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Asymptotic analysis of some spectral problems in high contrast homogenisation and in thin domains

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Asymptotic analysis of some spectral problems in high contrast homogenisation and in thin domains

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Contents

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Summary

We study the spectral properties of two problems involving small parameters. The first one is an eigenvalue problem for a divergence form elliptic operator A_ε with high contrast periodic coefficients of period ε in each coordinate, where ε is a small parameter. The coefficients are perturbed on a bounded domain of 'order one' size. The local perturbation of coefficients for such operator could result in emergence of localised waves in the gaps of the Floquet-Bloch spectrum. We prove that, for the so-called double porosity type scaling, the eigenfunctions decay exponentially at infinity, uniformly in ε. Then, using the tools of two-scale convergence for high contrast homogenisation, we prove the strong two-scale convergence of the eigenfunctions of A_ε to the eigenfunctions of a two-scale limit homogenised operator A_0, consequently establishing 'asymptotic one-to-one correspondence' between the eigenvalues and the eigenfunctions of these two operators. We also prove by direct means the stability of the essential spectrum of the homogenised operator with respect to the local perturbation of its coefficients. That allows us to establish not only the strong two-scale resolvent convergence of A_ε to A_0 but also the Hausdorff convergence of the spectra of A_ε to the spectrum of A_0, preserving the multiplicity of the isolated eigenvalues.

As the second problem we consider the eigenvalue problem for the Laplacian in a network of thin domains with Dirichlet boundary conditions. We construct an asymptotic solution to the problem using the method of matched asymptotic expansions to obtain appropriate boundary conditions for the terms of the asymptotics near the junctions of thin domains. We justify the asymptotics and prove the error bound of order $h^{3/2}$, where h is the width of thin domains. We then derive a limiting model on the graph (which serves as a frame for such domain) and prove that it gives a proper approximation for the eigenvalues end eigenfunctions of the original problem. An important new result is that the boundary conditions at the vertices of the graph are mixed boundary conditions involving the small parameter h. This type of conditions keeps the information about the interaction between the edges of the graph and at the same time provides a better approximation than previously known models. We also study the bottom of the spectrum of the problem, whose corresponding eigenfunctions are confined to the vertices.

Introduction: motivation, literature overview, and structure of the thesis

The present thesis consists of two parts studying two separate problems: spectral convergence in homogenisation of high contrast media with a defect, and spectral asymptotics for networks of thin domains. Both themes are unified by the need to develop asymptotic analysis for associated spectral problems, employing relevant tools from asymptotic methods, spectral theory, non-classical homogenisation, etc. In turn, both topics are motivated by applications such as photonics and phononics and quantum graphs.

The motivation for the first part of the thesis arises, in particular, from recent growth of interest to photonic and phononic crystals and crystal fibers. The photonic (phononic) crystals are composite materials that often have a periodic structure. The fundamental property of the photonic (phononic) crystals consists in the existence of special regions (bandgaps) of frequencies where no electromagnetic (elastic) waves can propagate. Mathematically this regions correspond to the gaps in the essential spectrum of the related elliptic operators. This effect opens large possibilities for various applications in physics. In particular introduction of a defect in a periodic photonic or phononic fiber can lead to a spatial localisation of waves near the defect. While photonic applications come from optics (with problems described mathematically by Maxwell's equation of electromagnetism) and phononic applications come from acoustics and elastodynamics, in both cases the key idea is that appropriate periodic media do not allow propagation of waves of certain frequency ranges. For example, the photonic crystal fibers, see e.g. [42], Figure 0-1, are typically represented by a core surrounded by a periodic cladding. Consequently, on the cross-section of the fiber the core itself represents a 'defect' with regards to the periodic cladding. In

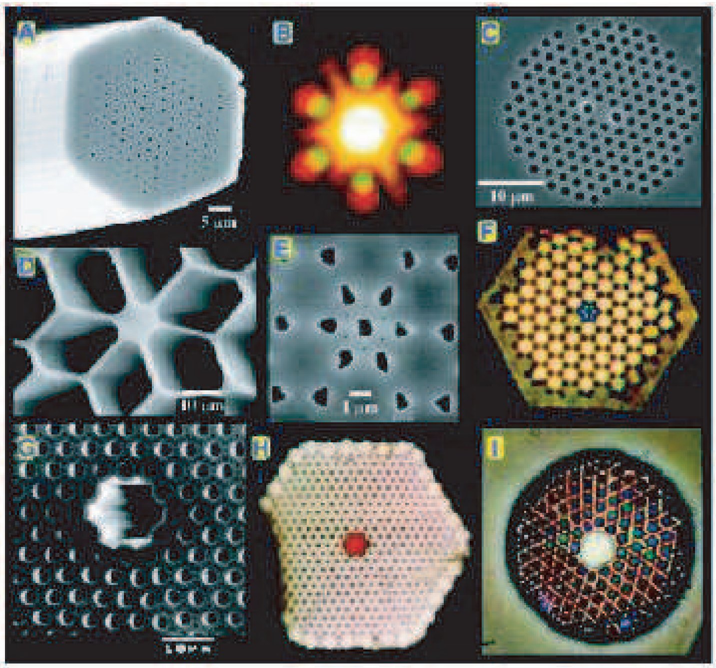

Figure 0-1: Photonic crystal fibers. Courtesy of P. Russell [42]

addition, it is well known that, in general, the width of the photonic (phononic) bandgap (which is obviously an important property of a crystal) increases when the contrast between physical characteristics of components becomes larger.

Inspired by the above mentioned facts we study a simplified mathematical model of a photonic/phononic crystal, described by a divergence type operator with high contrast periodic coefficients with a finite size defect and the periodicity size ε being a small parameter. (See [32] for a comprehensive review of mathematics of photonic crystals.)

There are several different mathematical aspects concerning the study of this sort of problems. First of all the above mentioned physical bandgap effect, in mathematical terms, is described by tools of spectral theory of differential operators with periodic coefficients, known as *Floquet-Bloch* theory. Namely, the above 'forbidden' frequencies precisely correspond to gaps in the spectra of such operators. Moreover, the emergence of localised modes due to defects in such periodic media corresponds in turn to eigenfunctions due to extra point spectrum appearing in the gaps. We hence first give a brief overview of the Floquet-Bloch theory, see e.g. [31], [41, v.4]. The Floquet-Bloch theory was originally developed by physicists to address problems involving periodic potentials, e.g. in solid state physics. It was probably first realised by Gelfand [24] that in the multi-dimensional case it can be described by means of the spectral theory of self-adjoint operators. The key point here is the *Floquet transform* of a function $f : \mathbb{R}^n \to \mathbb{R}$, initially defined for $f \in C_0^\infty(\mathbb{R}^n)$, and then extended by continuity to $L^2(\mathbb{R}^n)$:

$$(\mathcal{U}f)(x,k) = \sum_{\xi \in \mathbb{Z}^n} f(x-\xi)e^{ik\cdot\xi}.$$

It has two important properties: quasi-periodicity with respect to x,

$$(\mathcal{U}f)(x+m,k) = e^{ik\cdot m}(\mathcal{U}f)(x,k),\ \forall m \in \mathbb{Z}^n; \tag{1}$$

and periodicity with respect to k,

$$(\mathcal{U}f)(x,k+2\pi m) = (\mathcal{U}f)(x,k),\ \forall m \in \mathbb{Z}^n.$$

So, from considering a function defined on an unbounded domain ($\mathbb{R}^n$) one passes to considering a function of two variables defined on a bounded domain: $(x,k) \in Q \times Q^*$, where $Q = [0,1)^n$ is the *periodic cell*, and $Q^* = [0,2\pi)^n$ is the dual cell of 'quasimomenta' (the so-called Brillouin zone). Let us denote by $L(x,D)$ an elliptic differential operator $L(x,D)u = -\nabla \cdot A(x)\nabla u$, where $A(x)$ is a measurable periodic positive definite matrix, i.e. $\nu I \leq A(x) \leq \nu^{-1} I$ in the sense of quadratic forms for some $\nu > 0$, $A(x+m) = A(x)$, $\forall m \in \mathbb{Z}^n$. Due to its periodicity $L(x,D)$ commutes with the Floquet transform, i.e. for any $f \in C_0^\infty(\mathbb{R}^n)$

$$\mathcal{U}(Lf)(x,k) = L(x,D)(\mathcal{U}f)(x,k). \tag{2}$$

However now, on the right hand side of (2), for each k the operator $L(x,D)$ acts in a different domain of functions satisfying quasi-periodicity condition (1). So we have a family of operators $L(k)$ acting in spaces of functions defined on a compact domain. Hence each operator $L(k)$, appropriately extended to the self-adjoint one, has a discrete spectrum $\sigma(L(k)) = \bigcup_{i=1}^\infty \lambda_j(k)$. Then the following central spectral property can be shown for the spectrum $\sigma(L(x,D))$ of $L(x,D)$, see e.g. [31]:

$$\sigma(L(x,D)) = \bigcup_{k\in Q^*} \sigma(L(k)) = \bigcup_{j=1}^{\infty} \bigcup_{k\in Q^*} \lambda_j(k) = \bigcup_{j=1}^{\infty} B_j.$$

The spectrum of $L(x,D)$ has hence a band-gap structure: the bands $B_j,\ j \geq 1$, may cease to overlap, resulting thereby in the presence of the gaps. Moreover, if the periodic coefficients of $L(x, D)$ are compactly perturbed, which physically corresponds to introduction of a defect, the spectral theory assures that the essential spectrum remains unperturbed, and hence the only extra spectrum can be the discrete spectrum in the gaps.[1]

Therefore the spectral theory allows us to connect mathematical objects (e.g. band-gaps and point spectrum in the gaps) with physical effects (e.g. forbidden frequencies and localised modes). However, problems of the existence of the gaps, their location and width, the existence of point spectrum due to defects, the properties of the related eigenfunctions etc, have no general answer and require additional analytical or numerical investigation. Our key idea is to advance in these directions analytically using asymptotic methods, i.e. exploiting the presence of a small parameter. In our context, ε describes the size of the periodicity, which is the standard setup of the homogenisation theory[2] being reviewed next.

In the presence of a small parameter one normally looks for some asymptotic approximation to the problem. Namely, periodic rapidly oscillating problems are usually treated by the means of well developed theory of *homogenisation*, which was originated as mathematical discipline probably in the work of De Giorgi and Spagnolo [25][3]. The idea of homogenisation is to approximate a given operator by some homogenised operator (with constant or slowly varying coefficients). There are several different approaches to this theory, which often can supplement each other. One uses the method of asymptotic expansions, which assumes that the solution to an appropriate differential equation

$$L_\varepsilon u_\varepsilon := -\nabla \cdot A(x/\varepsilon)\nabla u_\varepsilon = f \tag{3}$$

can be represented in the form

$$u_\varepsilon = u_0(x, \varepsilon^{-1}x) + \varepsilon u_1(x, \varepsilon^{-1}x) + \varepsilon^2 u_2(x, \varepsilon^{-1}x) + \ldots, \tag{4}$$

where the terms are assumed to be periodic in the second variable, $u_i(x, y+m) = u_i(x, y)$, $m \in \mathbb{Z}^n$, $i = 0, 1, \ldots$. Substituting this ansatz into the equation and

[1]We do not discuss in this thesis the issue of whether *embedded* eigenvalues can emerge on the bands as a result of the perturbation.

[2]Note that there are other ways of applying asymptotics methods in the present context, not using homogenisation, see e.g. [26, 36].

[3]Condition of periodicity (on the ε-scale) can be relaxed in various ways or removed altogether, see e.g [27, 47], which we do not address in this thesis.

equating the coefficients at same powers of ε one arrives at a recurrent sequence of equations depending on two variables x and $y = \varepsilon^{-1}x$. One first observes that u_0 is function of x only, $u_0(x, y) = u_0(x)$. Then the corrector $u_1(x, y)$ is found in the form

$$u_1(x, y) = \sum_{k=1}^{n} N_k(y)\frac{\partial u_0}{\partial x_k},$$

where $N_k \in H^1_{loc}(\mathbb{R}^n)$ is a periodic solution of 'unit cell' problems

$$-\nabla_y \cdot A(y)\nabla_y N_k(y) = -\sum_{i=1}^{n} \frac{\partial}{\partial y_i} a_{ik}(y),$$

(a_{ij} are entries of the matrix A). Finally, the solvability condition for u_2 leads to the homogenised equation for u_0:

$$-\nabla \cdot A^{\text{hom}}\nabla u_0 = f,$$

where A^{hom} is the homogenised matrix of constant coefficients given by

$$A^{\text{hom}} = \int_Q A(y)(I + \nabla_y N)dy.$$

Here I is the unity matrix and $\nabla_y N$ is the matrix with columns $\nabla_y N_1$, $\nabla_y N_2, \ldots,$ $\nabla_y N_n$.

The problem of justification, or convergence of u_ε to u_0, has received considerable separate attention. The above procedure of asymptotic expansion can be advanced further, using the uniform ellipticity of L_ε, to obtain not only the convergence but also error bounds establishing the rate of convergence. For instance for bounded Ω with Lipschitz boundary $\partial\Omega$ and Dirichlet boundary conditions on $\partial\Omega$ one has $\|u_\varepsilon - (u_0 + \varepsilon u_1(x, x/\varepsilon))\|_{H^1(\Omega)} \leq C\varepsilon^{1/2}$, see e.g. [5, 10, 27, 43] and further references therein. An alternative method of directly passing to the limit is based on selecting appropriate oscillatory test functions in the weak formulation of (3), using methods of compensated compactness, see e.g. [37, 47].

Another approach to homogenisation is associated with the method of *two-scale convergence*. The idea of the two-scale convergence is to preserve in the limit the information about oscillations of elements of a sequence on ε scale. For example, in the sense of the usual convergence in L^2-norm a sequence $f(x)\sin(\varepsilon^{-1}x)$ weakly converges to zero (and there is no strong convergence). However, in the sense of two-scale convergence this sequence strongly converges to the function of two variables $f(x)\sin(y)$. In general, the strong two-scale convergence of $u_\varepsilon(x)$ to $u_0(x,y)$, $u_\varepsilon(x) \xrightarrow{2} u_0(x,y)$, loosely means $\|u_\varepsilon(x) - u_0(x, x/\varepsilon)\|_{L^2(\Omega)} \to 0$ as $\varepsilon \to 0$. The first and crucial step in this direction was made by Nguetseng in [38] where he actually introduced (what we now call) weak two-scale convergence, proved the weak two-scale compactness of a sequence of bounded in L^2-norm functions (which is a two-scale analogue of the Banach-Alaoglu theorem), and derived a formula for the weak two-scale limit of gradients of a bounded in H^1-norm sequence of functions. He applied these results to the homogenisation of a periodic uniformly elliptic problem obtaining classical results and also a new convergence formula for the gradient of the solution of the problem. Later Allaire [2], relying on the work of Nguetseng, introduced the notion of strong two-scale convergence and developed further the theory and its applications to some problems of homogenisation for operators with periodic coefficients. The theory of two-scale convergence was advanced thereafter, among others, by Zhikov [48], see below, who in particular extended it for (periodic) measures and applied it to study the convergence of spectra, see also [50].

Classical homogenisation is incapable of accounting for the above described effects: the homogenised operator has constant coefficients and therefore its spectrum is the whole positive semi-axis with no band gaps. However, certain versions of non-classical homogenisation do allow one to account for some of these effects, as we discuss next. On the other hand, the method of two-scale convergence is applicable not only to the above reviewed 'classical' homogenisation, but also to various 'non-classical' versions of homogenisation. The non-classical homogenisation includes higher-order homogenisation, see e.g. [5, 20, 46], exponential homogenisation [28], non-local homogenisation [9, 15, 17, 18, 19, 21], etc., which often refer to *high contrast* in the coefficients (see also [45]), as is the case in our model. (Notice that, as additional motivation for our simplified model, in e.g. photonic crystal fibers even low-contrast structures can sometimes display 'apparent' high contrast on a cross-section, see e.g. [11] for a physical discussion.)

Following e.g. [2] and [48] we consider a special scaling of the coefficients $a(x,\varepsilon)$ of the operator

$$-\nabla \cdot a(x,\varepsilon)\nabla.$$

Namely, we let $a(x,\varepsilon)$ be of order 1 in the *matrix* phase and of order δ in the *inclusion* phase. The asymptotic behaviour will then crucially depend on the relation between the two small parameters ε and δ, a phenomenon known in physical literature as 'noncommuting limits', see e.g. [40] and further references therein. It is well-known that $\delta(\varepsilon) \sim \varepsilon^2$ is a critical scaling in this context, often referred to as a 'double porosity-type' scaling.[4] This is the only scaling that leads in the limit to the dependence on the fast variable y (i.e. the main order term $u_0(x, y)$ in the asymptotic expansion (4) retains the dependence on y [5]), whereas other scalings of order ε^α where $\alpha \neq 2$ lead to either the classical homogenised problem with no dependence on y ($\alpha < 2$) or to a degenerate problem ($\alpha > 2$), see [2] and [48] for more detailed explanations. The problem of *high contrast* (or *double porosity-type*) homogenisation has become a popular subject in the past two decades, in particular, it was firstly treated by the two-scale convergence method in [2]. Since the principal interest of our study is in the spectral characteristics of the problem, we mainly refer to the two works by Zhikov [48, 49], where the author, in particular, developed further the method of two-scale convergence, including its application to the high contrast homogenisation, described the spectrum of the limit homogenised operator in an explicit way and proved convergence of the spectra of the periodic operators to the spectrum of the homogenised one in the sense of Hausdorff (see Section 1.1). The spectrum of the homogenised operator has an explicit band-gap structure, hence so does the spectrum of the periodic operator for small enough ε.

As it was mentioned earlier, due to the presence of gaps in the spectrum of the operator one can expect that an introduction of a defect into the periodic structure of the coefficients may lead to emergence of localised modes, i.e. eigenvalues in the gaps of the essential spectrum with corresponding eigenfunctions concentrated near the defect. Indeed, it was proven in [23] that for a given gap in the spectrum of a periodic operator one can introduce a defect in the periodic media, i.e. can perturb locally the coefficients of the operator, so that the perturbed operator will have at least one eigenvalue inside the gap. Moreover, as was also proven in [23], under the compact perturbation of coefficients the essential spectrum of the operator remains unperturbed and the eigenfunctions corresponding to the eigenvalues in the gaps decay exponentially at infinity. This type of results is actually quite general in the perturbation theory of self-adjoint operators (see e.g. [12, 41]).

We now describe our problem and the results in more detail. In the first chapter we study an elliptic divergence form operator A_ε with locally perturbed

[4] The term 'double porosity' originates from mathematically similar problems of fluid flows in fractured porous rocks [7].

[5] This relates asymptotically to phenomenon of 'micro-resonances', both in phononic and photonic contexts, found in physical literature, see e.g. [33, 34]

high contrast (of order ε^2) ε-periodic coefficients. The behaviour of A_ε and its spectral characteristics as $\varepsilon \to 0$ is the main topic of interest. A similar problem is considered in [29] using the method of asymptotic expansions, but the present study pursues different aims and approaches the problem from another direction, namely developing an appropriate version of the two-scale convergence technique [2, 38, 49]. As a result we obtain a complete description of the asymptotic (with respect to ε) behaviour of the localised modes and other spectral characteristics for the operator A_ε in terms of an explicitly described (two-scale) limit operator A_0. For other recent applications of the high contrast homogenisation techniques see also [4, 8, 13, 16, 19, 21, 30, 44].

In the absence of a defect, Zhikov considers in [48] a divergence form elliptic operator $\widehat{A}_\varepsilon$ (denoted in [48] by A_ε) with periodic coefficients corresponding to a double-porosity model [3, 14] (A_ε in our notation is obtained from $\widehat{A}_\varepsilon$ by a compact perturbation of its coefficients). Operators of such type have the Floquet-Bloch essential spectrum, displaying a band-gap structure. Zhikov proves that the spectra of $\widehat{A}_\varepsilon$ converge in the sense of Hausdorff to the spectrum of a certain two-scale homogenised operator $\widehat{A}_0$ with constant coefficients, see also [26, 49], and that $\widehat{A}_0$ is the limit of $\widehat{A}_\varepsilon$ in the sense of strong two-scale resolvent convergence. The spectrum of $\widehat{A}_0$ is purely essential and displays an explicit band-gap structure. As we already mentioned, in the case of a compact perturbation of periodic coefficients in the elliptic operator $\widehat{A}_\varepsilon$ its essential spectrum remains unperturbed, see e.g. [23, 41]. The only extra spectrum that can emerge in the gaps due to the perturbation is a discrete one (isolated eigenvalues with finite multiplicity). Such an extra spectrum does emerge at least under some assumptions, e.g. [23, 29]. This corresponds physically to localised modes emerging near the defect.

One of the main goals of the first part of the thesis is to establish the strong two-scale convergence of the eigenfunctions of A_ε corresponding to the eigenvalues in the gaps. In order to achieve this we need the strong two-scale compactness of the eigenfunctions. This requires in turn an exponential decay of the eigenfunctions *uniform in* ε.

