University of Bath

PHD

A study of normalisation through subatomic logic

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Award date:
2017

Awarding institution:
University of Bath

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submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Computer Science

January 2017

Abstract

We introduce subatomic logic, a new methodology where by looking *inside of atoms* we are able to represent a wide variety of proof systems in such a way that every rule is an instance of a single, regular, linear rule scheme. We show the generality of the subatomic approach by presenting how it can be applied to several different systems with very different expressivity.

In this thesis we use subatomic logic to study two normalisation procedures: cut-elimination and decomposition. In particular, we study cut-elimination by characterising a whole class of substructural logics and giving a generalised cut-elimination procedure for them, and we study decomposition by providing generalised rewriting rules for derivations that we can then apply to decompose derivations and to eliminate cycles.

Acknowledgements

First and foremost, I would like to thank my supervisor Alessio Guglielmi. I could not have asked for a more supportive, inspiring and cake-loving supervisor to guide me through my PhD. He introduced me to proof theory and to the world of research, and I couldn't have done this without his help, insight and encouragement. Thank you.

I would like to thank Benjamin Ralph for our fruitful discussions in front of the whiteboard, for his patience while listening to my convoluted explanations, and for all the help he has provided me throughout the writing of this thesis, typesetting atomic flows and giving greatly useful comments on various drafts.

I would also like to greatly thank my examiners Willem Heijltjes and Luca Roversi for their useful comments and observations.

I am also truly grateful to the whole mathfound group in Bath. Everyone has been helpful, kind, interesting and very willing to share their knowledge with me.

On a personal note, I would like to thank my parents and my sister for their love and support: it is what has led me here, and I couldn't be luckier to have you.

Thank you Laura for being the best friend there can ever be. Thank you to the Bath gang: D., L., M., C., J., Z., E., H., Z., A., A., you are my family away from home, and have made these three years the most wonderful time.

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Chapter 1

Introduction

Proof theorists have long been interested in the study of normalisation of proofs. From cut-elimination to proof identity, finding a normal form for proofs is a valuable research goal that includes questions such as which properties we would like for the normal form, and what the size of the normal form is in relation to the original proof. To study normalisation procedures generally is however very difficult: cut-elimination procedures for example are highly sensitive to variations on the form and structure of the rules of a system, where a single change in one of the rules or the addition of another warrant the need for a full new proof of cut-elimination in a new system. In this thesis, we provide a new approach within the setting of deep inference, which we call *subatomic*, that allows us to present a wide variety of propositional proof systems in such a way that *every rule* is an instance of a single linear rule scheme. We then exploit this generality to study normalisation procedures and their complexity, and in particular the role played by the interactions between the rules. These first applications of the *subatomic* methodology open an avenue for promising future research on the effect the interactions between rules have in different procedures, as well as in the realm of proof system design.

The first normalisation procedure that we set out to study is *cut-elimination*. Gentzen's proofs of cut-elimination [11] for classical and intuitionistic logic were only the first instance of a type of argument that has been long studied since. From that breakthrough, Gentzen-style cut elimination proofs abound in the literature, exploring on a system-by-system basis how to permute the cut-rules towards the premiss of a proof. The specificity needed for these cut-elimination arguments requires tricky case by case analyses, making it difficult to understand *how* cut-elimination works. Indeed, when designing a new proof system a complex trial and error phase is necessary to obtain cut admissibility. The fact that simple variations of a rule have so much influence on these arguments is the first hint that cut-elimination is in fact a combinatorial phenomenon, hinging mostly on the shape and interaction between the rules of a system. It is precisely this phenomenon that we set out to study, to understand how the interactions between the rules affect cut-elimination.

The second normalisation procedure that we analyse is *decomposition*. It is known that we can decompose a classical logic proof into a linear phase and a phase made-up only of contractions [29], or that we can decompose a first-order proof into a propositional phase and a quantified phase through a Herbrand theorem [7]. These type of results are called decomposition theorems, and they provide normal forms for proofs that are of great use since they allow us to separate proofs into different fragments that we can study independently from each other. Cut-elimination and decomposition often seem to be intertwined, since in the literature the proofs of both decomposition and cut-elimination theorems often rely on super rules, permuting contractions and cuts together in a single rule. However, in [29, 24] the decomposition of classical logic proofs into a linear phase followed by only atomic contractions is shown to be a purely local phenomenon independent of cut-elimination in proofs without *cycles*. By providing general rewriting schemes to permute rules, we will study the effect the interaction of rules has in decomposition and in the removal of cycles, showing that decomposition is a property fully independent from cut-elimination.

Since our main aim is to study the interactions between rules, we will do so in the setting of deep inference [18] where rules can be reduced to their atomic form providing great regularity in the inference rule schemes. In deep inference, proofs can be composed by the logical connectives that are used to compose formulae [25]. For example, if $\phi = \begin{array}{c} A \\ \| \\ B \end{array}$ and $\psi = \begin{array}{c} C \\ \| \\ D \end{array}$ are two proofs in propositional logic,

$$\phi \wedge \psi = \begin{array}{c} A \\ \| \\ B \end{array} \wedge \begin{array}{c} C \\ \| \\ D \end{array} \quad \text{and} \quad \phi \vee \psi = \begin{array}{c} A \\ \| \\ B \end{array} \vee \begin{array}{c} C \\ \| \\ D \end{array}$$

are two valid proofs with premisses $A \wedge C$ and $A \vee C$ and conclusions $B \wedge D$ and $B \vee D$ respectively. In deep inference, rules can be applied at any depth inside a formula and as a result every contraction and cut instance can be locally transformed into their atomic variants by a local procedure of polynomial-size complexity [4]. This provides a surprising regularity in the inference rule schemes: it can be observed that in most deep inference systems all rules besides the atomic ones can be expressed as

$$\frac{(A \ \alpha \ B) \ \beta \ (C \ \gamma \ D)}{(A \ \epsilon \ C) \ \zeta \ (B \ \eta \ D)} \quad ,$$

where A, B, C, D are formulae and $\alpha, \beta, \gamma, \epsilon, \zeta, \eta$ are logical relations. We call this rule shape a *medial shape*. Following this discovery, we will achieve an even greater regularity on the inference rules by looking even further, *inside the atoms*. We will introduce a new methodology through which we are able to represent *every rule* as an instance of a *single* inference rule scheme. This characterisation is not trivial: it is a delicate trade-off to impose restrictions on the possible assignments for $\alpha, \beta, \gamma, \epsilon, \zeta, \eta$ that allow us to characterise systems that enjoy cut-elimination and decompostion, but that are general enough to encompass the expressivity of a wide variety of logics. Indeed, the finding of these restrictions is the product of a long trial-and-error phase to obtain the desired generality together with the desired properties.

The main idea of this work is to consider atoms as self-dual, noncommutative binary logical relations and to build formulae by freely composing units by atoms and the other logical relations. We will consider the occurrences of an atom a as interpretations of more primitive expressions involving a noncommutative binary relation, still denoted by a. Two formulae A and B in the relation a, in this order, are denoted by $A\,a\,B$. Formulae are built over the units for the logical relations, denoted for example by t, f in the case of classical logic. We can think of it as a superposition of truth values: $\mathsf{f}\,a\,\mathsf{t}$ is the superposition of the two possible assignments for the atom a. We can for example have a projection onto a specific assignment by choosing which 'side' we read: if we read the values on the left of the atom we assign f to a and if we read the ones on the right we assign t to a. We call these formulae *subatomic*. For example,

$$((\mathsf{t}\,a\,\mathsf{f}) \wedge (\mathsf{f}\,b\,\mathsf{f})) \vee ((\mathsf{f} \wedge \mathsf{t})\,a\,\mathsf{t}) \quad \text{and} \quad (\mathsf{t}\,a\,\mathsf{t})\,b\,(\mathsf{f} \wedge \mathsf{f})$$

are subatomic formulae for classical logic.

In this way, we obtain an extended language of formulae which we can relate to the usual propositional formulae, or *interpret*, through an interpretation map $\stackrel{I}{\mapsto}$. A natural way to build such a map is to provide meaning to units inside the scope of an atom, by setting $\mathsf{f}\,a\,\mathsf{t} \stackrel{I}{\mapsto} a$ and $\mathsf{t}\,a\,\mathsf{f} \stackrel{I}{\mapsto} \bar{a}$, and extending it to all formulae in the natural way.

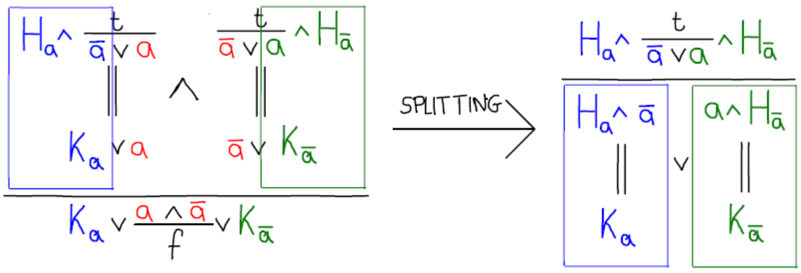
Subatomic formulae are much more than a clever representation. By using them, we are strikingly able to present proof systems in such a way that *every rule* has a medial shape, *including the atomic rules* that do not usually follow this scheme. For example, the rules for atomic introduction and atomic contraction can be represented as

$$\frac{(\mathsf{f} \vee \mathsf{t})\,a\,(\mathsf{t} \vee \mathsf{f})}{(\mathsf{f}\,a\,\mathsf{t}) \vee (\mathsf{t}\,a\,\mathsf{f})} \stackrel{I}{\mapsto} \frac{\mathsf{t}}{a \vee \bar{a}} \quad \text{and} \quad \frac{(\mathsf{f}\,a\,\mathsf{t}) \vee (\mathsf{f}\,a\,\mathsf{t})}{(\mathsf{f} \vee \mathsf{f})\,a\,(\mathsf{t} \vee \mathsf{t})} \stackrel{I}{\mapsto} \frac{a \vee a}{a} \quad .$$

This provides us with an extremely useful way to reason generally about proof systems: we need only focus on how the interaction of rules of this shape influences the cut-elimination and the decomposition procedures.

There are many different cut-elimination techniques in the deep inference literature [16, 3, 2, 37, 28], exploiting different aspects of the proof systems they work on. In this assortment, a particular methodology does however stand out for its generality: cut-elimination via *splitting* [21] can be achieved in the deep inference systems for linear logic [35], multiplicative exponential linear logic [37], the mixed commutative/non-commutative logic BV [21] and its extension with linear exponentials NEL [28], or classical predicate logic [3]. The generality of this procedure points towards the fact that it exploits some properties that are common to all these systems.

Splitting is based on a simple idea: to show that an atomic cut involving a and $\bar{a}$ is admissible, we follow a and $\bar{a}$ to the top of the proof to find two independent subproofs, the premiss of one containing the dual of a and the other one containing the dual of $\bar{a}$. In this way we obtain two independent 'pieces' that we can rearrange to get a new cut-free proof.



This type of argument has been used to prove the admissibility of rules other than the atomic cut [21], showing that it can be applied to any logical relation that we can follow upwards in a proof. Thus, the splitting procedure hinges strongly on the dualities present in propositional logical systems (to find the duals of a and $\bar{a}$) and on the regularity of deep inference rules (to follow the atoms in a proof), further confirming the suspicion that logical dualities and the shape of rules have a strong bearing on cut-elimination. Based on this intuition, we capitalise on the regularity of subatomic inference rules to generalise this process, studying which rules allow us to follow a connective in a proof. We show that in systems where the scope of relations only increases reading from bottom to top, called *splittable* systems, we can follow these relations through the proof and hence a whole class of rules is admissible via the splitting procedure. Splittable systems turn out to be the subatomic equivalent to propositional systems that we would characterise as linear, i.e., having no contractions. Unsurprisingly then, the class of rules shown admissible is precisely the class of rules that allow us to make the cut atomic in deep inference formalisms. Achieving this simple characterisation of splittable systems gives us a full understanding of *how* the splitting procedure works, and *why* it has been used with success to prove the admissibility of different rules in several systems. We note that splitting is a global procedure: we need to study the proof as a whole to obtain a cut-free proof through splitting. Furthermore, splitting does not create meaningful complexity: the size of the cut-free proofs obtained by general splitting is linear on the size of the proofs with cut they come from, and splitting is a procedure of polynomial-time complexity. This is an interesting observation for the further study of complexity, since deep inference proofs are as long or shorter than sequent proofs [6].

The generalised splitting procedure works in linear systems, but splitting theorems and in general cut-elimination have been proved in systems with contraction, such as for classical logic. It has long been suspected that this is due to the ability of these systems to be decomposed into a linear phase followed by a contractive phase. In classical Gentzen-style cut-elimination arguments, contractions are pushed to the top of a proof together with the cuts with the use of a mix rule, since permuting contraction rules and cut rules is not straightforward. This strategy can also be applied in deep inference systems, but in using it we lose sight of how the shapes and interactions of the rules influence cut-elimination and of when complexity is introduced, as we deal with the problematic case by conflating the rules together. This mismatch between cut-elimination procedures with and without contractions suggests that by moving the contractions together with the cut we conflate two different phenomena: the interactions of the contraction rules and the linear rules that generate complexity when they permute upwards in a proof, and the interactions between the linear rules and the cut rules that are straightforwardly taken care of via splitting. It has indeed been shown for classical logic and for multiplicative additive linear logic (MALL) [29, 35] that decomposing proofs into a linear phase followed by atomic contractions may generate an exponential increase in complexity.

We will study this phenomenon, providing general rewriting rules that correspond to the rewritings presented both for classical logic and for MALL in [29] and [35], proving that both decomposition results are a consequence of precisely the same properties. Additionally, it has long been conjectured [4] that it is possible to achieve a further decomposition of these systems, permuting not only the atomic contraction but a whole family of *contractive* rules towards the bottom of a derivation. The generalised rewriting rules that we present should be a significant step towards a proof of this conjecture.

Lastly, decomposition for classical logic has been proved to be independent from cut-elimination in the case of cycle-free proofs [29]. Cycles are a particular construction that might occur in a proof with cuts and contractions, and it is known that it is possible to remove them as a consequence of cut-elimination. Loops have been studied in the sequent calculus, and it has been shown that removing them might entail an exponential complexity growth [9]. Through our generalised rewriting rules we are able to present a purely local procedure based on permutations to remove the cycles in proofs, fully showing that decomposition in classical logic is independent from cut-elimination. Furthermore, this procedure will allow us to be able to study the complexity cost of the elimination of cycles in deep inference independently from cut-elimination, which is as of now unknown.

In this thesis we present and formalise subatomic logic and exploit its uniformity to study the effect of the interactions between rules in normalisation procedures. We present a generalisation of the splitting procedure, together with sufficient conditions for a system to enjoy splitting, that can be applied to a variety of logics to prove cut-elimination. We show a generalisation of decomposition reduction rules, together with sufficient conditions for a system to be decomposable into phases containing only atomic contractions/cocontractions and a linear phase. Furthermore, we show a cycle-eliminating procedure in classical logic. We obtain the following results:

Procedure	Splitting	Decomposition	Cycle-elimination
Logic	MLL BV KV^- CL^- Substructural Logic Class	CL MALL	CL MALL

In other words, we provide a new methodology that proves itself to be useful in its generality, allowing us to generalise and understand normalisation procedures in such a way that they capture several differently expressive logics. For this reason, this research aims to be only the start of the characaterisation of proof systems and their properties by the shape of their rules, as well as a useful reference for proof system design.

Chapter 2

Subatomic Logic

In this chapter, we will show how to achieve complete regularity on the shape of inference rules by introducing a new methodology, that we call *subatomic* because we look 'inside the atoms'. We will start by introducing subatomic formulae and giving tools to relate them to 'ordinary formulae'. Subatomic formulae are built by freely composing constants by connectives and atoms. For example,

$$A \equiv ((\mathsf{f}\ a\ \mathsf{t}) \vee \mathsf{t}) \wedge (\mathsf{t}\ b\ \mathsf{f}) \quad \text{and} \quad B \equiv ((\mathsf{t}\ b\ \mathsf{f}) \wedge \mathsf{t}) \vee \mathsf{f}$$

are two subatomic formulae for classical logic. The main idea is to interpret $\mathsf{f}\ a\ \mathsf{t}$ as a positive occurrence of the atom a, and $\mathsf{t}\ a\ \mathsf{f}$ as a negative occurrence of the same atom, denoted by $\bar{a}$. Intuitively, we can view subatomic formulae as a superposition of truth values. For example, $\mathsf{f}\ a\ \mathsf{t}$ is the superposition of the two possible assignments for the atom a, and $\mathsf{t}\ a\ \mathsf{f}$ is the superposition of the possible assignments for $\bar{a}$: if we read the value on the left of the atom we assign f to a and t to $\bar{a}$, and if we read the one on the right we assign t to a and f to $\bar{a}$.

Since we consider atoms as connectives, we will give a broad definition of what relations are, not assuming any logical characteristics or properties such as commutativity or associativity. We will therefore encompass logics with both commutative and non-commutative, associative and non-associative, dual and-self dual relations. This feature deserves to be highlighted since expressing self-dual non-commutative connectives into proof systems that enjoy cut-elimination is a challenge in Gentzen-style sequent calculi: it is impossible to have a complete analytic system with a self-dual non-commutative relation [38].

Using the new structure offered by subatomic formulae together with the regularity provided by deep inference we will then show that it is possible to achieve full regularity on the shape of inference rules in a wide variety of systems. In deep inference, the possibilty of composing proofs with the same connectives as formulae allows us to reduce most rules to their atomic form. The inference rules so obtained present a surprising regularity, that we can exploit towards obtaining a general rule scheme that encompasses every inference rule. We will show an underlying structure on the shape of the inference rules, using it to present all the rules of a system as instances of a single rule scheme, including the atomic ones.

Consider for example system SKS for classical logic [4].

$$
ai\!\downarrow \frac{\mathsf{t}}{a \vee \bar{a}} \qquad ai\!\uparrow \frac{a \wedge \bar{a}}{\mathsf{f}}
$$

$$
s \frac{(A \vee B) \wedge C}{(A \wedge C) \vee B} \qquad m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}
$$

$$
ac\!\downarrow \frac{a \vee a}{a} \qquad ac\!\uparrow \frac{a}{a \wedge a}
$$

$$
aw\!\downarrow \frac{\mathsf{f}}{a} \qquad aw\!\uparrow \frac{a}{\mathsf{t}}
$$

System SKS

We can derive the rule s from the rule

$$
\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \quad ,
$$

which has the same 'shape' as the rule m. In fact we will show that in many systems most non-atomic rules can be made to fit this scheme as well. By using the subatomic methodology, we are able to further extend this phenomenon to atomic rules in such a way that we can present a system for classical logic where every rule of the system has the same shape.

$$
a\!\downarrow \frac{(A \vee B)\, a\, (C \vee D)}{(A\, a\, C) \vee (B\, a\, D)} \qquad a\!\uparrow \frac{(A\, a\, B) \wedge (C\, a\, D)}{(A \wedge C)\, a\, (B \wedge D)}
$$

$$
\wedge\!\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \vee\!\uparrow \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}
$$

$$
m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}
$$

$$
ac \frac{(A\, a\, B) \vee (C\, a\, D)}{(A \vee C)\, a\, (B \vee D)} \qquad a\bar{c} \frac{(A \wedge B)\, a\, (C \wedge D)}{(A\, a\, C) \wedge (B\, a\, D)}
$$

System SAKS

We will present a characterisation of this rule shape, showcasing its generality by presenting examples of several such regular systems for different logics, which will be extended with further examples throughout the rest of the thesis.

Lastly, we will extend the notion of proof to subatomic systems, in order to relate them to 'usual' proof systems.

2.1 Subatomic formulae

Subatomic formulae are built by freely composing constants by connectives and atoms. For example, $A \equiv ((\mathsf{f}\, a\, \mathsf{t}) \vee \mathsf{t}) \wedge (\mathsf{t}\, b\, \mathsf{f})$ and $B \equiv ((\mathsf{t}\, b\, \mathsf{f}) \wedge \mathsf{t}) \vee \mathsf{f}$ are two subatomic formulae for Classical Logic. By considering atoms as relations we will work with an extended language of formulae, since we can have atoms in the scope of other atoms, something that does not occur in 'traditional' formulae.

Definition 2.1. Let $\mathscr{U}$ be a denumerable set of *constants* whose elements are denoted by $u, v, w, \ldots$. Let $\mathscr{R}$ be a denumerable partially ordered set of *relations* whose elements are denoted by α, β, γ, ... The set $\mathscr{F}$ of *subatomic formulae* (or SA formulae) contains terms defined by the grammar

$$\mathscr{F} ::= \mathscr{U} \mid \mathscr{F}\ \mathscr{R}\ \mathscr{F} \quad .$$

Formulae are denoted by A, B, C, ...

A *(formula) context* $K\{\ \}\cdots\{\ \}$ is a formula where some subformulae are substituted by holes; $K\{A_1\}\cdots\{A_n\}$ denotes a formula where the n holes in $K\{\ \}\cdots\{\ \}$ have been filled with $A_1, \ldots, A_n$.

The expression $A \equiv B$ means that the formulae A and B are syntactically equal. We omit parentheses when there is no ambiguity.

In $K\{A\ \alpha\ B\}$ we say that the subformulae of A and B are *in the scope of* α.

Example 2.2. The set $\mathscr{F}_{cl}$ of subatomic formulae for classical logic is given by the set of constants $\mathscr{U} = \{\mathsf{f}, \mathsf{t}\}$ and the set of relations $\mathscr{R} = \{\wedge, \vee\} \cup \mathscr{A}$ where $\mathscr{A}$ is a denumerable set of atoms, denoted by $a, b, \ldots$ with $\mathscr{A} \cap \{\wedge, \vee\} = \varnothing$. Two examples of subatomic formulae for classical logic are

$$A \equiv ((\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{t}\ a\ \mathsf{t})) \wedge (\mathsf{t}\ b\ \mathsf{f}) \quad \text{and} \quad B \equiv ((\mathsf{t}\ b\ \mathsf{f}) \wedge \mathsf{t}) \vee (\mathsf{f}\ a\ \mathsf{f}) \quad .$$

Example 2.3. The set $\mathscr{F}_{ll}$ of subatomic formulae for multiplicative linear logic is given by the set of constants $\mathscr{U} = \{\bot, 1\}$ and the set of relations $\mathscr{R} = \{⅋, \otimes\} \cup \mathscr{A}$ where $\mathscr{A}$ is a denumerable set of atoms, denoted by $a, b, \ldots$ with $\mathscr{A} \cap \{⅋, \otimes\} = \varnothing$. Two examples of subatomic formulae for linear logic are

$$C \equiv ((1 ⅋ \bot)\ a\ 1) \otimes \bot \quad \text{and} \quad D \equiv ((\bot ⅋ 1)\ b\ 1) \otimes (1\ a\ \bot) \quad .$$

Aside from classical logic and multiplicative linear logic, we will feature the logic BV [21] amongst the examples to showcase a well-studied logic with self-dual non-commutative connectives. For that, we define the logic BVU. BV will correspond to BVU with all the units identified.

Example 2.4. We define system BVU. The formulae of BVU are built from the units $\bot, \circ, 1$ by composing them with the relations $⅋, \triangleleft, \otimes$.

The relations $\invamp$ and $\otimes$ are dual to each other, associative, commutative and have units $\bot$ and 1 respectively. $\triangleleft$ is self-dual and associative, and has unit $\circ$.

Negation on BVU formulae is built respecting DeMorgan dualities, with $\bar{\circ} = \circ$, and $\bar{\bot} = 1$.

The units verify the equations $\circ \invamp \circ = 1$; $\circ \otimes \circ = \bot$ and $1 \triangleleft 1 = 1$; $\bot \triangleleft \bot = \bot$.

The inference rules for system BVU are given by the same rules as for system BV [21]. System BV corresponds to system BV with the three units identified, i.e. $1 = \circ = \bot$.

The set $\mathscr{F}_{bv}$ of subatomic formulae for the non-commutative logics BVU and BV is given by the set of constants $\mathscr{U} = \{\bot, 1, \circ\}$ and the set of relations $\mathscr{R} = \{\invamp, \triangleleft, \otimes\} \cup \mathscr{A}$ where $\mathscr{A}$ is a denumerable set of atoms, denoted by $a, b, \dots$ with $\mathscr{A} \cap \{\invamp, \triangleleft, \otimes\} = \varnothing$. Two examples of subatomic formulae for BV are

$$E \equiv (1\; a\; \bot) \triangleleft (\circ \otimes (\bot\; b\; \bot)) \quad \text{and} \quad F \equiv ((\circ \otimes 1)\; a\; 1) \invamp 1 \quad .$$

Just like for 'ordinary' formulae, we will define an equational theory and a negation map on the set of subatomic formulae. We will work in a classical setting, in the sense that we will consider an involutive negation that satisfies DeMorgan dualities. Furthermore, in order to keep track of the equational theory in the general results exposed in this thesis, we restrict the equalities that we allow.

Definition 2.5. We define *negation* as a pair of involutive maps $\bar{\cdot} : \mathscr{R} \mapsto \mathscr{R}$ and $\bar{\cdot} : \mathscr{U} \mapsto \mathscr{U}$. We define the *negation map on formulae* as the map inductively defined by setting $\overline{A\; \alpha\; B} := \overline{A}\; \overline{\alpha}\; \overline{B}$.

We define an equational theory $=$ on $\mathscr{F}$ as the minimal equivalence relation closed under negation (if $A = B$, then $\bar{A} = \bar{B}$) and under context (if $A = B$, then $K\{A\} = K\{B\}$ for any context $K\{\ \}$) defined by a set of axioms of the form:

(1) $\forall A, B, C \in \mathscr{F}.\ (A\; \alpha\; B)\; \alpha\; C = A\; \alpha\; (B\; \alpha\; C)$; *(Associativity of α)*

(2) $\forall A, B \in \mathscr{F}.\ A\; \alpha\; B = B\; \alpha\; A$; *(Commutativity of α)*

(3) $\forall A \in \mathscr{F}.\ A\; \alpha\; u_\alpha = A = u_\alpha\; \alpha\; A$ for a fixed $u_\alpha \in \mathscr{U}$; *(Unit of α)*

(4) $v\; \alpha\; w = u$ for fixed $v, w, u \in \mathscr{U}$; *(Constant assignment for α)*

(5) $u = v$ for fixed $u, v \in \mathscr{U}$. *(Constant identification)*

If there is an axiom of the form (1) for α, we say that α is *associative*. If there is an axiom of the form (2) for α, we say that α is *commutative*. If there is an axiom of the form (3) for α we say that α is *unitary*, and we call u_α the *unit of* α.

Remark 2.6. Since the equational theory is closed under negation, if α is unitary with unit u_α, then $\overline{\alpha}$ is unitary and its unit is $\overline{u_\alpha}$.

Example 2.7. For the set of subatomic formulae for classical logic $\mathscr{F}_{cl}$ defined in example 2.2, we define negation through:

$$\bar{\wedge} := \vee \ ; \qquad \bar{a} := a \text{ for all } a \in \mathscr{A} \ ; \qquad \bar{\mathsf{t}} := \mathsf{f} \ .$$

We define the equational theory $=$ on $\mathscr{F}_{cl}$ as the minimal equivalence relation closed under negation and under context defined by:

For all $A, B, C \in \mathscr{F}$:

$$
\begin{array}{ll}
(A \wedge B) \wedge C = A \wedge (B \wedge C) \ ; & (A \vee B) \vee C = A \wedge (B \vee C) \ ; \\
A \wedge B = B \wedge A \ ; & A \vee B = B \vee A \ ; \\
A \wedge \mathsf{t} = A \ ; & A \vee \mathsf{f} = A \ ; \\
\mathsf{f} \wedge \mathsf{f} = \mathsf{f} \ ; & \mathsf{t} \vee \mathsf{t} = \mathsf{t} \ ; \\
\forall a \in \mathscr{A}. \ \mathsf{f} \, a \, \mathsf{f} = \mathsf{f} \ ; & \forall a \in \mathscr{A}. \ \mathsf{t} \, a \, \mathsf{t} = \mathsf{t} \ .
\end{array}
$$

Example 2.8. For the set of subatomic formulae for linear logic $\mathscr{F}_{ll}$ defined in example 2.3, we define negation through:

$$\bar{\otimes} = \parr \ ; \qquad \bar{a} := a \text{ for all } a \in \mathscr{A} \ ; \qquad \bar{1} := \bot \ .$$

We define the equational theory $=$ on $\mathscr{F}_{ll}$ as the minimal equivalence relation closed under negation and under context defined by:

For all $A, B, C \in \mathscr{F}$:

$$
\begin{array}{ll}
(A \otimes B) \otimes C = A \otimes (B \otimes C) \ ; & (A \parr B) \parr C = A \parr (B \parr C) \ ; \\
A \otimes B = B \otimes A \ ; & A \parr B = B \parr A \ ; \\
A \otimes 1 = A \ ; & A \parr \bot = A \ ; \\
\forall a \in \mathscr{A}. \ \bot \, a \, \bot = \bot \ ; & \forall a \in \mathscr{A}. \ 1 \, a \, 1 = 1 \ .
\end{array}
$$

Example 2.9. For both BVU and BV we will define the same negation map. They will differ only on the equational theory, since all the units are identified in BV.

For the set of subatomic formulae for BVU and for BV $\mathscr{F}_{bv}$ defined in example 2.4, we define negation through:

$$\begin{aligned}
\bar{\otimes} &:= \parr \ ; \\
\bar{\lhd} &:= \lhd \ ; \\
\bar{a} &:= a \text{ for all } a \in \mathscr{A} \ ; \\
\bar{\circ} &:= \circ \ ; \\
\bar{\bot} &:= 1 \ .
\end{aligned}$$

For the logic BVU we define an equational theory $=$ on $\mathscr{F}_{bv}$ as the minimal equivalence relation closed under negation and under context defined by:

For all $A, B, C \in \mathscr{F}$:

$$\begin{array}{ll}
(A \otimes B) \otimes C = A \otimes (B \otimes C) \ ; & (A \parr B) \parr C = A \parr (B \parr C) \ ; \\
A \otimes B = B \otimes A \ ; & A \parr B = B \parr A \ ; \\
(A \lhd B) \lhd C = A \lhd (B \lhd C) \ ; & \\
A \otimes 1 = A \ ; & A \parr \bot = A \ ; \\
A \lhd \circ = A \ ; & \circ \lhd A = A \ ; \\
\circ \otimes \circ = \bot \ ; & \circ \parr \circ = 1 \ ; \\
\forall a \in \mathscr{A}.\ \bot\, a\, \bot = \bot \ ; & \forall a \in \mathscr{A}.\ 1\, a\, 1 = 1 \ ; \\
\bot \lhd \bot = \bot \ ; & 1 \lhd 1 = 1 \ .
\end{array}$$

The equational theory for the logic BV defined on the set of subatomic formulae $\mathscr{F}_{bv}$ is given by the previous equations, together with the added axioms:

$$1 = \circ \ ; \qquad\qquad \bot = \circ \ .$$

Given a propositional logic with certain relations and constants, its subatomic counterpart is therefore composed of an extended language of formulae, made up from the same relations but with the added possibility of having atoms in the scope of other atoms. To translate the subatomic formulae into the 'usual' formulae, we can define a simple interpretation map.

The intuitive idea behind the translation is to interpret a certain assignment of units inside an atom as a positive occurrence of the atom, and the dual assignment as a negative occurrence of the atom. For example, for classical logic we interpret $\mathsf{f}\, a\, \mathsf{t}$ as a positive occurrence of the atom a and $\mathsf{t}\, a\, \mathsf{f}$ as a negative one. In this way, the formula $A \equiv ((\mathsf{f}\, a\, \mathsf{t}) \vee \mathsf{t}) \wedge (\mathsf{t}\, b\, \mathsf{f})$ is interpreted as $A' \equiv (a \vee \mathsf{t}) \wedge \bar{b}$.

We can view the constants in the scope of an atom as a superposition of truth values. $\mathsf{f}\, a\, \mathsf{t}$ is the superposition of the two possible assignments for the atom a and $\mathsf{t}\, a\, \mathsf{f}$ the superposition of the two assignments for $\bar{a}$. We can project onto a specific assignment by choosing which 'side' we read: if we read the values on the left of the atom we assign f to a and t to $\bar{a}$ and if we read the ones on the right we assign t to a and f to $\bar{a}$.

In order to define an intepretation map following this idea, subatomic formulae must be built from the same relations as the 'original' formulae, with the addition of the atoms as connectives.

Definition 2.10. Let $\mathcal{G}$ be the set of formulae of a propositional logic L. We say that the set of subatomic formulae $\mathcal{F}$ is *natural* for L if there is a partition on the set of relations $\mathcal{R} = \mathcal{A} \cup \mathcal{R}'$ with $\mathcal{A} \cap \mathcal{R}' = \varnothing$,such that:

- there is an injective map from the constants of $\mathcal{G}$ to the constants in $\mathcal{U}$;
- there is a one to one correspondence between the relations in $\mathcal{G}$ and the relations in $\mathcal{R}'$;
- there is a one to one correspondence between the set of unordered pairs of dual atoms $\{a, \bar{a}\}$ in $\mathcal{G}$ and the set of relations $\mathcal{A}$.

We call the relations in $\mathcal{A}$ *atoms* as well. For each distinct pair of dual atoms we give a polarity assignment: we call one atom of the pair *positive*, and the other one *negative*. We will denote the atom of $\mathcal{A}$ corresponding to each pair with the same letter as the positive atom of the pair.

We will denote the constants of $\mathcal{U}$ and the relations in $\mathcal{R}'$ with the same symbols as their counterparts in $\mathcal{G}$.

Example 2.11. The sets of subatomic formulae defined in examples 2.7, 2.8 and 2.9 are natural for classical logic, multiplicative linear logic and BV respectively.

The notion of interpretation map is easily extended to all logics for which we define a subatomic logic in the natural way. This interpretation will allow us to go back and forth between subatomic systems and 'regular' propositional systems.

Definition 2.12. Let $\mathcal{G}$ be the set of formulae of a propositional logic L with negation denoted by $\bar{\cdot}$ and equational theory denoted by $=$. Let $\mathcal{F}$ be the set of subatomic formulae with constants $\mathcal{U}$ and relations $\mathcal{R}$ with negation denoted by $\bar{\cdot}$ and equational theory denoted by $=$. A surjective partial function $I : \mathcal{F} \to \mathcal{G}$ is called *interpretation map*. The domain of definition of I is the *set of interpretable formulae* and is denoted by $\mathcal{F}^i$. If $F \equiv I(A)$, we say that F is the *interpretation* of A, and that A is a *representation* of F.

We extend the notion of interpretability to contexts: we say that $S\{\ \}$ is interpretable if $S\{A\}$ is interpretable for every interpretable A.

If $\mathcal{F}$ is natural for L, we say that an interpretation $i : \mathcal{F}^i \to \mathcal{G}$ is *natural* when:

- $I(u) \equiv u$ for every constant u of $\mathcal{G}$;
- $\forall \alpha \in \mathcal{R}'$, if $A, B \in \mathcal{F}^i$ then $A\ \alpha\ B \in \mathcal{F}^i$ and $I(A\ \alpha\ B) \equiv I(A)\ \alpha\ I(B)$;
- For some distinguished constants $u_1, u_2 \in \mathcal{U}$, for all $a \in \mathcal{A}$, $I(u_1\ a\ u_2) \equiv a$ and $I(u_2\ a\ u_1) \equiv \bar{a}$.

We define the *natural representation* $R : \mathcal{G} \to \mathcal{F}$ *associated to* I for every formula $G \in \mathcal{G}$ inductively on the structure of G by:

- $R(u) \equiv u$ if u is a constant;
- $R(a) \equiv u_1 \, a \, u_2$ if a is a positive atom;
- $R(b) \equiv u_2 \, a \, u_1$ if $b \equiv \bar{a}$ is a negative atom;
- $R(A \, \alpha \, B) \equiv R(A) \, \alpha \, R(B)$ for every relation α of $\mathscr{G}$.

For every formula $A \in \mathscr{F}$, $I(R(A)) \equiv A$.

Example 2.13. A natural interpretation for the set of subatomic formulae for classical logic defined in example 2.2 is given by considering the assignments:

$$
\begin{array}{ll}
- \; I(\mathsf{t}) \equiv \mathsf{t} \; ; & - \; I(\mathsf{f}) \equiv \mathsf{f} \; ; \\
- \; \forall a \in \mathscr{A}. \; I(\mathsf{f} \, a \, \mathsf{f}) \equiv \mathsf{f} \; ; & - \; \forall a \in \mathscr{A}. \; I(\mathsf{t} \, a \, \mathsf{t}) \equiv \mathsf{t} \; ; \\
- \; \forall a \in \mathscr{A}. \; I(\mathsf{f} \, a \, \mathsf{t}) \equiv a \; ; & - \; \forall a \in \mathscr{A}. \; I(\mathsf{t} \, a \, \mathsf{f}) \equiv \bar{a} \; ; \\
- \; I(A \vee B) \equiv I(A) \vee I(B) \; ; & - \; I(A \wedge B) \equiv I(A) \wedge I(B) \; ;
\end{array}
$$

where $A, B \in \mathscr{F}^i$, and extending it in such a way that $A \, a \, B$ is interpretable iff $A = u, B = v$ with $u, v \in \{\mathsf{f}, \mathsf{t}\}$ and then $I(A \, a \, B) \equiv I(u \, a \, v)$.

For example, if $A \equiv (((\mathsf{f} \wedge \mathsf{t}) \, a \, \mathsf{t}) \vee \mathsf{t}) \wedge (\mathsf{t} \, b \, \mathsf{f})$, its interpretation is $I(A) = (a \vee \mathsf{t}) \wedge \bar{b}$.

Note that the set $\mathscr{F}^i$ of interpretable formulae is composed by all formulae equal to a formula where an atom does not occur in the scope of another atom. Every other formula is not interpretable, such as $B \equiv ((\mathsf{t} \, b \, \mathsf{f}) \wedge \mathsf{t}) \, a \, \mathsf{f}$.

Example 2.14. A natural interpretation for the set of subatomic formulae for multiplicative additive linear logic defined in example 2.3 is given by considering the assignments:

$$
\begin{array}{ll}
- \; I(1) \equiv 1 \; ; & - \; I(\bot) \equiv \bot \; ; \\
- \; \forall a \in \mathscr{A}. \; I(\bot \, a \, \bot) \equiv \bot \; ; & - \; \forall a \in \mathscr{A}. \; I(1 \, a \, 1) \equiv 1 \; ; \\
- \; \forall a \in \mathscr{A}. \; I(\bot \, a \, 1) \equiv a \; ; & - \; \forall a \in \mathscr{A}. \; I(1 \, a \, \bot) \equiv \bar{a} \; ; \\
- \; I(A \parr B) \equiv I(A) \parr I(B) \; ; & - \; I(A \otimes B) \equiv I(A) \otimes I(B) \; ;
\end{array}
$$

where $A, B \in \mathscr{F}^i$, and extending it in such a way that $A \, a \, B$ is interpretable iff $A = u, B = v$ with $u, v \in \{\bot, 1\}$ and then $I(A \, a \, B) \equiv I(u \, a \, v)$.

For example, for $C \equiv ((1 \parr \bot) \, a \, 1) \otimes \bot$, $I(C) = a \otimes \bot$.

The formulae that are not interpretable are not only those equal to a formula where an atom occurs in the scope of another atom, but also those where a formula made up of units not equal to 1 or $\bot$ occurs in the scope of an atom, such as $(1 \parr 1) \, a \, \bot$.

Example 2.15. A natural interpretation for the set of subatomic formulae $\mathscr{F}_{bv}$ into the set of formulae of BVU is given by considering the assignments:

- $I(\bot) \equiv \bot$;
- $I(1) \equiv 1$;
- $I(\circ) \equiv \circ$;
- $\forall a \in \mathscr{A}.\ I(\bot\ a\ \bot) \equiv \bot$;
- $\forall a \in \mathscr{A}.\ I(1\ a\ 1) \equiv 1$;
- $\forall a \in \mathscr{A}.\ I(\bot\ a\ 1) \equiv a$;
- $\forall a \in \mathscr{A}.\ I(1\ a\ \bot) \equiv \bar{a}$;
- $I(A \parr B) \equiv I(A) \parr I(B)$;
- $I(A \otimes B) \equiv I(A) \otimes I(B)$;
- $I(A \triangleleft B) \equiv I(A) \triangleleft I(B)$;

where $A, B \in \mathscr{F}^i$, and extending it in such a way that $A\ a\ B$ is interpretable iff $A = u, B = v$ with $u, v \in \{\bot, 1\}$ and then $I(A\ a\ B) \equiv I(u\ a\ v)$.

The formulae that are not interpretable are not only those equal to a formula where an atom occurs in the scope of another atom, but also those where a formula made-up of units not equal to $\bot$ or 1 occurs in the scope of an atom, such as $(1 \parr 1)\ a \circ$.

This interpretation is also natural as an interpretation into the set of formulae of BV. Note that even though $\bot\ a\ 1 = \circ\ a\ 1$ in BV, the former is interpretable, while the latter is not. Interpretability is not necessarily preserved by equality.

2.2 Subatomic proof systems

The useful properties of subatomic formulae become apparent when we extend the principle to atomic inference rules. Let us consider, for example, the usual contraction rule for an atom in classical logic given by

$$\frac{a \vee a}{a} \quad .$$

We could obtain this rule subatomically through the interpretation map defined in example 2.13 as follows:

$$\frac{(\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{f}\ a\ \mathsf{t})}{(\mathsf{f} \vee \mathsf{f})\ a\ (\mathsf{t} \vee \mathsf{t})} \overset{I}{\mapsto} \frac{a \vee a}{a} \quad \text{and} \quad \frac{(\mathsf{t}\ a\ \mathsf{f}) \vee (\mathsf{t}\ a\ \mathsf{f})}{(\mathsf{t} \vee \mathsf{t})\ a\ (\mathsf{f} \vee \mathsf{f})} \overset{I}{\mapsto} \frac{\bar{a} \vee \bar{a}}{\bar{a}} \quad .$$

These rules are therefore generated by the linear scheme

$$\frac{(A\ a\ B) \vee (C\ a\ D)}{(A \vee C)\ a\ (B \vee D)} \quad \text{, where } A, B, C, D \text{ are formulae.}$$

Strikingly, the non-linearity of the contraction rule has been pushed from the atoms to the units.

