Parabolic Projection and Generalized Cox Configurations

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Abstract

Building on the work of Longuet-Higgins in 1972 and Calderbank and Macpherson in 2009, we study the combinatorics of symmetric configurations of hyperplanes and points in projective space, called generalized Cox configurations.

To do so, we use the formalism of morphisms between incidence systems. We notice that the combinatorics of Cox configurations are closely related to incidence systems associated to certain Coxeter groups. Furthermore, the incidence geometry of projective space $\mathbb{P}(V)$, where V is a vector space, can be viewed as an incidence system of maximal parabolic subalgebras in a semisimple Lie algebra $\mathfrak{g}$, in the special case $\mathfrak{g} = \mathfrak{pgl}(V)$ the projective general linear Lie algebra of V. Using Lie theory, the Coxeter incidence system for the Coxeter group, whose Coxeter diagram is the underlying diagram of the Dynkin diagram of the $\mathfrak{g}$, can be embedded into the parabolic incidence system for $\mathfrak{g}$. This embedding gives a symmetric geometric configuration which we call a standard parabolic configuration of $\mathfrak{g}$. In order to construct a generalized Cox configuration, we project a standard parabolic configuration of type D_n into the parabolic incidence system of projective space using a process called parabolic projection, which maps a parabolic subalgebra of the Lie algebra to a parabolic subalgebra of a lower dimensional Lie algebra.

As a consequence of this construction, we obtain Cox configurations and their analogues in higher dimensional projective spaces. We conjecture that the generalized Cox configurations we construct using parabolic projection are non-degenerate and, furthermore, any non-degenerate Cox configuration is obtained in this way. This conjecture yields a formula for the dimension of the space of non-degenerate generalized Cox configurations of a fixed type, which enables us to develop a recursive construction for them. This construction is closely related to Longuet-Higgins' recursive construction of (generalized) Clifford configurations but our examples are more general and involve the extra parameters.

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Introduction

1.1 Motivation

In projective geometry, the elementary objects are projective subspaces such as points, lines, and planes. A finite projective configuration is a finite collection of these objects with a prescribed incidence relation. For example:

- In $\mathbb{P}^2$, a complete quadrangle (Figure 1.1.2 (a)) is a collection of four points and six lines such that each point is incident with three lines and each line is incident with two points;

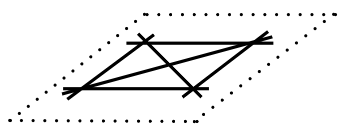


Figure 1.1.1: A complete quadrangle in $\mathbb{P}^2$.

- In $\mathbb{P}^3$, a tetrahedron (Figure 1.1.2 (b)) is a collection of four points, six lines and four planes such that each point is incident with three lines and three planes, any line is incident with two points and two planes, and any plane is incident with three points and three lines.

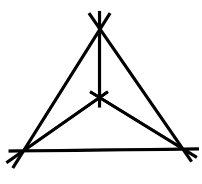


Figure 1.1.2: A tetrahedron in $\mathbb{P}^3$.

Given a finite projective configuration, the set of its all objects together with its prescribed (symmetric) incidence relation is a multipartite graph, i.e., a graph equipped with a type function on vertices such that distinct vertices of the same type are not incident, with types determined by the dimensions of objects. The concept is shown in Figure 1.1.3 where the multipartite graph associated with a tetrahedron is drawn. In general, we define an *incidence*

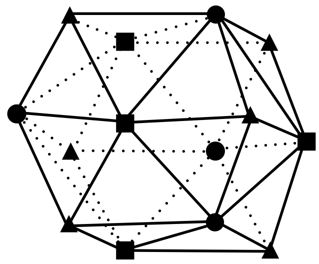


Figure 1.1.3: The multipartite graph associated with a tetrahedron ([BC13], page 5).

system to be a multipartite graph with the convention that each vertex is considered to be incident with itself. For example:

- Given an n-dimensional vector space V, the set of all non-zero proper subspaces of V, denoted by $\mathsf{Proj}\,(V)$, is an incidence system with the incidence relation determined by containment and the type function

$$\begin{aligned} d : \mathsf{Proj}\,(V) &\to \{1, 2, \ldots, n-1\} \\ V' &\mapsto \dim\left(V'\right). \end{aligned}$$

  We call $\mathsf{Proj}\,(V)$ the projective incidence system of $\mathbb{P}\,(V)$;

- The set of all non-empty proper subsets of $\{1, 2, \ldots, n\}$, denoted by $\mathcal{P}_\star\,(n)$, is an incidence system with the incidence relation determined by containment and the type function

$$\begin{aligned} t : \mathcal{P}_\star\,(n) &\to \{1, 2, \ldots, n-1\} \\ S &\mapsto |S|, \end{aligned}$$

  where $|S|$ is the number of elements in S.

The point of view that we will adopt henceforth is that a finite projective configuration is a realization of a finite incidence system such as $\mathcal{P}_\star(n)$ inside the incidence system $\mathsf{Proj}(V)$ for some vector space V. For example, an abstract $(n-1)$-simplex in $\mathbb{P}(V)$, where V is an n-dimensional vector space with a basis $\{v_1, v_2, \ldots, v_n\}$, is the image of the realization

$$\begin{aligned} \Psi : \mathcal{P}_\star(n) &\rightarrow \mathsf{Proj}(V) \\ S &\mapsto \langle v_i \,|\, i \in S \rangle, \end{aligned} \tag{1.1.1}$$

where $\langle v_i \,|\, i \in S\rangle$ is a vector subspace of V spanned by v_i, for all $i \in S$. Formally, this realization is a *strict incidence system morphism*, i.e., a map between two incidence systems having the same set of types which preserves types and the incidence relation.

So, in general, a geometric configuration may be treated as a strict incidence system morphism between two incidence systems. The co-domain of a geometric configuration is the space in which the configuration is realized, while the domain of the configuration determines the combinatorics of the configuration. Notice that the domain of a geometric configuration also provides a labelling of the configuration. For example, when $n = 4$, the realization of $\mathcal{P}_\star(4)$ by Ψ in $\mathbb{P}(V)$ is a labelled tetrahedron as shown in Figure 1.1.4. Its points are labelled by the numbers $1, 2, 3$, and 4, i.e., one element subsets of $\mathcal{P}_\star(4)$. Then lines and planes are then automatically labelled by two and three element subsets of $\mathcal{P}_\star(4)$ respectively.

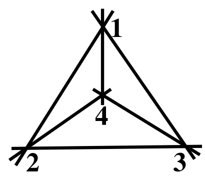


Figure 1.1.4: A tetrahedron in $\mathbb{P}^3$ labelled by elements in $\mathcal{P}_\star(4)$

More complicated geometric configurations we are interested in are finite projective configurations in $\mathbb{P}(V)$, where V is a four dimensional vector space, and governed by a well known chain of theorems, studied by H.Cox ([Cox91]) in 1891, as follows:

- Suppose given four general planes a, b, c, d through a point p_0 in $\mathbb{P}(V)$. Since every pair of planes, say a and b, determines a line, by choosing a point, say ab on such line, there are six such points. Since any three points like ab, bc, ac generate a plane, say *abc*, there are four such planes which, by Möbius theorem ([Möb28]), intersect in a point, say *abcd*;

- Suppose given five planes a, b, c, d, e through a point p_0 in $\mathbb{P}(V)$. Then any four of them, such as a, b, c, d give a point abcd. There are totally five such points, all of which lie on a plane, say *abcde*;

- By introducing a new plane through p_0 in $\mathbb{P}(V)$ in each step and continuing in this manner inductively, we obtain Cox's chain of theorems.

These geometric configurations are called Cox configurations, consisting of 2^{n-1} planes and 2^{n-1} points in a three dimensional projective space with n planes passing through each point and n points lying on each planes. A Cox configuration in $\mathbb{P}(V)$ consisting of of 2^{n-1} planes and 2^{n-1} points is a realization of an abstract n-hypercube, which is a bipartite graph $\mathsf{Hcube}(n)$ of an n-hypercube, into $\mathsf{Proj}(V)_{\{1,3\}}$, the subset of $\mathsf{Proj}(V)$ containing all but two dimensional vector subspaces of V. As the vertices of $\mathsf{Hcube}(n)$ are partitioned into two types, they can be colored by black and white. For example, $\mathsf{Hcube}(3)$ is shown in 1.1.5. Thus $\mathsf{Hcube}(n)$ can be considered an incidence system whose elements are vertices of the

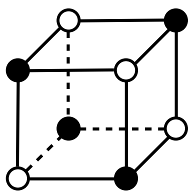


Figure 1.1.5: An abstract cube (3-hypercube).

graph with the incidence relation determined by edges of the graph and the type function $\{\text{vertices of } \mathsf{Hcube}(n)\} \to \{\text{black, white}\}$. The combinatorics between points and planes of the Cox configuration is same as the combinatorics between two types of vertices of the graph. By postcomposing the type function of $\mathsf{Hcube}(n)$ by the bijective map

$$\{\text{black, white}\} \to \{1, 3\}\,; \text{black} \mapsto 1, \text{white} \mapsto 3,$$

$\mathsf{Hcube}(n)$ can be considered as an incidence system over $\{1, 3\}$. Therefore the Cox configuration is a strict incidence system morphism from $\mathsf{Hcube}(n)$ to $\mathsf{Proj}(V)_{\{1,3\}}$ mapping the black vertices of the graph to points of the configuration and the white vertices of the graph to planes of the configuration.

The strategy we use in this thesis to understand complicated geometric configurations is by studying the projection of simpler geometric configurations to lower dimensional geometrical spaces. It is well known that, by projecting a tetrahedron in $\mathbb{P}^3$ away from a suitable point onto $\mathbb{P}^2$, one obtains a complete quadrangle as in Figure 1.1.6.

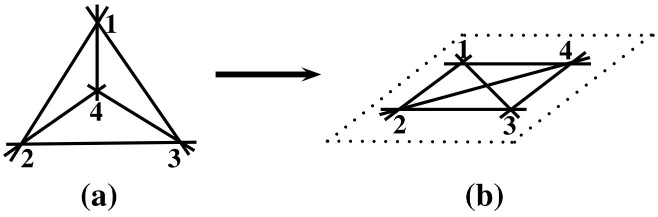


Figure 1.1.6: A complete quadrangle obtained from a tetrahedron.

Formally, let V be an n-dimensional vector space and p be a point in $\mathbb{P}(V)$, i.e., p is a one-dimensional vector subspace of V. The map

$$\begin{aligned} \varphi_p : \{\text{vector subspaces of } V\} &\rightarrow \{\text{vector subspaces of } V/p\} \\ L &\mapsto (L+p)/p\,, \end{aligned}$$

maps a vector subspace of V to a vector subspace of the lower dimensional vector space V/p. We call φ_p the projection away from p.

Denote $\mathsf{Proj}^p(V)$ a subset of $\mathsf{Proj}(V)$ consisting all projective subspaces which are generic to p, i.e., they do not contain p. Then $\mathsf{Proj}^p(V)$ is an incidence system. Notice that if $L \subseteq L'$ in $\mathsf{Proj}^p(V)$, then $\varphi_p(L) \subseteq \varphi_p(L')$, and moreover if L and L' in $\mathsf{Proj}^p(V)$ have the same dimension, then $\varphi_p(L)$ and $\varphi_p(L')$ also have the same dimension. However $\varphi_p(\mathsf{Proj}^p(V)) \nsubseteq \mathsf{Proj}(V/p)$ because if L is a maximal proper subspace of V generic to p then $(L+p)/p = V/p$. Thus in order to make φ_p a strict incidence system whose co-domain is $\mathsf{Proj}(V/p)$, we need to restrict φ_p to the subset of $\mathsf{Proj}^p(V)$ consists of all but $(n-1)$-dimensional vector subspaces in $\mathsf{Proj}^p(V)$; this subset is an incidence sub-system of $\mathsf{Proj}^p(V)$.

This idea motivates us to define an incidence system morphism between two incidence system having different sets of types. In general, given A and A' incidence systems over N and N' respectively, $\Phi : \mathsf{A} \rightarrow \mathsf{A}'$ is an incidence system morphism over a map $\nu : N' \rightarrow N$ if it is a strict incidence system morphism $\phi : \bigsqcup_{i \in N'} \mathsf{A}_{\nu(i)} \rightarrow \mathsf{A}'$. In particular, a strict incidence system morphism is an incidence system morphism over the identity map.

Thus φ_p induces the incidence system morphism

$$\Phi_p : \mathsf{Proj}^p(V) \to \mathsf{Proj}(V/p)$$

over the map $\nu_p : \{1, 2 \ldots, n-2\} \to \{1, 2, \ldots, n-1\}\,;i \mapsto i$. Given an abstract $(n-1)$-simplex $\Psi : \mathcal{P}_\star(n) \to \mathsf{Proj}^p(V)$ such that objects in the image of Ψ are all generic to the point p, the postcomposition of Ψ by this projection Φ_p is an incidence system morphism

$$\Phi_p \circ \Psi : \mathcal{P}_\star(n) \to \mathsf{Proj}(V/p)$$

over the map ν_p. The case $n = 4$ is shown in Figure 1.1.6.

On the other hand, by choosing a hyperplane P in $\mathbb{P}(V)$, i.e., P is an $(n-1)$-dimensional vector subspace of V, one can define an incidence system morphism projecting a non-empty proper vector subspace of V generic to P, i.e., not contained in P, to a vector subspace of P as follows

$$\begin{aligned} \Phi_P : \mathsf{Proj}^P(V) &\to \mathsf{Proj}(P) \\ L &\mapsto L \cap P \end{aligned}$$

over the map $\nu_P : \{1, 2 \ldots, n-2\} \to \{1, 2, \ldots, n-1\}\,;i \mapsto i+1$, where $\mathsf{Proj}^P(V)$ is a subset of $\mathsf{Proj}(V)$ consisting all projective subspaces which are generic to P. Similarly, given an abstract $(n-1)$-simplex $\Psi : \mathcal{P}_\star(n) \to \mathsf{Proj}(V)^{\circ P}$ such that objects in the image of Ψ are all generic to P, the postcomposition of Ψ by this projection Φ_P is an incidence system morphism

$$\Phi_P \circ \Psi : \mathcal{P}_\star(n) \to \mathsf{Proj}(P)$$

over the map ν_p. In the case $n = 4$, the projection Φ_P sends a tetrahedron in $\mathbb{P}(V)$, generic to P, to a complete quadrilateral in $\mathsf{Proj}(P)$ as in Figure 1.1.7. Its dual configuration in $\mathsf{Proj}(P)$ is a complete quadrangle.

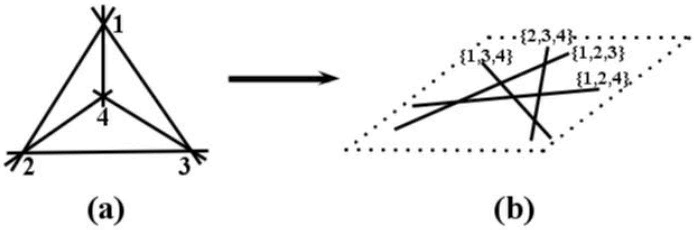


Figure 1.1.7: A complete quadrilateral obtained from a tetrahedron.

A complete quadrilateral can also be obtained from an octahedron in the Klein quadric $Q^4 \subseteq \mathbb{P}\left(\bigwedge^2 V\right)$, where V is a four dimensional vector space, composed of six points, four α-planes, and four β-planes. Let $\mathsf{Klein}\,(V)$ denote the incidence system consisting of the points and the two types α and β of planes in Q^4 where planes of the same type meet in a point and planes in different types meet in a line or in the empty set. The points, lines and planes in $\mathbb{P}(V)$ correspond to α-planes, points, and β-planes in Q^4, respectively. By choosing a β-plane P' in Q^4, we define a projection map away from the chosen plane onto the space $\mathbb{P}(V/P')$ via the incidence system morphism

$$\begin{aligned} \Phi_{P'} : \mathsf{Klein}^{P'}(V) &\rightarrow \mathsf{Proj}\left(\left(\bigwedge^2 V\right)/P'\right) \\ L &\mapsto \left(L + P'\right)/P' , \end{aligned}$$

over a map $\nu_{P'} : \{1,2\} \rightarrow \{\text{point}, \alpha\text{- plane}, \beta\text{-plane}\}; 1 \mapsto \text{point}, 2 \mapsto \beta\text{-plane}$, where $\mathsf{Klein}^{P'}(V)$ is a subset of $\mathsf{Klein}\,(V)$ containing all objects generic to P', i.e., they are not contained in P'. Let

$$\Psi' : \mathcal{P}_\star(4) \rightarrow \mathsf{Klein}^{P'}(V),$$

over $\nu : \{\text{point}, \alpha\text{-plane}, \beta\text{- plane}\} \rightarrow \{1,2,3\}; \text{point} \mapsto 2, \alpha\text{-plane} \mapsto 1, \beta\text{-plane} \mapsto 3$, be an octahedron such that objects in the image are generic to P'. Then the postcomposition of Ψ' by this projection $\Phi_{P'}$ is an incidence system morphism

$$\Phi_{P'} \circ \Psi' : \mathcal{P}_\star(4) \rightarrow \mathsf{Proj}\left(\left(\bigwedge^2 V\right)/P'\right)$$

over the map $\nu \circ \nu_{P'} : \{1,2\} \rightarrow \{1,2,3\}$.

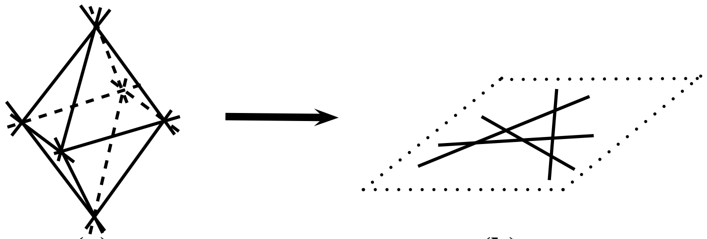


Figure 1.1.8: A complete quadrilateral obtained from an octahedron in Q^4.

Indeed, $\mathsf{Proj}\,(V) \cong \mathsf{Klein}\,(V)$. Therefore, a complete quadrilateral is obtained from essentially the same geometric configurations when we associate each object in $\mathsf{Proj}\,(V)$ to the corresponding object in $\mathsf{Klein}\,(V)$, even though from the geometric viewpoint, these two geometric configuration are different. To see that $\mathsf{Proj}\,(V) \cong \mathsf{Klein}\,(V)$, we use Lie theory. Given a Lie algebra $\mathfrak{g}$, the set of all maximal parabolic subalgebras of $\mathfrak{g}$, denoted by $\mathsf{Para}\,(\mathfrak{g})$, is an incidence system with the incidence relation between any two maximal parabolic subalgebras determined by their intersection being again a parabolic subalgebra, and the type function

$$\begin{aligned} t' : \mathsf{Para}\,(\mathfrak{g}) &\to \mathscr{D} \\ \mathfrak{p} &\mapsto \text{the crossed nodes of } \mathscr{D}_{\mathfrak{p}}, \end{aligned}$$

where $\mathscr{D}$ is the Dynkin diagram (and also the set of its vertices) of $\mathfrak{g}$ and $\mathscr{D}_{\mathfrak{p}}$ is the decorated Dynkin diagram representing $\mathfrak{p}$, as defined in Section 2.2.4. The projective incidence system of a projective space $\mathbb{P}\,(V)$ is isomorphic to the set of all maximal parabolic subalgebras of the projective general linear Lie algebra $\mathfrak{pgl}\,(V)$ via the incidence system isomorphism

$$\mathsf{Proj}\,(V) \to \mathsf{Para}\,(\mathfrak{pgl}\,(V))\,; V' \mapsto \mathrm{Stab}_{\mathfrak{pgl}(V)}\left(V'\right).$$

Similarly, $\mathsf{Klein}\,(V) \cong \mathsf{Para}\,(\mathfrak{pgl}\,(V))$. Therefore $\mathsf{Proj}\,(V) \cong \mathsf{Klein}\,(V)$.

The symmetry group S_n acts transitively on the flags, i.e., sets of mutually incident elements, of the same type, i.e., the set of all types of elements in a flag, of $\mathcal{P}_\star\,(n)$. We say that $\mathcal{P}_\star\,(n)$ is S_n-homogeneous. Consider the geometric configuration $\Psi : \mathcal{P}_\star\,(n) \to \mathsf{Proj}\,(V)$ defined as in (1.1.1), the n points $\langle v_1\rangle, \langle v_2\rangle, \ldots, \langle v_n\rangle$ in $\mathbb{P}\,(V)$ determine the maximal torus T of $\mathrm{PGL}\,(V)$ fixing each of these points. Hence $N_G\,(T)\,/T \cong S_n$ permutes all these points.

In other words, S_n is the Weyl group of PGL (V). In Section 1.2, we exploit this relationship to give a construction of a standard configuration in $\mathsf{Para}(\mathfrak{g})$ generalizing this example in $\mathsf{Proj}(V) \cong \mathsf{Para}(\mathfrak{pgl}(V))$.

1.2 Standard parabolic configurations

Let (W, S) be a Coxeter group, i.e., a free group W generated by elements in S modulo the relations $(ss')^{m(s,s')} = 1$ where $m(s, s') \in \{3, 4, 5, \ldots\} \cup \{\infty\}$ and $m(s, s) = 1$ for all $s, s' \in S$, with the Coxeter diagram $\mathscr{D}$, i.e., the nodes of $\mathscr{D}$ correspond to the generators in $\{s_i \,|1 \leq i \leq n-1\}$ and edges joining s and s' are determined $m(s, s')$. The set

$$\mathsf{C}(W) := \{wW_i \,|w \in W \text{ and } i \in \mathscr{D}\},$$

where W_i is a subgroup of W generated by all simple reflections except one corresponding to the node $i \in \mathscr{D}$, equipped with the incidence relation given by the relation having non-empty intersection is an incidence system over $\mathscr{D}$ (see Section 3.2). We call $\mathsf{C}(W)$ the *Coxeter incidence system* for W.

Suppose that W is finite and crystallographic. Then there exists a finite-dimensional simple algebraic group G over an algebraically closed field $\mathbb{F}$ of characteristic zero with Lie algebra $\mathfrak{g}$ such that its Dynkin diagram $\mathscr{D}_{\mathfrak{g}}$ has $\mathscr{D}$ as the underlying diagram. Each node of the Dynkin diagram $\mathscr{D}_{\mathfrak{g}}$ corresponds to a conjugacy class (resp. an adjoint orbit) of maximal parabolic subgroups (resp. maximal parabolic subalgebras) of the Lie group (resp. the associated Lie algebra $\mathfrak{g}$).

Any pairs $(\mathfrak{t}, \mathfrak{b})$, where $\mathfrak{t}$ is a Cartan subalgebra and $\mathfrak{b}$ is a Borel subalgebra containing $\mathfrak{t}$ of $\mathfrak{g}$, determines a specific isomorphism from W to $N_G(T)/T$, where T is the maximal torus of G with the Lie algebra $\mathfrak{t}$. Under this isomorphism, we can define an action of W on the set of all parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{t}$ and identify each W_i, where $i \in \mathscr{D}$, as the stabilizer of the parabolic subalgebra $\mathfrak{p}_i$, of type i, containing $\mathfrak{b}$. Hence it induces a well-defined injective map

$$\begin{aligned} \Upsilon_{(\mathfrak{t},\mathfrak{b})} : \mathsf{C}(W) &\rightarrow \mathsf{Para}(\mathfrak{g}) \\ wW_i &\mapsto w \cdot \mathfrak{p}_i, \end{aligned} \tag{1.2.1}$$

where $\mathsf{Para}(\mathfrak{g})$ is the set of all maximal parabolic subalgebras of $\mathfrak{g}$, which is actually a strict incidence system morphism; we call $\Upsilon_{(\mathfrak{t},\mathfrak{b})}$ a *standard parabolic configuration.*

For example, given a pair $(\mathfrak{t},\mathfrak{b})$ of the Lie algebra $\mathfrak{pgl}(V)$, where V is an n-dimensional vector space, the choice of the Cartan subalgebra $\mathfrak{t}$ determines n points $\langle v_1\rangle, \langle v_2\rangle, \ldots, \langle v_n\rangle$ in $\mathbb{P}(V)$ each of which is fixed under the action of T, where T is the maximal torus of $\mathfrak{pgl}(V)$ with Lie algebra $\mathfrak{t}$, while the choice of the Borel subalgebra $\mathfrak{b}$ determines a full flag of subspaces of V, and hence an ordering on the set of these n points. Define a strict incidence system morphism

$$\mathcal{P}_\star(n) \to \mathsf{Proj}(V); X \mapsto \mathrm{Span}\{v_i \,|i \in X\}.$$

The symmetry group S_n together with the generating set $\{s_i\,|1 \le i \le n-1\}$, where s_i is the permutation swapping i and $i+1$, is a Coxeter group. Since $\mathcal{P}_\star(n)$ has a full flag $\{\{1\},\{1,2\},\ldots,\{1,2,\ldots,n\}\}$ and it is S_n-homogeneous, one can show that $\mathcal{P}_\star(n) \cong \mathsf{C}(S_n)$ (see Proposition 3.2.2). Since $\mathcal{P}_\star(n) \cong \mathsf{C}(S_n)$ and $\mathsf{Proj}(V) \cong \mathsf{Para}(\mathfrak{g})$, it induces a strict incidence system morphism $\Upsilon_{(\mathfrak{t},\mathfrak{b})} : \mathsf{C}(S_n) \to \mathsf{Para}(\mathfrak{pgl}(V))$.

Denote $\mathscr{U}$ the set of all the pairs $(\mathfrak{t},\mathfrak{b})$ consisting of a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{t}$ and

$$\mathrm{Mor}^{\mathrm{inj}}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g})) := \{\Upsilon : \mathsf{C}(W) \to \mathsf{Para}(\mathfrak{g}) \text{ is injective}\}.$$

Then we have the following.

Theorem 1.2.1. $\mathscr{U} \cong \mathrm{Mor}^{\mathrm{inj}}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g}))$.

We can obtain more complicated geometric configurations from standard parabolic configurations by projection. The projection of standard parabolic configurations can be explained in more abstract approach by using parabolic subalgebras.

1.3 Parabolic projection

For any parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$, in Section 2.2.3, we show that $\left((\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{q}^\perp\right)/\mathfrak{q}^\perp$, where $\mathfrak{q}^\perp$ is the orthogonal complement of $\mathfrak{q}$ with respect to the Killing form on $\mathfrak{g}$, is a parabolic subalgebra of the reductive Lie algebra $\mathfrak{q}/\mathfrak{q}^{\perp}$. Thus we have a well-defined map

$$\begin{aligned} \varphi_{\mathfrak{q}} : \mathscr{P}(\mathfrak{g}) &\to \mathscr{P}\left(\mathfrak{q}/\mathfrak{q}^{\perp}\right) \\ \mathfrak{p} &\mapsto \left((\mathfrak{p}\cap\mathfrak{q})+\mathfrak{q}^{\perp}\right)/\mathfrak{q}^{\perp}, \end{aligned} \tag{1.3.1}$$

where $\mathscr{P}(\mathfrak{g})$ (resp. $\mathscr{P}(\mathfrak{q}/\mathfrak{q}^{\perp})$) is the set of all parabolic subalgebras of $\mathfrak{g}$ (resp. $\mathfrak{q}_0 := \mathfrak{q}/\mathfrak{q}^{\perp}$). We call this map *parabolic projection.*

In Section 4.2, by using Dynkin diagram automorphisms, we introduce a procedure (Proposition 4.2.2) to compute the type of $\varphi_{\mathfrak{q}}(\mathfrak{p})$ where $\mathfrak{p}$ is weakly opposite to $\mathfrak{q}$, i.e., $\mathfrak{g}=\mathfrak{p}+\mathfrak{q}$. By using the procedure we introduced, we have the following.

Theorem 1.3.1. *The parabolic projection $\varphi_{\mathfrak{q}}$ induces an incidence system morphism*

$$\Phi_{\mathfrak{q}} : \mathsf{Para}(\mathfrak{g})^{\mathfrak{q}} \to \mathsf{Para}(\mathfrak{q}_0), \tag{1.3.2}$$

over the map ν (computed by the procedure). Furthermore, the diagram

$$\begin{array}{ccc} \mathcal{F}(\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})) & \xrightarrow{\tau^{\mathfrak{q}}} & \mathscr{P}^{\mathfrak{q}}(\mathfrak{g}) \\ \downarrow{\scriptstyle \mathcal{F}(\Phi_{\mathfrak{q}})} & & \downarrow{\scriptstyle \varphi_{\mathfrak{q}}} \\ \mathcal{F}(\mathsf{Para}(\mathfrak{q}_0)) & \xrightarrow[\tau]{} & \mathscr{P}(\mathfrak{q}_0) \end{array}$$

commutes, where

$$\begin{aligned} \mathcal{F}(\Phi_{\mathfrak{q}}) : \mathcal{F}(\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})) &\to \mathcal{F}(\mathsf{Para}(\mathfrak{q})) \\ f &\mapsto \{\Phi_{\mathfrak{q}}(\mathfrak{p}) \,|\, \mathfrak{p}\in f\} \end{aligned}$$

is the flag extension map of $\Phi_{\mathfrak{q}}$, and τ (resp. $\tau^{\mathfrak{q}}$) is the isomorphism identifying $\mathcal{F}(\mathsf{Para}(\mathfrak{q}_0))$ (resp. $\mathcal{F}(\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}))$) with $\mathscr{P}(\mathfrak{q}_0)$ (resp.$\mathscr{P}^{\mathfrak{q}}(\mathfrak{g})$) defined in (3.4.1) (resp. (3.4.2)).

For any parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$, the set

$$\mathscr{U}^{\mathfrak{q}} := \{(\mathfrak{t},\mathfrak{b})\in\mathscr{U} \mid \text{any parabolic subalgebra } \mathfrak{p} \text{ of } \mathfrak{g} \text{ containing } \mathfrak{t} \text{ satisfying } \mathfrak{g}=\mathfrak{p}+\mathfrak{q}\} \tag{1.3.3}$$

is non-empty. Thus Theorem 1.2.1 implies that

$$\mathscr{U}^{\mathfrak{q}} \cong \mathrm{Mor}^{\mathrm{inj}}\left(\mathsf{C}\left(W\right), \mathsf{Para}^{\mathfrak{q}}\left(\mathfrak{g}\right)\right) := \left\{\Upsilon : \mathsf{C}\left(W\right) \to \mathsf{Para}^{\mathfrak{q}}\left(\mathfrak{g}\right) \text{ is injective}\right\},$$

where $\mathsf{Para}\left(\mathfrak{g}\right)^{\mathfrak{q}}$ is the incidence system consisting of all maximal parabolic subalgebras of $\mathfrak{g}$ weakly opposite to $\mathfrak{q}$. We call each element in $\mathrm{Mor}^{\mathrm{inj}}\left(\mathsf{C}\left(W\right), \mathsf{Para}^{\mathfrak{q}}\left(\mathfrak{g}\right)\right)$ a $\mathfrak{q}$-*generic standard parabolic configuration*. Therefore the postcomposition of a $\mathfrak{q}$-generic standard configurations by parabolic projection

$$\begin{array}{ccc} \mathsf{C}\left(W\right) & \xrightarrow{\Upsilon(\mathfrak{t},\mathfrak{b})} & \mathsf{Para}^{\mathfrak{q}}\left(\mathfrak{g}\right) \\ & \searrow & \downarrow \Phi_{\mathfrak{q}} \\ & & \mathsf{Para}\left(\mathfrak{q}_0\right), \end{array}$$

gives rise a geometric configuration

$$\Phi_{\mathfrak{q}} \circ \Upsilon_{(\mathfrak{t},\mathfrak{b})} : \mathsf{C}\left(W\right) \to \mathsf{Para}\left(\mathfrak{q}_0\right),$$

over the map ν, for all $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}^{\mathfrak{q}}$. In the case that $\mathfrak{q}_0$ has a simple component which is isomorphic to $\mathfrak{pgl}\left(V\right)$, for some vector space V, we shall see that a further projection yields a projective configuration from $\mathsf{C}\left(W\right)$ to $\mathsf{Proj}\left(V\right)$.

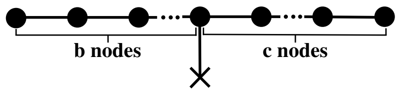
1.4 Generalized Cox configurations

In Chapter 5, we will use the postcomposition method by parabolic projection to study Cox configurations and their analogue in higher dimensions. Let us first gives some historical background about Cox configurations.

Another chain of theorems governing points and circles, studied by W.K. Clifford ([Cli71]), called Clifford's chain. It shows that, given n general planes through a point p_0 on a non-singular quadric in $\mathbb{P}^3$, such as a sphere, this quadric determines a point on any pair of planes different from the point p_0; by the same construction as in Cox's chain, the theorems in Clifford's chain imply that we will obtain a Cox's configuration whose points lie on the quadric. Its associated configuration on the quadric is called a Clifford configuration consisting of 2^{n-1} circles and 2^{n-1} points with the same combinatorics as those of Cox's configuration. More accurately, Clifford studied the stereographic projection of these configurations of points and circles in the quadric. Some years later, J.H. Grace ([Gra98]) and

L.M. Brown ([Bro54]) generalized Clifford's chain to higher dimension.

In 1972, Longuet-Higgins ([LH72]) introduced an approach to study these generalized Clifford configurations corresponding to Clifford's chain and its analogues. He inductively investigated a correspondence between generalized Clifford configurations and polytopes (described by the formalism developed by Coxeter [Cox73], Section 5.7) with the decorated Coxeter diagrams



.

In other word, such a polytope is the convex hull of an orbit of a particular point under the reflection action of the corresponding Coxeter group W. The incidence system $\mathsf{C}(W)$ represents an incidence sub-system of the faces of the polytope. For example, given a basis $\{e_1, e_2, \ldots, e_n\}$ of a Euclidean space, a convex simplex in the Euclidean space is a strict incidence system morphism

$$\begin{aligned} \mathcal{P}_\star(n) &\rightarrow \{\text{convex hulls of some elements in the Euclidean space}\} \\ X &\mapsto \operatorname{Conv}(\{e_i \,|i \in X\}), \end{aligned}$$

where $\operatorname{Conv}(\{e_i \,|i \in X\})$ is the convex hull of elements in $\{e_i \,|i \in X\}$. As $\mathcal{P}_\star(n) \cong \mathsf{C}(S_n)$, each maximal coset of $\mathsf{C}(S_n)$ represents a face of the simplex.

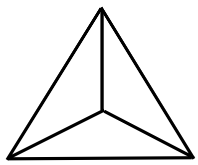


Figure 1.4.1: A convex tetrahedron in a Euclidean space.

Longuet-Higgins showed that such polytopes parametrize the generalized Clifford configurations in the following sense: the points of a (generalized) Clifford configuration correspond to the vertices of the polytope and its hyperspheres correspond to the facets of a certain type. It is implicit in his recursive construction that all other objects, i.e. lower dimensional spheres, in the configuration correspond to the certain faces of polytopes in between the vertices and the facets of that type, each of which corresponds to an element in $\mathsf{C}(W)$. In other words, there exist strict incidence system morphisms, preserving the incidence relation and types from $\mathsf{C}(W)$ to the incidence system of the quadric which a generalized Clifford configuration lies on; the incidence system of the quadric consists of circles and points on the quadric with the incidence relation determined by lying on. By using this correspondence, he was able to show that the finiteness of Clifford configurations depends on the necessary condition for the finiteness of their corresponding polytope given by

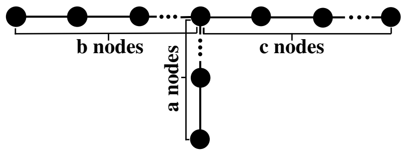
$$\frac{1}{2}+\frac{1}{b}+\frac{1}{c}>1.$$

In 2009, A.W. Macpherson and D.M.J. Calderbank ([Mac09]) explored the relation between (generalized) Cox configurations and the flag varieties associated to representations on Lie groups. They introduced a class of maps, called collapsing maps, with the property that each maps the weight polytope of a representation of a certain Lie group into a flag variety. Any weight polytope Γ of G is determined by a pair (T, B), consisting of a maximal torus T and a Borel subgroup B containing T, and the standard parabolic subgroup P containing B of G. Faces or even flags, i.e., chains of faces, of Γ are classified by W-orbits (types) in the set of all parabolic subgroups of G containing T, where $W = N_G(T)/T$ is a Weyl group. In particular, the set of points of Γ are actually $\{g \cdot P \,|g \in W\}$ which can be embedded into the flag variety G/P. Note that the pair (T, B) turns the Weyl group W into a Coxeter group, and so the set faces of Γ of each type is considered as a coset space of W. Their construction involved choosing a suitable parabolic subgroup Q of G which is weakly opposite to all the Borel subgroups B containing T, i.e., $G = QB$. Then they obtain a collapsing map as a composite map such that, for any parabolic subgroup $P' \supseteq T$, there exists a parabolic subgroup R' of Q such that $P' \cap Q \subseteq R'$, and the collapsing map maps

$$\{g \cdot P' \,|g \in W\} \hookrightarrow G/P' \xrightarrow{\cong} Q/(P' \cap Q) \longrightarrow Q/R' \;.$$

The images of collapsing maps gives a large family of configurations in generalized flag varieties. By choosing an appropriate parabolic subgroup Q so that Q/R' is a flag variety of type A, the images are elements or even flags in projective configurations, including generalized Cox configurations.

Generalizing Longuet-Higgins' work, suppose that the Coxeter diagram $\mathscr{D}$ for W is



where

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1. \tag{1.4.1}$$

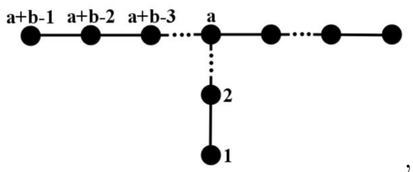
Let V a vector space of dimension $a+b$. Then a *generalized Cox configuration of type* (a,b,c) is then a geometric configuration

$$\Psi : \mathsf{C}(W) \to \mathsf{Proj}(V),$$

over the map

$$\varrho : \{1,2,\ldots,a+b-1\} \to \mathscr{D} \tag{1.4.2}$$

given by the following labelling



,

is defined by the strict incidence system morphism

$$\psi : \varrho^{\star}(\mathsf{C}(W)) \to \mathsf{Proj}(V),$$

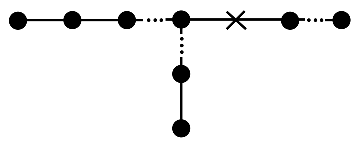
where $\varrho^{\star}(\mathsf{C}(W)) := \left\{wW_{\varrho(i)} \,|1 \le i \le a+b-1\right\}$. For example, in the case that $b=1$ and $c=2$, the geometric configuration is a complete quadrangle. Denote by

$$\mathrm{GCO}_{(a,b,c)}(W,V) := \{\Psi : \mathsf{C}(W) \to \mathsf{Proj}(V) \text{ over the map } \varrho\},$$

and

$$\mathrm{GCO}^{inj}_{(a,b,c)}(W,V) := \{\text{injective } \Psi : \mathsf{C}(W) \to \mathsf{Proj}(V) \text{ over the map } \varrho\}.$$

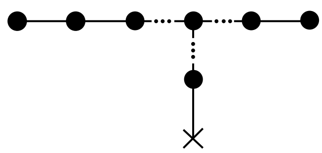
In order to construct a generalized Cox configuration, let Q be a parabolic subgroup of G with the Lie algebra $\mathfrak{q}$ such that it is in the conjugacy class complementary to one represented by the decorated Dynkin diagram



Then by above construction, we have a well-defined map

$$\begin{array}{rcl} \mathscr{U}^{\mathfrak{q}} & \to & \mathrm{Mor}\left(\mathsf{C}\left(W\right), \mathsf{Para}\left(\mathfrak{q}_0\right)\right) \\ \left(\mathfrak{t}, \mathfrak{b}\right) & \mapsto & \Phi_{\mathfrak{q}} \circ \Upsilon_{\left(\mathfrak{t}, \mathfrak{b}\right)}, \end{array}$$

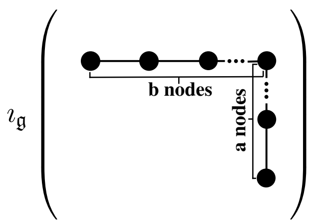
where $\Upsilon_{(\mathfrak{t},\mathfrak{b})}$, $\Phi_{\mathfrak{q}}$, and $\mathscr{U}^{\mathfrak{q}}$ are defined as in (1.2.1), (1.3.2), and (1.3.3), respectively. According to [Mac09], each $(\mathfrak{t}, \mathfrak{b})$ determines a labelled weight polytope of a representation of $\mathfrak{g}$ inside G/P which is a conjugacy class of parabolic subgroups where P is a parabolic subgroup of G with the decorated Dynkin diagram



.

Hence $\mathscr{U}$ may be regarded as the set of all weight polytopes of representations of $\mathfrak{g}$.

Let $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{q}$ such that the quotient $\overline{\mathfrak{q}} := \mathfrak{q}/\mathfrak{k}$ is a simple Lie algebra with the Dynkin diagram



where $\imath_{\mathfrak{g}}$ is the dual involution of the Dynkin diagram $\mathscr{D}_{\mathfrak{g}}$. We have an incidence system morphism

$$\Theta : \mathsf{Para}\left(\mathfrak{q}_0\right) \to \mathsf{Para}\left(\overline{\mathfrak{q}}\right),$$

over the inclusion map $\mathrm{i} : \mathscr{D}_{\overline{\mathfrak{q}}} \to \mathscr{D}_{\mathfrak{q}_0}$. Let V be a vector space of dimension $a + b$. For any $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}^{\mathfrak{q}}$, choose an isomorphism from $\overline{\mathfrak{q}}$ to $\mathfrak{pgl}(V)$ so that V is a representation of $\overline{\mathfrak{q}}$ with the highest fundamental weight $\lambda_{\varsigma^{-1}(1)}$, where $\varsigma : \mathscr{D}_{\overline{\mathfrak{q}}} \to \{1, 2, \ldots, a + b\}$ is bijective and satisfying $\nu \circ \mathrm{i} = \varrho \circ \varsigma$. Since V is a representation of $\overline{\mathfrak{q}}$, we have an incidence system isomorphism

$$\Xi : \mathsf{Proj}\left(V\right) \to \mathsf{Para}\left(\overline{\mathfrak{q}}\right),$$

over the map $\varsigma : \mathscr{D}_{\overline{\mathfrak{q}}} \to [a+b]$.

$$\begin{array}{ccccc}
\mathsf{C}(W) & & \xrightarrow{\Upsilon(\mathfrak{t},\mathfrak{b})} & & \mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}) \\
\Big\downarrow \Psi_{(\mathfrak{t},\mathfrak{b})} & \mathscr{D}_{\mathfrak{g}} \xleftarrow{\ \mathrm{id}\ } \mathscr{D}_{\mathfrak{g}} & & & \Big\downarrow \Phi_{\mathfrak{q}} \\
 & \varrho \uparrow \qquad \uparrow \nu & & & \\
 & [a+b] \xleftarrow{\ \varsigma\ } \mathscr{D}_{\overline{\mathfrak{q}}} \xrightarrow{\ \mathrm{i}\ } \mathscr{D}_{\mathfrak{q}_0} & & & \\
\mathsf{Proj}(V) & \xrightarrow{\ \Xi\ } & \mathsf{Para}(\overline{\mathfrak{q}}) & \xleftarrow{\ \Theta\ } & \mathsf{Para}(\mathfrak{q}_0)\,,
\end{array}$$

The composite incidence system morphism

$$\Psi_{(\mathfrak{t},\mathfrak{b})} : \mathsf{C}(W) \to \mathsf{Para}(\mathfrak{pgl}(V)) \cong \mathsf{Proj}(V) \tag{1.4.3}$$

over the map ϱ as in (1.4.2), is in $\mathrm{GCo}_{(a,b,c)}(W,V)$.

Compared with Calderbank and Macpherson's approach, in our approach, we fix a projective space (a flag variety in [Mac09]) by choosing Q and project the set of all faces we interested in of any possible polytope (a weight polytope in [Mac09]) into the fixed projective space.

In the case $a = b = 2$, each element in $\mathrm{GCo}_{(2,2,c)}(W,V)$ is a Cox configuration; its image in the projective space can be constructed by using a theorem in Cox's chain. The incidence system morphism $\Psi_{(\mathfrak{t},\mathfrak{b})} : \mathsf{C}(W) \to \mathsf{Proj}(V)$ defined in (1.4.3) shows the correspondence between $\mathsf{C}(W)$ and its corresponding generalized Cox configuration; any coset in $\mathsf{C}(W)$ of the type labelled by i, where $i \in \{1, 2, \ldots, a+b-1\}$ is mapped to an $(i-1)$-dimensional projective subspace of $\mathbb{P}^{a+b-1}(V)$.

Therefore there exists a well-defined map

$$\begin{array}{rcl}
\Psi : \mathscr{U}^{\mathfrak{q}} & \to & \mathrm{GCo}_{(a,b,c)}(W,V) \\
(\mathfrak{t},\mathfrak{b}) & \mapsto & \Psi_{(\mathfrak{t},\mathfrak{b})}.
\end{array}$$

In the case $a = 1$, $b = 1$, or $c = 1$, classical facts imply that $K \backslash \mathscr{U}^{\mathfrak{q}} \cong \mathrm{GCo}^{inj}_{(a,b,c)}(W,V)$, where K is the connected algebraic subgroup of Q with the Lie algebra $\mathfrak{k}$. However, we hope that this is also true in general. We thus make the following conjecture.

Conjecture 1.4.1. *For arbitrary positive integer a, b, and c satisfying (1.4.1),*

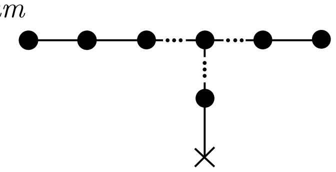
$$K \backslash \mathscr{U}^{\mathfrak{q}} \cong \mathrm{GCo}^{inj}_{(a,b,c)}(W, V).$$

Let $C(a,b,c) := \dim(K \backslash \mathscr{U}^{\mathfrak{q}})$. Then we obtain the following inductive formula.

Theorem 1.4.2. *For any $a, b, c \in \mathbb{N}$ such that $a \geq 2$,*

$$C(a,b,c) = (a+b-1) + C(a-1,b,c) + dim\left(\mathfrak{p}^{\perp}\right).$$

where $\mathfrak{p}$ is the Lie algebra of the parabolic subgroup P in the conjugacy class represented by the decorated Dynkin diagram



containing a pair $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}^{\mathfrak{q}}$.

Theorem 1.4.2 shows that if Conjecture 1.4.1 is true then the following Conjecture is automatically true.

Conjecture 1.4.3. *For any $a, b, c \in \mathbb{N}$ such that $a \geq 2$,*

$$dim\left(\mathrm{GCo}^{inj}_{(a,b,c)}(W,V)\right) = (a+b-1) + dim\left(\mathrm{GCo}^{inj}_{(a-1,b,c)}(W,V)\right) + dim\left(\mathfrak{p}^{\perp}\right), \quad (1.4.4)$$

where $\mathfrak{p}$ is the parabolic subalgebra of P defined as in Theorem 1.4.2.

The equation (1.4.4) suggests that if we choose a point p_0 in a $(a+b-1)$-dimensional projective space $\mathbb{P}(V)$ and a residual generalized Cox configuration at the point p_0 (i.e., a generalized Cox configuration of type $(a-1,b,c)$ in the projective space $\mathbb{P}(V/p_0)$), then, by choosing $\dim\left(\mathfrak{p}^{\perp}\right)$ more parameters, a generalized Cox configuration of type (a,b,c) could be constructed. Compared with the recursive construction of (generalized) Clifford configurations (when $a = 2$) given by Longuet-Higgins ([LH72], Section 7), when the diagram $\mathscr{D}$ is of type A or D, $\dim\left(\mathfrak{p}^{\perp}\right)$ represents the number of choices of points on the lines through the point p_0; these choices do not appear in Longuet-Higgins' construction due to the constraint of lying on a quadric surface. However, when $\mathscr{D}$ is of type E, there must be additional parameters, apart from those appearing when $\mathscr{D}$ is of type A and D. By carefully analyzing Conjecture 1.4.3, we found some further cases for which it holds.

Theorem 1.4.4. *Conjecture 1.4.3 is true when* (a, b, c) *is equal to* $(a, 1, c)$, $(a, b, 1)$, $(2, b, 2)$, $(2, 2, c)$, $(2, 3, 3)$, $(2, 3, 4)$, *or* $(2, 4, 3)$ *for any* $a, b, c \in \mathbb{N}$ *such that* $a \geq 2$.

Chapter 2

Review of basic materials

We begin with a chapter introducing the basic objects and terminology used throughout this thesis.

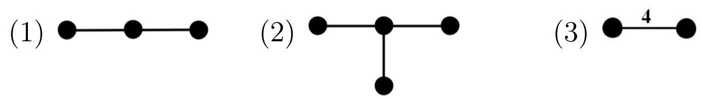
2.1 Coxeter groups and root systems

Coxeter groups were studied first in [Cox34]. Since then they have become an important class of groups which is used in many branches of Mathematics. In this section, we begin by giving basic definitions of Coxeter groups and root systems. Then we consider special subgroups in Coxeter groups. For further details on the fundamental theory of Coxeter groups, see [CM57], [Dav08], and [Hum92].

2.1.1 Coxeter groups

Definition 2.1.1. A COXETER DIAGRAM $\mathscr{D}$ is an undirected graph with each edge labelled by an element of $\{3, 4, 5, \ldots\} \cup \{\infty\}$; the label 3 is usually suppressed. A Coxeter diagram is said to be CONNECTED if it is a connected graph.

Example 2.1.2. These are some examples of Coxeter diagrams:



Definition 2.1.3. Let W be a group. A COXETER SYSTEM S ON W WITH COXETER DIAGRAM $\mathscr{D}$ is an injective map $S : \mathscr{D} \to W$ such that

$$W \cong \left\langle S_{\mathscr{D}} \,\middle|\, (S_i S_j)^{m(i,j)} = 1 \text{ for } i, j \in \mathscr{D} \right\rangle,$$

where, by slightly abuse notation, $\mathscr{D}$ is also considered as the set of vertices of the diagram $\mathscr{D}$ and $S_i := S(i)$ for all $i \in \mathscr{D}$; for any $i, j \in \mathscr{D}$, $m(i,i) = 1$, $m(i,j) =$ the label of the edge connecting the vertices i and j, and $m(i,j) = 2$ otherwise.

A Coxeter system is **INDECOMPOSABLE** or **IRREDUCIBLE** if its Coxeter diagram is connected. We call (W,S) a **COXETER GROUP**; we sometimes abuse terminology and denote (W,S) by W. The number of vertices of $\mathscr{D}$ is called the **RANK** of the Coxeter group.

Remark 2.1.4. For any $w \in W$, define the map $w \cdot S : \mathscr{D} \to W; i \mapsto wS_iw^{-1}$. One can check that $w \cdot S$ is also a Coxeter system on W with Coxeter diagram $\mathscr{D}$.

If (W,S) is a Coxeter system, it may be possible to express $w \in W$ as a product of S_i's in more than one way. This leads us to the following definition:

Definition 2.1.5. Let (W,S) be a Coxeter group with Coxeter diagram $\mathscr{D}$. For each element w in a Coxeter group W, let $\ell(w)$ be the smallest number of S_i's, where $i \in \mathscr{D}$, in an expression of w. $\ell(w)$ is called the **LENGTH** of w. Any expression of w as a product of $\ell(w)$ elements of $\{S_i \,|i \in \mathscr{D}\}$ is called a **REDUCED EXPRESSION** of w.

Proposition 2.1.6. *Let (W,S) be a finite Coxeter group. Then W has a unique longest element w_0 and for any $w \in W$,*

$$\ell(w_0 w) = \ell(w_0) - \ell(w).$$

Proof. See [Hum92], Section 1.8. □

Let (W,S) be a Coxeter group with Coxeter diagram $\mathscr{D}$. For any $w \in W$, define

$$\begin{aligned} \boldsymbol{r} : W &\to \mathcal{P}(\{S_i \,|i \in \mathscr{D}\}) \\ w &\mapsto \{S_i\text{'s in a reduced expression of } w\}, \end{aligned} \tag{2.1.1}$$

where $\mathcal{P}(\{S_i \,|i \in \mathscr{D}\})$ is the power set of $\{S_i \,|i \in \mathscr{D}\}$. The solution to the word problem of Coxeter groups tells us that $\boldsymbol{r}$ is a well-defined map (see [Dav08], Proposition 4.1.1).

Theorem 2.1.7. (*Deletion Condition*) *For all $w \in W$, if $\ell(w) < k$ and $w = S_{i_1} S_{i_2} \cdots S_{i_k}$, for some $i_1, i_2, \ldots, i_k \in \mathscr{D}$, then there exists indices $1 \leq j < l \leq k$ such that*

$$w = S_{i_1} \cdots S_{i_{j-1}} S_{i_{j+1}} \cdots S_{i_{l-1}} S_{i_{l+1}} \cdots S_{i_k}.$$

Proof. See [Hum92], p.117. □

The Deletion Condition shows that a reduced expression for any element $w \in W$ can be obtained from any expression for w by omitting an even number of generators S_i's.

2.1.2 Root systems

A root system is a tool to understand the associated reflection group; it describes the reflections in the group. Root systems are also important in the theory of finite Coxeter groups and Lie algebras. In this section, we will state some crucial facts about root systems, omitting standard proofs which can be found in [Bou02], [Hum92], and [Ser66].

Throughout this section, let V be a finite-dimensional real vector space and B be a positive definite symmetric bilinear form.

Definition 2.1.8. Let $v \in V$ be a non-zero element. The **REFLECTION** of v is the endomorphism τ_v of V such that $\tau_v(v) = -v$ and τ_v fixes the hyperplane $H_v = \{v' \in V \mid B(v, v') = 0\}$.

Definition 2.1.9. A finite spanning subset $\mathcal{R}$ of V, which does not contain 0, is a **ROOT SYSTEM** in V if for any $\alpha, \beta \in \mathcal{R}$,

$$\tau_\alpha(\mathcal{R}) = \mathcal{R}$$

and $\tau_\alpha(\beta) - \beta$ is an integer multiple of α. The elements of $\mathcal{R}$ are called **ROOTS** of V and the dimension of V is called the **RANK** of $\mathcal{R}$.

A subset Δ of $\mathcal{R}$ is called a **SIMPLE SYSTEM** of $\mathcal{R}$ if it is a basis for V and each root β can be written as $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ with integral coefficients k_α all non-negative or all non-positive. The elements of Δ are called the **SIMPLE ROOTS** for Δ.

A subset $\mathcal{R}^+$ of $\mathcal{R}$ is called a **POSITIVE ROOT SYSTEM** if it is closed under addition, i.e., for $\alpha, \beta \in \mathcal{R}^+$, if $\alpha + \beta \in \mathcal{R}$ then $\alpha + \beta \in \mathcal{R}^+$, and for any $\alpha \in \mathcal{R}$, either α or $-\alpha$ is in $\mathcal{R}^+$.

Definition 2.1.10. Let $\mathcal{R}$ be a root system in V. The subgroup $W(\mathcal{R})$ of $\mathrm{GL}(V)$ generated by the reflections τ_α for $\alpha \in \mathcal{R}$ is called the **WEYL GROUP** of $\mathcal{R}$.

For general results about root systems, we state them in the following Proposition:

Proposition 2.1.11. *Suppose that $\mathcal{R}$ is a root system in V. Then*

(1) *$\mathcal{R}$ contains a simple system,*

(2) *there is one-to-one correspondence between positive root systems in $\mathcal{R}$ and simple systems of $\mathcal{R}$,*

(3) *any two positive (resp. simple) systems in a root system $\mathcal{R}$ in V are conjugate under $W(\mathcal{R})$,*

(4) *if Δ is a simple system of $\mathcal{R}$, then $W(\mathcal{R})$ is generated by the $\{\tau_\alpha \,|\alpha \in \Delta\}$ subject to the relations*:

$$(\tau_\alpha \tau_\beta)^{m(\alpha,\beta)} = 1,$$

where $m(\alpha, \beta) = 2, 3, 4$ or 6 in the case when $B(\alpha, \beta) = 0, -1, -2$ or -3 respectively, and $W(\mathcal{R}) \cdot \Delta = \mathcal{R}$.

Proof. The proof of (1) can be found in [Bou02], Chapter VI, §1.5, Theorem 2. For (2) and (3), their proofs is in [Hum92], p.8 and p. 10, respectively. Finally the proof of (4) can be found in [Ser66], p.33. □

Proposition 2.1.11 (4) implies that if $\mathcal{R}$ is a root system in V, there is a Coxeter system $\tau : \mathscr{D} \to W(\mathcal{R})$, where $\mathscr{D}$ is a Coxeter system whose set of vertices is a simple system Δ and edges are determined by $m(\alpha, \beta)$ for all $\alpha, \beta \in \Delta$, making $(W(\mathcal{R}), \tau)$ a Coxeter group with Coxeter diagram $\mathscr{D}$.

Remark 2.1.12. On the other hand, given a finite Coxeter group (W, S) with Coxeter diagram $\mathscr{D}$, then the group W is actually a Weyl group of the root system $\mathcal{R}$ in a real vector space of dimension $|\mathscr{D}|$ with a basis $\Delta_{\mathscr{D}} = \{\alpha_i \,|i \in \mathscr{D}\}$ (see [Hum92], Section 5.3). The basis $\Delta_{\mathscr{D}}$ is a simple system of $\mathcal{R}$ contained in the basis $\mathcal{R}^+$, and each S_i is the reflection of α_i, where i is a node of $\mathscr{D}$.

2.1.3 Coxeter polytopes and parabolic subgroups of Coxeter groups

Let (W, S) be a Coxeter system with the Coxeter diagram $\mathscr{D}$. For $I \subseteq \mathscr{D}$, let W_I be the subgroup of W generated by all elements in $S_I = \{S_i \,|i \in I\}$; in particular if I is a maximal proper subset of $\mathscr{D}$, we will write W_i, where $\mathscr{D} \backslash I = \{i\}$, in place of W_I for convenience.

Definition 2.1.13. The subgroup W_I, for some $I \subseteq \mathscr{D}$, is called a STANDARD PARABOLIC SUBGROUP of W with respect to S. A PARABOLIC SUBGROUP of W is a W-conjugate of a standard parabolic subgroup of W.

A useful notation for a standard parabolic subgroup W_I of W for some $I \subseteq \mathscr{D}$ is to use a **DECORATED COXETER DIAGRAM** $\mathscr{D}_I$ obtained by the rule: all nodes in $\mathscr{D} \backslash I$ of the Coxeter diagram $\mathscr{D}$ for W are crossed. W_I is indeed a Coxeter group with Coxeter diagram obtained from $\mathscr{D}_I$ by removing all the crossed nodes and edges adjacent to them.

According to Remark 2.1.12, W is the Weyl group of a root system $\mathcal{R}$ of a real vector space V with the simple system $\Delta_{\mathscr{D}} = \{\alpha_i \,|i \in \mathscr{D}\}$ corresponding to the system of standard generators $S_{i:i\in\mathscr{D}}$ of W. Since $\mathcal{R}$ span V and W acts on $\mathcal{R}$, thus W acts on V. For any $I \subseteq \mathscr{D}$, in order to represent the elements in $W/W_I := \{wW_I \,|w \in W\}$, as points in V, we have to find a point $v \in V$ such that

$$W_I = \{w \in W \,|w \cdot v = v\}.$$

Any element $v \in V$ satisfying

$$\frac{(v, \alpha_i)}{(\alpha_i, \alpha_i)} \begin{cases} < 0 & \text{for } i \notin I, \\ = 0 & \text{for } i \in I, \end{cases} \tag{2.1.2}$$

has the stabilizer W_I. Since $\Delta_{\mathscr{D}}$ is a basis of V, we can find $v \in V$ satisfying (2.1.2) but it is not unique. Therefore we have a well-defined map

$$W/W_I \to V; wW_I \mapsto w \cdot v.$$

This map identifies the set W/W_I with the orbit $W \cdot v$. We call the convex hull of points in $W \cdot v$ a **COXETER POLYTOPE**. The orbits of W on V have parabolic subgroups as stabilizers. This is a motivation for studying their stabilizers, the parabolic subgroups.

Proposition 2.1.14. *For each $I \subseteq \mathscr{D}$,*

$$W_I = \{w \in W \,|\boldsymbol{r}(w) \subseteq S_I\},$$

where $\boldsymbol{r}$ is defined as in (2.1.1).

Proof. For any $w, w' \in W_I$, we have $\boldsymbol{r}(w) = \boldsymbol{r}\left(w^{-1}\right)$ and $\boldsymbol{r}(ww') \subseteq \boldsymbol{r}(w)\,\boldsymbol{r}(w')$, by Theorem 2.1.7. Thus $X := \{w \in W \,|\boldsymbol{r}(w) \subseteq S_I\}$ is a subgroup contained in W_I. Since $S_I \subseteq X$

and W_I is generated by elements in S_I, therefore $W_I = X$. □

Corollary 2.1.15. *Let $I, J \subseteq \mathscr{D}$. $W_I \cap W_J = W_{I \cap J}$.*

Proof. This is true by Proposition 2.1.14. □

Corollary 2.1.16. *If $W_I w_1 \cap W_J w_2 \neq \phi$ where $w_1, w_2 \in W$, then there exists $w \in W$ such that $W_I w_1 \cap W_J w_2 = W_{I \cap J} w$.*

Proof. Let $w_1, w_2 \in W$ and $W_I w_1 \cap W_J w_2 \neq \phi$. Then there exists $w \in W$ such that $w \in W_I w_1 \cap W_J w_2$, whence $W_I w_1 = W_I w$ and $W_J w_2 = W_J w$. Therefore, by Corollary 2.1.15,

$$W_I w_1 \cap W_J w_2 = W_I w \cap W_J w = (W_I \cap W_J)\, w = W_{I \cap J} w.$$

□

2.2 Lie algebras

In this section, we review the fundamental theory of finite-dimensional Lie algebras and analyze the properties of parabolic subalgebras of Lie algebras. Additional information about Lie algebras can be found in [Hum72], [Jac79], [Kna02], and [Mil12b].

2.2.1 Definitions and examples

Here we define Lie algebras over an arbitrary field $\mathbb{F}$ and give some basic examples of them.

Definition 2.2.1. A **Lie algebra** $\mathfrak{g}$ is a vector space over $\mathbb{F}$ equipped with a skew symmetric bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **Lie bracket**, satisfying the **Jacobi identity**, i.e., $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in \mathfrak{g}$. In particular, a Lie algebra $\mathfrak{g}$ is said to be **abelian** if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

A **Lie subalgebra** $\mathfrak{s}$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ which is closed under the Lie bracket on $\mathfrak{g}$, i.e., $[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{s}$.

As the Lie bracket is skew symmetric, it implies that $[x, x] = 0$ for all $x \in \mathfrak{g}$. Unless specified otherwise, we shall consider only finite-dimensional Lie algebras.

Example 2.2.2. (Examples of Lie algebras)

1. For any associative algebra A with multiplication $\star : A \times A \to A$, the algebra $\mathfrak{g}_A := A$ equipped with a bilinear form

$$\begin{aligned} [\cdot,\cdot] : \mathfrak{g}_A \times \mathfrak{g}_A &\to \mathfrak{g}_A \\ (x,y) &\mapsto x \star y - y \star x, \end{aligned} \tag{2.2.1}$$

is a Lie algebra. It is the Lie algebra associated to A.

2. Let V be a finite dimensional vector space. Then the set of endomorphisms of V is an associative algebra. Therefore, equipped with the Lie bracket defined as (2.2.1), it corresponds to a Lie algebra, denoted by $\mathfrak{gl}(V)$.

3. If $\mathbb{F} \subseteq \mathbb{F}'$ is a field extension, then for any Lie algebra $\mathfrak{g}$ over $\mathbb{F}$,

$$\mathfrak{g}' := \mathbb{F}' \otimes_{\mathbb{F}} \mathfrak{g}$$

is a Lie algebra over $\mathbb{F}'$ with the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}'} : \mathfrak{g}' \times \mathfrak{g}' \to \mathfrak{g}'$, $(\lambda \otimes x, \mu \otimes y) \mapsto \lambda\mu \otimes [x,y]$.

Definition 2.2.3. A $\mathbb{F}$-linear map $f : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras over $\mathbb{F}$ is called a **Lie algebra homomorphism** if

$$[f(x), f(y)] = f([x,y])$$

for all $x, y \in \mathfrak{g}$. Further, we say that $\mathfrak{g} \cong \mathfrak{g}'$ if and only if there exists a homomorphism $f' : \mathfrak{g}' \to \mathfrak{g}$ such that $f' \circ f = \mathrm{id}_{\mathfrak{g}}$ and $f \circ f' = \mathrm{id}_{\mathfrak{g}'}$; we call f (and f') an **Lie algebra isomorphism**.