The problem of wave localisation (i.e. of the existence of eigenvalues with corresponding eigenfunctions decaying exponentially) in the gaps of the essential spectrum has been intensively investigated for a wide range of differential operators over the last decades. The results obtained up to date ensure the exponential decay of eigenfunctions of A_ε for a *fixed* ε, see e.g. [23]. However this is insufficient for establishing the required compactness. Moreover, the developed methods, e.g. [6] and [23] (the latter using the method of Agmon[1]), seem to be insufficient for the present purpose. The reason is that in order to obtain the *uniform* exponential decay one has to perform some kind of two-scale asymptotic analysis, investigating the behaviour of the eigenfunctions on small and large scales simultaneously. To achieve this we supplement the method of [1] by the related two-scale techniques, which play a crucial role. As a result, we obtain a uniform estimate with the decay exponent α (see (1.24) and (1.13) below) which ensures the compactness, but may also be of an independent interest. On one hand, it is sharp in a sense as $\varepsilon \to 0$. On the other hand, it behaves qualitatively entirely differently compared to e.g. the one in [6]: while the one in [6] is proportional to the square root of the distance to the gap end, the decay exponent we derive becomes large on approaching the left end of the gap and small near the right end.

The structure of the first part is the following. We first define the problem in Section 1.1, describe the two-scale limit operator A_0 and state the main result. We then consider a subsequence of eigenvalues of A_ε converging to some point λ_0 lying in a gap of the spectrum of $\widehat{A}_0$. In Section 1.2 we prove (Theorem 1.2.2) the uniform exponential decay for the eigenfunctions of A_ε. Section 1.3 is devoted to the proof of a main auxiliary lemma that is employed in the previous section, which may also be of an independent interest. In Section 2.1 we list some properties of the two-scale convergence and several related statements which we use in the next section. Employing the uniform exponential decay, we establish in Section 2.2 (see Theorem 2.2.1) the strong two-scale compactness of (normalised) eigenfunctions of A_ε, see e.g. [48, 49]. This implies that, up to a subsequence, the eigenfunctions two-scale converge to a function, which is eventually proved to be an eigenfunction of the two-scale limit operator A_0 with a defect, which could be considered as a perturbation of $\widehat{A}_0$. Accordingly λ_0 is an eigenvalue of A_0. The two-scale convergence of the eigenfunctions together with the results of [29] on the existence of the eigenvalues in the gaps and related error bounds allow us to make a conclusion about the 'asymptotic one-to-one correspondence' between eigenfunctions and eigenvalues of the operators A_ε and A_0 as $\varepsilon \to 0$. In the Section 2.3 we prove by direct means (via Weyl sequences) the stability of the essential spectrum of $\widehat{A}_0$ with respect to the local perturbation of its coefficients (see Theorem 2.3.1). Thereby this establishes the convergence of the spectra of A_ε to the spectrum of A_0 in the sense of Hausdorff (Theorem 1.1.1).

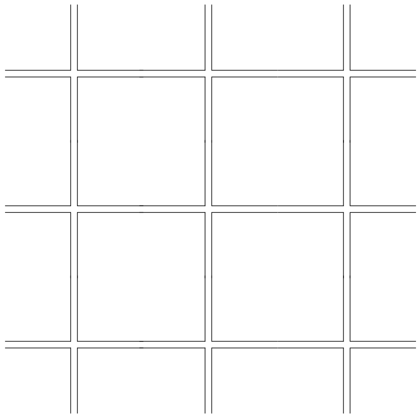

Figure 0-2: Graph-like domain Ω_h.

Another interesting topic in asymptotic analysis relates to various problems in thin structures which naturally arise in physics, chemistry, engineering, when one considers for instance propagation of waves in a network of thin domains. When the cross-sectional size of such an object is much smaller than its length it is natural to try to approximate the original problem by a differential (or sometimes more general) equation on a graph eliminating the transversal dimensions. In this case one obtains a so-called "quantum graph", i.e. one-dimensional differential equation posed on the graph. Probably one of the first quantum graph models was employed in chemistry where one considered a model of free electron motion along a carcass of a molecule (see e.g. [79]). Other examples can be found in nanotechnology and mesoscopic physics where several dimensions of physical objects are reduced to a size of a few nanometers [54]. Problems in thin domains appear in many other areas of mathematics and have been studied in different contexts, see e.g. [52, 53, 55, 56, 57, 61, 63, 75].

First we recall some results obtained for models related to graphs with straight edges. Consider a domain Ω_h given as an h-neighbourhood of a planar graph, where h is a small parameter, see Figure 0-2. Let A_h be an elliptic self-adjoint differential operator in Ω_h with some boundary conditions. We are interested in the spectrum of such operator. It is natural to try reducing the given problem in Ω_h to some problem on the graph. In the limit as h tends to zero one normally obtains a differential operator A, e.g. $A = -\frac{\partial^2}{\partial s^2}$, where s is an arclength along the edges, which must be equipped with appropriate *boundary conditions at vertices*. The latter is not always a trivial question, and for some boundary value problems on Ω_h it is still (or was until recently) essentially open, see e.g. [64] or [71] for the relevant discussion.

The case of Neumann boundary conditions is probably the easiest one. In this case the first transversal eigenvalue is zero and bounded states confined to the junctions of Ω_h are absent. The limiting operator is equipped with the so-called Kirchhoff boundary conditions at each vertex v_l:

$$\sum_{\{j|v_l\in e_j\}} \frac{df}{dx_j}(v_l) = 0.$$

Here e_j denotes edges of the graph. The following result in a more or less general way was obtained by different researchers (see e.g. [59, 70, 77, 78, 80]):

$$\lambda_n(A_h) \to \lambda_n(A_0) \text{ as } h \to 0,$$

where λ_n is the n-th eigenvalue of a corresponding operator in the ascending order, counted with multiplicities. The idea of the proof is the following: using the minimax definition of the eigenvalues one needs to construct a mapping from the H^1 space on graph into the H^1 space on Ω_h and vice versa such that the ratio $\frac{\|\nabla f\|^2}{\|f\|^2}$ does not increase substantially.

The case of the Dirichlet Laplacian is considerably more difficult. There are two reasons for that. The first one is that the spectrum of the Dirichlet Laplacian behaves completely differently compared to the Neumann Laplacian case. The first transversal eigenvalue ν_0 for Dirichlet boundary conditions is non-zero. Hence the corresponding eigenvalues of A_h should behave essentially as $h^{-2}\nu_0$. Additionally there may be bounded states living below the part of the spectrum generated by the transversal eigenvalues. Another difficulty lies in finding appropriate conditions at vertices of the graph for the limiting problem, as was already mentioned.

A classical and very popular tool for dealing with problems in graph-like domains is the method of matched asymptotic expansions. Employing this method one considers an inner problem in a neighbourhood of a junction and an outer problem in an adjacent strip and then one must match the corresponding solutions in some intermediate region in the vicinity of the junction.

Consider the outer problem, i.e. eigenvalue problem for the Dirichlet Laplacian in a thin (of width h) strip. One can introduce a stretched transversal variable $\eta = h^{-1}y$ so that the problem is considered in a fixed rectangle. Then

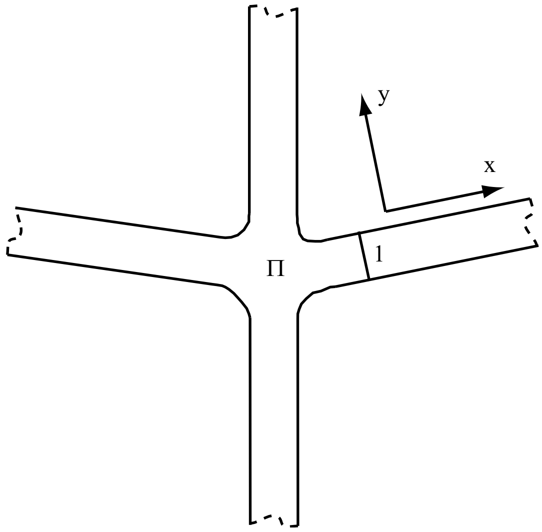

Figure 0-3: Junction

the Laplacian in the new variables reads

$$-\Delta = -h^{-2}\frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial s^2}. \tag{5}$$

One can then seek a solution to the outer problem in the separation of variables form

$$u_h(s,\eta) = v_i(s)\varphi_j(\eta), \qquad \lambda_h = h^{-2}\nu_j + \mu_i, \tag{6}$$

where ν_j and φ_j are the transversal eigenvalues and eigenfunctions satisfying Dirichlet boundary conditions and μ_i and v_i are eigenvalues and eigenfunctions of the operator $-\frac{d^2}{ds^2}$ which is not fully defined due to uncertain boundary conditions at the vertices of the graph. We restrict ourself to considering the eigenvalues of A_h generated by the first transversal eigenvalue ν_0.

The inner problem is set in the 'spider domain' Π obtained from the rescaled (by h^{-1}) neighbourhood of a junction by attaching straight strips Π_j of infinite length and width 1, see Figure 0-3. In order to match a solution of the inner problem with the solution of outer problem one needs to consider the following equation

$$-\Delta g = (\nu_0 + h^2\mu_i)g \text{ in } \Pi,$$
$$g = 0 \text{ on } \partial\Pi.$$

In general an L^2-solution to this problem does not exist. Assuming the absence of L^2-solutions one is interested in a 'scattering solution' which is defined as follows. Let m be the number of the adjoint strips. In each strip Π_j, $j = 1, \ldots, m$, we introduce local coordinates (x, y), so that y is the transversal coordinate. A function $g = g_p$ is called a solution of the scattering problem in Π if it has the following asymptotic behaviour in each infinite strip Π_j, $j = 1, \ldots, m$:

$$g_p = \delta_{pj}\ e^{-ih\sqrt{\mu_i}x}\varphi_0(y) + s_{pj}\ e^{ih\sqrt{\mu_i}x}\varphi_0(y) + O(e^{-\beta x}), \tag{7}$$

where $\beta > 0$ is some constant (which depends on h and μ_i), δ_{pj} is the Kronecker symbol and φ_0 is transversal eigenfunction corresponding to ν_0 (in our case $\varphi_0(y) = \sin(\pi y)$). The first term in (7) can be interpreted as an incident wave coming from infinity along the strip Π_p and all the remaining terms describe the transmitted (including reflected, $j = p$) waves. The matrix

$$S = [s_{pj}]$$

is called the scattering matrix. S is unitary and depends on h analytically.

Matching the asymptotics of the inner and the outer solutions, one can obtain a description of the spectrum of Ω_h in terms of spectrum of the operator $-\frac{d^2}{ds^2}$ acting on the graph with some boundary conditions (gluing conditions) at the vertices which depends on the scattering matrix. This program was carried out in recent works [64, 72, 73, 74].

The existence of bound states in strip-like domains is well known from the waveguide theory, see e.g. [58, 60] (see also [68, 76] for the similar effect in a thin plates), where bounded states are proven to exist in L-shaped domains or as induced by a curvature. Thus, below the part of the spectrum induced by the transversal eigenvalues there can exist eigenvalues of Ω_h corresponding to the bound states with eigenfunctions confined to the junctions of Ω_h or 'sharp' bends of its channels. Apparently, these eigenvalues cannot be described in terms of the limiting operator on the graph.

In the present work we study a spectral problem for the negative Dirichlet Laplacian in a simplified graph-like domain $\widehat{\Omega}_h$ with non-straight strips. Our main goal is to obtain a delicate asymptotic description of the spectrum of $\widehat{A}_h$ in terms of limiting operators on the graph. One can start with considering a symmetric graph that consists of only two edges joining in a single vertex at an angle less than π. We assume that the edges are straight in some neighbourhood of the vertex. The corresponding $\widehat{\Omega}_h$ is symmetric with respect to the bisectrix of the angle between edges of the graph, see Figure 0-4. This implies that the eigenfunctions of $\widehat{A}_h$ satisfy either Dirichlet or Neumann boundary conditions

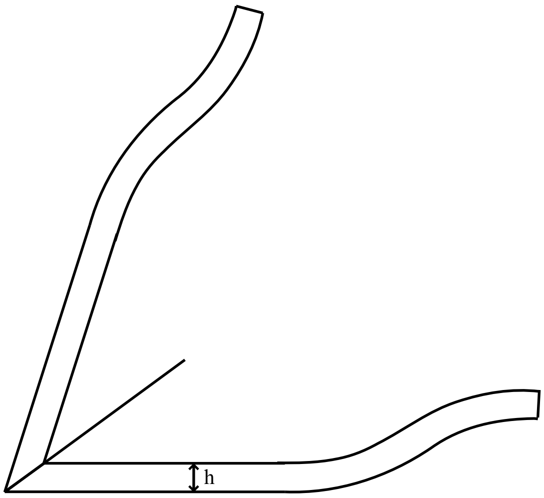

Figure 0-4: Simple symmetric graph-like domain $\widehat{\Omega}_h$

at the bisectrix. Then it is sufficient to consider the eigenvalue problem for the negative Laplacian in Ω_h, which is a part of $\widehat{\Omega}_h$ lying on one side of the bisectrix, with Dirichlet or Neumann boundary conditions at the bisectrix (being now a slanted end of Ω_h) and Dirichlet conditions elsewhere. The limiting graph for Ω_h in this case is a simple curve. The case of Dirichlet boundary conditions on the slanted end of Ω_h is very similar to the Neumann one and yet is simpler. So we consider only the Neumann case, and denote the corresponding operator by A_h, see the precise definitions and illustrations in Chapter 3.

We implement the general scheme of matched asymptotic expansions outlined above. Considering the outer problem we change the variables so as to flatten the domain and scale the transversal variable by h^{-1}. The main terms of the asymptotic solution the eigenvalue problem in the new domain have form (6), however μ_i, ν_i solve now the eigenvalue problem for the operator $-\frac{d^2}{ds^2} - \frac{1}{4}\kappa^2$, where $\kappa \not\equiv 0$. Hence some eigenvalues μ_i can be negative (which is different from already studied problems in [64, 72, 73, 74]). We construct further terms of the asymptotics to obtain more accurate approximations to the eigenelements of A_h (namely, the error bound of order $h^{3/2}$ is proven).

Matching the asymptotics of the outer solution with the asymptotics of the scattering solution of the inner problem we derive the boundary conditions for the limiting operator. The scattering matrix S (which is merely a complex number in our setting of the problem) depends analytically on h. We use its asymptotic expansion obtained in [66], which is given in terms of the scattering matrix corresponding to the threshold case

$$-\Delta g = \nu_0 g.$$

Normally, one has Dirichlet boundary conditions for v_i at the end of the curve corresponding to the slanted end of Ω_h. However for some geometric configurations (i.e. for some values of the angle of the slant) a Neumann boundary condition is possible. Namely, this is the case when the scattering solution at the threshold is bounded in L^∞-norm (i.e. being the so-called generalised bounded state); we call this the critical case.

We also consider the eigenvalues corresponding to the bounded states in the semi-infinite straight strip obtained from Ω_h by rescaling it in the neighbourhood of the slanted end. It is well known that there exists at least one bounded state lying below the transversal eigenvalue ν_0, see [69]. We provide some new estimates on the number of such bounded states with respect to the value of the angle of the slant.

In the case of Dirichlet boundary conditions on the slanted end of Ω the limiting problem always has Dirichlet boundary conditions at the corresponding end of the curve. There do not exist bounded states for the rescaled semi-infinite strip in this case.

The structure of the second part is the following. In Chapter 3 we construct the asymptotics of the problem in Ω_h. We state the problem in Section 3.1. In Section 3.2 we derive a formal asymptotic solution to the outer problem. In the next section we recall some results on scattering solutions of the inner problem from [66], also deriving order h term in the asymptotics of the scattering matrix in the critical case. Then we match the asymptotics of the inner and the outer solutions and consequently obtain the boundary conditions for auxiliary problems on the limiting graph for the terms of the asymptotic expansion in Section 3.4. Section 3.5 is devoted to the justification of the asymptotics. Chapter 4 is devoted to the construction of the limiting model graph, which is probably the most important result of the second part. In the last two sections we study properties of the bottom of the spectrum of the operator A_h, which is related to the bound states in the rescaled semi-infinite strip lying below the first transversal eigenvalue ν_0. Notice that the notation that we use in Part II may be different from the one used in the present introduction.

Part I

Spectral convergence for high contrast media with a defect via homogenisation

Chapter 1

Uniform exponential decay of eigenfunctions

In this chapter we first formulate the problem studied in the first part of the thesis. We then review already known results related to our problem, namely, the results on spectral convergence for the high contrast homogenisation when there is no defect [48, 49], and on existence of the eigenvalues of the operator A_ε near the eigenvalues of the limit homogenised operator A_0 [29]. We also describe the structure of the homogenised operators and properties of their spectra. In the end of Section 1.1 we state the main result of the chapter on the exponential decay uniform with respect to ε of the eigenfunctions of A_ε, which subsequently implies the two-scale compactness of the sequence of eigenvalues (Chapter 2). The rest of the chapter is devoted to the proof of the uniform exponential decay.

1.1 Notation, problem formulation, limit operator and the main result

We will use the following notation for the geometric configuration visualised on Figure 1-1, cf. [29]. Consider a periodic set of unit cubes

$$\{Q : Q = [0,1)^n + \xi,\, \xi \in \mathbb{Z}^n\}. \tag{1.1}$$

Let F_0 be an open periodic set with period one in each coordinate such that $F_0 \cap Q \Subset Q$ is a connected domain with infinitely smooth boundary. We denote $F_0 \cap Q$ by Q_0 and its complement $Q \backslash \overline{Q}_0$ by Q_1. Notice that the position of the

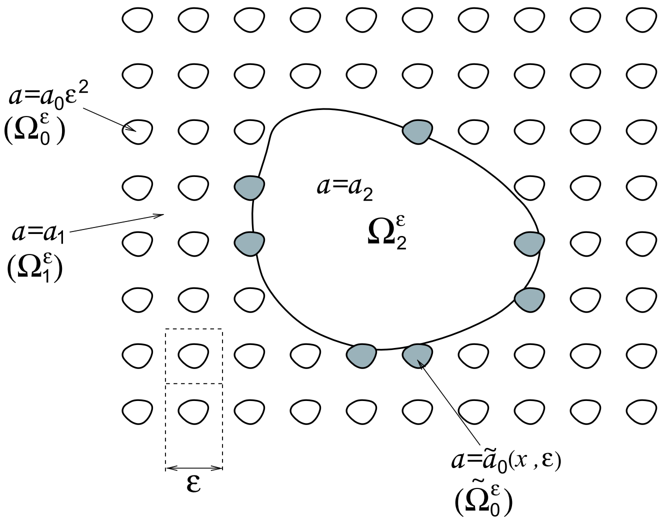

Figure 1-1: A defect in a rapidly oscillating high contrast periodic medium, cf. [29, Fig. 1].

particular set Q_0, Q_1 or Q depends on $\xi \in \mathbb{Z}^n$, however we will not reflect this in the notation to simplify the latter. Regularity assumptions on the boundary could be relaxed.[1] Let Ω_2 be a bounded domain containing the origin and with a sufficiently smooth boundary; its complement is denoted by Ω_1, $\Omega_1 = \mathbb{R}^n \backslash \overline{\Omega}_2$.

We define the 'inclusion phase' or the 'soft phase' Ω_0^ε as

$$\Omega_0^\varepsilon = \bigcup_{\varepsilon Q_0 \subset \Omega_1} \varepsilon Q_0,$$

where $\varepsilon > 0$ is a small parameter. The set of inclusions εQ_0 which intersect the boundary of Ω_2 is denoted by $\widetilde{\Omega}_0^\varepsilon$. The 'matrix phase', denoted by Ω_1^ε, is the complement to the inclusions in Ω_1, i.e. $\Omega_1^\varepsilon = \Omega_1 \backslash \overline{(\Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon)}$. 'Defect domain' Ω_2^ε is defined by $\Omega_2 \backslash \overline{\widetilde{\Omega}_0^\varepsilon}$. We also use the notation Θ_Ω for the characteristic function of a set Ω and B_R for the open ball of radius R centred at the origin.