Similarly, we can consider the atomic identity rule

$$\frac{\mathsf{t}}{a \vee \bar{a}} \quad .$$

It can be obtained subatomically as follows:

$$\frac{(\mathsf{f} \vee \mathsf{t})\; a\; (\mathsf{t} \vee \mathsf{f})}{(\mathsf{f}\; a\; \mathsf{t}) \vee (\mathsf{t}\; a\; \mathsf{f})} \overset{I}{\mapsto} \frac{\mathsf{t}}{a \vee \bar{a}} \quad .$$

Similarly to the contraction rule, it is generated by the linear scheme

$$\frac{(A \vee B)\; a\; (C \vee D)}{(A\; a\; C) \vee (B\; a\; D)} \quad \text{, where } A, B, C, D \text{ are formulae.}$$

It is quite plain to see that both the subatomic contraction rule and the subatomic introduction rule have the same shape. This surprising regularity is made very useful in combination with the observation that in fact the linear rule scheme

$$\frac{(A\; \alpha\; B)\; \nu\; (C\; \beta\; D)}{(A\; \nu\; C)\; \alpha\; (B\; \gamma\; D)} \quad ,$$

where $\alpha, \nu, \beta, \gamma$ are relations, and A, B, C, D are formulae is typical of logical rules in deep inference. We refer to it as a *medial shape*. For example, consider system SKS for classical logic:

$ai{\downarrow}\,\dfrac{\mathsf{t}}{a \vee \bar{a}}$	$ai{\uparrow}\,\dfrac{a \wedge \bar{a}}{\mathsf{f}}$
$s\,\dfrac{(A \vee B) \wedge C}{(A \wedge C) \vee B}$	$m\,\dfrac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$
$ac{\downarrow}\,\dfrac{a \vee a}{a}$	$ac{\uparrow}\,\dfrac{a}{a \wedge a}$
$aw{\downarrow}\,\dfrac{\mathsf{f}}{a}$	$aw{\uparrow}\,\dfrac{a}{\mathsf{t}}$

System SKS

We can see that the rule m follows this scheme as well, and we can derive the rule s from the rule

$$\wedge{\downarrow}\,\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \quad ,$$

which follows this scheme. We have therefore uncovered an underlying sturucture behind the shape of inference rules, that we will exploit to obtain a general characterisation of rules.

To make use of the general characterisation, we will impose some restrictions on $\alpha, \nu, \beta, \gamma$. These conditions strike a balance between being general enough to encompass a wide variety of logics and being explicit enough to enable us to generalise procedure such as cut-elimination and decomposition. They are the result of a trial-and-error phase comprised of the comparison of different proof systems together with the study of the properties necessary for cut-elimination and decomposition results.

The restrictions on the relations of the rule scheme stem from the observation that certain dualities between the relations are maintained in every rule. For example, we can write the rule $\wedge\downarrow$ as

$$\wedge\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \mathbin{\bar\wedge} D)}$$

and the subatomic identity rule as

$$\frac{(A \vee B)\, a\, (C \vee D)}{(A\, a\, C) \vee (B\, \bar{a}\, D)} \quad .$$

We will generalise this observation, considering rules with a medial shape and certain dualities between the connectives involved and show that this shape is enough to represent a wide variety of logics. With the subatomic methodology, we are therefore able to represent proof systems in such a way that every rule has the same shape. This full regularity gives us a newly gained ability to characterise proof systems that enjoy properties such as decomposition and cut-elimination.

To characterise the dualities present in the inference rules, we introduce a notion of polarity in the pairs of dual relations. This notion of polarity can be reminiscent of the polarities assigned to connectives in linear logic [13], but the idea behind it is rather to assign which of the relations in the pair is 'stronger' than the other. Intuitively, it loosely corresponds to assigning which relation of the pair will imply the other. For example, in classical logic $A \wedge B$ implies $A \vee B$, and thus we will assign $\wedge$ to be *strong* and $\vee$ to be *weak*.

Definition 2.16. For each pair of relations $\{\alpha, \overline{\alpha}\}$, we give a polarity assignment: we call one relation of the pair *strong* and the other one *weak*.

If α is strong and $\overline{\alpha}$ is weak, we will write $\alpha^M{=}\overline{\alpha}^M{=}\alpha$ and $\alpha^m{=}\overline{\alpha}^m{=}\overline{\alpha}$. Self-dual relations are both strong and weak.

Definition 2.17. A *subatomic proof system* SA with set of formulae $\mathscr{F}$ is

- a collection of inference rules of the shape $\dfrac{(A\,\beta\,B) \;\; \alpha \;\; (C\,\beta\,\;D)}{(A\,\alpha\,C) \;\; \beta \;\; (B\,\alpha^m\,D)}$, $\alpha, \beta {\in}\, \mathscr{R}$, called *down-rules*,
- a collection of inference rules of the shape $\dfrac{(A\,\beta\,B) \;\; \alpha \;\; (C\,\beta^M\,D)}{(A\,\alpha\,C) \;\; \beta \;\; (B\,\alpha\;\;D)}$, $\alpha, \beta {\in}\, \mathscr{R}$, called *up-rules*,

- a collection of rules $= \frac{A}{B}$ and $= \frac{\overline{A}}{\overline{B}}$, for every axiom $A = B$ of the equational theory $=$ on $\mathscr{F}$, called *equality rules.*

Note that the non-invertible rules are linear: surprisingly, non-linearity can be pushed from the atoms to the units.

Remark 2.18. Since we will not always work modulo equality, we define the equality rules to be inference steps just like the inference rules, rather than focusing on equality as equations between formulae. Two formulae A and B will be equal if and only if there is a derivation from A to B composed only of equality rules.

We could have just as well defined equality between formulae directly in this way, but chose to define it initially as an equivalence relation for the sake of clearer exposition when defining the interpretation map.

The rules $= \frac{A}{B}$ are invertible and correspond to equivalence by mutual implication. Every non-invertible rule with logical significance is therefore an instance of the general rule scheme with medial shape.

Remark 2.19. We will often use the notation

$$\frac{(A\ \beta\ B)\ \alpha^{M}\ (C\ \beta\ D)}{(A\ \alpha\ B)\ \beta\ (C\ \overline{\alpha}\ D)}$$

for down-rules with a strong relation in the premiss where β is commutative.

Example 2.20. We consider $\wedge$ as strong and $\vee$ as weak in classical logic. The subatomic proof system SAKS is given by the inference rules in Figure 2-1, together with the equality rules given by $= \frac{A}{B}$ for every A, B on opposite sides of the equality axioms provided in example 2.7.

Rules labeled with a $\downarrow$ are down-rules, and rules labeled by a $\uparrow$ are up-rules. The medial rule labeled by m is self-dual, and is both a down-rule and an up-rule.

Example 2.21. We consider $\otimes$ as strong and $\parr$ as weak in multiplicative linear logic. The subatomic proof system SAMLLS is given by the inference rules in Figure 2-3 together with the equality rules given by $= \frac{A}{B}$ for every A, B on opposite sides of the equality axioms provided in example 2.8.

Example 2.22. We consider $\otimes$ as strong and $\parr$ as weak in BVU and BV. The subatomic proof system SABVU is given by the inference rules in Figure 2-5 together with the equality rules given by $= \frac{A}{B}$ for every A, B on opposite sides of the equality axioms for BVU provided in example 2.9.

$$a\downarrow \frac{(A \vee B)\; a\; (C \vee D)}{(A\; a\; C) \vee (B\; a\; D)} \qquad a\uparrow \frac{(A\; a\; B) \wedge (C\; a\; D)}{(A \wedge C)\; a\; (B \wedge D)}$$

$$\wedge\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \vee\uparrow \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$ac \frac{(A\; a\; B) \vee (C\; a\; D)}{(A \vee C)\; a\; (B \vee D)} \qquad a\bar{c} \frac{(A \wedge B)\; a\; (C \wedge D)}{(A\; a\; C) \wedge (B\; a\; D)}$$

Figure 2-1: SAKS

$$i\downarrow \frac{\mathsf{t}}{a \vee \bar{a}} \qquad i\uparrow \frac{a \wedge \bar{a}}{\mathsf{f}}$$

$$s \frac{(A \vee B) \wedge C}{(A \wedge C) \vee B} \qquad m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$c\downarrow \frac{a \vee a}{a} \qquad c\uparrow \frac{a}{a \wedge a}$$

$$aw\downarrow \frac{\mathsf{f}}{a} \qquad aw\uparrow \frac{a}{\mathsf{t}}$$

Figure 2-2: SKS *[4]*

$$a\downarrow \frac{(A \parr B)\; a\; (C \parr D)}{(A\; a\; C) \parr (B\; a\; D)} \qquad a\uparrow \frac{(A\; a\; B) \otimes (C\; a\; D)}{(A \otimes C)\; a\; (B \otimes D)}$$

$$\otimes\downarrow \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\uparrow \frac{(A \parr B) \otimes (C \otimes D)}{(A \otimes C) \parr (B \otimes D)}$$

Figure 2-3: SAMLLS

$$ai\downarrow \frac{1}{a \parr \bar{a}} \qquad ai\uparrow \frac{a \otimes \bar{a}}{\bot}$$

$$s \frac{(A \parr B) \otimes C}{(A \otimes C) \parr B}$$

Figure 2-4: SMLLS *[36]*

$$
a{\downarrow}\,\frac{(A \parr B)\ a\ (C \parr D)}{(A\ a\ C) \parr (B\ a\ D)} \qquad a{\uparrow}\,\frac{(A\ a\ B) \otimes (C\ a\ D)}{(A \otimes C)\ a\ (B \otimes D)}
$$

$$
\otimes{\downarrow}\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr{\uparrow}\,\frac{(A \parr B) \otimes (C \otimes D)}{(A \otimes C) \parr (B \otimes D)}
$$

$$
\lhd{\downarrow}\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} \qquad \lhd{\uparrow}\,\frac{(A \lhd B) \otimes (C \lhd D)}{(A \otimes C) \lhd (B \otimes D)}
$$

Figure 2-5: SABV

$$
ai{\downarrow}\,\frac{\circ}{a \parr \bar{a}} \qquad a{\uparrow}\,\frac{a \otimes \bar{a}}{\circ}
$$

$$
s\,\frac{(A \parr B) \otimes C)}{(A \otimes C) \parr B}
$$

$$
q{\downarrow}\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} \qquad q{\uparrow}\,\frac{(A \lhd B) \otimes (C \lhd D)}{(A \otimes C) \lhd (B \otimes D)}
$$

Figure 2-6: SBV *[21]*

Likewise, the subatomic proof system SABV is given by the same inference rules and equality rules, together with the equality rules given by $=\frac{\bot}{\circ}$, $=\frac{1}{\circ}$ and their converse.

Remark 2.23. An interesting future line of work is to characterise sound rules based on a partial order on relations. Some preliminary research in this direction has yielded very encouraging results. We assign a partial order based on implication to the relations of classical logic: $\vee < a < \wedge$.

Then, all down-rules in systems SAKS obey the scheme $\frac{(A\ \beta\ B)\ \ \alpha\ \ (C\ \beta\ \ D)}{(A\ \alpha\ C)\ \ \beta\ \ (B\ \alpha^m\ D)}$, $\bar{\beta} \geq \alpha$.

Dually, all up-rules obey the scheme $\frac{(A\ \beta\ B)\ \ \ \alpha\ \ (C\ \beta^M\ D)}{(A\ \alpha\ C)\ \ \beta\ \ (B\ \alpha\ \ \ D)}$, $\bar{\alpha} \geq \beta$.

Furthermore, every rule following this scheme is sound in classical logic.

We can similarly assign partial orders to the relations of multiplicative additive linear logic and BV ($\parr < \oplus < a < \& < \otimes$ and $\parr < \lhd, a < \otimes$). Then, the rules of systems SAMALLS (Figure 4-3) and SBV verify this scheme as well.

To reduce rules to their subatomic form, we will work in the setting of deep inference [18], where proofs can be composed with the same connectives as formulae. The deep inference methodology has been exploited in many ways, such as shortening analytic proofs by exponential factors with respect to Gentzen proofs [6, 10], modeling process algebras [5, 31, 33, 34] or typing optimised versions of the λ-calculus that provide a novel treatment of sharing and duplication [30]. The particular property that most interests us is that rules can be applied at any depth inside a formula and as a result every contraction and cut instances can be locally transformed into their atomic variants by a local procedure of polynomial-size complexity [4]. Therefore they can be transformed into their subatomic variants straightforwardly.

We will present proofs in the open deduction formalism [25], which is a logic-independent formalism, allowing us to reach the desired level of generality.

Definition 2.24. Given a subatomic systems SA and formulae A and B, a *derivation* ϕ in SA from *premiss* A to *conclusion* B denoted by $\boxed{\begin{array}{c} A \\ \phi \Vert \mathsf{SA} \\ B \end{array}}$ is defined to be:

- a formula $\phi \equiv A \equiv B$;
- a *composition by inference*

$$\phi \equiv \rho \frac{\boxed{\begin{array}{c} A \\ \phi_1 \Vert \mathsf{SA} \\ A' \end{array}}}{\boxed{\begin{array}{c} B' \\ \phi_2 \Vert \mathsf{SA} \\ B \end{array}}}$$

  where ρ is an instance of an inference rule in SA and ϕ_1 and ϕ_2 are derivations in SA;
- a *composition by relations*

$$\phi \equiv \boxed{\begin{array}{c} A_1 \\ \phi_1 \Vert \mathsf{SA} \\ B_1 \end{array}} \; \alpha \; \boxed{\begin{array}{c} A_2 \\ \phi_2 \Vert \mathsf{SA} \\ B_2 \end{array}}$$

  where $\alpha \in \mathscr{R}$, $A \equiv A_1 \; \alpha \; A_2$, $B \equiv B_1 \; \alpha \; B_2$, ϕ_1 and ϕ_2 are derivations in SA.

We denote by

$$\boxed{\begin{array}{c} A \\ \phi \Vert \{\rho_1,\ldots,\rho_n\} \\ B \end{array}}$$

a derivation where only the rules $\rho_1, \ldots, \rho_n$ appear.

Sometimes we omit the name of a derivation or the name of the proof system if there is no ambiguity.

To improve readability sometimes we remove the boxes around derivations.

Notation 2.25. We consider the two derivations

$$\rho_2 \frac{\left(\begin{array}{c} A_1 \\ \phi_1 \, \| \, \mathsf{SA} \\ \rho_1 \dfrac{A_2}{B_1} \\ \phi_2 \, \| \, \mathsf{SA} \\ B_2 \end{array}\right)}{\begin{array}{c} C_1 \\ \phi_3 \, \| \, \mathsf{SA} \\ C_2 \end{array}} \qquad \text{and} \qquad \rho_1 \frac{\begin{array}{c} A_1 \\ \phi_1 \, \| \, \mathsf{SA} \\ A_2 \end{array}}{\left(\begin{array}{c} B_1 \\ \phi_2 \, \| \, \mathsf{SA} \\ \rho_2 \dfrac{B_2}{C_1} \\ \phi_3 \, \| \, \mathsf{SA} \\ C_2 \end{array}\right)}$$

to be equal and we denote them both by

$$\begin{array}{c} A_1 \\ \phi_1 \, \| \, \mathsf{SA} \\ \rho_1 \dfrac{A_2}{B_1} \\ \phi_2 \, \| \, \mathsf{SA} \\ \rho_2 \dfrac{B_2}{C_1} \\ \phi_2 \, \| \, \mathsf{SA} \\ C_2 \end{array} .$$

Example 2.26. The following is a SAKS derivation with premiss $(\mathsf{f} \vee \mathsf{t}) a (\mathsf{t} \vee \mathsf{f}) \wedge ((\mathsf{f} b \mathsf{t}) \vee \mathsf{t}) \wedge \mathsf{t}$ and conclusion $((\mathsf{f}\ a\ \mathsf{t}) \wedge (\mathsf{f}\ b\ \mathsf{t})) \vee ((\mathsf{t}\ a\ \mathsf{f}) \vee \mathsf{t}) \wedge \mathsf{t}$:

$$\boxed{s \frac{\boxed{ai{\downarrow} \dfrac{(\mathsf{f} \vee \mathsf{t})\ a\ (\mathsf{t} \vee \mathsf{f})}{(\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{t}\ a\ \mathsf{f})}} \wedge ((\mathsf{f}\ b\ \mathsf{t}) \vee \mathsf{t})}{((\mathsf{f}\ a\ \mathsf{t}) \wedge (\mathsf{f}\ b\ \mathsf{t})) \vee ((\mathsf{t}\ a\ \mathsf{f}) \vee \mathsf{t})}} \wedge \mathsf{t}$$

Definition 2.27. Let $\begin{array}{c} A \\ \phi \, \| \, \mathsf{SA} \\ B \end{array}$ and $\begin{array}{c} B \\ \psi \, \| \, \mathsf{SA} \\ C \end{array}$ be two derivations. We define their *composition* $\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}$ as the derivation constructed as follows:

- if ϕ is a formula then $\begin{array}{c} \phi \\ \vdots \\ \psi \end{array} \equiv \psi$; likewise if ψ is a formula then $\begin{array}{c} \phi \\ \vdots \\ \psi \end{array} \equiv \phi$;

- if $\phi \equiv \dfrac{\phi_1}{\phi_2}$ then $\begin{array}{c} \phi \\ \vdots \\ \psi \end{array} \equiv \dfrac{\phi_1}{\left(\begin{array}{c} \phi_2 \\ \vdots \\ \psi \end{array}\right)}$; likewise if $\psi \equiv \dfrac{\psi_1}{\psi_2}$ then $\begin{array}{c} \phi \\ \vdots \\ \psi \end{array} \equiv \dfrac{\left(\begin{array}{c} \phi \\ \vdots \\ \psi_1 \end{array}\right)}{\psi_2}$;

- if $\phi \equiv \phi_1 \; \alpha \; \phi_2$ and $\psi \equiv \psi_1 \; \alpha \; \psi_2$ then $\begin{array}{c}\phi \\ \vdots \\ \psi\end{array} \equiv \begin{array}{c}\phi_1 \\ \vdots \\ \psi_1\end{array} \; \alpha \; \begin{array}{c}\phi_2 \\ \vdots \\ \psi_2\end{array}$.

Definition 2.28. Let $\begin{array}{c}A \\ \phi \Vert \mathsf{SA} \\ B\end{array}$ be a derivation, and $K\{\ \}$ a context. We define the derivation $K\{\phi\}$ from $K\{A\}$ to $K\{B\}$ as the derivation obtained by inserting ϕ in the place of the hole in $K\{\ \}$.

Example 2.29. If $\phi = ai{\downarrow}\dfrac{(\mathsf{f} \vee \mathsf{t}) \; a \; (\mathsf{t} \vee \mathsf{f})}{(\mathsf{f} \; a \; \mathsf{t}) \vee (\mathsf{t} \; a \; \mathsf{f})}$ and $K\{\ \} = (\mathsf{t} \wedge \{\ \}) \vee (\mathsf{f} \wedge \mathsf{f})$, then

$$K\{\phi\} = \left(\mathsf{t} \wedge ai{\downarrow}\dfrac{(\mathsf{f} \vee \mathsf{t}) \; a \; (\mathsf{t} \vee \mathsf{f})}{(\mathsf{f} \; a \; \mathsf{t}) \vee (\mathsf{t} \; a \; \mathsf{f})}\right) \vee (\mathsf{f} \wedge \mathsf{f}) \quad .$$

Sometimes we will work by induction on the number of rules on a derivation. For that, it is useful to impose an order on the rules to have a notion of which one is the 'last' rule of the derivation. We impose this order by sequentialising the derivation.

Definition 2.30. Let $\begin{array}{c}A \\ \phi \Vert \\ B\end{array}$ be a derivation. We define the sequential form of ϕ as follows by structural induction on ϕ:

- if $\phi \equiv A$ is a formula, then its sequential form is given by A ;

- if $\phi \equiv \rho \dfrac{\boxed{\begin{array}{c}A \\ \phi_1 \Vert \\ A'\end{array}}}{\boxed{\begin{array}{c}B' \\ \phi_2 \Vert \\ B\end{array}}}$, then we consider ϕ_1 and ϕ_2 in sequential form:

$$\phi_1 = \begin{array}{c} \rho_1 \dfrac{A}{A_2} \\ \vdots \\ \rho_n \dfrac{A_n}{A'} \end{array} \qquad \text{and} \qquad \phi_2 = \begin{array}{c} \rho_{n+1} \dfrac{B'}{B_2} \\ \vdots \\ \rho_m \dfrac{B_m}{B} \end{array}$$

and the sequential form of ϕ is given by

$$\phi = \begin{array}{r@{}c} & A \\ \rho_1 & \overline{\overline{}} \\ & \vdots \\ \rho_n & \overline{\overline{}} \\ & A' \\ \rho & \overline{\overline{}} \\ & B' \\ \rho_{n+1} & \overline{\overline{}} \\ & \vdots \\ \rho_{n+m} & \overline{\overline{}} \\ & B \end{array} .$$

- if $\phi \equiv \boxed{\begin{array}{c} A_1 \\ \phi_1 \Vert \\ B_1 \end{array}} \; \alpha \; \boxed{\begin{array}{c} A_2 \\ \phi_2 \Vert \\ B_2 \end{array}}$, then we sequentialise ϕ_1 and ϕ_2 to obtain

$$\phi_1 = \begin{array}{c} \rho_1 \dfrac{A_1}{C_2} \\ \vdots \\ \rho_n \dfrac{C_n}{B_1} \end{array} \qquad \text{and} \qquad \phi_2 = \begin{array}{c} \rho_{n+1} \dfrac{A_2}{D_2} \\ \vdots \\ \rho_{n+m} \dfrac{D_m}{B_2} \end{array}$$

and the sequential form of ϕ is given by

$$\phi = \begin{array}{c} \rho_1 \dfrac{A_1 \; \alpha \; A_2}{C_2 \; \alpha \; A_2} \\ \vdots \\ \rho_n \dfrac{C_n \; \alpha \; A_2}{B_1 \; \alpha \; A_2} \\ \rho_{n+1} \overline{\overline{B_1 \; \alpha \; D_2}} \\ \vdots \\ \rho_{n+m} \dfrac{B_1 \; \alpha \; D_m}{B_1 \; \alpha \; B_2} \end{array} .$$

To simplify readability, when there is no ambiguity we will represent the sequential form through single lines $\dfrac{A}{B}$ instead of double lines $\begin{array}{c} A \\ \overline{\overline{}} \\ B \end{array}$.

The sequential form is not a normal form: we can choose how to sequentialise a composition by relation, by starting from either side of the relation. However we make this choice, the number of rules in the sequential form of the derivation stays nonetheless equal to the number of inference rules in its open deduction form.

Example 2.31. The sequential form of the derivation ϕ of example 2.26 is:

$$\phi = \begin{array}{c} ai\downarrow \dfrac{(((\mathsf{f} \vee \mathsf{t})\, a\, (\mathsf{t} \vee \mathsf{f})) \wedge ((\mathsf{f}\, b\, \mathsf{t}) \vee \mathsf{t})) \wedge \mathsf{t}}{s\,\dfrac{(((\mathsf{f}\, a\, \mathsf{t}) \vee (\mathsf{t}\, a\, \mathsf{f})) \wedge ((\mathsf{f}\, b\, \mathsf{t}) \vee \mathsf{t})) \wedge \mathsf{t}}{(((\mathsf{f}\, a\, \mathsf{t}) \wedge (\mathsf{f}\, b\, \mathsf{t})) \vee ((\mathsf{t}\, a\, \mathsf{f}) \vee \mathsf{t})) \wedge \mathsf{t}}} \end{array} .$$

For some results, such as the splitting theorem in Section 3 it is useful to consider proofs modulo certain equalities. To simplify the presentation and the case analysis, we define the Calculus of Structures presentation. This presentation provides us with a natural way of extending an equivalence relation between formulae to an equivalence relation between derivations.

Definition 2.32. Let $\sim$ be an equivalence relation on $\mathscr{F}$ obtained from a subset of the axioms that define $=$ as per Definition 2.5.

If $C \sim C'$, there is a derivation $\begin{array}{c} C \\ \zeta \,\| \\ C' \end{array}$ where ζ is composed only of equality rules corresponding to the axioms of $\sim$. We will denote such derivations by $\sim \dfrac{C}{C'}$.

A derivation in sequential form

$$\phi = \begin{array}{c} \sim \dfrac{A_0}{\rho_1 \dfrac{A_1}{\sim \dfrac{A_2}{A_3}}} \\ \vdots \\ \sim \dfrac{A_n}{\rho_m \dfrac{A_{n+1}}{\sim \dfrac{A_{n+2}}{A_{n+3}}}} \end{array}$$

has *Calculus of Structures (CoS) notation* for $\sim$ given by

$$\phi = \begin{array}{c} \rho_1 \dfrac{A_0}{A_3} \\ \vdots \\ \rho_m \dfrac{A_{n+1}}{A_{n+3}} \end{array} .$$

We define the equivalence relation $\sim$ on derivations as $\phi_1 \sim \phi_2$ if

$$\phi_1 = \begin{array}{c} \rho_1 \dfrac{A_0}{A_1} \\ \vdots \\ \rho_{n+1} \dfrac{A_n}{A_{n+1}} \end{array} \quad \text{and} \quad \phi_2 = \begin{array}{c} \rho_1 \dfrac{A'_0}{A'_1} \\ \vdots \\ \rho_{n+1} \dfrac{A'_n}{A'_{n+1}} \end{array}$$

in CoS notation for $\sim$, with $A_i \sim A'_i$ for every $0 \leq i \leq n+1$.

Example 2.33. If $\sim$ is the equivalence relation on the set of formulae $\mathscr{F}_{cl}$ for classical logic defined by the axiom $A \wedge \mathsf{t} = A$, then

$$\begin{array}{c} ai{\downarrow} \dfrac{(((\mathsf{f} \vee \mathsf{t})\ a\ (\mathsf{t} \vee \mathsf{f})) \wedge ((\mathsf{f}\ b\ \mathsf{t}) \vee \mathsf{t})) \wedge \mathsf{t}}{s \dfrac{(((\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{t}\ a\ \mathsf{f})) \wedge ((\mathsf{f}\ b\ \mathsf{t}) \vee \mathsf{t})) \wedge \mathsf{t}}{(((\mathsf{f}\ a\ \mathsf{t}) \wedge (\mathsf{f}\ b\ \mathsf{t})) \vee ((\mathsf{t}\ a\ \mathsf{f}) \vee \mathsf{t})) \wedge \mathsf{t}}} \end{array} \sim \begin{array}{c} ai{\downarrow} \dfrac{((\mathsf{f} \vee \mathsf{t})\ a\ (\mathsf{t} \vee \mathsf{f})) \wedge ((\mathsf{f}\ b\ \mathsf{t}) \vee \mathsf{t})}{s \dfrac{((\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{t}\ a\ \mathsf{f})) \wedge ((\mathsf{f}\ b\ \mathsf{t}) \vee \mathsf{t})}{((\mathsf{f}\ a\ \mathsf{t}) \wedge (\mathsf{f}\ b\ \mathsf{t})) \vee ((\mathsf{t}\ a\ \mathsf{f}) \vee \mathsf{t})}} \end{array} .$$

2.3 Proofs

To study proof theory through subatomic proof systems, we need to have a notion of proofs equivalent to that of the 'regular' theory. For that, we will establish a notion of correspondence between subatomic systems and deep inference systems. In a *correct* proof system every 'ordinary' proof will have a corresponding subatomic proof, and every subatomic proof where every step has an interpretation will correspond to an 'ordinary' proof.

Definition 2.34. Let $1 \in \mathscr{U}$ be a distinguished constant. A *proof* of A is a derivation ϕ whose premiss is 1. We denote proofs by $\begin{array}{c} \phi \Vert \\ A \end{array}$.

For reasons of convention, the distinguished unit for each proof system might be denoted with a different symbol, as is the case for classical logic.

Example 2.35. A proof in SAKS is a derivation with premiss t.

Example 2.36. A proof in SAMLLS is a derivation with premiss $\mathsf{1}$.

Example 2.37. A proof in SABV is a derivation with premiss $\mathsf{1}$.

Definition 2.38. Given an interpretation map I for SA, a derivation is *interpretable* if every formula appearing in its sequential form is interpretable.

Definition 2.39. Let SA be a subatomic system with a natural interpretation I into the set $\mathscr{G}$ of formulae of a complete proof system S for a propositional logic L, with associated representation map R.

We say that SA is *correct* for S when:

- for every interpretable SA derivation ψ with premiss P and conclusion C, there is a derivation ψ' in S with premiss $I(P)$ and conclusion $I(C)$; and
- for every derivation ϕ in S with premiss P' and conclusion C', there is an interpretable derivation ϕ' in SA with premiss $R(P')$ and conclusion $R(C')$.

Lemma 2.40. *Let* SA *be a subatomic system with a natural interpretation I into the set $\mathscr{G}$ of formulae of a complete proof system* S *for a propositional logic L, with associated representation map R.*

SA *is correct for S if, and only if:*

- *for every interpretable instance of an inference rule of* SA

$$\rho \frac{A}{B} \quad ,$$

there is a derivation

$$\begin{array}{c} I(A) \\ \| \mathsf{S} \\ I(B) \end{array} \quad ;$$

- *for every interpretable instance of derivations of the form*

$$\rho \frac{A}{B} \, a \, C \qquad \textit{and} \qquad D \, a \, \rho \frac{A}{B}$$

with $a \in \mathscr{A}$ and ρ an inference rule of SA*, there are derivations*

$$\begin{array}{c} I(A \, a \, C) \\ \| \mathsf{S} \\ I(B \, a \, C) \end{array} \qquad \textit{and} \qquad \begin{array}{c} I(D \, a \, A) \\ \| \mathsf{S} \\ I(D \, a \, B) \end{array} \qquad ; \textit{ and}$$

- *for every inference rule*

$$r \frac{A}{B}$$

of S*, there is an interpretable derivation*

$$\begin{array}{c} R(A) \\ \| \mathsf{SA} \\ R(B) \end{array} \quad .$$

Proof. It is clear from how derivations are built and from the fact that $I(A \; \alpha \; B) = I(A) \; \alpha \; I(B)$ for $\alpha \in \mathscr{R}'$ and that $R(A \; \alpha \; B) = R(A) \; \alpha \; R(B)$ for $\alpha \in \mathscr{R}'$.

□

Example 2.41. System SAKS of Figure 2-1 is correct for system SKS of Figure 2-2.

Every interpretable assignment of units in the inference rules has a corresponding derivation in SKS. For example, for rule $a\!\downarrow$ we have the following interpretable assignments:

$$
\begin{array}{ll}
{\scriptstyle a\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{t})\,a\,(\mathsf{t}\vee\mathsf{t})}{(\mathsf{t}\,a\,\mathsf{t})\vee(\mathsf{t}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{t}\vee\mathsf{t}} &
{\scriptstyle a\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{f})\,a\,(\mathsf{f}\vee\mathsf{f})}{(\mathsf{f}\,a\,\mathsf{f})\vee(\mathsf{f}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\mathsf{f}}{\mathsf{f}\vee\mathsf{f}} \\
{\scriptstyle a\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{t})\,a\,(\mathsf{t}\vee\mathsf{f})}{(\mathsf{f}\,a\,\mathsf{t})\vee(\mathsf{t}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{a\vee\bar{a}} &
{\scriptstyle a\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{f})\,a\,(\mathsf{f}\vee\mathsf{t})}{(\mathsf{t}\,a\,\mathsf{f})\vee(\mathsf{f}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\bar{a}\vee a} \\
{\scriptstyle a\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{t})\,a\,(\mathsf{f}\vee\mathsf{t})}{(\mathsf{f}\,a\,\mathsf{f})\vee(\mathsf{t}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{f}\vee\mathsf{t}} &
{\scriptstyle a\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{f})\,a\,(\mathsf{t}\vee\mathsf{f})}{(\mathsf{t}\,a\,\mathsf{t})\vee(\mathsf{f}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{t}\vee\mathsf{f}} \\
{\scriptstyle a\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{f})\,a\,(\mathsf{t}\vee\mathsf{t})}{(\mathsf{f}\,a\,\mathsf{t})\vee(\mathsf{f}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{a}{a\vee a} &
{\scriptstyle a\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{t})\,a\,(\mathsf{f}\vee\mathsf{f})}{(\mathsf{t}\,a\,\mathsf{f})\vee(\mathsf{t}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\bar{a}}{\bar{a}\vee\bar{a}} \\
{\scriptstyle a\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{t})\,a\,(\mathsf{t}\vee\mathsf{t})}{(\mathsf{f}\,a\,\mathsf{t})\vee(\mathsf{t}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{a\vee\mathsf{t}} &
{\scriptstyle a\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{t})\,a\,(\mathsf{f}\vee\mathsf{t})}{(\mathsf{t}\,a\,\mathsf{f})\vee(\mathsf{t}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\bar{a}\vee\mathsf{t}} \\
{\scriptstyle a\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{f})\,a\,(\mathsf{t}\vee\mathsf{t})}{(\mathsf{t}\,a\,\mathsf{t})\vee(\mathsf{f}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{t}\vee a} &
{\scriptstyle a\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{t})\,a\,(\mathsf{t}\vee\mathsf{f})}{(\mathsf{t}\,a\,\mathsf{t})\vee(\mathsf{t}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{t}\vee\bar{a}} \\
{\scriptstyle a\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{t})\,a\,(\mathsf{f}\vee\mathsf{f})}{(\mathsf{f}\,a\,\mathsf{f})\vee(\mathsf{t}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\bar{a}}{\mathsf{f}\vee\bar{a}} &
{\scriptstyle a\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{f})\,a\,(\mathsf{f}\vee\mathsf{t})}{(\mathsf{f}\,a\,\mathsf{f})\vee(\mathsf{f}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{a}{\mathsf{f}\vee a} \\
{\scriptstyle a\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{f})\,a\,(\mathsf{f}\vee\mathsf{f})}{(\mathsf{t}\,a\,\mathsf{f})\vee(\mathsf{f}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\bar{a}}{\bar{a}\vee\mathsf{f}} &
{\scriptstyle a\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{f})\,a\,(\mathsf{t}\vee\mathsf{f})}{(\mathsf{f}\,a\,\mathsf{t})\vee(\mathsf{f}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{a}{a\vee\mathsf{f}} \quad .
\end{array}
$$

It is easy to see that for each of them there is an SKS derivation with the same premiss and conclusion as the interpretation.

Likewise, we can check every interpretable instance of a rule inside the scope of an atom:

$$
\begin{array}{ll}
\dfrac{\mathsf{f}}{\mathsf{f}}\,a\,\mathsf{f} \overset{I}{\mapsto} \dfrac{\mathsf{f}}{\mathsf{f}} &
\dfrac{\mathsf{f}}{\mathsf{f}}\,a\,\mathsf{t} \overset{I}{\mapsto} \dfrac{a}{a} \\
\dfrac{\mathsf{t}}{\mathsf{t}}\,a\,\mathsf{f} \overset{I}{\mapsto} \dfrac{\bar{a}}{\bar{a}} &
\dfrac{\mathsf{t}}{\mathsf{t}}\,a\,\mathsf{t} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{t}} \\
\mathsf{f}\,a\,\dfrac{\mathsf{f}}{\mathsf{f}} \overset{I}{\mapsto} \dfrac{\mathsf{f}}{\mathsf{f}} &
\mathsf{t}\,a\,\dfrac{\mathsf{f}}{\mathsf{f}} \overset{I}{\mapsto} \dfrac{\bar{a}}{\bar{a}} \\
\mathsf{f}\,a\,\dfrac{\mathsf{t}}{\mathsf{t}} \overset{I}{\mapsto} \dfrac{a}{a} &
\mathsf{t}\,a\,\dfrac{\mathsf{t}}{\mathsf{t}} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{t}} \\
\dfrac{\mathsf{f}}{\mathsf{t}}\,a\,\mathsf{f} \overset{I}{\mapsto} \dfrac{\mathsf{f}}{\bar{a}} &
\dfrac{\mathsf{f}}{\mathsf{t}}\,a\,\mathsf{t} \overset{I}{\mapsto} \dfrac{a}{\mathsf{t}} \\
\mathsf{f}\,a\,\dfrac{\mathsf{f}}{\mathsf{t}} \overset{I}{\mapsto} \dfrac{\mathsf{f}}{a} &
\mathsf{t}\,a\,\dfrac{\mathsf{f}}{\mathsf{t}} \overset{I}{\mapsto} \dfrac{\bar{a}}{\mathsf{t}} \quad .
\end{array}
$$

It is easy to see that for each of them there is an SKS derivation with the same premiss and conclusion as the interpretation.

Furthermore, every inference rule of system SAKS trivially corresponds to the representation of an inference rule of system SKS , except for the rules $aw{\downarrow}$ and $aw{\uparrow}$.

$aw{\downarrow}$ corresponds to

$$=\frac{\mathsf{f}}{=\dfrac{\mathsf{f}\;a\;\boxed{m\,\dfrac{(\mathsf{f}\wedge\mathsf{t})\vee(\mathsf{t}\wedge\mathsf{f})}{(\mathsf{f}\vee\mathsf{t})\wedge(\mathsf{t}\vee\mathsf{f})}}}{\mathsf{f}\;a\;\mathsf{t}}} \overset{I}{\mapsto} \frac{\mathsf{f}}{a} \quad \text{and} \quad =\frac{\mathsf{f}}{=\dfrac{\boxed{m\,\dfrac{(\mathsf{f}\wedge\mathsf{t})\vee(\mathsf{t}\wedge\mathsf{f})}{(\mathsf{f}\vee\mathsf{t})\wedge(\mathsf{t}\vee\mathsf{f})}}\;a\;\mathsf{f}}{\mathsf{t}\;a\;\mathsf{f}}} \overset{I}{\mapsto} \frac{\mathsf{f}}{\bar{a}} \quad ,$$

and $aw{\uparrow}$ is the image of the dual derivations.

Furthermore, $\vee$ and $\wedge$ are associative and commutative in SAKS and their units are f and t respectively, and so the conditions are trivially verified for the equality inference rules.

Example 2.42. System SAMLLS of Figure 2-3 is correct for the multiplicative fragment of system SLLS given in Figure 2-4..

Every interpretable assignment of units in the inference rules has a corresponding derivation in the multiplicative fragment of SLLS. For example, for rule $a{\downarrow}$ we have the following interpretable assignments:

$$a{\downarrow}\frac{(\bot\parr\bot)\;a\;(\bot\parr\bot)}{(\bot\;a\;\bot)\parr(\bot\;a\;\bot)} \overset{I}{\mapsto} \frac{\bot}{\bot\parr\bot}$$

$$a{\downarrow}\frac{(\bot\parr 1)\;a\;(1\parr\bot)}{(\bot\;a\;1)\parr(1\;a\;\bot)} \overset{I}{\mapsto} \frac{1}{a\parr\bar{a}} \qquad a{\downarrow}\frac{(1\parr\bot)\;a\;(\bot\parr 1)}{(1\;a\;\bot)\parr(\bot\;a\;1)} \overset{I}{\mapsto} \frac{1}{\bar{a}\parr a}$$

$$a{\downarrow}\frac{(\bot\parr 1)\;a\;(\bot\parr 1)}{(\bot\;a\;\bot)\parr(1\;a\;1)} \overset{I}{\mapsto} \frac{1}{\bot\parr 1} \qquad a{\downarrow}\frac{(1\parr\bot)\;a\;(1\parr\bot)}{(1\;a\;1)\parr(\bot\;a\;\bot)} \overset{I}{\mapsto} \frac{1}{1\parr\bot}$$

$$a{\downarrow}\frac{(\bot\parr\bot)\;a\;(\bot\parr 1)}{(\bot\;a\;\bot)\parr(\bot\;a\;1)} \overset{I}{\mapsto} \frac{a}{\bot\parr a} \qquad a{\downarrow}\frac{(\bot\parr 1)\;a\;(\bot\parr\bot)}{(\bot\;a\;\bot)\parr(1\;a\;\bot)} \overset{I}{\mapsto} \frac{\bar{a}}{\bot\parr\bar{a}}$$

$$a{\downarrow}\frac{(\bot\parr\bot)\;a\;(1\parr\bot)}{(\bot\;a\;1)\parr(\bot\;a\;\bot)} \overset{I}{\mapsto} \frac{a}{a\parr\bot} \qquad a{\downarrow}\frac{(1\parr\bot)\;a\;(\bot\parr\bot)}{(1\;a\;\bot)\parr(\bot\;a\;\bot)} \overset{I}{\mapsto} \frac{\bar{a}}{\bar{a}\parr\bot} \quad .$$

It is easy to see that for each of them there is a derivation in the multiplicative fragment of SLLS with the same premiss and conclusion as the interpretation.

Every interpretable instance of a rule ρ inside the scope of an atom is necessarily an instance where the premiss and conclusion of ρ are interpreted as constants. The only such instances are of the form $\dfrac{u}{u}$ with $u \in \{\bot, 1\}$ and therefore every interpretable instance of a rule inside the scope of an atom trivially corresponds to a derivation in the multiplicative fragment of SLLS.

Every inference rule of SAMLLS of Figure 2-3 trivially corresponds to the representation of an inference rule of the multiplicative fragment of system SLLS.

$\parr$ and $\otimes$ are associative and commutative in SAMLLS and their units are $\bot$ and 1

respectively. Therefore, the equality rules trivially verify the conditions.

Example 2.43. System SABV of Figure 2-5 is correct for system SBV given in Figure 2-6.

Every interpretable assignment of units in the inference rules has a corresponding derivation in SBV. For example, for rule $a\!\downarrow$ we have the following interpretable assignments:

$$a\!\downarrow\frac{(\bot \parr \bot)\; a\; (\bot \parr \bot)}{(\bot\; a\; \bot) \parr (\bot\; a\; \bot)} \overset{I}{\mapsto} \frac{\bot}{\bot \parr \bot}$$

$$a\!\downarrow\frac{(\bot \parr 1)\; a\; (1 \parr \bot)}{(\bot\; a\; 1) \parr (1\; a\; \bot)} \overset{I}{\mapsto} \frac{1}{a \parr \bar{a}} \qquad a\!\downarrow\frac{(1 \parr \bot)\; a\; (\bot \parr 1)}{(1\; a\; \bot) \parr (\bot\; a\; 1)} \overset{I}{\mapsto} \frac{1}{\bar{a} \parr a}$$

$$a\!\downarrow\frac{(\bot \parr 1)\; a\; (\bot \parr 1)}{(\bot\; a\; \bot) \parr (1\; a\; 1)} \overset{I}{\mapsto} \frac{1}{\bot \parr 1} \qquad a\!\downarrow\frac{(1 \parr \bot)\; a\; (1 \parr \bot)}{(1\; a\; 1) \parr (\bot\; a\; \bot)} \overset{I}{\mapsto} \frac{1}{1 \parr \bot}$$

$$a\!\downarrow\frac{(\bot \parr \bot)\; a\; (\bot \parr 1)}{(\bot\; a\; \bot) \parr (\bot\; a\; 1)} \overset{I}{\mapsto} \frac{a}{\bot \parr a} \qquad a\!\downarrow\frac{(\bot \parr 1)\; a\; (\bot \parr \bot)}{(\bot\; a\; \bot) \parr (1\; a\; \bot)} \overset{I}{\mapsto} \frac{\bar{a}}{\bot \parr \bar{a}}$$

$$a\!\downarrow\frac{(\bot \parr \bot)\; a\; (1 \parr \bot)}{(\bot\; a\; 1) \parr (\bot\; a\; \bot)} \overset{I}{\mapsto} \frac{a}{a \parr \bot} \qquad a\!\downarrow\frac{(1 \parr \bot)\; a\; (\bot \parr \bot)}{(1\; a\; \bot) \parr (\bot\; a\; \bot)} \overset{I}{\mapsto} \frac{\bar{a}}{\bar{a} \parr \bot} \quad .$$

It is easy to see that for each of them there is a derivation in SBV with the same premiss and conclusion as the interpretation.

Every interpretable inference rule in the scope of an atom corresponds to a rule $\frac{u}{u}$ with $u \in \{\bot, \circ, 1\}$ and therefore trivially corresponds to an SBV derivation.

Every inference rule of system SABV is trivially the representation of an inference rule of system SBV, and the equality axioms are trivially represented by the equational theory for SABV we defined in example 2.9 where the units are identified.