A **representation** of a Lie algebra $\mathfrak{g}$ is a vector space V together with a Lie algebra homomorphism $f : \mathfrak{g} \to \mathfrak{gl}(V)$. A **sub-representation** of a representation $f : \mathfrak{g} \to \mathfrak{gl}(V)$ is a subspace W satisfying

$$f(x)(W) \subseteq W$$

for all $x \in \mathfrak{g}$. A representation of a Lie algebra $\mathfrak{g}$ is said to be **irreducible** if it contains no proper sub-representation. If $W \subseteq W' \subseteq V$ are sub-representations of a representation $f : \mathfrak{g} \to \mathfrak{gl}(V)$, then we can define a representation $f' : \mathfrak{g} \to \mathfrak{gl}(V/W)$, called a **quotient representation**, of f and a representation $f'' : \mathfrak{g} \to \mathfrak{gl}(W'/W)$, called a **subquotient representation**, of f.

The **NILPOTENT RADICAL**, denoted by $\mathfrak{nr}(\mathfrak{g})$, of a Lie algebra $\mathfrak{g}$ is the intersection of the kernels of the irreducible representations of $\mathfrak{g}$.

Example 2.2.4. The Lie bracket of a Lie algebra $\mathfrak{g}$ defines a representation, called the **ADJOINT REPRESENTATION**, of $\mathfrak{g}$,

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ x &\to \mathrm{ad}(x), \end{aligned}$$

where $\mathrm{ad}(x)(y) = [x, y]$, for all $y \in \mathfrak{g}$.

Definition 2.2.5. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{s}$ be a Lie subalgebra of $\mathfrak{g}$. The **CENTRALIZER** of $\mathfrak{s}$ in $\mathfrak{g}$ denoted by

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) := \{x \in \mathfrak{g} \,|\, [x, y] = 0 \text{ for all } y \in \mathfrak{s}\}$$

is a Lie subalgebra of $\mathfrak{g}$. We will call the centralizer of $\mathfrak{g}$ in $\mathfrak{g}$ the **CENTER** of $\mathfrak{g}$ and denote it by $\mathfrak{z}(\mathfrak{g})$. Note that if $\mathfrak{z}(\mathfrak{g}) = \mathfrak{g}$ if and only if $\mathfrak{g}$ is abelian.

The **NORMALIZER** of $\mathfrak{s}$ in $\mathfrak{g}$ is the subalgebra

$$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{s}) := \{x \in \mathfrak{g} \,|\, [x, \mathfrak{s}] \subseteq \mathfrak{s}\}.$$

And $\mathfrak{s}$ is called an **IDEAL** if $[\mathfrak{g}, \mathfrak{s}] \subseteq \mathfrak{s}$.

Example 2.2.6. 1. Let $\mathfrak{g}$ be a Lie algebra. For any subsets $\mathfrak{s}_1$, $\mathfrak{s}_2$ of $\mathfrak{g}$, define

$$[\mathfrak{s}_1, \mathfrak{s}_2] = \left\{ \sum_{i=1}^{n} c_i [x_i, y_i] \,|\, c_i \in \mathbb{F}, x_i \in \mathfrak{s}_1, y_i \in \mathfrak{s}_2 \text{ and } n \in \mathbb{N} \right\}.$$

This obviously is a subspace of $\mathfrak{g}$. Similarly,

$$\mathfrak{s}_1 + \mathfrak{s}_2 = \left\{ \sum_{i=1}^{n} c_i x_i \,|\, c_i \in \mathbb{F}, x_i \in \mathfrak{s}_1 \cup \mathfrak{s}_2 \text{ and } n \in \mathbb{N} \right\}$$

is a subspace of $\mathfrak{g}$. If $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are ideals of $\mathfrak{g}$, then so are $[\mathfrak{s}_1, \mathfrak{s}_2]$ and $\mathfrak{s}_1 + \mathfrak{s}_2$.

2. If $\mathfrak{s}$ is a Lie subalgebra of a Lie algebra $\mathfrak{g}$, then its normalizer $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{s})$ is the largest subalgebra of $\mathfrak{g}$ containing $\mathfrak{s}$ as an ideal.

3. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras and $f : \mathfrak{g} \to \mathfrak{g}'$ be a Lie algebra homomorphism. The kernel

$$\mathfrak{ker}(f) = \{x \in \mathfrak{g} \,|\, f(x) = 0\}$$

of f is then an ideal of $\mathfrak{g}$.

4. Let $\mathfrak{g}$ be a Lie algebra. Then the nilpotent radical $\mathfrak{nr}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra. By skew symmetry of the Lie bracket, any ideal is two-sided. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then the Lie bracket on $\mathfrak{g}$ induces a Lie bracket, $[\cdot,\cdot]_{\mathfrak{g}/\mathfrak{a}}$, on the quotient space $\mathfrak{g}/\mathfrak{a}$ by

$$[x+\mathfrak{a}, y+\mathfrak{a}]_{\mathfrak{g}/\mathfrak{a}} := [x,y] + \mathfrak{a},$$

hence $\mathfrak{g}/\mathfrak{a}$ is a Lie algebra.

Definition 2.2.7. A symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ on a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is called **INVARIANT** if

$$B([z,x], y) + B(x, [z,y]) = 0$$

for any $x, y, z \in \mathfrak{g}$, and is called **NON-DEGENERATE** when

$$\{x \in \mathfrak{g} \,|\, B(x,y) = 0, \forall y \in \mathfrak{g}\} = \{0\}.$$

For a subspace $\mathfrak{s}$ of $\mathfrak{g}$, let $\mathfrak{s}^{\perp} := \{x \in \mathfrak{g} \,|\, B(x,\mathfrak{s}) = 0\}$ the **ORTHOGONAL SUBSPACE** of $\mathfrak{s}$ with respect to B.

Any Lie algebra $\mathfrak{g}$ admits an important invariant bilinear form, called the **KILLING FORM**, defined by

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F} : (x,y) \mapsto \operatorname{tr}(\operatorname{ad}(x) \circ \operatorname{ad}(y)),$$

where ad is the adjoint representation.

Theorem 2.2.8. (ENGEL'S THEOREM) *Let $\mathfrak{g}$ be Lie subalgebra of $\mathfrak{gl}(V)$ such that every element x in $\mathfrak{g}$ is a nilpotent endomorphism of V, i.e., there exists a positive integer k such that $x^k(V) = 0$, for all $x \in \mathfrak{g}$. Then there exists a nonzero vector $v \in V$ such that $x(v) = 0$ for all $x \in \mathfrak{g}$.*

Proof. See [FH91], Theorem 9.9. □

Corollary 2.2.9. *If $\mathfrak{n}$ is an ideal of a Lie algebra $\mathfrak{g}$ and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a finite dimensional representation of $\mathfrak{g}$. Then the following are equivalent*:

(1) *For any $x \in \mathfrak{n}$, the endomorphism $\rho(x)$ of V is nilpotent,*

(2) *V has a filtration*

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k = V,$$

such that $\rho(\mathfrak{n})(V_i) \subseteq V_{i-1}$,

(3) $\rho(\mathfrak{n})$ acts trivially on any irreducible subquotient of ρ.

Proof. (1) $\Rightarrow$ (2) Let $V_1 := \{v \in V \,|\rho(x)(v) = 0 \text{ for all } x \in \mathfrak{n}\}$. Then, by Theorem 2.2.8, $V_1 \neq \{0\}$ and $\rho(\mathfrak{n})(V_1) \subseteq \{0\}$. If $V_1 = V$, then we are done and $k = 1$.

Suppose that $V_1 \neq V$. Then V/V_1 is a nontrivial quotient representation of $\mathfrak{n}$. Now let $V_2 := \{v \in V \,|\rho(x)(v) \in V_1 \text{ for all } x \in \mathfrak{n}\}$. Since $\rho(x)$ is a nilpotent endomorphism of V, it is also a nilpotent endomorphism of V/V_1. Again, by Theorem 2.2.8, $V_2 \neq \{0\}$ and $\rho(\mathfrak{n})(V_2) \subseteq V_1$.

By the same manner, we finally obtain a filtration $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k = V$ of V because V is finite-dimensional.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Let $x \in \mathfrak{n}$. Since V is finite-dimensional, there exists a positive integer k such that

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k = V$$

a filtration of sub-representations and V_i/V_{i-1} is an irreducible subquotient representation of ρ, for all $1 \leq i \leq k$. Therefore $\rho(x)$ is nilpotent because it acts trivially on any irreducible subquotient of ρ. ☐

Remark 2.2.10. Let $\mathfrak{g}$ be a Lie algebra. Any representation f of $\mathfrak{g}$ gives an ideal

$$\mathfrak{n}_f := \bigcap \text{kernel of irreducible subquotients of } f. \tag{2.2.2}$$

Then

$$\mathfrak{nr}(\mathfrak{g}) = \bigcap_{\text{representations } f} \mathfrak{n}_f.$$

Definition 2.2.11. Let $\mathfrak{g}'$ and $\mathfrak{g}''$ be Lie algebras over $\mathbb{F}$. An EXTENSION of $\mathfrak{g}''$ by $\mathfrak{g}'$ is a

Lie algebra $\mathfrak{g}$ together with an exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}'' \longrightarrow 0\ .$$

The extension is said to be **CENTRAL** if $[\mathfrak{g}', \mathfrak{g}] = 0$.

Remark 2.2.12. If $\mathfrak{g}$ is an extension of $\mathfrak{g}''$ by $\mathfrak{g}'$, then, by abusing notation, we may identify $\mathfrak{g}'$ with its image in $\mathfrak{g}$ and consider it as an ideal of $\mathfrak{g}$; whence $\mathfrak{g}'' \cong \mathfrak{g}/\mathfrak{g}'$.

Definition 2.2.13. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then $\mathfrak{g}$ is a **SEMI-DIRECT SUM** of subalgebras $\mathfrak{a}$ and $\mathfrak{s}$, denoted by $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{s}$, if $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ and the canonical quotient map $\mathfrak{a} \to \mathfrak{g}/\mathfrak{a}$ induces a Lie algebra isomorphism $\mathfrak{s} \to \mathfrak{g}/\mathfrak{a}$.

Remark 2.2.14. The above definition is equivalent to say that the exact sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\ i\ } \mathfrak{g} \xrightarrow{\ \pi\ } \mathfrak{s} \longrightarrow 0$$

is split, i.e., there is a Lie algebra homomorphism $\beta : \mathfrak{s} \to \mathfrak{g}$ such that $\pi \circ \beta = \mathrm{id}_{\mathfrak{s}}$.

2.2.2 Basic structure theory of Lie algebras

We will assume henceforth that any Lie algebra we mention is finite-dimensional. The underlying field is still arbitrary, unless otherwise stated.

Definition 2.2.15. Let $\mathfrak{g}$ be a Lie algebra. The **LOWER CENTRAL SERIES** of $\mathfrak{g}$ is a sequence of ideals of $\mathfrak{g}$:

$$C_1(\mathfrak{g}) = \mathfrak{g} \text{ and } C_{n+1}(\mathfrak{g}) = [\mathfrak{g}, C_n(\mathfrak{g})]\,,$$

for integer $n \geq 1$. The **DERIVED SERIES** of a Lie algebra $\mathfrak{g}$ is a sequence of ideals of $\mathfrak{g}$;

$$D_1(\mathfrak{g}) = \mathfrak{g} \text{ and } D_{n+1}(\mathfrak{g}) = [D_n(\mathfrak{g}), D_n(\mathfrak{g})]\,,$$

for integer $n \geq 1$.

A Lie algebra $\mathfrak{g}$ is called **NILPOTENT** (resp. **SOLVABLE**) if there exists a positive integer $n \geq 2$ such that $C_n(\mathfrak{g}) = \{0\}$ (resp. $D_n(\mathfrak{g}) = \{0\}$).

Remark 2.2.16. As $D_n(\mathfrak{g}) \subseteq C_n(\mathfrak{g})$ for all $n \geq 1$, if $\mathfrak{g}$ is a nilpotent Lie algebra, then it is solvable.

If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are ideals of a Lie algebra $\mathfrak{g}$, one can show, by induction, that for $n \geq 1$, both $C_n(\mathfrak{a}_1)$ and $D_n(\mathfrak{a}_1)$ are ideals of $\mathfrak{g}$; moreover

$$C_{2n}(\mathfrak{a}_1 + \mathfrak{a}_2) \subseteq C_n(\mathfrak{a}_1) + C_n(\mathfrak{a}_2)$$

and

$$C_{2n+1}(\mathfrak{a}_1 + \mathfrak{a}_2) \subseteq C_{n+1}(\mathfrak{a}_1) + C_{n+1}(\mathfrak{a}_2)$$

for all $n \geq 1$; whence if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are nilpotent, then so is $\mathfrak{a}_1 + \mathfrak{a}_2$. On the other hand, if $\mathfrak{a}_1$ is a solvable ideal and $\mathfrak{a}_2$ is a solvable subalgebra of a Lie algebra $\mathfrak{g}$, then $\mathfrak{a}_1 + \mathfrak{a}_2$ is a solvable subalgebra because there must be a positive integer m such that $D_m(\mathfrak{a}_1 + \mathfrak{a}_2) \subseteq \mathfrak{a}_1$. This implies that if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are solvable ideals of a Lie algebra $\mathfrak{g}$, then so is the ideal $\mathfrak{a}_1 + \mathfrak{a}_2$. Therefore the maximal nilpotent (resp.solvable) ideal of $\mathfrak{g}$ exists and equals to the sum of all nilpotent (resp. solvable) ideals of $\mathfrak{g}$.

Definition 2.2.17. The **NILRADICAL** (resp. **RADICAL**), denoted by $\mathfrak{nil}(\mathfrak{g})$ (resp. $\mathfrak{rad}(\mathfrak{g})$), of a Lie algebra $\mathfrak{g}$ is the maximal nilpotent (resp. solvable) ideal of $\mathfrak{g}$.

Remark 2.2.18. Since any nilpotent Lie algebra is solvable, $\mathfrak{nil}(\mathfrak{g}) \subseteq \mathfrak{rad}(\mathfrak{g})$.

Corollary 2.2.9 implies that a Lie algebra $\mathfrak{g}$ has a filtration

$$\{0\} = \mathfrak{g}_0 \subsetneq \mathfrak{g}_1 \subsetneq \cdots \subsetneq \mathfrak{g}_k = \mathfrak{g},$$

such that $\mathrm{ad}(\mathfrak{nil}(\mathfrak{g}))(\mathfrak{g}_i) \subseteq \mathfrak{g}_{i-1}$; hence, by Remark 2.2.10,

$$\mathfrak{nil}(\mathfrak{g}) = \bigcap_{i=1}^{\dim(\mathfrak{g})} \mathfrak{ker}(\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}_{i+1} / \mathfrak{g}_i).$$

Therefore $\mathfrak{nr}(\mathfrak{g}) \subseteq \mathfrak{n}_{\mathrm{ad}} = \mathfrak{nil}(\mathfrak{g}) \subseteq \mathfrak{rad}(\mathfrak{g})$, where $\mathfrak{n}_{\mathrm{ad}}$ is defined as in Equation (2.2.2); whence $\mathfrak{nr}(\mathfrak{g})$ is nilpotent.

Definition 2.2.19. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Then

1. $\mathfrak{g}$ is **SIMPLE** if it is non-abelian and has no proper ideals,

2. $\mathfrak{g}$ is **SEMISIMPLE** if $\mathfrak{rad}(\mathfrak{g}) = \{0\}$; equivalently, $\mathfrak{g}$ splits into the direct sum of simple ideals, called **SIMPLE COMPONENTS**, of $\mathfrak{g}$,

3. $\mathfrak{g}$ is **REDUCTIVE** if $\mathfrak{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$.

A **CARTAN SUBALGEBRA** of a Lie algebra over a field $\mathbb{F}$ is a nilpotent subalgebra equal to its own normalizer. A **BOREL SUBALGEBRA** of a Lie algebra is a maximal solvable subalgebra.

Remark 2.2.20. Any Borel subalgebra of a Lie algebra $\mathfrak{g}$ contains $\mathfrak{rad}(\mathfrak{g})$.

Lemma 2.2.21. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{m}$ *be an ideal of* $\mathfrak{g}$. *Then* $\mathfrak{rad}(\mathfrak{g}/\mathfrak{m}) \subseteq (\mathfrak{rad}(\mathfrak{g}) + \mathfrak{m})/\mathfrak{m}$. *In particular, if* $\mathfrak{m}$ *is solvable, then* $\mathfrak{rad}(\mathfrak{g}/\mathfrak{m}) = \mathfrak{rad}(\mathfrak{g})/\mathfrak{m}$.

Proof. $\mathfrak{rad}(\mathfrak{g}) + \mathfrak{m}$ is an ideal of $\mathfrak{g}$ containing $\mathfrak{m}$ because both $\mathfrak{rad}(\mathfrak{g})$ and $\mathfrak{m}$ are ideals of $\mathfrak{g}$. Since $\mathfrak{rad}(\mathfrak{g})$ is solvable, there exists a positive integer n such that $D_n(\mathfrak{rad}(\mathfrak{g})) = \{0\}$; whence $D_n(\mathfrak{rad}(\mathfrak{g}) + \mathfrak{m}) \subseteq D_n(\mathfrak{rad}(\mathfrak{g})) + \mathfrak{m} \subseteq \mathfrak{m}$. Thus $(\mathfrak{rad}(\mathfrak{g}) + \mathfrak{m})/\mathfrak{m}$ is a solvable ideal of $\mathfrak{g}/\mathfrak{m}$, and therefore $(\mathfrak{rad}(\mathfrak{g}) + \mathfrak{m})/\mathfrak{m} \subseteq \mathfrak{rad}(\mathfrak{g}/\mathfrak{m})$.

Suppose that $\mathfrak{m}$ is solvable and $\mathfrak{rad}(\mathfrak{g}/\mathfrak{m}) = \mathfrak{a}/\mathfrak{m}$ for some ideal $\mathfrak{a}$ of $\mathfrak{g}$. Then there exists a positive integer k such that $D_k(\mathfrak{a}) \subseteq \mathfrak{m}$. As $\mathfrak{m}$ is also solvable, thus $\mathfrak{a}$ is solvable and $\mathfrak{a} \subseteq \mathfrak{rad}(\mathfrak{g})$. Therefore $\mathfrak{rad}(\mathfrak{g}/\mathfrak{m}) = \mathfrak{rad}(\mathfrak{g})/\mathfrak{m}$. ☐

Proposition 2.2.22. *Let* $\mathfrak{g}$ *be a Lie algebra* over a field $\mathbb{F}$ *of characteristic zero. Then*

$$\mathfrak{nr}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{rad}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{rad}(\mathfrak{g})].$$

Moreover $\mathfrak{g}/\mathfrak{nr}(\mathfrak{g})$ *is reductive.*

Proof. Proposition 7.5 in [Mil12b] shows the first part of the Theorem. Now we will show that $\mathfrak{g}/\mathfrak{nr}(\mathfrak{g})$ is reductive. Since $[\mathfrak{g}, \mathfrak{rad}(\mathfrak{g})] \subseteq \mathfrak{nr}(\mathfrak{g})$, Lemma 2.2.21 implies that

$$\mathfrak{rad}(\mathfrak{g}/\mathfrak{nr}(\mathfrak{g})) = \mathfrak{rad}(\mathfrak{g})/\mathfrak{nr}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}/\mathfrak{nr}(\mathfrak{g})).$$

Therefore $\mathfrak{g}/\mathfrak{nr}(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{rad}(\mathfrak{g})]$ is reductive. ☐

Remark 2.2.23. If $\mathfrak{g}$ is a solvable Lie algebra, it immediately follows from Proposition 2.2.22 that $\mathfrak{nr}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Theorem 2.2.24. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{s}$ be a semisimple Lie algebra over a field $\mathbb{F}$ *of characteristic zero.* If $\pi : \mathfrak{g} \to \mathfrak{s}$ is a surjective homomorphism, then there is a splitting $\beta : \mathfrak{s} \to \mathfrak{g}$ such that $\pi \circ \beta = \mathrm{id}_{\mathfrak{s}}$.

Proof. See [Pro07], p.305. □

Corollary 2.2.25. (*Semisimple Levi Decomposition*) *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic zero. Then there exists a semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{rad}(\mathfrak{g})$ (direct sum of vector spaces).*

Proof. By Lemma 2.2.21, $\mathfrak{rad}(\mathfrak{g}/\mathfrak{rad}(\mathfrak{g}))$ is trivial, and so $\mathfrak{g}/\mathfrak{rad}(\mathfrak{g})$ is semisimple. Consider the exact sequence

$$0 \longrightarrow \mathfrak{rad}(\mathfrak{g}) \longrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathfrak{rad}(\mathfrak{g}) \longrightarrow 0$$

Theorem 2.2.24 implies that the exact sequence is split. Let $\mathfrak{s} = \beta(\mathfrak{g}/\mathfrak{rad}(\mathfrak{g})) \subseteq \mathfrak{g}$, where $\beta : \mathfrak{g}/\mathfrak{rad}(\mathfrak{g}) \to \mathfrak{g}$ is a Lie algebra homomorphism such that $\pi \circ \beta = \mathrm{id}_{\mathfrak{g}/\mathfrak{rad}(\mathfrak{g})}$. Then $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{rad}(\mathfrak{g})$ because $\mathfrak{s} \cap \mathfrak{rad}(\mathfrak{g}) = \mathfrak{s} \cap \mathfrak{im}(i) = \mathfrak{s} \cap \mathfrak{ker}(\pi) = \{0\}$ and $\mathfrak{s} \cong \mathfrak{g}/\mathfrak{rad}(\mathfrak{g})$. □

Remark 2.2.26. The Lie subalgebra $\mathfrak{s}$ is called a semisimple part of $\mathfrak{g}$. It is not uniquely determined. However, any two such subalgebras of a Lie algebra are conjugate under an inner automorphism.

Therefore any finite-dimensional Lie algebra is an extension of a semisimple Lie algebra by a solvable Lie algebra. There is not always a subalgebra complementary to $\mathfrak{nr}(\mathfrak{g})$ in $\mathfrak{g}$; to have such complementary subalgebra, we need $\mathfrak{rad}(\mathfrak{g})$ to be split as a direct sum of $\mathfrak{nr}(\mathfrak{g})$ and a subspace of $\mathfrak{rad}(\mathfrak{g})$.

Corollary 2.2.27. *Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra over a field $\mathbb{F}$ of characteristic zero. Then there exists a semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{s}$ (direct sum of vector spaces).*

Proof. This follows immediately from the fact that $\mathfrak{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$ and Corollary 2.2.25. □

2.2.3 Parabolic subalgebras

Let us fix a field $\mathbb{F}$ of characteristic zero. The Lie algebras we are going to talk about throughout this section will be finite dimensional Lie algebras over $\mathbb{F}$. For any Lie subalgebra

$\mathfrak{s}$ of a Lie algebra $\mathfrak{g}$ with an invariant bilinear form B and an ideal $\mathfrak{n}$ of $\mathfrak{s}$, set $\mathfrak{s}_{(0)} = \mathfrak{s}$, $\mathfrak{s}_{(-i)} := C_i(\mathfrak{n})$ and $\mathfrak{s}_{(i)} := \left(\mathfrak{s}_{(-i-1)}\right)^{\perp}$ for all positive integer $i \geq 1$.

Lemma 2.2.28. *Let $\mathfrak{s}$ be a Lie subalgebra of a Lie algebra $\mathfrak{g}$ and B be an invariant bilinear form on $\mathfrak{g}$. If $\left[\mathfrak{s}_{(-1)}, \mathfrak{s}_{(0)}\right] \subseteq \mathfrak{s}_{(-1)}$, then $\left[\mathfrak{s}_{(i)}, \mathfrak{s}_{(j)}\right] \subseteq \mathfrak{s}_{(i+j)}$.*

Proof. Suppose that $\left[\mathfrak{s}_{(-1)}, \mathfrak{s}_{(0)}\right] \subseteq \mathfrak{s}_{(-1)}$. For $j < 0$, we have $\left[\mathfrak{s}_{(-1)}, \mathfrak{s}_{(j)}\right] \subseteq \mathfrak{s}_{(j-1)}$. If $j > 0$, then

$$B\left(\left[\mathfrak{s}_{(-1)}, \mathfrak{s}_{(j)}\right], \mathfrak{s}_{(-j)}\right) = B\left(\mathfrak{s}_{(j)}, \left[\mathfrak{s}_{(-j)}, \mathfrak{s}_{(-1)}\right]\right) \subseteq B\left(\mathfrak{s}_{(j)}, \mathfrak{s}_{(-j-1)}\right) = 0,$$

and so $\left[\mathfrak{s}_{(-1)}, \mathfrak{s}_{(-j)}\right] \subseteq \left(\mathfrak{s}_{(-j)}\right)^{\perp} = \mathfrak{s}_{(j-1)}$. This implies that $\left[\mathfrak{s}_{(-1)}, \mathfrak{s}_{(j)}\right] \subseteq \mathfrak{s}_{(j-1)}$ for all $j \in \mathbb{Z}$.

For $i < 0$ and $j \in \mathbb{Z}$, the Jacobi identity and the definition of $\mathfrak{s}_{(i)}$ imply that $\left[\mathfrak{s}_{(i)}, \mathfrak{s}_{(j)}\right] \subseteq \mathfrak{s}_{(i+j)}$. Now there is only one case left which is when $i, j \geq 0$. If $i, j \geq 0$, then

$$B\left(\left[\mathfrak{s}_{(i)}, \mathfrak{s}_{(j)}\right], \mathfrak{s}_{(-i-j-1)}\right) = B\left(\mathfrak{s}_{(j)}, \left[\mathfrak{s}_{(-i-j-1)}, \mathfrak{s}_{(i)}\right]\right) \subseteq B\left(\mathfrak{s}_{(j)}, \left[\mathfrak{s}_{(-j-1)}\right]\right) = 0,$$

and so $\left[\mathfrak{s}_{(i)}, \mathfrak{s}_{(j)}\right] \subseteq \left(\mathfrak{s}_{(-i-j-1)}\right)^{\perp} = \mathfrak{s}_{(i+j)}$. □

Definition 2.2.29. Let $\mathfrak{g}$ be a Lie algebra and (V, ρ) be a representation of $\mathfrak{g}$. The **TRACE FORM** associated to (V, ρ) is the symmetric invariant bilinear form

$$(x, y)_{\rho} := \operatorname{tr}(\rho(x)\rho(y)).$$

Furthermore, the trace form is said to be **ADMISSIBLE** if $\mathfrak{g}^{\perp} = \mathfrak{nr}(\mathfrak{g})$, where $\mathfrak{nr}(\mathfrak{g})$ is the nilpotent radical of $\mathfrak{g}$.

Proposition 2.2.30. *Let $\mathfrak{g}$ be a Lie algebra with a trace form B associated to (V, ρ). Then $\mathfrak{nr}(\mathfrak{g}) \subseteq \mathfrak{g}^{\perp}$ and the induced invariant bilinear form on $\mathfrak{g}/\mathfrak{nr}(\mathfrak{g})$ is a trace form.*

Proof. Let $(V_i)_{i=0,\ldots,k}$ be a finite chain of $\mathfrak{g}$-submodules of V such that $\rho(\mathfrak{nr}(\mathfrak{g}))(V_i) \subseteq V_{i-1}$ and let $V' = \bigoplus_{i=1}^{k} V_i/V_{i-1}$ be the associated graded representation. Then the representation V and V' induce the same invariant form on $\mathfrak{g}$. Since $\mathfrak{nr}(\mathfrak{g})$ acts trivially on V', we have $\mathfrak{nr}(\mathfrak{g}) \subseteq \mathfrak{g}^{\perp}$. Thus the bilinear form

$$\begin{aligned} B' : \mathfrak{g}/\mathfrak{nr}(\mathfrak{g}) \times \mathfrak{g}/\mathfrak{nr}(\mathfrak{g}) &\to \mathbb{F} \\ (x + \mathfrak{nr}(\mathfrak{g}), y + \mathfrak{nr}(\mathfrak{g})) &\mapsto B(x, y) \end{aligned}$$

is a well-defined invariant bilinear form. Since V' descends to a representation of $\mathfrak{g}/\mathfrak{nr}(\mathfrak{g})$, therefore B' is the trace form of $\mathfrak{g}/\mathfrak{nr}(\mathfrak{g})$ associated to the representation V'. □

Corollary 2.2.31. *Let $\mathfrak{g}$ be a Lie algebra with a trace form B associated to (V,ρ) and $\mathfrak{q}$ be a Lie subalgebra of $\mathfrak{g}$. If $\mathfrak{q}^{\perp} = \mathfrak{nr}(\mathfrak{q})$, then the induced trace form on $\mathfrak{q}/\mathfrak{q}^{\perp}$ is admissible.*

Proof. This is an immediate result from Proposition 2.2.30. □

Lemma 2.2.32. *Any Lie algebra $\mathfrak{g}$ admits an admissible trace form.*

Proof. This comes from applying the fact that a Lie algebra is reductive if and only if it admits a faithful finite-dimensional semisimple representation with associated nondegenerate trace form (see [Bou89]) to the reductive Lie algebra $\mathfrak{g}/\mathfrak{nr}(\mathfrak{g})$. □

For any Lie subalgebra $\mathfrak{p}$ of a Lie algebra $\mathfrak{g}$ with an admissible trace from B, let $\mathfrak{g}_{(0)} := \mathfrak{p}$, let $\mathfrak{g}_{(-i)} := C_i(\mathfrak{nr}(\mathfrak{p}))$, and let $\mathfrak{g}_{(i)} := \left(\mathfrak{g}_{(-i-1)}\right)^{\perp}$ for all positive integer $i \geq 1$. By Lemma 2.2.28, these give a filtration of $\mathfrak{g}$. Define $\mathfrak{g}_i := \mathfrak{g}_{(i)}/\mathfrak{g}_{(i-1)}$ for all $i \in \mathbb{Z}$ and $gr_{\mathfrak{p}}(\mathfrak{g}) := \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$.

Definition 2.2.33. Let $\mathfrak{p}$ be a Lie subalgebra of a Lie algebra $\mathfrak{g}$ with an admissible trace form B. A **grading element** for $\mathfrak{p}$ in $\mathfrak{g}$ is an element $\delta \in gr_{\mathfrak{p}}(\mathfrak{g})$ with $[\delta, x] = ix$ for all $i \in \mathbb{Z}$ and $x \in \mathfrak{g}_i$.

Remark 2.2.34. Any reductive Lie algebra has 0 as a grading element.

Definition 2.2.35. A Lie subalgebra $\mathfrak{p}$ of a Lie algebra $\mathfrak{g}$ is called **parabolic** if it contains $\mathfrak{rad}(\mathfrak{g})$ and $\mathfrak{p}^{\perp}$ (with respect to some admissible trace form of $\mathfrak{g}$) is a nilpotent subalgebra of $\mathfrak{p}$.

Remark 2.2.36. A Lie algebra $\mathfrak{g}$ is a (improper) parabolic subalgebra of itself because $\mathfrak{nr}(\mathfrak{g})$ is nilpotent. If $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, then $\mathfrak{nr}(\mathfrak{g}) = \mathfrak{g}^{\perp} \subseteq \mathfrak{p}^{\perp} \subseteq \mathfrak{p}$. Moreover, if $\mathfrak{q}$ is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{p}$, then $\mathfrak{q}$ is also a parabolic subalgebra of $\mathfrak{g}$ because $\mathfrak{rad}(\mathfrak{g}) \subseteq \mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{q}^{\perp} \subseteq \mathfrak{p}^{\perp} \subseteq \mathfrak{p} \subseteq \mathfrak{q}$.

Proposition 2.2.37. *Let $\mathbb{F} \subseteq \mathbb{F}'$ be a field extension. Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{p}' := \mathbb{F}' \otimes_{\mathbb{F}} \mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}' := \mathbb{F}' \otimes_{\mathbb{F}} \mathfrak{g}$.*

Proof. Since $\mathfrak{rad}(\mathfrak{g}') = \mathbb{F}' \otimes_{\mathbb{F}} \mathfrak{rad}(\mathfrak{g})$, a Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{rad}(\mathfrak{g})$ if and only if $\mathfrak{p}'$ contains $\mathfrak{rad}(\mathfrak{g}')$. Moreover, there is a one-to-one correspondence between invariant bilinear forms of $\mathfrak{g}$ and those of $\mathfrak{g}'$ in such a way that $(\mathfrak{g}')^{\perp} = \mathbb{F}' \otimes_{\mathbb{F}} \mathfrak{g}^{\perp}$. Thus $\mathfrak{g}^{\perp} = \mathfrak{nr}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{rad}(\mathfrak{g})]$ if and only if $(\mathfrak{g}')^{\perp} = \mathbb{F}' \otimes_{\mathbb{F}} [\mathfrak{g}, \mathfrak{rad}(\mathfrak{g})] = [\mathbb{F}' \otimes_{\mathbb{F}} \mathfrak{g}, \mathbb{F}' \otimes_{\mathbb{F}} \mathfrak{rad}(\mathfrak{g})]$, and $\mathfrak{p}^{\perp}$ is a nilpotent subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{p}'$ is a nilpotent subalgebra of $\mathfrak{g}'$. □

Proposition 2.2.38. *Let $\mathfrak{g}$ be a Lie algebra and (V, ρ) be a finite-dimensional semisimple representation of $\mathfrak{g}$. Suppose that $x \in [\mathfrak{g}, \mathfrak{g}]$ is ad-nilpotent. Then $\rho(x)$ is nilpotent.*

Proof. The Jordan Chevalley decomposition of $\rho(x)$ is $\rho(x)_s + \rho(x)_n$. Then $\mathrm{ad}(\rho(x)_s) \circ \rho = \rho \circ \mathrm{ad}(x)_s = 0$ because x is ad-nilpotent, and so $\rho(x)_s \in \mathfrak{z}_{\mathfrak{gl}(V)}(\rho(\mathfrak{g}))$. On the other hand, $\rho(x)_s = \rho(x) - \rho(x)_n$ and $\rho(x) \in [\rho(\mathfrak{g}), \rho(\mathfrak{g})]$. By restricting to any simple component of V and extending the base field, $\rho(x)_s$ is a trace-free multiple of the identity. Thus the restriction of $\rho(x)_s$ to each simple component of V is zero; whence $\rho(x)_s = 0$. Therefore $\rho(x)$ is nilpotent. □

Theorem 2.2.39. *Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Then $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{p}$ and $\mathfrak{p}^{\perp} = \mathfrak{nr}(\mathfrak{p})$.*

Proof. Since $\mathfrak{nr}(\mathfrak{p}) \subseteq \mathfrak{p}^{\perp} \cap \mathfrak{p} = \mathfrak{p}^{\perp}$, it suffices to show that $\mathfrak{p}^{\perp} \subseteq \mathfrak{nr}(\mathfrak{p})$. As $\mathfrak{p}^{\perp}$ is an ideal of $\mathfrak{p}$, we have $\mathfrak{p} \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}^{\perp})$ and so $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}^{\perp})^{\perp} \subseteq \mathfrak{p}^{\perp}$. Moreover, we have $\mathfrak{p}^{\perp} \subseteq [\mathfrak{g}, \mathfrak{g}]$ because $[\mathfrak{g}, \mathfrak{g}]^{\perp} \subseteq \mathfrak{rad}(\mathfrak{g}) \subseteq \mathfrak{p}$ by Cartan's criterion. Since $\mathfrak{p}^{\perp} \subseteq [\mathfrak{g}, \mathfrak{g}]$ is a nilpotent ideal of $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}^{\perp})$, Proposition 2.2.38 implies that $\mathfrak{p}^{\perp} \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}^{\perp})^{\perp}$. Hence $\mathfrak{p}^{\perp} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}^{\perp})^{\perp}$.

Suppose that there exists $0 \neq x \in [\mathfrak{p}, \mathfrak{p}^{\perp}]^{\perp} \setminus \mathfrak{p}$. Since $\mathfrak{p} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}^{\perp})$, there exists $b \in \mathfrak{p}^{\perp}$ such that $[x, b] \notin \mathfrak{p}^{\perp}$. Thus there exists $a \in \mathfrak{p}$ such that $0 \neq B(a, [x, b]) = B([a, b], x) = 0$ which is a contradiction. So $[\mathfrak{p}, \mathfrak{p}^{\perp}]^{\perp} \subseteq \mathfrak{p}$ and whence $\mathfrak{p}^{\perp} \subseteq [\mathfrak{p}, \mathfrak{p}^{\perp}] \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p})^{\perp} \subseteq \mathfrak{p}^{\perp}$. Therefore $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{p}$. Furthermore, $\mathfrak{p}^{\perp} \subseteq [\mathfrak{p}, \mathfrak{p}^{\perp}] \subseteq [\mathfrak{p}, \mathfrak{rad}(\mathfrak{p})] = \mathfrak{nr}(\mathfrak{p})$. □

Corollary 2.2.40. *Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{p}$ be a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{rad}(\mathfrak{g})$. Then the following are equivalent:*

(1) *$\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$.*

(2) *$dim(\mathfrak{nr}(\mathfrak{p})) = dim(\mathfrak{g}) - dim(\mathfrak{p})$.*

(3) *For any admissible form on $\mathfrak{g}$, $\mathfrak{p}^{\perp} = \mathfrak{nr}(\mathfrak{p})$.*

Proof. (1) ⇒ (2) If $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, then, by Theorem 2.2.39, there exists an admissible trace form such that $\mathfrak{p}^{\perp} = \mathfrak{nr}(\mathfrak{p})$. So

$$\dim(\mathfrak{nr}(\mathfrak{p})) = \dim\left(\mathfrak{p}^{\perp}\right) = \dim(\mathfrak{g}) - \dim(\mathfrak{p}).$$

(2) ⇒ (3) Suppose that $\dim(\mathfrak{nr}(\mathfrak{p})) = \dim(\mathfrak{g}) - \dim(\mathfrak{p})$. Given an admissible trace form on $\mathfrak{g}$, we have $\mathfrak{nr}(\mathfrak{p}) \subseteq \mathfrak{p}^{\perp} \cap \mathfrak{p} \subseteq \mathfrak{p}^{\perp}$. Hence

$$\dim(\mathfrak{nr}(\mathfrak{p})) \leq \dim\left(\mathfrak{p}^{\perp}\right) = \dim(\mathfrak{g}) - \dim(\mathfrak{p}) = \dim(\mathfrak{nr}(\mathfrak{p})).$$

Therefore $\mathfrak{p}^{\perp} = \mathfrak{nr}(\mathfrak{p})$.

(3) ⇒ (1) Since $\mathfrak{nr}(\mathfrak{p})$ is nilpotent subalgebra of $\mathfrak{p}$, if $\mathfrak{p}^{\perp} = \mathfrak{nr}(\mathfrak{p})$, then $\mathfrak{p}^{\perp}$ is a nilpotent subalgebra of $\mathfrak{p}$. □

Proposition 2.2.41. *Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Then $gr_{\mathfrak{p}}(\mathfrak{g})$ is reductive and has a unique inner derivation δ, called a grading element, in $\mathfrak{z}(\mathfrak{p}_0) \cap [gr(\mathfrak{g}), gr(\mathfrak{g})]$, with $[\delta, x] = ix$ for all $-m \leq i \leq m$ and $x \in \mathfrak{p}_i$.*

Proof. Let B be a nondegenerate trace form associated to a faithful finite-dimensional semisimple representation (V, ρ) of $\mathfrak{g}$. Since $\mathfrak{p}^{\perp} = \mathfrak{nr}(\mathfrak{p})$, it acts nilpotently on V. This gives a finite chain $(V_i)_{i=0,\ldots,k}$ of $\mathfrak{g}$-submodules of V such that $\rho\left(\mathfrak{p}^{\perp}\right)(V_i) \subseteq V_{i-1}$. Since $\mathfrak{p}_{(i-1)} = \mathfrak{p}^{\perp}_{(-i)}$ for $i \leq 0$, the induced trace form on $gr_{\mathfrak{p}}(\mathfrak{g})$ is nondegenerate. Therefore $gr_{\mathfrak{p}}(\mathfrak{g})$ is reductive.

The derivation D of $gr_{\mathfrak{p}}(\mathfrak{g})$ defined by $Dx = ix$ for all $x \in \mathfrak{p}_i$ vanishes on the centre of $gr_{\mathfrak{p}}(\mathfrak{g})$ and preserves its semisimple complement. Hence D is an inner derivation, i.e., $D = \mathrm{ad}(\delta)$ for a grading element $\delta \in \mathfrak{z}(\mathfrak{p}_0)$ which is uniquely determined by requiring it is in the complement $\left[gr_{\mathfrak{p}}(\mathfrak{g}), gr_{\mathfrak{p}}(\mathfrak{g})\right]$ to the centre of $gr_{\mathfrak{p}}(\mathfrak{g})$. □

Remark 2.2.42. Let $\mathfrak{p}$ be a parabolic subalgebra of a Lie algebra $\mathfrak{g}$. Any lift $\widetilde{\delta}$, called an **ALGEBRAIC WEYL STRUCTURE**, in $\mathfrak{p}$ of δ with respect to the canonical quotient map $\pi : \mathfrak{p} \to \mathfrak{p}_0$ splits the filtration of $\mathfrak{g}$. Therefore the exact sequence

$$0 \longrightarrow \mathfrak{nr}(\mathfrak{p}) \longrightarrow \mathfrak{p} \longrightarrow \mathfrak{p}_0 \longrightarrow 0,$$

where $\mathfrak{p}_0 := \mathfrak{p} / \mathfrak{nr}(\mathfrak{p})$, splits. We call a subalgebra $\widetilde{\mathfrak{p}}_0$ of $\mathfrak{p}$ such that $\mathfrak{p} = \widetilde{\mathfrak{p}}_0 \oplus \mathfrak{nr}(\mathfrak{p})$ and $\widetilde{\mathfrak{p}}_0 \cong \mathfrak{p}_0$ a **LEVI SUBALGEBRA** of $\mathfrak{p}$, and call $\mathfrak{p}_0$ the **LEVI FACTOR** of $\mathfrak{p}$.

Definition 2.2.43. Let $\mathfrak{g}$ be a semisimple Lie algebra. Any two parabolic subalgebras $\mathfrak{p}$ and $\mathfrak{q}$ of $\mathfrak{g}$ are said to be

1. **CO-STANDARD** iff $\mathfrak{p} \cap \mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$; equivalently, $\mathfrak{nr}(\mathfrak{p})$ is a nilpotent subalgebra of $\mathfrak{q}$,

2. **WEAKLY OPPOSITE** iff $\mathfrak{p} + \mathfrak{q} = \mathfrak{g}$; equivalently, $\mathfrak{nr}(\mathfrak{p}) \cap \mathfrak{nr}(\mathfrak{q}) = \{0\}$,

3. **COMPLEMENTARY** if $\mathfrak{p} \cap \mathfrak{q}$ is a common Levi subalgebra of both $\mathfrak{p}$ and $\mathfrak{q}$.

Proposition 2.2.44. *Suppose that $\mathfrak{q}$ is a parabolic subalgebra of a reductive Lie algebra $\mathfrak{g}$ and $\mathfrak{p}$ is a Lie subalgebra of $\mathfrak{g}$. Then the following are equivalent:*

(1) $\mathfrak{p}$ *is a parabolic subalgebra of* $\mathfrak{q}$.

(2) $\mathfrak{p}$ *is a parabolic subalgebra of* $\mathfrak{g}$.

(3) $\mathfrak{p}$ *contains* $\mathfrak{nr}(\mathfrak{q})$ *and* $\mathfrak{p}/\mathfrak{nr}(\mathfrak{q})$ *is a parabolic subalgebra of* $\mathfrak{q}_0$.

Proof. Fix an admissible trace form on $\mathfrak{g}$. By Corollary 2.2.40, its restriction to $\mathfrak{q}$ is also admissible.

(1) $\Rightarrow$ (2) Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{q}$. Then $\mathfrak{rad}(\mathfrak{g}) \subseteq \mathfrak{rad}(\mathfrak{q}) \subseteq \mathfrak{p}$ and, by Corollary 2.2.40, $\mathfrak{q}^{\perp} = \mathfrak{nr}(\mathfrak{q}) \subseteq \mathfrak{rad}(\mathfrak{q}) \subseteq \mathfrak{p}$. Thus $\mathfrak{p}^{\perp} \subseteq \mathfrak{q}$, and so $\mathfrak{p}^{\perp}$ is a nilpotent subalgebra of $\mathfrak{p}$.

(2) $\Rightarrow$ (3) Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. By Corollary 2.2.40, we have $\mathfrak{nr}(\mathfrak{g}) \subseteq \mathfrak{nr}(\mathfrak{q}) \subseteq \mathfrak{nr}(\mathfrak{p}) \subseteq \mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{g}$. And, by Proposition 2.2.41, $\mathfrak{p}^{\perp} = \mathfrak{nr}(\mathfrak{p}) = [\mathfrak{nr}(\mathfrak{p}), \mathfrak{p}] \subseteq [\mathfrak{q}, \mathfrak{q}]$; whence $\mathfrak{rad}(\mathfrak{q}) \subseteq [\mathfrak{q}, \mathfrak{q}]^{\perp} \subseteq \mathfrak{p}$. Then, by Lemma 2.2.21, $\mathfrak{rad}(\mathfrak{q}_0) = \mathfrak{rad}(\mathfrak{q})/\mathfrak{nr}(\mathfrak{q}) \subseteq \mathfrak{p}/\mathfrak{nr}(\mathfrak{q})$. The induced invariant form on $\mathfrak{q}_0$ is admissible by Proposition 2.2.30. Moreover, $(\mathfrak{p}/\mathfrak{nr}(\mathfrak{q}))^{\perp} = \mathfrak{nr}(\mathfrak{p})/\mathfrak{nr}(\mathfrak{q})$ is a nilpotent subalgebra of $\mathfrak{p}/\mathfrak{nr}(\mathfrak{q})$.

(3) $\Rightarrow$ (1) Suppose that $\mathfrak{p}$ contains $\mathfrak{nr}(\mathfrak{q})$ and $\mathfrak{p}/\mathfrak{nr}(\mathfrak{q})$ is a parabolic subalgebra of $\mathfrak{q}_0$. Then, by Definition 2.2.35, $\mathfrak{rad}(\mathfrak{q}_0) \subseteq \mathfrak{p}/\mathfrak{nr}(\mathfrak{q})$ and there exists an admissible trace form B of $\mathfrak{q}_0$ such that $(\mathfrak{p}/\mathfrak{nr}(\mathfrak{q}))^{\perp}$ is the nilpotent radical of $\mathfrak{p}/\mathfrak{nr}(\mathfrak{q})$. By Lemma 2.2.21, $\mathfrak{rad}(\mathfrak{q})/\mathfrak{nr}(\mathfrak{q}) = \mathfrak{rad}(\mathfrak{q}_0) \subseteq \mathfrak{p}/\mathfrak{nr}(\mathfrak{q})$; whence $\mathfrak{rad}(\mathfrak{q}) \subseteq \mathfrak{p}$. Define an invariant bilinear form

$$\begin{aligned} B' : \mathfrak{q} \times \mathfrak{q} &\rightarrow \mathbb{F} \\ (x, y) &\mapsto B(x + \mathfrak{nr}(\mathfrak{q}), y + \mathfrak{nr}(\mathfrak{q})). \end{aligned}$$

Therefore $\mathfrak{nr}(\mathfrak{q}) \subseteq \mathfrak{p}^{\perp}$ and $\mathfrak{p}^{\perp} = [\mathfrak{p}^{\perp}, \mathfrak{p}] + \mathfrak{nr}(\mathfrak{q})$ is a solvable ideal of $\mathfrak{p}$. Hence $[\mathfrak{p}, \mathfrak{p}^{\perp}] \subseteq [\mathfrak{p}, \mathfrak{rad}(\mathfrak{p})] = \mathfrak{nr}(\mathfrak{p})$. It follows that $\mathfrak{p}^{\perp}$ is a sum of two nilpotent ideals; whence it is nilpotent. □

By means of Corollary 2.2.25 and Proposition 2.2.44, to study parabolic subalgebras of

Lie algebras, it suffices to focus on those of semisimple Lie algebras. If $\mathfrak{p}$ is a parabolic subalgebra of a Lie algebra $\mathfrak{g}$, then

$$\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{s}) \oplus \mathfrak{rad}(\mathfrak{g}),$$

where $\mathfrak{s}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{p} \cap \mathfrak{s}$ is a parabolic subalgebra of $\mathfrak{s}$. Recall that if a Lie algebra $\mathfrak{g}$ is semisimple, then any invariant bilinear form of $\mathfrak{g}$ is determined by the Killing forms on the simple components of $\mathfrak{g}$; actually it is the direct sum of scalar multiples of the Killing forms of the components of $\mathfrak{g}$. Thus B is non-degenerate which implies that $\mathfrak{g}^{\perp} = \{0\} = [\mathfrak{g}, \mathfrak{rad}(\mathfrak{g})]$.

Proposition 2.2.45. *Assume that $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{k}$ is a simple component of $\mathfrak{g}$. If $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{k}$, then $\mathfrak{p}/\mathfrak{k}$ is a parabolic subalgebra of $\mathfrak{g}/\mathfrak{k}$.*

Proof. Let $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{k}$. It suffices to show that $(\mathfrak{nr}(\mathfrak{p}) + \mathfrak{k})/\mathfrak{k}$ is a nilpotent subalgebra of $\mathfrak{p}/\mathfrak{k}$. Since $C_n((\mathfrak{nr}(\mathfrak{q}) + \mathfrak{k})/\mathfrak{k}) \subseteq (C_n(\mathfrak{nr}(\mathfrak{q})) + \mathfrak{k})/\mathfrak{k}$, for all $n \in \mathbb{N}$, and $\mathfrak{nr}(\mathfrak{q})$ is a nilpotent subalgebra of $\mathfrak{p}$, thus $(\mathfrak{nr}(\mathfrak{q}) + \mathfrak{k})/\mathfrak{k}$ is a nilpotent subalgebra of $\mathfrak{p}/\mathfrak{k}$. Therefore $\mathfrak{p}/\mathfrak{k}$ is a parabolic subalgebra of $\mathfrak{g}/\mathfrak{k}$. □

Proposition 2.2.46. *Suppose that $\mathfrak{p}$ and $\mathfrak{q}$ are parabolic subalgebras of a reductive Lie algebra $\mathfrak{g}$. Then $(\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{nr}(\mathfrak{q})$ is a parabolic subalgebra of $\mathfrak{q}$, and so of $\mathfrak{g}$.*

Proof. According to the Proposition 2.2.44, it suffices to show that $((\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{nr}(\mathfrak{q}))/\mathfrak{nr}(\mathfrak{q})$ is a parabolic subalgebra of $\mathfrak{q}_0$. Denote $\mathfrak{r} := (\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{nr}(\mathfrak{q})$. Since $\mathfrak{p} \cap \mathfrak{q}$ is a subalgebra and $\mathfrak{nr}(\mathfrak{q})$ is an ideal of $\mathfrak{q}$, $\mathfrak{r}$ is a subalgebra of $\mathfrak{g}$. As $\mathfrak{p}$ and $\mathfrak{q}$ are parabolic subalgebras of $\mathfrak{g}$, thus $\mathfrak{rad}(\mathfrak{g}) \subseteq \mathfrak{p} \cap \mathfrak{q} \subseteq \mathfrak{r}$. Given an admissible trace form on $\mathfrak{g}$ and the induced admissible trace form on $\mathfrak{q}_0$, we will show that $(\mathfrak{r}/\mathfrak{nr}(\mathfrak{q}))^{\perp}$ is a nilpotent Lie subalgebra of $\mathfrak{r}/\mathfrak{nr}(\mathfrak{q})$. Consider

$$\begin{aligned} \mathfrak{r}^{\perp} &= \left((\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{q}^{\perp}\right)^{\perp} \\ &= (\mathfrak{p} \cap \mathfrak{q})^{\perp} \cap \mathfrak{q} \\ &= \left(\mathfrak{p}^{\perp} + \mathfrak{q}^{\perp}\right) \cap \mathfrak{q} \\ &= \left(\mathfrak{p}^{\perp} \cap \mathfrak{q}\right) + \mathfrak{q}^{\perp} \subseteq \mathfrak{r} \end{aligned} \tag{2.2.3}$$

because $\mathfrak{q}^{\perp} \subseteq \mathfrak{q}$. Since $\mathfrak{nr}(\mathfrak{q}) = \mathfrak{q}^{\perp} \subseteq \mathfrak{r}^{\perp}$, hence $(\mathfrak{r}/\mathfrak{nr}(\mathfrak{q}))^{\perp} = \mathfrak{r}^{\perp}/\mathfrak{nr}(\mathfrak{q}) \subseteq \mathfrak{r}/\mathfrak{nr}(\mathfrak{q})$. Since

$\mathfrak{p}^{\perp}$ is nilpotent, $\mathfrak{r}^{\perp}/\mathfrak{nr}(\mathfrak{q})$ is nilpotent. $\square$

Proposition 2.2.47. *If $\mathfrak{g}$ is semisimple, then any two parabolic subalgebras $\mathfrak{p}$ and $\mathfrak{q}$ of $\mathfrak{g}$ admit algebraic Weyl structures $\widetilde{\xi}_{\mathfrak{p}}$ and $\widetilde{\xi}_{\mathfrak{q}}$ with $\left[\widetilde{\xi}_{\mathfrak{p}}, \widetilde{\xi}_{\mathfrak{q}}\right] = 0$, and so they contain a common Cartan subalgebra of $\mathfrak{g}$.*

Proof. Let $\mathfrak{r} = (\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{nil}(\mathfrak{q})$. Then, by Proposition 2.2.46, $\mathfrak{r}$ is a parabolic subalgebra of $\mathfrak{g}$ contained in $\mathfrak{q}$; whence $\mathfrak{r}/\mathfrak{nil}(\mathfrak{q})$ is a parabolic subalgebra of $\mathfrak{q}/\mathfrak{nil}(\mathfrak{q})$. As any Cartan subalgebra of $\mathfrak{r}/\mathfrak{nil}(\mathfrak{q})$ is a Cartan subalgebra of $\mathfrak{q}/\mathfrak{nil}(\mathfrak{q})$, it contains $\mathfrak{z}(\mathfrak{q}/\mathfrak{nil}(\mathfrak{q}))$. Hence there is an algebraic Weyl structure $\widetilde{\xi}$ of $\mathfrak{q}$ in $\mathfrak{p} \cap \mathfrak{q}$ uniquely modulo $\mathfrak{p} \cap \mathfrak{nil}(\mathfrak{q})$. Similarly, there is an algebraic Weyl structure $\widetilde{\xi}_{\mathfrak{p}}$ of $\mathfrak{p}$ in $\mathfrak{p} \cap \mathfrak{q}$. As $\left[\widetilde{\xi}, \widetilde{\xi}_{\mathfrak{p}}\right] \in \mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{nil}(\mathfrak{q})$ and $\mathrm{ad}(\xi_{\mathfrak{p}})$ is invertible on $\mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{nil}(\mathfrak{q})$, there exists $x \in \mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{nil}(\mathfrak{q})$ such that $\left[\widetilde{\xi}, x\right] = \left[\widetilde{\xi}, \widetilde{\xi}_{\mathfrak{p}}\right]$. Now let $\widetilde{\xi}_{\mathfrak{q}} := \widetilde{\xi} - x$. Then $\widetilde{\xi}_{\mathfrak{q}}$ is an algebraic Weyl structure of $\mathfrak{q}$ which commutes with $\widetilde{\xi}_{\mathfrak{p}}$.

The span of $\widetilde{\xi}_{\mathfrak{p}}$ and $\widetilde{\xi}_{\mathfrak{q}}$ consists only of semisimple elements in $\mathfrak{g}$; whence it lies in a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Since all elements of $\mathfrak{t}$ commutes with $\widetilde{\xi}_{\mathfrak{p}}$ and $\widetilde{\xi}_{\mathfrak{q}}$, so $\mathfrak{t} \subseteq \mathfrak{p} \cap \mathfrak{q}$. $\square$

Corollary 2.2.48. *If $\mathfrak{g}$ is semisimple and $\mathfrak{p}$ and $\mathfrak{q}$ are parabolic subalgebras of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$ if and only if there exists a parabolic subalgebra $\widehat{\mathfrak{p}}$ of $\mathfrak{g}$ complementary to $\mathfrak{p}$ and co-standard with $\mathfrak{q}$.*

Proof. Assume that $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$. Given an admissible trace form on $\mathfrak{g}$, then $\mathfrak{p}^{\perp} \cap \mathfrak{q}^{\perp} = (\mathfrak{p} + \mathfrak{q})^{\perp} = \{0\}$. By Proposition 2.2.47, choose an algebraic Weyl structure $\widetilde{\xi}$ of $\mathfrak{p}$ in $\mathfrak{p} \cap \mathfrak{q}$ and determine $\widehat{\mathfrak{p}}$ be the parabolic subalgebra of $\mathfrak{g}$ complementary to $\mathfrak{p}$ by using $\widetilde{\xi}$. Since $\mathfrak{p}^{\perp} \cap \mathfrak{q}^{\perp} = \{0\}$, thus $\mathfrak{q}^{\perp}$ has nonnegative eigenvalues for $\widetilde{\xi}$. Therefore $\mathfrak{q}^{\perp} \subseteq \widehat{\mathfrak{p}}$. This implies that $\widehat{\mathfrak{p}}$ and $\mathfrak{q}$ are co-standard.

Conversely, let $\widehat{\mathfrak{p}}$ be a parabolic subalgebra of $\mathfrak{g}$ complementary to $\mathfrak{p}$ and co-standard with $\mathfrak{q}$. Then $\mathfrak{q}^{\perp} \subseteq \widehat{\mathfrak{p}}$ and $\mathfrak{p}^{\perp} \cap \widehat{\mathfrak{p}} = \{0\}$. So $\mathfrak{p}^{\perp} \cap \mathfrak{q}^{\perp} = \{0\}$. This implies that

$$\mathfrak{g} = \{0\}^{\perp} = \left(\mathfrak{p}^{\perp} \cap \mathfrak{q}^{\perp}\right)^{\perp} = \mathfrak{p} + \mathfrak{q}.$$

$\square$

Remark 2.2.49. The parabolic subalgebras $\widehat{\mathfrak{p}}$ complementary to $\mathfrak{p}$ are parametrized by algebraic Weyl structures in $\mathfrak{p}$; in particular, $\widehat{\mathfrak{p}}$ contains a Cartan subalgebra $\mathfrak{t}$ if and only if the algebraic Weyl structure which parametrizes $\widehat{\mathfrak{p}}$ is contained in $\mathfrak{t}$.

By Corollary 2.2.48, we have that if $\mathfrak{g}$ is semisimple and $\mathfrak{p}$ is parabolic subalgebra of $\mathfrak{g}$, then a minimal parabolic subalgebra $\mathfrak{p}'$ is weakly opposite to $\mathfrak{p}$ if and only if there is a parabolic subalgebra containing $\mathfrak{p}'$ such that it is complementary to $\mathfrak{p}$.

2.2.4 Split semisimple Lie algebras

In this section, we will study some aspects of split semisimple Lie algebras over a field of characteristic zero. Assume, throughout this section, that $\mathbb{F}$ is a field of characteristic zero.

Definition 2.2.50. A Cartan subalgebra $\mathfrak{t}$ of a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ is said to be **splitting** if the eigenvalues of $\mathrm{ad}\,(x)$ are in $\mathbb{F}$ for all $x \in \mathfrak{t}$. A **split semisimple Lie algebra** $\mathfrak{g}$ over a field $\mathbb{F}$ is a semisimple Lie algebra containing a splitting Cartan subalgebra.

Remark 2.2.51. If $\mathbb{F}$ is algebraically closed, any finite-dimensional Lie algebra is split and any Cartan subalgebra is splitting.

Let $\mathfrak{g}$ be a finite-dimensional split semisimple Lie algebra over $\mathbb{F}$ and $\mathfrak{t}$ be a splitting Cartan subalgebra of $\mathfrak{g}$. As $\mathfrak{g}$ is semisimple, we automatically have $\{\mathrm{ad}\,(x)\,|x \in \mathfrak{t}\}$ is simultaneously diagonalizable. It gives a vector space decomposition, called the **root space decomposition**, of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in (\mathfrak{t})^*} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha := \{x \in \mathfrak{g}\,|[t,x] = \alpha\,(t)\,x \text{ for all } t \in \mathfrak{t}\}$. The Jacobi identity yields $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{t}$ for all $\alpha, \beta \in \mathfrak{t}^*$. Since the Killing form κ of $\mathfrak{g}$ is invariant, we have

$$0 = \kappa\left([\mathfrak{t}, \mathfrak{g}_\alpha], \mathfrak{g}_\beta\right) + \kappa\left(\mathfrak{g}_\alpha, [\mathfrak{t}, \mathfrak{g}_\beta]\right) = \kappa\left(\alpha \mathfrak{g}_\alpha, \mathfrak{g}_\beta\right) + \kappa\left(\mathfrak{g}_\alpha, \beta \mathfrak{g}_\beta\right) = (\alpha + \beta)\,\kappa\left(\mathfrak{g}_\alpha, \mathfrak{g}_\beta\right),$$

for any $\alpha, \beta \in \mathfrak{t}^\star$. Consequently, if $\alpha \neq -\beta$, then $\kappa\left(\mathfrak{g}_\alpha, \mathfrak{g}_\beta\right) = 0$. Each $\alpha \in (\mathfrak{t})^* \setminus \{0\}$, such that $\mathfrak{g}_\alpha \neq \{0\}$, is called a **root** of $\mathfrak{g}$. Denote $\mathcal{R}\,(\mathfrak{g}, \mathfrak{t})$ the set of all roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$. For convenient, write $\mathcal{R}$ for $\mathcal{R}\,(\mathfrak{g}, \mathfrak{t})$. Clearly $\mathcal{R}$ is finite. Then the root space composition of $\mathfrak{g}$ is actually in the form

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha.$$

For each $\alpha \in \mathcal{R}$, the space $\mathfrak{g}_\alpha$ is called a **root space** of $\mathfrak{g}$ and $\dim\left(\mathfrak{g}_\alpha\right) = 1$. Since the Killing form is non-degenerate, therefore, for any root α of $\mathfrak{g}$, we can associate the unique element t_α of $\mathfrak{t}$ such that

$$\alpha(t) = \kappa(t_\alpha, t),$$

for all $t \in \mathfrak{t}$.

For any Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ containing $\mathfrak{t}$, we have

$$\mathfrak{s} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}_\mathfrak{s}} \mathfrak{g}_\alpha,$$

where $\mathcal{R}_\mathfrak{s} := \{\alpha \in \mathcal{R} \,|\, \mathfrak{g}_\alpha \subseteq \mathfrak{s}\}$, because $\dim(\mathfrak{g}_\alpha) = 1$ for all $\alpha \in \mathcal{R}$; moreover, $\mathcal{R}_\mathfrak{s}$ is additively closed since $\mathfrak{s}$ is a Lie subalgebra. The additively closed property of subsets of $\mathcal{R}$ plays an important role in determining a Lie subalgebra of $\mathfrak{g}$ as we shall see in the following.

Proposition 2.2.52. *For any additively closed subset A of $\mathcal{R}$ and any subspace $\mathfrak{t}'$ of $\mathfrak{t}$ containing $\mathfrak{t}_{A\cap -A} = \sum_{\alpha \in A\cap -A} \mathfrak{t}_\alpha$, the subspace*

$$\mathfrak{s} := \mathfrak{t}' \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$$

is a Lie subalgebra of $\mathfrak{g}$. Furthermore, $\mathfrak{s}$ is semisimple if and only if $A = -A$ and $\mathfrak{t}' = \mathfrak{t}_{A\cap -A}$, and $\mathfrak{s}$ is solvable if and only if $A \cap (-A) = \emptyset$.

Proof. See [Bou05], Chapter VIII, §3.1, Proposition 1 and Proposition 2. □

Let $\mathfrak{t}^\star_\mathbb{Q}$ be the $\mathbb{Q}$-space spanned by the roots. Then $\mathfrak{t}^\star_\mathbb{Q} \subseteq \mathfrak{t}^\star$. It was shown (see [Jac79], Chapter IV, Section 2) that $\mathcal{R}$ is a root system in the vector space $\mathfrak{t}^\star_\mathbb{Q}$, called the **ROOT SYSTEM** in $\mathfrak{g}$ with respect to $\mathfrak{t}$. Let $\Delta \subseteq \mathcal{R}$ be a simple system and the set $\mathcal{R}^+$ be the positive root system of $\mathcal{R}$ corresponding to Δ as in Theorem 2.1.11. Define the set of **FUNDAMENTAL WEIGHTS** $\{\lambda_i \in \mathfrak{t}^\star | 1 \leq i \leq |\Delta|\}$ be such that

$$\lambda_i\left(t_{\alpha_j}\right) = \delta_{ij}, \qquad \alpha_j \in \Delta. \tag{2.2.4}$$

The Weyl group $W(\mathcal{R})$ of $\mathcal{R}$ is then the subgroup of $\operatorname{Aut}\left(\mathfrak{t}^\star_\mathbb{Q}\right)$ generated by $\{s_\alpha \,|\alpha \in \Delta\}$, where s_α is the reflection of α in $\mathfrak{t}^\star$. As already discussed in the end of Section 2.1.2, $W(\mathcal{R})$ is a Coxeter group with a simple system $s : \Delta \to W(\mathcal{R})$. The nodes of the Coxeter diagram of $W(\mathcal{R})$ correspond to elements of Δ. The **DYNKIN DIAGRAM** $\mathscr{D}_\mathfrak{g}$ of $\mathfrak{g}$ is constructed from the Coxeter diagram of $W(\mathcal{R})$ by replacing all the edges labelled by 3, 4, and 6 with single, double, and triple edges, respectively, and decorating with an arrow on double and triple edges pointing toward the shorter roots. In particular, the Dynkin diagram of a split simple Lie algebra over arbitrary field of characteristic zero is of type A_n where $n \geq 1$, B_n where $n \geq 2$, C_n where $n \geq 3$, D_n where $n \geq 4$, G_2, F_4, E_6, E_7, or E_8 (see [Jac79], Chapter IV, Section 6).