We consider an eigenvalue problem

$$A_\varepsilon u^\varepsilon \;=\; \lambda_\varepsilon u^\varepsilon \tag{1.2}$$

[1]In particular, the results on the two-scale convergence stated in the present work remain valid at least under the assumption of Lipschitz regular boundaries. The $\varepsilon^{1/2}$-order bounds, as obtained in [29], require higher regularity.

for the point spectrum of an elliptic operator A_ε, self-adjoint in $L^2(\mathbb{R}^n)$,

$$A_\varepsilon u^\varepsilon := -\nabla \cdot \Big(a(x,\varepsilon)\nabla u^\varepsilon(x) \Big), \quad x \in \mathbb{R}^n, \tag{1.3}$$

with coefficient $a(x,\varepsilon)$ given by the formula

$$a(x,\varepsilon) = \begin{cases} a_0\varepsilon^2, & x \in \Omega_0^\varepsilon, \\ a_1, & x \in \Omega_1^\varepsilon, \\ a_2, & x \in \Omega_2^\varepsilon, \\ \widetilde{a}_0(x,\varepsilon), & x \in \widetilde{\Omega}_0^\varepsilon, \end{cases} \tag{1.4}$$

where measurable $\widetilde{a}_0(x,\varepsilon)$ is such that

$$\text{either } \widetilde{A}_0\,\varepsilon^{2-\theta} \le \widetilde{a}_0(x,\varepsilon) \le \sigma_0\,\varepsilon^{2-\theta} \text{ for all } \varepsilon, \text{ or } \widetilde{a}_0(x,\varepsilon) = a_0\,\varepsilon^2 \text{ for all } \varepsilon. \tag{1.5}$$

Here a_0, a_1, a_2, $\widetilde{A}_0$, σ_0 and θ are some positive constants independent of ε, $\theta \in (0,2]$. Notice that this includes as particular cases e.g. the case of 'removed' boundary inclusions, i.e. $a(x,\varepsilon) = a_1$ if $x \in \widetilde{\Omega}_0^\varepsilon \cap \Omega_1$, $a(x,\varepsilon) = a_2$ if $x \in \widetilde{\Omega}_0^\varepsilon \cap \Omega_2$, and the case of the 'full' inclusions, $\widetilde{a}_0(x,\varepsilon) = a_0\,\varepsilon^2$. The domain of A_ε is defined in a standard way via Friedrichs extension procedure with a bilinear form

$$B_\varepsilon(u,w) = \int_{\mathbb{R}^n} a(x,\varepsilon)\nabla u \cdot \nabla w \, dx$$

defined on $H^1(\mathbb{R}^n)$. By definition, $u^\varepsilon \in H^1(\mathbb{R}^n)$, $u^\varepsilon \not\equiv 0$, is an eigenfunction of the eigenvalue problem (1.2) with an eigenvalue λ_ε if

$$B_\varepsilon(u^\varepsilon, w) = \lambda_\varepsilon \int_{\mathbb{R}^n} u^\varepsilon w \, dx \tag{1.6}$$

for all $w \in H^1(\mathbb{R}^n)$.

Properties of A_ε are closely associated with properties of a corresponding purely periodic high contrast self-adjoint operator $\widehat{A}_\varepsilon$, i.e. with no defect present. The operator $\widehat{A}_\varepsilon$ is generated (via Friedrichs extension procedure) by a bilinear form

$$\widehat{B}_\varepsilon(u,w) = \int_{\mathbb{R}^n} \widehat{a}(x,\varepsilon)\nabla u \cdot \nabla w \, dx$$

acting on $H^1(\mathbb{R}^n)$ with a coefficient

$$\widehat{a}(x,\varepsilon) = \begin{cases} a_0\varepsilon^2, & x \in \varepsilon F_0, \\ a_1, & x \in \varepsilon(\mathbb{R}^n \backslash \overline{F}_0). \end{cases}$$

It is well known that the spectrum of a periodic operator is so called Floquet-Bloch spectrum, it is purely essential and has a band-gap structure. This operator was considered by Zhikov in [48, 49]. He proves that the spectra of $\widehat{A}_\varepsilon$ converge in the sense of Hausdorff to the spectrum of a certain homogenised operator $\widehat{A}_0$.

By definition, the Hausdorff convergence of spectra, $\sigma(\widehat{A}_\varepsilon) \xrightarrow{H} \sigma(\widehat{A}_0)$ as $\varepsilon \to 0$, means that

- for any $\lambda \in \sigma(\widehat{A}_0)$ there exists a sequence $\lambda_\varepsilon \in \sigma(\widehat{A}_\varepsilon)$ such that $\lambda_\varepsilon \to \lambda$;
- if $\lambda_\varepsilon \in \sigma(\widehat{A}_\varepsilon)$ and $\lambda_\varepsilon \to \lambda$, then $\lambda \in \sigma(\widehat{A}_0)$.

The limiting operator $\widehat{A}_0$ is of a 'two-scale' nature. It acts in a Hilbert space

$$\widehat{\mathcal{H}}_0 := \left\{ u(x,y) \in L^2\left(\mathbb{R}^n \times Q\right) \,\middle|\, u(x,y) = u_0(x) + v(x,y), \right. \\ \left. u_0 \in L^2\left(\mathbb{R}^n\right), v \in L^2\left(\mathbb{R}^n;\, L^2(Q_0)\right) \right\}, \tag{1.7}$$

with the natural inner product inherited from $L^2(\mathbb{R}^n \times Q)$ and $\widehat{\mathcal{H}}_0$ being its closed subspace. At this point we suppose for definiteness that $Q = [0,1)^n$. It is implied that v is extended by zero for $y \in Q_1$. In what follows we will assume that a function defined for $y \in Q$ is extended by periodicity to the whole $\mathbb{R}^n$. The operator $\widehat{A}_0$ is defined as generated by a (closed) symmetric and bounded from below bilinear form $\widehat{B}_0(u,w)$ acting in a dense subspace

$$\widehat{\mathcal{V}} = H^1\left(\mathbb{R}^n\right) + L^2\left(\mathbb{R}^n, H_0^1(Q_0)\right) \tag{1.8}$$

of $\widehat{\mathcal{H}}_0 = L^2\left(\mathbb{R}^n\right) + L^2\left(\mathbb{R}^n, L^2(Q_0)\right)$, which is defined as follows: for $u = u_0 + v, w = w_0 + z \in \mathcal{V}$,

$$\widehat{B}_0(u,w) = \int_{\mathbb{R}^n} A^{\text{hom}} \nabla u_0 \cdot \nabla w_0 \, dx + a_0 \int_{\mathbb{R}^n} \int_{Q_0} \nabla_y v \cdot \nabla_y z \, dy \, dx. \tag{1.9}$$

Here $A^{\text{hom}} = \left(A_{ij}^{\text{hom}}\right)$ is the standard "porous" homogenised (symmetric, positive-definite) matrix for the periodic medium as described above but when no defect is present and with $a_0 = 0$, see e.g. [27, §3.1]:

$$A_{ij}^{\hom}\xi_i\xi_j = \inf_{w \in C^\infty_{\rm per}(Q)} \int\limits_{Q_1} a_1|\xi + \nabla w|^2\,dy \quad (\xi \in \mathbb{R}^n)\,. \tag{1.10}$$

Notation $C^\infty_{per}(Q)$ stands for the set of infinitely smooth functions with periodic boundary conditions. Then one can see that the form is indeed bounded from below, densely defined and closed. Hence, according to the standard Friedrichs extension procedure, e.g. [41], $\widehat{A}_0$ can be defined as a self-adjoint operator with a domain $\mathcal{D}(\widehat{A}_0) \subset \widehat{\mathcal{V}}$.

It is also proved in [49] that the spectrum of $\widehat{A}_0$ is purely essential and has a band-gap structure. It can be described in terms of a function $\beta(\lambda)$ which we introduce next, see [48, 49] (cf. also [13]). First we define an operator T as follows,

$$Tf := -a_0\Delta f, \quad f(y) \in H^1_0(Q_0) \cap H^2(Q_0). \tag{1.11}$$

Denote by b the solution to

$$Tb - \lambda b = -a_0\Delta b - \lambda b = 1,\ b(y) \in H^1_0(Q_0). \tag{1.12}$$

Then the function $\beta(\lambda)$ is defined by the formula

$$\beta(\lambda) := \lambda\big(1 + \lambda\langle b\rangle_y\big), \tag{1.13}$$

where $\langle f\rangle_y := \int\limits_Q f(y)\,dy$ denotes a mean value of a function in a unit cell.

One can get a more transparent notion of $\beta(\lambda)$ by applying a spectral decomposition to problem (1.12). Let λ_i, λ'_j and φ_i, φ'_j, $i,j = 1,2,\dots,$ be all eigenvalues (repeated accordingly to their multiplicity) and corresponding orthonormalised eigenfunctions of T, where eigenfunctions φ'_j have zero mean, $\langle\varphi'_j\rangle_y = 0$. The set of eigenfunctions of T makes a basis in $H^1_0(Q_0)$. Hence we can write b as

$$b = \sum_{i=1}^\infty c_i\varphi_i + \sum_{j=1}^\infty c'_j\varphi'_j.$$

We substitute this expansion into (1.12) to obtain

$$\sum_{i=1}^\infty(\lambda_i - \lambda)c_i\varphi_i + \sum_{j=1}^\infty(\lambda'_j - \lambda)c'_j\varphi'_j = 1.$$

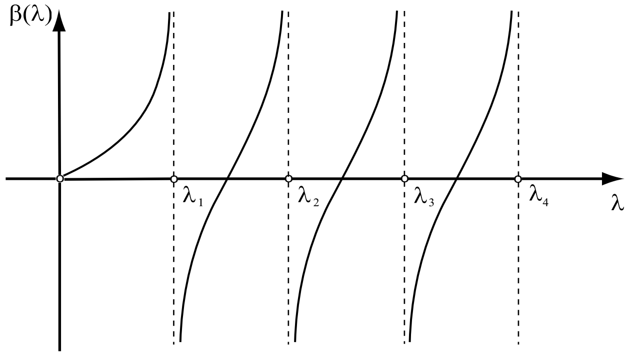

Figure 1-2: $\beta(\lambda)$, cf. [48].

Multiplying the latter by φ_i or φ'_j and integrating we arrive at

$$c_i = \frac{\langle \varphi_i \rangle_y}{\lambda_i - \lambda}, \quad c'_j = 0.$$

This yields us the following expression for $\beta(\lambda)$,

$$\beta(\lambda) = \lambda + \lambda^2 \sum_{i=1}^{\infty} \frac{\langle \varphi_i \rangle_y^2}{\lambda_i - \lambda}, \tag{1.14}$$

see Figure 1-2. The intervals where $\beta(\lambda) \geq 0$ correspond to the bands of the spectrum of $\widehat{A}_0$. Isolated points of the spectrum of $\widehat{A}_0$, i.e. λ'_j such that $\beta(\lambda'_j) < 0$, can also be regarded as degenerate bands. The intervals on which $\beta(\lambda) < 0$ (excluding λ'_j) are gaps.

The operator A_ε is obtained from $\widehat{A}_\varepsilon$ by a compact perturbation of its coefficient. It was shown in [23] (cf. also [41]) that in this case the essential spectrum of A_ε coincides with the spectrum of $\widehat{A}_\varepsilon$ and only extra eigenvalues can emerge, in particular in the gaps. We do not consider possible emergence of embedded eigenvalues, i.e. eigenvalues in the bands of essential spectrum. Existence of embedded eigenvalues is believed to be very unlikely, but this supposition has not been proved. In this work we are interested in convergence properties of the eigenvalues of A_ε lying in the gaps of its spectrum and the corresponding eigenfunctions. We will prove that if a sequence of eigenvalues converges to a point lying in the gap of $\sigma_{\text{ess}}(\widehat{A}_0)$, then the latter is an eigenvalue of the 'limit' homogenised operator A_0 . The operator A_0 can be obtained from $\widehat{A}_0$ by a compact perturbation of the coefficients. Its definition, analogous to the definition of

$\widehat{A}_0$, is the following. The operator A_0 acts in a Hilbert space

$$\mathcal{H}_0 := \left\{ u(x,y) \in L^2\left(\mathbb{R}^n \times Q\right) \,\middle|\, u(x,y) = u_0(x) + v(x,y), \right. \\ \left. u_0 \in L^2\left(\mathbb{R}^n\right), v \in\ L^2\left(\Omega_1;\ L^2(Q_0)\right) \right\}, \tag{1.15}$$

It is defined via a Friedrichs extension procedure by a closed symmetric and bounded from below bilinear form

$$B_0(u,w) \;=\; a_2 \int\limits_{\Omega_2} \nabla u_0 \cdot \nabla w_0 \, dx + \int\limits_{\Omega_1} A^{\text{hom}} \nabla u_0 \cdot \nabla w_0 \, dx + a_0 \int\limits_{\Omega_1} \int\limits_{Q_0} \nabla_y v \cdot \nabla_y z \, dy \, dx \tag{1.16}$$

acting in a dense subspace

$$\mathcal{V} \;=\; H^1\left(\mathbb{R}^n\right) \;+\; L^2\left(\Omega_1, H^1_0(Q_0)\right) \tag{1.17}$$

of $\mathcal{H}_0 = L^2\left(\mathbb{R}^n\right) \;+\; L^2\left(\Omega_1, L^2(Q_0)\right)$, $u = u_0 + v, w = w_0 + z \in\ \mathcal{V}$. By definition λ_0 is an eigenvalue of A_0 and $u^0(x,y) = u_0(x) + v(x,y) \in \mathcal{V}$ is corresponding eigenfunction if

$$B_0(u^0, w) \;=\lambda_0(u^0, w)_{\mathcal{H}_0}, \tag{1.18}$$

for any $w = w_0 + z \in\ \mathcal{V}$. The eigenfunction solves the following problem:

$$\begin{aligned} -\nabla \cdot a_2 \nabla u_0(x) &= \lambda_0 u_0(x), \quad x \in \Omega_2, \\ -\nabla \cdot A^{\text{hom}} \nabla u_0(x) &= \lambda_0 \left(u_0 + \langle v \rangle_y\right), \quad x \in \mathbb{R}^n \backslash \Omega_2, \\ -a_0 \Delta_y v &= \lambda_0 \left(u_0 \;+\; v\right), y \in Q_0; \;\; v = 0,\; y \in \partial Q_0 \;\; \left(x \in \mathbb{R}^n \backslash \Omega_2\right), \\ \left(u_0\right)_- = \left(u_0\right)_+, \;\; a_2 \left(\frac{\partial u_0}{\partial n}\right)_- &= \left(\sum_{i,j} A^{\text{hom}}_{ij} \frac{\partial u_0}{\partial x_j} n_i\right)_+, \; x \in\ \partial \Omega_2. \end{aligned} \tag{1.19}$$

Here

$$\langle v \rangle_y(x) \;:=\; |Q|^{-1} \int_Q v(x,y)\, dy$$

denotes the averaging with respect to y over the periodicity cell Q (extending v by zero outside Q_0); $(\cdot)_-$ and $(\cdot)_+$ denote respectively the interior and exterior limit values of the appropriate entities at the boundary $\partial\Omega_2$ of Ω_2, n is the interior unit normal to $\partial\Omega_2$.

A similar problem is considered in [29]. The authors use an asymptotic expansion approach seeking a solution to problem (1.2) in the form

$$\begin{aligned} u^\varepsilon(x) &= u^0(x, x/\varepsilon) + \varepsilon u^{(1)}(x, x/\varepsilon) + \varepsilon^2 u^{(2)}(x, x/\varepsilon) + \dots, \\ \lambda(\varepsilon) &= \lambda_0 + o(1). \end{aligned}$$

They prove that if there exists an eigenvalue λ_0 satisfying $\beta(\lambda_0) < 0$, $\lambda_0 \neq \lambda'_j$, then there exists $\varepsilon_0 > 0$ and a constant $C_1 > 0$ independent of ε such that for any $0 < \varepsilon \leq \varepsilon_0$ there exists an isolated eigenvalue λ_ε of operator A_ε of finite multiplicity, such that

$$|\lambda_\varepsilon - \lambda_0| < C_1 \varepsilon^{1/2}. \tag{1.20}$$

Moreover if (u_0, v) is an eigenfunction of A_0 which corresponds to λ_0 then the function

$$u^{\mathrm{appr}}(x, \varepsilon) := \begin{cases} u_0(x) + v(x, x/\varepsilon), & x \in \Omega_0^\varepsilon, \\ u_0(x), & x \in \Omega_1^\varepsilon \cup \Omega_2 \cup \tilde{\Omega}_0^\varepsilon, \end{cases} \tag{1.21}$$

is an approximate eigenfunction for A_ε at least in the following sense: there exist constants $c_j(\varepsilon)$ such that

$$\|u^{\mathrm{appr}} - \sum_{j \in J_\varepsilon} c_j(\varepsilon) u_j^\varepsilon\|_{L_2(\mathbb{R}^n)} < C_2 \varepsilon^{1/2}, \tag{1.22}$$

where $J_\varepsilon = \{j : |\lambda_{\varepsilon,j} - \lambda_0| < C\varepsilon^{1/2}\}$ is a finite set of indices (for each ε), and $\lambda_{\varepsilon,j}$, $u_j^\varepsilon(x)$ are eigenvalues and L_2-normalised eigenfunctions of A_ε, and the constants C_1 and C_2 are independent of ε.

This assertion partly answers the problem of asymptotic behaviour of the discrete spectrum of A_ε. Thus, we know that any eigenvalue of A_0 has converging to it a sequence of eigenvalues of A_ε. In this work we study an open question that consists in the following. Suppose there is a sequence of eigenvalues of A_ε converging to a point in the gap of $\sigma(\widehat{A}_0)$, $\lambda_\varepsilon \to \lambda_0$. Is the limit λ_0 an eigenvalue of A_0 or not? To answer this question affirmatively one firstly needs to show a compactness (in the sense of two-scale convergence) of the corresponding eigenfunctions. Once having the compactness proved one can pass to a limit in the spectral problem (1.2) to get eventually the spectral problem for the homogenised operator. In its turn the proof of compactness requires uniform with respect to ε exponential decay at infinity of eigenfunctions u_ε corresponding to a convergent sequence of eigenvalues.

Now we formulate our main result.

Theorem 1.1.1. *The operator* A_ε *converges to* A_0 *in the sense of the strong two-scale resolvent convergence. Hence the spectral projectors also strongly two-scale converge away from the point spectrum of* A_0. *The spectrum of* A_ε *converges in the sense of Hausdorff to the spectrum of* A_0. *Let* λ_0 *be an isolated eigenvalue of multiplicity* m *of the operator* A_0 *in the gap of its essential spectrum. Then, for small enough* ε, *there exist exactly* m *eigenvalues* $\lambda_{\varepsilon,i}$ *of* A_ε *(counted with their multiplicities) such that*

$$|\lambda_{\varepsilon,i} - \lambda_0| \leq C\varepsilon^{1/2},\ i = 1, \ldots, m, \tag{1.23}$$

with a constant C *independent of* ε.[2] *If for some sequence* $\varepsilon_k \to 0$ *a sequence of eigenvalues* λ_{ε_k} *of* A_ε *converges to* λ_0 *which is in the gap of the essential spectrum of* A_0, *then* λ_0 *is an isolated eigenvalue of* A_0 *of a finite multiplicity* m *and for large enough* k, $\lambda_{\varepsilon_k} \in \{\lambda_{\varepsilon_k,i},\ i = 1, \ldots, m\}$.

1.2 Uniform exponential decay of the eigenfunctions of A_ε

The phenomenon of exponential decay of eigenfunctions of various differential operators corresponding to the eigenvalues in the gaps of essential spectra has been extensively investigated for the few last decades, see e.g. [6, 22, 23, 39]. For example, in [6] a Schrödinger operator H with random perturbation is considered. It is shown that the rate of the exponential decay is proportional to $\sqrt{\Delta_+(E)\Delta_-(E)}$, where E is an eigenvalue in a gap, $\Delta_+(E)$ and $\Delta_-(E)$ are distances from E to the right and left edges of the gap respectively. Roughly speaking, the exponent obtained in [6] is proportional to the distance from the essential spectrum near the centre of the gap and to the square root from the distance near the both edges of the gap. This estimate is the best of the sort known at present. Another result, obtained in [23], can be straightforwardly applied to the operator A_ε when ε is fixed. It follows from [23] that an eigenfunction u^ε decays exponentially with an exponent proportional to $\operatorname{dist}(\lambda_\varepsilon, \sigma_{\mathrm{ess}}(A_\varepsilon))$. Nevertheless, this is not sufficient for our purposes. To gain compactness of a sequence

[2] The error bound (1.23) employs the results of [29] requiring, as stated, higher regularity of ∂Q_0. The rest of the statement of the theorem applies potentially to less regular boundaries.

u^ε we need uniform decay, i.e. exponential decay with exponent independent of ε, which previous results do not guarantee. In this section we prove ε-uniform exponential decay of sequence of u^ε corresponding to converging in the gap λ_ε.

Remark 1.2.1. We would like to draw attention to the qualitative difference between the previously obtained estimates for the rate of exponential decay and the one we prove in this work. As we mentioned above, known results give the rate of decay proportional to the distance to gap edges or to the square root of the distance. Our estimate (1.24) is entirely different. For small enough ε the rate of decay is $O\left(\mathrm{dist}(\lambda_\varepsilon, \sigma_{\mathrm{ess}}(A_\varepsilon))^{1/2}\right)$ at the right end of each interval $\beta(\lambda) < 0$ and proportional to $(\mathrm{dist}(\lambda_\varepsilon, \sigma_{\mathrm{ess}}(A_\varepsilon))^{-1/2}$ at the left end, cf. Figure 1-2 and (1.14).

