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New Vacuum Solutions for Quadratic Metric–affine Gravity

submitted by

Vedad Pašić

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

February 2009

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Summary

In this thesis we deal with *quadratic metric–affine gravity*, which is an alternative theory of gravity. We present *new vacuum solutions* for this theory and attempt to give their *physical interpretation* on the basis of comparison with existing classical models.

These new explicit vacuum solutions of quadratic metric–affine gravity are constructed using *generalised pp-waves*. A classical pp-wave is a 4-dimensional Lorentzian spacetime which admits a non–vanishing parallel spinor field. We *generalise* this definition to metric compatible spacetimes with torsion, describe basic properties of such spacetimes and eventually use them to construct new solutions to the field equations of quadratic metric–affine gravity.

The physical interpretation of these solutions we propose in this thesis is that these new solutions represent a *conformally invariant metric–affine model for a massless elementary particle*. We give a comparison with a classical model describing the interaction of gravitational and massless neutrino fields, namely *Einstein-Weyl theory*.

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Chapter 1

Introduction

This thesis is our attempt in furthering the study of alternative theories of gravity. In this introductory chapter we provide background knowledge about an alternative theory of gravity, namely metric–affine gravity. We will then introduce *quadratic* metric–affine gravity and present known solutions of this theory. Finally, certain other alternative theories of gravity are mentioned and very briefly explained.

Please note that in this introductory Chapter we might use and mention some constructs and concepts that are only properly introduced and defined in Chapter 2 which deals with background and notation. However, as we don't want to get entangled in too much rigorous detail right at the beginning, we will try to keep the 'gory details' to a minimum in this introductory Chapter. Please refer Chapter 2 for the details on constructs and concepts used within this thesis.

1.1 General relativity – a very brief introduction

In 1905, *Albert Einstein* published his work on the theory of *special relativity*. Classical mechanics and classical electromagnetism provide models that are good representations of two sets of actual experiences. As Einstein noted in [23], it is not possible to combine these into a single self–consistent model. The construction of the simplest possible self–consistent model by Einstein is the achievement of Einstein's theory of special relativity.

This theory gave a very satisfactory representation of the electromagnetic interaction between charged particles. Special relativity itself does not deal with gravitational interaction.

General relativity is a theory of gravitation that was developed by Einstein between 1907 and 1915. Hermann Minkowski put Einstein's special relativity model into geometrical terms, and it is widely believed that Einstein constructed his theory of general relativity by experimenting with the generalisation of the geometric model.

Two problems with general relativity became apparent quite quickly. Einstein considered that what are recognised locally as inertial properties of local matter must be determined by the properties of the rest of the universe. To what extent general relativity manages to do this is still unclear to this day, although Einstein's efforts to discover this extent basically founded the modern study of *cosmology*.

The second problem of general relativity was that, although electromagnetism pointed the way to general relativity, it is not included in the theory itself. As Einstein said in the only 'textbook' he ever wrote, the *Meaning of Relativity*¹ [24]:

"A theory in which the gravitational field and the electromagnetic field do not enter as logically distinct structures would be much preferable".

As is evident from his remarks, see Einstein [24], Einstein expected much more from general relativity than 'just' the amalgamation of gravitation and electromagnetism at the macroscopic level. He thought the theory should explain the existence of elementary particles and should provide a treatment for nuclear forces. He spent most of the second part of his life in pursuit of this aim, but with no real success. It is humbling to think how much the theory of general relativity progressed in the first few years of its existence and how comparatively little had been done in its advancement for many decades afterwards. Only after Einstein's death did the subject again really become 'fashionable'.

Paraphrasing John Wheeler [94], Einstein's geometric theory of gravity can be summarised thus: spacetime tells matter how to move; matter tells spacetime how to curve. In order to comprehend this, we have to understand the following three things:

- the motion of particles which are so small that their effect on the gravitational field they move in is negligible;
- the nature of matter as a source for gravity;
- Einstein's equation, which shows how this matter source is related to the curvature of spacetime.

¹The first edition consisted of Einstein's Stafford Little Lectures, delivered in May 1921 at Princeton University

Einstein's equation is the centerpiece of general relativity. It provides a formulation of the relationship between spacetime geometry and the properties of matter. This equation is formulated using the language of *Riemannian geometry*, in which the geometric properties of spacetime are described by the metric. The metric encodes the information needed to compute the fundamental geometric notions of distance and angle in a curved spacetime.

The vacuum Einstein's equation

$$Ric_{\alpha\beta} - \frac{1}{2}\mathcal{R}g_{\alpha\beta} = 0$$

is obtained by varying the Einstein–Hilbert action

$$\frac{1}{2k} \int \mathcal{R} \sqrt{|\det g|},$$

where \mathcal{R} is the scalar curvature, Ric the Ricci curvature and k is a universal constant, with respect to the metric g . For more details, see for example Landau and Lifshitz [57] and Section 5.2.1 of this thesis.

The full field equation is then obtained by adding the matter Lagrangian to the Einstein–Hilbert action, which gives us Einstein's equation in tensor form

$$Ric_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = kT_{\mu\nu}, \tag{1.1}$$

where T is the stress energy tensor that arises from the matter Lagrangian (see Landau and Lifshitz [57]), Ric is the Ricci curvature, \mathcal{R} is the scalar curvature, g is the metric, G the gravitational constant (5.10) and c is the speed of light. So, up to a constant multiple, the quantity that measures curvature is equated with the quantity that measures the matter content. The simplest solution of this equation is the Minkowski spacetime from special relativity.

General relativity is very successful in providing an accurate model for an impressive array of physical phenomena, but there are many interesting open questions - in particular, the theory as a whole is almost certainly incomplete.

1.2 Metric–affine gravity

There are a number of different *alternative* theories of gravity that try to further the completion of Einstein’s theory of gravity. One such theory, propagated by Einstein himself for some time, is *metric–affine gravity*, which is the theory employed by this thesis.

A number of developments in physics in the last several decades have evoked the possibility that the treatment of spacetime might involve more than just the Riemannian spacetime of Einstein’s general relativity, as stated by Hehl et al. in [42]. Some of these are, for example:

- our failure so far to quantize gravity is probably the strongest reason for going beyond a geometry which is dominated by the classical distance concept;
- the generalisation of the three-dimensional theory of elastic continua with microstructure to the four-dimensional spacetime of gravity suggests physical interpretations for the newly emerging structures in post-Riemannian spacetime geometry;
- the description of hadron (or nuclear) matter in terms of extended structures;
- the study of the early universe;
- the accelerating universe;
-

The smallest departure from a Riemannian spacetime of Einstein’s general relativity would consist of admitting *torsion* (2.2), arriving thereby at a Riemann–Cartan spacetime, and, furthermore, possible nonmetricity (2.5), resulting in a ‘*metric–affine*’ spacetime.

Metric–affine gravity is a natural generalisation of Einstein’s general relativity, which is based on a spacetime with a Riemannian metric g of Lorentzian signature. Similarly, in metric–affine gravity we consider spacetime to be a connected real 4–manifold M equipped with a Lorentzian metric g and an affine connection Γ . Note that the characterisation of the spacetime manifold by an *independent* linear connection Γ initially distinguishes metric–affine gravity from general relativity. The connection incorporates the inertial properties of spacetime and it can be viewed, according

to Hermann Weyl [93], as the guidance field of spacetime. The metric describes the structure of spacetime with respect to its spacio-temporal distance relations.

The spacetime of metric–affine gravity reduces to that of general relativity provided that the *torsion* (2.2) of the connection Γ vanishes and that the connection is *metric-compatible* (i.e. the covariant derivative of the metric g vanishes, $\nabla g \equiv 0$). In this case the connection is uniquely determined by the metric (*Levi-Civita* connection) and the same is true for the curvature. Consequently, the metric g is the only unknown quantity of Einstein’s equation. In contrast, the metric–affine approach does not involve any a priori assumptions about the connection Γ and thus the metric g and the connection Γ are viewed as two totally independent unknown quantities. For a very comprehensive review of metric–affine gravity, see Hehl et al. [42].

The 10 independent components of the (symmetric) metric tensor $g_{\mu\nu}$ and the 64 connection coefficients $\Gamma^\lambda_{\mu\nu}$ are the unknowns of metric–affine gravity.

As stated by Hehl et al. in [42], in Einstein’s general relativity theory the linear connection of its Riemannian spacetime is metric–compatible and symmetric. The symmetry of the Levi–Civita connection translates into the closure of infinitesimally small parallelograms. Already the transition from the flat gravity-free Minkowski spacetime to the Riemannian spacetime in Einstein’s theory can locally be understood as a deformation process.

The lifting of the constraints of metric–compatibility and symmetry yields non-metricity and torsion, respectively. The continuum under consideration, here classical spacetime, is thereby assumed to have a non-trivial microstructure, similar to that of a liquid crystal or a dislocated metal or ferromagnetic material etc. It is gratifying, though, to have the geometrical concepts of nonmetricity and torsion already arising in the (three-dimensional) continuum theory of lattice defects, see [51, 52].

The most important distinction between spacetimes used in this thesis is given by the following definition of a *Riemannian spacetime*:

Definition 1.2.1. We call a spacetime $\{M, g, \Gamma\}$ *Riemannian* if the connection is Levi–Civita (i.e. $\Gamma^\lambda_{\mu\nu} = \left\{ \begin{smallmatrix} \lambda \\ \mu\nu \end{smallmatrix} \right\}$), and *non-Riemannian* otherwise.

Remark 1.2.2. It is important to stress that throughout this thesis the metric is assumed to be Lorentzian (not positive definite). In particular, in our Definition 1.2.1 of Riemannian spacetimes the metric is also assumed to be Lorentzian. This is the convention used in theoretical physics literature. Note that the terminology used in

mathematical literature is different. In mathematical literature ‘Riemannian’ indicates positivity of the metric. We follow the terminology of theoretical physics.

1.3 Quadratic metric–affine gravity

In quadratic metric–affine gravity, we define our action as

$$S := \int q(R) \quad (1.2)$$

where q is a quadratic form on curvature R . The coefficients of this quadratic form are assumed to depend only on the metric, and the form itself is assumed to be $O(1, 3)$ invariant. Note that $\int f := \int_M f \sqrt{|\det g|} dx^0 dx^1 dx^2 dx^3$, $\det g := \det(g_{\mu\nu})$.

The quadratic form $q(R)$ has 16 R^2 terms with 16 real coupling constants, and it can be represented as

$$\begin{aligned} q(R) = & b_1 \mathcal{R}^2 + b_1^* \mathcal{R}_*^2 \\ & + \sum_{l,m=1}^3 b_{6lm}(\mathcal{A}^{(l)}, \mathcal{A}^{(m)}) + \sum_{l,m=1}^2 b_{9lm}(\mathcal{S}^{(l)}, \mathcal{S}^{(m)}) + \sum_{l,m=1}^2 b_{9lm}^*(\mathcal{S}_*^{(l)}, \mathcal{S}_*^{(m)}) \\ & + b_{10}(R^{(10)}, R^{(10)})_{\text{YM}} + b_{30}(R^{(30)}, R^{(30)})_{\text{YM}} \end{aligned}$$

with some real constants $b_1, b_1^*, b_{6lm} = b_{6ml}, b_{9lm} = b_{9ml}, b_{9lm}^* = b_{9ml}^*, b_{10}, b_{30}$. Here $\mathcal{R}, \mathcal{R}_*, \mathcal{A}^{(l)}, \mathcal{S}^{(l)}, \mathcal{S}_*^{(l)}, R^{(10)}, R^{(30)}$ are tensors defined in Section 2.3 and the inner-products (\cdot, \cdot) and $(\cdot, \cdot)_{\text{YM}}$ are defined in equations (2.37) and (2.38) respectively. For the detailed explanation of irreducible pieces of curvature and quadratic forms on curvature, please see Sections 2.3 and 2.4.

Independent variation of (1.2) with respect to the metric g and the connection Γ produces the system of Euler–Lagrange equations which we will write symbolically as

$$\partial S / \partial g = 0, \quad (1.3)$$

$$\partial S / \partial \Gamma = 0. \quad (1.4)$$

Our objective is the study of the combined system of field equations (1.3), (1.4). This is a system of $10 + 64$ real nonlinear partial differential equations with $10 + 64$ real

unknowns.

The study of equations (1.3), (1.4) for specific purely quadratic curvature Lagrangians has a long history. Quadratic curvature Lagrangians were first discussed by Weyl [93], Pauli [70], Eddington [22] and Lanczos [54, 55, 56] in an attempt to include the electromagnetic field in Riemannian geometry.

Our motivation comes from Yang–Mills theory. The Yang–Mills action for the affine connection is a special case of (1.2) with

$$q(R) = q_{\text{YM}}(R) := R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_{\kappa}{}^{\mu\nu}. \quad (1.5)$$

With this choice of $q(R)$, equation (1.4) is the Yang–Mills equation for the affine connection, which was analysed by Yang [96]. Yang was looking for Riemannian solutions, so he specialised equation (1.4) to the Levi–Civita connection and arrived at the equation

$$\nabla_\lambda \text{Ric}_{\kappa\mu} - \nabla_\kappa \text{Ric}_{\lambda\mu} = 0. \quad (1.6)$$

Here ‘specialisation’ means that one sets $\Gamma^\lambda{}_{\mu\nu} = \left\{ \begin{smallmatrix} \lambda \\ \mu\nu \end{smallmatrix} \right\}$ *after* the variation in Γ is carried out. For more on the solutions in Yang–Mills theory, see Section 1.4.1.

The idea of using a purely quadratic action in General Relativity goes back to Hermann Weyl, as given at the end of his paper [93], where he argued that the most natural gravitational action should be quadratic in curvature and involve all possible invariant quadratic combinations of curvature, like the square of Ricci curvature, the square of scalar curvature, etc. Unfortunately, Weyl himself never afterwards pursued this analysis. There were quite a few other authors who did pursue this idea.

Stephenson [82], for example, looked at three different quadratic invariants: scalar curvature squared, Ricci curvature squared and $R_\kappa{}^{\lambda\mu\nu} R^\kappa{}_{\lambda\mu\nu}$ (Yang–Mills Lagrangian (1.5)) and varied with respect to the metric and the affine connection. He concluded that every equation arising from the above mentioned quadratic Lagrangians has the Schwarzschild solution and that the equations give the same results for the three ‘crucial tests’ of general relativity, i.e. the bending of light, the advance of the perihelion of Mercury and the red-shift.

Higgs [47] continued in a similar fashion to show that in scalar squared and Ricci squared cases, one set of equations may be transformed into field equations of the Einstein type with an arbitrary ‘cosmological constant’ in terms of the ‘new gauge–invariant metric’.

One can get more information and form an idea on the historical development of the quadratic metric–affine theory of gravity from [14, 26, 27, 47, 58, 68, 71, 82, 84, 85, 95, 96] to name but a few works in this field.

In the metric–affine setting there are 11 irreducible pieces of curvature, as given in much more detail in Section 2.3. Since the end of the 1980’s researchers squared all of these pieces of curvature in the study of quadratic metric–affine gravity, thus obtaining 11 terms in the quadratic action. It was later shown however that things were more complicated than that, as certain irreducible subspaces of the 96–dimensional vector space of real rank 4 tensors $R^\kappa_{\lambda\mu\nu}$ are isomorphic to each other. Hence there are in fact 16 different ways of squaring the 11 pieces of curvature, as pointed out by Esser in [25] and Hehl and Macias in [43].

Remark 1.3.1. It should be noted that the action (1.2) contains only purely quadratic curvature terms, so it excludes the Einstein–Hilbert term (linear in curvature) and any terms quadratic in torsion and nonmetricity. By choosing a purely quadratic curvature Lagrangian we are hoping to describe phenomena whose characteristic wavelength is sufficiently small and curvature sufficiently large.

Remark 1.3.2. We should also point out that the action (1.2) is conformally invariant, i.e. it does not change if we perform a Weyl rescaling of the metric $g \rightarrow e^{2f}g$, $f : M \rightarrow \mathbb{R}$, without changing the connection Γ . Here it is important to stress once again that in the metric–affine setting the metric and the connection are viewed independently, i.e. the connection is not assumed to be Levi–Civita, see Definition 1.2.1.

We should also note that the classical Einstein–Hilbert Lagrangian is not invariant under conformal changes, except for the trivial case with dimension two, see Hehl et al. [42].

Remark 1.3.3. In view of Remark 1.3.2, the actual number of independent equations in (1.3) is not 10 but 9.

1.4 Known solutions for quadratic metric–affine gravity

In this Section we shall give an overview of previously known solutions of the problem described in Section 1.3 and some of the history related to the subject of finding solutions for quadratic metric–affine gravity. This Section is divided into two parts due

to the two clearly distinct types of spacetimes considered, namely Riemannian and non-Riemannian solutions, see Definition 1.2.1.

1.4.1 Known Riemannian solutions

Here we present known Riemannian solutions of the system (1.3), (1.4) and some of the work previously done on the subject.

It is important to note at this point that although we consider Riemannian solutions, the variations of the action that produce the field equations (1.3), (1.4) are still performed independently. Only *after* these variations have been performed do we set the connection to be Levi–Civita and consider Riemannian solutions of the field equations.

As previously mentioned in Section 1.3, Yang studied the equation (1.6) and a direct consequence of this equation is that Einstein spaces satisfy the equation (1.4).

It was shown later by a number of authors that in the special case (1.5) Einstein spaces satisfy both equations (1.3) and (1.4); see for example Mielke [58] in which the author of the review noticed a fact missed in the paper under review [82], namely, that Einstein spaces are stationary points of the Yang–Mills action with respect to the variation of both the metric and the connection, a fact repeatedly rediscovered in later years.

Apart from [58, 82], there has been substantial other work devoted to the study of the system of equations (1.3), (1.4) for the special case (1.5); see, for example, [14, 26, 27, 47, 68, 71, 84, 85, 95], with many authors rediscovering known results independently.

There have also been attempts in the past to establish a uniqueness result. The problem of uniqueness is a very delicate matter, even in the simpler Yang–Mills case (1.6). A particularly interesting attempt at establishing uniqueness was done by Fairchild [26], who wanted to show that Einstein spaces were the only solutions of (1.3), (1.4). However the result and the proof were incorrect and the author had to publish an erratum [27].

A comprehensive study of equations (1.3), (1.4) for the most general case of quadratic action with sixteen R^2 terms given at the beginning of section 1.3 was done relatively recently. Vassiliev [90] solved the problem of existence and uniqueness for the most general case in 2005. He showed the following spacetimes are solutions of the

equations (1.3), (1.4):

- Einstein spaces;
- pp-waves with parallel Ricci curvature;
- Riemannian spacetimes which have zero scalar curvature and are locally a product of Einstein 2-manifolds.

Furthermore, in the same paper [90] Vassiliev showed that the above spacetimes are the *only* Riemannian solutions of the system of field equations (1.3), (1.4), finally solving this problem. It is also interesting that before this paper it had not been noticed that pp-waves were solutions of the problem, although they were well known spacetimes in theoretical physics. The paper by Vassiliev [90] was also the main motivation behind this thesis, and because of the uniqueness result most of the work done to produce this thesis has been to establish new non-Riemannian solutions of the system (1.3), (1.4). For a detailed description of (classical) pp-waves, see Section 3.1.

1.4.2 Known non-Riemannian solutions

There is a significant amount of work by various authors devoted to finding spacetimes that solve the system (1.3), (1.4) which are non-Riemannian, i.e. which incorporate torsion. However, there is a further subdivision in this class of spacetimes, as metric-compatibility of the connection is no longer guaranteed, so we shall view these separately.

Metric-compatible non-Riemannian solutions

In this subsection we assume that the connection is metric-compatible, i.e. $\nabla g \equiv 0$ and give known non-Riemannian solutions that satisfy the above condition.

Several papers of Vassiliev [49, 87, 90] give us a substantial insight into this problematic as well as present some non-Riemannian solutions of the system (1.3), (1.4).

The following construction, as presented by Vassiliev in [90], provides one method of obtaining such solutions.

Definition 1.4.1. We call a spacetime $\{M, g, \Gamma\}$ a *pseudoinstanton* if the connection is metric-compatible and curvature is irreducible and simple.

Irreducibility of curvature means that all of the 11 irreducible pieces of curvature but one are identically equal to zero. Simplicity means that the given irreducible subspace that provides the non-zero piece of curvature is not isomorphic to any other irreducible subspace.

It is the case that there are only three types of pseudoinstantons, namely the following:

1. *scalar* pseudoinstanton, where only the scalar piece of curvature $R^{(1)}$ is not identically zero,
2. *pseudoscalar* pseudoinstanton, where only the pseudoscalar piece of curvature $R_*^{(1)}$ is not identically zero, and
3. *Weyl* pseudoinstanton, where only the Weyl piece of curvature $R^{(10)}$ is not identically zero.

Vassiliev [90] proved the following theorem:

Theorem 1.4.2. *A pseudoinstanton is a solution of the field equations (1.3), (1.4).*

Note that the pseudoinstanton construction is ideologically very similar to the double duality ansatz of [5, 58], as the curvature of a pseudoinstanton is an eigenvector of the double duality map, $*R* = \pm R$. However, unlike [5, 58], Vassiliev dealt with the most general case where the quadratic form contains 16 R^2 terms.

Of course, a pseudoinstanton need not necessarily be non-Riemannian, but as the uniqueness result for Riemannian solutions was already presented in Section 1.4.1, we are primarily interested in non-Riemannian pseudoinstantons. The pseudoinstanton technique is in any case limited in the sense that it does not provide all the Riemannian solutions, as an Einstein space or a pp-wave need not necessarily be pseudoinstantons.

Vassiliev [90] presented one non-Riemannian pseudoinstanton solution of the system (1.3), (1.4) and it is constructed in the following way. Consider the trivial manifold $M = \mathbb{R}^4$ equipped with global coordinates (x^0, x^1, x^2, x^3) and Minkowski metric $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. Let $l \neq 0$ be a constant real vector, $m \neq 0$ a constant complex vector, and let

$$A(x) = m e^{-il \cdot x} \tag{1.7}$$

be a plane wave solution of the polarized Maxwell equation (3.15). Define torsion as $T := \frac{1}{2} \text{Re}(A \otimes dA)$, and let Γ be the corresponding metric-compatible connection.

Then, as shown in [87], the above described spacetime is a Weyl pseudoinstanton, and hence by Theorem 1.4.2 a solution of the field equations (1.3), (1.4).

For the Yang–Mills case (1.5) the ‘torsion wave’ solution described above was first obtained by Singh and Griffiths: see last paragraph of Section 5 in [81] and put $k = 0$, $N = e^{-il \cdot x}$ and the same solution was later rediscovered by King and Vassiliev in [49]. It should be pointed out that the torsion wave solution of King and Vassiliev is a highly specialised version of the solution obtained by Singh and Griffiths [81], which is a solution of algebraic type III, where the Riemannian spacetime is a Kundt plane-fronted gravitational wave and the contortion is purely tensor. Vassiliev’s contribution in [90] was to show that these spacetimes satisfy equations (1.3), (1.4) in the most general case of the purely quadratic action (1.2).

This work of Vassiliev went on to conclude that this torsion wave was a non-Riemannian analogue of a pp-wave, whence came the motivation for generalising the notion of a pp-wave to spacetimes with torsion in such a way as to incorporate this non-Riemannian pseudoinstanton into the construction, which led to the work presented in Sections 3.2 and 4.2.

In view of the work presented in [81], the two papers of Singh [78, 79] are also of interest to us. In [78] Singh presents solutions of the vacuum field equations with *purely axial torsion*, which is a class of solutions unobtainable by the double duality ansatz of [5, 58]. In fact Singh uses the ‘spin coefficient technique’ of [81] in the construction of his solutions.

In the second paper [79], Singh similarly uses the spin coefficient technique of [81] to construct solutions unobtainable by the double duality ansatz, but this time that have *purely trace torsion*. These solutions are similar in many ways, as the metric and the hence the Riemannian pieces of curvature are the same - which leads the author to stipulate that it might be possible to combine these two solutions but he however shows that this is unfortunately not possible.

It should be pointed out that in [78, 79] Singh was not working within the setting of the most general 16-term purely quadratic action and the solutions were obtained for the Yang–Mills case (1.5).

Non-metric-compatible non-Riemannian solutions

In this subsection we give a very brief overview of known solutions in metric-affine gravity where the spacetime is not metric-compatible, i.e. nonmetricity (2.5) appears ($\nabla g \neq 0$). Note that in this thesis we only deal with metric-compatible spacetimes.

The first set of solutions we view in this subsection arises from the *triplet ansatz*, see [43, 64]. The main difference between the triplet approach and the approach in this thesis is that the triplet ansatz uses only a special form of the Lagrangian, namely it is only applicable to the those quadratic forms on curvature where only one non-trivial coupling constant is allowed to be non-zero.

As Obukhov states in [67], the triplet ansatz might be considered a useful tool which helps avoid a possible ill-posedness of the field equations by reducing them to the effective Einstein-Maxwell system of equations.

According to [64], the triplet ansatz uses the irreducible decompositions of torsion and nonmetricity to provide a pattern for the decomposition of the gravitational gauge field momenta which enter the field equations of metric-affine gravity.

Obukhov et al. demonstrated in the equivalence theorem of Obukhov [64] that a triplet of torsion and nonmetricity 1-forms describes the general and unique vacuum solutions of the field equations of quadratic metric-affine gravity (under certain assumptions).

For details about this construction and results related to the triplet ansatz see [20, 28, 43, 64, 65, 92].

One other and more recent non-metric-compatible result comes from Obukhov [67] which is a result that does not fall into the triplet ansatz. The quadratic form on curvature considered is the most general, and identical to the 16-term quadratic form used in this thesis and in [69, 89, 90], see Section 2.4. However, unlike the solutions presented in these works, Obukhov constructs new solutions that have non-zero nonmetricity, which are generalisations of pp-waves. Obukhov presents solutions that have not only torsion waves present but the nonmetricity has a non-trivial wave behaviour as well, which is different from the generalised pp-waves presented in this thesis. Moreover, Obukhov suggests that his solutions provide a minimal generalisation of the pseudoinstanton, see Definition 1.4.1. However, it should be pointed out that solutions presented in [67] are *not* non-metric-compatible generalisations of solutions presented in this thesis, but we hope to be able to respond to this work of Obukhov's in the near future.

The class of solutions presented in [67] is discussed in more detail in Section 5.3, especially in view of comparing them to the result given in this thesis.

1.5 Other alternative theories of gravity

There are many other alternative theories of gravity apart from metric–affine gravity. We do not go into much detail in describing these other alternative theories of gravity, as none of these theories are used in this thesis – which is not to say that we are not interested in trying to employ them in advancing our study in the immediate future, especially with regard to teleparallelism. Hence this section provides only the basic information about these theories and the main references concerning their development and some recent results in the field.

General relativity was not the first classical relativistic field theory of gravitation, or even the first metric theory of gravitation – that was Nordström’s theory of gravitation, proposed by the Finnish theoretical physicist Gunnar Nordström in 1912–1913. In fact, there are two theories of Nordström’s – the first one was quickly dismissed, but the second became the first known example of a metric theory of gravitation. See [62, 63] for Nordström’s work. Both theories are now known to be incompatible with observation.

One very interesting alternative theory of gravity is *teleparallelism*, a theory initially used by Einstein to try to unify electromagnetism and gravity². The subject of teleparallelism has a long history dating back to the 1920s. Its origins lie in the pioneering works of Élie Cartan, Albert Einstein and Roland Weitzenböck. Modern reviews of the physics of teleparallelism are given in [38, 44, 60, 76].

The basic idea of teleparallelism is to work with a Lorentzian metric, vanishing curvature and *non-vanishing torsion*, so it could be viewed as a special case of metric–affine gravity. However, in practice instead of using the metric as the unknown of this theory, one uses a quartet of covectors (a *coframe*).

An interesting recent result in teleparallel gravity related to the result of this thesis can be found in [91], where a new (teleparallel) representation for the Weyl Lagrangian is given. The advantage of the teleparallel approach is that it does not require the use of spinors, Pauli matrices or covariant differentiation. The only geometric concepts used are those of a metric, differential form, wedge product and exterior derivative.

²English translations of Einstein’s original papers on teleparallelism are now available, see [86]

One other alternative theory of gravity, quite popular today, is $f(\mathcal{R})$ modified gravity, which works with Lagrangians which are functions of scalar curvature \mathcal{R} . This theory was originally suggested by H. A. Buchdahl [15] and its further immediate development can be found in [6, 21, 48]. This particular alternative theory of gravity is quite popular today and especially after the publication by Carroll et al. [16] it became quite topical. An interesting recent publication on the topic by Bertolami et al. [8] derives the equation of motion for test particles in $f(\mathcal{R})$ modified theories of gravity and shows that in a coupling between an arbitrary function of scalar curvature and the Lagrangian density of matter an extra force arises. It goes on to discuss the connections with modified Newtonian dynamics (MOND) and the Pioneer anomaly.