Remark 2.2.53. Suppose that $\mathfrak{g}$ is reductive. Then, by Corollary 2.2.27, $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{s}$, where $\mathfrak{s}$ is a semisimple Lie algebra. We may allow using the Dynkin diagram $\mathscr{D}_{\mathfrak{s}}$ to represent $\mathfrak{g}$, i.e., $\mathscr{D}_{\mathfrak{g}} = \mathscr{D}_{\mathfrak{s}}$, but the reader need to keep in mind that $\mathfrak{g}$ is also equipped with $\mathfrak{z}(\mathfrak{g})$.

Since $\mathcal{R}^+$ is a maximal additively closed subset of $\mathcal{R}$ such that $\mathcal{R}^+ \cap -\mathcal{R}^+ = \emptyset$, then, by Proposition 2.2.52, any subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ of the form

$$\mathfrak{b} := \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}^+} \mathfrak{g}_\alpha,$$

is a Borel subalgebra of $\mathfrak{g}$. Conversely, a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{t}$ determines a maximal additively closed subset A of $\mathcal{R}$ such that $A \cap -A = \emptyset$. Then A is a positive root system. Theorem 2.1.11 implies that there is a unique simple system contained in A. Therefore the Borel subalgebras $\mathfrak{b}$ containing $\mathfrak{t}$ correspond bijectively to the simple systems $\Delta := \Delta_{(\mathfrak{t},\mathfrak{b})}$ of $\mathcal{R}$. Note that, with respect to the Killing form of $\mathfrak{g}$, we have $\mathfrak{b}^\perp = \bigoplus_{\alpha \in \mathcal{R}^+} \mathfrak{g}_\alpha = \mathfrak{nil}(\mathfrak{b}) \subseteq \mathfrak{b}$. Therefore every Borel subalgebra of $\mathfrak{g}$ is a parabolic subalgebra.

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Then

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}_\mathfrak{p}} \mathfrak{g}_\alpha,$$

and, with respect to the Killing from of $\mathfrak{g}$,

$$\mathfrak{p}^\perp = \bigoplus_{\alpha \in \mathcal{R}_\mathfrak{p} \setminus (-\mathcal{R}_\mathfrak{p})} \mathfrak{g}_\alpha$$

is a nilpotent ideal of $\mathfrak{p}$. Since $\mathcal{R}_\mathfrak{p} \cap (-\mathcal{R}_\mathfrak{p})$ is additively closed, so by Proposition 2.2.52,

$$\widetilde{\mathfrak{p}}_0 := \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}_\mathfrak{p} \cap (-\mathcal{R}_\mathfrak{p})} \mathfrak{g}_\alpha$$

is a Lie subalgebra of $\mathfrak{g}$. Notice that $\mathfrak{p} = \widetilde{\mathfrak{p}}_0 \oplus \mathfrak{p}^\perp$. By Corollary 2.2.40, $\widetilde{\mathfrak{p}}_0 \cong \mathfrak{p} / \mathfrak{p}^\perp$ is reductive. We call $\widetilde{\mathfrak{p}}_0$ the **Levi subalgebra** of $\mathfrak{p}$ with respect to $\mathfrak{t}$.

Proposition 2.2.54. *If $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ containing a splitting Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, then $\mathfrak{p}^{\perp} = \mathfrak{nil}(\mathfrak{p})$.*

Proof. Let $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ containing a splitting Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Since $\mathfrak{p}^{\perp}$ is a nilpotent ideal of $\mathfrak{p}$, so $\mathfrak{p}^{\perp} \subseteq \mathfrak{nil}(\mathfrak{p})$. Notice that $\mathfrak{t} \subseteq \mathfrak{p} \setminus \mathfrak{nil}(\mathfrak{p})$. Thus it is sufficient to show that $\mathfrak{g}_\alpha \subsetneq \mathfrak{nil}(\mathfrak{p})$ for all $\alpha \in \mathcal{R}_{\mathfrak{p}} \setminus (-\mathcal{R}_{\mathfrak{p}})$. Let $\alpha \in \mathcal{R}$. Then $\mathfrak{g}_\alpha + \mathfrak{p}^{\perp}$ is a subalgebra of $\mathfrak{p}$ but not an ideal of $\mathfrak{p}$ because if $x \in \mathfrak{g}_{-\alpha}$, then

$$\left[x, \mathfrak{g}_\alpha + \mathfrak{p}^{\perp}\right] = [x, \mathfrak{g}_\alpha] + \left[x, \mathfrak{p}^{\perp}\right] \subseteq \mathfrak{t}_\alpha + \mathfrak{p}^{\perp},$$

where $\mathfrak{t}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{t}$. Therefore $\mathfrak{p}^{\perp} = \mathfrak{nil}(\mathfrak{p})$. □

The following Lemma gives us an idea to characterize parabolic subalgebras containing $\mathfrak{t}$ by using a certain subset of $\mathcal{R}$.

Lemma 2.2.55. *Let $A \subseteq \mathcal{R}$. There exists a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{t}$ with $\mathcal{R}_{\mathfrak{p}} = \mathcal{R} \setminus A$ if and only if $A \cap (-A) = \emptyset$ and $\mathcal{R} \setminus A$ is additively closed.*

Proof. Assume that $A \cap (-A) = \emptyset$ and $\mathcal{R} \setminus A$ is additively closed. Then A is additively closed; for $\alpha, \beta \in A$ such that $\alpha + \beta \in \mathcal{R}$, suppose that $\alpha + \beta \notin A$, then $\alpha = (\alpha + \beta) + (-\beta) \in \mathcal{R} \setminus A$ which is a contradiction because $-\beta \in -A \subseteq \mathcal{R} \setminus A$. Let $\mathfrak{p} := \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R} \setminus A} \mathfrak{g}_\alpha$. Then, by Proposition 2.2.52, $\mathfrak{p}$ is a Lie subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{p}^{\perp} \subseteq \bigoplus_{\alpha \in -A} \mathfrak{g}_\alpha$. Since $A \cap (-A) = \emptyset$, $\mathfrak{p}^{\perp}$ is a nilpotent subalgebra of $\mathfrak{p}$.

Conversely, let $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. Denote $A := \mathcal{R} \setminus \mathcal{R}_{\mathfrak{p}}$. Then $\mathcal{R} \setminus A = \mathcal{R}_{\mathfrak{p}}$ is additively closed. Suppose, for a contradiction, that $A \cap (-A) \neq \emptyset$. Assume that $\alpha \in A \cap (-A)$. Then $\mathfrak{g}_{\pm\alpha} \nsubseteq \mathfrak{p}$ but $\mathfrak{g}_{\pm\alpha} \subseteq \mathfrak{p}^{\perp}$. This implies that $\mathfrak{p}^{\perp} \nsubseteq \mathfrak{p}$, a contradiction. Therefore $A \cap (-A) = \emptyset$. □

Remark 2.2.56. Let $\mathfrak{p}$ be a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{t}$. Then Lemma 2.2.55 tells us the subset $A := \mathcal{R} \setminus \mathcal{R}_{\mathfrak{p}}$ have the property that $A \cap -A = \emptyset$. Since $\mathcal{R} \setminus A$ is additively closed, so is $\mathcal{R} \setminus (-A)$. Hence, by the proof of Lemma 2.2.55, we see that

$$\widehat{\mathfrak{p}} := \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R} \setminus (-A)} \mathfrak{g}_\alpha = \mathfrak{t} \oplus \bigoplus_{\alpha \in -\mathcal{R}_{\mathfrak{p}}} \mathfrak{g}_\alpha$$

is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Moreover, $\widetilde{\mathfrak{p}}_0 = \mathfrak{p} \cap \widehat{\mathfrak{p}} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}_\mathfrak{p} \cap (-\mathcal{R}_\mathfrak{p})} \mathfrak{g}_\alpha$ is a Levi subalgebra. Therefore $\mathfrak{p}$ and $\widehat{\mathfrak{p}}$ are complementary.

Corollary 2.2.57. *Any parabolic subalgebra* $\mathfrak{p}$ *of* $\mathfrak{g}$ *containing* $\mathfrak{t}$ *contains a Borel subalgebra containing* $\mathfrak{t}$.

Proof. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. By Lemma 2.2.55, there exists a subset $A \in \mathcal{R}$ such that $\mathcal{R} \setminus A = \mathcal{R}_\mathfrak{p}$ where $A \cap (-A) = \emptyset$ and $\mathcal{R} \setminus A$ is additively closed. For any $\alpha \in \mathcal{R} \setminus (A \cup (-A))$, one can construct an additively closed subset B of $\mathcal{R}$ containing both $-A$ and α such that $B \cap -B = \emptyset$. Let A' be the maximal additively closed subset of $\mathcal{R}$ containing $-A$ such that $A' \cap -A' = \emptyset$ and $A' \cup -A' = \mathcal{R}$, i.e., A' is a positive root system of $\mathcal{R}$. Then $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in A'} \mathfrak{g}_\alpha$ is a Borel subalgebra contained in $\mathfrak{p}$ because $A' \subseteq \mathcal{R} \setminus A = \mathcal{R}_\mathfrak{p}$. □

Remark 2.2.58. Given a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{t}$. Therefore Lemma 2.2.55 tells us that the parabolic subalgebras $\mathfrak{p}$ containing $\mathfrak{b}$, called **STANDARD PARABOLIC SUBALGEBRAS**, correspond bijectively to the subsets $\Delta_{(\mathfrak{t},\mathfrak{b})} \cap (\mathcal{R} \setminus (-\mathcal{R}_\mathfrak{p}))$ of the simple system $\Delta_{(\mathfrak{t},\mathfrak{b})}$. Thus the parabolic subalgebras $\mathfrak{p}$ containing $\mathfrak{b}$ correspond bijectively to the **DECORATED DYNKIN DIAGRAMS** $\mathscr{D}_\mathfrak{p}$; the decoration of the Dynkin diagram is obtained by crossing the vertices corresponding to the simple roots in $\Delta_{(\mathfrak{t},\mathfrak{b})} \cap (\mathcal{R} \setminus (-\mathcal{R}_\mathfrak{p}))$. Note that if $\mathfrak{g}$ is reductive, we still be able to use the same terminology for the decorated diagram representing its parabolic subalgebras because parabolic subalgebras of $\mathfrak{g}$ contain $\mathfrak{z}(\mathfrak{g})$.

According to Remark 2.2.53, we also represent the Levi subalgebra $\widetilde{\mathfrak{p}}_0$ (and the Levi factor $\mathfrak{p}_0$) of a standard parabolic subalgebra $\mathfrak{p}$ by the Dynkin diagram of its semisimple part which is a sub-diagram of $\mathscr{D}_\mathfrak{g}$ obtained from removing all the crossed nodes in $\mathscr{D}_\mathfrak{p}$ and edges adjacent to it. The number of crossed nodes in $\mathscr{D}_\mathfrak{p}$ is equal to the dimension of the center of $\widetilde{\mathfrak{p}}_0$.

Lemma 2.2.59. *Let* $\mathfrak{p}$ *and* $\mathfrak{q}$ *be parabolic subalgebras of* $\mathfrak{g}$ *containing a splitting Cartan subalgebra* $\mathfrak{t}$. *Then* $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$ *if and only if* $\mathfrak{p}$ *contains a Borel subalgebra of* $\mathfrak{g}$ *complementary to one contained in* $\mathfrak{q}$.

Proof. Suppose that $\mathfrak{p}$ contains a Borel subalgebra $\mathfrak{b}$ complementary to a Borel subalgebra $\widehat{\mathfrak{b}}$ contained in $\mathfrak{q}$. Then $\mathfrak{g} = \mathfrak{b} + \widehat{\mathfrak{b}} \subseteq \mathfrak{p} + \mathfrak{q}$.

Conversely, assume that $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$. Let $\mathcal{R}$ be the root system in $\mathfrak{g}$ with respect to $\mathfrak{t}$. By Corollary 2.2.48, $\widehat{\mathfrak{p}} = \mathfrak{t} \oplus \bigoplus_{\alpha \in -\mathcal{R}_\mathfrak{p}} \mathfrak{g}_\alpha$ is a parabolic subalgebra of $\mathfrak{g}$ complementary to $\mathfrak{p}$ but co-standard with $\mathfrak{q}$. Then $\widehat{\mathfrak{p}} \cap \mathfrak{q}$ is a parabolic subalgebra, and so by Corollary 2.2.57, there is a positive root system $\mathcal{R}^+$ of $\mathcal{R}$ such that $\mathcal{R}^+ \subseteq \mathcal{R}_{\widehat{\mathfrak{p}}} \cap \mathcal{R}_{\mathfrak{q}} = (-\mathcal{R}_{\mathfrak{p}}) \cap \mathcal{R}_{\mathfrak{q}}$. Hence $-\mathcal{R}^+ \subseteq \mathcal{R}_{\mathfrak{p}}$. Therefore $\widehat{\mathfrak{b}} = \mathfrak{t} \oplus \bigoplus_{\alpha \in -\mathcal{R}^+} \mathfrak{g}_\alpha$ is a Borel subalgebra contained in $\mathfrak{p}$ and complementary to the Borel subalgebra $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}^+} \mathfrak{g}_\alpha \subseteq \mathfrak{q}$. □

Remark 2.2.60. Assume that $\mathfrak{p}$ and $\mathfrak{q}$ are parabolic subalgebras of $\mathfrak{g}$ containing the splitting Cartan subalgebra $\mathfrak{t}$. Denote $\mathfrak{r} := (\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{q}^\perp$. Then we have seen that the nilradical of $\mathfrak{r}$ is $\mathfrak{r}^\perp = \left(\mathfrak{p}^\perp \cap \mathfrak{q}\right) + \mathfrak{q}^\perp$ (see Equation (2.2.3)). We now compute $\widetilde{\mathfrak{r}}_0$. Since $\mathfrak{t} \subseteq \mathfrak{p} \cap \mathfrak{q}$, hence $\mathfrak{t} \subseteq \mathfrak{r}$. Let $\mathcal{R}$ be the root system in $\mathfrak{g}$ with respect to $\mathfrak{t}$. Consider the root space decomposition of $\mathfrak{r}$:

$$\mathfrak{r} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \mathcal{R}_{\mathfrak{p}} \cap \mathcal{R}_{\mathfrak{q}}} \mathfrak{g}_\alpha + \bigoplus_{\alpha \in \mathcal{R}_{\mathfrak{q}^\perp}} \mathfrak{g}_\alpha \right).$$

So

$$\begin{aligned} \mathcal{R} &= (\mathcal{R}_{\mathfrak{p}} \cap \mathcal{R}_{\mathfrak{q}}) \cup \mathcal{R}_{\mathfrak{q}^\perp} \\ &= \left(\mathcal{R}_{\mathfrak{p}} \cap \mathcal{R}_{\widetilde{\mathfrak{q}}_0}\right) \sqcup \mathcal{R}_{\mathfrak{q}^\perp} \\ &= \left(\mathcal{R}_{\widetilde{\mathfrak{p}}_0} \cap \mathcal{R}_{\widetilde{\mathfrak{q}}_0}\right) \sqcup \left(\mathcal{R}_{\mathfrak{p}^\perp} \cap \mathcal{R}_{\widetilde{\mathfrak{q}}_0}\right) \sqcup \mathcal{R}_{\mathfrak{q}^\perp}. \end{aligned}$$

Since $\left(\mathcal{R}_{\mathfrak{p}^\perp} \cap \mathcal{R}_{\widetilde{\mathfrak{q}}_0}\right) \sqcup \mathcal{R}_{\mathfrak{q}^\perp} = \left(\mathcal{R}_{\mathfrak{p}^\perp} \cap \mathcal{R}_{\mathfrak{q}}\right) \cup \mathcal{R}_{\mathfrak{q}^\perp}$, we have $\mathcal{R}_{\mathfrak{r}} = \left(\mathcal{R}_{\widetilde{\mathfrak{p}}_0} \cap \mathcal{R}_{\widetilde{\mathfrak{q}}_0}\right) \sqcup \left(\mathcal{R}_{\mathfrak{p}^\perp} \cap \mathcal{R}_{\mathfrak{q}}\right) \cup \mathcal{R}_{\mathfrak{q}^\perp}$. Consider

$$\begin{aligned} \left(\mathcal{R}_{\mathfrak{p}^\perp} \cap \mathcal{R}_{\mathfrak{q}}\right) \cap - \left(\mathcal{R}_{\mathfrak{p}^\perp} \cap \mathcal{R}_{\mathfrak{q}}\right) &\subseteq \mathcal{R}_{\mathfrak{p}^\perp} \cap -\mathcal{R}_{\mathfrak{p}^\perp} = \emptyset, \\ \left(\mathcal{R}_{\mathfrak{p}^\perp} \cap \mathcal{R}_{\mathfrak{q}}\right) \cap -\mathcal{R}_{\mathfrak{q}^\perp} &= \left(\mathcal{R}_{\mathfrak{p}^\perp} \cap \mathcal{R}_{\mathfrak{q}}\right) \cap \left(-\mathcal{R}_{\mathfrak{q}} \setminus \mathcal{R}_{\mathfrak{q}}\right) = \emptyset, \\ \mathcal{R}_{\mathfrak{q}^\perp} \cap -\mathcal{R}_{\mathfrak{q}^\perp} &= \emptyset. \end{aligned}$$

Hence if $x \in \left(\mathcal{R}_{\mathfrak{p}^\perp} \cap \mathcal{R}_{\mathfrak{q}}\right) \cup \mathcal{R}_{\mathfrak{q}^\perp}$, then $-x \notin \left(\mathcal{R}_{\mathfrak{p}^\perp} \cap \mathcal{R}_{\mathfrak{q}}\right) \cup \mathcal{R}_{\mathfrak{q}^\perp}$. This implies that

$$\mathcal{R}_{\widetilde{\mathfrak{r}}_0} = \mathcal{R}_{\widetilde{\mathfrak{p}}_0} \cap \mathcal{R}_{\widetilde{\mathfrak{q}}_0} \text{ and } \mathcal{R}_{\mathfrak{r}^\perp} = \left(\mathcal{R}_{\mathfrak{p}^\perp} \cap \mathcal{R}_{\mathfrak{q}}\right) \cup \mathcal{R}_{\mathfrak{q}^\perp}.$$

Therefore $\widetilde{\mathfrak{r}}_0 = \widetilde{\mathfrak{p}}_0 \cap \widetilde{\mathfrak{q}}_0$.

Example 2.2.61. Let V be an n-dimensional vector space and $\mathfrak{g} := \mathfrak{pgl}_n(V)$ and . Let $\mathfrak{p}$ and $\mathfrak{q}$ be maximal parabolic subalgebras of $\mathfrak{g}$ such that $\mathfrak{p} \neq \mathfrak{q}$ and $\mathfrak{r} := (\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{q}^\perp$.

According to Proposition 2.2.47, we can choose $\mathfrak{t} \subseteq \mathfrak{p} \cap \mathfrak{q}$ a Cartan subalgebra.

In the case that $\mathfrak{p}$ and $\mathfrak{q}$ are co-standard, we choose a suitable basis $\{e_i \,|1 \leq i \leq n\}$ of V so that $\mathfrak{p}$ and $\mathfrak{q}$ are standard parabolic subalgebra. Then there are two sequences of nested subspaces

$$\{0\} \subset U \subset V \text{ and } \{0\} \subset W \subset V$$

where $U := \langle e_1, \ldots, e_s \rangle$ and $W := \langle e_1, \ldots, e_t \rangle$, such that their stabilizers in $\mathfrak{g}$ are $\mathfrak{p}$ and $\mathfrak{q}$, respectively. $\mathfrak{p}$ (resp. $\mathfrak{q}$) consists of upper-triangular block diagonal matrices of the form

$$\begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix} \left(\text{resp. } \begin{pmatrix} B_1 & * \\ 0 & B_2 \end{pmatrix} \right)$$

where A_1 and A_2 (resp. B_1 and B_2) are respectively $s \times s$ and $(n-s) \times (n-s)$ (resp. $t \times t$ and $(n-t) \times (n-t)$) blocks. Thus the Lie subalgebra $\mathfrak{r}$ consisting of upper-triangular block diagonal matrices of the form

$$\begin{pmatrix} M_1 & * & * \\ 0 & M_2 & * \\ 0 & 0 & M_3 \end{pmatrix} \text{ if } s < t$$

where M_1, M_2, and M_3 are respectively $s \times s$, $(t-s) \times (t-s)$, and $(n-t) \times (n-t)$ blocks if $s < t$, or respectively $t \times t$, $(s-t) \times (s-t)$, and $(n-s) \times (n-s)$ blocks if $s > t$, is a standard parabolic subalgebra. Moreover, we see that $\mathfrak{r}$ is the stabilizer in $\mathfrak{g}$ of the sequence of nested subspaces

$$\{0\} \subset U \subset W \subset V \text{ if } s < t \text{ or } \{0\} \subset W \subset U \subset V \text{ if } s > t.$$

In the case that $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$, we choose a suitable basis $\{e_i \,|1 \leq i \leq n\}$ of V so that $\mathfrak{q}$ are standard parabolic subalgebra and, by Lemma 2.2.59, $\mathfrak{p}$ contain the complementary standard Borel subalgebra. Then there are two sequences of nested subspaces

$$\{0\} \subset U \subset V \text{ and } \{0\} \subset W \subset V$$

where $U := \langle e_{n-s+1}, \ldots, e_n \rangle$ and $W := \langle e_1, \ldots, e_t \rangle$, such that their stabilizers in $\mathfrak{g}$ are $\mathfrak{p}$ and $\mathfrak{q}$, respectively. $\mathfrak{p}$ (resp. $\mathfrak{q}$) consists of lower-triangular (resp. upper-triangular) block diagonal matrices of the form

$$\begin{pmatrix} A_1 & 0 \\ * & A_2 \end{pmatrix} \left(\text{resp.} \begin{pmatrix} B_1 & * \\ 0 & B_2 \end{pmatrix}\right)$$

where A_1 and A_2 (resp. B_1 and B_2) are respectively $(n-s)\times(n-s)$ and $s\times s$ (resp. $t\times t$ and $(n-t)\times(n-t)$) blocks. Thus the Lie subalgebra $\mathfrak{r}$ consists of block matrices of the form

$$\begin{pmatrix} M_1 & * & * \\ 0 & M_2 & 0 \\ 0 & * & M_3 \end{pmatrix} \text{ if } U\cap W = \{0\}$$

where M_1, M_2, and M_3 are respectively $t\times t$, $(n-s-t)\times(n-s-t)$, and $s\times s$ blocks, or

$$\begin{pmatrix} M_1 & 0 & * \\ * & M_2 & * \\ 0 & 0 & M_3 \end{pmatrix} \text{ if } U + W = V$$

where M_1, M_2, and M_3 are respectively $(n-s)\times(n-s)$, $(s+t-n)\times(s+t-n)$, and $(n-t)\times(n-t)$ blocks. Moreover, we see that $\mathfrak{r}$ is the stabilizer in $\mathfrak{g}$ of the sequence of nested subspaces

$$\{0\}\subset W \subset U+W \subset V \quad \text{if } U\cap W=\{0\},$$

or

$$\{0\}\subset U\cap W \subset W \subset V \quad \text{if } U+W=V.$$

Remark 2.2.62. From Example 2.2.61, $\mathfrak{r}$ is in general the stabilizer in $\mathfrak{g}$ of the sequence of nested subspaces

$$\{0\}\subseteq U\cap W\subseteq W\subseteq U+W\subseteq V.$$

2.3 Algebraic groups

This section, we introduce algebraic groups from the functorial viewpoint review the basic results (without proof), and discuss a method for passing from algebraic groups to Lie algebras. We also discuss about algebraic subgroups of an algebraic group and their corresponding Lie algebras that are needed later in this thesis. For more details about algebraic groups, we refer the reader to [Ayo], [Mil12a], and [Mil11]. Here we define algebraic groups over an arbitrary field $\mathbb{F}$, otherwise stated, and state some basic properties of them.

2.3.1 Basic definitions and properties

Definition 2.3.1. An ALGEBRA over a commutative ring R or an R-ALGEBRA is an R-module A together with a map, called the multiplication map, $m : A \times A \to A$ such that (A, m) is a commutative ring and m is R-bilinear, i.e., for all $a, b \in A$ and $r \in R$

$$r \cdot (ab) = (r \cdot a)\, b = a\, (r \cdot b)\,,$$

where $\cdot$ denotes the R-action on A.

We say that an R-algebra A is FINITELY GENERATED if it is isomorphic to the quotient of a polynomial algebra $R\,[X_1, X_2, \ldots, X_n]$. If A is a finitely generated R-algebra, we denote

$$\operatorname{spm}(A) := \left\{ A' \subsetneq A \,\middle|\, A' \text{ is a maximal ideal of } A \right\},$$

endowed with the topology for which the closed sets are those of the form

$$\left\{ A' \in \operatorname{spm}(A) \,\middle|\, A' \supseteq A'' \right\},$$

for any ideal A'' in A.

Definition 2.3.2. An ALGEBRAIC GROUP over a field $\mathbb{F}$ is a functor $G : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$ from $\mathbb{F}$-algebras to groups such that its composition with the forgetful functor $F : \mathrm{Grp} \to \mathrm{Set}$ is representable, i.e., there exists a natural isomorphism such that

$$F \circ G \cong \operatorname{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(A, -)\,,$$

for some finitely generated $\mathbb{F}$-algebra A. Any such A is called the COORDINATE RING of G and we will denote it by $\mathcal{O}(G)$.

An ALGEBRAIC SUBGROUP (resp. NORMAL ALGEBRAIC SUBGROUP) H of an algebraic group G is a subfunctor of G such that $H(A)$ is a subgroup (resp. normal subgroup) of $G(A)$, for all $A \in \mathrm{Alg}_{\mathbb{F}}$, and $\mathcal{O}(H)$ is a quotient of $\mathcal{O}(G)$.

Remark 2.3.3. [Mil12a], Chapter IX, Theorem 2.1, Theorem 4.4, and Theorem 5.1 show that the standard isomorphisms in group theory hold for algebraic groups.

Example 2.3.4. (Examples of algebraic groups)

1. Let $\mathbb{G}_a$ be the functor from $\mathrm{Alg}_{\mathbb{F}}$ to Grp given by $A \mapsto (A, +)$. Then

$$F \circ \mathbb{G}_a (A) \cong \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}} (\mathbb{F}[x], A).$$

Then $\mathbb{G}_a$ is an algebraic group, called the **additive group**.

2. Let $\mathbb{G}_m$ be the functor from $\mathrm{Alg}_{\mathbb{F}}$ to Grp given by $A \mapsto (A^{\times}, m)$, where $A^{\times}$ is the group of elements with a multiplicative inverse in A. Then

$$F \circ \mathbb{G}_m (A) \cong \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}} \left(\frac{\mathbb{F}[x, y]}{(xy - 1)}, A \right).$$

Then $\mathbb{G}_m$ is an algebraic group, called the **multiplicative group**.

3. Let GL_n be the functor from $\mathrm{Alg}_{\mathbb{F}}$ to Grp sending an $\mathbb{F}$-algebra A to the set of all invertible $n \times n$ matrices with entries in A. Then

$$F \circ \mathrm{GL}_n (A) \cong \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}} \left(\frac{\mathbb{F}[x_{11}, x_{22}, \ldots, x_{nn}, y]}{(\det (x_{ij}) y - 1)}, A \right),$$

where $\det (x_{ij}) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \cdot x_{1\sigma(1)} \cdots x_{n\sigma(n)}$. Therefore GL_n is an algebraic group, called the **general linear group**.

Let G be an $\mathbb{F}$-algebraic group. For any $g \in G(\mathbb{F})$,

$$g : \mathcal{O}(G) \to \mathbb{F};$$

we let $\mathcal{O}(G)_{\ker(g)}$ be the ring of fractions obtained from $\mathcal{O}(G)$ by inverting the elements of the set $\{f \in \mathcal{O}(G) \,|\, f(g) \neq 0\}$.

Definition 2.3.5. An algebraic group G is said to be **finite** if $\mathcal{O}(G)$ is a finite $\mathbb{F}$-algebra, i.e., finitely generated as a vector space. An algebraic group G is said to be **smooth** if $\mathrm{spm}(\mathcal{O}(G))$ is smooth, i.e., $\mathbb{F}_{\mathrm{al}} \otimes_{\mathbb{F}} \mathcal{O}(G)$ is regular (see [Mil12a], Chapter VI, 7.3), where $\mathbb{F}_{\mathrm{al}}$ is an algebraic closure of $\mathbb{F}$. An algebraic group G is said to be **connected** if $\mathrm{spm}(\mathcal{O}(G))$ is connected (as a topological space). The **identity component** of G is denoted by G°.

Theorem 2.3.6. *Every algebraic group over a field of characteristic zero is smooth.*

Proof. [Mil12a], Chapter VI, Theorem 9.3. □

Let G be an algebraic group over $\mathbb{F}$. Let R be an $\mathbb{F}$-algebra. An R-algebra A can be regard as an $\mathbb{F}$-algebra, thus

$$G_R : \mathrm{Alg}_R \to \mathrm{Grp}; A \mapsto G(A)$$

is a functor. If G is an algebraic group, then so is G_R with the coordinate ring $\mathcal{O}(G_R) = R \otimes_{\mathbb{F}} \mathcal{O}(G)$ because, for any $A \in \mathrm{Alg}_R$,

$$\mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathcal{O}(G_R), A) \cong \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(R \otimes_{\mathbb{F}} \mathcal{O}(G), A).$$

The algebraic group G_R is an **EXTENSION BY SCALARS** of G.

Definition 2.3.7. Let G and G' be algebraic groups. a map $f : G \to G'$ is an **ALGEBRAIC GROUP HOMOMORPHISM** if it is a natural transformation of functors and $f(A) : G(A) \to G'(A)$ is a group homomorphism for all $A \in \mathrm{Alg}_{\mathbb{F}}$.

Proposition 2.3.8. *For any algebraic group homomorphism $f : G \to G'$, there is an algebraic group N of G such that*

$$N : Alg_{\mathbb{F}} \to Grp; A \mapsto Ker\big(f(A) : G(A) \to G'(A)\big)$$

and its coordinate ring is $\mathcal{O}(G)/I_{G'}\mathcal{O}(G)$, where $I_{G'}$ is the kernel of the identity element $\epsilon : \mathcal{O}(G') \to \mathbb{F}$ of $G'(\mathbb{F})$, and $I_{G'}\mathcal{O}(G)$ is the ideal generated by its image in $\mathcal{O}(G)$.

Proof. See [Mil12a], Chapter VII, Proposition 4.1. □

Remark 2.3.9. The algebraic group N is called the **KERNEL** of the homomorphism f.

Definition 2.3.10. An algebraic group homomorphism $f : G \to G'$ is said to be **SURJECTIVE** if for every $\mathbb{F}$-algebra A' and $g' \in G'(A')$, there exists a faithfully flat A'-algebra A, i.e., taking the tensor over A' with A through a sequence gives an exact sequence if and only if the original sequence is exact, and $g \in G(A)$ mapping to the image of g' in $G'(A)$.

$$\begin{array}{ccc} G(A) & \xrightarrow{f(A)} & G'(A) \\ \uparrow & & \uparrow \\ G(A') & \xrightarrow[f(A')]{} & G'(A') \end{array} \qquad \begin{array}{c} \exists g \longmapsto \star \\ \uparrow \\ g'. \end{array}$$

A surjective algebraic group homomorphism $f : G \to G'$ with the kernel N is called the QUOTIENT of G by N, and G' is denoted by G/N.

Proposition 2.3.11. *If $f : G \to G'$ and $f' : G \to G''$ are quotient maps with the same kernel. Then there is a unique algebraic group isomorphism $u : G' \to G''$ such that $u \circ f = f'$.*

Proof. See [Mil12a], Chapter VII, Corollary 7.9. □

Proposition 2.3.11 implies that the quotient is uniquely determined up to a unique algebraic group isomorphism.

Proposition 2.3.12. *Quotients of smooth algebraic groups over a field are smooth algebraic groups.*

Proof. [Mil12a], Chapter VII, Proposition 10.1 and Chapter VIII, Proposition 8.6. □

Theorem 2.3.13. *For any normal algebraic subgroup N of an algebraic group G, there exists a quotient map with the kernel N.*

Proof. See [Mil12a], Chapter VIII, Theorem 19.4. □

Definition 2.3.14. A LINEAR REPRESENTATION of G on an $\mathbb{F}$-vector space V is an algebraic group homomorphism $r : G \to \mathrm{GL}_V$, where

$$\mathrm{GL}_V : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}; A \mapsto \mathrm{GL}(A \otimes_{\mathbb{F}} V).$$

Let $r : G \to \mathrm{GL}_V$ be a representation of G, and let W be a vector subspace of V. The functor

$$\begin{array}{rcl} Stab_G(W) : \mathrm{Alg}_{\mathbb{F}} & \to & \mathrm{Grp} \\ A & \mapsto & \{g \in G(A) \,|\, g \cdot (A \otimes_{\mathbb{F}} V) = A \otimes_{\mathbb{F}} V\} \end{array}$$

is an algebraic subgroup of G (see [Mil12a], Chapter VIII, Proposition 12.1)

For a subgroup H of an $\mathbb{F}$-algebraic group G and $g \in G(A)$, let ${}^gH : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$ be a functor defined by

$$ {}^gH(A) := g \cdot H(A) \cdot g^{-1}, $$

for all $A \in \mathrm{Alg}_{\mathbb{F}}$. Define

$$ \begin{aligned} N_G(H) : \mathrm{Alg}_{\mathbb{F}} &\to \mathrm{Grp} \\ A &\mapsto \left\{ g \in G(A) \,|\, {}^gH(A) = H(A) \right\}. \end{aligned} $$

Proposition 7.39, Chapter I in [Mil11] shows that $N_G(H)$ is an algebraic subgroup of G. For each $n \in N_G(H)$, we have a natural transformation

$$ i_n : H(A) \to H(A) : h \mapsto nhn^{-1}, $$

of H. Define

$$ \begin{aligned} Z_G(H) : \mathrm{Alg}_{\mathbb{F}} &\to \mathrm{Grp} \\ A &\mapsto \left\{ n \in N_G(H)(A) \,|\, i_n = \mathrm{id}_H \right\}. \end{aligned} $$

Proposition 7.44, Chapter I in [Mil11] shows that $Z_G(H)$ is an algebraic subgroup of G if H is locally free.

Definition 2.3.15. For any locally free subgroup H of an $\mathbb{F}$-algebraic group G, the $\mathbb{F}$-algebraic group $N_G(H)$ is called the **normalizer** of H in G and the $\mathbb{F}$-algebraic group $Z_G(H)$ is called the **centralizer** of H in G.

2.3.2 Lie algebras of algebraic groups

Let G be an algebraic group over a field $\mathbb{F}$, and let $\mathbb{F}[\varepsilon] := \mathbb{F}[x] / (x^2)$ be the ring of dual numbers. Then $\mathbb{F}[\varepsilon] = \mathbb{F} \oplus \mathbb{F}\varepsilon$ as a vector space. We have a short exact sequence

$$ 0 \longrightarrow \mathbb{F} \xrightarrow{i} \mathbb{F}[\varepsilon] \xrightarrow{\pi} \mathbb{F} \longrightarrow 0, $$

where $i(a) = a + 0\varepsilon$, and $\pi(a + b\varepsilon) = a$, and so

$$0 \longrightarrow G(\mathbb{F}) \xrightarrow{G(i)} G(\mathbb{F}[\varepsilon]) \xrightarrow{G(\pi)} G(\mathbb{F}) \longrightarrow 0\,.$$

Denote

$$\mathrm{Lie}(G) := \ker\left(G(\pi) : G(\mathbb{F}[\varepsilon]) \to G(\mathbb{F})\right).$$

Remark 2.3.16. $\mathrm{Lie}(G)$ is the tangent space of $\mathcal{O}(G)$ at 1_G.

Proposition 2.3.17. *Let $\mathbb{F}'$ be a field containing $\mathbb{F}$. Then $Lie(G_{\mathbb{F}'}) \cong \mathbb{F}' \otimes_{\mathbb{F}} Lie(G)$.*

Proof. [Mil12a], Chapter XI, Proposition 6.1. □

Proposition 1.11 in [Mil11] shows that $\mathrm{Lie}(G)$ has the structure of $\mathbb{F}$-vector space. Moreover, Lie is a functor from the category of algebraic groups to the category of $\mathbb{F}$-vector spaces.

For any $\mathbb{F}$-algebra A, we have an exact sequence

$$0 \longrightarrow A \xrightarrow{i} A[\varepsilon] \xrightarrow{\pi} A \longrightarrow 0\,,$$

where $i(a) = a + 0\varepsilon$, and $\pi(a + b\varepsilon) = a$, and so

$$0 \longrightarrow G(A) \xrightarrow{G(i)} G(A[\varepsilon]) \xrightarrow{G(\pi)} G(A) \longrightarrow 0\,.$$

Let $\mathfrak{g}(A) := \ker(G(\pi) : G(A[\varepsilon]) \to G(A))$, where $A[\varepsilon] := \mathbb{F}[\varepsilon] \otimes_{\mathbb{F}} A \cong A[X] / (X^2)$. Then $\mathfrak{g}(A) \cong A \otimes_{\mathbb{F}} \mathfrak{g}(\mathbb{F})$ (See [Mil11], Chapter II, Remark 1.29). Define

$$\mathrm{Ad} : G(A) \to \mathrm{Aut}(\mathfrak{g}(A))\,,$$

where $\mathrm{Ad}(g)(x) = (G(i)(g)) \cdot x \cdot (G(i)(g))^{-1}$, for all $g \in G(A)$ and $x \in \mathfrak{g}(A)$. Then $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g}(A))$. This gives a natural transformation, called the **ADJOINT MAP**,

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}) = \mathrm{GL}_{\mathfrak{g}},$$

i.e., it is a homomorphism of algebraic groups. By applying the functor Lie to the algebraic group homomorphism, we have a homomorphism of vector spaces

$$\mathrm{ad} := \mathrm{Lie}(\mathrm{Ad}) : \mathrm{Lie}(G) \to \mathrm{Lie}(\mathrm{GL}_{\mathfrak{g}})$$

In [Mil12a], Chapter XI, Section 8, it is shown that ad defines a Lie bracket on $\mathrm{Lie}(G)$, and so $\mathrm{Lie}(G)$ is a Lie algebra.

Definition 2.3.18. The **Lie algebra** of an algebraic group G is the vector space

$$\mathrm{Lie}(G) := \ker(G(\pi) : G(\mathbb{F}[\varepsilon]) \to G(\mathbb{F})),$$

together with the Lie bracket $[\cdot,\cdot] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathbb{F}; (x,y) \mapsto [x,y] := \mathrm{ad}(x)(y)$.

For a standard convention, we will write $\mathfrak{g}$ for $\mathrm{Lie}(G)$, $\mathfrak{h}$ for $\mathrm{Lie}(H)$, and so on.

Example 2.3.19. Let $G = \mathrm{GL}_n$ and I_n be the identity $n \times n$ matrix. For any $n \times n$ matrix A,

$$I_n + \varepsilon A \in \mathrm{GL}_n(\mathbb{F}(\varepsilon)),$$

and

$$(I_n + \varepsilon A)(I_n - \varepsilon A) = I_n.$$

Thus $I_n + \varepsilon A \in \mathrm{Lie}(\mathrm{GL}_n)$. Moreover,

$$\mathfrak{gl}_n := \mathrm{Lie}(\mathrm{GL}_n) = \{I_n + \varepsilon A \,|A \in M_n\} \cong M_n,$$

where $\mathrm{M}_n : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}; A \mapsto \mathrm{M}_n(A)$ is a functor sending $\mathbb{F}$-algebra A to the set of all invertible $n \times n$ matrices with entries in A. Then $\mathfrak{gl}_n$ is a Lie algebra over $\mathbb{F}$, with the Lie bracket $[A,B] = AB - BA$.

Definition 2.3.20. Let G be a connected algebraic group over a field of characteristic zero. Then G is said to be **unipotent**, **solvable**, **reductive**, **semisimple**, or **simple** if its Lie algebra $\mathfrak{g}$ is nilpotent, solvable, reductive, semisimple, or simple, respectively.

Example 2.3.21. The algebraic group GL_n is reductive. Its quotient by its center,

$$\mathrm{PGL}_n := \mathrm{GL}_n / \mathbb{G}_m$$

is semisimple.

Definition 2.3.22. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. The **algebraic adjoint group** of the Lie algebra $\mathfrak{g}$ is the smallest algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$ where its Lie algebra contains $\mathrm{ad}(\mathfrak{g})$.

Theorem 2.3.23. *Assume that* $\mathbb{F}$ *is an algebraically closed field of characteristic zero. For every finite-dimensional Lie algebra* $\mathfrak{g}$ *over* $\mathbb{F}$*, there exists a connected algebraic group* G *with the unipotent centre such that* $Lie\,(G) = \mathfrak{g}$.

Proof. See [Hoc71]. □

Proposition 2.3.24. *For any* $\mathbb{F}$*-algebraic group* G*,*

(1) $dim\,(Lie\,(G)) \geq dim\,(G)$*, with the equality if and only if* G *is smooth.*

(2) If V *is a representation of* G *and* $W \subseteq V$*, then*

$$Lie\,(Stab_G\,(W)) = Stab_{Lie(G)}\,(W)\,.$$

In particular, if W *is stable under* G*, then it is stable under* $Lie\,(G)$.

Proof. (1)[Mil12a], Chapter XI, Proposition 16.2. (2) [Mil12a], Chapter XI, Proposition 16.15 and Corollary 16.16. □

Proposition 2.3.25. *Let* G*,* K*, and* Q *be algebraic groups over a field* $\mathbb{F}$ *of characteristic zero. If*

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is exact, then

$$0 \longrightarrow Lie\,(K) \longrightarrow Lie\,(G) \longrightarrow Lie\,(Q) \longrightarrow 0$$

is exact.

Proof. [Mil12a], Chapter XI, Proposition 16.7. □

Theorem 2.3.26. *Let* G *be a connected algebraic group over a field* $\mathbb{F}$ *of characteristic zero. Then we have that*:

(1) The correspondence $H \mapsto \mathfrak{h} := Lie\,(H)$ *is injective and inclusion preserving between the collection of closed connected subgroups* H *of* G *and the collection of their Lie algebras, regarded as subalgebras of* $\mathfrak{g} := Lie\,(G)$.

(2) Let f *and* f' *are algebraic group homomorphism from* G *to an* $\mathbb{F}$*-algebraic group* H*. If* $\mathrm{Lie}\,(f) = \mathrm{Lie}\,(f')$, then $f = f'$.

Proof. [Mil12a], Chapter XI, Theorem16.11 and Proposition 16.13. □

Definition 2.3.27. A **SPLITTING TORUS** is an algebraic group isomorphic to a finite product of copies $\mathbb{G}_m$. A **TORUS** is an algebraic group T such that $T_{\mathbb{F}_s}$ is a split torus. A **MAXIMAL TORUS** of an algebraic group is a subgroup which is a torus and is maximal among all tori contained in G.

Theorem 2.3.28. *Let G be a split reductive algebraic group, i.e., a reductive algebraic group containing a splitting maximal torus . All split maximal tori in G are conjugate by an element of $G(\mathbb{F})$.*

Proof. [Mil11], Chapter V, Theorem 2.19. □

Definition 2.3.29. Let G be a reductive algebraic group over a field $\mathbb{F}$ and let T be a maximal torus of G. The quotient

$$W(G,T) := N_G(T)(\mathbb{F})/T(\mathbb{F})$$

is called a **WEYL GROUP** of G.

Definition 2.3.30. A **BOREL SUBGROUP** of an $\mathbb{F}$-algebraic group G is a smooth subgroup B such that $B_{\mathbb{F}_{al}}$ is a maximal smooth connected solvable subgroup $G_{\mathbb{F}_{\mathrm{al}}}$, where $\mathbb{F}_{\mathrm{al}}$ is an algebraic closure of $\mathbb{F}$.

Theorem 2.3.31. *Let G be a reductive group over a field $\mathbb{F}$.*

(1) If B is a Borel subgroup of G, then G/B is a projective variety.

(2) Any two Borel subgroups of G are conjugate by an element of $G(\mathbb{F}_{al})$.

Proof. [Mil11], Chapter V, Theorem 3.21. □

Corollary 2.3.32. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero and G be a connected algebraic group over $\mathbb{F}$ with the Lie algebra $\mathfrak{g}$. Any two Borel subalgebras of $\mathfrak{g}$ are conjugate by an element $Ad(g)$ for some $g \in G$.*

Proof. This is a consequence of Theorem 2.3.31, (2). □

Theorem 2.3.33. (BRUHAT DECOMPOSITION) *Let G be a split connected reductive algebraic group over a field $\mathbb{F}$. Let B be a Borel subgroup of G and T be a maximal splitting torus of G contained in B. Then*

$$G = \bigsqcup_{\omega \in W(G.T)} B\overline{\omega}B$$

where $\overline{\omega} \in N_G(T)$ is a representative of ω. In particular, each $B\overline{\omega}B$, where $\omega \in W(G.T)$, is called **Bruhat cells**.

Proof. [Spr98], Theorem 8.3.8. □

Remark 2.3.34. Since $B = UT$, where U is the unipotent radical of B, if ω_0 is the longest element in $W(G,T)$, then

$$B\overline{\omega}_0 B = UT\overline{\omega}_0 B = U\overline{\omega}_0 TB = U\overline{\omega}_0 B = \overline{\omega}_0 U^- B$$

where U^- is the unipotent radical of the Borel subgroup B^- opposite to B. [Spr98], Corollary 8.3.11 shows that $B\overline{\omega}_0 B$ is an open subset of G; it is called the **big cell**.

2.3.3 Parabolic subgroups of algebraic groups

In this section, we assume that $\mathbb{F}$ is algebraically closed and has characteristic zero.

Definition 2.3.35. Let G be a connected algebraic group over $\mathbb{F}$. An algebraic subgroup P of G is **parabolic** if it contains a Borel subgroup of G.

Theorem 2.3.36. *Let G be a connected algebraic group over $\mathbb{F}$. An algebraic subgroup P of G is parabolic if and only if G/P is a projective variety,called a* **flag variety**.

Proof. [Mil11], Chapter V, Theorem 3.27. □

Proposition 2.3.37. *Let G be a connected algebraic group over $\mathbb{F}$. Two parabolic subgroups containing the same Borel subgroup and conjugate under $G(\mathbb{F})$ are equal.*

Proof. Suppose that P is a parabolic subgroup of G such that

$$B \subseteq P \cap gPg^{-1},$$

where B is a Borel subgroup of G and $g \in G(\mathbb{F})$. Then $B \cup gBg^{-1} \subseteq P$. Since B is connected, B and gBg^{-1} are Borel subgroups of P°. By Theorem 2.3.31, there exists $p \in P^\circ(\mathbb{F})$ such that $B = pgBg^{-1}p^{-1}$. Since $N_G(B) = B$ (see [TY05], Theorem 28.4.2), $pg \in B(\mathbb{F}) \subseteq P(\mathbb{F})$. Therefore $g \in P(\mathbb{F})$, and so $P = gPg^{-1}$. □

Suppose that G be a connected semisimple algebraic group. We choose B be a Borel subgroup of G and T be a maximal torus of G contained in B. For any dominant weight

$\lambda \in \mathfrak{t}^{\star}$, i.e. a non-negative linear combination of fundamental weights defined in Equation (2.2.4), let V_{λ} be the irreducible G-representation with the highest weight λ and $1 \otimes_{\mathbb{F}} v_{\lambda}$ a highest weight vector. Then, for any $A \in \mathrm{Alg}_{\mathbb{F}}$, the stabilizer of $A \otimes_{\mathbb{F}} v_{\lambda}$ in $G(A)$ contains $B(A)$; thus it is a parabolic subgroup. Therefore we can obtained the flag varieties in terms of the fundamental weights.

Chapter 3

Incidence geometries and buildings

In this chapter, we will introduce the notion of a parabolic configuration which is a morphism between two certain incidence geometries. We will also show how to construct parabolic configurations from a given algebraic group.

3.1 Incidence systems and geometries

We recall some terminology on incidence systems and geometries. More detailed overviews of incidence systems and geometries can be found in the books [BC13], [Bue95], and [Pas94]

Definition 3.1.1. An INCIDENCE SYSTEM over a set N (of types) is a set A equipped with a reflexive symmetric relation $\mathsf{I} \subseteq \mathsf{A} \times \mathsf{A}$, called the INCIDENCE RELATION, and a surjective map $t : \mathsf{A} \to N$, called the TYPE FUNCTION, such that for each $a, b \in \mathsf{A}$,

$$\text{if } (a, b) \in \mathsf{I} \text{ and } t(a) = t(b), \text{ then } a = b. \tag{3.1.1}$$

In particular, an incidence system A is said to be FINITE if $|\mathsf{A}| < \infty$. The cardinality of N is called the RANK of A.

Remark 3.1.2. By the property of an incidence system A over N, we have A is a disjoint union of fibres

$$\mathsf{A}_i := t^{-1}(i),$$

and I is a disjoint union of incidence relations between elements of types

$$\mathsf{I}_{ij} := \mathsf{I} \cap (\mathsf{A}_i \times \mathsf{A}_j) \subseteq \mathsf{A}_i \times \mathsf{A}_j,$$

for $i, j \in N$.

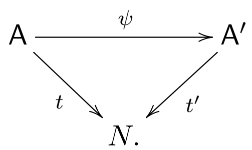
One may think of an incidence system A over N as an N-partite graph on A defined by the incidence relation. An incidence system A is said to be connected if its incidence graph is connected.

Example 3.1.3. Let V be a vector space of dimension $n \geq 2$ over a field $\mathbb{F}$ and $\mathsf{Proj}(V)$ be the set of nonzero, proper subspaces of V. Define

$$\mathsf{I} = \{(a, b) \in \mathsf{Proj}(V) \times \mathsf{Proj}(V) \,|a \subseteq b \text{ or } b \subseteq a\}$$

and $t : \mathsf{Proj}(V) \to [n] := \{1, 2, \ldots, n-1\}$, $a \mapsto \dim(a)$. Then $\mathsf{Proj}(V)$ is an incidence system over N called the **PROJECTIVE INCIDENCE SYSTEM** of V.

Definition 3.1.4. Let A and A' be incidence systems over N. A **STRICT INCIDENCE SYSTEM MORPHISM** ψ from A to A' is a morphism from A to A' preserving the incidence relation and types, i.e., for all $a, b \in \mathsf{A}$, if $(a, b) \in \mathsf{I}$ then $(\psi(a), \psi(b)) \in \mathsf{I}'$, and the following diagram commutes:



Remark 3.1.5. If $\psi : \mathsf{A} \longrightarrow \mathsf{A}'$ is a strict incidence system morphism between incidence systems over N, then by the property of ψ, we have

$$\psi_i := \psi\,|_{\mathsf{A}_i} : \mathsf{A}_i \to \mathsf{A}'_i,$$

and

$$\psi_{ij} := (\psi_i, \psi_j)\,|_{\mathsf{I}} : \mathsf{I}_{ij} \to \mathsf{I}'_{ij},$$

for $i, j \in N$.

In particular, the identity morphism $\mathrm{id} : \mathsf{A} \to \mathsf{A}$ is a strict incidence system morphism and the composition of strict incidence system morphisms $\psi : \mathsf{A} \to \mathsf{A}'$ and $\psi' : \mathsf{A}' \to \mathsf{A}''$, defined by

$$\begin{aligned} \psi' \circ \psi : \mathsf{A} &\rightarrow \mathsf{A}'' \\ a &\mapsto \psi'(\psi(a)), \end{aligned}$$

is again a strict morphism. Thus the incidence systems over N together with their strict morphisms form a category called the **CATEGORY OF INCIDENCE SYSTEMS OVER** N, denoted by $\mathbf{ISys}_N$. Note that a strict isomorphism is not just a bijective morphism. Indeed, we have the following.

Lemma 3.1.6. *Let* A *and* A' *be incidence systems over* N*. A map* $\psi : \mathsf{A} \longrightarrow \mathsf{A}'$ *is a strict incidence system isomorphism if and only if* ψ_i *and* ψ_{ij} *are bijective for all* $i, j \in N$*.*

Proof. Assume that ψ is a strict incidence system isomorphism. By remark 3.1.5, for each $i, j \in N$, the subscripts i and ij may be respectively considered as functors $\mathsf{A} \rightarrow \mathsf{A}_i$ and $\mathsf{A} \rightarrow \mathsf{I}_{ij}$ from the category of incidence systems A over N to the categories of sets. Therefore if $\psi : \mathsf{A} \longrightarrow \mathsf{A}'$ is an incidence system isomorphism, then both ψ_i and ψ_{ij} are bijective for all $i, j \in N$ because A_i and I_{ij} are functorial.

Conversely, assume that ψ_i and ψ_{ij} are bijective for all $i, j \in N$. We will show that $\psi : \mathsf{A} \longrightarrow \mathsf{A}'$ is a strict incidence system isomorphism. Define $\psi' : \mathsf{A}' \longrightarrow \mathsf{A}$ be a map such that ψ'_i is the inverse of ψ_i for all $i \in N$. Let $i, j \in N$. If $(a', b') \in \mathsf{I}'_{ij}$, then there exists $(a, b) \in \mathsf{I}_{ij}$ such that $\psi_{ij}(a, b) = (a', b')$ because ψ_{ij} is surjective; whence

$$\psi'_{ij}\left(a', b'\right) = \psi'_{ij} \circ \psi_{ij}(a, b) = \left(\psi'_i \circ \psi_i(a), \psi'_j \circ \psi_j(b)\right) = (a, b) \in \mathsf{I}_{ij}.$$

Hence $\psi' : \mathsf{A}' \longrightarrow \mathsf{A}$ is a strict incidence system morphism. One can easily check that

$$\begin{aligned} \psi' \circ \psi &= \mathrm{id} : \mathsf{A} \rightarrow \mathsf{A} \\ \psi \circ \psi' &= \mathrm{id} : \mathsf{A}' \rightarrow \mathsf{A}'. \end{aligned}$$

Therefore $\psi : \mathsf{A} \longrightarrow \mathsf{A}'$ is a strict incidence system isomorphism. □

Therefore $\mathrm{Aut}(\mathsf{A}) \subseteq \prod_{i \in N} \mathrm{Sym}(\mathsf{A}_i)$ where $\mathrm{Aut}(\mathsf{A})$ is the set of all incidence system automorphisms of an incidence system A over N. Given an incidence system A over N and a map $\nu : N' \to N$, one can construct an incidence system over N' as follows: set

$$\nu^\star \mathsf{A}_i := \mathsf{A}_{\nu(i)},$$

and

$$\nu^\star \mathsf{I}_{ij} := \mathsf{I}_{\nu(i)\nu(j)}.$$

Then the set $\nu^\star \mathsf{A} := \bigsqcup_{i \in N'} \nu^\star \mathsf{A}_i$ is an incidence system over N' with the incidence relation $\nu^\star \mathsf{I} := \bigsqcup_{i,j \in N'} \nu^\star \mathsf{I}_{ij}$ and the type function $\nu^\star t : \nu^\star \mathsf{A} \to N'$ given by $\nu^\star t\,(a) = i$ for all $a \in \nu^\star \mathsf{A}_i$ and $i \in N'$.

Definition 3.1.7. Let A be an incidence system over N and $\nu : N' \to N$ be a map. Then the incidence system $\nu^\star \mathsf{A}$ is called the **PULL BACK INCIDENCE SYSTEM** of A over N' induced by ν. In particular, if ν is the inclusion map of a subset $N' \subseteq N$, then $\nu^\star \mathsf{A} \subseteq \mathsf{A}$; whence we call $\nu^\star \mathsf{A}$ an **INCIDENCE SUB-SYSTEM** of A over N', denoted precisely by $(\mathsf{A})_{N'}$.

If $\psi : \mathsf{A} \longrightarrow \mathsf{A}'$ is a strict incidence system morphism between two incidence systems over N and $\nu : N' \to N$ is a map, then the map

$$\nu^\star \psi : \nu^\star \mathsf{A} \longrightarrow \nu^\star \mathsf{A}',$$

defined by

$$\nu^\star \psi_i := \nu^\star \psi\,|_{\nu^\star \mathsf{A}_i} = \psi_{\nu(i)} : \mathsf{A}_{\nu(i)} \to \mathsf{A}'_{\nu(i)},$$

for all $i \in N'$, is well-defined and preserves the incidence relation because ψ does; whence $\nu^\star \psi$ is an incidence system morphism. Therefore $\nu^\star$ is a functor from $\mathbf{ISys}_N$ to $\mathbf{ISys}_{N'}$. We shall now use such functors to define an incidence system morphism between any two incidence systems over different set of types.

Definition 3.1.8. Let A and A' be incidence systems over N and N', respectively. An **INCIDENCE SYSTEM MORPHISM** $\Psi : \mathsf{A} \to \mathsf{A}'$ over a map $\nu : N' \to N$ is a strict incidence system morphism $\psi : \nu^\star \mathsf{A} \to \mathsf{A}'$. In particular, we say that the morphism Ψ is **INJECTIVE** (resp. **SURJECTIVE**) if $\psi : \nu^\star \mathsf{A} \to \mathsf{A}'$ is injective (resp. surjective).

Remark 3.1.9. A strict incidence system morphism between two incidence systems over N is indeed an incidence system morphism over the identity map $\mathrm{id} : N \to N$.

Notation 3.1.10. Denote $\mathrm{Mor}_\nu(\mathsf{A}, \mathsf{A}')$ the set of all incidence system morphisms from A to A' over the map $\nu : N' \to N$. In particular, if $\nu = \mathrm{id}$, then we will write $\mathrm{Mor}(\mathsf{A}, \mathsf{A}')$ instead of $\mathrm{Mor}_{\mathrm{id}}(\mathsf{A}, \mathsf{A}')$.

If $\Psi : \mathsf{A} \to \mathsf{A}'$ and $\Psi' : \mathsf{A}' \to \mathsf{A}''$ are incidence system morphisms over maps $\nu : N' \to N$ and $\nu' : N'' \to N'$ respectively, then we define that the composite morphism $\Psi' \circ \Psi : \mathsf{A} \to \mathsf{A}''$ over the map $\nu \circ \nu' : N'' \to N$ is given by the strict incidence morphism

$$\psi' \circ (\nu')^\star \psi : (\nu')^\star (\nu^\star (\mathsf{A})) \to (\nu')^\star (\mathsf{A}') \to \mathsf{A}''.$$

One can check that the composition of incidence system morphisms is associative, and so incidence systems over arbitrary sets together with their incidence system morphisms over arbitrary maps form a category called the **category of incidence systems**, denoted by **ISys**.

Definition 3.1.11. Let A be an incidence system over N. A **flag** f of A is a set of mutually incident elements of A. If f is a flag of A, then we say that f is of **type** $t(f) := \{t(x) \,|x \in f\}$ and of **rank** $|t(f)|$. A **full flag** of A is a flag of type N. The **residue** of a flag f of A, denoted by $\mathrm{Res}(f)$ is a subset of A consisting of all $x \in \mathsf{A} \backslash f$ such that $(x, y) \in \mathsf{I}$ for all $y \in f$. The **type** of $\mathrm{Res}(f)$ is $N \backslash t(f)$. We will denote the set of all flags of A by $\mathcal{F}(\mathsf{A})$.

Remark 3.1.12. Any flag f of A may be thought of as an injective map

$$f : t(f) \longrightarrow \mathsf{A},$$

such that $t \circ f = \mathrm{id} : t(f) \to t(f)$ and $(f(i), f(j)) \in \mathsf{I}$ for $i, j \in t(f)$.

One can simply check that if f is a flag of an incidence system A over N, then $\mathrm{Res}(f)$ is also an incidence system, called a **residual incidence system**, over $N \backslash t(f)$.

Example 3.1.13. Let V be a vector space of dimension $n \geq 2$ over a field $\mathbb{F}$. A flag in $\mathsf{Proj}(V)$ (in Example 3.1.3) is a chain of subspaces $W_1 \subseteq W_2 \subseteq \ldots \subseteq W_k$.

Definition 3.1.14. Let A be an incidence system over N. We call A **flag regular** if every maximal flag of A is full and **homogeneous** if $\mathrm{Aut}(\mathsf{A})$ acts transitively on the flags of all types, i.e., if f and f' are flags of A with $t(f) = t(f')$, then there exists $\psi \in \mathrm{Aut}(\mathsf{A})$ such that $f' = \psi(f) := \{\psi(x) \,|x \in f\}$. In particular, if there exists a subgroup G of $\mathrm{Aut}(\mathsf{A})$

which acts transitively on the flags of all types, then we say A is G-**HOMOGENEOUS**; this clearly implies A is homogeneous. If A is flag regular and homogeneous, then we say that A is an **INCIDENCE GEOMETRY** over N.

Remark 3.1.15. The definition of "incidence geometry" used here is a little stronger than what may be found in [BC13] because we impose that its automorphism group must acts transitively on the flags of all types. However, it is still a geometry in the Buekenhout's sense.

Let A be a G-homogeneous incidence system over N. By using the orbit-stabilizer theorem, we have that, for any $i, j \in N$, if $x \in \mathsf{A}_i$ and $y \in \mathsf{A}_j$ such that $(x, y) \in \mathsf{I}_{ij}$, then

$$\left|(\mathsf{Res}\left(\{x\}\right))_j\right| = \frac{|Stab_G\left(x\right)|}{|Stab_G\left(x\right) \cap Stab_G\left(y\right)|},$$

where $Stab_G\left(a\right) := \{g \in G \,|\, g \cdot a = a\}$ for all $a \in \mathsf{A}$. Moreover, for any $z \in \mathsf{A}_i$, there is $g \in G$ such that $z = g \cdot x$, and so

$$\begin{aligned}
(\mathsf{Res}\left(\{z\}\right))_j &= \{a \in \mathsf{A}_j \,|(z, a) \in \mathsf{I}_{ij}\} \\
&= \{g \cdot a \in \mathsf{A}_j \,|(x, a) \in \mathsf{I}_{ij}\} \\
&= g \cdot (\mathsf{Res}\left(\{x\}\right))_j
\end{aligned}$$

because $G \subseteq \mathrm{Aut}\left(\mathsf{A}\right)$; therefore $\left|(\mathsf{Res}\left(\{x\}\right))_j\right| = \left|(\mathsf{Res}\left(\{z\}\right))_j\right|$. Hence each element in A_i is incident with a certain number of elements in A_j. This tells us that the incidence structure of the incidence sub-system $\mathsf{A}_{\{i,j\}}$ is symmetric and we can explicitly write the incidence structure as follows.

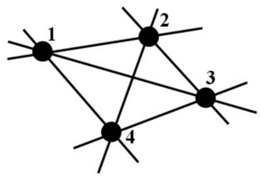
Definition 3.1.16. Let A be a incidence system over N and G be a subgroup of $\mathsf{Aut}\left(\mathsf{A}\right)$ such that A_i is G-homogeneous for all $i \in N$. For any $i, j \in N$, we will denote the incidence sub-system $\left(\mathsf{A}\right)_{\{i,j\}}$ of A by

$$a \overset{c \qquad\qquad d}{\overline{\qquad\qquad\quad}} b\,,$$

where $|\mathsf{A}_i| = a$, $|\mathsf{A}_j| = b$, $\left|(\mathrm{Res}\left(\{x\}\right))_j\right| = c$, $\left|(\mathrm{Res}\left(\{y\}\right))_i\right| = d$ for any $x \in \mathsf{A}_i$ and $y \in \mathsf{A}_j$. We will call this the **SUMMARY** of $\mathsf{A}_{\{i,j\}}$.

Remark 3.1.17. Let A be a incidence system over N and G be a subgroup of $\mathsf{Aut}\left(\mathsf{A}\right)$ such that A_i is G-homogeneous for all $i \in N$. For any $i, j \in N$, if $x \in \mathsf{A}_i$ and $H := Stab_G(x) \subseteq G$, then the action H on A_i (resp. A_j) decomposes A_i (resp. A_j) into disjoint orbits. Moreover each orbit in A_i/H has certain incidence relation with each orbit in A_j/H. We are thus able to construct an incidence graph, called the **BRANCHED SUMMARY** for $(\mathsf{A})_{\{i,j\}}$, by writing the elements in A_i/H on the right and the elements in A_j/H on the left and drew the line joining between any orbits if they are incident.

Example 3.1.18. Consider a complete quadrangle whose vertices are labelled by 1, 2, 3, and 4. We will assume that the line joining between the vertices labelled by i and j is labelled by (i, j) for all $i, j \in \{1, 2, 3, 4\}$.