We formulate the main result of this section (and also one of the principal results of the first part) in the following

Theorem 1.2.2. *Let λ_{ε_k} and u^{ε_k} be sequences of eigenvalues of the operator A_ε and corresponding eigenfunctions normalised in $L^2(\mathbb{R}^n)$, where ε_k is some positive sequence converging to zero as $k \to \infty$. Let λ_0 be such that $\beta(\lambda_0)$ is negative and λ_0 is not an eigenvalue of the operator T given by (1.11). Suppose that λ_{ε_k} converges to λ_0. Then for small enough ε_k eigenfunctions u^{ε_k} decay uniformly exponentially at infinity, namely, for*

$$0 < \alpha < \sqrt{-\beta(\lambda_0)/a_1} \tag{1.24}$$

the following holds:

$$\|e^{\alpha|x|}u^{\varepsilon_k}\|_{L^2(\mathbb{R}^n)} \leq C,$$

uniformly in ε_k, i.e. for any $0 < \varepsilon_k < \varepsilon(\alpha)$, with $C = C(\alpha)$ independent of ε.

Proof. We drop the index k in ε_k for the sake of simplification of notation. So, when we say, for instance, 'sequence λ_ε' we actually mean 'subsequence λ_{ε_k}'.

The plan of the proof is the following. We first derive 'elementary' a priori estimates for the eigenfunction u^ε outside the set of inclusions $\Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon$. Next we study the structure of the eigenfunction at the small scale and deduce some vital inequalities for $\varepsilon\nabla u^\varepsilon$ inside the inclusions. As a central technical step, we then employ in the integral identity (1.6) a test function with exponentially growing weight $g^2(|x|)$, see (1.37)–(1.38) below, and perform some delicate two-scale uniform estimates to achieve the result. Introducing a test function with exponentially growing weight we use the idea of Agmon, [1]. This on its own does not lead to a straightforward conclusion. We have to develop some delicate two-scale analysis, studying the properties of eigenfunctions of A_ε both at large and small scales at the same time. The main auxiliary technical results are proven in Lemma 1.2.4 and Proposition 1.3.1.

Step 1. Due to the nature of the operator A_ε (coefficient $a(x,\varepsilon)$ is very small on the inclusion phase) one can expect that the eigenfunctions (more precise, their gradients) oscillate wildly on the inclusion phase. Nevertheless it is possible to control u^ε outside the inclusions. Setting $w = u^\varepsilon$ in (1.6) we have

$$\begin{aligned} \varepsilon^2 a_0 \|\nabla u^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon)} + a_1 \|\nabla u^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)} + a_2 \|\nabla u^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)} + \\ + \|\widetilde{a}_0^{1/2}(x,\varepsilon)\,\nabla u^\varepsilon\|^2_{L^2(\widetilde{\Omega}_0^\varepsilon)} = \lambda_\varepsilon \|u^\varepsilon\|^2_{L^2(\mathbb{R}^n)} = \lambda_\varepsilon. \end{aligned} \tag{1.25}$$

Therefore

$$\|u^\varepsilon\|_{H^1(\mathbb{R}^n \setminus (\Omega_0^\varepsilon \bigcup \widetilde{\Omega}_0^\varepsilon))} \le C \tag{1.26}$$

uniformly in ε. From now on C denotes a generic constant whose precise value is insignificant and can change from line to line.

Step 2. Now we will represent u^ε as a sum of two functions, one of them has ε-uniformly bonded norm in H^1, another preserves the 'uncontrollable' oscillations of the gradient of u^ε. Let us consider u^ε in a cell εQ corresponding to such $\xi = \xi(\varepsilon) \in \mathbb{Z}^n$, see 1.1, that the respective 'inclusion' εQ_0 has a nonempty intersection with Ω_1. There exists an extension $\widetilde{u}^\varepsilon$ of $u^\varepsilon|_{\varepsilon Q_1}$ to the whole cell εQ such that

$$\|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \le C \|u^\varepsilon\|_{L^2(\varepsilon Q_1)}, \quad \|\nabla \widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \le C \|\nabla u^\varepsilon\|_{L^2(\varepsilon Q_1)}, \tag{1.27}$$

where C does not depend on ε or ξ, see e. g. [35, Ch. 3, §4, Th. 1], which is a version of the so-called 'extension lemma', see also e.g. [27, §3.1, L. 3.2]. In particular, we can choose the following extension:

$$\begin{aligned} \widetilde{u}^\varepsilon \equiv u^\varepsilon, \qquad & x \in \Omega_1^\varepsilon \cup \Omega_2^\varepsilon, \\ -\nabla \cdot \big(a(x,\varepsilon) \nabla \widetilde{u}^\varepsilon(x)\big) = 0, \quad & x \in \Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon, \end{aligned}$$

which minimises $\|a^{1/2}(x,\varepsilon)\nabla \widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)}$ subject to the prescribed boundary conditions, with (1.4) and (1.5) ensuring that (1.27) still holds. From (1.26) and (1.27) we conclude that

$$\|\widetilde{u}^\varepsilon\|_{H^1(\mathbb{R}^n)} \le C. \tag{1.28}$$

We represent u^ε in the form

$$u^\varepsilon(x) = \widetilde{u}^\varepsilon(x) + v^\varepsilon(x) \tag{1.29}$$

and consider the function $v^\varepsilon \in H_0^1(\Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon)$. We assume that v^ε is extended by zero to the whole $\mathbb{R}^n$. In each inclusion $\varepsilon Q_0 \subset \Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon$ we have the following boundary value problem for $v^\varepsilon(x)$:

$$\begin{aligned} -\nabla \cdot (a(x,\varepsilon)\nabla v^\varepsilon) - \lambda_\varepsilon v^\varepsilon &= \lambda_\varepsilon \widetilde{u}^\varepsilon, \ x \in \varepsilon Q_0, \\ v^\varepsilon(x) &= 0, \ x \in \partial(\varepsilon Q_0). \end{aligned} \tag{1.30}$$

When $a(x,\varepsilon) = a_0\varepsilon^2$, i.e. everywhere in Ω_0^ε and also in $\widetilde{\Omega}_0^\varepsilon$ in the case $\widetilde{a}_0(x,\varepsilon) = a_0\varepsilon^2$, after changing the variables $x \to y = x/\varepsilon$ we obtain

$$\begin{aligned} -a_0\Delta_y v^\varepsilon(\varepsilon y) - \lambda_\varepsilon v^\varepsilon(\varepsilon y) &= \lambda_\varepsilon \widetilde{u}^\varepsilon(\varepsilon y), \ y \in Q_0, \\ v^\varepsilon(\varepsilon y) &= 0, \ y \in \partial Q_0. \end{aligned} \tag{1.31}$$

Since $\lambda_0 \neq \lambda_j$ by the assumptions of the theorem, λ_ε is separated uniformly from the spectrum of the operator T, (1.11), for small enough ε. Hence the resolvent of T at λ_ε is bounded uniformly in ε and (1.31) implies

$$\|v^\varepsilon(\varepsilon y)\|_{H^1(Q_0)} \le C\|\widetilde{u}^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}. \tag{1.32}$$

In the case when $\widetilde{A}_0\,\varepsilon^{2-\theta} \le \widetilde{a}_0(x,\varepsilon) \le \sigma_0\,\varepsilon^{2-\theta}$, $\theta \in (0,2]$, we multiply equation (1.30) by v^ε and integrate by parts to obtain after rescaling

$$\varepsilon^{-2}\int\limits_{Q_0} \widetilde{a}_0(\varepsilon y,\varepsilon)|\nabla_y v^\varepsilon(\varepsilon y)|^2 dy - \lambda_\varepsilon \int\limits_{Q_0} \big(v^\varepsilon(\varepsilon y)\big)^2\, dy = \lambda_\varepsilon \int\limits_{Q_0} \widetilde{u}^\varepsilon(\varepsilon y) v^\varepsilon(\varepsilon y)\, dy. \tag{1.33}$$

Notice that $\varepsilon^{-2}\widetilde{a}_0(\varepsilon y,\varepsilon) \ge \widetilde{A}_0\varepsilon^{-\theta} \to \infty$ as $\varepsilon \to 0$. Then using Poincaré inequality for functions from $H^1(Q_0)$

$$\int\limits_{Q_0} f^2\, dy \le C\int\limits_{Q_0} |\nabla_y f|^2 dy,$$

and Hölder inequality

$$\left[\int_{Q_0} fg\,dy\right]^2 \le \int_{Q_0} f^2\,dy \int_{Q_0} g^2\,dy,$$

we derive from (1.33) that

$$(C\varepsilon^{-\theta} - \lambda_\varepsilon)\|v^\varepsilon(\varepsilon y)\|^2_{L^2(Q_0)} \le \lambda_\varepsilon \|v^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}\|\widetilde{u}^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}.$$

The latter immediately implies that

$$\|v^\varepsilon(\varepsilon y)\|_{L^2(Q_0)} \le C\|\widetilde{u}^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}. \tag{1.34}$$

In fact an even stronger relation is valid, $\|v^\varepsilon(\varepsilon y)\|_{L^2(Q_0)} = o\left(\|\widetilde{u}^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}\right)$. From (1.33) and (1.34) one directly obtains

$$\varepsilon^{-2}\|\widetilde{a}_0^{1/2}\nabla_y v^\varepsilon(\varepsilon y)\|^2_{L^2(Q_0)} \le C\|\widetilde{u}^\varepsilon(\varepsilon y)\|^2_{L^2(Q_0)}, \tag{1.35}$$

for small enough ε. Returning in (1.32) and in (1.34), (1.35) to the variable x we arrive at the following inequality that describes the behaviour of v^ε and its gradient in $\Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon$,

$$\|a^{1/2}\nabla v^\varepsilon(x)\|^2_{L^2(\varepsilon Q_0)} + \|v^\varepsilon(x)\|^2_{L^2(\varepsilon Q_0)} \le C\|\widetilde{u}^\varepsilon(x)\|^2_{L^2(\varepsilon Q_0)}, \tag{1.36}$$

with an ε-independent constant C.

Step 3. In order to get the uniform exponential decay of the eigenfunctions we next substitute in (1.6) a test function of a special form:

$$w = g^2(|x|)\widetilde{u}^\varepsilon(x). \tag{1.37}$$

Here we define function g as follows

$$g(t) = \begin{cases} e^{\alpha t}, & t \in [0, R], \\ e^{\alpha R}, & t \in (R, +\infty), \end{cases} \tag{1.38}$$

where R is some arbitrary positive number. The exponent α will be chosen later. This method was employed e.g. by Agmon, see [1], but in the present case its realisation is not straightforward. Namely, to obtain the desired estimates we have to implement the approach of [1] in the context of the two-scale analysis. We will show that $g(|x|)\widetilde{u}^\varepsilon(x)$, and consequently $g(|x|)u^\varepsilon(x)$, are bounded in $L^2(\mathbb{R}^n)$ uniformly with respect to R and ε. Then we will show via passing to the limit as $R\to\infty$ that we can replace $g(|x|)$ by $e^{\alpha|x|}$.

Remark 1.2.3. We cannot use $e^{2\alpha|x|}\widetilde{u}^\varepsilon(x)$ as a test function directly, since it is not known at this stage that functions $e^{\alpha|x|}\widetilde{u}^\varepsilon(x)$ and $e^{\alpha|x|}u^\varepsilon(x)$ are square integrable.

The following identity holds by direct inspection

$$\nabla\widetilde{u}^\varepsilon\nabla(g^2\widetilde{u}^\varepsilon)=|\nabla(g\widetilde{u}^\varepsilon)|^2-|\nabla g|^2(\widetilde{u}^\varepsilon)^2. \tag{1.39}$$

Notice that the Euclidian norm of ∇g is bounded by g with constant α (uniformly with respect to R):

$$\big|\nabla g(|x|)\big|\le\alpha g(|x|). \tag{1.40}$$

After the substitution of (1.37) into (1.6) we have, via (1.29) and (1.39),

$$\begin{aligned}\varepsilon^2a_0\int\limits_{\Omega_0^\varepsilon}\nabla u^\varepsilon\cdot\nabla(g^2\widetilde{u}^\varepsilon)\,dx+\int\limits_{\widetilde{\Omega}_0^\varepsilon}\widetilde{a}_0\nabla v^\varepsilon\cdot\nabla(g^2\widetilde{u}^\varepsilon)\,dx+\int\limits_{\mathbb{R}^n\setminus\Omega_0^\varepsilon}a(x,\varepsilon)|\nabla(g\widetilde{u}^\varepsilon)|^2\,dx-\\
-a_1\int\limits_{\Omega_1^\varepsilon}|\nabla g|^2(\widetilde{u}^\varepsilon)^2\,dx-\lambda_\varepsilon\int\limits_{\Omega_0^\varepsilon\cup\Omega_1^\varepsilon}g^2(\widetilde{u}^\varepsilon)^2\,dx-\lambda_\varepsilon\int\limits_{\Omega_0^\varepsilon}g^2v^\varepsilon\widetilde{u}^\varepsilon\,dx=\\
=\lambda_\varepsilon\int\limits_{\widetilde{\Omega}_0^\varepsilon}g^2u^\varepsilon\widetilde{u}^\varepsilon\,dx+\lambda_\varepsilon\int\limits_{\Omega_2^\varepsilon}g^2(\widetilde{u}^\varepsilon)^2dx+\int\limits_{\Omega_2^\varepsilon\cup\widetilde{\Omega}_0^\varepsilon}a(x,\varepsilon)|\nabla g|^2(\widetilde{u}^\varepsilon)^2\,dx.\end{aligned} \tag{1.41}$$

Notice that the right hand side is bounded by some constant C independent of ε and R due to (1.26), (1.28), (1.36) and the boundedness of the domains of integration.

The rough idea of the remaining part of the proof is the following. We argue that the second term on the left hand side of the latter is small and the first one is relatively small (compared to the other terms of the identity). One can notice that in equation (1.31) the right hand side is 'almost' a constant for every fixed Q_0. Then one can expect that the solution of (1.31) is 'approximately equal' to the solution of (1.12) corresponding to $\lambda = \lambda_\varepsilon$ multiplied by $\lambda_\varepsilon \widetilde{u}^\varepsilon(\varepsilon y)$,

$$v^\varepsilon(\varepsilon y) \sim \lambda_\varepsilon b(y) \widetilde{u}^\varepsilon(\varepsilon y).$$

Rearranging the integrated entities in the last two terms on the left hand side of (1.41) one obtain 'approximately'

$$-\lambda_\varepsilon g^2 (\widetilde{u}^\varepsilon)^2 - \lambda_\varepsilon g^2 v^\varepsilon \widetilde{u}^\varepsilon \sim -\left[\lambda_\varepsilon (1 + \lambda_\varepsilon b(y))\right] g^2 (\widetilde{u}^\varepsilon(\varepsilon y))^2. \tag{1.42}$$

The expression in the square brackets resembles the definition of $\beta(\lambda_\varepsilon)$, see (1.13). Rescaling back to variable x and integrating the right hand side of (1.42) one can obtain

$$-\beta(\lambda_\varepsilon) \|g \widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}. \tag{1.43}$$

Notice that as $\lambda_\varepsilon \to \lambda_0$, $\beta(\lambda_\varepsilon) \to \beta(\lambda_0) < 0$. Hence we obtain $\|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}$ multiplied by a uniformly positive coefficient, end we need only to choose appropriate exponent α, see (1.38), to ensure that the fourth term on the left hand side of (1.41),

$$-a_1 \int\limits_{\Omega_1^\varepsilon} |\nabla g|^2 (\widetilde{u}^\varepsilon)^2 \, dx > -a_1 \alpha^2 \|g \widetilde{u}^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)},$$

is compensated by (1.43).

Let us continue the proof. Consider the second term on the left hand side of (1.41). Since the coefficient $\widetilde{a}_0(x, \varepsilon)$ is bounded uniformly in ε and the sequence of domains $\widetilde{\Omega}_0^\varepsilon$ is also bounded (so $g^2|_{\widetilde{\Omega}_0^\varepsilon}, \nabla g^2|_{\widetilde{\Omega}_0^\varepsilon} \le C$ uniformly) we derive that

$$\left| \int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 \nabla v^\varepsilon \cdot \nabla (g^2 \widetilde{u}^\varepsilon) \, dx \right| \le C \int\limits_{\widetilde{\Omega}_0^\varepsilon} a_0^{1/2} |\nabla v^\varepsilon| \big(|\nabla \widetilde{u}^\varepsilon| + |\widetilde{u}^\varepsilon| \big) \, dx \le$$

$$\le C \|a_0^{1/2} \nabla v^\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \big(\|\nabla \widetilde{u}^\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} + \|\widetilde{u}^\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \big).$$

Then from (1.28), (1.36) follows that

$$\left| \int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 \nabla v^\varepsilon \cdot \nabla (g^2 \widetilde{u}^\varepsilon) \, dx \right| \le C \|\widetilde{u}^\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)}.$$

The right hand side of the latter converges to zero. Indeed, let us take an arbitrary subsequence $\widetilde{u}^\varepsilon$. Since $\|\widetilde{u}^\varepsilon\|_{H^1(\mathbb{R}^n)}$ is bounded uniformly in ε, see (1.28), the set of functions $\widetilde{u}^\varepsilon$ is weakly compact in $H^1(B_R)$, hence strongly compact in $L^2(B_R)$ for any R; we take R large enough so that $\Omega_2 \Subset B_R$. Then there exists a further subsequence $\widetilde{u}^\varepsilon$ that converges to some function u_0 strongly in $L^2(B_R)$. Then

$$\|\widetilde{u}^\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \leq \|u_0\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} + \|\widetilde{u}^\varepsilon - u_0\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \to 0 \tag{1.44}$$

as the Lebesgue measure of the set $\widetilde{\Omega}_0^\varepsilon$ tends to zero. Since we have chosen in the beginning an arbitrary subsequence $\widetilde{u}^\varepsilon$, (1.44) holds for any sequence of ε. Hence

$$\left| \int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 \nabla v^\varepsilon \cdot \nabla(g^2 \widetilde{u}^\varepsilon)\, dx \right| \to 0. \tag{1.45}$$

From (1.36) and (1.44) we also obtain

$$\|v^\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \to 0. \tag{1.46}$$

Step 4. The following Lemma approximates the last two terms and bounds the first term (both, in a sense, of a 'two-scale' nature) on the left hand side of (1.41).

Lemma 1.2.4. *There exists $\varepsilon_0 > 0$ such that for all positive $\varepsilon < \varepsilon_0$ the following estimates are valid*

$$\begin{aligned} \left| \lambda_\varepsilon \int\limits_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} g^2 (\widetilde{u}^\varepsilon)^2\, dx + \lambda_\varepsilon \int\limits_{\Omega_0^\varepsilon} g^2 v^\varepsilon \widetilde{u}^\varepsilon\, dx - \beta(\lambda_\varepsilon) \int\limits_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} g^2 (\widetilde{u}^\varepsilon)^2\, dx \right| \leq \\ \leq C\, \varepsilon \left(\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} + \|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)} \right) + C, \end{aligned} \tag{1.47}$$

and

$$\left| \varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla(g^2 \widetilde{u}^\varepsilon)\, dx \right| \leq C\, \varepsilon \left(\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} + \|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)} + C \right), \tag{1.48}$$

where C does not depend on ε and R.

