingale, W_λ, for the killed BBM. A comparison with Z_λ allows us to determine the convergence properties of W_λ, and then martingale methods enable us to find the almost-sure exponential growth rate of the number of particles near spatial rays of gradient $\lambda \in (0, \sqrt{2\beta} - \rho)$ in the super-critical region. Turning our attention to the sub-critical region, we find an asymptotic expression for the probability that there is a particle in the BBM near the spatial ray with gradient $\lambda > \sqrt{2\beta} - \rho$. Combining this with the arguments (for standard BBM) of Chauvin and Rouault [20], we prove a Yaglom-type conditional limit theorem for killed BBM. We also show that the martingale W_λ appears as the limiting Radon-Nikodým derivative when conditioning a particle in the killed BBM to travel at asymptotic speed $\lambda > \sqrt{2\beta} - \rho$.

Much of the work in this chapter has appeared in Harris *et al.* [45].

3.1 W_λ and statement of results

The central role of the Z_λ additive martingales in the study of branching Brownian motion is long established and, as already seen in Chapter 2, we can also exploit them in studying killed BBM. However, Z_λ is not a martingale for killed branching Brownian motion, and so we define a family of (positive) additive martingales W_λ.

Lemma 3.1.1. *For each $\lambda > 0$, the process*

$$W_\lambda(t) := \sum_{u \in N_t^{-\rho}} (1 - e^{-2\lambda Y_u(t)}) e^{(\lambda+\rho)Y_u(t) - (\frac{1}{2}(\lambda^2-\rho^2)+\beta)t} \tag{3.1}$$

defines a martingale for the $(-\rho, \beta; \mathbb{R}^+)$-BBM.

Proof. Let $Y(t)$ be a Brownian motion started at $x > 0$ with drift λ under the measure $\mathbb{P}_\lambda^x$. Defining $\tau_0 := \inf\{t \geq 0 : Y(t) = 0\}$, it follows from the Many-to-One lemma and a single-particle change of measure that

$$\begin{aligned} E^x(W_\lambda(t)) &= e^{-\frac{1}{2}(\lambda^2-\rho^2)t} \mathbb{E}_{-\rho}^x((1 - e^{-2\lambda Y_t})e^{(\lambda+\rho)Y_t}; \tau_0 > t) \\ &= \mathbb{P}_\lambda^x((1 - e^{-2\lambda Y_t}); \tau_0 > t)e^{(\lambda+\rho)x} \\ &= (1 - e^{-2\lambda x})e^{(\lambda+\rho)x}. \end{aligned}$$

Note the last equality is a consequence of the useful fact that $\mathbb{P}_\lambda^x(\tau_0 = \infty) = 1 - e^{-2\lambda x}$ is a scale function for Y killed at the origin under $\mathbb{P}_\lambda^x$.

We now apply the branching property to see that

$$E^x(W_\lambda(t+s)|\mathcal{F}_t) = \sum_{u \in N_t^{-\rho}} E^x(W_\lambda^{(u)}(s)|\mathcal{F}_t)e^{-(\frac{1}{2}(\lambda^2-\rho^2)+\beta)t}$$

where given $\mathcal{F}_t$, each of the terms $W_\lambda^{(u)}(s)$ are independent copies of $W_\lambda(s)$ under $P^{Y_u(t)}$. The conclusion of the previous paragraph now completes the proof. □

These martingales not only prove to be a useful tool in our later analysis, they appear fundamental to the study of the killed BBM. In Section 3.3, we discuss their convergence properties and show how they can be used to deduce growth rates of particles moving at speeds $\lambda < \sqrt{2\beta} - \rho$.

Like Z_λ, the martingale W_λ can be used to change measure to yield a spine decomposition. We define a measure $\mathbb{Q}_\lambda$ by

$$\left.\frac{\mathrm{d}\mathbb{Q}_\lambda^x}{\mathrm{d}P^x}\right|_{\mathcal{F}_t} = \frac{W_\lambda(t)}{W_\lambda(0)} = \frac{e^{-(\lambda+\rho)x}}{(1 - e^{-2\lambda x})} W_\lambda(t), \tag{3.2}$$

and it can be shown (see Section 3.2) that, under $\mathbb{Q}_\lambda^x$, the branching Brownian motion can be constructed path-wise as under π_λ^x, except that the spine now moves like a Brownian motion with drift λ that is additionally *conditioned to avoid the origin.* The next result gives the range of λ values for which W_λ is uniformly integrable.

Theorem 3.1.2. *W_λ is a uniformly integrable martingale if both $\rho < \sqrt{2\beta}$ and $\lambda \in (0, \sqrt{2\beta} - \rho)$, otherwise W_λ has a P^x-almost-sure zero limit.*

Whenever W_λ is uniformly integrable, $\{W_\lambda(\infty) > 0\}$ and $\{\zeta = \infty\}$ agree up to a P^x-null set.

To state our result for the exponential growth rates of particles in the super-critical region we define the counting function

$$N_t^{-\rho}(a,b) := \sum_{u \in N_t^{-\rho}} \mathbf{1}_{\{Y_u(t) \in (a,b)\}}, \tag{3.3}$$

for the number of particles found in the interval (a,b) at time t.

Theorem 3.1.3. *For $x > 0$, under each P^x law, the limit*

$$G(\lambda) := \lim_{t \to \infty} t^{-1} \ln N_t^{-\rho}(\lambda t, \infty)$$

exists almost surely and is given by

$$G(\lambda) = \begin{cases} \Delta(\lambda) & \text{if } 0 < \lambda < \sqrt{2\beta} - \rho \text{ and } \{\zeta = \infty\}, \\ -\infty & \text{otherwise,} \end{cases}$$

where $\Delta(\lambda) := \beta - \frac{1}{2}(\lambda + \rho)^2$.

Remark 3.1.4. Note that we gain the right-most particle speed of Lemma 2.1.2 as a corollary to Theorem 3.1.3. It is immediate from Theorem 3.1.3 that

$$\liminf_{t \to \infty} \frac{R_t}{t} \geq \sqrt{2\beta} - \rho$$

P^x-almost everywhere on $\{\zeta = \infty\}$; and that

$$\limsup_{t \to \infty} \frac{R_t}{t} \leq \sqrt{2\beta} - \rho$$

P^x-almost everywhere on $\{\zeta = \infty\}$ follows from Theorem 3.1.3 and the fact that $N_t^{-\rho}(\lambda t, \infty)$ is integer valued.

Remark 3.1.5. The almost-sure growth rate above is the same as the growth rate 'in expectation', by which we mean that for $0 < \lambda < \sqrt{2\beta} - \rho$,

$$\lim_{t\to\infty} t^{-1} \ln E^x\Big(N_t^{-\rho}(\lambda t, \infty)\Big) = \Delta(\lambda). \tag{3.4}$$

To see this, let $Y(t)$ be a Brownian motion started at $x > 0$ with drift $-\rho$ under the measure $\mathbb{P}^x_{-\rho}$ and define $\tau_0 := \inf\{t \geq 0 : Y(t) = 0\}$. The Many-to-One lemma states that, for measurable f,

$$E^x\bigg(\sum_{u\in N_t^{-\rho}} f(Y_u(t))\bigg) = e^{\beta t}\mathbb{E}^x_{-\rho}(f(Y_t); \tau_0 > t). \tag{3.5}$$

The expected growth rate (3.4) is now an easy consequence of (3.5) and the one-particle calculation

$$\mathbb{P}^x_{-\rho}(Y(t) \geq \lambda t; \tau_0 > t) \underset{t\to\infty}{\sim} \frac{1}{(\lambda+\rho)\sqrt{2\pi t}}(1 - e^{-2\lambda x})e^{(\lambda+\rho)x - \frac{1}{2}(\lambda+\rho)^2 t}.$$

In Section 3.4, we investigate the probability that the right-most particle has travelled at faster speeds than usual.

Theorem 3.1.6. *For $\lambda > \sqrt{2\beta} - \rho$ and all $x > 0, \theta \geq 0$, there exists a constant $C > 0$ such that*

$$\lim_{t\to\infty} P^x(R_t \geq \lambda t + \theta)\frac{\sqrt{2\pi t}}{(1 - e^{-2\lambda x})}e^{-(\lambda+\rho)(x-\theta) + (\frac{1}{2}(\lambda+\rho)^2 - \beta)t} = C, \tag{3.6}$$

or equivalently,

$$P^x(R_t > \lambda t + \theta) \underset{t\to\infty}{\sim} (\lambda+\rho)C \times E^x(N_t^{-\rho}(\lambda t + \theta, \infty)). \tag{3.7}$$

Considering particles with spatial position $Y_u(t) > \lambda t + \theta$, for $\lambda > \sqrt{2\beta} - \rho$, a Yaglom-type result also holds.

Theorem 3.1.7. *For $\lambda > \sqrt{2\beta} - \rho$ there is a probability distribution $(\Pi_i)_{i\geq 1}$ defined on $\mathbb{N}$ such that*

$$\lim_{t\to\infty} P^x\Big(N_t^{-\rho}(\lambda t, +\infty) = i \Big| N_t^{-\rho}(\lambda t, +\infty) > 0\Big) = \Pi_i,$$

and this distribution has (finite) expectation equal to $1/(\lambda + \rho)C$.

Finally, we can see the fundamental nature of the W_λ martingale that we introduced into the killed BBM story: it appears as the Radon-Nikodým derivative linking P^x with the limit-law of the conditioned process.

Theorem 3.1.8. *For $\lambda > \sqrt{2\beta} - \rho$, $s \in (0, \infty)$ fixed, and $A \in \mathcal{F}_s$,*

$$P^x\Big(A\Big|N_{s+t}^{-\rho}(\lambda(s+t), \infty) > 0\Big) \xrightarrow[t \to +\infty]{} E^x\Big(\mathbf{1}_A \frac{W_\lambda(s)}{W_\lambda(0)}\Big).$$

Chauvin and Rouault [20] proved analogous results to Theorems 3.1.6–3.1.8 in the context of standard branching Brownian motion. Although guided by their approach when we prove Theorem 3.1.6, there are a number of significant complications and novelties caused by the killing at the origin. However, once these additional difficulties are overcome and we have proven Theorem 3.1.6, the proofs of Chauvin and Rouault [20] adapt almost unchanged for Theorems 3.1.7 and 3.1.8.

Branching processes with absorbing barriers have been seen before. Kesten [58] considers some related questions on survival probabilities, $P^x(R_t > 0)$, and population growth-rates in fixed subsets of $\mathbb{R}$ for a similar branching Brownian motion with an absorbing barrier. This work is discussed in much greater detail in Chapter 4. Biggins *et al.* [12], and Biggins and Kyprianou [11] considered a branching random walk with a barrier.

Remark 3.1.9. Recall that, when it exists, the two-sided travelling wave solution of the system (1.6) can be expressed as $f(x) = E^x e^{-Z_\lambda(\infty)}$, where λ satisfies $\frac{1}{2}(\lambda+\rho)^2 - \rho(\lambda+\rho) + \beta = 0$. Given the importance of W_λ in the analysis of the $(-\rho, \beta; \mathbb{R}^+)$-BBM, it is natural to ask whether $f_\lambda(x) := E^x e^{-W_\lambda(\infty)}$ is the one-sided solution of the system (2.4) (for a suitable choice of λ). The boundary condition $f_\lambda(0+) = 1$ is satisfied, but f_λ is not the one-sided solution. We can see this by writing

$$f_\lambda(x) = E^x(e^{-W_\lambda(\infty)}; \zeta = \infty) + E^x(e^{-W_\lambda(\infty)}; \zeta < \infty),$$

and then for any $\lambda \in (0, \sqrt{2\beta} - \rho)$,

$$E^x(e^{-W_\lambda(\infty)}; \zeta < \infty) = E^x(1; \zeta < \infty) = P^x(\zeta < \infty).$$

It follows from Theorem 3.1.2 that

$$E^x(e^{-W_\lambda(\infty)}; \zeta = \infty) \in (0, 1),$$

whence $f_\lambda(x) \neq P^x(\zeta < \infty)$. If, on the other hand, $\lambda > \sqrt{2\beta} - \rho$, then $f_\lambda(x) \equiv 1$.

3.2 The Hardy and Harris spine construction

In this thesis we construct our spines using the approach detailed in Hardy and Harris [41]. This is a much more natural construction than that seen for Galton-Watson processes in the earlier work of Lyons [71], Lyons *et al.* [72], and for branching diffusions in Chauvin and Rouault [20], and Kyprianou [66]. These approaches all involved certain measures that did not have finite mass, and so could not be normalised to probability measures. In the Hardy and Harris approach, we add extra structure to the probability space of the BBM in the form of a 'finer' filtration of the space. We then define a single-particle martingale with respect to this largest filtration; and we recover an additive martingale for the whole branching particle process when we look at the conditional expectation, with respect to a different filtration, of the single-particle martingale.

3.2.1 Spine notation

In this section we give the background results and notation required for the spine set-up. We give the most general presentation here, but remark that certain minor modifications are required in later chapters — for instance when we consider typed branching diffusions. Note in particular that the notation in this section is generalised to allow each particle u to give birth to $1+A_u$ offspring, where each A_u is an independent copy of a random variable taking values in $\{0, 1, 2, \ldots\}$. The spine techniques developed in this thesis could readily be applied to such models.

All probability measures are to be defined on the space $\tilde{\mathcal{T}}$ of *marked Galton-Watson trees with spines*; before defining this space precisely we need to set up some other notation. We recall the set of Ulam-Harris labels, Ω, defined by

$$\Omega := \{\varnothing\} \cup \bigcup_{n \in \mathbb{N}} (\mathbb{N})^n,$$

where $\mathbb{N} := \{1, 2, 3, \ldots\}$. For two words $u, v \in \Omega$, uv denotes the concatenated word, where we take $u\varnothing = \varnothing u = u$. So, for example, Ω contains elements such as $\varnothing 412$, which represents 'the individual being the second child of the first child of the fourth child of the initial ancestor $\varnothing$. For two labels $u, v \in \Omega$ the notation $v < u$ means that v is an ancestor of u, and $|u|$ denotes the length of u.

We define a *Galton-Watson tree* to be a set $\tau \subset \Omega$ such that:

(i) $\varnothing \in \tau$: there is the unique initial ancestor;

(ii) if $u, v \in \Omega$, then $vu \in \tau \Rightarrow v \in \tau$: this means that τ contains all of its ancestors of its nodes;

(iii) for all $u \in \tau$, there exists $A_u \in \{0, 1, 2, \ldots\}$ such that for $j \in \mathbb{N}$, $uj \in \tau$ if and only if $1 \leq j \leq 1 + A_u$.

The set of all such trees is $\mathbb{T}$, and we will use the symbol τ for a particular tree. As our work concerns branching diffusions, we shall often refer to the labels of τ as particles. Note that for the binary branching mechanisms considered in this thesis we have $P(A_u = 1) \equiv 1$, and so there is exactly one $\tau \in \mathbb{T}$ — the binary tree.

A Galton-Watson tree by itself only records the family structure of the individuals, so to each individual $u \in \tau$ we give a mark (Y_u, σ_u) which contains the following information: $\sigma_u \in [0, \infty)$ is the *lifetime* of particle u, which also determines the *fission* or *death time* of the particle as $S_u := \sum_{v \leq u} \sigma_v$; and the function $Y_u(t) : [S_u - \sigma_u, S_u) \to \mathbb{R}$ describes the particle's *spatial* motion in $\mathbb{R}$ during its lifetime. For clarity we must decide whether or not a particle is in existence at its death time; our convention will be that a particle dies 'infinitesimally before' its death time — this is why Y_u is defined on $[S_u - \sigma_u, S_u)$ and not $[S_u - \sigma_u, S_u]$ — so that at time S_u the particle u has disappeared and has been replaced by its two children.

We denote a particular marked tree by (τ, Y, σ), or the abbreviation (τ, M), and the set of all marked Galton-Watson trees by $\mathcal{T}$. For each $(\tau, Y, \sigma) \in \mathcal{T}$, the set of particles alive at time t is defined as $N_t := \{u \in \tau : S_u - \sigma_u \leq t < S_u\}$; note, however, that for the $(-\rho, \beta; \mathbb{R})$-BBM and $(-\rho, \beta; \mathbb{R}^+)$-BBM we use the symbols $\mathcal{N}_t^{-\rho}$ and $N_t^{-\rho}$ respectively. For any given marked tree $(\tau, M) \in \mathcal{T}$ we can distinguish individual lines of descent from the initial ancestor: $\varnothing, u_1, u_2, u_3, \ldots \in \tau$ where u_i is a child of u_{i-1} for all $i \in \{2, 3, \ldots\}$ and u_1 is a child of the initial individual $\varnothing$. We call such a line of descent a *spine* and denote it by ξ. In a slight abuse of notation we refer to ξ_t as the unique node in ξ that is alive at time t, and also for the position of the particle that makes up the spine at time t; that is $\xi_t := Y_u(t)$, where $u \in \xi \cap N_t$. However, although the interpretation of ξ_t should always be clear from the context, we introduce the following notation for use where some ambiguity may still arise: $\text{node}_t((\tau, M, \xi)) := u$ if $u \in \xi$ is the node in the spine alive at time t. It is natural to think of the spine as a single diffusing particle ξ_t.

We define n_t to be a counting function that tells us which generation of the spine is currently alive, or equivalently the number of fission times there have been on the spine: $n_t := |\text{node}_t(\xi)|$. The collection of all marked trees with a distinguished spine is the space $\tilde{\mathcal{T}}$ on which our probability measures will eventually be defined, but first we define four filtrations on this space, each containing different levels of information about the branching diffusion.

Filtration $(\mathcal{F}_t)_{t\geq 0}$

We define a filtration of $\tilde{\mathcal{T}}$ made up of the σ-algebras

$$\mathcal{F}_t := \sigma\bigg((u, Y_u, \sigma_u) : S_u \leq t; (u, Y_u(s) : s \in [S_u - \sigma_u, t]) : t \in [S_u - \sigma_u, S_u)\bigg),$$

which means that $\mathcal{F}_t$ is generated by the information concerning all particles that have lived and died before time t, and also those that are still alive at time t. This is what is usually referred to in the literature as the 'natural filtration'. Each of these σ-algebras is a subset of the limit

$$\mathcal{F}_\infty := \sigma\bigg(\bigcup_{t\geq 0} \mathcal{F}_t\bigg).$$

Filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$

We define the filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ by augmenting the filtration $\mathcal{F}_t$ with the knowledge of which node is the spine at time t:

$$(\tilde{\mathcal{F}}_t)_{t\geq 0} := \sigma(\mathcal{F}_t, \text{node}_t(\xi)), \qquad \tilde{\mathcal{F}}_\infty := \sigma\bigg(\bigcup_{t\geq 0} \tilde{\mathcal{F}}_t\bigg),$$

so that this filtration knows everything about the branching diffusion and everything about the spine. It is the filtration that contains the most information.

Filtration $(\mathcal{G}_t)_{t\geq 0}$

$(\mathcal{G}_t)_{t\geq 0}$ is a filtration of $\tilde{\mathcal{T}}$ defined by

$$\mathcal{G}_t := \sigma(\xi_s : 0 \leq s \leq t), \qquad \mathcal{G}_\infty := \sigma\bigg(\bigcup_{t\geq 0} \mathcal{G}_t\bigg).$$

These σ-algebras are generated only by the spine's motion and so do not contain the information about which nodes of the tree τ make up the spine.

Filtration $(\tilde{\mathcal{G}}_t)_{t\geq 0}$

As we did in going from $\mathcal{F}_t$ to $\tilde{\mathcal{F}}_t$ we create $(\tilde{\mathcal{G}}_t)_{t\geq 0}$ from $(\mathcal{G}_t)_{t\geq 0}$ by including knowledge of which nodes make up the spine:

$$(\tilde{\mathcal{G}}_t)_{t\geq 0} := \sigma(\mathcal{G}_t, \mathrm{node}_t(\xi)), \qquad \tilde{\mathcal{G}}_\infty := \sigma\Big(\bigcup_{t\geq 0} \tilde{\mathcal{G}}_t\Big).$$

This means that $\tilde{\mathcal{G}}_t$ also knows when the fission times on the spine occurred, whereas $\mathcal{G}_t$ does not.

The relationships between the filtrations of $\tilde{\mathcal{T}}$ may be summarised thus:

$$\mathcal{G}_t \subset \tilde{\mathcal{G}}_t \subset \tilde{\mathcal{F}}_t$$

$$\mathcal{F}_t \subset \tilde{\mathcal{F}}_t$$

Importantly, we have $\mathcal{G}_t \not\subset \mathcal{F}_t$, since $\mathcal{F}_t$ does not know which line of descent makes up the spine and so it cannot know the spine's motion.

Now that we have defined the underlying space and the filtrations of it that we require, we can define the probability measures for branching diffusions. Recalling the notation from Chapter 2, we have the measures $\{P^x : x \in \mathbb{R}\}$ on $(\tilde{\mathcal{T}}, \mathcal{F}_\infty)$ for the law of the $(-\rho, \beta; \mathbb{R})$-BBM; and we now extend the measure P to a measure $\tilde{P}$ on $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}}_\infty)$, which is the joint law of the $(-\rho, \beta; \mathbb{R})$-BBM with a spine. To achieve this we use the following result of Lyons [71].

Theorem 3.2.1. *If f is an $\tilde{\mathcal{F}}_t$-measurable function then we can write*

$$f = \sum_{u \in N_t} f_u \mathbf{1}_{\{\xi_t = u\}}, \tag{3.8}$$

where f_u is $\mathcal{F}_t$-measurable.

Now we extend $P^{x,y}$ to a measure $\tilde{P}^{x,y}$ on $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}}_\infty)$ by choosing the particle that makes up the spine uniformly at each fission time on the spine; more precisely, for any

$f \in m\tilde{\mathcal{F}}_t$ with a representation like (3.8) we have

$$\int_{\tilde{\mathcal{T}}} f \, \mathrm{d}\tilde{P}^x(\tau, M, \xi) := \int_{\mathcal{T}} \sum_{u \in N_t} f_u \prod_{v<u} \frac{1}{2} \mathrm{d}P^x(\tau, M).$$

$$\int_{\tilde{\mathcal{T}}} f \, \mathrm{d}\tilde{P}^x(\tau, M, \xi) := \int_{\mathcal{T}} \sum_{u \in N_t} f_u \prod_{v<u} \frac{1}{2} \mathrm{d}P^x(\tau, M).$$

Remark 3.2.2. Empty products are taken as equal to 1. Note that $P = \tilde{P}|_{\mathcal{F}_\infty}$.

In calculations we think of the spine as the 'backbone' of the BBM. This idea is made precise by the following decomposition of the measure $\tilde{P}$, which was first seen in Chauvin and Rouault [20]. We state the decomposition here in the form it appeared in Kyprianou [66].

Theorem 3.2.3. *The measure $\tilde{P}$ on $\tilde{\mathcal{F}}_t$ can be decomposed as:*

$$\mathrm{d}\tilde{P}(\tau, M, \xi) = \mathrm{d}\mathbb{P}_{-\rho}(\xi_t)\mathrm{d}\mathbb{L}^\beta(n_t) \prod_{v<\xi_t} \frac{1}{2} \mathrm{d}P((\tau, M)^v), \tag{3.9}$$

where ξ_t is a Brownian motion with drift $-\rho$ under $\mathbb{P}_{-\rho}$, and $\mathbb{L}^\beta$ is the law of a Poisson process with rate β.

This result means that the BBM may be constructed path-wise under $\tilde{P}$ by:

- the spine's spatial motion is determined by the single-particle measure $\mathbb{P}_{-\rho}$;
- the fission times on the spine occur as a Poisson process of rate β that is independent of the spine's motion;
- at each fission time on the spine two particles are produced;
- one of these is chosen uniformly at random to be the spine and it repeats stochastically the behaviour of its parent;
- the other particle initiates, from its birth position, an independent $(-\rho, \beta; \mathbb{R})$-BBM with law $P^{\cdot}$, giving rise to the subtree $(\tau, M)^v$, which is not part of the spine.

3.2.2 New measures for BBM

Now that we have set up the notation for the spine approach and defined the space $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}}_t, \tilde{P})$, we can give the changes of measure that we use to alter the behaviour of the spine. We first give details of some well-known martingales for one-dimensional diffusions, which we shall eventually combine to produce new additive martingales to change measure on the probability space of the BBM.

Proposition 3.2.4 (Change of measure for Poisson processes). *Let $g(t) : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-negative, bounded, continuous function and suppose that the Poisson process $(n, \mathbb{L}^g)$ has instantaneous rate $g(t)$, where $n = \{\{S_i : i = 1, \dots, n_t\} : t \geq 0\}$. Further, assume that n is adapted to $(\mathcal{G}_t)_{t\geq 0}$. Then under the change of measure*

$$\left.\frac{\mathrm{d}\mathbb{L}^{2g}}{\mathrm{d}\mathbb{L}^g}\right|_{\mathcal{G}_t} = 2^{n_t} \exp\left(- \int_0^t g(s)\,\mathrm{d}s \right),$$

the process $(n, \mathbb{L}^{2g})$ is a Poisson process with rate $2g(t)$ (See Jacod and Shiryaev [54, Chapter 3], and Engländer and Kyprianou [32]).

Proposition 3.2.5 (Conditioning a drifting Brownian motion to stay positive).
Let $x > 0$ and X_t be a Brownian motion (started at the point x) with drift $\lambda > 0$ under $\mathbb{P}^x_\lambda$ that is adapted to some filtration $(\mathcal{H}_t)_{t\geq 0}$. Then $1 - e^{-2\lambda X_t}$ is a $\mathbb{P}_\lambda$-martingale and so we can define a measure $\mathbb{P}^x_{(\mathrm{B},\lambda)}$ via the Radon-Nikodým derivative

$$\left.\frac{\mathrm{d}\mathbb{P}^x_{(\mathrm{B},\lambda)}}{\mathrm{d}\mathbb{P}^x_\lambda}\right|_{\mathcal{H}_t} = \frac{1 - e^{-2\lambda X_{t\wedge\tau_0}}}{1 - e^{-2\lambda x}}, \qquad \text{where } \tau_0 := \inf\{t > 0 : X_t = 0\},$$

and, under $\mathbb{P}_{(\mathrm{B},\lambda)}$, X is a Bessel-3 process with drift λ.

Remark 3.2.6. Proposition 3.2.5 can also be viewed as a Doob h-transform, since

$$\mathbb{P}^x_\lambda(\tau_0 = \infty) = 1 - e^{-2\lambda x}.$$

We build a single-particle martingale on $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}}_\infty, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{P}^x)$ from the results in Propositions 3.2.4 and 3.2.5, and the usual exponential martingale for changing the drift of a Brownian motion:

$$\tilde{W}_\lambda(t) := 2^{n_t} e^{-\beta t}(1 - e^{-2\lambda\xi_t}) e^{(\lambda+\rho)\xi_t - \frac{1}{2}(\lambda^2 - \rho^2)t} \mathbf{1}_{\{\tau_0 > t\}},$$

where $\tau_0 := \inf\{t > 0 : \xi_t = 0\}$. Using this we can define a new measure $\tilde{\mathbb{Q}}_\lambda$, for $\lambda > 0$, on $\tilde{\mathcal{F}}_t$ via

$$\left.\frac{\mathrm{d}\tilde{\mathbb{Q}}^x_\lambda}{\mathrm{d}\tilde{P}^x}\right|_{\tilde{\mathcal{F}}_t} := \frac{\tilde{W}_\lambda(t)}{\tilde{W}_\lambda(0)}. \tag{3.10}$$

Recalling the decomposition (3.9) of $\tilde{P}$, we can see the effect of changing measure with $\tilde{W}_t$, and can find a similar decomposition for $\tilde{\mathbb{Q}}$:

$$\begin{aligned}
\mathrm{d}\tilde{\mathbb{Q}}^x_\lambda &= \frac{\tilde{W}_\lambda(t)}{\tilde{W}_\lambda(0)}\mathrm{d}\tilde{P}^x \\
&= \frac{e^{-(\lambda+\rho)x}}{1-e^{-2\lambda x}}(1-e^{-2\lambda\xi_t})e^{(\lambda+\rho)\xi_t-\frac{1}{2}(\lambda^2-\rho^2)t}\mathbf{1}_{\{\tau_0>t\}} \times 2^{n_t}e^{-\beta t} \\
&\qquad \times \mathrm{d}\mathbb{P}_{-\rho}(\xi_t)\mathrm{d}\mathbb{L}^\beta(n_t)\prod_{v<\xi_t}\frac{1}{2}\mathrm{d}P((\tau,M)^v) \\
&= \mathrm{d}\mathbb{P}_{(\mathrm{B},\lambda)}(\xi_t)\mathrm{d}\mathbb{L}^{2\beta}(n_t)\prod_{v<\xi_t}\frac{1}{2}\mathrm{d}P((\tau,M)^v).
\end{aligned} \tag{3.11}$$

As was the case with $\tilde{P}$, the decomposition (3.11) tells us how to construct the BBM in law under $\tilde{\mathbb{Q}}_\lambda$:

- the spine's spatial motion is determined by the single-particle measure $\mathbb{P}_{(\mathrm{B},\lambda)}$;
- the fission times on the spine occur as a Poisson process of rate 2β that is independent of the spine's motion;
- at each fission time on the spine two particles are produced;
- one of these is chosen uniformly at random to be the spine and it repeats stochastically the behaviour of its parent;
- the other particle initiates, from its birth position, an independent $(-\rho,\beta;\mathbb{R})$-BBM with law $P^\cdot$, giving rise to the subtree $(\tau,M)^v$, which is not part of the spine.

If we define a measure $\mathbb{Q}_\lambda$ on $(\tilde{\mathcal{T}},\mathcal{F}_\infty,\{\mathcal{F}_t\}_{t\geq 0})$ as the restriction of $\tilde{\mathbb{Q}}_\lambda$, that is

$$\mathbb{Q}_\lambda := \tilde{\mathbb{Q}}_\lambda|_{\mathcal{F}_\infty},$$

we have that that, under $\mathbb{Q}_\lambda$, the BBM has the same path-wise construction as under $\tilde{\mathbb{Q}}_\lambda$. We now check that this is consistent with equation (3.2), by showing that W_λ is in fact obtained from $\tilde{W}_\lambda$ by taking a suitable conditional expectation.

Proposition 3.2.7. *The measure $\mathbb{Q}_\lambda$ satisfies*

$$\left.\frac{\mathrm{d}\mathbb{Q}^x_\lambda}{\mathrm{d}P^x}\right|_{\mathcal{F}_t} = \frac{W_\lambda(t)}{W_\lambda(0)}. \tag{3.12}$$

Proof. We first make the more general observation (see Williams [88]) that if $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are two measures defined on a measure space $(\Omega, \tilde{\mathcal{S}})$ with Radon-Nikodým derivative

$$\frac{\mathrm{d}\tilde{\mu}_2}{\mathrm{d}\tilde{\mu}_1} = g,$$

and if $\mathcal{S}$ is a sub-σ-algebra of $\tilde{\mathcal{S}}$, then the measures $\mu_1 := \tilde{\mu}_1|_{\mathcal{S}}$ and $\tilde{\mu}_2|_{\mathcal{S}}$ on $(\Omega, \mathcal{S})$ satisfy

$$\frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_1} = \tilde{\mu}_1(g|\mathcal{S}).$$

Applying this to the measures $\tilde{P}$ and $\tilde{\mathbb{Q}}_\lambda$, the change of measure (3.10) projects on to the sub-algebra $\mathcal{F}_t$ as a conditional expectation:

$$\left.\frac{\mathrm{d}\tilde{\mathbb{Q}}^x_\lambda}{\mathrm{d}\tilde{P}^x}\right|_{\mathcal{F}_t} = \frac{1}{W_\lambda(0)}\tilde{P}^x\left(2^{n_t}e^{-\beta t}(1-e^{-2\lambda\xi_t})e^{(\lambda+\rho)\xi_t-\frac{1}{2}(\lambda^2-\rho^2)t}\mathbf{1}_{\{\tau_0>t\}}\middle|\,\mathcal{F}_t\right).$$

Now we can write $2^{n_t} = \prod_{v<\xi_t} 2$ and then if we use the representation (3.8) we get

$$\begin{aligned}
&\tilde{P}^x\left(2^{n_t}e^{-\beta t}(1-e^{-2\lambda\xi_t})e^{(\lambda+\rho)\xi_t-\frac{1}{2}(\lambda^2-\rho^2)t}\mathbf{1}_{\{\tau_0>t\}}\middle|\,\mathcal{F}_t\right)\\
&= \tilde{P}^x\left(e^{-\beta t}\sum_{u\in\mathcal{N}_t^{-\rho}}(1-e^{-2\lambda\mathcal{Y}_u(t)})e^{(\lambda+\rho)\mathcal{Y}_u(t)-\frac{1}{2}(\lambda^2-\rho^2)t}\times\mathbf{1}_{\{\tau_u>t\}}\times\prod_{v<u}2\times\mathbf{1}_{\{\xi_t=u\}}\middle|\,\mathcal{F}_t\right)\\
&= e^{-\beta t}\sum_{u\in\mathcal{N}_t^{-\rho}}(1-e^{-2\lambda\mathcal{Y}_u(t)})e^{(\lambda+\rho)\mathcal{Y}_u(t)-\frac{1}{2}(\lambda^2\rho^2)t}\times\mathbf{1}_{\{\tau_u>t\}}\times\prod_{v<u}2\times\tilde{P}^x(\xi_t=u|\mathcal{F}_t),
\end{aligned}$$

where $\tau_u := \inf\{t>0 : \mathcal{Y}_u(t)=0\}$. But since we are considering binary splitting only, $\tilde{P}^x(\xi_t = u|\mathcal{F}_t) = \prod_{v<u}\frac{1}{2}$ and hence

$$\begin{aligned}
&\tilde{P}^x\left(2^{n_t}e^{-\beta t}(1-e^{-2\lambda\xi_t})e^{(\lambda+\rho)\xi_t-\frac{1}{2}(\lambda^2-\rho^2)t}\mathbf{1}_{\{\tau>t\}}\middle|\,\mathcal{F}_t\right)\\
&\qquad= e^{-\beta t}\sum_{u\in\mathcal{N}_t^{-\rho}}(1-e^{-2\lambda\mathcal{Y}_u(t)})e^{(\lambda+\rho)\mathcal{Y}_u(t)-\frac{1}{2}(\lambda^2-\rho^2)t}\mathbf{1}_{\{\tau_u>t\}}\\
&\qquad= \sum_{u\in N_t^{-\rho}}(1-e^{-2\lambda Y_u(t)})e^{(\lambda+\rho)Y_u(t)-(\frac{1}{2}(\lambda^2-\rho^2)+\beta)t}\\
&\qquad= W_\lambda(t),
\end{aligned}$$

as required. □

Remark 3.2.8. Proposition 3.2.7 on its own implies that W_λ is a martingale, although we already knew this from Lemma 3.1.1.