In the next chapter we will focus on showing the admissibility of certain distinguished rules.

Definition 2.44. We say that an inference rule ρ is *admissible* for a proof system SA if $\rho \notin$ SA and for every proof $\overset{\Pi\, \mathsf{SA}\cup\{\rho\}}{A}$ there exists a proof $\overset{\Pi\, \mathsf{SA}}{A}$.

Chapter 3

Splitting

Cut-elimination via splitting has been shown to work in a vast array of deep inference systems: linear logic [35], multiplicative exponential linear logic [37], the mixed commutative/non-commutative logic BV [21] and its extension with linear exponentials NEL [28] and classical predicate logic [3]. This generality points towards the fact that the splitting procedure hinges on some fundamental properties required for cut-elimination rather than on the specificities of each system.

In particular, cut-elimination proofs via splitting are very straightforward in those systems without contractions, as we will show in Section 3.1 with the example of multiplicative linear logic. This suggests that it is the properties of linear rules (as opposed to contraction rules) that enable us to prove cut-elimination. Indeed, the generalisation of the splitting procedure that we show in Section 3.2 allows us to fully confirm these suspicions: it is precisely because of the properties of the linear rules that we are able to prove cut-elimination for systems where they are present. In this way, we will give sufficient conditions that guarantee cut-elimination for a full class of substructural logics, similarly to [1, 39, 15] where conditions for a display calculus to enjoy cut elimination are presented, or to [32] where conditions for propositional based logics in the sequent calculus are presented.

3.1 Splitting for MLL

Linear logic was developed by Girard [14] as a refinement of classical logic by introducing restrictions on the structural rules of contraction and weakening. The core propositional connectives of linear logic are divided into additive and multiplicative connectives, exemplifying perfectly the distinction we will be making in this thesis between contractive systems and linear systems (that we will call *splittable*). The introduction rules for the additive conjunction & (with) and the multiplicative conjunction ⊗ (tensor) are given in their sequent calculus presentation as follows:

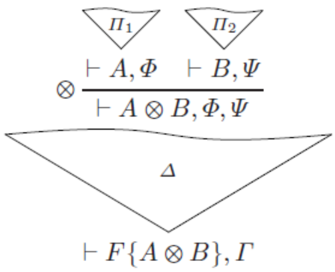
$$\frac{\vdash A, \Phi \quad \vdash B, \Phi}{\vdash A \mathbin{\&} B, \Phi} \; , \qquad \frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \; .$$

$$a\downarrow \frac{(A \parr B)\ a\ (C \parr D)}{(A\ a\ C) \parr (B\ a\ D)}$$

$$\otimes\downarrow \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)}$$

Figure 3-1: *System* $\mathsf{SAMLLS}^{\downarrow}$

Reading bottom up, we see that the additive conjunction & requires a duplication of the context whereas the multiplicative conjunction $\otimes$ requires that the context be divided between its hypotheses. There is no communication between Φ and Ψ in the proof above the tensor rule where they are united.



It is precisely this multiplicative rule shape that splitting hinges on. In the sequent calculus, the presence of a main connective allows us to know exactly which rules can be applied above a cut. In deep inference, this is not possible since any rule can be applied at any depth, and we therefore focus on the behaviour of the context around a cut to tackle cut-elimination. This allows us to have a better understanding of how the cut-elimination procedure changes the proof globally. If all the connectives of a system require a splitting of the context like the multiplicative tensor does, then we can keep track of exactly how the context around a connective behaves. This allows us to split a proof into independent subproofs above every rule, just like in the example above the proof is divided into Π_1 and Π_2 above the $\otimes$ introduction rule. Cut-elimination is then only a matter of rearranging the independent subproofs into a cut-free proof.

Multiplicative linear logic (MLL) is the fragment of linear logic comprising only the multiplicative connectives and their units. It is a very simple system in which every connective requires such a splitting of the context, and therefore ideal to provide an example of a proof of cut-elimination via splitting. In what follows we will present a proof of cut-elimination via splitting for MLL, as an example of an application of the generalised theorem of Section 3.2.

We will present this proof in the subatomic proof system for multiplicative linear logic SAMLLS to help the reader become accustomed to the subatomic notation, as well as to relate it better to the generalised theorem. We present subatomic system SAMLLS for MLL in Figure 2-3, together with the equations of example 2.8 and the interpretation map in example 2.14.

As is usual in deep inference systems, the sequent calculus cut rule is divided into several rules, corresponding to the up rules indicated by $\uparrow$. The splitting method allows us to prove the admissibility of all of these rules. The reduced cut-free system is denoted by $\mathsf{SAMLLS}^{\downarrow}$, and is shown in Figure 3-1.

By simple observation, we can notice that in $\mathsf{SAMLLS}^{\downarrow}$ the scope of the relations a and $\otimes$ only decreases when reading top to bottom. The widening scope of relations from bottom to top is the main property used to prove splitting. If we follow a particular instance of the tensor $\otimes$ through a proof, its scope will be at its widest in the premiss. Therefore, if we have a proof of $F\{A \otimes B\}$, we can follow $\otimes$ up in the proof to obtain two independent proofs $\begin{array}{c}\Pi_1 \Vert \\ Q_A\{A\}\end{array}$ and $\begin{array}{c}\Pi_2 \Vert \\ Q_B\{B\}\end{array}$.

$$\frac{\boxed{\begin{array}{c}\Pi_1 \Vert \\ A \parr K_1 \parr Q_1\end{array}} \otimes \boxed{\begin{array}{c}\Pi_2 \Vert \\ B \parr K_2 \parr Q_2\end{array}}}{\boxed{\begin{array}{c}(A \parr K_1) \otimes (B \parr K_2) \\ \Vert \\ (A \otimes B) \parr K_1 \parr K_2\end{array}} \parr Q_1 \parr Q_2}$$

If we do this for every occurrence of $\otimes$ and a in the conclusion of a proof, starting from the outermost, we obtain a series of subproofs independent from each other. This is the gist of the splitting theorem, and cut-elimination comes as a corollary, by showing that we are free to rearrange these independent subproofs in such a way that the cut is no longer necessary.

We will show that this cut-elimination procedure corresponds to cut-elimination in the non-subatomic system SMLLS. For that, we will pay particular attention to *tame* proofs.

Definition 3.1. We say that an interpretable derivation ϕ in SA is *tame* if the only instances of rules in the scope of an atom are equality rules.

Note that the composition of tame derivations by any relation that is not an atom yields a tame derivation.

Example 3.2. The derivation

$$a{\downarrow}\frac{(\bot \parr 1)\ a\ (\bot \parr 1)}{(\bot\ a\ \bot) \parr (1\ a\ 1)}\ a\ \bot$$

in SAMLLS is interpretable but is not tame.

The derivation

$$=\frac{1}{(\bot \parr 1)}\ a\ \bot$$

is tame.

Every proof in SMLLS corresponds to a tame proof in SAMLLS since every rule of SMLLS corresponds to a tame derivation in SAMLLS. This is trivial for every rule, except for the atomic introduction and cut rules

$$\frac{1}{a \parr \bar{a}} \quad \text{and} \quad \frac{a \otimes \bar{a}}{\bot} \quad .$$

The introduction rule corresponds to the tame derivation

$$a\downarrow \frac{= \frac{= \frac{1}{1\, a\, 1}}{(\bot \parr 1)\; a\; (1 \parr \bot)}}{(\bot\; a\; 1) \parr (1\; a\; \bot)} \quad ,$$

and dually the cut rule corresponds to a tame derivation as well.

Tameness is preserved by splitting and therefore it is preserved by the cut-elimination procedure. The cut-free proofs obtained from proofs in the 'original' system will therefore be tame and correspond to cut-free proofs in SMLLS.

In what follows we will present the splitting theorem for $\mathsf{SAMLLS}^{\downarrow}$. The form of the statement follows the standard scheme for splitting theorems, stemming from the original proof in [21]: it is therefore divided in two results for ease of reading, called *shallow splitting* and *context reduction.* Guided from the generalisation we present in Section 3.2, we use a simple induction measure. We will work modulo associativity, commutativity and unit of $\parr$.

Notation 3.3. We will abuse notation and refer to a derivation ϕ composed only of equality rules as an *equality.*

Definition 3.4. Given a proof ϕ in $\mathsf{SAMLLS}^{\downarrow}$, we define the *length of* ϕ as the number of inference rules in ϕ different from the equality rules for the associativity, commutativity and unit of $\parr$. We denote it by $|\phi|_{\parr}$.

Definition 3.5. We define $=_{\parr}$ as the equivalence relation on formulae defined by the axioms for the associativity, commutativity and unit of $\parr$.

We define the equivalence relation $=_{\parr}$ on derivations following Definition 2.32.

It is straightforward that if $\phi =_{\parr} \psi$, then $|\phi|_{\parr} = |\psi|_{\parr}$.

Theorem 3.6 (Shallow splitting)**.** *For all formulae A, B, C:*

1. *If there is a proof ϕ of $(A \otimes B) \parr C$ in $\mathsf{SAMLLS}^{\downarrow}$, there exist Q_1, Q_2 and*

$$\begin{array}{c} Q_1 \parr Q_2 \\ \psi \| \\ C \end{array} \quad , \quad \begin{array}{c} \phi_1 \| \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \| \\ B \parr Q_2 \end{array}$$

   such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi|_{\parr}$.

   Furthermore, if ϕ is tame, then ϕ_1, ϕ_2 and ψ are tame.

2. If there is a proof ϕ *of* $(A \mathbin{a} B) \parr C$ *in* $\mathsf{SAMLLS}^{\downarrow}$*, there exist* Q_1*,* Q_2 *and*

$$\begin{array}{c} Q_1 \mathbin{a} Q_2 \\ \psi \, \| \\ C \end{array} \quad , \quad \begin{array}{c} \phi_1 \, \| \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \, \| \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi|_{\parr}$.

Furthermore, if ϕ *is tame, then* ϕ_1, ϕ_2 *are equalities and* ψ *is tame.*

Proof. Given a proof ϕ of $(A \otimes B) \parr C$ in $\mathsf{SAMLLS}^{\downarrow}$ we reduce it to CoS notation for $=_{\parr}$. We proceed by induction on $|\phi|_{\parr}$.

1. If $|\phi|_{\parr} = 0$, then $(A \otimes B) \parr C =_{\parr} 1$. Then, either:

   – $A =_{\parr} B =_{\parr} 1, C =_{\parr} \bot$ and we take

$$\psi \equiv {=_{\parr}}\dfrac{{=}\dfrac{\bot \parr \bot}{\bot}}{C} \quad , \quad \phi_1 \equiv {=_{\parr}}\dfrac{1}{\boxed{{=_{\parr}}\dfrac{1}{A}} \parr \bot} \quad , \quad \phi_2 \equiv {=_{\parr}}\dfrac{1}{\boxed{{=_{\parr}}\dfrac{1}{B}} \parr \bot} \quad ; \text{or}$$

   – $A = \bot, B = C = 1$ and we take

$$\psi \equiv {=_{\parr}}\dfrac{{=}\dfrac{1 \parr \bot}{1}}{C} \quad , \quad \phi_1 \equiv {=_{\parr}}\dfrac{1}{\boxed{{=_{\parr}}\dfrac{\bot}{A}} \parr 1} \quad , \quad \phi_2 \equiv {=_{\parr}}\dfrac{1}{\boxed{{=_{\parr}}\dfrac{1}{B}} \parr \bot} \quad ; \text{or}$$

   – $B = \bot, A = C = 1$ and we take $Q_1 = \bot, Q_2 = 1$

$$\psi \equiv {=_{\parr}}\dfrac{{=}\dfrac{\bot \parr 1}{1}}{C} \quad , \quad \phi_1 \equiv {=_{\parr}}\dfrac{1}{\boxed{{=_{\parr}}\dfrac{1}{A}} \parr \bot} \quad , \quad \phi_2 \equiv {=_{\parr}}\dfrac{1}{\boxed{{=_{\parr}}\dfrac{\bot}{B}} \parr 1} \quad .$$

If $|\phi|_{\parr} = n > 0$, inspection of the rules provides us the following possible cases:

(1) $\phi =_{\parr} \boxed{\begin{array}{c} \phi' \, \| \\ r\dfrac{(A' \otimes B) \parr C}{(A \otimes B) \parr C} \end{array}}$;

(2) $\phi =_{\parr} \boxed{\begin{array}{c} \phi' \, \| \\ r\dfrac{(A \otimes B') \parr C}{(A \otimes B) \parr C} \end{array}}$;

(3) $$\phi =_{\parr} \begin{array}{c}\phi'\,\Pi\\ \boxed{r\,\dfrac{(A\otimes B)\parr C'}{(A\otimes B)\parr C}}\end{array} \quad ;$$

(4) $$\phi =_{\parr} \begin{array}{c}\phi'\,\Pi\\ \boxed{\otimes\!\downarrow\dfrac{((A\parr C_1)\otimes(B\parr C_2))\parr C_3}{(A\otimes B)\parr C_1\parr C_2\parr C_3}}\end{array} \quad \text{with } C =_{\parr} C_1\parr C_2\parr C_3\ ;$$

(5) $$\phi =_{\parr} \begin{array}{c}\phi'\,\Pi\\ \boxed{\otimes\!\downarrow\dfrac{(((A\otimes B)\parr C_1)\otimes(C_2\parr C_3))\parr C_4}{(A\otimes B)\parr C_2\parr(C_1\otimes C_3)\parr C_4}}\end{array} \quad \text{with } C =_{\parr} C_2\parr(C_1\otimes C_3)\parr C_4\ ;$$

(6) $$\phi =_{\parr} \begin{array}{c}\phi'\,\Pi\\ \boxed{=\,\dfrac{(A_1\otimes(A_2\otimes B))\parr C}{((A_1\otimes A_2)\otimes B)\parr C}}\end{array} \quad ;$$

(7) $$\phi =_{\parr} \begin{array}{c}\phi'\,\Pi\\ \boxed{=\,\dfrac{(A\otimes B)\parr C}{(B\otimes A)\parr C}}\end{array} \quad ;$$

(8) $$\phi =_{\parr} \begin{array}{c}\phi'\,\Pi\\ \boxed{\dfrac{(((A\otimes B)\parr C_1)\otimes 1)\parr C_2}{(A\otimes B)\parr C_1\parr C_2}}\end{array} \quad \text{with } C =_{\parr} C_1\parr C_2\ ;$$

(9) $$\phi =_{\parr} \begin{array}{c}\phi'\,\Pi\\ \boxed{\dfrac{(1\otimes((A\otimes B)\parr C_1))\parr C_2}{(A\otimes B)\parr C_1\parr C_2}}\end{array} \quad \text{with } C =_{\parr} C_1\parr C_2\ ;$$

(10) $$\phi =_{\parr} \begin{array}{c}\phi'\,\Pi\\ \boxed{\dfrac{A\parr C}{(A\otimes 1)\parr C}}\end{array} \quad \text{with } B =_{\parr} 1\ ;$$

(11) $$\phi =_{\parr} \begin{array}{c}\phi'\,\Pi\\ \boxed{\dfrac{A\parr C}{(1\otimes B)\parr C}}\end{array} \quad \text{with } A =_{\parr} 1\ .$$

(1) Since $|\phi'|_{\parr} = n-1$, we apply the induction hypothesis to ϕ'. There exist Q_1, Q_2 and

$$\begin{array}{c} Q_1\parr Q_2\\ \psi\,\Vert\\ C\end{array} \quad , \qquad \phi_1 \equiv \begin{array}{c}\phi_1'\,\Pi\\ \boxed{r\,\dfrac{A'}{A}}\end{array}\parr Q_1 \quad , \qquad \begin{array}{c}\phi_2\,\Pi\\ B\parr Q_2\end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} = |\phi_1'|_{\parr} + |\phi_2|_{\parr} + 1 \leq |\phi'|_{\parr} + 1 = |\phi|_{\parr}$.

If ϕ is tame, then ψ, ϕ_1' and ϕ_2 are tame. Furthermore, since ϕ is tame r is tame, and therefore ϕ_1 is interpretable.

(2) This case is analogous to (1).

(3) We apply the induction hypothesis to ϕ'. There exist Q_1, Q_2 and

$$\begin{array}{c} Q_1 \parr Q_2 \\ {\scriptstyle \psi'} \, \| \\ \boxed{{\scriptstyle r}\,\dfrac{C'}{C}} \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1} \, \Pi \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2} \, \Pi \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi'|_{\parr} \leq |\phi|_{\parr}$.

If ϕ is tame, then ψ', ϕ_1 and ϕ_2 are tame. Furthermore, since ϕ is tame r is tame. Therefore ψ is tame.

(4) We apply the induction hypothesis to ϕ'. There exist Q'_1, Q'_2 and

$$\begin{array}{c} Q'_1 \parr Q'_2 \\ {\scriptstyle \psi'} \, \| \\ C_3 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1} \, \Pi \\ A \parr C_1 \parr Q'_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2} \, \Pi \\ B \parr C_2 \parr Q'_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi'|_{\parr} \leq |\phi|_{\parr}$.

If ϕ is tame, then ψ', ϕ_1 and ϕ_2 are tame.

We take $Q_1 = C_1 \parr Q'_1$, $Q_2 = C_2 \parr Q'_2$ and we have

$$\psi \equiv \quad \begin{array}{c} C_1 \parr Q'_1 \parr C_2 \parr Q'_2 \\ =_{\parr} \overline{\overline{\hspace{6em}}} \\ C_1 \parr C_2 \parr \boxed{\begin{array}{c} Q'_1 \parr Q'_2 \\ {\scriptstyle \psi'} \, \| \\ C_3 \end{array}} \\ =_{\parr} \overline{\overline{\hspace{6em}}} \\ C \end{array} \quad .$$

If ϕ is tame, since ϕ_1 and ϕ_2 are tame, C_1, Q'_1 and C_2, Q'_2 are interpretable. Then, since ψ' is tame, ψ is tame.

(5) We apply the induction hypothesis to ϕ'. There exist Q'_1, Q'_2 and

$$\begin{array}{c} Q'_1 \parr Q'_2 \\ {\scriptstyle \psi_1} \, \| \\ C_4 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi'_1} \, \Pi \\ (A \otimes B) \parr C_1 \parr Q'_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi'_2} \, \Pi \\ C_2 \parr C_3 \parr Q'_2 \end{array}$$

such that $|\phi'_1|_{\parr} + |\phi'_2|_{\parr} \leq |\phi'|_{\parr}$.

We apply the induction hypothesis to ϕ'_1. There exist Q_1, Q_2 and

$$\psi \equiv \begin{array}{c} \otimes\!\downarrow \dfrac{\boxed{\begin{array}{c} Q_1 \parr Q_2 \\ {\scriptstyle \psi_2} \| \\ C_1 \parr Q'_1 \end{array}} \otimes \boxed{\begin{array}{c} {\scriptstyle \phi'_2} \Vert \\ C_2 \parr C_3 \parr Q'_2 \end{array}}}{(C_1 \otimes C_3) \parr C_2 \parr \boxed{\begin{array}{c} Q'_1 \parr Q'_2 \\ {\scriptstyle \psi_1} \| \\ C_4 \end{array}}} \\ =_{\parr} \overline{\overline{\qquad C \qquad}} \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1} \Vert \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2} \Vert \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi'_1|_{\parr} \leq |\phi'|_{\parr} \leq |\phi|_{\parr}$.

If ϕ is tame, then ψ_1, ϕ'_1 and ϕ'_2 are tame. Therefore, ψ_2, ϕ_1 and ϕ_2 are tame and thus ψ is tame.

(6) We apply the induction hypothesis to ϕ'. There exist Q'_1, Q'_2 and

$$\begin{array}{c} Q'_1 \parr Q'_2 \\ {\scriptstyle \psi_1} \| \\ C \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi'_1} \Vert \\ A_1 \parr Q'_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi'_2} \Vert \\ (A_2 \otimes B) \parr Q'_2 \end{array}$$

such that $|\phi'_1|_{\parr} + |\phi'_2|_{\parr} \leq |\phi'|_{\parr}$.

We apply the induction hypothesis to ϕ'_2. There exist M, Q_2 and

$$\begin{array}{c} M \parr Q_2 \\ {\scriptstyle \psi_2} \| \\ Q'_2 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \zeta} \Vert \\ A_2 \parr M \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2} \Vert \\ B \parr Q_2 \end{array}$$

such that $|\zeta|_{\parr} + |\phi_2|_{\parr} \leq |\phi'_2|_{\parr}$.

We take $Q_1 \equiv Q'_1 \parr M$ and

$$\psi \equiv \begin{array}{c} (Q'_1 \parr M) \parr Q_2 \\ =_{\parr} \overline{\overline{\qquad\qquad\qquad}} \\ Q'_1 \parr \boxed{\begin{array}{c} M \parr Q_2 \\ {\scriptstyle \psi_2} \| \\ Q'_2 \end{array}} \\ {\scriptstyle \psi_1} \| \\ C \end{array} \quad , \quad \phi_1 \equiv \otimes\!\downarrow \dfrac{\boxed{\begin{array}{c} {\scriptstyle \phi'_1} \Vert \\ A_1 \parr Q'_1 \end{array}} \otimes \boxed{\begin{array}{c} {\scriptstyle \zeta} \Vert \\ A_2 \parr M \end{array}}}{(A_1 \otimes A_2) \parr (Q'_1 \parr M)} \quad .$$

We have:
$|\phi_1|_{\parr} + |\phi_2|_{\parr} = |\phi'_1|_{\parr} + |\zeta|_{\parr} + 1 + |\phi_2|_{\parr} \leq |\phi'_1|_{\parr} + |\phi'_2|_{\parr} + 1 \leq |\phi'|_{\parr} + 1 = |\phi|_{\parr}$.

If ϕ is tame, ψ_1, ϕ'_1 and ϕ'_2 are tame. Then, ψ_2, ζ and ϕ_2 are tame. Therefore, ψ and ϕ_1 are tame.

(7) We apply the induction hypothesis to ϕ'. There are Q'_1, Q'_2 and

$$\begin{array}{c} Q_2 \parr Q_1 \\ \psi' \,\| \\ C \end{array} \quad , \quad \begin{array}{c} \phi_2 \,\Pi \\ B \parr Q_2 \end{array} \quad , \quad \begin{array}{c} \phi_1 \,\Pi \\ A \parr Q_1 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi|_{\parr}$.

We take

$$\psi \equiv \begin{array}{c} \Rightarrow_{\parr} \dfrac{Q_1 \parr Q_2}{Q_1 \parr Q_1} \\ \psi' \,\| \\ C \end{array} \quad .$$

If ϕ is tame, ψ', ϕ_1 and ϕ_2 are tame, and thus ψ is tame as well.

(8) We apply the induction hypothesis to ϕ'. There exist Q'_1, Q'_2 and

$$\begin{array}{c} Q'_1 \parr Q'_2 \\ \psi_1 \,\| \\ C_2 \end{array} \quad , \quad \begin{array}{c} \phi'_1 \,\Pi \\ (A \otimes B) \parr C_1 \parr Q'_1 \end{array} \quad , \quad \begin{array}{c} \phi'_2 \,\Pi \\ 1 \parr Q'_2 \end{array}$$

such that $|\phi'_1|_{\parr} + |\phi'_2|_{\parr} \leq |\phi'|_{\parr}$.

We apply the induction hypothesis to ϕ'_1. There exist Q_1, Q_2 and

$$\psi \equiv \begin{array}{c} \otimes\!\downarrow \dfrac{\boxed{\begin{array}{c} Q_1 \parr Q_2 \\ \psi_2 \,\| \\ C_1 \parr Q'_1 \end{array}} \otimes \boxed{\begin{array}{c} \phi'_2 \,\Pi \\ 1 \parr Q'_2 \end{array}}}{(C_1 \otimes 1) \parr \boxed{\begin{array}{c} Q'_1 \parr Q'_2 \\ \psi_1 \,\| \\ C_2 \end{array}}} \\ \Rightarrow_{\parr} \overline{\overline{\quad C \quad}} \end{array} \quad , \quad \begin{array}{c} \phi_1 \,\Pi \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \,\Pi \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi'_1|_{\parr} \leq |\phi'|_{\parr} \leq |\phi|_{\parr}$.

If ϕ is tame, then so are ψ_1, ϕ'_1 and ϕ'_2. Therefore, ψ_2, ϕ_1 and ϕ_2 are tame, and so is ψ.

(9) This case is analogous to case (8).

(10) We take

$$\psi \equiv \Rightarrow_{\parr} \frac{C \parr \bot}{C} \quad , \quad \phi_1 \equiv \begin{array}{c} \phi' \,\Pi \\ A \parr C \end{array} \quad , \quad \phi_2 \equiv \Rightarrow_{\parr} \frac{1}{\boxed{\Rightarrow_{\parr} \dfrac{1}{B}} \parr \bot} \quad .$$

We have $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi|_{\parr}$.

If ϕ is tame, then C is interpretable and ϕ' is tame and thus ψ and ϕ_1 are tame. ψ_2 is tame.

(11) This case is analogous to case (10).

2. If $|\phi|_{\parr} = 0$, then either

– $A =_{\parr} B =_{\parr} 1, C =_{\parr} \perp$ and we take

$$\psi \equiv {=}_{\parr}\frac{{=}\dfrac{\perp \mathbin{a} \perp}{\perp}}{C} \quad , \qquad \phi_1 \equiv {=}_{\parr}\frac{1}{\boxed{{=}_{\parr}\dfrac{1}{A}} \parr \perp} \quad , \qquad \phi_2 \equiv {=}_{\parr}\frac{1}{\boxed{{=}_{\parr}\dfrac{1}{B}} \parr \perp} \quad ,$$

with $|\phi_1|_{\parr} = |\phi_2|_{\parr} = 0$;

– or $A =_{\parr} B =_{\parr} \perp, C =_{\parr} 1$ and we take

$$ {=}_{\parr}\frac{{=}\dfrac{1 \mathbin{a} 1}{1}}{C} \quad , \qquad \phi_1 \equiv {=}_{\parr}\frac{1}{\boxed{{=}_{\parr}\dfrac{\perp}{A}} \parr 1} \quad , \qquad \phi_2 \equiv {=}_{\parr}\frac{1}{\boxed{{=}_{\parr}\dfrac{\perp}{B}} \parr 1} \quad .$$

with $|\phi_1|_{\parr} = |\phi_2|_{\parr} = 0$.

If $|\phi|_{\parr} = n > 0$ and $A \mathbin{a} B \neq_{\parr} u$, inspection of the rules provides us the following possible cases:

(1) $\phi =_{\parr} \boxed{r\,\dfrac{\overset{\phi' \,\Pi}{(A' \mathbin{a} B) \parr C}}{(A \mathbin{a} B) \parr C}}$;

(2) $\phi =_{\parr} \boxed{r\,\dfrac{\overset{\phi' \,\Pi}{(A \mathbin{a} B') \parr C}}{(A \mathbin{a} B) \parr C}}$;

(3) $\phi =_{\parr} \boxed{r\,\dfrac{\overset{\phi' \,\Pi}{(A \mathbin{a} B) \parr C'}}{(A \mathbin{a} B) \parr C}}$;

(4) $\phi =_{\parr} \boxed{\otimes\!\downarrow\,\dfrac{\overset{\phi' \,\Pi}{((A \parr C_1) \mathbin{a} (B \parr C_2)) \parr C_3}}{(A \mathbin{a} B) \parr (C_1 \mathbin{a} C_2) \parr C_3}}$ with $C =_{\parr} (C_1 \mathbin{a} C_2) \parr C_3$;

(5) $$\phi =_{\parr} \begin{array}{c}\phi' \,\Pi \\ \boxed{{\scriptstyle \otimes\downarrow}\,\dfrac{(((A \mathbin{a} B) \parr C_1) \otimes (C_2 \parr C_3)) \parr C_4}{(A \mathbin{a} B) \parr C_2 \parr (C_1 \otimes C_3) \parr C_4}}\end{array} \quad \text{with } C =_{\parr} C_2 \parr (C_1 \otimes C_3) \parr C_4 \;;$$

(6) $$\phi =_{\parr} \begin{array}{c}\phi' \,\Pi \\ \boxed{{\scriptstyle =}\,\dfrac{(((A \mathbin{a} B) \parr C_1) \otimes 1) \parr C_2}{(A \mathbin{a} B) \parr C_1 \parr C_2}}\end{array} \quad \text{with } C =_{\parr} C_1 \parr C_2 \;;$$

(7) $$\phi =_{\parr} \begin{array}{c}\phi' \,\Pi \\ \boxed{{\scriptstyle =}\,\dfrac{(1 \otimes ((A \mathbin{a} B) \parr C_1)) \parr C_2}{(A \mathbin{a} B) \parr C_1 \parr C_2}}\end{array} \quad \text{with } C =_{\parr} C_1 \parr C_2 \;;$$

(8) $$\phi =_{\parr} \begin{array}{c}\phi' \,\Pi \\ \boxed{{\scriptstyle =}\,\dfrac{1 \parr C}{(1 \mathbin{a} 1) \parr C}}\end{array} \quad \text{with } A =_{\parr} B =_{\parr} 1 \;;$$

(9) $$\phi =_{\parr} \begin{array}{c}\phi' \,\Pi \\ \boxed{{\scriptstyle =}\,\dfrac{\bot \parr C}{(\bot \mathbin{a} \bot) \parr C}}\end{array} \quad \text{with } A =_{\parr} B =_{\parr} 1 \;.$$

(1) We apply the induction hypothesis to ϕ'. There exist Q_1, Q_2 and

$$\begin{array}{c} Q_1 \mathbin{a} Q_2 \\ \| \\ C \end{array} \quad , \qquad \phi_1 \equiv \begin{array}{c}\phi_1' \,\Pi \\ \boxed{{\scriptstyle r}\,\dfrac{A'}{A}} \parr Q_1\end{array} \quad , \qquad \begin{array}{c}\phi_2 \,\Pi \\ B \parr Q_2\end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} = |\phi_1'|_{\parr} + 1 + |\phi_2|_{\parr} \leq |\phi'|_{\parr} + 1 = |\phi|_{\parr}$.

If ϕ is tame, ψ is tame and ϕ_1' and ϕ_2 are equalities. r is an equality, and therefore ϕ_1 is an equality.

(2) This case is analogous to (1).

(3) We apply the induction hypothesis to ϕ'. There exist Q_1, Q_2 and

$$\begin{array}{c} Q_1 \mathbin{a} Q_2 \\ \psi' \, \| \\ \boxed{{\scriptstyle r}\,\dfrac{C'}{C}} \end{array} \quad , \qquad \begin{array}{c}\phi_1 \,\Pi \\ A \parr Q_1\end{array} \quad , \qquad \begin{array}{c}\phi_2 \,\Pi \\ B \parr Q_2\end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi'|_{\parr} \leq |\phi|_{\parr}$.

If ϕ is tame, so are ψ' and r and thus so is ψ. ϕ_1 and ϕ_2 are equalities.

(4) We apply the induction hypothesis to ϕ'. There exist Q_1', Q_2' and

$$\begin{array}{c} Q_1' \; a \; Q_2' \\ \psi' \, \| \\ C_3 \end{array} \quad , \quad \begin{array}{c} \phi_1 \, \Vert \\ A \parr C_1 \parr Q_1' \end{array} \quad , \quad \begin{array}{c} \phi_2 \, \Vert \\ B \parr C_2 \parr Q_2' \end{array}$$

such that $|\phi_1|_\parr + |\phi_2|_\parr \leq |\phi'|_\parr \leq |\phi|_\parr$.

We take $Q_1 = C_1 \parr Q_1'$, $Q_2 = C_2 \parr Q_2'$ and

$$\psi \equiv \quad \dfrac{a\!\downarrow \dfrac{(C_1 \parr Q_1') \; a \; (C_2 \parr Q_2')}{(C_1 \; a \; C_2) \parr \boxed{\begin{array}{c} Q_1' \; a \; Q_2' \\ \psi' \, \| \\ C_3 \end{array}}}}{C} {=_\parr} \quad .$$

If ϕ is tame, then ψ' is tame and ϕ_1 and ϕ_2 are equalities. Then $C_1 \parr Q_1' = 1$ or $C_1 \parr Q_1' = \bot$ and $C_2 \parr Q_2' = 1$ or $C_2 \parr Q_2' = \bot$. Therefore, $(C_1 \parr Q_1') \, a \, (C_2 \parr Q_2')$ and $C_1 \; a \; C_2$ are interpretable and ψ is tame.

(5) We apply the induction hypothesis to ϕ'. There exist Q_1', Q_2' and

$$\begin{array}{c} Q_1' \parr Q_2' \\ \psi_1 \, \| \\ C_4 \end{array} \quad , \quad \begin{array}{c} \phi_1' \, \Vert \\ (A \; a \; B) \parr C_1 \parr Q_1' \end{array} \quad , \quad \begin{array}{c} \phi_2' \, \Vert \\ C_2 \parr C_3 \parr Q_2' \end{array}$$

such that $|\phi_1'|_\parr + |\phi_2'|_\parr \leq |\phi'|_\parr$.

We apply the induction hypothesis to ϕ_1'. There exist Q_1, Q_2 and

$$\psi \equiv \quad \dfrac{\otimes\!\downarrow \dfrac{\boxed{\begin{array}{c} Q_1 \; a \; Q_2 \\ \psi_2 \, \| \\ C_1 \parr Q_1' \end{array}} \otimes \boxed{\begin{array}{c} \phi_2' \, \Vert \\ C_2 \parr C_3 \parr Q_2' \end{array}}}{(C_1 \otimes C_3) \parr C_2 \parr \boxed{\begin{array}{c} Q_1' \parr Q_2' \\ \psi_1 \, \| \\ C_4 \end{array}}}}{C} {=_\parr} \quad , \quad \begin{array}{c} \phi_1 \, \Vert \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \, \Vert \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_\parr + |\phi_2|_\parr \leq |\phi_1'|_\parr \leq |\phi'|_\parr \leq |\phi|_\parr$.

If ϕ is tame, then ψ_1, ϕ_1', ϕ_2' and ψ_2 are tame. Therefore ψ is tame. Furthermore, by the induction hypothesis ϕ_1 and ϕ_2 are equalities.

(6) We apply the induction hypothesis to ϕ'. There exist Q_1', Q_2' and

$$\begin{array}{c} Q_1' \parr Q_2' \\ \psi_1 \, \| \\ C_2 \end{array} \quad , \qquad \begin{array}{c} \phi_1' \, \| \\ (A \, a \, B) \parr C_1 \parr Q_1' \end{array} \quad , \quad \begin{array}{c} \phi_2' \, \| \\ 1 \parr Q_2' \end{array}$$

such that $|\phi_1'|_\parr + |\phi_2'|_\parr \leq |\phi'|_\parr$.

We apply the induction hypothesis to ϕ_1'. There exist Q_1, Q_2 and

$$\psi \equiv \cfrac{\cfrac{\boxed{\begin{array}{c} Q_1 \, a \, Q_2 \\ \psi_2 \, \| \\ C_1 \parr Q_1' \end{array}} \otimes \boxed{\begin{array}{c} \phi_2' \, \| \\ 1 \parr Q_2' \end{array}}}{(C_1 \otimes 1) \parr \boxed{\begin{array}{c} Q_1' \parr Q_2' \\ \psi_1 \, \| \\ C_2 \end{array}}} \otimes\!\downarrow}{C} =_\parr \quad , \qquad \begin{array}{c} \phi_1 \, \| \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \, \| \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_\parr, |\phi_2|_\parr \leq |\phi_1'|_\parr \leq |\phi'|_\parr \leq |\phi|_\parr$.

If ϕ is tame, then ψ_1, ϕ_1', ϕ_2' and ψ_2 are tame. Therefore ψ is tame. Furthermore, by the induction hypothesis ϕ_1 and ϕ_2 are equalities.

(7) This case is analogous to case (5).

(8) We take

$$\phi_1 \equiv =_\parr \frac{1}{\boxed{=_\parr \frac{1}{A}} \parr \bot} \quad , \qquad \phi_2 \equiv =_\parr \frac{1}{\boxed{=_\parr \frac{1}{B}} \parr \bot} \qquad \text{and}$$

$$\psi \equiv \otimes\!\downarrow \frac{\left(= \frac{\bot \, a \, \bot}{\bot} \parr \bot \right) \otimes \boxed{\begin{array}{c} \phi \, \| \\ 1 \parr C \end{array}}}{\boxed{= \frac{\bot \otimes 1}{\bot}} \parr \bot \parr C} \quad ,$$

with $|\phi_1|_\parr = |\phi_2|_\parr = 0$.

If ϕ is tame, ψ is tame. Furthermore, ϕ_1 and ϕ_2 are equalities.

(9) We take

$$\phi_1 \equiv =_\parr \frac{1}{\boxed{=_\parr \frac{\bot}{A}} \parr 1} \quad , \qquad \phi_2 \equiv =_\parr \frac{1}{\boxed{=_\parr \frac{\bot}{B}} \parr 1} \qquad \text{and}$$

$$\psi \equiv \otimes\!\downarrow \frac{\left(= \frac{1\,a\,1}{1} \parr \bot\right) \otimes \boxed{\begin{array}{c}\phi \Vert \\ \bot \parr C\end{array}}}{\boxed{= \frac{1 \otimes \bot}{\bot}} \parr \bot \parr C} \quad ,$$

with $|\phi_1|_{\parr} = |\phi_2|_{\parr} = 0$.

If ϕ is tame, ψ is tame. Furthermore, ϕ_1 and ϕ_2 are equalities.

□

Note the big similarities in the case analysis for both clauses of the theorem. In fact, in the general splitting theorem we will provide a case analysis that holds for every connective.

To grasp the generalization, it is important to note that the base cases rely on the dualities in the equational theory. If A and B are equal to constants v and w respectively, there need to be dual constants $\bar{v}$ and $\bar{w}$ such that $v \parr \bar{v} = 1$ and $w \parr \bar{w} = 1$. Furthermore, tameness is preserved by splitting because of some properties of the interpretation map, most importantly those that allow us to guarantee the interpretability of the premiss in case 2.(4). These will be fundamental requirements for the generalised splitting theorem.

Shallow splitting tells us that from 'shallow' contexts where the main connective is $\parr$ we can follow occurrences of $\otimes$ and of the atoms up in the proof and obtain independent subproofs. We can now apply shallow splitting starting from the outermost occurrences of $\otimes$ or the atoms, and apply this process recursively on every subproof to obtain a series of nested subproofs that in a way make-up the original proof. We formalise this recursive process in the following theorem.

Definition 3.7. We say that a context $H\{\ \}$ is *provable* if $H\{1\} = 1$.

Definition 3.8. Given a context $S\{\ \}$ we define its *height* as the number of instances of $\otimes$ and a that $\{\ \}$ is in the scope of. We denote it by $|S|_{\parr}$.

Example 3.9. The height of $S\{\ \} = (\bot\ a\ (1 \otimes \{\ \})) \parr (1\ a\ \bot)$ is 2.

Theorem 3.10 (Context Reduction)**.** *For any formula A and any context S, given a proof* $\begin{array}{c}\phi \Vert \mathsf{SAMLLS}{\downarrow} \\ S\{A\}\end{array}$ *there exist a provable context $H\{\ \}$, a formula K and derivations*

$$\begin{array}{c}\zeta \Vert \mathsf{SAMLLS}{\downarrow} \\ K \parr A\end{array} \quad , \qquad \begin{array}{c} H\{K \parr \{\ \}\} \\ \chi \Vert \\ S\{\ \}\end{array} \quad ,$$

such that if ϕ is tame, then ζ is tame.

Furthermore, if $\{\ \}$ is not in the scope of an atom in $S\{\ \}$ and ϕ is tame, then χ is tame.

Proof. We proceed by induction on $|S|_{\parr}$.

- If $S\{A\} =_{\parr} A \parr K$, it is clear.

- If $S\{A\} =_{\parr} (S'\{A\} \otimes L) \parr M$, we apply Theorem 3.6. There exist Q_1, Q_2 and

$$\begin{array}{c} Q_1 \parr Q_2 \\ \psi \Vert \\ M \end{array} \quad , \quad \begin{array}{c} \phi_1 \Vert \\ S'\{A\} \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \Vert \\ L \parr Q_2 \end{array} \quad .$$

We apply the induction hypothesis to $S'\{A\} \parr Q_1$. There exist a provable context $H\{\ \}$, a formula K and derivations

$$\begin{array}{c} \zeta \Vert \mathsf{SAMLLS}^{\downarrow} \\ K \parr A \end{array} \quad , \quad \chi \equiv \otimes\!\downarrow \frac{\boxed{\begin{array}{c} H\{K \parr \{\ \}\} \\ \chi' \Vert \\ S'\{\ \} \parr Q_1 \end{array}} \otimes \boxed{\begin{array}{c} \phi_2 \Vert \\ L \parr Q_2 \end{array}}}{(S'\{\ \} \otimes L) \parr \boxed{\begin{array}{c} Q_1 \parr Q_2 \\ \psi \Vert \\ M \end{array}}} \quad .$$

We take $H\{\ \} \equiv H'\{\ \} \otimes 1$.

If ϕ is tame, then ζ is tame. If $\{\ \}$ is not in the scope of an atom in $S\{\ \}$ and ϕ is tame, then χ' is tame. Furthermore, ϕ_2 and ψ are tame, and therefore χ is tame.

- If $S\{A\} =_{\parr} (S'\{A\}\; a\; L) \parr M$, we apply Theorem 3.6. There exist Q_1, Q_2 and

$$\begin{array}{c} Q_1 \; a \; Q_2 \\ \psi \Vert \\ M \end{array} \quad , \quad \begin{array}{c} \phi_1 \Vert \\ S'\{A\} \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \Vert \\ L \parr Q_2 \end{array} \quad .$$

We apply the induction hypothesis to $S'\{A\} \parr Q_1$. There exist a provable context H', a formula K and derivations

$$\begin{array}{c} \zeta \Vert \mathsf{SAMLLS}^{\downarrow} \\ K \parr A \end{array} \quad , \quad \chi \equiv a\!\downarrow \frac{\boxed{\begin{array}{c} H'\{K \parr \{\ \}\} \\ \chi' \Vert \\ S'\{\ \} \parr Q_1 \end{array}} \; a \; \boxed{\begin{array}{c} \phi_2 \Vert \\ L \parr Q_2 \end{array}}}{(S'\{\ \} \otimes L) \parr \boxed{\begin{array}{c} Q_1 \; a \; Q_2 \\ \psi \Vert \\ M \end{array}}} \quad .$$

We take $H\{\ \} \equiv H'\{\ \}\; a\; 1$.

If ϕ is tame, then ζ is tame.

□

The splitting results are stronger than cut-elimination: they give us information about the structure of a proof and the 'pieces' from which it's built. Cut-elimination is a corollary of these results, stemming from our ability to rearrange these pieces in a way that suits us and still obtain a proof.

To show that the cut is admissible in a proof we will follow the relations a and $\otimes$ that take part in the cut to find what independent subproofs they belong to. We will then rearrange them in such a way that the cut is no longer needed.