1.6 Structure of the thesis

This thesis is divided into several chapters:

- Chapter 2 presents the background and notation employed in this thesis. Section 2.1 presents the notation used throughout the thesis and some basic definitions, Section 2.2 describes the irreducible pieces of torsion, Section 2.3 the irreducible pieces of curvature, Section 2.4 the quadratic forms on curvature, while Section 2.5 gives the spinor formalism used in the main body of the thesis. Finally, Section 2.7 gives a brief explanation of the main result of the thesis, working with a particular choice of local coordinates in which all of the practical calculations have been performed.
- Chapter 3 deals in its entirety with pp-waves and is divided into several Sections. In Section 3.1 we recall the notion of the classical pp-wave (without torsion) and list the basic properties of these spacetimes. Section 3.2 defines the notion of a generalised pp-wave (with torsion) and lists the main properties of these new metric-compatible spacetimes. At the end of the Chapter, in Section 3.3 we give the specifics of the spinor formalism for generalised pp-waves
- In Chapter 4 we use generalised pp-waves described in the Section 3.2 to present a class of new solutions for quadratic metric-affine gravity. In Section 4.1 we write down explicitly our field equations (1.3), (1.4) and in Section 4.2 we present pp-wave type solutions of these equations. Section 4.2 contains the proof of Theorem 2.7.1 which is the main result of the thesis.

- Chapter 5 is the discussion of the results obtained in Chapters 3 and 4. In Section 5.1 we give the physical interpretation of our solutions, i.e. we attempt to understand the mathematical and physical significance of generalised pp-waves with parallel Ricci curvature. In Section 5.2 we explore this matter further by explicitly constructing pp-wave type solutions of Einstein–Weyl theory, a known classical model describing the interaction of gravitational and massless neutrino fields, and we compare our new metric–affine solutions to the Einstein–Weyl solutions. We also provide references to earlier work by several authors on this subject. In Section 5.3 we compare our solutions to similar results in existing literature.
- Finally, Appendices provide some auxiliary mathematical facts, but also a lot of original work that enabled the construction of the results obtained. Appendix A gives the brief derivation of the Bianchi identity for curvature, Appendix B gives details of the variations of Weyl’s action and finally Appendix C gives the variations of quadratic forms on curvature used in this thesis.

Chapter 2

Notation and Background

This chapter provides the notation and the description of the main structures used in the thesis, in particular regarding the irreducible pieces of torsion and curvature, quadratic forms on curvature and the spinor formalism. At the end of the chapter, after the introduction of all the relevant material, we present the main result of the thesis in a simplified form.

2.1 Notation and basic definitions

This introductory Section provides the notation used in this thesis and some basic definitions. Our notation follows [49, 69, 87, 90].

In particular, we denote local coordinates by x^μ , $\mu = 0, 1, 2, 3$, and write $\partial_\mu := \partial/\partial x^\mu$.

We define the covariant derivative of a vector field as

$$\nabla_\mu v^\lambda := \partial_\mu v^\lambda + \Gamma^\lambda_{\mu\nu} v^\nu. \quad (2.1)$$

Torsion is defined to be as

$$T^\lambda_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \quad (2.2)$$

We define contortion as

$$K^\lambda_{\mu\nu} := \frac{1}{2}(T^\lambda_{\mu\nu} + T_\mu{}^\lambda{}_\nu + T_\nu{}^\lambda{}_\mu) \quad (2.3)$$

(see formula (7.35) in [61]). It is easy to see that contortion has the antisymmetry property $K_{\lambda\mu\nu} = -K_{\nu\mu\lambda}$ and that

$$T^\lambda_{\mu\nu} = K^\lambda_{\mu\nu} - K^\lambda_{\nu\mu}. \quad (2.4)$$

Formulae (2.3), (2.4) allow us to express torsion and contortion via one another.

We say that our connection Γ is *metric compatible* if $\nabla g \equiv 0$.

We define *nonmetricity* Q by

$$Q_{\mu\alpha\beta} := \nabla_\mu g_{\alpha\beta}. \quad (2.5)$$

The Christoffel symbol is

$$\left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} := \frac{1}{2} g^{\lambda\kappa} (\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}). \quad (2.6)$$

We denote by $\{\nabla\}$ the covariant derivative with respect to the Levi-Civita connection, i.e.

$$\{\nabla\}_\mu v^\lambda := \partial_\mu v^\lambda + \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} v^\nu, \quad (2.7)$$

so our convention is to use the curly brackets for this purpose.

The interval is $ds^2 := g_{\mu\nu} dx^\mu dx^\nu$.

We define curvature as

$$R^\kappa_{\lambda\mu\nu} := \partial_\mu \Gamma^\kappa_{\nu\lambda} - \partial_\nu \Gamma^\kappa_{\mu\lambda} + \Gamma^\kappa_{\mu\eta} \Gamma^\eta_{\nu\lambda} - \Gamma^\kappa_{\nu\eta} \Gamma^\eta_{\mu\lambda}, \quad (2.8)$$

Ricci curvature as

$$Ric_{\lambda\nu} := R^\kappa_{\lambda\kappa\nu}, \quad (2.9)$$

scalar curvature as

$$\mathcal{R} := Ric^\lambda_{\lambda} \quad (2.10)$$

and trace-free Ricci curvature as

$$Ric := Ric - \frac{1}{4} \mathcal{R} g. \quad (2.11)$$

We denote Weyl curvature by \mathcal{W} ; here, as in [69, 87, 90], Weyl curvature is understood

as the irreducible piece of curvature defined by conditions

$$R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}, \quad (2.12)$$

$$\varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0, \quad (2.13)$$

$$Ric = 0. \quad (2.14)$$

We employ the standard convention of raising and lowering tensor indices by means of the metric tensor, i.e.

$$g_{\alpha\beta}v^\beta = v_\alpha, \quad g^{\alpha\beta}v_\beta = v^\alpha.$$

Some care is, however, required when performing covariant differentiation: the operations of raising and lowering of indices do not commute with the operation of covariant differentiation unless the connection is metric compatible.

Given a scalar function $f : M \rightarrow \mathbb{R}$ we write for brevity

$$\int f := \int_M f \sqrt{|\det g|} dx^0 dx^1 dx^2 dx^3, \quad \det g := \det(g_{\mu\nu}). \quad (2.15)$$

A spacetime $\{M, g, \Gamma\}$ is called *Riemannian* if the connection is Levi–Civita (i.e. $\Gamma^\lambda_{\mu\nu} = \left\{ \begin{smallmatrix} \lambda \\ \mu\nu \end{smallmatrix} \right\}$), and *non-Riemannian* otherwise, as already stated in the Introduction.

Definition 2.1.1. An *Einstein space* is a Riemannian spacetime with $Ric = \Lambda g$ where Λ is some real constant.

We define the action of the Hodge star on a rank q antisymmetric tensor as

$$(*Q)_{\mu_{q+1}\dots\mu_4} := (q!)^{-1} \sqrt{|\det g|} Q^{\mu_1\dots\mu_q} \varepsilon_{\mu_1\dots\mu_4}, \quad (2.16)$$

where ε is the totally antisymmetric quantity, $\varepsilon_{0123} := +1$.

When we apply the Hodge star to curvature we have a choice between acting either on the first or the second pair of indices, so we introduce two different Hodge stars: the left Hodge star

$$(*R)_{\kappa\lambda\mu\nu} := \frac{1}{2} \sqrt{|\det g|} R^{\kappa'\lambda'}_{\mu\nu} \varepsilon_{\kappa'\lambda'\kappa\lambda} \quad (2.17)$$

and the right Hodge star

$$(R^*)_{\kappa\lambda\mu\nu} := \frac{1}{2} \sqrt{|\det g|} R_{\kappa\lambda}{}^{\mu'\nu'} \varepsilon_{\mu'\nu'\mu\nu}. \quad (2.18)$$

Note that the star in (2.16) is inline, whereas in (2.17), (2.18) it comes as a superscript. The right Hodge star (2.18) is the Hodge star normally used in Yang–Mills theory and in relevant literature it is usually written inline to the left of curvature. As to the left Hodge star (2.17), it is important to observe that in the general metric–affine setting curvature is not antisymmetric in the first pair of indices so the linear map $R \rightarrow *R$ is not an automorphism of the vector space of curvatures. Hence, use of the left Hodge star really makes sense only in the metric compatible setting when we are guaranteed antisymmetry in the first pair of indices.

We use the term ‘parallel’ to describe the situation when the covariant derivative of some spinor or tensor field is identically zero.

We do not assume that our spacetime admits a (global) spin structure, cf. Section 11.6 of [61]. In fact, our only topological assumption is connectedness. This does not prevent us from defining and parallel transporting spinors or tensors locally.

2.2 Irreducible Pieces of Torsion

This Section gives the explanation of the irreducible pieces of torsion. Torsion as an extension to Einstein’s version of general relativity was introduced by Élie Cartan in 1922. He recognized that the torsion was characterized by a tensor and developed a differential geometric formulation, see [17]. His idea was that the torsion of spacetime might be connected to the intrinsic angular momentum (spin) of matter. As such, he believed that it should vanish in matter-free regions of spacetime.

In [45] Hehl and Obukhov give a review of the use of torsion in field theory in which they state that Cartan’s investigations started by analyzing Einstein’s general relativity theory and by taking recourse to the theory of Cosserat continua.

Our notation in this Section differs from the more modern exposition, as given for example in Appendix B.2 from [42]. We use holonomic notation so for the sake of convenience we present everything in this form, hence this Section follows the exposition from Appendix C from [87]. According to [42, 87] the irreducible pieces of

torsion are

$$T^{(1)} = T - T^{(2)} - T^{(3)}, \quad (2.19)$$

$$T^{(2)}_{\lambda\mu\nu} = g_{\lambda\mu}v_\nu - g_{\lambda\nu}v_\mu, \quad (2.20)$$

$$T^{(3)} = *w, \quad (2.21)$$

where

$$v_\nu = \frac{1}{3}T^\lambda{}_{\lambda\nu}, \quad w_\nu = \frac{1}{6}\sqrt{|\det g|}T^{\kappa\lambda\mu}\varepsilon_{\kappa\lambda\mu\nu}. \quad (2.22)$$

The pieces $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ are called *tensor torsion*, *trace torsion*, and *axial torsion* respectively.

We define the action of the Hodge star on torsions as

$$(*T)_{\lambda\mu\nu} := \frac{1}{2}\sqrt{|\det g|}T_{\lambda\mu'\nu'}\varepsilon^{\mu'\nu'}{}_{\mu\nu}. \quad (2.23)$$

The Hodge star maps tensor torsions to tensor torsions, trace to axial, and axial to trace:

$$(*T)^{(1)} = *(T^{(1)}), \quad (2.24)$$

$$(*T)^{(2)}_{\lambda\mu\nu} = g_{\lambda\mu}w_\nu - g_{\lambda\nu}w_\mu, \quad (2.25)$$

$$(*T)^{(3)} = -*v. \quad (2.26)$$

Note that the $*$ appearing in the RHS's of formulae (2.21) and (2.26) is the standard Hodge star (2.16) which should not be confused with the Hodge star on torsions (2.23).

The decomposition described above assumes torsion to be real and metric to be Lorentzian. If torsion is complex or if $\det g > 0$ then the subspace of tensor torsions decomposes further into eigenspaces of the Hodge star.

Substituting formulae (2.19)–(2.21) into formula (2.3), and formula (2.4) into formulae (2.22) we obtain the irreducible decomposition of contortion:

$$K^{(1)} = K - K^{(2)} - K^{(3)}, \quad (2.27)$$

$$K^{(2)}_{\lambda\mu\nu} = g_{\lambda\mu}v_\nu - g_{\nu\mu}v_\lambda, \quad (2.28)$$

$$K^{(3)} = \frac{1}{2}*w, \quad (2.29)$$

where

$$v_\nu = \frac{1}{3} K^\lambda{}_{\lambda\nu}, \quad w_\nu = \frac{1}{3} \sqrt{|\det g|} K^{\kappa\lambda\mu} \varepsilon_{\kappa\lambda\mu\nu}. \quad (2.30)$$

The irreducible pieces of torsion (2.19)–(2.21) and contortion (2.27)–(2.29) are related as

$$T^{(j)}{}_{\lambda\mu\nu} = K^{(j)}{}_{\mu\lambda\nu}, \quad j = 1, 2, \quad T^{(3)}{}_{\lambda\mu\nu} = 2K^{(3)}{}_{\lambda\mu\nu}$$

(note the order of indices).

2.3 Irreducible Pieces of Curvature

This Section provides facts about the irreducible pieces of curvature, and it follows the notation and exposition from [90].

A curvature generated by a general affine connection has only one (anti)symmetry, namely,

$$R^\kappa{}_{\lambda\mu\nu} = -R^\kappa{}_{\lambda\nu\mu}. \quad (2.31)$$

For a fixed $x \in M$ we denote by \mathbf{R} the 96-dimensional vector space of real rank 4 tensors $R^\kappa{}_{\lambda\mu\nu}$ satisfying condition (2.31).

Let g be the Lorentzian metric at the point $x \in M$ and let $O(1, 3)$ be the corresponding full Lorentz group, i.e. the group of linear transformations of coordinates in the tangent space $T_x M$ which preserve the metric. It is known, see Appendix B.4 from [42], that the vector space \mathbf{R} decomposes into a direct sum of 11 subspaces which are invariant and irreducible under the action of $O(1, 3)$. These subspaces are listed in Table 2.1. Note that our notation differs from that of [42]: we want to emphasize the fact that there are 3 groups of isomorphic subspaces, namely,

$$\{\mathbf{R}^{(6,l)}, l = 1, 2, 3\}, \quad \{\mathbf{R}^{(9,l)}, l = 1, 2\}, \quad \{\mathbf{R}_*^{(9,l)}, l = 1, 2\}. \quad (2.32)$$

Two subspaces are said to be isomorphic if there is a linear bijection between them which commutes with the action of $O(1, 3)$.

In what follows we lower and raise tensor indices using the metric and we also use the right Hodge star (2.18). The linear map $R \rightarrow R^*$ is an automorphism of \mathbf{R} ; note that as we are working in the real Lorentzian setting this map has *no* eigenvalues.

The explicit description of irreducible subspaces of dimension < 10 is given in Table 2.2. Here \mathcal{R} , \mathcal{R}_* are arbitrary scalars, $\mathcal{A}^{(l)}$ are arbitrary rank 2 antisymmetric

Table 2.1: List of irreducible subspaces

Dimension	Number of subspaces	Notation for subspaces
1	2	$\mathbf{R}^{(1)}, \mathbf{R}_*^{(1)}$
6	3	$\mathbf{R}^{(6,l)}, l = 1, 2, 3$
9	4	$\mathbf{R}^{(9,l)}, \mathbf{R}_*^{(9,l)}, l = 1, 2$
10	1	$\mathbf{R}^{(10)}$
30	1	$\mathbf{R}^{(30)}$

tensors, and $\mathcal{S}^{(l)}, \mathcal{S}_*^{(l)}$ are arbitrary rank 2 symmetric trace-free tensors, with ‘arbitrary’ meaning that the quantity in question spans its vector space. The a ’s in Table 2.2 are some fixed real constants, the only condition being that $a_1, a_1^*, \det(a_{6lm})_{l,m=1}^3, \det(a_{9lm})_{l,m=1}^2$ and $\det(a_{9lm}^*)_{l,m=1}^2$ are non-zero. The freedom in choosing irreducible subspaces of dimension 6 and 9 is due to the fact that we have groups of isomorphic subspaces (2.32).

Table 2.2: Explicit description of irreducible subspaces of dimension < 10

Subspace	Formula for curvature R
$\mathbf{R}^{(1)}$	$R_{\kappa\lambda\mu\nu} = a_1(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu})\mathcal{R}$
$\mathbf{R}_*^{(1)}$	$(R^*)_{\kappa\lambda\mu\nu} = a_1^*(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu})\mathcal{R}_*$
$\mathbf{R}^{(6,l)}$	$R_{\kappa\lambda\mu\nu} = a_{6l1}(g_{\kappa\mu}\mathcal{A}^{(l)}_{\lambda\nu} - g_{\kappa\nu}\mathcal{A}^{(l)}_{\lambda\mu}) + a_{6l2}(g_{\lambda\mu}\mathcal{A}^{(l)}_{\kappa\nu} - g_{\lambda\nu}\mathcal{A}^{(l)}_{\kappa\mu}) + a_{6l3}g_{\kappa\lambda}\mathcal{A}^{(l)}_{\mu\nu}$
$\mathbf{R}^{(9,l)}$	$R_{\kappa\lambda\mu\nu} = a_{9l1}(g_{\kappa\mu}\mathcal{S}^{(l)}_{\lambda\nu} - g_{\kappa\nu}\mathcal{S}^{(l)}_{\lambda\mu}) + a_{9l2}(g_{\lambda\mu}\mathcal{S}^{(l)}_{\kappa\nu} - g_{\lambda\nu}\mathcal{S}^{(l)}_{\kappa\mu})$
$\mathbf{R}_*^{(9,l)}$	$(R^*)_{\kappa\lambda\mu\nu} = a_{9l1}^*(g_{\kappa\mu}\mathcal{S}_*^{(l)}_{\lambda\nu} - g_{\kappa\nu}\mathcal{S}_*^{(l)}_{\lambda\mu}) + a_{9l2}^*(g_{\lambda\mu}\mathcal{S}_*^{(l)}_{\kappa\nu} - g_{\lambda\nu}\mathcal{S}_*^{(l)}_{\kappa\mu})$

It is convenient to choose the following a ’s:

$$a_1 = a_1^* = \frac{1}{12}, \quad (a_{6lm}) = \begin{pmatrix} \frac{5}{12} & -\frac{1}{12} & -\frac{1}{6} \\ -\frac{1}{12} & \frac{5}{12} & -\frac{1}{6} \\ -\frac{1}{12} & -\frac{1}{12} & \frac{1}{3} \end{pmatrix},$$

$$(a_{9lm}) = (a_{9lm}^*) = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix}. \quad (2.33)$$

Then the lower rank tensors \mathcal{R} , \mathcal{R}_* , $\mathcal{A}^{(l)}$, $\mathcal{S}^{(l)}$, $\mathcal{S}_*^{(l)}$ appearing in Table 2.2 are expressed via the full (rank 4) curvature tensor R according to the following simple formulae:

$$\begin{aligned}
\mathcal{R} &:= R^{\kappa\lambda}{}_{\kappa\lambda}, \\
Ric^{(1)}{}_{\lambda\nu} &:= R^\kappa{}_{\lambda\kappa\nu}, & Ric^{(2)}{}_{\kappa\nu} &:= R_\kappa{}^\lambda{}_{\lambda\nu}, \\
Ric^{(1)} &:= Ric^{(1)} - \frac{1}{4}\mathcal{R}g, & Ric^{(2)} &:= Ric^{(2)} + \frac{1}{4}\mathcal{R}g, \\
\mathcal{S}^{(l)}{}_{\mu\nu} &:= \frac{Ric^{(l)}{}_{\mu\nu} + Ric^{(l)}{}_{\nu\mu}}{2}, & \mathcal{A}^{(l)}{}_{\mu\nu} &:= \frac{Ric^{(l)}{}_{\mu\nu} - Ric^{(l)}{}_{\nu\mu}}{2}, \quad l = 1, 2, \\
\mathcal{A}^{(3)}{}_{\mu\nu} &:= R^\kappa{}_{\kappa\mu\nu}, \quad (2.34)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}_* &:= (R^*)^{\kappa\lambda}{}_{\kappa\lambda}, \\
Ric_*^{(1)}{}_{\lambda\nu} &:= (R^*)^\kappa{}_{\lambda\kappa\nu}, & Ric_*^{(2)}{}_{\kappa\nu} &:= (R^*)_\kappa{}^\lambda{}_{\lambda\nu}, \\
Ric_*^{(1)} &:= Ric_*^{(1)} - \frac{1}{4}\mathcal{R}_*g, & Ric_*^{(2)} &:= Ric_*^{(2)} + \frac{1}{4}\mathcal{R}_*g, \\
\mathcal{S}_*^{(l)}{}_{\mu\nu} &:= \frac{Ric_*^{(l)}{}_{\mu\nu} + Ric_*^{(l)}{}_{\nu\mu}}{2}, & \mathcal{A}_*^{(l)}{}_{\mu\nu} &:= \frac{Ric_*^{(l)}{}_{\mu\nu} - Ric_*^{(l)}{}_{\nu\mu}}{2}, \quad l = 1, 2, \\
\mathcal{A}_*^{(3)}{}_{\mu\nu} &:= (R^*)^\kappa{}_{\kappa\mu\nu}. \quad (2.35)
\end{aligned}$$

Note that the tensors $\mathcal{A}_*^{(l)}$ are not used in Table 2.2. This is not surprising as the tensors $\mathcal{A}^{(l)}$ and $\mathcal{A}_*^{(l)}$ are not independent: the $\mathcal{A}^{(l)}$ are linear combinations of the Hodge duals of $\mathcal{A}_*^{(l)}$ and vice versa. Also note that the tensor $Ric^{(1)}$ is equivalent to the usual Ricci tensor Ric , as given in equation (2.9).

Finally, let us give an explicit description of the 10- and 30-dimensional irreducible subspaces. $\mathbf{R}^{(10)}$ is the subspace of curvatures R such that

$$R^\kappa{}_{\lambda\kappa\nu} = (R^*)^\kappa{}_{\lambda\kappa\nu} = 0, \quad R_\kappa{}^\lambda{}_{\lambda\nu} = (R^*)_\kappa{}^\lambda{}_{\lambda\nu} = 0, \quad R^\kappa{}_{\kappa\mu\nu} = 0 \quad (2.36)$$

(all possible traces are zero) and $R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}$. $\mathbf{R}^{(30)}$ is the subspace of curvatures R satisfying (2.36) and $R_{\kappa\lambda\mu\nu} = R_{\lambda\kappa\mu\nu}$.

Given a decomposition

$$\mathbf{R} = \mathbf{R}^{(1)} \oplus \mathbf{R}_*^{(1)} \oplus_{l=1}^3 \mathbf{R}^{(6,l)} \oplus_{l=1}^2 \mathbf{R}^{(9,l)} \oplus_{l=1}^2 \mathbf{R}_*^{(9,l)} \oplus \mathbf{R}^{(10)} \oplus \mathbf{R}^{(30)}$$

any $R \in \mathbf{R}$ can be uniquely written as

$$R = R^{(1)} + R_*^{(1)} + \sum_{l=1}^3 R^{(6,l)} + \sum_{l=1}^2 R^{(9,l)} + \sum_{l=1}^2 R_*^{(9,l)} + R^{(10)} + R^{(30)}$$

where the R 's in the RHS are from the corresponding irreducible subspaces. We will call these R 's the *irreducible pieces of curvature*.

We call the irreducible subspaces $\mathbf{R}^{(1)}$, $\mathbf{R}_*^{(1)}$, $\mathbf{R}^{(10)}$, $\mathbf{R}^{(30)}$ *simple* because they are not isomorphic to any other subspaces. Accordingly, we call the irreducible pieces $R^{(1)}$, $R_*^{(1)}$, $R^{(10)}$, $R^{(30)}$ *simple*.

Remark 2.3.1. ‘Starred’ and ‘unstarred’ subspaces of same dimension are not isomorphic under the action of the group $O(1, 3)$ because the Hodge star is a linear map which depends on the choice of the element of the group. This dependence is encoded in the normalisation of the totally antisymmetric quantity: $\varepsilon_{0123} = +1$ or $\varepsilon_{0123} = -1$ depending on whether the orientation of the coordinate system is positive or negative.

Remark 2.3.2. The global definition of the Hodge star requires the orientability of our manifold M . However, for the purpose of decomposing curvatures orientability is not needed: any abstract vector subspace is preserved under inversion (vector \mapsto $-\text{vector}$), so when writing explicit formulae for subspaces it does not matter whether $\varepsilon_{0123} = +1$ or $\varepsilon_{0123} = -1$. The delicate features of the Hodge star come to light only when we examine the relationship between *pairs* of different subspaces, see Remark 2.3.1.

Remark 2.3.3. If we complexify our problem then our 11 subspaces will still remain irreducible under the action of $O(1, 3)$. In order to justify this claim we argue as follows. Replace the full Lorentz group $O(1, 3)$ by the proper orthochronous Lorentz group $SO(1, 3)^\uparrow$. Then we have the standard algorithm (see, for example, Section 1.2 in [13]), for finding irreducible subspaces in terms of spinors. Applying this algorithm we see that the complexified subspaces $\mathbf{R}^{(6,l)}$, $\mathbf{R}^{(10)}$ and $\mathbf{R}^{(30)}$ split into eigenspaces of the right Hodge star (2.18), and these ‘halves’ are the only proper $SO(1, 3)^\uparrow$ -invariant subspaces of the original subspaces. However, the ‘halves’ are not invariant under change of orientation.

2.4 Quadratic Forms on Curvature

This Section provides information about the quadratic forms on curvature, and it follows the notation and exposition from [90]. We provide here the Lagrangian used in obtaining the solutions in this thesis.

Let us define an inner product on rank 2 tensors

$$(K, L) := K_{\mu\nu} L^{\mu\nu}, \quad (2.37)$$

and a Yang–Mills inner product on curvatures

$$(R, Q)_{\text{YM}} := R^\kappa{}_{\lambda\mu\nu} Q^\lambda{}_{\kappa}{}^{\mu\nu}. \quad (2.38)$$

Lemma 2.4.1. *Let $q : \mathbf{R} \rightarrow \mathbb{R}$ be an $O(1, 3)$ -invariant quadratic form on curvature. Then*

$$\begin{aligned} q(R) = & b_1 \mathcal{R}^2 + b_1^* \mathcal{R}_*^2 \\ & + \sum_{l,m=1}^3 b_{6lm}(\mathcal{A}^{(l)}, \mathcal{A}^{(m)}) + \sum_{l,m=1}^2 b_{9lm}(\mathcal{S}^{(l)}, \mathcal{S}^{(m)}) + \sum_{l,m=1}^2 b_{9lm}^*(\mathcal{S}_*^{(l)}, \mathcal{S}_*^{(m)}) \\ & + b_{10}(R^{(10)}, R^{(10)})_{\text{YM}} + b_{30}(R^{(30)}, R^{(30)})_{\text{YM}} \end{aligned} \quad (2.39)$$

with some real constants $b_1, b_1^*, b_{6lm} = b_{6ml}, b_{9lm} = b_{9ml}, b_{9lm}^* = b_{9ml}^*, b_{10}, b_{30}$. Here $\mathcal{R}, \mathcal{R}_*, \mathcal{A}^{(l)}, \mathcal{S}^{(l)}, \mathcal{S}_*^{(l)}, R^{(10)}, R^{(30)}$ are tensors defined in Section 2.3.

Proof. Let us equip each of the 11 irreducible subspaces of curvature, see Table 2.1, with an $O(1, 3)$ -invariant non-degenerate inner product. For 1-dimensional subspaces we employ the usual multiplication of scalars, and for 6- and 9-dimensional subspaces we employ (2.37); here scalars and rank 2 tensors are related to irreducible pieces of curvature in accordance with Table 2.2. Note that these inner products are well defined even if the manifold M is non-orientable: the fact that in a ‘starred’ subspace our scalar or rank 2 tensor may be defined up to sign has no bearing on the inner product because both entries in the inner product would simultaneously retain or change sign upon continuation over a loop in M . See also Remark 2.3.2. Our inner products on 1-, 6- and 9-dimensional subspaces are clearly non-degenerate.

For 10- and 30-dimensional subspaces we employ the inner product (2.38). It is

not *a priori* clear that this inner product is non-degenerate on these subspaces. We establish non-degeneracy as follows. The inner product (2.38) is clearly non-degenerate on the whole vector space \mathbf{R} . It is easy to check that any pair of non-isomorphic subspaces is orthogonal with respect to the inner product (2.38), so \mathbf{R} decomposes into a direct sum of 7 orthogonal subspaces

$$\mathbf{R}^{(1)}, \quad \mathbf{R}_*^{(1)}, \quad \bigoplus_{l=1}^3 \mathbf{R}^{(6,l)}, \quad \bigoplus_{l=1}^2 \mathbf{R}^{(9,l)}, \quad \bigoplus_{l=1}^2 \mathbf{R}_*^{(9,l)}, \quad \mathbf{R}^{(10)}, \quad \mathbf{R}^{(30)}.$$

Hence, the inner product (2.38) is non-degenerate on each of these 7 subspaces. In particular, it is non-degenerate on $\mathbf{R}^{(10)}$ and $\mathbf{R}^{(30)}$.

Further on in the proof we deal with the bilinear form $b : \mathbf{R} \times \mathbf{R} \rightarrow \mathbb{R}$ associated with the quadratic form q , i.e. $q(R) = b(R, R)$.

Let V and W be irreducible subspaces of \mathbf{R} and let $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_W$ be their $O(1, 3)$ -invariant non-degenerate inner products. Here V and W are not necessarily distinct. Consider the $O(1, 3)$ -invariant bilinear form $b_{VW} := b|_{V \times W}$. Then there is a unique linear map $B_{VW} : V \rightarrow W$ such that

$$(B_{VW}v, w)_W = b_{VW}(v, w), \quad \forall v \in V, \quad \forall w \in W,$$

and this map commutes with the action of $O(1, 3)$. By Schur's lemma B_{VW} is either zero or a bijection, in which case V and W are isomorphic. Thus, only pairs of isomorphic irreducible subspaces can give non-zero contributions to the bilinear form b .