The proof of this lemma is quite technical and we give it in the next section. We make use of Lemma 1.2.4 and convergence (1.45) to transform identity (1.41) into the following inequality, valid for small enough ε:

$$a_1\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} - a_1\|(\nabla g)\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)} - \beta(\lambda_\varepsilon)\|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon\cup\Omega_1^\varepsilon)} -$$
$$- 2C\varepsilon\left(\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} + \|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon\cup\Omega_1^\varepsilon)}\right) \le C,$$

where C is independent of ε and R. Notice that $\beta(\lambda_\varepsilon)$ is negative and uniformly bounded away from zero as $\lambda_\varepsilon \to \lambda_0$. Applying (1.40) to the second term on the left hand side we arrive at

$$(a_1 - 2C\varepsilon)\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} + \left(-\beta(\lambda_\varepsilon) - \alpha^2 a_1 - 2C\varepsilon\right)\|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon\cup\Omega_1^\varepsilon)} \le C. \quad (1.49)$$

Hence we should choose α such that $-\beta(\lambda_0) - \alpha^2 a_1$ is positive, i.e.

$$\alpha < \sqrt{-\beta(\lambda_0)/a_1}.$$

Since $g(|x|)$ coincides with $e^{\alpha|x|}$ on the ball B_R, taking ε small enough and restricting the L^2-norms to B_R we arrive at

$$\left\|e^{\alpha|x|}\widetilde{u}^\varepsilon\right\|_{L^2(B_R)} \le C,$$

where C does not depend on ε and R. Then passing to the limit as $R \to \infty$ we obtain

$$\left\|e^{\alpha|x|}\widetilde{u}^\varepsilon\right\|_{L^2(\mathbb{R}^n)} \le C. \quad (1.50)$$

Step 5. Despite the fact that the sequence of ∇v^ε is unbounded in L^2-norm, the function v^ε itself is controlled by $\widetilde{u}^\varepsilon$, see (1.36). Therefore we can get the estimate for the function u^ε analogous to (1.50).

$$\left\|e^{\alpha|x|}u^\varepsilon\right\|^2_{L^2(\mathbb{R}^n)} \le \left\|e^{\alpha|x|}\widetilde{u}^\varepsilon\right\|^2_{L^2(\mathbb{R}^n)} + \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon} \left\|e^{\alpha|x|}v^\varepsilon\right\|^2_{L^2(\varepsilon Q_0)}.$$

In each cell we use inequality (1.36) and

$$\sup_{x' \in \varepsilon Q} e^{\alpha|x'|} \le e^{\alpha\sqrt{n}\varepsilon} e^{\alpha|x|}, \qquad \forall x \in \varepsilon Q, \quad (1.51)$$

to obtain

$$\left\|e^{\alpha|x|}v^\varepsilon\right\|_{L^2(\varepsilon Q_0)} \le Ce^{\alpha\sqrt{n}\varepsilon}\left\|e^{\alpha|x|}\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q_0)} \le C\left\|e^{\alpha|x|}\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q_0)},$$

and hence, finally,

$$\left\|e^{\alpha|x|}u^\varepsilon\right\|_{L^2(\mathbb{R}^n)} \le C$$

uniformly in ε. ☐

Remark 1.2.5. It is easy to see that $\nabla\widetilde{u}^\varepsilon$ (unlike ∇u^ε) decays exponentially uniformly in ε with the same rate as u^ε. Indeed, from (1.51) and (1.27) it follows that

$$\|g\nabla\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon\cup\widetilde{\Omega}_0^\varepsilon)} \le \sum_{\varepsilon Q_0\subset\Omega_0^\varepsilon\cup\widetilde{\Omega}_0^\varepsilon} \sup_{x'\in\varepsilon Q} g\|\nabla\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon Q_0)} \le C\|g\nabla\widetilde{u}^\varepsilon\|^2_{L^2(\Pi^\varepsilon)},$$

where $\Pi^\varepsilon := \left\{\bigcup \varepsilon Q_1 | \varepsilon Q_1 \text{ is such that correcponding } \varepsilon Q_0 \subset \Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon\right\}$. Since $\|\nabla\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)}$ and hence $\|g\nabla\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)}$ are bounded uniformly, we have

$$\|g\nabla\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^n)} \le C + C\|g\nabla\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)} = C + C\|\nabla(g\widetilde{u}^\varepsilon) - \nabla g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)} \le$$
$$\le C + C\|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)} + C\alpha\|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)}.$$

The latter is bounded uniformly in ε and R due to (1.49) and (1.50). Hence, passing to the limit as $R\to\infty$, we finally arrive at

$$\left\|e^{\alpha|x|}\nabla\widetilde{u}^\varepsilon\right\|_{L^2(\mathbb{R}^n)} \le C \tag{1.52}$$

uniformly in ε.

Remark 1.2.6. Estimate (1.24) is sharp in a sense. As we will show later, u^ε strongly two-scale converges to u_0, for which $\sqrt{-\beta(\lambda_0)/a_1}$ is the optimal estimate for its decay exponent, cf. (2.51).

1.3 Proof of Lemma 1.2.4.

Proof. **Step 1.** First we decompose the function v^ε in Ω_0^ε into the sum of two functions:

$$v^\varepsilon = \widetilde{v}^\varepsilon + \widehat{v}^\varepsilon, \tag{1.53}$$

solving the following equations (cf. (1.31)):

$$\begin{aligned} -a_0\Delta_y\widetilde{v}^\varepsilon(\varepsilon y) - \lambda_\varepsilon\widetilde{v}^\varepsilon(\varepsilon y) &= \lambda_\varepsilon\langle\widetilde{u}^\varepsilon(\varepsilon y)\rangle_y,\ y\in Q_0,\\ \widetilde{v}^\varepsilon(\varepsilon y) &= 0,\ y\in\partial Q_0, \end{aligned} \tag{1.54}$$

$$
\begin{gathered}
-a_0\Delta_y\widehat{v}^\varepsilon(\varepsilon y) - \lambda_\varepsilon\widehat{v}^\varepsilon(\varepsilon y) = \lambda_\varepsilon\left(\widetilde{u}^\varepsilon(\varepsilon y) - \langle\widetilde{u}^\varepsilon(\varepsilon y)\rangle_y\right),\ y \in Q_0, \\
\widehat{v}^\varepsilon(\varepsilon y) = 0,\ y \in \partial Q_0.
\end{gathered} \tag{1.55}
$$

The solution of (1.54) could by written in the form

$$
\widetilde{v}^\varepsilon(\varepsilon y) = \lambda_\varepsilon\langle\widetilde{u}^\varepsilon\rangle_y b_\varepsilon(y), \tag{1.56}
$$

where b_ε is a solution of (1.12) with $\lambda = \lambda_\varepsilon$. Due to the uniform (with respect to ε) boundedness of the resolvent of the operator T in the neighbourhood of λ_0, the solution of (1.55) is bounded as follows,

$$
\begin{aligned}
\|\widehat{v}^\varepsilon(\varepsilon y)\|_{L^2(Q_0)} = \left\|(T-\lambda_\varepsilon)^{-1}(\widetilde{u}^\varepsilon(\varepsilon y) - \langle\widetilde{u}^\varepsilon\rangle_y)\right\|_{L^2(Q_0)} \le \\
\le C\,\|\widetilde{u}^\varepsilon(\varepsilon y) - \langle\widetilde{u}^\varepsilon\rangle_y\|_{L^2(Q_0)} \le C\,\|\nabla_y\widetilde{u}^\varepsilon\|_{L^2(Q_0)},
\end{aligned}
$$

here we employed the version of Poincaré inequality for functions from $H^1(Q_0)$,

$$
\|f - \langle f\rangle_y\|_{L^2(Q_0)} \le C\,\|\nabla_y f\|_{L^2(Q_0)}.
$$

After the rescaling we obtain that $\widehat{v}^\varepsilon$ is relatively small compared to $\nabla\widetilde{u}^\varepsilon$,

$$
\|\widehat{v}^\varepsilon(x)\|_{L^2(\varepsilon Q_0)} \le \varepsilon\,C\|\nabla\widetilde{u}^\varepsilon(x)\|_{L^2(\varepsilon Q)}, \tag{1.57}
$$

where C in the inequality does not depend on ε or $\xi \in \mathbb{Z}^n$.

Step 2. At this stage we will need several inequalities which follow from the properties of g and $\widetilde{u}^\varepsilon$.

Proposition 1.3.1. *The following estimates are valid for small enough ε with constants independent of ε and the choice of particular εQ:*

$$
\left\|g^2\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q)}\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \le C\left(\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\varepsilon Q_1)} + \|g\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon Q)}\right), \tag{1.58}
$$

$$
\|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\left\|\nabla(g^2\widetilde{u}^\varepsilon)\right\|_{L^2(\varepsilon Q)} \le C\left(\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\varepsilon Q_1)} + \|g\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon Q)}\right), \tag{1.59}
$$

$$
\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\left\|\nabla(g^2\widetilde{u}^\varepsilon)\right\|_{L^2(\varepsilon Q)} \le C\left(\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\varepsilon Q_1)} + \|g\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon Q)}\right). \tag{1.60}
$$

Proof. We remind that

$$
\sup_{\varepsilon Q} g \le e^{\alpha\sqrt{n}\varepsilon}g(x) \le Cg(x), \qquad x \in \varepsilon Q, \tag{1.61}
$$

for small enough ε. We apply (1.27), (1.40) and (1.61) to get (1.58):

$$\left\|g^2\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q)}\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \le C\left\|\left(\sup_{\varepsilon Q} g\right) g\,\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q)}\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_1)} \le$$

$$\le C\,\|g\,\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\left\|\left(\sup_{\varepsilon Q} g\right)\nabla\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q_1)} \le C\,\|g\,\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\|g\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_1)} =$$

$$= C\,\|g\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\|\nabla(g\widetilde{u}^\varepsilon)-(\nabla g)\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_1)} \le$$

$$\le C\left(\|g\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)}+\|g\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon Q)}\right) \le$$

$$\le C\left(\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\varepsilon Q_1)}+\|g\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon Q)}\right).$$

The last inequality in the chain follows from the elementary

$$|ab| \le \frac{1}{2}(a^2+b^2).$$

The proof of (1.59) and (1.60) is analogous:

$$\|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\left\|\nabla(g^2\widetilde{u}^\varepsilon)\right\|_{L^2(\varepsilon Q)} = \|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\left\|\nabla(g^2)\widetilde{u}^\varepsilon+g^2\nabla\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q)} \le$$

$$\le C\left(\left\|g^2\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q)}\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}+\|g\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon Q_1)}\right) \le$$

$$\le C\left(\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\varepsilon Q_1)}+\|g\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon Q)}\right).$$

$$\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\left\|\nabla(g^2\widetilde{u}^\varepsilon)\right\|_{L^2(\varepsilon Q)} = \|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\left\|\nabla(g^2)\widetilde{u}^\varepsilon+g^2\nabla\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q)}$$

$$\le C\left(\left\|g^2\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q)}\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}+\left\|\left(\sup_{\varepsilon Q} g\right)\nabla\widetilde{u}^\varepsilon\right\|^2_{L^2(\varepsilon Q_1)}\right) \le$$

$$\le C\left(\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\varepsilon Q_1)}+\|g\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon Q)}\right).$$

□

Substituting (1.53) into the second term on the left hand side of (1.47)) we obtain

$$\lambda_\varepsilon\int\limits_{\Omega_0^\varepsilon} g^2 v^\varepsilon\widetilde{u}^\varepsilon\,dx = \lambda_\varepsilon\int\limits_{\Omega_0^\varepsilon} g^2\widetilde{v}^\varepsilon\widetilde{u}^\varepsilon\,dx+\lambda_\varepsilon\int\limits_{\Omega_0^\varepsilon} g^2\widehat{v}^\varepsilon\widetilde{u}^\varepsilon\,dx$$

Let us show that $\lambda_\varepsilon\int\limits_{\Omega_0^\varepsilon} g^2\widehat{v}^\varepsilon\widetilde{u}^\varepsilon\,dx$ is relatively small. Indeed, applying inequalities

(1.57) and (1.58) in each cell we obtain

$$\lambda_\varepsilon \int\limits_{\Omega_0^\varepsilon} g^2 \widehat{v}^\varepsilon \widetilde{u}^\varepsilon \, dx \le \lambda_\varepsilon \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \left\| g^2 \widetilde{u}^\varepsilon \right\|_{L^2(\varepsilon Q_0)} \|\widehat{v}^\varepsilon\|_{L^2(\varepsilon Q_0)} \le$$
$$\le \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon\, C \left(\|\nabla (g\widetilde{u}^\varepsilon)\|^2_{L^2(\varepsilon Q_1)} + \|g\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon Q)} \right). \tag{1.62}$$

Considering sets

$$\bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q \qquad \text{and} \qquad \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q_1,$$

one can notice that they are "nearly" equal to

$$\Omega_0^\varepsilon \cup \Omega_1^\varepsilon \qquad \text{and} \qquad \Omega_1^\varepsilon,$$

respectively. Namely,

$$\Omega_0^\varepsilon \cup \Omega_1^\varepsilon = \left(\bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q \right) \cup \Omega_{1,+}^\varepsilon \setminus \Omega_{1,-}^\varepsilon,$$
$$\Omega_1^\varepsilon = \left(\bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q_1 \right) \cup \Omega_{1,+}^\varepsilon \setminus \Omega_{1,-}^\varepsilon, \tag{1.63}$$

where

$$\Omega_{1,-}^\varepsilon = \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q_0 \cap \Omega_2,$$
$$\Omega_{1,+}^\varepsilon = \bigcup_{\varepsilon Q_0 \cap \Omega_2 \neq \emptyset} \varepsilon Q_0 \cap \Omega_1^\varepsilon.$$

We introduce two 'correctors'

$$r^\varepsilon = \|\nabla (g\widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_{1,-}^\varepsilon)} + \|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_{1,-}^\varepsilon)}, \tag{1.64}$$

and

$$r_1^\varepsilon = \|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_{1,+}^\varepsilon \cup \Omega_{1,-}^\varepsilon)}.$$

Then inequality (1.62) transforms into

$$\int\limits_{\Omega_0^\varepsilon} g^2 \widehat{v}^\varepsilon \widetilde{u}^\varepsilon \, dx \le \varepsilon C \left(\|\nabla (g\widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} + \|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)} + r^\varepsilon \right). \tag{1.65}$$

Step 3. Now we consider the term $\lambda_\varepsilon \int\limits_{\Omega_0^\varepsilon} g^2 \widetilde{v}^\varepsilon \widetilde{u}^\varepsilon \, dx$. We substitute (1.56) to obtain

$$\lambda_\varepsilon \int\limits_{\Omega_0^\varepsilon} g^2 \widetilde{v}^\varepsilon \widetilde{u}^\varepsilon \, dx = \lambda_\varepsilon^2 \varepsilon^n \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \int\limits_Q g^2 \widetilde{u}^\varepsilon(\varepsilon y) b_\varepsilon(y) \langle \widetilde{u}^\varepsilon \rangle_y \, dy,$$

where b_ε is considered as a periodic function on $\mathbb{R}^n$, $b_\varepsilon(y+\xi) = b_\varepsilon(y)$, $\xi \in \mathbb{Z}^n$, and $\langle \widetilde{u}^\varepsilon \rangle_y = \langle \widetilde{u}^\varepsilon \rangle_y(y) = \int\limits_{Q: y \in Q} \widetilde{u}^\varepsilon(\varepsilon y') dy'$ is a step function that takes constant values on each cell Q. Notice also that $\beta(\lambda_\varepsilon) - \lambda_\varepsilon = \lambda_\varepsilon^2 \langle b_\varepsilon \rangle_y$. Then, keeping in mind (1.63), we obtain

$$\begin{aligned}
\Lambda_\varepsilon &:= \left| \lambda_\varepsilon \int\limits_{\Omega_0^\varepsilon} g^2 \widetilde{v}^\varepsilon \widetilde{u}^\varepsilon \, dx - (\beta(\lambda_\varepsilon) - \lambda_\varepsilon) \int\limits_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} g^2 (\widetilde{u}^\varepsilon)^2 \, dx \right| \leq \\
&\leq C\varepsilon^n \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \left| \int\limits_Q g^2 \widetilde{u}^\varepsilon(\varepsilon y) b_\varepsilon(y) \langle \widetilde{u}^\varepsilon \rangle_y \, dy - \langle b_\varepsilon \rangle_y \int\limits_Q g^2(\varepsilon y) (\widetilde{u}^\varepsilon(\varepsilon y))^2 \, dy \right| + \\
&+ C\, r_1^\varepsilon \leq C\varepsilon^n \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \left| \langle \widetilde{u}^\varepsilon \rangle_y \int\limits_Q \left(g^2 \widetilde{u}^\varepsilon - \langle g^2 \widetilde{u}^\varepsilon \rangle_y \right) b_\varepsilon \, dy \right| + \\
&\qquad + C\varepsilon^n \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \left| \langle b_\varepsilon \rangle_y \int\limits_Q \left(g^2 \widetilde{u}^\varepsilon - \langle g^2 \widetilde{u}^\varepsilon \rangle_y \right) \widetilde{u}^\varepsilon \, dy \right| + C\, r_1^\varepsilon.
\end{aligned} \tag{1.66}$$

We will separately estimate terms contained in the last expression. The mean value of $\widetilde{u}^\varepsilon$ is bounded by its norm in L^2 by Hölder inequality

$$\langle \widetilde{u}^\varepsilon(\varepsilon y) \rangle_y^2 = \left[\int\limits_Q \widetilde{u}^\varepsilon \, dy \right]^2 \leq \int\limits_Q (\widetilde{u}^\varepsilon)^2 \, dy \int\limits_Q 1 \, dy = \|\widetilde{u}^\varepsilon(\varepsilon y)\|_{L^2(Q)}^2. \tag{1.67}$$

Similarly,

$$\langle b_\varepsilon \rangle_y \leq \|b_\varepsilon\|_{L^2(Q_0)} \leq C,$$

where C does not depend on ε due to the uniform boundedness of the resolvent

$(T-\lambda)^{-1}$ in the neighbourhood of λ_0. Via the Poincaré inequality we derive

$$\left|\int\limits_Q \left(g^2\widetilde{u}^\varepsilon - \langle g^2\widetilde{u}^\varepsilon\rangle_y\right)\widetilde{u}^\varepsilon\,dy\right| \le \left\|g^2\widetilde{u}^\varepsilon - \langle g^2\widetilde{u}^\varepsilon\rangle_y\right\|_{L^2(Q)}\|\widetilde{u}^\varepsilon\|_{L^2(Q)} \le$$

$$\le C\left\|\nabla_y\left(g^2\widetilde{u}^\varepsilon\right)\right\|_{L^2(Q)}\|\widetilde{u}^\varepsilon\|_{L^2(Q)},$$

and, similarly,

$$\left|\int\limits_Q \left(g^2\widetilde{u}^\varepsilon - \langle g^2\widetilde{u}^\varepsilon\rangle_y\right) b_\varepsilon\,dy\right| \le C\left\|\nabla_y\left(g^2\widetilde{u}^\varepsilon\right)\right\|_{L^2(Q)}, \tag{1.68}$$

with constants independent of ε and ξ (see (1.1)). Applying inequalities (1.67)–(1.68) and then (1.59) to (1.66) we arrive at

$$\begin{aligned}\Lambda_\varepsilon &\le \varepsilon^n C\sum_{\varepsilon Q_0\subset\Omega_0^\varepsilon}\|\widetilde{u}^\varepsilon(\varepsilon y)\|_{L^2(Q)}\left\|\nabla_y(g^2(\varepsilon y)\widetilde{u}^\varepsilon(\varepsilon y))\right\|_{L^2(Q)} + C\,r_1^\varepsilon \le\\ &\le \varepsilon\,C\sum_{\varepsilon Q_0\subset\Omega_0^\varepsilon}\|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\left\|\nabla(g^2\widetilde{u}^\varepsilon)\right\|_{L^2(\varepsilon Q)} + C\,r_1^\varepsilon \le\\ &\le \varepsilon\,C\sum_{\varepsilon Q_0\subset\Omega_0^\varepsilon}\left(\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\varepsilon Q_1)} + \|g\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon Q)}\right) + C\,r_1^\varepsilon \le\\ &\le \varepsilon\,C\left(\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} + \|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon\cup\Omega_1^\varepsilon)} + r^\varepsilon\right) + C\,r_1^\varepsilon,\end{aligned} \tag{1.69}$$

where C is ε-independent. At the last step we used formulas (1.63) and (1.64). Since the correctors r^ε, r_1^ε are uniformly bounded, inequalities (1.65) and (1.69) together imply the validity of (1.47).

Step 4. Finally, it is not difficult to obtain similarly (1.48) via (1.36), (1.59) and (1.60):

$$\left|\varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla (g^2 \widetilde{u}^\varepsilon)\, dx\right| \le \varepsilon^2 C \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \|\nabla u^\varepsilon\|_{L^2(\varepsilon Q_0)} \left\|\nabla (g^2 \widetilde{u}^\varepsilon)\right\|_{L^2(\varepsilon Q_0)} \le$$

$$\le \varepsilon C \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \big(\|\varepsilon \nabla v^\varepsilon\|_{L^2(\varepsilon Q_0)} + \varepsilon \|\nabla \widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)}\big) \left\|\nabla (g^2 \widetilde{u}^\varepsilon)\right\|_{L^2(\varepsilon Q_0)} \le$$

$$\le \varepsilon C \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \Big(\|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \left\|\nabla (g^2 \widetilde{u}^\varepsilon)\right\|_{L^2(\varepsilon Q_0)} + \varepsilon \|\nabla \widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \left\|\nabla (g^2 \widetilde{u}^\varepsilon)\right\|_{L^2(\varepsilon Q_0)}\Big) \le$$

$$\le \varepsilon C \left(\|\nabla (g \widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} + \|g \widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)} + r^\varepsilon\right) \le$$

$$\le \varepsilon C \left(\|\nabla (g \widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} + \|g \widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)} + C\right)$$

for small enough ε.

Notice that all the estimates obtained in this section are independent of R.

□

Chapter 2

Two-scale convergence of eigenfunctions and convergence of spectra

In this chapter we study the convergence properties of the localised eigenfunctions of the operator A_ε and convergence of its spectrum. We list the definitions and some properties of the two-scale convergence, see [2, 38, 48, 49], in the first section. We also formulate several auxillary statements (analogous to those in [49]) which are necessary for obtaining the two-scale convergence of the eigenfunctions of A_ε and for the derivation of the limit equation. In Section 2.2 we prove, relying on the uniform exponential decay, the main results on the two-scale convergence of the eigenfunctions and the subsequent convergence of the point spectrum of A_ε. In Section 2.3 we provide a proof of stability of the essential spectrum of the two-scale homogenised operator with respect to the compact perturbation of its coefficients, thereby establishing the Hausdorff convergence of the spectra of A_ε to the spectrum of the homogenised operator A_0.