Remark 3.2.9. Events containing information about the spine are neither P- nor $\mathbb{Q}_\lambda$-measurable. However, for any $A \in \mathcal{F}_\infty$, we have (by definition) $\tilde{P}(A) = P(A)$ and $\tilde{\mathbb{Q}}_\lambda(A) = \mathbb{Q}_\lambda(A)$.

3.3 Proofs of the martingale results

Before proving the main results for W_λ described earlier, we first prove a result which identifies the speed of the particles that contribute to the limit of W_λ.

Proposition 3.3.1. *For any $\varepsilon > 0$ we have, P^x-almost surely,*

$$W_\lambda(\infty) = \lim_{t\to\infty} \sum_{u\in N_t^{-\rho}} (1 - e^{-2\lambda Y_u(t)}) e^{(\lambda+\rho)Y_u(t) - (\frac{1}{2}(\lambda^2-\rho^2)+\beta)t} \mathbf{1}_{\left\{\left|\frac{Y_u(t)}{t} - \lambda\right| < \varepsilon\right\}}$$

Proof. Define $E_\lambda = E(\lambda) := \frac{1}{2}(\lambda^2 - \rho^2) + \beta$, let $\varepsilon > 0$, and set $\mu := \lambda - \varepsilon$. Our method of proof is to show that particles at order t (or greater) displacements from the spatial ray of gradient λ make no contribution to the martingale limit.

$$\begin{aligned}
&\sum_{u\in N_t^{-\rho}} (1 - e^{-2\lambda Y_u(t)}) e^{(\lambda+\rho)Y_u(t) - E_\lambda t} \mathbf{1}_{\{0<Y_u(t)\le(\lambda-\varepsilon)t\}} \\
&\qquad \le \sum_{u\in N_t^{-\rho}} e^{(\mu+\rho)Y_u(t) - E_\mu t} \mathbf{1}_{\{0<Y_u(t)\le(\lambda-\varepsilon)t\}} \times e^{(\lambda-\mu)Y_u(t) - (E_\lambda - E_\mu)t} \\
&\qquad \le e^{-\frac{1}{2}\varepsilon^2 t} \sum_{u\in \mathcal{N}_t^{-\rho}} e^{(\mu+\rho)\mathcal{Y}_u(t) - E_\mu t} = e^{-\frac{1}{2}\varepsilon^2 t} Z_\mu(t),
\end{aligned}$$

where $Z_\mu(t)$ is the martingale given in equation (2.3), which is positive and hence almost surely convergent. Hence, P^x-almost surely,

$$\limsup_{t\to\infty} \sum_{u\in N_t^{-\rho}} (1 - e^{-2\lambda Y_u(t)}) e^{(\lambda+\rho)Y_u(t) - E_\lambda t} \mathbf{1}_{\{0<Y_u(t)\le(\lambda-\varepsilon)t\}} = 0.$$

Similarly, on setting $\mu := \lambda + \varepsilon$, we obtain

$$\sum_{u\in N_t^{-\rho}} (1 - e^{-2\lambda Y_u(t)}) e^{(\lambda+\rho)Y_u(t) - E_\lambda t} \mathbf{1}_{\{Y_u(t)\ge(\lambda+\varepsilon)t\}} \le e^{-\frac{1}{2}\varepsilon^2 t} Z_\mu(t)$$

and the result follows. □

Remark 3.3.2. It is implicit in the proof of Proposition 3.3.1 that we also have another representation for $W_\lambda(\infty)$:

$$W_\lambda(\infty) = \lim_{t\to\infty} \sum_{u\in N_t^{-\rho}} e^{(\lambda+\rho)Y_u(t)-(\frac{1}{2}(\lambda^2-\rho^2)+\beta)t}. \tag{3.13}$$

The limit of W_λ is equal to that part of the limit of Z_λ contributed by particles that avoided the origin.

Recall that $P^x(\zeta = \infty) > 0$ if and only if $\rho < \sqrt{2\beta}$, and note that if this condition does not hold there can be no values of λ for which W_λ is uniformly integrable. The critical λ value of $\sqrt{2\beta} - \rho$ in Theorem 3.1.2 corresponds to the right-most particle asymptotic of Lemma 2.1.2, tallying with the intuitive notion that the W_λ martingale limit counts particles travelling at speed λ.

We will now prove Theorem 3.1.2 on the convergence properties of W_λ. The part of Theorem 3.1.2 concerning uniform integrablity will follow from the stronger result given in the next lemma.

Lemma 3.3.3. *For $\lambda, x > 0$ and for any $p \in (1, 2]$:*

(i) the martingale W_λ is $\mathcal{L}^p(P^x)$-convergent if $\frac{1}{2}p(\lambda+\rho)^2 < \beta$;

(ii) almost surely under P^x, $W_\lambda(\infty) = 0$ when $\frac{1}{2}(\lambda+\rho)^2 \geq \beta$.

Proof. (i) Using Doob's $\mathcal{L}^p$-inequality we need only show that W_λ is bounded in $\mathcal{L}^p(P^x)$ when $\frac{1}{2}p(\lambda+\rho)^2 < \beta$; this follows immediately from the inequality $W_\lambda \leq Z_\lambda$ and known results for $\mathcal{L}^p(P^x)$-boundedness of Z_λ.

(ii) We have $0 \leq W_\lambda(t) \leq Z_\lambda(t)$ for all $t \geq 0$, and it is well known that $Z_\lambda(t) \to 0$ almost surely when $\frac{1}{2}(\lambda+\rho)^2 \geq \beta$. □

We now prove Theorem 3.1.2.

Proof of Theorem 3.1.2. It follows from Lemma 3.3.3 that if $\frac{1}{2}(\lambda+\rho)^2 < \beta$ there exists a $p > 1$ such that W_λ converges in $\mathcal{L}^p(P^x)$, whence W_λ is uniformly integrable.

It remains to check that process survival is equivalent to a strictly positive limit for W_λ. From the definition for W_λ, it is clear that $\{\zeta < \infty\} \subseteq \{W_\lambda(\infty) = 0\}$, so that $P^x(W_\lambda(\infty) = 0; \zeta < \infty) = P^x(\zeta < \infty)$. We can also write $P^x(W_\lambda(\infty) = 0)$ as

$$P^x(W_\lambda(\infty) = 0) = P^x(W_\lambda(\infty) = 0; \zeta < \infty) + P^x(W_\lambda(\infty) = 0; \zeta = \infty),$$

and the result follows if we can show that $P^x(W_\lambda(\infty) = 0) = P^x(\zeta < \infty)$. Define $g(x) := P^x(W_\lambda(\infty) = 0)$, and then, by a similar argument to that used in the proof of Theorem 2.5.1

$$g(x) = E^x\Big(P^x(W_\lambda(\infty) = 0|\mathcal{F}_t)\Big) = E^x\Big(\prod_{u\in N_t^{-\rho}} g(Y_u(t))\Big), \tag{3.14}$$

and hence $g(x)$ satisfies both the ODE in the system (2.4) and the boundary condition

$$\lim_{x\downarrow 0} g(x) = 1.$$

With the representation of equation (3.13) in mind, considering the $(-\rho,\beta;\mathbb{R})$-BBM path-wise we see that $g(x)$ is monotone decreasing in x, and so $g(x)\downarrow g(\infty)$ as $x\to\infty$. Now for any fixed time $t>0$, we have $N_t^{-\rho}\uparrow \mathcal{N}_t^{-\rho}$ as $x\to\infty$, and, looking at the process path-wise again, we also have $Y_u(t)\uparrow\infty$ as $x\to\infty$, for all $u\in N_t^{-\rho}$. Thus we may take limits on both sides of (3.14) to obtain $g(\infty) = E^0\left(\prod_{u\in\mathcal{N}_t^{-\rho}} g(\infty)\right)$, whence $g(\infty)=0$ or 1. Since W_λ is uniformly integrable for the values of λ under consideration we must have $g(\infty) = 0$. Hence $g(x)$ satisfies the ODE and boundary conditions in (2.4), and uniqueness of the one-sided travelling wave completes the argument. □

We conclude this section by proving our result for exponential growth of particles in the super-critical region.

Proof of Theorem 3.1.3. The key idea in the proof is to overestimate the indicator functions in (3.3) by exponentials and then re-arrange the expression to obtain martingale terms.

Bounding $N_t^{-\rho}(\lambda t,\infty)$ above, we have

$$N_t^{-\rho}(\lambda t,\infty) = \sum_{u\in N_t^{-\rho}} \mathbf{1}_{\{Y_u(t)-\lambda t\geq 0\}} \leq \sum_{u\in\mathcal{N}_t^{-\rho}} e^{(\lambda+\rho)(\mathcal{Y}_u(t)-\lambda t)} = e^{\Delta(\lambda)t} Z_\lambda(t).$$

Now if $\lambda\geq\sqrt{2\beta}-\rho$ then Z_λ has an almost-sure zero limit and $\Delta(\lambda)\leq 0$, whence, noting that $N_t^{-\rho}(\lambda t,\infty)$ is integer valued,

$$\sum_{u\in N_t^{-\rho}} \mathbf{1}_{\{Y_u(t)\geq\lambda t\}} = 0$$

eventually, with probability one. On the other hand if $0<\lambda<\sqrt{2\beta}-\rho$, then $Z_\lambda(\infty)>0$

almost surely, and $\Delta(\lambda) > 0$, so

$$\limsup_{t\to\infty} t^{-1} \ln N_t^{-\rho}(\lambda t, \infty) \le \Delta(\lambda), \qquad \text{a.s.}$$

For the reverse inequality, let $\varepsilon > 0$ be small and $0 < \lambda < \sqrt{2\beta} - \rho$. Then

$$\begin{aligned}\sum_{u\in N_t^{-\rho}} &(1 - e^{-2\lambda Y_u(t)})e^{(\lambda+\rho)Y_u(t) - E_\lambda t}\mathbf{1}_{\{(\lambda-\varepsilon)t<Y_u(t)\le(\lambda+\varepsilon)t\}}\\ &\le e^{((\lambda+\rho)\lambda - E_\lambda)t}e^{\varepsilon(\lambda+\rho)t}\sum_{u\in N_t^{-\rho}}\mathbf{1}_{\{(\lambda-\varepsilon)t<Y_u(t)\}}.\end{aligned}$$

Noting that $(\lambda+\rho)\lambda - E_\lambda = -\Delta(\lambda)$, we obtain

$$\begin{aligned}t^{-1}\ln \sum_{u\in N_t^{-\rho}} &(1 - e^{-2\lambda Y_u(t)})e^{(\lambda+\rho)Y_u(t) - E_\lambda t}\mathbf{1}_{\{(\lambda-\varepsilon)t<Y_u(t)\le(\lambda+\varepsilon)t\}}\\ &\le -\Delta(\lambda) + \varepsilon(\lambda+\rho) + t^{-1}\ln\sum_{u\in N_t^{-\rho}}\mathbf{1}_{\{(\lambda-\varepsilon)t<Y_u(t)\}}.\end{aligned} \tag{3.15}$$

Now as $t \to \infty$, it follows from the crucial facts that $W_\lambda(\infty) > 0$ (from Theorem 3.1.2), and that the limit only 'sees' particles of speed λ (from Proposition 3.3.1) that, on $\{\zeta = \infty\}$, the left-hand side of (3.15) tends to zero. Since $\varepsilon > 0$ is arbitrary, we find that

$$\liminf_{t\to\infty} t^{-1}\ln\sum_{u\in N_t^{-\rho}}\mathbf{1}_{\{Y_u(t)\ge\lambda t\}} \ge \Delta(\lambda), \qquad \text{a.s.}$$

which completes the proof. □

3.4 Proof of Theorem 3.1.6

The proof of Theorem 3.1.6 rests on the close links between branching diffusions and partial differential equations; essentially, the assertion of Theorem 3.1.6 is that as $t \to \infty$ the solution of the non-linear equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial u}{\partial x} + \beta u(1-u), \tag{3.16}$$

with $u \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^+)$ and an indicator function initial condition, is asymptotically equal to the solution of the linearised equation

$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2} - \rho\frac{\partial w}{\partial x} + \beta w, \tag{3.17}$$

for $w \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^+)$ and with the same initial condition.

Considering $y > 0$ fixed and $x, t > 0$, the link with PDEs can be seen from the observation that $v(t,x) := P^x(R_t < y)$ has the representation

$$v(t,x) = E^x\Big(\prod_{u \in N_s^{-\rho}} v(t-s, Y_u(s))\Big), \qquad s \in [0,t],$$

from which it follows that $\prod_{u \in N_s^{-\rho}} v(t-s, Y_u(s))$ is a product martingale on $[0,t]$ and so, as before, standard arguments using Kolmogorov's equations show that $v \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^+)$ and solves

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} - \rho\frac{\partial v}{\partial x} + \beta v(v-1),$$

with $v(0,x) = \mathbf{1}_{\{x<y\}}$. Defining

$$u(t,x,y) := P^x(R_t > y),$$

so $u = 1 - v$, we see that u satisfies the non-linear partial differential equation (3.16) for $(t,x,y) \in (0,\infty) \times (0,\infty) \times (0,\infty)$, with initial condition $u(0,x,y) = \mathbf{1}_{\{x>y\}}$. See Ikeda *et al.* [50, 51, 52] for extensive discussion of probabilistic solutions of ordinary and partial differential equations in a very general setting. Before proving Theorem 3.1.6 we shall establish some useful inequalities.

Proposition 3.4.1. *Let $w(t,x,y)$ be the solution of the linearised equation* (3.17) *with the same initial condition, that is $w(0,x,y) = \mathbf{1}_{\{x>y\}}$. Then $u(t,x,y) \le w(t,x,y)$ for all $t,x,y > 0$.*

Proof. Itô's formula implies that, for Y_t a Brownian motion with drift $-\rho$ under $\mathbb{P}^x_{-\rho}$ and $\tau_0 := \inf\{t > 0 : Y_t = 0\}$,

$$M_t(s) := u(t-(s\wedge\tau_0), Y_{s\wedge\tau_0}, y)\exp\left(\beta\int_0^{s\wedge\tau_0}(1-u(t-\phi, Y(\phi\wedge\tau_0), y))\,\mathrm{d}\phi\right)$$

is a $\mathbb{P}^x_{-\rho}$-local martingale on $[0,t]$, and in fact M_t is a uniformly integrable martingale on $[0,t]$. In particular

$$u(t,x,y) = \mathbb{E}^x_{-\rho}\left(\mathbf{1}_{\{Y_t>y\}}e^{\beta\int_0^t(1-u(t-\phi,Y_\phi,y))\,\mathrm{d}\phi}; \tau_0 > t\right). \tag{3.18}$$

The Feynman-Kac formula gives

$$w(t,x,y) = e^{\beta t}\mathbb{E}^x_{-\rho}(\mathbf{1}_{\{Y_t>y\}}; \tau_0 > t), \tag{3.19}$$

and it is clear that $u(t,x,y) \leq w(t,x,y)$ for all $t,x,y > 0$. □

Remark 3.4.2. The probabilistic interpretation of Proposition 3.4.1 is that, for all $t, x > 0$,

$$P^x(R_t > \lambda t + \theta) \leq E^x(N_t^{-\rho}(\lambda t + \theta, \infty)).$$

Proof of Theorem 3.1.6. Let $\{\mathbb{B}^t_{x_1,x_2}(\phi) : s \in [0,t]\}$ be the Brownian bridge that travels from point x_1 to x_2 over time period $[0,t]$, and let τ_0^t to be the first hitting of the origin by the bridge; then we may re-write (3.18) as

$$u(t,x,y) = e^{\beta t}\int_0^\infty \mathbb{P}^x_{-\rho}(Y_t - y \in \mathrm{d}z)\mathbb{E}\left(e^{-\beta\int_0^t u(t-\phi,\mathbb{B}^t_{x,z+y}(\phi),y)\,\mathrm{d}\phi}; \tau_0^t > t\right).$$

Hence

$$\begin{aligned} u(t,x,\lambda t+\theta) = \frac{e^{-(\frac{1}{2}(\lambda+\rho)^2-\beta)t}}{\sqrt{2\pi t}} e^{(\lambda+\rho)(x-\theta)} \int_0^\infty e^{-(\lambda+\rho)z}\exp\left(\frac{-(x-\theta-z)^2}{2t}\right) \\ \times\,\mathbb{E}\left(e^{-\beta\int_0^t u(t-\phi,\mathbb{B}^t_{x,z+\lambda t+\theta}(\phi),\lambda t+\theta)\,\mathrm{d}\phi}; \tau_0^t > t\right)\mathrm{d}z, \end{aligned}$$

and then, by dominated convergence, to complete the proof of (3.6) it suffices to show that there exists some function $g : (0,\infty) \to (0,1]$ such that

$$\mathbb{E}\left(e^{-\beta\int_0^t u(t-\phi,\mathbb{B}^t_{x,z+\lambda t+\theta}(\phi),\lambda t+\theta)\,\mathrm{d}\phi}; \tau_0^t > t\right) \xrightarrow[t\to\infty]{} g(z)(1-e^{-2\lambda x}),$$

since this would ensure that

$$\begin{aligned} &\int_0^\infty e^{-(\lambda+\rho)z}\exp\left(\frac{-(x-\theta-z)^2}{2t}\right) \\ &\quad\times\,\mathbb{E}\left(e^{-\beta\int_0^t u(t-\phi,\mathbb{B}^t_{x,z+\lambda t+\theta}(\phi),\lambda t+\theta)\,\mathrm{d}\phi}; \tau_0^t > t\right)\mathrm{d}z \xrightarrow[t\to\infty]{} C(1-e^{-2\lambda x}). \end{aligned}$$

Although we are essentially following the strategy of Chauvin and Rouault [20] here, extra effort is required to deal with the complications arising from the introduction of the absorbing barrier. In particular, Lemma 3.4.3 below uses a similar argument to show that a certain expectation converges, but additional work is needed to identify the limit; after this, the remaining difficulties arising from the introduction of the barrier are overcome using a carefully chosen construction of the family of Brownian bridges from two independent Brownian motions.

Letting $B := \{B(s) : s \geq 0\}$ be a standard Brownian motion started at the origin, we recall that a Brownian bridge $\{\mathbb{B}^t_{x_1,x_2}(\phi) : \phi \in [0,t]\}$ from positions x_1 to x_2 can be constructed by taking

$$\mathbb{B}^t_{x_1,x_2}(\phi) = B(\phi) - \frac{\phi}{t}B(t) + x_1 + \frac{\phi}{t}(x_2 - x_1) \qquad 0 \leq \phi \leq t. \tag{3.20}$$

This representation turns out to be exceptionally useful in the sequel. We now complete the proof of Theorem 3.1.6 in a series of lemmas.

Lemma 3.4.3. *Fix x, z, and θ. As $t \to \infty$,*

$$\mathbb{E}\left(e^{-\beta \int_0^t u(t-\phi,\mathbb{B}^t_{x,z+\lambda t+\theta}(\phi),\lambda t+\theta)\,\mathrm{d}\phi}\right) \to \mathbb{E}\left(e^{-\beta \int_0^\infty \tilde{u}(\phi,B(\phi)+z-\lambda\phi,0)\,\mathrm{d}\phi}\right), \tag{3.21}$$

where

$$\tilde{u}(t,x,y) := P^x(\mathcal{R}_t > y), \tag{3.22}$$

and $\mathcal{R}_t$ is the right-most particle in the unkilled $(-\rho,\beta;\mathbb{R})$-BBM. Note that the limit is independent of θ and x.

Proof of Lemma 3.4.3. Since the left-hand side of (3.21) is bounded in $[0,1]$ it is sufficient to prove almost-sure convergence of $\int_0^t u(t-\phi,\mathbb{B}^t_{x,z+\lambda t+\theta}(\phi),\lambda t+\theta)\,\mathrm{d}\phi$, which is equal in law to $\int_0^t u(\phi,\mathbb{B}^t_{z+\lambda t+\theta,x}(\phi),\lambda t+\theta)\,\mathrm{d}\phi$.

Consider $\phi > 0$ fixed and construct the Brownian bridges $\mathbb{B}^t_{z+\lambda t+\theta,x}(\phi)$, parameterised by t, using (3.20). Then as $B(t)/t \to 0$ almost surely, for any $\varepsilon > 0$ there exists some $t' > 0$ such that

$$\left|\mathbb{B}^t_{z+\lambda t+\theta,x}(\phi) - (B(\phi) + z + \theta + \lambda(t-\phi))\right| < \varepsilon$$

for all $t > t'$. Define $h(t,\phi) := B(\phi) + z + \theta + \lambda(t-\phi)$, and then since $u(\phi,x,y)$ is monotone increasing in x

$$u(\phi,h(t,\phi)-\varepsilon,\lambda t+\theta) \leq u(\phi,\mathbb{B}^t_{z+\lambda t+\theta,x}(\phi),\lambda t+\theta) \leq u(\phi,h(t,\phi)+\varepsilon,\lambda t+\theta),$$

for all $t > t'$. Recalling that $\tilde{u}(t,x,y) := P^x(\mathcal{R}_t > y)$, for fixed ϕ we have

$$\lim_{t\to\infty} u(\phi, h(t,\phi) - \varepsilon, \lambda t + \theta) = \tilde{u}(\phi, B(\phi) + z - \lambda\phi - \varepsilon, 0)$$

almost surely, since the effect of killing at the origin vanishes as the particle's start position goes to infinity. Similarly

$$\lim_{t\to\infty} u(\phi, h(t,\phi) + \varepsilon, \lambda t + \theta) = \tilde{u}(\phi, B(\phi) + z - \lambda\phi + \varepsilon, 0)$$

almost surely. Thus, since $u(\phi, x, y)$ is continuous in x and ε can be arbitrarily small, we have shown that

$$\mathbf{1}_{[0,t]}(\phi) u(\phi, \mathbb{B}^t_{z+\lambda t+\theta,x}(\phi), \lambda t + \theta) \xrightarrow[t\to\infty]{} \tilde{u}(\phi, B(\phi) + z - \lambda\phi, 0)$$

almost surely for fixed ϕ.

So to prove almost-sure convergence of $\int_0^t u(\phi, \mathbb{B}^t_{z+\lambda t+\theta,x}(\phi), \lambda t + \theta)\,\mathrm{d}\phi$ it is enough to prove almost sure domination by an integrable random variable. We now recall Chernov's inequality.

Lemma 3.4.4 (Chernov's inequality). *Let X be a (discrete or continuous) random variable and $a > 0$. Then $\mathbb{P}(X \geq a) \leq \min_{t\geq 0} e^{-at}\mathbb{E}(e^{tX})$.*

Applying this to the function w from Proposition 3.4.1 we see that, for $\phi \in [0,t]$ any $B \in \mathbb{R}$,

$$\begin{aligned} w(\phi, B, \lambda t + \theta) &= e^{\beta\phi}\mathbb{E}^B_{-\rho}(\mathbf{1}_{\{Y_t > \lambda t+\theta\}}; \tau_0 > t) \\ &\leq e^{\beta\phi}\mathbb{P}^0_0(Y_\phi > \lambda t + \theta - B + \rho\phi) \\ &\leq \exp\left(\beta\phi - \frac{(\lambda t + \theta - B + \rho\phi)^2}{2\phi}\right), \end{aligned} \tag{3.23}$$

and combining this with Proposition 3.4.1 gives

$$\mathbf{1}_{[0,t]}(\phi) u(\phi, \mathbb{B}^t_{z+\lambda t+\theta,x}(\phi), \lambda t + \theta) \leq \mathbf{1}_{[0,t]}(\phi) \exp\left(\beta\phi - \frac{(\lambda t + \theta - \mathbb{B}^t_{z+\lambda t+\theta,x}(\phi) + \rho\phi)^2}{2\phi}\right).$$

Next, using the representation of the Brownian bridge (3.20), we have

$$\lambda t + \rho\phi + \theta - \mathbb{B}^t_{z+\lambda t+\theta,x}(\phi) = (\lambda + \rho)\phi + \phi\left(\frac{B(t)}{t} - \frac{B(\phi)}{\phi}\right) - \left(1 + \frac{\phi}{t}\right) z - \frac{\phi}{t}(x + \theta),$$

and by the law of large numbers there exists, for any $\varepsilon > 0$, a $t_0 > 0$ such that for any $t \geq \phi \geq t_0$

$$\frac{B(t)}{t} - \frac{B(\phi)}{\phi} \geq -\varepsilon$$

almost surely. Hence, almost surely for any $t \geq \phi \geq t_0$,

$$u(\phi, \mathbb{B}^t_{z+\lambda t+\theta,x}(\phi), \lambda t + \theta) \leq e^{\delta(\lambda+\rho-\varepsilon)} e^{-(\frac{1}{2}(\lambda+\rho-\varepsilon)^2-\beta)\phi}, \tag{3.24}$$

where $\delta := (2z+\theta+x)$. Thus $u(\phi, \mathbb{B}^t_{z+\lambda t+\theta,x}(\phi), \lambda t+\theta)$ decays exponentially in ϕ for all $t \geq \phi \geq t_0$, provided ε is taken sufficiently small that $\frac{1}{2}(\lambda + \rho - \varepsilon)^2 - \beta > 0$. Trivially, $u(\phi, \mathbb{B}^t_{z+\lambda t+\theta,x}(\phi), \lambda t + \theta) \leq 1$ for $\phi \leq t_0$ and so we have almost-sure domination by an integrable function, as required. □

Construction of the bridges

We make the following simultaneous construction of all the Brownian bridges $\mathbb{B}^t_{x,z}(\cdot)$ with parameters $t > 0$, $x, z > 0$ using two *independent* Brownian motions started at the origin, $W = \{W(s) : s > 0\}$ and $X = \{X(s) : s > 0\}$:

$$\mathbb{B}^t_{x,z}(\phi) := \begin{cases} W(\phi) - \frac{\phi}{t}W(t) + x + \frac{\phi}{t}(z-x) & \text{for } \phi \in [0, \tau^t_0) \\ \widetilde{\mathbb{B}}^{t-\tau^t_0}_{z,0}(t-\phi) & \text{for } \phi \in [\tau^t_0, t] \end{cases} \tag{3.25}$$

where

$$\tau^t_0 = \tau^t_0(x,z) = \inf\left\{\phi \geq 0 : W(\phi) - \frac{\phi}{t}W(t) + x + \frac{\phi}{t}(z-x) = 0\right\},$$

and for any $s > 0$

$$\widetilde{\mathbb{B}}^s_{z,0}(u) := X(u) - \frac{u}{s}X(s) + z - \frac{u}{s}z \qquad \text{for } u \in [0,s]. \tag{3.26}$$

Note that τ^t_0 is determined entirely from the path of W, and, almost surely as $t \to \infty$, we have

$$\tau^t_0(x, z + \lambda t + \theta) \to \tau_0 := \inf\{\phi : W(\phi) + \lambda\phi + x = 0\}. \tag{3.27}$$

Further,

$$\mathbb{B}^t_{x,z+\lambda t+\theta}(\phi) \to W(\phi) + \lambda\phi + x \tag{3.28}$$

almost surely, and this convergence is uniform for $\phi \in [0,s]$, where $s < \tau_0$.

Lemma 3.4.5. *On the event $\{\tau_0 < \infty\}$,*

$$\int_0^{\tau_0^t} u(t-\phi, \mathbb{B}^t_{x,z+\lambda t+\theta}(\phi), \lambda t + \theta)\,\mathrm{d}\phi \to 0$$

almost surely as $t \to \infty$.

Proof of Lemma 3.4.5. Let $s < \tau_0 < \infty$ and then, since $\tau_0^t \to \tau_0$ almost surely, there exists some t_0 such that $s < \tau_0^t$ for all $t \geq t_0$. Using Proposition 3.4.1 and the inequality at (3.23), we have, for any $B \in \mathbb{R}$,

$$u(t-\phi, B, \lambda t + \theta) \leq e^{(\beta - \frac{1}{2}(\lambda+\rho)^2)t} \times e^{-\beta\phi + \frac{1}{2}(\lambda+\rho)^2\phi + (B-\theta)(\lambda+\rho)}.$$

Then, combining these facts with (3.28), we see that

$$\begin{aligned} e^{(\frac{1}{2}(\lambda+\rho)^2-\beta)t} &\int_0^s u(t-\phi, \mathbb{B}^t_{x,z+\lambda t+\theta}(\phi), \lambda t + \theta)\,\mathrm{d}\phi \\ &\leq \int_0^s e^{-\beta\phi + \frac{1}{2}(\lambda+\rho)^2\phi + (\mathbb{B}^t_{x,z+\lambda t+\theta}(\phi)-\theta)(\lambda+\rho)}\,\mathrm{d}\phi \\ &\to \int_0^s e^{-\beta\phi + \frac{1}{2}(\lambda+\rho)^2\phi + (W(\phi)+\lambda\phi+x-\theta)(\lambda+\rho)}\,\mathrm{d}\phi < \infty \end{aligned}$$

as $t \to \infty$. Since $\lambda > \sqrt{2\beta} - \rho$ and the above holds for all $s < \tau_0 < \infty$, the lemma follows. □

Define

$$I(z) := \int_0^\infty \tilde{u}(\phi, X(\phi) + z - \lambda\phi, 0)\,\mathrm{d}\phi \tag{3.29}$$

and note that this definition is *independent* of W (hence also of each τ_0^t and τ_0).

Lemma 3.4.6. *On the event $\{\tau_0 < \infty\}$,*

$$\int_0^t u(t-\phi, \mathbb{B}^t_{x,z+\lambda t+\theta}(\phi), \lambda t + \theta)\,\mathrm{d}\phi \to I(z), \tag{3.30}$$

almost surely as $t \to \infty$. In particular, $I(z) \in (0, \infty)$ and is independent of θ and x.

Proof. From Lemma 3.4.5, we note that it is sufficient to prove the integral from τ_0^t to

t converges to $I(z)$. Using the construction at (3.25), we see that

$$\int_{\tau_0^t}^t u(t-\phi, \mathbb{B}^t_{x,z+\lambda t+\theta}(\phi), \lambda t+\theta)\,\mathrm{d}\phi = \int_{\tau_0^t}^t u(t-\phi, \widetilde{\mathbb{B}}^{t-\tau_0^t}_{z+\lambda t+\theta,0}(t-\phi), \lambda t+\theta)\,\mathrm{d}\phi$$
$$= \int_0^{t-\tau_0^t} u(\phi, \widetilde{\mathbb{B}}^{t-\tau_0^t}_{z+\lambda t+\theta,0}(\phi), \lambda t+\theta)\,\mathrm{d}\phi$$

We now note that, because of our construction of $\widetilde{\mathbb{B}}^{t-\tau_0^t}_{z+\lambda t+\theta,0}$ at (3.26), we may almost exactly mirror the proof of Lemma 3.4.3 (noting, however, that we do not need to use any distributional equivalence) to give the required convergence result. □

Immediately from Lemma 3.4.6, we see that

$$\mathbb{E}\left(e^{-\beta\int_0^t u(t-\phi,\mathbb{B}^t_{x,z+\lambda t+\theta}(\phi),\lambda t+\theta)\,\mathrm{d}\phi}; \tau_0^t < t\right) \xrightarrow[t\to\infty]{} \mathbb{E}\left(e^{-\beta I(z)}; \tau_0 < \infty\right)$$

To complete the proof of (3.6), we note that $I(z)$ and τ_0 are independent by construction, $\mathbb{P}^x(\tau_0 < \infty) = e^{-2\lambda x}$, and the right-hand side of equation (3.21) is the same as $\mathbb{E}(e^{-\beta I(z)})$, hence

$$\mathbb{E}\left(e^{-\beta\int_0^t u(t-\phi,\mathbb{B}^t_{x,z+\lambda t+\theta}(\phi),\lambda t+\theta)\,\mathrm{d}\phi}; \tau_0^t > t\right) \xrightarrow[t\to\infty]{} (1-e^{-2\lambda x})\mathbb{E}\left(e^{-\beta I(z)}\right),$$

as required.