The complete quadrangle is an incidence system consisting of a collection of vertices and a collection of lines. The symmetric group S_4 is the automorphism group of this incidence system. Moreover the set of all vertices (resp. lines) are S_4-homogeneous. Since

$$Stab_{S_4}(1) = \{e, (23), (24), (34), (234), (243)\},$$

we will see that this group acts on the set of all vertices (resp. lines) and decomposes it in to disjoint orbits. Thus we have the branched summary and the summary for this incidence system as in Figure 3.1.1.

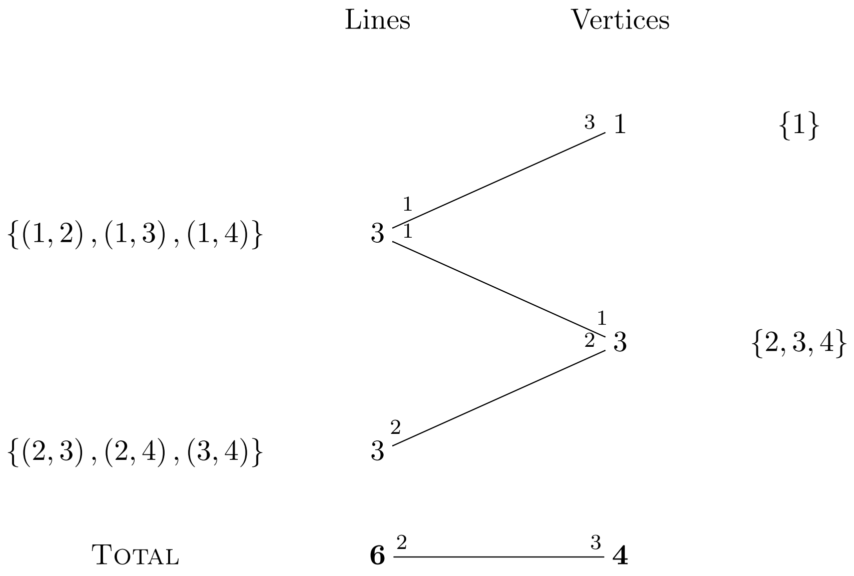


Figure 3.1.1: The branched summary for a complete quadrangle.

Remark 3.1.19. Each line in the branched summary in the Figure 3.1.1 shows the incidence relation between the orbit on the left and the orbit on the right. The left (resp. right) number appearing on each line tells us the number of elements in the orbit on the right (resp. left) of the line incident with each element in the orbit on the left (resp. right) of the line.

Lemma 3.1.20. *Let f be a flag of an incidence geometry A over N. Then the residual incidence system $Res(f)$ is an incidence geometry, called a* **RESIDUAL INCIDENCE GEOMETRY**.

Proof. Since A is flag regular, $\mathrm{Res}(f)$ is automatically flag regular. Let

$$G := \left\{ \psi \left|_{\mathrm{Res}(f)} \right| \psi \in Stab_{\mathrm{Aut}(\mathsf{A})}(f) \right\} \subseteq \mathrm{Aut}(\mathrm{Res}(f)).$$

Let f_1 and f_2 be flag of $\mathrm{Res}(f)$ of the same type. By the definition of $\mathrm{Res}(f)$, we have $f_1 \cup f$ and $f_2 \cup f$ are full flag of A of the same type. As A is homogeneous, there exists $\psi \in \mathsf{Aut}(\mathsf{A})$ such that $f_2 \cup f = \psi(f_1 \cup f)$. Hence, since ψ preserves types, $\psi \in Stab_{\mathrm{Aut}(\mathsf{A})}(f)$ and $f_2 = \psi(f_1)$; thus $\mathrm{Res}(f)$ is G-homogeneous. Therefore $\mathrm{Res}(f)$ is an incidence geometry. $\square$

The incidence geometries over arbitrary sets together with incidence system morphisms then form a full subcategory of **ISys**, called the **CATEGORY OF INCIDENCE GEOMETRIES**.

Example 3.1.21. Let V be a vector space of dimension $n \geq 2$ over a field $\mathbb{F}$. By elementary facts of linear algebra, $\mathsf{Proj}(V)$ (in Example 3.1.3) is clearly flag regular and also homogeneous; whence it is an incidence geometry.

Lemma 3.1.22. *Let A be an incidence geometry over N and $\nu : N' \to N$ be a map. Then $\nu^\star\mathsf{A}$ is also an incidence geometry, called a* **PULL BACK INCIDENCE GEOMETRY**, *over N'.*

Proof. Let $G := \{\psi|_{\nu^\star\mathsf{A}} | \psi \in \mathrm{Aut}(\mathsf{A})\} \subseteq \mathrm{Aut}(\nu^\star\mathsf{A})$. Suppose f and f' be flags of $\nu^\star\mathsf{A}$ of the same type $\nu^\star t(f) = \nu^\star t(f')$. By the property 3.1.1 of an incidence system, f and f' may be regarded as flags of A of type $t(f) = t(f')$ with some duplicate elements. Since A is homogeneous, then there exists $\psi \in \mathrm{Aut}(\mathsf{A})$ such that $f' = \nu^\star\psi(f)$. Therefore $\nu^\star\mathsf{A}$ is G-homogeneous. Moreover, f is contained in a full flag of $\nu^\star\mathsf{A}$ because A is flag regular. □

Therefore the category of incidence geometries is closed under pullbacks.

Proposition 3.1.23. *Let A be a homogeneous incidence system over N. If A has a full flag, then A is an incidence geometry.*

Proof. Assume that A has a full flag f. We need to show that A is flag regular. Let f' be a flag of A. Since f is full, there exists a $f'' \subseteq f$ such that $t(f'') = t(f')$. As A is homogeneous, there exists a $\psi \in \mathrm{Aut}(\mathsf{A})$ such that $f' = \psi(f'') \subseteq \psi(f)$, whence the result. □

3.2 Coset incidence systems and geometries

We dedicate this section to investigate some properties of one of the most elementary but interesting example of incidence systems which is one constructed from a group and some of its subgroups. This was first introduced by Jacques Tits in [Tit62].

Let G be a group and N a finite set. Suppose that, for each $i \in N$, H_i is a non-empty subgroup of G. Then

$$\mathsf{C}(G; H_{i:i\in N}) := \{H_i g | g \in G, i \in N\}$$

is an incidence system over N with the type function

$$t^{\mathsf{c}} : \mathsf{C}(G; H_{i:i\in N}) \to N : H_i g \mapsto i,$$

and the incidence relation

$$\mathsf{I}^{\mathsf{c}} := \{(H_i g, H_j g) \in \mathsf{C}(G; H_{i:i\in N}) \times \mathsf{C}(G; H_{i:i\in N}) \,|\, g \in G, i, j \in N\},$$

i.e., two cosets are incident if and only if they have nonempty intersection. We call the incidence system $\mathsf{C}(G; H_{i:i\in N})$ a (right) **COSET INCIDENCE SYSTEM** for G.

Remark 3.2.1. As there is a bijection from right cosets of H_i in G to its left cosets, for all $i \in N$, any following result which applies to the coset incidence systems (constructed from the right cosets) also applies to the coset incidence system (constructed from the left cosets).

Notice that $\mathsf{C}(G; H_{i:i\in N})$ contains a full flag $\{H_i g \,|\, i \in N\}$, where $g \in G$. Hence if $\mathsf{C}(G; H_{i:i\in N})$ is homogeneous, then, by Proposition 3.1.23, it is an incidence geometry, called a **COSET INCIDENCE GEOMETRY** for G. There is a condition that when we impose it to a homogeneous incidence system, the homogeneous incidence system is then a coset incidence geometry.

Proposition 3.2.2. *Let* A *be a* G*-homogeneous incidence system over* N*. If* A *has a full flag, then* A *is isomorphic to a coset incidence geometry for* G*.*

Proof. Assume that A has a full flag f. Denote $H_i = \mathrm{Stab}(f(i)) \leq G$ for $i \in N$. We will show that A is isomorphic to $\mathsf{C}(G; H_{i:i\in N})$. Define $\psi : \mathsf{A} \longrightarrow \mathsf{C}(G; H_{i:i\in N})$ by

$$\begin{aligned} \psi_j : \mathsf{A}_j &\longrightarrow \mathsf{C}(G; H_{i:i\in N}) \\ a &\longmapsto H_j \gamma, \end{aligned}$$

for some $\gamma \in G$ such that $\gamma(f(j)) = a$, for all $j \in N$. Then ψ is well-defined because, for each $j \in N$, if $\gamma, \gamma' \in G$ and $\gamma(f(j)) = \gamma'(f(j))$, then $\gamma^{-1} \circ \gamma'(f(i)) = f(i)$; whence $\gamma^{-1} \circ \gamma' \in H_i$ and so $H_i \gamma = H_i \gamma'$. Assume that $j, k \in N$ and $(a, b) \in \mathsf{I}_{jk}$. Since A is regular, there exists $\gamma \in G$ such that $\gamma \circ f(j) = a$ and $\gamma \circ f(k) = b$. Hence

$$(\psi(a), \psi(b)) = (H_i \gamma, H_j \gamma) \in \mathsf{I}^{\mathsf{c}}.$$

Hence ψ is an incidence system morphism. One can check that ψ_i is bijective for all $i \in N$ and thus ψ_{ij} is bijective for $i, j \in N$. Therefore, by Lemma 3.1.6, ψ is an incidence system isomorphism. Since A is G-homogeneous, then so is $\mathsf{C}(G; H_{i:i\in N})$ and whence the result. □

Next we will consider the symmetry groups of coset incidence systems. Let $\mathsf{C}\left(G; H_{i:i\in N}\right)$ be a coset incidence system.

Lemma 3.2.3. *The map*

$$\begin{aligned} \sigma : G &\longrightarrow Aut\left(\mathsf{C}\left(G; H_{i:i\in N}\right)\right) \\ g &\longmapsto r_g, \end{aligned} \tag{3.2.1}$$

where

$$\begin{aligned} r_g : \mathsf{C}\left(G; H_{i:i\in N}\right) &\longrightarrow \mathsf{C}\left(G; H_{i:i\in N}\right) \\ H_i h &\longmapsto H_i h g \end{aligned} \tag{3.2.2}$$

i.e., the right multiplication by g on cosets, is a homomorphism, and so

$$G\,/Ker\sigma \subseteq Aut\left(\mathsf{C}\left(G; H_{i:i\in N}\right)\right).$$

Proof. It suffices to check that $r_g \in \mathrm{Aut}\left(\mathsf{C}\left(G; H_{i:i\in N}\right)\right)$ for all $g \in G$. Let $g \in G$. Clearly, for any $i, j \in N$, if $(H_i h, H_j h) \in \mathsf{I}^{\mathsf{c}}$, then $(H_i hg, H_j hg) \in \mathsf{I}^{\mathsf{c}}$. Hence r_g is an incidence system morphism. Since $g \in G$, so $g^{-1} \in G$ and $r_{g^{-1}}$ is also an incidence system morphism. One can check that

$$\begin{aligned} r_g \circ r_{g^{-1}} &= \mathrm{id} : \mathsf{C}\left(G; H_{i:i\in N}\right) \longrightarrow \mathsf{C}\left(G; H_{i:i\in N}\right), \\ r_{g^{-1}} \circ r_g &= \mathrm{id} : \mathsf{C}\left(G; H_{i:i\in N}\right) \longrightarrow \mathsf{C}\left(G; H_{i:i\in N}\right). \end{aligned}$$

So $r_g \in \mathrm{Aut}\left(\mathsf{C}\left(G; H_{i:i\in N}\right)\right)$. Moreover, for any $g_1, g_2 \in G$,

$$\sigma\left(g_1 g_2\right) = r_{g_1 g_2} = r_{g_1} \circ r_{g_2} = \sigma\left(g_1\right) \circ \sigma\left(g_2\right).$$

□

In general, any coset incidence geometry is not necessarily G-homogeneous as we shall see from the following example.

Example 3.2.4. Consider the symmetric group S_5, by the above construction, we have

$$\mathsf{A} : \mathsf{C}\left(S_5; \langle(12)\rangle, \langle(12),(13)\rangle, \langle(12),(45)\rangle\right)$$

is an incidence system over $N = \{1, 2, 3\}$. There is no element $s \in S_5$ such that

$$\{\langle(12)\rangle, \langle(12),(13)\rangle\} \cdot s = \{\langle(12)\rangle, \langle(12),(45)\rangle\}.$$

In general, a coset incidence system $\mathsf{C}(G; H_{i:i\in N})$ is not necessary G-homogeneous. So we would like to know when it is G-homogeneous; this will turn $\mathsf{C}(G; H_{i:i\in N})$ into an incidence geometry.

Theorem 3.2.5. *Let $\mathsf{C}(G; H_{i:i\in N})$ be a coset incidence system for G. Then $\mathsf{C}(G; H_{i:i\in N})$ is $(G/Ker\sigma)$-homogeneous if and only if*

$$\textit{\textbf{(P}}_1\textit{\textbf{)}} \text{ for any flag } f \text{ of } \mathsf{C}(G; H_{i:i\in N}), \text{ the intersection } \bigcap_{X\in f} X \neq \emptyset.$$

*That is $\mathsf{C}(G; H_{i:i\in N})$ is an incidence geometry if and only if the condition **(P$_1$)** is satisfied.*

Proof. Assume that $\mathsf{C}(G; H_{i:i\in N})$ is $(G/\mathrm{Ker}\sigma)$-homogeneous. Let $\emptyset \neq I \subseteq N$. Then $f := \{H_i g \,|\, i \in J\}$ is a flag of $\mathsf{C}(G; H_{i:i\in N})$ of type I. Since $\mathsf{C}(G; H_{i:i\in N})$ is $(G/\mathrm{Ker}\sigma)$-homogeneous, any flag of type I of $\mathsf{C}(G; H_{i:i\in N})$ is then of the form $\{H_i gh \,|\, i \in I\}$, whence the result.

Conversely, assume that, for each flag f of $\mathsf{C}(G; H_{i:i\in N})$, the intersection $\bigcap_{X\in f} X$ is nonempty. Let f and f' be flags of $\mathsf{C}(G; H_{i:i\in N})$ with $t^{\mathsf{c}}(f) = t^{\mathsf{c}}(f')$. Choose $a \in \bigcap_{X\in f} X$ and $a' \in \bigcap_{X'\in f'} X'$. Then we have

$$\sigma\left(a^{-1}a'\right)(f) = f'.$$

Thus $\mathsf{C}(G; H_{i:i\in N})$ is $(G/\mathrm{Ker}\sigma)$-homogeneous. The final result follows immediately from Proposition 3.2.2. □

Proposition 3.2.6. *Let $\mathsf{C}(G; H_{i:i\in N})$ be a coset incidence geometry. For any $i \in N$,*

$$Res(\{H_i\}) \cong \mathsf{C}\left(H_i; (H_i \cap H_j)_{:j\in N\setminus\{i\}}\right).$$

Proof. Let $i \in N$. For any $j \in N\setminus\{i\}$ and $g \in G$, if $(H_i, H_j g) \in \mathsf{I}^{\mathsf{c}}_{ij}$, then $H_j g = H_i h$ for some $h \in H_i$; whence

$$(\text{Res}(\{H_i\}))_j = \{H_j h \,|h \in H_i\}.$$

Define $\psi : \text{Res}(\{H_i\}) \longrightarrow \mathsf{C}\left(H_i; (H_i \cap H_j)_{:j\in N\setminus\{i\}}\right)$ by

$$\begin{aligned} \psi_j : (\text{Res}(\{H_i\}))_j &\longrightarrow (H_i \cap H_j)\backslash H_i \\ H_j h &\longmapsto (H_i \cap H_j)\, h, \end{aligned}$$

for all $j \in N\setminus\{i\}$. Then for each $j \in N\setminus\{i\}$ and any $h_1, h_2 \in H_i$,

$$\begin{aligned} H_j h_1 = H_j h_2 &\Leftrightarrow h_1 h_2^{-1} \in H_j \\ &\Leftrightarrow h_1 h_2^{-1} \in H_i \cap H_j \\ &\Leftrightarrow (H_i \cap H_j)\, h_1 = (H_i \cap H_j)\, h_1. \end{aligned}$$

Thus ψ is well-defined and ψ_j is injective; since ψ_j is automatically surjective, thus ψ_j is bijective.

For any $j, k \in N\setminus\{i\}$, if $h, h' \in H_i$ and $(H_j h, H_k h') \in \mathsf{I}^{\mathsf{c}}_{ij}$, then, by Theorem 3.2.5, there exists $h'' \in H_i$ such that $(H_j h, H_k h') = (H_j h'', H_k h'')$; whence

$$\psi_{jk}\left(H_j h'', H_k h''\right) = \left((H_i \cap H_j)\, h'', (H_i \cap H_k)\, h''\right).$$

So ψ preserves the incidence relation, and so it is an incidence system morphism.

Moreover, for any $j, k \in N\setminus\{i\}$, we have ψ_{jk} is injective because ψ_j and ψ_k are injective, and ψ_{jk} is clearly surjective because $H_i \subseteq G$. Thus ψ_{jk} is bijective for all $j, k \in N\setminus\{i\}$. Therefore, by Lemma 3.1.6, ψ is an incidence system isomorphism. □

Proposition 3.2.7. *Let A, B, and C be subgroups of a group G. Then the following are equivalent.*

(i) Any cosets Ax, By, and Cz which intersect pairwise have the non-empty intersection

$Ax \cap By \cap Cz$.

(*ii*) $(A \cap B)(A \cap C) = A \cap BC$.

(*iii*) $A(B \cap C) = AB \cap AC$.

Proof. (*i*) $\Rightarrow$ (*ii*) It suffices to show that $A \cap BC \subseteq (A \cap B)(A \cap C)$. Let $a \in A \cap BC$. Then $a = bc$ for some $b \in B$ and $c \in C$. For any $x \in G$, the cosets Ax, Bax, and $Cb^{-1}ax$ intersect pairwise; whence $Ax \cap Bax \cap Cb^{-1}ax \neq \emptyset$. Choose $u \in Ax \cap Bax \cap Cb^{-1}ax$. Then $axu^{-1} \in A \cap B$ and $ux^{-1} \in A \cap C$. Therefore

$$a = \left(axu^{-1}\right)\left(ux^{-1}\right) \in (A \cap B)(A \cap C).$$

(*ii*) $\Rightarrow$ (*iii*)

$$\begin{aligned} AB \cap AC &= (AB \cap A)(AB \cap C) \\ &= A(B \cap A)(A \cap C)(B \cap C) \\ &= A(B \cap C). \end{aligned}$$

(*iii*) $\Rightarrow$ (*i*) If Ax, By, and Cz intersect pairwise, then $xy^{-1} \in AB$, $zy^{-1} \in CB$, and $zx^{-1} \in CA$. Since $\left(zx^{-1}\right)\left(xy^{-1}\right) = zy^{-1} \in CB$, there exists $b \in B$ and $c \in C$ such that $xy^{-1}b^{-1} = xz^{-1}c$. But $xy^{-1}b^{-1} \in AB$ and $xz^{-1}c \in AC$. This implies that there exists $a \in A$ and $u \in B \cap C$ such that $xy^{-1}b^{-1} = xz^{-1}c = au$; whence $a^{-1}x = uc^{-1}z = uby$. Therefore $Ax \cap By \cap Cz \neq \emptyset$. ☐

Define

$$(\mathbf{P}_2): \text{ for any } \emptyset \neq I \subseteq N \text{ with } |I| \geq 3, \text{ there exist } i, j \in I \text{ such that}$$

$$K \cap H_i H_j = (K \cap H_i)(K \cap H_j),$$

where $K = \bigcap_{k \in I \setminus \{i,j\}} H_k$, i.e., $k \in K \Rightarrow k = k_i k_j$ for some $k_i \in K \cap H_i$ and $k_j \in K \cap H_j$.

Proposition 3.2.8. *For any coset incidence system, the condition* $(\boldsymbol{P}_2)$ *implies the condition* $(\boldsymbol{P}_1)$.

Proof. Let $\mathsf{C}(G; H_{i:i \in N})$ be a coset incidence system with the condition $(\mathbf{P}_2)$ satisfied and f be a flag of $\mathsf{C}(G; H_{i:i \in N})$. We will proceed this by induction on $t^{\mathsf{c}}|f|$.

If $t^{\mathsf{c}}|f| = 1$ or 2, then the result is obvious.

Suppose that $t^{\mathsf{c}}|f| > 2$. Let $I := t^{\mathsf{c}}|f|$ and $i, j \in I$ be such that the condition $(\mathbf{P}_2)$ satisfied. Denote $K = \bigcap_{k \in I \setminus \{i,j\}} H_k$ and $A, B \in f$ be such that $t^{\mathsf{c}}(A) = i$ and $t^{\mathsf{c}}(B) = j$. Then by induction, $\bigcap_{X \in f \setminus \{A,B\}} X = Kg$ for some $g \in G$; moreover $A \cap Kg \neq \emptyset$ and $B \cap Kg \neq \emptyset$. Hence A, B, and Kg intersect pairwise. Since

$$K \cap H_i H_j = (K \cap H_i)(K \cap H_j),$$

Proposition 3.2.7 implies that $A \cap B \cap Kg \neq \emptyset$. □

Corollary 3.2.9. *Let $\mathsf{C}(G; H_{i:i\in N})$ be a coset incidence system with the condition* $(\mathbf{P}_2)$ *satisfied. Then $\mathsf{C}(G; H_{i:i\in N})$ is a coset incidence geometry for G.*

Proof. We need to show that $\mathsf{C}(G; H_{i:i\in N})$ is homogeneous but it is an immediate consequence from Theorem 3.2.5 and Proposition 3.2.8. □

3.3 Coxeter incidence geometries

In this section, we explore a crucial example of incidence geometries we interested in in this thesis.

Let W be a Coxeter group with the Coxeter diagram $\mathscr{D}$. Note that when we say W is a Coxeter group here, we mean that it is already equipped with a simple system S. Recall from Section 2.1.3, the parabolic subgroup W_I is a subgroup of W for all $I \subseteq \mathscr{D}$. In particular, we denote maximal parabolic subgroups $W_i := W_{\mathscr{D}\setminus\{i\}}$ for all $i \in \mathscr{D}$.

Definition 3.3.1. Let W be a Coxeter group with the Coxeter diagram $\mathscr{D}$. Then $\mathsf{C}(W) := \mathsf{C}(W; W_{i:i\in\mathscr{D}})$ is a (left) coset incidence system, called a **Coxeter incidence system** for W.

Consider the homomorphism

$$\sigma : W \longrightarrow \operatorname{Aut}(\mathsf{C}(W)) : w \longmapsto l_w,$$

where

$$l_w : \mathsf{C}(W) \longrightarrow \mathsf{C}(W) : w'W_i \longmapsto ww'W_i.$$

The following shows a condition that makes Kerσ trivial; whence we can think of W as a subgroup of Aut $(\mathsf{C}(W))$.

Proposition 3.3.2. *Let W be a Coxeter group with the Coxeter diagram $\mathscr{D}$. Then the homomorphism $\sigma : W \to Aut(\mathsf{C}(W))$ is injective.*

Proof. For any $i \in \mathscr{D}$, the kernel of the action of W on W/W_i under σ is contained in W_i. As, by Corollary 2.1.15, $\bigcap_{i\in\mathscr{D}} W_i = \{1\}$, thus W acts effectively on $\prod_{i\in\mathscr{D}} W/W_i$. Therefore the kernel of the homomorphism σ is $\{1\}$, i.e., σ is injective. □

To summarize, for any Coxeter group (W, S), the group W is a subgroup of Aut $(\mathsf{C}(W))$.

Corollary 3.3.3. *Let W be a Coxeter group with the Coxeter diagram $\mathscr{D}$ and $\mathsf{C}(W)$ be the Coxeter incidence system for W. Then $\mathsf{C}(W)$ is an incidence geometry, the so-called* **Coxeter incidence geometry**.

Proof. Let $\emptyset \neq I \subseteq S$ with $|I| \geq 3$. Suppose that $|I| = 3$ and the vertices corresponding to i_1, i_2, i_3 in I are pairwise adjacent in $\mathscr{D}$. Let $w \in W_{i_1} \cap W_{i_2}W_{i_3}$. Since $w \in W_{i_2}W_{i_3}$, thus

$$x = (S_{j_1}S_{j_2}\cdots S_{j_m})(S_{k_1}S_{k_2}\cdots S_{k_n}),$$

where $S_{j_1}S_{j_2}\cdots S_{j_m}$ is a reduced expression in W_{i_2} and $S_{k_1}S_{k_2}\cdots S_{k_n}$ is a reduced expression in W_{i_3}. We can obtain a reduced expression from this expression of w by using the Deletion Condition (Theorem 2.1.7) and so again it is in $W_{i_2}W_{i_3}$. Since $w \in W_{i_1}$, there is no S_{i_1} appearing in such reduced expression. Therefore $w \in (W_{i_1} \cap W_{i_2})(W_{i_1} \cap W_{i_3})$. Hence $W_{i_1} \cap W_{i_2}W_{i_3} = (W_{i_1} \cap W_{i_2})(W_{i_1} \cap W_{i_3})$, i.e., the condition $(\mathbf{P}_2)$ is satisfied.

Otherwise there exists $i, j \in I$ such that there is no line joining the nodes i and j, i.e., $(S_iS_j)^2 = 1$. Denote $K := \bigcap_{k\in I\setminus\{i,j\}} W_k$. Then $K = W_{\{i,j\}}$, i.e., the subgroup of W generated by S_i and S_j. Since $(S_iS_j)^2 = 1$, we have $S_iS_j = S_jS_i$, and so

$$K = W_{\{j\}}W_{\{i\}} = (K \cap W_i)(K \cap W_j).$$

Then the condition $(\mathbf{P}_2)$ is satisfied. By Corollary 3.2.9, $\mathsf{C}(W)$ is a coset incidence geometry. □

3.4 Parabolic incidence geometries

In this section, we explore another class of incidence geometries which is a key ingredient in this thesis.

For this section, let $\mathfrak{g}$ be a finite-dimensional split reductive Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero, and let G be a connected reductive algebraic group over $\mathbb{F}$ with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ and Q be the parabolic subgroup of G with the associated Lie algebra $\mathfrak{q}$. Recall, from Remark 2.2.53, $\mathfrak{g}$ has the diagram $\mathscr{D}_{\mathfrak{g}}$ which is the Dynkin diagram of a semisimple part of $\mathfrak{g}$. Let $\mathscr{P}(\mathfrak{g})$ be the set of all parabolic subalgebras of $\mathfrak{g}$ and $\mathscr{P}^{\mathfrak{q}}(\mathfrak{g}) := \{\mathfrak{p} \in \mathscr{P}(\mathfrak{g}) \,|\, \mathfrak{g} = \mathfrak{p} + \mathfrak{q}\}$, i.e., the set of all parabolic subalgebras weakly opposite to $\mathfrak{q}$. The adjoint action of G on $\mathscr{P}(\mathfrak{g})$ decomposes $\mathscr{P}(\mathfrak{g})$ into disjoint G-orbits.

Let $\mathfrak{b}_0$ be a Borel subalgebra of $\mathfrak{g}$. By Corollary 2.2.57, a parabolic subalgebra $\mathfrak{p}$ contains a Borel subalgebra $\mathfrak{b}$, and, by Corollary 2.3.32, there exists $g \in G$ such that $\mathfrak{b}_0 = g\cdot\mathfrak{b}$; whence each orbit $G\cdot\mathfrak{p}$ contains the standard parabolic subalgebra $\mathfrak{p}_0 := g\cdot\mathfrak{p}$ containing $\mathfrak{b}_0$. Denote

$$\mathscr{P}_I(\mathfrak{g}) := G\cdot\mathfrak{p}_0,$$

where $\mathfrak{p}_0 \supseteq \mathfrak{b}_0$ and I corresponds to the set of crossed nodes of the decorated Dynkin diagram $\mathscr{D}_{\mathfrak{p}_0}$ as in Remark 2.2.58. Hence

$$\mathscr{P}(\mathfrak{g}) = \bigsqcup_{I\subseteq\mathscr{D}_{\mathfrak{g}}} \mathscr{P}_I(\mathfrak{g}),$$

and

$$\mathscr{P}^{\mathfrak{q}}(\mathfrak{g}) = \bigsqcup_{I\subseteq\mathscr{D}_{\mathfrak{g}}} \mathscr{P}_I^{\mathfrak{q}}(\mathfrak{g}),$$

where $\mathscr{P}_I^{\mathfrak{q}}(\mathfrak{g}) := \mathscr{P}_I(\mathfrak{g}) \cap \mathscr{P}^{\mathfrak{q}}(\mathfrak{g})$. Note that, for any $I \subseteq \mathscr{D}_{\mathfrak{g}}$, the set $\mathscr{P}_I^{\mathfrak{q}}(\mathfrak{g})$ is nonempty. To see this, for any Borel subalgebra $\mathfrak{b}$ of $\mathfrak{q}$ containing a Cartan subalgebra $\mathfrak{t}$, let $\widehat{\mathfrak{b}}$ be the Borel subalgebra complementary to $\mathfrak{b}$ with respect to $\mathfrak{t}$, i.e., $\mathfrak{b}\cap\widehat{\mathfrak{b}} = \mathfrak{t}$. Then there exists a parabolic subalgebra $\mathfrak{p} \in \mathscr{P}_I^{\mathfrak{q}}(\mathfrak{g})$ containing $\widehat{\mathfrak{b}}$ the Borel subalgebra of $\mathfrak{g}$ complementary to $\mathfrak{b}$ with respect to $\mathfrak{t}$ because all Borel subalgebras of $\mathfrak{g}$ are $\mathrm{Ad}(G)$-conjugate, and so

$$\mathfrak{g} = \mathfrak{b}^- + \mathfrak{b} \subseteq \mathfrak{p} + \mathfrak{q} \subseteq \mathfrak{g}.$$

In particular, there is a one-to-one correspondence between G-adjoint orbits of maximal parabolic subalgebras and the nodes of $\mathscr{D}_{\mathfrak{g}}$; for any $i \in \mathscr{D}_{\mathfrak{g}}$, the orbit $\mathscr{P}_i^{\mathfrak{q}}(\mathfrak{g}) := \mathscr{P}_{\{i\}}^{\mathfrak{q}}(\mathfrak{g})$ corresponds to the node i in the diagram. Denote

$$\mathsf{Para}(\mathfrak{g}) := \bigsqcup_{i \in \mathscr{D}_{\mathfrak{g}}} \mathscr{P}_i(\mathfrak{g}),$$

and

$$\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}) := \mathscr{P}^{\mathfrak{q}}(\mathfrak{g}) \cap \mathsf{Para}(\mathfrak{g}) \subseteq \mathsf{Para}(\mathfrak{g}).$$

Define a relation $\mathsf{I}^{\mathsf{Para}(\mathfrak{g})}$ on $\mathsf{Para}(\mathfrak{g})$ by

$$\left(\mathfrak{p}, \mathfrak{p}'\right) \in \mathsf{I}^{\mathsf{Para}(\mathfrak{g})} \Leftrightarrow \mathfrak{p} \text{ and } \mathfrak{p}' \text{ are co-standard, i.e., } \mathfrak{p} \cap \mathfrak{p}' \text{ is a parabolic subalgebra,}$$

for all $\mathfrak{p}, \mathfrak{p}' \in \mathsf{Para}(\mathfrak{g})$, and

$$t^{\mathsf{Para}(\mathfrak{g})} : \mathsf{Para}(\mathfrak{g}) \to \mathscr{D}_{\mathfrak{g}}; \mathfrak{p} \mapsto \text{the node of } \mathscr{D}_{\mathfrak{g}} \text{corresponding to the adjoint orbit of } \mathfrak{p}.$$

Then $\mathsf{Para}(\mathfrak{g})$ is an incidence system, and so is $\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})$.

Definition 3.4.1. Let $\mathfrak{g}$ be a semisimple Lie algebra over a field $\mathbb{F}$ of characteristic zero and $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$. Then $\mathsf{Para}(\mathfrak{g})$ and $\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})$ are incidence systems over $\mathscr{D}_{\mathfrak{g}}$ called the PARABOLIC INCIDENCE SYSTEM and $\mathfrak{q}$-GENERIC PARABOLIC INCIDENCE SYSTEM for $\mathfrak{g}$, respectively.

Lemma 3.4.2. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be parabolic subalgebras of $\mathfrak{g}$ which are pairwise co-standard. Then $\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n$ is a parabolic subalgebra.*

Proof. For any two parabolic subalgebras $\mathfrak{p}$ and $\mathfrak{q}$ of $\mathfrak{g}$, we have $\mathfrak{p} \cap \mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{p}^{\perp} \subseteq \mathfrak{q}$. We will proceed from this by using induction on n.

If $n = 1$, then result is trivial.

Suppose that $n \geq 2$. By induction hypothesis, $\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_{n-1}$ is a parabolic subalgebra. Since $\mathfrak{p}_n \cap \mathfrak{p}_i$ is a parabolic subalgebra for all $1 \leq i \leq n-1$, we have $\mathfrak{p}_n^{\perp} \subseteq \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_{n-1}$, and so $(\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_{n-1})^{\perp} = \mathfrak{p}_1^{\perp} + \ldots + \mathfrak{p}_{n-1}^{\perp} \subseteq \mathfrak{p}_n$. Hence $\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n$ is a parabolic subalgebra. □

Corollary 3.4.3. *The map*

$$\tau : \mathcal{F}\left(\mathsf{Para}\left(\mathfrak{g}\right)\right) \to \mathscr{P}\left(\mathfrak{g}\right) : f \mapsto \bigcap_{\mathfrak{p}\in f} \mathfrak{p} \tag{3.4.1}$$

is a bijection map preserving types.

Proof. Let $f \in \mathcal{F}\left(\mathsf{Para}\left(\mathfrak{g}\right)\right)$. Lemma 3.4.2 implies that $\mathfrak{r} = \tau\left(f\right) = \bigcap_{\mathfrak{p}\in f} \mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{b}$ be a Borel subalgebra contained in $\mathfrak{r}$ and $\mathfrak{t}$ a Cartan subalgebra contained in $\mathfrak{b}$. Denote $\mathcal{R} := \mathcal{R}\left(\mathfrak{g}, \mathfrak{t}\right)$. Any $\mathfrak{p}_1, \mathfrak{p}_2 \in f$, the parabolic subalgebra $\mathfrak{p}_1 \cap \mathfrak{p}_2$ corresponds to the subset

$$\Delta_{(\mathfrak{t},\mathfrak{b})} \cap \left(\mathcal{R} \setminus \left(-\mathcal{R}_{\mathfrak{p}_1\cap\mathfrak{p}_2}\right)\right) = \left(\Delta_{(\mathfrak{t},\mathfrak{b})} \cap \left(\mathcal{R} \setminus \left(-\mathcal{R}_{\mathfrak{p}_1}\right)\right)\right) \cup \left(\Delta_{(\mathfrak{t},\mathfrak{b})} \cap \left(\mathcal{R} \setminus \left(-\mathcal{R}_{\mathfrak{p}_2}\right)\right)\right)$$

of $\Delta_{(\mathfrak{t},\mathfrak{b})}$; whence $\mathfrak{p}_1 \cap \mathfrak{p}_2 \in \mathscr{P}_{\{t(\mathfrak{p}_1),t(\mathfrak{p}_1)\}}\left(\mathfrak{g}\right)$. By induction, we have $\mathfrak{r} \in \mathscr{P}_{t(f)}\left(\mathfrak{g}\right)$. Therefore, τ is well-defined and preserves types.

It suffices to show that, for any $I \subseteq \mathscr{D}_{\mathfrak{g}}$,

$$\tau_I := \tau\left|_{\mathcal{F}_I(\mathsf{Para}(\mathfrak{g}))} \mathcal{F}_I\left(\mathsf{Para}\left(\mathfrak{g}\right)\right) \to \mathscr{P}_I\left(\mathfrak{g}\right)\right.,$$

where $\mathcal{F}_I\left(\mathsf{Para}\left(\mathfrak{g}\right)\right) := \{\text{flags in } \mathsf{Para}\left(\mathfrak{g}\right) \text{ of types } I\}$, is bijective.

Let $I \subseteq \mathscr{D}_{\mathfrak{g}}$. Let $f, f' \in \mathcal{F}_I\left(\mathsf{Para}\left(\mathfrak{g}\right)\right)$ be such that $\tau_I\left(f\right) = \tau_I\left(f'\right)$. Suppose that $f \neq f'$. Then there exists $i \in I$ such that $f\left(i\right) \neq f'\left(i\right)$ and $f\left(i\right) \cap f'\left(i\right) \supseteq \tau_I\left(f\right)$, a contradiction. Therefore τ_I is injective. Next let $\mathfrak{p} \in \mathscr{P}_I\left(\mathfrak{g}\right)$. Then, for each $i \in I$, there exists a unique $\mathfrak{p}_i \in \mathscr{P}_i\left(\mathfrak{g}\right)$ such that $\mathfrak{p} \subseteq \mathfrak{p}_i$. So $f := \{\mathfrak{p}_i \,|i \in I\} \in \mathcal{F}_I\left(\mathsf{Para}\left(\mathfrak{g}\right)\right)$ and $\mathfrak{p} \subseteq \tau_I\left(f\right) \in \mathscr{P}_I\left(\mathfrak{g}\right)$; whence $\tau_I\left(f\right) = \mathfrak{p}$. Therefore τ_I is surjective. □

The adjoint action of G on $\mathsf{Para}\left(\mathfrak{g}\right)$ makes $G \leq \mathrm{Aut}\left(\mathsf{Para}\left(\mathfrak{g}\right)\right)$. Since G acts transitively on each orbit $\mathscr{P}_I\left(\mathfrak{g}\right)$, thus $\mathsf{Para}\left(\mathfrak{g}\right)$ is G-homogeneous. However, G doesn't act on $\mathsf{Para}^{\mathfrak{q}}\left(\mathfrak{g}\right)$. To show that $\mathsf{Para}^{\mathfrak{q}}\left(\mathfrak{g}\right)$ is also homogeneous, we need to find a subgroup of G which acts transitively on the flags of all types of $\mathsf{Para}^{\mathfrak{q}}\left(\mathfrak{g}\right)$.

Proposition 3.4.4. *If* $\mathfrak{p} \in \mathscr{P}^{\mathfrak{q}}\left(\mathfrak{g}\right)$*, then* $Q \cdot \mathfrak{p}$ *is an open dense orbit in* $G \cdot \mathfrak{p}$.

Proof. Let $\mathfrak{p} \in \mathscr{P}^{\mathfrak{q}}\left(\mathfrak{g}\right)$. Thus we have $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$, and so

$$\mathfrak{g}/\mathfrak{p} = \left(\mathfrak{p} + \mathfrak{q}\right)/\mathfrak{p} \cong \mathfrak{q}/\left(\mathfrak{p} \cap \mathfrak{q}\right).$$

This implies that

$$\dim(G/P) = \dim(\mathfrak{g}/\mathfrak{p}) = \dim(\mathfrak{q}/(\mathfrak{p}\cap\mathfrak{q})) = \dim(Q/(Q\cap P))$$

where $P = \{g \in G \,|\, g\cdot\mathfrak{p} = \mathfrak{p}\}$. Since $G\cdot\mathfrak{p} \cong G/P$ and $Q\cdot\mathfrak{p} \cong Q/(Q\cap P)$, we see that $Q\cdot\mathfrak{p}$ is open in $G\cdot\mathfrak{p}$. Moreover $Q\cdot\mathfrak{p}$ is dense in $G\cdot\mathfrak{p}$ because $G\cdot\mathfrak{p}$ is irreducible. □

Lemma 3.4.5. For each $I \subseteq \mathscr{D}_{\mathfrak{g}}$, Q acts transitively on $\mathscr{P}_I^{\mathfrak{q}}(\mathfrak{g})$.

Proof. If $\mathfrak{p} \in \mathscr{P}_I^{\mathfrak{q}}(\mathfrak{g})$, then, for each $q \in Q$, $q\cdot\mathfrak{p} \in \mathscr{P}_I^{\mathfrak{q}}(\mathfrak{g})$ because

$$\begin{aligned}\mathfrak{g} &= q\cdot\mathfrak{g}\\ &= q\cdot\mathfrak{p} + q\cdot\mathfrak{q}\\ &= q\cdot\mathfrak{p} + \mathfrak{q}.\end{aligned}$$

Now let $\mathfrak{p}, \mathfrak{p}' \in \mathscr{P}_I^{\mathfrak{q}}(\mathfrak{g})$. By Proposition 3.4.4, $Q\cdot\mathfrak{p}$ and $Q\cdot\mathfrak{p}'$ are open and dense in $\mathscr{P}_I^{\mathfrak{q}}(\mathfrak{g})$. Thus $Q\cdot\mathfrak{p}\cap Q\cdot\mathfrak{p}' \neq \emptyset$ which implies that $\mathfrak{p}' = q\cdot\mathfrak{p}$ for some $q \in Q$. □

Proposition 3.4.4 and Lemma 3.4.5 show that, $Q \leq \mathrm{Aut}(\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}))$ and, for any subset $I \subseteq \mathscr{D}_{\mathfrak{g}}$, the set $\mathscr{P}_I^{\mathfrak{q}}(\mathfrak{g})$ is the unique open dense Q-orbit in $\mathscr{P}_I(\mathfrak{g})$.

Lemma 3.4.6. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be parabolic subalgebras of $\mathfrak{g}$ which are pairwise co-standard and weakly opposite to $\mathfrak{q}$. Then $\mathfrak{p}_1\cap\ldots\cap\mathfrak{p}_n$ is a parabolic subalgebra of $\mathfrak{g}$ which is weakly opposite to $\mathfrak{q}$.*

Proof. We will prove this by induction on n. If $n = 1$, then the result is trivial. Suppose that $n > 1$. Let $\mathfrak{r} := \mathfrak{p}_1\cap\ldots\cap\mathfrak{p}_n$. Then $\mathfrak{r}$ is a parabolic subalgebra by Lemma 3.4.2. Let $\mathfrak{t} \subseteq \mathfrak{r}\cap\mathfrak{q}$ be a Cartan subalgebra. Choose an algebraic Weyl structure $\widetilde{\xi} \in \mathfrak{t}$ of $\mathfrak{q}$ and determine the parabolic subalgebra $\widehat{\mathfrak{q}}$ complementary to $\mathfrak{q}$. By induction hypothesis, $(\mathfrak{r}')^{\perp}\cap\mathfrak{q}^{\perp} = \{0\}$, where $\mathfrak{r} := \mathfrak{p}_1\cap\ldots\cap\mathfrak{p}_{n-1}$. Thus $(\mathfrak{r}')^{\perp}$ has nonnegative eigenvalues for $\widetilde{\xi}$, and so $(\mathfrak{r}')^{\perp} \subseteq \widehat{\mathfrak{q}}$. By the same argument for $\mathfrak{p}_n$, we have $\mathfrak{p}_n^{\perp} \subseteq \widehat{\mathfrak{q}}$. Thus $\mathfrak{r}^{\perp} = (\mathfrak{r}')^{\perp} + \mathfrak{p}_n^{\perp} \subseteq \widehat{\mathfrak{q}}$, i.e., $\mathfrak{r}$ and $\widehat{\mathfrak{q}}$ are co-standard. Therefore, by Corollary 2.2.48, $\mathfrak{r}$ is weakly opposite to $\mathfrak{q}$. □

Corollary 3.4.7. *The map*

$$\tau^{\mathfrak{q}} := \tau\,\big|_{\mathcal{F}(\mathsf{Para}(\mathfrak{g}))} : \mathcal{F}(\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})) \to \mathscr{P}^{\mathfrak{q}}(\mathfrak{g}), \tag{3.4.2}$$

where τ is defined as (3.4.1), is a bijection preserving types.

Proof. This is a consequence of Corollary 3.4.3 and Lemma 3.4.6. □

Corollary 3.4.8. $\mathsf{Para}(\mathfrak{g})$ *and* $\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})$ *are incidence geometries.*

Proof. Given a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, one can find a parabolic subalgebra $\mathfrak{p}' \in \mathscr{P}_j(\mathfrak{g})$ (resp. $\mathfrak{p}' \in \mathscr{P}_j^{\mathfrak{q}}(\mathfrak{g})$) containing $\mathfrak{b}$, for all $j \in \mathscr{D}$. Hence Lemma 3.4.2 implies that $\mathsf{Para}(\mathfrak{g})$ is flag regular, and similarly Lemma 3.4.6 implies that $\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})$ is flag regular. Let Q be the parabolic subgroup of G with the Lie algebra $\mathfrak{q}$. Then we have seen that $\mathsf{Para}(\mathfrak{g})$ is G-homogeneous and $\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})$ is Q-homogeneous. Therefore both $\mathsf{Para}(\mathfrak{g})$ and $\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})$ are incidence geometries. □

3.5 Incidence systems and labelled simplicial complexes

This section is included for completeness to provide a bridge between the previous section on incidence systems and the next section on buildings. No essential use will be made of it in later chapters. We begin this section by introducing labelled simplicial complexes; it will be seen later in this section that there are some correspondences between them and incidence systems.

Definition 3.5.1. A LABELLED SIMPLICIAL COMPLEX Σ over a set N is a disjoint union of nonempty collections Σ_I of sets indexed by $I \subseteq N$ satisfying the following conditions:

(SC1) for each $I \subseteq J \subseteq N$, there is a map $\partial_{J,I} : \Sigma_J \to \Sigma_I$ such that for any $I \subseteq J \subseteq K \subseteq N$,

$$\partial_{J,I} \circ \partial_{K,J} = \partial_{K,I} : \Sigma_K \to \Sigma_I,$$

and $\partial_{I,I}$ is the identity on Σ_I.

(SC2) for each $I \subseteq N$, any element $\sigma \in \Sigma_I$ is uniquely determined by $\left\{\partial_{I,\{i\}}(\sigma) \,|\, i \in I\right\}$.

The elements of Σ are called SIMPLICES; in particular, the elements of Σ_I, where $I \subseteq N$, are called I-SIMPLICES. The $\{i\}$-simplices, where $i \in N$, are called VERTICES and N-simplices are called CHAMBERS. For any $\sigma, \sigma' \in \Sigma$, if there exists an inclusion $I \subseteq J \subseteq N$ such that $\partial_{J,I}(\sigma) = \sigma'$, then we say that σ' is a FACE of σ. Any two simplices $\sigma \in \Sigma_I$ and $\sigma' \in \Sigma_J$ are INCIDENT if and only if there exists $\tau \in \Sigma_{I\cup J}$ such that $\partial_{I\cup J,I}(\tau) = \sigma$ and $\partial_{I\cup J,J}(\tau) = \sigma'$.

Definition 3.5.2. Suppose that Σ and Σ' are labelled simplicial complexes over N. A **STRICT LABELLED SIMPLICIAL MORPHISM** $\psi : \Sigma \to \Sigma'$ over N is a map from Σ to Σ' such that, for each $I \subseteq J \subseteq N$, $\psi_I := \psi|_{\Sigma_I} : \Sigma_I \to \Sigma'_I$, and $\partial'_{J,I} \circ \psi_J = \psi_I \circ \partial_{J,I} : \Sigma_J \to \Sigma'_I$.

Let Σ and Σ' are labelled simplicial complexes over N and N', respectively. A **LABELLED SIMPLICIAL MORPHISM** $\Psi : \Sigma \to \Sigma'$ over a map $\nu : N \to N'$ is a strict labelled simplicial morphism $\psi : \nu^\star \Sigma \to \Sigma'$, where $\nu^\star \Sigma$ is a a labelled simplicial complex, which is a disjoint union of $\nu^\star \Sigma_I := \Sigma_{\nu(I)}$, over N' and $\nu^\star \partial_{I,J} = \partial_{\nu(I),\nu(J)}$.

Remark 3.5.3. By **(SC1)**, a labelled simplicial system over N can be consider as a presheaf, i.e., a functor, $\Sigma : \mathcal{P}(N)^{op} \to \mathbf{Set}$. The labelled simplicial morphisms are natural transformations of functors.

Labelled simplicial complexes over N and their morphisms form a category; we denote this category by $\mathbf{SC}_N$.

There is a connection between labelled simplicial complexes and incidence systems. Suppose A is an incidence system over N. Then $\mathcal{F}(\mathsf{A})$, i.e., the set of all flags, of A is a labelled simplicial complex over N with, for each $I \subseteq J \subseteq N$,

$$\begin{aligned} \partial_{J,I} : \mathcal{F}(\mathsf{A})_J &\to \mathcal{F}(\mathsf{A})_I \\ f &\mapsto f \cap t^{-1}(I), \end{aligned}$$

where $\mathcal{F}(\mathsf{A})_J$ is the subset of $\mathcal{F}(\mathsf{A})$ consisting of all flags of type J, and t is the type function of A. Full flags of A are chambers, and singleton subsets of A are vertices.

If $\psi : \mathsf{A} \to \mathsf{A}'$ is a strict incidence morphism between incidence systems over N, then

$$\begin{aligned} \mathcal{F}(\psi) : \mathcal{F}(\mathsf{A}) &\to \mathcal{F}(\mathsf{A}') \\ f &\mapsto \psi(f) := \{\psi(a) \,|\, a \in f\}, \end{aligned}$$

is a simplicial morphism because ψ preserves types. This implies that $\mathcal{F} : \mathbf{ISys}_N \to \mathbf{SC}_N$ is a functor. Moreover, if $\Psi : \mathsf{A} \to \mathsf{A}'$ is an incidence morphism over a map $\nu : N' \to N$, then $\mathcal{F}(\Psi) : \mathcal{F}(\mathsf{A}) \to \mathcal{F}(\mathsf{A}')$ defined by

$$(f : t(f) \to \mathsf{A}) \mapsto \big(\psi(f) : \nu^\star(t(f)) \to \mathsf{A}' : x \mapsto \psi(f(\nu(x)))\big)$$

is a labelled simplicial morphism over the map $\nu : N' \to N$.

Example 3.5.4. Given a Coxeter group (W, S) with the Coxeter diagram $\mathscr{D}$, let $\mathsf{C}(W)$ be a Coxeter incidence system as defined in Definition 3.6. Corollary 2.1.16 and Proposition 3.2.8 imply that each flag f of type $J \subseteq \mathscr{D}$ can be considered as a coset $W_J w$ for some $w \in \bigcap_{X \in f} X$. Hence

$$\mathcal{F}(\mathsf{C}(W)) \cong \mathsf{C}(W; W_{J:J\subseteq\mathscr{D}})$$

is a labelled simplicial complex over $\mathscr{D}$ whose simplices are cosets in $\mathsf{C}(W; W_{J:J\subseteq\mathscr{D}})$. The vertices of $\mathsf{C}(W; W_{J:J\subseteq\mathscr{D}})$ correspond to wW_i where $w \in W$ and $i \in \mathscr{D}$, and the chambers of $\mathsf{C}(W; W_{J:J\subseteq\mathscr{D}})$ are the singleton sets $\{w\}$ where $w \in W$.

Example 3.5.5. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero with the Dynkin diagram $\mathscr{D}_{\mathfrak{g}}$. Then $\mathsf{Para}(\mathfrak{g})$, as defined in Section 3.4, is an incidence system. By Corollary 3.4.3,

$$\mathcal{F}(\mathsf{Para}(\mathfrak{g})) \cong \mathscr{P}(\mathfrak{g})$$

is a labelled simplicial complex over $\mathscr{D}_{\mathfrak{g}}$ whose simplices corresponding to proper parabolic subalgebras of $\mathfrak{g}$. The vertices of $\mathscr{P}(\mathfrak{g})$ correspond to maximal proper parabolic subalgebras of $\mathfrak{g}$, and the vertices $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ form vertices of a simplex if and only if $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_n$ is a parabolic subalgebra corresponding to such simplex.

Conversely, suppose Σ is a labelled simplicial complex over N. Let $\mathcal{E}(\Sigma) \subseteq \Sigma$ be the set of all vertices of Σ, i.e., the disjoint union of $\Sigma_{\{i\}}$ where $i \in N$. Then the relation $\sim$ on $\mathcal{E}(\Sigma)$ defined by $\sigma \sim \sigma'$ if and only if σ and σ' are incident and the trivial type function $t : \mathcal{E}(\Sigma) \to N$ give an incidence structure on $\mathcal{E}(\Sigma)$. Therefore $\mathcal{E}(\Sigma)$ is an incidence system. For any labelled simplicial morphism $\psi : \Sigma \to \Sigma'$, the map $\mathcal{E}(\psi) : \mathcal{E}(\Sigma) \to \mathcal{E}(\Sigma')$ defined by restricting ψ to $\mathcal{E}(\Sigma)$ will then preserve the incidence relation and types because $\partial_{I,\{i\}} \circ \psi_I = \psi_{\{i\}} \circ \partial_{I,\{i\}}$ for all $i \in I \subseteq N$. Thus $\mathcal{E} : \mathbf{SC}_N \to \mathbf{ISys}_N$ is a functor.

Example 3.5.6. Given a Coxeter group (W, S) with the Coxeter diagram $\mathscr{D}$, from example 3.5.4, $\mathsf{C}(W; W_{J:J\subseteq\mathscr{D}})$ is a labelled simplicial complex over $\mathscr{D}$, and so

$$\mathcal{E}(\mathsf{C}(W; W_{J:J\subseteq\mathscr{D}})) = \{wW_i \,|w \in W \text{ and } i \in \mathscr{D}\} = \mathsf{C}(W).$$

Example 3.5.7. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero with the Dynkin diagram $\mathscr{D}_{\mathfrak{g}}$. Then, from example 3.5.5, $\mathscr{P}(\mathfrak{g})$ is a labelled simplicial complex over $\mathscr{D}_{\mathfrak{g}}$ and so

$$\mathcal{E}(\mathscr{P}(\mathfrak{g})) = \{\text{maximal proper parabolic subalgebras of } \mathfrak{g}\} = \mathsf{Para}(\mathfrak{g}).$$

In general, $\mathcal{E} \circ \mathcal{F} : \mathbf{ISys}_N \to \mathbf{ISys}_N$ is naturally isomorphic to the identity $\mathrm{id}_{\mathbf{ISys}_N} : \mathbf{ISys}_N \to \mathbf{ISys}_N$ via the natural isomorphism $\varepsilon : \mathcal{E} \circ \mathcal{F} \to \mathrm{id}_{\mathbf{ISys}_N}$ such that

$$\begin{aligned} \varepsilon_{\mathsf{A}} : \mathcal{E} \circ \mathcal{F}(\mathsf{A}) &\to \mathsf{A} \\ \{a\} &\mapsto a, \end{aligned}$$

for all $\mathsf{A} \in \mathbf{ISys}_N$. On the other hand, there is a natural transformation $\eta : \mathrm{id}_{\mathbf{SC}_N} \to \mathcal{F} \circ \mathcal{E}$ such that

$$\begin{aligned} \eta_\Sigma : \Sigma &\to \mathcal{F} \circ \mathcal{E}(\Sigma) \\ \sigma &\mapsto \left\{\partial_{I,\{i\}}(\sigma) \,|\, i \in I\right\}, \end{aligned}$$

for all $\Sigma \in \mathbf{SC}_N$. One can show that $\mathcal{F}$ and $\mathcal{E}$ are respectively right and left adjoint functors, i.e., $\mathrm{Hom}_{\mathbf{SC}_N}(\Sigma, \mathcal{F}(\mathsf{A})) \cong \mathrm{Hom}_{\mathbf{ISys}_N}(\mathcal{E}(\Sigma), \mathsf{A})$, where $\mathsf{A} \in \mathbf{ISys}_\mathsf{N}$ and $\Sigma \in \mathbf{SC}_N$.

Definition 3.5.8. A labelled simplicial complex over N is called a **Coxeter complex** if it is isomorphic to $\mathcal{F}(\mathsf{C}(W))$ of some Coxeter group W; in particular, $\mathcal{F}(\mathsf{C}(W))$ itself is called a **standard Coxeter complex** associated to the Coxeter group W.

3.6 Buildings

In this sections we state some basic notions and facts about buildings; for more details, we refer to the books by [AB08], [Gar97], [Ron09], and [Tit81]. The approach that we will use to study buildings here is using graphs. So we will begin this section by introducing some terminologies about graphs, and then, at the end of this section, we will see that under certain conditions buildings and incidence systems are related.

In the following, a graph Δ is a set Δ equipped with a symmetric irreflexive relation $E(\Delta)$; so elements of Δ and $E(\Delta)$ are respectively nodes and edges of the graph Δ. The set $E(\Delta)$ may be considered as a collection of two element subsets of Δ.

Definition 3.6.1. Let Δ be an edge colored graph with index set N, i.e., Δ is a graph equipped with a surjective map $c : E(\Delta) \to N$, called an edge coloring. Let $a, b \in V(\Delta)$ and $I \subseteq N$. We will write

$$a \sim_i b$$

if $c(\{a, b\}) = i$. The vertices a and b are called i-**ADJACENT** if $a \sim_i b$ and I-**ADJACENT** if they are i-adjacent for some $i \in I$.

A **PATH** of length n from a to b is a sequence of $n+1$ vertices $v_0, v_1, v_2, \ldots, v_n$ such that

$$a = v_0 \sim_{i_1} v_1 \sim_{i_2} v_2 \sim_{i_3} \cdots \sim_{i_n} v_n = b,$$

for some $i_1, i_2, \ldots, i_n \in N$, and we will denote this path by

$$a \to_w b,$$

where $w := i_1 i_2 \cdots i_n$; the **TYPE** of the path is w. A I-**PATH** is a path whose type is a sequence of elements in I.

The **DISTANCE** from a to b, denoted by $\operatorname{dist}(a, b)$, is the length of a shortest path from a to b if there is a path form a to b, and ∞ otherwise. A **MINIMAL PATH** from a to b is a path whose length is $\operatorname{dist}(a, b)$.

The **DIAMETER** of Δ, denoted by $\operatorname{diam}(\Delta)$, is the supremum of the set

$$\left\{\operatorname{dist}\left(a', b'\right) \middle| a', b' \in \Delta\right\}.$$

a and b are said to be **OPPOSITE** if $\operatorname{dist}(a, b) = \operatorname{diam}(\Delta) < \infty$; so if $\operatorname{diam}(\Delta) = \infty$, there are no opposite elements.

Δ is said to be **CONNECTED** (resp. I-**CONNECTED**) if for any two vertices of Δ, there exists a path (resp. I-path) from a to b. A **CONNECTED COMPONENT** of Δ is the subgraph spanned by an equivalence class with respect to the equivalence relation that there exists a gallery from x to y in Δ.

An I-**RESIDUE** of Δ is a connected component of the subgraph of Δ obtained from Δ by removing all edges whose color is not in I, and they have **RANK** $|I|$. The J-**RESIDUE**

CONTAINING a is denoted by $[a]_J$; in particular $[a]_\emptyset$ is just a.

Definition 3.6.2. A **CHAMBER GRAPH** Δ over a set N is an edge-colored graph Δ with index set N such that for each $i \in N$, all $\{i\}$-residues, called i**-PANELS**, of Δ are complete graph with at least two vertices. We call the vertices of Δ **CHAMBERS**. A **SUB-CHAMBER GRAPH** of a chamber graph Δ is a subgraph of Δ which is also a chamber graph.

A chamber graph is **THIN** if every panel contains exactly two chambers, and **THICK** if every panel contains at least three chambers.

Remark 3.6.3. The term "chamber graph" is non-standard in context of buildings. Ronan ([Ron09]) defines a chamber system over N as a set equipped with equivalence relations $\sim_i$, one for each $i \in N$. However, a chamber system can be viewed as an edge-colored graph over N with a loop on every vertex; its maximal subgraph without loops is a chamber graph. Under the convention that there is a loop on every vertex of a chamber graph which is not drawn, we will use chamber graphs in place of chamber systems. The "paths" in a chamber graph we use here correspond to the "non-stuttering galleries" of a chamber system.

Example 3.6.4. Let (W, S) be a Coxeter group. Define an irreflexive symmetric relation $E(W)$ on W by

$$E(W) := \left\{ \left\{ w, w' \right\} \subseteq W \,\middle|\, w' = ws \text{ for some } s \in S \right\}.$$

Then W is a graph. Now define the edge coloring

$$\begin{aligned} c : E(W) &\rightarrow S \\ \left\{ w, w' \right\} &\mapsto w^{-1}w'; \end{aligned}$$

this is well-defined, i.e., $w^{-1}w' = (w')^{-1} w$, because $s^2 = 1$ for all $s \in S$. For any $s \in S$ and $w \in W$, the set of chambers in the $\{s\}$-panel $[w]_{\{s\}}$ is $\{w, ws\}$. Therefore W is a thin chamber graph over S. We call W a **COXETER CHAMBER GRAPH**.

Remark 3.6.5. If W is finite and w_0 is the longest element of W, then w_0 sends any chamber w of W to its opposite chamber ww_0. Moreover, Proposition 2.1.6 implies that any $w \in W$ is on a minimal path from 1 to w_0.

Example 3.6.6. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ with the Dynkin diagram $\mathscr{D}_{\mathfrak{g}}$. Recall form Section 3.4, $\mathscr{P}_I(\mathfrak{g})$ is an adjoint orbit of parabolic subalgebras corresponding to the subset I (of crossed nodes) of $\mathscr{D}_{\mathfrak{g}}$. Let $\mathscr{B}(\mathfrak{g})$ is the set of all Borel subalgebras of $\mathfrak{g}$. Then the relation $E(\mathscr{B}(\mathfrak{g}))$, consisting of the collection of all distinct pairs of Borel subalgebras $\{\mathfrak{b}, \mathfrak{b}'\}$ having a subminimal parabolic subalgebra $\mathfrak{p}_i \in \mathscr{P}_{\mathscr{D}_{\mathfrak{g}}\setminus\{i\}}(\mathfrak{g})$, i.e. the minimal parabolic subalgebra which is not a Borel subalgebra, for some $i \in \mathscr{D}_{\mathfrak{g}}$, such that $\mathfrak{b} \cup \mathfrak{b}' \subseteq \mathfrak{p}_{\mathfrak{i}}$, is a symmetric irreflexive relation on $\mathscr{B}(\mathfrak{g})$. Thus $\mathscr{B}(\mathfrak{g})$ is an edge-colored graph.

For any $\mathfrak{b} \in \mathscr{B}(\mathfrak{g})$ and $i \in \mathscr{D}_{\mathfrak{g}}$, the chambers in the $\{i\}$-panel $[\mathfrak{b}]_{\{i\}}$ are all Borel subalgebras contained in the subminimal parabolic subalgebra $\mathfrak{p}_i \in \mathscr{P}_{\mathscr{D}_{\mathfrak{g}}\setminus\{i\}}(\mathfrak{g})$ containing $\mathfrak{b}$. Hence $[\mathfrak{b}]_{\{i\}}$ has at least two elements, otherwise $\mathfrak{p}_i = \mathfrak{b}$. Therefore $\mathscr{B}(\mathfrak{g})$ is a chamber graph over $\mathscr{D}_{\mathfrak{g}}$.

Definition 3.6.7. A CHAMBER GRAPH MORPHISM $\psi : \Delta \to \Delta'$ of chamber systems over N is a map from Δ to Δ' that preserves i-adjacence for all $i \in N$, i.e., if $a \sim_i b$ in Δ then $\psi(a) = \psi(b)$ or $\psi(a) \sim_i \psi(b)$ in Δ'.

Chamber systems over N and their chamber graph morphisms form a category $\mathbf{CG}_N$. Next we will show that there is a connection between labelled simplicial complexes and chamber systems.

Assume that Σ is a labelled simplicial complex over N. Let $\mathcal{M}(\Sigma)$ denote the set of all chambers of Σ. For each $i \in N$, define a relation $\sim_i$ on $\mathcal{M}(\Sigma)$ by

$$\sigma \sim_i \sigma' \Leftrightarrow \partial_{N,N\setminus\{i\}}(\sigma) = \partial_{N,N\setminus\{i\}}(\sigma');$$

clearly $\sim_i$ is an equivalence relation. Therefore an edge-colored graph, all $\{i\}$-residues of $\mathcal{M}(\Sigma)$ are complete. Hence $\mathcal{M}(\Sigma)$ is a chamber graph over N. For any labelled simplicial morphism $\psi : \Sigma \to \Sigma'$, the map $\mathcal{M}(\psi) : \mathcal{M}(\Sigma) \to \mathcal{M}(\Sigma')$ defined by restricting ψ to $\mathcal{M}(\Sigma)$ is then preserve the incidence and labels because $\partial_{I,\{i\}} \circ \psi_I = \psi_{\{i\}} \circ \partial_{I,\{i\}}$ for all $i \in I \subseteq N$. Thus $\mathcal{M} : \mathbf{SC}_N \to \mathbf{CG}_N$ is a functor.