2.1 Some properties of two-scale convergence

Let Ω be an arbitrary region in $\mathbb{R}^n$, in particular $\Omega = \mathbb{R}^n$. Denote by $\square$ the unit cube $[0,1)^n$. We consider all functions of the form $u(x,y)$ to be 1-periodic in y in each coordinate.

Definition 2.1.1. *We say that a bounded in $L^2(\Omega)$ sequence v_ε is weakly two-*

scale convergent to a function $v \in L^2(\Omega \times \square)$ *,* $v_\varepsilon(x) \stackrel{2}{\rightharpoonup} v(x,y)$ *, if*

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega} v_\varepsilon(x)\varphi(x)b\left(\frac{x}{\varepsilon}\right)\, dx = \int\limits_{\Omega}\int v(x,y)\varphi(x)b(y)\, dydx$$

for all $\varphi \in C_0^\infty(\Omega)$ *and all* $b \in C^\infty_{\rm per}(\square)$ *(where* $C^\infty_{\rm per}(\square)$ *is the set of 1-periodic functions from* $C^\infty(\mathbb{R}^n)$ *).*

Definition 2.1.2. *We say that a bounded in* $L^2(\Omega)$ *sequence* u_ε *is strongly two-scale convergent to a function* $u \in L^2(\Omega \times \square)$ *,* $u_\varepsilon(x) \stackrel{2}{\to} u(x,y)$ *, if*

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega} u_\varepsilon(x)v_\varepsilon(x)\, dx = \int\limits_{\Omega}\int u(x,y)v(x,y)\, dy\, dx$$

for all $v_\varepsilon(x) \stackrel{2}{\rightharpoonup} v(x,y)$ *.*

Proposition 2.1.3. *(Properties of the two-scale convergence.)*
(i) If $u_\varepsilon(x) \stackrel{2}{\rightharpoonup} u(x,y)$ *and* $a \in L^\infty_{\rm per}(\square)$ *then*

$$a(x/\varepsilon)u_\varepsilon(x) \stackrel{2}{\rightharpoonup} a(y)u(x,y).$$

(ii) $v_\varepsilon(x) \stackrel{2}{\to} v(x,y)$ *if and only if* $v_\varepsilon(x) \stackrel{2}{\rightharpoonup} v(x,y)$ *and*

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega} v_\varepsilon^2\, dx = \int\limits_{\Omega}\int v^2\, dy\, dx.$$

(iii) If $f_\varepsilon(x) \to f(x)$ *in* $L^2(\Omega)$ *, then* $f_\varepsilon(x) \stackrel{2}{\to} f(x)$ *.*
(iv) A sequence u_ε *bounded in* $L^2(\Omega)$ *is compact in the sense of weak two-scale convergence.*

Proposition 2.1.4. *(The mean value property of periodic functions.) Let* $\Phi(y) \in L^1_{\rm per}(\square)$ *. Then for each* $\phi(x) \in C_0^\infty(\mathbb{R}^n)$ *we have*

$$\lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^n} \phi(x)\Phi(x/\varepsilon)dx = \langle \Phi \rangle_y \int\limits_{\mathbb{R}^n} \phi(x)dx.$$

The potential vector space $V_{\rm pot}$ is defined as a closure of the set $\{\nabla\varphi : \varphi \in C^\infty_{\rm per}(\square)\}$ in $L^2(\square)^n$. We say that a vector $b \in L^2(\square)^n$ is solenoidal ($b \in V_{\rm sol}$)

if it is orthogonal to all potential vectors. Thus,

$$L^2(\square)^n = V_{\text{pot}} \oplus V_{\text{sol}},$$

and

$$L^2(\Omega \times \square)^n = L^2(\Omega, V_{\text{pot}}) \oplus L^2(\Omega, V_{\text{sol}}).$$

Lemma 2.1.5. *Let u_ε and $\varepsilon \nabla u_\varepsilon$ be bounded in $L^2(\mathbb{R}^n)$. Then (up to a subsequence)*

$$u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x,y) \in L^2(\mathbb{R}^n, H^1_{\text{per}}),$$

$$\varepsilon \nabla u_\varepsilon(x) \overset{2}{\rightharpoonup} \nabla_y u(x,y),$$

where $H^1_{\text{per}} = H^1_{\text{per}}(\square)$ is the Sobolev space of periodic functions.

Lemma 2.1.6. *Let $u_\varepsilon \in H^1(\mathbb{R}^n)$,*

$$u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x) \in H^1(\mathbb{R}^n), \tag{2.1}$$

and ∇u_ε is bounded in $L^2(\mathbb{R}^n)$. Then, up to a subsequence,

$$\nabla u_\varepsilon(x) \overset{2}{\rightharpoonup} \nabla u(x) + v(x,y), \text{ where } v \in L^2(\mathbb{R}^n, V_{\text{pot}}). \tag{2.2}$$

Lemma 2.1.7. *Let (2.1) and (2.2) be valid. Let also*

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega_1^\varepsilon} a_1 \nabla u_\varepsilon(x) \cdot \nabla_y w(\varepsilon^{-1}x) \varphi(x)\, dx = 0 \tag{2.3}$$

for any $\varphi \in C_0^\infty(\Omega_1)$ and $w \in C^\infty_{\text{per}}(\square)$. Then the following weak convergence of the flows takes place:

$$a_1 \Theta_{Q_1}(\varepsilon^{-1}x) \nabla u_\varepsilon(x) \rightharpoonup A^{\text{hom}} \nabla u(x) \text{ in } \Omega_1,$$

where homogenised matrix A^{hom} is defined by (1.10).

The proofs of the listed statements repeat the proofs of the corresponding assertions in [48] with no or only small alterations, and are not given here. The following is an important definition of the strong two-scale resolvent convergence of operators.

Definition 2.1.8. *Let A_ε, $\varepsilon > 0$, and A_0 be non-negative self-adjoint operators in $L^2(\mathbb{R}^n)$ and $\mathcal{H}_0 \subset L^2(\mathbb{R}^n \times Q)$, see (1.15), respectively. We say that $A_\varepsilon \overset{2}{\to}$*

A_0 in the sense of the strong two-scale resolvent convergence if for any $\lambda > 0$ $(A_\varepsilon + \lambda I)^{-1} f_\varepsilon \xrightarrow{2} (A_0 + \lambda I)^{-1} f_0$ as long as $f_\varepsilon \xrightarrow{2} f_0$.

2.2 Strong two-scale convergence of the eigenfunctions and multiplicity of the eigenvalues of A_ε

In this section we will show that the normalised eigenfunctions u_ε are compact in the sense of strong two-scale convergence. Namely, provided $\lambda_\varepsilon \to \lambda_0$, a sequence of normalised eigenfunctions u_ε of the operator A_ε strongly two-scale converges, up to a subsequence, to a function $u^0(x, y)$. Using the properties of two-scale convergence we then pass to a limit in integral identity (1.6) with a specially chosen test function. As a result we obtain in the limit integral identity (1.18) which implies that λ_0 and $u^0(x, y)$ are an eigenvalue and a corresponding eigenfunction of A_0. This, together with the results of [29], allows us to establish an 'asymptotic one-to-one correspondence' between isolated eigenvalues and corresponding eigenfunctions of the operators A_ε and A_0.

Theorem 2.2.1. *Under the assumptions of Theorem 1.2.2 λ_0 is an eigenvalue of the operator A_0. Moreover, there exists a subsequence ε such that eigenfunctions u^ε of the operator A_ε strongly two-scale converge to an eigenfunction $u^0(x, y)$ of A_0 corresponding to the eigenvalue λ_0.*

Proof. **Step 1.** In order to establish strong two-scale convergence of the eigenfunctions $u^\varepsilon = \widetilde{u}^\varepsilon + v^\varepsilon$ we prove it for each of its components separately. The gradient of $\widetilde{u}^\varepsilon$ is bounded in L^2-norm uniformly in ε. Naively speaking, this means that $\widetilde{u}^\varepsilon$ itself is a function of slow variation and its two-scale limit should not depend on the fast variable y. Then one can expect that the sequence $\widetilde{u}^\varepsilon$ is compact in a usual L^2-norm sense.

From (1.50) it follows that

$$\|e^{\alpha R}\widetilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n\setminus B_R)} \leq \|e^{\alpha|x|}\widetilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n\setminus B_R)} \leq C.$$

Then

$$\|\widetilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n\setminus B_R)} \leq Ce^{-\alpha R} \tag{2.4}$$

with C independent of ε and R. From this one can easily conclude that $\widetilde{u}^\varepsilon$ is weakly compact in $H^1(\mathbb{R}^n)$ and strongly compact in $L^2(\mathbb{R}^n)$. Indeed, since $\widetilde{u}^\varepsilon$ is bounded in $H^1(\mathbb{R}^n)$ uniformly in ε (see (1.28)),

$$\widetilde{u}^\varepsilon(x) \rightharpoonup u_0(x) \text{ in } H^1(\mathbb{R}^n), \tag{2.5}$$

up to a subsequence due to the weak compactness of a bounded set in $H^1(\mathbb{R}^n)$. It is well known that $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$ when Ω is bounded. Hence, for any fixed R function $\widetilde{u}^\varepsilon$ converges to u_0 weakly in $H^1(B_R)$ and strongly in $L^2(B_R)$ up to a subsequence. Considering a sequence of balls B_R, $R \in \mathbb{N}$, one can use the method of extracting a diagonal subsequence to obtain a sequence converging in any ball B_R,

$$\widetilde{u}^\varepsilon \to u_0 \text{ in } L^2(B_R) \tag{2.6}$$

for any $R > 0$.

For any $\delta > 0$ we can choose R such that $\|u_0\|^2_{L^2(\mathbb{R}^n \setminus B_R)} < \delta/3$ and $\|\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^n \setminus B_R)} < \delta/3$ for all sufficiently small ε (the latter follows from (2.4)). From (2.6) it follows that $\|u_0 - \widetilde{u}^\varepsilon\|^2_{L^2(B_R)} < \delta/3$ for sufficiently small ε. So we conclude that

$$\|u_0 - \widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^n)} \leq \|u_0 - \widetilde{u}^\varepsilon\|^2_{L^2(B_R)} + \|u_0\|^2_{L^2(\mathbb{R}^n \setminus B_R)} + \|\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^n \setminus B_R)} < \delta$$

for small enough ε. Hence, up to a subsequence, we have proved convergence

$$\widetilde{u}^\varepsilon \to u_0 \text{ in } L^2(\mathbb{R}^n). \tag{2.7}$$

Then from the properties of two-scale convergence (Proposition 2.1.3 *(iii)*) we conclude that

$$\widetilde{u}^\varepsilon(x) \xrightarrow{2} u_0(x). \tag{2.8}$$

Step 2. Now let us consider v^ε. Formulas (1.53) and (1.56) show that v^ε is of a two-scale nature. One can expect that its two-scale limit depends both on x and y. Since the coefficient $a(x, \varepsilon)$ on $\widetilde{\Omega}_0^\varepsilon$ is defined very loosely we consider the behaviour of v^ε on $\widetilde{\Omega}_0^\varepsilon$ separately. We denote by v_1^ε and v_2^ε the restrictions $v_\varepsilon|_{\Omega_0^\varepsilon}$ and $v_\varepsilon|_{\widetilde{\Omega}_0^\varepsilon}$ respectively, extending them by zero to the rest of $\mathbb{R}^n$.

Lemma 2.2.2. *The following convergence properties are valid for v_1^ε (up to a* *subsequence):*

$$v_1^\varepsilon(x) \xrightarrow{2} v(x,y) \in L^2(\Omega_1, H_0^1(Q_0)),$$
$$\varepsilon\nabla v_1^\varepsilon(x) \xrightarrow{2} \nabla_y v(x,y),$$

where $v(x,y)$ *is a solution to the following problem:*

$$-a_0\Delta_y v - \lambda_0 v = \lambda_0 u_0, \quad y \in Q_0. \tag{2.9}$$

Here u_0 *is a function from (2.8).*

Proof. The function $v_1^\varepsilon \in H^1(\Omega_0^\varepsilon)$ satisfies the following differential equation:

$$-\varepsilon^2 a_0 \Delta v_1^\varepsilon - \lambda_\varepsilon v_1^\varepsilon = \lambda_\varepsilon \widetilde{u}^\varepsilon, \ \ x \in \Omega_0^\varepsilon. \tag{2.10}$$

Let us rewrite it in the form

$$-\varepsilon^2 a_0 \Delta v_1^\varepsilon - \lambda_\varepsilon v_1^\varepsilon = \lambda_\varepsilon \widetilde{u}^\varepsilon \left(\Theta_{Q_0}(x/\varepsilon) - \Theta_{\widetilde{\Omega}_0^\varepsilon}(x)\right), \ \ x \in \Omega_1. \tag{2.11}$$

We understand the term $\Theta_{Q_0}(y)$ as a characteristic function of Q_0 in Q extended by periodicity on $\mathbb{R}^n$. Since $\widetilde{u}^\varepsilon$ is bounded in $L^2(\mathbb{R}^n)$ and the Lebesgue measure of $\widetilde{\Omega}_0^\varepsilon$ tends to zero, we have

$$\|\widetilde{u}^\varepsilon\, \Theta_{\widetilde{\Omega}_0^\varepsilon}\|^2_{L^2(\mathbb{R}^n)} \le \|\widetilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n)} \|\Theta_{\widetilde{\Omega}_0^\varepsilon}\|_{L^2(\mathbb{R}^n)} \to 0.$$

By Proposition 2.1.3 *(i)* for any $f_\varepsilon(x) \xrightarrow{2} f(x,y)$ it is true that $f_\varepsilon(x)\Theta_{Q_0}(x/\varepsilon) \xrightarrow{2} f(x,y)\Theta_{Q_0}(y)$. Since $\widetilde{u}^\varepsilon$ strongly two-scale converges to u_0, by the above and the definition of the strong two-scale convergence we have

$$\int\limits_{\mathbb{R}^n} \widetilde{u}^\varepsilon(x) f_\varepsilon(x)\Theta_{Q_0}(x/\varepsilon)dx \to \int\limits_{\mathbb{R}^n}\int\limits_{Q} u_0(x) f(x,y)\Theta_{Q_0}(y)dx$$

with arbitrary $f_\varepsilon(x) \xrightarrow{2} f(x,y)$. But this implies $\widetilde{u}^\varepsilon(x)\Theta_{Q_0}(x/\varepsilon) \xrightarrow{2} u_0(x)\Theta_{Q_0}(y)$. Hence we conclude that

$$\lambda_\varepsilon \widetilde{u}^\varepsilon \left(\Theta_{Q_0}(x/\varepsilon) - \Theta_{\widetilde{\Omega}_0^\varepsilon}(x)\right) \xrightarrow{2} \lambda_0 \Theta_{Q_0}(y) u_0(x) \in L^2(\Omega_1 \times \square). \tag{2.12}$$

Following [48] we consider the more general problem

$$z_\varepsilon \in H^1(\Omega_0^\varepsilon), \quad -\varepsilon^2 a_0 \Delta z_\varepsilon - \lambda_\varepsilon z_\varepsilon = f_\varepsilon, \quad f_\varepsilon \in L^2(\Omega_0^\varepsilon). \tag{2.13}$$

(It is implicit that $f_\varepsilon = z_\varepsilon = 0$ in $\mathbb{R}^n \backslash \Omega_0^\varepsilon$.)

Proposition 2.2.3. *Let*

$$f^\varepsilon(x) \xrightharpoonup{2} f(x,y).$$

Then

$$z^\varepsilon(x) \xrightharpoonup{2} z(x,y) \in L^2(\Omega_1, H_0^1(Q_0)),$$
$$\varepsilon \nabla z^\varepsilon(x) \xrightharpoonup{2} \nabla_y z(x,y),$$

where function $z(x,y)$ *solves the following equation:*

$$-a_0 \Delta_y z - \lambda_0 z = f, \quad y \in Q_0. \tag{2.14}$$

Proof. One can easily derive an estimate for z^ε analogous to (1.36), applying to (2.13) a reasoning similar to those for the solution of equation (1.30):

$$a_0 \varepsilon^2 \|\nabla z^\varepsilon(x)\|^2_{L^2(\Omega_0^\varepsilon)} + \|z^\varepsilon(x)\|^2_{L^2(\Omega_0^\varepsilon)} \leq C \|f^\varepsilon(x)\|^2_{L^2(\Omega_0^\varepsilon)},$$

with C independent of ε. Since f^ε weakly two-scale converges, it is bounded. Then z^ε and $\varepsilon \nabla z^\varepsilon$ are also bounded, and we can apply Lemma 2.1.5:

$$z^\varepsilon(x) \xrightharpoonup{2} z(x,y) \in L^2(\Omega_1, H^1_{\text{per}}),$$
$$\varepsilon \nabla z^\varepsilon(x) \xrightharpoonup{2} \nabla_y z(x,y).$$

Equation (2.14) follows by a straightforward passing to the limit in the integral identity corresponding to (2.13) with appropriately chosen test functions. The full proof can be found in [48] and applies to the present situation with no alteration.

Remark 2.2.4. Validity of $z \in L^2(\Omega_1, H_0^1(Q_0))$, i.e. that z vanishes on the boundary of Q_0, follows from $z \in L^2(\Omega_1, H^1_{\text{per}})$ and the obvious convergence property

$$0 \equiv z^\varepsilon(x) \Theta_{Q_1}(x/\varepsilon) \xrightharpoonup{2} z(x,y) \Theta_{Q_1}(y).$$

□

The above proposition together with (2.12) establishes a "weak" form of the statement of the lemma, i.e. weak two-scale convergence of v_1^ε to the solution of (2.9). We now prove that the convergence is actually strong, following again [48]. Multiply (2.10) and (2.13) by z^ε and v_1^ε respectively and integrate by parts. The left hand sides of the resulting equalities are identical. So, equating the right hand sides, we obtain the following identity

$$\int_{\Omega_1} f^\varepsilon v_1^\varepsilon \, dx = \lambda_\varepsilon \int_{\Omega_1} \widetilde{u}^\varepsilon z^\varepsilon \, dx. \tag{2.15}$$

Since $\widetilde{u}^\varepsilon$ strongly two-scale converges, then by the definition we have

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon \int_{\Omega_1} \widetilde{u}^\varepsilon z^\varepsilon \, dx = \lambda_0 \int_{\Omega_1}\int_{Q_0} u_0(x) z(x,y)\, dy\, dx.$$

Multiplying (2.9) and (2.14) by z and v respectively and integrating by parts it is easy to see that

$$\lambda_0 \int_{\Omega_1}\int_{Q_0} u_0(x) z(x,y)\, dydx = \int_{\Omega_1}\int_{Q_0} f(x,y) v(x,y)\, dy\, dx. \tag{2.16}$$

Since the right hand side of (2.15) converges to the left hand side of (2.16) we conclude that:

$$\lim_{\varepsilon\to 0} \int_{\Omega_1} f^\varepsilon v_1^\varepsilon \, dx = \int_{\Omega_1}\int_{Q_0} f(x,y) v(x,y)\, dy\, dx$$

for any weakly two-scale convergent sequence f^ε. Hence, by the definition of the strong two-scale convergence,

$$v_1^\varepsilon(x) \xrightarrow{2} v(x,y).$$

□

Lemma 2.2.5. *The sequence of functions* v_2^ε *converges to zero in the sense of strong two-scale convergence:*

$$v_2^\varepsilon \xrightarrow{2} 0 \text{ as } \varepsilon \to 0.$$

Proof. Straightforward from (1.46) and Proposition 2.1.3 *(iii).* □

Combining (2.8) with Lemmas 2.2.2 and 2.2.5, we arrive at

$$u^\varepsilon(x) \xrightarrow{2} u^0(x,y) = u_0(x) + v(x,y), \tag{2.17}$$

where $u_0 \in H^1(\mathbb{R}^n)$, $v \in L^2(\Omega_1, H_0^1(Q_0))$.