Finally, the equivalent statement of equation (3.7) can be deduced from equation (3.6), the Many-to-One lemma (3.5), and the one-particle calculation

$$\mathbb{P}^x_{-\rho}(Y(t) \geq \lambda t+\theta; \tau_0 > t) \underset{t\to\infty}{\sim} \frac{1}{(\lambda+\rho)\sqrt{2\pi t}}(1-e^{-2\lambda x})e^{(\lambda+\rho)(x-\theta)-\frac{1}{2}(\lambda+\rho)^2 t}.$$

This completes the proof of Theorem 3.1.6. □

3.5 A Yaglom-type limit theorem

This proof is just the adaptation to our situation of the Chauvin and Rouault [20] proof for standard branching Brownian motion.

Proof of Theorem 3.1.7. Let $x, y, \gamma > 0$, $t \in [0, \infty)$ and define

$$u_\gamma(t,x,y) := 1 - E^x(e^{-\gamma N_t^{-\rho}(y,\infty)})$$
$$u_\infty(t,x,y) := P^x(N_t^{-\rho}(y,\infty) > 0).$$

For γ fixed,

$$v(t,x,y) := 1 - u_\gamma(t,x,y) = E^x\bigg(\prod_{u \in N_t^{-\rho}} e^{-\gamma \mathbf{1}_{\{Y_u(t)>y\}}}\bigg) \tag{3.31}$$

is the McKean representation for the solution of

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} - \rho\frac{\partial v}{\partial x} + \beta v(v-1) \tag{3.32}$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ with initial condition $v(0,x,y) = e^{-\gamma \mathbf{1}_{\{x>y\}}}$, so $u_\gamma(t,x,y)$ solves

$$\frac{\partial u_\gamma}{\partial t} = \frac{1}{2}\frac{\partial^2 u_\gamma}{\partial x^2} - \rho\frac{\partial u_\gamma}{\partial x} + \beta u_\gamma(1-u_\gamma), \tag{3.33}$$

with initial condition $u_\gamma(0,x,y) = (1-e^{-\gamma})\mathbf{1}_{\{x>y\}}$. The representation at (3.31) is justified for the $(-\rho, \beta; \mathbb{R}^+)$-BBM because for $s \in [0,t]$ we have

$$\begin{aligned}
v(t,x,y) &= E^x\Big(E^x\Big(e^{-\gamma N_t^{-\rho}(y,\infty)}\Big|\mathcal{F}_s\Big)\Big)\\
&= E^x\bigg(E^x\bigg(\prod_{u \in N_s^{-\rho}} \exp\Big(-\gamma \sum_{v \in N_t^{-\rho}, u<v} \mathbf{1}_{\{Y_v(t)>y\}}\Big)\bigg|\mathcal{F}_s\bigg)\bigg)\\
&= E^x\bigg(\prod_{u \in N_s^{-\rho}} E^x\bigg(\exp\Big(-\gamma \sum_{v \in N_t^{-\rho}, u<v} \mathbf{1}_{\{Y_v(t)>y\}}\Big)\bigg|\mathcal{F}_s\bigg)\bigg)\\
&= E^x\bigg(\prod_{u \in N_s^{-\rho}} v(t-s, Y_u(s), y)\bigg),
\end{aligned}$$

whence $M_t(s) := \prod_{u \in N_s^{-\rho}} v(t-s, Y_u(s), y)$ is a product martingale on $[0,t]$. Therefore $v(t,x,y)$ solves the partial differential equation (3.32) and

$$v(t,x,y) = E^x M_t(0) = E^x M_t(t) = E^x\bigg(\prod_{u \in N_t^{-\rho}} e^{-\gamma \mathbf{1}_{\{Y_u(s)>y\}}}\bigg).$$

Define for $\lambda > \sqrt{2\beta} - \rho$

$$F(\gamma) := \lim_{t\to\infty} E^x\left(e^{-\gamma N_t^{-\rho}(\lambda t,\infty)}\middle| N_t^{-\rho}(\lambda t,\infty) > 0\right).$$

We prove that F is the Laplace transform of some distribution on $\mathbb{N}$; this means we must show that the limit as $t \to \infty$ above is well defined, and that

$$\lim_{\gamma\to 0} F(\gamma) = 1 \qquad \text{and} \qquad \lim_{\gamma\to\infty} F(\gamma) = 0.$$

Noting that

$$E^x\left(e^{-\gamma N_t^{-\rho}(\lambda t,\infty)}\middle| N_t^{-\rho}(\lambda t,\infty) > 0\right) = 1 - \frac{u_\gamma(t,x,\lambda t)}{u_\infty(t,x,\lambda t)},$$

this is equivalent to showing

$$\lim_{\gamma\to 0}\lim_{t\to\infty} \frac{u_\gamma(t,x,\lambda t)}{u_\infty(t,x,\lambda t)} = 0, \tag{3.34}$$

$$\lim_{\gamma\to \infty}\lim_{t\to\infty} \frac{u_\gamma(t,x,\lambda t)}{u_\infty(t,x,\lambda t)} = 1. \tag{3.35}$$

We can express $u_\gamma(t,x,y)$ as

$$u_\gamma(t,x,y) = (1-e^{-\gamma})\mathbb{E}^x_{-\rho}\left(\mathbf{1}_{\{Y_t>y\}} e^{\beta\int_0^t (1-u_\gamma(t-\phi,Y_\phi,y))\,\mathrm{d}\phi}; \tau_0 > t\right), \tag{3.36}$$

and by comparison with (3.19) we see that $0 \le u_\gamma(t,x,y) \le (1-e^{-\gamma})w(t,x,y)$. From Theorem 3.1.6 we know that $u_\infty(t,x,\lambda t) \sim (\lambda+\rho)Cw(t,x,\lambda t)$ as $t\to\infty$, so

$$\lim_{\gamma\to 0}\lim_{t\to\infty} \frac{u_\gamma(t,x,\lambda t)}{u_\infty(t,x,\lambda t)} \le \lim_{\gamma\to 0}(1-e^{-\gamma})\lim_{t\to\infty}\frac{w(t,x,\lambda t)}{u_\infty(t,x,\lambda t)} = \lim_{\gamma\to 0}\frac{(1-e^{-\gamma})}{(\lambda+\rho)C}$$

and (3.34) is verified.

Now let $R(u) := \beta u(1-u)$ and define $v_\gamma := u_\infty - u_\gamma$. Then $v_\gamma(t,x,y)$ solves

$$\frac{\partial v_\gamma}{\partial t} = \frac{1}{2}\frac{\partial^2 v_\gamma}{\partial x^2} - \rho\frac{\partial v_\gamma}{\partial x} + \frac{R(u_\infty)-R(u_\gamma)}{u_\infty - u_\gamma} v_\gamma$$

with initial condition $v_\gamma(0,x,y) = e^{-\gamma}\mathbf{1}_{\{x>y\}}$. Equation (3.35) then becomes

$$\lim_{\gamma\to\infty}\lim_{t\to\infty}\frac{v_\lambda(t,x,\lambda t)}{u_\infty(t,x,\lambda t)} = 0$$

and, since

$$\frac{R(u_\infty) - R(u_\gamma)}{u_\infty - u_\gamma} \leq R'(0) = \beta,$$

the Feynman-Kac formula yields the upper bound $v_\gamma(t,x,\lambda t) \leq e^{-\gamma} w(t,x,\lambda t)$. Hence

$$\lim_{\gamma\to\infty}\lim_{t\to\infty}\frac{v_\lambda(t,x,\lambda t)}{u_\infty(t,x,\lambda t)} \leq \lim_{\gamma\to\infty} e^{-\gamma}\lim_{t\to\infty}\frac{w(t,x,\lambda t)}{u_\infty(t,x,\lambda t)} = \lim_{\gamma\to\infty}\frac{e^{-\gamma}}{(\lambda+\rho)C} = 0.$$

To prove that the limit law has expectation equal to $1/(\lambda+\rho)C$ we must show that

$$\lim_{\gamma\to 0}\lim_{t\to\infty}\frac{u_\gamma(t,x,\lambda t)}{\gamma u_\infty(t,x,\lambda t)} = \frac{1}{(\lambda+\rho)C}.$$

In light of (3.33) and (3.36) it follows from Theorem 3.1.6 that

$$u_\gamma(t,x,\lambda t) \underset{t\to\infty}{\sim} (\lambda+\rho)C_\gamma(1-e^{-\gamma})w(t,x,\lambda t),$$

where

$$C_\gamma = \int_0^\infty e^{-(\lambda+\rho)z}\mathbb{E}\left(e^{-\beta\int_0^\infty \tilde{u}_\gamma(\phi,B(\phi)+z-\lambda\phi,0)\,\mathrm{d}\phi}\right)\,\mathrm{d}z, \tag{3.37}$$

and $\tilde{u}_\gamma(t,x,y) := 1 - E^x(e^{-\gamma\mathcal{N}_t^{-\rho}(y,\infty)})$, but this time the expectation is with respect to the law of the unkilled $(-\rho,\beta;\mathbb{R})$-BBM. Replacing u with u_γ and setting $\theta = 0$ does not significantly affect the proof of Theorem 3.1.6:

- the inequality $0 \leq u_\gamma(t,x,y) \leq (1-e^{-\gamma})w(t,x,y)$ ensures that Lemma 3.4.5 holds for u_γ;
- $u_\gamma(t,x,y)$ is continuous and monotone increasing in x, so in Lemma 3.4.3 we have

$$\mathbf{1}_{[0,t]}(\phi)u_\gamma(\phi,\mathbb{B}^t_{z+\lambda t,x}(\phi),\lambda t) \xrightarrow[t\to\infty]{} \tilde{u}_\gamma(\phi,B(\phi)+z-\lambda\phi,0);$$

- the inequality $0 \leq u_\gamma(t,x,y) \leq (1-e^{-\gamma})w(t,x,y)$ means that the domination part of the argument in proving Lemma 3.4.3 remains unchanged;
- with Lemmas 3.4.3 and 3.4.5 established for u_γ the proof of Lemma 3.4.6 remains unchanged.

So

$$\lim_{t\to\infty}\frac{u_\gamma(t,x,\lambda t)}{\gamma u_\infty(t,x,\lambda t)} = \frac{1-e^{-\gamma}}{\gamma}\frac{C_\gamma}{C},$$

and hence to complete the proof it is sufficient to show that

$$C_\gamma \xrightarrow[\gamma\to 0]{} \frac{1}{\lambda+\rho}.$$

Noting again that $u_\gamma(t,x,y) \le (1-e^{-\gamma})e^{\beta t}\mathbb{P}^x_{-\rho}(Y_t > y)$, we have, for $\phi, z > 0$ fixed

$$\lim_{\gamma\to 0} \tilde{u}_\gamma(\phi, B(\phi) + z - \lambda\phi, 0) = 0 \qquad \text{a.s.}$$

and then using dominated convergence at (3.37) gives the result. □

3.6 An alternative interpretation of $\mathbb{Q}_\lambda$

This proof is virtually unchanged from the Chauvin and Rouault [20] proof for standard branching Brownian motion.

Proof of Theorem 3.1.8. For $\lambda > \sqrt{2\beta} - \rho$, $s \in (0,\infty)$ fixed, and $A \in \mathcal{F}_s$, we have

$$P^x\Big(A\Big|N^{-\rho}_{t+s}(\lambda(t+s),\infty) > 0\Big) = \frac{P^x\Big(\mathbf{1}_A P^x\Big(N^{-\rho}_{t+s}(\lambda(t+s),\infty) > 0\Big|\mathcal{F}_s\Big)\Big)}{P^x\Big(N^{-\rho}_{t+s}(\lambda(t+s),\infty) > 0\Big)}. \tag{3.38}$$

Now set

$$\begin{aligned} Y &:= P^x\Big(N^{-\rho}_{t+s}(\lambda(t+s),\infty) > 0\Big|\mathcal{F}_s\Big) \\ &= 1 - \prod_{u\in N^{-\rho}_s}\Big(1 - P^{Y_u(s)}\Big(\exists u \in N^{-\rho}_t : Y_u(t) \ge \lambda(s+t)\Big)\Big). \end{aligned} \tag{3.39}$$

Since for all $x > 0$, $P^x\Big(\exists u \in N^{-\rho}_t : Y_u(t) \ge \lambda(s+t)\Big) \to 0$ as $t\to\infty$, we have

$$Y \underset{t\to\infty}{\sim} \sum_{u\in N^{-\rho}_s} P^{Y_u(s)}\Big(\exists u \in N^{-\rho}_t : Y_u(t) \ge \lambda(s+t)\Big). \tag{3.40}$$

Theorem 3.1.6 and (3.40) imply that

$$Y \underset{t\to\infty}{\sim} \frac{C}{\sqrt{2\pi t}} \sum_{u\in N^{-\rho}_s} (1 - e^{-2\lambda Y_u(s)})e^{(\lambda+\rho)Y_u(s)}e^{-(\frac{1}{2}(\lambda+\rho)^2-\beta)t-\lambda^2 s-\lambda\rho s},$$

and combining this with the asymptotic for $P^x\Big(N^{-\rho}_{t+s}(\lambda(t+s),\infty) > 0\Big)$ we have, almost surely,

$$\mathbf{1}_A \frac{P^x\Big(N^{-\rho}_{t+s}(\lambda(t+s),\infty) > 0\Big|\mathcal{F}_s\Big)}{P^x\Big(N^{-\rho}_{t+s}(\lambda(t+s),\infty) > 0\Big)} \xrightarrow[t\to+\infty]{} \mathbf{1}_A \frac{W_\lambda(s)}{W_\lambda(0)}. \tag{3.41}$$

Then to take the limit inside the expectation in (3.38) we find an integrable random variable (independent of t) that dominates the left-hand side of (3.41) and use dominated convergence.

From (3.39) it follows that

$$Y \le \sum_{u\in N_s^{-\rho}} P^{Y_u(s)}\Big(\exists u \in N_t^{-\rho} : Y_u(t) \ge \lambda(s+t)\Big),$$

and then from Theorem 3.1.6 we have that, for some constant $A > 0$,

$$\begin{aligned} Y &\le \frac{A}{\sqrt{2\pi t}} \sum_{u\in N_s^{-\rho}} e^{(\lambda+\rho)Y_u(s)} e^{-(\frac{1}{2}(\lambda+\rho)^2-\beta)t-\lambda^2 s-\lambda\rho s} \\ &= \frac{A}{\sqrt{2\pi t}} \sum_{u\in N_s^{-\rho}} e^{(\lambda+\rho)Y_u(s)} e^{-(\frac{1}{2}(\lambda+\rho)^2-\beta)(t+s)-(\frac{1}{2}(\lambda^2-\rho^2)+\beta)s} \\ &= \frac{A}{\sqrt{2\pi t}} e^{-(\frac{1}{2}(\lambda+\rho)^2-\beta)(t+s)} Z_\lambda(s) \end{aligned} \tag{3.42}$$

To bound the denominator of the left-hand side of (3.41) below we have from Theorem 3.1.6 that there exists some constant $B > 0$ such that for all $t, s \ge 0$

$$P^x\Big(N^{-\rho}_{t+s}((\lambda(t+s),\infty) > 0)\Big) \ge \frac{B}{\sqrt{2\pi(t+s)}}(1-e^{-2\lambda x})e^{(\lambda+\rho)x-(\frac{1}{2}(\lambda+\rho)^2-\beta)(s+t)},$$

and together with (3.42) this means that there exists a $t_0 > 0$ and a constant $B' > 0$ such that for $t > t_0$

$$\mathbf{1}_A \frac{P^x\Big(N^{-\rho}_{t+s}((\lambda(t+s),\infty) > 0)\Big|\mathcal{F}_s\Big)}{P^x\Big(N^{-\rho}_{t+s}((\lambda(t+s),\infty) > 0)\Big)} \le B' Z_\lambda(s) \in \mathcal{L}^1(P^x),$$

which provides the required domination. $\square$

Chapter 4

Survival probabilities for sub-critical BBM with absorption

In this chapter we extend some of the ideas of Chapter 3 and study survival probabilities for the $(-\rho, \beta; \mathbb{R}^+)$-BBM in the case $\rho > \sqrt{2\beta}$ — recall from Chapter 3 that the extinction time, ζ, for the process is almost surely finite in this parameter range. The main result is a sharp t-asymptotic for $P^x(R_t > 0)$, the probability that the process survives until time t. We also define another new additive martingale, V, for the $(-\rho, \beta; \mathbb{R}^+)$-BBM. We use V in a spine change of measure, and show that under the changed measure the spine diffuses as a Bessel-3 process. Finally, we interpret the spine change of measure (with V) in terms of 'conditioning the BBM to survive forever' when $\rho > \sqrt{2\beta}$, in the sense that it is the t-limit of the conditional probabilities $P^x(A|R_{t+s} > 0)$, for $A \in \mathcal{F}_s$.

The work in this chapter appears in Harris and Harris [44].

4.1 Introduction and summary of results

We saw in Chapter 2 that, if $\rho \geq \sqrt{2\beta}$, the $(-\rho, \beta; \mathbb{R}^+)$-BBM is sub-critical, and $\zeta < \infty$ almost surely. There are several examples in the branching-process literature of spine constructions being interpreted as null conditionings of a process — one conditions on an event that has probability zero in the limit as $t \to \infty$, and then in the limit as $t \to \infty$ the law of the conditioned process can be shown to be related to the original law via a Radon-Nikodým derivative, in a spine change of measure that causes the spine to perform the conditioning event with probability one. As well as Chauvin and

Rouault's results ([20], [21]) for branching Brownian motion, Evans [35], for example, gives 'immortal particle' constructions for conditioned superprocesses. In light of Theorem 3.1.8, then, it is entirely natural to ask what can be said about the limit as $t \to \infty$ of the conditional probabilities $P^x(A|R_{s+t} > 0)$, for $A \in \mathcal{F}_s$.

Inspection of the proof of Theorem 3.1.8 reveals that there is nothing special about the conditioning event $\{N_t^{-\rho}(\lambda t + \theta, \infty) > 0\}$, and it could be replaced with the event $\{N_t^{-\rho}(0,\infty) > 0\} = \{R_t > 0\}$. Provided that $P^x(R_t > 0) \to 0$ as $t \to \infty$, the argument that worked to prove Theorem 3.1.8 should also work for the conditioned law $P^x(\cdot|R_{s+t} > 0)$. However, in order to use this argument as we did in Chapter 3, we will need a sharp t-asymptotic for the survival probability $P^x(R_t > 0)$.

Survival probabilities for branching Brownian motion with an absorbing barrier at zero were studied in Kesten [58], and the behaviour of the process when $\rho > \sqrt{2\beta}$ is very different to that when $\rho = \sqrt{2\beta}$ (look ahead to Theorems 4.1.2 and 4.1.3). Kesten [58] primarily studied the critical case $\rho = \sqrt{2\beta}$; here, in contrast, we study the survival probability in the case $\rho > \sqrt{2\beta}$ — note the strict inequality here.

The major result of this chapter is an asymptotic result for the large-t behaviour of the probability $P^x(R_t > 0)$ when $\rho > \sqrt{2\beta}$. Clearly $P^x(R_t > 0) \le E^x N_t^{-\rho}(0,\infty)$, and Kesten [58] gave the the following result for this expectation.

Theorem 4.1.1 (Kesten [58, Theorem 1.1]). *There exist constants $0 < C = C(x,\rho) < \infty$ such that for $x > 0$ and $t \to \infty$*

$$E^x N_t^{-\rho}(0,\infty) \sim \begin{cases} Ct^{-\frac{3}{2}} e^{-(\frac{1}{2}\rho^2 - \beta)t} & \text{if } \rho > 0, \\ Ct^{-\frac{1}{2}} e^{\beta t} & \text{if } \rho = 0, \\ Ce^{\beta t} & \text{if } \rho < 0. \end{cases}$$

This result is a consequence of the Many-to-One lemma and standard estimates for Brownian motion. The case of interest to us, $\rho > \sqrt{2\beta}$, is detailed below.

$$\begin{aligned} E^x N_t^{-\rho}(0,\infty) &= e^{\beta t} \mathbb{E}^x_{-\rho}(\mathbf{1}_{\{\tau_0 > t\}}) \\ &= e^{\beta t} \int_t^\infty \frac{x}{\sqrt{2\pi s^3}} \exp\left(- \frac{(x-\rho s)^2}{2s} \right) \mathrm{d}s \\ &\sim \sqrt{\frac{2}{\pi(\rho^2 t)^3}}\, x e^{\rho x - (\frac{1}{2}\rho^2 - \beta)t} \qquad \text{as } t \to \infty, \end{aligned}$$

where Y_t is a Brownian motion with drift $-\rho$ under $\mathbb{P}^x_{-\rho}$ and for $\tau_0 := \inf\{t > 0 : Y_t = 0\}$

we have

$$\mathbb{P}^x_{-\rho}(\tau_0 \in \mathrm{d}s) = \frac{x}{\sqrt{2\pi s^3}} \exp\left(- \frac{(x-\rho s)^2}{2s} \right) \mathrm{d}s.$$

As was the case in Theorem 3.1.6, $E^x N_t^{-\rho}(0,\infty)$ decays exponentially in t, and so we might expect that $P^x(R_t > 0)$ is asymptotically equal to a constant multiple of $E^x N_t^{-\rho}(0,\infty)$. The next result, which we will prove in Section 4.2, shows that this is indeed the case.

Theorem 4.1.2. *For $\rho > \sqrt{2\beta}$ and $x > 0$,*

$$\lim_{t\to\infty} P^x(R_t > 0) \frac{\sqrt{2\pi t^3}}{x} e^{-\rho x + (\frac{1}{2}\rho^2 - \beta)t} = K,$$

for some constant $K > 0$ that is independent of x. This is equivalent to

$$P^x(R_t > 0) \underset{t\to\infty}{\sim} \frac{1}{2}\rho^{\frac{3}{2}} K \times E^x(N_t^{-\rho}(0,\infty)).$$

We remark in passing the behaviour of the process in the 'critical case' of $\rho = \sqrt{2\beta}$ is significantly different to that seen so far in Chapters 3 and 4. The probabilities $P^x(R_t > \lambda t + \theta)$ and $P^x(R_t > 0)$ both decay in t like the related expected numbers of particles; this is not true when $\rho = \sqrt{2\beta}$. One of the main results of Kesten [58] gives bounds for $P^x(R_t > 0)$ in the critical case.

Theorem 4.1.3 (Kesten [58, Theorem 1.3]). *Let $\rho = \sqrt{2\beta}$. Then there exist constants $C_1, C_2, C_3 \in (0,\infty)$, depending only on β, such that for $x > 0$*

$$x \exp(\rho x - C_1(\log t)^2) \leq P^x(R_t > 0) \exp\left(\frac{3\rho^2\pi^2}{2}\right)^{\frac{1}{3}} t^{\frac{1}{3}} \leq (1+x)\exp(\rho x - C_1(\log t)^2).$$

Moreover, as $t \to \infty$,

$$P^x\Big(N_t^{-\rho}(C_2 t^{\frac{2}{9}}(\log t)^{\frac{2}{3}}, \infty) > 0 \Big| N_t^{-\rho}(0,\infty) > 0\Big) \to 0 \tag{4.1}$$

and

$$P^x\Big(N_t^{-\rho}(0,\infty) \geq \exp(C_3 t^{\frac{2}{9}}(\log t)^{\frac{2}{3}}) \Big| N_t^{-\rho}(0,\infty) > 0\Big) \to 0. \tag{4.2}$$

Even though this result does not give an exact asymptotic for $P^x(R_t > 0)$, it is still difficult to prove. Note that it follows from the Many-to-One lemma that $E^x N_t^{-\rho}(0,\infty)$ is of order $t^{-\frac{3}{2}}$ when $\rho = \sqrt{2\beta}$, which is a slower decay in t than $P^x(R_t > 0)$. No precise asymptotic for $P^x(R_t > 0)$ is known in this case, and it is also an open question what the analogue of Yaglom's theorem should be. Equations (4.1) and (4.2) give only a partial answer.

If we condition the sub-critical process (with $\rho > \sqrt{2\beta}$) to survive up to time $t+s$, and then take the limit as $t \to \infty$, our intuition tells us to expect the conditioned process to do the minimum possible to satisfy the conditioning event — that is to make a single particle avoid the origin forever. Recalling that a Brownian motion conditioned to avoid the origin is a Bessel-3 process, we now make an additive martingale that can be used to change measure to cause the spine to diffuse as a Bessel-3 process; this is constructed from single-particle changes of measure in the same way as W_λ.

Lemma 4.1.4. *For all $\beta, \rho > 0$ the process*

$$V(t) := \sum_{u \in N_t^{-\rho}} Y_u(t) e^{\rho Y_u(t) + (\frac{1}{2}\rho^2 - \beta)t}$$

defines an additive martingale for the $(-\rho, \beta; \mathbb{R}^+)$-BBM.

Proof. Let $Y(t)$ be a Brownian motion started at $x > 0$ with drift $-\rho$ under the measure $\mathbb{P}^x_{-\rho}$. By the Many-to-One lemma (equation (3.5)) and a Girsanov change of measure it follows that

$$\begin{aligned} E^x(V(t)) &= \mathbb{E}^x_{-\rho}(Y_t e^{\rho Y_t + \frac{1}{2}\rho^2 t}; \tau_0 > t) \\ &= \mathbb{P}^x_0(Y_t; \tau_0 > t) e^{\rho x} = x e^{\rho x}. \end{aligned}$$

Applying the branching Markov property we see that

$$E^x(V(t+s)|\mathcal{F}_t) = \sum_{u \in N_t^{-\rho}} E^x(V^{(u)}(s)|\mathcal{F}_t) e^{(\frac{1}{2}\rho^2 - \beta)t}$$

where, conditional on $\mathcal{F}_t$, the $V^{(u)}(s)$ are independent copies of $V(s)$ under $P^{Y_u(t)}$. The result now follows from the previous calculation □

We can now use the asymptotic expression for $P^x(R_t > 0)$ from Theorem 4.1.2 to show that changing measure with V does indeed correspond to conditioning the sub-critical process to survive forever — the result below will be proved in Section 4.4.

Theorem 4.1.5. *Let $\rho > \sqrt{2\beta}$. For $s > 0$ fixed and $A \in \mathcal{F}_s$,*

$$P^x(A|R_{s+t} > 0) \xrightarrow[t \to \infty]{} \mathbb{Q}^x(A), \tag{4.3}$$

where the measure $\mathbb{Q}$ is defined by

$$\left.\frac{\mathrm{d}\mathbb{Q}^x}{\mathrm{d}P^x}\right|_{\mathcal{F}_s} = \frac{1}{x}e^{-\rho x}\sum_{u\in N_s^{-\rho}} Y_u(s)e^{\rho Y_u(s)+(\frac{1}{2}\rho^2-\beta)s} = \frac{V(s)}{V(0)}.$$

As was the case for changing measure with W_λ, this can be interpreted by means of a spine construction. Under $\mathbb{Q}^x$, the $(-\rho,\beta;\mathbb{R}^+)$-BBM can be reconstructed in law as:

- starting from position x, the initial ancestor diffuses as a Bessel-3 process;
- at rate 2β the initial ancestor undergoes fission producing two particles;
- one of these particles is selected at random with probability one half;
- this chosen particle (the spine) repeats stochastically the behaviour of their parent;
- the other particle initiates from its birth position an independent copy of a $(-\rho,\beta;\mathbb{R}^+)$-BBM with law $P^{\cdot}$; we denote by $(\tau, M)^u$ the subtree resulting from the fission of $u\in\xi$, which is not part of the spine.

This description of the $\mathbb{Q}$-law of the $(-\rho,\beta;\mathbb{R})$-BBM facilitates the following result on uniform integrability for V.

Theorem 4.1.6. *If $0<\rho<\sqrt{2\beta}$, then V is uniformly integrable and the events $\{V(\infty)>0\}$ and $\{\zeta=\infty\}$ agree up to a P^x-null set. If $\rho\geq\sqrt{2\beta}$ then, P^x-almost surely, $V(\infty)=0$.*

Remark 4.1.7 *(Relationship between W_λ and V).* Consider conditioning a standard Brownian motion, Y_t, started at $x>0$ to avoid the origin 'forever'. This can be done in two ways: either condition Y to hit some fixed $M>x$ before hitting 0, and then let $M\to\infty$; or alternatively condition Y to avoid the origin up to time t, and then let $t\to\infty$. The limiting process, a 3-dimensional Bessel, is the same in both cases.

There is a similar relationship between the two ways in which we have conditioned the $(-\rho,\beta;\mathbb{R}^+)$-BBM on a null event. Recall from Chapter 3 that, for $\lambda>0$,

$$W_\lambda(t) := \sum_{u\in N_t^{-\rho}}(1-e^{-2\lambda Y_u(t)})e^{(\lambda+\rho)Y_u(t)-(\frac{1}{2}(\lambda^2-\rho^2)+\beta)t},$$

and changing measure with W_λ made the spine diffuse as a Brownian motion with drift λ, conditioned to avoid the origin. We find that, for any $s > 0$,

$$\lim_{\lambda \to 0} \frac{W_\lambda(s)}{W_\lambda(0)} = \frac{V(s)}{V(0)},$$

and the construction of the $(-\rho, \beta; \mathbb{R}^+)$-BBM under $\mathbb{Q}^x$ matches the construction under $\mathbb{Q}^x_\lambda$ (see Section 3.2 in the previous chapter) if we set $\lambda = 0$. Loosely speaking, then, V is in some sense the '$\lambda = 0$' case for W_λ.

In Section 4.2 below we prove the asymptotic result for the survival probability (Theorem 4.1.2), although a lot of the detail is left until Section 4.3. The remaining proofs follow in Section 4.4.

4.2 Survival probabilities

As the result is similar in appearance to Theorem 3.1.6, it is not surprising that the proof uses some of the same ideas. The Feynman-Kac representation and the Brownian bridge are again fundamental to the method, but this time we have to look at a Brownian bridge that is conditioned to avoid the origin. The significant differences between this proof and the proof of Theorem 3.1.6 are contained in Proposition 4.2.1 and Lemma 4.3.2.

Proof of Theorem 4.1.2. Define $u(t, x) := P^x(R_t > 0)$. Using a product martingale it can be shown that $u(t, x)$ satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial u}{\partial x} + \beta u(1 - u),$$

with initial condition $u(0, x) = \mathbf{1}_{\{x > 0\}}$, whence for Y a Brownian motion (started at x) with drift $-\rho$ under $\mathbb{P}^x_{-\rho}$ and $\tau_0 := \inf\{s > 0 : Y_s = 0\}$,

$$M_t(s) := u(t - (s \wedge \tau_0), Y(s \wedge \tau_0)) \exp\left(\beta \int_0^{s \wedge \tau_0} (1 - u(t - \phi, Y(\phi)) \,\mathrm{d}\phi\right)$$

is a uniformly integrable $\mathbb{P}^x_{-\rho}$-martingale on $[0, t]$. As a consequence of the optional stopping theorem, we can write

$$u(t, x) = \mathbb{E}^x_{-\rho}\left(\mathbf{1}_{\{Y_t > 0\}} e^{\beta \int_0^t (1 - u(t - \phi, Y_\phi))\,\mathrm{d}\phi}; \tau_0 > t\right).$$

For later use, we note that applying Chernov's inequality here gives an upper bound on u:

$$u(t,x) \le e^{\beta t}\mathbb{E}^x_{-\rho}(\mathbf{1}_{\{Y_t>0\}}) \le e^{\rho x-(\frac{1}{2}\rho^2-\beta)t}, \qquad \text{for all } t,x>0. \tag{4.4}$$

Still following the idea of the proof of Theorem 3.1.6, we re-write our expression for u in terms of the Brownian bridge

$$u(t,x) = e^{\beta t}\int_0^\infty \mathbb{P}^x_{-\rho}(Y_t\in \mathrm{d}z)\mathbb{E}\Big(e^{-\beta\int_0^t u(t-\phi,\mathbb{B}^t_{x,z}(\phi))\,\mathrm{d}\phi};\tau_0^t>t\Big),$$

where $\{\mathbb{B}^t_{x,z}(\phi):\phi\in[0,t]\}$ is the Brownian bridge from x to z and $\tau_0^t=\tau_0^t(x,z):=\inf\{\phi>0:\mathbb{B}^t_{x,z}(\phi)=0\}$. Hence

$$\begin{aligned} u(t,x)\frac{\sqrt{2\pi t^3}}{xe^{\rho x}}e^{(\frac{1}{2}\rho^2-\beta)t} = \int_0^\infty &\frac{t}{x}e^{-\rho z}\exp\left(-\frac{(x-z)^2}{2t}\right) \\ &\times\mathbb{E}\Big(e^{-\beta\int_0^t u(t-\phi,\mathbb{B}^t_{x,z}(\phi))\,\mathrm{d}\phi};\tau_0^t>t\Big)\,\mathrm{d}z. \end{aligned} \tag{4.5}$$

Using the distributional equivalence

$$\int_0^t u(t-\phi,\mathbb{B}^t_{x,z}(\phi))\,\mathrm{d}\phi \overset{d}{=} \int_0^t u(\phi,\mathbb{B}^t_{z,x}(\phi))\,\mathrm{d}\phi,$$

obtained by a time reversal, and the explicit form for the probability that the Brownian bridge avoids the origin,

$$\mathbb{P}(\tau_0^t>t) = 1-\exp\Big(-\frac{2xz}{t}\Big),$$

we can re-write the right-hand side of (4.5) as

$$\int_0^\infty \frac{t}{x}(1-e^{-\frac{2xz}{t}})e^{-\rho z}\exp\left(-\frac{(x-z)^2}{2t}\right)\mathbb{E}\Big(e^{-\beta\int_0^t u(\phi,\mathbb{B}^t_{z,x}(\phi))\,\mathrm{d}\phi}\Big|\tau_0^t>t\Big)\,\mathrm{d}z. \tag{4.6}$$

Now as $t\to\infty$,

$$0\le\frac{t}{x}(1-e^{-\frac{2xz}{t}})\uparrow 2z,$$

and so it is sufficient (by dominated convergence) to show that for some function $g:[0,\infty)\to(0,1]$,

$$\mathbb{E}\Big(e^{-\beta\int_0^t u(\phi,\mathbb{B}^t_{z,x}(\phi))\,\mathrm{d}\phi}\Big|\tau_0^t>t\Big)\xrightarrow[t\to\infty]{} g(z),$$

since then by dominated convergence the expression at (4.6) tends to $\int_0^\infty 2ze^{-\rho z} g(z)\,\mathrm{d}z$ as $t \to \infty$. This follows from the following proposition.