The proof of Lemma 2.4.1 has been reduced to proving the following fact: if V is an irreducible subspace of \mathbf{R} and $B_V : V \rightarrow V$ is a linear operator which commutes with the action of $O(1, 3)$ then B_V is a multiple of the identity map. In order to prove this fact we complexify our problem, noting that by Remark 2.3.3 this does not affect the irreducibility of V . After complexification the fact we are proving becomes a special case of a well known abstract result. ■

Formula (2.39) in different (anholonomic) notation was first established in [25], [43].

2.5 Spinor Formalism

This Section provides the spinor formalism used throughout the thesis. Unless otherwise stated, we work in a general metric compatible spacetime with torsion.

When introducing our spinor formalism, we were faced with the problem that there doesn't seem to exist a uniform convention in the existing literature on how to treat spinors. Optimally, we would have wanted to achieve the following:

- (i) consecutive raising and lowering of a spinor index does not change the sign of a rank 1 spinor;
- (ii) the metric spinor ϵ^{ab} is the raised version of ϵ_{ab} and vice versa;
- (iii) the spinor inner product is invariant under raising and lowering of indices, i.e.

$$\xi_a \eta^a = \xi^a \eta_a.$$

Unfortunately, it becomes clear that it is not possible to satisfy all three desired properties, as shown in [75]. This inconsistency is related to the well known fact (see for example Section 19 in [7] or Section 3–5 in [83]), that a spinor does not have a particular sign – for example, a spatial rotation of the coordinate system by 2π leads to a change of sign. Also see [73] for more helpful insight about the problem of choice of the spinor formalism.

There are various conventions and different authors defined their spinor formalisms in different ways.

Remark 2.5.1. Pirani [75] notes that in his spinor formalism the spinor inner product is not invariant under raising and lowering of indices, hence property (iii) sacrificed. Namely, in Pirani's spinor formalism one gets that $\zeta_a \eta^a = -\eta_a \zeta^a$. The other properties are satisfied.

Remark 2.5.2. Yet another convention is found in Buchbinder and Kuzenko [13], where properties (ii) and (iii) are not satisfied, due to defining the metric spinor so that $\epsilon_{ab} = -\epsilon^{ab}$. Hence in this formalism ϵ^{ab} is not the raised version of ϵ_{ab} and vice versa and the inner product is not invariant under raising and lowering of indices. However, property (i) is satisfied.

Remark 2.5.3. Griffiths [30, 81] defines his formalism so that only property (i) is sacrificed, i.e. the consecutive raising and lowering of indices leads to a change of sign. The other properties are satisfied.

Remark 2.5.4. Landau and Lifshitz [7] also give their version of spinor formalism, but we are not completely sure what their precise convention is as their textbook does not provide sufficient details.

Remark 2.5.5. Blagojevic [9] follows Pirani’s formalism, hence only property (iii) is not satisfied.

We decided to define our spinor formalism in the following way, as was done in Appendix A from [69].

We define the ‘metric spinor’ as

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.40)$$

with the first index enumerating rows and the second enumerating columns. We raise and lower spinor indices according to the formulae

$$\xi^a = \epsilon^{ab}\xi_b, \quad \xi_a = \epsilon_{ab}\xi^b, \quad \eta^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\eta_{\dot{b}}, \quad \eta_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}\eta^{\dot{b}}. \quad (2.41)$$

Our definition (2.40), (2.41) has the following advantages:

- The spinor inner product is invariant under the operation of raising and lowering of indices, i.e. $(\epsilon_{ac}\xi^c)(\epsilon^{ad}\eta_d) = \xi^a\eta_a$.
- The ‘contravariant’ and ‘covariant’ metric spinors are ‘raised’ and ‘lowered’ versions of each other, i.e. $\epsilon^{ab} = \epsilon^{ac}\epsilon_{cd}\epsilon^{bd}$ and $\epsilon_{ab} = \epsilon_{ac}\epsilon^{cd}\epsilon_{bd}$.

The disadvantage of our definition (2.40), (2.41) is that the consecutive raising and lowering of a single spinor index leads to a change of sign, i.e. $\epsilon_{ab}\epsilon^{bc}\xi_c = -\xi_a$. In formulae where the sign is important we will be careful in specifying our choice of sign; see, for example, (2.42), (2.47). We in a sense intentionally ‘sacrificed’ this property in order to guarantee that the other two properties, which in our view have greater physical significance, are satisfied.

Let \mathfrak{v} be the real vector space of Hermitian 2×2 matrices σ_{ab} . Pauli matrices σ^α_{ab} , $\alpha = 0, 1, 2, 3$, are a basis in \mathfrak{v} satisfying $\sigma^\alpha_{ab}\sigma^{\beta cb} + \sigma^\beta_{ab}\sigma^{\alpha cb} = 2g^{\alpha\beta}\delta_a^c$ where

$$\sigma^{\alpha ab} := \epsilon^{ac}\sigma^\alpha_{cd}\epsilon^{bd}. \quad (2.42)$$

At every point of the manifold M Pauli matrices are defined uniquely up to a Lorentz transformation.

Define

$$\sigma_{\alpha\beta ac} := \frac{1}{2} (\sigma_{\alpha ab} \epsilon^{bd} \sigma_{\beta cd} - \sigma_{\beta ab} \epsilon^{bd} \sigma_{\alpha cd}). \quad (2.43)$$

These ‘second order Pauli matrices’ are polarized, i.e.

$$* \sigma = \pm i \sigma \quad (2.44)$$

depending on the orientation of ‘basic’ Pauli matrices σ^α_{ab} , $\alpha = 0, 1, 2, 3$.

We define the covariant derivatives of spinor fields as

$$\begin{aligned} \nabla_\mu \xi^a &= \partial_\mu \xi^a + \Gamma^a_{\mu b} \xi^b, & \nabla_\mu \xi_a &= \partial_\mu \xi_a - \Gamma^b_{\mu a} \xi_b, \\ \nabla_\mu \eta^{\dot{a}} &= \partial_\mu \eta^{\dot{a}} + \bar{\Gamma}^{\dot{a}}_{\mu b} \eta^{\dot{b}}, & \nabla_\mu \eta_{\dot{a}} &= \partial_\mu \eta_{\dot{a}} - \bar{\Gamma}^{\dot{b}}_{\mu \dot{a}} \eta_{\dot{b}}, \end{aligned}$$

where $\bar{\Gamma}^{\dot{a}}_{\mu \dot{b}} = \overline{\Gamma^a_{\mu b}}$. The explicit formula for the spinor connection coefficients $\Gamma^a_{\mu b}$ can be derived from the following two conditions:

$$\nabla_\mu \epsilon^{ab} = 0, \quad (2.45)$$

$$\nabla_\mu j^\alpha = \sigma^\alpha_{ab} \nabla_\mu \zeta^{ab}, \quad (2.46)$$

where ζ is an arbitrary rank 2 mixed spinor field and $j^\alpha := \sigma^\alpha_{ab} \zeta^{ab}$ is the corresponding vector field (current). Conditions (2.45), (2.46) give a system of linear algebraic equations for $\text{Re } \Gamma^a_{\mu b}$, $\text{Im } \Gamma^a_{\mu b}$ the unique solution of which is

$$\Gamma^a_{\mu b} = \frac{1}{4} \sigma_\alpha^{ac} (\partial_\mu \sigma^\alpha_{bc} + \Gamma^\alpha_{\mu\beta} \sigma^\beta_{bc}). \quad (2.47)$$

See for example section 3 of [31] for more background on covariant differentiation of spinors.

2.6 Pauli matrices in Minkowski space

Here we give the standard choice of Pauli matrices for Minkowski space, a 4-dimensional manifold $M = \mathbb{R}^4$ equipped with global coordinates (x^0, x^1, x^2, x^3) and Minkowski metric $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. The Pauli matrices σ^μ_{ab} in this setting are given

by:

$$\begin{aligned}\sigma^0_{ab} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma^1_{ab} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma^2_{ab} &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, & \sigma^3_{ab} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned}\quad (2.48)$$

We also provide the explicit formulae for the ‘second order Pauli matrices’ (2.43) for the Minkowski metric. Note that as the second order Pauli matrices $\sigma^{\alpha\beta}_{ab}$ are anti-symmetric over the tensor indices, i.e. $\sigma^{\alpha\beta}_{ab} = -\sigma^{\beta\alpha}_{ab}$, we only give the independent non-zero terms.

$$\begin{aligned}\sigma^{01}_{ab} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \sigma^{02}_{ab} &= \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, & \sigma^{03}_{ab} &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \\ \sigma^{12}_{ab} &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & \sigma^{13}_{ab} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & \sigma^{23}_{ab} &= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.\end{aligned}\quad (2.49)$$

2.7 The main result of the thesis

This Section gives a brief explanation of the main result of the thesis. We first need to introduce and briefly describe a new class of spacetimes, namely *generalised pp-waves*. The whole of Chapter 3 is devoted to the properties of pp-waves and their generalisations, where we give a much more detailed description of these spacetimes. In order to give a simplified but more straightforward and ‘visual’ version of the main theorem, in this introductory Section we shall be working with a particular choice of local coordinates in which all of the practical calculations have been performed. These coordinates arise from the following definition of (classical) pp-waves:

A *pp-wave* is a Riemannian spacetime whose metric can be written locally in the form

$$ds^2 = 2 dx^0 dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1, x^2, x^3) (dx^3)^2$$

in some local coordinates (x^0, x^1, x^2, x^3) .

The choice of local coordinates in which the pp-metric assumes the above form is not unique. We choose a particular set of such coordinates in this Section.

It has been discovered recently by Vassiliev [90] that pp-waves of parallel Ricci

curvature are solutions of the system (1.3), (1.4). We want to generalise the concept of these spacetimes to create new solutions of our field equations. One natural way of generalising the notion of a pp-wave is simply to extend it to general metric-compatible spacetimes. However, this would give us a class of spacetimes which is too wide and difficult to work with. We choose to extend the classical definition in a more special way, and in this Section we do it by introducing torsion explicitly.

Let A be a complex vector field defined by

$$A = h(x^3)m + k(x^3)l$$

where l is a parallel null light-like vector and m is a complex isotropic vector field orthogonal to l . We choose the set of local coordinates for which $l^\mu = (1, 0, 0, 0)$ and $m^\mu = (0, 1, \mp i, 0)$. The functions $h, k : \mathbb{R} \rightarrow \mathbb{C}$ are arbitrary.

We can then define a *generalised pp-wave* as a metric-compatible spacetime with pp-metric and torsion

$$T := \frac{1}{2}\text{Re}(A \otimes dA).$$

Torsion can be expressed more explicitly in our local coordinates as

$$T^\alpha{}_{\beta\gamma} = \frac{1}{2}\text{Re} [(k(x^3)h'(x^3)l^\alpha + h(x^3)h'(x^3)m^\alpha) (l \wedge m)_{\beta\gamma}].$$

Torsion is purely tensor (see Lemma 3.2.4) and it has 4 non-zero independent components. The formula for curvature in our local coordinates is

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{2}(l \wedge \partial)_{\alpha\beta}(l \wedge \partial)_{\gamma\delta}f + \frac{1}{4}\text{Re} ((h(x^3)^2)'' (l \wedge m)_{\alpha\beta} (l \wedge m)_{\gamma\delta}).$$

Curvature only has two irreducible pieces, namely symmetric trace-free Ricci and Weyl and it can be written down as

$$R_{\kappa\lambda\mu\nu} = \frac{1}{2}(g_{\kappa\mu}Ric_{\lambda\nu} - g_{\lambda\mu}Ric_{\kappa\nu} + g_{\lambda\nu}Ric_{\kappa\mu} - g_{\kappa\nu}Ric_{\lambda\mu}) + \mathcal{W}_{\kappa\lambda\mu\nu}.$$

The Ricci and Weyl curvatures are given by

$$\begin{aligned} Ric_{\mu\nu} &= \frac{1}{2}(f_{11} + f_{22})l_\mu l_\nu, \\ \mathcal{W}_{\kappa\lambda\mu\nu} &= \sum_{j,k=1}^2 w_{jk}(l \wedge m_j) \otimes (l \wedge m_k), \end{aligned}$$

where $m_1 = \text{Re}(m)$, $m_2 = \text{Im}(m)$, $f_{\alpha\beta} := \partial_\alpha \partial_\beta f$ and w_{jk} are real scalars given by

$$\begin{aligned} w_{11} &= \frac{1}{4}[-f_{11} + f_{22} + \text{Re}((h^2)'')], & w_{22} &= -w_{11}, \\ w_{12} &= \pm \frac{1}{2}f_{12} - \frac{1}{4}\text{Im}((h^2)''), & w_{21} &= w_{12}. \end{aligned}$$

Note that our generalised pp-waves have the same irreducible pieces of curvature as classical pp-waves and that their curvature has all the usual symmetries of curvature in the Riemannian case (see equations (3.25)-(3.28)). For more details on the properties of generalised pp-waves, see Section 3.2.

Now we can state the main result of the thesis:

Theorem 2.7.1. *Generalised pp-waves of parallel Ricci curvature are solutions of the system of equations (1.3), (1.4).*

Note that when using Theorem 2.7.1 it does not really matter whether the condition ‘parallel Ricci curvature’ is understood in the non-Riemannian sense $\nabla Ric = 0$, the Riemannian sense $\{\nabla\}\{Ric\} = 0$, or any combination of the two ($\{\nabla\}Ric = 0$ or $\nabla\{Ric\} = 0$). Here curly brackets refer to the Levi–Civita connection, see equation (2.7).

In special local coordinates, the condition that Ricci curvature is parallel is written as $f_{11} + f_{22} = \text{const}$, where $f_{\alpha\beta} := \partial_\alpha \partial_\beta f$. Hence, generalised pp-waves of parallel Ricci curvature admit a simple explicit description.

The proof of the main theorem is done by ‘brute force’. We write down our field equations (1.3), (1.4) explicitly under certain assumptions on the properties of the spacetime, which generalised pp-waves automatically possess, see Section 4.1. The proof of the theorem is then quite straightforward, as we explicitly show that the field equations are satisfied by inserting the formulae for the irreducible pieces of curvature and torsion of generalised pp-waves.

Chapter 3

PP-waves With Torsion

In this chapter we introduce *pp-waves*, starting with the notion of a classical pp-wave, with historical background and its various mathematical and physical properties. Then we introduce a generalisation with the addition of torsion. Lastly, we deal with the particular spinor formalism of pp-waves.

3.1 Classical pp-waves

PP-waves are well known spacetimes in general relativity, first discovered by Brinkmann [11] in 1923, and subsequently rediscovered by several authors, for example Peres¹ [72] in 1959. They are used in this thesis as the basis for constructing new solutions for quadratic metric–affine gravity. Hence, an introduction to classical pp-waves is required in order to fully understand this construction.

There are differing views on what the ‘pp’ stands for. According to Griffiths [29] and Kramer et al. [50] for example, ‘pp’ is an abbreviation for ‘plane-fronted gravitational waves with parallel rays’, the reasons for which will be explained below. According to Peres [72], ‘pp’ is an abbreviation for ‘plane polarized gravitational waves’, which we now believe to be wrong.

We define pp-waves in a different, much more geometric way from the one originally used.

As we are not dealing with torsion in this Section, please note that spacetime is assumed to be Riemannian (see Definition 1.2.1), i.e. the connection is assumed to be

¹Note that when we first constructed the solutions presented in [69], we were not aware of Brinkmann’s work and believed Peres invented pp-waves

Levi-Civita.

Definition 3.1.1. A *pp-wave* is a Riemannian spacetime which admits a nonvanishing parallel spinor field.

Note that we use the term ‘parallel’ to describe the situation when the covariant derivative of some tensor or spinor field is identically equal to zero.

The metric of a pp-wave is called the *metric of the pp-wave* or simply the *pp-metric*.

The nonvanishing parallel spinor field appearing in the definition of pp-waves will be denoted throughout this thesis by

$$\chi = \chi^a$$

and we assume this spinor field to be *fixed*. However, it should be noted that there is no natural normalisation, i.e. a nonvanishing spinor field can be scaled by a non-zero complex factor. Also, in flat space there are two linearly independent nonvanishing spinor fields. We fix the spinor field χ in order to be able to give the following arguments without ambiguity.

Put

$$l^\alpha := \sigma^\alpha_{ab} \chi^a \bar{\chi}^b \tag{3.1}$$

where the σ^α are Pauli matrices, see Section 2.5 for notation and Section 3.3 for spinor formalism for pp-waves. Then l is a nonvanishing parallel real null vector field.

Now we define the real scalar function

$$\varphi : M \rightarrow \mathbb{R}, \quad \varphi(x) := \int l \cdot dx. \tag{3.2}$$

This function is called the *phase*. It is defined uniquely up to the addition of a constant and possible multi-valuedness resulting from a nontrivial topology of the manifold.

The 3-manifolds $\tilde{M} = \{\varphi = \text{const}\}$ are called *wave fronts*. Let us fix a particular wave front \tilde{M} , take a pair of points $\tilde{p}, \tilde{q} \in \tilde{M}$, and a curve $\tilde{\gamma} \subset \tilde{M}$ connecting these points. Take a 4-vector tangent to \tilde{M} at \tilde{p} and parallel transport it in accordance with the Levi-Civita connection along $\tilde{\gamma}$. It is easy to see that the resulting 4-vector will be tangent to \tilde{M} at \tilde{q} . This means that the Levi-Civita connection Γ over TM admits a natural restriction to a connection $\tilde{\Gamma}$ over $T\tilde{M}$. The latter cannot be interpreted as the Levi-Civita connection corresponding to the restriction of our Lorentzian 4-metric to the 3-manifold \tilde{M} as this restricted metric is degenerate.

An important property of pp-waves is that the connection $\tilde{\Gamma}$ is flat. This gives the explanation for the first meaning of ‘pp’ in pp-waves, i.e. ‘plane-fronted gravitational waves with parallel rays’. The fact that the wave fronts are flat motivates the following definitions:

Definition 3.1.2. We say that a complex vector field u is *transversal* if $l_\alpha u^\alpha = 0$.

Definition 3.1.3. We say that a complex vector field v is a *plane wave* if $u^\alpha \nabla_\alpha v^\beta = 0$ for any transversal vector field u .

Remark 3.1.4. Note that Definition 3.1.3 is far from standard, see for example Griffiths [29]. We however use Definition 3.1.3 to more clearly present our arguments in this chapter.

Clearly, l itself is both transversal and a plane wave. Put

$$F_{\alpha\beta} := \sigma_{\alpha\beta ab} \chi^a \chi^b \quad (3.3)$$

where the $\sigma_{\alpha\beta}$ are ‘second order Pauli matrices’ (2.43). Then F is a nonvanishing parallel complex 2-form with the additional properties $*F = \pm iF$ and $\det F = 0$. It can be written as

$$F = l \wedge m \quad (3.4)$$

where m is a complex vector field satisfying $m_\alpha m^\alpha = l_\alpha m^\alpha = 0$, $m_\alpha \bar{m}^\alpha = -2$. The vector field m is defined uniquely up to the addition of

$$\{\text{arbitrary complex valued scalar function}\} \times l .$$

We can impose an additional restriction on our choice of m requiring that m be a plane wave. Under this restriction the vector field m is defined uniquely up to the addition of

$$\{\text{arbitrary complex valued scalar function of } \varphi\} \times l$$

and

$$\nabla_\alpha m_\beta = p l_\alpha l_\beta. \quad (3.5)$$

where $p : M \rightarrow \mathbb{C}$ is some scalar function.

Our choice of the vector field m is assumed to be fixed. This implies, in particular, that the function p appearing in (3.5) is fixed.

It is known (see Alekseevsky [1], Bryant [12]), that Definition 3.1.1 is equivalent to the following

Definition 3.1.5. A *pp-wave* is a Riemannian spacetime whose metric can be written locally in the form

$$ds^2 = 2 dx^0 dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1, x^2, x^3) (dx^3)^2 \quad (3.6)$$

in some local coordinates (x^0, x^1, x^2, x^3) .

This was the definition originally used by Peres when he introduced pp-waves in [72, 74]. His main motivation for the introduction of the concept of a pp-wave was the simplicity of the formula for curvature, as the remarkable property of the metric (3.6) is that the corresponding curvature tensor R is linear in f :

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{2}(l \wedge \partial)_{\alpha\beta} (l \wedge \partial)_{\gamma\delta} f \quad (3.7)$$

where $(l \wedge \partial)_{\alpha\beta} := l_\alpha \partial_\beta - \partial_\alpha l_\beta$.

The advantage of Definition 3.1.5 is that it gives an explicit formula for the metric of a pp-wave. Its disadvantage is that it relies on a particular choice of local coordinates in each coordinate patch. Although our preferred definition of pp-waves is the much more geometrical Definition 3.1.1, we do all our practical calculations in coordinates (3.6) and with Pauli matrices (3.31). Of course, the choice of local coordinates in which the pp-metric assumes the form (3.6) is not unique. We will restrict our choice to those coordinates in which

$$\chi^a = (1, 0), \quad l^\mu = (1, 0, 0, 0), \quad m^\mu = (0, 1, \mp i, 0). \quad (3.8)$$

With such a choice formula (3.2) reads $\varphi(x) = x^3 + \text{const.}$

Observe now that in our special local coordinates f satisfies the equations

$$l^\mu \partial_\mu f = 0, \quad m^\mu \partial_\mu f = 2p \quad (3.9)$$

where p is the function from (3.5). Equations (3.9) are invariantly defined equations for a scalar function $f : M \rightarrow \mathbb{R}$. These equations allow us to give an invariant interpretation of our function f as a *potential* for a pp-wave. Equations (3.9) specify

the gradient of f along wave fronts, and, consequently, they define f uniquely up to the addition of an arbitrary real valued scalar function of φ .

Formula (3.7) can now be rewritten in invariant form

$$R = -\frac{1}{2}(l \wedge \nabla) \otimes (l \wedge \nabla)f \quad (3.10)$$

where $l \wedge \nabla := l \otimes \nabla - \nabla \otimes l$. Indeed, in our special local coordinates all the terms with connection coefficients in the RHS of (3.10) cancel out and (3.10) becomes (3.7). As both sides of (3.10) are tensors, formula (3.10) holds in any coordinate system.

The curvature of a pp-wave has the following irreducible pieces: (symmetric) trace-free Ricci and Weyl. The curvature tensor R can therefore be represented as

$$R_{\kappa\lambda\mu\nu} = \frac{1}{2}(g_{\kappa\mu}Ric_{\lambda\nu} - g_{\lambda\mu}Ric_{\kappa\nu} + g_{\lambda\nu}Ric_{\kappa\mu} - g_{\kappa\nu}Ric_{\lambda\mu}) + \mathcal{W}_{\kappa\lambda\mu\nu}, \quad (3.11)$$

where \mathcal{W} denotes Weyl curvature, i.e. the irreducible piece of curvature that satisfies the symmetries (2.12), (2.13) and (2.14). Note that in view of Section 2.3 and Table 2.2, which provide the description of the irreducible pieces of curvature, the only non-zero irreducible pieces of curvature come from the $\mathbf{R}^{(9,1)}$, $\mathbf{R}^{(9,2)}$ and $\mathbf{R}^{(10)}$ irreducible subspaces of the 96-dimensional vector space of rank 4 tensors satisfying $R^\kappa{}_{\lambda\mu\nu} = -R^\kappa{}_{\lambda\nu\mu}$.

Ricci curvature is proportional to $l \otimes l$ whereas Weyl curvature is a linear combination of $\text{Re}((l \wedge m) \otimes (l \wedge m))$ and $\text{Im}((l \wedge m) \otimes (l \wedge m))$. In our special local coordinates (3.6), (3.8), we can express these as

$$Ric_{\mu\nu} = \frac{1}{2}(f_{11} + f_{22})l_\mu l_\nu, \quad (3.12)$$

$$\mathcal{W}_{\kappa\lambda\mu\nu} = \sum_{j,k=1}^2 w_{jk}(l \wedge m_j) \otimes (l \wedge m_k), \quad (3.13)$$

where $m_1 = \text{Re}(m)$, $m_2 = \text{Im}(m)$, $f_{\alpha\beta} := \partial_\alpha \partial_\beta f$ and w_{jk} are real scalars given by

$$\begin{aligned} w_{11} &= \frac{1}{4}(-f_{11} + f_{22}), & w_{22} &= -w_{11}, \\ w_{12} &= \pm \frac{1}{2}f_{12}, & w_{21} &= w_{12}. \end{aligned} \quad (3.14)$$

3.2 Generalised pp-waves

One natural way of generalising the concept of a classical pp-wave is simply to extend Definition 3.1.1 to general metric compatible spacetimes, i.e. spacetimes whose connection is not necessarily Levi-Civita. However, this gives a class of spacetimes which is too wide and difficult to work with. We choose to extend the classical definition in a more special way better suited to the study of the system of field equations (1.3), (1.4).

Consider the polarized Maxwell equation

$$*dA = \pm idA \quad (3.15)$$

in a classical pp-space, see Section 3.1. Here A is the unknown complex vector field. We seek plane wave solutions of (3.15), see Definition 3.1.3. These can be written down explicitly:

$$A = h(\varphi)m + k(\varphi)l. \quad (3.16)$$

Here l and m are the vector fields defined in Section 3.1, $h, k : \mathbb{R} \rightarrow \mathbb{C}$ are arbitrary functions, and φ is the phase (3.2).

Definition 3.2.1. A *generalised pp-wave* is a metric compatible spacetime with pp-metric and torsion

$$T := \frac{1}{2}\text{Re}(A \otimes dA) \quad (3.17)$$

where A is a vector field of the form (3.16).

We list below the main properties of generalised pp-waves. Note that here and further on we denote by $\{\nabla\}$ the covariant derivative with respect to the Levi-Civita connection which should not be confused with the full covariant derivative ∇ incorporating torsion, see equations (2.1) and (2.7).

The curvature of a generalised pp-wave is

$$R = -\frac{1}{2}(l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\})f + \frac{1}{4}\text{Re}((h^2)''(l \wedge m) \otimes (l \wedge m)). \quad (3.18)$$

and the torsion of a generalised pp-wave is

$$T = \text{Re}((a l + b m) \otimes (l \wedge m)), \quad (3.19)$$

where

$$a := \frac{1}{2}h'(\varphi) k(\varphi), \quad b := \frac{1}{2}h'(\varphi) h(\varphi).$$

Remark 3.2.2. Here and further on, the prime simply stands for the derivative of a function of one real variable, e.g. in equations (3.30), (3.32), (5.2), (5.18).

Torsion can be written down even more explicitly in the following form

$$T = \sum_{j,k=1}^2 t_{jk} m_j \otimes (l \wedge m_k) + \sum_{j=1}^2 t_j l \otimes (l \wedge m_j), \quad (3.20)$$

where

$$t_{11} = -t_{22} = \frac{1}{2}\text{Re}(b), \quad t_{12} = t_{21} = -\frac{1}{2}\text{Im}(b), \quad t_1 = \frac{1}{2}\text{Re}(a), \quad t_2 = -\frac{1}{2}\text{Im}(a),$$

$m_1 = \text{Re}(m)$, $m_2 = \text{Im}(m)$ and a and b at the same functions of the phase φ appearing in equation (3.19).

Remark 3.2.3. From equation (3.20) we can clearly see that torsion has 4 independent non-zero components.

Lemma 3.2.4. *The torsion (3.17) of a generalised pp-wave is purely tensor, i.e.*

$$T^\alpha{}_{\alpha\gamma} = 0, \quad \varepsilon_{\alpha\beta\gamma\delta} T^{\alpha\beta\gamma} = 0. \quad (3.21)$$

Proof. The first equation $T^\alpha{}_{\alpha\gamma} = 0$ follows directly from equation (3.19) and the fact that we have that $l_\alpha l^\alpha = m_\alpha l^\alpha = m_\alpha m^\alpha = 0$, as stated in Section 3.1.

The second equation $\varepsilon_{\alpha\beta\gamma\delta} T^{\alpha\beta\gamma} = 0$ follows from the fact that

$$*(l \wedge m) = \pm i(l \wedge m).$$

Using equation (2.16) that defines the action of the Hodge star on a rank q antisymmetric tensor and the equation above, clearly we have

$$\varepsilon_{\alpha\beta\gamma\delta} (l \wedge m)^{\beta\gamma} = Z(l \wedge m)_{\alpha\delta},$$

where $Z \in \mathbb{C}$ is some constant. Then using the formula for torsion (3.19) we have

$$\varepsilon_{\alpha\beta\gamma\delta} T^{\alpha\beta\gamma} = \text{Re}(Z(a l + b m)^\alpha (l \wedge m)_{\alpha\delta}) = 0,$$

using the same argument as before, i.e. the fact that $l_\alpha l^\alpha = m_\alpha l^\alpha = m_\alpha m^\alpha = 0$. ■

In the beginning of Section 3.1 we introduced the spinor field χ satisfying $\{\nabla\}\chi = 0$. It becomes clear that this spinor field also satisfies $\nabla\chi = 0$.