For example, we consider the following simple proof:

$$\otimes\!\downarrow \frac{\boxed{a\!\downarrow \dfrac{\boxed{1 \parr \bot} \; a \; \boxed{\bot \parr 1}}{(1 \; a \; \bot) \parr (\bot \; a \; 1)}} \otimes \boxed{a\!\downarrow \dfrac{\boxed{\bot \parr 1} \; a \; \boxed{1 \parr \bot}}{(\bot \; a \; 1) \parr (1 \; a \; \bot)}}}{a\!\uparrow \dfrac{(1 \; a \; \bot) \otimes (\bot \; a \; 1)}{(1 \otimes \bot) \; a \; (\bot \otimes 1)} \parr (\bot \; a \; 1) \parr (1 \; a \; \bot)} \quad .$$

We follow the relations participating in the cut (in red) to find the boxed independent subproofs via context reduction and splitting. We can then rearrange them to obtain the following cut-free proof:

$$a\!\downarrow \frac{\boxed{\otimes\!\downarrow \dfrac{\boxed{1 \parr \bot} \otimes \boxed{\bot \parr 1}}{(1 \otimes \bot) \parr (\bot \parr 1)}} \; a \; \boxed{\otimes\!\downarrow \dfrac{\boxed{\bot \parr 1} \otimes \boxed{1 \parr \bot}}{(\bot \otimes 1) \parr 1 \parr \bot}}}{(1 \otimes \bot) \; a \; (\bot \otimes 1) \parr \boxed{a\!\downarrow \dfrac{(\bot \parr 1) \; a \; (1 \parr \bot)}{(\bot \; a \; 1) \parr (1 \; a \; \bot)}}} \quad .$$

Through the following corollary we will show that such a rearrangement is always possible, and therefore the cut is admissible.

Corollary 3.11 (Cut Elimination)**.** *For any formulae A, B, C, D, any context S and any proof*

$$\phi \equiv \begin{array}{c} \| \mathsf{SAMLLS}^{\downarrow} \\ S\left\{ a\!\uparrow \dfrac{(A \; a \; B) \otimes (C \; a \; D)}{(A \otimes C) \; a \; (B \otimes D)} \right\} \end{array} \quad ,$$

there is a proof

$$\begin{array}{c} \phi' \, \| \mathsf{SAMLLS}^{\downarrow} \\ S\{(A \otimes C) \; a \; (B \otimes D)\} \end{array} \quad .$$

Furthermore, if ϕ is tame then ϕ' is tame.

Proof. Given a proof $\begin{array}{c} \| \mathsf{SAMLLS}^{\downarrow} \\ S\{(A \; a \; B) \otimes (C \; a \; D)\} \end{array}$, we apply Theorem 3.10.

There exist a provable context H, a formula K and derivations

$$\begin{array}{c} \zeta \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ K \parr ((A \, a \, B) \otimes (C \, a \, D)) \end{array} \quad , \quad \begin{array}{c} H\{K \parr \{\ \}\} \\ \chi \,\Vert \\ S\{\ \} \end{array} \quad .$$

We apply Theorem 3.6 to ζ. There are formulae Q_1, Q_2 and derivations

$$\begin{array}{c} Q_1 \parr Q_2 \\ \psi \,\Vert \\ K \end{array} \quad , \quad \begin{array}{c} \phi_1 \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ (A \, a \, B) \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ (C \, a \, D) \parr Q_2 \end{array} \quad .$$

We apply Theorem 3.6 to ϕ_1. There are formulae Q_A, Q_B and derivations

$$\begin{array}{c} Q_A \, a \, Q_B \\ \psi_1 \,\Vert \\ Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_A \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ A \parr Q_A \end{array} \quad , \quad \begin{array}{c} \phi_B \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ B \parr Q_B \end{array} \quad .$$

We apply Theorem 3.6 to ϕ_2. There are formulae Q_C, Q_D and derivations

$$\begin{array}{c} Q_C \, a \, Q_D \\ \psi_2 \,\Vert \\ Q_2 \end{array} \quad , \quad \begin{array}{c} \phi_C \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ C \parr Q_C \end{array} \quad , \quad \begin{array}{c} \phi_D \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ D \parr Q_D \end{array} \quad .$$

Finally then, there exists a proof in $\mathsf{SAMLLS}^{\downarrow}$:

$$\phi' = \begin{array}{c} H\left\{ a\!\downarrow \dfrac{ \otimes\!\downarrow \dfrac{ \begin{array}{c} \phi_A \Vert \\ A \parr Q_A \end{array} \otimes \begin{array}{c} \phi_C \Vert \\ C \parr Q_C \end{array} }{(A \otimes C) \parr Q_A \parr Q_C} \; a \; \otimes\!\downarrow \dfrac{ \begin{array}{c} \phi_B \Vert \\ B \parr Q_B \end{array} \otimes \begin{array}{c} \phi_D \Vert \\ D \parr Q_D \end{array} }{(B \otimes D) \parr Q_B \parr Q_D} }{ ((A \otimes C) \, a \, (B \otimes D)) \parr \begin{array}{c} a\!\downarrow \dfrac{(Q_A \parr Q_C) \, a \, (Q_B \parr Q_D)}{ \begin{array}{c} Q_A \, a \, Q_B \\ \psi_1 \Vert \\ Q_1 \end{array} \parr \begin{array}{c} Q_C \, a \, Q_D \\ \psi_2 \Vert \\ Q_2 \end{array} } \\ \psi \Vert \\ K \end{array} } \right\} \\ \chi \,\Vert \\ S\{(Q_A \parr Q_C) \, a \, (Q_B \parr Q_D)\} \end{array} \quad .$$

If ϕ is tame, then $\{\ \}$ is not in the scope of an atom in $S\{\ \}$. Then ζ and χ are tame. ψ_1, ψ_2, ψ_3 are tame as well. ϕ_1 and ϕ_2 are equalities. Furthermore, since $(A \otimes C) \, a \, (B \otimes D)$ is interpretable, then $(A \otimes C)$ and $(B \otimes C)$ are of the form $1 \otimes 1$ or $\bot \otimes 1$. Therefore, the instances of $\otimes\!\downarrow$ are trivially of the form $\dfrac{1}{1}$ and can be replaced by equalities. ϕ' is then tame.

□

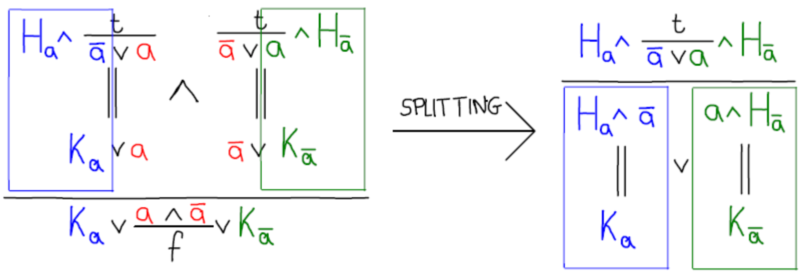
Note that in this last proof we have implicitly made use of the associativity and commutativity of $\parr$. In fact this will be a requirement in the generalised splitting theorem.

Since every proof of SMLLS corresponds to a tame proof in SAMLLS, the cut-free proof obtained from it will be tame and therefore interpretable. This cut-elimination procedure therefore corresponds to cut-elimination in SMLLS.

It is interesting to observe that at no point in the reasoning leading us to cut-elimination have we required formulae to be interpretable. Splitting and the admissibility of up-rules hold for the full subatomic language, and in particular for interpretable proofs.

3.2 General splitting

Splitting is based on a simple idea: to show that an atomic cut involving a and $\bar{a}$ is admissible, we follow a and $\bar{a}$ to the top of the derivation to find two independent subderivations, the premisses of which contain the dual of a and the dual of $\bar{a}$ respectively. In this way we obtain two proofs that don't interact above the cut, that we can rearrange to get a new cut-free proof.



Proofs of cut-elimination by splitting therefore rely on two main properties of a proof system: the *dualities* present in it to ensure that each of the independent subproofs contains the dual of an atom involved in the cut, and the *shape* of the linear rules ensuring that the two proofs remain independent above the cut. It is precisely a formal characterisation of these properties that we will provide, enabling us to understand *why* they are enough to guarantee cut-elimination. We therefore show how the interaction of linear rules and the cut affects cut-elimination.

Since the splitting proof consists on being able to follow relations through a proof to obtain the subproofs that compose it, its generalisation will be based on a characterisation of the relations that we can follow in such a way. In a system with only these relations, cut-elimination will be a mere corollary of splitting as is the case in $\mathsf{SAMLLS}^{\downarrow}$.

To follow a relation through the proof from the bottom to the top, we require their scope to widen. As we observed in $\mathsf{SAMLLS}^{\downarrow}$, the scope of $\otimes$ and a in the inference rules only widens when reading bottom-up. Accordingly, we will consider systems where the shape of the rules ensures the widening of the scope.

Notation 3.12. In what follows we will consider a subatomic system $\mathsf{SA}^{\downarrow}$ with set of formulae $\mathscr{F}$, set of relations $\mathscr{R}$, set of constants $\mathscr{U}$ and a natural interpretation I whose inference rules are *all down-rules.*

A proof in SA is a derivation with premiss $1 \in \mathscr{U}$.

Definition 3.13. We say that a relation α is *contractive* in $\mathsf{SA}^{\downarrow}$ if there is an inference rule

$$\frac{(A\ \alpha\ B)\ \ \nu\ \ (C\ \alpha\ \ D)}{(A\ \nu\ C)\ \ \alpha\ \ (B\ \nu^m\ D)}\quad \text{for some } \nu \in \mathscr{R}$$

in $\mathsf{SA}^{\downarrow}$.

Otherwise, we say that the relation α is *non-contractive.*

Example 3.14. In $\mathsf{SAMLLS}^{\downarrow}$ (Figure 3-1), $\otimes$ and a are non-contractive.

Example 3.15. In SAKS (Figure 2-1), a is contractive since in the rule ac its scope shrinks from bottom to top. Likewise, $\wedge$ is contractive.

In $\mathsf{SAMLLS}^{\downarrow}$ the only contractive relation is $\parr$. The property distinguishing $\parr$ from a and $\otimes$ is in fact that it is the *minimal* relation: it is the relation that appears in the excluded middle rules that introduce the dualities. In particular, the fact that $u \parr \bar{u} = 1$, for every constant u is fundamental to prove the base cases of Theorem 3.6. In every propositional system with an identity rule that introduces dualities there is such a distinguished relation. We will characterise *splittable systems*, i.e., systems with sufficient conditions to ensure cut-elimination through a splitting procedure.

In splittable systems, mimicking the case of MLL, we will require that all relations except for a distinguished relation $+$ be non-contractive so that we are able to follow them in a proof, and that there be a rule $u + \bar{u} = 1$ for every constant u.

Furthermore, when looking for the nested subproofs provided by context reduction in Theorem 3.10, we start from the outermost occurrence of a or $\otimes$ in the conclusion of a proof, and apply shallow splitting recursively. To piece together all the subproofs in such a way that we obtain a provable context, we can see that a fundamental property of a and $\otimes$ is that $1\ a\ 1 = 1$ and $1 \otimes 1 = 1$. In splittable systems we will follow the same procedure, and will therefore require that $1\ \alpha^M\ 1 = 1$ for every α.

Lastly, we implicitly made use of the associativity and commutativity of $\parr$. We will in the same way require associativity and commutativity of $+$.

Definition 3.16. A system $\mathsf{SA}^{\downarrow}$ is *splittable* if:

1. There is a strong relation $\times$ with unit 1 and dual $+$ with unit 0,

2. Every relation $\alpha \neq +$ is non-contractive,

3. There is a constant assignment $u + \bar{u} = 1$ for every unit $u \in \mathscr{U}$,

4. $+$ is associative and commutative,

5. $1\ \alpha^M\ 1 = 1$ for every α.

$$a\!\downarrow \frac{(A \vee B)\ a\ (C \vee D)}{(A\ a\ C) \vee (B\ a\ D)}$$

$$\wedge\!\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)}$$

Figure 3-2: $\mathsf{SAKS}^{\downarrow}$

$$a\!\downarrow \frac{(A \parr B)\ a\ (C \parr D)}{(A\ a\ C) \parr (B\ a\ D)}$$

$$\lhd\!\downarrow \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$$\otimes\!\downarrow \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)}$$

Figure 3-3: *Systems* $\mathsf{SABVU}^{\downarrow}$ *and* $\mathsf{SABV}^{\downarrow}$

Example 3.17. $\mathsf{SAMLLS}^{\downarrow}$ is splittable, and the minimal relation + introducing dualities is $\parr$.

Example 3.18. The linear down fragment of classical logic $\mathsf{SAKS}^{\downarrow}$ of Figure 3-2 together with the equality rules corresponding to the axioms of example 2.7 is splittable. The minimal relation + introducing dualities is $\vee$.

Example 3.19. The down fragment of SABVU given in Figure 3-3 $\mathsf{SABVU}^{\downarrow}$ together with the equality rules corresponding to the axioms of example 2.9 is splittable. The minimal relation + introducing dualities is $\parr$.

Likewise, the down fragment of SABV given in the same figure is splittable.

Remark 3.20. From condition 3 in Definition 3.16 and the closure of = under negation, $\times$ is associative and commutative.

Notation 3.21. As all relations $\alpha \neq +$ are non-contractive, all the inference rules of a splittable system are of the form

$$\alpha\!\downarrow \frac{(A + B)\ \alpha\ (C + D)}{(A\ \alpha\ C) + (B\ \alpha^m\ D)} \quad .$$

We denote this rule by $\alpha\!\downarrow$.

The idea behind the generalisation of splitting is simple: if a relation α is non-contractive, its scope only widens when following it from the bottom to the top of a proof. Therefore, given a proof

$$\begin{array}{c} \phi \| \\ S\{A\ \alpha\ B\} \end{array} \quad ,$$

we can follow α all the way to the top of π we will find that its scope only widens and that ϕ is of the form

$$\alpha\downarrow \frac{\boxed{\begin{array}{c}\Vert\\ A + Q_1\end{array}} \;\alpha\; \boxed{\begin{array}{c}\Vert\\ B + Q_2\end{array}}}{\boxed{\begin{array}{c}(A \;\alpha\; B) + (Q_1 \;\alpha^m\; Q_2)\\ \Vert\\ S\{A \;\alpha\; B\}\end{array}}} \quad .$$

In other words, the proof ϕ splits into two subproofs that have no interaction above $\alpha\downarrow$.

We will obtain the admissibility of certain rules as a corollary of splitting. In particular, we will show that the subatomic rule that corresponds to the atomic cut rule is admissible. To prove that this result corresponds to cut-elimination in the original systems, we will need to show that the cut-free proofs obtained from proofs of the non-subatomic original system via this procedure are interpretable themselves, and therefore correspond to proofs in the original system. For that, we will pay particular attention to tame proofs, in which no inference rule occurs in the scope of an atom. If the interpretation I is built in a natural way, every proof of the original system will be represented by a tame proof in SA. The interpretability of tame proofs is preserved by splitting as long as interpretability is preserved by duals. In that case, as a corollary, interpretability will be preserved by the cut-elimination procedure.

Definition 3.22. We define $=_+$ as the equivalence relation on formulae defined by the axioms for the associativity, commutativity, unit of $+$ and constant assignments for $+$. We define the equivalence relation $=_+$ on derivations following Definition 2.32.

Definition 3.23. We say that a system SA with a natural interpretation I, negation $\bar{\cdot}$ and an equational theory $=$ is *preservable* when:

1. If A is interpretable and $A =_+ B$, then B is interpretable ;
2. If $A \;\alpha\; B$ is interpretable, $\alpha \in \mathscr{R}$, then A and B are interpretable ;
3. If $A \;a\; B$ is interpretable and $A + A' = 1$, $B + B' = 1$ then $A' \;a\; B'$ is interpretable for $a \in \mathscr{A}$;
4. If A is interpretable, then $\overline{A}$ is interpretable ;
5. The atoms of $\mathscr{A}$ are non-commutative, non-associative and non-unitary.

These conditions ensure that interpretability is preserved by duality, meaning that if an instance of a rule is interpretable, the same rule instantiated with the duals of the formulae involved is interpretable as well.

The proof of the splitting result is done in two steps for ease of reading: shallow splitting and context reduction, just as in the example in Section 3.1. As noted in [21] and in [36], the main difficulty of splitting is finding the right induction measure for every system. In the literature, each splitting theorem for each proof system uses a different induction measure tailored specifically for it. By providing a general splitting theorem, we not only give a formal definition of what a splitting theorem is, but also give a new one-size-fits-all induction measure that works for every splittable system, taking the search for an induction measure out of the process for designing a proof system.

Lemma 3.24. *If* $\mathsf{SA}^{\downarrow}$ *is splittable, then for every proof*

$$\begin{array}{c} \phi \,\Vert\, \mathsf{SA}^{\downarrow} \\ u + C \end{array}$$

where $u \in \mathcal{U}$*, there is a derivation*

$$\begin{array}{c} \bar{u} \\ \psi \,\Vert\, \mathsf{SA}^{\downarrow} \\ C \end{array} .$$

Furthermore, if $\mathsf{SA}^{\downarrow}$ *is preservable, then if* ϕ *is tame we have that* ψ *is tame.*

Proof. We take

$$\psi \equiv {}^{\times\downarrow}\frac{(\bar{u} + 0) \times \boxed{\begin{array}{c} \phi \,\Vert \\ u + C \end{array}}}{\boxed{=\frac{\bar{u} \times u}{0}} + 0 + C} .$$

□

Definition 3.25. Given a derivation ϕ, we define the *length* of ϕ as the number of rules in ϕ different from the equality rules for the associativity and commutativity of $+$, the unit rule for $+$ and the unit assignments for $+$. We denote it by $|\phi|_+$.

It is straightforward that if $\phi =_+ \psi$, then $|\phi|_+ = |\psi|_+$. It is clear as well that if SA is preservable and ϕ is tame, then ψ is as well, since interpretability is preserved by $=_+$ and we cannot add or remove non-equality rules in the scope of atoms from a formula through the equalities of $=_+$.

Notation 3.26. We will abuse notation and refer to derivations made up only of equality rules rules as *equalities.*

Theorem 3.27 (Shallow Splitting)**.** *If* $\mathsf{SA}^{\downarrow}$ *is splittable, for every formulae* A*,* B*,* C*, for every relation* $\alpha \neq +$*, for every proof*

$$\begin{array}{c} \phi \,\Vert\, \mathsf{SA}^{\downarrow} \\ (A \;\alpha\; B) + C \end{array}$$

there exist formulae Q_1, Q_2 *and derivations*

$$\begin{array}{c} Q_1 \; \bar{\alpha} \; Q_2 \\ \psi \, \Vert \, \mathsf{SA}^{\downarrow} \\ C \end{array} \quad , \quad \begin{array}{c} \phi_1 \, \Vert \, \mathsf{SA}^{\downarrow} \\ A + Q_1 \end{array} \quad \textit{and} \quad \begin{array}{c} \phi_2 \, \Vert \, \mathsf{SA}^{\downarrow} \\ B + Q_2 \end{array} \quad ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\phi|_+$.

If $SA^{\downarrow}$ *is preservable and* ϕ *is tame, then* ϕ_1, ϕ_2 *and* ψ *are tame. Furthermore, if* α *is an atom then* ϕ_1 *and* ϕ_2 *are equalities.*

Proof. Given a proof ϕ in SA of $(A \; \alpha \; B) + C$ we reduce it to CoS notation for $=_+$. We will proceed by induction on $|\phi|_+$.

If $|\phi|_+ = 1$, then $A =_+ v, B =_+ w$ and $v \; \alpha \; w =_+ u$, with $u + C =_+ 1$. By Lemma 3.24, there is a derivation $\begin{array}{c} \bar{u} \\ \psi' \, \Vert \, \mathsf{SA}^{\downarrow} \\ C \end{array}$ and we take:

$$\psi \equiv \begin{array}{c} = \dfrac{\bar{v} \; \bar{\alpha} \; \bar{w}}{\bar{u}} \\ \psi' \, \Vert \\ C \end{array} \quad , \quad \phi_1 \equiv =_+ \dfrac{1}{\boxed{=_+ \dfrac{v}{A}} + \bar{v}} \quad \text{and} \quad \phi_2 \equiv =_+ \dfrac{1}{\boxed{=_+ \dfrac{w}{B}} + \bar{w}} \quad .$$

ψ' is tame and $\bar{v} \; \bar{\alpha} \; \bar{w}$ is interpretable, and therefore ψ is tame. Furthermore, ϕ_1 and ϕ_2 are tame and equalities.

If $|\phi|_+ = |\phi'|_+ > 1$, we prove the inductive step for all the possible cases of the bottom inference rule ρ of ϕ.

Inspection of the rules provides us with the following possible cases:

(1) $$\phi =_+ \begin{array}{c} \phi' \, \Vert \, \mathsf{SA}^{\downarrow} \\ \rho \dfrac{(A \; \alpha \; B) + C'}{(A \; \alpha \; B) + C} \end{array} \quad ;$$

(2) $$\phi =_+ \begin{array}{c} \phi' \, \Vert \, \mathsf{SA}^{\downarrow} \\ \times\!\downarrow \dfrac{(((A \; \alpha \; B) + C_1) \times (C_2 + C_3)) + C_4}{(A \; \alpha \; B) + C_2 + (C_1 \times C_3) + C_4} \end{array} \quad ;$$

(3) $$\phi =_+ \begin{array}{c} \phi' \, \Vert \, \mathsf{SA}^{\downarrow} \\ = \dfrac{(((A \; \alpha \; B) + C_1) \; \beta \; u_\beta) + C_2}{(A \; \alpha \; B) + C_1 + C_2} \end{array} \quad ;$$

(4) $$\phi =_+ \begin{array}{c} \phi' \, \Vert \, \mathsf{SA}^{\downarrow} \\ = \dfrac{(u_\beta \; \beta \; ((A \; \alpha \; B) + C_1)) + C_2}{(A \; \alpha \; B) + C_1 + C_2} \end{array} \quad ;$$

(5) $$\phi =_{+} {}_{\rho}\overset{\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}}{\dfrac{(A' \;\alpha\; B) + C}{(A \;\alpha\; B) + C}} \qquad ;$$

(6) $$\phi =_{+} {}_{\rho}\overset{\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}}{\dfrac{(A \;\alpha\; B') + C}{(A \;\alpha\; B) + C}} \qquad ;$$

(7) $$\phi =_{+} {}_{\alpha\downarrow}\overset{\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}}{\dfrac{((A + C_1) \;\alpha\; (B + C_2)) + C_3}{((A \;\alpha\; B) + (C_1 \;\overline{\alpha}\; C_2)) + C_3}} \qquad \text{if } \alpha \text{ is strong ;}$$

(8) $$\phi =_{+} {}_{\alpha\downarrow}\overset{\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}}{\dfrac{((A + C_1) \;\overline{\alpha}\; (B + C_2)) + C_3}{((A \;\alpha\; B) + (C_1 \;\overline{\alpha}\; C_2)) + C_3}} \qquad \text{if } \alpha \text{ is weak ;}$$

(9) $$\phi =_{+} {}_{\alpha\downarrow}\overset{\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}}{\dfrac{((A + C_1) \;\alpha\; (B + C_2)) + C_3}{((A \;\alpha\; B) + (C_1 \;\alpha\; C_2)) + C_3}} \qquad \text{if } \alpha \text{ is weak ;}$$

(10) $$\phi =_{+} {}_{=}\overset{\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}}{\dfrac{(B \;\alpha\; A) + C}{(A \;\alpha\; B) + C}} \qquad \text{if } \alpha \text{ is commutative ;}$$

(11) $$\phi =_{+} {}_{=}\overset{\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}}{\dfrac{((A \;\alpha\; B_1) \;\alpha\; B_2) + C}{(A \;\alpha\; (B_1 \;\alpha\; B_2)) + C}} \qquad \text{if } \alpha \text{ is associative ;}$$

(12) $$\phi =_{+} {}_{=}\overset{\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}}{\dfrac{(A_1 \;\alpha\; (A_2 \;\alpha\; B)) + C}{((A_1 \;\alpha\; A_2) \;\alpha\; B) + C}} \qquad \text{if } \alpha \text{ is associative ;}$$

(13) $$\phi =_{+} {}_{=}\overset{\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}}{\dfrac{A + C}{(A \;\alpha\; u_{\alpha}) + C}} \qquad \text{if } \alpha \text{ is unitary, with } B =_{+} u_{\beta} \text{ ;}$$

(14) $$\phi =_{+} {}_{=}\overset{\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}}{\dfrac{B + C}{(u_{\alpha} \;\alpha\; B) + C}} \qquad \text{if } \alpha \text{ is unitary, with } A =_{+} u_{\beta} \text{ ;}$$

(15) $$\phi =_{+} {}_{=}\overset{\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}}{\dfrac{u + C}{(v \;\alpha\; w) + C}} \qquad \text{with } A =_{+} v \text{ and } B =_{+} w \text{ .}$$

We proceed as follows:

(1) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$.

There are derivations

$$\psi =_+ \begin{array}{c} Q_1 \;\overline{\alpha}\; Q_2 \\ {\scriptstyle \psi' \Vert \mathsf{SA}^{\downarrow}} \\ C' \\ \rho\,\overline{} \\ C \end{array} \quad , \qquad \begin{array}{c} {\scriptstyle \phi_1 \overline{\Vert} \mathsf{SA}^{\downarrow}} \\ A + Q_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} {\scriptstyle \phi_2 \overline{\Vert} \mathsf{SA}^{\downarrow}} \\ B + Q_2 \end{array} \quad ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\phi|_+ < |\phi|_+$.

If ϕ is tame, then ρ and ϕ_1, ϕ_2 and ψ' are tame. Hence ψ is tame.

Furthermore, if α is an atom then by the induction hypothesis ϕ_1 and ϕ_2 are equalities.

(2) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$.

There are derivations

$$\psi =_+ \begin{array}{c} H_1 + H_2 \\ {\scriptstyle \psi' \Vert \mathsf{SA}^{\downarrow}} \\ C_4 \end{array} \quad , \qquad \begin{array}{c} {\scriptstyle \omega_1 \overline{\Vert} \mathsf{SA}^{\downarrow}} \\ (A \;\alpha\; B) + C_1 + H_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} {\scriptstyle \omega_2 \overline{\Vert} \mathsf{SA}^{\downarrow}} \\ C_2 + C_3 + H_2 \end{array} \quad ,$$

with $|\omega_1|_+ + |\omega_2|_+ \leq |\phi''|_+$.

If ϕ is tame, then ϕ' is tame and ω_1, ω_2 and ψ' are tame.

We apply the induction hypothesis to ω_1 as $|\omega_1|_+ \leq |\phi'|_+ < |\phi|_+$.

There are derivations

$$\begin{array}{c} Q_1 \;\overline{\alpha}\; Q_2 \\ {\scriptstyle \psi'' \Vert \mathsf{SA}^{\downarrow}} \\ C_1 + H_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1 \overline{\Vert} \mathsf{SA}^{\downarrow}} \\ A + Q_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2 \overline{\Vert} \mathsf{SA}^{\downarrow}} \\ B + Q_2 \end{array} \quad ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\omega_1|_+ < |\phi|_+$.

We take:

$$\psi =_+ \times\!\downarrow \frac{\boxed{\begin{array}{c} Q_1 \;\overline{\alpha}\; Q_2 \\ {\scriptstyle \psi'' \Vert} \\ C_1 + H_1 \end{array}} \times \boxed{\begin{array}{c} {\scriptstyle \omega_2 \overline{\Vert}} \\ C_2 + C_3 + H_2 \end{array}}}{(C_1 \times C_3) + C_2 + \boxed{\begin{array}{c} H_1 + H_2 \\ {\scriptstyle \psi' \Vert} \\ C_4 \end{array}}} \quad .$$

If ϕ is tame, then ω_1 is tame and ϕ_1, ϕ_2 and ψ'' are tame. ψ' and ω_2 are tame as well, and since I is preservable, C_1, C_2, C_3 are interpretable. Therefore ψ is tame.

Furthermore, if α is an atom then by the induction hypothesis ϕ_1 and ϕ_2 are equalities.

(3) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} H_1 \; \overline{\beta} \; H_2 \\ \psi' \, \| \, \mathsf{SA}^{\downarrow} \\ C_2 \end{array} \quad , \quad \begin{array}{c} \omega_1 \, \overline{\|} \, \mathsf{SA}^{\downarrow} \\ (A \; \alpha \; B) + C_1 + H_1 \end{array} \quad , \quad \begin{array}{c} \omega_2 \, \overline{\|} \, \mathsf{SA}^{\downarrow} \\ u_\beta + H_2 \end{array} \quad ,$$

with $|\omega_1|_+ + |\omega_2|_+ \leq |\phi'|_+$.

By Lemma 3.24, there is a derivation

$$\begin{array}{c} \bar{u}_\beta \\ \psi'' \, \| \, \mathsf{SA}^{\downarrow} \\ H_2 \end{array} \quad .$$

If ϕ is tame, then ω_2 is tame and thus ψ'' is tame.

We apply the induction hypothesis to ω_1 as $|\omega_1|_+ \leq |\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} Q_1 \; \overline{\alpha} \; Q_2 \\ \psi''' \, \| \, \mathsf{SA}^{\downarrow} \\ C_1 + H_1 \end{array} \quad , \quad \begin{array}{c} \phi_1 \, \overline{\|} \, \mathsf{SA}^{\downarrow} \\ A + Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \, \overline{\|} \, \mathsf{SA}^{\downarrow} \\ B + Q_2 \end{array} \quad ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\omega_1|_+ < |\phi|_+$.

We take:

$$\psi =_+ \begin{array}{c} Q_1 \; \overline{\alpha} \; Q_2 \\ \psi''' \, \| \\ C_1 + \boxed{\begin{array}{c} H_1 \; \overline{\beta} \; \boxed{\begin{array}{c} \bar{u}_\beta \\ \psi'' \, \| \\ H_2 \end{array}} \\ \psi' \, \| \\ C_2 \end{array}} \end{array} \quad .$$

Atoms are not unitary, and thus β is not an atom. If ϕ is tame, then ω_1 is tame and ϕ_1, ϕ_2 and ψ''' are tame. ψ'' and ψ' are tame as well, and hence ψ is tame.

Furthermore, if α is an atom then by the induction hypothesis ϕ_1 and ϕ_2 are equalities.

(4) This case is analogous to (3).

(5) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} Q_1\ \alpha^m\ Q_2 \\ \scriptstyle \psi \,\Vert\, \mathsf{SA}^{\downarrow} \\ C \end{array}, \qquad \phi_1 \equiv \boxed{\begin{array}{c}\scriptstyle \phi_1' \,\overline{\Vert}\, \mathsf{SA}^{\downarrow} \\ \rho\,\dfrac{A'}{A}\end{array}} + Q_1 \qquad \text{and} \qquad \begin{array}{c} \scriptstyle \phi_2 \,\overline{\Vert}\, \mathsf{SA}^{\downarrow} \\ B + Q_2 \end{array},$$

with $|\phi_1'|_+ + |\phi_2|_+ \leq |\phi'|_+$.

We have $|\phi_1|_+ + |\phi_2|_+ = |\phi_1'|_+ + 1 + |\phi_2|_+ \leq |\phi'|_+ + 1 = |\phi|_+$.

If ϕ is tame, then ϕ' is tame and ϕ_1', ϕ_2 and ψ are tame. ρ is tame as well, and thus ϕ_1 is tame.

Furthermore, if α is an atom the only allowed instances of ρ are equalities and ϕ_1' is an equality, and thus ϕ_1 is an equality. By induction hypothesis, ϕ_2 is an equality.

(6) This case is analogous to (5).

(7) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} H_1\ \overline{\alpha}\ H_2 \\ \scriptstyle \psi' \,\Vert\, \mathsf{SA}^{\downarrow} \\ C_3 \end{array}, \qquad \begin{array}{c} \scriptstyle \phi_1 \,\overline{\Vert}\, \mathsf{SA}^{\downarrow} \\ A + C_1 + H_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} \scriptstyle \phi_2 \,\overline{\Vert}\, \mathsf{SA}^{\downarrow} \\ B + C_2 + H_2 \end{array},$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\phi'|_+ < |\phi|_+$.

We take $Q_1 \equiv C_1 + H_1$, $Q_2 \equiv C_2 + H_2$ and

$$\psi =_+ \ \overline{\alpha}{\downarrow}\,\frac{(C_1 + H_1)\ \overline{\alpha}\ (C_2 + H_2)}{(C_1\ \overline{\alpha}\ C_2) + \boxed{\begin{array}{c} H_1\ \overline{\alpha}\ H_2 \\ \scriptstyle \psi' \,\Vert \\ C_3 \end{array}}}\ .$$

If ϕ is tame, then ϕ' is tame and by induction hypothesis ϕ_1, ϕ_2 and ψ' are tame.

If α is an atom, then by the induction hypothesis ϕ_1 and ϕ_2 are equalities. Then $(C_1 + H_1)\ \overline{\alpha}\ (C_2 + H_2)$ is interpretable by condition 3 of preservability. Therefore, ψ is tame.

If ϕ is tame and α is not an atom, then ψ is trivially tame since C_1, H_1, C_2, H_2 are interpretable and ψ' is tame.

(8) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} H_1 \; \alpha \; H_2 \\ \psi' \, \| \, \mathsf{SA}^{\downarrow} \\ C_3 \end{array} , \quad \begin{array}{c} \phi_1 \, \|\, \mathsf{SA}^{\downarrow} \\ A + C_1 + H_1 \end{array} \quad \text{and} \quad \begin{array}{c} \phi_2 \, \| \, \mathsf{SA}^{\downarrow} \\ B + C_2 + H_2 \end{array} ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\phi'|_+ < |\phi|_+$.

We take $Q_1 \equiv C_1 + H_1$, $Q_2 \equiv C_2 + H_2$ and

$$\psi =_+ \quad \overline{\alpha}{\downarrow} \frac{(C_1 + H_1) \; \overline{\alpha} \; (C_2 + H_2)}{(C_1 \; \overline{\alpha} \; C_2) + \boxed{\begin{array}{c} H_1 \; \alpha \; H_2 \\ \psi' \, \| \\ F_3 \end{array}}} .$$

If ϕ is tame, then ϕ' is tame and by induction hypothesis ϕ_1, ϕ_2 and ψ' are tame.

If α is an atom, then by the induction hypothesis ϕ_1 and ϕ_2 are equalities. Then $(C_1 + H_1) \; \overline{\alpha} \; (C_2 + H_2)$ is interpretable by condition 3 of preservability. Therefore, ψ is tame.

If ϕ is tame and α is not an atom, then ψ is trivially tame since C_1, H_1, C_2, H_2 are interpretable and ψ' is tame.

(9) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} H_1 \; \overline{\alpha} \; H_2 \\ \psi' \, \| \, \mathsf{SA}^{\downarrow} \\ F_3 \end{array} , \quad \begin{array}{c} \phi_1 \, \|\, \mathsf{SA}^{\downarrow} \\ A + C_1 + H_1 \end{array} \quad \text{and} \quad \begin{array}{c} \phi_2 \, \| \, \mathsf{SA}^{\downarrow} \\ B + C_2 + H_2 \end{array} ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\phi'|_+ < |\phi|_+$.

We take $Q_1 \equiv C_1 + H_1$, $Q_2 \equiv C_2 + H_2$ and

$$\psi =_+ \quad \alpha^m{\downarrow} \frac{(C_1 + H_1) \; \overline{\alpha} \; (C_2 + H_2)}{(C_1 \; \alpha \; C_2) + \boxed{\begin{array}{c} H_1 \; \overline{\alpha} \; H_2 \\ \psi' \, \| \\ C_3 \end{array}}} .$$

If ϕ is tame, then ϕ' is tame and by induction hypothesis ϕ_1, ϕ_2 and ψ' are tame.

If α is an atom, then by the induction hypothesis ϕ_1 and ϕ_2 are equalities. Then $(C_1 + H_1) \; \overline{\alpha} \; (C_2 + H_2)$ is interpretable by condition 3 of preservability. Therefore, ψ is tame.

If ϕ is tame and α is not an atom, then ψ is trivially tame since C_1, H_1, C_2, H_2 are interpretable and ψ' is tame.

(10) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} H_1 \mathbin{\overline{\alpha}} H_2 \\ {\scriptstyle \psi'} \Vert {\scriptstyle \mathsf{SA}^{\downarrow}} \\ C \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \omega_1} \Vert {\scriptstyle \mathsf{SA}^{\downarrow}} \\ B + H_1 \end{array} \quad \text{and} \quad \begin{array}{c} {\scriptstyle \omega_2} \Vert {\scriptstyle \mathsf{SA}^{\downarrow}} \\ A + H_2 \end{array} \quad ,$$

with $|\omega_1|_+ + |\omega_2|_+ \leq |\phi'|_+$.

We take $Q_1 \equiv H_2$, $Q_2 \equiv H_1$, $\phi_1 \equiv \omega_2$, $\phi_2 \equiv \omega_1$ and

$$\psi \equiv \begin{array}{c} \boxed{= \dfrac{H_2 \mathbin{\overline{\alpha}} H_1}{H_1 \mathbin{\overline{\alpha}} H_2}} \\ {\scriptstyle \psi'} \Vert \\ C \end{array} \quad .$$

Atoms are not commutative and thus α is not an atom.

If ϕ is tame, then ϕ' is tame and by induction hypothesis ψ_1, ψ_2 and ψ' are tame. Then H_1 and H_2 are interpretable and hence ψ is tame as well.

(11) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} H_1 \mathbin{\overline{\alpha}} H_2 \\ {\scriptstyle \psi'} \Vert {\scriptstyle \mathsf{SA}^{\downarrow}} \\ C \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \omega_1} \Vert {\scriptstyle \mathsf{SA}^{\downarrow}} \\ (A \mathbin{\alpha} B_1) + H_1 \end{array} \quad \text{and} \quad \begin{array}{c} {\scriptstyle \omega_2} \Vert {\scriptstyle \mathsf{SA}^{\downarrow}} \\ B_2 + H_2 \end{array} \quad ,$$

with $|\omega_1|_+ + |\omega_2|_+ \leq |\phi'|_+$.

If ϕ is tame, then ϕ' is tame and by induction hypothesis ω_1, ω_2 and ψ' are tame.

We apply the induction hypothesis to ω_1 as $|\omega_1|_+ \leq |\phi'|_+ < |\phi|_+$. There are

$$\begin{array}{c} Q_1 \mathbin{\overline{\alpha}} H_3 \\ {\scriptstyle \psi''} \Vert {\scriptstyle \mathsf{SA}^{\downarrow}} \\ H_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1} \Vert {\scriptstyle \mathsf{SA}^{\downarrow}} \\ A + Q_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \omega_3} \Vert {\scriptstyle \mathsf{SA}^{\downarrow}} \\ B_1 + H_3 \end{array} \quad ,$$

with $|\phi_1|_+ + |\omega_3|_+ \leq |\omega_1|_+$.

We take $Q_2 \equiv H_3 \mathbin{\overline{\alpha}} H_2$ and

$$\phi_2 \equiv \;{\scriptstyle \alpha\downarrow}\frac{\boxed{\begin{array}{c}{\scriptstyle \omega_3}\,\Vert \\ B_1 + H_3\end{array}}\;\alpha^M\;\boxed{\begin{array}{c}{\scriptstyle \omega_2}\,\Vert \\ B_2 + H_2\end{array}}}{(B_1\;\alpha\;B_2) + (H_3\;\overline{\alpha}\;H_2)} \quad , \quad \psi \equiv \begin{array}{c}\boxed{{\scriptstyle =}\frac{Q_1\;\overline{\alpha}\;(H_3\;\overline{\alpha}\;H_2)}{\boxed{\begin{array}{c}Q_1\;\overline{\alpha}\;H_3 \\ {\scriptstyle \psi''}\,\Vert \\ H_1\end{array}}\;\overline{\alpha}\;H_2}} \\ {\scriptstyle \psi'}\,\Vert \\ C\end{array} \quad .$$

We have $|\phi_1|_+ + |\phi_2|_+ = |\phi_1|_+ + |\omega_3|_+ + |\omega_2|_+ + 1 \leq |\omega_1|_+ + |\omega_2|_+ + 1 \leq |\phi'|_+ + 1 = |\phi|_+$.

Atoms are not associative, thus α is not an atom. If ϕ is tame, then ω_2, ω_3, ψ' and ψ'' are tame and so Q_1, H_2, H_3 are interpretable. Therefore ϕ_1, ϕ_2 and ψ are tame.

(12) This case is analogous to (11).

(13) We take $Q_1 \equiv C$, $Q_2 \equiv \bar{u}_\alpha$ and

$$\psi \equiv {\scriptstyle =}\frac{C\;\overline{\alpha}\;\bar{u}_\alpha}{C} \quad , \quad \phi_1 \equiv \begin{array}{c}{\scriptstyle \phi'}\,\Vert \\ A + C\end{array} \quad , \phi_2 \equiv {\scriptstyle =_+}\frac{1}{\boxed{{\scriptstyle =_+}\frac{u_\alpha}{\overline{\overline{B}}}} + \bar{u}_\alpha} \quad .$$

Then, $|\phi_1|_+ + |\phi_2|_+ = |\phi'|_+ < |\phi|_+$.

If ϕ is tame, then C is interpretable and ϕ' is tame, and therefore ϕ_1, ϕ_2 and ψ are tame.

(14) This case is analogous to (13).

(15) By Lemma 3.24, there is a derivation $\begin{array}{c}\bar{u} \\ {\scriptstyle \psi'}\,\Vert\,{\scriptstyle \mathsf{SA}\downarrow} \\ C\end{array}$ and we take:

$$\psi \equiv \begin{array}{c}{\scriptstyle =}\frac{\bar{v}\;\overline{\alpha}\;\bar{w}}{\bar{u}} \\ {\scriptstyle \psi'}\,\Vert \\ C\end{array} \quad , \quad \phi_1 \equiv {\scriptstyle =_+}\frac{1}{\boxed{{\scriptstyle =_+}\frac{v}{\overline{\overline{A}}}} + \bar{v}} \quad \text{and} \quad \phi_2 \equiv {\scriptstyle =_+}\frac{1}{\boxed{{\scriptstyle =_+}\frac{w}{\overline{\overline{B}}}} + \bar{w}} \quad .$$

If ϕ is tame, then ψ' is tame and ϕ_1 and ϕ_2 are tame. Since $v\;\alpha\;w$ is interpretable, by condition 4 of preservability $\bar{v}\;\overline{\alpha}\;\bar{w}$ is interpretable. Therefore ψ is tame. Furthermore, ϕ_1 and ϕ_2 are equalities.

□

We can see that shallow splitting hinges precisely on the non-contractiveness of relations and on the duality between constants.

Remark 3.28. The requirement for + to be associative and commutative can be relaxed, with the condition that the rule $\times\!\downarrow$ be restricted in such a way that it corresponds to two rules

$$\frac{(A+B)\times C}{(A\times C)+B} \quad \text{and} \quad \frac{A\times(B+C)}{B+(A\times C)} \quad .$$

Since all relations are non-contractive, we can apply shallow splitting to the outermost relation in any context S, and continue applying it inductively to split any proof completely. This process is formalised in the following Theorem 3.29, which is a generalisation of Theorem 4.1.5 in [21].