Proposition 4.2.1. *For each $z > 0$,*

$$\mathbb{E}\left(e^{-\beta \int_0^t u(\phi, \mathbb{B}^t_{z,x}(\phi))\,\mathrm{d}\phi} \middle| \tau_0^t > t\right) \xrightarrow[t\to\infty]{} \mathbb{E}^z_B\left(e^{-\beta\int_0^\infty u(\phi, X(\phi))\,\mathrm{d}\phi}\right) \in (0,1],$$

where $\{X(\phi) : \phi \geq 0\}$ is a Bessel-3 process under $\mathbb{P}^z_B$.

□

The intuition behind Proposition 4.2.1 becomes clear when we consider how the Brownian bridge behaves for large t. We can represent the Brownian bridge $\mathbb{B}^t_{z,x}(\phi)$ as

$$\mathbb{B}^t_{z,x}(\phi) = Y(\phi) - \frac{\phi}{t}Y(t) + \frac{\phi}{t}(x - z) + z,$$

where Y is a standard Brownian motion started at the origin. Since $Y_t/t \to 0$ almost surely as $t \to \infty$, for $0 < \phi \ll t$ the bridge is approximately $\mathbb{B}^t_{z,x}(\phi) \approx z + Y(\phi)$, that is the conditioning event $\{Y(t) \in \mathrm{d}x\}$ does not significantly affect the Brownian motion for small ϕ. When we additionally condition the bridge to avoid the origin, we might expect that $\mathbb{B}^t_{z,x}(\phi) \approx X(\phi)$ for $\phi \ll t$, where X is a Bessel process started at z (in fact, we will see later that a Brownian bridge conditioned to avoid the origin is equal in law to a Bessel-3 bridge with the same start and end points). To make this idea watertight we will show that $\{\mathbb{B}^t_{z,x}(\phi) | \tau_0^t > t\}_{\phi \geq 0}$ converges in distribution to $\{X(\phi)\}_{\phi\geq 0}$ as $t \to \infty$. For all $t > 0$ we define $\mathbb{B}^t_{z,x}(\phi) \equiv x$ for $\phi > t$ to ensure that the conditioned processes have paths in $D_{[0,\infty)}[0,\infty)$ — the set of càdlàg paths in $[0,\infty)$.

The exponential decay of $u(t,x)$ with respect to t — recall equation (4.4) — allows us to essentially neglect the contribution from the tail of the integral to the conditional expectation in Proposition 4.2.1. This is vitally important, as it means that the limit as $t \to \infty$ in the conditional expectation is independent of x. In the next section we turn this intuitive explanation into a rigorous proof.

4.3 Proof of Proposition 4.2.1

To simplify notation, we will use the family of measures indexed by t, $\mathbb{P}^{z,x}_t$, with associated expectation $\mathbb{E}^{z,x}_t$, for the laws of the Brownian bridges of length t conditioned to avoid the origin (but recall that we have extended the bridge definitions to include times $\phi > t$). For the remainder of this section we will just use the notation $\{X\}_{\phi \geq 0}$ for a process with paths in $D_{[0,\infty)}[0,\infty)$, and remember that under $\mathbb{P}_t^{z,x}$, X is the conditioned Brownian bridge, while under $\mathbb{P}_B^z$, X is a Bessel-3 process.

We now give a useful characterisation of weak convergence. The main reference for the theory on weak convergence in this section is Ethier and Kurtz [34].

Theorem 4.3.1 (Ethier and Kurtz [34, Theorem 3.1]). *Let (S,d) be a separable metric space and let $\mathcal{P}(S)$ be the set of Borel measures on S, with $\{\mathbb{P}_n\} \subset \mathcal{P}(S)$ and $\mathbb{P} \in \mathcal{P}(S)$. Then the following are equivalent.*

(i) $\mathbb{P}_n \Rightarrow \mathbb{P}$.

(ii) $\lim_{n\to\infty} \int f \,\mathrm{d}\mathbb{P}_n = \int f \,\mathrm{d}\mathbb{P}$ *for all uniformly continuous bounded* $f : S \to \mathbb{R}$.

(iii) $\limsup_{n\to\infty} \mathbb{P}_n(F) \leq \mathbb{P}(F)$ *for all closed sets* $F \subset S$.

(iv) $\liminf_{n\to\infty} \mathbb{P}_n(G) \geq \mathbb{P}(G)$ *for all open sets* $G \subset S$.

(v) $\lim_{n\to\infty} \mathbb{P}_n(A) = \mathbb{P}(A)$ *for all* $\mathbb{P}$*-continuity sets* A *— these are sets* A *such that* $\mathbb{P}(\partial A) = 0$.

Denote by $D_S[0,\infty)$ the set of càdlàg paths in S. The Skorohod metric (see Ethier and Kurtz [34, Chapter 3] for the definition) can be defined on $D_S[0,\infty)$, and this space is complete and separable with respect to the Skorohod metric.

Lemma 4.3.2. *As $t \to \infty$, $\mathbb{P}_t^{z,x} \Rightarrow \mathbb{P}_B^z$.*

Proving that a sequence of processes (or, equivalently, the laws of those processes) converges in distribution generally involves showing first that the finite dimensional distributions converge, and then proving a tightness condition. In proving tightness it matters that the measures $\mathbb{P}_t^{z,x}$ are not indexed by a discrete parameter, but we use a result from Ethier and Kurtz [34] to deal with this. The proof of Lemma 4.3.2 is given at the end of this section.

Proof of Proposition 4.2.1. Fix $T > 0$ and let $t > T$. For notational convenience we define, for $a, b \geq 0$ and X a process with càdlàg paths in $D_{[0,\infty)}[0,\infty)$,

$$I(a,b) := \int_a^b u(\phi, X(\phi))\,\mathrm{d}\phi,$$

Now let $\varepsilon > 0$ and define

$$A(t) := \mathbb{E}_t^{z,x}\Big(e^{-\beta I(0,T)-\beta I(T,t)}; \sup_{T\leq\phi} \frac{X(\phi)}{\phi} > \varepsilon\Big)$$
$$B(t) := \mathbb{E}_t^{z,x}\Big(e^{-\beta I(0,T)-\beta I(T,t)}; \sup_{T\leq\phi} \frac{X(\phi)}{\phi} \leq \varepsilon\Big),$$

so that $\mathbb{E}_t^{z,x}(e^{-\beta I(0,t)}) = A(t) + B(t)$. Now by Lemma 4.3.2 and Theorem 4.3.1,

$$A(t) \leq \mathbb{P}_t^{z,x}\Big(\sup_{T\leq\phi} \frac{X(\phi)}{\phi} > \varepsilon\Big) \to \mathbb{P}_B^{z}\Big(\sup_{T\leq\phi} \frac{X(\phi)}{\phi} > \varepsilon\Big) \qquad \text{as } t \to \infty.$$

Since $X(\phi)/\phi \to 0$ $\mathbb{P}_B^z$-almost surely as $t \to \infty$, the final probability in the line above can be made arbitrarily small (for any $\varepsilon > 0$) by letting $T \to \infty$.

To deal with the term $B(t)$, we bound it above and below with expressions that are equal (in the limit) to the required expectation as we first let $t \to \infty$, and then let $T \to \infty$. For the upper bound we have

$$B(t) \leq \mathbb{E}_t^{z,x}\Big(e^{-\beta I(0,T)}\Big) \to \mathbb{E}_B^{z}\Big(e^{-\beta I(0,T)}\Big)$$

as $t \to \infty$. This follows from Lemma 4.3.2 because $e^{-\beta\int_0^T u(\phi,\cdot)\,\mathrm{d}\phi}$ is a continuous bounded function of the sample paths. Letting $T \to \infty$, and using bounded convergence, we obtain

$$\limsup_{t\to\infty} B(t) \leq \mathbb{E}_B^{z}\Big(e^{-\beta I(0,\infty)}\Big). \tag{4.7}$$

For the lower bound, recall that $\rho > \sqrt{2\beta}$, and so we can take $\varepsilon > 0$ sufficiently small that $\delta := -\rho\varepsilon + (\frac{1}{2}\rho^2 - \beta) > 0$. Then if $X(\phi) \leq \varepsilon\phi$, using the upper bound for $u(t,x)$ at equation (4.4) we have $u(\phi, X(\phi)) \leq e^{-\delta\phi}$ and hence

$$\begin{aligned}
B(t) &\geq \mathbb{E}_t^{z,x}\Big(e^{-\beta I(0,T)-\beta\int_T^t e^{-\delta\phi}\,\mathrm{d}\phi}; \sup_{T\leq\phi} \frac{X(\phi)}{\phi} \leq \varepsilon\Big) \\
&= \exp\Big(-\frac{\beta}{\delta}(e^{-\delta T} - e^{-\delta t})\Big)\mathbb{E}_t^{z,x}\Big(e^{-\beta I(0,T)}; \sup_{T\leq\phi} \frac{X(\phi)}{\phi} \leq \varepsilon\Big) \\
&\to \exp\Big(-\frac{\beta}{\delta}e^{-\delta T}\Big)\mathbb{E}_B^{z}\Big(e^{-\beta I(0,T)}; \sup_{T\leq\phi} \frac{X(\phi)}{\phi} \leq \varepsilon\Big)
\end{aligned}$$

as $t \to \infty$. Note that, although

$$e^{-\beta \int_0^T u(\phi, v(\phi))\, \mathrm{d}\phi} \mathbf{1}_{\{\sup_{T\le\phi} \frac{v(\phi)}{\phi} \le \varepsilon\}}$$

is not continuous as a function of $v(\cdot) \in D_{[0,\infty)}[0,\infty)$, the set of discontinuities,

$$\left\{ v(\cdot) \in D_{[0,\infty)}[0,\infty) : \sup_{T\le\phi} \frac{v(\phi)}{\phi} = \varepsilon \right\},$$

has $\mathbb{P}_B^z$-measure zero, and so the expectation does converge (see Billingsley [13, Theorem 5.1]). On letting $T \to \infty$, using bounded convergence we have

$$\liminf_{t\to\infty} B(t) \ge \mathbb{E}_B^z\left(e^{-\beta I(0,\infty)}\right), \tag{4.8}$$

and it follows from (4.7) and (4.8) that $\lim_{t\to\infty} B(t) = \mathbb{E}_B^z\left(e^{-\beta I(0,\infty)}\right)$. Recalling that $A(t) \to 0$ as $t \to \infty$, we have shown that

$$\mathbb{E}_t^{z,x}\left(e^{-\beta \int_0^t u(\phi, X(\phi))\, \mathrm{d}\phi}\right) \xrightarrow[t\to\infty]{} \mathbb{E}_B^z\left(e^{-\beta \int_0^\infty u(\phi, X(\phi))\, \mathrm{d}\phi}\right),$$

as required.

It remains to show that the limit in the line above is strictly positive. Since the limit is bounded in $[0,1]$, it is sufficient to prove a $\mathbb{P}_B^z$-almost sure domination of $\int_0^\infty u(\phi, X(\phi))\, \mathrm{d}\phi$ by some finite quantity. By the Law of Large Numbers, there exists a random $T_0 < \infty$ such that $\mathbb{P}_B^z$-almost surely, for all $\phi > T_0$, $X(\phi)/\phi < \varepsilon$. Then $\mathbb{P}_B^z$-almost surely

$$u(\phi, X(\phi)) \le e^{-\delta\phi} \qquad \text{for all } \phi > T_0,$$

and as $u(\phi, X(\phi)) \le 1$ for $0 \le \phi \le T_0$ we have an almost sure domination. □

The remainder of this section is devoted to the proof of Lemma 4.3.2. We will first give a criterion for tightness of the measures $\mathbb{P}_t^{z,x}$.

We state the condition for tightness of measures on $D_S[0,\infty)$ in terms of the following modulus of continuity. For $v(s) \in D_S[0,\infty)$, $\delta > 0$, and $T > 0$, define

$$w'(v, \delta, T) := \inf_{\{t_i\}} \max_i \sup_{r,s\in[t_{i-1},t_i)} d(v(r), v(s)),$$

where $\{t_i\}$ ranges over all partitions of the form $0 = t_0 < t_1 < \ldots < t_m = T$ with $m \ge 1$ and $\min_{1\le i\le m}(t_i - t_{i-1}) > \delta$.

For a compact set $K \subset S$, we denote by K^ε the ε-expansion of K, that is the set $\{x \in S : \inf_{y\in K} d(x,y) < \varepsilon\}$.

Theorem 4.3.3 (Ethier and Kurtz [34, Theorem 7.2]). *Let (S,d) be complete and separable, and let $\{\mathbb{P}_\alpha\}$ be a family of laws of processes with sample paths in $D_S[0,\infty)$. Then $\{\mathbb{P}_\alpha\}$ is relatively compact if and only if the following two conditions hold.*

(i) For every $\varepsilon > 0$ and rational $T \geq 0$, there exists a compact set $\Gamma_{\varepsilon,T} \subset S$ such that

$$\inf_\alpha \mathbb{P}_\alpha(X(T) \in \Gamma^\varepsilon_{\varepsilon,T}) \geq 1 - \varepsilon.$$

(ii) For every $\varepsilon > 0$ and $T > 0$, there exists a $\delta > 0$ such that

$$\sup_\alpha \mathbb{P}_\alpha(w'(X,\delta,T) \geq \varepsilon) \leq \varepsilon.$$

(Recall that tightness and relative compactness are equivalent in complete separable spaces.)

Observe that both the conditions in Theorem 4.3.3 involve only the paths of the processes on fixed intervals $[0,T]$. Our strategy for proving tightness of the conditioned Brownian bridges rests on the fact that the conditions of Theorem 4.3.3 are satisfied by the Bessel process, and that for large enough t, the process X under the law $\mathbb{P}^{z,x}_t$ is sufficiently 'close' in law to the Bessel process on $[0,T]$ for the conditions to hold for it also.

There two steps to the proof of Lemma 4.3.2: direct calculation with the transition densities shows that convergence holds in the sense of finite dimensional distributions, and then tightness of the measures implies full convergence in distribution. We give the two sections of this proof separately.

Proof of Lemma 4.3.2: convergence of finite dimensional distributions. For $y_1, y_2 \in \mathbb{R}$ and $s > 0$ introduce the notation

$$p_s(y_1,y_2) := \frac{1}{\sqrt{2\pi s}} \exp\left(- \frac{(y_1 - y_2)^2}{2s}\right),$$

for the standard Brownian transition density (with respect to Lebesgue measure), and

$$q_s(y_1,y_2) := p_s(y_1,y_2) - p_s(y_1,-y_2),$$

the transition density for Brownian motion killed at the origin. With this notation, the transition density for a Bessel-3 process is $\frac{y_2}{y_1} q_s(y_1, y_2)$. Now for any finite set of times $0 < t_1 < t_2 < \ldots < t_k < t$, we can re-write the $\mathbb{P}_t^{z,x}$-law of $(X(t_1), \ldots, X(t_k))$ in terms of a standard Brownian motion Y (started at z) under the law $\mathbb{P}_0^z$, conditioned on its position at time t — we remark that rigorous justification for this slight abuse of notation can be found in Revuz and Yor [79, Chapter XI] or Bramson [16]. See also Borodin and Salminen [14, IV.20–IV.26] for some very similar calculations.

$$
\begin{aligned}
&\mathbb{P}_t^{z,x}(X(t_1) \in \mathrm{d}y_1; \ldots; X(t_k) \in \mathrm{d}y_k) \\
&\quad = \frac{\mathbb{P}_0^z(Y(t_1) \in \mathrm{d}y_1; \ldots; Y(t_k) \in \mathrm{d}y_k; \tau_0 > t \,|\, Y(t) \in \mathrm{d}x)}{\mathbb{P}_0^z(\tau_0 > t \,|\, Y(t) \in \mathrm{d}x)} \\
&\quad = \frac{\mathbb{P}_0^z(Y(t_1) \in \mathrm{d}y_1; \ldots; Y(t_k) \in \mathrm{d}y_k; Y(t) \in \mathrm{d}x; \tau_0 > t)}{\mathbb{P}_0^z(Y(t) \in \mathrm{d}x; \tau_0 > t)} \\
&\quad = \frac{\mathbb{P}_0^z(Y(t_1) \in \mathrm{d}y_1; \tau_0 > t_1) \ldots \mathbb{P}_0^{y_k}(Y(t - t_k) \in \mathrm{d}x; \tau_0 > t - t_k)}{\mathbb{P}_0^z(Y(t) \in \mathrm{d}x; \tau_0 > t)},
\end{aligned}
$$

where the final equality follows from the Markov property, and this probability density is equal to

$$
\frac{\frac{y_1}{z} q_{t_1}(z, y_1) \cdots \frac{y_k}{y_{k-1}} q_{t_k - t_{k-1}}(y_{k-1}, y_k) \frac{x}{y_k} q_{t - t_k}(y_k, x)}{\frac{x}{z} q_t(z, x)}.
$$

From this we conclude that the finite dimensional distributions of the Brownian bridge conditioned to avoid the origin are the finite dimensional distributions of the Bessel-3 bridge, and hence these two bridges are equal in law. This means that we can use $\mathbb{P}_t^{z,x}$ for the law of a Bessel-3 bridge from z to x over the time interval $[0, t]$, with the extension $X(\phi) \equiv x$ for $\phi > t$. Further, a calculation with the explicit expressions for the transition densities shows that

$$
\frac{z}{y_k} \frac{q_{t-t_k}(y_k, x)}{q_t(z, x)} = \frac{z}{y_k} \sqrt{\frac{t}{t - t_k}} \frac{\exp\left(-\frac{(y_k - x)^2}{2(t - t_k)}\right)\left(1 - e^{\frac{-2y_k x}{t - t_k}}\right)}{\exp\left(-\frac{(z - x)^2}{2t}\right)\left(1 - e^{\frac{-2zx}{t}}\right)} \longrightarrow 1
$$

as $t \to \infty$, and so

$$
\mathbb{P}_t^{z,x}(X(t_1) \in \mathrm{d}y_1; \ldots; X(t_k) \in \mathrm{d}y_k) \xrightarrow[t \to \infty]{} \mathbb{P}_B^z(X(t_1) \in \mathrm{d}y_1; \ldots; X(t_k) \in \mathrm{d}y_k).
$$

Hence the Bessel bridge from z to x on time interval $[0, t]$ converges in the sense of finite dimensional distributions to a Bessel process started at z as $t \to \infty$. $\square$

Proof of Lemma 4.3.2: tightness. We break this proof down into a series of lemmas. The first lemma expresses formally the the intuitive notion that, for $0 < T \ll t$, the Bessel bridge behaves almost like a Bessel process on $[0, T]$.

Lemma 4.3.4 (Revuz and Yor [79, Chapter XI, Exercise(3.10)]). *Fix $T > 0$, let $t > T$, and define the σ-algebra $\mathcal{F}_T := \sigma(X_s : 0 \le s \le T)$. $\mathbb{P}_t^{z,x}$ has a density $M^{(T)}(t)$ on $\mathcal{F}_T$ with respect to $\mathbb{P}_B^z$, given by*

$$\left.\frac{\mathrm{d}\mathbb{P}_t^{z,x}}{\mathrm{d}\mathbb{P}_B^z}\right|_{\mathcal{F}_T} = M^{(T)}(t) = \frac{\mathbb{P}_B^z(X(t) = x|\mathcal{F}_T)}{\mathbb{P}_B^z(X(t) = x)}.$$

As $t \to \infty$, $M^{(T)}(t) \to 1$ point-wise and in $\mathcal{L}^1(\mathbb{P}_B^z)$.

Proof. Let $A \in \mathcal{F}_T$.

$$\mathbb{P}_t^{z,x}(A) = \mathbb{P}_B^z(A|X(t) = x) = \frac{\mathbb{P}_B^z(\mathbf{1}_A \mathbb{P}_B^z(X(t) = x|\mathcal{F}_T))}{\mathbb{P}_B^z(X(t) = x)},$$

and then

$$M^{(T)}(t) = \frac{\mathbb{P}_B^z(X(t) = x|\mathcal{F}_T)}{\mathbb{P}_B^z(X(t) = x)} = \frac{z}{X(T)} \frac{q_{t-T}(X(T), x)}{q_t(z, x)}.$$

Another calculation with the transition densities shows that this converges to 1 point-wise as $t \to \infty$. Since $\mathbb{P}_B^z(M^{(T)}(t)) = 1$ for all $t > T$, $M^{(T)}(t) \to 1$ in $\mathcal{L}^1(\mathbb{P}_B^z)$ also. □

Lemma 4.3.5. *Retaining the notation of the previous result, let $A \in \mathcal{F}_T$. As $t \to \infty$, $\mathbb{P}_t^{z,x}(A) \to \mathbb{P}_B^z(A)$.*

Proof. Note first that, by Lemma 4.3.4, $\mathbf{1}_A M^{(T)}(t) \to \mathbf{1}_A$ almost surely with respect to $\mathbb{P}_B^z$ as $t \to \infty$. Also $\mathbf{1}_A M^{(T)}(t) \le M^{(T)}(t)$, and we now bound $M^{(T)}(t)$ uniformly in $X(T)$. Noting that $1 - e^{-x} \le x$, we have

$$\begin{aligned}
M^{(T)}(t) &= \frac{z}{X(T)} \frac{q_{t-T}(X(T), x)}{q_t(z, x)} \\
&= \frac{z}{X(T)} \sqrt{\frac{t}{t-T}} \frac{\exp\left(-\frac{(X(T)-x)^2}{2(t-T)}\right)\left(1 - e^{\frac{-2X(T)x}{t-T}}\right)}{\exp\left(-\frac{(z-x)^2}{2t}\right)\left(1 - e^{\frac{-2zx}{t}}\right)} \\
&\le \frac{2xz}{t-T} \sqrt{\frac{t}{t-T}} \left(\exp\left(-\frac{(z-x)^2}{2t}\right)\left(1 - e^{\frac{-2zx}{t}}\right)\right)^{-1} \\
&\to 1 \qquad \text{as } t \to \infty.
\end{aligned}$$

This deterministic bound is continuous on the interval $[T+\varepsilon,\infty)$, for any $\varepsilon > 0$, and so there exists a constant $0 < C(x,z) < \infty$ such that $M^{(T)}(t) \leq C(x,z)$ on $[T+\varepsilon,\infty)$. Bounded convergence now finishes the argument.

Alternatively, since $\mathbf{1}_A M^{(T)}(t) \leq M^{(T)}(t)$ and $\mathbb{P}^z_B(M^{(T)}(t)) = 1$ for all $t > T$, dominated convergence (as stated in Kallenberg [56, Theorem 1.21]) gives

$$\lim_{t\to\infty} \mathbb{P}^{z,x}_t(A) = \lim_{t\to\infty} \mathbb{P}^z_B(\mathbf{1}_A M^{(T)}(t)) = \mathbb{P}^z_B(A).$$

□

Since the Bessel-3 process has continuous paths, it can be easily checked that the conditions of Theorem 4.3.3 hold for the single law $\mathbb{P}^z_B$; another way to see this is that by Ethier and Kurtz [34, Lemma 2.1] a single measure on a complete separable space is tight, and so $\mathbb{P}^z_B$ satisfies the conditions of Theorem 4.3.3. Hence

(i) for every $\varepsilon > 0$ and rational $T \geq 0$, we can choose a compact set $\Gamma_{\varepsilon,T} \subset [0,\infty)$ such that
$$\mathbb{P}^z_B(X(T) \in \Gamma^\varepsilon_{\varepsilon,T}) \geq 1 - \frac{\varepsilon}{2}; \qquad \text{and} \tag{4.9}$$

(ii) for every $\varepsilon > 0$ and $T > 0$, we can choose a $\delta > 0$ such that
$$\mathbb{P}^z_B(w'(X,\delta,T) \geq \varepsilon) \leq \frac{\varepsilon}{2}. \tag{4.10}$$

The events $\{X(T) \in \Gamma^\varepsilon_{\varepsilon,T}\}$ and $\{w'(X,\delta,T) \geq \varepsilon\}$ are both $\mathcal{F}_T$-measurable, and it follows from Lemma 4.3.5, and equations (4.9) and (4.10), that there exists a $t_0(z) < \infty$ such that, for all $t > t_0(z)$,

$$\mathbb{P}^{z,x}_t(X(T) \in \Gamma^\varepsilon_{\varepsilon,T}) \geq 1 - \varepsilon \qquad \text{and} \qquad \mathbb{P}^{z,x}_t(w'(X,\delta,T) \geq \varepsilon) \leq \varepsilon.$$

Here $\Gamma^\varepsilon_{\varepsilon,T}$ and δ are those chosen at, respectively, (4.9) and (4.10). Theorem 4.3.3 now gives us that the family of laws $\{\mathbb{P}^{z,x}_t\}_{t>t_0(z)}$ is tight, and this is sufficient to complete the proof of Lemma 4.3.2. □

4.4 Martingale results and spine decompositions

Introduced in Lyons [71], and also seen in, for example, Kyprianou [66], a very useful method for analysing additive martingales such as Z_λ and V is to use a spine decomposition — this involves taking a $\tilde{\mathbb{Q}}$-conditional expectation with respect to $\tilde{\mathcal{G}}_\infty$ (the

σ-algebra generated by the spine's spatial path and fission times), which reduces the analysis of the martingale to one-particle calculations on the spine. We use this to prove Theorem 4.1.6, but, as was the case in Chapter 3, we must first justify the spine construction rigorously by defining a measure on the largest filtration $\tilde{\mathcal{F}}_t$. We retain the notation of Section 3.2, and remind the reader in particular that $\tilde{P}^x$ is the law of the $(-\rho, \beta; \mathbb{R})$-BBM with a distinguished spine; that ξ_t is the spine, which is a Brownian motion with drift $-\rho$ under $\tilde{P}^x$; and that n_t is the number of births on the spine.

The process

$$\tilde{V}(t) := 2^{n_t} e^{-\beta t} \times e^{\rho \xi_t + \frac{1}{2}\rho^2 t} \times \xi_t \mathbf{1}_{\{\tau_0 > t\}},$$

where $\tau_0 := \inf\{t > 0 : \xi_t = 0\}$, defines a $\tilde{P}^x$-martingale that is adapted to $\{\tilde{\mathcal{F}}_t\}_{t \geq 0}$. Note that this is the product of single-particle Radon-Nikodým derivatives that will — reading from left to right — increase the branching rate on the spine to 2β, remove the drift from the spine, and condition the spine to avoid to origin. We use this to define a new measure $\tilde{\mathbb{Q}}^x$

$$\left.\frac{\mathrm{d}\tilde{\mathbb{Q}}^x}{\mathrm{d}\tilde{P}^x}\right|_{\tilde{\mathcal{F}}_t} = \frac{\tilde{V}(t)}{\tilde{V}(0)},$$

for $x > 0$. Using this change of measure with the decomposition of $\tilde{P}^x$ at equation (3.9) we see that $\tilde{\mathbb{Q}}$ can be decomposed as

$$\mathrm{d}\tilde{\mathbb{Q}} = \mathrm{d}\mathbb{P}_B(\xi_t)\mathrm{d}\mathbb{L}^{2\beta}(n_t) \prod_{v < \xi_t} \frac{1}{2} \mathrm{d}P((\tau, M)^v),$$

where ξ_t is a Bessel-3 process under $\mathbb{P}_B$. This means that, under $\tilde{\mathbb{Q}}$, the $(-\rho, \beta; \mathbb{R}^+)$-BBM may be re-constructed in law in the manner described in Section 4.1. We now let $\mathbb{Q}$ be the restriction $\tilde{\mathbb{Q}}|_{\mathcal{F}_\infty}$, and it can be shown that $\mathbb{Q}$ satisfies

$$\left.\frac{\mathrm{d}\mathbb{Q}^x}{\mathrm{d}P^x}\right|_{\mathcal{F}_t} = \frac{V(t)}{V(0)}.$$

This follows from a calculation very similar to the proof of Proposition 3.2.7, and it justifies the construction of the process under $\mathbb{Q}$ given earlier. As noted in Remark 3.2.8, this is also another way of showing that V is a martingale.

To prove Theorem 4.1.6, we adapt some of the spine techniques of Hardy and Harris [42]. This involves full use of the different filtrations introduced in Section 3.2, beginning with the spine decomposition for additive martingales. We give the spine decomposition for V below, with a full proof because it is an important idea. We note, however, that a similar calculation appears in Kyprianou [66], and a much more general formulation of the spine decomposition is given in Hardy and Harris [41].

Proposition 4.4.1 (Spine decomposition of $V(t)$).

$$\tilde{\mathbb{Q}}(V(t)|\tilde{\mathcal{G}}_\infty) = \sum_{u<\xi_t} \xi(S_u)e^{\rho\xi(S_u)+(\frac{1}{2}\rho^2-\beta)S_u} + \xi_t e^{\rho\xi_t+(\frac{1}{2}\rho^2-\beta)t}$$

Proof. This decomposition follows from the fact that V can be split into a contribution from the particle that is the spine at time t, plus the contribution from each of the n_t sub-trees that have branched from the spine by time t. These sub-trees that branch off the spine behave as if under the original measure P, and so have constant $\tilde{\mathbb{Q}}$-expectation, because V is a P-martingale. More formally, under $\tilde{\mathbb{Q}}$ we can write

$$\begin{aligned} V(t) &= \sum_{u\in N_t^{-\rho}, u\notin\xi} Y_u(t)e^{\rho Y_u(t)+(\frac{1}{2}\rho^2-\beta)t} + \xi_t e^{\rho\xi_t+(\frac{1}{2}\rho^2-\beta)t} \\ &= \sum_{u<\xi_t} e^{(\frac{1}{2}\rho^2-\beta)S_u}\Bigg(\sum_{v\in N_t^{-\rho}\cap(\tau,M)^u} Y_v(t)e^{\rho Y_v(t)+(\frac{1}{2}\rho^2-\beta)(t-S_u)}\Bigg) + \xi_t e^{\rho\xi_t+(\frac{1}{2}\rho^2-\beta)t}. \end{aligned} \tag{4.11}$$

Here we have just decomposed $V(t)$ into the contribution from the spine, plus the sum of the contributions from the sub-trees which branched off from the spine before time t. We have

$$\begin{aligned} &\tilde{\mathbb{Q}}\Bigg(\sum_{u<\xi_t} e^{(\frac{1}{2}\rho^2-\beta)S_u}\Bigg(\sum_{v\in N_t^{-\rho}\cap(\tau,M)^u} Y_v(t)e^{\rho Y_v(t)+(\frac{1}{2}\rho^2-\beta)(t-S_u)}\Bigg)\Bigg|\tilde{\mathcal{G}}_\infty\Bigg) \\ &= \sum_{u<\xi_t} e^{(\frac{1}{2}\rho^2-\beta)S_u}\tilde{\mathbb{Q}}\Bigg(\sum_{v\in N_t^{-\rho}\cap(\tau,M)^u} Y_v(t)e^{\rho Y_v(t)+(\frac{1}{2}\rho^2-\beta)(t-S_u)}\Bigg|\tilde{\mathcal{G}}_\infty\Bigg) \\ &= \sum_{u<\xi_t} e^{(\frac{1}{2}\rho^2-\beta)S_u} \times \xi(S_u)e^{\rho\xi(S_u)}, \end{aligned}$$

since the sub-trees branching off from the spine behave like the process under the measure P. In view of this, taking the $\tilde{\mathbb{Q}}$-conditional expectation of (4.11) completes the proof. □

Remark 4.4.2. The spine decomposition can be used with any branching-particle martingale once a spine structure has been added to the probability space in which one is working.

We now give the measure-theoretic dichotomy, mentioned in Chapter 1, that has been seen to be so useful in proving convergence results for additive martingales. We then use it to prove Theorem 4.1.6 by adapting the method of Hardy and Harris [42].