Lemma 3.2.5. *The generalised pp-wave and the underlying classical pp-wave admit the same nonvanishing parallel spinor field.*

Proof. To see that $\nabla\chi = 0$, we look at the only remaining torsion generated term from formula (2.47) for the spinor connection coefficients $\Gamma^a{}_{\mu b}$, namely

$$\nabla_\mu \chi^a = \{\nabla\}_\mu \chi^a + \frac{1}{4} \sigma^{\alpha\dot{a}\dot{c}} T_{\mu\alpha\beta} \sigma^\beta{}_{b\dot{c}} \chi^b.$$

In view of equation (3.19), it is sufficient to show that the term involving $(l \wedge m)$ contracted with the Pauli matrices gives zero. To see this, we rewrite the term in the following form

$$\begin{aligned} \sigma^{\alpha\dot{a}\dot{c}} (l \wedge m)_{\alpha\beta} \sigma^\beta{}_{b\dot{c}} &= \frac{1}{2} (l \wedge m)_{\alpha\beta} (\sigma^{\alpha\dot{a}\dot{c}} \sigma^\beta{}_{b\dot{c}} - \sigma^{\beta\dot{a}\dot{c}} \sigma^\alpha{}_{b\dot{c}}), \\ &= (l \wedge m)_{\alpha\beta} \sigma^{\alpha\beta a}{}_b = 0, \end{aligned}$$

which can be checked directly using our local coordinates (3.6), (3.8) and the second order Pauli matrices (3.35), i.e.

$$-\sigma^{13}{}_{ab} - i\sigma^{23}{}_{ab} + \sigma^{31}{}_{ab} + i\sigma^{31}{}_{ab} = 0.$$

Hence, $\nabla\chi = 0$. ■

Remark 3.2.6. In view of Lemma 3.2.5, it is clear that both the generalised pp-wave and the underlying classical pp-wave admit the same nonvanishing parallel real null vector field l and the same nonvanishing parallel complex 2-form $l \wedge m$.

Examination of formula (3.18) for the curvature of a generalised pp-wave reveals the following remarkable properties of generalised pp-waves:

- The curvatures generated by the Levi-Civita connection and torsion simply add up (compare formulae (3.10) and (3.18)).
- The second term in the RHS of (3.18) is purely Weyl. Consequently, the Ricci curvature of a generalised pp-wave is completely determined by the pp-metric.

- Clearly, generalised pp-waves have the same non-zero irreducible pieces of curvature as classical pp-waves, namely symmetric trace-free Ricci and Weyl. Using special local coordinates (3.6), (3.8), these can be expressed explicitly as

$$Ric_{\mu\nu} = \frac{1}{2}(f_{11} + f_{22})l_\mu l_\nu, \quad (3.22)$$

$$\mathcal{W}_{\kappa\lambda\mu\nu} = \sum_{j,k=1}^2 w_{jk}(l \wedge m_j) \otimes (l \wedge m_k), \quad (3.23)$$

where $m_1 = \text{Re}(m)$, $m_2 = \text{Im}(m)$, $f_{\alpha\beta} := \partial_\alpha \partial_\beta f$ and w_{jk} are real scalars given by

$$\begin{aligned} w_{11} &= \frac{1}{4}[-f_{11} + f_{22} + \text{Re}((h^2)'')], & w_{22} &= -w_{11}, \\ w_{12} &= \pm \frac{1}{2}f_{12} - \frac{1}{4}\text{Im}((h^2)''), & w_{21} &= w_{12}. \end{aligned} \quad (3.24)$$

Compare these to the corresponding equations (3.12) and (3.13) for classical pp-waves.

- The curvature of a generalised pp-wave has all the usual symmetries of curvature in the Riemannian case, that is,

$$R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}, \quad (3.25)$$

$$\epsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0, \quad (3.26)$$

$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}, \quad (3.27)$$

$$R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu}. \quad (3.28)$$

Of course, (3.28) is true for any curvature whereas (3.27) is a consequence of metric compatibility. Also, (3.27) follows from (3.25) and (3.28).

- The second term in the RHS of (3.16) is pure gauge in the sense that it does not affect curvature (3.18). It does, however, affect torsion (3.17).
- The Ricci curvature of a generalised pp-wave is zero if and only if

$$f_{11} + f_{22} = 0 \quad (3.29)$$

and the Weyl curvature is zero if and only if

$$f_{11} - f_{22} = \operatorname{Re}((h^2)''), \quad f_{12} = \pm \frac{1}{2} \operatorname{Im}((h^2)''). \quad (3.30)$$

Here we use special local coordinates (3.6), (3.8) and denote $f_{\alpha\beta} := \partial_\alpha \partial_\beta f$.

- The curvature of a generalised pp-wave is zero if and only if we have both (3.29) and (3.30). Clearly, for any given function h we can choose a function f such that $R = 0$: this f is a quadratic polynomial in x^1, x^2 with coefficients depending on x^3 . Thus, as a spin-off, we get a class of examples of Weitzenböck spaces ($T \neq 0, R = 0$).

3.3 Spinor formalism for generalised pp-waves

In this Section we provide the particular spinor formalism for generalised pp-waves. For the general spinor formalism used in this thesis see Section 2.5.

For the pp-metric (3.6) we choose Pauli matrices

$$\begin{aligned} \sigma^0_{ab} &= \begin{pmatrix} 1 & 0 \\ 0 & -f \end{pmatrix}, \quad \sigma^1_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma^2_{ab} &= \begin{pmatrix} 0 & \mp i \\ \pm i & 0 \end{pmatrix}, \quad \sigma^3_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}. \end{aligned} \quad (3.31)$$

Our two choices of Pauli matrices differ by orientation. When dealing with a classical pp-wave the choice of orientation of Pauli matrices does not really matter, but for a generalised pp-wave it is convenient to choose orientation of Pauli matrices in agreement with the sign in (3.15) and (5.4) as this simplifies the resulting formulae.

Remark 3.3.1. In the case $f = 0$, formulae (3.31) do not turn into the Minkowski space Pauli matrices (2.48), since we write the metric in the form (3.6). This is a matter of convenience in calculations.

Now we want to describe the spinor connection coefficients $\Gamma^a_{\mu b}$. For a generalised pp-wave, formula (2.47) given in Section 2.5 which describes the spinor formalism used throughout this thesis reads as follows: the non-zero coefficients of

$$\Gamma^a_{\mu b} = \frac{1}{4} \sigma_\alpha^{ac} (\partial_\mu \sigma^\alpha_{bc} + \Gamma^\alpha_{\mu\beta} \sigma^\beta_{bc}).$$

are

$$\Gamma^1_{12} = \frac{1}{2}hh', \quad \Gamma^1_{22} = \mp \frac{i}{2}hh', \quad \Gamma^1_{32} = \frac{1}{2} \left(\frac{\partial f}{\partial x^1} \pm i \frac{\partial f}{\partial x^2} \right) - \frac{1}{2}kh'. \quad (3.32)$$

Here we use special local coordinates (3.6), (3.8) and Pauli matrices (3.31). Note that with our choice of Pauli matrices the signs in formulae (3.31) and (2.44) agree.

In a generalised pp-wave torsion is purely tensor, see (3.21), so Weyl's equation (B.2) takes the form

$$\sigma^\mu_{ab} \nabla_\mu \xi^a = 0, \quad (3.33)$$

or equivalently

$$\sigma^\mu_{ab} \{\nabla\}_\mu \xi^a = 0, \quad (3.34)$$

see Appendix B for more on Weyl's equation.

Remark 3.3.2. In view of equations (3.33) and (3.34), it is easy to see that $\chi^F(\varphi)$ is a solution of Weyl's equation. Here F is an arbitrary function of the phase φ (3.2) and χ is the parallel spinor introduced in Section 3.1.

We also provide the explicit formulae for the 'second order Pauli matrices' (2.43) for the pp-metric (3.6).

$$\begin{aligned} \sigma^{01}_{ab} &= \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}, & \sigma^{02}_{ab} &= \begin{pmatrix} \mp i & 0 \\ 0 & \pm i f \end{pmatrix}, & \sigma^{03}_{ab} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma^{12}_{ab} &= \begin{pmatrix} 0 & \mp i \\ \mp i & 0 \end{pmatrix}, & \sigma^{13}_{ab} &= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, & \sigma^{23}_{ab} &= \begin{pmatrix} 0 & 0 \\ 0 & \pm 2i \end{pmatrix} \end{aligned} \quad (3.35)$$

Note that as the second order Pauli matrices $\sigma^{\alpha\beta}_{ab}$ are antisymmetric over the tensor indices, i.e. $\sigma^{\alpha\beta}_{ab} = -\sigma^{\beta\alpha}_{ab}$, we only give the independent non-zero terms.

Chapter 4

New Vacuum Solutions for Quadratic Metric–Affine Gravity

This Chapter provides the construction of new solutions of quadratic metric–affine gravity and contains the proof of the main result of the thesis, Theorem 2.7.1. As the proof of the main theorem relies on the explicit representation of the field equations (1.3), (1.4), Section 4.1 contains the derivation of these equations, while Section 4.2 shows that generalised pp-waves with parallel Ricci curvature are solutions of the field equations.

4.1 Explicit representation of the field equations

We write down explicitly our field equations (1.3), (1.4) under the following assumptions:

- (i) our spacetime is metric compatible;
- (ii) torsion is purely tensor, see (3.21);
- (iii) curvature has symmetries (3.25), (3.26);
- (iv) scalar curvature is zero.

Note that a generalised pp-space automatically possesses these properties - property (i) is clear as $\nabla g \equiv 0$, property (ii) was shown in Lemma 3.2.4, property (iii) is clear from formula (3.18) for curvature of a generalised pp-wave and (iv) is satisfied as

scalar curvature for a generalised pp-wave is indeed zero (see equation (3.22)). The main result of this Section is the following

Lemma 4.1.1. *Under the above assumptions (i)–(iv), the field equations (1.3), (1.4) are*

$$d_1 \mathcal{W}^{\kappa\lambda\mu\nu} Ric_{\kappa\mu} + d_3 \left(Ric^{\lambda\kappa} Ric_{\kappa}{}^{\nu} - \frac{1}{4} g^{\lambda\nu} Ric_{\kappa\mu} Ric^{\kappa\mu} \right) = 0, \quad (4.1)$$

$$\begin{aligned} 0 &= d_6 \nabla_\lambda Ric_{\kappa\mu} - d_7 \nabla_\kappa Ric_{\lambda\mu} \\ &+ d_6 \left(Ric^\eta{}_\kappa (T_{\eta\mu\lambda} - T_{\lambda\mu\eta}) + \frac{1}{2} g_{\mu\lambda} \mathcal{W}^{\eta\zeta}{}_{\kappa\xi} (T_\eta{}^\xi{}_\zeta - T_\zeta{}^\xi{}_\eta) + g_{\mu\lambda} Ric^\eta{}_\xi T_\eta{}^\xi{}_\kappa \right) \\ &- d_7 \left(Ric^\eta{}_\lambda (T_{\eta\mu\kappa} - T_{\kappa\mu\eta}) + \frac{1}{2} g_{\kappa\mu} \mathcal{W}^{\eta\zeta}{}_{\lambda\xi} (T_\eta{}^\xi{}_\zeta - T_\zeta{}^\xi{}_\eta) + g_{\kappa\mu} Ric^\eta{}_\xi T_\eta{}^\xi{}_\lambda \right) \\ &+ b_{10} \left((g_{\kappa\mu} \mathcal{W}^{\eta\zeta}{}_{\lambda\xi} - g_{\mu\lambda} \mathcal{W}^{\eta\zeta}{}_{\kappa\xi}) (T_\eta{}^\xi{}_\zeta - T_\zeta{}^\xi{}_\eta) + Ric^\eta{}_\xi (g_{\kappa\mu} T_\eta{}^\xi{}_\lambda - g_{\mu\lambda} T_\eta{}^\xi{}_\kappa) \right) \\ &+ 2b_{10} \left(\mathcal{W}^\eta{}_{\mu\kappa\xi} (T_\eta{}^\xi{}_\lambda - T_\lambda{}^\xi{}_\eta) + \mathcal{W}^\eta{}_{\mu\lambda\xi} (T_\kappa{}^\xi{}_\eta - T_\eta{}^\xi{}_\kappa) - \mathcal{W}^{\xi\eta}{}_{\kappa\lambda} T_{\eta\mu\xi} \right), \end{aligned} \quad (4.2)$$

where

$$\begin{aligned} d_1 &= b_{912} - b_{922} + b_{10}, & d_3 &= b_{922} - b_{911}, \\ d_6 &= b_{912} - b_{911} + b_{10}, & d_7 &= b_{912} - b_{922} + b_{10}, \end{aligned}$$

the b 's being coefficients from formula (2.39).

Remark 4.1.2. It is important to stress once more that all the independent variations with respect to the metric and the affine connection of the action that produce the field equations (1.3), (1.4) are performed without *any* of the above assumptions. Only *after* the variations are performed do we apply the assumptions (i)–(iv) in order to look for solutions of this type. Hence, non-metricity (2.5) for example is allowed in the variations.

Remark 4.1.3. The number of independent equations in (4.1) is 9 in view of Remarks 1.3.3 and 1.3.2. The number of independent equations in (4.2) is 64. There are 5 different independent parameters appearing in equations (4.1), (4.2).

Remark 4.1.4. An effective technique for writing down the field equations explicitly can be found in [38, 42]. Namely, according to formulae (142), (143) of [38], our system of field equations reads

$$e_\alpha \rfloor V - (e_\alpha \rfloor R_\beta{}^\gamma \wedge \frac{\partial V}{\partial R_\beta{}^\gamma}) = 0, \quad (4.3)$$

$$D \frac{\partial V}{\partial R_\alpha{}^\beta} = 0. \quad (4.4)$$

Here the notation is *anholonomic* (as opposed to the holonomic notation of this thesis), $V := *q(R)$ is the Lagrangian, e_α is the frame and D is the covariant exterior differential, \lrcorner is the interior product and the exterior product is \wedge . Equation (4.4) is the explicit form of equation (1.4), but equations (4.3) and (1.3) are somewhat different: the difference is that (4.3) is the result of variation with respect to the frame rather than the metric. It is known, however, that the systems (1.3), (1.4) and (4.3), (4.4) are equivalent.

Before proving Lemma 4.1.1, let us have a brief discussion of this result. The LHS of equation (4.1) and the RHS of equation (4.2) are respectively the components of tensors A and B from the formula

$$\delta S = \int (2A^{\lambda\nu} \delta g_{\lambda\nu} + 2B^{\kappa\mu}{}_\lambda \delta \Gamma^\lambda{}_{\mu\kappa}).$$

Here δg and $\delta \Gamma$ are the (independent) variations of the metric and the connection, and δS is the resulting variation of the action. In (4.2) the first two indices of B have been lowered to make the expression easier to read.

Note that the LHS of equation (4.1) is trace-free, which is a consequence of the conformal invariance of our action (1.2), as noted before in Remark 1.3.2.

Equation (4.1) is equation (12) of [90] but with $\mathcal{R} = 0$. This is not surprising because when we vary the metric it does not matter whether the curvature tensor $R^\kappa{}_{\lambda\mu\nu}$ was generated by a Levi-Civita connection or a general affine connection. What matters are the symmetries (3.25), (3.26) which in our case are the same as in the Riemannian case. In fact, our case is simpler because scalar curvature is zero.

Equation (4.2) is similar to equation (13) of [90] but is not exactly the same. Namely,

- the first line of the RHS of (4.2) coincides with the LHS of equation (13) of [90] with $\mathcal{R} \equiv 0$,
- the remaining lines of the RHS of (4.2) contain extra algebraic terms generated by torsion.

Note also that the covariant derivatives in (4.2) and in equation (13) of [90] are different: we use the notation ∇ for the *full* covariant derivative, so the ∇ in (4.2) itself incorporates torsion.

We will now prove Lemma 4.1.1.

In deriving explicit formulae for tensors A and B we simplified our calculations by adopting the following argument. Formula (2.39) can be rewritten as

$$\begin{aligned} q(R) &= \sum_{l,m=1}^2 b_{9lm}(\mathcal{S}^{(l)}, \mathcal{S}^{(m)}) + b_{10}(R^{(10)}, R^{(10)})_{\text{YM}} + \dots \\ &= \sum_{l,m=1}^2 b_{9lm}(\mathcal{R}ic^{(l)}, \mathcal{R}ic^{(m)}) + b_{10}(R^{(10)}, R^{(10)})_{\text{YM}} + \dots \end{aligned}$$

where by \dots we denote terms which do not contribute to δS when we start our variation using the assumptions from the beginning of this Section and $(\cdot, \cdot)_{\text{YM}}$ is the Yang–Mills inner product on curvatures $(R, Q)_{\text{YM}} := R^\kappa{}_{\lambda\mu\nu} Q^\lambda{}_{\kappa}{}^{\mu\nu}$. Recall that the $\mathcal{R}ic^{(l)}$ are defined in accordance with (2.34) and the b 's are coefficients from formula (2.39). In accordance with the convention of [69, 90], put

$$\begin{aligned} P_- &:= \frac{1}{2}(\mathcal{R}ic^{(1)} - \mathcal{R}ic^{(2)}), \\ P_+ &:= \frac{1}{2}(\mathcal{R}ic^{(1)} + \mathcal{R}ic^{(2)}) = \frac{1}{2}(\mathcal{R}ic^{(1)} + \mathcal{R}ic^{(2)}). \end{aligned}$$

Note that in a metric compatible spacetime $\mathcal{R}ic^{(2)} = -\mathcal{R}ic^{(1)}$, hence $P_- = \mathcal{R}ic$ and $P_+ = 0$, so the tensor P_+ is generated by nonmetricity. Our quadratic form can now be rewritten as

$$q(R) = b_{10}(R^{(10)}, R^{(10)})_{\text{YM}} + (b_{911} - 2b_{912} + b_{922})(P_-, P_-) + 2(b_{911} - b_{922})(P_-, P_+) + \dots \quad (4.5)$$

We also provide another version of this formula which is in accordance with the notation of [87], where most of these terms were studied in detail. The equation (4.5) can be rewritten as

$$q(R) = c_1(R^{(1)}, R^{(1)})_{\text{YM}} + c_3(R^{(3)}, R^{(3)})_{\text{YM}} + 2(b_{911} - b_{922})(P_-, P_+) + \dots \quad (4.6)$$

where

$$c_1 = -\frac{1}{2}(b_{911} - 2b_{912} + b_{922}), \quad c_3 = b_{10}, \quad (4.7)$$

and the $R^{(j)}$ s are the irreducible pieces of curvature labeled in accordance with [87]; note that the labeling of irreducible pieces in [87] differs from that used in this thesis.

4.1.1 Variation with respect to the connection

The variations of $\int (R^{(j)}, R^{(j)})_{\text{YM}}$ were computed in Section 4 of [87]:

$$\delta \int (R^{(j)}, R^{(j)})_{\text{YM}} = 4 \int ((\delta_{\text{YM}} R^{(j)})^\mu (\delta \Gamma)_\mu) \quad (4.8)$$

where $(\delta_{\text{YM}} R)^\mu := \frac{1}{\sqrt{|\det g|}} (\partial_\nu + [\Gamma_\nu, \cdot]) (\sqrt{|\det g|} R^{\mu\nu})$ is the Yang–Mills divergence. Here we hide the Lie algebra indices of curvature by using matrix notation; say, $[\Gamma_\xi, R_{\mu\nu}]$ stands for

$$[\Gamma_\xi, R_{\mu\nu}]^\kappa{}_\lambda = \Gamma^\kappa{}_{\xi\eta} R^\eta{}_{\lambda\mu\nu} - R^\kappa{}_{\eta\mu\nu} \Gamma^\eta{}_{\xi\lambda}. \quad (4.9)$$

Now, in our case $R^{(1)}{}_{\kappa\lambda\mu\nu} = \frac{1}{2} (g_{\kappa\mu} Ric_{\lambda\nu} - g_{\lambda\mu} Ric_{\kappa\nu} - g_{\kappa\nu} Ric_{\lambda\mu} + g_{\lambda\nu} Ric_{\kappa\mu})$, $R^{(3)} = \mathcal{W}$, with the other $R^{(j)}$'s being zero. Substituting these expressions into (4.8) we get

$$\begin{aligned} \delta \int (R^{(1)}, R^{(1)})_{\text{YM}} &= 2 \int (\nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu} + g_{\kappa\mu} \nabla_\eta Ric_{\lambda\eta}^\eta - g_{\lambda\mu} \nabla_\eta Ric_{\kappa\eta}^\eta \\ &\quad + Ric_{\kappa\eta}^\eta (T_{\eta\mu\lambda} - T_{\lambda\mu\eta}) + Ric_{\lambda\eta}^\eta (T_{\kappa\mu\eta} - T_{\eta\mu\kappa})) \delta \Gamma^{\lambda\mu\kappa}, \end{aligned} \quad (4.10)$$

$$\delta \int (R^{(3)}, R^{(3)})_{\text{YM}} = 4 \int (\nabla_\eta \mathcal{W}_{\kappa\lambda\mu}{}^\eta + \mathcal{W}_{\kappa\lambda}{}^{\eta\xi} T_{\eta\mu\xi}) \delta \Gamma^{\lambda\mu\kappa}. \quad (4.11)$$

The variation of $\int (P_-, P_+)$ is

$$\begin{aligned} \delta \int (P_-, P_+) &= \int (Ric, \delta P_+) = \frac{1}{2} \int (Ric, \delta Ric) + \frac{1}{2} \int (Ric, \delta Ric^{(2)}) \\ &= -\frac{1}{2} \int [\nabla_\lambda Ric_{\kappa\mu} + \nabla_\kappa Ric_{\lambda\mu} - g_{\mu\kappa} \nabla_\eta Ric_{\lambda\eta}^\eta - g_{\mu\lambda} \nabla_\eta Ric_{\kappa\eta}^\eta \\ &\quad + Ric_{\kappa\eta}^\eta (T_{\eta\mu\lambda} - T_{\lambda\mu\eta}) + Ric_{\lambda\eta}^\eta (T_{\eta\mu\kappa} - T_{\kappa\mu\eta})] \delta \Gamma^{\lambda\mu\kappa}. \end{aligned} \quad (4.12)$$

Compare this with the corresponding formula in Section 3 of [90] and see Appendix C for the explicit variations of quadratic forms used.

Combining formulae (4.6), (4.7), (4.10)–(4.12) we arrive at the explicit form of the field equation (1.4):

$$\begin{aligned}
& d'_6(\nabla_\lambda Ric_{\kappa\mu} - g_{\mu\lambda}\nabla_\eta Ric^\eta{}_\kappa - T_{\lambda\mu\eta}Ric^\eta{}_\kappa + T_{\eta\mu\lambda}Ric^\eta{}_\kappa) \\
& - d'_7(\nabla_\kappa Ric_{\lambda\mu} - g_{\kappa\mu}\nabla_\eta Ric^\eta{}_\lambda - T_{\kappa\mu\eta}Ric^\eta{}_\lambda + T_{\eta\mu\kappa}Ric^\eta{}_\lambda) \\
& + 2b_{10}(\nabla_\eta \mathcal{W}^\eta{}_{\mu\lambda\kappa} - \mathcal{W}^{\eta\xi}{}_{\kappa\lambda}T_{\xi\mu\eta}) = 0
\end{aligned} \tag{4.13}$$

where

$$d'_6 = b_{912} - b_{911}, \quad d'_7 = b_{912} - b_{922}.$$

Let us now make use of the Bianchi identity for curvature¹

$$(\partial_\xi + [\Gamma_\xi, \cdot])R_{\mu\nu} + (\partial_\nu + [\Gamma_\nu, \cdot])R_{\xi\mu} + (\partial_\mu + [\Gamma_\mu, \cdot])R_{\nu\xi} = 0, \tag{4.14}$$

where we hide the Lie algebra indices of curvature by using matrix notation as in (4.9). Making one contraction in (4.14) and using the four assumptions listed in the beginning of Section 4.1 we get

$$\begin{aligned}
& \nabla_\kappa Ric_{\mu\lambda} - \nabla_\lambda Ric_{\mu\kappa} + g_{\mu\kappa}\nabla_\eta Ric^\eta{}_\lambda - g_{\mu\lambda}\nabla_\eta Ric^\eta{}_\kappa \\
& + Ric^\eta{}_\xi(g_{\mu\kappa}T_\eta{}^\xi{}_\lambda - g_{\mu\lambda}T_\eta{}^\xi{}_\kappa) + Ric^\eta{}_\kappa(T_{\eta\lambda\mu} - T_{\lambda\eta\mu}) + Ric^\eta{}_\lambda(T_{\kappa\eta\mu} - T_{\eta\kappa\mu}) \\
& + 2[\nabla_\eta \mathcal{W}^\eta{}_{\mu\lambda\kappa} + \mathcal{W}^\eta{}_{\mu\kappa\xi}(T_\lambda{}^\xi{}_\eta - T_\eta{}^\xi{}_\lambda) + \mathcal{W}^\eta{}_{\mu\lambda\xi}(T_\eta{}^\xi{}_\kappa - T_\kappa{}^\xi{}_\eta)] = 0.
\end{aligned} \tag{4.15}$$

Another contraction in (4.15) yields

$$\nabla_\eta Ric^\eta{}_\lambda = -Ric^\eta{}_\xi T_\eta{}^\xi{}_\lambda - \frac{1}{2}\mathcal{W}^{\eta\zeta}{}_{\lambda\xi}(T_\eta{}^\xi{}_\zeta - T_\zeta{}^\xi{}_\eta). \tag{4.16}$$

Substitution of (4.16) into (4.15) gives

$$\begin{aligned}
& \nabla_\eta \mathcal{W}^\eta{}_{\mu\lambda\kappa} = \mathcal{W}^\eta{}_{\mu\kappa\xi}(T_\eta{}^\xi{}_\lambda - T_\lambda{}^\xi{}_\eta) + \mathcal{W}^\eta{}_{\mu\lambda\xi}(T_\kappa{}^\xi{}_\eta - T_\eta{}^\xi{}_\kappa) \\
& \quad + \frac{1}{4}(T_\zeta{}^\xi{}_\eta - T_\eta{}^\xi{}_\zeta)(g_{\mu\lambda}\mathcal{W}^{\eta\zeta}{}_{\kappa\xi} - g_{\mu\kappa}\mathcal{W}^{\eta\zeta}{}_{\lambda\xi}) \\
& + \frac{1}{2}[\nabla_\lambda Ric_{\mu\kappa} - \nabla_\kappa Ric_{\mu\lambda} + Ric^\eta{}_\kappa(T_{\lambda\eta\mu} - T_{\eta\lambda\mu}) + Ric^\eta{}_\lambda(T_{\eta\kappa\mu} - T_{\kappa\eta\mu})].
\end{aligned} \tag{4.17}$$

Formulae (4.16) and (4.17) allow us to exclude the terms with $\nabla_\eta Ric^\eta{}_\kappa$, $\nabla_\eta Ric^\eta{}_\lambda$ and $\nabla_\eta \mathcal{W}^\eta{}_{\mu\lambda\kappa}$ from equation (4.13) reducing the latter to (4.2).

¹For the detailed calculations used in this Section, see Appendix A

4.1.2 Variation with respect to the metric

Although it has been already remarked that the field equation (4.1) is identical to the one in the Riemannian case as given in [90], only with the scalar curvature being zero, here we give an outline of the derivation.

A lengthy but straightforward calculation shows that

$$\delta \int (R^{(1)}, R^{(1)})_{\text{YM}} = -2 \int \mathcal{W}^{\kappa\beta\alpha\nu} Ric_{\kappa\nu} \delta g_{\alpha\beta} \quad (4.18)$$

$$\delta \int (R^{(3)}, R^{(3)})_{\text{YM}} = -2 \int \mathcal{W}^{\kappa\beta\alpha\nu} Ric_{\kappa\nu} \delta g_{\alpha\beta} \quad (4.19)$$

The variation of $\int (P_-, P_+)^2$ is

$$\begin{aligned} \delta \int (P_-, P_+) &= \int (Ric, \delta P_+) = \frac{1}{2} \int (Ric, \delta Ric) + \frac{1}{2} \int (Ric, \delta Ric^{(2)}) \\ &= -\frac{1}{4} \int (4Ric^{\kappa\alpha} Ric_{\kappa}{}^{\beta} - g^{\alpha\beta} Ric^{\kappa\nu} Ric_{\kappa\nu} + 2\mathcal{W}^{\kappa\alpha\beta\nu} Ric_{\kappa\nu}) \delta g_{\alpha\beta}, \end{aligned} \quad (4.20)$$

see Appendices C.2.2, C.3.2.

Combining formulae (4.6), (4.7), (4.18)–(4.20) we arrive at the explicit form of the field equation

$$d_1 \mathcal{W}^{\kappa\alpha\nu\beta} Ric_{\kappa\nu} + d_3 (Ric^{\kappa\alpha} Ric_{\kappa}{}^{\beta} - \frac{1}{4} g^{\alpha\beta} Ric^{\kappa}{}_{\nu} Ric_{\kappa}{}^{\nu}) = 0,$$

which is exactly equation (4.1).

This ends the proof of Lemma 4.1.1.

4.2 PP–wave type solutions of the field equations

The main result of this thesis is given in Theorem 2.7.1, which we recall here:

Generalised pp-spaces of parallel Ricci curvature are solutions of the system of field equations (1.3), (1.4).