Theorem 3.29 (Context Reduction)**.** *Let* $\mathsf{SA}^{\downarrow}$ *be a splittable system. For any formula* A *and for any context* $S\{\ \}$*, given a proof* $\begin{array}{c} \phi \| \mathsf{SA}^{\downarrow} \\ S\{A\} \end{array}$*, there exist a formula* K*, a provable context* $H\{\ \}$ *and derivations*

$$\begin{array}{c} \zeta \| \mathsf{SA}^{\downarrow} \\ A+K \end{array} \quad \textit{and} \quad \begin{array}{c} H\{\{\ \}+K\} \\ \chi \| \mathsf{SA}^{\downarrow} \\ S\{\ \} \end{array}$$

such that if ϕ *is tame, then* ζ *is tame.*

Furthermore, if $\{\ \}$ *is not in the scope of an atom in* $S\{\ \}$ *and* ϕ *is tame, then* χ *is tame.*

Proof. We proceed by induction on the number of relations $\alpha \neq +$ that $\{\ \}$ is in the scope of in $S\{\ \}$. We denote it by $|S|_+$.

If $|S|_+ = 0$, then $S\{A\} =_+ A + K$ and we take $\zeta =_+ \phi$ and $H\{\ \} = \{\ \}$.

If $S\{A\} =_+ (S'\{A\}\ \beta\ B) + C$ with $\beta \neq +$, we apply Theorem 3.27 to ϕ. There exist derivations

$$\begin{array}{c} Q_1\ \overline{\beta}\ Q_2 \\ \psi \| \mathsf{SA}^{\downarrow} \\ C \end{array} \quad , \quad \begin{array}{c} \phi_1 \| \mathsf{SA}^{\downarrow} \\ S'\{A\}+Q_1 \end{array} \quad \text{and} \quad \begin{array}{c} \phi_2 \| \mathsf{SA}^{\downarrow} \\ B+Q_2 \end{array}$$

such that ϕ_1, ϕ_2 and ψ are tame if ϕ is tame.

We apply the induction hypothesis to ϕ_1 since $|S'|_+ < |S|_+$. There are derivations

$$\begin{array}{c} \zeta \| \mathsf{SA}^{\downarrow} \\ A+K \end{array} \quad , \quad \begin{array}{c} H'\{\{\ \}+K\} \\ \chi' \| \mathsf{SA}^{\downarrow} \\ S'\{\ \}+Q_1 \end{array} \quad ,$$

with H' a provable context, such that ζ is tame if ϕ_1 is tame.

We take $H\{\ \} = H'\{\ \}\ \beta^M\ 1$. We have $H\{1\} = H'\{1\}\ \beta^M\ 1 = 1\ \beta^M\ 1 = 1$, and we can build in $\mathsf{SA}^{\downarrow}$

$$\chi \equiv \beta\!\downarrow \frac{\boxed{\begin{array}{c} H'\{\{\ \} + K\} \\ \chi' \Vert \\ S'\{\ \} + Q_1 \end{array}}\ \beta^M\ \boxed{\begin{array}{c} \phi_2 \Vert \\ B + Q_2 \end{array}}}{(S'\{\ \}\ \beta\ B) + \boxed{\begin{array}{c} Q_1\ \overline{\beta}\ Q_2 \\ \psi \Vert \\ C \end{array}}} \quad .$$

If $\{\ \}$ is not in the scope of an atom in $S\{\ \}$ and ϕ is tame, then by the induction hypothesis χ' is tame and $\{\ \}$ is not in the scope of an atom in $H'\{\ \}$. Since β is not an atom, $\{\ \}$ is not in the scope of an atom in $H\{\ \}$ and χ is tame.

We proceed likewise if $S\{A\} =_+ (B\ \beta\ S'\{A\}) + C$.

□

As a corollary of shallow splitting and context reduction we can show the admissibility of a class of up-rules. The main idea is that through splitting we can separate a proof into "building blocks" that are independently provable. We can then easily combine these building blocks differently to obtain a new proof with the same conclusion.

Since tameness is preserved by splitting, cut-free proofs obtained from tame proofs will be tame themselves. The cut-free proofs obtained from non-subatomic proofs will therefore be interpretable, and we can ensure that this cut-elimination result corresponds to cut-elimination in the original system.

When designing a proof system that enjoys cut-elimination, we will therefore only have to ensure that the interpretation map is preservable. This is quite an easy task, since the conditions for an interpretation map to be natural are very lenient, and therefore there is much freedom to design an interpretation to suit many needs.

Definition 3.30. Rules of the form $\alpha\!\uparrow \dfrac{(A\ \alpha\ B) \times \left(C\ \alpha^M\ D\right)}{(A \times C)\ \alpha\ (B \times D)}$ are *cuts.*

Corollary 3.31 (Admissibility of cuts)**.** *Let* SA *be a splittable proof system.*

For any formulae A, B, C, D, any context S, any relation $\alpha \neq +$, given a proof

$$\phi \equiv \begin{array}{c} \phi' \Vert \mathsf{SA}^{\downarrow} \\ S\left\{ \alpha\!\uparrow \dfrac{(A\ \alpha\ B) \times \left(C\ \alpha^M\ D\right)}{(A \times C)\ \alpha\ (B \times D)} \right\} \end{array} \quad ,$$

there is a proof

$$\begin{array}{c} \pi \Vert \mathsf{SA}^{\downarrow} \\ S\{(A \times C)\ \alpha\ (B \times D)\} \end{array} \quad .$$

Furthermore, if ϕ is tame and α is not an atom, π is tame.

Proof. We apply Theorem 3.29 to ϕ.

There are derivations

$$\begin{array}{c} \zeta \Vert \mathsf{SA}^{\downarrow} \\ \big((A\;\alpha\;B)\times\big(C\;\alpha^M\;D\big)\big)+K \end{array} \qquad \text{and} \qquad \begin{array}{c} H\{\{\;\}+K\} \\ \chi \Vert \mathsf{SA}^{\downarrow} \\ S\{\;\} \end{array} \;,$$

with $H\{1\}=1$.

We apply Theorem 3.27 to ζ. There exist derivations

$$\begin{array}{c} Q_1+Q_2 \\ \psi \Vert \mathsf{SA}^{\downarrow} \\ K \end{array} \;, \quad \begin{array}{c} \phi_1 \Vert \mathsf{SA}^{\downarrow} \\ (A\;\alpha\;B)+Q_1 \end{array} \quad \text{and} \quad \begin{array}{c} \phi_2 \Vert \mathsf{SA}^{\downarrow} \\ \big(C\;\alpha^M\;D\big)+Q_2 \end{array} \;.$$

We apply Theorem 3.27 to ϕ_3 and ϕ_4 and we obtain

$$\begin{array}{c} Q_A\;\overline{\alpha}\;Q_B \\ \psi_1 \Vert \mathsf{SA}^{\downarrow} \\ Q_1 \end{array} \;, \quad \begin{array}{c} \phi_3 \Vert \mathsf{SA}^{\downarrow} \\ Q_A+A \end{array} \quad \text{and} \quad \begin{array}{c} \phi_4 \Vert \mathsf{SA}^{\downarrow} \\ Q_B+B \end{array} \;,$$

$$\begin{array}{c} Q_C\;\alpha^m\;Q_D \\ \psi_2 \Vert \mathsf{SA}^{\downarrow} \\ Q_2 \end{array} \;, \quad \begin{array}{c} \phi_5 \Vert \mathsf{SA}^{\downarrow} \\ Q_C+C \end{array} \quad \text{and} \quad \begin{array}{c} \phi_6 \Vert \mathsf{SA}^{\downarrow} \\ Q_D+D \end{array} \;.$$

We can then build the following proof in $\mathsf{SA}^{\downarrow}$

$$\pi = \begin{array}{c} H\left\{ \alpha^M\!\!\downarrow \dfrac{ \times\!\!\downarrow \dfrac{ \begin{array}{c}\phi_3\Vert\\ A+Q_A\end{array} \times \begin{array}{c}\phi_5\Vert\\ C+Q_C\end{array} }{(A\times C)+Q_A+Q_C} \;\alpha^M\; \times\!\!\downarrow \dfrac{ \begin{array}{c}\phi_4\Vert\\ B+Q_B\end{array} \times \begin{array}{c}\phi_6\Vert\\ D+Q_D\end{array} }{(B\times D)+Q_B+Q_D} }{ ((A\times C)\;\alpha\;(B\times D)) + \begin{array}{c} \overline{\alpha}\!\downarrow \dfrac{(Q_A+Q_C)\;\overline{\alpha}\;(Q_B+Q_D)}{\begin{array}{c}Q_A\;\overline{\alpha}\;Q_B\\ \psi_1\Vert\\ Q_1\end{array} + \begin{array}{c}Q_C\;\alpha^m\;Q_D\\ \psi_2\Vert\\ Q_2\end{array}} \\ \psi\Vert \\ K \end{array} } \right\} \\ \chi\Vert \\ S\{(A\times C)\;\alpha\;(B\times D)\} \end{array} \;.$$

If ϕ is tame, then $\{\;\}$ is not in the scope of an atom in $S\{\;\}$ and $\phi_3,\phi_4,\phi_5,\phi_6,\psi_1,\psi_2$ and χ are tame. Therefore, if α is not an atom, π is tame.

□

Remark 3.32. The rule $+\!\uparrow \dfrac{(A+B)\times(C\times D)}{(A\times C)+(B\times D)}$ is always admissible in systems with the rule $\times\!\downarrow$ where $\times$ is associative. We obtain it as follows:

$$
\times\!\downarrow \dfrac{=\dfrac{\times\!\downarrow \dfrac{=\dfrac{=\dfrac{(A+B)\times(C\times D)}{((A+B)\times C)\times D}}{((A+B)\times(C+0))\times D}}{((A\times C)+(B+0))\times D}}{((A\times C)+B)\times(0+D)}}{(A\times C)+0+(B\times D)} \quad .
$$

Example 3.33. We can apply this theorem to show the admissibility of the up fragment of SAMLLS.

Example 3.34. We have shown the admissibility of the up rules

$$
a\!\uparrow \frac{(A\; a\; B)\wedge(C\; a\; D)}{(A\wedge C)\; a\; (B\wedge D)} \quad \text{and} \quad \vee\!\uparrow \frac{(A\vee B)\wedge(C\wedge D)}{(A\vee C)\wedge(B\vee D)}
$$

in system $\mathsf{SAKS}^{\downarrow}$.

We can show the admissibility of these rules in system $\mathsf{SAKS}^{\downarrow}$ where $\wedge$ is associative and commutative, or we could use the splitting procedure to show the admissibilty of commutativity and associativity of $\wedge$ as well, if we consider them as given by the rule

$$
\wedge\!\uparrow \frac{(A\wedge B)\wedge(C\wedge D)}{(A\wedge C)\wedge(B\wedge D)} \quad .
$$

Every rule of the linear fragment of system KS for classical logic corresponds to a tame derivation in SAKS. Therefore every proof in that fragment corresponds to a tame proof in SAKS.

Tameness is preserved when eliminating rule $a\!\uparrow$ since every instance of a rule $\wedge\!\downarrow$ with the premiss equal to t has conclusion equal to t and can therefore be replaced by an equality to obtain a tame cut-free proof. Therefore, if α is an atom and ϕ is tame in Theorem 3.31, π is tame as well.

Example 3.35. We have shown the admissibility of the up rules of system SABVU, $a\!\uparrow$ and $\lhd\!\uparrow$. Just as above, we can likewise choose to show the admissibility of commutativity and associativity of $\otimes$. The cut-free proofs obtained from tame proofs are tame, since identically to the case of SAMLLS, if there is an interpretable instance of $a\!\uparrow$, then the instances of $\otimes\!\downarrow$ in the cut-free proof can be replaced by equalities to obtain a tame proof (see the proof of Theorem 3.11).

This extends to system BV where the units are identified. Even though system SABV does not verify condition 3 of preservability, in a tame proof there are no instances of the equality axioms $0=\circ$ and $1=\circ$ in the scope of an atom since $\circ$ in the scope of an atom is not interpretable. Therefore, in Theorem 3.27, if ϕ is tame and α is an atom then ϕ_1 and ϕ_2 are equalities that do not contain any instance of these axioms. Tameness is preserved since in the absence of these axioms condition 3 of preservability holds.

The splitting procedure is therefore a very general phenomenon: it can be applied to systems with any number of relations and units as long as certain basic equations are satisfied, and is maintained by the identification of any of these units.

3.3 The robustness of splitting: adding a modality

As we have shown in the previous section, splitting hinges only on the shape of rules and on dualities. In the general splitting theorem that we presented we considered only binary relations, but it will be the focus of future research to extend this result to include relations of different arities: splitting can be applied to different types of unary operators, as is shown by the splitting theorems for exponentials in [36] or for a self-dual binder in [34]. In this section we will show a starting point in the direction of such a generalisation, by extending the general procedure to a system with a self-dual modality. The fact that it is possible to do so shows the robustness of the general splitting methodology: it is based on properties that are present in systems with very different expressiveness and therefore it can be expanded to include an extremely wide variety of relations as long as they are introduced by rules of non-contractive shape.

We will present system SAKV^- [22], a system with a self-dual modality. SAKV^- combines a linear splittable core with a self-dual commutative connective (therefore being outside the realm of what is achievable with Gentzen-style calculi) and the simplest case of a modality in terms of the further study of decomposition, the self-dual modality $\star$.

Definition 3.36. We define the set $\mathscr{R} = \mathscr{A} \cup \{⅋, \triangleleft, \otimes\}$ where $\mathscr{A}$ is a denumerable set with $\mathscr{A} \cap \{⅋, \triangleleft, \otimes\} = \varnothing$. We define the set $\mathscr{U} = \{\bot, \circ, 1\}$ of constants. The set $\mathscr{F}$ of *formulae* of SAKV^- contains terms defined by the grammar

$$\mathscr{F} ::= \mathscr{U} \mid \star\mathscr{F} \mid \mathscr{F}\ \alpha\ \mathscr{F} \quad ,$$

with $\alpha \in \mathscr{R}$.

We define *negation* as an involutive map $\bar{\cdot}$ on $\mathscr{F}$ by setting:

$$\begin{aligned}
&\bar{\otimes} := ⅋\ ;\\
&\bar{\triangleleft} := \triangleleft\ ;\\
&\bar{a} := a \text{ for all } a \in \mathscr{A}\ ;\\
&\bar{\circ} := \circ\ ;\\
&\bar{\bot} := 1
\end{aligned}$$

$$a\downarrow \frac{(A \parr B)\; a\; (C \parr D)}{(A\; a\; C) \parr (B\; a\; D)} \qquad a\uparrow \frac{(A\; a\; B) \otimes (C\; a\; D)}{(A \otimes C)\; a\; (B \otimes D)}$$

$$\otimes\downarrow \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\uparrow \frac{(A \parr B) \otimes (C \otimes D)}{(A \otimes C) \parr (B \otimes D)}$$

$$\triangleleft\downarrow \frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)} \qquad \triangleleft\uparrow \frac{(A \triangleleft B) \otimes (C \triangleleft D)}{(A \otimes C) \triangleleft (B \otimes D)}$$

$$\star\downarrow \frac{\star(A \parr B)}{\star A \parr \star B} \qquad \star\uparrow \frac{\star A \otimes \star B}{\star(A \otimes B)}$$

Figure 3-4: *System* SAKV$^-$

and

$$\overline{A\; \alpha\; B} := \overline{A}\; \overline{\alpha}\; \overline{B} \;;$$
$$\overline{\star A} := \star\overline{A} \;.$$

We define an equational theory $=$ on $\mathscr{F}$ as the minimal equivalence relation closed under negation and under context defined by:

For all $A, B, C \in \mathscr{F}$:

$(A \otimes B) \otimes C = A \otimes (B \otimes C)$; $\qquad (A \parr B) \parr C = A \parr (B \parr C)$;
$A \otimes B = B \otimes A$; $\qquad A \parr B = B \parr A$;
$(A \triangleleft B) \triangleleft C = A \triangleleft (B \triangleleft C)$;

$A \otimes 1 = A$; $\qquad A \parr \bot = A$;
$A \triangleleft \circ = A$; $\qquad \circ \triangleleft A = A$;

$\circ \otimes \circ = \bot$; $\qquad \circ \parr \circ = 1$;
$\bot \triangleleft \bot = \bot$; $\qquad 1 \triangleleft 1 = 1$;
$\forall a \in \mathscr{A}.\ \bot\; a\; \bot = \bot$; $\qquad \forall a \in \mathscr{A}.\ 1\; a\; 1 = 1$;
$\star\circ = \circ$;

$1 = \circ$; $\qquad \bot = \circ$.

The subatomic proof system SAKV$^-$ is given by the inference rules in Figure 3-4, together with the equality rules given by $= \frac{A}{B}$ for every A, B on opposite sides of the equality axioms above.

A proof in SAKV$^-$ is a derivation with premiss 1.

We define SAKV$^\downarrow$ as the system given by the down-rules of system SAKV$^-$.

We can observe that the rules $\star\downarrow$ and $\star\uparrow$ correspond to the unary versions of the rules $\alpha\downarrow$ considered in the previous section. Furthermore, the constants verify the same equations than for BV and therefore they verify the duality conditions necessary for the splitting theorem. For these reasons, extending this result to SKV^- is a straightforward task, showcasing the generality of the conditions that allow us to obtain splitting.

For the sake of brevity we omit considerations about tameness, that are done identically to the previous section.

Theorem 3.37.

1. *For every formulae A, B, C, for every relation $\alpha \neq \parr$, for every proof*

$$\begin{array}{c} \phi \,\|\, \mathsf{SAKV}^{\downarrow} \\ (A \;\alpha\; B) \parr C \end{array}$$

there exist formulae Q_1, Q_2 and derivations

$$\begin{array}{c} Q_1 \;\overline{\alpha}\; Q_2 \\ \psi \,\|\, \mathsf{SAKV}^{\downarrow} \\ C \end{array} \quad , \quad \begin{array}{c} \phi_1 \,\|\, \mathsf{SAKV}^{\downarrow} \\ A \parr Q_1 \end{array} \quad \text{and} \quad \begin{array}{c} \phi_2 \,\|\, \mathsf{SAKV}^{\downarrow} \\ B \parr Q_2 \end{array} \quad ,$$

with $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi|_{\parr}$.

2. *For every formulae A, C, for every proof*

$$\begin{array}{c} \phi \,\|\, \mathsf{SAKV}^{\downarrow} \\ \star A \parr C \end{array}$$

there exists a formula Q and derivations

$$\begin{array}{c} \star Q \\ \psi \,\|\, \mathsf{SAKV}^{\downarrow} \\ C \end{array} \quad , \quad \begin{array}{c} \phi_1 \,\|\, \mathsf{SAKV}^{\downarrow} \\ A \parr Q \end{array} \quad ,$$

with $|\phi_1|_{\parr} \leq |\phi|_{\parr}$.

Proof.

1. This case is an instance of the general splitting theorem 3.27, since it is straightforward that the presence of rule $\star\!\downarrow$ does not introduce any new cases and that the conditions are satisfied.

2. We proceed by induction on $|\phi|_{\parr}$. The base case is an instance of case (7) below.

   We prove the inductive step for all the possible cases of the bottom inference rule ρ of ϕ.

   Identically to the proof of 3.27, inspection of the rules provides us with the following possible cases:

(1) $$\phi =_{\parr} \;{}_{\rho}\dfrac{\begin{array}{c}\phi' \,\|\, \mathsf{SAKV}^{\downarrow}\\ \star A \parr C'\end{array}}{\star A \parr C} \quad ;$$

(2) $$\phi =_{\parr} \;{}_{\otimes\downarrow}\dfrac{\begin{array}{c}\phi' \,\|\, \mathsf{SAKV}^{\downarrow}\\ ((\star A \parr C_1) \otimes (C_2 \parr C_3)) \parr C_4\end{array}}{\star A \parr C_2 \parr (C_1 \otimes C_3) \parr C_4} \quad ;$$

(3) $$\phi =_{\parr} \;{}_{=}\dfrac{\begin{array}{c}\phi' \,\|\, \mathsf{SAKV}^{\downarrow}\\ ((\star A \parr C_1)\ \beta\ u_\beta) \parr C_2\end{array}}{\star A \parr C_1 \parr C_2} \quad ;$$

(4) $$\phi =_{\parr} \;{}_{=}\dfrac{\begin{array}{c}\phi' \,\|\, \mathsf{SAKV}^{\downarrow}\\ (u_\beta\ \beta\ (\star A \parr C_1)) \parr C_2\end{array}}{\star A \parr C_1 \parr C_2} \quad ;$$

(5) $$\phi =_{\parr} \;{}_{\rho}\dfrac{\begin{array}{c}\phi' \,\|\, \mathsf{SAKV}^{\downarrow}\\ \star A' \parr C\end{array}}{\star A \parr C} \quad ;$$

(6) $$\phi =_{\parr} \;{}_{\parr\downarrow}\dfrac{\begin{array}{c}\phi' \,\|\, \mathsf{SAKV}^{\downarrow}\\ \star(A \parr C_1) \parr C_2\end{array}}{\star A \parr \star C_1 \parr C_2} \quad ;$$

(7) $$\phi =_{\parr} \;{}_{=}\dfrac{\begin{array}{c}\phi' \,\|\, \mathsf{SAKV}^{\downarrow}\\ \circ \parr C\end{array}}{\star\circ \parr C} \quad ;$$

We proceed as follows:

(1) This case corresponds to case (1) of Theorem 3.27. We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$.

There are derivations

$$\psi =_{\parr} \;{}_{\rho}\dfrac{\begin{array}{c}\star Q\\ \psi' \,\|\, \mathsf{SAKV}^{\downarrow}\\ C'\end{array}}{C} \quad , \qquad \begin{array}{c}\phi_1 \,\|\, \mathsf{SAKV}^{\downarrow}\\ A \parr Q\end{array} \quad ,$$

with $|\phi_1|_{\parr} \leq |\phi'|_{\parr} < |\phi'|_{\parr}$.

(2) This case corresponds to case (2) of Theorem 3.27. We can apply case 1 of this Theorem 3.37 to ϕ'.

There are derivations

$$\begin{array}{c} H_1 \parr H_2 \\ \psi' \, \| \, \mathsf{SAKV}^{\downarrow} \\ C_4 \end{array} \quad , \qquad \begin{array}{c} \omega_1 \, \widetilde{\|} \, \mathsf{SAKV}^{\downarrow} \\ \star A \parr C_1 \parr H_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} \omega_2 \, \widetilde{\|} \, \mathsf{SAKV}^{\downarrow} \\ C_2 \parr C_3 \parr H_2 \end{array} \quad ,$$

with $|\omega_1|_{\parr} + |\omega_2|_{\parr} \leq |\phi'|_{\parr}$.

We apply the induction hypothesis to ω_1 as $|\omega_1|_{\parr} \leq |\phi'|_{\parr} < |\phi|_{\parr}$.

There are derivations

$$\begin{array}{c} \star Q \\ \psi'' \, \| \, \mathsf{SAKV}^{\downarrow} \\ C_1 + H_1 \end{array} \quad , \qquad \begin{array}{c} \phi_1 \, \widetilde{\|} \, \mathsf{SAKV}^{\downarrow} \\ A \parr Q \end{array} \quad ,$$

with $|\phi_1|_{\parr} \leq |\omega_1|_{\parr} < |\phi|_{\parr}$.

We take:

$$\psi =_{\parr} \otimes\!\!\downarrow \frac{\boxed{\begin{array}{c} \star Q \\ \psi'' \, \| \\ C_1 \parr H_1 \end{array}} \otimes \boxed{\begin{array}{c} \omega_2 \, \widetilde{\|} \\ C_2 \parr C_3 \parr H_2 \end{array}}}{(C_1 \otimes C_3) \parr C_2 \parr \boxed{\begin{array}{c} H_1 \parr H_2 \\ \psi' \, \| \\ C_4 \end{array}}} \quad .$$

(3) This corresponds to case (3) of Theorem 3.27. We can apply case 1 of this Theorem 3.37 to ϕ'. There are derivations

$$\begin{array}{c} H_1 \; \overline{\beta} \; H_2 \\ \psi' \, \| \, \mathsf{SAKV}^{\downarrow} \\ C_2 \end{array} \quad , \qquad \begin{array}{c} \omega_1 \, \widetilde{\|} \, \mathsf{SAKV}^{\downarrow} \\ \star A \parr C_1 \parr H_1 \end{array} \quad , \qquad \begin{array}{c} \omega_2 \, \widetilde{\|} \, \mathsf{SAKV}^{\downarrow} \\ u_\beta \parr H_2 \end{array} \quad ,$$

with $|\omega_1|_{\parr} + |\omega_2|_{\parr} \leq |\phi'|_{\parr}$.

By Lemma 3.24, there is a derivation

$$\begin{array}{c} \bar{u}_\beta \\ \psi'' \, \| \, \mathsf{SAKV}^{\downarrow} \\ H_2 \end{array} \quad .$$

We apply the induction hypothesis to ω_1 as $|\omega_1|_{\parr} \leq |\phi'|_{\parr} < |\phi|_{\parr}$. There are derivations

$$\begin{array}{c} \star Q \\ \psi''' \, \| \, \mathsf{SAKV}^{\downarrow} \\ C_1 \parr H_1 \end{array} \quad , \qquad \begin{array}{c} \phi_1 \, \widetilde{\|} \, \mathsf{SAKV}^{\downarrow} \\ A \parr Q \end{array} \quad ,$$

with $|\phi_1|_{\parr} \leq |\omega_1|_{\parr} < |\phi'|_{\parr}$.

We take:

$$\psi =_{\parr} \begin{array}{c} \star Q \\ \psi''' \parallel \\ C_1 \parr \boxed{\begin{array}{c} H_1\ \overline{\beta}\ \boxed{\begin{array}{c} \bar{u}_\beta \\ \psi'' \parallel \\ H_2 \end{array}} \\ \psi' \parallel \\ C_2 \end{array}} \end{array} .$$

(4) This case is analogous to (3).

(5) This corresponds to case (5) of Theorem 3.27. We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} \star Q \\ \psi \parallel \mathsf{SAKV}^{\downarrow} \\ C \end{array} , \qquad \phi_1 \equiv \begin{array}{c} \phi_1' \,\overline{\parallel}\, \mathsf{SAKV}^{\downarrow} \\ \boxed{\rho\, \dfrac{A'}{A}} \parr Q \end{array} ,$$

with $|\phi_1|_{\parr} = |\phi_1'|_{\parr} + 1 \leq |\phi'|_{\parr} + 1 = |\phi|_{\parr}$.

(6) This corresponds to case (7) of Theorem 3.27. We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} \star H \\ \psi' \parallel \mathsf{SAKV}^{\downarrow} \\ C_2 \end{array} , \qquad \begin{array}{c} \phi_1 \,\overline{\parallel}\, \mathsf{SAKV}^{\downarrow} \\ A \parr C_1 \parr H \end{array} ,$$

with $|\phi_1|_{\parr} \leq |\phi'|_{\parr} < |\phi|_{\parr}$.

We take $Q \equiv C_1 \parr H$, and

$$\psi =_{\parr} \star\!\downarrow \frac{\star(C_1 \parr H)}{\star C_1 \parr \boxed{\begin{array}{c} \star H \\ \psi' \parallel \\ C_2 \end{array}}} .$$

(7) This corresponds to case (15) of Theorem 3.27. By Lemma 3.24, there is a

derivation $\begin{array}{c} \circ \\ \psi' \Vert \mathsf{SAKV}^{\downarrow} \\ C \end{array}$ and we take:

$$\psi \equiv \begin{array}{c} = \dfrac{\star\circ}{\circ} \\ \psi' \Vert \\ C \end{array} \quad , \qquad \phi_1 \equiv =_{\parr} \dfrac{1}{\circ \parr \circ} \quad ,$$

with $|\phi_1|_{\parr} = 0 \leq |\phi|_{\parr}$.

$\square$

Theorem 3.38. *For any formula A and any context $S\{\ \}$, given a proof* $\begin{array}{c} \phi \Vert \mathsf{SAKV}^{\downarrow} \\ S\{A\} \end{array}$*, there exist a formula K, a provable context $H\{\ \}$ and derivations*

$$\begin{array}{c} \zeta \Vert \mathsf{SAKV}^{\downarrow} \\ A + K \end{array} \qquad \textit{and} \qquad \begin{array}{c} H\{\{\ \} + K\} \\ \chi \Vert \mathsf{SAKV}^{\downarrow} \\ S\{\ \} \end{array} \quad .$$

Proof. We proceed by induction on the number of relations $\alpha \neq \parr$ that $\{\ \}$ is in the scope of in $S\{\ \}$. We denote it by $|S|_{\parr}$.

If $|S|_+ = 0$, then $S\{A\} =_{\parr} A \parr K$ and we take $\zeta =_{\parr} \phi$ and $H\{\ \} = \{\ \}$.

If $S\{A\} =_{\parr} (S'\{A\}\ \beta\ B) \parr C$ we proceed as in Theorem 3.29.

If $S\{A\} =_{\parr} \star S'\{A\} \parr C$, we apply Theorem 3.27 to ϕ. There exist derivations

$$\begin{array}{c} \star Q \\ \psi \Vert \mathsf{SAKV}^{\downarrow} \\ C \end{array} \quad , \quad \begin{array}{c} \phi_1 \Vert \mathsf{SAKV}^{\downarrow} \\ S'\{A\} \parr Q \end{array} \quad .$$

We apply the induction hypothesis to ϕ_1 since $|S'|_+ < |S|_+$. There are derivations

$$\begin{array}{c} \zeta \Vert \mathsf{SAKV}^{\downarrow} \\ A \parr K \end{array} \quad , \quad \begin{array}{c} H'\{\{\ \} \parr K\} \\ \chi' \Vert \mathsf{SAKV}^{\downarrow} \\ S'\{\ \} \parr Q_1 \end{array} \quad ,$$

with H' a provable context.

We take $H\{\ \} = \star H'\{\ \}$. We have $H\{1\} = \star H'\{1\} = \star 1 = \star\circ = \circ = 1$, and we can build in $\mathsf{SAKV}^{\downarrow}$

$$\chi \equiv \star\!\downarrow \frac{\star\boxed{\begin{array}{c} H'\{\{\ \}\parr K\} \\ \chi' \| \\ S'\{\ \}\parr Q \end{array}}}{\star S'\{\ \}\parr \boxed{\begin{array}{c} \star Q \\ \psi \| \\ C \end{array}}} \quad .$$

□

Elimination of the rules $\parr\!\uparrow$, $a\!\uparrow$, $\triangleleft\!\uparrow$ is a consequence of Theorem 3.31. We will focus on showing the admissibility of the rule $\star\!\uparrow$ in an identical argument, showcasing tha fact that admissibility is a broad phenomenon related to the particular shape of rules and extending beyond the cut.

Corollary 3.39 (Admissibility of $\star\!\uparrow$)**.** *Let* SA *be a splittable proof system.*

For any formulae A, B, C, D, any context S, given a proof

$$\phi \equiv \begin{array}{c} \phi' \| \mathsf{SAKV}^{\downarrow} \\ S\left\{ \alpha\!\uparrow \dfrac{\star A \otimes \star B}{\star(A \otimes B)} \right\} \end{array} \quad ,$$

there is a proof

$$\begin{array}{c} \pi \| \mathsf{SAKV}^{\downarrow} \\ S\{\star(A \otimes B)\} \end{array} \quad .$$

Proof. We apply Theorem 3.38 to ϕ.

There are derivations

$$\begin{array}{c} \zeta \| \mathsf{SAKV}^{\downarrow} \\ (\star A \otimes \star B) \parr K \end{array} \quad \text{and} \quad \begin{array}{c} H\{\{\ \}\parr K\} \\ \chi \| \mathsf{SAKV}^{\downarrow} \\ S\{\ \} \end{array} \quad ,$$

with $H\{1\} = 1$.

We apply Theorem 3.37 to ζ. There exist derivations

$$\begin{array}{c} Q_1 \parr Q_2 \\ \psi \| \mathsf{SAKV}^{\downarrow} \\ K \end{array} \quad , \quad \begin{array}{c} \phi_1 \| \mathsf{SAKV}^{\downarrow} \\ \star A \parr Q_1 \end{array} \quad \text{and} \quad \begin{array}{c} \phi_2 \| \mathsf{SAKV}^{\downarrow} \\ \star B \parr Q_2 \end{array} \quad .$$

We apply Theorem 3.37 to ϕ_3 and ϕ_4 and we obtain

$$\begin{array}{c} \star Q_A \\ \psi_1 \| \mathsf{SAKV}^{\downarrow} \\ Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_3 \| \mathsf{SAKV}^{\downarrow} \\ Q_A \parr A \end{array} \quad ,$$

$$
\begin{array}{c} \star Q_B \\ \psi_2 \,\Vert\, \mathsf{SAKV}^{\downarrow} \\ Q_2 \end{array}
\quad , \quad
\begin{array}{c} \phi_4 \,\overline{\Vert}\, \mathsf{SAKV}^{\downarrow} \\ Q_B \parr B \end{array}
\quad ,
$$

We can then build the following proof in $\mathsf{SAKV}^{\downarrow}$

$$
\pi = \begin{array}{c}
H \left\{ \begin{array}{c}
\star\!\downarrow \dfrac{\star\, \boxed{\otimes\!\downarrow \dfrac{\boxed{\begin{array}{c}\phi_3 \,\overline{\Vert} \\ A \parr Q_A\end{array}} \otimes \boxed{\begin{array}{c}\phi_4 \,\overline{\Vert} \\ B \parr Q_B\end{array}}}{(A \otimes B) \parr Q_A \parr Q_B}}}{\star(A \otimes B) \parr \boxed{\begin{array}{c} \star\!\downarrow \dfrac{\star(Q_A \parr Q_B)}{\boxed{\begin{array}{c}\star Q_A \\ \psi_1 \Vert \\ Q_1\end{array}} \parr \boxed{\begin{array}{c}\star Q_B \\ \psi_2 \Vert \\ Q_2\end{array}}} \\ \psi \Vert \\ K \end{array}}}
\end{array} \right\} \\
\chi \Vert \\
S\{\star(A \otimes B)\}
\end{array}
\quad .
$$

□

3.4 Conclusions

The general splitting procedure gives us a full understanding of *how* the splitting procedure works, and *why* it has been shown to work in every linear system expressed in deep inference so far. We have shown that dualities and the interactions between linear rules are the fundamental phenomena behind admissibility. In this way, we come to see admissibility as a property resulting from the shape of rules that extends beyond the cut: we can show the admissibility of a whole class of inference rules. Furthermore, the understanding that we gain from the generalised theorem allows us to showcase just how broad this methodology is. We have given sufficient properties verified by a whole class of substructural logics that are enough to prove cut-elimination.

Splitting is a global procedure: we have to take into consideration the whole proof to find independent subproofs and rearrange them. This comes only at a polynomial-time complexity cost, and the size of the cut-free proof is at most linear on the size of the original proof. Therefore we see that linear rules do not contribute towards the complexity cost of cut-elimination procedures.

Last, the generalisation of splitting does not only contribute to the understanding of the procedure, it also provides guidelines for the design of logical systems. By providing a generalised theorem, we are able to remove the search for cut-elimination from the design process.

Chapter 4

Decomposition

It is a well known phenomenon in proof theory that in many systems derivations can be arranged into consecutive subderivations made up of only certain rules. For example, we can decompose a first-order proof into a propositional phase and a quantified phase through a Herbrand theorem [7]. This phenomenon has long been explored in deep inference [4, 26, 29, 36, 16], presenting decomposition by means of specific permutations of rules or super-rules, permuting the contractions and cuts together.

Decomposition theorems provide a way to normalise proofs and divide derivations into independent subsystems that can be studied independently. Furthermore, they give the possibility of dividing cut-elimination into several different procedures: decomposition, which introduces complexity, and cut-elimination on a proper linear fragment which does not.

Although decomposition theorems abound, it is the separation of a particular subsystem that we are after: it has long been conjectured that classical logic and linear logic proofs can be decomposed into a splittable phase and a contractive phase independently from cut-elimination, as happens for example in the logic NEL [27] or in the mutiplicative exponential fragment of linear logic [36].

In fact, obtaining a total decomposition into a splittable phase followed by a contractive phase is equivalent to showing that general contractions such as the inference

$$\frac{A \vee A}{A}$$

in classical logic can be permuted to the bottom of linear proofs. However, as is pointed out in [36], it is not always clear whether (and how) this general rule permutes with other rules of the system.

The locality awarded by deep inference allows us to advance towards this result, since we can permute atomic contractions to the bottom of a proof in both classical logic [29] and linear logic [36] through reduction rules for proofs. The decomposition procedures that yield these results are independent from cut-elimination in the case of proofs that do not contain a particular type of subderivation, called a *cycle*.

The decomposition results for atomic contractions in this thesis are a significant step towards proving these conjectures, but need to be expanded in two ways to obtain a full decomposition result independent from cut-elimination. The first one is that for both classical logic and linear logic cut-elimination is used to prove the termination of the decomposition procedure, to show that cycles can be removed from proofs. The second one is that it is unclear how rules involved in making contractions atomic, such as the rule m of SKS, should be permuted with other rules.

In this chapter we will present general reduction rules for systems that achieve four goals:

- We are able to show that the existing decomposition results for classical logic and linear logic are obtained via reductions that are in fact instances of a more general reduction coming from the interactions of contractive rules with other rules;
- We present sufficient conditions for two rules to permute with each other, reducing the analysis usually necessary to obtain decomposition results;
- We show that decomposition and cut-elimination are independent procedures by providing a local procedure to remove cycles through these reduction rules;
- We present tools for future work on achieving a full decomposition theorem for both classical logic and linear logic.

We will start by introducing the reduction rules given in [29] to obtain the decomposition result for atomic contractions in classical logic. We will introduce *atomic flows*, an invariant of proofs that allows us to intuitively follow these reductions and the measure used to prove the termination of the reduction system in the absence of cycles. Following that, we will present a generalisation of the notion of contraction, and characterise a type of rules, called *contractive*, which we can permute downwards in a proofs through the general reduction rules we present. In the last chapter we will use these generalised reduction rules to present a procedure allowing us to remove cycles from proofs without recurring to cut-elimination.

4.1 Preliminaries: atomic decomposition in classical logic and multiplicative additive linear logic

In system SKS (Figure 2-2) it is possible to obtain reduction rules to permute atomic contractions $ac{\downarrow}$ and atomic cocontractions $ac{\uparrow}$ towards the bottom or the top of a derivation respectively. We will introduce the rewriting system for derivations presentes in [29] to achieve that.

Definition 4.1. A *reduction rule* r is a couple (ϕ', ψ') where ϕ' and ψ' are derivations in SKS with $\mathsf{pr}\,\phi' \equiv \mathsf{pr}\,\psi'$ and $\mathsf{cn}\,\phi' \equiv \mathsf{cn}\,\psi'$. We write $r : \phi' \to \psi'$.

For every reduction rule $r : \phi' \to \psi'$ we define the reduction $\to_r$ such that $\phi \to_r \psi$ if and only if ψ' is a subderivation of ϕ and ψ is obtained from ϕ by replacing ϕ' by ψ'.

We call a finite set R of reduction rules a *rewriting system*. Given a set S of derivations, we say that rewriting system R is *terminating on S* if there is no infinite chain $\phi \to_{r_1} \phi_1 \to_{r_2} \ldots$ with $r_i \in R$ for any $\phi \in S$.

Definition 4.2. We define the following reduction rules for SKS:

- $c{\downarrow}-c{\downarrow}$:

$$\mathsf{ac}{\uparrow}\dfrac{\mathsf{ac}{\downarrow}\dfrac{a \vee a}{a}}{a \wedge a} \quad\longrightarrow\quad m\,\dfrac{\boxed{\mathsf{ac}{\uparrow}\dfrac{a}{a \wedge a}} \vee \boxed{\mathsf{ac}{\uparrow}\dfrac{a}{a \wedge a}}}{\boxed{\mathsf{ac}{\downarrow}\dfrac{a \vee a}{a}} \wedge \boxed{\mathsf{ac}{\downarrow}\dfrac{a \vee a}{a}}}$$

- $c{\downarrow}-i{\uparrow}$:

$$\mathsf{ai}{\uparrow}\dfrac{\boxed{\mathsf{ac}{\downarrow}\dfrac{a \vee a}{a}} \wedge \bar{a}}{\mathsf{f}} \quad\longrightarrow\quad =\dfrac{s\,\dfrac{s\,\dfrac{(a \vee a) \wedge \boxed{\mathsf{ac}{\uparrow}\dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}}}{(a \wedge (\bar{a} \wedge \bar{a})) \vee a}}{\boxed{\mathsf{ai}{\uparrow}\dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee \boxed{\mathsf{ai}{\uparrow}\dfrac{a \wedge \bar{a}}{\mathsf{f}}}}}{\mathsf{f}}$$

- $c{\downarrow}-w{\uparrow}$:

$$\mathsf{aw}{\uparrow}\dfrac{\mathsf{ac}{\downarrow}\dfrac{a \vee a}{a}}{\mathsf{t}} \quad\longrightarrow\quad =\dfrac{\boxed{w{\uparrow}\dfrac{a}{\mathsf{t}}} \vee \boxed{w{\uparrow}\dfrac{a}{\mathsf{t}}}}{\mathsf{t}}$$

And their duals:

- $i{\downarrow}-c{\uparrow}$:

$$\mathsf{ai}{\downarrow}\dfrac{\mathsf{t}}{\boxed{\mathsf{ac}{\uparrow}\dfrac{a}{a \wedge a}} \vee \bar{a}} \quad\longrightarrow\quad =\dfrac{\mathsf{t}}{s\,\dfrac{\boxed{\mathsf{ac}{\downarrow}\dfrac{\mathsf{t}}{a \vee \bar{a}}} \wedge \boxed{\mathsf{ac}{\downarrow}\dfrac{\mathsf{t}}{a \vee \bar{a}}}}{s\,\dfrac{((a \vee \bar{a}) \wedge a) \vee \bar{a}}{(a \wedge a) \vee \boxed{\mathsf{ac}{\downarrow}\dfrac{\bar{a} \vee \bar{a}}{\bar{a}}}}}}$$

- $w\!\downarrow - c\!\uparrow$:

$$ac\!\uparrow \frac{aw\!\downarrow \dfrac{\mathsf{f}}{a}}{a \wedge a} \quad \longrightarrow \quad = \frac{\mathsf{f}}{\boxed{aw\!\downarrow \dfrac{\mathsf{f}}{a}} \wedge \boxed{aw\!\downarrow \dfrac{\mathsf{f}}{a}}}$$

Last, we define the trivial family of reduction rules:

- $c\!\downarrow - \rho_H$:

$$\rho \frac{H\left\{ ac\!\downarrow \dfrac{a \vee a}{a} \right\}}{H'\{a\}} \quad \longrightarrow \quad \rho \frac{H\{a \vee a\}}{H'\left\{ ac\!\downarrow \dfrac{a \vee a}{a} \right\}}$$

- $\rho_H - c\!\uparrow$:

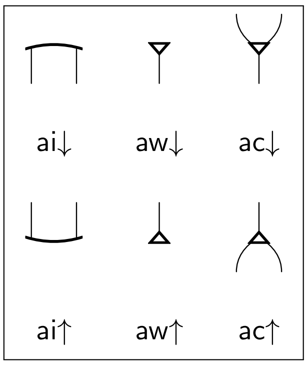
$$\rho \frac{H'\{a\}}{H\left\{ ac\!\uparrow \dfrac{a}{a \wedge a} \right\}} \quad \longrightarrow \quad \rho \frac{H'\left\{ ac\!\uparrow \dfrac{a}{a \wedge a} \right\}}{H\{a \wedge a\}}$$

It is clear that if the rewriting system obtained from the reduction rules of definition 4.1 terminates, then we will obtain a derivation with three phases: a top phase made up only of rules $ac\!\uparrow$, a phase made-up of rules s,m, $ai\!\uparrow$, $ai\!\downarrow$, $w\!\uparrow$, $w\!\downarrow$ and a bottom phase made up only of rules $ac\!\downarrow$.