Example 3.6.8. Given a Coxeter group (W, S) with the Coxeter diagram $\mathscr{D}$, from example 3.5.4, $\mathsf{C}(W; W_{J:J\subseteq\mathscr{D}})$ is a labelled simplicial complex over $\mathscr{D}$, and so

$$\mathcal{M}(\mathsf{C}(W; W_{J:J\subseteq\mathscr{D}})) = \{\{w\} \,|\, w \in W\} \cong W.$$

Example 3.6.9. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero with the Dynkin diagram $\mathscr{D}_{\mathfrak{g}}$. Then, from example 3.5.5, $\mathscr{P}(\mathfrak{g})$ is a labelled simplicial complex over $\mathscr{D}_{\mathfrak{g}}$ and so

$$\mathcal{M}(\mathscr{P}(\mathfrak{g})) = \{\text{Borel subalgebras of } \mathfrak{g}\} = \mathscr{B}(\mathfrak{g}).$$

Conversely, let $\mathcal{C}$ be a chamber graph over N. Define a labelled simplicial complex $\mathcal{S}(\mathcal{C})$ over N by letting $\mathcal{S}(\mathcal{C})$ be the disjoint union of collections $\mathcal{S}(\mathcal{C})_J$ of J-residues of the labelled graph $\mathcal{C}$ where $J \subseteq N$, and for any $I \subseteq J \subseteq N$, defining $\partial_{J,I} : \mathcal{S}(\mathcal{C})_J \to \mathcal{S}(\mathcal{C})_I$ mapping $[c]_J$ to $[c]_I$. For any chamber morphism $\psi : \mathcal{C} \to \mathcal{C}'$, the map

$$\mathcal{S}(\psi) : \mathcal{S}(\mathcal{C}) \to \mathcal{S}(\mathcal{C}'),$$

defined by $\mathcal{S}(\psi)(\sigma) = \psi(\sigma)$ is then satisfied the condition $\partial_{J,I} \circ \psi_J = \psi_I \circ \partial_{J,I}$ for all $I \subseteq J \subseteq N$. Thus $\mathcal{S} : \mathbf{CG}_N \to \mathbf{SC}_N$ is a functor.

Example 3.6.10. Given a Coxeter group (W, S) with the Coxeter diagram $\mathscr{D}$, from example 3.6.8, W is a chamber graph over $\mathscr{D}$, and so

$$\mathcal{S}(W) = \{wW_I \,|w \in W \text{ and } I \subseteq \mathscr{D}\} = \mathsf{C}(W; W_{J:J\subseteq\mathscr{D}}).$$

Example 3.6.11. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero with the Dynkin diagram $\mathscr{D}_{\mathfrak{g}}$. Then, from example 3.5.5, $\mathscr{B}(\mathfrak{g})$ is a chamber graph over $\mathscr{D}_{\mathfrak{g}}$ and so

$$\mathcal{S}(\mathscr{B}(\mathfrak{g})) \cong \mathscr{P}(\mathfrak{g}).$$

In general, $\mathcal{M} \circ \mathcal{S} : \mathbf{CG}_N \to \mathbf{CG}_N$ is naturally isomorphic to the identity $\mathrm{id}_{\mathbf{CG}_N} : \mathbf{CG}_N \to \mathbf{CG}_N$ via the natural isomorphism $\eta' : \mathrm{id}_{\mathbf{CG}_N} \to \mathcal{M} \circ \mathcal{S}$ such that

$$\begin{aligned} \eta'_\Delta : \Delta &\to \mathcal{M} \circ \mathcal{S}(\Delta) \\ a &\mapsto \{a\}, \end{aligned}$$

for all $\Delta \in \mathbf{CG}_N$. On the other hand, there is a natural transformation $\varepsilon' : \mathcal{S} \circ \mathcal{M} \to \mathrm{id}_{\mathbf{SC}_N}$

such that

$$\begin{array}{rcl}\varepsilon'_{\Sigma} : \mathcal{S} \circ \mathcal{M}(\Sigma) & \rightarrow & \Sigma \\ [\sigma]_{N \setminus J} & \mapsto & \partial_{N,J}(\sigma),\end{array}$$

for all $\Sigma \in \mathbf{SC}_N$. One can show that $\mathcal{M}$ and $\mathcal{S}$ are respectively right and left adjoint functors, i.e., $\mathrm{Hom}_{\mathbf{SC}_N}(\Sigma, \mathcal{S}(\Delta)) \cong \mathrm{Hom}_{\mathbf{ISys}_N}(\mathcal{M}(\Sigma), \Delta)$, where $\Delta \in \mathbf{CG}_N$ and $\Sigma \in \mathbf{SC}_N$.

Definition 3.6.12. A BUILDING of type (W, S), where (W, S) is a Coxeter group, is a chamber graph Δ over S such that:

(B1) every panel of Δ contains at least two chambers;

(B2) Δ has a W-valued metric $\delta : \Delta \times \Delta \rightarrow W$ such that if $w = s_1 s_2 \cdots s_k$ is a reduced expression of w in W then

$$\delta(c, c') = w \Leftrightarrow \text{there is a path } c \rightarrow_w c' \text{ in } \mathcal{C}.$$

If (W, S) is a finite Coxeter group, then we call Δ a SPHERICAL building.

Lemma 3.6.13. *For any Coxeter group* (W, S)*,*

$$\mathcal{M}(\mathcal{F}(\mathsf{C}(W))) \cong W$$

is a building of type (W, S)*.*

Proof. Let (W, S) be a Coxeter group. Then W is a chamber graph over S. In Example 3.6.4, we have seen that the panels of W are in the form $\{w, ws\}$, where $s \in S$ and $w \in W$; whence **(B1)** is satisfied. Define

$$\begin{array}{rcl}\delta_W : W \times W & \rightarrow & W \\ (w, w') & \mapsto & w^{-1}w'.\end{array}$$

Then, for any $w, w' \in W$, we have

$$\begin{aligned} \delta_W\left(w, w'\right) = w'' &\Leftrightarrow w^{-1}w' = w'' \\ &\Leftrightarrow w' = ww'' \\ &\Leftrightarrow \text{there is a gallery } w \rightarrow_{w''} w'. \end{aligned}$$

In particular, if w'' is reduced, then $w \rightarrow_{w''} w'$ is the minimal gallery. Thus **(B2)** is satisfied. □

Lemma 3.6.14. *Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field of characteristic zero with the Dynkin diagram $\mathscr{D}_{\mathfrak{g}}$. The chamber graph $\mathscr{B}(\mathfrak{g})$ over $\mathscr{D}_{\mathfrak{g}}$ is a building of type (W, S), where (W, S) is the Coxeter group whose Coxeter diagram is the underlying diagram of $\mathscr{D}_{\mathfrak{g}}$.*

Proof. From Example 3.6.6, we have seen that **(B1)** is satisfied. Let G be an algebraic group over $\mathbb{F}$ with Lie algebra $\mathfrak{g}$. Since G acts transitively on $\mathscr{B}(\mathfrak{g})$, hence

$$\mathscr{B}(\mathfrak{g}) \cong G/B$$

for some Borel subgroup B with Lie algebra $\mathfrak{b} \in \mathscr{B}(\mathfrak{g})$; each coset $gB \in G/B$ corresponds to the Borel subalgebra $g \cdot \mathfrak{b}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$. Then $\mathfrak{t}$ and $\mathfrak{b}$ defines a specific isomorphism from W to $N_G(T)/T$, where T is the maximal torus of G with Lie algebra $\mathfrak{t}$. This induces a right action of W on G/B. Define

$$\begin{aligned} \delta : G/B \times G/B &\rightarrow W \\ \left(gB, g'B\right) &\mapsto w, \end{aligned}$$

such that $g^{-1}g' \in BwB$.

Let $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ be a reduced expression of w in W. If $\delta(gB, g'B) = w$, then $g' = gbs_{i_1}s_{i_2}\cdots s_{i_k}b'$ for some $b, b' \in B$. For any $1 \leq j \leq k$, $B\left\langle s_{i_j}\right\rangle B$ is a minimal parabolic subgroup of G (see [Hum75]), denoted by P_{i_j}, because it is a subgroup of G containing B. Hence there is a gallery

$$g \cdot \mathfrak{b} \sim_{i_1} (gbs_{i_1}) \cdot \mathfrak{b} \sim_{i_2} (gbs_{i_1}s_{i_2}) \cdot \mathfrak{b} \cdots \left(gbs_{i_1}s_{i_2}\cdots s_{i_{k-1}}\right) \cdot \mathfrak{b} \sim_{i_k} (gbs_{i_1}s_{i_2}\cdots s_{i_k}) \cdot \mathfrak{b} = g' \cdot \mathfrak{b}$$

such that $\sim_{i_j}$ is the equivalence relation defined by being contained in the minimal proper parabolic subalgebra $\left(gbs_{i_1}\cdots s_{i_j}\right)\cdot\mathfrak{p}_{i_j}$, where $\mathfrak{p}_{i_j}$ is the Lie algebra of P_{i_j}.

Conversely, if $gB \to_w g'B$, then $g'B = gBw$, and hence $g^{-1}g' \in BwB$. Thus **(B2)** is satisfied. □

Definition 3.6.15. An APARTMENT of a building Δ of type (W, S) is an isometric image, i.e. the image of a chamber graph morphism $\psi : W \to \Delta$ such that $\delta_W(a, b) = \delta(\psi(a), \psi(b))$, of the Coxeter chamber W in Δ.

Proposition 3.6.16. *For any two chambers a and b in a building Δ, there is an apartment of Δ containing both a and b.*

Proof. See [AB08], Corollary 5.74. □

Proposition 3.6.17. *If A is a chamber subgraph of a building Δ of type (W, S) and isomorphic to W, then A is an apartment of W.*

Proof. See [AB08], Proposition 4.59. □

Example 3.6.18. Given a Coxeter group (W, S), there is just one apartment in W which is W itself.

Example 3.6.19. Let G be a connected semisimple algebraic group over an algebraically closed field of characteristic zero with Lie algebra $\mathfrak{g}$. Let $\mathscr{D}_{\mathfrak{g}}$ be the Dynkin diagram of $\mathfrak{g}$. By Lemma 3.6.14, $\mathscr{B}(\mathfrak{g})$ is a building of type (W, S), where W is isomorphic to the Weyl group of G. Let T be a maximal torus of G with Lie algebra $\mathfrak{t}$. By choosing a Borel subalgebra $\mathfrak{b} \in \mathscr{B}(\mathfrak{g})$, there is an isomorphism from W to $N_G(T)/T$ determined by $\mathfrak{t}$ and $\mathfrak{b}$. Since $N_G(T)$ acts transitively on $A_{\mathfrak{t}} := \{\mathfrak{b} \in \mathscr{B}(\mathfrak{g}) \,|\mathfrak{t} \subseteq \mathfrak{b}\}$ with the stabilizer T, the set $A_{\mathfrak{t}} \cong N_G(T)/T$ is an apartment of $\mathscr{B}(\mathfrak{g})$.

Theorem 3.6.20. (*Convexity of apartments*) *In a building Δ, let A be an apartment containing a and b. Then any minimal gallery in Δ from a to b lies inside A.*

Proof. See [Tha11], Proposition 4.3, and [AB08], Proposition 4.40. □

We will use this convexity result to prove an important fact in the next section.

3.7 Parabolic configurations

Definition 3.7.1. Let $\mathsf{C}(W)$ be the Coxeter incidence geometry for a Coxeter group W with the Coxeter diagram $\mathscr{D}$ and $\mathsf{Para}(\mathfrak{g})$ be the parabolic incidence geometry for a semisimple Lie algebra $\mathfrak{g}$ with the Dynkin diagram $\mathscr{D}_{\mathfrak{g}}$. An incidence system morphism

$$\Psi : \mathsf{C}(W) \to \mathsf{Para}(\mathfrak{g}),$$

over a map $\nu : \mathscr{D}_{\mathfrak{g}} \to \mathscr{D}$ is called a **PARABOLIC CONFIGURATION**.

Denote $\mathrm{Mor}_{\nu}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g}))$ the set of all parabolic configurations over ν. In particular, if $\mathscr{D}_{\mathfrak{g}} = \mathscr{D}$ and ν is the identity map, then we may write $\mathrm{Mor}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g}))$ instead of $\mathrm{Mor}_{\mathrm{id}}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g}))$.

Let G be a connected algebraic group over an algebraically closed field $\mathbb{F}$ of characteristic zero with the Lie algebra $\mathfrak{g}$, and let Q be a parabolic subgroup of G with the Lie algebra $\mathfrak{q}$.

Notation 3.7.2. Denote by $\mathscr{U}$ the set of all pairs $(\mathfrak{t}, \mathfrak{b})$ where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$. Let $\mathscr{T}$ be the set of all Cartan subalgebras of $\mathfrak{g}$ and

$$\begin{aligned} \pi_1 : \mathscr{U} &\to \mathscr{T} \\ (\mathfrak{t}, \mathfrak{b}) &\mapsto \mathfrak{t} \end{aligned}$$

be the first projection map.

Since G acts on the set of Cartan subalgebras and the set of Borel subalgebras by the adjoint action, G also acts on $\mathscr{U}$,

$$\begin{aligned} G \times \mathscr{U} &\to \mathscr{U} \\ (g, (\mathfrak{t}, \mathfrak{b})) &\mapsto (g \cdot \mathfrak{t}, g \cdot \mathfrak{b}). \end{aligned}$$

Lemma 3.7.3. *G acts transitively on $\mathscr{U}$. Moreover,*

$$Stab_G((\mathfrak{t}, \mathfrak{b})) := \{g \in G \,|\, g \cdot (\mathfrak{t}, \mathfrak{b}) = (\mathfrak{t}, \mathfrak{b})\} = T,$$

where $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}$ and T is the maximal torus in G with the associated Lie algebra $\mathfrak{t}$.

Proof. Let $(\mathfrak{t}_1, \mathfrak{b}_1)$ and $(\mathfrak{t}_2, \mathfrak{b}_2)$ be in $\mathscr{U}$. Then there exists $g \in G$ such that $\mathfrak{t}_1 = g \cdot \mathfrak{t}_2$. So $\mathfrak{t}_1 = g' \cdot (g \cdot \mathfrak{b}_2)$ and $\mathfrak{b}_1 = g' \cdot (g \cdot \mathfrak{b}_2)$ for some $g' \in N_G(T_1)$ where T_1 is a maximal torus in G with the associated Lie algebra $\mathfrak{t}_1$. Therefore $(\mathfrak{t}_1, \mathfrak{b}_1) = g'g \cdot (\mathfrak{t}_2, \mathfrak{b}_2)$. Moreover

$$\begin{aligned} g \cdot (\mathfrak{t}_1, \mathfrak{b}_1) = (\mathfrak{t}_1, \mathfrak{b}_1) \quad &\Leftrightarrow \quad g \cdot \mathfrak{t}_1 = \mathfrak{t}_1 \text{ and } g \cdot \mathfrak{b}_1 = \mathfrak{b}_1 \\ &\Leftrightarrow \quad g \in N_G(T_1) \cap B_1 = T_1, \end{aligned}$$

where B_1 is the Borel subgroup of G with the associated Lie algebra $\mathfrak{b}_1$. □

Thus, by the Orbit Stabilizer Theorem, a choice of $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}$ determines an isomorphism $\mathscr{U} \cong G/T$, where T is a maximal torus of G with the Lie algebra $\mathfrak{t}$. Moreover $\mathscr{T}$ can be parametrized by the quotient $G/N_G(T)$ because G acts transitively on $\mathscr{T}$ by conjugation. Therefore π_1 can be identified with the quotient map

$$\begin{aligned} \pi_1 : G/T \quad &\to \quad G/N_G(T) \\ gT \quad &\mapsto \quad gN_G(T), \end{aligned}$$

and each fibre can be identified with the Weyl group $N_G(T)/T$ of G.

Let $(\mathfrak{t}_0, \mathfrak{b}_0) \in \mathscr{U}$ and T_0 be the maximal torus of G with the associated Lie algebra $\mathfrak{t}_0$. Denote $W = N_G(T_0)/T_0$. According to Theorem 8.28 in [Spr98], $\mathfrak{b}_0$ turns the Weyl group W to a Coxeter group with Coxeter diagram $\mathscr{D}_{\mathfrak{g}}$.

Proposition 3.7.4. *For any $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}$, there is a well-defined strict incidence system morphism*

$$\begin{aligned} \Upsilon_{(\mathfrak{t},\mathfrak{b})} : \mathsf{C}(W) \quad &\to \quad \mathsf{Para}(\mathfrak{g}) \\ \overline{w}W_i \quad &\mapsto \quad gw \cdot \mathfrak{p}_i, \end{aligned} \tag{3.7.1}$$

where $(\mathfrak{t}, \mathfrak{b}) = g \cdot (\mathfrak{t}_0, \mathfrak{b}_0)$ for some $g \in G$, where $\mathfrak{p}_i \in \mathscr{P}_i(\mathfrak{g})$ is the one containing $\mathfrak{b}_0$, and where $w \in N_G(T_0)$ is a coset representative of $\overline{w}$.

Proof. Let $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}$. For any $w, w' \in N_G(T_0)$ such that $w^{-1}w' \in W_i$, we have

$$gw' \cdot \mathfrak{p}_i = gww^{-1}w' \cdot \mathfrak{p}_i = gw \cdot \mathfrak{p}_i,$$

because $w^{-1}w' \cdot \mathfrak{p}_i = \mathfrak{p}_i$. Moreover, for any $g' \in G$ such that $g' \cdot (\mathfrak{t}_0, \mathfrak{b}_0) = (\mathfrak{t}, \mathfrak{b}) = g \cdot (\mathfrak{t}_0, \mathfrak{b}_0)$, we have $g^{-1}g' \in T_0$, and so

$$g'w \cdot \mathfrak{p}_i = gww^{-1}g^{-1}g'w \cdot \mathfrak{p}_i = gw \cdot \mathfrak{p}_i,$$

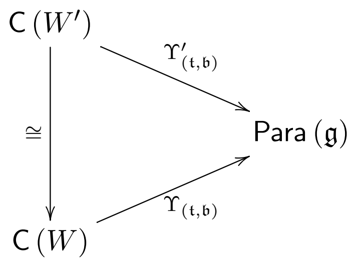
because $w^{-1}g^{-1}gw \in T_0$. Thus $\Upsilon_{(\mathfrak{t},\mathfrak{b})}$ is well-defined.

From (3.7.1), the map $\Upsilon_{(\mathfrak{t},\mathfrak{b})}$ is automatically preserves types. For any $i, j \in \mathscr{D}_{\mathfrak{g}}$, if $\overline{w}_i W_i \cap \overline{w}_j W_j \neq \emptyset$, then

$$\Upsilon_{(\mathfrak{t},\mathfrak{b})}\left(\overline{w}_i W_i\right) \cap \Upsilon_{(\mathfrak{t},\mathfrak{b})}\left(\overline{w}_j W_j\right) = gw \cdot \mathfrak{p}_i \cap gw \cdot \mathfrak{p}_j \supseteq gw \cdot \mathfrak{b}_0,$$

for some $w \in \overline{w}_i W_i \cap \overline{w}_j W_j$, is a parabolic subalgebra. Therefore $\Upsilon_{(\mathfrak{t},\mathfrak{b})}$ is a strict incidence system morphism. □

Remark 3.7.5. Since G acts transitively on $\mathscr{U}$, W is defined up to canonical isomorphism and so is $\mathsf{C}(W)$. Furthermore, if we choose a different base point $(\mathfrak{t}_0', \mathfrak{b}_0') \in \mathscr{U}$, then the diagram



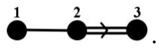
commutes, where $W' = N_G\left(T_0'\right)/T_0'$ and T_0' is a maximal torus of G with Lie algebra $\mathfrak{t}_0'$. In this way, Υ does not depend essentially on the pair $(\mathfrak{t}_0, \mathfrak{b}_0)$.

Definition 3.7.6. Let

$$\begin{aligned} \Upsilon : \mathscr{U} &\rightarrow \operatorname{Mor}\left(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g})\right) \\ (\mathfrak{t}, \mathfrak{b}) &\mapsto \Upsilon_{(\mathfrak{t},\mathfrak{b})}, \end{aligned} \tag{3.7.2}$$

where $\Upsilon_{(\mathfrak{t},\mathfrak{b})}$ is defined in (3.7.1). We call each $\Upsilon_{(\mathfrak{t},\mathfrak{b})}$, where $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}$, a STANDARD PARABOLIC CONFIGURATION for $\mathfrak{g}$.

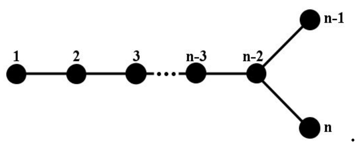
Example 3.7.7. Let V be a complex vector space of dimension 7. The group $\mathfrak{so}(V)$ has the Dynkin diagram $\mathscr{D}$ given by



Let $\{e_1, e_2, \ldots, e_7\}$ be a basis of V and W be the Weyl group of $\mathfrak{so}(V)$. The basis turns W into a Coxeter group. The set $\mathsf{C}(W)_1$, $\mathsf{C}(W)_2$, and $\mathsf{C}(W)_3$ can be identified with the orbits of fundamental weights λ_1, λ_2, and λ_3 corresponding to the nodes labelled by 1, 2, and 3 respectively; whence each element in $\mathsf{C}(W)$ can be assigned by the standard parabolic configuration with respect to the basis $\{e_1, e_2, \ldots, e_7\}$ to the parabolic subalgebra of $\mathfrak{so}(V)$ stabilizing its corresponding weight.

Geometrically, the set $\mathsf{C}(W)_1$, $\mathsf{C}(W)_2$, and $\mathsf{C}(W)_3$ are mapped into $\mathsf{Para}(\mathfrak{so}(V))_1$, $\mathsf{Para}(\mathfrak{so}(V))_2$, and $\mathsf{Para}(\mathfrak{so}(V))_3$, which are the sets of orthogonal Grassmannians of isotropic lines, 2-planes, and 3-planes respectively in V. The image of the standard configuration is then an octahedron lying in $Q^5 \subseteq \mathbb{P}(V)$ such that the set $\mathsf{C}(W)_1$, $\mathsf{C}(W)_2$, and $\mathsf{C}(W)_3$ correspond to the collection of points, collection of lines, and the collection of planes of the octahedron.

Example 3.7.8. Let V be a complex vector space of dimension $2n$. The group $\mathfrak{so}(V)$ has the Dynkin diagram $\mathscr{D}$ given by



Let $\{e_1, e_2, \ldots, e_{2n}\}$ be a basis of V and W be the Weyl group of $\mathfrak{so}(V)$. The basis turns W into a Coxeter group. The set $\mathsf{C}(W)_i$ can be identified with the orbits of fundamental weights λ_i corresponding to the nodes labelled by i for $1 \leq i \leq n$; whence each each element in $\mathsf{C}(W)$ can be assigned by the standard parabolic configuration with respect to the basis $\{e_1, e_2, \ldots, e_{2n}\}$ to the parabolic subalgebra of $\mathfrak{so}(V)$ stabilizing its corresponding weight.

Geometrically, the set $\mathsf{C}(W)_i$, for $1 \leq i \leq n-3$, is mapped into $\mathsf{Para}(\mathfrak{so}(V))_i$, which is the Grassmannian of isotropic i-planes in V. While the set $\mathsf{C}(W)_{n-1}$ and $\mathsf{C}(W)_n$ are mapped respectively into $\mathsf{Para}(\mathfrak{so}(V))_{n-1}$ and $\mathsf{Para}(\mathfrak{so}(V))_n$ which are two connected components of the Grassmannian of isotropic n-planes.

Lemma 3.7.9. *For any* $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}$*, the strict incidence morphism* $\Upsilon_{(\mathfrak{t},\mathfrak{b})} : \mathsf{C}(W) \to \mathsf{Para}(\mathfrak{g})$ *is injective.*

Proof. Let $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}$. Then there exists $g \in G$ such that $(\mathfrak{t}, \mathfrak{b}) = g \cdot (\mathfrak{t}_0, \mathfrak{b}_0)$. Since $\Upsilon_{(\mathfrak{t},\mathfrak{b})}$ preserves types, it suffices to show that $\left(\Upsilon_{(\mathfrak{t},\mathfrak{b})}\right)_i$ is injective. Suppose that $\Upsilon_{(\mathfrak{t},\mathfrak{b})}(\overline{w}_1) = \Upsilon_{(\mathfrak{t},\mathfrak{b})}(\overline{w}_2)$. Then $gw_1 \cdot \mathfrak{p}_i = gw_2 \cdot \mathfrak{p}_i$, and so $w_1^{-1}w_2 \in P_i$. We have $w_1^{-1}w_2 \in N_{P_i}(T_0)$. Therefore $\overline{w}_1 W_i = \overline{w}_2 W_i$. □

Consider a G-action on $\mathrm{Mor}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g}))$ given by $(g, \psi) \mapsto g \cdot \psi := \mathrm{Ad}(g) \circ \psi$. We will show that the map Υ is G-equivariant.

Lemma 3.7.10. Υ *is equivariant under the G-action.*

Proof. By Lemma 3.7.3, it suffices to show that, for any $g \in G$, $g \cdot \Upsilon_{(\mathfrak{t}_0,\mathfrak{b}_0)} = \Upsilon_{g\cdot(\mathfrak{t}_0,\mathfrak{b}_0)}$. For any $i \in \mathscr{D}_{\mathfrak{g}}$ and $\overline{w} \in W(G, T_0)$,

$$\begin{aligned} \Upsilon_{g\cdot(\mathfrak{t}_0,\mathfrak{b}_0)}(\overline{w}W_i) &= gw \cdot \mathfrak{p}_i \\ &= g \cdot (w \cdot \mathfrak{p}_i) \\ &= g \cdot \Psi_{(\mathfrak{t}_0,\mathfrak{b}_0)}(\overline{w}W_i), \end{aligned}$$

where $\mathfrak{t}_0 \subseteq \mathfrak{b}_0 \subseteq \mathfrak{p}_i \in \mathscr{P}_i(\mathfrak{g})$ and $w \in N_G(T_0)$ is a coset representative of $\overline{w}$. □

Proposition 3.7.11. *The map $\Upsilon : \mathscr{U} \to Mor(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g}))$ is injective.*

Proof. By Lemma 3.7.10, it suffices to show that, for any $g \in G$, if $\Upsilon_{(\mathfrak{t}_0,\mathfrak{b}_0)} = g \cdot \Upsilon_{(\mathfrak{t}_0,\mathfrak{b}_0)}$, then $(\mathfrak{t}_0, \mathfrak{b}_0) = g \cdot (\mathfrak{t}_0, \mathfrak{b}_0)$. Let $g \in G$ be such that $\Upsilon_{(\mathfrak{t}_0,\mathfrak{b}_0)} = g \cdot \Upsilon_{(\mathfrak{t}_0,\mathfrak{b}_0)}$. Then we have

$$w \cdot \mathfrak{p}_i = gw \cdot \mathfrak{p}_i,$$

for all $w \in N_G(T_0)$ and $i \in \mathscr{D}_{\mathfrak{g}}$. This implies that

$$g \in \bigcap_{w \in N_G(T_0)} \left(\bigcap_{i \in \mathscr{D}_{\mathfrak{g}}} \mathfrak{p}_i \right) = \bigcap_{w \in N_G(T_0)} \mathfrak{b}_0 = \mathfrak{t}_0.$$

Therefore $(\mathfrak{t}_0, \mathfrak{b}_0) = g \cdot (\mathfrak{t}_0, \mathfrak{b}_0)$. □

Proposition 3.7.12. *For any incidence system morphism $\psi : \mathsf{C}(W) \to \mathsf{Para}(\mathfrak{g})$, if ψ is injective, then there exists $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}$ such that $\psi = \Upsilon_{(\mathfrak{t},\mathfrak{b})}$.*

Proof. Let $\psi \in \mathrm{Mor}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g}))$ be such that it is injective. Since $\mathcal{M}$ and $\mathcal{F}$ are functors (see Section 3.5 and Section 3.6),

$$\mathcal{M} \circ \mathcal{F}(\psi) : \mathcal{M} \circ \mathcal{F}(\mathsf{C}(W)) \to \mathcal{M} \circ \mathcal{F}(\mathsf{Para}(\mathfrak{g})) = \mathscr{B}(\mathfrak{g})$$

is a chamber graph morphism. Furthermore $\mathcal{M} \circ \mathcal{F}(\psi)$ is an injective chamber graph morphism because ψ is injective.

Since $\mathcal{M} \circ \mathcal{F}(\mathsf{C}(W)) \cong W$ and $\mathcal{M} \circ \mathcal{F}(\psi)$ is injective, by Proposition 3.6.17, $\mathcal{M} \circ \mathcal{F}(\psi)(\mathcal{M} \circ \mathcal{F}(\mathsf{C}(W)))$ is an apartment of $\mathscr{B}(\mathfrak{g})$. Let w_0 be the longest element of W and denote $\mathcal{M} \circ \mathcal{F}(\psi)(\{1\})$ (resp. $\mathcal{M} \circ \mathcal{F}(\psi)(\{w_0\})$) by $\mathfrak{b}$ (resp. $\mathfrak{b}'$). By Theorem 3.6.20,

$$\mathrm{dist}\left(\mathfrak{b}, \mathfrak{b}'\right) = \ell(w_0),$$

where $\ell(w_0)$ is the length of w_0. Hence $\mathfrak{b}$ and $\mathfrak{b}'$ are opposite Borel subalgebras. Let $\mathfrak{t} = \mathfrak{b} \cap \mathfrak{b}'$. Then $\mathfrak{t}$ is a Cartan subalgebra. By Remark 3.6.5, any $\mathcal{M} \circ \mathcal{F}(\psi)(\{w\})$, where $w \in W$, is on a minimal path from $\mathfrak{b}$ to $\mathfrak{b}'$; whence $\mathcal{M} \circ \mathcal{F}(\psi)(\{w\})$ is in $A_{\mathfrak{t}}$ (as defined in Example 3.6.19) by Theorem 3.6.20. Therefore $\psi(\mathsf{C}(W)) = \Upsilon_{(\mathfrak{t},\mathfrak{b})}(\mathsf{C}(W))$. Since ψ is an incidence system morphism, $\mathfrak{t} \subseteq \mathcal{M} \circ \mathcal{F}(\psi)(\{w\}) \subseteq \psi(wW_i) \in \mathscr{P}_i(\mathfrak{g})$ for all $i \in \mathscr{D}_{\mathfrak{g}}$, and so $\psi = \Upsilon_{(\mathfrak{t},\mathfrak{b})}$. □

Theorem 3.7.13. *Let* $\mathrm{Mor}^{\mathrm{inj}}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g})) := \{\Upsilon : \mathsf{C}(W) \to \mathsf{Para}(\mathfrak{g}) \text{ is injective}\}$. Then

$$\mathscr{U} \cong Mor^{inj}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g})).$$

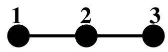
Proof. This follows from Lemma 3.7.9, Proposition 3.7.11 and Proposition 3.7.12. □

Remark 3.7.14. By Proposition 2.3.37, $\mathsf{Para}_i(\mathfrak{g})$ is a projective variety, for all $i \in \mathscr{D}_{\mathfrak{g}}$; whence $\prod_{X \in \mathsf{C}(W)} \mathsf{Para}_{t^c(X)}(\mathfrak{g})$ is a projective variety (e.g. [Har92], Example2.21). Notice that

$$\mathrm{Mor}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g})) \subseteq \prod_{X \in \mathsf{C}(W)} \mathsf{Para}_{t^c(X)}(\mathfrak{g}).$$

As the incidence relation in $\mathrm{Mor}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g}))$ is a closed condition, $\mathrm{Mor}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g}))$ is a closed subvariety of $\prod_{X \in \mathsf{C}(W)} \mathsf{Para}_{t^c(X)}(\mathfrak{g})$. Therefore $\mathrm{Mor}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g}))$ is a projective variety. Furthermore, since $\mathscr{U} \cong G/T$, we have that $\mathrm{Mor}^{\mathrm{inj}}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g}))$ is an open irreducible subvariety by Theorem 3.7.13; however it is not necessarily dense.

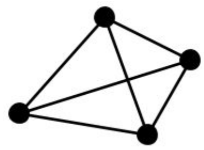
Example 3.7.15. Let $G = \mathrm{SL}_4(\mathbb{C})$. Then its Lie algebra is $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ and the Dynkin diagram $\mathscr{D}_{\mathfrak{g}}$ corresponding to $\mathfrak{g}$ is



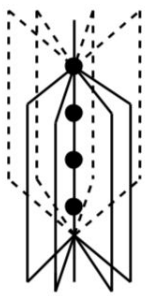
The incidence system $\mathsf{C}(W)$ has the incidence structure of an abstract 3-simplex. Then

$$\mathsf{Para}_i(\mathfrak{g}) \cong \left\{ i - \text{dimensional subspaces in } \mathbb{C}^4 \right\},$$

i.e., $\mathsf{Para}(\mathfrak{g}) \cong \mathsf{Proj}\left(\mathbb{C}^4\right)$. A standard parabolic configuration $\mathsf{C}(W) \to \mathsf{Proj}\left(\mathbb{C}^4\right)$ has image looking like



which is a non-degenerate 3-simplex in $\mathbb{P}^3$. While a degenerate 3-simplex in $\mathbb{P}^3$, e.g.



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may arise from a parabolic configuration $\mathsf{C}(W) \to \mathsf{Proj}\left(\mathbb{C}^4\right)$ which is not injective. In fact, such degenerate simplices form another irreducible component of $\mathrm{Mor}(\mathsf{C}(W), \mathsf{Para}(\mathfrak{g}))$ different from the standard one.

Notation 3.7.16. Let $\mathscr{T}^{\mathfrak{q}}$ be the subset of $\mathscr{T}$ containing Cartan subalgebras $\mathfrak{t}$ such that all Borel subalgebras $\mathfrak{b} \supseteq \mathfrak{t}$ are weakly opposite to $\mathfrak{q}$. Let $\mathscr{U}^{\mathfrak{q}} := \pi_1^{-1}(\mathscr{T}^{\mathfrak{q}})$ be the inverse image of $\mathscr{T}^{\mathfrak{q}}$ under the projection map $\pi_1 : \mathscr{U} \to \mathscr{T}$.

Note that

$$\mathscr{U}^{\mathfrak{q}} \subsetneq \{(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U} \,|\mathfrak{b} \text{ is weakly opposite to } \mathfrak{q}\}.$$

Proposition 3.7.17. *$\mathscr{T}^{\mathfrak{q}}$ is an open dense subset of $\mathscr{T}$, and hence $\mathscr{U}^{\mathfrak{q}}$ is an open dense subset of $\mathscr{U}$.*

Proof. If $\mathscr{T}^{\mathfrak{q}}$ is an open dense subset of $\mathscr{T}$, then the result that $\mathscr{U}^{\mathfrak{q}}$ is an open dense subset of $\mathscr{U}$ follows immediately. We now will show the first part of the proposition. Let $\mathfrak{b}_0$ be a Borel subalgebra contained in $\mathfrak{q}$ and $\mathfrak{t}_0$ be a Cartan subalgebra contained in $\mathfrak{b}_0$. Denote T the maximal torus of G with the associated Lie algebra $\mathfrak{t}_0$. Let $\overline{\omega}_0$ be the longest element in $W(G, T_0)$ with respect to some length function defined using the simple system Δ corresponding to $\mathfrak{b}_0$. Then $\omega_0 \cdot \mathfrak{b}_0 \in \mathscr{P}^{\mathfrak{q}}(\mathfrak{g})$, and so by Lemma 3.4.5,

$$Q \cdot (\omega_0 \cdot \mathfrak{b}_0) = \mathscr{P}^{\mathfrak{q}}(\mathfrak{g}),$$

where $\omega_0 \in N_G(T_0)$ is the coset representative of $\overline{\omega}_0$. For each $w \in N_G(T')$, denote

$$G_w = \{g \in G \,|\, g \cdot (w \cdot \mathfrak{b}_0) \in \mathscr{P}^{\mathfrak{q}}(\mathfrak{g})\}.$$

Then $Q\omega_0 B_0 w^{-1} \subseteq G_w$, where B_0 is the Borel subgroup of G whose associated Lie algebra is $\mathfrak{b}_0$, for all $w \in N_G(T_0)$ because

$$\left(Q\omega_0 B_0 w^{-1}\right) \cdot (w \cdot \mathfrak{b}_0) = Q \cdot (\omega_0 \cdot \mathfrak{b}_0) = \mathscr{P}^{\mathfrak{q}}(\mathfrak{g}).$$

Since $B_0\omega_0 B_0$ is an open dense subset of G, then $Q\omega_0 B_0$ contains an open dense subset of G, and so does $Q\omega_0 B_0 w^{-1}$. Hence G_w contains an open dense subset of G for all $w \in N_G(T_0)$. This implies that $\bigcap_{w \in N_G(T_0)} G_w$ contains an open dense subset of G because the finite intersection of open dense sets is again open and dense. By choosing $g \in \bigcap_{w \in N_G(T_0)} G_w$, we have $\mathfrak{t} := g \cdot \mathfrak{t}_0$ as a required Cartan subalgebra because if $\mathfrak{b}$ is a Borel subalgebra containing $\mathfrak{t}$, then $g^{-1} \cdot \mathfrak{b}$ contains $\mathfrak{t}_0$, and so $g^{-1} \cdot \mathfrak{b} = w \cdot \mathfrak{b}_0$ for some $w \in N_G(T_0)$; whence $\mathfrak{b} = g \cdot (w \cdot \mathfrak{b}_0) \in \mathscr{P}^{\mathfrak{q}}(\mathfrak{g})$. □

Remark 3.7.18. For any $\mathfrak{t} \in \mathscr{T}^{\mathfrak{q}}$, we know that $\mathfrak{t} \nsubseteq \mathfrak{q}$. Otherwise, as appears in the proof of Proposition 3.7.17, $g \in Q$, and so $g \cdot \mathfrak{b}' \notin \mathscr{P}^{\mathfrak{q}}(\mathfrak{g})$, i.e., $g \notin G_1$, which is a contradiction.

Lemma 3.7.19. *Q acts locally freely on $\mathscr{U}^{\mathfrak{q}}$.*

Proof. Let $\mathfrak{t} \in \mathscr{T}^{\mathfrak{q}}$ and T be a maximal torus of G with Lie algebra $\mathfrak{t}$. Firstly, we will show that $\{\mathfrak{p} \in G \cdot \mathfrak{q} \,|\, \mathfrak{t} \subseteq \mathfrak{p}\} \subseteq \overline{T \cdot \mathfrak{q}}$. Let $\mathfrak{t} \subseteq \mathfrak{p} \in G \cdot \mathfrak{q}$ and $\widehat{\mathfrak{p}}$ be the parabolic subalgebra of $\mathfrak{q}$ complementary to $\mathfrak{p}$ parametrized by the algebraic Weyl structure $\widetilde{\xi} \in \mathfrak{t}$. Then $\mathfrak{q}$ is weakly opposite to $\widehat{\mathfrak{p}}$ because $\mathfrak{t} \subseteq \widehat{\mathfrak{p}}$ and $\mathfrak{t} \in \mathscr{T}^{\mathfrak{q}}$. So $\mathfrak{q}$ is complementary to $\widehat{\mathfrak{p}}$. Since $\exp\left(\left(\widehat{\mathfrak{p}}\right)^{\perp}\right)$ acts freely and transitively on all parabolic subalgebras complementary to $\widehat{\mathfrak{p}}$, thus $\mathfrak{q} = \exp(x) \cdot \mathfrak{p}$, for some $x \in \left(\widehat{\mathfrak{p}}\right)^{\perp}$. Therefore

$$\exp\left(\widetilde{\xi}\right) \cdot \mathfrak{q} = \exp\left(\left[\widetilde{\xi}, x\right]\right) \cdot \mathfrak{p}.$$

Since $\widetilde{\xi}$ has negative eigenvalues on $\left(\widehat{\mathfrak{p}}\right)^{\perp}$, this implies that $x = 0$ is in the closure of the T orbit on $\left(\widehat{\mathfrak{p}}\right)^{\perp}$. Therefore $\mathfrak{p} \in \overline{T \cdot \mathfrak{q}}$.

Let $\mathfrak{p} \in G \cdot \mathfrak{q}$ containing $\mathfrak{t}$ and

$$\lambda := \sum_i \lambda_i,$$

where λ_i is the fundamental weight corresponding to a crossed node of the decorated Dynkin diagram $\mathscr{D}_{\mathfrak{p}}$. By Proposition 1, Section 5.2, in [GS87], we have

$$\dim\left(\mathrm{Stab}_T(\mathfrak{q})\right) = \dim(T) - \dim\left(\overline{T \cdot \mathfrak{q}}\right) = \mathrm{codim}\left(\mathrm{conv}\left(L_Q\right)\right),$$

where $L_Q := \{\alpha \in W \cdot \lambda \,|p^{\alpha}(Q) \neq 0\}$ and $W = N_G(T)/T$. Since $\{\mathfrak{p} \in G \cdot \mathfrak{p} \,| \mathfrak{t} \subseteq \mathfrak{p}\} \subseteq \overline{T \cdot \mathfrak{q}}$, this implies that $p^{\alpha}(Q) \neq 0$ for all $\alpha \in W \cdot \lambda$, and so $\mathrm{codim}(\mathrm{conv}(L_Q)) = 0$. Therefore $\mathrm{Stab}_T(\mathfrak{q}) = \{1\}$, and whence $\mathfrak{q} \cap \mathfrak{t} = \{0\}$. By Lemma 3.7.3, Q acts locally freely on $\mathscr{U}^{\mathfrak{q}}$. □

Define

$$\begin{aligned} \Upsilon^{\mathfrak{q}} : \mathscr{U}^{\mathfrak{q}} &\to \mathrm{Mor}\left(\mathsf{C}(W), \mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})\right) \\ (\mathfrak{t}, \mathfrak{b}) &\mapsto \Upsilon_{(\mathfrak{t},\mathfrak{b})} \end{aligned}$$

where $W = W(G, T_0)$ is a Coxeter group determined by some $(\mathfrak{t}_0, \mathfrak{b}_0) \in \mathscr{U}^{\mathfrak{q}}$. Then $\Upsilon^{\mathfrak{q}}$ is well-defined. To see this, for any $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}^{\mathfrak{q}}$ and $\mathfrak{p} \in \mathrm{im}\left(\Upsilon_{(\mathfrak{t},\mathfrak{b})}\right)$, we have $\mathfrak{t} \subseteq w \cdot \mathfrak{b} \subseteq \mathfrak{p}$, for some $w \in N_G(T)$, and so

$$\mathfrak{g} = w \cdot \mathfrak{b} + \mathfrak{q} \subseteq \mathfrak{p} + \mathfrak{q} \subseteq \mathfrak{g}.$$

Corollary 3.7.20. *Let* $\mathrm{Mor}^{\mathrm{inj}}\left(\mathsf{C}(W), \mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})\right) := \{\psi : \mathsf{C}(W) \to \mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}) \text{ is injective}\}$. *Then*

$$\mathscr{U}^{\mathfrak{q}} \cong Mor^{inj}\left(\mathsf{C}(W), \mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})\right).$$

Proof. By Theorem 3.7.13, Υ is injective. Thus $\Upsilon^{\mathfrak{q}}$ is injective because $\Upsilon^{\mathfrak{q}} = \Upsilon\,|_{\mathscr{U}^{\mathfrak{q}}}$.

Let $\psi \in \mathrm{Mor}^{\mathrm{inj}}(\mathsf{C}(W), \mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}))$. Then $\psi = \Upsilon_{(\mathfrak{t},\mathfrak{b})}$ for some $(\mathfrak{t},\mathfrak{b}) \in \mathscr{U}$. Let $\mathfrak{b}'$ be a Borel subalgebra containing $\mathfrak{t}$. The $\mathfrak{b}'$ is of the form

$$\mathfrak{b}' = \bigcap_{\mathfrak{p} \in f} \mathfrak{p},$$

for some full flag f of $\mathsf{Para}(\mathfrak{g})$. Any maximal parabolic subalgebra $\mathfrak{p}$ containing $\mathfrak{t}$ is in $\mathrm{im}(\psi)$; whence $\mathfrak{p} \in \mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})$. This implies that f is a full flag of $\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})$, and so, by Lemma 3.4.6, $\mathfrak{b}'$ is weakly opposite to $\mathfrak{q}$. Therefore $(\mathfrak{t},\mathfrak{b}) \in \mathscr{U}^{\mathfrak{q}}$. □

Definition 3.7.21. Each element in the image of $\Upsilon^{\mathfrak{q}}$ is called a $\mathfrak{q}$-GENERIC STANDARD PARABOLIC CONFIGURATION for $\mathfrak{g}$.

Chapter 4

Parabolic projection

4.1 Definitions and properties

Let G be a connected reductive algebraic group over an algebraically closed field $\mathbb{F}$ of characteristic zero with the Lie algebra $\mathfrak{g}$, and let Q be a parabolic subgroup of G with the Lie algebra $\mathfrak{q}$. Define a map

$$\begin{aligned} \varphi_{\mathfrak{q}} : \mathscr{P}^{\mathfrak{q}}(\mathfrak{g}) &\rightarrow \mathscr{P}(\mathfrak{q}_0) \\ \mathfrak{p} &\mapsto ((\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{nr}(\mathfrak{q}))/\mathfrak{nr}(\mathfrak{q}) . \end{aligned} \tag{4.1.1}$$

By Proposition 2.2.44 and Proposition 2.2.46, $\varphi_{\mathfrak{q}}$ is well-defined.

Definition 4.1.1. $\varphi_{\mathfrak{q}}$ is called the **PARABOLIC PROJECTION** of $\mathscr{P}^{\mathfrak{q}}(\mathfrak{g})$ into $\mathscr{P}(\mathfrak{q}_0)$.

Q acts on both $\mathscr{P}^{\mathfrak{q}}(\mathfrak{g})$ and $\mathscr{P}(\mathfrak{q}_0)$ and we have the following.

Lemma 4.1.2. *For any $\mathfrak{p}, \mathfrak{p}' \in \mathscr{P}^{\mathfrak{q}}(\mathfrak{g})$, if $\mathfrak{p}$ and $\mathfrak{p}'$ are co-standard, then so are $\varphi_{\mathfrak{q}}(\mathfrak{p})$ and $\varphi_{\mathfrak{q}}(\mathfrak{p}')$.*

Proof. Let $\mathfrak{p}, \mathfrak{p}' \in \mathscr{P}^{\mathfrak{q}}(\mathfrak{g})$ be such that $\mathfrak{p} \cap \mathfrak{p}'$ is a parabolic subalgebra of $\mathfrak{g}$. Then

$$\varphi_{\mathfrak{q}}(\mathfrak{p} \cap \mathfrak{p}') = ((\mathfrak{p} \cap \mathfrak{p}' \cap \mathfrak{q}) + \mathfrak{nr}(\mathfrak{q}))/\mathfrak{nr}(\mathfrak{q})$$

is a parabolic subalgebra of $\mathfrak{q}_0$. Moreover,

$$\varphi_{\mathfrak{q}}(\mathfrak{p} \cap \mathfrak{p}') \subseteq \varphi_{\mathfrak{q}}(\mathfrak{p}) \cap \varphi_{\mathfrak{q}}(\mathfrak{p}') ;$$

whence $\varphi_{\mathfrak{q}}(\mathfrak{p}) \cap \varphi_{\mathfrak{q}}(\mathfrak{p}')$ is a parabolic subalgebra of $\mathfrak{q}_0$. □

Lemma 4.1.3. *$\varphi_{\mathfrak{q}}$ is equivariant under the Q-action.*

Proof. For each $q \in Q$ and $\mathfrak{p} \in \mathscr{P}^{\mathfrak{q}}(\mathfrak{g})$,

$$\varphi_{\mathfrak{q}}(q \cdot \mathfrak{p}) = (((q \cdot \mathfrak{p}) \cap \mathfrak{q}) + \mathfrak{nr}(\mathfrak{q}))/\mathfrak{nr}(\mathfrak{q}) = q \cdot ((\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{nr}(\mathfrak{q}))/\mathfrak{nr}(\mathfrak{q}) = q \cdot (\varphi_{\mathfrak{q}}(\mathfrak{p}))$$

because $\mathfrak{nr}(\mathfrak{q})$ is an ideal of $\mathfrak{q}$. □

Lemma 3.4.5 and Lemma 4.1.3 imply that $\varphi_{\mathfrak{q}}$ induces a map

$$\mu : \mathcal{P}(\mathscr{D}_{\mathfrak{g}}) \to \mathcal{P}(\mathscr{D}_{\mathfrak{q}_0})$$

between the power sets of $\mathscr{D}_{\mathfrak{g}}$ and $\mathscr{D}_{\mathfrak{q}_0}$, respectively in such a way that the diagram

$$\begin{array}{ccc} \mathscr{P}^{\mathfrak{q}}(\mathfrak{g}) & \xrightarrow{\varphi_{\mathfrak{q}}} & \mathscr{P}(\mathfrak{q}_0) \\ {\scriptstyle t_{\mathfrak{g}}}\downarrow & & \downarrow{\scriptstyle t_{\mathfrak{q}_0}} \\ \mathcal{P}(\mathscr{D}_{\mathfrak{g}}) & \xrightarrow[\mu]{} & \mathcal{P}(\mathscr{D}_{\mathfrak{q}_0}) \end{array}$$

commutes, where

$$t_{\mathfrak{g}} : \mathscr{P}^{\mathfrak{q}}(\mathfrak{g}) \to \mathcal{P}(\mathscr{D}_{\mathfrak{g}}) : \mathfrak{p} \in \mathscr{P}^{\mathfrak{q}}_I(\mathfrak{g}) \mapsto I,$$

and

$$t_{\mathfrak{q}_0} : \mathscr{P}(\mathfrak{q}_0) \to \mathcal{P}(\mathscr{D}_{\mathfrak{q}_0}) : \mathfrak{p} \in \mathscr{P}_I(\mathfrak{q}_0) \mapsto I.$$

In the following section, we will investigate the map μ.

4.2 The induced map on types

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$, and $\mathfrak{r} := (\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{nr}(\mathfrak{q})$. By Corollary 2.2.48, let $\widehat{\mathfrak{p}}$ be a parabolic subalgebra of $\mathfrak{g}$ complementary to $\mathfrak{p}$ and co-standard with $\mathfrak{q}$. Denote $\widehat{\mathfrak{r}} := \left(\widehat{\mathfrak{p}} \cap \mathfrak{q}\right) + \mathfrak{nr}(\mathfrak{q})$.

Lemma 4.2.1. *The parabolic subalgebras $\mathfrak{r}/\mathfrak{nr}(\mathfrak{q})$ and $\widehat{\mathfrak{r}}/\mathfrak{nr}(\mathfrak{q})$ of $\mathfrak{q}_0$ are complementary.*

Proof. It suffice to show that $(\mathfrak{nr}(\mathfrak{r})/\mathfrak{nr}(\mathfrak{q})) \cap (\widehat{\mathfrak{r}}/\mathfrak{nr}(\mathfrak{q})) = \mathfrak{nr}(\mathfrak{q})/\mathfrak{nr}(\mathfrak{q})$, i.e., $\mathfrak{nr}(\mathfrak{r}) \cap \widehat{\mathfrak{r}} = \mathfrak{nr}(\mathfrak{q})$ and $(\mathfrak{nr}(\widehat{\mathfrak{r}})/\mathfrak{nr}(\mathfrak{q})) \cap (\mathfrak{r}/\mathfrak{nr}(\mathfrak{q})) = \mathfrak{nr}(\mathfrak{q})/\mathfrak{nr}(\mathfrak{q})$, i.e., $\mathfrak{nr}(\widehat{\mathfrak{r}}) \cap \mathfrak{r} = \mathfrak{nr}(\mathfrak{q})$; whence

$(\mathfrak{r} \cap \widehat{\mathfrak{r}})/\mathfrak{nr}(\mathfrak{q})$ is a common Levi subalgebra of $\mathfrak{r}/\mathfrak{nr}(\mathfrak{q})$ and $\widehat{\mathfrak{r}}/\mathfrak{nr}(\mathfrak{q})$.

Since $\mathfrak{p}$ and $\widehat{\mathfrak{p}}\cap\mathfrak{q}$ are parabolic subalgebras of $\mathfrak{g}$, they contain a common Cartan subalgebra that makes $\mathfrak{nr}(\mathfrak{q})$ be a direct sum of one-dimensional root spaces, each of which must lie in either $\mathfrak{p}$ or $\mathfrak{nr}(\widehat{\mathfrak{p}})$. Therefore $\mathfrak{nr}(\mathfrak{q}) = \left(\mathfrak{nr}(\mathfrak{q}) \cap \mathfrak{nr}(\widehat{\mathfrak{p}})\right) \oplus (\mathfrak{nr}(\mathfrak{q}) \cap \mathfrak{p})$, and so

$$\begin{aligned} \mathfrak{nr}(\widehat{\mathfrak{r}}) \cap \mathfrak{r} &= \left(\left(\mathfrak{nr}(\widehat{\mathfrak{p}}) + \mathfrak{nr}(\mathfrak{q})\right) \cap \mathfrak{p}\right) + \mathfrak{nr}(\mathfrak{q}) \\ &= \left(\left(\mathfrak{nr}(\widehat{\mathfrak{p}}) + (\mathfrak{nr}(\mathfrak{q}) \cap \mathfrak{p})\right) \cap \mathfrak{p}\right) + \mathfrak{nr}(\mathfrak{q}) \\ &= \left(\left(\mathfrak{nr}(\widehat{\mathfrak{p}}) \cap \mathfrak{p}\right) + (\mathfrak{nr}(\mathfrak{q}) \cap \mathfrak{p})\right) + \mathfrak{nr}(\mathfrak{q}) \\ &= \mathfrak{nr}(\mathfrak{q}). \end{aligned}$$

Moreover $\mathfrak{nr}(\mathfrak{r}) \cap \widehat{\mathfrak{r}} = \left(\mathfrak{p} \cap \widehat{\mathfrak{p}} \cap \mathfrak{q}\right) + \mathfrak{nr}(\mathfrak{q}) = \mathfrak{nr}(\mathfrak{q})$. □

This gives us a four-step procedure to obtain the Dynkin diagram representing $\mathfrak{r}$ as follows:

Proposition 4.2.2. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be parabolic subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$. Then the Dynkin diagram representing the parabolic subalgebra $\mathfrak{r} := (\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{nr}(\mathfrak{q})$ of $\mathfrak{g}$ is obtained from the following procedure:*

1. Apply the dual involution $\imath_{\mathfrak{g}}$ of the Dynkin diagram of $\mathfrak{g}$ to the decorated Dynkin diagram representing $\mathfrak{p}$. (According to the proof of Corollary 2.2.48, this gives the decorated Dynkin diagram representing $\widehat{\mathfrak{p}}$ a parabolic subalgebra of $\mathfrak{g}$ complementary to $\mathfrak{p}$ and co-standard with $\mathfrak{q}$)

2. Remove all the nodes in the decorated Dynkin diagram representing $\widehat{\mathfrak{p}}$ obtained from step *1. which are crossed in the decorated Dynkin diagram representing $\mathfrak{q}$.* (This gives the decorated Dynkin diagram representing $\left(\left(\widehat{\mathfrak{p}} \cap \mathfrak{q}\right) + \mathfrak{nr}(\mathfrak{q})\right)/\mathfrak{nr}(\mathfrak{q})$)

3. Apply the dual involution $\imath_{\mathfrak{q}_0}$ of the Dynkin diagram of $\mathfrak{q}_0$ to the diagram obtained from step 2. (This gives the decorated Dynkin diagram representing $\mathfrak{r}/\mathfrak{nr}(\mathfrak{q})$ by Lemma 4.2.1)

4. Add all nodes we removed from step 2. to the diagram obtained from step 3. and cross all such nodes. This yields the decorated Dynkin diagram representing $\mathfrak{r}$.

Remark 4.2.3. In the case that the Dynkin diagram of a simple Lie algebra $\mathfrak{g}$ is of type B_n, D_{2n}, C_n, E_7, E_8, F_4, or G_4, the dual involution $\imath_{\mathfrak{g}}$ is the identity.

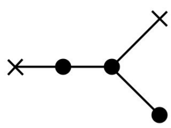
Example 4.2.4. Let $\mathfrak{g} := \mathfrak{sl}_8(\mathbb{C})$. Denote $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ represented by the decorated Dynkin diagram

and $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ represented by the decorated Dynkin diagram

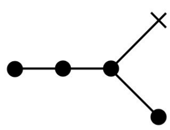
such that $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$. Apply above procedure to get the diagram representing the parabolic subalgebra $\mathfrak{r} := (\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{nil}(\mathfrak{q})$:

	$\mathfrak{q}$	
	$\mathfrak{p}$	
Step 1.	$\widehat{\mathfrak{p}}$	
Step 2.	$\left(\left(\widehat{\mathfrak{p}} \cap \mathfrak{q}\right) + \mathfrak{nr}(\mathfrak{q})\right) / \mathfrak{nr}(\mathfrak{q})$	
Step 3.	$\mathfrak{r} / \mathfrak{nr}(\mathfrak{q})$	
Step 5.	$\mathfrak{r}$	

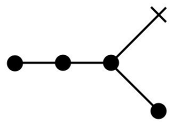
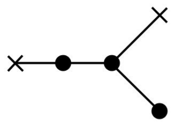
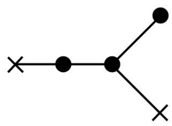
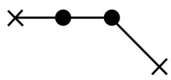
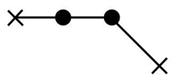
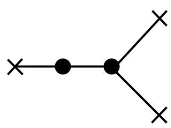
Example 4.2.5. Let $\mathfrak{g} := \mathfrak{so}_{10}(\mathbb{C})$. Denote $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ represented by the decorated Dynkin diagram



and $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ represented by the decorated Dynkin diagram



such that $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$. Apply above procedure to get the diagram representing the parabolic subalgebra $\mathfrak{r} := (\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{q}^{\perp}$:

	$\mathfrak{q}$	
	$\mathfrak{p}$	
Step 1.	$\widehat{\mathfrak{p}}$	
Step 2.	$\left(\left(\widehat{\mathfrak{p}} \cap \mathfrak{q}\right) + \mathfrak{nr}\left(\mathfrak{q}\right)\right) / \mathfrak{nr}\left(\mathfrak{q}\right)$	
Step 3.	$\mathfrak{r} / \mathfrak{nil}\left(\mathfrak{q}\right)$	
Step 5.	$\mathfrak{r}$	

In particular if $\mathfrak{g}$ is semisimple, together with describing $\mathfrak{r}$ by a decorated Dynkin diagram, one can also describe $\mathfrak{r}$ more precisely as a subalgebra of $\mathfrak{g}$. By Lemma 2.2.59, let $\mathfrak{b}$ be a Borel subalgebra contained in $\mathfrak{q}$ and complementary to one $\widehat{\mathfrak{b}}$ contained in $\mathfrak{p}$. Then $\mathfrak{r}$ contains the Cartan subalgebra $\mathfrak{t} = \mathfrak{b} \cap \widehat{\mathfrak{b}} \subseteq \mathfrak{q} \cap \mathfrak{p}$. Denote $\mathcal{R} := \mathcal{R}(\mathfrak{g}, \mathfrak{t})$ be the root system of $\mathfrak{g}$ associated to $\mathfrak{t}$. The Borel subalgebra $\mathfrak{b}$ determines a choice of positive roots $\mathcal{R}^+$, and consequently a choice of simple systems Δ, of $\mathcal{R}$. By Remark 2.2.60, we see that

$$\mathcal{R}_{\mathfrak{r}} = \left(\mathcal{R}_{\widetilde{\mathfrak{p}}_0} \cap \mathcal{R}_{\widetilde{\mathfrak{q}}_0}\right) \sqcup \left(\left(\mathcal{R}_{\mathfrak{nr}(\mathfrak{p})} \cap \mathcal{R}_{\mathfrak{q}}\right) \cup \mathcal{R}_{\mathfrak{nr}(\mathfrak{q})}\right),$$

with $\mathcal{R}_{\widetilde{\mathfrak{r}}_0} = \mathcal{R}_{\widetilde{\mathfrak{p}}_0} \cap \mathcal{R}_{\widetilde{\mathfrak{q}}_0}$ and $\mathcal{R}_{\mathfrak{nr}(\mathfrak{r})} = \left(\mathcal{R}_{\mathfrak{nr}(\mathfrak{p})} \cap \mathcal{R}_{\mathfrak{q}}\right) \cup \mathcal{R}_{\mathfrak{nr}(\mathfrak{q})}$. Denote

$$\Gamma := \{\alpha \in \Delta \,|{-\alpha} \in \mathcal{R}_{\mathfrak{q}}\} = \Delta \cap \mathcal{R}_{\widetilde{\mathfrak{q}}_0}.$$

Then

$$\widetilde{\mathfrak{q}}_0 = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R} \cap \operatorname{span}(\Gamma)} \mathfrak{g}_\alpha \text{ and } \mathfrak{nr}(\mathfrak{q}) = \bigoplus_{\alpha \in \mathcal{R}^+ \setminus (\mathcal{R} \cap \operatorname{span}(\Gamma))} \mathfrak{g}_\alpha.$$

The Weyl group $W(\widetilde{\mathfrak{q}}_0, \mathfrak{t})$ of $\widetilde{\mathfrak{q}}_0$ with respect to $\mathfrak{t}$ is identified as a subgroup of $W(\mathfrak{g}, \mathfrak{t})$ generated by $\{s_\alpha \,|\alpha \in \Gamma\}$. Let $\omega_0^{\widetilde{\mathfrak{q}}_0}$ be the longest elements of the Weyl groups $W(\widetilde{\mathfrak{q}}_0, \mathfrak{t})$. It is well known that $\omega_0^{\widetilde{\mathfrak{q}}_0}$ is an involution on $\mathcal{R}_{\widetilde{\mathfrak{q}}_0}$ sending positive roots of $\widetilde{\mathfrak{q}}_0$ to negative roots of $\widetilde{\mathfrak{q}}_0$. Therefore

$$\omega_0^{\widetilde{\mathfrak{q}}_0} : \Gamma \to -\Gamma.$$

Lemma 4.2.6. *The simple system $\omega_0^{\widetilde{\mathfrak{q}}_0}(\Delta)$ is contained in $\mathcal{R}_{\mathfrak{r}}$. Thus* $\mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R} \cap \operatorname{span}\left(\omega_0^{\widetilde{\mathfrak{q}}_0}(\Delta)\right)} \mathfrak{g}_\alpha$ *is a Borel subalgebra of $\mathfrak{g}$ contained in $\mathfrak{r}$.*

Proof. Since $\Delta \subseteq \mathcal{R}_{\mathfrak{q}}$,

$$\Delta = \left(\Delta \cap \mathcal{R}_{\widetilde{\mathfrak{q}}_0}\right) \sqcup \left(\Delta \cap \mathcal{R}_{\mathfrak{nr}(\mathfrak{q})}\right) = \Gamma \sqcup (\Delta \setminus \Gamma).$$

Furthermore we have

$$\omega_0^{\widetilde{\mathfrak{q}}_0}(\Gamma) = -\Gamma \subseteq \mathcal{R}_{\mathfrak{p}} \cap \mathcal{R}_{\widetilde{\mathfrak{q}}_0} \subseteq \mathcal{R}_{\mathfrak{r}}$$

because $-\Gamma \subseteq -\Delta \subseteq \mathcal{R}_{\mathfrak{p}}$, and

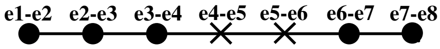
$$\omega_0^{\widetilde{\mathfrak{q}}_0}(\Delta \setminus \Gamma) \subseteq \omega_0^{\widetilde{\mathfrak{q}}_0}\left(\mathcal{R}_{\mathfrak{nr}(\mathfrak{q})}\right) \subseteq \mathcal{R}_{\mathfrak{nr}(\mathfrak{q})} \subseteq \mathcal{R}_{\mathfrak{r}}$$

since $\mathfrak{nr}(\mathfrak{q})$ is an ideal of $\mathfrak{q}$. Therefore $\omega_0^{\widetilde{\mathfrak{q}_0}}(\Delta) = \omega_0^{\widetilde{\mathfrak{q}_0}}(\Gamma) \cup \omega_0^{\widetilde{\mathfrak{q}_0}}(\Delta \setminus \Gamma) \subseteq \mathcal{R}_{\mathfrak{r}}$. □

Example 4.2.7. Let $\mathfrak{g} := \mathfrak{sl}_8(\mathbb{C})$ with the standard diagonal Cartan subalgebra $\mathfrak{t}$. Then $\mathfrak{t}$ gives the root system

$$\mathcal{R} = \{e_i - e_j \,|1 \leq i, j \leq 8 \text{ and } i \neq j\},$$

where $e_i(t) = t_{ii}$ for $1 \leq i \leq 8$. Let $\Delta = \{e_i - e_{i+1} \,|1 \leq i \leq 7\}$. Then Δ is a simple system of $\mathcal{R}$. Let $\mathfrak{q}$ be a standard parabolic subalgebra (with respect to Δ) of $\mathfrak{g}$ represented by the Dynkin diagram

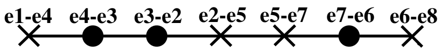


Then $\mathfrak{q}$ contains the standard Borel subalgebra $\mathfrak{b}$ determined by Δ.