Proof. The theorem is proved by direct substitution of formulae for torsion, Ricci curvature and Weyl curvature of a generalised pp-space into the field equations (4.1), (4.2).

²Here we use the fact that the Ricci curvature does not depend on the metric

The ∇Ric terms in the LHS of (4.2) vanish as Ricci curvature is assumed to be parallel, so it remains to check the vanishing of the remaining purely algebraic terms in the LHS's of (4.1), (4.2).

According to Section 3.2, torsion, Ricci curvature and Weyl curvature of a generalised pp-space are of the form

$$T = \sum_{j,k=1}^2 t_{jk} m_j \otimes (l \wedge m_k) + \sum_{j=1}^2 t_j l \otimes (l \wedge m_j), \quad (4.21)$$

$$Ric = s l \otimes l, \quad (4.22)$$

$$\mathcal{W} = \sum_{j,k=1}^2 w_{jk} (l \wedge m_j) \otimes (l \wedge m_k), \quad (4.23)$$

where t_{jk} , t_j , s , w_{jk} are some real scalars satisfying

$$t_{jk} = t_{kj}, \quad w_{jk} = w_{kj}, \quad t_{11} + t_{22} = w_{11} + w_{22} = 0,$$

l and m are vectors introduced in Section 3.1, and $m_1 = \text{Re } m$, $m_2 = \text{Im } m$. Note that the real vectors l , m_1 , m_2 satisfy

$$l \cdot l = l \cdot m_1 = l \cdot m_2 = m_1 \cdot m_2 = 0, \quad m_1 \cdot m_1 = m_2 \cdot m_2 = -1.$$

All the algebraic terms containing Ric in the LHS's of (4.1), (4.2) vanish because they involve contractions with at least one of the indices of Ric , the latter being of the form (4.22) with vector l orthogonal to all other vectors appearing in (4.21)–(4.23). It remains to consider the $\mathcal{W} \times T$ terms in the LHS of (4.2). The terms with 3 contractions vanish because in view of (4.21) at least one of the contractions involves the vector l . The term $\mathcal{W}^{\xi\eta}_{\kappa\lambda} T_{\eta\mu\xi}$ also vanishes because in view of (4.23) at least one of the contractions involves the vector l . Thus, the proof of Theorem 2.7.1 reduces to checking that

$$\mathcal{W}^{\eta}_{\mu\kappa\xi} (T_{\eta}^{\xi\lambda} - T_{\lambda}^{\xi\eta}) + \mathcal{W}^{\eta}_{\mu\lambda\xi} (T_{\kappa}^{\xi\eta} - T_{\eta}^{\xi\kappa}) = 0. \quad (4.24)$$

The tensor in the LHS of (4.24) is proportional to $l_{\lambda} l_{\mu} l_{\kappa}$. In special local coordinates we can write this as

$$\mathcal{W}^{\eta}_{\mu\kappa\xi} (T_{\eta}^{\xi\lambda} - T_{\lambda}^{\xi\eta}) = \left(-\frac{1}{2} \text{Re}(b) f_{11} + \frac{1}{2} \text{Re}(b) f_{22} + \text{Im}(b) f_{12} \right) l_{\mu} l_{\kappa} l_{\lambda},$$

where $b = \frac{1}{2}h'(\varphi) h(\varphi)$ as in equation (3.19). Since the LHS of (4.24) is antisymmetric in κ, λ , it is therefore zero. ■

Remark 4.2.1. We know (see the list of properties at the end of Section 3.2) that in a generalised pp-space $Ric = \{Ric\}$. Moreover, it is easy to see that in a generalised pp-space $\nabla Ric = \{\nabla\} Ric$. This means that when using Theorem 2.7.1 it does not really matter whether the condition “parallel Ricci curvature” is understood in the non-Riemannian sense $\nabla Ric = 0$, the Riemannian sense $\{\nabla\}\{Ric\} = 0$, or any combination of the two ($\{\nabla\} Ric = 0$ or $\nabla\{Ric\} = 0$).

Remark 4.2.2. In special local coordinates (3.6), (3.8) the condition that Ricci curvature is parallel in Theorem 2.7.1 is written as $f_{11} + f_{22} = \text{const}$. Compare this to equation (3.29).

Chapter 5

Discussion

In this Chapter we attempt to give a physical interpretation of our new vacuum solutions of field equations (1.3), (1.4) obtained in Chapters 3 and 4. As noted before in Section 1.4.1, the following two classes of Riemannian spacetimes are solutions of our field equations:

- Einstein spaces ($Ric = \Lambda g$), and
- classical pp-spaces of parallel Ricci curvature.

In general relativity, Einstein spaces are an accepted mathematical model for vacuum. However, classical pp-spaces of parallel Ricci curvature do not have an obvious physical interpretation. This Chapter gives an attempt at understanding whether our newly constructed spacetimes are of mathematical or physical significance.

5.1 Physical interpretation of the new solutions

Our analysis of vacuum solutions of quadratic metric–affine gravity shows, see Theorem 2.7.1, that classical pp-spaces of parallel Ricci curvature should not be viewed on their own. They are a particular (degenerate) representative of a wider class of solutions, namely, generalised pp-spaces of parallel Ricci curvature. Indeed, according to formula (3.18) the curvature of a generalised pp-space is a sum of two curvatures: the curvature

$$-\frac{1}{2}(l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\})f \tag{5.1}$$

of the underlying classical pp-space and the curvature

$$\frac{1}{4}\text{Re}((h^2)''(l \wedge m) \otimes (l \wedge m)) \quad (5.2)$$

generated by a torsion wave traveling over this classical pp-space. Our torsion (3.17), (3.16) and corresponding curvature (5.2) are waves traveling at speed of light because h and k are functions of the phase φ which plays the role of a null coordinate, $g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi = 0$, see formula (3.2). The underlying classical pp-space of parallel Ricci curvature can now be viewed as the ‘gravitational imprint’ created by a wave of some massless matter field. Such a situation occurs in Einstein–Maxwell theory¹ and Einstein–Weyl theory². The difference with our model is that Einstein–Maxwell and Einstein–Weyl theories contain the gravitational constant which dictates a particular relationship between the strengths of the fields in question, whereas our model is conformally invariant and the amplitudes of the two curvatures (5.1) and (5.2) are totally independent.

In the remainder of this subsection we outline an argument in favour of interpreting our torsion wave (3.17), (3.16) as a mathematical model for some massless particle.

We base our interpretation on the analysis of the curvature (5.2) generated by our torsion wave. Examination of formula (5.2) indicates that it is more convenient to deal with the complexified curvature

$$\mathfrak{R} := r(l \wedge m) \otimes (l \wedge m) \quad (5.3)$$

where $r := \frac{1}{4}(h^2)''$ (this r is a function of the phase φ); note also that complexification is in line with the traditions of quantum mechanics. Our complex curvature is polarized,

$$*\mathfrak{R} = \mathfrak{R}^* = \pm i\mathfrak{R}, \quad (5.4)$$

and purely Weyl, hence it is equivalent to a (symmetric) rank 4 spinor ω . The relationship between \mathfrak{R} and ω is given by the formula

$$\mathfrak{R}_{\alpha\beta\gamma\delta} = \sigma_{\alpha\beta ab}\omega^{abcd}\sigma_{\gamma\delta cd} \quad (5.5)$$

¹Einstein–Maxwell theory is a classical model describing the interaction of gravitational and electromagnetic fields

²Einstein–Weyl theory is a classical model describing the interaction of gravitational and massless neutrino fields

where the $\sigma_{\alpha\beta}$ are “second order Pauli matrices” (2.43). Resolving (5.5) with respect to ω we get, in view of (5.3), (3.4), (3.3),

$$\omega = \xi \otimes \xi \otimes \xi \otimes \xi \quad (5.6)$$

where

$$\xi := r^{1/4} \chi \quad (5.7)$$

and χ is the spinor field introduced in the beginning of Section 3.1.

Formula (5.6) shows that our rank 4 spinor ω has additional algebraic structure: it is the 4th tensor power of a rank 1 spinor ξ . Consequently, the complexified curvature generated by our torsion wave is completely determined by the rank 1 spinor field ξ .

We claim that the spinor field (5.7) satisfies Weyl’s equation, see (3.33) or (3.34). Indeed, as χ is parallel checking that ξ satisfies Weyl’s equation reduces to checking that $(r^{1/4})' \sigma^{\mu}_{ab} l_{\mu} \chi^a = 0$. The latter is established by direct substitution of the explicit formula for l , see (3.1).

5.2 Comparison with Einstein–Weyl theory

The aim of this Section is first to provide a reminder of Einstein–Weyl theory and the field equations arising from this classical model describing the interaction of gravitational and massless neutrino fields, then to provide pp-wave type solutions within this model, provide the previously known solutions of this type and, finally, to compare them to the pp-wave type solutions of our conformally invariant metric–affine model of gravity.

5.2.1 Einstein–Weyl field equations

In Einstein–Weyl theory the action is given by

$$S_{EW} := 2i \int \left(\xi^a \sigma^{\mu}_{ab} (\{\nabla\}_{\mu} \bar{\xi}^b) - (\{\nabla\}_{\mu} \xi^a) \sigma^{\mu}_{ab} \bar{\xi}^b \right) + k \int \mathcal{R}, \quad (5.8)$$

where the constant k can be chosen so that the non-relativistic limit yields the usual form of Newton's gravity law. According to Brill and Wheeler [10] for example

$$k = \frac{c^4}{16\pi G}, \quad (5.9)$$

where G is the gravitational constant. In SI units the recommended numerical value of the gravitational constant³ is

$$G := (6.6742 \pm 0.001) \times 10^{-11} \text{Nm}^2\text{kg}^{-2}. \quad (5.10)$$

Remark 5.2.1. Note that in Einstein–Weyl theory the connection is assumed to be Levi-Civita, so we only vary the action (5.8) with respect to the metric and the spinor.

We obtain the well known Einstein–Weyl field equations

$$\frac{\delta S_{EW}}{\delta g} = 0, \quad (5.11)$$

$$\frac{\delta S_{EW}}{\delta \xi} = 0. \quad (5.12)$$

The first term of the action S depends on the spinor ξ and the metric g while the second depends on the metric g only. Hence the formal variation of the action (5.8) with respect to the spinor just yields the Weyl equation, see Appendix B.

The variation with respect to the metric is somewhat more complicated as both terms of the action (5.8) depend on the metric g . The variation of the first term of the action with respect to the metric yields the energy momentum tensor of the Weyl action (B.1), i.e.

$$\begin{aligned} E^{\mu\nu} &= \frac{i}{2} \left[\sigma^{\nu}_{ab} \left(\bar{\xi}^b \{\nabla\}^{\mu} \xi^a - \xi^a \{\nabla\}^{\mu} \bar{\xi}^b \right) + \sigma^{\mu}_{ab} \left(\bar{\xi}^b \{\nabla\}^{\nu} \xi^a - \xi^a \{\nabla\}^{\nu} \bar{\xi}^b \right) \right] \\ &+ i \left(\xi^a \sigma^{\eta}_{ab} (\{\nabla\}_{\eta} \bar{\xi}^b) g^{\mu\nu} - (\{\nabla\}_{\eta} \xi^a) \sigma^{\eta}_{ab} \bar{\xi}^b g^{\mu\nu} \right). \end{aligned} \quad (5.13)$$

Note that the energy momentum tensor is not a priori trace-free.

³According to the Committee on Data for Science and Technology

Variation with respect to the metric of the Einstein–Hilbert term of the action yields

$$\delta \int \mathcal{R} = - \int (Ric^{\mu\nu} - \frac{1}{2} \mathcal{R} g^{\mu\nu}) \delta g_{\mu\nu},$$

which is a straightforward calculation, see e.g. Landau and Lifshitz [57].

Hence we get the explicit representation of the Einstein–Weyl field equations (5.11), (5.12):

$$\begin{aligned} \frac{i}{2} \left[\sigma^{\nu}_{ab} \left(\bar{\xi}^b \{\nabla\}^{\mu} \xi^a - \xi^a \{\nabla\}^{\mu} \bar{\xi}^b \right) + \sigma^{\mu}_{ab} \left(\bar{\xi}^b \{\nabla\}^{\nu} \xi^a - \xi^a \{\nabla\}^{\nu} \bar{\xi}^b \right) \right] \\ + i \left(\xi^a \sigma^{\eta}_{ab} (\{\nabla\}_{\eta} \bar{\xi}^b) g^{\mu\nu} - (\{\nabla\}_{\eta} \xi^a) \sigma^{\eta}_{ab} \bar{\xi}^b g^{\mu\nu} \right) \\ - k Ric^{\mu\nu} + \frac{k}{2} \mathcal{R} g^{\mu\nu} = 0, (5.14) \\ \sigma^{\mu}_{ab} \{\nabla\}_{\mu} \xi^a = 0. (5.15) \end{aligned}$$

Remark 5.2.2. Note that when equation (5.15) is satisfied, we have that the energy-momentum tensor (5.13) is trace free and the second line of (5.14) vanishes, see e.g. end of section 2 of Griffiths and Newing [32].

5.2.2 Known solutions of Einstein–Weyl theory

The examination of the Einstein–Weyl field equations has a long and fruitful history with many interesting results from a number of authors. Here we would like to give a brief literature review of this theory, in particular in view of finding exact solutions.

In one of the early works on this subject, Griffiths and Newing [31] show how the solutions of Einstein–Weyl equations can be constructed and present five examples of solutions. The authors define a neutrino field as the particle that satisfies the Dirac equation and has zero mass and zero charge. They then generalise the Dirac equation by using four matrices that satisfy the equation $\sigma^{\alpha}_{ab} \sigma^{\beta cb} + \sigma^{\beta}_{ab} \sigma^{\alpha cb} = 2g^{\alpha\beta} \delta_a^c$ (Pauli matrices) and explain the covariant derivative of spinors, as done similarly in this thesis in Section 2.5. The authors then develop the tensor method of incorporating Dirac’s equation into general relativity and use it to then obtain a number of exact solutions of the neutrino-gravitational field equations corresponding to the particular case of pure radiation fields. Although this paper is presented in a non-standard notation, it is of great interest to us, as in particular the solutions contain the plane-wave solution.

A later work by the same authors [32] presents a more general solution of Kundt's class, by exhibiting the Weyl neutrino equations as equivalent tetrad equations, constructing a tetrad of null vectors from the spinors and Pauli matrices (in a very similar fashion to equation (3.1) of this thesis) and employing this setting to show that Weyl's equation is equivalent to a tensor equation previously derived.

Audretsch and Graf [3] derive a differential equation representing radiation solutions of the general relativistic Weyl equation and study the corresponding energy-momentum tensor. In section 6, an exact solution of Einstein–Weyl equations in the form of pp-waves is presented. The authors conclude that gravitational and neutrino pp-waves taken together, represent an exact solution of the Einstein–Weyl system of field equations. In [2] Audretsch continues to study the asymptotic behaviour of the neutrino energy-momentum tensor in curved space-time with the sole aid of generally covariant assumptions about the nature of the Weyl field. In particular, the author shows that these Weyl fields behave asymptotically like neutrino radiation.

The work presented in [32] was further investigated by Griffiths [33] and this paper is of particular interest to us as in section 5 of [33] the author presents solutions whose metric is the pp-wave metric (3.6) and the author also presents a condition on the function f from the pp-metric (3.6). In view of the pp-wave type solutions presented in Section 5.2.3, it should be stated that these were obtained independently and we only became aware of the work in [33] recently.

In [34], Griffiths identifies a class of neutrino fields with zero energy momentum tensor and stipulates that these spacetimes may also be interpreted as describing gravitational waves. Collinson and Morris [18] showed that these could be either pp-waves or Robinson-Trautman type N solutions presented in [31]. Subsequently these were called '*ghost neutrinos*' by Davis and Ray in [19] where the authors state that neutrinos, as before, yield a zero energy momentum tensor and therefore the gravitational field is the same as for the vacuum case.

Kuchowicz and Żebrowski [53] expand on the work on ghost neutrinos trying to resolve this anomaly by considering non-zero torsion in the framework of Einstein–Cartan theory, showing that the extra terms specific to this theory remove the incompatibility found in general relativity. Griffiths [36] also considers the possibility of non-zero torsion and in a more general work [37] he showed that neutrino fields in Einstein-Cartan theory must have metrics that belong to the family of solutions of Kundt's class, which include the pp-waves.

Singh and Griffiths [80] corrected several mistakes from [37] and showed that neutrino fields in Einstein–Cartan theory also include the Robinson–Trautman type N solutions and that any solution of the Einstein–Weyl equations in general relativity has a corresponding solution in Einstein–Cartan theory. Thus pp-wave type solutions of Einstein–Weyl equations have corresponding solutions in Einstein–Cartan theory. This paper was one of the main inspirations behind Section 5.2.3.

5.2.3 PP–wave type solutions of Einstein–Weyl theory

The aim of this Section is to point out again the fact that the nonlinear system of equations (5.14), (5.15) has solutions in the form of pp-waves. Throughout this Section we use the set of local coordinates (3.6), (3.8) and Pauli matrices (3.31). We now present a class of explicit solutions of (5.14), (5.15) where the metric g is in the form of a pp-metric and the spinor ξ as in (5.7). As shown in the Section 5.1, the spinor (5.7) satisfies the equation (5.15). In the setting of a pp-space scalar curvature vanishes and as the spinor χ appearing in formula (5.7) is parallel, in view of Remark 5.2.2 equation (5.14) becomes

$$\frac{i}{2}\sigma^\nu{}_{ab}\left(\bar{\xi}^{\dot{b}}\{\nabla\}^\mu\xi^a - \xi^a\{\nabla\}^\mu\bar{\xi}^{\dot{b}}\right) + \frac{i}{2}\sigma^\mu{}_{ab}\left(\bar{\xi}^{\dot{b}}\{\nabla\}^\nu\xi^a - \xi^a\{\nabla\}^\nu\bar{\xi}^{\dot{b}}\right) - kRic^{\mu\nu} = 0. \quad (5.16)$$

We now need to determine under what condition the equation (5.16) is satisfied. In our local coordinates, we have

$$Ric = \left(\frac{1}{2}\frac{\partial^2 f}{\partial(x^1)^2} + \frac{1}{2}\frac{\partial^2 f}{\partial(x^2)^2}\right)(l \otimes l). \quad (5.17)$$

Substituting formulae (5.7), (5.17) into equation (5.16), and using the fact that the spinor χ is parallel, we obtain the equality

$$i(\sigma^\nu{}_{ab}l^\mu + \sigma^\mu{}_{ab}l^\nu)\left((r^{1/4})' \overline{r^{1/4}} - r^{1/4} (\overline{r^{1/4}})'\right)\chi^a\bar{\chi}^{\dot{b}} = \frac{1}{2}kl^\mu l^\nu \left(\frac{\partial^2 f}{\partial(x^1)^2} + \frac{\partial^2 f}{\partial(x^2)^2}\right).$$

Since we know that $\sigma^\mu{}_{ab}\chi^a\bar{\chi}^{\dot{b}} = l^\mu$, we obtain the condition for a pp-wave type solution of the Einstein–Weyl model

$$\frac{1}{2}\frac{\partial^2 f}{\partial(x^1)^2} + \frac{1}{2}\frac{\partial^2 f}{\partial(x^2)^2} = \frac{i}{k}\left((r^{1/4})' \overline{r^{1/4}} - r^{1/4} (\overline{r^{1/4}})'\right). \quad (5.18)$$

Thus, the complex valued function r of one real variable x^3 can be chosen arbitrarily and it uniquely determines the RHS of (5.18). From (5.18) one recovers the pp-metric by solving Poisson's equation.

5.2.4 Comparison of metric–affine and Einstein–Weyl solutions

To make our comparison clearer, let us compare these models in the case of monochromatic solutions of both models using local coordinates (3.6), (3.8) and Pauli matrices (3.31).

Monochromatic metric–affine solutions

In the case of the metric–affine model, from Theorem 2.7.1 we know that generalised pp-waves of parallel Ricci curvature are solutions of the equations (1.3), (1.4). Whether we are viewing monochromatic solutions or not, the condition on the solution of the model remains unchanged, namely Ricci curvature (5.17) has to be parallel. In our special local coordinates the condition of parallel Ricci curvature is written as

$$\frac{1}{2} \frac{\partial^2 f}{\partial (x^1)^2} + \frac{1}{2} \frac{\partial^2 f}{\partial (x^2)^2} = C, \quad (5.19)$$

where C is an arbitrary real constant.

However, the construction of torsion simplifies in the monochromatic case. Namely, we can choose the function h of the phase (3.2) so that the plane wave (3.16) becomes

$$A = \frac{ic^2}{2a} e^{2i(ax^3+b)} m,$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$. Torsion (3.17) then takes the form

$$T = -\frac{c^4}{4a} \operatorname{Re} \left(ie^{4i(ax^3+b)} m \otimes (l \wedge m) \right).$$

Hence the complexified curvature (5.3) generated by the torsion wave becomes

$$\mathfrak{R} = c^4 e^{4i(ax^3+b)} (l \wedge m) \otimes (l \wedge m),$$

and r from (5.3) becomes

$$r = c^4 e^{4i(ax^3+b)}.$$

The spinor ξ from (5.7) is explicitly given by

$$\xi = c \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i(ax^3+b)}. \quad (5.20)$$

Monochromatic Einstein–Weyl solutions

Let us now look for monochromatic solutions in Einstein–Weyl theory. We take the spinor field as in formula (5.20) in which case condition (5.18) simplifies to

$$\frac{1}{2} \frac{\partial^2 f}{\partial(x^1)^2} + \frac{1}{2} \frac{\partial^2 f}{\partial(x^2)^2} = -\frac{2ac^2}{k}. \quad (5.21)$$

Comparison of monochromatic metric–affine and Einstein–Weyl solutions

The main difference between the two models is that in the metric–affine model our generalised pp-waves solutions have parallel Ricci curvature, whereas in the Einstein–Weyl model the pp-wave type solutions do not necessarily have parallel Ricci curvature. However, when we look at monochromatic pp-wave type solutions in the Einstein–Weyl model their Ricci curvature also becomes parallel. The only remaining difference is in the right-hand sides of equations (5.19) and (5.21): in (5.19) the constant is arbitrary whereas in (5.21) the constant is expressed via the characteristics of the spinor wave and the gravitational constant.

In other words, comparing equations (5.19) and (5.21) we see that while in the metric–affine case the Laplacian of f can be *any* constant, in the Einstein–Weyl case it is required to be a *particular* constant. This should not be surprising as our metric–affine model is conformally invariant, see Remark 1.3.2, while the Einstein–Weyl model is not.

We also want to clarify that f and the quantities a, b, c appearing in this Section 5.2.4 are generally arbitrary functions of the null coordinate x^3 . As such, if these quantities are non-zero only for a short finite interval of x^3 , the solutions represent spinors, curvature and torsion components which propagate at the speed of light.

We want to point out a very interesting fact that that generalised pp-waves of parallel Ricci curvature are very similar to pp-type solutions of the Einstein–Weyl model, which is a classical model describing the interaction of massless neutrino and gravitational fields, to suggest that

Generalised pp-waves of parallel Ricci curvature represent a metric–affine model for some massless particle.

Which particle this is remains object of discussion and we hope to be able to address this question with more certainty in the near future.

5.3 Comparison with existing literature

There are a number of publications in which authors suggested various generalisations of the concept of a classical pp-wave. These generalisations were performed within the Riemannian setting and usually involved the incorporation of a constant non-zero scalar curvature; see [66] and extensive further references therein. Our construction described in Section 3.2 generalises the concept of a classical pp-wave in a different direction: we add torsion while retaining zero scalar curvature.

A powerful method which in the past has been used for the construction of vacuum solutions of quadratic metric–affine gravity is the so-called *double duality ansatz* [4, 5, 58, 59, 87, 90]. Its basic version [87] is as follows. For certain types of quadratic actions (see item (b) below) the following is known to be true: if the spacetime is metric compatible and curvature is irreducible (i.e. all irreducible pieces except one are identically zero) then this spacetime is a solution of (1.3), (1.4). This fact is referred to as the double duality ansatz because the proof is based on the use of the double duality transform $R \mapsto {}^*R^*$ (this idea is due to Mielke [58]) and because the above conditions imply ${}^*R^* = \pm R$. However, solutions presented in Theorem 2.7.1 do not fit into the double duality scheme. This is due to the following reasons.

- (a) The curvature of a pp-wave, classical or generalised, contains trace-free Ricci and Weyl pieces, hence this curvature is not necessarily irreducible and not necessarily an eigenvector of the double duality operator. Namely, for a pp-wave the following statements are equivalent:

$$R \text{ is purely trace-free Ricci} \iff \text{condition (3.30) is satisfied} \iff {}^*R^* = +R,$$

$$R \text{ is purely Weyl} \iff \text{condition (3.29) is satisfied} \iff {}^*R^* = -R.$$

Furthermore, the curvature of a pp-wave, classical or generalised, does not necessarily satisfy the conditions of the modified double duality ansatz [4, 5, 59].

- (b) The double duality ansatz in its basic [87] or modified [4, 5, 59] forms does not work for the most general 16-parameter actions introduced in [25, 43, 90] and considered in our current paper. It works only for more special actions with up to 11 coupling constants. The fundamental difference between the 11-parameter and 16-parameter models is best seen if one considers the specialisation of the field equation (1.4) to the Levi-Civita connection:

$$\partial S / \partial \Gamma |_{L-C} = 0. \quad (5.22)$$

Equation (5.22) arises when one looks for Riemannian solutions of (1.4). Here it is important to understand the logical sequence involved in the derivation of (5.22): we set $\Gamma^\lambda_{\mu\nu} = \left\{ \begin{smallmatrix} \lambda \\ \mu\nu \end{smallmatrix} \right\}$ *after* the variation of the connection has been carried out. It is known [87] that for a generic 11-parameter action equation (5.22) reduces to

$$\nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu} = 0, \quad (5.23)$$

whereas according to [90] for a generic 16-parameter action equation (5.22) reduces to

$$\nabla Ric = 0. \quad (5.24)$$

The field equations (5.23) and (5.24) are very much different, with (5.24) being by far more restrictive. In particular, Nordström–Thompson spacetimes (Riemannian spacetimes with $*R^* = +R$) satisfy (5.23) but do not necessarily satisfy (5.24).

- (c) The basic double duality ansatz [87] can be reformulated in a way that makes it applicable to 16-parameter actions: one has to impose the additional condition that curvature is *simple*, i.e. the given irreducible subspace of the vector space of curvatures is not isomorphic to any other irreducible subspace, see Section 6 of [90] for details. According to formula (44) of [90] the (symmetric) trace-free Ricci piece of curvature is not simple, hence the version of the double duality ansatz from [90] works for a pp-wave, classical or generalised, only when curvature is purely Weyl.

The new vacuum solutions of quadratic metric–affine gravity presented in Theorem 2.7.1 are similar to those of Singh and Griffiths [81]. The main differences are as follows.

- The solutions in [81] satisfy the condition $\{Ric\} = 0$ whereas our solutions satisfy the weaker condition $\{\nabla\}\{Ric\} = 0$, see also Remark 4.2.1.
- The solutions in [81] were obtained for the Yang–Mills case (1.5) whereas we deal with a general $O(1, 3)$ -invariant quadratic form q with 16 coupling constants.

The two papers of Singh [78, 79] construct solutions for the Yang–Mills case (1.5) with purely axial and purely trace torsion respectively and unlike the solution of [81], $\{Ric\}$ is not assumed to be zero. It is obvious that these solutions differ from the ones presented in this thesis, as our torsion is assumed to be purely tensor. It would however be of interest to us to see whether this construction of Singh’s can be expanded to our most general $O(1, 3)$ -invariant quadratic form q with 16 coupling constants. This is a question we hope to answer sometime in the future.

The observation that one can construct vacuum solutions of quadratic metric–affine gravity in terms of pp-waves is a recent development. The fact that classical pp-waves of parallel Ricci curvature are solutions was first pointed out in [88, 89, 90].

One interesting generalisation of the concept of a pp-wave was presented by Obukhov in [67], which is a more recent result from the one presented in [69]. Obukhov’s motivation comes from his previous work [66] which is the Riemannian case. In fact, the ansatz for the metric and the coframe of [67] is exactly the same as in the Riemannian case. However, the connection extends the Christoffel connection in such a way that torsion and *nonmetricity* (2.5) are present, and are determined by this extension of the connection.

Obukhov studies the same general quadratic Lagrangian as studied in this thesis with 16 terms, see Section 2.4, and the result of [67] does not belong to the triplet ansatz, see [43, 64]. Similarly to the result of this thesis, his gravitational wave solutions have only two non-trivial pieces of curvature. However, unlike our solution, the two non-zero pieces of curvature in [67] are equivalent to the pieces of curvature coming from the 10-dimensional $\mathbf{R}^{(10)}$ and the 30-dimensional $\mathbf{R}^{(30)}$ irreducible curvature subspaces, as given in Section 2.3.

Hence the main differences between our result and Obukhov’s generalisation of [67] are the following:

- In Obukhov’s plane-fronted waves not only are the torsion waves present, but the non-metricity has a non-trivial wave behaviour as well. As we are only looking

at metric-compatible spacetimes in this thesis, nonmetricity cannot appear in our construction. However, as Obukhov states, his construction yields a number of interesting mathematical and physical applications.