Theorem 4.4.3. *Let $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0})$ be a filtered space. Suppose that P and $\mathbb{Q}$ are two probability measures thereon, which are related by the Radon-Nikodým derivative*

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}P}\right|_{\mathcal{F}_t} = Z_t,$$

for some strictly-positive martingale Z_t. Defining $Z_\infty := \limsup_{t\to\infty} Z_t$, for any $A \in \mathcal{F}_\infty$ we have

$$\mathbb{Q}(A) = \int_A Z_\infty \,\mathrm{d}P + \mathbb{Q}(A \cap \{Z_\infty = \infty\}), \tag{4.12}$$

and hence:

(i) $\mathbb{Q}$ is absolutely continuous with respect to P if and only if $\int_\Omega Z_\infty \,\mathrm{d}P = 1$ if and only if $\mathbb{Q}(Z_\infty = \infty) = 0$;

(ii) $\mathbb{Q}$ is singular with respect to P if and only if $P(Z_\infty = 0) = 1$ if and only if $\mathbb{Q}(Z_\infty = \infty) = 1$.

A proof of (4.12) can be found in Athreya [3]. See also Durrett [26, Exercise 3.6]. In a similar manner to our result on the convergence properties of W_λ (Theorem 3.1.2), we actually prove a stronger result for V than the $\mathcal{L}^1(P^x)$-convergence stated in Theorem 4.1.6.

Proposition 4.4.4. *For $x > 0$ and any $p \in (1, 2]$,*

(i) the martingale V is $\mathcal{L}^p(P^x)$-convergent if $p\rho^2/2 < \beta$;

(ii) almost surely under P^x, $V(\infty) = 0$ when $\rho \geq \sqrt{2\beta}$.

Proof. (i) For $p \in (1, 2]$ we have

$$P^x\left(\frac{V(t)^p}{V(0)^p}\right) = P^x\left(\frac{V(t)^{p-1}}{V(0)^{p-1}}\frac{V(t)}{V(0)}\right) = \frac{1}{V(0)^q}\mathbb{Q}^x(V(t)^q),$$

where $q := p-1$. That $V(t)^q$ is a $\mathbb{Q}$-submartingale follows from the fact that, by Jensen's inequality, $V(t)^{1+q}$ is a P-submartingale. We prove that $\mathbb{Q}^x(V(t)^q)$ is bounded for all $t > 0$, since then $V(t)$ is bounded in $\mathcal{L}^p(P^x)$ and hence converges in $\mathcal{L}^p(P^x)$ by Doob's $\mathcal{L}^p$ inequality (see Rogers and Williams [80, Theorem II.70.2]).

The conditional form of Jensen's inequality gives

$$\tilde{\mathbb{Q}}(V(t)^q|\tilde{\mathcal{G}}_\infty) \leq \tilde{\mathbb{Q}}(V(t)|\tilde{\mathcal{G}}_\infty)^q, \tag{4.13}$$

and we have the following inequality for sums, used in Neveu [76].

Proposition 4.4.5. *If* $q \in (0,1]$ *and* $x, y > 0$, *then* $(x+y)^q \leq x^q + y^q$.

Using this, with equation (4.13) and the spine decomposition for V (recall Proposition 4.4.1), we have

$$\begin{aligned}\mathbb{Q}^x(V(t)^q) &= \tilde{\mathbb{Q}}^x(\tilde{\mathbb{Q}}^x(V(t)^q|\tilde{\mathcal{G}}_\infty)) \leq \tilde{\mathbb{Q}}^x(\tilde{\mathbb{Q}}^x(V(t)|\tilde{\mathcal{G}}_\infty)^q)\\ &\leq \tilde{\mathbb{Q}}^x\Big(\sum_{u<\xi_t} \xi(S_u)^q e^{q\rho\xi(S_u)+q(\frac{1}{2}\rho^2-\beta)S_u}\Big) + \tilde{\mathbb{Q}}^x\Big(\xi_t^q e^{q\rho\xi_t+q(\frac{1}{2}\rho^2-\beta)t}\Big).\end{aligned}$$

We will refer to the two expectations on the line above as, respectively, $\tilde{\mathbb{Q}}^x(\mathbf{sum}(t))$ and $\tilde{\mathbb{Q}}^x(\mathbf{spine}(t))$, and we now find upper bounds for these two expressions.

The transition density of the Bessel-3 process is

$$p_B(t;x,y) := \frac{1}{\sqrt{2\pi t}}\frac{y}{x}\Big(\exp\Big(-\frac{(x-y)^2}{2t}\Big) - \exp\Big(-\frac{(x+y)^2}{2t}\Big)\Big),$$

and so

$$\begin{aligned}\tilde{\mathbb{Q}}^x(\xi_t^q e^{q\rho\xi_t}) &= \int_0^\infty y^q e^{q\rho y} p_B(t;x,y)\,\mathrm{d}y\\ &\leq \frac{1}{\sqrt{2\pi t}}\int_0^\infty y^q e^{q\rho y}\frac{y}{x}\exp\Big(-\frac{(x-y)^2}{2t}\Big)\,\mathrm{d}y,\\ &= \frac{e^{\rho q x+\frac{1}{2}\rho^2 q^2 t}}{x\sqrt{2\pi t}}\int_0^\infty y^{q+1}\exp\Big(-\frac{(y-(\rho q t+x))^2}{2t}\Big)\,\mathrm{d}y\\ &\leq \frac{e^{\rho q x+\frac{1}{2}\rho^2 q^2 t}}{x}\Big(\mathbb{P}^x_{\rho q}(Y_t \in [0,1]) + \mathbb{E}^x_{\rho q}(Y_t^2)\Big),\end{aligned}$$

where Y is a Brownian motion with drift $\rho q > 0$ under $\mathbb{P}_{\rho q}$. Noting that $\mathbb{E}^x_{\rho q}(Y_t^2) = t + (x+\rho q t)^2 =: k(t)$, we have

$$\tilde{\mathbb{Q}}^x(\mathbf{spine}(t)) \leq \frac{e^{\rho q x}}{x}(1+k(t))e^{q(\frac{1}{2}(q+1)\rho^2-\beta)t},$$

which decays as $t \to \infty$ if and only if $\frac{1}{2}p\rho^2 < \beta$.

To complete the proof of (i) we must show that $\tilde{\mathbb{Q}}^x(\mathbf{sum}(t))$ is also bounded for all

$t > 0$. The fission-times on the spine occur as a Poisson process of rate 2β, and standard theory for Poisson processes — see Kallenberg [56] — lets us re-write $\tilde{\mathbb{Q}}^x(\mathbf{sum}(t))$ as an integral:

$$\begin{aligned}\tilde{\mathbb{Q}}^x(\mathbf{sum}(t)) &= \tilde{\mathbb{Q}}^x\left(\tilde{\mathbb{Q}}^x\left(\sum_{u<\xi_t} \xi(S_u)^q e^{q\rho\xi(S_u)+q(\frac{1}{2}\rho^2-\beta)S_u}\middle|\mathcal{G}_\infty\right)\right)\\ &= \tilde{\mathbb{Q}}^x\left(\int_0^t 2\beta\xi_s^q e^{q\rho\xi_s+q(\frac{1}{2}\rho^2-\beta)s}\,\mathrm{d}s\right).\end{aligned}$$

Applying Fubini's Theorem with our upper bound for $\tilde{\mathbb{Q}}^x(\mathbf{spine}(t))$ gives

$$\tilde{\mathbb{Q}}^x(\mathbf{sum}(t)) \le \int_0^t \frac{2\beta e^{\rho q x}}{x}(1+k(s))e^{q(\frac{1}{2}p\rho^2-\beta)s}\,\mathrm{d}s.$$

This is bounded for all $t > 0$ if $\frac{1}{2}p\rho^2 < \beta$.

(ii) Observe that

$$V(t) \ge \xi_t e^{\rho\xi_t+(\frac{1}{2}\rho^2-\beta)t},$$

since one of the particles alive at time t is the spine. Then as ξ_t is a $\tilde{\mathbb{Q}}^x$-Bessel-3 process, and so is transient, we have

$$\mathbb{Q}^x(\limsup_{t\to\infty} V(t) = +\infty) = 1$$

if $\rho \ge \sqrt{2\beta}$. Applying Theorem 4.4.3 gives $P^x(\limsup_{t\to\infty} V(t) = 0) = 1$ and completes the proof. Alternatively one could note that $\{\zeta < \infty\} \subseteq \{V(\infty) = 0\}$, and since $P^x(\zeta < \infty) = 1$ when $\rho \ge \sqrt{2\beta}$ the result follows. □

We now prove Theorem 4.1.6.

Proof of Theorem 4.1.6. If $\rho \ge \sqrt{2\beta}$ then, as we have just seen in Proposition 4.4.4, $V(\infty) = 0$ almost surely.

If $\rho < \sqrt{2\beta}$ then it follows from Proposition 4.4.4 that there exists a $p > 1$ such that V converges in $\mathcal{L}^p(P^x)$, and so V is uniformly integrable. It remains to show that $V(\infty) > 0$ almost surely.

To prove that $P^x(V(\infty) = 0; \zeta = \infty) = 0$ when $\rho < \sqrt{2\beta}$, we note that

$$\begin{aligned}P^x(V(\infty) = 0) &= P^x(V(\infty) = 0; \zeta = \infty) + P^x(V(\infty) = 0; \zeta < \infty)\\ &= P^x(V(\infty) = 0; \zeta = \infty) + P^x(\zeta < \infty)\end{aligned}$$

and hence it suffices to show that $P^x(V(\infty) = 0) = P^x(\zeta < \infty)$. As in the proof of Theorem 3.1.2, we use uniqueness of the one-sided travelling wave in the case $\rho < \sqrt{2\beta}$ to prove this. Define $p(x) := P^x(V(\infty) = 0)$, and then

$$p(x) = E^x\Big(P^x(V(\infty) = 0|\mathcal{F}_t)\Big) = E^x\bigg(\prod_{u \in N_t^{-\rho}} p(Y_u(t))\bigg), \tag{4.14}$$

whence $p(x)$ satisfies the travelling-wave ODE. Since extinction in a finite time guarantees that $V(\infty) = 0$, we also have $\lim_{x \downarrow 0} p(x) = 1$. We do not have a representation for $V(\infty)$ like the one we gave for $W_\lambda(\infty)$ in Remark 3.3.2, but, considering the process path-wise, we see that increasing x increases the value of V under the law P^x. Recall here that $\rho > 0$, so $xe^{\rho x}$ is increasing in x. Hence $p(x)$ is monotone decreasing in x and $p(x) \downarrow p(\infty)$ as $x \to \infty$.

Now consider taking any fixed infinite BBM tree started at x. For any fixed time $t > 0$, we have $N_t^{-\rho} \uparrow \mathcal{N}_t^{-\rho}$ as $x \to \infty$. Looking at the process path-wise again, for all $u \in N_t^{-\rho}$ we have $Y_u(t) \uparrow \infty$ as $x \to \infty$. Taking the limit $x \to \infty$ in (4.14) we then have $p(\infty) = E^0\Big(\prod_{u \in \mathcal{N}_t^{-\rho}} p(\infty)\Big)$, whence $p(\infty) \in \{0, 1\}$ and now uniform integrability of V forces $p(\infty) = 0$. Uniqueness of the one-sided travelling wave (Theorem 2.1.1) now finishes the argument. □

Remark 4.4.6. As we did at Remark 3.1.9 for W_λ, we can see that $f(x) := E^x(e^{-V(\infty)})$ is not the one-sided travelling wave solution (when it exists) for the system (2.4), since

$$\begin{aligned} E^x(e^{-V(\infty)}) &= E^x(e^{-V(\infty)}; \zeta < \infty) + E^x(e^{-V(\infty)}; \zeta = \infty) \\ &= P^x(\zeta < \infty) + E^x(e^{-V(\infty)}; \zeta = \infty). \end{aligned}$$

V is uniformly integrable if $\rho < \sqrt{2\beta}$, and so $E^x(e^{-V(\infty)}; \zeta = \infty) > 0$ for this parameter range.

Proof of Theorem 4.1.5. This is the same method as the proof of Theorem 3.1.8 — we only have to check the details.

Let $\rho > \sqrt{2\beta}$, $s \in (0, \infty)$ fixed, and $A \in \mathcal{F}_s$. Then

$$P^x(A|R_{t+s} > 0) = \frac{P^x(\mathbf{1}_A P^x(R_{t+s} > 0|\mathcal{F}_s))}{P^x(R_{t+s} > 0)}. \tag{4.15}$$

Setting off along the same path we took in the proof of Theorem 3.1.8, we have

$$Y_s(t) := P^x(R_{t+s} > 0|\mathcal{F}_s) = 1 - \prod_{u \in N_s^{-\rho}} \left(1 - P^{Y_u(s)}(R_t > 0)\right),$$

and, using the asymptotic (Theorem 4.1.2) for the (decaying) probability $P^x(R_t > 0)$,

$$Y_s(t) \underset{t\to\infty}{\sim} \frac{C}{\sqrt{2\pi t^3}} \sum_{u \in N_s^{-\rho}} Y_u(s) e^{\rho Y_u(s) - (\frac{1}{2}\rho^2 - \beta)t}.$$

Combining this with the asymptotic for $P^x(R_{t+s} > 0)$ we have, almost surely,

$$\mathbf{1}_A \frac{P^x(R_{t+s} > 0|\mathcal{F}_s)}{P^x(R_{t+s} > 0)} \xrightarrow[t\to+\infty]{} \mathbf{1}_A \frac{V(s)}{V(0)}. \tag{4.16}$$

It remains to dominate the left-hand side of (4.16) by a random variable that is independent of t. Using the inequality (4.4) gives

$$P^x(R_{t+s} > 0|\mathcal{F}_s) \leq \sum_{u \in N_s^{-\rho}} e^{\rho Y_u(s) - (\frac{1}{2}\rho^2 - \beta)t},$$

and, in light of Theorem 4.1.2, there exists a constant $A(x) > 0$ such that

$$P^x(R_{t+s} > 0) \geq A(x) e^{-(\frac{1}{2}\rho^2 - \beta)(t+s)}$$

for $t > 0$. Putting these last two inequalities together we find that

$$\mathbf{1}_A \frac{P^x(R_{t+s} > 0|\mathcal{F}_s)}{P^x(R_{t+s} > 0)} \leq A(x)^{-1} \sum_{u \in N_s^{-\rho}} e^{\rho Y_u(s) + (\frac{1}{2}\rho^2 - \beta)s} \leq A(x)^{-1} Z_0(s) \in \mathcal{L}^1(P^x),$$

and then dominated convergence gives

$$P^x(A|R_{t+s} > 0) = P^x\left(\mathbf{1}_A \frac{P^x(R_{t+s} > 0|\mathcal{F}_s)}{P^x(R_{t+s} > 0)}\right) \xrightarrow[t\to\infty]{} P^x\left(\mathbf{1}_A \frac{V(s)}{V(0)}\right) = \mathbb{Q}^x(A),$$

as required. $\square$

Chapter 5

Exponential growth rates for a typed branching diffusion

In this chapter we study the typed branching diffusion introduced by Harris and Williams [48], and studied further in Git and Harris [47]. Particles in this process diffuse spatially as driftless Brownian motions on $\mathbb{R}$, and they also have a type, which takes values in $\mathbb{R}$ and evolves as an ergodic Ornstein-Uhlenbeck process. A particle's type determines both its breeding rate and the infinitesimal variance of the Brownian motion driving its spatial movement. The main result of this chapter gives the almost-sure rate of exponential growth of the number of particles at particular space-type positions

The work in this chapter is developed from the preprint by Git and S. C. Harris [47]. The main result of this chapter — Theorem 5.1.1 — appeared in the preprint [47] as a conjecture, and in this thesis we give the first rigorous proof of this result. Our new proof relies on martingale arguments and spine techniques, rather than classical large-deviations methods in the style of the original preprint [47] and Git [38]. However the heuristics presented in Git and Harris [47] are vital in building intuition for our proofs, and so we include a review of these in Section 5.2. Sections 5.4–5.6 contain the original work of this chapter — the proof of the lower bound of Theorem 5.1.1.

This work has appeared in Git *et al.* [39], which in addition contains full details of the original results of Git and Harris [47] that are used without proof in the sequel.

5.1 Introduction

We define N_t to be the set of particles alive at time $t \geq 0$. For a particle $u \in N_t$, $X_u(t) \in \mathbb{R}$ is the spatial position of u, and $Y_u(t) \in \mathbb{R}$ is the *type* of u. The configuration of the branching diffusion at time t is given by the point process $\mathbb{X}_t := \{(X_u(t), Y_u(t)) : u \in N_t\}$. The type moves on the real line as an Ornstein-Uhlenbeck process with a (standard normal) invariant density $\phi(y) := (2\pi)^{-1/2} \exp(-y^2/2)$ and associated differential operator (generator)

$$Q_\theta := \frac{\theta}{2}\left(\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y}\right),$$

where θ is a positive real parameter considered as the *temperature* of the system. The spatial motion of a particle of type y is a driftless Brownian motion with variance

$$A(y) := ay^2, \qquad \text{where } a \geq 0.$$

A particle of type y undergoes dyadic branching at rate

$$R(y) := ry^2 + \rho, \qquad \text{where } r, \rho \geq 0.$$

Each offspring inherits its parent's current type and spatial position, and then moves off independently of all others. We use $P^{x,y}$ and $E^{x,y}$, where $x, y \in \mathbb{R}$, to represent probability and expectation when the Markov process starts with a single particle at position (x, y).

The quadratic breeding rate is a critical rate in terms of *explosions* in the population of particles. In a branching Brownian motion on $\mathbb{R}$ with binary splitting occurring at rate x^p at position x, the population will almost surely explode in a finite time if $p > 2$; whereas for $p = 2$ the expected number of particles blows up in a finite time, but there almost surely remains a finite population at all times (see Itô and McKean [53, pp 200–211]). In Chapter 6 we study the spread of BBM with a quadratic breeding potential.

The Ornstein-Uhlenbeck process has exactly the right drift to help counteract the quadratic breeding rates: for large enough temperature values $\theta > 8r$, there is a sufficiently strong mean-reversion to keep the particles well behaved but, for low temperatures, the large breeding rate outweighs the pull back toward the origin and the particles behave very differently. In fact for $\theta < 8r$ the expected number of particles blows up in finite time, but the total number of particles alive remains almost surely finite for all time — see Harris and Williams [48]. These properties the 'low temperature' typed branching diffusion shares with the quadratic-breeding BBM, which is the object of study in Chapter 6, and in future work we hope to be able to extend some of the methods developed in Chapter 6 to enable a study of this typed process in the low temperature case. Throughout this chapter we consider only values of θ above the critical temperature, that is $\theta > 8r$.

Note that if the spatial motion is ignored, we have a binary branching Ornstein-Uhlenbeck process in a quadratic breeding potential. Enderle and Hering [30] considered a branching Ornstein-Uhlenbeck with constant branching rate but random offspring distribution. In contrast to our exponential growth results, Enderle and Hering studied the convergence properties as $t \to \infty$ of the proportion of the total population in an arbitrary Borel set.

The main result of this chapter is the almost-sure asymptotic rate of exponential growth in the number of particles with given space and type position. For $\gamma \geq 0$ and $C \subset \mathbb{R}$, define

$$N_t(\gamma; C) := \sum_{u \in N_t} \mathbf{1}_{\{X_u(t) \leq -\gamma t;\, Y_u(t) \in C\}}. \tag{5.1}$$

For $\gamma, \kappa \geq 0$, it can be shown that the limit

$$\lim_{t \to \infty} t^{-1} \ln E^{x,y}\left(N_t(\gamma; [\kappa\sqrt{t}, \infty))\right)$$

exists and takes the value

$$\Delta(\gamma, \kappa) := \rho + \frac{(\theta - \kappa^2)}{4} - \frac{1}{4\theta a}\sqrt{\theta(\theta - 8r)(4a\theta\gamma^2 + a^2(\theta + \kappa^2)^2)}. \tag{5.2}$$

An outline for this calculation was given in Git and Harris [47]. This motivates the following almost-sure result.

Theorem 5.1.1. *Let $\gamma, \kappa \geq 0$ with $\Delta(\gamma, \kappa) \neq 0$. Under each $P^{x,y}$ law, the limit*

$$D(\gamma, \kappa) := \lim_{t \to \infty} t^{-1} \ln N_t(\gamma; [\kappa\sqrt{t}, \infty))$$

exists almost surely and is given by

$$D(\gamma, \kappa) = \begin{cases} \Delta(\gamma, \kappa) & \text{if } \Delta(\gamma, \kappa) > 0, \\ -\infty & \text{if } \Delta(\gamma, \kappa) < 0. \end{cases} \tag{5.3}$$

To prove the upper bound on the growth rate, we will use a result from Git and Harris [47] on the exact rate of convergence of some additive martingales. To prove the trickier lower bound, we will exhibit an explicit mechanism whereby the branching diffusion can build up at least the required exponential number of particles near to $-\gamma t$ in space and $\kappa\sqrt{t}$ in type position by large times t. This mechanism involves particles spending almost all of their time building up in large numbers at a certain proportion of the required spatial position, and then a small proportion of this group of particles succeeding in making a very rapid ascent out to the required final position. Finding a lower bound for the probability that a single particle makes this rapid ascent is an application of spine change-of-measure techniques, and represents the bulk of the original work of this chapter. The following two corollaries are immediate consequences of Theorem 5.1.1.

Corollary 5.1.2. *For any $F \subset \mathbb{R}^2$, define*

$$\mathcal{N}_t(F) := \sum_{u \in N_t} \mathbf{1}_{\left\{\left(\frac{X_u(t)}{t}, \frac{Y_u(t)}{\sqrt{t}}\right) \in F\right\}}.$$

If $B \subset \mathbb{R}^2$ is any open set and $C \subset \mathbb{R}^2$ is any closed set, then almost surely under any $P^{x,y}$

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathcal{N}_t(B) \geq \sup_{(\gamma,\kappa)\in B} D(\gamma,\kappa),$$
$$\limsup_{t\to\infty} \frac{1}{t} \log \mathcal{N}_t(C) \leq \sup_{(\gamma,\kappa)\in C} D(\gamma,\kappa),$$

with the growth rate given by $D(\gamma,\kappa)$ as defined above.

Corollary 5.1.3. *Let $B \subset \mathbb{R}^2$ be any open set. Almost surely under each $P^{x,y}$ law,*

$$\mathcal{N}_t(B) \to \begin{cases} 0 & \text{if } \mathcal{S} \cap B = \varnothing \\ +\infty & \text{if } \mathcal{S} \cap B \neq \varnothing, \end{cases}$$

where $\mathcal{S} \subset \mathbb{R}^2$ is the set given by $\mathcal{S} := \{(\gamma,\kappa) \in \mathbb{R}^2 | \Delta(\gamma,\kappa) > 0\}$.

5.2 Review of previous work on this model

In this section we review some of the results of Harris and Williams [48], and Git and Harris [47]. These results will be needed as intermediate steps in the proof of

Theorem 5.1.1. As well as this, we will give brief details of some heuristic arguments that Git and Harris gave to gain an intuition for the dominant behaviour of particles that are found at space-type position $(-\gamma t, \kappa\sqrt{t})$ at time t. Although not strictly essential for our later arguments, these heuristics are useful motivation for our new spine approach to Theorem 5.1.1; and they go some way toward justifying our later parameter choices.

First, we give some key definitions. Let

$$\lambda_{\min} := -\sqrt{\frac{\theta - 8r}{4a}},$$

and then let $\lambda < 0$ be a parameter such that

$$\lambda_{\min} < \lambda < 0.$$

Also, define

$$\mu_\lambda := \frac{1}{2}\sqrt{\theta(\theta - 8r - 4a\lambda^2)} \qquad \psi_\lambda^{\pm} := \frac{1}{4} \pm \frac{\mu_\lambda}{2\theta} \qquad E_\lambda^{\pm} := \rho + \theta\psi_\lambda^{\pm} \quad c_\lambda^{\pm} := -E_\lambda^{\pm}/\lambda.$$

Note that $\lambda_{\min}$ is the point beyond which μ_λ is no longer a real number.

Martingales

The main tools in our analysis of this typed branching diffusion, and indeed also in Git and Harris's proofs of their results that we quote in this section, are two families of additive martingales. These are defined as

$$Z_\lambda^{\pm}(t) := \sum_{u \in N_t} v_\lambda^{\pm}(Y_u(t)) e^{\lambda X_u(t) - E_\lambda^{\pm} t} \qquad \lambda \in (\lambda_{\min}, 0), \tag{5.4}$$

where $v_\lambda^{\pm}(y) := \exp(\psi_\lambda^{\pm} y^2)$ are strictly-positive eigenfunctions of the self-adjoint operator

$$\frac{1}{2}\lambda^2 A + R + Q_\theta,$$

with corresponding eigenvalues $E_\lambda^- < E_\lambda^+$ (note that $v_\lambda^+ \notin L^2(\phi)$ so is *not* normalisable).

The spatial growth rate

A vital intermediate step in our proof of the lower bound of Theorem 5.1.1 is to find the asymptotic growth-rate of particles in the spatial dimension only. For $\gamma \geq 0$, the limit giving the *expected* rate of growth,

$$\lim_{t\to\infty} t^{-1} \log E(N_t(\gamma;\mathbb{R})),$$

can be shown to exist and its value can be calculated to be

$$\Delta(\gamma) := \rho + \frac{\theta}{4} - \frac{1}{4}\sqrt{\theta(\theta-8r)\,(1+4\gamma^2/(\theta a))}.$$

An outline for this calculation was given in Git and Harris [47]. It is now tempting to guess that the asymptotic speed of the spatially left-most particle, $\tilde{c}(\theta)$, is given by

$$\begin{aligned} \tilde{c}(\theta) &:= \sup\{\gamma : \Delta(\gamma) > 0\} \\ &= \sqrt{2a\left(r+\rho+\frac{2(2r+\rho)^2}{\theta-8r}\right)}. \end{aligned} \tag{5.5}$$

In this particular situation, this guess that 'expectation' and 'particle' wave-speeds agree was proved using a change-of-measure technique in Harris and Williams [48]. In this thesis, we extend this connection and prove that the 'expected' and 'almost sure' rates of growth of particles (in both space and type dimensions) actually agree.

The following almost-sure result is proved in Git *et al.* [39].

Theorem 5.2.1. *Let $\gamma \geq 0$ and $y_0 < y_1$. Under each $P^{x,y}$ law, the limit*

$$D(\gamma) := \lim_t t^{-1} \log N_t(\gamma; [y_0, y_1])$$

exists almost surely and is given by

$$D(\gamma) = \begin{cases} \Delta(\gamma) & \text{if } 0 \leq \gamma < \tilde{c}(\theta), \\ -\infty & \text{if } \gamma \geq \tilde{c}(\theta). \end{cases}$$

Note that symmetry in the process means that there is a corresponding result for particles with spatial velocities greater than $+\gamma$. We may occasionally make use of spatial symmetries without further comment.

Heuristics for the lower bound of Theorem 5.1.1

Following Git and Harris [47], we now try to present an intuitive picture that explains why the almost-sure asymptotic exponential growth of $N_t(\gamma, [\kappa\sqrt{t}, \infty))$ is *at least* of size $\Delta(\gamma, \kappa)$, whenever $\Delta(\gamma, \kappa) > 0$. We emphasise that this is not meant to be precise: it is solely to provide valuable intuition and motivation for our rigorous approach later on.

Our interest is in the number of particles, $N_t(\gamma, \kappa)$, that have space-type positions near to $(\gamma t, \kappa\sqrt{t})$ at a large time t. In the case $\kappa \neq 0$, we require particles at a large distance from the type origin. However, when $\theta > 8r$, the attraction to the type origin is much stronger than the quadratic growth rate, and it is very unlikely that any individual particle u satisfies $Y_u(t) > \varepsilon\sqrt{t}$ for a prolonged period of time. The majority of particles that contribute to $N_t(\gamma, [\kappa\sqrt{t}, \infty))$ at some large time t_0 will fail to contribute to $N_t(\gamma, [\kappa\sqrt{t}, \infty))$ at times soon after; particles come and go very quickly from the wave front near $(\gamma t, \kappa\sqrt{t})$. There will be many possible trajectories these particles have travelled along to get to a position $(\gamma t, \kappa\sqrt{t})$ by large time t, but our proof of the lower bound in Theorem 5.1.1 relies on finding the form of a trajectory that a dominant number of particles will follow. It turns out that such dominant particles will, with a high probability, have had a history made up of two distinct phases.

(i) The *long tread*, taking up almost all of the available time, in which the particles drift spatially with speed $\gamma\theta/(\theta + \kappa^2)$.

(ii) The *short climb*, when, over a fixed time $[t, t + \tau]$, some particles make a rapid ascent to type position $\kappa\sqrt{t}$ whilst additionally gaining $\{\gamma\kappa^2/(\theta + \kappa^2)\}t$ in spatial positioning. These particles will then stay near $(\gamma t, \kappa\sqrt{t})$ for only a short period of time before their type decays back to 0 again. Note that if $\kappa = 0$, then only the first phase is applicable.

For the following heuristics, we can think of $\tau \ll t$ as large and fixed (although in our rigorous approach we will subsequently choose τ as a particular function growing with $\ln t$).

Git and Harris gave a heuristic calculation to find the probability that a single particle, starting near to the origin $(0, 0)$, has at least one descendent in the vicinity of $(-\beta t, \kappa\sqrt{t})$ during a small interval of time just before time τ. This probability turns out to be roughly $\exp(-\Theta(\beta,\kappa)t)$ as t gets very large, where

$$\Theta(\beta,\kappa) = \frac{\kappa^2}{4} + \frac{\sqrt{\theta(\theta-8r)(a^2\kappa^4+4a\theta\beta^2)}}{4a\theta}.$$

This will be formulated precisely as a large-deviation lower bound in Section 5.4, but we will now give very brief details of the Git and Harris heuristic calculation, because the optimal paths that arise for the long tread and short climb will be the ones that we later use in our spine arguments. For the full details of this heuristic argument, we refer the reader to Git *et al.* [39].

Suppose we start the branching diffusion with a single particle at $(0,0)$, and let t be arbitrarily large. We wish to know the probability that there is at least one particle at time τ that has a spatial position near $-\beta t$, having followed close to the path $x(s)$ for $0 \le s \le \tau$, *and* a type position near $\kappa\sqrt{t}$, having closely followed the path $y(s)$ for $0 \le s \le \tau$.

We recall from the large-deviation theory of Freidlin and Wentzell (see Dembo and Zeitouni [24, Chapter 5], for example) that the probability that a *single* particle manages to follow closely both the type path $y(s)$ and the spatial path $x(s)$ for $0 \le s \le \tau$ is roughly given by

$$\exp\bigg(-\frac{1}{2\theta}\int_0^\tau \Big(\dot{y}(s)+\frac{\theta}{2}y(s)\Big)^2 \mathrm{d}s - \frac{1}{2}\int_0^\tau \frac{\dot{x}(s)^2}{ay(s)^2}\mathrm{d}s\bigg),$$

when $x(0)=0, x(\tau)=-\beta t$, $y(0)=0, y(\tau)=\kappa\sqrt{t}$, and t is very large. This probability will typically be very small; but, if such paths are followed by particles in the branching diffusion, we have to also take account of the large breeding rates that are found far from the type origin.

Git and Harris used heuristic arguments involving a birth-death process to show that the probability that a single particle in the branching diffusion manages to make the difficult rapid ascent along path (x,y), to finish near $(\gamma t, \kappa\sqrt{t})$ at time τ, may be roughly estimated by

$$\exp\bigg(-\inf_{x,y}\sup_{w\in[0,\tau]} J(x,y,w)\bigg), \tag{5.6}$$

where

$$J(x,y,w) := \int_0^w \bigg(\frac{1}{2\theta}\Big(\dot{y}(s)+\frac{\theta}{2}y(s)\Big)^2 + \frac{1}{2}\frac{\dot{x}(s)^2}{ay(s)^2} - ry(s)^2 - \rho\bigg)\mathrm{d}s.$$

Git and Harris showed that, in fact,

$$\inf_{x,y} \sup_{w\in[0,\tau]} J(x,y,w) = \inf_{x,y} J(x,y,\tau),$$

where the intuition for this comes from noticing that the paths (x, y) are 'steep', and so the point on the paths that is the most difficult for a particle to reach is the point right at the very end, i.e. at time τ.