Definition 4.3. We define *rewriting system* C for SKS as the rewriting system given by the reduction rules of Definition 4.1.

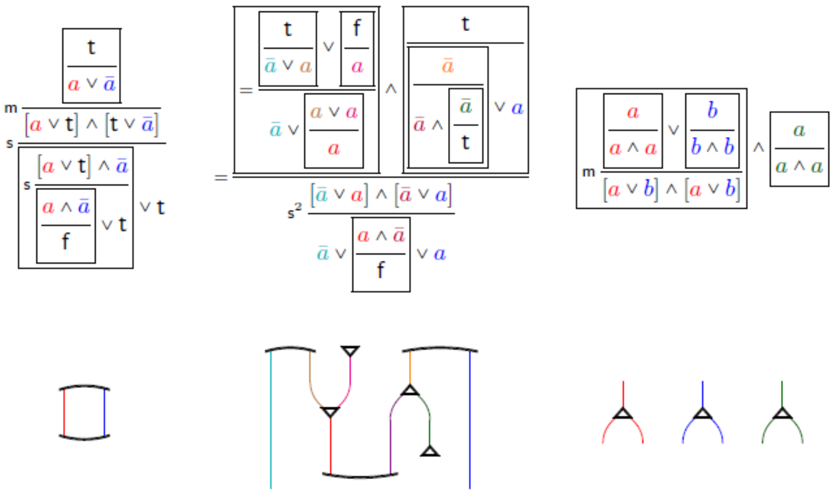
We will see that in the absence of a certain construction inside a derivation, called cycle, the termination of rewriting system C is guaranteed. To provide a measure for termination, we will introduce the *atomic flows*, a graphical invariant of proofs that allows us to intuitively follow these reductions.

Atomic flows are specialised Buss flow graphs [8] that follow the occurrences of atoms in a derivation in SKS. They can be seen as composite diagrams that are freely generated from a set of six elementary diagrams, or as labelled directed graphs, where the six possible labels for the vertices are given in the following figure.



We can associate an atomic flow to every derivation in SKS in a natural way: every edge follows the occurrence of an atom in the derivation, and each vertex label corresponds to the occurrence of a critical rule where atoms are created or destroyed ($ai\!\downarrow, ai\!\uparrow, aw\!\downarrow, aw\!\uparrow, ac\!\downarrow, ac\!\uparrow$). The direction of the edges corresponds to the up-down direction in a derivation. The units f and t are not represented in the flow.

Example 4.4. Below are several examples of derivations and the flows associated to them. Every edge represents an occurrence of the atom of the same colour.



Technically, there are some restrictions on the construction of the flows to guarantee that for every flow there is an associated SKS derivation. However, only an intuitive understanding of the flows is required to follow the graphical representation of the rewriting rules and the measure presented in this section and this is what we are seeking to provide. The interested reader is invited to refer to [29] for further details on the definition of the atomic flows and on the definitions and results presented in what follows.

The measure used to prove termination can be easily followed in a flow: it corresponds to the length of a certain type of paths.

Definition 4.5. Given an edge ϵ in an atomic flow, we define $\mathsf{up}(\epsilon)$ as the upper vertex it is connected to, and $\mathsf{lo}(\epsilon)$ as the lower vertex it is connected to.

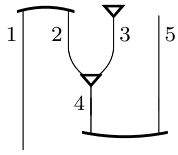
Given a sequence of distinct edges $\epsilon_1, \dots, \epsilon_n$ such that $\mathsf{lo}(\epsilon_i) = \mathsf{up}(\epsilon_{i+1})$ for $1 \le i < n$, we say that $\epsilon_1, \dots, \epsilon_n$ is a *path of length n from* $\mathsf{up}(\epsilon_1)$ *to* $\mathsf{lo}(\epsilon_n)$, and that $\epsilon_n, \dots, \epsilon_1$ is a *path of length n from* $\mathsf{lo}(\epsilon_n)$ *to* $\mathsf{up}(\epsilon_1)$.

Given a sequence of distinct edges $\epsilon_1, \dots, \epsilon_n$, we say that $\epsilon_1, \dots, \epsilon_n$ is an *ai-path of length n from vertex v_1 to vertex v_2* if it is a path from v_1 to v_2 or if there exists a vertex v labelled by $ai\!\uparrow$ or $ai\!\downarrow$ such that $\epsilon_1, \dots, \epsilon_h$ is an ai-path from v_1 to v and $\epsilon_{h+1}, \dots, \epsilon_n$ is an ai-path from v to v_2.

An ai-path of length n is *maximal* if no ai-path containing its edges has length greater than n. An ai-path of length n from v is *maximal* if no ai-path from v containing its edges has length greater than n.

Intuitively, paths correspond to any non-empty sequence of edges from v_1 to v_2 that does not change direction (it either only 'goes downwards' or 'goes upwards'). ai-paths are allowed to change direction, but only at ai-vertices: they are zig-zag paths that change direction at ai-nodes.

Example 4.6.



Some examples of paths of this flow are $2, 4$ and 5.
Some examples of ai-paths in this flow are given by $1, 2$ and $3, 4, 5$.
The maximal ai-paths of this flow are $1, 2, 4, 5$ and $3, 4, 5$ and their reverse.
The maximal ai-paths from the $ac\downarrow$ vertex are $2, 1$ and 3 and $4, 5$.

If we consider the maximal ai-paths from an $ac\downarrow$ vertex starting with its lower edge, we can see that their length corresponds to the number of critical rules the contraction it corresponds to will have to "go through" when applying the reduction rules. For example, in a derivation whose flow is the flow of example 4.6, when we apply the reduction rules to move the atomic contraction downwards, it will permute with one instance of the rule $ai\uparrow$.

More precisely, we can assign a *rank* to every contraction and to every cocontraction of a derivation by refering to its flow. The rank of a contraction will be given by the sum of the lengths of the maximal ai-paths starting with the lower edge of its corresponding vertex in the flow. Dually, the rank of a cocontraction will be given by the sum of the lengths of the maximal ai-paths starting with the upper edge of its corresponding vertex in a flow. We will see that the reduction rules of system C reduce the sum of the ranks of the contractions and cocontractions in a derivation, effectively providing a termination measure when these ranks are finite.

Definition 4.7. Given a vertex v labelled with $ac\downarrow$ in a flow, we define its *rank* as the sum of the lengths of the maximal ai-paths $\epsilon_1, \ldots, \epsilon_n$ from v such that $\mathsf{up}(\epsilon_1) = v$.

Dually, given a vertex v labelled with $ac\uparrow$ in a flow, we define its *rank* as the sum of the lengths of the maximal ai-paths $\epsilon_1, \ldots, \epsilon_n$ from v such that $\mathsf{lo}(\epsilon_1) = v$.

Example 4.8. The rank of the $ac\downarrow$ vertex of the flow of example 4.6 is 2: it corresponds to the length of the ai-path $4, 5$.

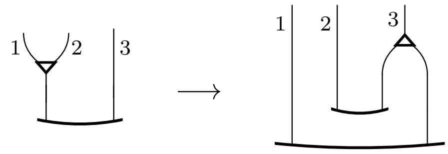
Definition 4.9. Given an occurrence of the rule $ac\downarrow$ in a derivation ϕ with flow ψ, we define its *rank* as the rank of its corresponding vertex in ψ.

Likewise, we define the rank of an occurrence of the rule $ac\uparrow$ as the rank of its corresponding vertex.

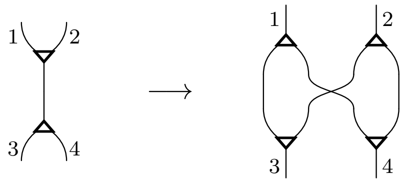
The reductions of system C will reduce the sum of the ranks of the contractions and cocontractions in a derivation except when a certain construction is present, that we call an *ai*-cycle.

This can perhaps best be seen by considering the atomic flow reductions associated to the reductions on derivations:

$c\!\downarrow\!-i\!\uparrow$:



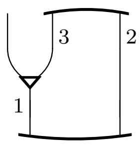
$c\!\downarrow\!-c\!\uparrow$:



It is easy to check that the sum of the ranks of $ac\!\downarrow$ and $ac\!\uparrow$ vertexes is decreased by these reductions, when the cycles defined in what follows are not present.

Definition 4.10. An *ai*-path from v to v is called an *ai-cycle*.

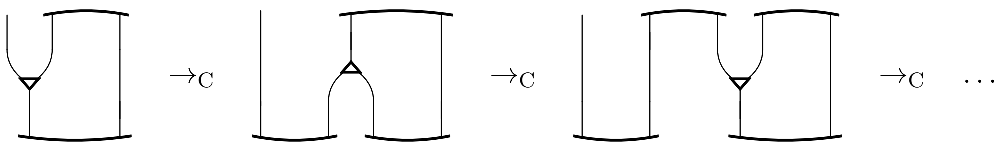
Example 4.11.



The *ai*-path $1, 2, 3$ is an *ai*-cycle.

Definition 4.12. We say that a derivation contains an *ai*-cycle if its atomic flow contains an *ai*-cycle.

When we apply the reductions in C to atomic contractions that belong to a cycle, the rewriting system is not terminating:



In the absence of *ai*-cycles however, the rewriting system terminates as is proved in [29]. We simply outline that proof here to give the reader an idea of the proof and to show that the termination measure and arguments can easily be extended to the rewriting system for MALL that we will present next.

Theorem 4.13. *Rewriting system* C *is terminating on the set of ai-cycle-free derivations.*

Proof. The first observation is that it is clear by inspection of the reduction rules that the rank of (co)contractions not involved in the reduction stays the same.

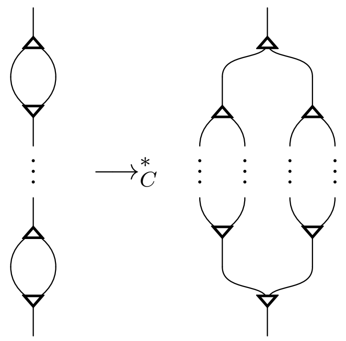
Given an ai-cycle-free derivation ϕ, we consider the lexicographic order on (r, d). r is the sum of the ranks of the contractions and cocontractions in ϕ, and d is the sum of the number of rules below each contraction and the number of rules above each cocontraction when sequentialising ϕ.

We will show that each application of a reduction of C reduces (r, d).

- Applications of the rules $c{\downarrow} - c{\uparrow}$, $c{\downarrow} - i{\uparrow}$ and $i{\downarrow} - c{\uparrow}$ reduce r in the absence of ai-cycles as is shown in the proof of Theorem 7.2.3 of [29] .
- Applications of the rules $c{\downarrow} - w{\uparrow}$ and $w{\downarrow} - c{\uparrow}$ reduce r since they remove contractions and cocontractions.
- Applications of the rules $c{\downarrow} - \rho_H$ and $\rho_H - c{\uparrow}$ trivially maintain r and reduce d.

□

The decomposition procedure may increase the size of a proof exponentially, through the crossings of contractions and cocontractions in the following configuration:



The formula corresponding to the middle line of the diagram on the right will contain a number of atoms exponentially larger than any of the formulae corresponding to the diagram on the left.

This poses a stark contrast with the polynomial cost of cut-elimination via splitting: by separating the two procedures we are able to isolate the source of the complexity cost of cut-elimination in cycle-free proofs.

ai-cycles are evidently removed through cut-elimination, since they are caused by the connexion of a cut and an introduction. In Chapter 5 we will present a procedure to remove loops that does not involve cut-elimination, thus proving the independence of decomposition from cut-elimination. The complexity cost of that procedure is as of yet unknown, and is the last missing element in understanding and separating the causes of the complexity cost of cut-elimination.

Weakenings and coweakenings can be permuted to the bottom/top of a derivation easily through the following reductions, presented in [29] as well.

Definition 4.14. We define the following reduction rules for SKS:

- $w{\downarrow}-c{\downarrow}$:

$$ac{\downarrow}\,\frac{\boxed{aw{\downarrow}\,\dfrac{\mathsf{f}}{a}} \vee a}{a} \quad\longrightarrow\quad =\frac{\mathsf{f} \vee a}{a}$$

- $w{\downarrow}-i{\uparrow}$:

$$ai{\uparrow}\,\frac{\boxed{aw{\downarrow}\,\dfrac{\mathsf{f}}{a}} \wedge \bar{a}}{\mathsf{f}} \quad\longrightarrow\quad =\frac{\mathsf{f} \wedge \boxed{aw{\uparrow}\,\dfrac{\bar{a}}{\mathsf{t}}}}{\mathsf{f}}$$

- $w{\downarrow}-w{\uparrow}$:

$$\begin{array}{l} aw{\downarrow}\,\dfrac{\mathsf{f}}{a} \\ aw{\uparrow}\,\dfrac{}{\mathsf{t}} \end{array} \quad\longrightarrow\quad \begin{array}{l} =\dfrac{\mathsf{f}}{\mathsf{f} \wedge (\mathsf{f} \vee \mathsf{t})} \\ s\,\dfrac{}{(\mathsf{f} \wedge \mathsf{f}) \vee \mathsf{t}} \\ =\dfrac{}{\mathsf{t}} \end{array}$$

And their duals:

- $c{\uparrow}-w{\uparrow}$:

$$ac{\uparrow}\,\frac{a}{\boxed{aw{\uparrow}\,\dfrac{a}{\mathsf{t}}}} \quad\longrightarrow\quad =\frac{a}{a \wedge \mathsf{t}}$$

- $i{\downarrow}-w{\uparrow}$:

$$ai{\downarrow}\,\frac{\mathsf{t}}{\boxed{aw{\uparrow}\,\dfrac{a}{\mathsf{t}}} \vee \bar{a}} \quad\longrightarrow\quad =\frac{\mathsf{t}}{\mathsf{t} \vee \boxed{aw{\downarrow}\,\dfrac{\mathsf{f}}{\bar{a}}}}$$

And the trivial reductions:

- $w{\downarrow}-\rho_H$:

$$\rho\,\frac{H\left\{aw{\downarrow}\,\dfrac{\mathsf{f}}{a}\right\}}{H'\{\mathsf{f}\}} \quad\longrightarrow\quad \rho\,\frac{H\{\mathsf{f}\}}{H'\left\{aw{\downarrow}\,\dfrac{\mathsf{f}}{a}\right\}}$$

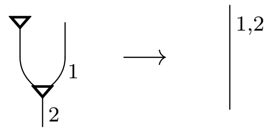
- $\rho_H - w\uparrow$:

$$\rho\,\frac{H'\{a\}}{H\left\{ aw\uparrow\,\frac{a}{\mathsf{t}} \right\}} \quad\longrightarrow\quad \rho\,\frac{H'\left\{ aw\uparrow\,\frac{a}{\mathsf{t}} \right\}}{H\{\mathsf{t}\}}$$

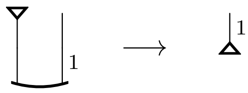
Definition 4.15. We define rewriting system W as the rewriting system given by the reductions in Definition 4.14.

By observing the corresponding flow reductions, it is easy to see that the non-trivial reductions of W remove edges of atomic flows:

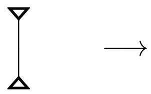
$w\downarrow - c\downarrow$:



$w\downarrow - i\uparrow$:



$w\downarrow - u\uparrow$:



Termination is then clear, since every application of a non-trivial reduction rule reduces the number of edges of the associated flow to a derivation, and the trivial rules reduce the number of rules below weakenings and above coweakenings. By a similar argument to the one used for Theorem 4.13, we will then obatin termination.

Theorem 4.16. *Rewriting system* W *is terminating.*

Note that the reductions of system W do not introduce atomic (co)contractions or medials: only splittable rules. By applying system C followed by system W to a derivation, we obtain an SKS derivation of the form

$$\begin{array}{l} A \\ \| \, w\uparrow \\ A_1 \\ \| \, ac\uparrow \\ A_2 \\ \| \, s,m,ai \\ A_3 \\ \| \, ac\downarrow \\ A_4 \\ \| \, w\downarrow \\ B \end{array} \quad .$$

$$
ai\downarrow \frac{1}{a \parr \bar{a}} \qquad\qquad ai\uparrow \frac{a \otimes \bar{a}}{\bot}
$$

$$
s \frac{(A \parr B) \otimes C}{(A \otimes C) \parr B}
$$

$$
d\downarrow \frac{(A \parr B) \mathbin{\&} (C \parr D)}{(A \mathbin{\&} C) \parr (B \oplus D)} \qquad\qquad d\uparrow \frac{(A \oplus B) \otimes (C \mathbin{\&} D)}{(A \otimes C) \oplus (B \otimes D)}
$$

$$
\oplus\downarrow \frac{(A \parr B) \oplus (C \parr D)}{(A \oplus C) \parr (B \oplus D)} \qquad\qquad \&\uparrow \frac{(A \mathbin{\&} B) \otimes (C \mathbin{\&} D)}{(A \otimes C) \mathbin{\&} (B \otimes D)}
$$

$$
m \frac{(A \mathbin{\&} B) \oplus (C \mathbin{\&} D)}{(A \oplus C) \mathbin{\&} (B \oplus D)}
$$

$$
ac\downarrow \frac{a \oplus a}{a} \qquad\qquad ac\uparrow \frac{a}{a \mathbin{\&} a}
$$

$$
m_2\downarrow \frac{(A \otimes B) \oplus (C \otimes D)}{(A \oplus C) \otimes (B \oplus D)} \qquad\qquad m_2\uparrow \frac{(A \mathbin{\&} B) \parr (C \mathbin{\&} D)}{(A \parr C) \mathbin{\&} (B \parr D)}
$$

$$
at\downarrow \frac{0}{a} \qquad\qquad at\uparrow \frac{a}{\top}
$$

Figure 4-1: *System* SMALLS

Extremely similar rewriting systems can be presented for linear logic [36] to permute atomic (co)contractions with the other rules. We will particularly focus on the multiplicative additive fragment of linear logic (MALL) given by the subsystem SMALLS (Figure 4-1) corresponding to the MALL fragment of the system SLLS in [36]. The exponentials are expected to be included in future research as unary relations.

We will briefly introduce the rewriting systems, to highlight the similarities between the reduction rules in classical logic and in linear logic, and to observe that an identical termination argument than that made for Theorem 4.13 holds for derivations without ai-cycles in multiplicative additive linear logic.

Definition 4.17. We present the following reduction rules for SMALLS:

- $c\downarrow - c\downarrow$:

$$
\begin{array}{c} ac\downarrow \dfrac{a \oplus a}{ac\uparrow \dfrac{a}{a \mathbin{\&} a}} \end{array}
\quad \longrightarrow \quad
m \frac{\boxed{ac\uparrow \dfrac{a}{a \mathbin{\&} a}} \oplus \boxed{ac\uparrow \dfrac{a}{a \mathbin{\&} a}}}{\boxed{ac\downarrow \dfrac{a \oplus a}{a}} \mathbin{\&} \boxed{ac\downarrow \dfrac{a \oplus a}{a}}}
$$

- $c{\downarrow}-i{\uparrow}$:

$$ai{\uparrow}\,\frac{\boxed{ac{\downarrow}\,\frac{a \oplus a}{a}} \otimes \bar{a}}{\bot} \quad\longrightarrow\quad = \frac{d{\uparrow}\,\dfrac{(a \oplus a) \otimes \boxed{ac{\uparrow}\,\frac{\bar{a}}{\bar{a} \mathbin{\&} \bar{a}}}}{\boxed{ai{\uparrow}\,\frac{a \mathbin{\&} \bar{a}}{\bot}} \oplus \boxed{ai{\uparrow}\,\frac{a \mathbin{\&} \bar{a}}{\bot}}}}{\bot}$$

- $c{\downarrow}-w{\uparrow}$:

$$aw{\uparrow}\,\frac{ac{\downarrow}\,\dfrac{a \oplus a}{a}}{\top} \quad\longrightarrow\quad = \frac{\boxed{w{\uparrow}\,\frac{a}{\top}} \oplus \boxed{w{\uparrow}\,\frac{a}{\top}}}{\top}$$

Just like for classical logic, we can define the duals of these reductions and the trivial reduction rules.

Definition 4.18. Rewriting system Q for SMALLS is given by the reduction rules presented in Definition 4.17 and their duals.

We can define the rank of atomic contractions and atomic cocontractions in an identical fashion to classical logic, and present the exact same argument for the termination of Q in the absence of *ai*-cycles.

Theorem 4.19. *Rewriting system* Q *is terminating on the set of ai-cycle-free* SMALLS *derivations.*

Again, this decomposition procedure may increase the size of a proof exponentially, through the exact same phenomenon as in classical logic.

We can define reduction rules for the permutation of weakenings and coweakenings.

Definition 4.20. We define the following reduction rules for SMALLS:

- $w{\downarrow}-c{\downarrow}$:

$$ac{\downarrow}\,\frac{\boxed{aw{\downarrow}\,\frac{\mathsf{0}}{a}} \oplus a}{a} \quad\longrightarrow\quad = \frac{\mathsf{0} \oplus a}{a}$$

- $w{\downarrow}-i{\uparrow}$:

$$ai{\uparrow}\,\frac{\boxed{aw{\downarrow}\,\frac{\mathsf{0}}{a}} \otimes \bar{a}}{\bot} \quad\longrightarrow\quad = \frac{\mathsf{0} \otimes \boxed{aw{\uparrow}\,\frac{\bar{a}}{\top}}}{\bot}$$

- $w\downarrow - w\uparrow$:

$$\frac{aw\uparrow \dfrac{aw\downarrow \dfrac{\mathsf{0}}{a}}{\top}}{} \quad \longrightarrow \quad = \dfrac{d\downarrow \dfrac{= \dfrac{\mathsf{0}}{(\bot \parr \top) \mathbin{\&} (\bot \parr \mathsf{0})}}{(\bot \mathbin{\&} \bot) \oplus (\top \oplus \mathsf{0})}}{\top}$$

We can define the dual reductions and the trivial reductions identically to clasical logic.

Definition 4.21. Rewriting system Y for SMALLS is given by the reduction rules of Definition 4.20 together with their duals and the trivial reduction rules.

Just like for classical logic, these reduction rules remove atoms from a derivation. Therefore, the rewriting system is clearly terminating.

Theorem 4.22. *Rewriting system Y is terminating.*

Again, we can remark that the reductions of system Y do not introduce atomic (co)contractions or other contractive rules: only splittable rules $d\downarrow$ and $d\uparrow$.

We have thus shown that it is possible to decompose SKS and SMALLS derivations in extremely similar ways. In the next section we will show that both decomposition theorems correspond to the same phenomenon: the interaction of contractive rules. Furthermore, in the last section of this chapter we will present a procedure to remove ai-cycles from derivations, effectively showing the independence of decomposition and cut-elimination.

4.2 General rewriting system

Decomposition theorems obtained by permutations of rules, being a local phenomenon, are as different as different logics are. Therefore, generalising decomposition is not a straightforward task. However, permuting atomic contractions to the bottom of a proof has been proved possible in both classical logic and in linear logic (Section 4.1). The reduction rules to achieve it are extremely similar in both logics, suggesting that they are heavily dependant on the shape of the rules rather than being system-specific.

Furthermore, it has long been a conjecture that it is possible to further decompose proofs into a splittable phase followed by the other rules in classical logic [4] and in linear logic, suggesting that we can permute rules other than atomic contractions downwards in a proof as well.

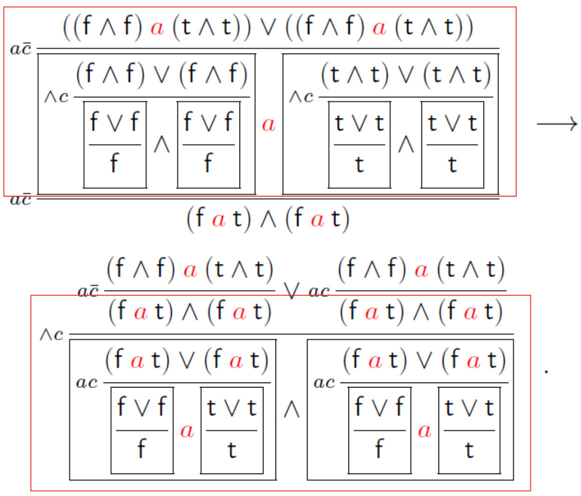
Both these arguments indicate that it should be possible to characterise the rules that can be permuted downwards in proofs and generalise the reduction rules. This is what we set out to do in this section: we will present generalised reduction rules that encompass the existing reduction rules for classical logic and linear logic, as well as allow us to permute other contractive rules downwards in a proof. It is expected that future research will yield a full decomposition theorem for classical logic by means of these reductions.

In addition, these reduction rules will be fundamental in the *ai*-cycle removal procedure that we will present in Chapter 5.

The main problem we face when permuting contractive rules such as the rule m of SKS downwards in a proof is that it is not clear how to proceed, since by permuting it through certain rules we may create an unbounded number of cocontractions and medials, making it extremely difficult to guarantee that we are in fact advancing towards a medial-free proof and to find a measure that will show the termination of the procedure.

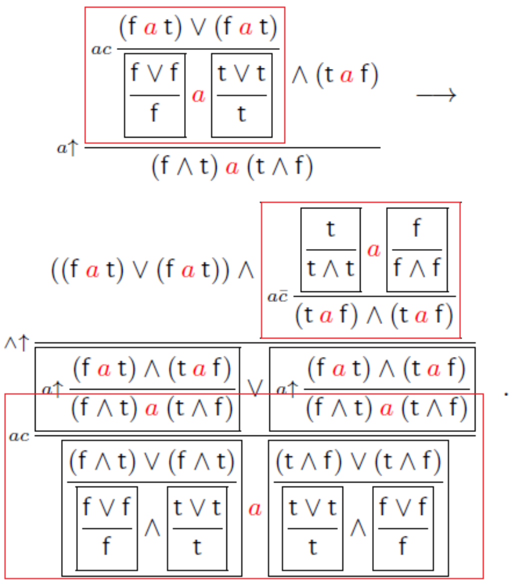
By observing the subatomic reduction rules corresponding to the reductions presented in the previous section, a novel way of controlling this phenomenon arises: we will show that it is possible to move 'blocks' of nested contractive rules together, in such a way that we are no longer concerned by the number of cocontractions and medials created by the decomposition procedure.

The reduction $c\!\downarrow\to c\!\uparrow$ for SKS can for example be written subatomically as



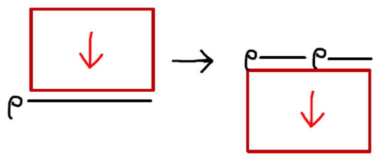
This reduction corresponds to moving a block of nested contractions (surrounded by a red box) by creating another block of nested contractions lower in the proof.

The rule $c\!\downarrow\!-i\!\uparrow$ can be written subatomically as

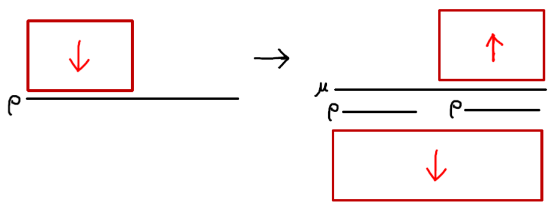


In this case we move a block of nested contractions by creating another block of nested contractions lower in the proof and a block of nested cocontractions.

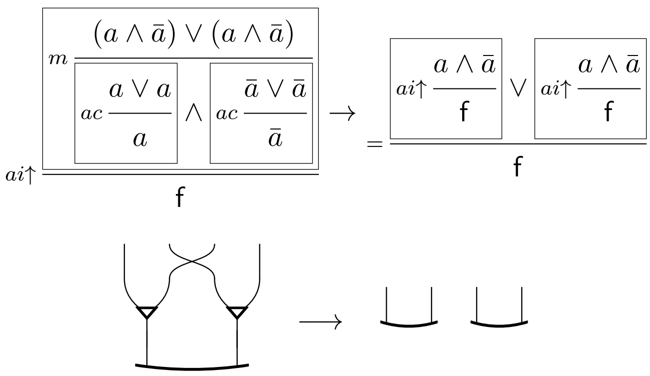
We will call *generic contractions* the blocks of nested contractions, and define general reductions to permute them downwards just like in these examples. We will present two types of reductions, corresponding to the two types of reductions that we have just shown as examples: a reduction s given by



and a reduction t given by



In this way we obtain novel reductions for derivations, such as the reduction



that is fundamental for the cycle-elimination procedure that we will present in the next chapter.

In this section we will use classical logic and multiplicative additive linear logic as examples. However, instead of taking associativity and commutativity as equality axioms, we will present them as instances of rules $\dfrac{(A \mathrel{\alpha} B) \mathrel{\alpha} (C \mathrel{\alpha} D)}{(A \mathrel{\alpha} C) \mathrel{\alpha} (B \mathrel{\alpha} D)}$ (Figures 4-2 and 4-3). This small change does not warrant a change of name for the system, and therefore we will refer to this system for classical logic as SAKS as well.

Definition 4.23 (System $\mathsf{SAMALLS}$)**.** Subatomic formulae for multiplicative additive linear logic $\mathscr{F}$ are given by the set of constants $\mathscr{U} = \{\bot, 0, \top, 1\}$ and the set of relations $\mathscr{R} = \{⅋, \oplus, \&, \otimes\} \cup \mathscr{A}$ where $\mathscr{A}$ is a denumerable set of atoms, denoted by a, b, ... Two examples of subatomic formulae for linear logic are

$$C \equiv ((1 ⅋ \bot) \; a \; 1) \otimes 0 \quad \text{and} \quad D \equiv ((0 \mathbin{\&} \top) \; b \; 1) \; a \; (1 ⅋ \bot) \quad .$$

For the set of subatomic formulae for linear logic $\mathscr{F}$, we define negation through:

$$\begin{aligned} &\bar{\otimes} = ⅋ \\ &\bar{\&} := \oplus \; ; \\ &\bar{a} := a \text{ for all} a \in \mathscr{A} \; ; \\ &\bar{1} := \bot \; ; \\ &\bar{\top} := 0 \; . \end{aligned}$$

We define the equational theory $=$ on $\mathscr{F}$ as the minimal equivalence relation closed under negation and under context defined by:

$$\forall A, B, C \in \mathscr{F},$$

$$\begin{array}{ll}
A \otimes 1 = A\ ; & A \parr \bot = A\ ; \\
A \mathbin{\&} \top = A\ ; & A \oplus 0 = A\ ; \\
\bot \mathbin{\&} \bot = \bot\ ; & 1 \mathbin{\&} 1 = 1\ ; \\
\bot \oplus \bot = \bot\ ; & 1 \oplus 1 = 1\ ; \\
0 \otimes 0 = 0\ ; & \top \parr \top = \top\ ; \\
0 \parr 0 = 0\ ; & \top \otimes \top = \top\ ; \\
0 \mathbin{\&} 0 = 0\ ; & \top \oplus \top = \top\ ; \\
\forall a \in \mathscr{A}.\ \bot\, a \,\bot = \bot\ ; & \forall a \in \mathscr{A}.\ 1\, a\, 1 = 1\ ; \\
\forall a \in \mathscr{A}.\ 0\, a\, 0 = 0\ ; & \forall a \in \mathscr{A}.\ \top\, a\, \top = \top\ ; \\
\forall a \in \mathscr{A}.\ \bot\, a\, \top = \top\ ; & \forall a \in \mathscr{A}.\ 1\, a\, 0 = 0\ ; \\
\forall a \in \mathscr{A}.\ \top\, a\, \bot = \top\ ; & \forall a \in \mathscr{A}.\ 0\, a\, 1 = 0\ ; \\
\forall a \in \mathscr{A}.\ 1\, a\, \top = \top\ ; & \forall a \in \mathscr{A}.\ \bot\, a\, 0 = 0\ ; \\
\forall a \in \mathscr{A}.\ \top\, a\, 1 = \top\ ; & \forall a \in \mathscr{A}.\ 0\, a\, \bot = 0\ ;
\end{array}$$

A natural interpretation is given by considering the assignments:

$$\begin{array}{ll}
-\ I(1) \equiv 1\ ; & -\ I(\bot) \equiv \bot\ ; \\
-\ I(\top) \equiv \top\ ; & -\ I(0) \equiv 0\ ; \\
-\ \forall a \in \mathscr{A}.\ I(\bot\, a\, \bot) \equiv \bot\ ; & -\ \forall a \in \mathscr{A}.\ I(1\, a\, 1) \equiv 1\ ; \\
-\ \forall a \in \mathscr{A}.\ I(\bot\, a\, 1) \equiv a\ ; & -\ \forall a \in \mathscr{A}.\ I(1\, a\, \bot) \equiv \bar{a}\ ; \\
-\ \forall a \in \mathscr{A}.\ I(0\, a\, 0) \equiv 0\ ; & -\ \forall a \in \mathscr{A}.\ I(\top\, a\, \top) \equiv \top\ ; \\
-\ \forall a \in \mathscr{A}.\ I(\bot\, a\, \top) \equiv \top\ ; & -\ \forall a \in \mathscr{A}.\ I(\top\, a\, \bot) \equiv \top\ ; \\
-\ \forall a \in \mathscr{A}.\ I(\top\, a\, 1) \equiv \top\ ; & -\ \forall a \in \mathscr{A}.\ I(1\, a\, \top) \equiv \top\ ; \\
-\ \forall a \in \mathscr{A}.\ I(0\, a\, 1) \equiv 0\ ; & -\ \forall a \in \mathscr{A}.\ I(1\, a\, 0) \equiv 0\ ; \\
-\ \forall a \in \mathscr{A}.\ I(\bot\, a\, 0) \equiv 0\ ; & -\ \forall a \in \mathscr{A}.\ I(0\, a\, \bot) \equiv 0\ ; \\
-\ I(A \parr B) \equiv I(A) \parr I(B)\ ; & -\ I(A \otimes B) \equiv I(A) \otimes I(B)\ ; \\
-\ I(A \oplus B) \equiv I(A) \oplus I(B)\ ; & -\ I(A \mathbin{\&} B) \equiv I(A) \mathbin{\&} I(B)\ .
\end{array}$$

where $A, B \in \mathscr{F}^i$, extending it in such a way that $A\, a\, B$ is interpretable iff $A = u, B = v$ with $u, v \in \{\bot, 0, \top, 1\}$ and $u\, a\, v$ is interpretable. Then, $I(A\, a\, B) \equiv I(u\, a\, v)$.

System SAMALLS for multiplicative additive linear logic is given by the inference rules of Figure 4-3 together with an equality rule for each pair of formulae on opposite sides of an equality in the equations above.

System SAMALLS is correct for the multiplicative additive fragment of system SLLS in [35]. Every rule of that fragment trivially corresponds to a rule of SAMALLS, except for the rules $at{\downarrow}$ and $at{\uparrow}$ that are obtained identically to the rules $aw{\downarrow}$ and $aw{\uparrow}$ of classical logic in example 2.41.

$$a\!\downarrow \frac{(A \vee B)\ a\ (C \vee D)}{(A\ a\ C) \vee (B\ a\ D)} \qquad a\!\uparrow \frac{(A\ a\ B) \wedge (C\ a\ D)}{(A \wedge C)\ a\ (B \wedge D)}$$

$$\wedge\!\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \vee\!\uparrow \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\vee\!\downarrow \frac{(A \vee B) \vee (C \vee D)}{(A \vee C) \vee (B \vee D)} \qquad \wedge\!\uparrow \frac{(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

$$m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$ac \frac{(A\ a\ B) \vee (C\ a\ D)}{(A \vee C)\ a\ (B \vee D)} \qquad a\bar{c} \frac{(A \wedge B)\ a\ (C \wedge D)}{(A\ a\ C) \wedge (B\ a\ D)}$$

Figure 4-2: SAKS

$$a\!\downarrow \frac{(A \parr B)\ a\ (C \parr D)}{(A\ a\ C) \parr (B\ a\ D)} \qquad a\!\uparrow \frac{(A\ a\ B) \otimes (C\ a\ D)}{(A \otimes C)\ a\ (B \otimes D)}$$

$$\otimes\!\downarrow \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\!\uparrow \frac{(A \parr B) \otimes (C \otimes D)}{(A \otimes C) \parr (B \otimes D)}$$

$$\&\!\downarrow \frac{(A \parr B)\ \&\ (C \parr D)}{(A\ \&\ C) \parr (B \oplus D)} \qquad \oplus\!\uparrow \frac{(A \oplus B) \otimes (C\ \&\ D)}{(A \otimes C) \oplus (B \otimes D)}$$

$$\oplus\!\downarrow \frac{(A \parr B) \oplus (C \parr D)}{(A \oplus C) \parr (B \oplus D)} \qquad \&\!\uparrow \frac{(A\ \&\ B) \otimes (C\ \&\ D)}{(A \otimes C)\ \&\ (B \otimes D)}$$

$$\parr\!\downarrow \frac{(A \parr B) \parr (C \parr D)}{(A \parr C) \parr (B \parr D)} \qquad \otimes\!\uparrow \frac{(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

$$m \frac{(A\ \&\ B) \oplus (C\ \&\ D)}{(A \oplus C)\ \&\ (B \oplus D)}$$

$$ac \frac{(A\ a\ B) \oplus (C\ a\ D)}{(A \oplus C)\ a\ (B \oplus D)} \qquad a\bar{c} \frac{(A\ \&\ B)\ a\ (C\ \&\ D)}{(A\ a\ C)\ \&\ (B\ a\ D)}$$

$$\otimes c \frac{(A \otimes B) \oplus (C \otimes D)}{(A \oplus C) \otimes (B \oplus D)} \qquad \parr\bar{c} \frac{(A\ \&\ B) \parr (C\ \&\ D)}{(A \parr C)\ \&\ (B \parr D)}$$

$$\oplus c \frac{(A \oplus B) \oplus (C \oplus D)}{(A \oplus C) \oplus (B \oplus D)} \qquad \&\bar{c} \frac{(A\ \&\ B)\ \&\ (C\ \&\ D)}{(A\ \&\ C)\ \&\ (B\ \&\ D)}$$

Figure 4-3: SAMALLS

The first step in the generalisation is to characterise the *contractions*, the rules that will be permuted. Unsurprisingly, the rules that we will be able to permute downwards/upwards in a derivation correspond to the rules involved in making contraction atomic. We will call them *contractions* as well.

ν-contractive systems will then be defined in such a way that they correspond to those systems where we can always recover *general contractions* of the form

$$\frac{A \;\nu\; A}{A} \quad .$$

Definition 4.24. Let ν be a relation with unit $\triangledown$, and $\overline{\nu}$ its dual with unit $\vartriangle$. A *ν-contractive system* SA is a subatomic proof system where:

- For every relation α there is a down rule of the form

$$\alpha c \frac{(A \;\alpha\; B) \;\nu\; (C \;\alpha\; D)}{(A \;\nu\; C) \;\alpha\; (B \;\nu\; D)} \quad ,$$

  that we call *contraction for α*.

- Dually, for every relation α there is an up-rule of the form

$$\alpha \bar{c} \frac{(A \;\overline{\nu}\; B) \;\alpha\; (C \;\overline{\nu}\; D)}{(A \;\alpha\; B) \;\overline{\nu}\; (C \;\alpha\; D)} \quad ,$$

  that we call *cocontraction for α*.

- The only unit assignments for ν are of the form $u \;\nu\; u = u$ for every constant $u \in \mathcal{U}$. We call the equality rule $= \frac{u \;\nu\; u}{u}$ the *contraction equality rule for u*.

- Dually, the only unit assignments for $\overline{\nu}$ are of the form $u \;\overline{\nu}\; u = u$ for every constant $u \in \mathcal{U}$. We call the equality rule $= \frac{u \;\overline{\nu}\; u}{u}$ the *cocontraction equality rule for u*.

- For every constant $u \in \mathcal{U}$, $\frac{\triangledown}{u}$ is derivable in SA. We will denote these unitary instances of contractive rules by $w \frac{\triangledown}{u}$ and call them *weakenings*.

- Dually, for every constant $u \in \mathcal{U}$, $\frac{u}{\vartriangle}$ is derivable in SA. We will denote these unitary instances of contractive rules by $\bar{w} \frac{\triangledown}{u}$ and call them *coweakenings*.

- For every relation α there is an equality axiom $\triangledown \; \alpha \; \triangledown = \triangledown$.
- Dually, for every relation α there is an equality axiom $\vartriangle\alpha\vartriangle=\vartriangle$.

We call ν the *contracting relation*, and $\overline{\nu}$ the *cocontracting relation*.

Remark 4.25. Note that this definition implies that ν is weak.

Example 4.26. System SAKS (Figure 4-2) is a $\vee$-contractive system.

Example 4.27. System SAMALLS (Figure 4-3) is a $\oplus$-contractive system.

Furthermore, general contractions correspond to a very particular arrangement of these rules: they can be recovered through derivations made-up of nested contraction rules, just like the 'blocks' we highlighted in the introductory example. This type of derivation, that we call *generic contraction*, is the type of derivation that we will show it is possible to permute downwards in a proof.

Definition 4.28. *Generic contractions* are defined recursively as follows:

- The empty derivation is a generic contraction ;
- A contraction equality rule is a generic contraction ;
- A derivation

$$c\,\frac{(A \;\alpha\; B)\;\nu\;(C\;\alpha\;D)}{\boxed{\begin{array}{c}A\;\nu\;C\\ \phi_1 \,\|\, c\\ R\end{array}}\;\alpha\;\boxed{\begin{array}{c}B\;\nu\;D\\ \phi_2 \,\|\, c\\ S\end{array}}}$$

  is a generic contraction if c is a contraction and ϕ_1 and ϕ_2 are generic contractions. In this case, we say that it is a generic contraction with *main relation* α.

We label generic contractions with a c, as in $\begin{array}{c}A\\ \phi \,\|\, c\\ A'\end{array}$.

Generic cocontractions are defined dually, and are labeled with $\bar{c}$.

Lemma 4.29. *In a ν-contractive system, for any formula A there is a generic contraction*

$$\begin{array}{c}A\;\nu\;A\\ \phi \,\|\, c\\ A\end{array}\quad .$$

Dually, there is a generic cocontraction

$$\begin{array}{c}A\\ \psi \,\|\, \bar{c}\\ A\;\overline{\nu}\;A\end{array}\quad .$$

Proof. We proceed by structural induction on A.

If $A \equiv u$, with u a constant, we take $\phi \equiv = \dfrac{u \; \nu \; u}{u}$.