- Our construction of generalised pp-waves only has, like Obukhov's construction, two non-zero irreducible pieces of curvature. However, the non-zero pieces in our construction differ from the non-zero pieces in Obukhov's construction (see above).
- One more very interesting property of Obukhov's gravitational wave solutions is that they provide a minimal generalisation of the pseudoinstanton (see Definition 1.4.1), in the sense that nonmetricity does not vanish and that curvature has two non-zero pieces.

As these generalised pp-waves with nonmetricity constructed by Obukhov were a response to the results presented by Pasic and Vassiliev in [69], we hope to also respond in the near future to his work by trying to further expand the generalised pp-waves presented in Chapter 3.2 to non-metric-compatible spacetimes and to then compare these to Obukhov's spacetimes from [67].

Appendix A

Bianchi Identity for Curvature

In this Appendix we present the derivation of the Bianchi identity for curvature (4.15) used in producing the explicit form of the second field equation (1.4) in Section 4.1. We will use the assumptions on the curvature used in deriving the explicit representation of field equations (1.3), (1.4) given in the beginning of Section 4.1, namely

- (i) Our spacetime is metric compatible.
- (ii) Torsion is purely tensor, see (3.21).
- (iii) Curvature has symmetries (3.25), (3.26).
- (iv) Scalar curvature is zero.

We start from the Bianchi identity for curvature

$$(\partial_\xi + [\Gamma_\xi, \cdot])R_{\mu\nu} + (\partial_\nu + [\Gamma_\nu, \cdot])R_{\xi\mu} + (\partial_\mu + [\Gamma_\mu, \cdot])R_{\nu\xi} = 0$$

where we hide the Lie algebra indices of curvature by using matrix notation as in (4.9). Using the assumptions (i)-(iv), we can represent the curvature as

$$R_{\kappa\lambda\mu\nu} = \frac{1}{2}(g_{\kappa\mu}Ric_{\lambda\nu} - g_{\lambda\mu}Ric_{\kappa\nu} + g_{\lambda\nu}Ric_{\kappa\mu} - g_{\kappa\nu}Ric_{\lambda\mu}) + \mathcal{W}_{\kappa\lambda\mu\nu}. \quad (\text{A.1})$$

Remark A.1. The following calculation is done here to demonstrate how the result was obtained. For an alternative and shorter exposition of the Bianchi's identity, see for example Schouten [77].

Remark A.2. Note that in equation (A.1) we can use Ric or $\mathcal{R}ic$ interchangeably, as under assumption (iv) scalar curvature \mathcal{R} is zero.

Substituting (A.1) into the Bianchi identity for curvature, after a lengthy but straightforward calculation, we obtain that the Bianchi identity becomes

$$\begin{aligned}
& \frac{1}{2} [\delta^\kappa_\mu (\nabla_\xi Ric_{\lambda\nu} + \Gamma^\eta_{\xi\nu} Ric_{\lambda\eta}) + g_{\lambda\mu} (-\nabla_\xi Ric^\kappa_\nu - \Gamma^\eta_{\xi\nu} Ric^\kappa_\eta) \\
& + g_{\lambda\nu} (\nabla_\xi Ric^\kappa_\mu + \Gamma^\eta_{\xi\mu} Ric^\kappa_\eta) + \delta^\kappa_\nu (-\nabla_\xi Ric_{\lambda\mu} - \Gamma^\eta_{\xi\mu} Ric_{\lambda\eta}) \\
& + \delta^\kappa_\xi (\nabla_\nu Ric_{\lambda\mu} + \Gamma^\eta_{\nu\mu} Ric_{\lambda\eta}) + \delta^\kappa_\mu (-\nabla_\nu Ric_{\lambda\xi} - \Gamma^\eta_{\nu\xi} Ric_{\lambda\eta}) \\
& + g_{\lambda\mu} (\nabla_\nu Ric^\kappa_\xi + \Gamma^\eta_{\nu\xi} Ric^\kappa_\eta) + g_{\lambda\xi} (-\nabla_\nu Ric^\kappa_\mu - \Gamma^\eta_{\nu\mu} Ric^\kappa_\eta) \\
& + \delta^\kappa_\nu (\nabla_\mu Ric_{\lambda\xi} + \Gamma^\eta_{\mu\xi} Ric_{\lambda\eta}) + \delta^\kappa_\xi (-\nabla_\mu Ric_{\lambda\nu} - \Gamma^\eta_{\mu\nu} Ric_{\lambda\eta}) \\
& + g_{\lambda\xi} (\nabla_\mu Ric^\kappa_\nu + \Gamma^\eta_{\mu\nu} Ric^\kappa_\eta) + g_{\lambda\nu} (-\nabla_\mu Ric^\kappa_\xi - \Gamma^\eta_{\mu\xi} Ric^\kappa_\eta) \\
& + \Gamma^\kappa_{\nu\eta} (\delta^\eta_\xi Ric_{\lambda\mu} - \delta^\eta_\mu Ric_{\lambda\xi}) + \Gamma^\eta_{\nu\lambda} (g_{\eta\xi} Ric^\kappa_\mu - g_{\eta\mu} Ric^\kappa_\xi) \\
& + \Gamma^\kappa_{\xi\eta} (\delta^\eta_\mu Ric_{\lambda\nu} - \delta^\eta_\nu Ric_{\lambda\mu}) + \Gamma^\eta_{\xi\lambda} (g_{\eta\mu} Ric^\kappa_\nu - g_{\eta\nu} Ric^\kappa_\mu) \\
& + \Gamma^\kappa_{\mu\eta} (\delta^\eta_\nu Ric_{\lambda\xi} - \delta^\eta_\xi Ric_{\lambda\nu}) + \Gamma^\eta_{\mu\lambda} (g_{\eta\nu} Ric^\kappa_\xi - g_{\eta\xi} Ric^\kappa_\nu)] \\
& + \partial_\xi \mathcal{W}^\kappa_{\lambda\mu\nu} + \partial_\nu \mathcal{W}^\kappa_{\lambda\xi\mu} + \partial_\mu \mathcal{W}^\kappa_{\lambda\nu\xi} + \Gamma^\kappa_{\nu\eta} \mathcal{W}^\eta_{\lambda\xi\mu} \\
& - \Gamma^\eta_{\nu\lambda} \mathcal{W}^\kappa_{\eta\xi\mu} + \Gamma^\kappa_{\xi\eta} \mathcal{W}^\eta_{\lambda\mu\nu} - \Gamma^\eta_{\xi\lambda} \mathcal{W}^\kappa_{\eta\mu\nu} + \Gamma^\kappa_{\mu\eta} \mathcal{W}^\eta_{\lambda\nu\xi} - \Gamma^\eta_{\mu\lambda} \mathcal{W}^\kappa_{\eta\nu\xi} = 0.
\end{aligned}$$

Now we make one contraction, between κ and μ . As this is a very long equality now, we break it down into three manageable pieces. First we handle the terms involving the Weyl piece of curvature \mathcal{W} .

$$\begin{aligned}
& \partial_\xi \mathcal{W}^\mu_{\lambda\mu\nu} + \partial_\nu \mathcal{W}^\mu_{\lambda\xi\mu} + \partial_\mu \mathcal{W}^\mu_{\lambda\nu\xi} + \Gamma^\mu_{\nu\eta} \mathcal{W}^\eta_{\lambda\xi\mu} \\
& - \Gamma^\eta_{\nu\lambda} \mathcal{W}^\mu_{\eta\xi\mu} + \Gamma^\mu_{\xi\eta} \mathcal{W}^\eta_{\lambda\mu\nu} - \Gamma^\eta_{\xi\lambda} \mathcal{W}^\mu_{\eta\mu\nu} + \Gamma^\mu_{\mu\eta} \mathcal{W}^\eta_{\lambda\nu\xi} - \Gamma^\eta_{\mu\lambda} \mathcal{W}^\mu_{\eta\nu\xi} \\
& = \nabla_\mu \mathcal{W}^\mu_{\lambda\nu\xi} + \mathcal{W}^\mu_{\lambda\xi\eta} (\Gamma^\eta_{\nu\mu} - \Gamma^\eta_{\mu\nu}) + \mathcal{W}^\mu_{\lambda\nu\eta} (\Gamma^\eta_{\mu\xi} - \Gamma^\eta_{\xi\mu}) \\
& = \nabla_\mu \mathcal{W}^\mu_{\lambda\nu\xi} + \mathcal{W}^\mu_{\lambda\xi\eta} (K^\eta_{\nu\mu} - K^\eta_{\mu\nu}) + \mathcal{W}^\mu_{\lambda\nu\eta} (K^\eta_{\mu\xi} - K^\eta_{\xi\mu})
\end{aligned}$$

completing the covariant derivative and using the fact that the Levi-Civita part of the connection disappears due to symmetry and using contortion for simplicity for the torsion generated part of the connection.

The terms involving the covariant derivative of Ricci ∇Ric become

$$\begin{aligned} & \frac{1}{2}[4\nabla_\xi Ric_{\lambda\nu} - g_{\lambda\mu}\nabla_\xi Ric^\mu{}_\nu + g_{\lambda\nu}\nabla_\xi Ric^\mu{}_\mu - \delta^\mu{}_\nu\nabla_\xi Ric_{\lambda\mu} + \delta^\mu{}_\xi\nabla_\nu Ric_{\lambda\mu} - 4\nabla_\nu Ric_{\lambda\xi} \\ & + g_{\lambda\mu}\nabla_\nu Ric^\mu{}_\xi - g_{\lambda\xi}\nabla_\nu Ric^\mu{}_\mu + \delta^\mu{}_\nu\nabla_\mu Ric_{\lambda\xi} - \delta^\mu{}_\xi\nabla_\mu Ric_{\lambda\nu} + g_{\lambda\xi}\nabla_\mu Ric^\mu{}_\nu - g_{\lambda\nu}\nabla_\mu Ric^\mu{}_\xi] \\ & = \frac{1}{2}[\nabla_\xi Ric_{\lambda\nu} - \nabla_\nu Ric_{\lambda\xi} + g_{\lambda\xi}\nabla_\mu Ric^\mu{}_\nu - g_{\lambda\nu}\nabla_\mu Ric^\mu{}_\xi] \end{aligned}$$

The other terms from the Bianchi identity are

$$\begin{aligned} & \frac{1}{2}[4Ric_{\lambda\eta}(\Gamma^\eta{}_{\xi\nu} - \Gamma^\eta{}_{\nu\xi}) + g_{\lambda\mu}Ric^\mu{}_\eta(\Gamma^\eta{}_{\nu\xi} - \Gamma^\eta{}_{\xi\nu}) + g_{\lambda\nu}Ric^\mu{}_\eta(\Gamma^\eta{}_{\xi\mu} - \Gamma^\eta{}_{\mu\xi}) \\ & + \delta^\mu{}_\nu Ric_{\lambda\eta}(\Gamma^\eta{}_{\mu\xi} - \Gamma^\eta{}_{\xi\mu}) + \delta^\mu{}_\xi Ric_{\lambda\eta}(\Gamma^\eta{}_{\nu\mu} - \Gamma^\eta{}_{\mu\nu}) + g_{\lambda\xi}Ric^\mu{}_\eta(\Gamma^\eta{}_{\mu\nu} - \Gamma^\eta{}_{\nu\mu}) \\ & + Ric_{\lambda\mu}(\Gamma^\mu{}_{\nu\xi} - \Gamma^\mu{}_{\xi\nu}) + Ric_{\lambda\xi}(\Gamma^\mu{}_{\mu\nu} - \Gamma^\mu{}_{\nu\mu}) + Ric^\mu{}_\mu(\Gamma_{\xi\nu\lambda} - \Gamma_{\nu\xi\lambda}) \\ & + Ric^\mu{}_\xi(\Gamma_{\nu\mu\lambda} - \Gamma_{\mu\nu\lambda}) + Ric_{\lambda\nu}(\Gamma^\mu{}_{\xi\mu} - \Gamma^\mu{}_{\mu\xi}) + Ric^\mu{}_\nu(\Gamma_{\mu\xi\lambda} - \Gamma_{\xi\mu\lambda})] \\ & = \frac{1}{2}[Ric^\mu{}_\eta(g_{\lambda\xi}K^\eta{}_{\mu\nu} - g_{\lambda\nu}K^\eta{}_{\mu\xi}) + Ric^\mu{}_\xi(K_{\nu\mu\lambda} - K_{\mu\nu\lambda}) + Ric^\mu{}_\nu(K_{\mu\xi\lambda} - K_{\xi\mu\lambda})], \end{aligned}$$

where again we can disregard the Levi-Civita part of the connection due to symmetry. Also, since we have assumed that torsion is purely tensor, the terms with a contraction in the contortion disappear.

Putting all these calculations together we get that the Bianchi identity for curvature is

$$\begin{aligned} & \frac{1}{2}[\nabla_\xi Ric_{\lambda\nu} - \nabla_\nu Ric_{\lambda\xi} + g_{\lambda\xi}\nabla_\mu Ric^\mu{}_\nu - g_{\lambda\nu}\nabla_\mu Ric^\mu{}_\xi \\ & + Ric^\mu{}_\eta(g_{\lambda\xi}K^\eta{}_{\mu\nu} - g_{\lambda\nu}K^\eta{}_{\mu\xi}) + Ric^\mu{}_\xi(K_{\nu\mu\lambda} - K_{\mu\nu\lambda}) + Ric^\mu{}_\nu(K_{\mu\xi\lambda} - K_{\xi\mu\lambda}) \\ & + \nabla_\mu \mathcal{W}^\mu{}_{\lambda\nu\xi} + \mathcal{W}^\mu{}_{\lambda\xi\eta}(K^\eta{}_{\nu\mu} - K^\eta{}_{\mu\nu}) + \mathcal{W}^\mu{}_{\lambda\nu\eta}(K^\eta{}_{\mu\xi} - K^\eta{}_{\xi\mu}) = 0. \end{aligned} \quad (\text{A.2})$$

As we assume that torsion is purely tensor, we have that $K_{\alpha\beta\gamma} = T_{\beta\alpha\gamma}$. If we substitute contortion for torsion in equation (A.2), and rename some indices, we obtain formula (4.15).

Making another contraction here between ξ and λ in equation (A.2) gives us the following formula

$$\nabla_\mu Ric^\mu{}_\nu = -Ric^\mu{}_\eta K^\eta{}_{\mu\nu} - \frac{1}{2}\mathcal{W}^{\mu\alpha}{}_{\nu\eta}(K^\eta{}_{\mu\alpha} - K^\eta{}_{\alpha\mu}). \quad (\text{A.3})$$

Substituting equation (A.3) into (A.2) yields

$$\begin{aligned}
\nabla_{\mu} \mathcal{W}^{\mu}{}_{\nu\kappa\lambda} &= \mathcal{W}^{\mu}{}_{\nu\lambda\eta} (K^{\eta}{}_{\mu\kappa} - K^{\eta}{}_{\kappa\mu}) + \mathcal{W}^{\mu}{}_{\nu\kappa\eta} (K^{\eta}{}_{\lambda\mu} - K^{\eta}{}_{\mu\lambda}) \\
&\quad + \frac{1}{4} (K^{\eta}{}_{\alpha\mu} - K^{\eta}{}_{\mu\alpha}) (g_{\nu\kappa} \mathcal{W}^{\mu\alpha}{}_{\lambda\eta} - g_{\nu\lambda} \mathcal{W}^{\mu\alpha}{}_{\kappa\eta}) \\
&\quad + \frac{1}{2} [\nabla_{\kappa} Ric_{\nu\lambda} - \nabla_{\lambda} Ric_{\nu\kappa} + Ric^{\mu}{}_{\lambda} (K_{\mu\kappa\nu} - K_{\kappa\mu\nu}) + Ric^{\mu}{}_{\kappa} (K_{\lambda\mu\nu} - K_{\mu\lambda\nu})],
\end{aligned}$$

which, after substituting torsion for contortion and renaming of some indices, is identical to equation (4.17), i.e.

$$\begin{aligned}
\nabla_{\eta} \mathcal{W}^{\eta}{}_{\mu\lambda\kappa} &= \mathcal{W}^{\eta}{}_{\mu\kappa\xi} (T_{\eta}{}^{\xi}{}_{\lambda} - T_{\lambda}{}^{\xi}{}_{\eta}) + \mathcal{W}^{\eta}{}_{\mu\lambda\xi} (T_{\kappa}{}^{\xi}{}_{\eta} - T_{\eta}{}^{\xi}{}_{\kappa}) \\
&\quad + \frac{1}{4} (T_{\zeta}{}^{\xi}{}_{\eta} - T_{\eta}{}^{\xi}{}_{\zeta}) (g_{\mu\lambda} \mathcal{W}^{\eta\zeta}{}_{\kappa\xi} - g_{\mu\kappa} \mathcal{W}^{\eta\zeta}{}_{\lambda\xi}) \\
&\quad + \frac{1}{2} [\nabla_{\lambda} Ric_{\mu\kappa} - \nabla_{\kappa} Ric_{\mu\lambda} + Ric^{\eta}{}_{\kappa} (T_{\lambda\eta\mu} - T_{\eta\lambda\mu}) + Ric^{\eta}{}_{\lambda} (T_{\eta\kappa\mu} - T_{\kappa\eta\mu})].
\end{aligned}$$

Appendix B

Weyl's Equation

The generally accepted point of view [30, 39, 40, 41, 46] is that a massless neutrino field in a metric compatible spacetime with or without torsion is described by the action, see formula (11) of [30].

$$S_{\text{neutrino}} := 2i \int \left(\xi^a \sigma^\mu{}_{ab} (\nabla_\mu \bar{\xi}^b) - (\nabla_\mu \xi^a) \sigma^\mu{}_{ab} \bar{\xi}^b \right). \quad (\text{B.1})$$

Note that in this Appendix we do not assume that torsion is zero, but is considered to be *fixed*.

We first vary the action (B.1) with respect to the spinor ξ , while keeping torsion and the metric *fixed*. A straightforward calculation produces Weyl's equation

$$\sigma^\mu{}_{ab} \nabla_\mu \xi^a - \frac{1}{2} T^\eta{}_{\eta\mu} \sigma^\mu{}_{ab} \xi^a = 0 \quad (\text{B.2})$$

which can be equivalently rewritten as

$$\sigma^\mu{}_{ab} \{\nabla\}_\mu \xi^a \pm \frac{i}{4} \varepsilon_{\alpha\beta\gamma\delta} T^{\alpha\beta\gamma} \sigma^\delta{}_{ab} \xi^a = 0 \quad (\text{B.3})$$

where $\{\nabla\}$ is the covariant derivative with respect to the Levi-Civita connection.

B.1 Energy momentum tensor

In this subsection we give the derivation of the energy momentum tensor of the action S_{neutrino} , where we vary the metric keeping the spinor fixed.

If the covariant metric changes in the following way

$$g_{\alpha\beta} \mapsto g_{\alpha\beta} + \delta g_{\alpha\beta}, \quad (\text{B.4})$$

then the contravariant metric changes as

$$g^{\alpha\beta} \mapsto g^{\alpha\beta} - g^{\alpha\alpha'} (\delta g_{\alpha'\beta'}) g^{\beta\beta'}. \quad (\text{B.5})$$

We are interested to see what happens to the Pauli matrices when varying the metric. It is easy to see that they transform in the following way

$$\sigma_\alpha \mapsto \sigma_\alpha + \frac{1}{2} \delta g_{\alpha\beta} g^{\beta\gamma} \sigma_\gamma, \quad (\text{B.6})$$

$$\sigma^\alpha \mapsto \sigma^\alpha - \frac{1}{2} g^{\alpha\beta} (\delta g_{\beta\gamma}) \sigma^\gamma. \quad (\text{B.7})$$

Formulae describe a ‘symmetric’ variation of the Pauli matrices caused by the (symmetric) variation of the (symmetric) metric. We did not include an ‘antisymmetric’ variation of the Pauli matrices in formulae (B.6), (B.7), as this would mean a variation of our choice of Pauli matrices for given metric (which would also have to be accompanied by a corresponding variation of the spinors).

Remark B.1. We do most of the following calculations, for brevity and clarity, under the assumption that the metric is the Minkowski metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and that the connection is Levi-Civita, using the standard choice of Pauli matrices (2.48).

These calculations we can of course perform for the general connection incorporating torsion. We do not present those calculations here as they are very cumbersome and technical – and for the purposes of the Section 5.2.1 (where this result is only used within the thesis) these are unnecessary, as Section 5.2.1 only considers the classical Einstein–Weyl model.

Now we need to look at the $\delta\Gamma^\alpha_{\beta\gamma}$. We will show that the connection transforms in the following way

$$\delta\Gamma^\kappa_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} (\nabla_\mu \delta g_{\lambda\nu} + \nabla_\nu \delta g_{\lambda\mu} - \nabla_\lambda \delta g_{\mu\nu}). \quad (\text{B.8})$$

Using the definition of the Levi-Civita connection, we have that

$$\begin{aligned}\delta\Gamma^\kappa_{\mu\nu} &= \frac{1}{2}\delta(g^{\kappa\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})) = \\ &= \frac{1}{2}\delta g^{\kappa\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) + \frac{1}{2}g^{\kappa\lambda}(\partial_\mu \delta g_{\lambda\nu} + \partial_\nu \delta g_{\lambda\mu} - \partial_\lambda \delta g_{\mu\nu}).\end{aligned}$$

Using equation (B.5) and using metric compatibility ($\nabla g \equiv 0$), we get that

$$\delta\Gamma^\kappa_{\mu\nu} = -g^{\kappa\lambda}\Gamma^\eta_{\mu\nu}\delta g_{\lambda\eta} + \frac{1}{2}g^{\kappa\lambda}(\partial_\mu \delta g_{\lambda\nu} + \partial_\nu \delta g_{\lambda\mu} - \partial_\lambda \delta g_{\mu\nu}),$$

which is equivalent to equation (B.8).

Next we need to see what happens to the covariant derivative of ξ under metric variation. We first want to prove the following

Lemma B.1.1. *The variation of the covariant derivative of ξ with respect to the metric is*

$$\delta\nabla_\mu \xi^a = \frac{1}{8}(\sigma_\alpha^{ad}\sigma^\beta_{cd} - \sigma^{\beta ad}\sigma_{\alpha cd})\xi^c \delta\Gamma^\alpha_{\mu\beta}. \quad (\text{B.9})$$

Proof. Using equation (2.47), we get that

$$\delta\nabla_\mu \xi^a = \delta \left(\partial_\mu \xi^a + \frac{1}{4}\sigma_\alpha^{ad} (\partial_\mu \sigma^\alpha_{cd} + \Gamma^\alpha_{\mu\beta}\sigma^\beta_{cd}) \xi^c \right),$$

and as ξ does not contribute to the variation, we get

$$\begin{aligned}4\delta\nabla_\mu \xi^a &= (\delta\sigma_\alpha^{ad}) (\partial_\mu \sigma^\alpha_{cd} + \Gamma^\alpha_{\mu\beta}\sigma^\beta_{cd}) \xi^c \\ &+ \sigma_\alpha^{ad} (\partial_\mu (\delta\sigma^\alpha_{cd}) + (\delta\Gamma^\alpha_{\mu\beta})\sigma^\beta_{cd} + \Gamma^\alpha_{\mu\beta}\delta\sigma^\beta_{cd}) \xi^c.\end{aligned}$$

Under the assumptions in Remark B.1 quite a few terms disappear and we obtain

$$4\delta\nabla_\mu \xi^a = \sigma_\alpha^{ad} (\partial_\mu (\delta\sigma^\alpha_{cd}) + (\delta\Gamma^\alpha_{\mu\beta})\sigma^\beta_{cd}) \xi^c.$$

We look at these terms separately. According to equation (B.5), we have that

$$\partial_\mu (\delta\sigma^\alpha_{cd}) = -\frac{1}{2}\partial_\mu (g^{\alpha\eta}(\delta g_{\eta\beta})\sigma^\beta_{cd}) = -\frac{1}{2}g^{\alpha\eta}(\delta\partial_\mu g_{\eta\xi})\sigma^\xi_{cd}$$

and since by metric compatibility we have $\nabla g = 0$, i.e.

$$\partial_\mu g_{\eta\xi} = \Gamma^\zeta_{\mu\eta} g_{\zeta\xi} + \Gamma^\zeta_{\mu\xi} g_{\eta\zeta}.$$

Putting this in the above equality we get that

$$\partial_\mu(\delta\sigma^\alpha_{cd}) = -\frac{1}{2}g^{\alpha\eta}\sigma_{\zeta cd}\delta\Gamma^\zeta_{\mu\eta} - \frac{1}{2}\delta^\alpha_\zeta\sigma^\xi_{cd}\delta\Gamma^\zeta_{\mu\xi}.$$

Combining this with the formula for the variation of $\nabla\xi$, we get

$$4\delta\nabla_\mu\xi^a = -\frac{1}{2}\sigma^{\beta ad}\sigma_{\alpha cd}\xi^c\delta\Gamma^\alpha_{\mu\beta} - \frac{1}{2}\sigma_\alpha^{ad}\sigma^\beta_{cd}\xi^c\delta\Gamma^\alpha_{\mu\beta} + (\delta\Gamma^\alpha_{\mu\beta})\sigma^\beta_{cd}\sigma_\alpha^{ad}\xi^c$$

or

$$4\delta\nabla_\mu\xi^a = \frac{1}{2}\sigma_\alpha^{ad}\sigma^\beta_{cd}\xi^c\delta\Gamma^\alpha_{\mu\beta} - \frac{1}{2}\sigma^{\beta ad}\sigma_{\alpha cd}\xi^c\delta\Gamma^\alpha_{\mu\beta}$$

which is exactly equation (B.9). ■

We now want to combine equations (B.8) and (B.9):

$$\delta\nabla_\mu\xi^a = \frac{1}{16}(\sigma_\alpha^{ad}\sigma^\beta_{cd} - \sigma^{\beta ad}\sigma_{\alpha cd})\xi^c g^{\alpha\lambda}(\partial_\mu\delta g_{\lambda\beta} + \partial_\beta\delta g_{\lambda\mu} - \partial_\lambda\delta g_{\mu\beta}),$$

or

$$\delta\nabla_\mu\xi^a = \frac{1}{16}(\sigma^{\lambda ad}\sigma^\beta_{cd} - \sigma^{\beta ad}\sigma^\lambda_{cd})\xi^c(\partial_\mu\delta g_{\lambda\beta} + \partial_\beta\delta g_{\lambda\mu} - \partial_\lambda\delta g_{\mu\beta}).$$

As the first derivative is symmetric over λ, β and the Pauli matrices are antisymmetric over these indices, we get

$$(\sigma^{\lambda ad}\sigma^\beta_{cd} - \sigma^{\beta ad}\sigma^\lambda_{cd})\partial_\mu\delta g_{\lambda\beta} = -(\sigma^{\lambda ad}\sigma^\beta_{cd} - \sigma^{\beta ad}\sigma^\lambda_{cd})\partial_\mu\delta g_{\beta\lambda} = 0.$$

Hence,

$$\delta\nabla_\mu\xi^a = \frac{1}{16}(\sigma^{\lambda ad}\sigma^\beta_{cd} - \sigma^{\beta ad}\sigma^\lambda_{cd})\xi^c\partial_\beta\delta g_{\lambda\mu} - \frac{1}{16}(\sigma^{\beta ad}\sigma^\lambda_{cd} - \sigma^{\lambda ad}\sigma^\beta_{cd})\xi^c\partial_\beta\delta g_{\mu\lambda}.$$

So finally, we get the formula for the variation of the covariant derivative:

$$\delta\{\nabla\}_\mu\xi^a = \frac{1}{8}\xi^c(\sigma^{\alpha ad}\sigma^\beta_{cd} - \sigma^{\beta ad}\sigma^\alpha_{cd})\partial_\beta\delta g_{\mu\alpha}. \quad (\text{B.10})$$

The variation of the covariant derivative of a dotted spinor is

$$\delta\{\nabla\}_\mu\bar{\xi}^b = \frac{1}{8}\bar{\xi}^d(\sigma^{\alpha cb}\sigma^\beta_{cd} - \sigma^{\beta cb}\sigma^\alpha_{cd})\partial_\beta\delta g_{\mu\alpha}. \quad (\text{B.11})$$

Note that in the above equations we are using the curly bracket notation for denoting the covariant derivative with respect to the Levi-Civita connection. After doing all these preliminary calculations, finally we get to vary the action (B.1) with respect to the metric.