We now outline a calculation to find $\inf_{x,y} J(x, y, \tau)$. It is straightforward to optimise over the choice of function x given y, finding that $\dot{x}(s) \propto ay(s)^2$, and hence

$$x(s) = \lambda a \int_0^s y(u)^2 \, \mathrm{d}u,$$

where λ is the constant of proportionality and must satisfy

$$\lambda = \frac{-\beta t}{a\int_0^\tau y(s)^2 \, \mathrm{d}s},$$

yielding

$$\frac{1}{2}\int_0^\tau \frac{\dot{x}(s)^2}{ay(s)^2} \, \mathrm{d}s = \frac{\beta^2 t^2}{2a\int_0^\tau y(s)^2 \, \mathrm{d}s}.$$

This is to be anticipated since, when following the path y in type space, the spatial position of the particle is a Brownian motion with total variance at time τ given by $a\int_0^\tau y(s)^2 \, \mathrm{d}s$, and so the probability that a particle following the path y in type space will also be found near to $-\beta t$ in space at time τ is roughly

$$\exp\left(\frac{-\beta^2 t^2}{2a\int_0^\tau y(s)^2 \, \mathrm{d}s}\right).$$

The optimisation problem for y is then

$$\begin{aligned}
&\inf_y \left\{ \int_0^\tau \left(\frac{1}{2\theta}\left(\dot{y}(s) + \frac{\theta}{2}y(s)\right)^2 + \frac{\beta^2 t^2}{2a\int_0^\tau y(s)^2 \, \mathrm{d}s} - ry(s)^2 \right) \mathrm{d}s \right\} \\
&= \inf_y \sup_\lambda \left\{ \int_0^\tau \left(\frac{1}{2\theta}\left(\dot{y}(s) + \frac{\theta}{2}y(s)\right)^2 - ry(s)^2 - \frac{1}{2}a\lambda^2 y(s)^2 \right) \mathrm{d}s - \lambda\beta t \right\} \\
&\geq \sup_\lambda \inf_y \left\{ \int_0^\tau \left(\frac{1}{2\theta}\left(\dot{y}(s) + \frac{\theta}{2}y(s)\right)^2 - ry(s)^2 - \frac{1}{2}a\lambda^2 y(s)^2 \right) \mathrm{d}s - \lambda\beta t \right\},
\end{aligned}$$

where the equality above follows on maximising the quadratic in λ. Some Euler-

Lagrange optimisation now gives the optimal path as

$$y_\lambda(s) = \kappa\sqrt{t}\frac{\sinh \mu_\lambda s}{\sinh \mu_\lambda \tau} \qquad (0 \leq s \leq \tau), \tag{5.7}$$

where

$$\mu_\lambda = \frac{\sqrt{\theta(\theta - 8r - 4a\lambda^2)}}{2},$$

and then

$$\begin{aligned}\sup_\lambda \inf_y &\left\{ \int_0^\tau \left(\frac{1}{2\theta}\left(\dot{y}(s) + \frac{\theta}{2}y(s)\right)^2 - ry(s)^2 - \frac{1}{2}a\lambda^2 y(s)^2 \right) \mathrm{d}s + \lambda\beta t \right\} \\ &= \sup_\lambda \left\{ \lambda\beta t + \kappa^2 t \left(\frac{1}{4} + \frac{\mu_\lambda}{2\theta}\coth \mu_\lambda \tau \right) \right\}.\end{aligned}$$

The optimal choice $\hat{\lambda}$ (which depends on τ as well as the model parameters) then satisfies

$$\frac{-\beta t}{a\hat{\lambda}} = \kappa^2 t \left(\frac{\coth \mu_{\hat{\lambda}}\tau}{2\mu_{\hat{\lambda}}} - \frac{\tau}{2\sinh^2 \mu_{\hat{\lambda}}\tau} \right) = \int_0^\tau y_{\hat{\lambda}}(s)^2 \,\mathrm{d}s,$$

and one can now check that the supremum and infimum could have been freely interchanged, maintaining equality in the previous expression. The optimal spatial path is then

$$x_\lambda(s) := \lambda a \int_0^s y_\lambda(u)^2 \,\mathrm{d}u = -\beta t \frac{\sinh 2\mu_\lambda s - 2\mu_\lambda s}{\sinh 2\mu_\lambda \tau - 2\mu_\lambda \tau}, \tag{5.8}$$

and defining $\hat{x} := x_{\hat{\lambda}}, \hat{y} := y_{\hat{\lambda}}$, we have

$$\begin{aligned}\inf_{x,y} J(x, y, \tau) &= J(\hat{x}, \hat{y}, \tau) \\ &= t \sup_\lambda \left\{ \kappa^2 \left(\frac{1}{4} + \frac{\mu_\lambda}{2\theta}\coth \mu_\lambda \tau \right) - \lambda\beta \right\} - \rho\tau \\ &= t \left(\kappa^2 \left(\frac{1}{4} + \frac{\mu_{\hat{\lambda}}}{2\theta}\coth \mu_{\hat{\lambda}}\tau \right) - \hat{\lambda}\beta \right) - \rho\tau.\end{aligned}$$

An important note on the optimal paths

As $\tau \to \infty$, we have

$$\sup_\lambda \left\{ \kappa^2 \left(\frac{1}{4} + \frac{\mu_\lambda}{\theta}\coth \mu_\lambda \tau \right) - \lambda\beta \right\} \uparrow \sup_\lambda \left\{ \kappa^2 \psi_\lambda^+ - \lambda\beta \right\} = \kappa^2 \psi_{\bar{\lambda}}^+ - \bar{\lambda}\beta,$$

where the optimising parameters of the supremums also converge with

$$\hat{\lambda} \to \bar{\lambda} = -\beta\sqrt{\frac{\theta(\theta - 8r)}{a^2\kappa^4 + 4a\theta\beta^2}} = \bar{\lambda}\left(\left(\frac{\kappa^2+\theta}{\kappa^2}\right)\beta, \kappa\right). \tag{5.9}$$

Then letting

$$\Theta(\beta, \kappa) := \sup_{\lambda}\left\{\kappa^2\psi_\lambda^+ - \lambda\beta\right\} = \frac{\kappa^2}{4} + \frac{\sqrt{\theta(\theta - 8r)(a^2\kappa^4 + 4a\theta\beta^2)}}{4a\theta} \tag{5.10}$$

and writing $\bar{x} := x_{\bar{\lambda}}$ and $\bar{y} := y_{\bar{\lambda}}$, we note that for all $\varepsilon, \delta > 0$ there exist $\tilde{\tau}, \mu > 0$ such that for all $t > 0$ and $\tau > \tilde{\tau}$

$$\begin{aligned}
\exp\left(-\inf_{x,y} J(x,y,\tau)\right) &\geq \exp\left(-J(\bar{x},\bar{y},\tau)\right) \\
&= \exp\left(-t\left(\kappa^2\left(\frac{1}{4} + \frac{\mu_{\bar{\lambda}}}{2\theta}\coth \mu_{\bar{\lambda}}\tau\right) - \bar{\lambda}\beta\right) + \rho\tau\right) \\
&\geq \exp\left(-t\left(\Theta(\beta,\kappa) + \varepsilon\right)\right).
\end{aligned}$$

Further (when $\kappa > 0$), for all $s \in [\tau - \mu, \tau]$,

$$\bar{y}(s) \geq (\kappa - \delta)\sqrt{t}, \qquad \bar{x}(s) \leq -(\beta - \delta)t.$$

In particular, the paths stay close to the required positions for some fixed length of time, with corresponding probability that is at least as large as required.

We now combine this estimate for the probability that a single particle has a descendent that performs the short climb with the spatial growth rate of Theorem 5.2.1. We recall from Theorem 5.2.1 that, for large times t, there will be approximately $\exp(\Delta(\alpha)t)$ particles near $-\alpha t$ in space. The number of particles that are near $(-\alpha t, 0)$ at time large time t, which then proceed to have *at least one* descendant alive near position $(-(\alpha + \beta)t, \kappa\sqrt{t})$ during the small time interval $[t + \tau - \mu, t + \tau]$, will be *approximately* Poisson with mean

$$\exp\left((\Delta(\alpha) - \Theta(\beta,\kappa))t\right).$$

By the Strong Law of Large Numbers, the actual number found will remain sufficiently close to this large mean with a very high probability, so we will eventually keep finding large enough numbers of particles close to the required positions.

Optimising over $\alpha+\beta=\gamma$, with $\alpha,\beta\geq 0$, some simple calculus reveals that with

$$\bar{\alpha}=\gamma\theta/(\theta+\kappa^2), \qquad \bar{\beta}=\gamma\kappa^2/(\theta+\kappa^2)$$

we have $\Delta(\alpha)-\Theta(\beta,\kappa)\leq\Delta(\bar{\alpha})-\Theta(\bar{\beta},\kappa)=\Delta(\gamma,\kappa)$.

To summarise, we have that the number of particles found in the vicinity of $(\gamma t,\kappa\sqrt{t})$ is at least roughly of order $\exp(\Delta(\gamma,\kappa)\,t)$ for all sufficiently large times t. This suggests a lower bound on the exponential growth rate of $\Delta(\gamma,\kappa)$, in agreement with the growth rate of expected number of particles.

5.3 Proof of Theorem 5.1.1: upper bound

In the interest of completeness we give the proof of the upper bound of Theorem 5.1.1 from Git and Harris [47].

Proof of Theorem 5.1.1: the upper bound. Simply observe that for $\lambda\in(\lambda_{\min},0)$,

$$\begin{aligned}
N_t(\gamma;[\kappa\sqrt{t},\infty)) &= \sum_{u\in N_t}\mathbf{1}_{\{X_u(t)\leq-\gamma t;\,Y_u(t)\geq\kappa\sqrt{t}\}}\\
&\leq \sum_{u\in N_t}\mathbf{1}_{\{X_u(t)\leq-\gamma t;\,Y_u(t)^2\geq\kappa^2 t\}}e^{\psi_\lambda^+(Y_u(t)^2-\kappa^2 t)+\lambda(X_u(t)+\gamma t)}\\
&\leq e^{(E_\lambda^+-\kappa^2\psi_\lambda^++\lambda\gamma)t}\sum_{u\in N_t}e^{\psi_\lambda^+Y_u(t)^2+\lambda X_u(t)-E_\lambda^+t}\\
&\leq e^{-\lambda(c_\lambda^+-c_\lambda^-)t}Z_\lambda^+(t)\,e^{(E_\lambda^-+\lambda\gamma-\kappa^2\psi_\lambda^+)t},
\end{aligned}\tag{5.11}$$

where $E_\lambda^\pm=-\lambda c_\lambda^\pm$.

It can be shown that

$$\Delta(\gamma,\kappa)=\inf_{\lambda\in(\lambda_{\min},0)}\left\{E_\lambda^-+\lambda\gamma-\kappa^2\psi_\lambda^+\right\},$$

where the infimum is attained at

$$\bar{\lambda}(\gamma,\kappa)=-\sqrt{\frac{\gamma^2\theta(\theta-8r)}{a^2(\kappa^2+\theta)^2+4a\gamma^2\theta}}.$$

In a separate result, Git and Harris showed that

$$\limsup_{t\to\infty}t^{-1}\log Z_\lambda^+(t)\leq\lambda(c_\lambda^+-c_\lambda^-)\tag{5.12}$$

almost surely for all $\lambda \in (\lambda_{\min}, 0)$, and hence in cases where $\Delta(\gamma, \kappa) < 0$, we can use the minimising value for λ, equation (5.12), and the fact that $N_t(\gamma; [\kappa\sqrt{t}, \infty))$ is integer valued to deduce that $N_t(\gamma; [\kappa\sqrt{t}, \infty)) = 0$ eventually, almost surely. Hence

$$\limsup_{t\to\infty} t^{-1} \ln N_t(\gamma; [\kappa\sqrt{t}, \infty)) = -\infty$$

almost surely if $\Delta(\gamma, \kappa) < 0$.

Otherwise we have $\Delta(\gamma, \kappa) \geq 0$, which in fact guarantees that $\gamma \in (0, \tilde{c}(\theta)]$ and hence $\bar{\lambda}(\gamma, \kappa) \in [\tilde{\lambda}(\theta), 0)$. Then since

$$\begin{aligned}&\limsup_{t\to\infty} t^{-1} \ln N_t(\gamma; [\kappa\sqrt{t}, \infty))\\ &\qquad \leq \limsup_{t\to\infty} t^{-1} \ln \left(e^{-\lambda(c_\lambda^+ - c_\lambda^-)t} Z_\lambda^+(t)\right) + (E_\lambda^- + \lambda\gamma - \kappa^2\psi_\lambda^+)\end{aligned}$$

we can again make use of equation (5.12) and the minimising λ value, $\bar{\lambda}(\gamma, \kappa)$, to get the bound

$$\limsup_{t\to\infty} t^{-1} \ln N_t(\gamma; [\kappa\sqrt{t}, \infty)) \leq \Delta(\gamma, \kappa) \qquad \text{almost surely,}$$

as desired. □

5.4 Proof of Theorem 5.1.1: lower bound

In this section we will state a precise short climb probability result and show how to combine it with almost-sure spatial (only) growth rates to prove the lower bound of the growth rate in Theorem 5.1.1. This will make rigorous the two-phase mechanism described in Section 5.2.

The first phase requires knowledge of the almost-sure rates of growth of particles in the spatial dimension only, which is provided by Theorem 5.2.1. The second phase requires a lower bound for the probability that a single particle makes a rapid ascent in both the type and space dimensions over the time interval $[0, \tau]$. This is the lower bound found in the heuristics of Section 5.2, but we require some further notation before the precise result can be stated. Note that, throughout this section, we will only be interested in the optimal parameter value $\lambda = \bar{\lambda}$ as introduced in Section 5.2.

We wish to fix the relationship between *sufficiently large* t and τ as

$$\sqrt{\frac{\theta}{2\mu_{\bar{\lambda}}}} e^{\mu_{\bar{\lambda}}\tau} = \kappa\sqrt{t}$$

and so define $\tau = \tau(t)$ by

$$\tau(t) := \begin{cases} (2\mu_{\bar{\lambda}})^{-1} \ln(2\mu_{\bar{\lambda}} t/\theta) & \text{for } 2\mu_{\bar{\lambda}} t > \theta, \\ 0 & \text{otherwise.} \end{cases} \tag{5.13}$$

Recall the optimal paths $(\bar{x}, \bar{y})$ over $s \in [0, \tau]$, where

$$\begin{aligned} \bar{y}(s) &= \kappa\sqrt{t}\frac{\sinh \mu_{\bar{\lambda}} s}{\sinh \mu_{\bar{\lambda}} \tau}, \\ \bar{x}(s) &= a\bar{\lambda}\int_0^s \bar{y}(w)^2 \,\mathrm{d}w = -\beta t \frac{\sinh 2\mu_{\bar{\lambda}} s - 2\mu_{\bar{\lambda}} s}{\sinh 2\mu_{\bar{\lambda}} \tau - 2\mu_{\bar{\lambda}} \tau}, \end{aligned} \tag{5.14}$$

with fixed end points $\bar{y}(\tau) = \kappa\sqrt{t}$ and $\bar{x}(\tau) = -\beta t$.

For large times t and $\delta, \varepsilon > 0$, let

$$A_t^{\varepsilon,\delta}(u) := \left\{ \sup_{s\in[0,\tau(t)]} |Y_u(s) - \bar{y}(s)| < \varepsilon\sqrt{t};\ \sup_{s\in[0,\tau(t)]} |X_u(s) - \bar{x}(s)| < \delta t \right\}. \tag{5.15}$$

We will use the notation

$$A_t^{\varepsilon,\delta} := \bigcup_{u\in N_{\tau(t)}} A_t^{\varepsilon,\delta}(u) \tag{5.16}$$

for the event that there exists a particle in the branching diffusion that makes the short climb. Finally, recalling $\Theta(\beta, \kappa)$ given at (5.10), we can now state the short climb theorem.

Theorem 5.4.1. *Fix any $y_1 > y_0 > 0$, $x \in \mathbb{R}$, and let $\varepsilon_0 > 0$. Then for any $\varepsilon, \delta > 0$, there exists $T > 0$ such that for all $y \in [y_0, y_1]$,*

$$t^{-1} \ln P^{x,y}(A_t^{\varepsilon,\delta}) \geq -(\Theta(\beta,\kappa) + \varepsilon_0)$$

for all $t > T$.

We will prove Theorem 5.4.1 using a spine change of measure. This requires us to introduce the notation for the spine set-up in detail before proceeding, so this and further technical issues are postponed to Sections 5.5 and 5.6.

Remark 5.4.2. We note that Theorem 5.4.1 is actually a stronger result than needed to prove Theorem 5.1.1 because we identify the specific *paths* followed by particles that are near position $(\beta t, \kappa\sqrt{t})$ at time $t+\tau$, rather than just considering the particle's *positions* close to time $t+\tau$.

In combining the two phases, we will have a huge number of independent trials each with a small probability of success, intuitively giving rise to a Poisson approximation for a large number of successful particles. In fact, in our proof of the lower bound of Theorem 5.1.1 below, we will actually use the following result about the behaviour of sequences of sums of independent Bernouilli random variables.

Lemma 5.4.3. *For each n, define the random variable $B_n := \sum_{u\in F_n} \mathbf{1}_{E_n(u)}$ where the events $\{E_n(u) : u \in F_n\}$ are independent. Let $p_n(u) := P(E_n(u))$ and $S_n := \sum_{u\in F_n} p_n(u)$ and suppose that, for some $\nu \in (1/2, 1)$,*

$$\sum_{n\in\mathbb{N}} \frac{1}{(S_n)^{2\nu-1}} < \infty. \tag{5.17}$$

Then almost surely the sequence of (possibly dependent) random variables $\{B_1, B_2, \dots\}$ has $|B_n - S_n| > (S_n)^\nu$ for only finitely many n.

In particular, for any $\varepsilon > 0$, there exists some (random) $N \in \mathbb{N}$ such that, with probability one,

$$\frac{B_n}{S_n} > 1-\varepsilon \qquad \text{for all } n > N. \tag{5.18}$$

Proof. For $\nu \in (1/2, 1)$, Chebychev's inequality yields

$$\mathbb{P}\Big(|B_n - S_n| > (S_n)^\nu\Big) \le \frac{\sum_{u\in F_n} p_n(u)(1-p_n(u))}{(S_n)^{2\nu}} \le \frac{1}{(S_n)^{2\nu-1}},$$

and hence the Borel-Cantelli lemmas, combined with hypothesis (5.17), imply that

$$|B_n - S_n| > (S_n)^\nu$$

for only finitely many n, almost surely. Equation (5.18) now follows on division by S_n, and noticing the assumption (5.17) implies that $\lim_{n\to\infty} S_n = \infty$. □

Proof of Theorem 5.1.1: lower bound. Define $f^{-1}(t) := t - \tau(t)$, and note that both $f(t)/t \to 1$ and $f^{-1}(t)/t \to 1$ as $t \to \infty$. Also, for $n \in \mathbb{N}$ and $\mu > 0$, define $T_n := (n+1)\mu$. We want to estimate the number of particles that are near the large space-type position $(-(\alpha+\beta)T_n, \kappa\sqrt{T_n})$ during time interval $[T_{n-1}, T_n]$. For this, we will consider particles that travel with a velocity $-\alpha$ over time period $[0, f^{-1}(T_n)]$ before commencing their rapid ascent of (relatively short) duration $\tau(T_n)$ to be in final position at time T_n. Then

$$\begin{aligned}
&\inf_{s\in[T_{n-1},T_n]} N_s\Big((\alpha+\beta-\delta)T_n; [(\kappa-\delta)\sqrt{T_n},\infty)\Big) \\
&\geq \sum_{u\in N_{T_n}} \mathbf{1}_{\left\{\bigcap_{s\in[T_{n-1},T_n]}\left\{X_u(s)\leq-(\alpha+\beta-\delta)T_n;\, Y_u(s)\geq(\kappa-\delta)\sqrt{T_n}\right\}\right\}} \\
&\geq \sum_{u\in F_n^\alpha} \mathbf{1}_{\left\{\bar{N}_n^{\beta,\kappa}(u)>0\right\}}
\end{aligned} \tag{5.19}$$

where

$$F_n^\alpha := \Big\{u\in N_{f^{-1}(T_n)} : X_u(f^{-1}(T_n)) \leq -\alpha T_n, Y_u(f^{-1}(T_n)) \in [y_0, y_1]\Big\}$$

and, for $u \in F_n^\alpha$,

$$\bar{N}_n^{\beta,\kappa}(u) := \sum_{\substack{v\in N_{T_n}\\ v\geq u}} \mathbf{1}_{\left\{\bigcap_{s\in[T_{n-1},T_n]}\left\{X_v(s)-X_v(f^{-1}(T_n))\leq-(\beta-\delta)T_n;\, Y_v(s)\geq(\kappa-\delta)\sqrt{T_n}\right\}\right\}}.$$

We will now show that the sum at (5.19) grows as fast as anticipated:

Lemma 5.4.4. *For any $\varepsilon > 0$, we may choose $\mu > 0$ such that there exists a random $N \in \mathbb{N}$ where*

$$\frac{1}{T_n}\ln\sum_{u\in F_n^\alpha}\mathbf{1}_{\left\{\bar{N}_n^{\beta,\kappa}(u)>0\right\}} \geq \Delta(\alpha) - \Theta(\beta,\kappa) - \varepsilon$$

for all $n > N$ with probability one.

Proof. We will be able to apply Lemma 5.4.3 given sufficient information about the growth of $|F_n^\alpha|$ and decay of the probabilities

$$p_n^{\beta,\kappa}(u) := P(\bar{N}_n^{\beta,\kappa}(u) > 0|\mathcal{F}_{f^{-1}(T_n)}),$$

where $u \in F_n^\alpha \subset N_{f^{-1}(T_n)}$.

It follows easily from Theorem 5.2.1, the fact that $f^{-1}(T_n)/T_n \to 1$, and the continuity of $\Delta(\alpha)$ that

$$\frac{\ln|F_n^\alpha|}{T_n} \geq \Delta(\alpha) - \frac{\varepsilon}{4}$$

for all sufficiently large n.

The definition of $\bar{N}_n^{\beta,\kappa}(u)$ and spatial translation invariance implies that, for each $u \in F_n^\alpha$, the rapid ascent probability $p_n^{\beta,\kappa}(u)$ depends only on the initial type position $Y_u(f^{-1}(T_n))$.

For $\delta, \mu > 0$, define

$$B_t^{\delta,\mu}(u) := \bigcap_{s \in [\tau(t)-\mu, \tau(t)]} \left\{ X_u(s) - X_u(0) < -(\beta - \delta)t; Y_u(s) \geq (\kappa - \delta)\sqrt{t} \right\},$$

and

$$B_t^{\delta,\mu} := \bigcup_{u \in N_{\tau(t)}} B_t^{\delta,\mu}(u). \tag{5.20}$$

Recalling the comments on the optimal paths given in Section 5.2, there exist $\varepsilon', \delta' > 0$ and we may choose $\mu > 0$ sufficiently small, such that

$$p_n^{\beta,\kappa}(u) = P^{0, Y_u(f^{-1}(T_n))}(B_{T_n}^{\delta,\mu}) \geq P^{0, Y_u(f^{-1}(T_n))}(A_{T_n}^{\varepsilon',\delta'}) =: \bar{p}_n(u)$$

for all $u \in F_n^\alpha$ whenever n is sufficiently large. Together with Theorem 5.4.1 and since $Y_u(f^{-1}(T_n)) \in [y_0, y_1]$ for $u \in F_n^\alpha$, this reveals

$$\frac{\ln p_n^{\beta,\kappa}(u)}{T_n} \geq \frac{\ln \bar{p}_n(u)}{T_n} \geq -\Theta(\beta,\kappa) - \frac{\varepsilon}{4}$$

for all for $u \in F_n^\alpha$ and all sufficiently large n, almost surely. Then we may combine the observations above to obtain

$$\frac{1}{T_n} \ln \sum_{u \in F_n^\alpha} p_n^{\beta,\kappa}(u) \geq \Delta(\alpha) - \Theta(\beta,\kappa) - \frac{\varepsilon}{2}.$$

Taking this last line together the assertion of Lemma 5.4.3 at equation (5.18) gives the result. □

It is now straightforward to combine Lemma 5.4.4 with the inequality at (5.19) to see that, given $\varepsilon, \delta > 0$, there exists $\mu > 0$ and a random time T such that

$$t^{-1} \ln N_t\Big((\alpha + \beta - \delta)t; [(\kappa - \delta)\sqrt{t}, \infty)\Big) \geq \Delta(\alpha) - \Theta(\beta,\kappa) - \varepsilon$$

for all $t > T$, almost surely. Since ε and δ can be taken arbitrarily small, using the optimal $\bar{\alpha}$ and $\bar{\beta}$ found in Section 5.2, we find

$$\liminf_{t\to\infty} t^{-1} \ln N_t(\gamma; [\kappa\sqrt{t}, \infty)) \geq \Delta(\gamma, \kappa) \qquad \text{almost surely,}$$

as required. □

5.5 The spine construction

In this section we recall the notation for the Hardy and Harris technique method of constructing spines. Much of this notation is identical to that used in Chapters 3 and 4, but for the reader's convenience we re-state some of the most important definitions. There are some very minor alterations to the notation of those earlier chapters, arising from the introduction of the types.

All probability measures are (again) to be defined on the space $\tilde{\mathcal{T}}$ of marked Galton-Watson trees with spines. For a Galton-Watson tree, τ, to each individual $u \in \tau$ we give a mark (X_u, Y_u, σ_u) which contains the following information:

- $\sigma_u \in [0, \infty)$ is the *lifetime* of particle u, which also determines the *fission time* of the particle as $S_u := \sum_{v \leq u} \sigma_v$. We may also refer to the S_u as *death times*;
- the function $X_u(t) : [S_u - \sigma_u, S_u) \to \mathbb{R}$ describes the particle's *spatial* motion in $\mathbb{R}$ during its lifetime;
- the function $Y_u(t) : [S_u - \sigma_u, S_u) \to \mathbb{R}$ describes the evolution of the particle's *type* in $\mathbb{R}$ during its lifetime.

We denote a particular marked tree by (τ, X, Y, σ), or the abbreviation (τ, M), and the set of all marked Galton-Watson trees by $\mathcal{T}$. For each $(\tau, X, Y, \sigma) \in \mathcal{T}$, the set of particles alive at time t is defined as $N_t := \{u \in \tau : S_u - \sigma_u \leq t < S_u\}$. The collection of all marked trees with a distinguished spine is the space $\tilde{\mathcal{T}}$ on which our probability measures will eventually be defined.

The set of particles making up the spine is denoted ξ, and we think of the spine as a single diffusing particle ξ_t, or, strictly speaking, the pair (ξ_t, η_t), where η_t is the type of the spine at time t. In our model η_t is an Ornstein-Uhlenbeck process.

We define four filtrations on this space that contain different levels of information about the branching diffusion.

- Filtration $(\mathcal{F}_t)_{t\geq 0}$. The natural σ-field.

- Filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$. We define the filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ by augmenting the filtration $\mathcal{F}_t$ with the knowledge of which node is the spine at time t:

$$(\tilde{\mathcal{F}}_t)_{t\geq 0} := \sigma(\mathcal{F}_t, \mathrm{node}_t(\xi)), \qquad \tilde{\mathcal{F}}_\infty := \sigma\Big(\bigcup_{t\geq 0} \tilde{\mathcal{F}}_t\Big),$$

so that this filtration knows everything about the branching diffusion and everything about the spine.

- Filtration $(\mathcal{G}_t)_{t\geq 0}$. $(\mathcal{G}_t)_{t\geq 0}$ is a filtration of $\tilde{\mathcal{T}}$ defined by

$$\mathcal{G}_t := \sigma(\xi_s : 0 \leq s \leq t), \qquad \mathcal{G}_\infty := \sigma\Big(\bigcup_{t\geq 0} \mathcal{G}_t\Big).$$

These σ-algebras are generated only by the spine's motion and so do not contain the information about which nodes of the tree τ make up the spine.

- Filtration $(\tilde{\mathcal{G}}_t)_{t\geq 0}$. As we did in going from $\mathcal{F}_t$ to $\tilde{\mathcal{F}}_t$ we create $(\tilde{\mathcal{G}}_t)_{t\geq 0}$ from $(\mathcal{G}_t)_{t\geq 0}$ by including knowledge of which nodes make up the spine:

$$(\tilde{\mathcal{G}}_t)_{t\geq 0} := \sigma(\mathcal{G}_t, \mathrm{node}_t(\xi)), \qquad \tilde{\mathcal{G}}_\infty := \sigma\Big(\bigcup_{t\geq 0} \tilde{\mathcal{G}}_t\Big).$$

This means that $\tilde{\mathcal{G}}_t$ also knows when the fission time on the spine occurred, whereas $\mathcal{G}_t$ does not.

Now that we have defined the underlying space and the filtrations of it that we require, we define the probability measures on that space that give us the typed branching diffusion. The measures $\{P^{x,y} : x, y \in \mathbb{R}\}$ on $(\tilde{\mathcal{T}}, \mathcal{F}_\infty)$ are the law of the typed branching diffusion described in Section 5.1, and the ideas of Lyons [71] allow us to extend the measure $P^{x,y}$ to a measure that also keeps track of the spine.

We recall that, if f is an $\tilde{\mathcal{F}}_t$-measurable function then we can write

$$f = \sum_{u\in N_t} f_u \mathbf{1}_{\{\xi_t = u\}}, \tag{5.21}$$

where f_u is $\mathcal{F}_t$-measurable. Now we extend $P^{x,y}$ to a measure $\tilde{P}^{x,y}$ on $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}}_\infty)$ by choosing the spine uniformly each time there is a birth on the spine; more precisely, for any $f \in m\tilde{\mathcal{F}}_t$ with representation like (5.21), we have:

$$\int_{\tilde{T}} f \,\mathrm{d}\tilde{P}^{x,y}(\tau, M, \xi) := \int_T \sum_{u\in N_t} f_u \prod_{v<u} \frac{1}{2} \mathrm{d}P^{x,y}(\tau, M).$$

The measure $\tilde{P}$ can be decomposed as

$$\mathrm{d}\tilde{P}(\tau, M, \xi) = \mathrm{d}\mathbb{P}(\xi)\,\mathrm{d}\mathbb{P}^{(\theta,\frac{\theta}{2})}(\eta)\,\mathrm{d}\mathbb{L}^{R(\eta_t)}(n_t) \prod_{v<\xi_t} \frac{1}{2} \times \mathrm{d}P((\tau, M)^v),$$

where $\mathbb{P}$, $\mathbb{P}^{(\theta,\frac{\theta}{2})}$ and $\mathbb{L}^{R(\eta_t)}$ are, respectively, the laws of the spine's spatial position ξ_t, type position η_t, and the Poisson process n_t of births on the spine.

We construct the $\tilde{\mathcal{F}}_t$-measurable martingale $\tilde{\zeta}(t)$ as

$$\begin{aligned}\tilde{\zeta}(t) &:= v_\lambda^+(\eta_t) e^{\int_0^t \left(R(\eta_s)+\frac{1}{2}\lambda^2 A(\eta_s)\right)\mathrm{d}s - E_\lambda^+ t} \times 2^{n_t} e^{-\int_0^t R(\eta_s)\,\mathrm{d}s} \times e^{\lambda\xi_t - \frac{1}{2}\lambda^2 \int_0^t A(\eta_s)\,\mathrm{d}s} \\ &= v_\lambda^+(\eta_t) 2^{n_t} e^{\lambda\xi_t - E_\lambda^+ t}.\end{aligned} \tag{5.22}$$

Observe that this is a product of single-particle martingales, the first of which is a change of measure for the spine's type process. These martingales can also be thought of as h-transforms.

Proposition 5.5.1 (Harris and Williams [48]). *The process*

$$v_\lambda^+(\eta_t) e^{\int_0^t \left(R(\eta_s)+\frac{1}{2}\lambda^2 A(\eta_s)\right)\mathrm{d}s - E_\lambda^+ t}$$

is a $\tilde{P}$-martingale that will change the drift direction of η_t. More precisely, we have

$$\frac{\mathrm{d}\mathbb{P}_t^{(\theta,-\mu_\lambda)}}{\mathrm{d}\mathbb{P}_t^{(\theta,\frac{\theta}{2})}} = v_\lambda^+(\eta_t) e^{\int_0^t \left(R(\eta_s)+\frac{1}{2}\lambda^2 A(\eta_s)\right)\mathrm{d}s - E_\lambda^+ t},$$

where, under $\mathbb{P}_t^{(\theta,\delta)}$, η_t moves on $\mathbb{R}$ as an OU-process with generator

$$\frac{\theta}{2}\frac{\partial^2}{\partial y^2} - \delta y \frac{\partial}{\partial y},$$

and we will refer to this as an OU(θ,δ) process.

The effect of the other changes of measure that make up $\tilde{\zeta}(t)$ is to increase the breeding rate of the spine by a factor of two, and add a spatial drift to the spine. Using the martingale $\tilde{\zeta}(t)$ we may define a measure $\tilde{\mathbb{Q}}_\lambda^{x,y}$ on $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}}_\infty)$ by

$$\left.\frac{\mathrm{d}\tilde{\mathbb{Q}}_\lambda^{x,y}}{\mathrm{d}\tilde{P}^{x,y}}\right|_{\tilde{\mathcal{F}}_t} = \frac{\tilde{\zeta}(t)}{\tilde{\zeta}(0)} = \frac{e^{-\lambda x}}{v_\lambda^+(y)} v_\lambda^+(\eta_t) 2^{n_t} e^{\lambda \xi_t - E_\lambda^+ t}. \tag{5.23}$$

If we now factor in the Radon-Nikodým derivatives from the martingale $\tilde{\zeta}(t)$ we will obtain decomposition for $\tilde{\mathbb{Q}}_\lambda$. On $\tilde{\mathcal{F}}_t$ we have:

$$\begin{aligned}
\mathrm{d}\tilde{\mathbb{Q}}_\lambda &= \tilde{\zeta}(t)\,\mathrm{d}\tilde{P} \\
&= \mathrm{d}\mathbb{P}_\lambda(\xi)\,\mathrm{d}\mathbb{P}^{(\theta, -\mu_\lambda)}(\eta)\,\mathrm{d}\mathbb{L}^{2R(\eta_t)}(n_t) \prod_{v < \xi_t} \frac{1}{2} \times \mathrm{d}P((\tau, M)^v),
\end{aligned}$$

and in view of this the branching diffusion may be constructed under $\tilde{\mathbb{Q}}_\lambda^{x,y}$ as:

- starting from spatial position x and type y the spine (ξ_t, η_t) diffuses spatially as a Brownian motion with infinitesimal variance $A(\eta_t)$ and infinitesimal drift $\lambda A(\eta_t)$;
- the type of the spine, η_t, begins at y and moves in type space as an outward-drifting Ornstein-Uhlenbeck process with generator

  $$\frac{\theta}{2}\frac{\partial^2}{\partial y^2} + \mu_\lambda y \frac{\partial}{\partial y}$$

  (notice that η_t has a drift driving it *away* from the origin in type space);
- the spine branches at rate $2R(\eta_t)$, producing 2 particles;
- one of these particles is selected uniformly at random;
- the chosen offspring (the spine) repeats stochastically the behaviour of its parent;
- the other offspring particle initiates a $P^{\cdot,\cdot}$-BBM from its birth position and type.