If $A \equiv A_1 \; \alpha \; A_2$, then by the induction hypothesis, there are generic contractions

$$\begin{array}{c} A_1 \; \nu \; A_1 \\ {\scriptstyle \phi_1} \| {\scriptstyle c} \\ A_1 \end{array} \quad \text{and} \quad \begin{array}{c} A_2 \; \nu \; A_2 \\ {\scriptstyle \phi_2} \| {\scriptstyle c} \\ A_2 \end{array} \quad .$$

We take

$$\phi \equiv \quad {\scriptstyle c}\,\dfrac{(A_1 \; \alpha \; A_2) \; \nu \; (A_1 \; \alpha \; A_2)}{\boxed{\begin{array}{c} A_1 \; \nu \; A_1 \\ {\scriptstyle \phi_1} \| {\scriptstyle c} \\ A_1 \end{array}} \; \alpha \; \boxed{\begin{array}{c} A_2 \; \nu \; A_2 \\ {\scriptstyle \phi_2} \| {\scriptstyle c} \\ A_2 \end{array}}} \quad .$$

□

Example 4.30. Consider $A \equiv (1 \; a \; 0) \wedge (0 \; b \; 1)$. Then the generic contraction given by Lemma 4.29 is

$${\scriptstyle \wedge c}\,\dfrac{((1 \; a \; 0) \wedge (0 \; b \; 1)) \vee ((1 \; a \; 0) \wedge (0 \; b \; 1))}{\boxed{{\scriptstyle ac}\,\dfrac{(1 \; a \; 0) \vee (1 \; a \; 0)}{\boxed{= \dfrac{1 \vee 1}{1}} \; a \; \boxed{= \dfrac{0 \vee 0}{0}}}} \wedge \boxed{{\scriptstyle ac}\,\dfrac{(0 \; b \; 1) \vee (0 \; b \; 1)}{\boxed{= \dfrac{0 \vee 0}{0}} \; b \; \boxed{= \dfrac{1 \vee 1}{1}}}}} \quad .$$

In contractive systems where formulae are built over the units of relations, weakenings come 'for free'. This is a consequence of the fact that the inferences ${\scriptstyle w}\,\dfrac{\triangledown}{u_\alpha}$ are always derivable in a ν-contractive system. If u_α is a unit for α, then we can consider the following instance of a contractive inference rule:

$${\scriptstyle \alpha c}\,\dfrac{(u_\alpha \; \alpha \; \triangledown) \; \nu \; (\triangledown \; \alpha \; u_\alpha)}{(u_\alpha \; \nu \; \triangledown) \; \alpha \; (\triangledown \; \nu \; u_\alpha)}$$

with premiss $\triangledown$ and conclusion u_α.

Through these unitary weakenings, we can recover general weakenings

$$\dfrac{\triangledown}{A}$$

as well.

In fact, we will not treat weakenings as instances of contractive rules, and will therefore not permute them downwards in a proof with the reductions presented in what follows. We will instead present different reduction rules for them, as is done for the weakenings in the previous section.

Lemma 4.31. *In a ν-contractive system, for every formula A there is a derivation*

$$\begin{array}{c}\nabla\\ \phi\,\Vert\, w\\ A\end{array}$$

made-up only of weakenings and equalities, that we will call generic weakening*.*

Proof. We proceed by structural induction on A.

If $A \equiv u$, then we take $w\,\dfrac{\nabla}{u}$.

If $A \equiv A_1 \;\alpha\; A_2$, then by induction hypothesis there are derivations

$$\begin{array}{c}\nabla\\ \phi_1\,\Vert\, w\\ A_1\end{array} \qquad \text{and} \qquad \begin{array}{c}\nabla\\ \phi_2\,\Vert\, w\\ A_2\end{array} \quad .$$

We take

$$\phi \equiv \;=\frac{\nabla}{\boxed{\begin{array}{c}\nabla\\ \phi_1\,\Vert\, w\\ A_1\end{array}}\;\alpha\;\boxed{\begin{array}{c}\nabla\\ \phi_2\,\Vert\, w\\ A_2\end{array}}} \quad .$$

□

Notation 4.32. We will anotate generic weakenings with w, as in $\begin{array}{c}\nabla\\ \Vert\, w\\ A\end{array}$.

To permute contractive rules with other rules we will sometimes need to create cocontractive rules, just as is the case in the reduction $c{\downarrow}-c{\uparrow}$ presented in Section 4.1. However, unlike the atomic contraction case, we might create an arbitrarily big number of cocontractive rules. This is an important hurdle towards proving termination of the reduction system. To address this problem, we will show that it is possible to "move" the cocontractions created all together as a block, rather than one by one, therefore eliminating concerns about the size and number of the cocontractions created. Dually, we will show that it is possible to permute generic contractions as a whole with other rules, rather than contraction by contraction.

The following Lemma is instrumental in showing that the structure of generic contractions allows us to move them as a single block.

Lemma 4.33. *In a ν-contractive system, for every generic contraction*

$$\begin{array}{c} A \; \nu \; B \\ {\scriptstyle c} \| {\scriptstyle \phi} \\ M \; \beta \; N \end{array} ,$$

there are derivations

$$\begin{array}{c} A \\ \| {\scriptstyle =,w} \\ A_1 \; \beta \; A_2 \end{array} , \qquad \begin{array}{c} B \\ \| {\scriptstyle =,w} \\ B_1 \; \beta \; B_2 \end{array} ,$$

and generic contractions

$$\begin{array}{c} A_1 \; \nu \; B_1 \\ {\scriptstyle \phi_1} \| {\scriptstyle c} \\ M \end{array} , \qquad \begin{array}{c} A_2 \; \nu \; B_2 \\ {\scriptstyle \phi_2} \| {\scriptstyle c} \\ N \end{array} .$$

Proof. We proceed by induction on the number of contractive rules in ϕ, that we refer to as *size*.

- If the size is 0, then $M \; \beta \; N \equiv A \; \nu \; B$. We take $A_1 \equiv A, A_2 \equiv B, B_1 \equiv A, B_2 \equiv B$ and

$$\begin{array}{c} \quad \triangledown \\ A \; \nu \; \| {\scriptstyle w} \\ \quad B \end{array} , \qquad \begin{array}{c} \triangledown \quad \\ \| {\scriptstyle w} \; \nu \; B \\ A \quad \end{array} ,$$

  and

$$\begin{array}{c} A \; \nu \; A \\ {\scriptstyle \phi_1} \| {\scriptstyle c} \\ A \end{array} , \qquad \begin{array}{c} B \; \nu \; B \\ {\scriptstyle \phi_2} \| {\scriptstyle c} \\ B \end{array} .$$

- If the size is greater than 0, then

$$\phi \equiv \frac{(A_1 \; \beta \; A_2) \; \nu \; (B_1 \; \beta \; B_2)}{\boxed{\begin{array}{c} A_1 \; \nu \; B_1 \\ {\scriptstyle \phi_1} \| {\scriptstyle c} \\ M \end{array}} \; \beta \; \boxed{\begin{array}{c} A_2 \; \nu \; B_2 \\ {\scriptstyle \phi_2} \| {\scriptstyle c} \\ N \end{array}}} ,$$

  with $A \equiv A_1 \; \beta \; A_2, B \equiv B_1 \; \beta \; B_2$, and it is clear.

□

Notation 4.34. We will write

$$\frac{(A \; \beta \; B) \; \gamma \; (C \; \beta' \; D)}{(A \; \gamma \; C) \; \beta \; (B \; \gamma' \; D)}$$

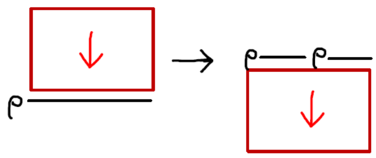
to represent both up and down-rules, i.e. either $\beta'{=}\beta$ and $\gamma'{=}\gamma^m$ or $\beta'{=}\beta^M$ and $\gamma'{=}\gamma$.

Definition 4.35. A *subatomic reduction rule* r for a system SA is a couple (ϕ', ψ') where ϕ' and ψ' are derivations in SA with $\mathsf{pr}\,\phi' \equiv \mathsf{pr}\,\psi'$ and $\mathsf{cn}\,\phi' \equiv \mathsf{cn}\,\psi'$. We write $r : \phi' \to \psi'$.

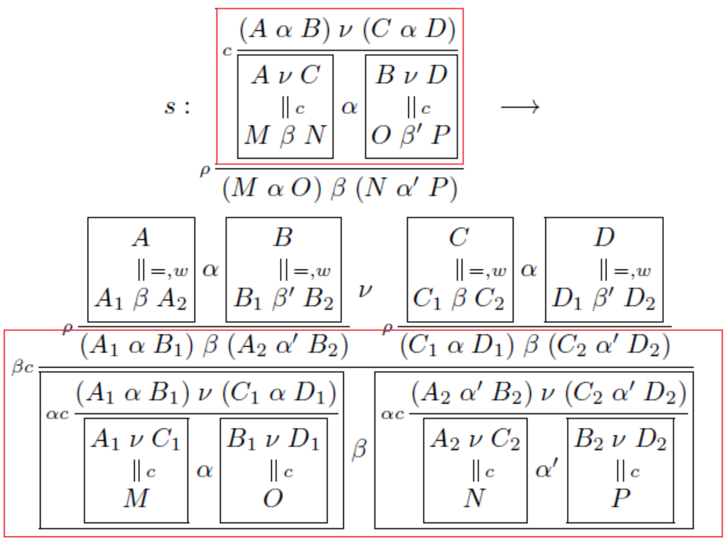
For every reduction rule $r : \phi' \to \psi'$ we define the reduction $\to_r$ such that $\phi \to_r \psi$ if and only if ψ' is a subderivation of ϕ and ψ is obtained from ϕ by replacing ϕ' by ψ'.

We call a finite set R of reduction rules a *rewriting system*. Given a set S of SA derivations, we say that rewriting system R is *weakly normalising on* S if for every $\phi \in S$ there is a finite chain $\phi \to_{r_1} \phi_1 \to_{r_2} \cdots \to_{r_n} \psi$ with $r_i \in R$ where no reduction rule of R can be applied to ψ.

The first family of reduction rules we present is akin to the rule $c{\uparrow} - c{\downarrow}$ for atomic flows.



Definition 4.36 (Decomposition rule s)**.** In a ν-contractive system, we define the following class of reduction rules:



where $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ are obtained from Lemma 4.33.

Since Lemma 4.33 holds for any ν-contractive system, this rewriting holds in any contractive system.

Example 4.37. The reduction rule $c{\uparrow} - c{\downarrow}$ for atomic flows is an instance of this reduction rule. Likewise, the reduction rule presented in [36] to permute atomic contractions and atomic cocontractions in linear logic is an instance of this reduction rule family:

$$
{\scriptstyle a\bar{c}}\dfrac{((\bot \mathbin{\&} \bot)\,a\,(1 \mathbin{\&} 1)) \oplus ((\bot \mathbin{\&} \bot)\,a\,(1 \mathbin{\&} 1))}{{\scriptstyle a\bar{c}}\dfrac{\boxed{{\scriptstyle \&c}\dfrac{(\bot \mathbin{\&} \bot) \oplus (\bot \mathbin{\&} \bot)}{\boxed{\dfrac{\bot \oplus \bot}{\bot}} \mathbin{\&} \boxed{\dfrac{\bot \oplus \bot}{\bot}}}} \; a \; \boxed{{\scriptstyle \&c}\dfrac{(1 \mathbin{\&} 1) \oplus (1 \mathbin{\&} 1)}{\boxed{\dfrac{1 \oplus 1}{1}} \mathbin{\&} \boxed{\dfrac{1 \oplus 1}{1}}}}}{(\bot\,a\,1) \mathbin{\&} (\bot\,a\,1)}} \quad \longrightarrow
$$

$$
{\scriptstyle \&c}\dfrac{\boxed{{\scriptstyle a\bar{c}}\dfrac{(\bot \mathbin{\&} \bot)\,a\,(1 \mathbin{\&} 1)}{(\bot\,a\,1) \mathbin{\&} (\bot\,a\,1)}} \oplus \boxed{{\scriptstyle ac}\dfrac{(\bot \mathbin{\&} \bot)\,a\,(1 \mathbin{\&} 1)}{(\bot\,a\,1) \mathbin{\&} (\bot\,a\,1)}}}{\boxed{{\scriptstyle ac}\dfrac{(\bot\,a\,1) \oplus (\bot\,a\,1)}{\boxed{\dfrac{\bot \oplus \bot}{\bot}} \; a \; \boxed{\dfrac{1 \oplus 1}{1}}}} \mathbin{\&} \boxed{{\scriptstyle ac}\dfrac{(\bot\,a\,1) \oplus (\bot\,a\,1)}{\boxed{\dfrac{\bot \oplus \bot}{\bot}} \; a \; \boxed{\dfrac{1 \oplus 1}{1}}}}}
$$

Example 4.38. We can apply an instance of this reduction rule to permute rule $\wedge c$ and rule $\wedge\!\downarrow$ of SAKS:

$$
{\scriptstyle \wedge\downarrow}\dfrac{{\scriptstyle \wedge c}\dfrac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}}{(A \wedge B) \vee (C \vee D)} \quad \longrightarrow
$$

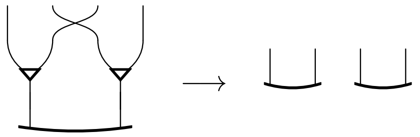
$$
{\scriptstyle \vee c}\dfrac{\boxed{{\scriptstyle \wedge\downarrow}\dfrac{\left(A \vee \begin{array}{c}\triangledown\\ \|{\scriptstyle w}\\ C\end{array}\right) \wedge \left(B \vee \begin{array}{c}\triangledown\\ \|{\scriptstyle w}\\ D\end{array}\right)}{(A \wedge B) \vee (C \vee D)}} \vee \boxed{{\scriptstyle \wedge\downarrow}\dfrac{\left(\begin{array}{l}\triangledown\\ \|{\scriptstyle w} \vee C\\ A\end{array}\right) \wedge \left(\begin{array}{l}\triangledown\\ \|{\scriptstyle w} \vee D\\ B\end{array}\right)}{(A \wedge B) \vee (C \vee D)}}}{\boxed{{\scriptstyle \wedge c}\dfrac{(A \wedge B) \vee (A \wedge B)}{\boxed{\begin{array}{c}A \vee A\\ \|{\scriptstyle c}\\ A\end{array}} \wedge \boxed{\begin{array}{c}B \vee B\\ \|{\scriptstyle c}\\ B\end{array}}}} \vee \boxed{{\scriptstyle \vee c}\dfrac{(C \vee D) \vee (C \vee D)}{\boxed{\begin{array}{c}C \vee C\\ \|{\scriptstyle c}\\ C\end{array}} \vee \boxed{\begin{array}{c}D \vee D\\ \|{\scriptstyle c}\\ D\end{array}}}}}
$$

Example 4.39. We can permute a generic contraction through a cut for example:

$$
{\scriptstyle \wedge c}\dfrac{((\mathsf{f}\,a\,\mathsf{t}) \wedge (\mathsf{t}\,a\,\mathsf{f})) \vee ((\mathsf{f}\,a\,\mathsf{t}) \wedge (\mathsf{t}\,a\,\mathsf{f}))}{{\scriptstyle a\uparrow}\dfrac{\boxed{{\scriptstyle ac}\dfrac{(\mathsf{f}\,a\,\mathsf{t}) \vee (\mathsf{f}\,a\,\mathsf{t})}{\boxed{{\scriptstyle =}\dfrac{\mathsf{f} \vee \mathsf{f}}{\mathsf{f}}} \; a \; \boxed{{\scriptstyle =}\dfrac{\mathsf{t} \vee \mathsf{t}}{\mathsf{t}}}}} \wedge \boxed{{\scriptstyle ac}\dfrac{(\mathsf{t}\,a\,\mathsf{f}) \vee (\mathsf{t}\,a\,\mathsf{f})}{\boxed{{\scriptstyle =}\dfrac{\mathsf{t} \vee \mathsf{t}}{\mathsf{t}}} \; a \; \boxed{{\scriptstyle =}\dfrac{\mathsf{f} \vee \mathsf{f}}{\mathsf{f}}}}}}{(\mathsf{f} \wedge \mathsf{t})\,a\,(\mathsf{t} \wedge \mathsf{f})}} \quad \longrightarrow
$$

$$
ac\,\dfrac{\boxed{a\!\uparrow\dfrac{(\mathsf{f}\;a\;\mathsf{t})\wedge(\mathsf{t}\;a\;\mathsf{f})}{(\mathsf{f}\wedge\mathsf{t})\;a\;(\mathsf{t}\wedge\mathsf{f})}}\vee\boxed{a\!\uparrow\dfrac{(\mathsf{f}\;a\;\mathsf{t})\wedge(\mathsf{t}\;a\;\mathsf{f})}{(\mathsf{f}\wedge\mathsf{t})\;a\;(\mathsf{t}\wedge\mathsf{f})}}}{\boxed{\wedge c\,\dfrac{(\mathsf{f}\wedge\mathsf{t})\vee(\mathsf{f}\wedge\mathsf{t})}{\boxed{=\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}\wedge\boxed{=\dfrac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}}}\;a\;\boxed{\wedge c\,\dfrac{(\mathsf{t}\wedge\mathsf{f})\vee(\mathsf{t}\wedge\mathsf{f})}{\boxed{=\dfrac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}\wedge\boxed{=\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}}}}
$$

We obtain the flow transformation:

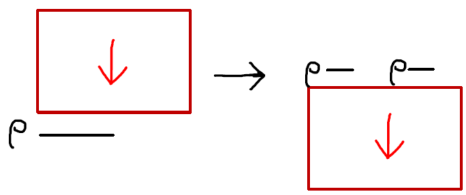


This transformation shows that permuting generic medials downwards in fact disconnects edges of an atomic flow. This is a fundamental advance allowing us to remove ai-cycles as we will show in the next chapter. This discovery has been made purely through the means of the subatomic methodology, and it suggests that by studying the behaviour of contractive rules in the same way that atomic flows study the behaviour of atomic contractions we can discover and characterise interesting properties of proof systems.

Since our aim is to permute generic contractions as a whole, we need to consider the case when a rule ρ occurs inside of them, such as

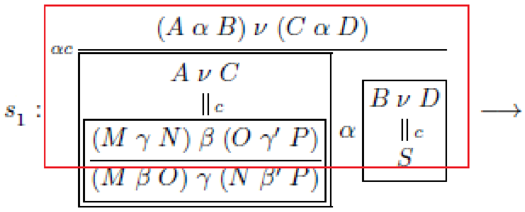
$$
c\,\dfrac{(A\;\alpha\;B)\;\nu\;(C\;\alpha\;D)}{\boxed{\begin{array}{c}A\;\nu\;C\\ c\,\|\\ \boxed{\dfrac{(M\;\gamma\;N)\;\beta\;(O\;\gamma'\;P)}{(M\;\beta\;O)\;\gamma\;(N\;\beta'\;P)}}\end{array}}\;\alpha\;\boxed{\begin{array}{c}B\;\nu\;D\\ c\,\|\\ S\end{array}}}\quad .
$$

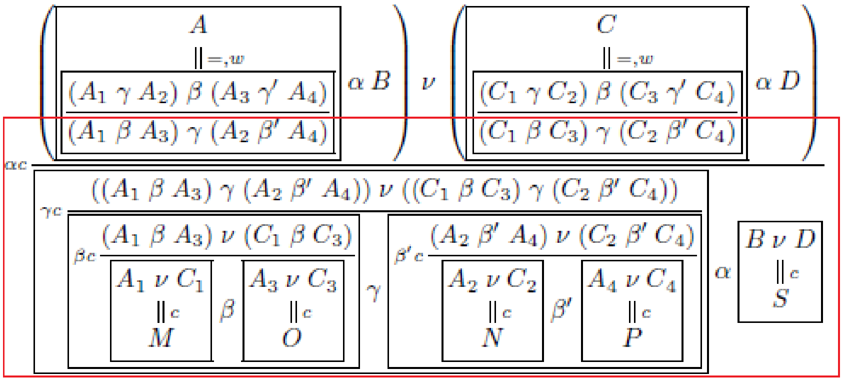
In this case, we could apply an instance of s and permute the generic contraction on the inside with ρ, followed by permuting the remaining rules of the generic contraction. However, to offer an advantage in termination arguments by being able to always move generic contractions as whole, we combine these two consecutive reductions in a single rule. We will name this rule s_n, where n is the depth at which the rule s is applied.



Definition 4.40. In a ν-contractive system, we define the following class of reduction

rules:

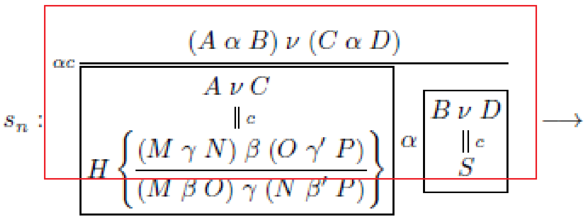


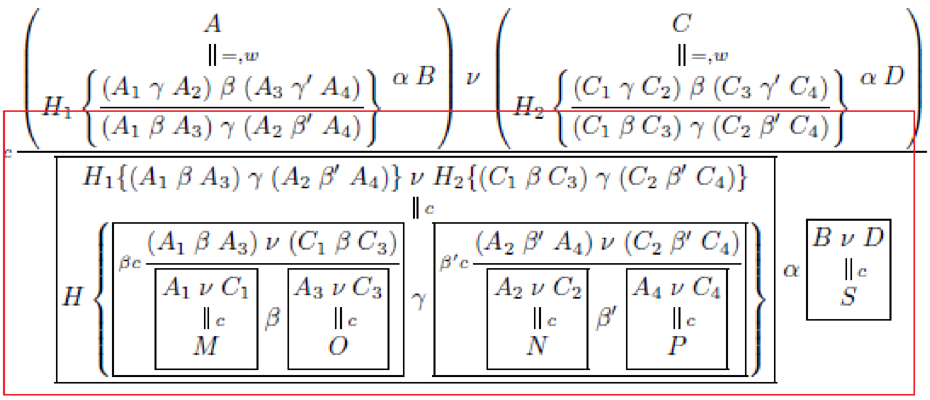


where $A_1, A_2, A_3, A_4, C_1, C_2, C_3, C_4$ are obtained from Lemma 4.33.

Likewise, we can extend it to the rule ρ being applied at any depth.

Definition 4.41. In a ν-contractive system, we define the following class of reduction rules:





where $A_1, A_2, A_3, A_4, C_1, C_2, C_3, C_4$ are obtained from Lemma 4.33.

Example 4.42. We can permute a generic contraction through an atomic contraction for example:

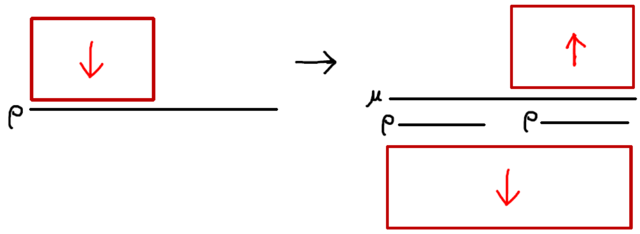
$$\alpha c\,\frac{(H_1\{(\mathsf{f}\,a_1\,\mathsf{t})\vee(\mathsf{f}\,a_2\,\mathsf{t})\}\ \alpha\ B)\vee(H_2\{(\mathsf{f}\,a_3\,\mathsf{t})\vee(\mathsf{f}\,a_4\,\mathsf{t})\}\ \alpha\ D)}{\boxed{\begin{array}{c}H_1\{(\mathsf{f}\,a_1\,\mathsf{t})\vee(\mathsf{f}\,a_2\,\mathsf{t})\}\vee H_2\{(\mathsf{f}\,a_3\,\mathsf{t})\vee(\mathsf{f}\,a_4\,\mathsf{t})\}\\ \|\,c\\ H\left\{\vee c\,\dfrac{((\mathsf{f}\,a_1\,\mathsf{t})\vee(\mathsf{f}\,a_2\,\mathsf{t}))\vee((\mathsf{f}\,a_3\,\mathsf{t})\vee(\mathsf{f}\,a_4\,\mathsf{t}))}{ac\,\dfrac{\boxed{c\,\dfrac{(\mathsf{f}\,a_1\,\mathsf{t})\vee(\mathsf{f}\,a_3\,\mathsf{t})}{\boxed{=\frac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}\ a_5\ \boxed{=\frac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}}}\vee\boxed{c\,\dfrac{(\mathsf{f}\,a_2\,\mathsf{t})\vee(\mathsf{f}\,a_4\,\mathsf{t})}{\boxed{=\frac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}\ a_6\ \boxed{=\frac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}}}}{(\mathsf{f}\vee\mathsf{f})\ a_7\ (\mathsf{t}\vee\mathsf{t})}}\right\}\end{array}}\ \alpha\ \boxed{\begin{array}{c}B\vee D\\ \|\,c\\ O\end{array}}}\quad\longrightarrow$$

$$\alpha c\,\frac{\left(H_1\left\{ac\,\dfrac{(\mathsf{f}\,a_1\,\mathsf{t})\vee(\mathsf{f}\,a_2\,\mathsf{t})}{(\mathsf{f}\vee\mathsf{f})\ a_8\ (\mathsf{t}\vee\mathsf{t})}\right\}\ \alpha\ B\right)\vee\left(H_2\left\{c\,\dfrac{(\mathsf{f}\,a_3\,\mathsf{t})\vee(\mathsf{f}\,a_4\,\mathsf{t})}{(\mathsf{f}\vee\mathsf{f})\ a_9\ (\mathsf{t}\vee\mathsf{t})}\right\}\ \alpha\ D\right)}{\boxed{\begin{array}{c}H_1\{(\mathsf{f}\vee\mathsf{f})\ a_8\ (\mathsf{t}\vee\mathsf{t})\}\vee H_2\{(\mathsf{f}\vee\mathsf{f})\ a_9\ (\mathsf{t}\vee\mathsf{t})\}\\ \|\,c\\ H\left\{ac\,\dfrac{((\mathsf{f}\vee\mathsf{f})\ a_8\ (\mathsf{t}\vee\mathsf{t}))\vee((\mathsf{f}\vee\mathsf{f})\ a_9\ (\mathsf{t}\vee\mathsf{t}))}{\boxed{\vee c\,\dfrac{(\mathsf{f}\vee\mathsf{f})\vee(\mathsf{f}\vee\mathsf{f})}{\boxed{=\frac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}\vee\boxed{=\frac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}}}\ a_7\ \boxed{\vee c\,\dfrac{(\mathsf{t}\vee\mathsf{t})\vee(\mathsf{t}\vee\mathsf{t})}{\boxed{=\frac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}\vee\boxed{=\frac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}}}}\right\}\end{array}}\ \alpha\ \boxed{\begin{array}{c}B\vee D\\ \|\,c\\ O\end{array}}}$$

where we numbered the occurrences of atoms for clarity.

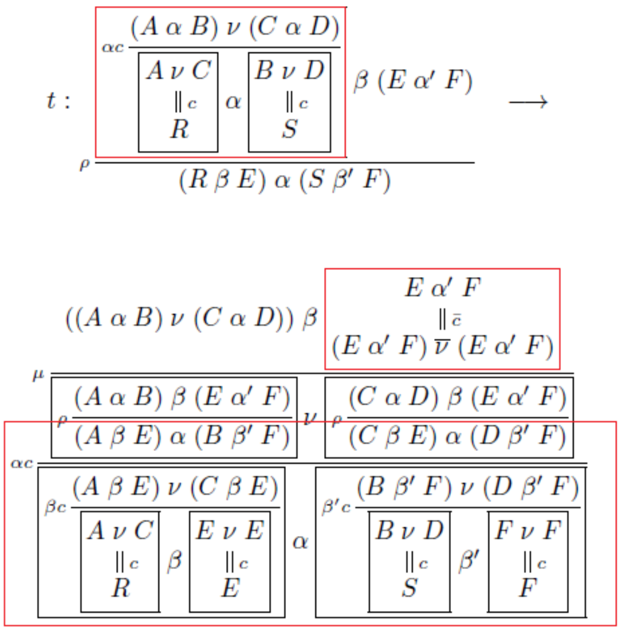
It is in the case where a generic contraction is "broken" by another rule where it has until now been unclear how to proceed. Just like in the reduction rule $c\!\downarrow -i\!\uparrow$, we might create cocontractions, but in this case we might obtain an arbitrarily big number of them.

The main contribution of this reduction rule is the fact that we can now consider all the cocontractions created as a single generic cocontraction block that we can move as a whole upwards in a proof, therefore not having to be concerned by its size.



Unlike for the previous reduction rules, the following rule is not always applicable. However, we can easily present sufficient conditions for its applicability, greatly simplifying the task of studying which contractions permute with every other rule.

Definition 4.43 (Reduction rule t)**.** If the rule $\mu \frac{(A \;\nu\; B) \;\beta\; (C \;\overline{\nu}\; D)}{(A \;\beta\; C) \;\nu\; (B \;\beta\; D)}$ is derivable in ν-contractive system SA we define the following family of rewriting rules:



Example 4.44. The reduction rule $c{\downarrow}{-}i{\uparrow}$ for classical logic is an instance of this reduction rule. Likewise, the reduction rule presented in [36] to permute atomic contractions and atomic cuts in linear logic is an instance of this reduction rule family:

$$ {\scriptstyle a\uparrow}\,\dfrac{\boxed{{\scriptstyle ac}\,\dfrac{(\bot\, a\, 1)\oplus(\bot\, a\, 1)}{\boxed{\dfrac{\bot\oplus\bot}{\bot}}\; a\; \boxed{\dfrac{1\oplus 1}{1}}}}\otimes(1\, a\,\bot)}{(\bot\otimes 1)\; a\;(1\otimes\bot)} \quad\longrightarrow\quad {\scriptstyle \oplus\uparrow}\,\dfrac{((\bot\, a\, 1)\oplus(\bot\, a\, 1))\otimes\boxed{{\scriptstyle a\bar{c}}\,\dfrac{\boxed{\dfrac{1}{1\mathbin{\&}1}}\; a\; \boxed{\dfrac{\bot}{\bot\mathbin{\&}\bot}}}{(1\, a\,\bot)\mathbin{\&}(1\, a\,\bot)}}}{{\scriptstyle ac}\,\dfrac{\boxed{{\scriptstyle a\uparrow}\,\dfrac{(\bot\, a\, 1)\otimes(1\, a\,\bot)}{(\bot\otimes 1)\; a\;(1\otimes\bot)}}\oplus\boxed{{\scriptstyle a\uparrow}\,\dfrac{(\bot\, a\, 1)\otimes(1\, a\,\bot)}{(\bot\otimes 1)\; a\;(1\otimes\bot)}}}{\boxed{\dfrac{(\bot\otimes 1)\oplus(\bot\otimes 1)}{\boxed{\dfrac{\bot\oplus\bot}{\bot}}\otimes\boxed{\dfrac{1\oplus 1}{1}}}}\; a\; \boxed{\dfrac{(1\otimes\bot)\oplus(1\otimes\bot)}{\boxed{\dfrac{1\oplus 1}{1}}\otimes\boxed{\dfrac{\bot\oplus\bot}{\bot}}}}}} \quad . $$

Example 4.45. In SAMALLS we have the following reduction rule:

$$ {\scriptstyle \otimes\uparrow}\,\dfrac{\boxed{{\scriptstyle \otimes c}\,\dfrac{((A\otimes B)\oplus(C\otimes D))}{((A\oplus C)\otimes(B\oplus D))}}\otimes(E\otimes F)}{((A\oplus C)\otimes E)\otimes((B\oplus D)\otimes F)} \quad\longrightarrow $$

$$ {\scriptstyle \oplus\uparrow}\,\dfrac{((A\otimes B)\oplus(C\otimes D))\otimes\boxed{{\scriptstyle \otimes\bar{c}}\,\dfrac{\boxed{\begin{array}{c}E\\ \|\,\bar{c}\\ E\mathbin{\&}E\end{array}}\otimes\boxed{\begin{array}{c}F\\ \|\,\bar{c}\\ F\mathbin{\&}F\end{array}}}{(E\otimes F)\mathbin{\&}(E\otimes F)}}}{{\scriptstyle \otimes c}\,\dfrac{\boxed{{\scriptstyle \otimes\uparrow}\,\dfrac{(A\otimes B)\otimes(E\otimes F)}{(A\otimes E)\otimes(B\otimes F)}}\oplus\boxed{{\scriptstyle \otimes\uparrow}\,\dfrac{(C\otimes D)\otimes(E\otimes F)}{(C\otimes E)\otimes(D\otimes F)}}}{\boxed{{\scriptstyle \otimes c}\,\dfrac{(A\otimes E)\oplus(C\otimes E)}{(A\oplus C)\mathbin{\&}\boxed{\begin{array}{c}E\oplus E\\ \|\,c\\ E\end{array}}}}\otimes\boxed{{\scriptstyle \otimes c}\,\dfrac{(B\otimes F)\oplus(D\otimes F)}{(B\oplus D)\mathbin{\&}\boxed{\begin{array}{c}F\oplus F\\ \|\,c\\ F\end{array}}}}}} $$

Thus, we can easily see if a contraction permutes through another rule in this way just by checking the existence of certain derivations, reducing the case by case analysis greatly. For example, we can see that in SAKS it is possible to move generic contractions with main relations $\wedge, a$ through every other possible rule. In SAMALLS it is possible to permute generic contractions with main relations $\otimes, a, \oplus$ through every rule.

Additionally, we present reduction rules regarding the interaction of generic contractions with equality rules and weakenings. Similarly to the reductions presented in Section 4.1, we replace them by equalities or weakenings. Since the only unit assignments for ν are the contraction equalities, we need only consider four cases:

$$e_1 : \quad {=}\genfrac{}{}{0.5pt}{}{\alpha c\genfrac{}{}{0.5pt}{}{(A\ \alpha\ u_\alpha)\ \nu\ (C\ \alpha\ u_\alpha)}{\boxed{\begin{matrix}A\ \nu\ C\\ \Vert\, c\\ M\end{matrix}}\ \alpha\ \boxed{=\genfrac{}{}{0.5pt}{}{u_\alpha\ \nu\ u_\alpha}{u_\alpha}}}}{M} \quad\rightarrow\quad \begin{matrix}\boxed{=\genfrac{}{}{0.5pt}{}{A\ \alpha\ u_\alpha}{A}}\ \nu\ \boxed{=\genfrac{}{}{0.5pt}{}{C\ \alpha\ u_\alpha}{C}}\\ \Vert\, c\\ M\end{matrix}$$

$$e_2 : \quad {=}\genfrac{}{}{0.5pt}{}{\alpha c\genfrac{}{}{0.5pt}{}{(v\ \alpha\ w)\ \nu\ (v\ \alpha\ w)}{\boxed{=\genfrac{}{}{0.5pt}{}{v\ \nu\ v}{v}}\ \alpha\ \boxed{=\genfrac{}{}{0.5pt}{}{w\ \nu\ w}{w}}}}{u} \quad\rightarrow\quad {=}\genfrac{}{}{0.5pt}{}{\boxed{=\genfrac{}{}{0.5pt}{}{v\ \alpha\ w}{u}}\ \nu\ \boxed{=\genfrac{}{}{0.5pt}{}{v\ \alpha\ w}{u}}}{u}$$

$$e_3 : \quad \alpha c\genfrac{}{}{0.5pt}{}{(A\ \alpha\ \triangledown)\ \nu\ (C\ \alpha\ D)}{\boxed{\begin{matrix}A\ \nu\ C\\ \Vert\, c\\ M\end{matrix}}\ \alpha\ \boxed{=\genfrac{}{}{0.5pt}{}{\triangledown\ \nu\ D}{D}}} \rightarrow \alpha c\genfrac{}{}{0.5pt}{}{\left(A\ \alpha\ \begin{matrix}\triangledown\\ \Vert\, w\\ D\end{matrix}\right)\ \nu\ (C\ \alpha\ D)}{\boxed{\begin{matrix}A\ \nu\ C\\ \Vert\, c\\ M\end{matrix}}\ \alpha\ \boxed{\begin{matrix}D\ \nu\ D\\ \Vert\, c\\ D\end{matrix}}}$$

$$e_4 : \quad {=}\genfrac{}{}{0.5pt}{}{\triangledown\ \nu\ \triangledown}{w\genfrac{}{}{0.5pt}{}{\triangledown}{u}} \quad\rightarrow\quad {=}\genfrac{}{}{0.5pt}{}{\boxed{w\genfrac{}{}{0.5pt}{}{\triangledown}{u}}\ \nu\ \boxed{w\genfrac{}{}{0.5pt}{}{\triangledown}{u}}}{u}$$

$$e_5 : \quad {=}\genfrac{}{}{0.5pt}{}{{=}\genfrac{}{}{0.5pt}{}{u\ \nu\ u}{u}}{v\ \alpha\ w} \quad\rightarrow\quad \alpha c\genfrac{}{}{0.5pt}{}{\boxed{=\genfrac{}{}{0.5pt}{}{u}{v\ \alpha\ w}}\ \nu\ \boxed{=\genfrac{}{}{0.5pt}{}{u}{v\ \alpha\ w}}}{\boxed{=\genfrac{}{}{0.5pt}{}{v\ \nu\ v}{v}}\ \alpha\ \boxed{=\genfrac{}{}{0.5pt}{}{w\ \nu\ w}{w}}}$$

Last, we can define the trivial reduction rules

$$i_1 : \quad \rho\genfrac{}{}{0.5pt}{}{H\left\{\begin{matrix}A\\ \Vert\, c\\ B\end{matrix}\right\}}{H'\{B\}} \quad\longrightarrow\quad \rho\genfrac{}{}{0.5pt}{}{H\{A\}}{H'\left\{\begin{matrix}A\\ \Vert\, c\\ B\end{matrix}\right\}}$$

$$i_2 : \quad \alpha c\genfrac{}{}{0.5pt}{}{(A'\ \alpha\ B)\ \nu\ (C\ \nu\ D)}{\left(\rho\genfrac{}{}{0.5pt}{}{A'}{A}\ \nu\ C\right)\ \alpha\ (B\ \nu\ D)} \quad\longrightarrow\quad \alpha c\genfrac{}{}{0.5pt}{}{\left(\rho\genfrac{}{}{0.5pt}{}{A'}{A}\ \alpha\ B\right)\ \nu\ (C\ \nu\ D)}{(A\ \nu\ C)\ \alpha\ (B\ \nu\ D)}$$

We can easily extend the rewriting rules presented to the symmetrical cases, such as

$$c\,\frac{(A\;\alpha\;B)\;\nu\;(C\;\alpha\;D)}{\boxed{\begin{array}{c}A\;\nu\;C\\ \Vert\, c\\ S\end{array}}\;\alpha\;\boxed{\begin{array}{c}B\;\nu\;D\\ c\,\Vert\\ \boxed{\dfrac{(M\;\gamma\;N)\;\beta\;(O\;\gamma'\;P)}{(M\;\beta\;O)\;\gamma\;(N\;\beta'\;P)}}\end{array}}}$$

for s_1 or

$$\rho\,\frac{(E\;\alpha'\;F)\;\beta\;\boxed{c\,\dfrac{(A\;\alpha\;B)\;\nu\;(C\;\alpha\;D)}{\boxed{\begin{array}{c}A\;\nu\;C\\ c\,\Vert\\ R\end{array}}\;\alpha\;\boxed{\begin{array}{c}B\;\nu\;D\\ c\,\Vert\\ S\end{array}}}}}{(E\;\beta\;R)\;\alpha\;(F\;\beta'\;S)}$$

for t.

Likewise, we can take the duals of these reductions to present a reduction rule system to permute generic cocontractions upwards in a derivation.

At this point, preservation of interpretability is not a concern: we want to permute generic contractions that correspond to generic contractions in the 'original' system. Interpretability is trivially preserved by these reductions applied to SAKS, since they do not introduce atoms in the scope of atoms, and every other configuration is interpretable in SAKS. However, in SAMALLS there are formulae such as 1⅋1 that are not interpretable when in the scope of an atom. For interpretability to be preserved, we need to ensure that changing the order of the application of rules does not introduce such uninterpretable formulae in the scope of atoms.

Nonetheless, for now we are not concerned about the general preservation of interpretability. For example, for system SKS we simply want to present reduction rules to permute generic contractions composed of medials m, associativity and commutativity of $\vee$ and of atomic contractions $ac\!\downarrow$. We can take the representations of these generic contractions in SAKS and study the specific reductions for them. It is easy to see that these reductions are all interpretable. If ρ does not involve atoms, then the generic contractions with main relation a remain untouched and are therefore still interpreted as $\dfrac{a \vee a}{a}$. Thus, we are only interested in studying reductions where contractions with main relation a appear, which is easily done. If ρ involves atoms, then the only possible cases are those coming from instances of the reductions in examples 4.37, 4.39, 4.42 and 4.44, which are all interpretable. Therefore, we can permute all generic contractions in SKS, and likewise in SMALLS. We will in fact use exactly these reductions in the next chapter to provide a procedure for cycle-elimination.

In this way, we can recover the rewriting systems C and Q of the previous section.

Definition 4.46. We define rewriting system C′ for SAKS as the system given by the instances of the general reductions s, t, e for generic contractions γ of the form

$$\gamma = \dfrac{(\mathsf{f}\; a\; \mathsf{t}) \vee (\mathsf{f}\; a\; \mathsf{t})}{\boxed{\dfrac{\mathsf{f} \vee \mathsf{f}}{\mathsf{f}}}\; a\; \boxed{\dfrac{\mathsf{t} \vee \mathsf{t}}{\mathsf{t}}}} \quad \text{and} \quad \gamma = \dfrac{(\mathsf{t}\; a\; \mathsf{f}) \vee (\mathsf{t}\; a\; \mathsf{f})}{\boxed{\dfrac{\mathsf{t} \vee \mathsf{t}}{\mathsf{t}}}\; a\; \boxed{\dfrac{\mathsf{f} \vee \mathsf{f}}{\mathsf{f}}}} \quad .$$

We define rewriting system Q′ for SAMALLS as the system given by the instances of the general reductions s, t, e for generic contractions γ of the form

$$\gamma = \dfrac{(\bot\; a\; 1) \oplus (\bot\; a\; 1)}{\boxed{\dfrac{\bot \oplus \bot}{\bot}}\; a\; \boxed{\dfrac{1 \oplus 1}{1}}} \quad \text{and} \quad \gamma = \dfrac{(1\; a\; \bot) \oplus (1\; a\; \bot)}{\boxed{\dfrac{1 \oplus 1}{1}}\; a\; \boxed{\dfrac{\bot \oplus \bot}{\bot}}} \quad .$$

These systems correspond exactly to the rewriting systems defined in the previous section, and therefore termination can be proved in the same way. We define ai-cycles for SAKS and SAMALLS in identical fashion as in the previous section: they correspond to the connexion of an atomic introduction and an atomic cut.

Theorem 4.47. *Rewriting system* C′ *is terminating on the set of ai-cycle-free derivations.*

Theorem 4.48. *Rewriting system* Q′ *is terminating on the set of ai-cycle-free derivations.*

Furthermore, with these rules we can consider rewriting systems for SAKS and for SAMALLS that would allow us to obtain full decompositon theorems for classical logic and for multiplicative additive linear logic.

As we showed in Section 4.1, in SAKS and SAMALLS there are derivations with *ai*-cycles where the reductions for atomic contractions do not terminate. When considering the reduction rules for other relations, we increase the type of cycles that can lead to non-termination. However, in both SAKS and SAMALLS every such cycle will originate from the presence of a "critical medial" which we will define in the next chapter. By permuting the widest generic (co)contraction first we can therefore guarantee that it is not in a cycle, and thus we obtain a normalisation strategy. To prove termination we only need to find an adequate notion of *rank* for generic (co)contractions, where the rank of the generic (co)contractions not involved in a reduction is maintained. Finding the appropriate notion of rank will be the focus of future research.

Definition 4.49. We define rewriting system D for SAKS as the system given by the general reductions s, t, e, the symmetric reductions, and the dual reductions for generic contractions with main relations $\wedge, \vee, a$.

We define rewriting system G for SAMALLS as the system given by the general reductions s, t, e, the symmetric reductions, and the dual reductions for generic contractions with main relations $\otimes, \&, a, \oplus, \parr$.