Note that, as usual

$$\delta\sqrt{|\det g|} = \frac{1}{2}\sqrt{|\det g|}g^{\alpha\beta}(\delta g_{\alpha\beta}).$$

Now looking at the variation of the whole action $S_{neutrino}$, we get

$$\begin{aligned} \delta S &= 2i\delta \int \left(\xi^a \sigma^\eta_{ab} (\{\nabla\}_\eta \bar{\xi}^b) - (\{\nabla\}_\eta \xi^a) \sigma^\eta_{ab} \bar{\xi}^b \right) \sqrt{|\det g|} \\ &= 2i \int \left(\xi^a (\delta\sigma^\eta_{ab}) (\{\nabla\}_\eta \bar{\xi}^b) + \xi^a \sigma^\eta_{ab} (\delta\{\nabla\}_\eta \bar{\xi}^b) \right. \\ &\quad \left. - (\delta\{\nabla\}_\eta \xi^a) \sigma^\eta_{ab} \bar{\xi}^b - (\{\nabla\}_\eta \xi^a) (\delta\sigma^\eta_{ab}) \bar{\xi}^b \right) \sqrt{|\det g|} \\ &\quad + 2i \int \left(\xi^a \sigma^\eta_{ab} (\{\nabla\}_\eta \bar{\xi}^b) - (\{\nabla\}_\eta \xi^a) \sigma^\eta_{ab} \bar{\xi}^b \right) \frac{1}{2} g^{\mu\nu} \delta g_{\mu\nu} \sqrt{|\det g|} \end{aligned}$$

and using equation (B.7) we get

$$\begin{aligned} \delta S &= 2i \int \left(-\frac{1}{4} \xi^a g^{\eta\mu} \sigma^\nu_{ab} (\{\nabla\}_\eta \bar{\xi}^b) - \frac{1}{4} \xi^a g^{\eta\nu} \sigma^\mu_{ab} (\{\nabla\}_\eta \bar{\xi}^b) + \frac{1}{4} (\{\nabla\}_\eta \xi^a) g^{\eta\mu} \sigma^\nu_{ab} \bar{\xi}^b \right. \\ &\quad \left. + \frac{1}{4} (\{\nabla\}_\eta \xi^a) g^{\eta\nu} \sigma^\mu_{ab} \bar{\xi}^b + \frac{1}{2} \xi^a \sigma^\eta_{ab} (\{\nabla\}_\eta \bar{\xi}^b) g^{\mu\nu} - \frac{1}{2} (\{\nabla\}_\eta \xi^a) \sigma^\eta_{ab} \bar{\xi}^b g^{\mu\nu} \right) \delta g_{\mu\nu} \\ &\quad + \xi^a \sigma^\eta_{ab} (\delta\{\nabla\}_\eta \bar{\xi}^b) - (\delta\{\nabla\}_\eta \xi^a) \sigma^\eta_{ab} \bar{\xi}^b. \end{aligned}$$

Now we look at the terms involving the variation of $\{\nabla\}\xi$ on their own. Using equations (B.10), (B.11) and renaming some indices we get that

$$\begin{aligned}
I_1 &= 2i \int \xi^a \sigma^\eta_{ab} (\delta\{\nabla\}_\eta \bar{\xi}^b) - 2i \int (\delta\{\nabla\}_\eta \xi^a) \sigma^\eta_{ab} \bar{\xi}^b \\
&= 2i \int \xi^a \sigma^\mu_{ab} \frac{1}{8} \bar{\xi}^d (\sigma^{\nu cb} \sigma^\eta_{cd} - \sigma^{\eta cb} \sigma^\nu_{cd}) \partial_\eta \delta g_{\mu\nu} \\
&\quad - 2i \int \frac{1}{8} \xi^c (\sigma^{\nu ad} \sigma^\eta_{cd} - \sigma^{\eta ad} \sigma^\nu_{cd}) \partial_\eta \delta g_{\mu\nu} \sigma^\mu_{ab} \bar{\xi}^b.
\end{aligned}$$

Now we integrate by parts and use our above simplifications from Remark B.1, to get

$$\begin{aligned}
I_1 &= -2i \int \frac{1}{8} (\xi^a \{\nabla\}_\eta \bar{\xi}^d + \bar{\xi}^d \{\nabla\}_\eta \xi^a) \sigma^\mu_{ab} (\sigma^{\nu cb} \sigma^\eta_{cd} - \sigma^{\eta cb} \sigma^\nu_{cd}) \delta g_{\mu\nu} \\
&\quad + 2i \int \frac{1}{8} (\xi^c \{\nabla\}_\eta \bar{\xi}^b + \bar{\xi}^b \{\nabla\}_\eta \xi^c) \sigma^\mu_{ab} (\sigma^{\nu ad} \sigma^\eta_{cd} - \sigma^{\eta ad} \sigma^\nu_{cd}) \delta g_{\mu\nu},
\end{aligned}$$

and renaming spinor indices to match we get

$$\begin{aligned}
I_1 &= -2i \int \frac{1}{8} (\xi^a \{\nabla\}_\eta \bar{\xi}^b + \bar{\xi}^b \{\nabla\}_\eta \xi^a) \sigma^\mu_{ad} (\sigma^{\nu cd} \sigma^\eta_{cb} - \sigma^{\eta cd} \sigma^\nu_{cb}) \delta g_{\mu\nu} \\
&\quad + 2i \int \frac{1}{8} (\xi^a \{\nabla\}_\eta \bar{\xi}^b + \bar{\xi}^b \{\nabla\}_\eta \xi^a) \sigma^\mu_{cb} (\sigma^{\nu cd} \sigma^\eta_{ad} - \sigma^{\eta cd} \sigma^\nu_{ad}) \delta g_{\mu\nu} \\
&= 2i \int \frac{1}{8} \bar{\xi}^b \{\nabla\}_\eta \xi^a (-\sigma^\mu_{ad} \sigma^{\nu cd} \sigma^\eta_{cb} + \sigma^\mu_{ad} \sigma^{\eta cd} \sigma^\nu_{cb} + \sigma^\eta_{ad} \sigma^{\nu cd} \sigma^\mu_{cb} - \sigma^\nu_{ad} \sigma^{\eta cd} \sigma^\mu_{cb}) \delta g_{\mu\nu} \\
&\quad + 2i \int \frac{1}{8} \bar{\xi}^b \{\nabla\}_\eta \xi^a (-\sigma^\mu_{ad} \sigma^{\nu cd} \sigma^\eta_{cb} + \sigma^\mu_{ad} \sigma^{\eta cd} \sigma^\nu_{cb} + \sigma^\eta_{ad} \sigma^{\nu cd} \sigma^\mu_{cb} - \sigma^\nu_{ad} \sigma^{\eta cd} \sigma^\mu_{cb}) \delta g_{\mu\nu}
\end{aligned}$$

and since we have

$$(\sigma^\mu_{ad} \sigma^{\eta cd} \sigma^\nu_{cb} - \sigma^\nu_{ad} \sigma^{\eta cd} \sigma^\mu_{cb}) \delta g_{\mu\nu} = 0,$$

as it is a product of symmetric and antisymmetric tensors, we get

$$I_1 = 2i \int \frac{1}{8} (\bar{\xi}^b \{\nabla\}_\eta \xi^a + \bar{\xi}^b \{\nabla\}_\eta \xi^a) (\sigma^\eta_{ad} \sigma^{\nu cd} \sigma^\mu_{cb} - \sigma^\mu_{ad} \sigma^{\nu cd} \sigma^\eta_{cb}) \delta g_{\mu\nu}.$$

Symmetrizing this we get

$$I_1 = 2i \int \frac{1}{16} (\bar{\xi}^b \{\nabla\}_\eta \xi^a + \bar{\xi}^b \{\nabla\}_\eta \xi^a) \\ (\sigma^\eta_{ad} \sigma^{\nu cd} \sigma^\mu_{cb} + \sigma^\eta_{ad} \sigma^{\mu cd} \sigma^\nu_{cb} - \sigma^\mu_{ad} \sigma^{\nu cd} \sigma^\eta_{cb} - \sigma^\nu_{ad} \sigma^{\mu cd} \sigma^\eta_{cb}) \delta g_{\mu\nu}.$$

A lengthy but straightforward calculation shows that

$$\sigma^\eta_{ad} \sigma^{\nu cd} \sigma^\mu_{cb} + \sigma^\eta_{ad} \sigma^{\mu cd} \sigma^\nu_{cb} - \sigma^\mu_{ad} \sigma^{\nu cd} \sigma^\eta_{cb} - \sigma^\nu_{ad} \sigma^{\mu cd} \sigma^\eta_{cb} = 0. \quad (\text{B.12})$$

Hence, we have shown that the terms involving $\delta\{\nabla\}\xi$ do not contribute to the variation, i.e.

$$I_1 = 2i \int \xi^a \sigma^\eta_{ab} (\delta\{\nabla\}_\eta \bar{\xi}^b) - 2i \int (\delta\{\nabla\}_\eta \xi^a) \sigma^\eta_{ab} \bar{\xi}^b = 0. \quad (\text{B.13})$$

Finally, we return to the variation of the whole action, which after some simplification becomes

$$\frac{\delta S}{\delta g} = 2i \int \left(-\frac{1}{4} \xi^a \sigma^\nu_{ab} \{\nabla\}^\mu \bar{\xi}^b - \frac{1}{4} \xi^a \sigma^\mu_{ab} \{\nabla\}^\nu \bar{\xi}^b + \frac{1}{4} (\{\nabla\}^\mu \xi^a) \sigma^\nu_{ab} \bar{\xi}^b \right. \\ \left. + \frac{1}{4} (\{\nabla\}^\nu \xi^a) \sigma^\mu_{ab} \bar{\xi}^b + \frac{1}{2} \xi^a \sigma^\eta_{ab} (\{\nabla\}_\eta \bar{\xi}^b) g^{\mu\nu} - \frac{1}{2} (\{\nabla\}_\eta \xi^a) \sigma^\eta_{ab} \bar{\xi}^b g^{\mu\nu} \right) \delta g_{\mu\nu},$$

or, after regrouping the terms

$$\frac{\delta S}{\delta g} = \frac{i}{2} \int \left(\sigma^\nu_{ab} ((\{\nabla\}^\mu \xi^a) \bar{\xi}^b - \xi^a \{\nabla\}^\mu \bar{\xi}^b) + \sigma^\mu_{ab} ((\{\nabla\}^\nu \xi^a) \bar{\xi}^b - \xi^a \{\nabla\}^\nu \bar{\xi}^b) \right) \delta g_{\mu\nu} \\ + i \int \left(\xi^a \sigma^\eta_{ab} (\{\nabla\}_\eta \bar{\xi}^b) g^{\mu\nu} - (\{\nabla\}_\eta \xi^a) \sigma^\eta_{ab} \bar{\xi}^b g^{\mu\nu} \right) \delta g_{\mu\nu}.$$

So the energy momentum tensor of the action (B.1) is

$$E^{\mu\nu} = \frac{i}{2} \left[\sigma^\nu_{ab} \left(\bar{\xi}^b \{\nabla\}^\mu \xi^a - \xi^a \{\nabla\}^\mu \bar{\xi}^b \right) + \sigma^\mu_{ab} \left(\bar{\xi}^b \{\nabla\}^\nu \xi^a - \xi^a \{\nabla\}^\nu \bar{\xi}^b \right) \right] \\ + i \left(\xi^a \sigma^\eta_{ab} (\{\nabla\}_\eta \bar{\xi}^b) g^{\mu\nu} - (\{\nabla\}_\eta \xi^a) \sigma^\eta_{ab} \bar{\xi}^b g^{\mu\nu} \right),$$

which is exactly equation (5.13).

Appendix C

Explicit Variations of Certain Quadratic Forms

In this Appendix we provide the details for the variations of the quadratic forms on curvature used in this thesis. Following the principles of metric–affine gravity, we give both the variations with respect to the metric and the connection.

The following result will be useful in all the calculations¹:

Proposition C.0.2. *Let (M, g) be a (pseudo-)Riemannian manifold. Under the variation $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta(g_{\mu\nu})$ and $g_{\mu\nu}, \det g$ change as*

$$1. \delta g^{\mu\nu} = -g^{\mu\kappa} g^{\lambda\nu} \delta g_{\kappa\lambda}$$

$$2. \delta \det g_{\mu\nu} = \det g_{\mu\nu} g^{\mu\nu} \delta g_{\mu\nu} \quad \delta \sqrt{|\det g_{\mu\nu}|} = \frac{1}{2} \sqrt{|\det g_{\mu\nu}|} g^{\mu\nu} \delta g_{\mu\nu}.$$

C.1 Variation of $\int R^\kappa_{\lambda\mu\nu} R^\lambda_{\kappa}{}^{\mu\nu}$

The first variation we look at is very well known, as the variation of the action with respect to Γ produces the Yang–Mills equation for the affine connection, whilst the variation with respect to the metric g produces the so-called complementary Yang–Mills equation for the affine connection.

¹For the proof, see, for example, [61]

C.1.1 Variation with respect to the connection

This is a well known calculation, see for example Section 4 of [87]. Here we provide the details.

As we are varying the curvature independently, it is clear that

$$\delta \int R^\kappa_{\lambda\mu\nu} R^\lambda_{\kappa}{}^{\mu\nu} = 2 \int (\delta R^\kappa_{\lambda\mu\nu}) R^\lambda_{\kappa}{}^{\mu\nu} \sqrt{|\det g|}.$$

From the definition of curvature (2.8), we get

$$\begin{aligned} \delta R^\kappa_{\lambda\mu\nu} &= \partial_\mu (\delta \Gamma^\kappa_{\nu\lambda}) - \partial_\nu (\delta \Gamma^\kappa_{\mu\lambda}) \\ &+ (\delta \Gamma^\kappa_{\mu\eta}) \Gamma^\eta_{\nu\lambda} + \Gamma^\kappa_{\mu\eta} \delta \Gamma^\eta_{\nu\lambda} - (\delta \Gamma^\kappa_{\nu\eta}) \Gamma^\eta_{\mu\lambda} - \Gamma^\kappa_{\nu\eta} \delta \Gamma^\eta_{\mu\lambda}. \end{aligned} \quad (\text{C.1})$$

Substituting equation (C.1) into the action and integrating the first two terms by parts, we get

$$\begin{aligned} \frac{1}{2} \delta \int R^\kappa_{\lambda\mu\nu} R^\lambda_{\kappa}{}^{\mu\nu} &= - \int (\delta \Gamma^\kappa_{\nu\lambda}) \partial_\mu \left(R^\lambda_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} \right) \frac{\sqrt{|\det g|}}{\sqrt{|\det g|}} \\ &+ \int (\delta \Gamma^\kappa_{\mu\lambda}) \partial_\nu \left(R^\lambda_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} \right) \frac{\sqrt{|\det g|}}{\sqrt{|\det g|}} \\ &+ \int (\delta \Gamma^\kappa_{\mu\eta}) \Gamma^\eta_{\nu\lambda} R^\lambda_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} \\ &+ \int \Gamma^\kappa_{\mu\eta} (\delta \Gamma^\eta_{\nu\lambda}) R^\lambda_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} \\ &- \int (\delta \Gamma^\kappa_{\nu\eta}) \Gamma^\eta_{\mu\lambda} R^\lambda_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} \\ &- \int \Gamma^\kappa_{\nu\eta} (\delta \Gamma^\eta_{\mu\lambda}) R^\lambda_{\kappa}{}^{\mu\nu} \sqrt{|\det g|}. \end{aligned}$$

Renaming some indices and using the fact that for any curvature $R^\kappa_{\lambda\mu\nu} = -R^\kappa_{\lambda\nu\mu}$, we get that

$$\begin{aligned} \delta \int R^\kappa_{\lambda\mu\nu} R^\lambda_{\kappa}{}^{\mu\nu} &= 4 \int (\delta \Gamma^\kappa_{\mu\lambda}) \partial_\nu \left(R^\lambda_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} \right) \frac{\sqrt{|\det g|}}{\sqrt{|\det g|}} \\ &+ 4 \int (\delta \Gamma^\kappa_{\mu\lambda}) (\Gamma^\lambda_{\nu\eta} R^\eta_{\kappa}{}^{\mu\nu} - \Gamma^\eta_{\nu\kappa} R^\lambda_{\eta}{}^{\mu\nu}) \sqrt{|\det g|} \\ &= 4 \int (\delta \Gamma_\mu) \frac{1}{\sqrt{|\det g|}} (\partial_\nu + [\Gamma_\nu, \cdot]) \left(\sqrt{|\det g|} R^{\mu\nu} \right) \sqrt{|\det g|}, \end{aligned}$$

which clearly yields the Yang–Mills equation for the affine connection, i.e.

$$\delta \int R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_{\kappa}{}^{\mu\nu} = 4 \int \left((\delta_{\text{YM}} R^{\mu\nu}) (\delta\Gamma)_\mu \right) \quad (\text{C.2})$$

where $(\delta_{\text{YM}} R)^\mu := \frac{1}{\sqrt{|\det g|}} (\partial_\nu + [\Gamma_\nu, \cdot]) \left(\sqrt{|\det g|} R^{\mu\nu} \right)$ is the Yang–Mills divergence.

C.1.2 Variation with respect to the metric

This calculation yields the so called *complementary* Yang–Mills equation for the affine connection. We have to look at all the terms involving the metric, i.e.

$$\begin{aligned} \delta \int R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_{\kappa}{}^{\mu\nu} &= \delta \int R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} \\ &= \int \delta \left(R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_{\kappa\mu'\nu'} g^{\mu\mu'} g^{\nu\nu'} \sqrt{|\det g_{\alpha\beta}|} \right) \\ &= \int R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_{\kappa\mu'\nu'} \left((\delta g^{\mu\mu'}) g^{\nu\nu'} \sqrt{|\det g_{\alpha\beta}|} \right. \\ &\quad \left. + g^{\mu\mu'} (\delta g^{\nu\nu'}) \sqrt{|\det g_{\alpha\beta}|} + g^{\mu\mu'} g^{\nu\nu'} \left(\delta \sqrt{|\det g_{\alpha\beta}|} \right) \right) \end{aligned}$$

Using Proposition C.0.2, we get that

$$\begin{aligned} \delta \int R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_{\kappa}{}^{\mu\nu} &= \int R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_{\kappa\mu'\nu'} (\delta g_{\alpha\beta}) \left[-g^{\nu\nu'} g^{\mu\alpha} g^{\beta\mu'} - g^{\mu\mu'} g^{\nu\alpha} g^{\beta\nu'} \right. \\ &\quad \left. + \frac{1}{2} g^{\mu\mu'} g^{\nu\nu'} g^{\alpha\beta} \right] \\ &= - \int (\delta g_{\alpha\beta}) \left(R^\kappa{}_{\lambda\nu}{}^\alpha R^\lambda{}_{\kappa}{}^{\nu\beta} + R^\kappa{}_{\lambda\mu}{}^\alpha R^\lambda{}_{\kappa}{}^{\mu\beta} - \frac{1}{2} g^{\alpha\beta} R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_{\kappa}{}^{\mu\nu} \right). \end{aligned}$$

Just renaming some contraction indices, we get that

$$\delta \int R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_{\kappa}{}^{\mu\nu} = -2 \int (\delta g_{\alpha\beta}) \left(R^\kappa{}_{\lambda\nu}{}^\alpha R^\lambda{}_{\kappa}{}^{\nu\beta} - \frac{1}{4} g^{\alpha\beta} R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_{\kappa}{}^{\mu\nu} \right). \quad (\text{C.3})$$

Hence we confirm that the complementary Yang–Mills equation is equivalent to

$$H - \frac{1}{4} (\text{tr } H) \delta = 0, \quad (\text{C.4})$$

where $H = H_\nu^\rho := R^\kappa_{\lambda\mu\nu} R^\lambda_{\kappa}{}^{\mu\rho}$ and $\delta = \delta_\nu^\rho$ is the identity tensor.

C.2 Variation of $\int Ric_{\mu\nu} Ric^{\mu\nu}$

Here we look at the variation of Ricci curvature squared. Note that, for brevity and clarity, contortion K (2.3) is used in the calculations.

C.2.1 Variation with respect to the connection

Using equation (C.1) and making one contraction to get Ricci curvature, we get

$$\begin{aligned} \delta Ric_{\mu\nu} &= \partial_\kappa \delta \Gamma^\kappa_{\nu\mu} - \delta^\lambda_{\kappa} \partial_\nu \delta \Gamma^\kappa_{\lambda\mu} \\ &+ \delta^\lambda_{\kappa} (\delta \Gamma^\kappa_{\lambda\eta}) \Gamma^\eta_{\nu\mu} + \Gamma^\kappa_{\kappa\eta} \delta \Gamma^\eta_{\nu\mu} - (\delta \Gamma^\kappa_{\nu\eta}) \Gamma^\eta_{\kappa\mu} - \Gamma^\kappa_{\nu\eta} \delta \Gamma^\eta_{\kappa\mu}. \end{aligned} \quad (C.5)$$

Now, since

$$\delta \int Ric_{\mu\nu} Ric^{\mu\nu} = \int (\delta Ric_{\mu\nu}) Ric^{\mu\nu} + \int Ric_{\mu\nu} (\delta Ric^{\mu\nu}) = 2 \int (\delta Ric_{\mu\nu}) Ric^{\mu\nu},$$

we get, substituting (C.5) into the above and integrating the first two terms by parts, that

$$\begin{aligned} \frac{1}{2} \delta \int Ric_{\mu\nu} Ric^{\mu\nu} &= - \int (\delta \Gamma^\kappa_{\nu\mu}) \partial_\kappa \left(\sqrt{|\det g|} Ric^{\mu\nu} \right) \frac{\sqrt{|\det g|}}{\sqrt{|\det g|}} \\ &+ \int (\delta \Gamma^\kappa_{\lambda\mu}) \delta^\lambda_{\kappa} \partial_\nu \left(\sqrt{|\det g|} Ric^{\mu\nu} \right) \frac{\sqrt{|\det g|}}{\sqrt{|\det g|}} \\ &+ \int \left[\delta^\lambda_{\kappa} (\delta \Gamma^\kappa_{\lambda\eta}) \Gamma^\eta_{\nu\mu} + \Gamma^\kappa_{\kappa\eta} \delta \Gamma^\eta_{\nu\mu} \right. \\ &\quad \left. - (\delta \Gamma^\kappa_{\nu\eta}) \Gamma^\eta_{\kappa\mu} - \Gamma^\kappa_{\nu\eta} \delta \Gamma^\eta_{\kappa\mu} \right] Ric^{\mu\nu} \sqrt{|\det g|} \\ &= \int (\delta \Gamma^\kappa_{\nu\mu}) \left[\left(-\partial_\kappa \left(\sqrt{|\det g|} Ric^{\mu\nu} \right) \right. \right. \\ &\quad \left. \left. + \delta^\nu_{\kappa} \partial_\lambda \left(\sqrt{|\det g|} Ric^{\mu\lambda} \right) \right) \frac{1}{\sqrt{|\det g|}} \right. \\ &\quad \left. + \delta^\nu_{\kappa} \Gamma^\mu_{\lambda\eta} Ric^{\eta\lambda} + \Gamma^\eta_{\eta\kappa} Ric^{\mu\nu} \right] \end{aligned}$$

$$- \Gamma^\mu_{\kappa\eta} Ric^{\eta\nu} - \Gamma^\nu_{\eta\kappa} Ric^{\mu\eta}] \sqrt{|\det g|},$$

by just renaming some indices to put everything together. Now, expanding the partial derivatives, expressing the connection as $\Gamma = \{\Gamma\} + K$ and using the fact that

$$\{\Gamma\}^\alpha_{\beta\alpha} = \frac{\partial_\beta |\det g|}{2|\det g|}, \quad (\text{C.6})$$

we get that

$$\begin{aligned} \frac{1}{2} \delta \int Ric_{\mu\nu} Ric^{\mu\nu} &= \int (\delta \Gamma^\kappa_{\nu\mu}) [-\partial_\kappa Ric^{\mu\nu} - \{\Gamma\}^\eta_{\eta\kappa} Ric^{\mu\nu} \\ &+ \delta^\nu_{\kappa} \partial_\lambda Ric^{\mu\lambda} + \delta^\nu_{\kappa} \{\Gamma\}^\eta_{\eta\lambda} Ric^{\mu\lambda} \\ &+ \delta^\nu_{\kappa} (\{\Gamma\}^\mu_{\lambda\eta} Ric^{\eta\lambda} + K^\mu_{\lambda\eta} Ric^{\eta\lambda}) + \{\Gamma\}^\eta_{\eta\kappa} Ric^{\mu\nu} + K^\eta_{\eta\kappa} Ric^{\mu\nu} \\ &- \{\Gamma\}^\mu_{\kappa\eta} Ric^{\eta\nu} - K^\mu_{\kappa\eta} Ric^{\eta\nu} - \{\Gamma\}^\nu_{\eta\kappa} Ric^{\mu\eta} - K^\nu_{\eta\kappa} Ric^{\mu\eta} \\ &+ \delta^\nu_{\kappa} (\{\Gamma\}^\lambda_{\lambda\eta} Ric^{\mu\eta} - \{\Gamma\}^\lambda_{\lambda\eta} Ric^{\mu\eta})] \\ &= \int (\delta \Gamma^\kappa_{\nu\mu}) [-(\partial_\kappa Ric^{\mu\nu} + \{\Gamma\}^\mu_{\kappa\eta} Ric^{\eta\nu} + \{\Gamma\}^\nu_{\kappa\eta} Ric^{\mu\eta}) \\ &+ \delta^\nu_{\kappa} (\partial_\lambda Ric^{\mu\lambda} + \{\Gamma\}^\mu_{\lambda\eta} Ric^{\eta\lambda} + \{\Gamma\}^\lambda_{\lambda\eta} Ric^{\mu\eta}) \\ &+ \delta^\nu_{\kappa} K^\mu_{\lambda\eta} Ric^{\eta\lambda} + K^\eta_{\eta\kappa} Ric^{\mu\nu} - K^\mu_{\kappa\eta} Ric^{\eta\nu} - K^\nu_{\eta\kappa} Ric^{\mu\eta}] \\ &= \int (\delta \Gamma^\kappa_{\nu\mu}) [-\{\nabla\}_{\kappa} Ric^{\mu\nu} + \delta^\nu_{\kappa} \{\nabla\}_{\lambda} Ric^{\mu\lambda} \\ &+ \delta^\nu_{\kappa} K^\mu_{\lambda\eta} Ric^{\eta\lambda} + K^\eta_{\eta\kappa} Ric^{\mu\nu} - K^\mu_{\kappa\eta} Ric^{\eta\nu} - K^\nu_{\eta\kappa} Ric^{\mu\eta}]. \end{aligned}$$

Alternatively, if we take the version with the covariant derivative with the full connection (∇ instead of $\{\nabla\}$), we have the following

$$\begin{aligned} \delta \int Ric_{\mu\nu} Ric^{\mu\nu} &= 2 \int (\delta \Gamma^\kappa_{\nu\mu}) [-\nabla_{\kappa} Ric^{\mu\nu} + \delta^\nu_{\kappa} \nabla_{\lambda} Ric^{\mu\lambda} \\ &+ K^\nu_{\kappa\eta} Ric^{\mu\eta} - \delta^\nu_{\kappa} K^\lambda_{\lambda\eta} Ric^{\mu\eta} + K^\eta_{\eta\kappa} Ric^{\mu\nu} - K^\nu_{\eta\kappa} Ric^{\mu\eta}]. \end{aligned} \quad (\text{C.7})$$

C.2.2 Variation with respect to the metric

Now we vary $\int Ric_{\mu\nu} Ric^{\mu\nu}$ with respect to the metric. It goes as follows:

$$\delta \int Ric_{\mu\nu} Ric^{\mu\nu} = \int \delta \left(Ric_{\mu\nu} Ric_{\mu'\nu'} g^{\mu\mu'} g^{\nu\nu'} \sqrt{|\det g_{\alpha\beta}|} \right),$$

which then becomes, using Proposition C.0.2,

$$\begin{aligned} \delta \int Ric_{\mu\nu} Ric^{\mu\nu} &= \int Ric_{\mu\nu} Ric_{\mu'\nu'} \left(\sqrt{|\det g|} \right) \left[(\delta g^{\mu\mu'}) g^{\nu\nu'} + g^{\mu\mu'} (\delta g^{\nu\nu'}) \right. \\ &\quad \left. + \frac{1}{2} g^{\mu\mu'} g^{\nu\nu'} g^{\alpha\beta} (\delta g_{\alpha\beta}) \right] \\ &= \int Ric_{\mu\nu} Ric_{\mu'\nu'} (\delta g_{\alpha\beta}) \left[-g^{\nu\nu'} g^{\mu\alpha} g^{\beta\mu'} - g^{\mu\mu'} g^{\nu\alpha} g^{\beta\nu'} \right. \\ &\quad \left. + \frac{1}{2} g^{\mu\mu'} g^{\nu\nu'} g^{\alpha\beta} \right] \\ &= \int (\delta g_{\alpha\beta}) \left(-Ric^{\alpha}_{\nu} Ric^{\beta\nu} - Ric_{\mu}^{\alpha} Ric^{\mu\beta} + \frac{1}{2} Ric_{\mu\nu} Ric^{\mu\nu} g^{\alpha\beta} \right). \end{aligned}$$

Hence, we get that the metric variation of Ricci squared is

$$\delta \int Ric_{\mu\nu} Ric^{\mu\nu} = \int (\delta g_{\alpha\beta}) \left(-Ric^{\alpha}_{\nu} Ric^{\beta\nu} - Ric_{\mu}^{\alpha} Ric^{\mu\beta} + \frac{1}{2} Ric_{\mu\nu} Ric^{\mu\nu} g^{\alpha\beta} \right). \quad (\text{C.8})$$

C.3 Variation of $\int Ric^{\mu\nu} Ric_{\mu\nu}^{(2)}$

These are the calculations used in determining the variation of $\int (P_-, P_+)$ in Section 4.1.1.