For future reference we emphasise that under $\tilde{\mathbb{Q}}_\lambda^{x,y}$ the spine has a spatial drift and a transient Ornstein-Uhlenbeck type process, as a consequence of which we will later see that the spine has a high probability of making the short climb. This is vital in the proof of Theorem 5.4.1.

Theorem 5.5.2. *If we define* $\mathbb{Q}_\lambda^{x,y} := \tilde{\mathbb{Q}}_\lambda^{x,y}|_{\mathcal{F}_\infty}$*, then* $\mathbb{Q}_\lambda^{x,y}$ *is a measure on* $\mathcal{F}_\infty$ *that satisfies*

$$\left.\frac{\mathrm{d}\mathbb{Q}_\lambda^{x,y}}{\mathrm{d}P^{x,y}}\right|_{\mathcal{F}_t} = \frac{Z_\lambda^+(t)}{Z_\lambda^+(0)} =: \hat{Z}_\lambda^+(t).$$

Moreover under $\mathbb{Q}_\lambda^{x,y}$, the path-wise construction of the branching diffusion is the same as under $\tilde{\mathbb{Q}}_\lambda$.

Proof. By definition of conditional expectation, the change of measure (5.23) projects onto the sub-algebra $\mathcal{F}_t$ as a conditional expectation:

$$\left.\frac{\mathrm{d}\tilde{\mathbb{Q}}_\lambda^{x,y}}{\mathrm{d}\tilde{P}^{x,y}}\right|_{\mathcal{F}_t} = \frac{e^{-\lambda x}}{v_\lambda^+(y)}\tilde{P}^{x,y}\Big(v_\lambda^+(\eta_t)2^{n_t}e^{\lambda\xi_t - E_\lambda^+ t}\Big|\mathcal{F}_t\Big).$$

To evaluate this conditional expectation we use the representation (5.21) and the fact that $2^{n_t} = \prod_{v<\xi_t} 2$, and we obtain

$$\begin{aligned}
&\tilde{P}^{x,y}\Big(v_\lambda^+(\eta_t)2^{n_t}e^{\lambda\xi_t - E_\lambda^+ t}\Big|\mathcal{F}_t\Big)\\
&\quad = \tilde{P}^{x,y}\Big(\sum_{u\in N_t} v_\lambda^+(Y_u(t))2^{n_t}e^{\lambda X_u(t) - E_\lambda^+ t} \times \prod_{v<u} 2 \times \mathbf{1}_{(\xi_t = u)}\Big|\mathcal{F}_t\Big)\\
&\quad = \sum_{u\in N_t} v_\lambda^+(Y_u(t))2^{n_t}e^{\lambda X_u(t) - E_\lambda^+ t} \times \prod_{v<u} 2 \times \tilde{P}^{x,y}(\xi_t = u|\mathcal{F}_t)\\
&\quad = \sum_{u\in N_t} v_\lambda^+(Y_u(t))2^{n_t}e^{\lambda X_u(t) - E_\lambda^+ t} \times \prod_{v<u} 2 \times \prod_{v<u}\frac{1}{2}\\
&\quad = \sum_{u\in N_t} v_\lambda^+(Y_u(t))2^{n_t}e^{\lambda X_u(t) - E_\lambda^+ t} = Z_\lambda^+(t).
\end{aligned}$$

□

Although the path-wise construction of the branching diffusion is the same under $\mathbb{Q}_\lambda$ and $\tilde{\mathbb{Q}}_\lambda$, only the measure $\tilde{\mathbb{Q}}_\lambda$ 'knows' about the spine. It is clear, however, that we have $\tilde{\mathbb{Q}}_\lambda(A) = \mathbb{Q}_\lambda(A)$ for any $A \in \mathcal{F}_\infty$.

Under the measure $\tilde{\mathbb{Q}}_\lambda$ only the behaviour of the spine is altered, and combining this observation with conditioning on the spine's path and fission-times gives us a spine decomposition for $Z_\lambda^+(t)$:

$$\tilde{\mathbb{Q}}_\lambda(Z_\lambda^+(t)|\tilde{\mathcal{G}}_\infty) = \sum_{u<\xi_t} v_\lambda^+(\eta_{S_u})e^{\lambda\xi_{S_u} - E_\lambda^+ S_u} + v_\lambda^+(\eta_t)e^{\lambda\xi_t - E_\lambda^+ t}. \tag{5.24}$$

Throughout the rest of this chapter we will refer to the two pieces of this decomposition as the 'sum term' and the 'spine term'. This decomposition is derived in a very similar manner to the decomposition of the martingale V in Proposition 4.4.1.

5.6 Proof of Theorem 5.4.1: the short climb probability

With the spine foundations firmly established in Section 5.5, we may proceed with the proof of the short climb probability lower bound from Theorem 5.4.1.

First, recall definitions (5.15) and (5.16), where $A_t^{\varepsilon,\delta}$ is the event that *there exists* a particle that makes the short climb along optimal path $(\bar{x}, \bar{y})$, and $A_t^{\varepsilon,\delta}(\xi)$ is the event that the *spine* makes the short climb. Note that ε controls the proximity to $\bar{x}$ and δ the proximity to $\bar{y}$. Importantly, we will *only* be interested in taking $\lambda = \bar{\lambda}$ throughout this section, although we will usually just write λ for notational simplicity. Also recall throughout that t and τ are related through $(\theta/2\mu_\lambda)\exp(2\mu_\lambda\tau) = \kappa^2 t$.

Proof of Theorem 5.4.1. The key step in the proof of this is the following use of the spine change of measure: for any function $g : \mathbb{R}^+ \to \mathbb{R}^+$ we have

$$\begin{aligned} P^{x,y}(A_t^{\varepsilon,\delta}) &= \mathbb{Q}_\lambda^{x,y}\left(\frac{\mathbf{1}_{A_t^{\varepsilon,\delta}}}{\hat{Z}_\lambda^+(\tau)}\right) = \tilde{\mathbb{Q}}_\lambda^{x,y}\left(\frac{\mathbf{1}_{A_t^{\varepsilon,\delta}}}{\hat{Z}_\lambda^+(\tau)}\right) \geq \tilde{\mathbb{Q}}_\lambda^{x,y}\left(\frac{\mathbf{1}_{A_t^{\varepsilon,\delta}(\xi)}}{\hat{Z}_\lambda^+(\tau)}\right) \\ &\geq \tilde{\mathbb{Q}}_\lambda^{x,y}\left(\frac{\mathbf{1}_{A_t^{\varepsilon,\delta}(\xi)}}{\hat{Z}_\lambda^+(\tau)}; \sup_{s\in[0,\tau]} \hat{Z}_\lambda^+(s) \leq g(\tau)\right) \\ &\geq g(\tau)^{-1}\tilde{\mathbb{Q}}_\lambda^{x,y}\left(A_t^{\varepsilon,\delta}(\xi); \sup_{s\in[0,\tau]} \hat{Z}_\lambda^+(s) \leq g(\tau)\right). \end{aligned} \tag{5.25}$$

Essentially we just have to make the 'correct' choice for both λ and g in expression (5.25), although there will still remain a number of technicalities to resolve.

The first idea is to ensure the (originally rare) event $A_t^{\varepsilon,\delta}$ actually occurs under the new measure $\tilde{\mathbb{Q}}_\lambda^{x,y}$ by making the spine follow close to the required path $(\bar{x}, \bar{y})$; this is achieved by choosing the optimal value $\bar{\lambda}$ for λ and choosing τ to be on the natural time scale it would would take the spine to reach position $\kappa\sqrt{t}$. In particular, this choice will mean that in the first line of the above set of inequalities there is no significant loss of mass when replacing the event $A_t^{\varepsilon,\delta}$ with $A_t^{\varepsilon,\delta}(\xi)$. Next, we wish to choose the smallest possible g that will still leave some positive probability on the last line of the above argument. So we need to identify the rate of growth of the martingale Z_λ^+ under $\tilde{\mathbb{Q}}_\lambda^{x,y}$, and this will essentially be governed by the contribution from the spine itself.

With this is mind, and recalling the various properties of the optimal paths and parameters from Section 5.2, for $\varepsilon_0 > 0$ we define

$$g_{\varepsilon_0}(\tau) := \exp\left(\left(\psi_\lambda^+ + \frac{\lambda^2 a}{2\mu_\lambda} + \frac{\varepsilon_0}{\kappa}\right)\left(\frac{\theta}{2\mu_\lambda}\right)e^{2\mu_\lambda\tau} - (\psi_\lambda^+ y^2 + \lambda x)\right)$$

and recall from (5.13) that the scaling between t and τ is fixed throughout, where $\kappa^2 t = (\theta/2\mu_{\bar\lambda})e^{2\mu_{\bar\lambda}\tau}$ for large t, hence $t+\tau \sim t$. Note that since we are only considering the optimal value $\lambda = \bar\lambda$, we have

$$\left(\psi_{\bar\lambda}^{+} + \frac{\bar\lambda^2 a}{2\mu_{\bar\lambda}}\right)\left(\frac{\theta}{2\mu_{\bar\lambda}}\right)e^{2\mu_{\bar\lambda}\tau} = (\kappa^2\psi_{\bar\lambda}^{+} - \bar\lambda\beta)t = \Theta(\beta,\kappa).$$

Then from (5.25) we have

$$P^{x,y}(A_t^{\varepsilon,\delta}) \geq g_{\varepsilon_0}(\tau)^{-1}\tilde{\mathbb{Q}}_{\lambda}^{x,y}\left(A_t^{\varepsilon,\delta}(\xi);\ \sup_{s\in[0,\tau]}\hat{Z}_{\lambda}^{+}(s) \leq g_{\varepsilon_0}(\tau)\right). \tag{5.26}$$

Our strategy for the rest of this proof is to show that the $\tilde{\mathbb{Q}}_{\lambda}^{x,y}$-probability in (5.26) is at least some $\varepsilon' > 0$ for all sufficiently large t, uniformly for $y \in [y_0, y_1]$, so that the decay rate part of (5.26) matches the desired rate in the statement of the theorem.

Conditioning on the spine's path and birth times, $\tilde{\mathcal{G}}_\infty$, and then making use of some standard properties of conditional expectation we have

$$\begin{aligned}
&\tilde{\mathbb{Q}}_{\lambda}^{x,y}\left(A_t^{\varepsilon,\delta}(\xi);\ \sup_{s\in[0,\tau]}\hat{Z}_{\lambda}^{+}(s) \leq g_{\varepsilon_0}(\tau)\right)\\
&\qquad = \tilde{\mathbb{Q}}_{\lambda}^{x,y}\left(\tilde{\mathbb{Q}}_{\lambda}^{x,y}\left(A_t^{\varepsilon,\delta}(\xi);\ \sup_{s\in[0,\tau]}\hat{Z}_{\lambda}^{+}(s) \leq g_{\varepsilon_0}(\tau)\,\middle|\,\tilde{\mathcal{G}}_\infty\right)\right)\\
&\qquad = \tilde{\mathbb{Q}}_{\lambda}^{x,y}\left(\mathbf{1}_{A_t^{\varepsilon,\delta}(\xi)}\tilde{\mathbb{Q}}_{\lambda}^{x,y}\left(\sup_{s\in[0,\tau]}\hat{Z}_{\lambda}^{+}(s) \leq g_{\varepsilon_0}(\tau)\,\middle|\,\tilde{\mathcal{G}}_\infty\right)\right),
\end{aligned}$$

since $A_t^{\varepsilon,\delta}(\xi)$ is $\tilde{\mathcal{G}}_\infty$-measurable. We next observe that, conditional on $\tilde{\mathcal{G}}_\infty$, we can write $\hat{Z}_{\lambda}^{+}(t)$ as

$$\hat{Z}_{\lambda}^{+}(t) = e^{-(\psi_{\lambda}^{+}y^2+\lambda x)}\left(\sum_{u<\xi_t} e^{-E_{\lambda}^{+}S_u}Z_{\lambda}^{(u)}(t-S_u) + f(t)\right), \tag{5.27}$$

where the $Z_{\lambda}^{(u)}$ are independent copies of Z_{λ}^{+} started from a single particle at (ξ_{S_u}, η_{S_u}); and $f(t)$ is the contribution to $Z_{\lambda}^{+}(t)$ from the spine, which, conditional on $\tilde{\mathcal{G}}_\infty$, is a known function of t. Now if we could show, for $0 < \tilde\varepsilon_0 < \varepsilon_0$,

$$\sup_{s\in[0,\tau]}\hat{f}(s) \leq \frac{g_{\tilde\varepsilon_0}(\tau)}{2} \quad\text{and}\quad \sup_{s\in[0,\tau]}\left(\hat{Z}_{\lambda}^{+}(s) - \hat{f}(s)\right) \leq \frac{g_{\varepsilon_0}(\tau)}{2},$$

where $\hat{f}(t) := e^{-(\psi_{\lambda}^{+}y^2+\lambda x)}f(t)$, we would have $\sup_{s\in[0,\tau]}\hat{Z}_{\lambda}^{+}(s) \leq g_{\varepsilon_0}(\tau)$. Hence, defining $\hat{\mathcal{Z}}^+_\lambda(s) := \hat{Z}^+_\lambda(s) - \hat{f}(s)$, we have

$$\begin{aligned}
&\tilde{\mathbb{Q}}^{x,y}_\lambda\left(\mathbf{1}_{A^{\varepsilon,\delta}_t(\xi)}\tilde{\mathbb{Q}}^{x,y}_\lambda\left(\sup_{s\in[0,\tau]}\hat{Z}^+_\lambda(s) \le g_{\varepsilon_0}(\tau)\middle|\tilde{\mathcal{G}}_\infty\right)\right)\\
&\ge \tilde{\mathbb{Q}}^{x,y}_\lambda\left(\mathbf{1}_{A^{\varepsilon,\delta}_t(\xi)}\tilde{\mathbb{Q}}^{x,y}_\lambda\left(\sup_{s\in[0,\tau]}\hat{f}(s) \le \frac{g_{\tilde{\varepsilon}_0}(\tau)}{2};\ \sup_{s\in[0,\tau]}\hat{\mathcal{Z}}^+_\lambda(s) \le \frac{g_{\varepsilon_0}(\tau)}{2}\middle|\tilde{\mathcal{G}}_\infty\right)\right)\\
&= \tilde{\mathbb{Q}}^{x,y}_\lambda\left(\mathbf{1}_{A^{\varepsilon,\delta}_t(\xi)}\mathbf{1}_{\{\sup_{s\in[0,\tau]}\hat{f}(s)\le\frac{g_{\tilde{\varepsilon}_0}(\tau)}{2}\}}\tilde{\mathbb{Q}}^{x,y}_\lambda\left(\sup_{s\in[0,\tau]}\hat{\mathcal{Z}}^+_\lambda(s) \le \frac{g_{\varepsilon_0}(\tau)}{2}\middle|\tilde{\mathcal{G}}_\infty\right)\right)
\end{aligned}$$

since, conditional on $\tilde{\mathcal{G}}_\infty$, the supremum of $\hat{f}$ on $[0,\tau]$ is known.

We see from (5.27) that, *conditional on* $\tilde{\mathcal{G}}_\infty$, $\hat{\mathcal{Z}}^+_\lambda(t)$ is a submartingale. This is because the $\tilde{\mathbb{Q}}^{x,y}_\lambda$-conditional expectation of each of the $Z^{(u)}_\lambda$ in the sum

$$e^{-(\psi^+_\lambda y^2+\lambda x)}\sum_{u<\xi_t} e^{-E^+_\lambda S_u} Z^{(u)}_\lambda(t-S_u) \tag{5.28}$$

is constant, so the expectation of the sum cannot decrease, and in fact this expectation increases every time there is a birth on the spine. Then by Doob's submartingale inequality we have

$$\begin{aligned}
\tilde{\mathbb{Q}}^{x,y}_\lambda\left(\sup_{s\in[0,\tau]}\hat{\mathcal{Z}}^+_\lambda(s) \le \frac{g_{\varepsilon_0}(\tau)}{2}\middle|\tilde{\mathcal{G}}_\infty\right) &= 1-\tilde{\mathbb{Q}}^{x,y}_\lambda\left(\sup_{s\in[0,\tau]}\hat{\mathcal{Z}}^+_\lambda(s) \ge \frac{g_{\varepsilon_0}(\tau)}{2}\middle|\tilde{\mathcal{G}}_\infty\right)\\
&\ge 1-\frac{2}{g_{\varepsilon_0}(\tau)}\tilde{\mathbb{Q}}^{x,y}_\lambda\left(\hat{\mathcal{Z}}^+_\lambda(\tau)\middle|\tilde{\mathcal{G}}_\infty\right).
\end{aligned}$$

We must note here that the expectation on the above line is not *a priori* finite. However, the expectation of each term in the sum (5.28) is bounded by $\sup_{s\in[0,\tau]}\hat{f}(s)$, which we have control over via an indicator function and so we do not have to worry about this expectation blowing up.

So we need to show that for all sufficiently large τ and all $y \in [y_0, y_1]$,

$$\tilde{\mathbb{Q}}^{x,y}_\lambda\left(\mathbf{1}_{A^{\varepsilon,\delta}_t(\xi)\cap\{\sup_{s\in[0,\tau]}\hat{f}(s)\le\frac{g_{\tilde{\varepsilon}_0}(\tau)}{2}\}}\left(1-\frac{2}{g_{\varepsilon_0}(\tau)}\tilde{\mathbb{Q}}^{x,y}_\lambda\left(\hat{\mathcal{Z}}^+_\lambda(\tau)\middle|\tilde{\mathcal{G}}_\infty\right)\right)\right) > \varepsilon',$$

since then we also have

$$\tilde{\mathbb{Q}}^{x,y}_\lambda\left(A^{\varepsilon,\delta}_t(\xi);\ \sup_{s\in[0,\tau]}\hat{Z}^+_\lambda(s) \le g_{\varepsilon_0}(\tau)\right) > \varepsilon'$$

as required. This will follow by combining both parts of the following result.

Lemma 5.6.1. *Fix $y_1 > y_0 > 0$ and $\varepsilon_0 > \tilde{\varepsilon}_0 > 0$.*

(i) For all sufficiently small $\varepsilon, \delta > 0$, there exists some $\varepsilon' > 0$ and $\tilde{T} > 0$ such that for all $y \in [y_0, y_1]$ and all $t > \tilde{T}$,

$$\tilde{\mathbb{Q}}_\lambda^{x,y}\left(A_t^{\varepsilon,\delta}(\xi);\ \sup_{s\in[0,\tau]} \hat{f}(s) \leq \frac{g_{\tilde{\varepsilon}_0}(\tau)}{2} \right) > \varepsilon'.$$

(ii) As $\tau \to \infty$,

$$\tilde{\mathbb{Q}}_\lambda^{x,y}\left(\frac{2}{g_{\varepsilon_0}(\tau)} \tilde{\mathbb{Q}}_\lambda^{x,y}\left(\hat{\mathcal{Z}}_\lambda^{+}(\tau) \middle| \tilde{\mathcal{G}}_\infty \right);\ \sup_{s\in[0,\tau]} \hat{f}(s) \leq \frac{g_{\tilde{\varepsilon}_0}(\tau)}{2} \right) \to 0$$

uniformly over $y \in [y_0, y_1]$.

Then we have shown that, for any $\varepsilon_0 > 0$, $y_1 > y_0 > 0$, and sufficiently small $\varepsilon, \delta > 0$, there exists a $T > 0$ such that, for all $y \in [y_0, y_1]$ and all $t > T$,

$$t^{-1} \ln P^{x,y}(A_t^{\varepsilon,\delta}) \geq -(\Theta(\beta, \kappa) + \varepsilon_0).$$

Finally, we observe that the probability $P^{x,y}(A_t^{\varepsilon,\delta})$ is trivially monotone increasing in both ε and δ, and so it follows that if the result is true for all sufficiently small ε and δ, it is in fact true for all $\varepsilon, \delta > 0$. This completes the proof of Theorem 5.4.1. □

Proof of Lemma 5.6.1(i). We will prove Lemma 5.6.1(i) in a sequence of other lemmas, using a convenient coupling for the spine's type process.

First recall that, under $\tilde{\mathbb{Q}}_\lambda^{x,y}$, η_s solves the SDE

$$\mathrm{d}\eta_s = \sqrt{\theta}\, \mathrm{d}B_s + \mu_\lambda \eta_s \,\mathrm{d}s,$$

where B_s is a $\tilde{\mathbb{Q}}_\lambda$-Brownian motion. Noting that $\mathrm{d}(e^{-\mu_\lambda s}\eta_s) = e^{-\mu_\lambda s}\sqrt{\theta}\,\mathrm{d}B_s$, we can construct $e^{-\mu_\lambda s}\eta_s$ as a time-change of a Brownian motion with

$$e^{-\mu_\lambda s}\eta_s - \eta_0 = \sqrt{\theta} \int_0^s e^{-\mu_\lambda w}\,\mathrm{d}B_w = \sqrt{\frac{\theta}{2\mu_\lambda}}\tilde{B}\Big(1 - e^{-2\mu_\lambda s}\Big),$$

where $\tilde{B}$ is also a $\tilde{\mathbb{Q}}_\lambda^{x,y}$-Brownian motion started at the origin.

In this way, for $y \in [y_0, y_1]$ we will construct processes η^y under $\mathbb{P}$ from Brownian motions B^y started at $y\sqrt{2\mu_\lambda/\theta}$ where, for $s \in [0,\infty)$,

$$\eta^y(s) = \sqrt{\frac{\theta}{2\mu_\lambda}} e^{\mu_\lambda s} B^y\Big(1 - e^{-2\mu_\lambda s}\Big).$$

To construct all type processes η^y under the same measure $\mathbb{P}$, we first construct the process B^{y_0} as an independent Brownian motion started at $y_0\sqrt{2\mu_\lambda/\theta}$. Secondly, we construct the process B^{y_1} by running an independent Brownian motion started at $y_1\sqrt{2\mu_\lambda/\theta}$ until it first hits the path of B^{y_0}, at which point we couple the two processes together. Next, for any other $y \in (y_0, y_1)$, we run an independent Brownian motion B^y until it first meets with either the process B^{y_0} below or B^{y_1} above, at which point we couple it to the process it first hits.

Finally, we construct all the corresponding spatial processes ξ^y under $\mathbb{P}$ from a single Brownian motion W by defining

$$\xi^y(s) = W\Big(a\int_0^s \eta^y(w)^2\,\mathrm{d}w\Big) + \lambda a \int_0^s \eta^y(w)^2\,\mathrm{d}w, \tag{5.29}$$

where W is started at x and is independent of the B^y processes. Constructed in this way, for each $y \in [y_0, y_1]$, the $\mathbb{P}$-law of (ξ^y, η^y) is the same as the $\tilde{\mathbb{Q}}_\lambda^{x,y}$-law of (ξ, η).

Fixing $\mu \in (0,1)$ and $K > \max\{y_1, 1\}$, we define the events and stopping times

$$\begin{aligned} A_\varepsilon^y &:= \Big\{B^y(s) \in \Big[1 - \frac{\varepsilon}{2\kappa}, 1 + \frac{\varepsilon}{2\kappa}\Big], \forall s \in (1-\mu, 1]\Big\} \\ T_0 &:= \inf\{t : B^{y_0}(t) = 0\}, \qquad T_K := \inf\{t : B^{y_1}(t) = K\} \\ \tilde{A}_{\varepsilon,K} &:= A_\varepsilon^{y_0} \cap A_\varepsilon^{y_1} \cap \{T_0 > 1\} \cap \{T_K > 1\}. \end{aligned}$$

Then clearly $\mathbb{P}(\tilde{A}_{\varepsilon,K}) > 0$, and, on the event $\tilde{A}_{\varepsilon,K}$, the coupling gives

$$0 < \eta^{y_0}(s) \le \eta^y(s) \le \eta^{y_1}(s) \le K\sqrt{\frac{\theta}{2\mu_\lambda}} e^{\mu_\lambda s},$$

for all $s \ge 0$ and $y \in [y_0, y_1]$. Note that our construction also ensures that if event $A_\varepsilon^{y_0} \cap A_\varepsilon^{y_1}$ occurs then so must A_ε^y for any $y \in [y_0, y_1]$, hence $A_\varepsilon^y \supset \tilde{A}_{\varepsilon,K}$.

Lemma 5.6.2. *Let $\varepsilon > 0$. On the event $\tilde{A}_{\varepsilon,K}$, there exists a deterministic time $s_0 =$*

$s_0(\varepsilon) > 0$ *such that for all* $\tau > s_0$,

$$\sup_{s\in[0,\tau]} |\eta^y(s) - \bar{y}(s)| \le \varepsilon\sqrt{t},$$

for all $y \in [y_0, y_1]$.

Proof. Set $s_1 = -\frac{1}{2\mu_\lambda}\ln\mu$ and then, on the event $\tilde{A}_{\varepsilon,K}$, for all $\tau \ge s > s_1$ we have

$$\left|\eta^y(s) - \sqrt{\frac{\theta}{2\mu_\lambda}}e^{\mu_\lambda s}\right| \le \frac{\varepsilon}{2\kappa}\sqrt{\frac{\theta}{2\mu_\lambda}}e^{\mu_\lambda s},$$

for all $y \in [y_0, y_1]$. Writing

$$\bar{y}(s) = \sqrt{\frac{\theta}{2\mu_\lambda}}e^{\mu_\lambda s}\left(\frac{1-e^{-2\mu_\lambda s}}{1-e^{-2\mu_\lambda \tau}}\right) \le \sqrt{\frac{\theta}{2\mu_\lambda}}e^{\mu_\lambda s},$$

we see that there exists $s_2 = s_2(\varepsilon) > 0$ such that, for $\tau \ge s > s_2$,

$$\left|\bar{y}(s) - \sqrt{\frac{\theta}{2\mu_\lambda}}e^{\mu_\lambda s}\right| \le \frac{\varepsilon}{2\kappa}\sqrt{\frac{\theta}{2\mu_\lambda}}e^{\mu_\lambda s}.$$

Taking $s_3(\varepsilon) = \max\{s_1, s_2(\varepsilon)\}$ now yields

$$|\eta^y(s) - \bar{y}(s)| < \frac{\varepsilon}{\kappa}\sqrt{\frac{\theta}{2\mu_\lambda}}e^{\mu_\lambda s} \le \varepsilon\sqrt{t} \tag{5.30}$$

for all $\tau \ge s > s_3$ and all $y \in [y_0, y_1]$.

Now consider $s \in [0, s_3]$. On $\tilde{A}_{\varepsilon,K}$ we have

$$|\eta^y(s) - \bar{y}(s)| \le \sqrt{\frac{\theta}{2\mu_\lambda}}e^{\mu_\lambda s_3}(1+K),$$

and hence for some $s_4(\varepsilon) > 0$ we have $|\eta^y(s) - \bar{y}(s)| \le \varepsilon\sqrt{t}$ for all $\tau > s_4$, all $s \in [0, s_3]$, and all $y \in [y_0, y_1]$. Taking $s_0(\varepsilon) = \max\{s_3, s_4\}$ yields the result. □

Lemma 5.6.3. *Let* $\delta > 0$. *Then for all sufficiently small* ε, *there exists a deterministic*

$\tau_0 = \tau_0(\varepsilon,\delta) > 0$ *such that, on* $\tilde{A}_{\varepsilon,K}$*, we have*

$$\sup_{s\in[0,\tau]} \left| \int_0^s \eta^y(w)^2 \,\mathrm{d}w - \int_0^s \bar{y}(w)^2 \,\mathrm{d}w \right| < \delta t \tag{5.31}$$

for all $\tau > \tau_0$ *and all* $y \in [y_0, y_1]$.

Proof. Given any $\delta > 0$, we first fix an $\varepsilon > 0$ sufficiently small such that $\varepsilon(2+\frac{\varepsilon}{\kappa})\frac{\kappa}{2\mu_\lambda} < \frac{\delta}{4}$; this yields a corresponding $s_3 = s_3(\varepsilon)$, which is chosen as at equation (5.30). Given this s_3, we find $\tau_1 = \tau_1(\varepsilon,\delta) > 0$ such that, for all $\tau > \tau_1$,

$$(K^2+1)\int_0^{s_3} \frac{\theta}{2\mu_\lambda} e^{2\mu_\lambda w}\,\mathrm{d}w < \frac{\delta}{4}t.$$

We now set $\tau_0 = \tau_0(\varepsilon,\delta) = \max\{s_3,\tau_1\}$. With this choice of ε and τ_0, we proceed to show that the inequality (5.31) is satisfied. Note that τ_0 is deterministic and independent of y.

From equation (5.30) we see that, on $\tilde{A}_{\varepsilon,K}$ and for $s > s_3$,

$$\begin{aligned}
\int_0^s \eta^y(w)^2\,\mathrm{d}w &\geq \int_0^{s_3} \eta^y(w)^2\,\mathrm{d}w + \int_{s_3}^s \left(\bar{y}(w) - \frac{\varepsilon}{\kappa}\sqrt{\frac{\theta}{2\mu_\lambda}}e^{\mu_\lambda w}\right)^2 \mathrm{d}w \\
&\geq \int_0^s \bar{y}(w)^2\,\mathrm{d}w - \int_0^{s_3} \bar{y}(w)^2\,\mathrm{d}w - 2\int_{s_3}^s \frac{\varepsilon\theta}{2\kappa\mu_\lambda}e^{2\mu_\lambda w}\,\mathrm{d}w \\
&\geq \int_0^s \bar{y}(w)^2\,\mathrm{d}w - \int_0^{s_3} \frac{\theta}{2\mu_\lambda}e^{2\mu_\lambda w}\,\mathrm{d}w - (2\varepsilon)\frac{\kappa t}{2\mu_\lambda} > \int_0^s \bar{y}(w)^2\,\mathrm{d}w - \frac{\delta}{2}t
\end{aligned}$$

for all $\tau > \tau_0$ and all $y \in [y_0, y_1]$. Similarly

$$\begin{aligned}
\int_0^s \eta^y(w)^2\,\mathrm{d}w &\leq \int_0^{s_3} \eta^y(w)^2\,\mathrm{d}w + \int_{s_3}^s \left(\bar{y}(w) + \frac{\varepsilon}{\kappa}\sqrt{\frac{\theta}{2\mu_\lambda}}e^{\mu_\lambda w}\right)^2 \mathrm{d}w \\
&\leq \int_0^s \bar{y}(w)^2\,\mathrm{d}w + \int_0^{s_3} \left(K\sqrt{\frac{\theta}{2\mu_\lambda}}e^{\mu_\lambda w}\right)^2 \mathrm{d}w \\
&\qquad + \int_{s_3}^s \frac{\varepsilon}{\kappa}\left(2+\frac{\varepsilon}{\kappa}\right)\left(\sqrt{\frac{\theta}{2\mu_\lambda}}e^{\mu_\lambda w}\right)^2 \mathrm{d}w \\
&\leq \int_0^s \bar{y}(w)^2\,\mathrm{d}w + K^2\int_0^{s_3} \frac{\theta}{2\mu_\lambda}e^{2\mu_\lambda w}\,\mathrm{d}w + \varepsilon\left(2+\frac{\varepsilon}{\kappa}\right)\frac{\kappa t}{2\mu_\lambda} \\
&< \int_0^s \bar{y}(w)^2\,\mathrm{d}w + \frac{\delta}{2}t
\end{aligned}$$

for all $\tau > \tau_0$ and all $y \in [y_0, y_1]$. Finally, for $s \in [0, s_3]$, on $\tilde{A}_{\varepsilon,K}$ we have

$$\begin{aligned}\left| \int_0^s \eta^y(w)^2 \,\mathrm{d}w - \int_0^s \bar{y}(w)^2 \,\mathrm{d}w \right| &\le \int_0^s \eta^y(w)^2 \,\mathrm{d}w + \int_0^s \bar{y}(w)^2 \,\mathrm{d}w \\ &\le (K^2 + 1) \int_0^{s_3} \frac{\theta}{2\mu_\lambda} e^{2\mu_\lambda w} \,\mathrm{d}w < \delta t\end{aligned}$$

for all $\tau > \tau_0$ and all $y \in [y_0, y_1]$. □

Lemma 5.6.4. *Let $\delta > 0$. Then for all sufficiently small $\varepsilon > 0$, there exists $\mathbb{P}$-almost everywhere on $\tilde{A}_{\varepsilon,K}$ a random time $S_0 = S_0(\delta, \varepsilon) < \infty$ such that*

$$\sup_{s \in [0,\tau]} \left| \xi^y(s) - \lambda a \int_0^s \eta^y(w)^2 \,\mathrm{d}w \right| < \delta t,$$

for all $y \in [y_0, y_1]$ and all $\tau > S_0$.