Step 3. Now it remains to show that λ_0 and $u^0(x,y)$ are an eigenvalue and the corresponding eigenfunction of the limit operator A_0, i.e. that $u^0(x,y)$ satisfies (1.19). In order to do that we need to choose an appropriate test-function ψ^ε and pass to the limit in the integral identity

$$\varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx + a_1 \int\limits_{\Omega_1^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx + \int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx + \\ + a_2 \int\limits_{\Omega_2^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx = \lambda_\varepsilon \int\limits_{\mathbb{R}^n} u^\varepsilon \psi^\varepsilon \, dx \tag{2.18}$$

corresponding to the original eigenvalue problem (1.2)–(1.3). Let us take

$$\psi^\varepsilon(x) = \psi_0(x) + \varphi(x) b(\varepsilon^{-1}x), \\ \psi_0 \in C_0^\infty(\mathbb{R}^n),\ \varphi \in C_0^\infty(\Omega_1),\ b(y) \in C_0^\infty(Q_0), \tag{2.19}$$

and consider each term of (2.18) separately. Let us expand the first term:

$$\varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla \psi^\varepsilon \, dx = \varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla \widetilde{u}^\varepsilon \nabla \psi^\varepsilon \, dx + \\ + \varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla v^\varepsilon \left(\nabla \psi_0 + b(\varepsilon^{-1}x) \nabla \varphi \right) dx + a_0 \int\limits_{\Omega_0^\varepsilon} \varepsilon \nabla v^\varepsilon \varphi \nabla_y b(\varepsilon^{-1}x) \, dx.$$

Since $\nabla \widetilde{u}^\varepsilon$ is bounded in L^2-norm and $|\nabla \psi^\varepsilon| \le C\varepsilon^{-1}$, the first term on the right hand side tends to zero. Consider the second term. By (1.36) we have $\varepsilon \|\nabla v^\varepsilon\|_{L^2(\Omega_0^\varepsilon)} \le C \|\widetilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon)} \le C$; then from the boundedness of $\nabla \psi_0 + b \nabla \varphi$ (in L^∞-norm) we conclude that the second term also converges to zero. By Lemma 2.2.2 $\varepsilon \nabla v^\varepsilon$ weakly two-scale converges to $\nabla_y v(x,y)$, hence, by the definition of the weak two-scale convergence, we obtain

$$\lim_{\varepsilon \to 0} \varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla \psi^\varepsilon \, dx = a_0 \int\limits_{\Omega_1} \int\limits_{Q_0} \nabla_y v(x,y) \varphi(x) \nabla_y b(y) \, dy \, dx. \tag{2.20}$$

The third term on the left hand side of (2.18) converges to zero due to the smallness of the domain of integration. Indeed, since for small enough ε the test function ψ^ε is equal to ψ_0 on $\widetilde{\Omega}_0^\varepsilon$, $\|\widetilde{a}_0^{1/2} \nabla u^\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \le C$ uniformly in ε (cf.

(1.25)), and $\left|\widetilde{\Omega}_0^\varepsilon\right| \to 0$ as $\varepsilon \to 0$, we derive for small enough ε

$$
\begin{aligned}
&\left| \int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 \nabla u^\varepsilon \nabla \psi^\varepsilon \, dx \right| = \left| \int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 \nabla u^\varepsilon \nabla \psi_0 \, dx \right| \le \\
&\le C \int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 |\nabla u^\varepsilon| \, dx \le C \left|\widetilde{\Omega}_0^\varepsilon\right|^{1/2} \widetilde{a}_0^{1/2} \, \|\widetilde{a}_0^{1/2} \, \nabla u^\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \to 0.
\end{aligned}
\tag{2.21}
$$

The eigenfunction u^ε coincides with $\widetilde{u}^\varepsilon$ on Ω_2^ε. Then, via (2.5) we have convergence of the last term on the left hand side of (2.18):

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} a_2 \int\limits_{\Omega_2^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx = \\
&= \lim_{\varepsilon \to 0} \left[a_2 \int\limits_{\Omega_2} \nabla \widetilde{u}^\varepsilon \cdot \nabla \psi_0 \, dx - a_2 \int\limits_{\widetilde{\Omega}_0^\varepsilon \cap \Omega_2} \nabla \widetilde{u}^\varepsilon \cdot \nabla \psi_0 \, dx \right] = \\
&= a_2 \int\limits_{\Omega_2} \nabla u_0 \cdot \nabla \psi_0 \, dx,
\end{aligned}
\tag{2.22}
$$

as

$$
\left| a_2 \int\limits_{\widetilde{\Omega}_0^\varepsilon \cap \Omega_2} \nabla \widetilde{u}^\varepsilon \cdot \nabla \psi_0 \, dx \right| \le C \int\limits_{\widetilde{\Omega}_0^\varepsilon \cap \Omega_2} |\nabla \widetilde{u}^\varepsilon| dx \le \|\nabla \widetilde{u}^\varepsilon\|_{\widetilde{\Omega}_0^\varepsilon \cap \Omega_2} \left| \widetilde{\Omega}_0^\varepsilon \cap \Omega_2 \right|^{1/2} \to 0.
$$

Now we will prove that the second term on the left hand side of (2.18) converges to the second term on the right hand side of (1.16) with $w_0 = \psi_0$. We need to show that u^ε satisfies the conditions of Lemma 2.1.7. Let us show that convergence property (2.3) holds for u^ε. To this end we substitute into (2.18) a test function of the form $\psi^\varepsilon = \varepsilon\, w(\varepsilon^{-1}x)\varphi(x)$, $\varphi \in C_0^\infty(\Omega_1)$, $w \in C_{\mathrm{per}}^\infty(\square)$, cf. [48]. Since $\nabla(\varepsilon\, w(\varepsilon^{-1}x)\varphi(x)) = O(1)$ and $\|\varepsilon \nabla u^\varepsilon\|_{L^2(\Omega_0^\varepsilon)}$ is bounded by (1.25) we have

$$
\left| \varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla u^\varepsilon \cdot \nabla(\varepsilon\, w\varphi) \, dx \right| \le \varepsilon a_0 \|\varepsilon \nabla u^\varepsilon\|_{L^2(\Omega_0^\varepsilon)} \|\nabla(\varepsilon\, w\varphi)\|_{L^2(\Omega_0^\varepsilon)} \to 0.
$$

$$\int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 \nabla u^\varepsilon \cdot \nabla(\varepsilon\, w\varphi)\, dx = 0$$

for small enough ε because $\varphi \in C_0^\infty(\Omega_1)$ equals zero in $\widetilde{\Omega}_0^\varepsilon$ for small ε and, obviously,

$$a_2 \int\limits_{\Omega_2^\varepsilon} \nabla u^\varepsilon \cdot \nabla(\varepsilon\, w\varphi)\, dx = 0.$$

Since $\varepsilon w\varphi = O(\varepsilon)$ as $\varepsilon \to 0$,

$$\lambda_\varepsilon \int\limits_{\mathbb{R}^n} u^\varepsilon \varepsilon w\varphi\, dx \to 0.$$

Thus all the terms in (2.18) with $\psi^\varepsilon = \varepsilon w\varphi$, except possibly

$$a_1 \int\limits_{\Omega_1^\varepsilon} \nabla u^\varepsilon \cdot \nabla(\varepsilon w\varphi)\, dx = a_1 \int\limits_{\Omega_1^\varepsilon} \left[\nabla u^\varepsilon \cdot \varepsilon w \nabla\varphi + \nabla u^\varepsilon(x) \cdot \nabla_y w(\varepsilon^{-1}x)\varphi(x)\right] dx,$$

converge to zero. Then the latter should also converge to zero. Since

$$a_1 \int\limits_{\Omega_1^\varepsilon} \nabla u^\varepsilon \cdot \varepsilon w \nabla\varphi\, dx \to 0,$$

we conclude the validity of (2.3).

The eigenfunction $\widetilde{u}^\varepsilon$ converges in the sense of the strong two-scale convergence and its gradient is bounded in L^2-norm, see (2.8) and (1.28). Then by Lemma 2.1.6

$$\nabla \widetilde{u}^\varepsilon \stackrel{2}{\rightharpoonup} \nabla u_0(x) + \widetilde{v}(x, y),$$

where $\widetilde{v} \in L^2(\mathbb{R}^n, V_{\text{pot}})$. As long as $\widetilde{u}^\varepsilon$ coincides with u^ε on Ω_1^ε, we now can apply Lemma 2.1.7 to obtain

$$\lim_{\varepsilon \to 0} a_1 \int\limits_{\Omega_1^\varepsilon} \nabla u^\varepsilon \cdot \nabla\psi^\varepsilon\, dx = \lim_{\varepsilon \to 0} a_1 \int\limits_{\Omega_1^\varepsilon} \nabla u^\varepsilon \cdot \nabla\psi_0\, dx = \int\limits_{\Omega_1} A^{\text{hom}} \nabla u_0 \cdot \nabla\psi_0\, dx, \quad (2.23)$$

where ψ^ε is as in (2.19).

Thus, passing to the limit as $\varepsilon \to 0$ on the left hand side of (2.18) via (2.20)–

(2.23), and on the right hand side via (2.17), we arrive at

$$a_0 \int_{\Omega_1} \int_{Q_0} \nabla_y v \cdot \varphi \nabla_y b \, dy \, dx + \int_{\Omega_1} A^{\hom} \nabla u_0 \cdot \nabla \psi_0 \, dx + a_2 \int_{\Omega_2} \nabla u_0 \cdot \nabla \psi_0 \, dx =$$

$$= \lambda_0 \int_{\mathbb{R}^n} \int_{Q} (u_0 + v)(\psi_0 + \varphi \, b) \, dy \, dx.$$

Since the space of functions from (2.19) is dense in $\mathcal{V}$ (see (1.17)), the latter is equivalent to (1.18). It follows from (2.17), Proposition 2.1.3 (*ii*) and the normalisation of u^ε that $u^0(x, y) \not\equiv 0$. Thus we have proved that λ_0 and $u^0(x, y)$ are respectively an eigenvalue and an eigenfunction of the operator A_0, completing the proof of the theorem. □

Remark 2.2.6. Let (a, b) be a gap in the spectrum of $\widehat{A}_0$ and I be an interval lying strictly inside the gap. As we mentioned earlier, due to results of [48, 49] $\sigma(\widehat{A}_\varepsilon) \to \sigma(\widehat{A}_0)$ in the sense of Hausdorff. This implies that for small enough ε the interval I belongs to the spectral gap of $\widehat{A}_\varepsilon$. Then we can implement Theorem 2 of [23] which claims that for large enough l small enough a_2, namely such that $l^2/a_2 > C$, the operator A_ε with $\Omega_2 = l\Omega$ has at least one localised eigenvalue λ_ε in I. The constant C depends only on the size and position of I and geometric properties of Ω. Hence one can extract a converging subsequence λ_ε satisfying conditions of Theorem 2.2.1. Then from the latter follows the existence of eigenvalues of A_0 in the gaps of its essential spectrum, provided Ω_2 is large enough and a_2 is small enough.

It is not hard to show that there holds the strong two-scale resolvent convergence $A_\varepsilon \xrightarrow{2} A_0$, see Definition 2.1.8. Consider the resolvent equation

$$A_\varepsilon w^\varepsilon + \lambda w^\varepsilon = f^\varepsilon, \tag{2.24}$$

where $-\lambda \notin \sigma(A_0)$. It is well posed for small enough ε since for such ε $\lambda \notin \sigma(A_\varepsilon)$. Suppose also that

$$f^\varepsilon(x) \xrightharpoonup{2} f^0(x, y).$$

Multiplying this equation by w^ε and integrating by parts we obtain

$$\|a_0^{1/2}(x, \varepsilon) \nabla w^\varepsilon\|^2_{L^2(\mathbb{R}^n)} + \lambda \|w^\varepsilon\|^2_{L^2(\mathbb{R}^n)} \leq \|w^\varepsilon\|_{L^2(\mathbb{R}^n)} \|f^\varepsilon\|_{L^2(\mathbb{R}^n)}.$$

The weakly two-scale converging sequence f^ε is bounded in L^2. If λ is positive, then we obviously get

$$\|w^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C, \tag{2.25}$$

and

$$\|a_0^{1/2}(x,\varepsilon)\nabla w^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C, \tag{2.26}$$

uniformly in ε. If λ is negative, then $\|w^\varepsilon\|_{L^2(\mathbb{R}^n)}$ could be bounded or unbounded. The case when $\|w^\varepsilon\|_{L^2(\mathbb{R}^n)}$ is unbounded we will consider later. Otherwise we also have (2.25), (2.26).

As when we considered the eigenvalue problem, we can represent the solution of (2.24) as $w^\varepsilon = \widetilde{w}^\varepsilon + z^\varepsilon$, where $\widetilde{w}^\varepsilon$ is a harmonic extension of $w^\varepsilon|_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon}$ to the whole $\mathbb{R}^n$. Obviously $\|\widetilde{w}^\varepsilon\|_{L^2(\mathbb{R}^n)}$ and $\|\nabla\widetilde{w}^\varepsilon\|_{L^2(\mathbb{R}^n)}$ are bounded by $\|w^\varepsilon\|_{L^2(\Omega_1^\varepsilon \cup \Omega_2^\varepsilon)}$ and $\|\nabla w^\varepsilon\|_{L^2(\Omega_1^\varepsilon \cup \Omega_2^\varepsilon)}$. Then applying Proposition 2.1.3 *(iv)*, Lemmas 2.1.5 and 2.1.6 we conclude that

$$w^\varepsilon = \widetilde{w}^\varepsilon + z^\varepsilon \stackrel{2}{\rightharpoonup} w^0(x,y) = w_0(x) + z(x,y) \in H^1(\mathbb{R}^n) + L^2(\Omega_1, H^1_{\text{per}}),$$

$$\varepsilon\nabla z^\varepsilon(x) \stackrel{2}{\rightharpoonup} \nabla_y z(x,y),$$

$$\nabla\widetilde{w}^\varepsilon(x) \stackrel{2}{\rightharpoonup} \nabla w_0(x) + v(x,y), \text{ where } v \in L^2(\mathbb{R}^n, V_{\text{pot}}).$$

As before, we can show that equality (2.3) holds with $u_\varepsilon = \widetilde{w}^\varepsilon$, and then, applying Lemma 2.1.7 and the above convergence properties, pass to a limit in the weak form of (2.24) with appropriately chosen test function to obtain

$$A_0 w^0 + \lambda w^0 = f^0.$$

Now suppose that $f^\varepsilon \stackrel{2}{\to} f^0$. We can carry out the same reasoning as in Lemma 2.1.6 (when we proved the strong two-scale convergence of v_1^ε) to prove that

$$w^\varepsilon \stackrel{2}{\to} w^0.$$

In order to complete the proof of the strong two-scale resolvent convergence we need to consider the case when λ is negative and the sequence $\|w^\varepsilon\|_{L^2(\mathbb{R}^n)}$ is unbounded. Then there is a subsequence w^ε with L^2-norms converging to infinity. We divide equation (2.24) by $\|w^\varepsilon\|_{L^2(\mathbb{R}^n)}$ and rename $\frac{w^\varepsilon}{\|w^\varepsilon\|_{L^2(\mathbb{R}^n)}}$ and $\frac{f^\varepsilon}{\|w^\varepsilon\|_{L^2(\mathbb{R}^n)}}$ again as w^ε and f^ε to simplify the notation. Then we arrive at (2.24) with $\|w^\varepsilon\|_{L^2(\mathbb{R}^n)} = 1$ and $\|f^\varepsilon\|_{L^2(\mathbb{R}^n)} \to 0$. By the properties of two-scale convergence $f^\varepsilon \stackrel{2}{\to} 0$. The by the above $w^\varepsilon \stackrel{2}{\to} w^0$, and $\|w^0\|_{L^2(\mathbb{R}^n,Q)} = \|w^\varepsilon\|_{L^2(\mathbb{R}^n)} = 1$, where w^0 satisfy the equation $A_0 w_0 + \lambda w_0 = 0$. This means that $-\lambda$ has to be an eigenvalue of A_0, which contradicts the initial assumption.

The strong two-scale resolvent convergence implies in particular the strong two-scale convergence of spectral projectors ($P_\varepsilon(\lambda) \xrightarrow{2} P_0(\lambda)$ if λ is not an eigenvalue of A_0), see [41, 48], and has other nice properties, however it does not imply in its own the convergence of the spectra. The latter requires an additional (two-scale) compactness property to hold, which Theorem 2.2.1 provides.

Remark 2.2.7. The function $v(x, y)$ could be represented as a product of $u_0(x)\big|_{\Omega_1}$ and $\lambda_0 b(y)$, where $b(y)$ solves (1.12) with $\lambda = \lambda_0$. Then $v(x, \varepsilon^{-1}x)$ strongly two-scale converges to $v(x, y)$ by the mean value property and the properties of two-scale convergence. Then

$$u^{\mathrm{appr}}(x, \varepsilon) := \begin{cases} u_0(x) + v(x, x/\varepsilon), & x \in \Omega_0^\varepsilon, \\ u_0(x), & x \in \mathbb{R}^n \backslash \Omega_0^\varepsilon, \end{cases} \tag{2.27}$$

also strongly two-scale converges to $u^0(x, y)$. Hence it approximates the eigenfunction $u^\varepsilon(x)$:

$$\|u^{\mathrm{appr}} - u^\varepsilon\|^2_{L_2(\mathbb{R}^n)} = \\ = \int\limits_{L_2(\mathbb{R}^n)} \left(u^{\mathrm{appr}}\right)^2 dx + \int\limits_{L_2(\mathbb{R}^n)} \left(u^\varepsilon\right)^2 dx - 2 \int\limits_{L_2(\mathbb{R}^n)} u^{\mathrm{appr}} u^\varepsilon dx \to 0. \tag{2.28}$$

Using the result of Theorem 2.2.1 we will discuss the multiplicity properties of the eigenvalues λ_ε and λ_0. Let us assume that the multiplicity of the eigenvalue λ_0 of A_0 is m. Suppose that for a subsequence $\varepsilon_k \to 0$ there exist l (accounting for multiplicities) eigenvalues of A_ε, $\lambda_{\varepsilon_k,1} \le \lambda_{\varepsilon_k,2}, \ldots \le \lambda_{\varepsilon_k,l}$, such that $\lambda_{\varepsilon_k,i} \to \lambda_0$, $i = 1, \ldots, l$. Let $u_i^{\varepsilon_k}$ be the corresponding eigenfunctions orthonormalised in $L^2(\mathbb{R}^n)$. It follows from Theorem 2.2.1 that there exists a subsequence k_m such that

$$u_i^{\varepsilon_{k_m}} \xrightarrow{2} u_i^0, \ i = 1, \ldots, l,$$

where u_i^0 are eigenfunctions of A_0 corresponding to λ_0. In particular, due to the strong two-scale convergence, we have convergence of the inner products:

$$(u_i^{\varepsilon_{k_m}}, u_j^{\varepsilon_{k_m}})_{L^2(\mathbb{R}^n)} \to (u_i^0, u_j^0)_{\mathcal{H}_0}. \tag{2.29}$$

However $(u_i^{\varepsilon_{k_m}}, u_j^{\varepsilon_{k_m}})_{L^2(\mathbb{R}^n)} = \delta_{ij}$. Then u_i^0, $i = 1, \ldots, l$ are also orthonormal (in $\mathcal{H}_0$), i.e. there exist at least l linearly independent eigenfunctions of A_0

corresponding to λ_0. Thus, $l \leq m$.

The results presented in [29] remain also valid in our setting of the problem, i.e. when the coefficients of the divergence form operator A_ε are of the form (1.4). By Theorem 4.1 of [29], if λ_0 is an eigenvalue of the limit operator A_0 lying in a gap of its essential spectrum, then for small enough ε, there exist eigenvalues (or at least one eigenvalue) of A_ε such that

$$|\lambda_{\varepsilon,i} - \lambda_0| \leq C\varepsilon^{1/2},\ i = 1, \ldots, l(\varepsilon). \tag{2.30}$$

Moreover, again by [29, Thm 4.1], for any eigenfunction u_i^0 of A_0 corresponding to λ_0 the related u_i^{appr}, see (2.27), can be approximated by a linear combination of the eigenfunctions of A_ε corresponding to $\lambda_{\varepsilon,i},\ i = 1, \ldots, l(\varepsilon)$:

$$\|u_i^{\mathrm{appr}} - \sum_{j=1}^{l(\varepsilon)} c_{ij}(\varepsilon_k) u_j^\varepsilon\|_{L_2(\mathbb{R}^n)} \leq C\varepsilon^{1/2}.$$

Then it is not hard to show that $l(\varepsilon) \geq m$. Assume, for contradiction, that it is not true. Then for some subsequence ε_k we have

$$\|u_i^{\mathrm{appr}} - \sum_{j=1}^{l} c_{ij}(\varepsilon_k) u_j^\varepsilon\|_{L_2(\mathbb{R}^n)} \leq C\varepsilon^{1/2}, \tag{2.31}$$

with $l < m$. Number of columns l of the matrix $(c_{ij}(\varepsilon_k))$ is less than number of its rows m, so the latter are linearly dependent vectors, and there exist coefficients $\alpha_i(\varepsilon_k),\ i = 1, \ldots, m$ not equal to zero simultaneously such that

$$\sum_{i=1}^{m} \alpha_i(\varepsilon_k) c_i(\varepsilon_k) = 0,$$

where $c_i(\varepsilon_k) = (c_{i1}(\varepsilon_k), \ldots, c_{il}(\varepsilon_k))$. Let coefficients $\alpha_i(\varepsilon_k)$ be normalised: $\sum_{i=1}^{m} |\alpha_i(\varepsilon_k)|^2 = 1$. It is obvious that then

$$\sum_{i=1}^{m} \alpha_i(\varepsilon_k) \sum_{j=1}^{l} c_{ij}(\varepsilon_k) u_j^{\varepsilon_k} \equiv 0. \tag{2.32}$$

From (2.31) and (2.32) it follows that

$$\left\|\sum_{i=1}^{m}\alpha_i(\varepsilon_k)u_i^{\rm appr}\right\|_{L_2(\mathbb{R}^n)} = \left\|\sum_{i=1}^{m}\alpha_i(\varepsilon_k)(u_i^{\rm appr}-\sum_{j=1}^{l}c_{ij}(\varepsilon_k)u_j^{\varepsilon_k})\right\|_{L_2(\mathbb{R}^n)} \to 0.$$

But on the other hand, by (2.29),

$$\left\|\sum_{i=1}^{m}\alpha_i(\varepsilon_k)u_i^{\rm appr}\right\|^2_{L_2(\mathbb{R}^n)} = \sum_{i,k=1}^{m}\alpha_i(\varepsilon_k)\alpha_k(\varepsilon_k)\left(u_i^{\rm appr},u_k^{\rm appr}\right)_{L_2(\mathbb{R}^n)}$$

$$= \sum_{i,k=1}^{m}\alpha_i(\varepsilon_k)\alpha_k(\varepsilon_k)(u_i^0,u_k^0)_{\mathcal{H}_0} + o(1) = \sum_{i,k=1}^{m}\alpha_i(\varepsilon_k)\alpha_k(\varepsilon_k)\delta_{ik} + o(1)$$

$$= \sum_{i=1}^{m}|\alpha_i(\varepsilon_k)|^2 + o(1) \to 1.$$

We get a contradiction. Thus, total multiplicity of $\lambda(\varepsilon)\to\lambda_0$ is at least m.