C.3.1 Variation with respect to the connection

We are only interested in viewing what happens to $\int (\delta Ric^{(2)}_{\mu\nu}) Ric^{\mu\nu}$, as the other variation doesn't give us anything new. This is also all we need for the purposes of the thesis.

Similarly to the approach used in Appendix C.2, using formula (C.1) and the fact that

$$\delta Ric^{(2)\kappa}_{\nu} = \delta R^{\kappa}_{\lambda\nu}{}^{\lambda} = g^{\lambda\mu} \delta R^{\kappa}_{\lambda\mu\nu}$$

we get that

$$\begin{aligned}
\int (\delta Ric^{(2)}_{\mu\nu}) Ric^{\mu\nu} &= \int (\delta Ric^{(2)\kappa}_{\nu}) Ric_{\kappa}{}^{\nu} \\
&= \int \frac{\sqrt{|\det g|}}{\sqrt{|\det g|}} (\delta \Gamma^{\kappa}_{\nu\lambda}) \\
&\quad \partial_{\mu} \left[\sqrt{|\det g|} (-g^{\lambda\mu} Ric_{\kappa}{}^{\nu} + g^{\lambda\nu} Ric_{\kappa}{}^{\mu}) \right] \\
&+ \int (\delta \Gamma^{\kappa}_{\nu\eta}) (g^{\lambda\nu} \Gamma^{\eta}_{\mu\lambda} Ric_{\kappa}{}^{\mu} - g^{\lambda\mu} \Gamma^{\eta}_{\mu\lambda} Ric_{\kappa}{}^{\nu}) \sqrt{|\det g|} \\
&+ \int (\delta \Gamma^{\eta}_{\nu\lambda}) (g^{\lambda\mu} \Gamma^{\kappa}_{\mu\eta} Ric_{\kappa}{}^{\nu} - g^{\lambda\nu} \Gamma^{\kappa}_{\mu\eta} Ric_{\kappa}{}^{\mu}) \sqrt{|\det g|} \\
&= \int (\delta \Gamma^{\kappa}_{\nu\lambda}) \\
&\quad \left\{ \frac{1}{\sqrt{|\det g|}} \partial_{\mu} \left[\sqrt{|\det g|} (-g^{\lambda\mu} Ric_{\kappa}{}^{\nu} + g^{\lambda\nu} Ric_{\kappa}{}^{\mu}) \right] \right. \\
&\quad + g^{\eta\nu} \Gamma^{\lambda}_{\mu\eta} Ric_{\kappa}{}^{\mu} - g^{\eta\mu} \Gamma^{\lambda}_{\mu\eta} Ric_{\kappa}{}^{\nu} \\
&\quad \left. + g^{\lambda\mu} \Gamma^{\eta}_{\mu\kappa} Ric_{\eta}{}^{\nu} - g^{\lambda\nu} \Gamma^{\eta}_{\mu\kappa} Ric_{\eta}{}^{\mu} \right\} \\
&= \int (\delta \Gamma^{\kappa}_{\nu\lambda}) \left\{ \frac{1}{\sqrt{|\det g|}} [-g^{\lambda\mu} \partial_{\mu} (\sqrt{|\det g|} Ric_{\kappa}{}^{\nu}) \right. \\
&\quad + g^{\lambda\nu} \partial_{\mu} (\sqrt{|\det g|} Ric_{\kappa}{}^{\mu})] \\
&\quad - \partial_{\mu} g^{\lambda\mu} Ric_{\kappa}{}^{\nu} + \partial_{\mu} g^{\lambda\nu} Ric_{\kappa}{}^{\mu} + g^{\eta\nu} \Gamma^{\lambda}_{\mu\eta} Ric_{\kappa}{}^{\mu} \\
&\quad \left. - g^{\eta\mu} \Gamma^{\lambda}_{\mu\eta} Ric_{\kappa}{}^{\nu} + g^{\lambda\mu} \Gamma^{\eta}_{\mu\kappa} Ric_{\eta}{}^{\nu} - g^{\lambda\nu} \Gamma^{\eta}_{\mu\kappa} Ric_{\eta}{}^{\mu} \right\} \\
&= \int (\delta \Gamma^{\kappa}_{\nu\lambda}) \left\{ \frac{1}{\sqrt{|\det g|}} [-g^{\lambda\mu} \partial_{\mu} (\sqrt{|\det g|} Ric_{\kappa}{}^{\nu}) \right. \\
&\quad + g^{\lambda\nu} \partial_{\mu} (\sqrt{|\det g|} Ric_{\kappa}{}^{\mu})] \\
&\quad - (\nabla_{\mu} g)^{\lambda\mu} Ric_{\kappa}{}^{\nu} + (\nabla_{\mu} g)^{\lambda\nu} Ric_{\kappa}{}^{\mu} \\
&\quad + g^{\lambda\eta} \Gamma^{\mu}_{\mu\eta} Ric_{\kappa}{}^{\nu} - g^{\lambda\eta} \Gamma^{\nu}_{\mu\eta} Ric_{\kappa}{}^{\mu} \\
&\quad \left. + g^{\lambda\mu} \Gamma^{\eta}_{\mu\kappa} Ric_{\eta}{}^{\nu} - g^{\lambda\nu} \Gamma^{\eta}_{\mu\kappa} Ric_{\eta}{}^{\mu} \right\},
\end{aligned}$$

completing the covariant derivative for g . Now, differentiating and using equation

(C.6), we get that

$$\begin{aligned}
\int (\delta Ric^{(2)}_{\mu\nu}) Ric^{\mu\nu} &= \int (\delta \Gamma^\kappa_{\nu\lambda}) \left[-g^{\lambda\mu} \frac{\partial_\mu |\det g|}{2|\det g|} Ric_\kappa^\nu - g^{\lambda\mu} \partial_\mu Ric_\kappa^\nu \right. \\
&+ g^{\lambda\nu} \frac{\partial_\mu |\det g|}{2|\det g|} Ric_\kappa^\mu + g^{\lambda\nu} \partial_\mu Ric_\kappa^\mu - (\nabla_\mu g)^{\lambda\mu} Ric_\kappa^\nu \\
&+ (\nabla_\mu g)^{\lambda\nu} Ric_\kappa^\mu + g^{\lambda\eta} \frac{\partial_\eta |\det g|}{2|\det g|} Ric_\kappa^\nu + g^{\lambda\eta} K^\mu_{\mu\eta} Ric_\kappa^\nu \\
&\left. - g^{\lambda\eta} \Gamma^\nu_{\mu\eta} Ric_\kappa^\mu + g^{\lambda\mu} \Gamma^\eta_{\mu\kappa} Ric_\eta^\nu - g^{\lambda\nu} \Gamma^\eta_{\mu\kappa} Ric_\eta^\mu \right] \\
&= \int (\delta \Gamma^\kappa_{\nu\lambda}) \left[-g^{\lambda\mu} \partial_\mu Ric_\kappa^\nu - g^{\lambda\mu} \Gamma^\nu_{\eta\mu} Ric_\kappa^\eta + g^{\lambda\mu} \Gamma^\eta_{\mu\kappa} Ric_\eta^\nu \right. \\
&+ g^{\lambda\nu} \frac{\partial_\mu |\det g|}{2|\det g|} Ric_\kappa^\mu + g^{\lambda\nu} \partial_\mu Ric_\kappa^\mu - (\nabla_\mu g)^{\lambda\mu} Ric_\kappa^\nu \\
&\left. + (\nabla_\mu g)^{\lambda\nu} Ric_\kappa^\mu + g^{\lambda\eta} K^\mu_{\mu\eta} Ric_\kappa^\nu - g^{\lambda\nu} \Gamma^\eta_{\mu\kappa} Ric_\eta^\mu \right] \\
&= \int (\delta \Gamma^\kappa_{\nu\lambda}) \left[-g^{\lambda\mu} (\{\nabla\}_\mu Ric)_\kappa^\nu + g^{\lambda\nu} \frac{\partial_\mu |\det g|}{2|\det g|} Ric_\kappa^\mu \right. \\
&+ g^{\lambda\nu} \partial_\mu Ric_\kappa^\mu - (\nabla_\mu g)^{\lambda\mu} Ric_\kappa^\nu + (\nabla_\mu g)^{\lambda\nu} Ric_\kappa^\mu \\
&- g^{\lambda\mu} K^\nu_{\eta\mu} Ric_\kappa^\eta + g^{\lambda\mu} K^\eta_{\mu\kappa} Ric_\eta^\nu + g^{\lambda\eta} K^\mu_{\mu\eta} Ric_\kappa^\nu \\
&\left. - g^{\lambda\nu} \Gamma^\eta_{\mu\kappa} Ric_\eta^\mu + g^{\lambda\nu} \{\Gamma\}^\mu_{\mu\eta} Ric_\kappa^\eta - g^{\lambda\nu} \{\Gamma\}^\mu_{\mu\eta} Ric_\kappa^\eta \right] \\
&= \int (\delta \Gamma^\kappa_{\nu\lambda}) \left[-g^{\lambda\mu} (\{\nabla\}_\mu Ric)_\kappa^\nu + g^{\lambda\nu} \frac{\partial_\mu |\det g|}{2|\det g|} Ric_\kappa^\mu \right. \\
&+ g^{\lambda\nu} (\partial_\mu Ric_\kappa^\mu + \{\Gamma\}^\mu_{\mu\eta} Ric_\kappa^\eta - \{\Gamma\}^\eta_{\mu\kappa} Ric_\eta^\mu) \\
&- (\nabla_\mu g)^{\lambda\mu} Ric_\kappa^\nu + (\nabla_\mu g)^{\lambda\nu} Ric_\kappa^\mu \\
&- g^{\lambda\mu} K^\nu_{\eta\mu} Ric_\kappa^\eta + g^{\lambda\mu} K^\eta_{\mu\kappa} Ric_\eta^\nu \\
&\left. + g^{\lambda\eta} K^\mu_{\mu\eta} Ric_\kappa^\nu - g^{\lambda\nu} K^\eta_{\mu\kappa} Ric_\eta^\mu - g^{\lambda\nu} \{\Gamma\}^\mu_{\mu\eta} Ric_\kappa^\eta \right] \\
&= \int (\delta \Gamma^\kappa_{\nu\lambda}) \left[-g^{\lambda\mu} (\{\nabla\}_\mu Ric)_\kappa^\nu + g^{\lambda\nu} \frac{\partial_\mu |\det g|}{2|\det g|} Ric_\kappa^\mu \right. \\
&+ g^{\lambda\nu} (\{\nabla\}_\mu Ric)_\kappa^\mu - (\nabla_\mu g)^{\lambda\mu} Ric_\kappa^\nu + (\nabla_\mu g)^{\lambda\nu} Ric_\kappa^\mu \\
&- g^{\lambda\mu} K^\nu_{\eta\mu} Ric_\kappa^\eta + g^{\lambda\mu} K^\eta_{\mu\kappa} Ric_\eta^\nu + g^{\lambda\eta} K^\mu_{\mu\eta} Ric_\kappa^\nu \\
&\left. - g^{\lambda\nu} K^\eta_{\mu\kappa} Ric_\eta^\mu - g^{\lambda\nu} \frac{\partial_\eta |\det g|}{2|\det g|} Ric_\kappa^\eta \right].
\end{aligned}$$

Hence we get that

$$\begin{aligned} \int (\delta Ric^{(2)}_{\mu\nu}) Ric^{\mu\nu} &= \int (\delta\Gamma^\kappa_{\nu\lambda}) \left[-g^{\lambda\mu} (\{\nabla\}_\mu Ric)_\kappa{}^\nu + g^{\lambda\nu} (\{\nabla\}_\mu Ric)_\kappa{}^\mu \right. \\ &\quad - (\nabla_\mu g)^{\lambda\mu} Ric_\kappa{}^\nu + (\nabla_\mu g)^{\lambda\nu} Ric_\kappa{}^\mu - g^{\lambda\mu} K^\nu_{\eta\mu} Ric_\kappa{}^\eta \\ &\quad \left. + g^{\lambda\mu} K^\eta_{\mu\kappa} Ric_\eta{}^\nu + g^{\lambda\eta} K^\mu_{\mu\eta} Ric_\kappa{}^\nu - g^{\lambda\nu} K^\eta_{\mu\kappa} Ric_\eta{}^\mu \right], \end{aligned}$$

and this is a tensor. We can also write this (using full covariant derivative ∇ instead of $\{\nabla\}$) as :

$$\begin{aligned} \int (\delta Ric^{(2)}_{\mu\nu}) Ric^{\mu\nu} &= \int (\delta\Gamma^\kappa_{\nu\lambda}) \left[-g^{\lambda\mu} (\nabla_\mu Ric)_\kappa{}^\nu + g^{\lambda\nu} (\nabla_\mu Ric)_\kappa{}^\mu \right. \quad (C.9) \\ &\quad - (\nabla_\mu g)^{\lambda\mu} Ric_\kappa{}^\nu + (\nabla_\mu g)^{\lambda\nu} Ric_\kappa{}^\mu - g^{\lambda\mu} K^\nu_{\eta\mu} Ric_\kappa{}^\eta \\ &\quad \left. + g^{\lambda\eta} K^\mu_{\mu\eta} Ric_\kappa{}^\nu + g^{\lambda\mu} K^\nu_{\mu\eta} Ric_\kappa{}^\eta - g^{\lambda\nu} K^\mu_{\mu\eta} Ric_\kappa{}^\eta \right]. \end{aligned}$$

C.3.2 Variation with respect to the metric

In an almost identical way as in part C.2.2, using Proposition C.0.2 we obtain that the metric variation is

$$\delta \int Ric^{(2)\kappa}_{\nu} Ric_\kappa{}^\nu = \int (\delta g_{\alpha\beta}) \left[-R^{\kappa\alpha\beta}_{\nu} Ric_\kappa{}^\nu - Ric^{(2)\kappa\alpha} Ric_\kappa{}^\beta + \frac{1}{2} g^{\alpha\beta} Ric^{(2)\kappa}_{\nu} Ric_\kappa{}^\nu \right]. \quad (C.10)$$

Bibliography

- [1] Alekseevsky D V 1974 Holonomy groups and recurrent tensor fields in Lorentzian spaces, in: *Problems of the Theory of Gravitation and Elementary Particles* issue 5 edited by Stanjukovich K P (Moscow: Atomizdat). 5–17 In Russian
- [2] Audretsch J 1971 Asymptotic behaviour of neutrino fields in curved space-time *Commun. math. Phys.* **21** 303–313
- [3] Audretsch J and Graf W 1970 Neutrino radiation in gravitational fields *Commun. math. Phys.* **19** 315–326
- [4] Baekler P, Hehl F W and Mielke E W 1982 Vacuum solutions with double duality properties of a quadratic Poincaré gauge field theory, in: *Proceedings of the Second Marcel Grossmann Meeting on General Relativity* edited by Ruffini R (Amsterdam: North–Holland Publishing Company) 413–453
- [5] Baekler P, Hehl F W and Mielke E W 1986 Nonmetricity and torsion: facts and fancies in gauge approaches to gravity, in: *Proceedings of the Fourth Marcel Grossmann Meeting on General Relativity* edited by Ruffini R (Amsterdam: Elsevier Science Publishers B.V.) 277–316
- [6] Barrow J D and Ottewill A C 1983 The stability of general relativistic cosmological theory *J. Phys. A: Math. Gen.* **16** 2757–2776
- [7] Berestetskii V B, Lifshitz E M, and Pitaevskii L P 1982 *Quantum Electrodynamics (Course of Theoretical Physics vol 4)* 2nd edn, Pergamon Press, Oxford.
- [8] Bertolami O, Böhmer CG, Harko T and Lobo F S N 2007 Extra force in $f(R)$ modified theories of gravity *Phys. Rev. D* **75** 104016 [5 pages]

- [9] Blagojevic M 2002 *Gravitation and Gauge Symmetries* Institute of Physics Publishing, Bristol
- [10] Brill, D R and Wheeler, J A 1957 Interaction of Neutrinos and Gravitational Fields in *Rev. Mod. Phys.* **29** 465–479
- [11] Brinkmann M W 1923 On Riemann spaces conformal to Euclidean space. *Proceedings of the National Academy of Sciences of USA* **9** 1-3
- [12] Bryant R L 2000 Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor, in: *Global Analysis and Harmonic Analysis (Marseille–Luminy, 1999)* Sémin. Congr. **4** (Paris: Soc. Math. France) 53–94
- [13] Buchbinder I L and Kuzenko S M 1998 *Ideas and Methods of Supersymmetry and Supergravity*, Institute of Physics Publishing, Bristol.
- [14] Buchdahl H A 1959 *Mathematical Reviews* **20** 7571.
- [15] Buchdahl H A 1970 Nonlinear Lagrangians and cosmological theory *Mon. Not. Roy. Astron. Soc.* **150** 1–8
- [16] Carroll S M, Duvvuri V, Trodden M and Turner M S 2004 Is cosmic speed-up due to new gravitational physics? *Phys. Rev. D* **70** 043528 [5 pages]
- [17] Cartan É 1986 *On Manifolds with an Affine Connection and the Theory of General Relativity*, Brill Academic Publishers, English translation of the French originals of 1923/1924.
- [18] Collinson C D and Morris P B 1972 Space-time admitting neutrino fields with zero energy and momentum *J. Phys.* **A6** 915–916
- [19] Davis T M Ray J R 1974 Ghost neutrinos in general relativity *Phys. Rev.* **D9** 331–333
- [20] Dereli T, Önder M, Schray J and Tucker R W (1996) Non-Riemannian gravity and the Einstein–Proca system *Class. Quantum Grav.* **13** L103–L109
- [21] Duruisseau J P, Kerner R and Eysseric P 1983 Non-Einsteinian gravitational Lagrangians assuring cosmological solutions without collapse *Gen. Rel. Grav.* **15** 797–807

- [22] Eddington A S *The Mathematical Theory of Relativity* Cambridge 1952
- [23] Einstein A 1905 Zur Elektrodynamik bewegter Körper, *Annalen der Physik* **17** 891–921
- [24] Einstein A 1967 *The Meaning of Relativity : Including the Relativistic Theory of the Non-Symmetric Field - With a new Foreword by W.H. McCrea, F.R.S.* Chapman and Hall Science Paperbacks
- [25] Esser W 1996 *Exact Solutions of the Metric-Affine Gauge Theory of Gravity* (University of Cologne: Diploma Thesis)
- [26] Fairchild E E Jr 1976 Gauge theory of gravitation *Phys. Rev. D* **14** 384–391.
- [27] Fairchild E E Jr 1976 Erratum: Gauge theory of gravitation *Phys. Rev. D* **14** 2833.
- [28] Garcia A, Hehl F W, Lämmerzahl C, Macias A and Socorro J 1998 Plebański–Demiański-like solutions in metric-affine gravity *Class. Quantum Grav.* **15** 1793–1799
- [29] Griffiths J B 1991 *Colliding Plane Waves in General Relativity* Oxford University Press
- [30] Griffiths J B 1981 Neutrino fields in Einstein–Cartan theory *Gen. Rel. Grav.* **13** 227–237
- [31] Griffiths J B and Newing R A 1970 The two-component neutrino field in general relativity *J. Phys.* **A3** 136–149
- [32] Griffiths J B and Newing R A 1970 Tetrad equations for the two-component neutrino field in general relativity *J. Phys.* **A3** 269–273
- [33] Griffiths J B 1972 Some physical properties of neutrino-gravitational fields *Int. J. Theoret. Phys.* **5** 141–150
- [34] Griffiths J B 1972 Gravitational radiation and neutrinos *Commun. Math. Phys.* **28** 295–299
- [35] Griffiths J B 1979 On Dirac fields in a curved space-time *J. Phys.* **A12** 2429–2435

- [36] Griffiths J B 1980 Ghost neutrinos in Einstein–Cartan theory *Phys. Lett.* **A75** 441–442
- [37] Griffiths J B 1981 Neutrino fields in Einstein–Cartan theory *Gen. Rel. Grav.* **13** 227–237
- [38] Gronwald F and Hehl F W 1996 On the gauge aspects of gravity, In: *Proc. of the 14th Course of the School of Cosmology and Gravitation on ‘Quantum gravity’ (Erice, Italy 1995)* World Scientific, Singapore, pp. 148–198, gr-qc/9602013
- [39] Hehl F W 1970 *Spin und Torsion in der Allgemeinen Relativitätstheorie oder die Riemann–Cartansche Geometrie der Welt* (Technischen Universität Clausthal: Habilitationsschrift)
- [40] Hehl F W 1973 Spin and torsion in general relativity I: Foundations *Gen. Rel. Grav.* **4** 333–349
- [41] Hehl F W 1974 Spin and torsion in general relativity II: Geometry and field equations *Gen. Rel. Grav.* **5** 491–516
- [42] Hehl F W, McCrea J D, Mielke E W and Ne’eman Y 1995 Metric–affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance, *Phys. Rep.* **258** 1–171
- [43] Hehl F W and Macías A 1999 Metric–affine gauge theory of gravity II. Exact solutions, *Int. J. Mod. Phys.* **D8** 399–416
- [44] Hehl F W, Nitsch J and von der Heyde P 1980 In: *General Relativity and Gravitation, Vol. 1*, Plenum Press, New York 329–355
- [45] Hehl F W, Obukhov, Yu N 2007 Elie Cartan’s torsion in geometry and in field theory, an essay gr-qc/0711.1535v1
- [46] Hehl F W, von der Heyde P, Kerlick G D and Nester J M 1976 General relativity with spin and torsion: Foundations and prospects *Rev. Mod. Phys.* **48** 393–416
- [47] Higgs P W 1959 Quadratic Lagrangians and general relativity *Nuovo Cimento* **11** 816–820.

- [48] Kerner R 1982 Cosmology without singularity and nonlinear gravitational Lagrangians *Gen. Rel. Grav.* **14** 453–469
- [49] King A D and Vassiliev D 2001 Torsion waves in metric–affine field theory, *Class. Quantum Grav.* **18** 2317–2329
- [50] Kramer D, Stephani H, Herlt E and MacCallum M 1980 *Exact Solutions of Einstein’s Field Equations*, Cambridge University Press, Cambridge
- [51] Kröner E 1980 Continuum theory of defects, *Physics of Defects, Les Houches, Session XXXV, 1980, R.Balian et al., eds.*, North–Holland, Amsterdam
- [52] Kröner E, 1986 The continuized crystal – a bridge between micro– and macromechanics *Gesellschaft angewandte Mathematik und Mechanik Jahrestagung Goettingen West Germany Zeitschrift Flugwissenschaften* Vol. 66
- [53] Kuchowicz C and Żebrowski J 1978 The presence of torsion enables a metric to allow a gravitational field *Phys. Lett.* **A67** 16–18
- [54] Lanczos C 1938 A remarkable property of the Riemann–Christoffel tensor in four dimensions *Ann. Math.* **39** 842–850
- [55] Lanczos C 1949 Lagrangian multiplier and Riemannian spaces *Rev. Mod. Phys.* **21** 497–502
- [56] Lanczos C 1957 Electricity and general relativity *Rev. Mod. Phys.* **29** 337–350
- [57] Landau L D and Lifshitz E M 1975 *The Classical Theory of Fields (Course of Theoretical Physics vol 2)* 4th edn, Pergamon Press, Oxford.
- [58] Mielke E W 1981 On pseudoparticle solutions in Yang’s theory of gravity, *Gen. Rel. Grav.* **13** 175–187
- [59] Mielke E W 2005 Duality in Yang’s theory of gravity, *Gen. Rel. Grav.* **37** 997–1007
- [60] Muench U, Gronwald F and Hehl F W 1998 A small guide to variations in teleparallel gauge theories of gravity and the Kaniel–Itin model *Gen. Rel. Grav.* **30** 933–961

- [61] Nakahara M 1998 *Geometry, Topology and Physics* Institute of Physics Publishing, Bristol
- [62] Nordström G 1912 Relativitätsprinzip und Gravitation *Phys. Zeitschr.* **13** 1126–1129
- [63] Nordström G 1913 Zur Theorie der Gravitation vom Standpunkt des Relativitätsprinzips *Annalen der Physik* **42** 533–554
- [64] Obukhov Yu N, Vlachynsky E J, Esser W and Hehl F W 1997 Effective Einstein theory from metric–affine gravity models via irreducible decompositions *Phys. Rev. D* **56** 7769–7778
- [65] Obukhov Yu N, Vlachynsky E J, Esser W, Tresguerres R and Hehl F W 1996 An exact solution of the metric–affine gauge theory with Dilation, shear, and spin charges *Phys. Lett. A* **220** 1–9
- [66] Obukhov Yu N 2004 Generalized plane–fronted gravitational waves in any dimension *Phys. Rev. D* **69** 024013 [7 pages]
- [67] Obukhov Yu N 2006 Plane waves in metric–affine gravity *Phys. Rev. D* **73** 024025 [6 pages]
- [68] Olesen P 1977 A relation between the Einstein and the Yang–Mills field equations *Phys. Lett. B* **71**, 189–190.
- [69] Pasic V and Vassiliev D 2005 PP–waves with torsion and metric–affine gravity *Class. Quantum Grav.* **22** 3961–3975
- [70] Pauli W 1919 Zur Theorie der Gravitation und der Elektrizität von Hermann Weyl *Physik. Zeitschr.* **20** 457–467
- [71] Pavelle R 1975 Unphysical solutions of Yang’s gravitational–field equations *Phys. Rev. Lett.* **34** 1114
- [72] Peres A 1959 Some gravitational waves, *Phys. Rev. Lett.* **3** 571–572
- [73] Penrose O and Rindler W 1984, 1986 *Propagating modes in gauge field theories of gravity*, 2 volumes, Cambridge University Press, Oxford.

- [74] Peres A 2002 PP – WAVES *preprint* hep–th/0205040 (reprinting of [72])
- [75] Pirani F A E 1964 *Introduction to Gravitational Radiation Theory*. In: *Lectures on General Relativity*, Prentice–Hall, Inc. Englewood Cliffs, New Jersey.
- [76] Sauer T 2004 Field equations in teleparallel spacetime: Einstein’s fernparallelismus approach towards unified field theory *preprint* physics/0405142v1
- [77] Schouten J A 1989 *Tensor Analysis for Physicists* Dover Publications, Inc., New York 2nd Edition ISBN 0486655822
- [78] Singh P 1990 On axial vector torsion in vacuum quadratic Poincaré gauge field theory *Phys. Lett.* **145A** 7–10
- [79] Singh P 1990 On null tratorial torsion in vacuum quadratic Poincaré gauge field theory *Class. Quantum Grav.* **7** 2125–2130
- [80] Singh P and Griffiths J B 1988 On neutrino fields in Einstein–Cartan theory *Phys. Lett.* **A132** 88–90
- [81] Singh P and Griffiths J B 1990 A new class of exact solutions of the vacuum quadratic Poincaré gauge field theory *Gen. Rel. Grav.* **22** 947–956
- [82] Stephenson G 1958 Quadratic Lagrangians and general relativity *Nuovo Cimento* **9** 263–269.
- [83] Streater R F and Wightman A S 2000 *PCT, spin and statistics, and all that*, Princeton Landmarks in Physics (Princeton University Press, Princeton, NJ), ISBN 0-691-07062-8, corrected third printing of the 1978 edition.
- [84] Thompson A H 1975 Yang’s gravitational field equations *Phys. Rev. Lett.* **34** 507–508.
- [85] Thompson A H 1975 Geometrically degenerate solutions of the Kilmister–Yang equations *Phys. Rev. Lett.* **35** 320–322.
- [86] Unzicker A and Case T 2005 Translation of Einstein’s attempt of a unified field theory with teleparallelism *preprint* physics/0503046v1
- [87] Vassiliev D 2002 Pseudoinstantons in metric–affine field theory *Gen. Rel. Grav.* **34** 1239–1265

- [88] Vassiliev D 2003 Pseudoinstantons in metric–affine field theory, in: *Quark Confinement and the Hadron Spectrum V* edited by Brambilla N and Proserpi G M (Singapore: World Scientific) 273–275.
- [89] Vassiliev D 2004 Quadratic non–Riemannian gravity *Journal of Nonlinear Mathematical Physics* **11**, Supplement, 204–216
- [90] Vassiliev D 2005 Quadratic metric–affine gravity *Ann. Phys. (Lpz.)* **14** 231–252
- [91] Vassiliev D 2006 Teleparallel model for the neutrino *Phys. Rev. D* **75** 025006 [6 pages]
- [92] Vlachynsky E J, Tresguerres R, Obukhov Yu N and Hehl F W 1996 An axially symmetric solution of metric–affine gravity *Class. Quantum Grav.* **13** 3253–3259
- [93] Weyl H 1919 Eine neue Erweiterung der Relativitätstheorie, *Ann. Phys. (Lpz.)* **59** 101–133
- [94] Wheeler J A 1990 *A Journey Into Gravity and Spacetime*, Scientific American Library, San Francisco
- [95] Wilczek F 1977 In: *Quark Confinement and Field theory*, eds. D. R. Stump and D. H. Weingarten, Wiley-Interscience, New York, 211–219.
- [96] Yang C N 1974 Integral Formalism for Gauge Fields *Phys. Rev. Lett.* **33** 445–447.