Let $\mathscr{P}_J(\mathfrak{g})$ be a component in $\mathscr{P}(\mathfrak{g})$ represented by the diagram

We can choose $\mathfrak{p} \in \mathscr{P}_J(\mathfrak{g})$ such that $\mathfrak{p}$ contains the Borel subalgebra complementary to $\mathfrak{b}$. Then $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$. By associating each node in the decorated Dynkin diagram

representing $\mathfrak{r} := \varphi_{\mathfrak{q}}(\mathfrak{p}) = (\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{nr}(\mathfrak{q})$ with the corresponding simple root in $\omega_0^{\widetilde{\mathfrak{q}_0}}(\Delta)$ (here $\omega_0^{\widetilde{\mathfrak{q}_0}} = (24)(67)$), we now are able to describe $\mathfrak{r}$ as a subalgebra in terms of the root spaces decomposition of $\mathfrak{g}$:



Therefore $\mathcal{R}_{\mathfrak{r}}$ is the set consisting of all positive linear combinations of the set

$$\omega_0^{\widetilde{\mathfrak{q}_0}}(\Delta) \cup \{e_3 - e_2, e_4 - e_3, e_7 - e_6\}$$

and

$$\mathfrak{r} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}_{\mathfrak{r}}} \mathfrak{g}_\alpha.$$

Theorem 4.2.8. *The parabolic projection $\varphi_{\mathfrak{q}}$ induces the map $\mu : \mathcal{P}(\mathscr{D}_{\mathfrak{g}}) \to \mathcal{P}(\mathscr{D}_{\mathfrak{q}_0})$ defined by*

$$\mu(S) = \imath_{\mathfrak{q}_0}(\imath_{\mathfrak{g}}(S) \cap \mathscr{D}_{\mathfrak{q}_0}).$$

Proof. This follows from Proposition 4.2.2. □

4.3 Parabolic projection as an incidence system morphism

For any two parabolic subalgebras $\mathfrak{p}$ and $\mathfrak{p}'$ of $\mathfrak{g}$, if $\mathfrak{p} \cap \mathfrak{p}'$ is a parabolic subalgebra of $\mathfrak{g}$, then $\varphi_{\mathfrak{q}}(\mathfrak{p}) \cap \varphi_{\mathfrak{q}}(\mathfrak{p}')$ is also a parabolic subalgebra of $\mathfrak{q}_0$ because it contains the parabolic subalgebra $\varphi_{\mathfrak{q}}(\mathfrak{p} \cap \mathfrak{p}')$. Therefore $\varphi_{\mathfrak{q}}$ preserves the incidence relation. It is worth to see under which condition $\varphi_{\mathfrak{q}}$ becomes an incidence system morphism between incidence systems.

Define a map

$$\nu := \imath_{\mathfrak{g}} \circ \mathrm{i} \circ \imath_{\mathfrak{q}_0} : \mathscr{D}_{\mathfrak{q}_0} \to \mathscr{D}_{\mathfrak{g}}$$

where i is the inclusion map from the diagram $\mathscr{D}_{\mathfrak{q}_0}$ to the diagram $\mathscr{D}_{\mathfrak{g}}$. Then ν is an injective map.

Lemma 4.3.1. *μ is the pull-back of ν.*

Proof. Let $S \in \mathcal{P}(\mathscr{D}_{\mathfrak{g}})$. Then

$$\mu(S) = \imath_{\mathfrak{q}_0}(\imath_{\mathfrak{g}}(S) \cap \mathscr{D}_{\mathfrak{q}_0}) = \imath_{\mathfrak{q}_0}^{\star} \circ \mathrm{i}^{\star} \circ \imath_{\mathfrak{g}}^{\star}(S) = (\imath_{\mathfrak{g}} \circ \mathrm{i} \circ \imath_{\mathfrak{q}_0})^{\star}(S) = \nu^{\star}(S).$$

□

Corollary 4.3.2. *$\varphi_{\mathfrak{q}}$ induces $\Phi_{\mathfrak{q}} : \mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}) \to \mathsf{Para}(\mathfrak{q}_0)$ an incidence system morphism over the map $\nu : \mathscr{D}_{\mathfrak{q}_0} \to \mathscr{D}_{\mathfrak{g}}$ given by the strict incidence system morphism*

$$\varphi_{\mathfrak{q}}\big|_{\nu^{\star}\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})} : \nu^{\star}\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}) \to \mathsf{Para}(\mathfrak{q}_0).$$

.

Proof. Consider

$$\nu^{\star}\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}) = \bigsqcup_{i \in \mathscr{D}_{\mathfrak{q}_0}} \mathsf{Para}^{\mathfrak{q}}_{\nu(i)}(\mathfrak{g}),$$

and

$$\mathsf{Para}(\mathfrak{q}_0) = \bigsqcup_{i \in \mathscr{D}_{\mathfrak{q}_0}} \mathsf{Para}_i(\mathfrak{q}_0).$$

For any $i \in \mathscr{D}_{\mathfrak{q}_0}$, if $\mathfrak{p} \in \mathsf{Para}^{\mathfrak{q}}_{\nu(i)}(\mathfrak{g})$, then $\varphi_{\mathfrak{q}}(\mathfrak{p}) \in \mathsf{Para}_i(\mathfrak{q}_0)$ because ν is injective and $\mu(\{\nu(i)\}) = \{i\}$. Thus $\varphi_{\mathfrak{q}}\big|_{\nu^{\star}\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})} : \nu^{\star}\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}) \to \mathsf{Para}(\mathfrak{q}_0)$ preserves types, and so, by Lemma 4.1.2, it preserves the incidence relation. Hence $\varphi_{\mathfrak{q}}\big|_{\nu^{\star}\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})}$ is a strict incidence system morphism. □

Theorem 4.3.3. *The parabolic projection arises from the flag extension of*

$$\Phi_{\mathfrak{q}} : \mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}) \to \mathsf{Para}(\mathfrak{q}_0)$$

in such a way that

$$\varphi_{\mathfrak{q}} = \tau \circ \mathcal{F}(\Phi_{\mathfrak{q}}) \circ (\tau^{\mathfrak{q}})^{-1} : \mathscr{P}^{\mathfrak{q}}(\mathfrak{g}) \to \mathscr{P}(\mathfrak{q}_0),$$

*where τ and $\tau^{\mathfrak{q}}$ are defined as (3.4.1) and (*3.4.2*), respectively, i.e., the diagram*

$$\begin{array}{ccc} \mathcal{F}(\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})) & \xrightarrow{\tau^{\mathfrak{q}}} & \mathscr{P}^{\mathfrak{q}}(\mathfrak{g}) \\ {\scriptstyle \mathcal{F}(\Phi)}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathfrak{q}}} \\ \mathcal{F}(\mathsf{Para}(\mathfrak{q}_0)) & \xrightarrow[\tau]{} & \mathscr{P}(\mathfrak{q}_0) \end{array}$$

commutes.

Proof. Let $\mathfrak{p} \in \mathscr{P}^{\mathfrak{q}}(\mathfrak{g})$. By Corollary 3.4.3. Denote $f := \tau^{-1}(\mathfrak{p})$ and I the type of f. Then (as discussed in Section 3.6)

$$\mathcal{F}(\Phi_{\mathfrak{q}})(f) = \varphi_{\mathfrak{q}}\big|_{\nu^{\star}\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})}(f) : \mu(I) \to \mathsf{Para}(\mathfrak{q}_0) : i \mapsto \varphi_{\mathfrak{q}}\big|_{\nu^{\star}\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})}(f(\nu(i))).$$

Therefore

$$\begin{aligned} \varphi_{\mathfrak{q}}(\mathfrak{p}) &= \varphi_{\mathfrak{q}}\left(\bigcap_{i \in I} f(i)\right) \\ &= \bigcap_{i \in \mu(I)} \varphi_{\mathfrak{q}}(f(\nu(i))) \\ &= \bigcap_{i \in \mu(I)} \varphi_{\mathfrak{q}}\big|_{\nu^{\star}\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})}(f(\nu(i))) \\ &= \bigcap_{i \in \mu(I)} \varphi_{\mathfrak{q}}\big|_{\nu^{\star}\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})}(f)((\nu(i))) \\ &= \bigcap_{i \in \mu(I)} \mathcal{F}(\Phi_{\mathfrak{q}})(f)((\nu(i))) \\ &= \tau(\mathcal{F}(\Phi_{\mathfrak{q}})(f)) \\ &= \tau \circ \mathcal{F}(\Phi_{\mathfrak{q}})\left(\tau^{-1}(\mathfrak{p})\right) \\ &= \tau \circ \mathcal{F}(\Phi_{\mathfrak{q}}) \circ \tau^{-1}(\mathfrak{p}), \end{aligned}$$

because, by Corollary 3.4.3, both $\varphi_{\mathfrak{q}}\left(\bigcap_{i\in I} f(i)\right)$ and $\bigcap_{i\in\mu(I)} \varphi_{\mathfrak{q}}(f(\nu(i)))$ are in $\mathscr{P}^{\mathfrak{q}}_{\mu(I)}(\mathfrak{g})$ and clearly $\varphi_{\mathfrak{q}}\left(\bigcap_{i\in I} f(i)\right) \subseteq \bigcap_{i\in\mu(I)} \varphi_{\mathfrak{q}}(f(\nu(i)))$. □

For any $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}^{\mathfrak{q}}$, we thus have the incidence system morphism

$$\varphi_{\mathfrak{q}} \circ \Upsilon_{(\mathfrak{t},\mathfrak{b})} : \mathsf{C}(W) \to \mathsf{Para}(\mathfrak{q}_0)$$

over the map $\nu : \mathscr{D}_{\mathfrak{q}_0} \to \mathscr{D}_{\mathfrak{g}} = \mathscr{D}$.

Example 4.3.4. Let V be a complex vector space of dimension 7. The group $\mathfrak{so}(V)$ has the Dynkin diagram $\mathscr{D}$ given by


.

According to Example 3.7.7, choose an isotropic line ℓ in V. Then $\ell \subseteq \ell^{\perp}$. Let $\mathfrak{q} = \mathrm{Stab}_{\mathfrak{so}(V)}(\ell)$ and $\mathfrak{p} = \mathrm{Stab}_{\mathfrak{so}(V)}(W)$, where W is an isotropic subspace of V. Then $\mathfrak{so}\left(\ell^{\perp}/\ell\right) \cong (\mathfrak{q}_0)_{ss}$, the semisimple Lie subalgebra of the reductive Lie algebra $\mathfrak{q}_0 = \mathfrak{q}/\mathfrak{nr}(\mathfrak{q})$. Since $\mathfrak{z}(\mathfrak{q}_0) = (\mathfrak{z}(\mathfrak{q}) + \mathfrak{nr}(\mathfrak{q}))/\mathfrak{nr}(\mathfrak{q})$, thus $\mathfrak{so}\left(\ell^{\perp}/\ell\right) \cong \mathfrak{q}/(\mathfrak{z}(\mathfrak{q}) + \mathfrak{nr}(\mathfrak{q}))$. Under the projection map, we have

$$\varphi_{\mathfrak{q}}(\mathfrak{p})/\mathfrak{z}(\mathfrak{q}_0) \cong ((\mathfrak{p}\cap\mathfrak{q}) + \mathfrak{nr}(\mathfrak{q}))/(\mathfrak{z}(\mathfrak{q}) + \mathfrak{nr}(\mathfrak{q})) \cong \mathrm{Stab}_{\mathfrak{so}\left(\ell^{\perp}/\ell\right)}\left(\left(W\cap\ell^{\perp} + \ell\right)/\ell\right). \tag{4.3.1}$$

Now let $\varrho : \{1, 2\} \to \mathscr{D}$ be the injective map given by the labelling

.

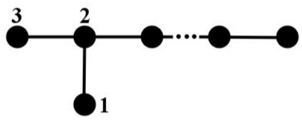
As in Example 3.7.7, the set $\mathsf{C}(W)_{\varrho(1)}$ and $\mathsf{C}(W)_{\varrho(2)}$ correspond to the collection of lines and the collection of planes of an octahedron lying in $Q^5 \subseteq \mathbb{P}(V)$. By choosing a suitable point in Q^5, one can send lines (resp. planes) in Q^5 to points (resp. lines) in Q^3 (as in Equation (4.3.1)).

By using parabolic projection, we can define the injective incidence system morphism $\Psi : \mathsf{C}(W) \to Q^3$ over the map $\varrho : \{1, 2\} \to \mathscr{D}$ such that Ψ maps $\mathsf{C}(W)_{\varrho(1)}$ to a collection of 12 points in Q^3, and $\mathsf{C}(W)_{\varrho(2)}$ to a collection of 8 lines in Q^3 with 2 lines through each point and 3 points on each line; it has the same incidence as edges and vertices of a cube.

Chapter 5

Generalized Cox Configurations

Suppose that V is a complex vector space of dimension four and (W, S) is a Coxeter group with Coxeter diagram $\mathscr{D}$ given by



.

Let $\varrho : \{1, 2, 3\} \to \mathscr{D}$ be the injective map given by the above labelling. According to Example 3.7.8, the set $\mathsf{C}(W)_{\varrho(1)}$ and $\mathsf{C}(W)_{\varrho(3)}$ can be identified with the orbits of fundamental weights λ_1 and λ_3 corresponding to the nodes labelled by 1 and 3 respectively. Each orbit is a demicube, i.e., a semi-regular polytope constructed from a hypercube with alternated vertices truncated. A hypercube may be considered as a bipartite graph between two types of vertices, black vertices and white vertices as in Figure 5.0.1, and thus an incidence system with two types. The sets $\mathsf{C}(W)_{\varrho(1)}$ and $\mathsf{C}(W)_{\varrho(3)}$ can be taken to be the black and white vertices with incidence pairs corresponding to edges. This is also the incidence system of a configuration associated with Cox's chain, consisting of 2^{n-1} points and 2^{n-1} planes in $\mathbb{P}(V)$, with n planes through each point and n points on each plane.

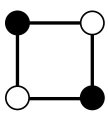


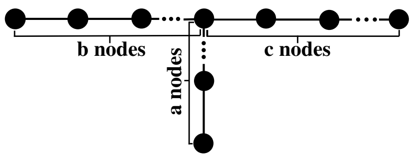
Figure 5.0.1: A 2-face of an n-hypercube.

Moreover, $\mathsf{C}(W)_{\varrho(2)}$ can be identified combinatorially with the space of 2-faces in the hypercube as in Figure 5.0.1. Any 2-face is incident with two white vertices and two black vertices. Thus we get an injective morphism $\Psi : \mathsf{C}(W) \to \mathsf{Proj}(V)$ over the map $\varrho : [4] \to \mathscr{D}$. Ψ maps $\mathsf{C}(W)_{\varrho(1)}$ to a collection of 2^{n-1} points in $\mathbb{P}(V)$, $\mathsf{C}(W)_{\varrho(2)}$ to a collection of $2^{n-2}\begin{pmatrix} n \\ 2 \end{pmatrix}$ lines in $\mathbb{P}(V)$, and $\mathsf{C}(W)_{\varrho(3)}$ to a collection of 2^{n-1} planes in $\mathbb{P}(V)$. We call the injective morphism Ψ a **Cox configuration**.

In this chapter, we will define generalized Cox configurations, construct them by using parabolic projection, and investigate their existence in some cases.

5.1 Generalized Cox configurations

Let W be a Coxeter group corresponding to the Coxeter diagram $\mathscr{D}$ of type

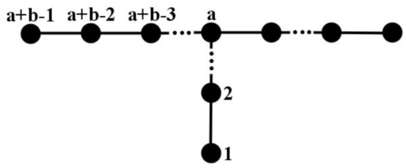
$T_{(a,b,c)} :=$ 

.

Definition 5.1.1. A **generalized Cox configuration of type** (a, b, c), where $a, b, c \in \mathbb{N}$, is a projective configuration

$$\Psi : \mathsf{C}(W) \to \mathsf{Proj}(V),$$

over the map $\varrho : [a + b] \to \mathscr{D}$, where V is a vector space of dimension $a + b$ and $[a + b]$ is the set of types of the incidence system $\mathsf{Proj}(V)$ (as defined in Example 3.1.3), given by the following labelling:



.

We will denote $\mathrm{Mor}_{\varrho}(\mathsf{C}(W), \mathsf{Proj}(V))$ by $\mathrm{GCo}_{(a,b,c)}(W, V)$. A generalized Cox configuration is said to be **non-degenerate** if it is an injective morphism and we will denote

$$\mathrm{GCo}^{inj}_{(a,b,c)}(W, V) := \mathrm{Mor}^{inj}_{\varrho}(\mathsf{C}(W), \mathsf{Proj}(V)).$$

When W and V can be understood from the context, we will suppress (W, V) and then write $\mathrm{GCo}_{(a,b,c)}$ for $\mathrm{GCo}_{(a,b,c)}(W, V)$.

Remark 5.1.2. Notice that we have an incidence system isomorphism from $\mathsf{Proj}(V)$ to $\mathsf{Para}(\mathfrak{pgl}(V))$ defined by

$$V' \mapsto \mathrm{Stab}_{\mathfrak{pgl}(V)}(V').$$

Therefore $\mathsf{Proj}(V) \cong \mathsf{Para}(\mathfrak{pgl}(V))$. Thus generalized Cox configurations are actually parabolic configurations.

As the image of a generalized Cox configuration of type (a, b, c) is in $\mathsf{Proj}(V)$, we can consider the dual of this configuration whose image is in $\mathsf{Proj}(V^\star) \cong \mathsf{Proj}(V)$. Furthermore, the dual of a generalized Cox configuration of type (a, b, c) is a generalized Cox configuration of type (b, a, c).

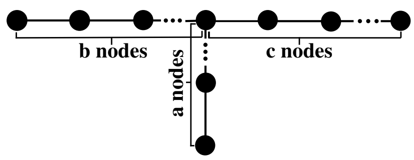
5.2 Generalized Cox configurations from parabolic projection

The notation of each algebraic group and each Lie algebra appearing in this section depends on (a, b, c). But for the rest of this chapter, we fix an (a, b, c) such that it satisfies the condition

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1, \tag{5.2.1}$$

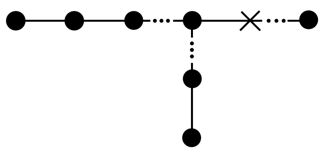
i.e., the diagram $T_{(a,b,c)}$ is a Dynkin diagram and so the Coxeter group is finite. It turns out into 3 classifications of the related diagram which is of A, D, or E-types. We say that a generalized Cox configuration of type (a, b, c) is of type A, D, or E if its related diagram is of that type.

For the rest of this chapter, let $\mathbb{F}$ be an algebraically closed field of characteristic zero. Let G be a connected simple algebraic group over $\mathbb{F}$ with the associated Dynkin diagram



.

Let Q be a parabolic subgroup of G in the conjugacy class *complementary* to the one represented by the decorated Dynkin diagram



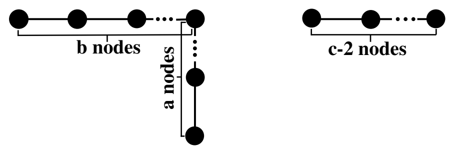
with the unipotent radical Q^u and the Levi factor $Q_0 := Q/Q^u$.

Let $\mathfrak{g}$ and $\mathfrak{q}$ be the associated Lie algebras of G and Q, respectively. Then $\mathfrak{q}^{\perp}$ is the Lie algebra of Q^u and $\mathfrak{q}_0$ is the Lie algebra of Q_0. By Corollary 2.2.27 and Proposition **??**,

$$\mathfrak{q}_0 = \mathfrak{z}(\mathfrak{q}_0) \oplus \mathfrak{s},$$

where $\mathfrak{s}$ is a semisimple Lie subalgebra of $\mathfrak{q}_0$ and $\dim(\mathfrak{z}(\mathfrak{q}_0)) = 1$. The Dynkin diagram $\mathscr{D}_{\mathfrak{s}} = \mathscr{D}_{\mathfrak{q}_0}$ (see Section 2.2.4) is given by

$\imath_{\mathfrak{g}} \Bigg($



$\Bigg).$

where $\imath_{\mathfrak{g}}$ is the dual involution of the Dynkin diagram $\mathscr{D}_{\mathfrak{g}}$.

As $\mathfrak{s}$ is semisimple and the Dynkin diagram $\mathscr{D}_{\mathfrak{s}}$ has two connected component,

$$\mathfrak{s} = \overline{\mathfrak{q}} \oplus \mathfrak{q}',$$

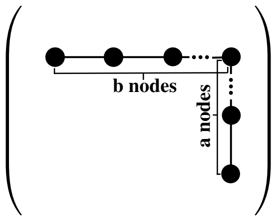
where $\overline{\mathfrak{q}} \cong \mathfrak{pgl}_{a+b}(\mathbb{F})$ and $\mathfrak{q}' \cong \mathfrak{pgl}_{c-1}(\mathbb{F})$ are simple ideals of $\mathfrak{s}$ such that $[\overline{\mathfrak{q}}, \mathfrak{q}'] = \{0\}$. Hence $\mathfrak{z}(\mathfrak{q}_0) \oplus \mathfrak{q}'$ is an ideal of $\mathfrak{q}_0$. Define $\pi' : \mathfrak{q} \to \mathfrak{q}_0$ and $\pi'' : \mathfrak{q}_0 \to \mathfrak{q}_0/(\mathfrak{z}(\mathfrak{q}_0) \oplus \mathfrak{q}') \cong \overline{\mathfrak{q}}$ be the canonical projection. Then

$$\pi'' \circ \pi' : \mathfrak{q} \to \overline{\mathfrak{q}}$$

is a surjective Lie algebra homomorphism. It induces a surjective algebraic group homomorphism $\pi : Q \to \overline{Q}$, where $\overline{Q}$ is the Lie subgroup of Q with the associated Lie algebra $\overline{\mathfrak{q}}$. Denote K be the kernel of the homomorphism π. So

$$0 \longrightarrow K \longrightarrow Q \xrightarrow{\pi} \overline{Q} \longrightarrow 0$$

is an exact sequence. Then K is a normal subgroup of Q containing Q^u and $\overline{Q} = Q/K$. Let $\mathfrak{k}$ be the Lie algebra of K. The Dynkin diagram $\mathscr{D}_{\overline{\mathfrak{q}}}$ of $\overline{\mathfrak{q}}$ is of type A_{a+b-1},

$\imath_{\mathfrak{g}}$  .

Then $\mathscr{D}_{\overline{\mathfrak{q}}}$ is a sub-diagram of $\mathscr{D}_{\mathfrak{q}_0}$. Let $\mathrm{i} : \mathscr{D}_{\overline{\mathfrak{q}}} \to \mathscr{D}_{\mathfrak{q}_0}$ be the inclusion map.

Proposition 5.2.1. *There is an incidence system morphism*

$$\Theta : \mathsf{Para}\left(\mathfrak{q}_0\right) \to \mathsf{Para}\left(\overline{\mathfrak{q}}\right), \tag{5.2.2}$$

over the inclusion map $\mathrm{i} : \mathscr{D}_{\overline{\mathfrak{q}}} \to \mathscr{D}_{\mathfrak{q}_0}$ *given by the strict incidence system*

$$\begin{aligned} \theta : \mathrm{i}^{\star}\mathsf{Para}\left(\mathfrak{q}_0\right) &\to \mathsf{Para}\left(\overline{\mathfrak{q}}\right) \\ \mathfrak{p} \Big/ \mathfrak{q}^{\perp} &\mapsto \mathfrak{p}\,/\mathfrak{k}\,, \end{aligned}$$

where $\mathfrak{p} \in \mathscr{P}\left(\mathfrak{q}\right)$.

Proof. First we will show that θ is well-defined. By Proposition 2.2.44, any parabolic subalgebra $\mathfrak{r}$ of $\mathfrak{q}_0$ is equal to $\mathfrak{p}\,/\mathfrak{q}^{\perp}$ for some $\mathfrak{p} \in \mathscr{P}\left(\mathfrak{q}\right)$. Furthermore, if $\mathfrak{r} \in \mathsf{Para}\left(\mathfrak{q}_0\right)_{(\mathrm{i}(j))}$, then the Dynkin diagram $\mathscr{D}_{\mathfrak{r}}$ representing $\mathfrak{r}$ has only one crossed node on the component of $\mathscr{D}_{\mathfrak{q}_0}$ of type A_{a+b-1}; whence $\mathfrak{k} \subseteq \mathfrak{r}$, and so $\theta\left(\mathfrak{r}\right) \in \mathsf{Para}\left(\overline{\mathfrak{q}}\right)_j$ by Proposition 2.2.45. Therefore θ is well-defined and preserves types.

Given two parabolic subalgebras $\mathfrak{p}\,/\mathfrak{q}^{\perp}$ and $\mathfrak{p}'\,/\mathfrak{q}^{\perp}$ in $\mathrm{i}^{\star}\mathsf{Para}\left(\mathfrak{q}_0\right)$ such that $\left(\mathfrak{p}\,/\mathfrak{q}^{\perp}\right) \cap \left(\mathfrak{p}'\,/\mathfrak{q}^{\perp}\right)$ is a parabolic subalgebra of $\mathfrak{q}_0$, then $\mathfrak{p} \cap \mathfrak{p}'$ is a parabolic subalgebra of $\mathfrak{q}$. Hence $\left(\mathfrak{p} \cap \mathfrak{p}'\right)/\mathfrak{r} = \left(\mathfrak{p}\,/\mathfrak{r}\right) \cap \left(\mathfrak{p}'\,/\mathfrak{r}\right)$ is a parabolic subalgebra of $\overline{\mathfrak{q}}$. □

Notice that Q acts on both quotient spaces $\mathfrak{q}_0$ and $\overline{\mathfrak{q}}$ by canonical adjoint actions. These induce actions of Q on $\mathsf{Para}\left(\mathfrak{q}_0\right)$ and $\mathsf{Para}\left(\overline{\mathfrak{q}}\right)$. Furthermore the subgroup $K \subseteq Q$ acts trivially on $\mathsf{Para}\left(\overline{\mathfrak{q}}\right)$. The morphism Θ as defined in Proposition 5.2.1 has the following property:

Lemma 5.2.2. *The morphism* $\Theta : \mathsf{Para}\left(\mathfrak{q}_0\right) \to \mathsf{Para}\left(\overline{\mathfrak{q}}\right)$ is Q-equivariant.

Proof. For any $q \in Q$ and $\mathfrak{p}\,/\mathfrak{q}^{\perp} \in \mathrm{i}^{\star}\mathsf{Para}\left(\mathfrak{q}_0\right)$,

$$q \cdot \theta\left(\mathfrak{p} \Big/ \mathfrak{q}^{\perp}\right) = q \cdot \left(\mathfrak{p}\,/\mathfrak{k}\right) = \left(q \cdot \mathfrak{p}\right)/\mathfrak{k} = \Theta\left(q \cdot \mathfrak{p}\right).$$

□

Recall that $\mathscr{T}$ is the set of all Cartan subalgebras of $\mathfrak{g}$, $\mathscr{U}$ is the set of all pairs $(\mathfrak{t}, \mathfrak{b})$ where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$, $\mathscr{T}^{\mathfrak{q}}$ is the subset of $\mathscr{T}$ containing Cartan subalgebras $\mathfrak{t}$ such that all Borel subalgebras $\mathfrak{b} \supseteq \mathfrak{t}$ are weakly opposite to $\mathfrak{q}$, and $\mathscr{U}^{\mathfrak{q}} := \pi_1^{-1}(\mathscr{T}^{\mathfrak{q}})$. So far we have, for any $u \in \mathscr{U}^{\mathfrak{q}}$,

$$\begin{array}{ccccccc} \mathsf{C}(W) & \xrightarrow{\Upsilon_u} & \mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}) & \xrightarrow{\Phi_{\mathfrak{q}}} & \mathsf{Para}(\mathfrak{q}_0) & \xrightarrow{\Theta} & \mathsf{Para}(\overline{\mathfrak{q}}) \\ \mathscr{D}_{\mathfrak{g}} & \xleftarrow{\mathrm{id}} & \mathscr{D}_{\mathfrak{g}} & \xleftarrow{\nu} & \mathscr{D}_{\mathfrak{q}_0} & \xleftarrow{\mathrm{i}} & \mathscr{D}_{\overline{\mathfrak{q}}}. \end{array}$$

As ν, i and ϱ are injective and $\mathrm{im}(\nu \circ \mathrm{i}) = \mathrm{im}(\varrho)$, there exists a unique bijective map $\varsigma : \mathscr{D}_{\overline{\mathfrak{q}}} \to [a+b]$ such that $\nu \circ \mathrm{i} = \varrho \circ \varsigma$. Let V be a vector space of dimension $a+b$. Choose an isomorphism from $\overline{\mathfrak{q}}$ to $\mathfrak{pgl}(V)$ so that V is a representation of $\overline{\mathfrak{q}}$ with the highest weight $\lambda_{\varsigma^{-1}(1)}$ (see Section 2.2.4). Define an incidence system isomorphism

$$\Xi : \mathsf{Proj}(V) \to \mathsf{Para}(\overline{\mathfrak{q}}),$$

over the map $\varsigma : \mathscr{D}_{\overline{\mathfrak{q}}} \to [a+b]$ given by the strict incidence system isomorphism

$$\begin{aligned} \xi : \varsigma^\star \mathsf{Proj}(V) &\to \mathsf{Para}(\overline{\mathfrak{q}}) \\ W &\mapsto \mathrm{Stab}_{\overline{\mathfrak{q}}}(W). \end{aligned}$$

Since $\overline{Q} \cong \mathrm{PGL}(V)$, this implies that Q acts on V and the subgroup $K \subseteq Q$ acts trivially on V.

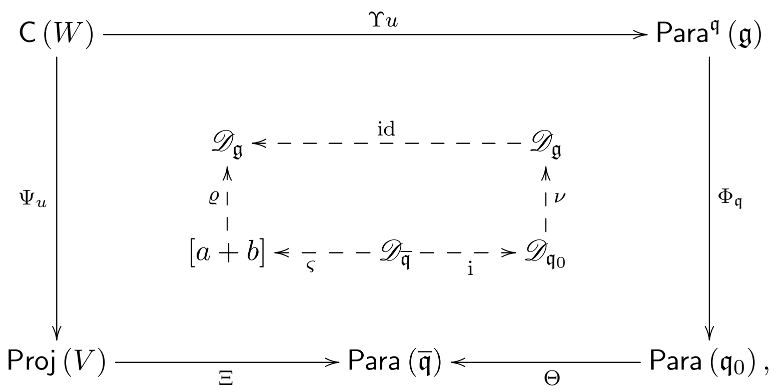
Lemma 5.2.3. *The morphism* $\Xi : \mathsf{Proj}(V) \to \mathsf{Para}(\overline{\mathfrak{q}})$ *is* Q*-equivariant.*

Proof. For any $q \in Q$, $x \in \overline{\mathfrak{q}}$, and $W \in \varsigma^\star \mathsf{Proj}(V)$, we have $(q \cdot x) \cdot (W) = q \cdot \left(x \cdot \left(q^{-1} \cdot W\right)\right)$. Therefore

$$\xi(q \cdot W) = \mathrm{Stab}_{\overline{\mathfrak{q}}}(q \cdot W) = q \cdot \mathrm{Stab}_{\overline{\mathfrak{q}}}(W) = q \cdot \xi(W).$$

□

Therefore, we have a commuting diagram



such that the composition map

$$\Psi_u := \Xi^{-1} \circ \Theta \circ \Phi_{\mathfrak{q}} \circ \Upsilon_u : \mathsf{C}(W) \to \mathsf{Proj}(V)$$

is an incidence system morphism over the injective map $\rho : [a+b] \to \mathscr{D}$ given by the strict incidence system morphism

$$\psi_u := \xi^{-1} \circ \left(\varsigma^{-1}\right)^{\star} \theta \circ \nu^{\star} \varphi_{\mathfrak{q}} \circ \Upsilon_u : \left(\varsigma^{-1}\right)^{\star} \left(\mathrm{i}^{\star}\left(\nu^{\star} \mathsf{C}(W)\right)\right) \to \mathsf{Proj}(V) .$$

5.3 A dimension formula and a conjecture

If A and A' are incidence systems over N and N' respectively, and Q acts on A' in such a way that this action preserves types and incidence relation in A', then the action of Q on A' induces an action of Q on $\mathrm{Mor}_{\vartheta}(\mathsf{A}, \mathsf{A}')$, for some map $\vartheta : N' \to N$, given by

$$(q \cdot \Lambda)(a) = q \cdot (\Lambda(a)) .$$

Thus the group Q acts on the spaces $\mathscr{U}^{\mathfrak{q}}$, $\mathrm{Mor}(\mathsf{C}(W), \mathsf{Para}^{\mathfrak{q}}(\mathfrak{g}))$, $\mathrm{Mor}_{\nu \circ \mathrm{i}}(\mathsf{C}(W), \mathsf{Para}(\overline{\mathfrak{q}}))$, and $\mathrm{GCo}(j,k,n)$. The subgroup $K \subseteq Q$ acts trivially on $\mathrm{Mor}_{\nu \circ \mathrm{i}}(\mathsf{C}(W), \mathsf{Para}(\overline{\mathfrak{q}}))$ and $\mathrm{GCo}_{(a,b,c)}(W,V)$ because it acts trivially on $\mathsf{Para}^{\mathfrak{q}}(\mathfrak{g})$ and $\mathsf{Proj}(V)$. Let $\mathscr{U}^{q}$ be the set defined in Notation 3.7.16. Consider the diagram

$$\begin{array}{ccc} \mathscr{U}^{\mathfrak{q}} & \xrightarrow{\Upsilon} & \operatorname{Mor}\left(\mathsf{C}\left(W\right), \mathsf{Para}^{\mathfrak{q}}\left(\mathfrak{g}\right)\right) \\ \Big\downarrow{\scriptstyle \Psi} & & \Big\downarrow{\scriptstyle (\Theta\circ\Phi_{\mathfrak{q}})_{\star}} \\ \operatorname{GCO}_{(a,b,c)}\left(W,V\right) & \xleftarrow[(\Xi^{-1})_{\star}]{} & \operatorname{Mor}_{\nu\circ\mathrm{i}}\left(\mathsf{C}\left(W\right), \mathsf{Para}\left(\overline{\mathfrak{q}}\right)\right), \end{array}$$

where

$$\left(\Theta \circ \Phi_{\mathfrak{q}}\right)_{\star}(\phi) = \Theta \circ \Phi_{\mathfrak{q}} \circ \phi,$$

for all $\phi \in \operatorname{Mor}\left(\mathsf{C}\left(W\right), \mathsf{Para}^{\mathfrak{q}}\left(\mathfrak{g}\right)\right)$, and

$$\left(\Xi^{-1}\right)_{\star}\left(\phi'\right) = \Xi^{-1} \circ \phi',$$

for all $\phi' \in \operatorname{Mor}_{\nu\circ\mathrm{i}}\left(\mathsf{C}\left(W\right), \mathsf{Para}\left(\overline{\mathfrak{q}}\right)\right)$. By Lemma 3.7.10, Lemma 4.1.3, Lemma 5.2.2, and Lemma 5.2.3, the morphism Ψ is Q-equivariant. As K acts trivially on $\operatorname{GCO}_{(a,b,c)}\left(W,V\right)$, so Ψ induces the map

$$\begin{aligned} [\Psi] : Q\backslash\mathscr{U}^{\mathfrak{q}} &\rightarrow \overline{Q}\backslash\operatorname{GCO}_{(a,b,c)}\left(W,V\right) \\ Q\cdot u &\mapsto \overline{Q}\cdot\Psi_u. \end{aligned} \tag{5.3.1}$$

We want to see if the map $[\Psi]$ in 5.3.1 is a bijective map. However, it is equivalent to see if the map

$$\begin{aligned} \overline{\Psi} : K\backslash\mathscr{U}^{\mathfrak{q}} &\rightarrow \operatorname{GCO}_{(a,b,c)}\left(W,V\right) \\ K\cdot u &\mapsto \Psi_u \end{aligned} \tag{5.3.2}$$

is bijective.

Consider the space $K\backslash\mathscr{U}^{\mathfrak{q}}$, a choice of $(\mathfrak{t},\mathfrak{b}) \in \mathscr{U}^{\mathfrak{q}}$ determines that

$$K\backslash\mathscr{U}^{\mathfrak{q}} \subseteq K\backslash\mathscr{U} \cong K\backslash G/T,$$

where T is a maximal torus of G with Lie algebra $\mathfrak{t}$. By Proposition 3.7.17, we have

$$\dim\left(K\backslash\mathscr{U}^{\mathfrak{q}}\right) = \dim\left(K\backslash G/T\right) = \dim\left(\mathfrak{g}/(\mathfrak{k}+\mathfrak{t})\right).$$

Then by Lemma 3.7.19, $\mathfrak{k}\cap\mathfrak{t} = \{0\}$, and so

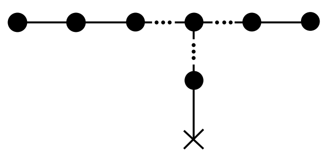
$$\dim\left(K\backslash\mathscr{U}^{\mathfrak{q}}\right) = \dim\left(\mathfrak{g}/(\mathfrak{k}+\mathfrak{t})\right) = \dim\left(\mathfrak{g}/(\mathfrak{k}\oplus\mathfrak{t})\right) = \dim\left(\mathfrak{g}/\mathfrak{k}\right) - \dim\left(\mathfrak{t}\right). \tag{5.3.3}$$

Definition 5.3.1. Denote $C\left(a,b,c\right) := \dim\left(K_{(a,b,c)}\backslash\mathscr{U}^{\mathfrak{q}}_{(a,b,c)}\right)$.

Theorem 5.3.2. *For any $a,b,c\in\mathbb{N}$ such that $a\geq 2$,*

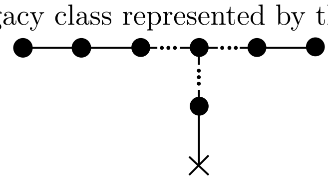
$$C\left(a,b,c\right) = (a+b-1) + C(a-1,b,c) + dim\left(\mathfrak{p}^{\perp}\right).$$

where $\mathfrak{p}$ is the Lie algebra of the parabolic subgroup P in the conjugacy class represented by the decorated Dynkin diagram



containing a pair $(\mathfrak{t},\mathfrak{b})\in\mathscr{U}^{\mathfrak{q}}$.

Proof. Let $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{q}$ be respectively the Lie algebras of the algebraic groups G, K, and Q as defined in Section 5.2 depending on (a,b,c). Let $(\mathfrak{t},\mathfrak{b})\in\mathscr{U}^{\mathfrak{q}}$ and P be a parabolic subgroup of G in the conjugacy class represented by the decorated Dynkin diagram



such that its Lie algebra $\mathfrak{p}$ contains $\mathfrak{b}$. Then $\mathfrak{k}/\mathfrak{q}^{\perp}\subseteq\left((\mathfrak{p}\cap\mathfrak{q})+\mathfrak{q}^{\perp}\right)/\mathfrak{q}^{\perp}$. This implies that $\left(\left(\mathfrak{p}^{\perp}\cap\mathfrak{q}\right)+\mathfrak{q}^{\perp}\right)/\mathfrak{q}^{\perp}\subseteq\left(\mathfrak{k}/\mathfrak{q}^{\perp}\right)^{\perp}$; whence $\mathfrak{p}^{\perp}\cap\mathfrak{k} = \left(\mathfrak{p}^{\perp}\cap\mathfrak{q}\right)\cap\mathfrak{k} = \{0\}$. The filtration

$$0\subseteq\mathfrak{p}^{\perp}\subseteq\mathfrak{p}\subseteq\mathfrak{g}$$

of $\mathfrak{g}$ induces a filtration

$$0\subseteq\mathfrak{p}^{\perp}\cong\left(\mathfrak{k}\oplus\mathfrak{p}^{\perp}\right)/\mathfrak{k}\subseteq(\mathfrak{k}+\mathfrak{p})/\mathfrak{k}\subseteq\mathfrak{g}/\mathfrak{k}$$

of $\mathfrak{g}/\mathfrak{k}$. Therefore, from equation (5.3.3),

$$\begin{aligned} C(a,b,c) &= \dim\left(\mathfrak{g}/(\mathfrak{k}\oplus\mathfrak{t})\right) \\ &= \dim\left(\mathfrak{g}/\mathfrak{k}\right) - \dim(\mathfrak{t}) \\ &= \dim\left(\mathfrak{g}/(\mathfrak{p}+\mathfrak{k})\right) + \dim\left((\mathfrak{k}+\mathfrak{p})\Big/\left(\mathfrak{k}\oplus\mathfrak{p}^{\perp}\right)\right) + \dim\left(\mathfrak{p}^{\perp}\right) - \dim(\mathfrak{t}) \\ &= \dim\left(\mathfrak{g}/(\mathfrak{p}+\mathfrak{k})\right) + \left(\dim\left((\mathfrak{k}+\mathfrak{p})\Big/\left(\mathfrak{k}\oplus\mathfrak{p}^{\perp}\right)\right) - \dim(\mathfrak{t})\right) + \dim\left(\mathfrak{p}^{\perp}\right). \end{aligned}$$

As $\mathfrak{g}/(\mathfrak{p}+\mathfrak{k}) = (\mathfrak{p}+\mathfrak{q})/(\mathfrak{p}+\mathfrak{k})$, and $\mathfrak{p}+\mathfrak{q}^{\perp}\subseteq\mathfrak{p}+\mathfrak{k}\subseteq\mathfrak{p}+(\mathfrak{p}\cap\mathfrak{q})+\mathfrak{q}^{\perp}\subseteq\mathfrak{p}+\mathfrak{q}^{\perp}$, so

$$\mathfrak{g}/(\mathfrak{p}+\mathfrak{k}) = (\mathfrak{p}+\mathfrak{q})\Big/\left(\mathfrak{p}+\mathfrak{q}^{\perp}\right) \cong \mathfrak{q}\Big/\left((\mathfrak{p}\cap\mathfrak{q})+\mathfrak{q}^{\perp}\right).$$

Note that $\mathfrak{q}/\left((\mathfrak{p}\cap\mathfrak{q})+\mathfrak{q}^{\perp}\right)$ is the tangent space of $Q/\left((P\cap Q)\,Q^{u}\right)$. As $Q/K \cong \mathrm{PGL}_{a+b}(\mathbb{F})$ and $\left((P\cap Q)\,Q^{u}\right)/K \cong \mathrm{GL}_{a+b-1}(\mathbb{F}) \ltimes \left(\mathbb{F}^{a+b-1}\right)^{\star}$, thus $Q/\left((P\cap Q)\,Q^{u}\right) \cong \mathbb{P}^{a+b-1}$. Hence $\dim\left(\mathfrak{g}/(\mathfrak{p}+\mathfrak{k})\right) = a+b-1$.

Let $\mathfrak{g}'$, $\mathfrak{k}'$, and $\mathfrak{q}'$ be respectively the Lie algebras of the algebraic groups G', K', and Q' as defined in Section 5.2 depending on $(a-1,b,c)$. Then

$$\dim\left(\mathfrak{p}\Big/\mathfrak{p}^{\perp}\right) = \dim\left(\mathfrak{g}'\right)+1,$$

$$\dim\left(\left((\mathfrak{q}\cap\mathfrak{p})+\mathfrak{p}^{\perp}\right)\Big/\mathfrak{p}^{\perp}\right) = \dim\left(\mathfrak{q}'\right)+1,$$

by Proposition 4.2.2, and

$$\dim\left(\left((\mathfrak{k}\cap\mathfrak{p})+\mathfrak{p}^{\perp}\right)\Big/\mathfrak{p}^{\perp}\right) = \dim\left(\mathfrak{k}'\right).$$

Since $(\mathfrak{k}+\mathfrak{p})/\left(\mathfrak{k}\oplus\mathfrak{p}^{\perp}\right) \cong \mathfrak{p}/\left((\mathfrak{k}\cap\mathfrak{p})+\mathfrak{p}^{\perp}\right)$, hence

$$\begin{aligned} \dim\left((\mathfrak{k}+\mathfrak{p})\Big/\left(\mathfrak{k}\oplus\mathfrak{p}^{\perp}\right)\right) &= \dim\left(\mathfrak{p}\Big/\mathfrak{p}^{\perp}\right) - \dim\left(\left((\mathfrak{k}\cap\mathfrak{p})+\mathfrak{p}^{\perp}\right)\Big/\mathfrak{p}^{\perp}\right) \\ &= \dim\left(\mathfrak{g}'\right)+1-\dim\left(\mathfrak{k}'\right) \\ &= \dim\left(\mathfrak{g}'/\mathfrak{k}'\right)+1. \end{aligned}$$

Thus $\dim\left((\mathfrak{k}+\mathfrak{p})/\left(\mathfrak{k}\oplus\mathfrak{p}^{\perp}\right)\right) - \dim(\mathfrak{t}) = \dim\left(\mathfrak{g}'/(\mathfrak{k}'\oplus\mathfrak{t}')\right) = C(a-1,b,c)$, where $\mathfrak{t}'$ is a

Cartan subalgebra of $\mathfrak{g}'$ contained in $\mathscr{T}^{\mathfrak{q}'}$ and $\dim(\mathfrak{t}') = \dim(\mathfrak{t}) - 1$. Therefore

$$C(a,b,c) = (a+b-1) + C(a-1,b,c) + \dim\left(\mathfrak{p}^{\perp}\right).$$

□

We shall see later that, in the cases $a = 1$,$b = 1$, or $c = 1$,

$$K\backslash\mathscr{U}^{\mathfrak{q}} \cong \mathrm{GCo}^{inj}_{(a,b,c)}(W,V).$$

Hence, we make the following conjecture for general (a,b,c).

Conjecture 5.3.3. $K\backslash\mathscr{U}^{\mathfrak{q}} \cong \mathrm{GCo}^{inj}_{(a,b,c)}(W,V)$.

To see whether Conjecture 5.3.3 is true or not, we need consider two following conditions:

(C1) $\overline{\Psi}$ is injective, i.e., $\Psi_u = \Psi_{u'} \Rightarrow u' \in K \cdot u$.

(C2) $\mathrm{im}(\Psi) = \mathrm{GCo}^{inj}_{(a,b,c)}(W,V)$.

If the Conjecture 5.3.3 is true, then Theorem 5.3.2 tells us that

$$\dim\left(\mathrm{GCo}^{inj}_{(a,b,c)}(W,V)\right) = (a+b-1) + \dim\left(\mathrm{GCo}^{inj}_{(a-1,b,c)}(W,V)\right) + \dim\left(\mathfrak{p}^{\perp}\right). \quad (5.3.4)$$

This suggests us that non-degenerate generalized Cox configurations should be recursively constructible.

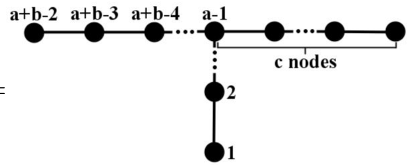
5.4 Recursive construction of generalized Cox configurations

In this section, we will explain a recursive construction of non-degenerate generalized Cox configurations of type (a,b,c) which will explain why (5.3.4) holds. Firstly, we recall all the notations used in Section 5.2 and Section 5.3, and we will use them throughout this section. We will see that this recursive construction is similar to Longuet-Higgins' construction for Clifford configuration (see [LH72], Section 7) but still somehow different.

By Proposition 3.2.6, the residual incidence geometry $\mathrm{Res}\left(\left\{W_{\varrho(1)}\right\}\right)$ in $\varrho^{\star}\mathsf{C}(W)$ is isomorphic to the coset incidence geometry

$$\mathsf{C}\left(W_{\varrho(1)}; \left(W_{\varrho(i)} \cap W_{\varrho(1)}\right)_{:i\in N\backslash\{1\}}\right),$$

which is the coset incidence geometry $(\varrho')^{\star}(\mathsf{C}(W'))$, where W' is a Coxeter group with the Coxeter diagram $T_{(a-1,b,c)}$,

$T_{(a-1,b,c)} :=$ 

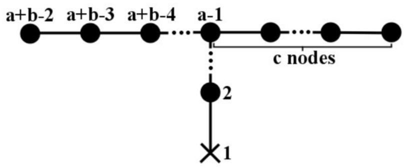
,

and $\varrho' : [a+b-1] \to T_{(a-1,b,c)}$ given by the above labelling.

Recall from Section 2.2.3, $\mathfrak{p}^{\perp}$ is nilpotent, and so it has a filtration

$$\{0\} = C_{m+1}\left(\mathfrak{p}^{\perp}\right) \subseteq C_m\left(\mathfrak{p}^{\perp}\right) \subseteq C_{m-1}\left(\mathfrak{p}^{\perp}\right) \subseteq \ldots \subseteq C_1\left(\mathfrak{p}^{\perp}\right) = \mathfrak{p}^{\perp},$$

where m is the depth of $\mathfrak{p}^{\perp}$, and we write $\mathfrak{p}_{-i} := C_i\left(\mathfrak{p}^{\perp}\right)/C_{i+1}\left(\mathfrak{p}^{\perp}\right)$. Since (a,b,c) satisfies the inequality (5.2.1), thus $\mathfrak{p}_{-1}$ is a miniscule representation with the highest weight corresponding to the crossed node



of the Levi factor of $\mathfrak{p}$ (see [Gre08], p.32), thus we have

$$\dim(\mathfrak{p}_{-1}) = \text{the number of the elements in } \mathsf{C}\left(W'\right)_{\varrho'(1)}. \tag{5.4.1}$$

Hence Equation 5.3.4 suggests a recursive procedure to construct a non-degenerate generalized Cox configuration of type (a,b,c) in the projective space $\mathbb{P}^{a+b-1}(V)$, as follows: for $a,b,c \in \mathbb{N}$ such that $a \geq 2$,

(1) Choose a point p_0 in $\mathbb{P}^{a+b-1}(V)$; this use $a+b-1$ parameters.

(2) Choose a generalized Cox configuration of type $(a-1,b,c)$ in the projective space $\mathbb{P}^{a+b-2}(V/p_0)$ of lines passing through p_0; this use $\dim\left(\mathrm{GCo}^{inj}_{(a-1,b,c)}(W,V)\right)$ parameters.

(3) Choose generally a point, different from p_0, on each line in the configuration passing through the point p_0; the number of parameters used in this step is equal to the number of lines passing through p_0, i.e., the number of elements in $\mathsf{C}(W')_{\varrho'(1)}$, and so by Equation 5.4.1, it is equal to $\dim(\mathfrak{p}_{-1})$.

(4) Finally, in order to complete the configuration, we should choose $\dim\left(\left[\mathfrak{p}^{\perp},\mathfrak{p}^{\perp}\right]\right)$ more parameters.

The table below lists $\dim\left(\mathfrak{p}^{\perp}\right)$ of some cases that (a,b,c) satisfied the inequality (5.2.1).

(a,b,c)	$\dim(\mathfrak{p}_{-1})$	$\dim(\mathfrak{p}_{-2})$	$\dim(\mathfrak{p}_{-3})$
$(1,b,c)$	bc	0	0
$(a,1,c)$	$a+c-1$	0	0
$(a,b,1)$	$a+b-1$	0	0
$(a,2,2)$	$2(a+1)$	0	0
$(2,b,2)$	$\binom{b+2}{b}$	0	0
$(2,2,c)$	$\binom{c+2}{c}$	0	0
$(2,3,3)$	20	1	0
$(2,3,4)$	35	7	0
$(2,3,5)$	56	28	8
$(2,4,3)$	35	7	0
$(2,5,3)$	56	28	8

Table 5.1: A table representing the dimension of $\mathfrak{p}^{\perp}_{(a,b,c)}$ for some (a,b,c)

Definition 5.4.1. We will call the configuration, obtained from Step (1), Step (2) and Step (3), an (a,b,c)-**STARTING CONFIGURATION**.

Therefore we make the following conjecture.

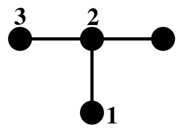
Conjecture 5.4.2. *For $a,b,c\in\mathbb{N}$ such that $a\geq 2$, Let V be a complex vector space of dimension $a+b$. Given an (a,b,c)-starting configuration in $\mathbb{P}^{a+b-1}(V)$, then constructing a non-degenerate generalized Cox configuration of type (a,b,c) requires $\dim\left(\left[\mathfrak{p}^{\perp},\mathfrak{p}^{\perp}\right]\right)$ more parameters.*

Remark 5.4.3. By showing that Conjecture 5.4.2 is true, the equation 5.3.4 is true.

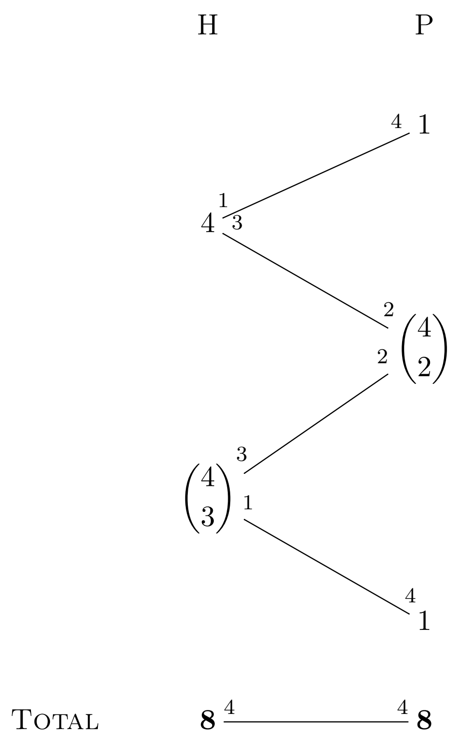
Let $\Psi:\mathsf{C}(W)\to\mathsf{Proj}(V)$ be a generalized Cox configuration of type (a,b,c). As $\mathrm{im}(\Psi)\subseteq\mathsf{Proj}(V)$, we can consider $\Psi(\mathsf{C}(W))$ as a configuration in $\mathbb{P}^{a+b-1}(V)$ and we can say about the incidence relation of Ψ by considering the incidence relation of $\Psi(\mathsf{C}(W))$, and so of $\mathsf{C}(W)$. We will focus on the incidence relation between $\varrho^{\star}(\mathsf{C}(W))_1$, corresponding to a collection of points in $\mathbb{P}^{a+b-1}(V)$, and $\varrho^{\star}(\mathsf{C}(W))_{a+b-1}$, corresponding to a collection of hyperplanes in $\mathbb{P}^{a+b-1}(V)$. To do this, we are going to use branched summaries (see Remark 3.1.17) and summaries (see Definition 3.1.16) to represent such incidence relation.

Any $\varrho^{\star}\left(\mathsf{C}\left(W\right)\right)_i$, where $1 < i < a+b-1$, corresponds to a collection of $(i-1)$-dimensional subspaces in $\mathbb{P}^{a+b-1}\left(V\right)$ which can be obtained by intersecting hyperplanes corresponding to $\varrho^{\star}\left(\mathsf{C}\left(W\right)\right)_{a+b-1}$ or being spanned by points corresponding to $\varrho^{\star}\left(\mathsf{C}\left(W\right)\right)_1$.

Example 5.4.4. Consider a generalized Cox configuration of type $(2,2,2)$,



its branched summary is represented by the graph



Any lines in this configuration can be obtained by the intersection of any two hyperplanes in the configuration or spanned by any two points in the configuration.

5.5 Generalized Cox configurations of A-type

This is the case of generalized Cox configurations of types $(1,b,c)$, $(a,1,c)$, or $(a,b,1)$. These types corresponds to classical configurations. The incidence geometry $\mathsf{C}\left(W\right)$ corresponding to the diagram of type A_n has the incidence structure of an abstract n-simplex. For this section, we denote $n := a + b + c - 2$.

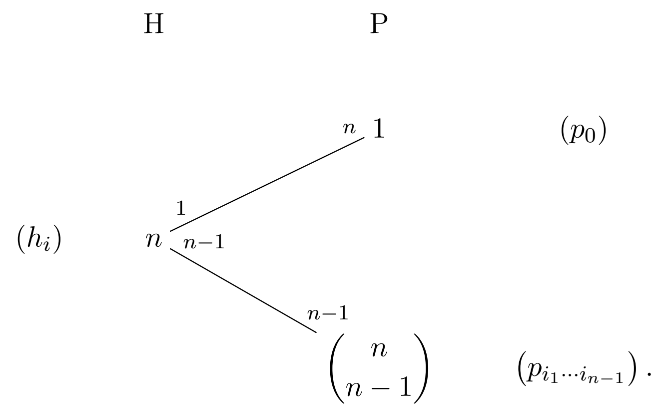
5.5.1 Generalized Cox configurations of type $(a, b, 1)$

Let V be a vector space of dimension $n+1$ and $G := \mathrm{PGL}_{n+1}(V)$. As in Section 5.2, the parabolic subgroup Q of G, we choose in this case, is actually G itself and K is $\{e\}$. Then $\mathscr{U}^{\mathfrak{q}} = \mathscr{U}$. Thus the map $\overline{\Psi}$ defined in (5.3.2) is actually the map Υ defined in (3.7.2), and so generalized Cox configurations of this type are actually standard parabolic configurations (see Definition 3.7.6). Any $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}$ gives us an ordered basis $\{v_0, v_1, \ldots, v_n\}$ (up to a scale) of V; this basis gives us an n-simplex, i.e., a collection of $n+1$ points whose homogeneous coordinates form a basis of V, in V.

By Proposition 3.7.11, the condition **(C1)** of Conjecture 5.3.3 is true. By Lemma 3.7.9 and Theorem 3.7.13, the condition **(C1)** of Conjecture 5.3.3 is true. Therefore Conjecture 5.3.3 is true.

Proposition 5.5.1. *Let V be a vector space of dimension $n+1$. Given a $(a, b, 1)$-starting configuration in $\mathbb{P}^n(V)$, a configuration in $\mathrm{GCO}^{inj}_{(a,b,1)}$ can be uniquely obtained from the starting configuration.*

Proof. The starting configuration has the following branched summary



The collection of n general points different from p_0 uniquely determines a new hyperplane, say h. Therefore we have $n+1$ hyperplanes and $n+1$ points, with n hyperplanes through each point and n points on each hyperplane, as in Figure 5.5.1.

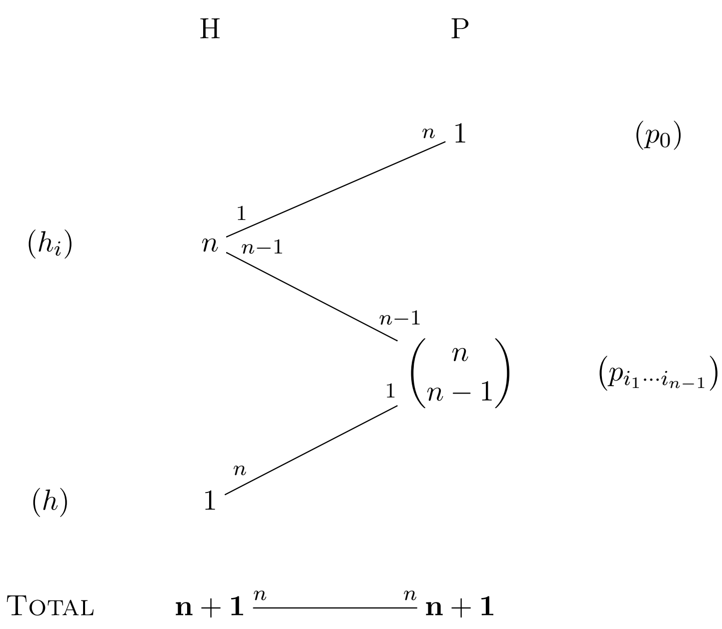


Figure 5.5.1: The branched summary for $\mathrm{GCO}_{(a,b,1)}$.

□

Hence Conjecture 5.4.2 is true and generalized Cox configurations of type $(a, b, 1)$ are n-simplices in $\mathbb{P}^n$.

5.5.2 Generalized Cox configurations of type $(a, 1, c)$

Let V be a vector space of dimension $n+1$ and $G := \mathrm{PGL}_{n+1}(V)$. Let U be a vector subspace of V of dimension $n-a$ and Q be the maximal parabolic subgroup of G stabilizing U. Let K be the subgroup of Q consisting of elements in $\mathrm{PGL}_{n+1}(V)$ fixing $\mathbb{P}^a(V/U)$ and $\overline{Q} = Q/K$. Then $\mathsf{Para}(\mathfrak{g})^{\mathfrak{q}} \subseteq \mathsf{Para}(\mathfrak{g}) = \mathsf{Proj}(V)$ and $\mathsf{Para}(\overline{\mathfrak{q}}) = \mathsf{Proj}(V/U)$. Any $\mathfrak{t} \in \mathscr{T}^{\mathfrak{q}}$ corresponds to the basis $\{v_0, v_1 \dots, v_n\}$ (up to a scale) of V.This basis $\{v_0, v_1 \dots, v_n\}$ forms an n-simplex in V. Any Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{t}$ corresponds to an ordering on $\{v_0, v_1 \dots, v_n\}$. As $\mathfrak{t} \in \mathscr{T}^{\mathfrak{q}}$, the subspace, spanned by any collection of $a+1$ vectors in $\{v_0, v_1 \dots, v_n\}$, intersects U trivially by Corollary 2.2.48. In particular,

$$\mathfrak{t} \in \mathscr{T}^{\mathfrak{q}} \Leftrightarrow \left\langle v_{i_1}, \dots, v_{i_{a+1}} \right\rangle \cap U = \{0\} \text{ for all } 0 \le i_1 < i_2 < \cdots < i_{a+1} \le n,$$

where $\{v_0, v_1 \dots, v_n\}$ is the basis (up to scale) of V corresponding to $\mathfrak{t}$.

As in Example 2.2.61, the incidence system morphism $\Theta \circ \Phi_{\mathfrak{q}} : \mathsf{Para}(\mathfrak{g})^{\mathfrak{q}} \to \mathsf{Para}(\overline{\mathfrak{q}})$

sending $Stab_{\mathfrak{g}}(W)$ to

$$Stab_{\mathfrak{g}}(U/U \subseteq (U+W)/U \subseteq V/U);$$

whence it is a classical projection $\mathbb{P}^n(V) \to \mathbb{P}^a(V/U)$.

Therefore the image of $\{v_0, v_1 \ldots, v_n\}$ under the map Ψ is a non-degenerate collection of $n+1$ points, i.e., any $a+1$ points in the collection span $\mathbb{P}^a(V/U)$, in $\mathbb{P}^a(V/U)$. . Next we will show that generalized Cox configurations of type $(a,1,c)$ are obtained by the classical projection of n-simplices in $\mathbb{P}^n(V)$ into $\mathbb{P}^a(V/U)$.

Theorem 5.5.2. *Any non-degenerate collection of $n+1$ points in $\mathbb{P}^a(V/U)$ is the projection of an n-simplex in some projection $\mathbb{P}^n(V) \to \mathbb{P}^a(V/U)$. Moreover, if any two simplices project to the same collection of points in $\mathbb{P}^a(V/U)$, then they are related by an element of $PGL_{n+1}(V)$ fixing $\mathbb{P}^a(V/U)$.*

Proof. The standard homogeneous coordinates of the $n+1$ points can be represented as the columns of a $(a+1) \times (n+1)$ matrix M. By the non-degeneracy condition, every $a+1$ columns span V/U, and so the rows of M are linearly independent. Thus the collection of rows of M can be extended to a basis of V, i.e., an $(n+1) \times (n+1)$ invertible matrix M'. The columns of M' are homogeneous coordinates of an n-simplex projected onto the collection $n+1$ points in $\mathbb{P}^a(V/U)$ as required.

Suppose that M'' is an $(n+1) \times (n+1)$ invertible matrix whose the first $a+1$ rows are same as the $a+1$ rows of M'. Then there exists an $(n+1) \times (n+1)$ matrix $B := (M'')(M')^{-1}$ of the form

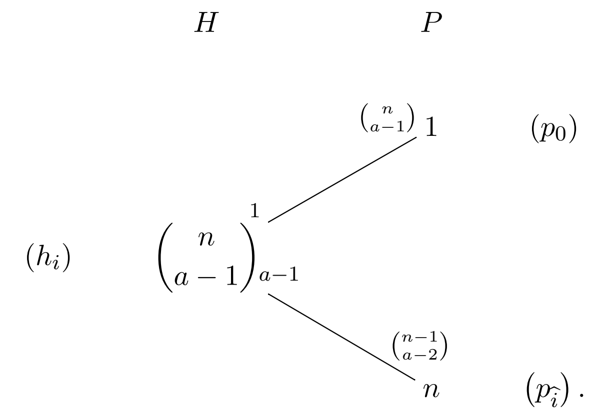
$$\begin{pmatrix} I_{a+1} & 0 \\ \star & \star \end{pmatrix}, \tag{5.5.1}$$

where I_{a+1} is the $(a+1) \times (a+1)$ identity matrix, such that $M'' = BM'$. □

From the proof of Theorem 5.5.2, we can extend the matrix M so that the span of any $a+1$ columns intersects the space U trivially; so the extended matrix M' corresponds to a $(\mathfrak{t}, \mathfrak{b}) \in \mathscr{U}^{\mathfrak{q}}$. Therefore Theorem 5.5.2 implies the Conjecture 5.3.3 is true. The following proposition shows that Conjecture 5.4.2 is also true.