Conjecture 4.50. *System* D *is weakly normalising on tame proofs.*

Normalisation through system G is slightly more complex: generic contractions with main relation $\parr$ do not permute with the associativity rule for $\parr$. Thus, the focus of the reduction should be to permute every other generic contraction. This should not be a problem, but the notion of rank of a generic contraction will have to be adapted to take that into account.

In both systems the decomposition results affecting atomic (co)weakenings are very simple, since every reduction rule reduces the number of atoms in a derivation. Therefore, once the reductions of D and G have been applied, atomic weakenings can be permuted since they do not introduce any new generic (co)contractions as we noted in the previous section. Unitary weakenings remain in the proof, but they can in most cases be replaced by instances of linear rules: in classical logic for example, the inference $\frac{\mathsf{f}}{\mathsf{t}}$ can be obtained from the rule $\wedge\!\downarrow$.

By presenting these general reduction rules we have shown that the atomic decomposition results for classical logic and linear logic correspond to the same phenomenon: both rewriting systems exploit the shape of atomic contractions to be able to permute them with other rules.

Furthermore, by being able to permute generic contractions together, we advance towards proving a full decomposition theorem for classical logic and multiplicative additive linear logic, which will be the focus of future research.

Another area of further research will be the exploration of the similarities between the general reduction rules that we presented and the duplication rules for sharing graphs [17]. In fact these similarities are perhaps not so surprising, since there is a Curry–Howard correspondence between well-formed interaction nets and a deep-inference deduction system based on linear logic [12]: decomposition in this system via the general rules of this chapter might well correspond to the duplication rules of sharing graphs.

In the next chapter we will present an application of the general reduction rules: the elimination of ai-cycles in both logics as a local procedure.

Chapter 5

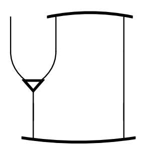
Removing cycles

As we saw in the previous chapter, atomic contractions and atomic cocontractions can be permuted downwards/upwards in a classical logic derivation in the absence of ai-cycles. Identically, the result holds for multiplicative additive linear logic.

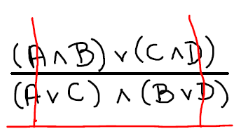
Our goal in this chapter is to take advantage of the reductions presented in the previous chapter to show that we can remove ai-cycles without recurring to cut-elimination, therefore proving the independence of the decomposition and the cut-elimination procedures.

Furthermore, the phenomenon of cycles has been studied in the sequent calculus, where it has been shown that it is possible to remove them through a procedure of quadratic-time complexity [9]. With the procedure we present in what follows, we hope to be able to study the complexity cost of cycle-elimination in deep inference in future research.

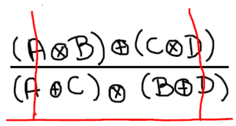
Cycles are a particular construction caused by the 'connection' of an introduction and a cut, as we saw in Section 4.1:



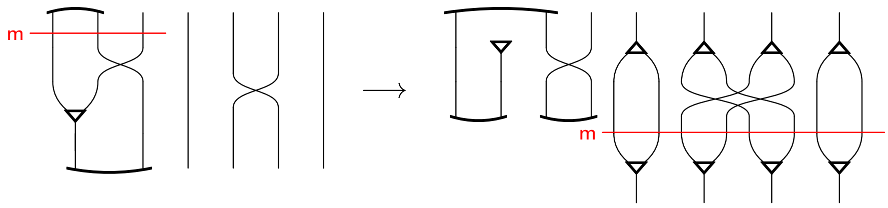
For an ai-cycle to occur in classical logic, two edges of an atomic flow that were related by $\vee$ at the top of the flow have to be connected by $\wedge$ at the bottom of the flow. Therefore, an instance of a rule that changes the main relation between formulae from $\alpha \neq \wedge$ to $\wedge$ needs to occur, containing the atoms involved in the cycle. In SKS, the only such rule is m.



Likewise, for an *ai*-cycle to occur in multiplicative additive linear logic, an instance of a rule that changes the main relation between formulae from $\alpha \neq \otimes$ to $\otimes$ has to occur. The only such rule is $\otimes c$.



Following this observation, and with the reduction rules of the previous section as tools, the procedure to remove cycles is very simple. We can easily permute these critical instances of generic contractions with main connective $\wedge$ or $\otimes$ downwards in a proof, together with all the generic contractions with main connective $\wedge$ or $\otimes$ between them and the cut-rule. When at the end of the procedure there are no remaining critical contractions above the cut, the cycle will have disappeared.



This idea of removing cycles by starting from the 'critical medial' has in fact yielded two methods for the elimination of cycles: the one presented in what follows, and the one presented in [23], that will both be studied to ascertain the complexity cost of each procedure.

To show the termination of our procedure, we only need to show that no new cycles are created by the application of the reduction rules. We will show it atomically rather than subatomically for ease of following the flows.

Definition 5.1. We define the rules

$$\vee c \frac{(A \vee B) \vee (C \vee D)}{(A \vee C) \vee (B \vee D)} \quad ,$$

$$\wedge \bar{c} \frac{(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)} \quad ,$$

$$\wedge\!\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \vee B) \vee (C \wedge D)} \quad .$$

Proposition 5.2. *In an* SKS *proof, we can replace every instance of associativity and commutatitvity of* $\vee$ *by instances of the rule* $\vee c$ *and the unit rule for* $\vee$*, and every instance of associativity and commutatitvity of* $\wedge$ *by instances of the rule* $\wedge \bar{c}$ *and the unit rule for* $\wedge$*. Furthermore, we can replace every instance of the rule* s *by instances of the rule* $\wedge\!\downarrow$ *and the unit rule for* $\vee$*.*

Proof. We replace

$$=\frac{(A \vee B) \vee C}{A \vee (B \vee C)}$$

by

$$=\frac{\vee c\frac{=\frac{(A \vee B) \vee C}{(A \vee B) \vee (\mathsf{f} \vee C)}}{(A \vee \mathsf{f}) \vee (B \vee C)}}{A \vee (B \vee C)} \quad .$$

We replace

$$=\frac{A \vee B}{B \vee A}$$

by

$$=\frac{\vee c\frac{=\frac{A \vee B}{(\mathsf{f} \vee A) \vee (B \vee \mathsf{f})}}{(\mathsf{f} \vee B) \vee (A \vee \mathsf{f})}}{B \vee A} \quad .$$

We proceed identically for $\wedge$.

We replace

$$=\frac{(A \vee B) \wedge C}{A \vee (B \wedge C)}$$

by

$$=\frac{\wedge\downarrow\frac{=\frac{(A \vee B) \wedge C}{(A \vee B) \wedge (\mathsf{f} \vee C)}}{(A \vee \mathsf{f}) \vee (B \wedge C)}}{A \vee (B \wedge C)} \quad .$$

□

We will proceed in system SKS with these replacements. This small change does not warrant a change of name, and we will therefore still refer to these derivations as SKS derivations.

Definition 5.3. *Generic contractions* are defined recursively as follows:

- The empty derivation is a generic contraction ;
- The rules $=\frac{\mathsf{f} \vee \mathsf{f}}{\mathsf{f}}$ and $=\frac{\mathsf{t} \vee \mathsf{t}}{\mathsf{t}}$ are generic contractions ;

- A derivation

$$c\,\frac{(A \;\alpha\; B) \vee (C \;\alpha\; D)}{\boxed{\begin{array}{c} A \vee C \\ \phi_1 \,\|\, c \\ R \end{array}} \;\alpha\; \boxed{\begin{array}{c} B \vee D \\ \phi_2 \,\|\, c \\ S \end{array}}}$$

  is a generic contraction if c is an instance of the rules m, $ac{\downarrow}$ or $\vee c$ and ϕ_1 and ϕ_2 are generic contractions. In this case, we say that it is a generic contraction with *main relation* α.

We will permute critical generic contractions with main relation $\wedge$ downwards in a proof, until they are no longer in a cycle. We will do so with the reduction rules defined in the previous chapter applied to SKS.

The reduction rules where ρ does not involve atoms are trivially applicable to SKS since they only involve switches $\wedge{\downarrow}$ and medials m. We will simply observe how the cases where ρ involves atoms are represented atomically, rather than subatomically.

If ρ involves atoms, we are either in the case of the reductions $c{\downarrow} - c{\uparrow}$ and $c{\downarrow} - i{\downarrow}$, in the cases of examples 4.39 and 4.42 or in the case of an equality reduction inside of an application of s_n.

In those cases, we obtain the atomic reductions

$$r_1 : \quad ai{\uparrow}\,\frac{\boxed{m\,\dfrac{(a \wedge \bar{a}) \vee (a \wedge \bar{a})}{\boxed{ac\,\dfrac{a \vee a}{a}} \wedge \boxed{ac\,\dfrac{\bar{a} \vee \bar{a}}{\bar{a}}}}}}{\mathsf{f}} \quad \longrightarrow \quad =\frac{\boxed{ai{\uparrow}\,\dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee \boxed{ai{\uparrow}\,\dfrac{a \wedge \bar{a}}{\mathsf{f}}}}{\mathsf{f}}$$

$$r_2 : \quad \vee c\,\frac{(a_1 \vee a_2) \vee (a_3 \vee a_4)}{\boxed{ac{\downarrow}\,\dfrac{\boxed{ac{\downarrow}\,\dfrac{a_1 \vee a_3}{a_5}} \vee \boxed{ac{\downarrow}\,\dfrac{a_2 \vee a_3}{a_6}}}{a_7}}} \quad \longrightarrow \quad ac{\downarrow}\,\frac{\boxed{ac{\downarrow}\,\dfrac{a_1 \vee a_2}{a_8}} \vee \boxed{ac{\downarrow}\,\dfrac{a_3 \vee a_4}{a_9}}}{a_7}$$

$$e_a : \quad aw{\uparrow}\,\frac{ac{\downarrow}\,\dfrac{a \vee a}{a}}{\mathsf{t}} \quad \longrightarrow \quad =\frac{\boxed{aw{\downarrow}\,\dfrac{a}{\mathsf{t}}} \vee \boxed{aw{\downarrow}\,\dfrac{a}{\mathsf{t}}}}{\mathsf{t}}$$

that we can apply at any depth inside of the generic contraction to obtain the atomic instances of the reduction rule s_n.

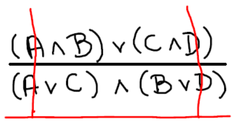
Theorem 5.4. *Given a derivation*

$$\begin{array}{c} A \\ \phi \, \| \, \mathsf{SKS} \\ B \end{array}$$

with an ai-cycle, there exists an ai-cycle-free derivation

$$\begin{array}{c} A \\ \phi' \, \| \, \mathsf{SKS} \\ B \end{array} \quad .$$

Proof. For an ai-cycle to occur in an SKS derivation, two atoms that were related by $\vee$ at the top of the derivation have to end up connected by $\wedge$ lower in the derivation. Therefore, an instance of a rule that changes the main relation between formulae from $\alpha \neq \wedge$ to $\wedge$ needs to occur, containing the atoms involved in the cycle. In SKS, the only such rule is m.



Therefore, there is at least an instance of a *critical medial* that contains the atoms involved in the cut on each side of the relation $\vee$ in the premiss. In particular, this medial is the top rule of a generic contraction with main relation $\wedge$.

We permute every critical generic contraction with main relation $\wedge$ downwards in the derivation via the reduction rules. These generic contractions permute with every rule except with other generic contractions with main relation $\wedge$. If there is such a generic contraction between them and the cut, we permute it downwards as well.

When the critical medials are permuted below the cut of the cycle, they no longer remain critical, and therefore the cycle disappears. We only need to show that when permuting a critical medial downwards we do not create new critical medials i.e. that we do not create new cycles.

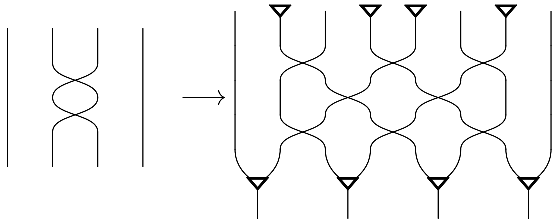
We remark that the flows of the atoms not involved in a reduction step remain unchanged, and therefore we only need to observe the flows of the atoms involved in each possible reduction step:

- It is easy to see in SAKS that instances of s where ρ is a rule that does not involve atoms do not change the links between the existing edges of a flow. They merely create two "smaller" instances of ρ that do not involve atoms and therefore do not break or change any existing connections.

For example, reductions of the form

$$
\begin{array}{c}
(A\wedge B)\vee(C\wedge D)\\
\hline
\boxed{\begin{array}{c}A\vee C\\ \| c\\ M\vee N\end{array}}\wedge\boxed{\begin{array}{c}B\vee D\\ \| c\\ O\vee P\end{array}}\\
\hline
(M\wedge O)\vee(N\vee P)
\end{array}
\begin{array}{l}\\ m\\ \\ s\end{array}
\longrightarrow
=\frac{\boxed{s\,\dfrac{A\wedge B}{(A_1\wedge B_1)\vee(A_2\vee B_2)}}\vee\boxed{s\,\dfrac{C\wedge D}{(C_1\wedge D_1)\vee(C_2\vee D_2)}}}{\boxed{m\,\dfrac{(A_1\wedge B_1)\vee(C_1\wedge D_1)}{\boxed{\begin{array}{c}A_1\vee C_1\\ \| c\\ M\end{array}}\wedge\boxed{\begin{array}{c}B_1\vee C_1\\ \| c\\ O\end{array}}}}\vee\boxed{=\dfrac{(A_2\vee B_2)\vee(C_2\vee D_2)}{\boxed{\begin{array}{c}A_2\vee C_2\\ \| c\\ N\end{array}}\vee\boxed{\begin{array}{c}B_2\vee C_2\\ \| c\\ P\end{array}}}}}
$$

create two switch rules instead of one, but do not change the links between the edges of the flow. They might introduce some weakenings and contractions, like in example 4.38.

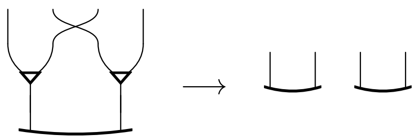


It is likewise for s applied at any depth, i.e. for any application of s^n where ρ is a rule that does not involve atoms.

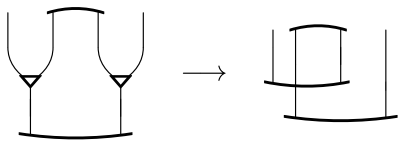
- Instances of s_n where ρ is a rule that involves atoms can only come from three cases:
  - From the reduction $c{\downarrow}-c{\uparrow}$ which does not introduce cycles,
  - From a reduction of the form

$$
m\,\frac{(H_1\{a\wedge\bar{a}\}\wedge B)\vee(H_2\{a\wedge\bar{a}\}\wedge D)}{\boxed{\begin{array}{c}H_1\{a\wedge\bar{a}\}\vee H_2\{a\wedge\bar{a}\}\\ \psi\,\| c\\ H\left\{\begin{array}{c} m\,\dfrac{(a\wedge\bar{a})\vee(a\wedge\bar{a})}{ai{\uparrow}\,\dfrac{\boxed{ac\,\dfrac{a\vee a}{a}}\wedge\boxed{ac\,\dfrac{\bar{a}\vee\bar{a}}{\bar{a}}}}{\mathsf{f}}}\end{array}\right\}\end{array}}\wedge\boxed{\begin{array}{c}B\vee D\\ \| c\\ O\end{array}}}
\longrightarrow
$$

$$
m\,\dfrac{\left(H_1\left\{{}_{ai\uparrow}\dfrac{a\wedge\bar{a}}{\mathsf{f}}\right\}\wedge B\right)\vee\left(H_2\left\{{}_{ai\uparrow}\dfrac{a\wedge\bar{a}}{\mathsf{f}}\right\}\wedge D\right)}{\boxed{\begin{array}{c}H_1\{\mathsf{f}\}\vee H_2\{\mathsf{f}\}\\ \psi\,\|\,c\\ H\left\{\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}\right\}\end{array}}\wedge\boxed{\begin{array}{c}B\vee D\\ \|\,c\\ O\end{array}}}
$$



where it is clear that we do not form new cycles, since the edges connected by a cut-rule after the reduction were already connected by a cut-rule before the reduction. It is precisely from an instance of this transformation that the cycle will be broken.

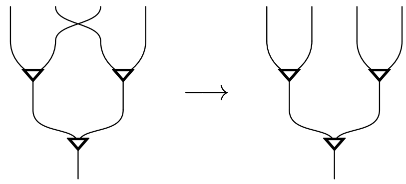


– Or from a reduction of the form

$$
m\,\dfrac{(H_1\{a_1\vee a_2\}\wedge B)\vee(H_2\{a_3\vee a_4\}\wedge D)}{\boxed{\begin{array}{c}H_1\{a_1\vee a_2\}\vee H_2\{a_3\vee a_4\}\\ \psi\,\|\,c\\ H\left\{{}_{c}\dfrac{=\dfrac{(a_1\vee a_2)\vee(a_3\vee a_4)}{\boxed{{}_{c}\dfrac{a_1\vee a_3}{a_5}}\vee\boxed{{}_{c}\dfrac{a_2\vee a_4}{a_6}}}}{a_7}\right\}\end{array}}\wedge\boxed{\begin{array}{c}B\vee D\\ \|\,c\\ O\end{array}}}\longrightarrow
$$

$$
m\,\dfrac{\left(H_1\left\{{}_{c}\dfrac{a_1\vee a_2}{a_8}\right\}\wedge B\right)\vee\left(H_2\left\{{}_{c}\dfrac{a_3\vee a_4}{a_9}\right\}\vee D\right)}{\boxed{\begin{array}{c}H_1\{a_8\}\vee H_2\{a_9\}\\ \psi\,\|\,c\\ H\left\{{}_{c}\dfrac{a_8\vee a_9}{a_7}\right\}\end{array}}\wedge\boxed{\begin{array}{c}B\vee D\\ \|\,c\\ O\end{array}}}
$$

where we numbered the occurrences of atoms for clarity, which clearly does not introduce cycles.

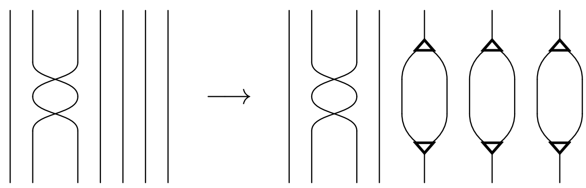


- Likewise, instances of t where ρ is a rule that does not involve atoms do not change the links between the existing edges of a flow. They might bifurcate previously "single" edges.

  For example, reductions of the form

$$
s\,\frac{\boxed{m\,\dfrac{(A\wedge B)\vee(C\wedge D)}{\boxed{\begin{array}{c}A\vee C\\ \|c\\ M\end{array}}\wedge\boxed{\begin{array}{c}B\vee D\\ \|c\\ N\end{array}}}}\wedge(E\vee F)}{(M\wedge E)\vee(N\wedge F)}
\quad\longrightarrow\quad
=\frac{s\,\dfrac{((A\wedge B)\vee(C\wedge D))\wedge\boxed{\begin{array}{c}E\vee F\\ \|\bar{c}\\ (E\vee F)\wedge(E\vee F)\end{array}}}{\boxed{s\,\dfrac{(A\wedge B)\wedge(E\vee F)}{(A\wedge E)\vee(B\wedge F)}}\vee\boxed{s\,\dfrac{(C\wedge D)\wedge(E\vee F)}{(C\wedge E)\vee(D\wedge F)}}}}{\boxed{m\,\dfrac{(A\wedge E)\vee(C\wedge E)}{\boxed{\begin{array}{c}A\vee C\\ \|c\\ M\end{array}}\wedge\boxed{\begin{array}{c}E\vee E\\ \|c\\ E\end{array}}}}\vee\boxed{m\,\dfrac{(B\wedge F)\vee(D\wedge F)}{\boxed{\begin{array}{c}B\vee D\\ \|c\\ N\end{array}}\wedge\boxed{\begin{array}{c}F\vee F\\ \|c\\ F\end{array}}}}}
$$

  simply create instances of the switch rule, and do not change any links between the edges of the flow.



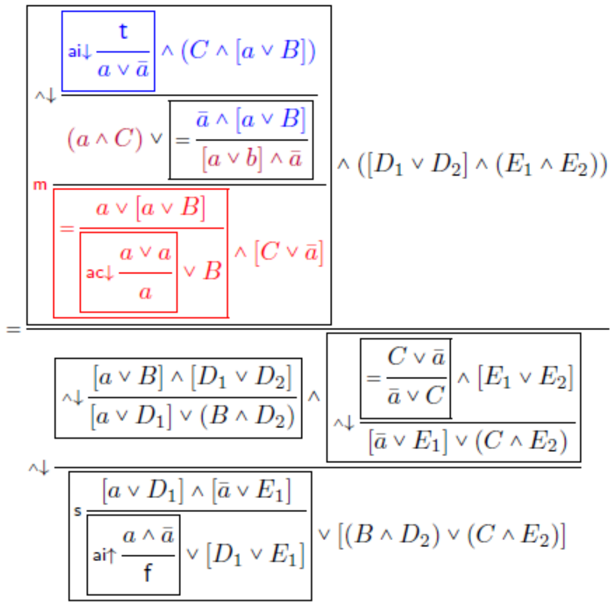
- Instances of t where ρ involves atoms do not occur when permuting generic medials with main relation $\vee$.

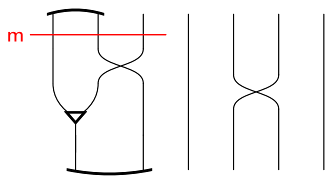
- Evidently, reduction rules e do not create new cycles since they only concern units and merely create weakenings, and reduction rules i do not change the flow of a derivation.

$\square$

Identically, we can check that the reductions for permuting generic contractions with main relation $\otimes$ in SAMALLS do not create new ai-cycles, and therefore remove cycles in SMALLS.

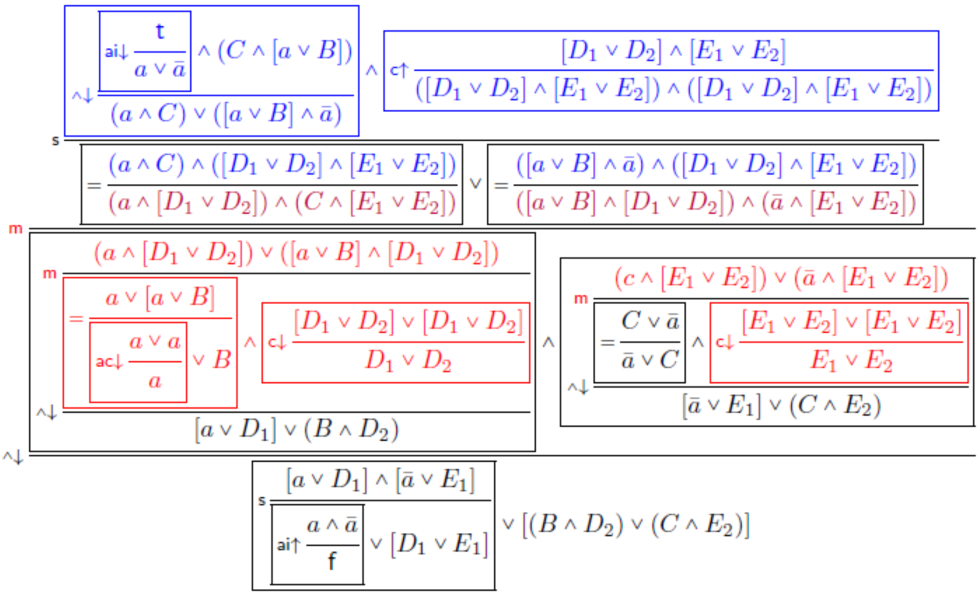
Example 5.5. We will remove the cycle in the following derivation:

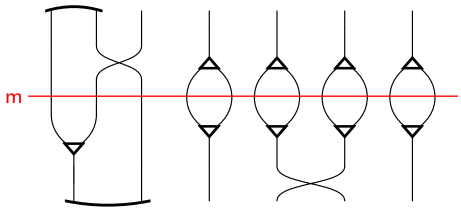




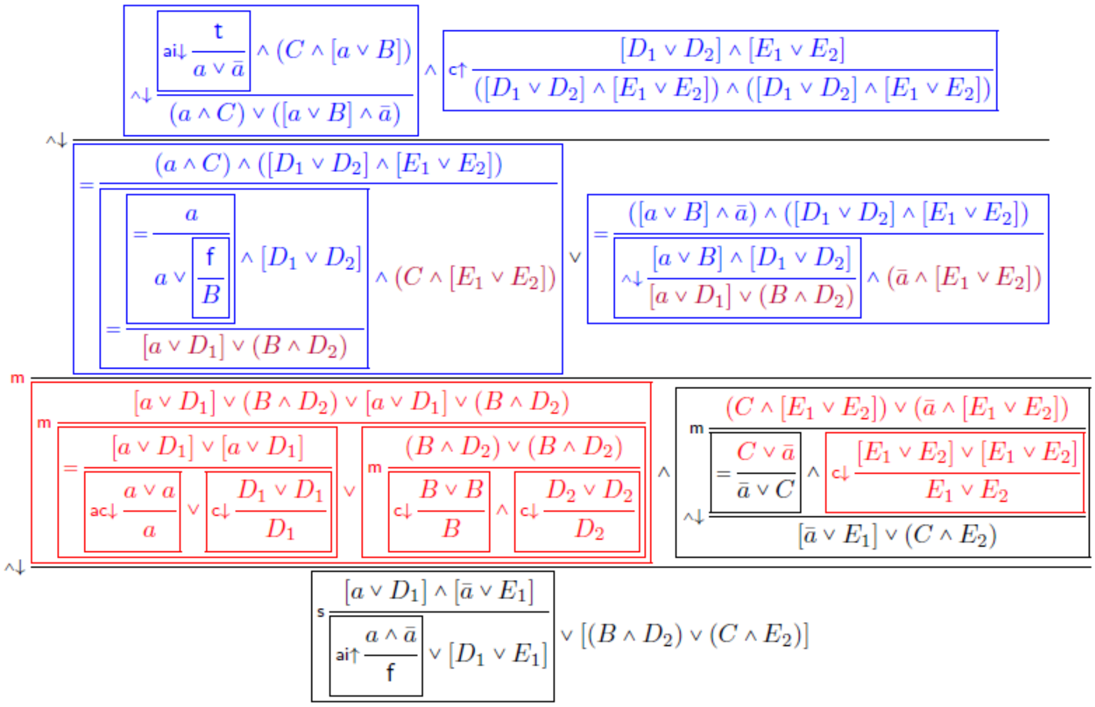
At every step, the part of the derivation that is above the critical contraction and therefore remains untouched by reductions is shown in blue. The premiss of the generic contraction that we permute is shown in purple, and the rest of it is shown in red.

We apply an instance of the reduction t to permute past the equality rule.

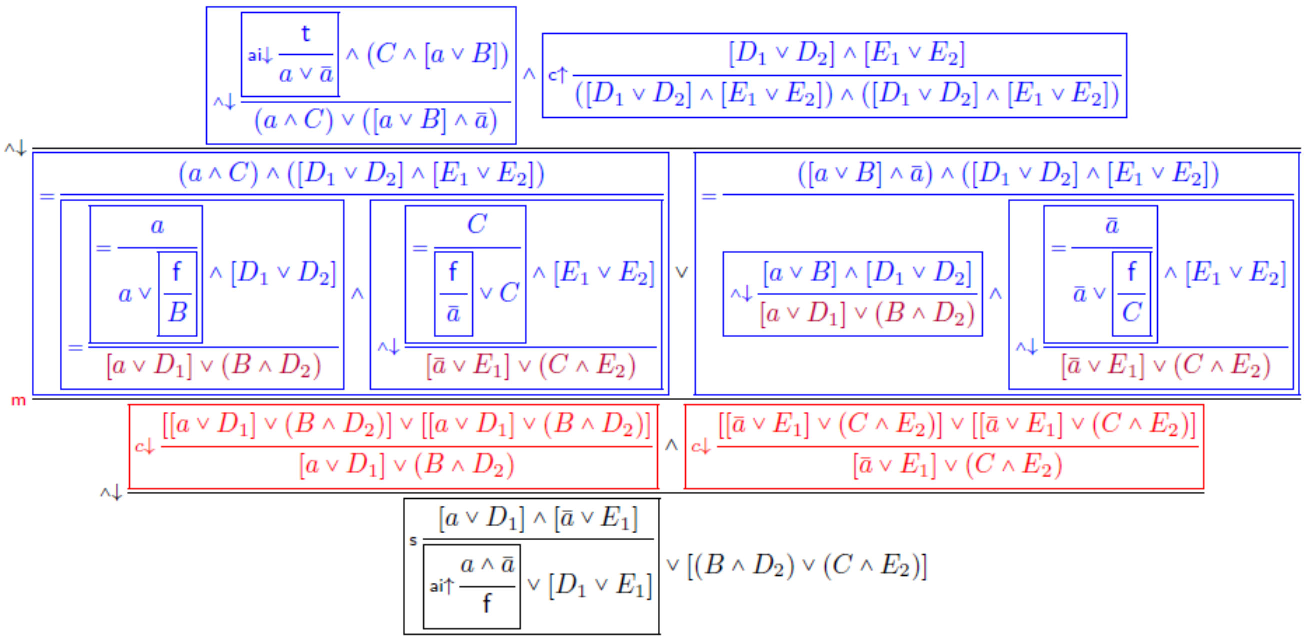


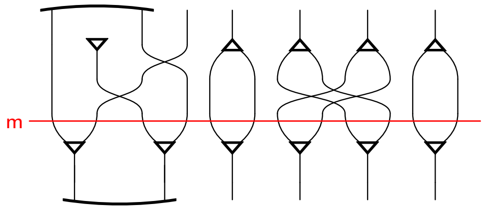


We apply an instance of e to permute past the equality $\mathsf{f} \vee B = B$ and an instance of s_1 to permute past $\wedge\downarrow$ (here, for brevity, they are shown together):



We apply instances of s_2 and s_1 to permute past the commutativity rule and the rule $\wedge\downarrow$.



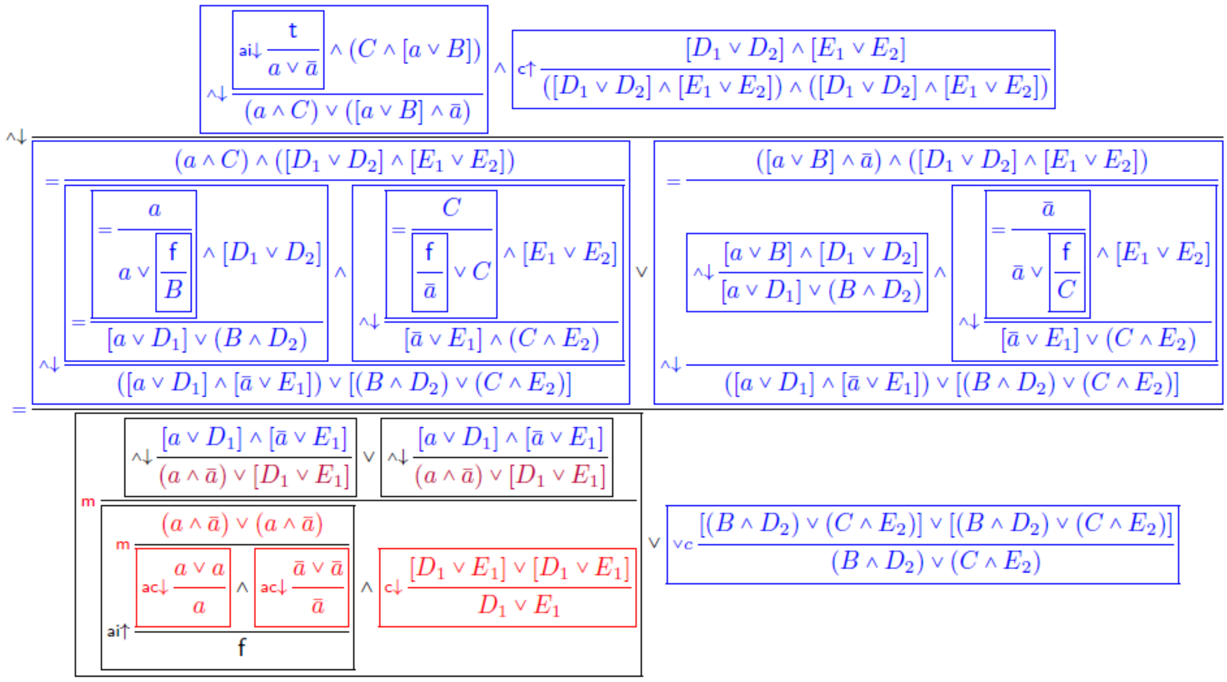


We apply an instance of s to permute past the rule $\wedge\!\downarrow$:

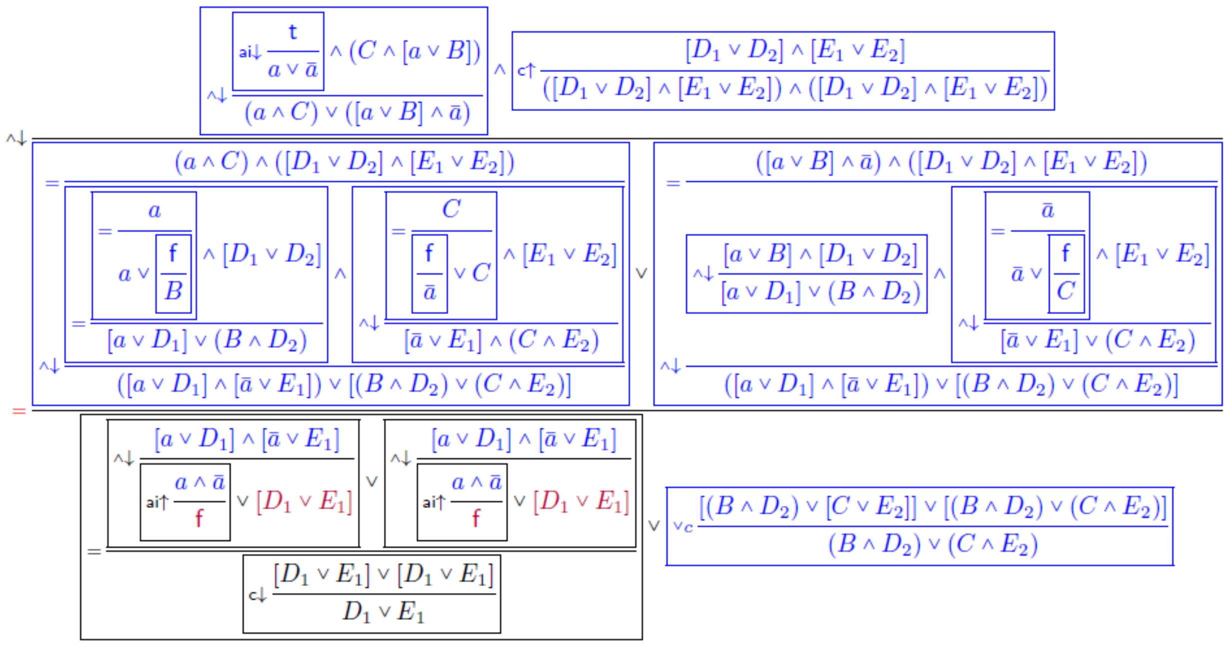
$$
\wedge\!\downarrow\dfrac{
\wedge\!\downarrow\dfrac{\left(\mathsf{ai}\!\downarrow\dfrac{\mathsf{t}}{a\vee\bar a}\right)\wedge(C\wedge[a\vee B])}{(a\wedge C)\vee([a\vee B]\wedge\bar a)}
\;\wedge\;
\mathsf{c}\!\uparrow\dfrac{[D_1\vee D_2]\wedge[E_1\vee E_2]}{([D_1\vee D_2]\wedge[E_1\vee E_2])\wedge([D_1\vee D_2]\wedge[E_1\vee E_2])}
}{
=\dfrac{(a\wedge C)\wedge([D_1\vee D_2]\wedge[E_1\vee E_2])}{\wedge\!\downarrow\dfrac{
\left(=\dfrac{\left(=\dfrac{a}{a\vee\left(\dfrac{\mathsf{f}}{B}\right)}\right)\wedge[D_1\vee D_2]}{[a\vee D_1]\vee(B\wedge D_2)}\right)
\wedge
\left(\wedge\!\downarrow\dfrac{\left(=\dfrac{C}{\left(\dfrac{\mathsf{f}}{\bar a}\right)\vee C}\right)\wedge[E_1\vee E_2]}{[\bar a\vee E_1]\vee(C\wedge E_2)}\right)
}{([a\vee D_1]\wedge[\bar a\vee E_1])\vee[(B\wedge D_2)\vee(C\wedge E_2)]}}
\;\vee\;
=\dfrac{([a\vee B]\wedge\bar a)\wedge([D_1\vee D_2]\wedge[E_1\vee E_2])}{\wedge\!\downarrow\dfrac{
\left(\wedge\!\downarrow\dfrac{[a\vee B]\wedge[D_1\vee D_2]}{[a\vee D_1]\vee(B\wedge D_2)}\right)
\wedge
\left(\wedge\!\downarrow\dfrac{\left(=\dfrac{\bar a}{\bar a\vee\left(\dfrac{\mathsf{f}}{C}\right)}\right)\wedge[E_1\vee E_2]}{[\bar a\vee E_1]\vee(C\wedge E_2)}\right)
}{([a\vee D_1]\wedge[\bar a\vee E_1])\vee[(B\wedge D_2)\vee(C\wedge E_2)]}}
}
$$

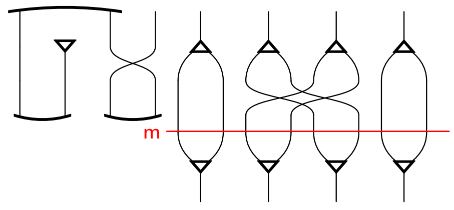
$$
=\;
m\dfrac{([a\vee D_1]\wedge[\bar a\vee E_1])\vee([a\vee D_1]\wedge[\bar a\vee E_1])}{\wedge\!\downarrow\dfrac{[a\vee D_1]\wedge\left(\left(ac\!\downarrow\dfrac{\bar a\vee\bar a}{\bar a}\right)\vee\left(c\!\downarrow\dfrac{E_1\vee E_1}{E_1}\right)\right)}{\left(\mathsf{ai}\!\uparrow\dfrac{a\wedge\bar a}{\mathsf{f}}\right)\vee[D_1\vee E_1]}}
\;\vee\;
\vee c\dfrac{[(B\wedge D_2)\vee(C\wedge E_2)]\vee[(B\wedge D_2)\vee(C\wedge E_2)]}{(B\wedge D_2)\vee(C\wedge E_2)}
$$

We then apply an instance of s to permute past the rule $\wedge\downarrow$:



Last, we apply an instance of s_1 to permute past the cut:




m

Chapter 6

Conclusion

In this thesis, we have achieved a series of technical results, by taking advantage of the generality provided by the subatomic methodology:

- We have provided a general characterisation of proof systems, in such a way that every rule is an instance of single, regular, linear, inference rule scheme. We showed how this characterisation encompasses such different systems as multiplicative additive linear logic, BV or classical logic, while remaining concise enough to be useful in generalising splitting and decomposition.
- We proved a generalised splitting theorem, allowing us to understand the properties of proof systems that the procedure hinges on. In this way, we prove cut-elimination for a whole class of substructural logics and show that splitting is a very general procedure that can be applied to many systems with any number of relations and units. Furthermore, we show that it is carried over by the identification of units, as happens in the case of BV. In addition, this generalisation provides useful guidelines for the design of linear proof systems, removing the search for cut-elimination from the design process.
- We have shown that the splitting procedure is not restricted to systems with binary connectives and can be extended to relations of different arities by proving a splitting theorem for SKV, a system with a modality.
- We have shown that admissibility is a property that goes beyond the cut-rule: as a corollary of splitting we have proved the admissibility of a whole class of rules that corresponds to those rules necessary to make the cut atomic, such as the rule $q\uparrow$ of BV or the associativity of $\wedge$ in classical logic.
- We provided general reduction rules for the permutation of generic contractions and cocontractions with other rules and a characterisation of the systems they can be applied to, including MALL and classical logic. By doing so, we showed that not only atomic contractions and cocontractions can be permuted downwards/upwards in a derivation, but that in fact it is possible to permute a whole class of rules. The ability to permute atomic contractions and cocontractions in MALL and

classical logic is an instance of this phenomenon, and is due to certain properties that both systems share.

- We used the general reduction rules to design a procedure to remove ai-cycles in SKS and SMALLS proofs, proving the independence of the decomposition procedure from cut-elimination, and advancing towards being able to ascertain the complexity cost of the removal of cycles.

These results leave room for future developments, some of which are currently being researched:

- It would be interesting to provide a characterisation of sound rules in terms of an order between the relations: the design of systems would be much simplified, and the characterisation of systems would be further improved, maintaining the properties of the characterisation we provided in this work while gaining in specificity.
- Generalising the characterisation of rules and the splitting result to relations of different arities to include modalities and exponentials is expected to be a close future development, since the study of the deep inference systems for linear logic (with exponentials) [37], for classical predicate logic [3] or for BV has yielded very encouraging results towards the characterisation of the rules involving the exponentials with a single shape.
- The notation for generic contractions and the rewriting rules can be simplified, particularly highlighting only those features that are necessary to prove termination of the rewriting system, as is done with the atomic flows for classical logic.
- Obtaining full decomposition for classical logic and for MALL in such a way that we can rewrite proofs into a splittable phase followed by a contractive phase is now a matter of finding the correct measure to prove that the permutations of generic contractions terminate.
- The removal of cycles from proofs has been proved to be a quadratic-time procedure in the sequent calculus [9]. By studying the procedure presented in this thesis, it will be possible to understand the complexity cost of cycle removal in deep inference.

The characterisation of rules through a single inference rule scheme was initially intended as a stepping stone towards the development of a graphical formalism that could be used to represent a wide variety of logics. The task however proved to be more daunting than we expected: to develop this formalism, a full understanding of the properties required for the normalisation procedures that we want to capture to isolate the complexity generating mechanisms (cut-elimination and decomposition) proved to be necessary. For that, a refinement of the general rule scheme was needed, and so the development of conditions on the relations that enable us to capture the normalisation procedures while maintaining generality came about. This characterisation was no easy task, since it needs to encompass both the linear and the contractive rules, that vary in behaviour and in shape in different non-subatomic systems.

Once the adequate characterisation was found, we proceeded to study cut-elimination and decomposition with this new methodology, with a strong focus on understanding the properties of the rules that are essential to obtain them. The generalisations of both of these procedures highlight which features should be captured by a graphical formalism: duality and contractiveness. When the final missing feature consisting of the extension of the notion of rank of an atomic contraction to generic contractions is found, we will have a description of all the elements that need to be featured in a graphical formalism in which cut-elimination and decomposition are naturally represented. I would very much like to continue towards this research direction: this thesis is a good start that provides many of the tools that I expect to use.

In short, in this work we have uncovered an underlying structure behind the shape of inference rules. This observation is truly surprising, and its generality can be exploited in many ways. Here, we used it to characterise proof systems and to study normalisation procedures, and it is expected that in the future the number of applications will only grow.

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