Proof. Given $\delta > 0$, choose any $\delta', \delta'' > 0$ such that $\delta'(|\beta/\lambda| + \delta'') < \delta$. Recalling the construction of ξ^y at (5.29), we see from standard properties of Brownian motion that there almost surely exists some $S_1 = S_1(\delta') < \infty$ such that

$$\frac{1}{t} \sup_{s \in [0,t]} |W(s)| < \delta', \qquad \text{for all } t > S_1.$$

Then

$$\sup_{s \in [0,\tau]} \left| W\left(a \int_0^s \eta^y(w)^2 \,\mathrm{d}w \right) \right| < \delta' \left(a \int_0^\tau \eta^y(w)^2 \,\mathrm{d}w \right) \tag{5.32}$$

for all τ such that $a \int_0^\tau \eta^y(w)^2 \,\mathrm{d}w > S_1$, and by the coupling construction, on $\tilde{A}_{\varepsilon,K}$ this is true for all $y \in [y_0, y_1]$ if $a \int_0^\tau \eta^{y_0}(w)^2 \,\mathrm{d}w > S_1$. Then there exists ($\mathbb{P}$-almost everywhere on $\tilde{A}_{\varepsilon,K}$) a random time $S_2 = S_2(\delta') < \infty$, which depends on B^{y_0} and S_1, such that $a \int_0^\tau \eta^y(w)^2 \,\mathrm{d}w > S_1$ for all $y \in [y_0, y_1]$ when $\tau > S_2$.

Now by Lemma 5.6.3, given δ'' and a sufficiently small ε, there exists a deterministic $\tau_0 = \tau_0(\varepsilon, \delta'') > 0$ such that, on $\tilde{A}_{\varepsilon,K}$,

$$a \int_0^\tau \eta^y(w)^2 \,\mathrm{d}w \le a \int_0^\tau \bar{y}(s)^2 \,\mathrm{d}s + \delta'' t = \left(\left| \frac{\beta}{\lambda} \right| + \delta'' \right) t \tag{5.33}$$

for all $\tau > \tau_0$ and all $y \in [y_0, y_1]$. Combining the inequalities at (5.32) and (5.33), we now see that, for $\tau > S_0 = S_0(\varepsilon, \delta', \delta'') = \max\{S_2, \tau_0\}$,

$$\sup_{s\in[0,\tau]} \left|\xi^y(s) - \lambda a \int_0^s \eta^y(w)^2 \,\mathrm{d}w\right| = \sup_{s\in[0,\tau]} \left|W\left(a \int_0^s \eta^y(w)^2 \,\mathrm{d}w\right)\right| < \delta t$$

for all $y \in [y_0, y_1]$. □

On combining Lemmas 5.6.3 and 5.6.4 and recalling the definition of optimal path $\bar{x}$ at (5.14), we obtain the following result.

Lemma 5.6.5. *Let $\delta > 0$. Then for all sufficiently small $\varepsilon > 0$, there exists $\mathbb{P}$-almost everywhere on $\tilde{A}_{\varepsilon,K}$ a random time $\tilde{S}_0 = \tilde{S}_0(\delta, \varepsilon) < \infty$ such that*

$$\sup_{s\in[0,\tau]} |\xi^y(s) - \bar{x}(s)| < \delta t,$$

for all $y \in [y_0, y_1]$, and all $\tau > \tilde{S}_0$.

We may now draw everything together to finish the proof of Lemma 5.6.1(i). First we observe that since $\lambda < 0$, on event $A_t^{\varepsilon,\delta}(\xi)$,

$$\begin{aligned}\sup_{s\in[0,\tau]} e^{-(\psi_\lambda^+ y^2 + \lambda x)} &\exp\left(\psi_\lambda^+ \eta_s^2 + \lambda \xi_s - E_\lambda^+ s\right) \\ &\le e^{-(\psi_\lambda^+ y^2 + \lambda x)} \exp\left(\psi_\lambda^+ (\kappa + \varepsilon)^2 t + \lambda(-\beta - \delta)t\right),\end{aligned}$$

and so, given $\tilde{\varepsilon}_0$, we can choose first δ and then ε sufficiently small so that

$$A_t^{\varepsilon,\delta}(\xi) \subset \left\{ \sup_{s\in[0,\tau]} \hat{f}(s) \le \frac{g_{\tilde{\varepsilon}_0}(\tau)}{2} \right\}$$

and, from Lemmas 5.6.2 and 5.6.5, there exists a random time $\tilde{T} = \tilde{T}(\delta, \varepsilon) < \infty$ such that on $\tilde{A}_{\varepsilon,K}$ we have

$$\sup_{s\in[0,\tau]} |\eta^y(s) - \bar{y}(s)| < \varepsilon\sqrt{t} \quad \text{and} \quad \sup_{s\in[0,\tau]} |\xi^y(s) - \bar{x}(s)| < \delta t$$

for all $\tau > \tilde{T}$ and all $y \in [y_0, y_1]$. That is, $\tilde{A}_{\varepsilon,K} \cap \{\tilde{T} < \tau\} \subset A_t^{\varepsilon,\delta}(\xi^y)$ for each $y \in [y_0, y_1]$, with the slight abuse of notation that

$$A_t^{\varepsilon,\delta}(\xi^y) = \left\{ \sup_{s\in[0,\tau(t)]} |\eta^y(s) - \bar{y}(s)| < \varepsilon\sqrt{t};\ \sup_{s\in[0,\tau(t)]} |\xi^y(s) - \bar{x}(s)| < \delta t \right\}.$$

Note also that $\mathbb{P}(\tilde{A}_{\varepsilon,K}) > \varepsilon'$ for some $\varepsilon' > 0$.

Combining the above, for any $y \in [y_0, y_1]$ we have

$$\begin{aligned}\tilde{\mathbb{Q}}_\lambda^{x,y}\Big(A_t^{\varepsilon,\delta}(\xi);\ \sup_{s\in[0,\tau]} \hat{f}(s) \le \frac{g_{\tilde{\varepsilon}_0}(\tau)}{2}\Big) &= \tilde{\mathbb{Q}}_\lambda^{x,y}(A_t^{\varepsilon,\delta}(\xi)) = \mathbb{P}(A_t^{\varepsilon,\delta}(\xi^y)) \\ &\ge \mathbb{P}(\tilde{A}_{\varepsilon,K}; \tilde{T} < \tau) \to \mathbb{P}(\tilde{A}_{\varepsilon,K})\end{aligned}$$

as $\tau \to \infty$, as required. □

Proof of Lemma 5.6.1(ii). Consider the expectation of the 'sum term'. We have

$$\begin{aligned}\tilde{\mathbb{Q}}_\lambda^{x,y}\Big(\hat{\mathcal{Z}}_\lambda^+(\tau)\Big|\tilde{\mathcal{G}}_\infty\Big) &= e^{-(\psi_\lambda^+ y^2+\lambda x)}\tilde{\mathbb{Q}}_\lambda^{x,y}\Big(\sum_{u<\xi_\tau} e^{-E_\lambda^+ S_u} Z_\lambda^{(u)}(t-S_u)\Big|\tilde{\mathcal{G}}_\infty\Big) \\ &= e^{-(\psi_\lambda^+ y^2+\lambda x)}\sum_{u<\xi_\tau} e^{-E_\lambda^+ S_u}\tilde{\mathbb{Q}}_\lambda^{x,y}\Big(Z_\lambda^{(u)}(t-S_u)\Big|\tilde{\mathcal{G}}_\infty\Big) \\ &\le e^{-(\psi_\lambda^+ y^2+\lambda x)} n_\tau \max\Big\{e^{\psi_\lambda^+ \eta(S_u)^2+\lambda\xi(S_u)-E_\lambda^+ S_u} : u<\xi_\tau\Big\} \\ &\le n_\tau \sup_{s\in[0,\tau]} \hat{f}(s).\end{aligned}$$

Hence

$$\begin{aligned}&\tilde{\mathbb{Q}}_\lambda^{x,y}\Big(\frac{2}{g_{\varepsilon_0}(\tau)}\tilde{\mathbb{Q}}_\lambda^{x,y}\Big(\hat{\mathcal{Z}}_\lambda^+(\tau)\Big|\tilde{\mathcal{G}}_\infty\Big);\ \sup_{s\in[0,\tau]}\hat{f}(s) \le \frac{g_{\tilde{\varepsilon}_0}(\tau)}{2}\Big) \\ &\quad\le \tilde{\mathbb{Q}}_\lambda^{x,y}\Big(n_\tau\frac{g_{\tilde{\varepsilon}_0}(\tau)}{g_{\varepsilon_0}(\tau)};\ \sup_{s\in[0,\tau]}\hat{f}(s) \le \frac{g_{\tilde{\varepsilon}_0}(\tau)}{2}\Big) \\ &\quad\le e^{-(\varepsilon_0-\tilde{\varepsilon}_0)t}\tilde{\mathbb{Q}}_\lambda^{x,y}(n_\tau),\end{aligned}\tag{5.34}$$

and we can now calculate $\tilde{\mathbb{Q}}_\lambda^{x,y}(n_\tau) = \tilde{\mathbb{Q}}_\lambda^{x,y}(\tilde{\mathbb{Q}}_\lambda^{x,y}(n_\tau|\mathcal{G}_\infty))$, where $\mathcal{G}_\infty$ the σ-algebra generated by the path of the spine (*not* including the birth times). Conditional on $\mathcal{G}_\infty$, n_τ is a Poisson random variable with mean given by $\int_0^\tau 2(r\eta_s^2+\rho)\,\mathrm{d}s$, and using Fubini's theorem we have

$$\begin{aligned}\tilde{\mathbb{Q}}_\lambda^{x,y}\left(\int_0^\tau 2(r\eta_s^2+\rho)\,\mathrm{d}s\right) &= \int_0^\tau 2r\tilde{\mathbb{Q}}_\lambda^{x,y}(\eta_s^2)\,\mathrm{d}s + 2\rho\tau \\ &= \frac{r}{\mu_\lambda}\left(\frac{\theta}{2\mu_\lambda}+y^2\right)e^{2\mu_\lambda\tau} - \left(\frac{\theta}{2\mu_\lambda}+y^2\right)\frac{r}{\mu_\lambda\tau} - \frac{r\theta\tau}{\mu_\lambda} + 2\rho\tau \\ &= \frac{r}{\mu_\lambda}\kappa^2 t + \frac{2y^2\kappa^2 r}{\theta}t + o(\tau).\end{aligned}$$

So the $\tilde{\mathbb{Q}}_\lambda^{x,y}$-expectation of n_τ only grows linearly in t. Then since $\varepsilon_0 - \tilde{\varepsilon}_0 > 0$, the expression at (5.34) tends to 0 as $t \to \infty$. Moreover, the expectation at (5.34) is bounded by the $\tilde{\mathbb{Q}}_\lambda^{x,y_1}$-expectation, and hence the convergence is uniform over $y \in [y_0, y_1]$, as claimed. $\square$

Remark 5.6.6. It is slightly surprising to see that we did not need to include the event $A_\tau^{\varepsilon,\delta}(\xi)$ in either the statement or the proof of Lemma 5.6.1(ii), although $\tilde{\mathbb{Q}}_\lambda^{x,y}$ has been constructed precisely to make $A_\tau^{\varepsilon,\delta}(\xi)$ a likely event.

Chapter 6

BBM in a quadratic breeding potential

6.1 Introduction

In this chapter we consider a branching Brownian motion with a quadratic breeding rate. Each particle diffuses as a driftless Brownian motion and splits into two particles at rate βy^2, where $\beta > 0$ and $y \in \mathbb{R}$ is the particle's spatial position. The set of particles alive at time t is N_t, and then, for each $u \in N_t$, $Y_u(t)$ is the spatial position of particle u at time t. In the sequel we refer to this process as a $(\beta y^2; \mathbb{R})$-BBM, with probabilities $\{P^x : x \in \mathbb{R}\}$, where P^x is the law of the process started from a single particle at the point $x \in \mathbb{R}$. This process is defined on the space $\tilde{\mathcal{T}}$ of marked Galton-Watson trees with spines introduced in Chapter 3, and we retain the notation from Section 3.2 for the filtrations of this space. We also remind the reader that the measure $\tilde{P}^x$ on $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}}_\infty, (\tilde{\mathcal{F}}_t)_{t\geq 0})$ is the law of the $(\beta y^2; \mathbb{R})$-BBM with a distinguished spine.

We noted in Chapter 5 that quadratic breeding is a critical rate for population explosions. If the breeding rate were instead βy^p for $p > 2$, the population would almost surely explode in a finite time. However, for the $(\beta y^2; \mathbb{R})$-BBM the *expected* number of particles blows up in a finite time, but the total number of particles alive remains finite almost surely, for all time — see Itô and McKean [53, pp 200–211]. The fact that expectations for this process are not well behaved adds to the difficulty of its study. Additive martingales and the spine technology are still available to us, however, and with these methods we are able to begin an analysis of the $(\beta y^2; \mathbb{R})$-BBM.

In this chapter we are interested in the spatial spread of the $(\beta y^2; \mathbb{R})$-BBM. To this end we define some families of additive martingales, and then study their convergence properties to prove the following theorem.

Theorem 6.1.1. *Defining $R_t := \sup_{u\in N_t} Y_u(t)$ to be the right-most particle,*

$$\liminf_{t\to\infty} \frac{\ln R_t}{t} \geq \sqrt{2\beta}$$

P^x-almost surely for the $(\beta y^2;\mathbb{R})$-BBM.

This result shows that the right-most particle in the $(\beta y^2;\mathbb{R})$-BBM has spatial displacement that is, asymptotically, at least of exponential order; contrast this with the linear spread of standard BBM. Although we do not prove an upper bound for the displacement of the right-most particle, we do conjecture that in fact $\lim_{t\to\infty} \frac{\ln R_t}{t}$ exists almost surely and equals $\sqrt{2\beta}$. Some heuristics to support this are given in Section 6.4.

The growth of spatial branching processes is an important question, both for theoretical purposes and applications. For examples of some biological applications we refer the reader to Biggins [8], and Kimmel and Axelrod [62]. The spatial spread of the branching random walk is studied in the work of Biggins [4, 6, 7, 9, 10]; and related results for branching diffusions and superdiffusions can be found in Bramson [15, 16], Engländer [31], and Kyprianou [67].

6.2 Additive martingales for the $(\beta y^2;\mathbb{R})$-BBM

Let $\lambda \in \mathbb{R}$ and suppose Y is an Ornstein-Uhlenbeck process with parameter λ, that is the solution of

$$\mathrm{d}Y_t = \mathrm{d}B_t - \lambda Y_t\,\mathrm{d}t,$$

under the measure $\mathbb{P}_{(\lambda)}$. Here B is a $\mathbb{P}_{(\lambda)}$-Brownian motion. We recall from Borodin and Salminen [14, Appendix 1] that $\mathbb{P}_{(\lambda)}$ is absolutely continuous with respect to the Wiener measure $\mathbb{P}^x$ and

$$\frac{\mathrm{d}\mathbb{P}^x_{(\lambda)}}{\mathrm{d}\mathbb{P}^x}\bigg|_{\mathcal{G}_t} = \exp\bigg(-\frac{\lambda}{2}(Y_t^2 - x^2) + \frac{\lambda t}{2} - \frac{\lambda^2}{2}\int_0^t Y_s^2\,\mathrm{d}s\bigg).$$

We can use this to build an additive martingale for the $(\beta y^2;\mathbb{R})$-BBM. We define a measure $\tilde{\mathbb{Q}}_\lambda$, for $\lambda > 0$, on the filtered probability space $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}}_\infty, (\tilde{\mathcal{F}}_t)_{t\geq 0})$ via

$$\frac{\mathrm{d}\tilde{\mathbb{Q}}^x_\lambda}{\mathrm{d}\tilde{P}^x}\bigg|_{\tilde{\mathcal{F}}_t} = \tilde{M}_\lambda(t),$$

where
$$\tilde{M}_\lambda(t) := e^{-\beta\int_0^t \xi_s^2\,\mathrm{d}s}2^{n_t} \times \exp\left(\frac{\lambda}{2}(\xi_t^2 - x^2) - \frac{\lambda t}{2} - \frac{\lambda^2}{2}\int_0^t \xi_s^2\,\mathrm{d}s\right).$$

Noting that this is the product of two martingale terms, which make the spine diffuse as a transient Ornstein-Uhlenbeck process and increase its breeding rate by a factor of two, we can re-construct the process in law under $\tilde{\mathbb{Q}}^x_\lambda$ as below:

- the spine's spatial motion is determined by the single-particle measure $\mathbb{P}_{(-\lambda)}$, so that
$$\mathrm{d}\xi_t = \mathrm{d}\tilde{B}_t + \lambda\xi_t\,\mathrm{d}t,$$
where $\tilde{B}$ is a $\tilde{\mathbb{Q}}_\lambda$-Brownian motion;
- the fission times on the spine occur as a Poisson process of instantaneous rate $2\beta\xi_t^2$, which is independent of the spine's motion;
- at each fission time on the spine two particles are produced;
- one of these is chosen uniformly at random to be the spine, and it repeats stochastically the behaviour of its parent;
- the other particle initiates, from its birth position, an independent $(\beta y^2;\mathbb{R})$-BBM with law $P^{\cdot}$.

We now define a $(P,(\mathcal{F}_t)_{t\geq 0})$-martingale by
$$M_\lambda(t) := \tilde{P}^x(\tilde{M}_\lambda(t)|\mathcal{F}_t). \tag{6.1}$$

This follows on noticing that
$$P^x(M_\lambda(t)) = \tilde{P}^x(M_\lambda(t)) = \tilde{P}^x\left(\tilde{P}^x(\tilde{M}_\lambda(t)|\mathcal{F}_t)\right) = \tilde{P}^x(\tilde{M}_\lambda(t)) = 1,$$

so that $M_\lambda(t) \in \mathcal{L}^1(P^x)$ and has constant expectation for all $t > 0$. The fact that the expected number of particles blows up in finite time does not prevent us from defining $(P,(\mathcal{F}_t)_{t\geq 0})$-martingales for the process.

Using the representation at (3.8), it can be shown that
$$M_\lambda(t) = \sum_{u\in N_t}\exp\left(\frac{\lambda}{2}(Y_u(t)^2 - x^2) - \frac{\lambda t}{2} - \left(\frac{\lambda^2}{2}+\beta\right)\int_0^t Y_u(s)^2\,\mathrm{d}s\right).$$

(See the proof of Theorem 5.5.2 for a very similar calculation.) Additionally, it follows from the definition (6.1) that

$$\left.\frac{\mathrm{d}\mathbb{Q}_\lambda^x}{\mathrm{d}P^x}\right|_{\mathcal{F}_t} = M_\lambda(t), \tag{6.2}$$

where $\mathbb{Q}_\lambda = \tilde{\mathbb{Q}}_\lambda|_{\mathcal{F}_\infty}$ (see also the comments at the beginning of the proof of Proposition 3.2.7). Since it is also true that $P^x = \tilde{P}^x|_{\mathcal{F}_\infty}$, it follows that the construction of the $(\beta y^2;\mathbb{R})$-BBM in $\mathbb{Q}_\lambda$-law is the same as that given above for $\tilde{\mathbb{Q}}_\lambda$. We now find the values of the parameter λ for which M_λ is uniformly integrable.

Theorem 6.2.1. *Let $x \in \mathbb{R}$.*

(i) If $0 < \lambda < \sqrt{2\beta}$, M_λ is uniformly integrable and $M_\lambda(\infty) > 0$ almost surely.

(ii) If $\lambda > \sqrt{2\beta}$ then $P^x(M_\lambda(\infty) = 0) = 1$.

Our method of proof does not readily extend to the case $\lambda = \sqrt{2\beta}$, and so we do not treat this case here because it is not needed to prove Theorem 6.1.1.

Proof. (i) To prove this result, we extend the $\mathcal{L}^1$-convergence method of Kyprianou [66]. The first step is to decompose the martingale M_λ by conditioning on the spine's path, $\{\xi_t\}_{t\geq 0}$, and fission times, $\{S_u : u \in \xi\}$. By conditioning on $\tilde{\mathcal{G}}_\infty$ (see the calculation of Proposition 4.4.1) we have

$$\begin{aligned}\tilde{\mathbb{Q}}_\lambda^x(M_\lambda(t)|\tilde{\mathcal{G}}_\infty) = \sum_{u<\xi_t} &\exp\left(\frac{\lambda}{2}(\xi_{S_u}^2 - x^2) - \frac{\lambda S_u}{2} - \left(\frac{\lambda^2}{2}+\beta\right)\int_0^{S_u} \xi_s^2\,\mathrm{d}s\right)\\ &+ \exp\left(\frac{\lambda}{2}(\xi_t^2 - x^2) - \frac{\lambda t}{2} - \left(\frac{\lambda^2}{2}+\beta\right)\int_0^{t} \xi_s^2\,\mathrm{d}s\right),\end{aligned}$$

and we refer to the two pieces of this decomposition as $\mathbf{sum}(t)$ and $\mathbf{spine}(t)$. We wish to show that the conditional expectation above is $\tilde{\mathbb{Q}}_\lambda^x$-almost surely bounded as $t \to \infty$.

Under $\tilde{\mathbb{Q}}_\lambda^x$ we have (from Borodin and Salminen [14, p. 136]) the construction

$$\xi_t = e^{\lambda t}\tilde{B}\left(\frac{1-e^{-2\lambda t}}{2\lambda}\right), \tag{6.3}$$

where $\tilde{B}$ is a $\tilde{\mathbb{Q}}_\lambda^x$-Brownian motion started at x. For notational convenience we define

$$\tau_t := \frac{1-e^{-2\lambda t}}{2\lambda} \qquad \text{and} \qquad \bar{B} := \tilde{B}\left(\frac{1}{2\lambda}\right).$$

Now for any $\varepsilon > 0$, there exists almost surely some random time $T_\varepsilon < \infty$ such that

$$|\bar{B} - \tilde{B}(\tau_t)| < \varepsilon,$$

for all $t > T_\varepsilon$; and then for $t > T_\varepsilon$ we can overestimate $\mathbf{spine}(t)$ by

$$\mathbf{spine}(t) \leq \exp\left(\frac{\lambda}{2}(|\bar{B}| + \varepsilon)^2 e^{2\lambda t} - \left(\frac{\lambda^2}{2} + \beta\right)\int_{T_\varepsilon}^t e^{2\lambda s}(|\bar{B}| - \varepsilon)^2 \,\mathrm{d}s\right).$$

We are only concerned with finding a decaying upper bound for the spine term, so the expressions that matter in the exponential above are

$$\frac{\lambda}{2}(|\bar{B}| + \varepsilon)^2 e^{2\lambda t} - \left(\frac{\lambda^2}{2} + \beta\right)\frac{1}{2\lambda}(|\bar{B}| - \varepsilon)^2 e^{2\lambda t} = \left(\frac{\lambda}{4} - \frac{\beta}{2\lambda}\right)|\bar{B}|^2 e^{2\lambda t} + C\varepsilon e^{2\lambda t},$$

where $C \in \mathbb{R}$ is a (possibly negative) constant that depends on λ, β, and $\bar{B}$. Since

$$\frac{\lambda}{4} - \frac{\beta}{2\lambda} < 0 \Leftrightarrow \lambda < \sqrt{2\beta},$$

and ε can be chosen arbitrarily small, there exists a constant $C' > 0$ and $T < \infty$ such that for all $t > T$

$$\mathbf{spine}(t) \leq \exp(-C' e^{2\lambda t})$$

almost surely.

The births on the spine are a Poisson process with instantaneous rate $2\beta\xi_t^2$, and so the number of births on the spine by time t is a Poisson random variable with expectation $2\beta\int_0^t \xi_s^2 \,\mathrm{d}s$. By the Strong Law of Large Numbers, this is almost surely $O(e^{2\lambda t})$, and so

$$\limsup_{t\to\infty} \tilde{\mathbb{Q}}_\lambda^x(M_\lambda(t)|\tilde{\mathcal{G}}_\infty) < +\infty, \tag{6.4}$$

$\tilde{\mathbb{Q}}_\lambda^x$-almost surely.

Using Fatou's lemma with (6.4) we have

$$\begin{aligned}\tilde{\mathbb{Q}}_\lambda^x(\liminf_{t\to\infty} M_\lambda(t)|\tilde{\mathcal{G}}_\infty) &\leq \liminf_{t\to\infty} \tilde{\mathbb{Q}}_\lambda^x(M_\lambda(t)|\tilde{\mathcal{G}}_\infty)\\ &\leq \limsup_{t\to\infty} \tilde{\mathbb{Q}}_\lambda^x(M_\lambda(t)|\tilde{\mathcal{G}}_\infty) < +\infty,\end{aligned}$$

$\tilde{\mathbb{Q}}^x_\lambda$-almost surely, which implies that

$$\liminf_{t\to\infty} M_\lambda(t) < +\infty \qquad \tilde{\mathbb{Q}}^x_\lambda\text{-a.s.}$$

Furthermore, $\liminf_{t\to\infty} M_\lambda(t)$ is $\mathcal{F}_\infty$-measurable, and so

$$\liminf_{t\to\infty} M_\lambda(t) < +\infty \qquad \mathbb{Q}^x_\lambda\text{-a.s.} \tag{6.5}$$

In light of (6.2), $1/M_\lambda(t)$ is a positive $\mathbb{Q}_\lambda$-martingale, which converges almost surely, whence $M_\lambda(t)$ converges $\mathbb{Q}_\lambda$-almost surely. Combining this observation with (6.5) we have

$$\lim_{t\to\infty} M_\lambda(t) = \liminf_{t\to\infty} M_\lambda(t) < +\infty$$

$\mathbb{Q}_\lambda$-almost surely. Applying Theorem 4.4.3 now shows that M_λ is a uniformly integrable martingale if $0 < \lambda < \sqrt{2\beta}$, so that $P^x(M_\lambda(\infty)) = 1$.

It remains to show that $P^x(M_\lambda(\infty) = 0) = 0$. We define $p(x) := P^x(M_\lambda(\infty) = 0)$, but we will now show that this probability is independent of x. To see this, apply the branching Markov property to obtain, for $t > 0$,

$$p(x) = E^x\Big(P^x(M_\lambda(\infty) = 0|\mathcal{F}_t)\Big) = E^x\bigg(\prod_{u\in N_t} p(Y_u(t))\bigg),$$

whence $M_t := \prod_{u\in N_t} p(Y_u(t))$ is a product martingale and since $0 \le M(t) \le 1$ for all $t > 0$, M_t is uniformly integrable and converges in $\mathcal{L}^1(P^x)$. It follows that $p \in C^2(\mathbb{R})$ and satisfies

$$\frac{1}{2}p'' + \beta x^2(p^2 - p) = 0. \tag{6.6}$$

In addition we note that, by symmetry, $p(x) = p(-x)$ for all $x \in \mathbb{R}$; and, since $0 \le p(x) \le 1$, we have $\beta x^2(p^2 - p) \le 0$ and so $p''(x) \ge 0$ on $\mathbb{R}$. We claim that these conditions on p force $p \equiv 0$ or 1.

The symmetry of p about the line $x = 0$ implies that $p'(0) = 0$; and further, since $p''(x) \ge 0$ on $\mathbb{R}$ we must have that $p'(x) \ge 0$ on $(0,\infty)$, whence p is increasing on $[0,\infty)$. Combining this observation with the fact that $p \le 1$ means that $\lim_{x\to\infty} p(x)$ exists, and so $\lim_{x\to\infty} p'(x) = 0$. Noting that p' is non-decreasing on $\mathbb{R}$, it must be the case that $p' \equiv 0$ on $[0,\infty)$. Symmetry of p now ensures that p must be constant on $\mathbb{R}$, and the only two constant solutions of the ordinary differential equation (6.6) are 0 and 1. Since M_λ is uniformly integrable when $\lambda \in (0, \sqrt{2\beta})$, we must have $p(x) \equiv 0$, as required.

(ii) Since one of the particles alive at time t is the spine, we have that

$$
\begin{aligned}
M_\lambda(t) &\geq \exp\left(\frac{\lambda}{2}(\xi_t^2 - x^2) - \frac{\lambda t}{2} - \left(\frac{\lambda^2}{2} + \beta\right)\int_0^t \xi_s^2 \, \mathrm{d}s\right) \\
&= \exp\left(\lambda \int_0^t \xi_s \, \mathrm{d}\xi_s - \left(\frac{\lambda^2}{2} + \beta\right)\int_0^t \xi_s^2 \, \mathrm{d}s\right).
\end{aligned} \tag{6.7}
$$

The equality above follows from Itô's formula, specifically

$$\mathrm{d}(\xi_s^2) = 2\xi_s \, \mathrm{d}\xi_s + \mathrm{d}s,$$

and so

$$\xi_t^2 - \xi_0^2 = 2\int_0^t \xi_s \, \mathrm{d}\xi_s + t.$$

Under $\tilde{\mathbb{Q}}_\lambda^x$, we can write

$$\mathrm{d}\xi_s = \mathrm{d}\tilde{B}_s + \lambda \xi_s \, \mathrm{d}s,$$

where $\tilde{B}$ is $\tilde{\mathbb{Q}}_\lambda^x$-Brownian motion, and so (6.7) is equal to

$$\exp\left(\lambda \int_0^t \xi_s \, \mathrm{d}\tilde{B}_s + \left(\frac{\lambda^2}{2} - \beta\right)\int_0^t \xi_s^2 \, \mathrm{d}s\right).$$

Now, $\tilde{\mathbb{Q}}_\lambda^x$-almost surely as $t \to \infty$,

$$\int_0^t \xi_s^2 \, \mathrm{d}s = O(e^{2\lambda t}) \qquad \text{and} \qquad \int_0^t \xi_s \, \mathrm{d}\tilde{B}_s = o(e^{2\lambda t}),$$

and consequently $\tilde{\mathbb{Q}}_\lambda^x(\limsup_{t\to\infty} M_\lambda(t) = +\infty) = 1$ if $\lambda > \sqrt{2\beta}$. This also holds $\mathbb{Q}_\lambda^x$-almost surely, and another application of Theorem 4.4.3 yields the result. □

6.3 A lower bound for the right-most particle

We can now prove Theorem 6.1.1.

Proof of Theorem 6.1.1. Let $0 < \lambda < \sqrt{2\beta}$ and define

$$A_\lambda := \{\exists u : \liminf_{t\to\infty} t^{-1} \ln |Y_u(t)| = \lambda\}.$$

Then $A_\lambda \in \mathcal{F}_\infty$ and $\mathbb{Q}_\lambda(A_\lambda) = \tilde{\mathbb{Q}}_\lambda(A_\lambda) = 1$, because under $\tilde{\mathbb{Q}}_\lambda$ the spine is a transient Ornstein-Uhlenbeck process — recall the representation (6.3). For $\lambda \in (0, \sqrt{2\beta})$, M_λ

is uniformly integrable by Theorem 6.2.1 and hence $P(M_\lambda(\infty)) = 1$. By definition we have

$$P(\mathbf{1}_{A_\lambda} M_\lambda(\infty)) = \mathbb{Q}_\lambda(A_\lambda) = 1,$$

and because both $P(M_\lambda(\infty) > 0) = 1$ and $P(M_\lambda(\infty)) = 1$, it follows that $P(A_\lambda) = 1$. Since this holds for any $\lambda \in (0, \sqrt{2\beta})$, and the spatial spread of the BBM is symmetric about the origin, we have that

$$\liminf_{t\to\infty} \frac{\ln R_t}{t} \geq \sqrt{2\beta}$$

P-almost surely. □

6.4 The asymptotic exponential speed of the right-most particle

In future work we hope to strengthen Theorem 6.1.1, and we conjecture that

$$\lim_{t\to\infty} \frac{\ln R_t}{t} = \sqrt{2\beta}$$

P-almost surely. Our intuition as to why we believe this to be the case arises from the consideration of other families of additive martingales for the $(\beta y^2;\mathbb{R})$-BBM. Using the Girsanov theorem, one can define an additive martingale, $Z_g(t)$ say, that gives the spine a general drift $g(s)$, for some $g : \mathbb{R}^+ \to \mathbb{R}$, under the changed measure $\mathbb{Q}$. In fact for the special case $g(s) = e^{\lambda s}$, where $\lambda > 0$, we obtain another family of additive martingales parameterised by λ, and it can be shown that these martingales have the same properties as M_λ in both the cases $\lambda \in (0, \sqrt{2\beta})$ and $\lambda \geq \sqrt{2\beta}$. These martingales thus offer an alternative route to the proof of Theorem 6.1.1. We can give a heuristic justification that, if the right-most particle satisfies

$$\limsup_{t\to\infty} \frac{\ln R(t)}{t} > \sqrt{2\beta}$$

almost surely, then certain (positive) additive martingales fail to converge, which is a contradiction. However making this into a rigorous proof will require certain subtleties to be resolved.

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