Proposition 5.5.3. *Let V be a complex vector space of dimension $a+1$. Given a $(a,1,c)$-starting configuration in $\mathbb{P}^a(V)$, a configuration in $\mathrm{GCO}^{inj}_{(a,1,c)}$ can be uniquely obtained from the starting configuration.*

Proof. The starting configuration has the following branched summary



Any a points in the collection of n general points which are different from p_0 uniquely determine a hyperplane; there are $\binom{n}{a}$ such hyperplanes. Therefore we have $\binom{n}{a-1} + \binom{n}{a} = \binom{n+1}{a}$ hyperplanes and $n+1$ points, with $\binom{n}{a-1}$ hyperplanes through each point and a points on each hyperplanes, as in Figure 5.5.2. □

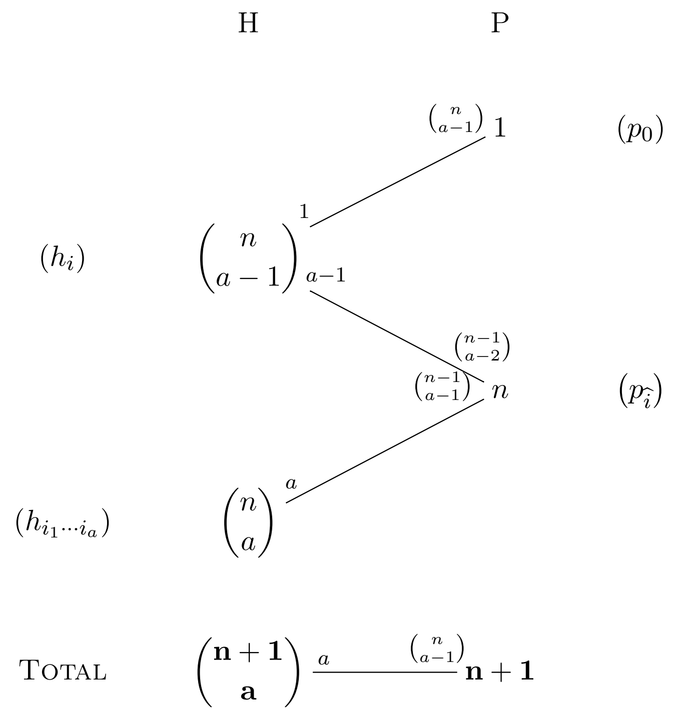


Figure 5.5.2: The branched summary for $\text{GCO}_{(a,1,c)}$.

5.5.3 Generalized Cox configurations of type $(1, b, c)$

Generalized Cox configurations of this type is the dual configurations of generalized Cox configurations of type $(b, 1, c)$. Thus generalized Cox configurations of this type are obtained by intersecting n-simplices with a projective subspace of dimension b. Since Conjecture 5.3.3 in the case $(b, 1, c)$ is true, by considering $\mathrm{GCo}^{inj}_{(1,b,c)}$ as the space of dual configurations of the configurations in $\mathrm{GCo}^{inj}_{(b,1,c)}$, both conjectures are also true in this case.

In fact, we can construct a configuration in $\mathrm{GCo}^{inj}_{(1,b,c)}$. To see this, let V be a complex vector space of dimension $b+1$. Given a point p_0 in $\mathbb{P}^b(V)$ and a configuration in the space $\mathbb{P}^{b-1}(V/p_0)$ of b hyperplanes (i.e., hyperplanes in $\mathbb{P}^b(V)$ through p_0) and b points (i.e., lines in $\mathbb{P}^b(V)$ through p_0) such that any $b-1$ hyperplanes passing through each point and any $b-1$ points lie on each hyperplane. Choose c general hyperplanes in $\mathbb{P}^b(V)$ different from those already exist; this use $\dim(\mathfrak{p}^{\perp}) = bc$ (see Table 5.1) parameters because, in order to choose a hyperplane in $\mathbb{P}^b(V)$, we need b parameters. Intersecting all the hyperplanes we obtain so far, it gives totally $\binom{n+1}{b}$ points in the configuration. Therefore we have $n+1$ hyperplanes and $\binom{n+1}{b}$ points, with b hyperplanes through each point and $\binom{n}{b-1}$ points on each hyperplanes.

5.6 Generalized Cox configurations of D-type

This is the case of generalized Cox configurations of types $(a, 2, 2)$, $(2, b, 2)$, or $(2, 2, c)$. In this section, we are going to show that Conjecture 5.4.2 is true for the case $(2, b, 2)$ and $(2, 2, c)$. According to Table 5.1, $\dim([\mathfrak{p}^{\perp}, \mathfrak{p}^{\perp}]) = 0$ for these cases. Hence Conjecture 5.4.2 states that the starting configurations of these types uniquely determine non-degenerate generalized Cox configurations of these types. The key ingredient to investigate these two cases is the following classical result.

Theorem 5.6.1. *(Möbius theorem) Let $PP_{12}P_{13}P_{23}$ and $P_{14}P_{24}P_{34}P_{1234}$ be two Tetrahedra in a projective 3-space such that the vertices P, P_{12}, P_{13}, and P_{23} lie on the faces $p_4 := \langle P_{14}, P_{24}, P_{34}\rangle$, $p_{124} := \langle P_{14}, P_{24}, P_{1234}\rangle$, $p_{134} := \langle P_{14}, P_{34}, P_{1234}\rangle$, and $p_{234} := \langle P_{24}, P_{34}, P_{1234}\rangle$ respectively of the second tetrahedron. If the vertices P_{14}, P_{24}, and P_{34} lie on the faces $p_1 := \langle P, P_{12}, P_{13}\rangle$, $p_2 := \langle P, P_{12}, P_{23}\rangle$, and $p_3 := \langle P, P_{13}, P_{23}\rangle$ respectively of the first tetrahedron, then the vertex P_{1234} must be incident with the plane $p_{123} := \langle P_{12}, P_{13}, P_{23}\rangle$.*

Proof. Assume that the vertices P_{14}, P_{24}, P_{34} are incident with the plane $p_1 := \langle P, P_{12}, P_{13} \rangle$, $p_2 := \langle P, P_{12}, P_{23} \rangle$, $p_3 := \langle P, P_{13}, P_{23} \rangle$, respectively. Let m be the intersection line of planes p_3 and p_{124}. As P is incident with p_3, $P_{34}P_{13}P_{23}P$ is a complete quadrangle giving a quadrangular set $\mathcal{Q}(ABC, DEF)$ on m.

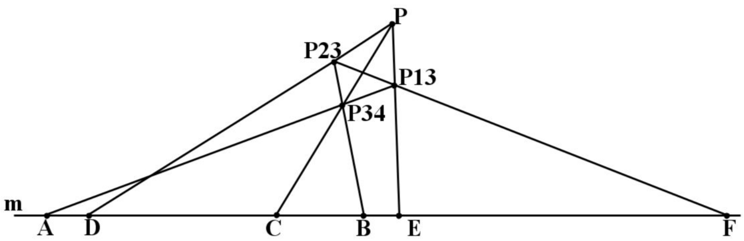


Figure 5.6.1: The quadrangular set $\mathcal{Q}(ABC, DEF)$ on m.

Then the point A is on the plane p_{134} and so the line $P_{14}P_{1234}$. Similarly, B, C, D, E are on the lines $P_{24}P_{1234}$, $P_{14}P_{24}$, $P_{12}P_{24}$, $P_{12}P_{24}$ respectively.

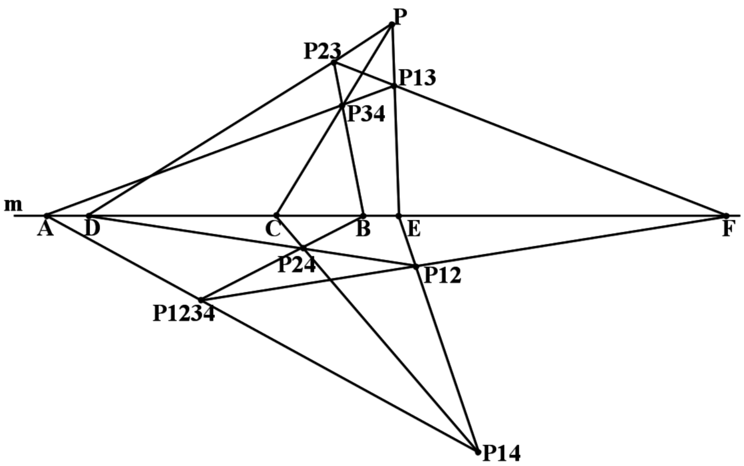


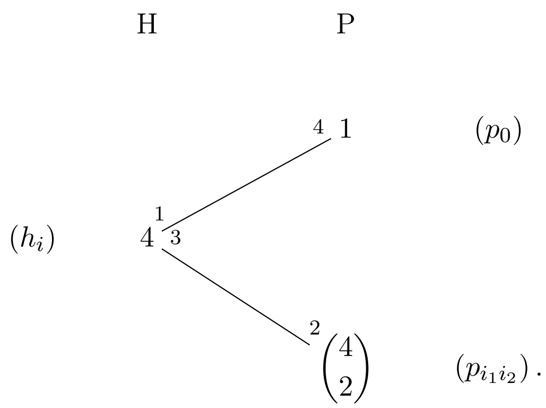
Figure 5.6.2: The quadrangular sets $\mathcal{Q}(ABC, DEF)$ and $\mathcal{Q}(DEF, ABC)$ on m.

Since $\mathcal{Q}(ABC, DEF)$ implies $\mathcal{Q}(DEF, ABC)$ and each point of a quadrangle set is uniquely determined by the rest points (see [Cox61], page 240-241), the point P_{1234} is on the line $P_{12}F$. Since the line $P_{12}P_{1234}$ pass through the vertex F on $P_{13}P_{23}$, P_{1234} is incident with the plane p_{123}. $\square$

According to Table 5.1, given a starting configuration of the case $(2, b, 2)$ or $(2, 2, c)$, we need to choose no more parameter to complete the configuration because $\dim\left(\left[\mathfrak{p}^{\perp}, \mathfrak{p}^{\perp}\right]\right) = 0$, i.e., the starting configuration uniquely determines a non-degenerate generalized configuration.

Proposition 5.6.2. *Let V be a complex vector space of dimension 4. Given a $(2,2,2)$-starting configuration in $\mathbb{P}^3(V)$, a configuration in* $\mathrm{GCo}^{inj}_{(2,2,2)}$ *can be uniquely obtained from the starting configuration.*

Proof. The starting configuration has the following branched summary



For $1 \le i_1 < i_2 < i_3 \le 4$, the points $p_{i_1 i_2}$, $p_{i_1 i_3}$ and $p_{i_2 i_3}$ uniquely determine a plane $h_{i_1 i_2 i_3}$; there are $\binom{4}{3} = 4$ such planes. Notice that the planes h_{124}, h_{134} and h_{234} meet in a point, say q.

Now consider a tetrahedron T formed by the planes h_1, h_2, h_3, h_{123} and a tetrahedron T$'$ formed by the planes h_4, h_{124}, h_{134}, h_{234}.

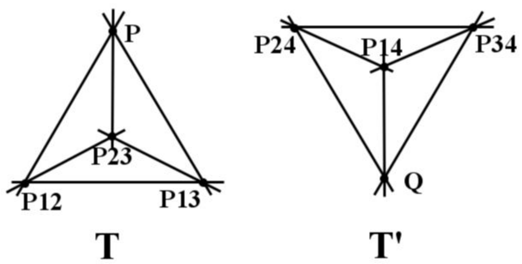


Figure 5.6.3: Tetrahedra T and T$'$.

From Figure 5.6.3, these two tetrahedra satisfy the condition in Theorem 5.6.1. This implies that the point q lies on the plane h_{123}. Thus the planes $h_{i_1 i_2 i_3}$'s, where $1 \le i_1 < i_2 < i_3 \le 4$, meet in a point, say p_{1234}. Therefore we have totally 8 planes and 8 points, with 4 planes through each point and 4 points on each planes, as in Figure 5.6.4.

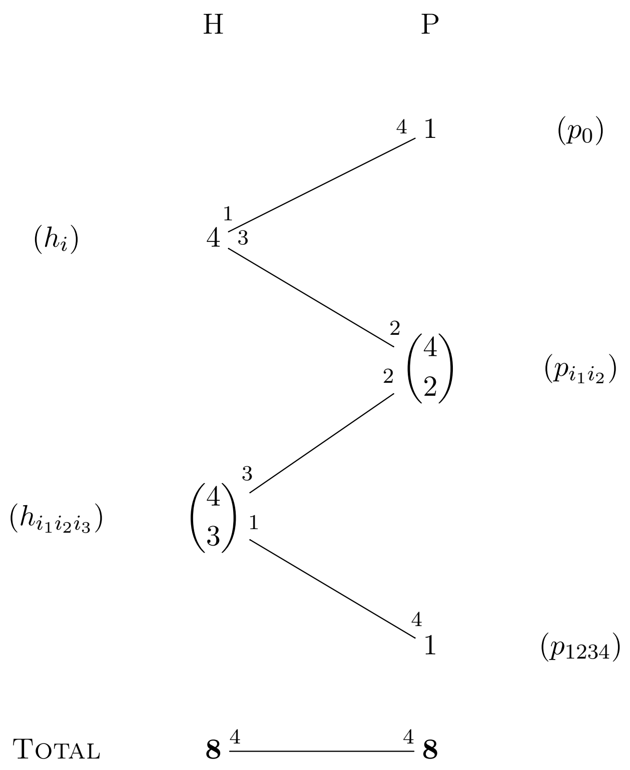


Figure 5.6.4: The branched summary for $\mathrm{GCo}_{(2,2,2)}$.

□

Remark 5.6.3. Proposition 5.6.2 is Cox's formulation of Möbius theorem. That is, given four general planes a, b, c, d through a point p_0 in a projective 3-space, choose a point, say ab, on the line of intersection of any pair of planes, say a and b; there are six such points. Since any three points like ab, bc, ac generate a plane, say abc, there are four such planes which, by Möbius theorem, intersect in a point, say $abcd$.

Proposition 5.6.2 is then the first Theorem in Cox's chain of Theorems (see [Cox50], p. 446-447).

Proposition 5.6.4. *Let V be a complex vector space of dimension* 4. *Given a* $(2, 2, c)$*-starting configuration in* $\mathbb{P}^3(V)$*, a configuration in* $\mathrm{GCo}^{inj}_{(2,2,c)}$ *can be uniquely obtained from the starting configuration.*

Proof. We claim that if n is even (resp. odd), we obtain the left (resp. right) branched summary as in Figure 5.6.5.

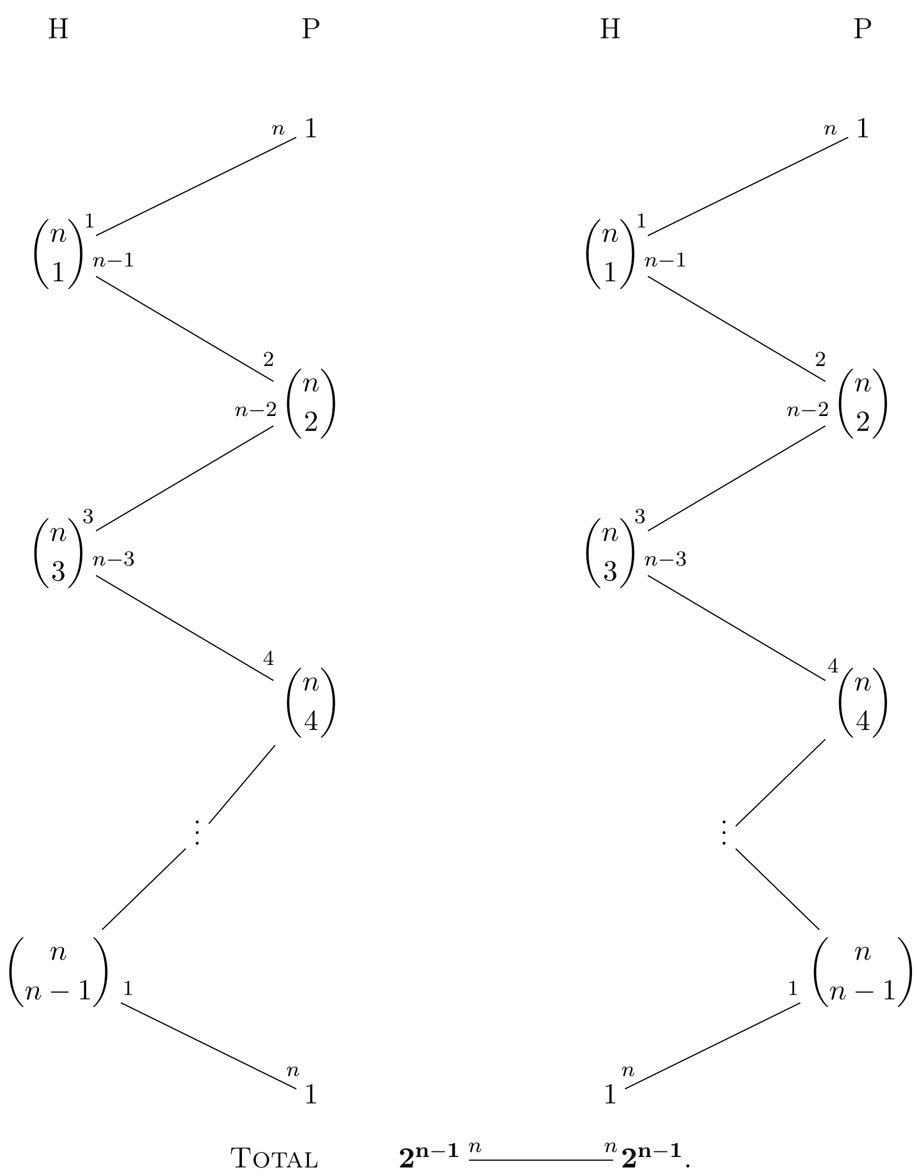
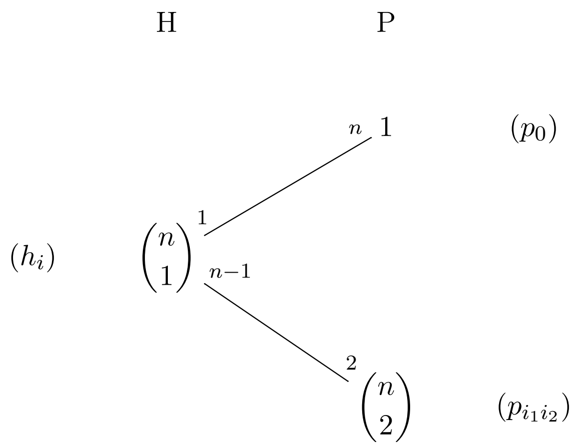


Figure 5.6.5: The branched summary for $\mathrm{GCo}_{(2,2,c)}$where n is even (on the left) and odd (on the right) respectively.

We will prove the proposition by induction on $n \geq 4$. The case when $n = 4$ is already proved in Proposition 5.6.2. Now suppose that $n > 4$.

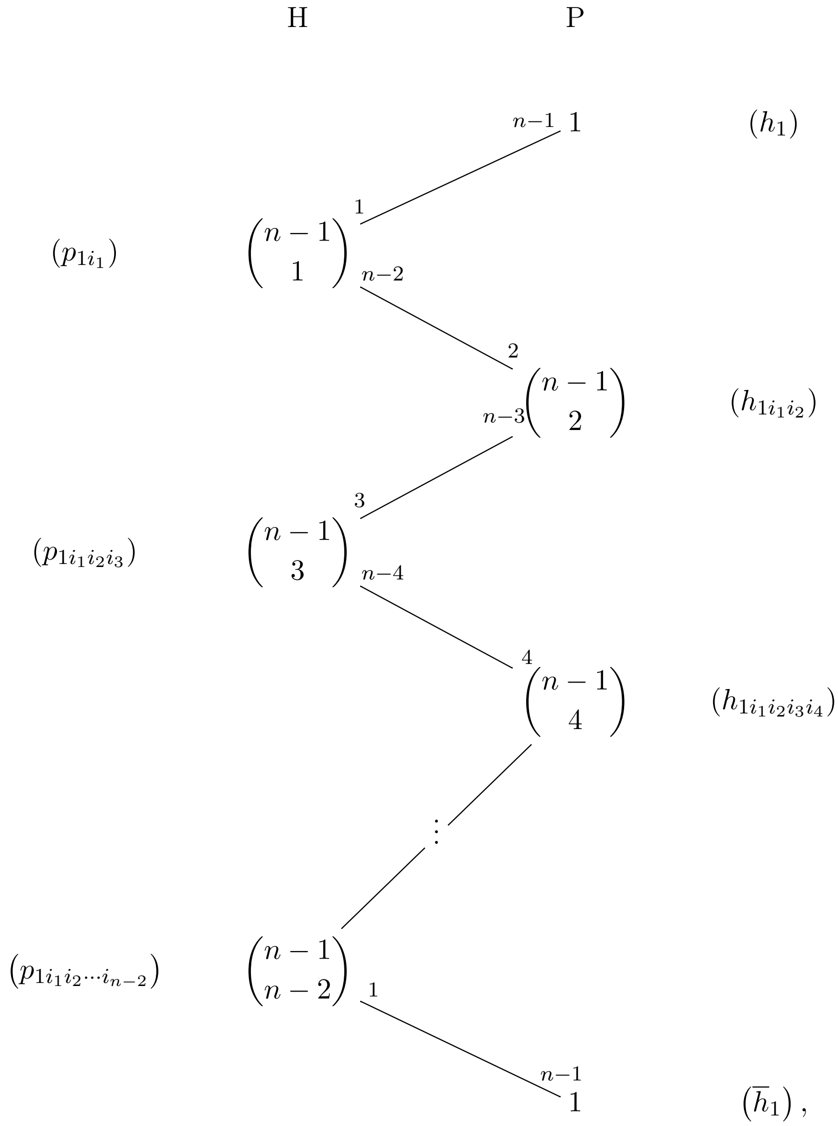
The starting configuration gives the branched summary



Case n is odd.

By eliminating the planes h_n, we have a structure of a configuration in $\mathrm{GCo}^{inj}_{(2,2,c-1)}$; by induction, there are $\binom{n-1}{3}$ planes $h_{i_1 i_2 i_3}$, where $1 \leq i_1 < i_2 < i_3 \leq n-1$, $\binom{n-1}{4}$ points $p_{i_1 i_2 i_3 i_4}$, where $1 \leq i_1 < i_2 < i_3 < i_4 \leq n-1$, ..., $\binom{n-1}{n-2} = n-1$ planes $h_{i_1 i_2 \cdots i_{n-2}}$, where $1 \leq i_1 < i_2 < \ldots < i_{n-2} \leq n-1$, and a point $p_{12\cdots(n-1)}$. By eliminating one plane h_i, $1 \leq i \leq n$, each turn, with the same argument as above, there are totally $\frac{n}{n-4}\binom{n-1}{4} = \binom{n}{4}$ points $p_{i_1 i_2 i_3 i_4}$, where $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$, $\frac{n}{n-5}\binom{n-1}{5} = \binom{n}{5}$ planes $h_{i_1 i_2 i_3 i_4 i_5}$, where $1 \leq i_1 < i_2 < \ldots < i_5 \leq n$, ..., $\frac{n}{2}\binom{n-1}{n-2} = \binom{n}{n-2}$ points $p_{i_1 i_2 \cdots i_{n-2}}$, where $1 \leq i_1 < i_2 < \ldots < i_{n-2} \leq n$, and $\binom{n}{n-1} = n$ planes $h_{i_1 i_2 \cdots i_{n-1}}$, where $1 \leq i_1 < i_2 < \ldots < i_{n-1} \leq n$.

Consider the structure of a configuration in $\mathrm{GCo}^{inj}_{(2,2,c-1)}$ in the dual projective space, the point h_1 has $n-1$ planes $p_{12}, \ldots, p_{1n}$ pass through and other incidences are given by



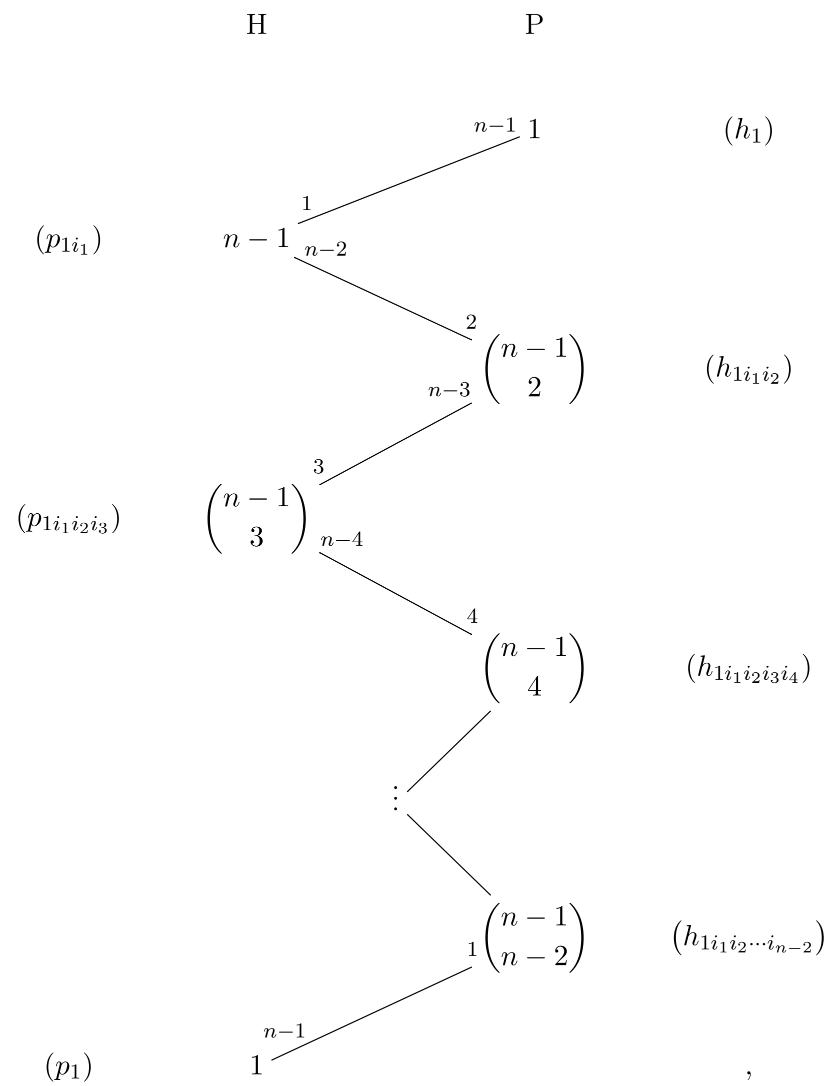
all the points $p_{1j_1j_2\cdots j_{n-2}}$, where $2 \leq j_1 < \ldots < i_{n-2} \leq n$, lie on a plane, say $\overline{h}_1$. Likewise for $2 \leq i \leq n$, there is a plane, say $\overline{h}_i$, which points $p_{j_1j_2\cdots j_{n-1}}$, where $1 \leq j_1 < \ldots < j_{n-1} \leq n$ and $i \in \{j_1, \ldots, j_{n-1}\}$, lie on. So all $\overline{h}_i$, where $1 \leq i \leq n$, are the same plane as required because, for $1 \leq i < k \leq n$, the planes $\overline{h}_i$ and $\overline{h}_k$ have $n-2 \geq 3$ points in common.

Case c is even.

By eliminating the planes h_{c+2}, we have a structure of a configuration in $\mathrm{GCO}^{inj}_{(2,2,c-1)}$; by induction, there are $\binom{n-1}{3}$ planes $h_{i_1i_2i_3}$, where $1 \leq i_1 < i_2 < i_3 \leq n-1$, $\binom{n-1}{4}$ points $p_{i_1i_2i_3i_4}$, where $1 \leq i_1 < i_2 < i_3 < i_4 \leq n-1$, …, and a plane $h_{12\cdots(n-1)}$. By eliminating one plane h_i, $1 \leq i \leq n$, each turn, there are thus $\frac{n}{n-4}\binom{n-1}{4} = \binom{n}{4}$ points $p_{i_1i_2i_3i_4}$, where

$1 \le i_1 < i_2 < i_3 < i_4 \le n$, because in n times of elimination, the point $p_{i_1 i_2 i_3 i_4}$ occurs $n-4$ times except when the hyperplane h_{i_1}, h_{i_2}, h_{i_3} or h_{i_4} is eliminated. By the same argument, there are thus $\dfrac{n}{n-5}\dbinom{n-1}{5} = \dbinom{n}{5}$ planes $h_{i_1 i_2 i_3 i_4 i_5}$, where $1 \le i_1 < i_2 < \ldots < i_5 \le n$, ..., and $\dbinom{n}{n-1} = n$ planes $h_{i_1 i_2 \cdots i_{n-1}}$, where $1 \le i_1 < i_2 < \ldots < i_{n-1} \le n$.

Consider the structure of a configuration in $\mathrm{GCo}^{inj}_{(2,2,c-1)}$ in the dual projective space, the point h_1 has $n-1$ planes $p_{12}, \ldots, p_{1n}$ pass through and other incidences are given by



,

all the planes $h_{1 j_1 j_2 \cdots j_{n-2}}$ where $2 \le j_1 < \ldots < j_{n-2} \le n$ pass through a point, say p_1. Likewise for $2 \le i \le n$, there is a point, say p_i, which lies on $n-1$ planes $h_{j_1 \cdots j_{n-1}}$'s for $1 \le j_1 < \ldots < j_{n-1} \le n$ such that $i \in \{j_1, \ldots, j_{n-1}\}$. Hence p_i, where $1 \le i \le n$, are the same point as required because, for $1 \le i < k \le n$, p_i and p_k lie on $n-2 \ge 3$ planes.

Therefore we have 2^{n-1} planes and 2^{n-1} points, with n planes through each point and n points on each planes, as in Figure 5.6.5. □

Remark 5.6.5. Proposition 5.6.4 is Cox's chain of Theorems (see [Cox50], p. 446-447).

Proposition 5.6.6. *Let V be a complex vector space of dimension n. Given a $(2, b, 2)$-starting configuration in $\mathbb{P}^{n-1}(V)$, a configuration in $\mathrm{GCo}^{inj}_{(2,b,2)}$ can be uniquely obtained from the starting configuration.*

Proof. We claim that if n is even (resp. odd), we obtain the left (resp. right) branched summary as in Figure 5.6.6.

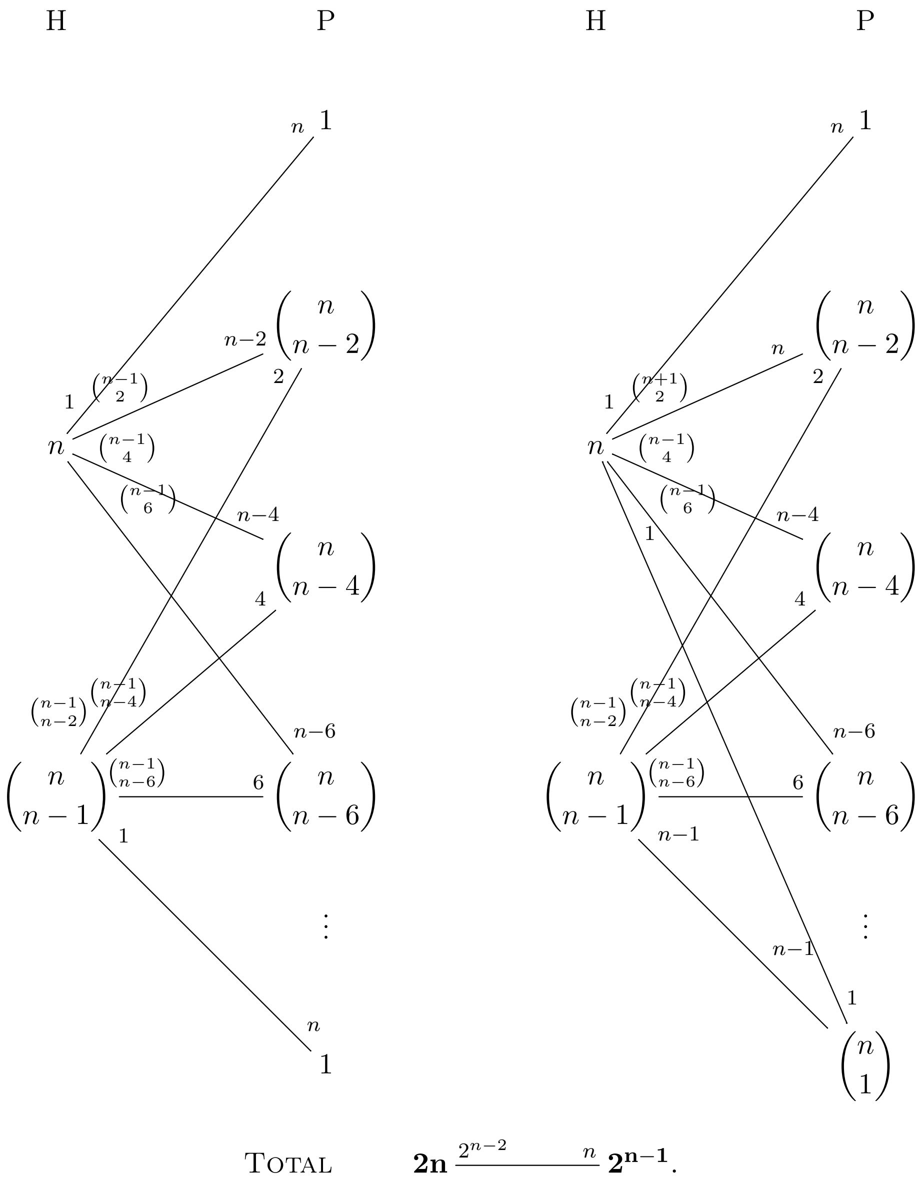
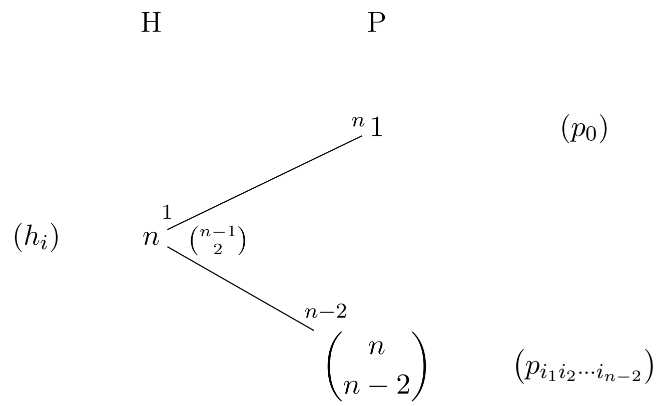


Figure 5.6.6: The branched summary for $\mathrm{GCo}_{(2,b,2)}$ where n is even (on the left) and odd (on the right) respectively.

We will prove the proposition by induction on $n \geq 4$. The case when $n = 4$ is already proved in Proposition 5.6.2. Now suppose that $n > 4$.

The starting configuration gives the branched summary



For $1 \leq i_1 < \ldots < i_{n-1} \leq n$, the set of points

$$\left\{ P_{j_1 j_2 \cdots j_{n-2}} \,|\, j_1 < \ldots < j_{n-2} \,\text{and}\, j_1, \ldots, j_{n-2} \in \{i_1, i_2, \ldots, i_{n-1}\} \right\}$$

uniquely determines a hyperplane, say $h_{\widehat{i}}$ such that $i \notin \{i_1, i_2, \ldots, i_{n-1}\}$; there are $\binom{n}{n-1} =$ n such hyperplanes.

Case n is odd.

By intersecting h_n with the other $n-1$ hyperplanes $h_1, \ldots, h_{n-1}$, we obtain $n-1$ hyperplanes in h_n through the point p_0 which gives the structure of a configuration in $\mathrm{GCo}^{inj}_{(2,b-1,2)}$ consisting of

$\binom{n-1}{n-3}$ points like $p_{i_1 \cdots i_{(n-3)} n} := h_n \cap \bigcap_{k \in \{i_1, \ldots, i_{n-3}\}} h_k \cap \bigcap_{k \notin \{i_1, \ldots, i_{n-3}, n\}} h_{\widehat{k}}$ where $1 \leq i_1 < \ldots < i_{n-3} \leq n-1$,

$\binom{n-1}{n-5}$ points like $p_{i_1 \cdots i_{(n-5)} n} := h_n \cap \bigcap_{k \in \{i_1, \ldots, i_{n-5}\}} h_k \cap \bigcap_{k \notin \{i_1, \ldots, i_{n-5}, n\}} h_{\widehat{k}}$ where $1 \leq i_1 < \ldots < i_{n-5} \leq n-1$,

$\vdots$

and a point p_n on $\bigcap_{k \neq n} h_{\widehat{k}}$.

Processing the same way for all hyperplanes h_i, $1 \leq i \leq n+2$, then we have totally $\frac{n}{n-4}\binom{n-1}{n-5} = \binom{n}{n-4}$ points like $p_{i_1 \cdots i_{n-4}}$ on $\bigcap_{k \in \{i_1, \ldots, i_{n-4}\}} h_k \cap \bigcap_{k \notin \{i_1, \ldots, i_{n-4}\}} h_{\widehat{k}}$, where $1 \leq i_1 < \ldots < i_{n-4} \leq n$, because for $1 \leq i_1 < \ldots < i_{n-4} \leq n$, in n choices of intersecting hyperplanes, the point $p_{i_1 \cdots i_{n-4}}$ occurs $n-4$ times except when the intersecting hyperplane h_j such that $j \notin \{i_1, \ldots, i_{n-4}\}$ is chosen. By the same argument, there are totally

$\dfrac{n}{n-6}\dbinom{n-1}{n-7} = \dbinom{n}{n-6}$ points like $p_{i_1\cdots i_{n-6}} := \bigcap_{k\in\{i_1,\ldots,i_{n-6}\}} h_k \cap \bigcap_{k\notin\{i_1,\ldots,i_{n-6}\}} h_{\widehat{k}}$ where $1 \leq i_1 < \ldots < i_{n-6} \leq n$,

$\vdots$

and $\dbinom{n}{1} = n$ points like $p_i := \bigcap_{k\neq i} h_{\widehat{k}}$ where $1 \leq i \leq n$, as required.

Case n is even.

Again by intersecting h_n with the other $n-1$ hyperplanes $h_1, \ldots, h_{n-1}$, we obtain $n-1$ hyperplanes in h_n through the point p_0 which gives the structure of a configuration in $\text{GCo}^{inj}_{(2,b-1,2)}$ consisting of

$\dbinom{n-1}{n-3}$ points like $p_{i_1\cdots i_{(n-3)}n} := h_n \cap \bigcap_{k\in\{i_1,\ldots,i_{n-3}\}} h_k \cap \bigcap_{k\notin\{i_1,\ldots,i_{n-3},n\}} h_{\widehat{k}}$ where $1 \leq i_1 < \ldots < i_{n-3} \leq n-1$,

$\dbinom{n-1}{n-5}$ points like $p_{i_1\cdots i_{(n-5)}n} := h_n \cap \bigcap_{k\in\{i_1,\ldots,i_{n-5}\}} h_k \cap \bigcap_{k\notin\{i_1,\ldots,i_{n-5},n\}} h_{\widehat{k}}$ where $1 \leq i_1 < \ldots < i_{n-5} \leq n-1$,

$\vdots$

and $\dbinom{n-1}{1} = n-1$ points like $p_{in} := \bigcap_{k\notin\{i,n\}} h_{\widehat{k}}$ where $1 \leq i \leq n-1$.

Processing the same way for all hyperplanes h_i, where $1 \leq i \leq n+2$, there are thus $\dfrac{n}{n-4}\dbinom{n-1}{n-5} = \dbinom{n}{n-4}$ points like $p_{i_1\cdots i_{n-4}}$ on $\bigcap_{k\in\{i_1,\ldots,i_{n-4}\}} h_k \cap \bigcap_{k\notin\{i_1,\ldots,i_{n-4}\}} h_{\widehat{k}}$, where $1 \leq i_1 < \ldots < i_{n-4} \leq n$, because for $1 \leq i_1 < \ldots < i_{n-4} \leq n$, in n choices of intersecting hyperplanes, the point $p_{i_1\cdots i_{n-4}}$ occurs $n-4$ times except when the intersecting hyperplane h_k such that $k \notin \{i_1, \ldots, i_{n-4}\}$ is chosen. By the same argument, there are totally

$\dfrac{n}{n-6}\dbinom{n-1}{n-7} = \dbinom{n}{n-6}$ points like $p_{i_1\cdots i_{n-6}} := \bigcap_{k\in\{i_1,\ldots,i_{n-6}\}} h_k \cap \bigcap_{k\notin\{i_1,\ldots,i_{n-6}\}} h_{\widehat{k}}$ where $1 \leq i_1 < \ldots < i_{n-6} \leq n$,

$\vdots$

and $\dfrac{n}{2}\dbinom{n-1}{1} = \dbinom{n}{2}$ points like $p_{i_1 i_2} := \bigcap_{k\notin\{i_1,i_2\}} h_{\widehat{k}}$ where $1 \leq i_1 < i_2 \leq n$.

Consider the intersection of $\bigcap_{k\notin\{1,2,3,4\}} h_{\widehat{k}}$ with the hyperplanes h_1, h_2, h_3 and h_4, respectively, we obtain 4 planes in $\bigcap_{k\notin\{1,2,3,4\}} h_{\widehat{k}}$, which is a 3-space, through the point P_{1234}. This give a structure of $\text{Co}(2,4)$ which implies that all the hyperplanes $h_{\widehat{k}}$, where $1 \leq k \leq n$, meet in a point, say $p_{\widehat{0}}$, as required.

Therefore we have $2n$ hyperplanes and 2^{n-1} points, with n hyperplanes through each point and 2^{n-2} points on each hyperplanes, as in Figure 5.6.6. □

Remark 5.6.7. Proposition 5.6.6 is considered to be a generalization of Cox's chain of Theorems (see [AD61]).

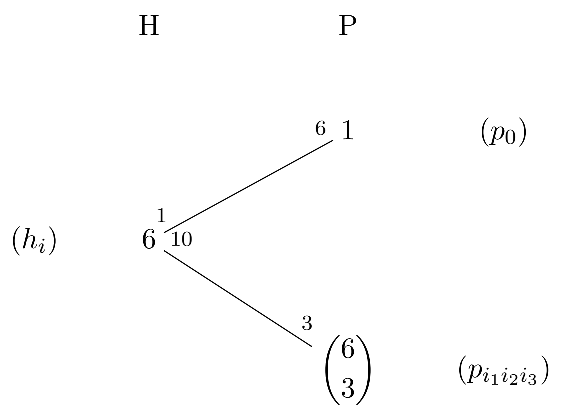
There is one more finite case of D-type that have not been proven yet which is the case $(a, 2, 2)$.

5.7 Generalized Cox configurations of E-type

We are going to show that Conjecture 5.4.2 is true for the case $(2,3,3)$, $(2,3,4)$, and $(2,4,3)$. According to Table 5.1, these three cases have $\dim\left([\mathfrak{p}^\perp, \mathfrak{p}^\perp]\right) \neq 0$. Hence, given a starting configuration of the case $(2,3,3)$, $(2,3,4)$, or $(2,4,3)$, we need to choose $\dim\left([\mathfrak{p}^\perp, \mathfrak{p}^\perp]\right)$ more parameters to complete the configuration.

Proposition 5.7.1. *Let V be a complex vector space of dimension* 5. *Given a* $(2,3,3)$*-starting configuration in in $\mathbb{P}^4(V)$, a configuration in $\mathrm{GCo}^{inj}_{(2,3,3)}$ can be obtained by choosing* 1 *more parameter.*

Proof. The starting configuration has the following branched summary

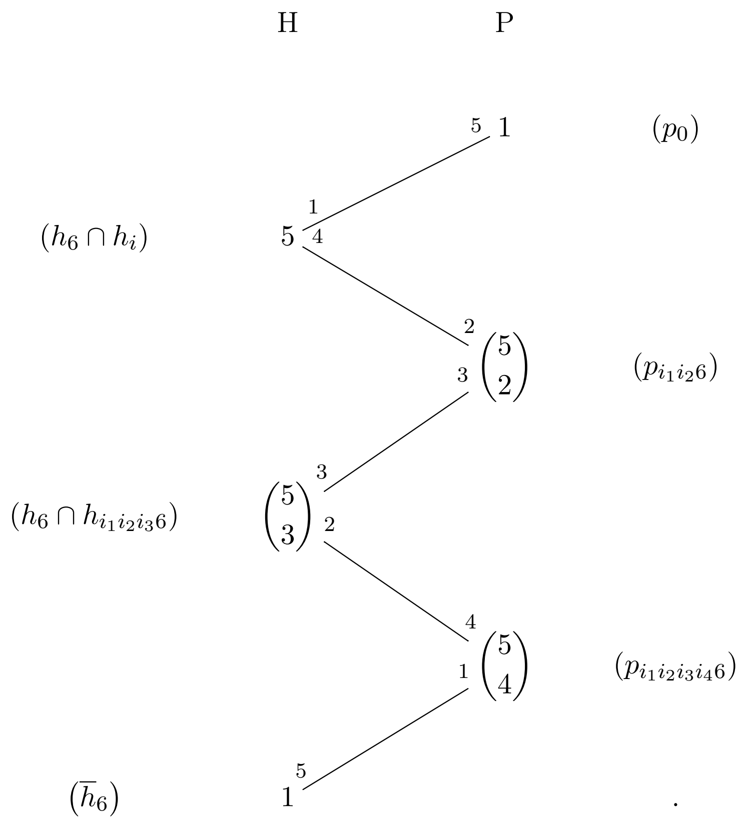


For $1 \leq i_1 < i_2 < i_3 < i_4 \leq 6$, the set of points

$$\{p_{j_1 j_2 j_3} \,|j_1 < j_2 < j_3 \,\text{and}\, j_1, j_2, j_3 \in \{i_1, \ldots, i_4\}\}$$

uniquely determines a hyperplane, say $h_{i_1 i_2 i_3 i_4}$; there are $\binom{6}{4} = 15$ such hyperplanes.

By intersecting h_6 with the other 5 hyperplanes $h_1, \ldots, h_5$, we obtain 5 planes in h_6 through p_0 which gives the structure a configuration in $\mathrm{GCo}^{inj}_{(2,2,3)}$ whose incidences are given by

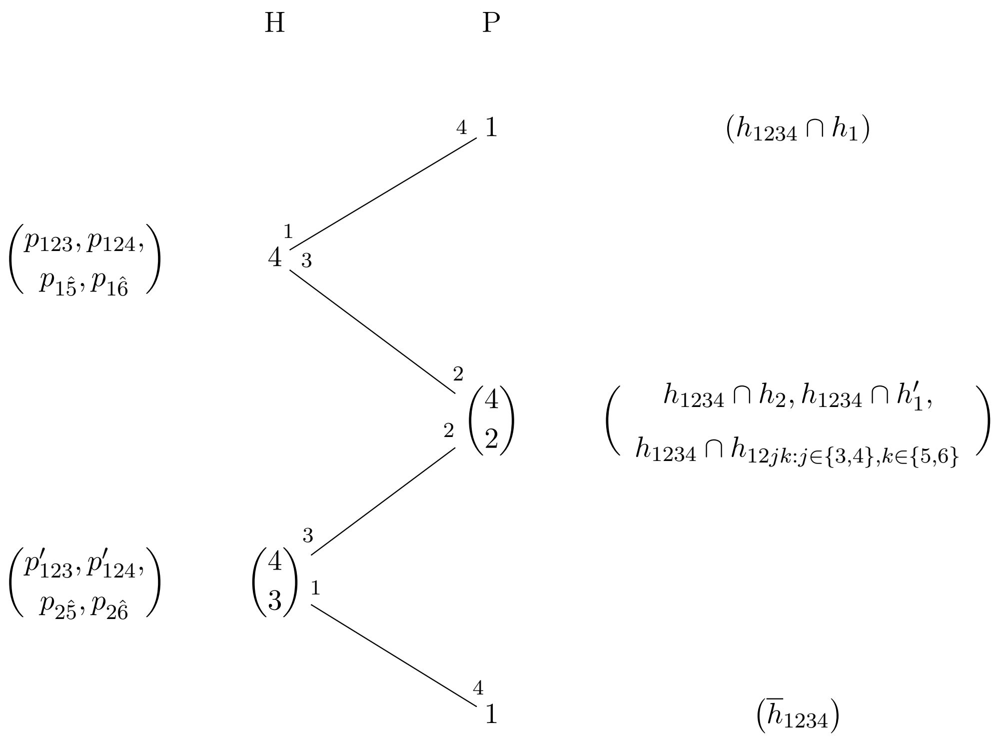


.

So there are $\binom{5}{4} = 5$ points like $p_{i_1 i_2 i_3 i_4 6} := h_6 \cap \bigcap_{\{k_1,k_2,k_3\} \subseteq \{i_1,i_2,i_3,i_4\}} h_{k_1 k_2 k_3 6}$,where $1 \le i_1 < i_2 < i_3 < i_4 \le 5$, and all of which lie on a plane, say $\overline{h}_6$, in h_6. For convenience, for each $1 \le i_1 < i_2 < i_3 < i_4 \le 5$, denote $p_{i_1 i_2 i_3 i_4 6}$ by $p_{6\hat{i}}$, where $i \notin \{i_1, i_2, i_3, i_4, 6\}$.

Processing the same way for all hyperplanes h_i where $1 \le i \le 6$, there are totally $(6)(5) = 30$ points like $p_{j\hat{i}} := \bigcap_{j \in \{k_1,\ldots,k_4\} \subset \{1,\ldots,6\} \setminus \{i\}} h_{k_1 k_2 k_3 k_4}$ where $1 \le i, j \le 6$ and $i \ne j$ and for each $j \in \{1, 2, \ldots, 6\}$, the points $p_{j\hat{i}}$, where $i \in \{1, \ldots, 6\} \setminus \{j\}$, lie on a plane $\overline{h}_j$ in h_j.

Now choose h'_1 be a hyperplane different from h_1 and containing $\overline{h}_1$; to do this, we need to use 1 parameter. For $2 \le i < j \le 6$, define $p'_{1ij} := h'_1 \cap \bigcap_{\{i,j\} \subseteq \{k_1,k_2,k_3\}} h_{1 k_1 k_2 k_3}$. In the dual space of h_{1234}, the point $h_{1234} \cap h_1$ has 4 planes p_{123}, p_{124}, $p_{1\hat{5}}$ and $p_{1\hat{6}}$ passing through it. This gives us a structure a configuration in $\mathrm{GCO}^{inj}_{(2,2,2)}$ which implies that the planes p'_{123}, p'_{124}, $p_{2\hat{5}}$ and $p_{2\hat{6}}$ meet in a point.



Therefore the points p'_{123}, p'_{124}, $p_{2\hat{5}}$ and $p_{2\hat{6}}$ lie on a plane, say $\overline{h}_{1234}$, in h_{1234}.

The plane $\overline{h}_{1234}$ and the plane $\overline{h}_2$ have the line, determined by $\{p_{2\hat{5}}, p_{2\hat{6}}\}$ in common. So let h'_2 be the hyperplane containing the plane $\overline{h}_2$ and the plane $\overline{h}_{1234}$. Again, by the same argument as above, $\langle p'_{123}, p'_{125}, p_{2\hat{4}}, p_{2\widehat{6}} \rangle$ is a plane which is obviously contained in h'_2 because

$$\langle p'_{123}, p'_{125}, p_{2\hat{4}}, p_{2\widehat{6}} \rangle = \langle p'_{123}, p_{2\hat{4}}, p_{2\widehat{6}} \rangle \subseteq h'_2.$$

Thus the points p'_{125} lies on h'_2. Similarly the points p'_{126} lies on h'_2. Now for $3 \leq i < j \leq 6$, define $p'_{2ij} := h'_2 \cap \bigcap_{\{2,i,j\} \subseteq \{k_1,k_2,k_3,k_4\}} h_{k_1k_2k_3k_4}$.

By the similar argument, $\langle p'_{123}, p'_{134}, p_{3\hat{5}}, p_{3\widehat{6}} \rangle$ is a plane in h_{1234}. The plane $\overline{h}_3$ and the plane $\langle p'_{123}, p'_{134}, p_{3\hat{5}}, p_{3\widehat{6}} \rangle$ have the line, determined by $\{p_{3\hat{5}}, p_{3\hat{6}}\}$ in common. So let h'_3 be the hyperplane containing the plane $\overline{h}_3$ and the plane $\langle p'_{123}, p'_{134}, p_{3\hat{5}}, p_{3\widehat{6}} \rangle$. The plane $\langle p'_{123}, p'_{135}, p_{3\hat{4}}, p_{3\widehat{6}} \rangle$ is a plane which is obviously contained in h'_3 because

$$\langle p'_{123}, p'_{135}, p_{3\hat{4}}, p_{3\widehat{6}} \rangle = \langle p'_{123}, p_{3\hat{4}}, p_{3\widehat{6}} \rangle \subseteq h'_3.$$

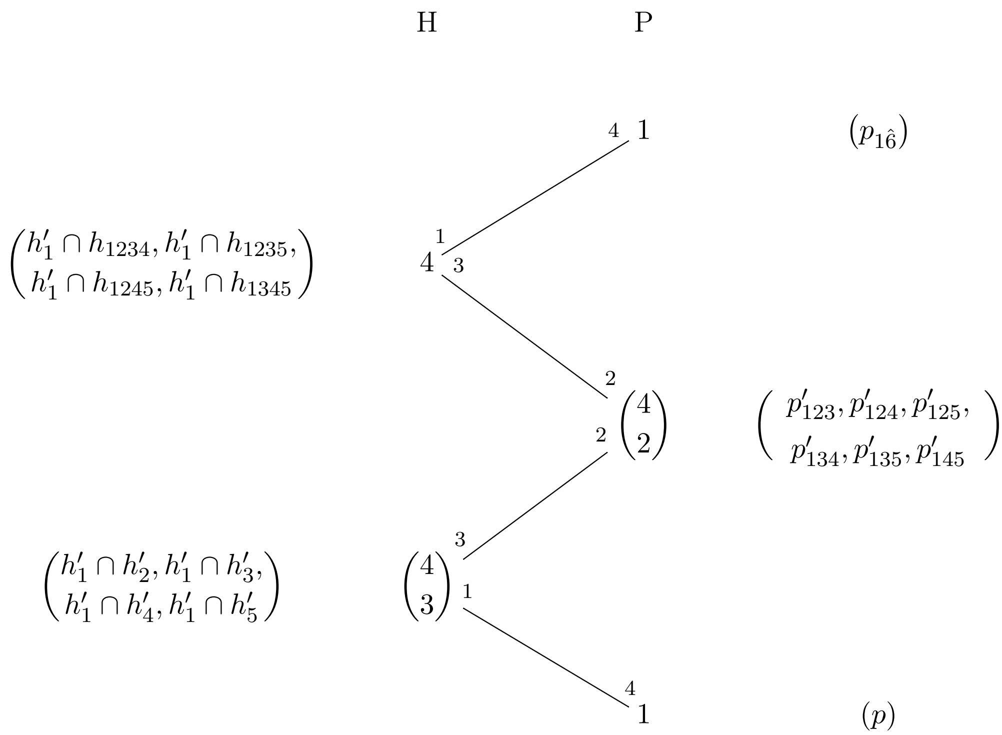
Thus the points p'_{135} lies on h'_3. By processing the same argument, we finally have the hyperplane h'_3 contains the plane $\overline{h}_3$ and possesses the points p'_{13j} (where $j \in \{2,4,5,6\}$) and p'_{23j} (where $4 \le j \le 6$). Now for $4 \le i < j \le 6$, define $p'_{3ij} := h'_3 \cap \bigcap_{\{3,i,j\} \subseteq \{k_1,k_2,k_3,k_4\}} h_{k_1k_2k_3k_4}$.

Let h'_4 be the hyperplane containing the plane $\overline{h}_4$ and possessing the points p'_{14j} (where $j \in \{2,3,5,6\}$), p'_{24j} (where $j \in \{3,5,6\}$) and p'_{34j} (where $5 \le j \le 6$). Define $p'_{456} := h'_4 \cap \bigcap_{1 \le k \le 3} h_{k456}$.

Let h'_5 be the hyperplane containing the plane $\overline{h}_5$ and possessing the points p'_{15j} (where $j \in \{2,3,4,6\}$), p'_{25j} (where $j \in \{3,4,6\}$), p'_{35j} (where $j \in \{4,6\}$) and p'_{456}.

Finally let h'_6 be the hyperplane containing the plane $\overline{h}_6$ and possessing the points p'_{1i6} (where $2 \le i \le 5$), p'_{2i6} (where $3 \le i \le 5$), p'_{3i6} (where $4 \le i \le 5$) and p'_{456}.

Consider 4 planes $h'_1 \cap h_{1234}$, $h'_1 \cap h_{1235}$, $h'_1 \cap h_{1245}$ and $h'_1 \cap h_{1345}$ through the point $p_{1\hat{6}}$ in h'_1. This gives a structure of a configuration in $\mathrm{GCo}^{inj}_{(2,2,2)}$ which tells us that the hyperplanes $h'_1, \ldots, h'_4$ and h'_5 meet in a point, say p.



However, by apply the same argument to 4 planes $h'_1 \cap h_{1345}$, $h'_1 \cap h_{1346}$, $h'_1 \cap h_{1356}$ and $h'_1 \cap h_{1456}$ through the point $p_{1\hat{2}}$ in h'_1, we have that the hyperplanes $h'_1, h'_3, \ldots, h'_5$ and h'_6

also meet in a point which must be p. Thus the hyperplanes $h'_1, \ldots, h'_5$ and h'_6 meets in the point p.

Therefore we have 27 hyperplanes and 72 points, with 6 hyperplanes through each point and 16 points on each hyperplanes as in Figure 5.7.1.

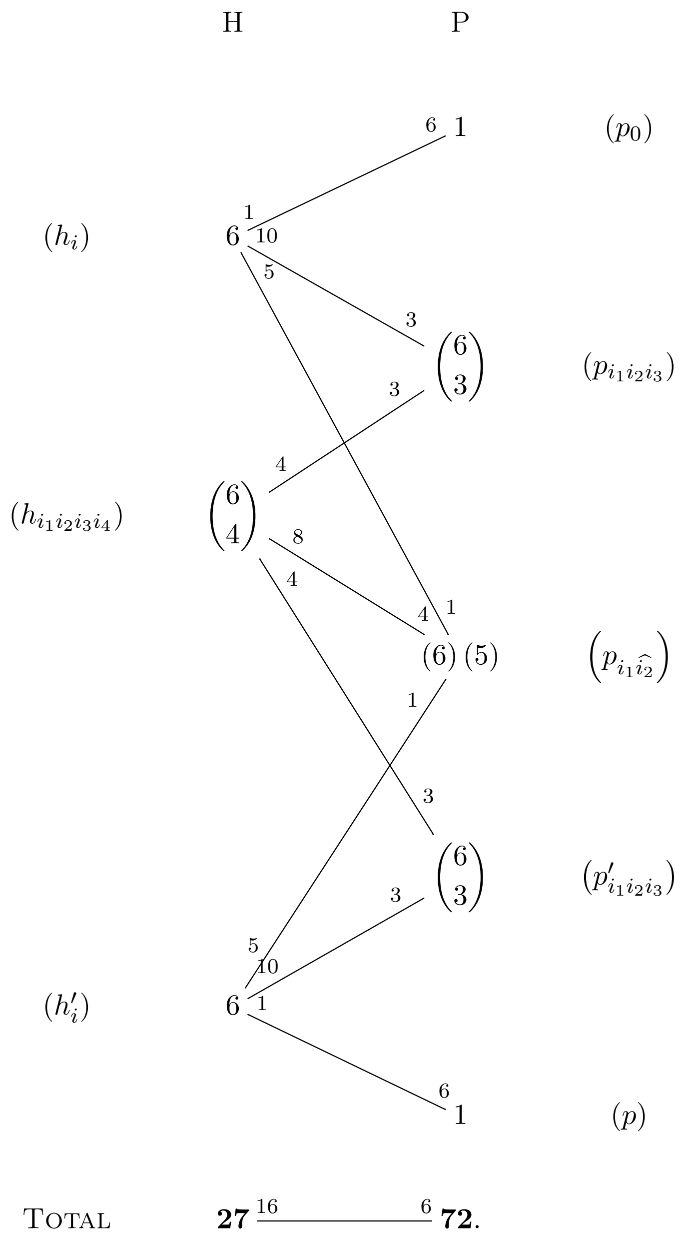


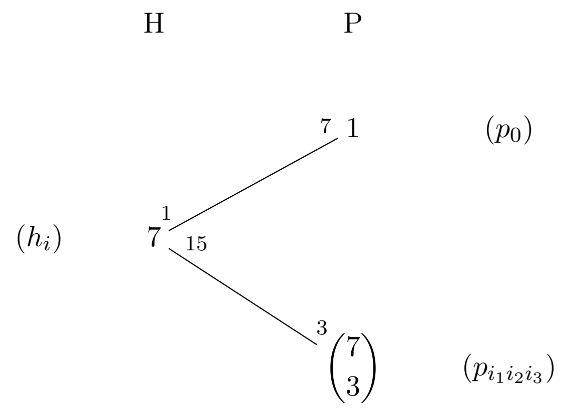
Figure 5.7.1: The branched summary for $\textsc{gCo}_{(2,3,3)}$.

□

Proposition 5.7.2. *Let V be a complex vector space of dimension* 5. *Given a* $(2, 3, 4)$-

starting configuration in in $\mathbb{P}^4(V)$*, a configuration in* $\text{GCo}^{inj}_{(2,3,4)}$ *can be obtained by choosing 7 more parameters.*

Proof. The starting configuration has the following branched summary

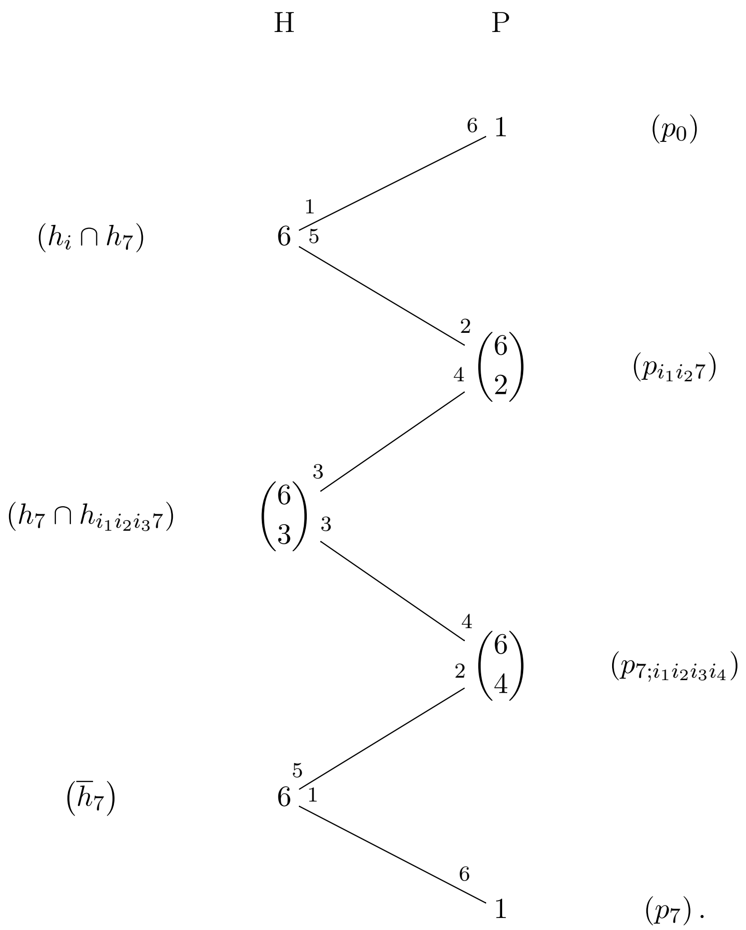


For $1 \leq i_1 < i_2 < i_3 < i_4 \leq 7$, the set of points

$$\{p_{j_1j_2j_3}\,|j_1 < j_2 < j_3 \,\text{and}\, j_1, j_2, j_3 \in \{i_1, \ldots, i_4\}\}$$

uniquely determine a hyperplane, say $h_{i_1i_2i_3i_4}$; there are $\binom{7}{4} = 35$ such hyperplanes.

The planes $h_1 \cap h_7$, $h_2 \cap h_7$, $h_3 \cap h_7$, $h_4 \cap h_7$, $h_5 \cap h_7$ and $h_6 \cap h_7$ passing through the point p_0, we obtain a structure of a configuration in $\text{GCo}^{inj}_{(2,2,4)}$, which we use 1 parameter (see Proposition 5.7.1). It's branched summary is given by

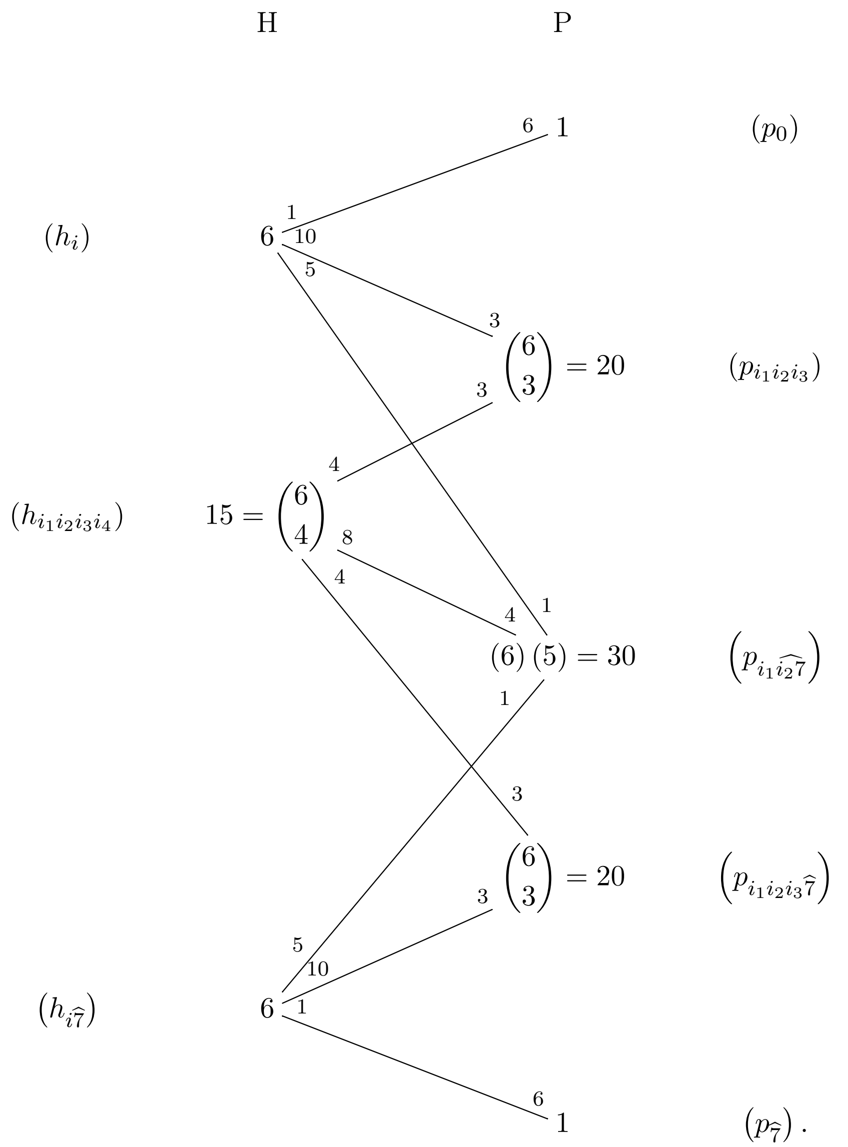


By applying this way for all hyperplanes h_i, $1 \leq i \leq 7$, there are $\binom{7}{1}\binom{6}{2} = 105$ points like $p_{i;k_1k_2k_3k_4} := h_i \cap \bigcap_{i \in \{k_1, \dots k_4\}} h_{k_1k_2k_3k_4}$ where $1 \leq i \leq 7$ and $k_1, \dots, k_4 \in \{1, 2, \dots 7\} \setminus \{i\}$; 7 points like p_i on h_i, for all $1 \leq i \leq 7$. Hence we use 7 parameters now. Moreover for each $1 \leq i \leq 7$, all points in the set

$$\{p_i, p_{i;k_1k_2k_3k_4} \,|k_1, \dots, k_4 \in \{1, 2, \dots 7\} \setminus \{i\} \text{ and } k_1 < k_2 < k_3 < k_4\}$$

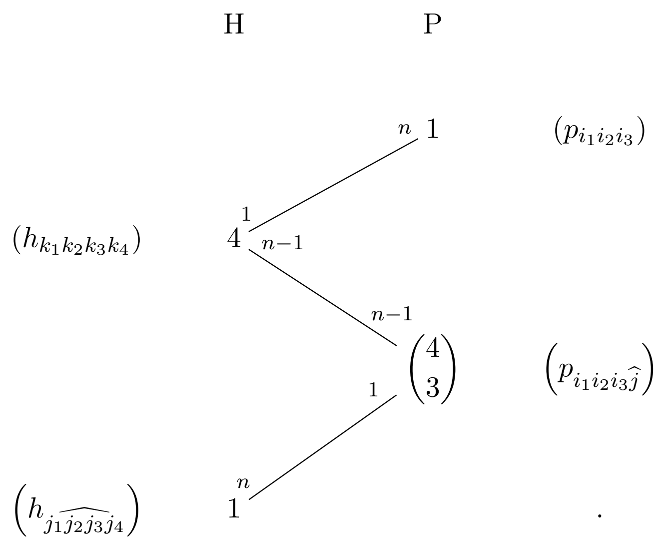
lie on a plane in h_i. For convenient, for each $1 \leq i \leq 7$, $k_1, \dots, k_4 \in \{1, 2, \dots 7\} \setminus \{i\}$ and $k_1 < k_2 < k_3 < k_4$, denote $p_{i;k_1k_2k_3k_4}$ by $p_{i\widehat{jl}}$ where $j, l \notin \{i, k_1, k_2, k_3, k_4\}$.

The hyperplanes $h_1, h_2, \dots, h_5$ and h_6 passing through the point p_0, we obtain a structure of a configuration in $\text{GCO}^{inj}_{(2,3,3)}$ whose incidences are given by



By eliminating one hyperplane each turn, there are $(6)(7) = 42$ hyperplanes like $h_{i\hat{j}}$ determined by $\left\{p_{i\widehat{k_1 k_2}} \,|1 \le k_1 < k_2 \le 7 \text{ and } j \in \{k_1, k_2\}\right\}$ where $1 \le i, j \le 7$ and $i \ne j$; $\binom{7}{3}\binom{4}{1} = 140$ points like $p_{i_1 i_2 i_3 \widehat{j}} := \bigcap_{\{i_1,i_2,i_3\} \subseteq \{k_1,\ldots,k_4\} \subset \{1,\ldots,7\} \setminus \{j\}} h_{k_1 k_2 k_3 k_4} \cap \bigcap_{k \in \{i_1,i_2,i_3\}} h_{k\widehat{j}}$ where $1 \le i_1 < i_2 < i_3 \le 7$ and $j \in \{1, \ldots, 7\} \setminus \{i_1, i_2, i_3\}$; 7 points like $p_{\widehat{i}} := \bigcap_{k \in \{1,\ldots,7\} \setminus \{i\}} h_{k\widehat{i}}$ where $1 \le i \le 7$.

For each $1 \le i, j \le 7$ and $i \ne j$, as $p_{i\widehat{k_1 k_2}}$, where $j \in \{k_1, k_2\}$, span the plane $h_i \cap h_{i\widehat{j}}$ and $\left\langle p_i, p_{i\widehat{k_1 k_2}} \,|j \in \{k_1, k_2\}\right\rangle$ is a plane, thus p_i is on $h_{i\widehat{j}}$. For $1 \le i_1 < i_2 < i_3 \le 7$, consider 4 hyperplanes $h_{k_1 k_2 k_3 k_4}$, where $1 \le k_1 < k_2 < k_3 < k_4 \le 7$ and $i_1, i_2, i_3 \in \{k_1, \ldots, k_4\}$, through the point $p_{i_1 i_2 i_3}$. This give a structure of a configuration in $\text{GCO}^{inj}_{(2,3,1)}$ whose incidences are given by



.

Hence there are $\binom{7}{4} = 35$ hyperplanes like $h_{\widehat{j_1 j_2 j_3 j_4}}$ determined by

$$\left\{ p_{i_1 i_2 i_3 \widehat{k}} \,|\{i_1, i_2, i_3\} = \{1, \ldots, 7\} \setminus \{j_1, \ldots j_4\},\ i_1 < i_2 < i_3 \text{ and } k \in \{j_1, \ldots j_4\} \right\}$$

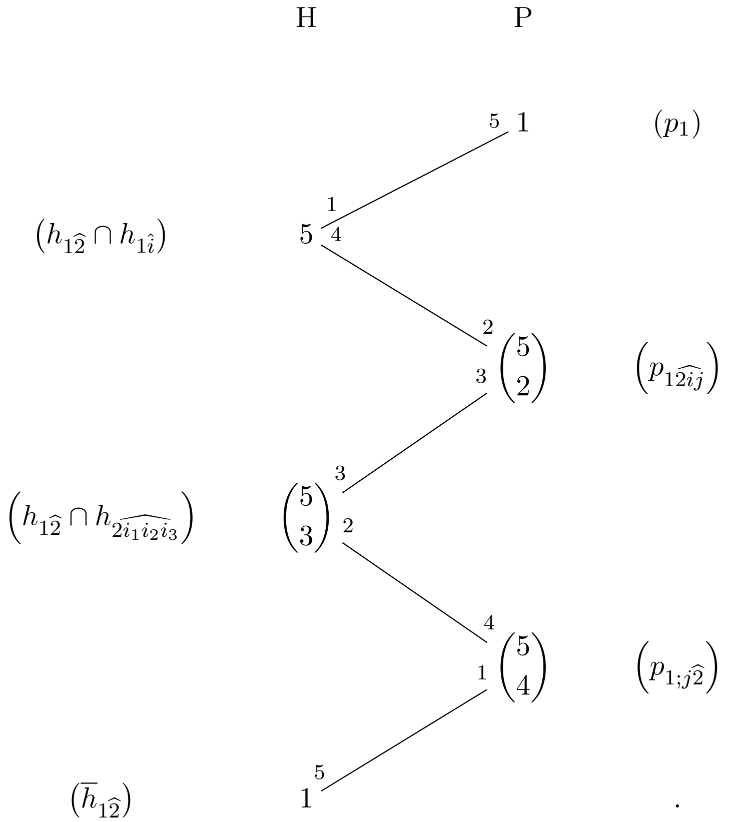
where $1 \leq j_1 < j_2 < j_3 < j_4 \leq 7$.

Now consider 4 planes $h_1 \cap h_{1234}$, $h_{1234} \cap h_{1235}$, $h_{1234} \cap h_{1236}$ and $h_{1234} \cap h_{1237}$ in h_{1234} through the point p_{123}. This gives us a structure of a configuration in $\text{GCO}^{inj}_{(2,2,2)}$ which implies that $h_{1234} \cap h_{1\widehat{5}}$, $h_{1234} \cap h_{1\widehat{6}}$, $h_{1234} \cap h_{1\widehat{7}}$ and $h_{1234} \cap h_{\widehat{4567}}$ meet in a point, say $p_{1\widehat{456}}$. Likewise $h_{1234} \cap h_{1\widehat{5}}$, $h_{1234} \cap h_{1\widehat{6}}$, $h_{1234} \cap h_{1\widehat{7}}$ and $h_{1234} \cap h_{\widehat{3567}}$ meet in a point and this point must be $p_{1\widehat{456}}$ because $h_{1234} \cap h_{1\widehat{5}} \cap h_{1\widehat{6}} \cap h_{1\widehat{7}} = \{p_{1\widehat{456}}\}$. By the same argument, the point $p_{1\widehat{456}}$ is also on the hyperplane $h_{\widehat{3567}}$. Hence there are totally $\binom{7}{1}\binom{6}{3} = 140$ points like

$$p_{i\widehat{k_1 k_2 k_3}} := \bigcap_{\{k_1, k_2, k_3\} \subseteq \{j_1, \ldots, j_4\}} h_{\widehat{j_1 j_2 j_3 j_4}} \cap \bigcap_{j \in \{k_1, k_2, k_3\}} h_{i\widehat{j}} \cap h_{i_1 i_2 i_3 i_4 (\{i_1, \ldots i_4\} = \{1, \ldots, 7\} \setminus \{k_1, k_2, k_3\})}$$

.

There are 5 planes $h_{1\widehat{2}} \cap h_{1\hat{i}}$ where $3 \leq i \leq 7$ through the point p_1. This gives a structure of a configuration of $\text{GCO}^{inj}_{(2,2,3)}$ whose incidences are given by



Notice that the point $p_{1;3\hat{2}}$ are determined by 5 planes $h_{1\hat{2}} \cap h_{1\hat{i}}$, where $3 \le i \le 7$, passing through the point p_1, is same as the point $p_{3;1\hat{2}}$, determined (by the same argument) by 5 planes $h_{3\hat{2}} \cap h_{3\hat{i}}$, where $1 \le i \le 7$ and $i \notin \{2,3\}$, through the point p_3 . We can omit the colon symbol in our notation $p_{i;j\hat{k}}$. So there are totally $\binom{7}{2}\binom{5}{1} = 105$ points like $p_{ij\hat{k}} := h_{i\hat{k}} \cap h_{j\hat{k}} \cap \bigcap_{k \in \{k_1,k_2,k_3,k_4\}} h_{\widehat{k_1k_2k_3k_4}}$ where $1 \le i < j \le 7$ and $k \in \{1,\dots,7\} \setminus \{i,j\}$.

Next consider 4 planes $h_{\widehat{1234}} \cap h_{4567}$, $h_{\widehat{1234}} \cap h_{5\hat{1}}$, $h_{\widehat{1234}} \cap h_{6\hat{1}}$ and $h_{\widehat{1234}} \cap h_{7\hat{1}}$ in $h_{\widehat{1234}}$ through the point $p_{567\hat{1}}$. This gives a structure of a configuration of $\mathrm{GCO}^{inj}_{(2,2,4)}$ which implies that the hyperplanes $h_{\widehat{1234}}$, $h_{\widehat{1235}}$, $h_{\widehat{1236}}$ and $h_{\widehat{1237}}$ meet in a point, say $p_{\widehat{123}}$, and $\langle p_{\widehat{123}}, p_{56\hat{1}}, p_{57\hat{1}}, p_{67\hat{1}} \rangle$ is a plane in $h_{\widehat{1234}}$. Likewise one can show that $\langle p_{\widehat{124}}, p_{56\hat{1}}, p_{57\hat{1}}, p_{67\hat{1}} \rangle$ and $\langle p_{\widehat{134}}, p_{56\hat{1}}, p_{57\hat{1}}, p_{67\hat{1}} \rangle$ are planes in $h_{\widehat{1234}}$. Then $\langle p_{\widehat{123}}, p_{\widehat{124}}, p_{\widehat{134}}, p_{56\hat{1}}, p_{57\hat{1}}, p_{67\hat{1}} \rangle$ is a plane in $h_{\widehat{1234}}$. There are thus $\binom{7}{3} = 35$ points like $p_{\widehat{i_1i_2i_3}} := \bigcap_{\{i_1,i_2,i_3\} \in \{k_1,k_2,k_3,k_4\}} h_{\widehat{k_1k_2k_3k_4}}$ where $1 \le i_1 < i_2 < i_3 \le 7$.

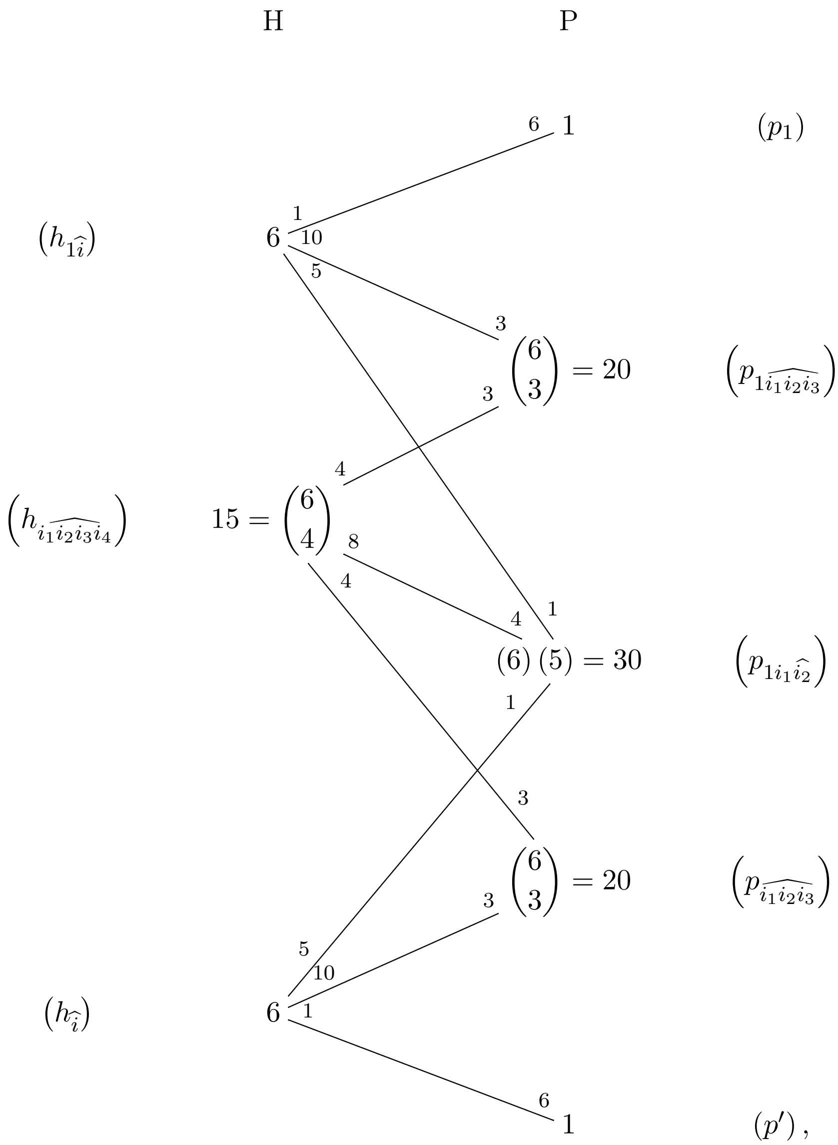
As $\langle p_{25\hat{1}}, p_{35\hat{1}}, p_{45\hat{1}}, p_{56\hat{1}}, p_{57\hat{1}} \rangle$ and $\langle p_{\widehat{123}}, p_{\widehat{124}}, p_{\widehat{134}}, p_{56\hat{1}}, p_{57\hat{1}}, p_{67\hat{1}} \rangle$ are planes having 2 points in common, let $h_{\hat{1}}$ be the hyperplane $\langle p_{25\hat{1}}, p_{35\hat{1}}, p_{45\hat{1}}, p_{56\hat{1}}, p_{57\hat{1}}, p_{67\hat{1}}, p_{\widehat{123}}, p_{\widehat{124}}, p_{\widehat{134}} \rangle$. Again since $\langle p_{45\hat{1}}, p_{47\hat{1}}, p_{57\hat{1}}, p_{\widehat{123}}, p_{\widehat{126}}, p_{\widehat{136}} \rangle$ is a plane, the points $p_{47\hat{1}}$, $p_{\widehat{126}}$ and $p_{\widehat{136}}$ lie on

$h_{\widehat{1}}$. Processing this way, we finally have $h_{\widehat{1}} := \left\langle p_{ij\widehat{1}}, p_{\widehat{1ij}} \,|2 \leq i < j \leq 7 \right\rangle$. By this argument, there are 7 hyperplanes like $h_{\widehat{i}}$ generated by

$$\left\{ p_{j_1 j_2 \widehat{i}}, p_{\widehat{k_1 k_2 k_3}} \,|1 \leq j_1 < j_2 \leq 7, 1 \leq k_1 < k_2 < k_3 \leq 7 \text{ and } i \in \{k_1, k_2, k_3\} \right\}$$

where $1 \leq i \leq 7$.

Consider a structure of a configuration in $\text{GCO}^{inj}_{(2,3,3)}$, the point p_1 has 6 hyperplanes $h_{1\widehat{i}}$, where $2 \leq i \leq 7$ pass through and other incidences are given by,



thus the hyperplanes $h_{\widehat{1}}, \ldots h_{\widehat{6}}$ and $h_{\widehat{7}}$ meet in a point, say p'. By considering a structure of a configuration in $\text{GCO}^{inj}_{(2,2,4)}$ consisting of 6 planes $h_{\widehat{1}} \cap h_{\widehat{i}}$, where $2 \leq i \leq 7$, through the point p', and so on, the point $p_{\widehat{1}}$ lies on the hyperplane $h_{\widehat{1}}$ because $\bigcap_{2 \leq i \leq 7} h_{i\widehat{1}} = \{p_{\widehat{1}}\}$. Hence, for each $1 \leq i \leq 7$, the point $p_{\widehat{i}}$ lies on the hyperplane $h_{\widehat{i}}$.

Therefore we have 126 hyperplanes and 576 points, with 7 hyperplanes through each point and 32 points on each hyperplanes, as in Figure 5.7.2.

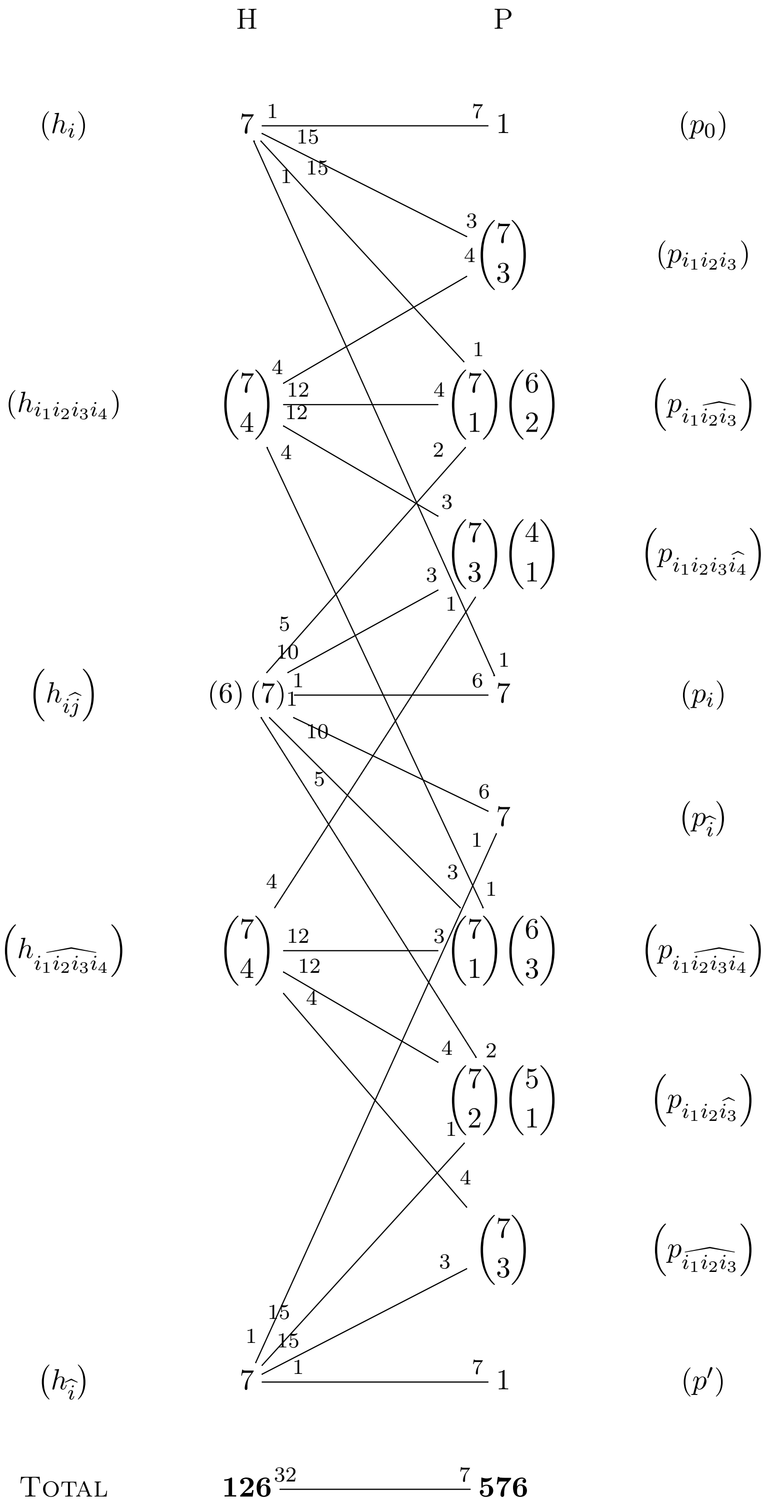
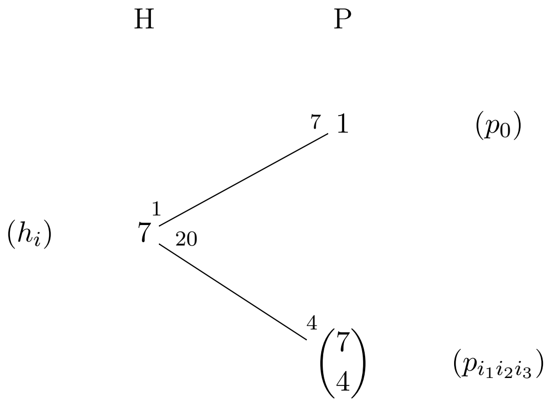


Figure 5.7.2: The branched summary for $\text{GCo}_{(2,3,4)}$

□

Proposition 5.7.3. *Let V be a complex vector space of dimension 6. Given a $(2,4,3)$-starting configuration in $\mathbb{P}^5(V)$, a configuration in* $\mathrm{GCo}^{inj}_{(2,4,3)}$ *can be obtained by choosing 7 more parameters.*

Proof. The starting configuration has the following branched summary

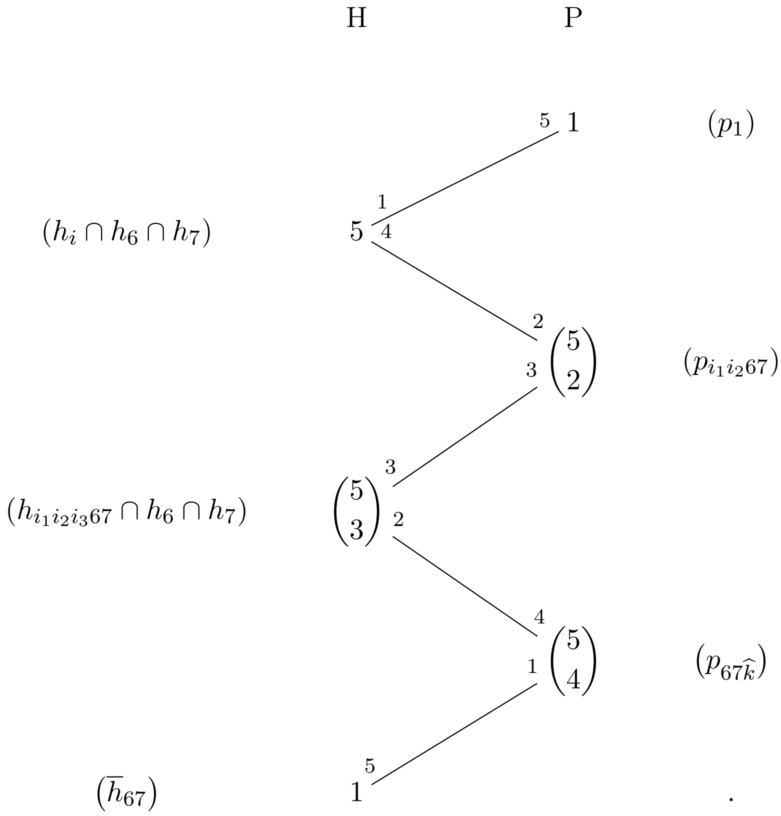


For $1 \le i_1 < \ldots < i_5 \le 7$, the set of points

$$\{p_{j_1 j_2 j_3 j_4} \,|\, j_1 < j_2 < j_3 < j_4 \text{ and } j_1, j_2, j_3 \in \{i_1, \ldots, i_5\}\}$$

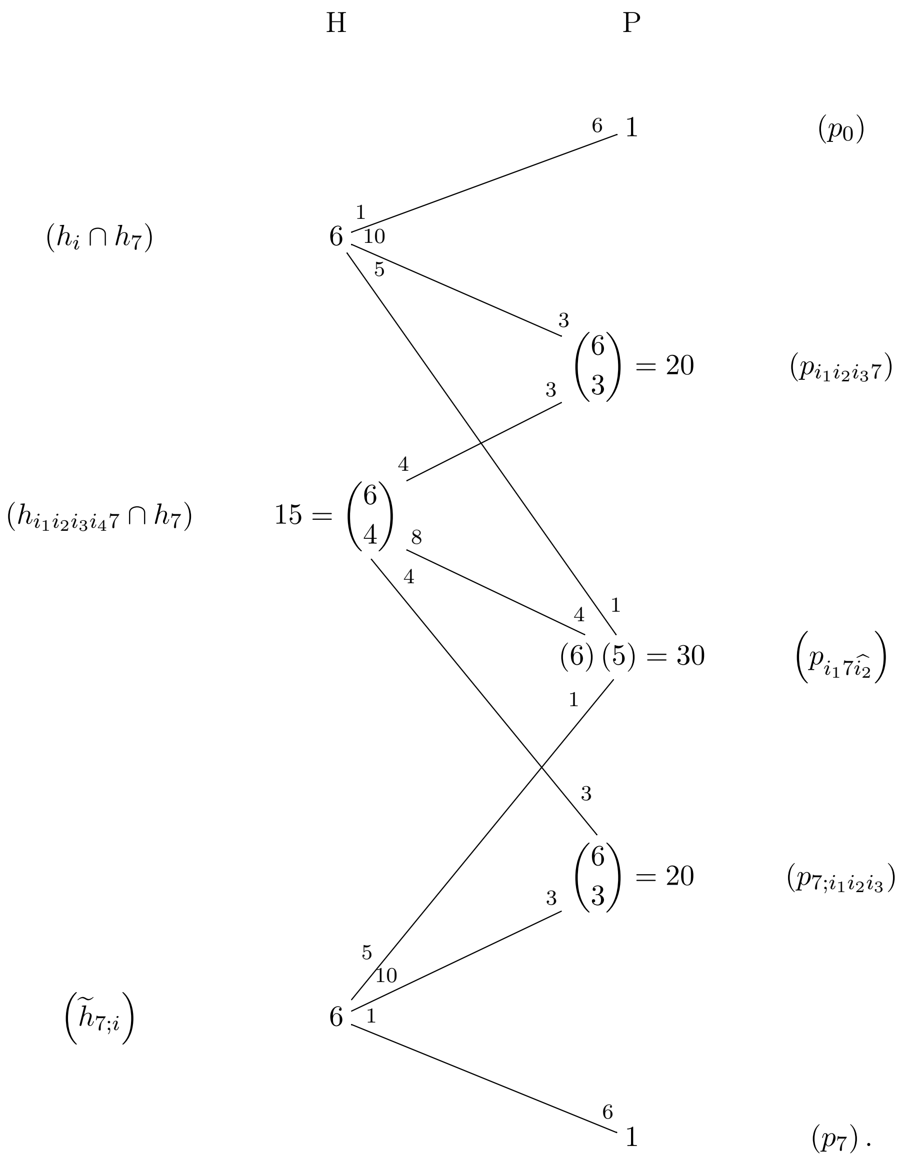
uniquely determines a hyperplane, say $h_{i_1 i_2 i_3 i_4}$; there are $\binom{7}{5} = 21$ such hyperplanes.

There are 5 planes $h_i \cap h_6 \cap h_7$ where $1 \le i \le 5$ through the point p_0. This gives a structure of a configuration in $\mathrm{GCo}^{inj}_{(2,2,3)}$ whose incidences are given by



So there are totally $\binom{7}{2}\binom{5}{1} = 105$ points like $p_{ij\widehat{k}}$ on $h_i \cap h_j \cap \bigcap_{k \in \{k_1,k_2,k_3,k_4,k_5\}} h_{k_1k_2k_3k_4k_5}$ where $1 \le i < j \le 7$ and $k \in \{1,\ldots,7\} \setminus \{i,j\}$.

By intersecting h_7 with the other 6 hyperplanes, we have 6 3-planes $h_i \cap h_7$ where $1 \le i \le 7$ pass through the point p_0 giving a structure of a configuration in $\mathrm{GCo}^{inj}_{(2,3,3)}$, which we use 1 parameter (see Proposition 5.7.1). It's branched summary is given by



By choosing a suitable 3-plane in h_7, we can define the points $p_{7;i_1 i_2 i_3}$, where $1 \le i_1 < i_2 < i_3 \le 6$, on h_7 such that, for each $1 \le i \le 6$,

$$\left\langle p_{j_1 7 \widehat{j_2}}, p_{7;k_1 k_2 k_3}, p_7 \,|1 \le j_1 < j_2 \le 6, 1 \le k_1 < k_2 < k_3 \le 6, j_2 \notin \{k_1, k_2, k_3\}, i \in \{k_1, k_2, k_3\} \right\rangle$$

is a 3-plane, say $\widetilde{h}_{7;i}$, in h_7. Processing the same way for all hyperplanes h_i where $1 \le i \le 7$, there are totally $\binom{7}{1}\binom{6}{3} = 140$ points like $p_{k;i_1 i_2 i_3} := h_k \cap \bigcap_{\{k,i_1,i_2,i_3\} \subseteq \{k_1,k_2,k_3,k_4,k_5\}} h_{k_1 k_2 k_3 k_4 k_5}$ where $1 \le i_1 < i_2 < i_3 \le 7$ and $k \in \{1, \ldots, 7\} \setminus \{i_1, i_2, i_3\}$ and for each $1 \le i, j \le 7$ such that $i \ne j$, $\widetilde{h}_{i;j}$ is a 3-plane in h_i. Hence we use 7 parameters now.

For any $1 \leq i < j \leq 7$, since $\widetilde{h}_{i;j} \cap \widetilde{h}_{j;i} = \overline{h}_{ij}$, define h_{ij} be a hyperplane generated $\widetilde{h}_{i;j}$ and $\widetilde{h}_{j;i}$. So there are $\binom{7}{2} = 21$ hyperplanes like h_{ij} where $1 \leq i < j \leq 7$; 7 points like $p_i := \bigcap_{i \in \{k_1, k_2\}} h_{k_1 k_2}$ where $1 \leq i \leq 7$.

The planes $h_{12345} \cap h_{12346} \cap h_i$ where $1 \leq i \leq 4$ through the point p_{1234} give a structure of a configuration in $\text{GCo}^{inj}_{(2,2,2)}$. This implies that the hyperplanes h_{12345}, h_{12346}, h_{12356}, h_{12456}, h_{13456} and h_{23456} meet in a point, say $p_{\widehat{7}}$. There are totally 7 points like $p_{\widehat{i}} := \bigcap_{\{k_1,\ldots,k_5\} \subseteq \{1,\ldots,7\} \setminus \{i\}} h_{k_1 k_2 k_3 k_4 k_5}$ where $1 \leq i \leq 7$.

Similarly the planes $h_{12345} \cap h_{12346} \cap h_{12347}$ and $h_{12345} \cap h_{12346} \cap h_i$ where $1 \leq i \leq 3$ through the point p_{1234} give again a structure of a configuration in $\text{GCo}^{inj}_{(2,2,2)}$ implying that the hyperplanes h_{12345}, h_{12346}, h_{12356}, h_{12}, h_{13} and h_{23} meets in a point, say $p_{123\widehat{7}}$. There are totally $\binom{7}{3}\binom{4}{1} = 140$ points like

$$p_{i_1 i_2 i_3 \widehat{k}} := \bigcap_{\{k_1,k_2\} \subseteq \{i_1,i_2,i_3\}} h_{k_1 k_2} \cap \bigcap_{\{i_1,i_2,i_3\} \subseteq \{k_1,\ldots,k_4\}, k \notin \{k_1,\ldots,k_4\}} h_{k_1 \cdots k_4},$$

where $1 \leq i_1 < i_2 < i_3 \leq 7$ and $k \in \{1, \ldots, 7\} \setminus \{i_1, i_2, i_3\}$.

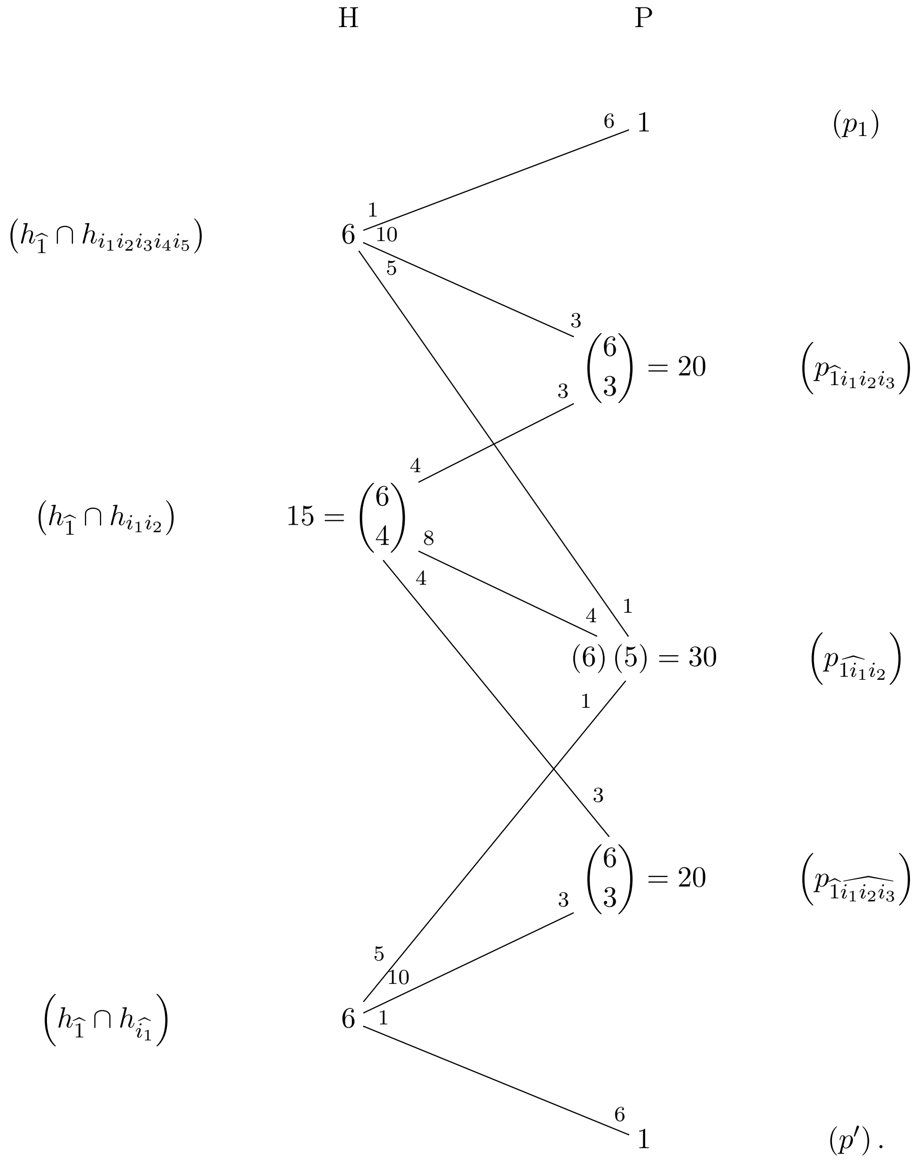
The planes $h_{12345} \cap h_{12346} \cap h_{12347}$ and $h_{12345} \cap h_{12346} \cap h_i$, where $1 \leq i \leq 4$, through the point p_{1234} give a structure of a configuration in $\text{GCo}^{inj}_{(2,2,3)}$ implying

$$\overline{h}_{\widehat{7}1234} := \langle p_{\widehat{7}}, p_{123\widehat{7}}, p_{124\widehat{7}}, p_{134\widehat{7}}, p_{234\widehat{7}} \rangle$$

is a plane in $h_{12345} \cap h_{12346}$. By the same reason, for any $1 \leq i_1 < i_2 < i_3 < i_4 \leq 7$ and $k \in \{1, \ldots, 7\} \setminus \{i_1, i_2, i_3, i_4\}$, the plane $\overline{h}_{\widehat{k} i_1 i_2 i_3 i_4}$ is a plane. Let $h_{\widehat{1}}$ be a hyperplane generated by the planes $\overline{h}_{\widehat{1}2345}$, $\overline{h}_{\widehat{1}2346}$, $\overline{h}_{\widehat{1}2456}$ and $\overline{h}_{\widehat{1}2457}$. Then it is not hard to see that $\overline{p}_{\widehat{1} i_1 i_2 i_3 i_4}$ is in $h_{\widehat{1}}$ for all $2 \leq i_1 < i_2 < i_3 < i_4 \leq 7$. Processing this way, there are totally 7 hyperplanes like $h_{\widehat{i}}$ containing all the planes $\overline{h}_{\widehat{i} k_1 k_2 k_3 k_4}$ ($1 \leq k_1 < k_2 < k_3 < k_4 \leq 7$ and $i \notin \{k_1, \ldots, k_4\}$), where $1 \leq i \leq 7$.

Next consider the planes $h_{\widehat{1}} \cap h_{34567} \cap h_{23456}$, $h_{\widehat{1}} \cap h_{34567} \cap h_{23457}$, $h_{\widehat{1}} \cap h_{34567} \cap h_{23467}$ and $h_{\widehat{1}} \cap h_{34567} \cap h_{23567}$ through the point $p_{\widehat{1}}$. This gives a structure of a configuration in $\text{GCo}^{inj}_{(2,2,2)}$ which implies that the hyperplanes $h_{\widehat{1}}$, h_{34567}, h_{34}, h_{35}, h_{36} and h_{37} meets in a point, say $p_{\widehat{1};\widehat{2}3}$. Processing this way, we will see that, for $1 \leq i < j < k \leq 7$and all i, j, k are distinct, $p_{\widehat{i};\widehat{j}k} = p_{\widehat{j};\widehat{i}k}$. So the colon sign does not have any meaning in our notation. For $1 \leq i < j \leq 7$ and $k \in \{1, \ldots, 7\} \setminus \{i, j\}$, denote the point $p_{\widehat{i};\widehat{j}k} = p_{\widehat{j};\widehat{i}k}$ by $p_{\widehat{i}\widehat{j}k}$. There are thus $\binom{7}{2}\binom{5}{1} = 105$ points like $p_{\widehat{ij}k}$ where $1 \leq i < j \leq 7$ and $k \in \{1, \ldots, 7\} \setminus \{i, j\}$.

Finally consider 3-planes $h_{\hat{1}} \cap h_{k_1k_2k_3k_4k_5}$ where $2 \leq k_1 < \ldots < k_5 \leq 7$ through the point p_1. This gives a structure of a configuration in $\mathrm{GCo}^{inj}_{(2,3,3)}$ whose incidences are given by



Processing this way, we will see that for all $1 \leq i_1 < i_2 < i_3 < i_4 \leq 7$, the point $p_{\hat{i_1}\widehat{i_2i_3i_4}} = p_{\hat{i_2}\widehat{i_1i_3i_4}}$. So for all $1 \leq i_1 < i_2 < i_3 < i_4 \leq 7$, denote $p_{\hat{i_1}\widehat{i_2i_3i_4}} = p_{\hat{i_2}\widehat{i_1i_3i_4}}$ by $p_{\widehat{i_1i_2i_3i_4}}$. There are $\binom{7}{4} = 35$ points like $p_{\widehat{i_1i_2i_3i_4}}$ where $1 \leq i_1 < i_2 < i_3 < i_4 \leq 7$. Moreover we will see that the hyperplanes $h_{\hat{i}}$ where $1 \leq i \leq 7$ meet in a point say p'.

Therefore we have 36 hyperplanes and 576 points, with 7 hyperplanes through each point and 72 points on each hyperplanes, as in Figure 5.7.3.

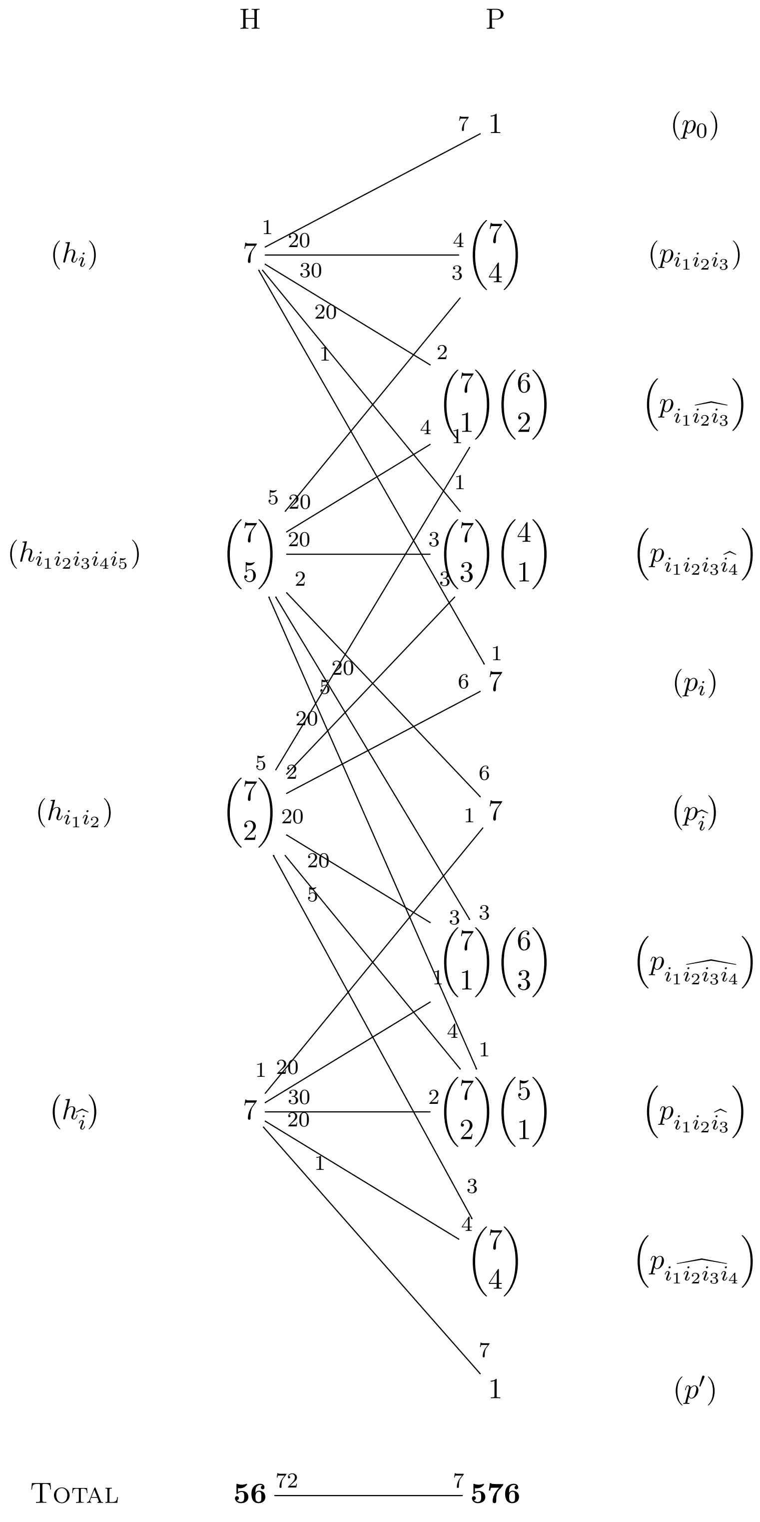


Figure 5.7.3: The branched summary for $\mathrm{GCo}_{(2,4,3)}$.

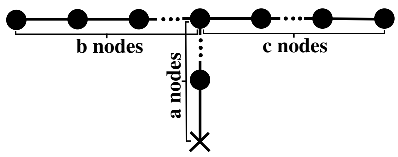
□

There are some finite cases of E-type that have not been proven yet in this thesis which are the cases of $(2,3,5)$, $(2,5,3)$, $(3,2,3)$, $(3,2,4)$, $(4,2,3)$, $(3,2,5)$, $(5,2,3)$, $(3,3,2)$, $(3,4,2)$, $(4,3,2)$, $(3,5,2)$ and $(5,3,2)$.

Chapter 6

Conclusion and outlook

A generalized Cox configuration is a symmetric configuration parametrized by a certain polytope with the decorated Coxeter diagram



.

It consists of points and hyperplanes, in a projective space with a certain incidence relation. We have interpreted such a labelled configuration as an incidence system morphism

$$\Psi_{(\mathfrak{t},\mathfrak{b})} : \mathsf{C}\,(W) \to \mathsf{Proj}\,(V)$$

over a certain map, where $\mathsf{C}\,(W)$ is a Coxeter incidence system and $(\mathfrak{t},\mathfrak{b}) \in \mathscr{U}^{\mathfrak{q}}$. This incidence system morphism shows explicitly the correspondence between maximal cosets in $\mathsf{C}\,(W)$ and the objects in the image of the configuration such as points, lines, planes.

While not all generalized Cox configurations are non-degenerate, we believe that the generalized Cox configurations constructed as in Section 5.2 are non-degenerate, i.e., injective. Moreover, we also claim that any non-degenerate generalized Cox configurations are constructed in this way. We investigate these claims (Conjecture 5.3.3) in the cases $(1, b, c)$, $(a, 1, c)$, and $(a, b, 1)$; all of these cases are A-type.

- A generalized Cox configuration of type $(a, b, 1)$ is a generic parabolic configuration (without projection). The image of this generalized Cox configuration is an $(a + b - 1)$-simplex in $\mathbb{P}^{a+b-1}$. In this case, Conjecture 5.3.3 is true.

- A generalized Cox configuration of type $(a, 1, c)$ is a classical projection of an $(a + c - 1)$-simplex in $\mathbb{P}^{a+c-1}$ into the lower dimensional space $\mathbb{P}^a$. In this case, Conjecture 5.3.3 is also true.

- A generalized Cox configuration of type $(1, b, c)$ is the dual configuration of a generalized Cox configuration of type $(a, 1, c)$. Thus it is intersecting a $(b + c - 1)$-simplex in $\mathbb{P}^{b+c-1}$ with a projective subspace of dimension b. In this case, Conjecture 5.3.3 is automatically true because of the duality.

There are still many cases in which Conjecture 5.3.3 has not been investigated yet here precisely in the case that $T_{(a,b,c)}$ is of types D and E. This requires some complicated work.

Base on Conjecture 5.3.3 and the dimension formula for $K\backslash \mathscr{U}^{\mathfrak{q}}$, we have a dimension formula

$$\dim\left(\mathrm{GCo}^{inj}_{(a,b,c)}(W,V)\right) = (a+b-1) + \dim\left(\mathrm{GCo}^{inj}_{(a-1,b,c)}(W,V)\right) + \dim\left(\mathfrak{p}^{\perp}\right). \quad (6.0.1)$$

This formula suggests a recursive construction for generalized Cox configurations of type (a, b, c), where $a \geq 2$. The recursive construction says that:

1. Choose a point p_0 in an $(a + b - 1)$-dimensional projective space $\mathbb{P}(V)$ and choose a residual generalized Cox configuration of the point p_0, i.e., a generalized Cox configuration of type $(a - 1, b, c)$ on the projective space $\mathbb{P}(V/p_0)$.

2. Now, there are $\dim\left(\mathfrak{p}^{\perp}/\left[\mathfrak{p}^{\perp}, \mathfrak{p}^{\perp}\right]\right)$ lines through the point p_0 in the construction. Generally choose a point on each such line different from p_0.

3. There are $\dim\left(\left[\mathfrak{p}^{\perp}, \mathfrak{p}^{\perp}\right]\right)$ more parameters to choose in order to complete a generalized Cox configuration of type (a, b, c).

Comparing this construction with the recursive construction for (generalized) Clifford configurations of type $(2, b, c)$, introduce by [LH72] in Section 7, we find that our recursive construction is closely related to Longuet-Higgins' recursive construction. Longuet-Higgins' recursive construction says that given a point p_0 on the surface of a $(b + 1)$-dimensional sphere and $b+c$ hyperspheres on the surface passing through the point p_0, there are $\binom{b+c}{b}$ points, obtained by the intersections of any b hyperspheres, on the surface different from p_0.

The symmetric configuration

$$\begin{array}{ccc} \text{Hyperspheres} & & \text{Points} \\ b+c & \overset{\binom{b+c-1}{b-1}}{\rule{5em}{0.4pt}} \, b & \binom{b+c}{b} \end{array}$$

corresponds to the residual configuration in the Coxeter incidence system $\mathsf{C}(W)$ of the coset corresponding to the point p_0 (in Longuet-Higgins' work). We see that the constraint of lying on the $(b+1)$-dimensional sphere uniquely determine a collection of $\dim\left(\mathfrak{p}^{\perp} / [\mathfrak{p}^{\perp}, \mathfrak{p}^{\perp}]\right) = \binom{b+c}{b}$ points on the lines in the residual configuration, while in our construction we have to choose those such points. Longuet-Higgins claimed that by following his recursive construction, one should be able to complete a (generalized) Clifford configuration type $(2, b, c)$. However, he did not mention anything about extra parameters which might have to be chosen in completing the configuration apart from those we have already seen. We conjecture (Conjecture 5.4.2) that our recursive construction works, which also implies that equation 6.0.1 is true, and we explore this conjecture in some cases as follows.

- In the cases $(a, 1, c)$, $(a, b, 1)$, $(2, b, 2)$, and $(2, 2, c)$,

  $$\dim\left(\left[\mathfrak{p}^{\perp}, \mathfrak{p}^{\perp}\right]\right) = 0.$$

  Thus a starting configuration uniquely determine the generalized Cox configuration. In particular, in the case $(2, 2, c)$, Cox's chain implies Conjecture 5.4.2.

- In the case $(2, 3, 3)$,

  $$\dim\left(\left[\mathfrak{p}^{\perp}, \mathfrak{p}^{\perp}\right]\right) = 1.$$

  This is the first case we face that a starting configuration does not uniquely determine a generalized Cox configuration.

- In the cases $(2, 3, 4)$, and $(2, 4, 3)$,

  $$\dim\left(\left[\mathfrak{p}^{\perp}, \mathfrak{p}^{\perp}\right]\right) = 7.$$

  A starting configuration definitely does not uniquely determine a generalized Cox configuration because the configuration is constructed from generalized Cox configurations of type $(2,3,3)$.

There are still some cases in which Conjecture 5.4.2 has not been investigated yet in this thesis. Moreover, compared with Longuet-Higgins' construction, the number $\dim\left([\mathfrak{p}^{\perp},\mathfrak{p}^{\perp}]\right)$ of parameters we found in the cases $(2,3,3)$, $(2,3,4)$, and $(2,4,3)$ may or may not appear in his construction. It's worth to investigate the remaining cases we haven't done in this thesis and even explore Longuet-Higgins' construction for Clifford configurations in the cases $(2,3,3)$, $(2,3,4)$, and $(2,4,3)$ to see whether it needs more parameters.

References

[AB08] P. Abramenko and K. S. Brown. *Buildings: Theory and applications*. Springer, August 2008. ISBN 9780387788340.

[AD61] M. W. Al-Dhahir. A generalization of Cox's chain of theorems. *Canadian Mathematical Bulletin*, 4(0):1–6, January 1961.

[AH06] P. Abramenko and V. M. Hendrik. Combinatorial characterizations of convexity and apartments in buildings. *Australasian Journal of Combinatorics*, 34:89–104, 2006.

[Ati82] M. F. Atiyah. Convexity and commuting hamiltonians. *Bulletin of the London Mathematical Society*, 14(1):1–15, January 1982.

[Ayo] J. Ayoub. Introduction to algebraic groups. Available online at: www.math.uzh.ch/index.php?file& key1=16119/.

[BC83] A. E. Brouwer and A. M. Cohen. *Local recognition of Tits geometries of classical type*. Math. Centrum, 1983.

[BC13] F. Buekenhout and A.M. Cohen. *Diagram geometry: Related to classical groups and buildings*. Springer London, Limited, 2013. ISBN 9783642344527.

[BE89] R. J. Baston and M. G. Eastwood. *The Penrose transform: Its interaction with representation theory*. Clarendon Press, 1989. ISBN 9780198535652.

[Bor56] A. Borel. Groupes linéaires algébriques. *Annals of Mathematics*, 64(1):20–82, July 1956.

[Bor91] A. Borel. *Linear algebraic groups*. Springer-Verlag, December 1991. ISBN 9783540973706.

[Bou89] N. Bourbaki. *Lie groups and Lie algebras: Chapters 1-3*. Springer, 1989. ISBN 9783540642428.

[Bou02] N. Bourbaki. *Lie groups and Lie algebras: Chapters 4-6*. Springer, November 2002. ISBN 9783540691716.

[Bou05] N. Bourbaki. *Lie groups and Lie algebras: Chapters 7-9*. Springer, November 2005. ISBN 9783540688518.

[Bro54] L. M. Brown. A configuration of points and spheres in four-dimensional space. *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics*, 64(02): 145–149, 1954.

[Bue79] F. Buekenhout. Diagrams for geometries and groups. *Journal of Combinatorial Theory, Series A*, 27(2):121–151, September 1979.

[Bue95] F. Buekenhout. *Handbook of incidence geometry: buildings and foundations*. Elsevier, March 1995. ISBN 9780444883551.

[CDS05] D. M. J. Calderbank, T. Diemer, and V. Souček. Ricci-corrected derivatives and invariant differential operators. *Differential Geometry and its Applications*, 23(2): 149–175, September 2005.

[CG67] H. S. M. Coxeter and S. L. Greitzer. *Geometry revisited*. MAA, 1967. ISBN 9780883856192.

[Che99] C. C. Chevalley. *Theory of Lie groups*. Princeton University Press, 1999. ISBN 9780691049908.

[Cli71] W. K. Clifford. Synthetic proof of Miquel's theorem. *Oxford, Cambridge and Dublin Messenger of Math.*, 5:124–141, 1871.

[CM57] H. S. M. Coxeter and W. O. J. Moser. *Generators and relations for discrete groups*. Springer, 1957.

[Cox91] H. Cox. Application of Grassmann's Ausdehnungslehre to properties of circles. *Quart. J. Math*, 25:1–71, 1891.

[Cox34] H. S. M. Coxeter. Discrete groups generated by reflections. *Ann. of Math*, 35: 588–612, 1934.

[Cox50] H. S. M. Coxeter. Self-dual configurations and regular graphs. *Bulletin of the American Mathematical Society*, 56(5):413–455, 1950.

[Cox56] H. S. M. Coxeter. *Review of Brown (1954)*, volume 17. 1956. 886 pp.

[Cox61] H. S. M. Coxeter. *Introduction to geometry.* Wiley, March 1961. ISBN 9780471504580.

[Cox73] H. S. M. Coxeter. *Regular polytopes.* Courier Dover Publications, 1973. ISBN 9780486614809.

[ČS09] A. Čap and J. Slovák. *Parabolic geometries I: Background and general theory.* American Mathematical Soc., 2009. ISBN 9780821875353.

[Dav08] M. Davis. *The geometry and topology of Coxeter groups.* Princeton University Press, 2008. ISBN 9780691131382.

[DK00] J. J. Duistermaat and J. A. C. Kolk. *Lie groups.* Springer, 2000. ISBN 9783540152934.

[FH91] W. Fulton and J. Harris. *Representation Theory: A First Course.* Springer, 1991. ISBN 9780387974958.

[Gar97] P. B. Garrett. *Buildings and classical groups.* Routledge Chapman & Hall, 1997. ISBN 9780412063312.

[Gra98] J.H. Grace. Circles, spheres and line complexes. *Trans. Camb. Phil. Soc.*, 16: 153–190, 1898.

[Gre08] R. M. Green. Representations of Lie algebras arising from polytopes. *International electronic journal of algebra*, 4:27–52, 2008.

[Gro57] A. Grothendieck. Sur la classification des fibres holomorphes sur la sphere de Riemann. *American Journal of Mathematics*, 79(1):121, January 1957.

[GS87] I. M. Gel'fand and V. V. Serganova. Combinatorial geometries and torus strata on homogeneous compact manifolds. *Russian Mathematical Surveys*, 42(2):133, April 1987.

[Har92] J. Harris. *Algebraic geometry: A first course.* Springer, September 1992. ISBN 9780387977164.

[HK80] R. B. Howlett and R. W. Kilmoyer. Principal series representation of finite groups with split BN pair. *Communications in Algebra*, 8(6):543–583, January 1980.

[HN12] J. Hilgert and K. H. Neeb. *Structure and geometry of Lie groups.* Springer, 2012. ISBN 9780387847948.

[Hoc71] G. Hochschild. Note on algebraic Lie algebras. *Proceedings of the American Mathematical Society*, 29(1):10, June 1971.

[Hoc81] G. P. Hochschild. *Basic theory of algebraic groups and Lie algebras.* Springer-Verlag, 1981. ISBN 9780387905419.

[Hum72] J. E. Humphreys. *Introduction to Lie algebras and representation theory.* Springer-Verlag GmbH, 1972. ISBN 9783540900535.

[Hum75] James E. Humphreys. *Linear algebraic groups.* Springer, May 1975. ISBN 9780387901084.

[Hum92] J. E. Humphreys. *Reflection groups and Coxeter groups.* Cambridge University Press, October 1992. ISBN 9780521436137.

[IS02] A. A. Ivanov and S. V. Shpectorov. *Geometry of sporadic groups: Volume 2, representations and amalgams.* Cambridge University Press, March 2002. ISBN 9780521623490.

[IT91] M. Ise and M. Takeuchi. *Lie groups: I, II.* American Mathematical Soc., 1991. ISBN 9780821845448.

[Jac79] N. Jacobson. *Lie algebras.* Courier Dover Publications, 1979. ISBN 9780486638324.

[Kas07] A. Kasikova. Characterization of some subgraphs of point-collinearity graphs of building geometries. *European Journal of Combinatorics*, 28(5):1493–1529, July 2007.

[Kha13] A. Khare. Faces and maximizer subsets of highest weight modules. arXiv e-print 1301.1140, January 2013.

[Kna02] A. W. Knapp. *Lie groups beyond an introduction.* Springer, August 2002. ISBN 9780817642594.

[LH72] M. S. Longuet-Higgins. Clifford's chain and its analogues in relation to the higher polytopes. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 330(1583):443–466, November 1972.

[Mac09] A. W. Macpherson. Symmetric configurations of spheres and points. *EPSRC vacation project (supervisor: D. M. J. Calderbank)*, 2009.

[Mes59] B. E. Meserve. *Fundamental concepts of geometry.* Courier Dover Publications, 1959. ISBN 9780486634159.

[Mil11] James S. Milne. Algebraic groups, Lie groups, and their arithmetic subgroups, 2011. Available online at www.jmilne.org/math/.

[Mil12a] James S. Milne. Basic theory of affine group schemes, 2012. Available online at: www.jmilne.org/math/.

[Mil12b] James S. Milne. Lie algebras, algebraic groups, and Lie groups, 2012. Available online at: www.jmilne.org/math/.

[Mil12c] James S. Milne. Reductive groups, 2012. Available online at: www.jmilne.org/math/.

[Möb28] A. F. Möbius. Kann von zwei dreiseitigen Pyramiden eine jede in Bezug auf die andere um- und eingeschrieben zugleich heißen? *Journal für die Reine und Angewandte Mathematik*, 1828(3):273–278, January 1828.

[MS90] P. McMullen and E. Schulte. Constructions for regular polytopes. *Journal of Combinatorial Theory, Series A*, 53(1):1–28, January 1990.

[Nev26] E. H. Neville. The inversion of the Miquel-Clifford configuration. *J. Indian Math. Soc.*, 16:241–247, 1926.

[Pas94] A. Pasini. *Diagram geometries.* Clarendon Press, 1994. ISBN 9780198534976.

[Pro07] C. Procesi. *Lie groups: An approach through invariants and representations.* Springer, 2007. ISBN 9780387289298.

[Ric41] H. W. Richmond. On a chain of theorems due to Homersham Cox. *Journal of the London Mathematical Society*, s1-16(2):105–107, April 1941.

[Ron09] M. Ronan. *Lectures on buildings: updated and revised.* University of Chicago Press, October 2009. ISBN 9780226724997.

[Sep07] M. R. Sepanski. *Compact Lie groups.* Springer, January 2007. ISBN 9780387302638.

[Ser66] J. P. Serre. *Algébres de Lie semi-simples complexes.* W.A. Benjamin, 1966.

[Spr98] T. A. Springer. *Linear Algebraic Groups.* Birkhäuser, January 1998. ISBN 9783764340216.

[Tha11] K. Thas. *Order in building theory, in Surveys in combinatorics 2011.* Cambridge University Press, June 2011. ISBN 9781139503686.

[Tit62] J. Tits. Géométries polyédriques et groupes simples. *Atti 2a Riunione Groupem. Math. Express. Lat. Firenze*, pages 66–88, 1962.

[Tit75] J. Tits. On buildings and their applications. volume 1, pages 209–220. Proc. Int. Congress of Mathematicians, 1975.

[Tit81] J. Tits. A local approach to buildings. In C. Davis, B. Grünbaum, and F. A. Sherk, editors, *The Geometric Vein*, pages 519–547. Springer New York, New York, NY, 1981. ISBN 978-1-4612-5650-2, 978-1-4612-5648-9.

[TY05] P. Tauvel and R. W. T. Yu. *Lie algebras and algebraic groups.* Springer, June 2005. ISBN 9783540241706.

[W.] Wallace W. ("Scoticus") 1804 question 86, article xxvi. *in The Mathematical repository (new series), See part C (1804), p. 22, and part I (1805), pp. 169–170*, 1 (1806) (ed. T. Leybourne).