As a result we come to a conclusion that if λ_0 is an eigenvalue of A_0 of multiplicity m then there exist exactly m eigenvalues (counted with their multiplicities) of A_ε converging to λ_0, and estimates (2.30) and (2.31) hold. In other words there is an “asymptotic one-to-one correspondence” between isolated eigenvalues and eigenfunctions of the operators A_ε and A_0.

2.3 Identity of the essential spectra of $\widehat{A}_0$ and A_0, convergence of the spectra of A_ε in the sense of Hausdorff

We recall that $\widehat{A}_\varepsilon$ and $\widehat{A}_0$ denote the ‘unperturbed’ operators corresponding to A_ε and A_0, see Section 1.1. It was shown in [48] that $\sigma(\widehat{A}_\varepsilon)\xrightarrow{H}\sigma(\widehat{A}_0)$ (the spectra of both $\widehat{A}_\varepsilon$ and $\widehat{A}_0$ are purely essential). In [23] it is proved that the essential spectrum of a divergence form operator $-\nabla\cdot a(x)\nabla$ (where $a(x)\geq\delta>0$ is a scalar function) remains unperturbed with respect to the local perturbation of the coefficient $a(x)$. Applying this assertion to the operator $\widehat{A}_\varepsilon$ and its perturbation A_ε we conclude that $\sigma(\widehat{A}_\varepsilon)=\sigma_{\rm ess}(A_\varepsilon)\xrightarrow{H}\sigma(\widehat{A}_0)$. Let us assume that $\sigma(\widehat{A}_0)=\sigma_{\rm ess}(A_0)$. Then $\sigma_{\rm ess}(A_\varepsilon)\xrightarrow{H}\sigma_{\rm ess}(A_0)$. In this case Theorem 2.2.1 together with the results of [29] imply the convergence of the discrete spectra in the gaps ($\sigma_{\rm disc}(A_\varepsilon) \xrightarrow{H} \sigma_{\rm disc}(A_0)$) and, consequently, we would have $\sigma(A_\varepsilon) \xrightarrow{H} \sigma(A_0)$. However, we cannot apply the result of [23] as it is stated to the case of the two-scale operators $\widehat{A}_0$ and A_0. In this section we prove the stability of the essential spectrum of $\widehat{A}_0$ with respect to the local perturbation of its coefficients, establishing thereby the missing part of the reasoning. We do this by direct means using the Weyl's criterion for the essential spectrum of an operator, see e.g. [12].

Theorem 2.3.1. *The essential spectra of the operators* $\widehat{A}_0$ *and* A_0 *coincide.*

Proof. **Step 1.** First we describe the domains of $\widehat{A}_0$ and A_0. According to the Friedrichs extension procedure, see e.g. [41], a function u belongs to $\mathcal{D}(A_0)$ if and only if $u = u_0(x) + v(x,y) \in \mathcal{V}$ and there exists $h = h_0(x) + g(x,y) \in \mathcal{H}_0$ such that

$$B_0(u,w) = (h,w)_{\mathcal{H}_0} \tag{2.33}$$

for all $w = w_0 + z \in \mathcal{V}$, see (1.15)–(1.17).

Let $u = u_0 + v \in \mathcal{D}(A_0)$. Then in order to $u_0 \in \mathcal{D}(A_0)$ be fulfilled there must be a function $f \in \mathcal{H}_0$ such that

$$a_2 \int\limits_{\Omega_2} \nabla u_0 \cdot \nabla w_0 \, dx + \int\limits_{\Omega_1} A^{\rm hom} \nabla u_0 \cdot \nabla w_0 \, dx = (f, w_0 + z)_{\mathcal{H}_0} \tag{2.34}$$

for all $w \in \mathcal{V}$. In particular, setting in (2.33) $z \equiv 0$ we obtain

$$a_2 \int\limits_{\Omega_2} \nabla u_0 \cdot \nabla w_0 \, dx + \int\limits_{\Omega_1} A^{\rm hom} \nabla u_0 \cdot \nabla w_0 \, dx = \int\limits_{\mathbb{R}^n} w_0 (h_0 + \langle g \rangle) \, dy \, dx. \tag{2.35}$$

Comparing (2.34) and (2.35) we infer that their right hand sides are equal and that f is orthogonal to $L^2(\Omega_1;\, L^2(Q_0))$. One can derive that f satisfying (2.34) is defined by

$$f = \begin{cases} h_0, & x \in \Omega_2, \\ |Q_1|^{-1} (h_0 + \langle g \rangle)\, \Theta_{Q_1}(y), & x \in \Omega_1. \end{cases}$$

Therefore u_0 and, hence, v belong to $\mathcal{D}(A_0)$ as soon as $u = u_0 + v \in \mathcal{D}(A_0)$. Due to the regularity properties of solutions of elliptic equations, $u_0 \in H^2_{\rm loc}$ everywhere away from the boundary of Ω_2.

Operator $\widehat{A}_0$ acting in the Hilbert space $\widehat{\mathcal{H}}_0$ was described in [48] and is generated by a (closed) symmetric and bounded from below bilinear form $\widehat{B}_0(u,w)$ on a dense subspace $\widehat{\mathcal{V}}$ of $\widehat{\mathcal{H}}_0$, where $\widehat{\mathcal{H}}_0$, $\widehat{\mathcal{V}}$ and $\widehat{B}_0(u,w)$ are defined by (1.7)–(1.9). A function u belongs to domain $\mathcal{D}(\widehat{A}_0)$ if and only if $u = u_0(x) + v(x,y) \in \widehat{\mathcal{V}}$

and there exists $h \in \widehat{\mathcal{H}}_0$ such that

$$\widehat{B}_0(u,w) = (h,w)_{\widehat{\mathcal{H}}_0}$$

for all $w \in \widehat{\mathcal{V}}$. Analogously to the case of $\mathcal{D}(A_0)$, if $u = u_0 + v \in \mathcal{D}(\widehat{A}_0)$ then $u_0, v \in \mathcal{D}(\widehat{A}_0)$, $u_0 \in H^2(\mathbb{R}^n)$.

Let A be a self-adjoint operator with domain $\mathcal{D}(A)$ acting in a Hilbert space H. By the Weyl's criterium, see e.g. [12], condition $\lambda \in \sigma_{\text{ess}}(A)$ is equivalent to the existence of a singular sequence $u^{(k)} \in \mathcal{D}(A)$, i.e. such that

$$0 < C_1 \le \|u^{(k)}\|_H \le C_2, \tag{2.36}$$

$$u^{(k)} \rightharpoonup 0 \text{ weakly in } H, \tag{2.37}$$

$$(A - \lambda)u^{(k)} \to 0 \text{ strongly in } H. \tag{2.38}$$

Employing this definition we will prove that $\lambda \in \sigma_{\text{ess}}(A_0)$ if and only if $\lambda \in \sigma_{\text{ess}}(\widehat{A}_0)$. The operators A_0 and $\widehat{A}_0$ possess very similar properties. The main difference between them consists in the fact that their domains differ. Luckily, a function which support does not intersect with $\overline{\Omega}_2$ belongs to $\mathcal{D}(A_0)$ and $\mathcal{D}(\widehat{A}_0)$ simultaneously. So the idea of the proof is the following. We consider arbitrary singular sequence of one operator and change it slightly to ensure that its elements belong the domain of another operator preserving all properties (2.36)–(2.38).

Step 2. Let $\lambda \in \sigma_{\text{ess}}(\widehat{A}_0)$ and $u^{(k)} = u_0^{(k)}(x) + v^{(k)}(x,y)$ be the corresponding singular sequence in $\mathcal{D}(\widehat{A}_0) \subset \widehat{\mathcal{H}}_0$. First notice that the gradient of $u_0^{(k)}$ is bounded in $L^2(\mathbb{R}^n)$. Indeed, from (1.9) and (2.38) we have

$$\|\nabla u_0^{(k)}\|^2_{L^2(\mathbb{R}^n)} \le C\widehat{B}_0(u^{(k)}, u^{(k)}) = C\lambda(u^{(k)}, u^{(k)})_{\widehat{\mathcal{H}}_0} + o(1) \le C. \tag{2.39}$$

Let us define a cut-off function

$$\eta_{k,R}(x) = \eta\left(\frac{1}{k}(|x| - R)\right),$$

where $\eta \in C^2(\mathbb{R})$ is such that

$$\eta(t) = \begin{cases} 1, & t \le 0, \\ 0, & t \ge 1. \end{cases}$$

So $\eta_{k,R}$ is 1 when $|x| \le R$, 0 when $|x| \ge R + k$ and has small gradient if k is large.

Consider the following sequence, $u^{(k)}\eta_{k,R_k}\in\mathcal{D}(\widehat{A}_0)$, where R_k is chosen large enough so that $\|u^{(k)}(1-\eta_{k,R_k})\|_{\widehat{\mathcal{H}}_0}\le\frac{1}{k}$. This sequence obviously satisfies (2.36) for large enough k regarding the operator $\widehat{A}_0$. Let us check property (2.38). The operator $\widehat{A}_0$ acts on a function $u\in H^2(\mathbb{R}^n)\subset\mathcal{D}(\widehat{A}_0)$ as follows[1], cf. [48]. Let

$$-\nabla\cdot A^{\mathrm{hom}}\nabla u(x)=f(x)\in L^2(\mathbb{R}^n).$$

Then, by the definition of $\widehat{A}_0$, we have

$$\widehat{A}_0u(x)=|Q_1|^{-1}\Theta_{Q_1}(y)f(x)\in\widehat{\mathcal{H}}_0.$$

Note that

$$\|\widehat{A}_0u\|_{\widehat{\mathcal{H}}_0}=|Q_1|^{-1/2}\|f\|_{L^2(\mathbb{R}^n)}.$$

For $u^{(k)}\eta_{k,R_k}$ we derive

$$\widehat{A}_0\left(u^{(k)}\eta_{k,R_k}\right)=\eta_{k,R_k}\widehat{A}_0u^{(k)}-$$
$$-|Q_1|^{-1}\Theta_{Q_1}(y)\left(2\nabla\eta_{k,R_k}\cdot A^{\mathrm{hom}}\nabla u_0^{(k)}+u_0^{(k)}\nabla\cdot A^{\mathrm{hom}}\nabla\eta_{k,R_k}\right).$$

The second term on the right hand side becomes small as $k\to\infty$. Thus we arrive at

$$\begin{aligned}\left\|(\widehat{A}_0-\lambda)(u^{(k)}\eta_{k,R_k})\right\|_{\widehat{\mathcal{H}}_0}&\le\left\|\eta_{k,R_k}(\widehat{A}_0-\lambda)u^{(k)}\right\|_{\widehat{\mathcal{H}}_0}+\\&+2|Q_1|^{-1/2}\left\|\nabla\eta_{k,R_k}\cdot A^{\mathrm{hom}}\nabla u_0^{(k)}\right\|_{L^2(\mathbb{R}^n)}+\\&+|Q_1|^{-1/2}\left\|u_0^{(k)}\nabla\cdot A^{\mathrm{hom}}\nabla\eta_{k,R_k}\right\|_{L^2(\mathbb{R}^n)}=\\&=o(1)+\frac{1}{k}O\left(\left\|\nabla u_0^{(k)}\right\|_{L^2(\mathbb{R}^n)}\right)+\frac{1}{k^2}O\left(\left\|u_0^{(k)}\right\|_{L^2(\mathbb{R}^n)}\right).\end{aligned}\tag{2.40}$$

Due to (2.36) and (2.39) the latter converges to 0 as $k\to\infty$. Hence (2.38) holds regarding $\widehat{A}_0$ and $u^{(k)}\eta_{k,R_k}$.

Now notice that if $\operatorname{supp}u\cap\overline{\Omega}_2=\emptyset$, then $u\in\mathcal{D}(\widehat{A}_0)$ if and only if $u\in\mathcal{D}(A_0)$; besides $\widehat{A}_0u=A_0u$. We hence next shift the supports of $u^{(k)}\eta_{k,R_k}$ away from Ω_2 ensuring also that the new sequence is weakly convergent to maintain (2.37).

[1] If $u=u_0(x)+v(x,y)$ then $\widehat{A}_0u=h\in\widehat{\mathcal{H}}_0$ implies $-\nabla\cdot A^{\mathrm{hom}}\nabla u_0=\langle h\rangle_y$ and $-a_0\Delta_yv=h(x,y)$, $y\in Q_0$.

Since $\operatorname{supp}\eta_{k,R_k}$ is a closed ball of radius $R_k + k$ centred at the origin, the shift of x by $\xi_k := (R_k + 2k + \operatorname{diam}(\Omega_2))\,\xi$ for every k, where ξ is an arbitrary unit vector from $\mathbb{R}^n$, will do the job. Hence, for the given λ we have constructed a singular sequence

$$w^{(k)}(x,y) = u^{(k)}(x+\xi_k, y)\,\eta_{k,R_k}(x+\xi_k),$$

satisfying all the properties (2.36)–(2.38) for the operator A_0. Namely, the translational invariance of $\widehat{A}_0$ in x ensures that (2.36) and (2.38) are satisfied. Finally, (2.37) follows from the pointwise convergence of $w^{(k)}$ to zero as $k \to \infty$ (since for any fixed x, $w^{(k)}(x,y) = 0$ for large enough k). Thus $\lambda \in \sigma_{\text{ess}}(A_0)$.

Step 3. Suppose now that $\lambda \in \sigma_{\text{ess}}(A_0)$ and $u^{(k)} = u_0^{(k)}(x) + v^{(k)}(x,y)$ is the corresponding singular sequence. Let R be such that $\overline{\Omega}_2 \subset B_R$. The situation now is more complicated. The elements of the sequence does not belong to $\mathcal{D}(\widehat{A}_0)$ because of the discontinuity of first derivative at the boundary of Ω_2. If we cut off the elements of the sequence in the neighbourhood of Ω_2 we may loose property (2.36). This may happen when functions $u^{(k)}$ mainly "concentrated" around Ω_2. In fact there are two possibilities: either functions $u^{(k)}$ decay uniformly at infinity in $\mathcal{H}_0$ or not. In the first case it is possible to prove the compactness of $u_0^{(k)}$ in $L^2(\mathbb{R}^n)$. Due to (2.37) the latter implies $u_0^{(k)} \to 0$. Then the sequence $v^{(k)}$ satisfies all the properties of Weyl sequence for $\widehat{A}_0$ and belongs its domain. In the second case we can cut off $u^{(k)}$ in the neighbourhood of Ω_2 to obtain Weyl sequence straight away. In the following we carry out this sketch in more precise way.

There are only two alternative possibilities[2]:

- There exists a sequence $\delta_i \to 0$ such that for any $i \in \mathbb{N}$

$$\|u^{(k)}(1 - \Theta_{B_{R+i}})\|_{\mathcal{H}_0} \leq \delta_i \tag{2.41}$$

  for all k.

- There exist a constant $M > 0$ and subsequences $k(j) \to \infty$, $i(j) \to \infty$ as $j \to \infty$ such that

$$\|u^{(k(j))}(1 - \Theta_{B_{R+i(j)}})\|_{\mathcal{H}_0} \geq M \tag{2.42}$$

[2]Let $A_{ki} := \|u^{(k)}(1 - \Theta_{B_{R+i}})\|_{\mathcal{H}_0}$ and let $\delta_i := \sup_k A_{ki}$. Then either $\delta_i \to 0$ giving (2.41) or $\delta_i \nrightarrow 0$ yielding (2.42).

for all j.

Let (2.41) take place. The sequence $\nabla u_0^{(k)}$ is bounded in $L^2(\mathbb{R}^n)$, cf. (2.39). From (2.41) and

$$\|f\|_{L^2(\mathbb{R}^n)} = \|f\|_{\mathcal{H}_0}, \text{ for all } f \in L^2(\mathbb{R}^n) \subset \mathcal{H}_0, \tag{2.43}$$

it follows that

$$u_0^{(k)} \to u(x) \text{ in } L^2(\mathbb{R}^n), \tag{2.44}$$

up to a subsequence. The reasoning leading to this assertion is essentially identical to the one in (2.4)–(2.7) and is not reproduced here. Since $u^{(k)} = u_0^{(k)} + v^{(k)}$ converges weakly in $\mathcal{H}_0$ to zero, from (2.44) we conclude that

$$v^{(k)}(x,y) \rightharpoonup -u(x) \text{ weakly in } \mathcal{H}_0.$$

Hence, on one hand, we have

$$\big(u, v^{(k)}\big)_{\mathcal{H}_0} \to -(u,u)_{\mathcal{H}_0} = -\int\limits_{\mathbb{R}^n} u^2\,dx$$

as $k \to \infty$. On the other hand,

$$\big(u, v^{(k)}\big)_{\mathcal{H}_0} = \int\limits_{\mathbb{R}^n}\int\limits_{Q_0} u\,v^{(k)}\,dy\,dx =$$

$$= \int\limits_{\mathbb{R}^n}\int\limits_{Q_0} u\,v^{(k)}\Theta_{Q_0}(y)\,dy\,dx = \big(u\,\Theta_{Q_0}(y), v^{(k)}\big)_{\mathcal{H}_0} \to$$

$$\to -\left(u\,\Theta_{Q_0}(y), u\right)_{\mathcal{H}_0} = -|Q_0|\int\limits_{\mathbb{R}^n} u^2\,dx.$$

Comparing the last two formulas, we conclude that $u \equiv 0$, i.e.

$$u_0^{(k)} \to 0 \text{ in } L^2(\mathbb{R}^n). \tag{2.45}$$

Moreover

$$v^{(k)}(x,y) \rightharpoonup 0 \text{ weakly in } \mathcal{H}_0. \tag{2.46}$$

Let us consider an arbitrary sequence $g^{(k)} = g_0^{(k)} + h^{(k)}$ from $\mathcal{H}_0$ converging to zero. It is simple to prove, but probably not entirely obvious that both $g_0^{(k)}$

and $h^{(k)}$ converge to zero. We can write the terms as $g^{(k)} = g_0^{(k)}\Theta_{Q_1}(y) + \left(g_0^{(k)}\Theta_{Q_0}(y) + h^{(k)}\right)$. We obtain

$$\|g^{(k)}\|^2_{\mathcal{H}_0} = |Q_1| \int\limits_{\mathbb{R}^n} \left(g_0^{(k)}\right)^2 dx + \int\limits_{\mathbb{R}^n}\int\limits_{Q_0} \left(g_0^{(k)}\Theta_{Q_0}(y) + h^{(k)}\right)^2 dydx \to 0.$$

Then it follows that $g_0^{(k)}$ converges to zero (in $L^2(\mathbb{R}^n)$ and $\mathcal{H}_0$), and, hence, $h^{(k)}$ converges to zero (in $\mathcal{H}_0$).

Now we denote $A_0 u^{(k)}$ by $g^{(k)}(x,y) = g_0^{(k)}(x) + h^{(k)}(x,y) \in \mathcal{H}_0$. From (2.38) we get the following convergence:

$$\|g_0^{(k)} - \lambda u_0^{(k)}\|_{L^2(\mathbb{R}^n)} \to 0,$$

$$\|h^{(k)} - \lambda v^{(k)}\|_{\mathcal{H}_0} \to 0. \tag{2.47}$$

Then (2.45) implies that

$$g_0^{(k)} \to 0 \text{ in } L^2(\mathbb{R}^n). \tag{2.48}$$

One might expect now that $v^{(k)}$ has to be a Weyl sequence for the operator A_0 (and also for the operator $\widehat{A}_0$, as $v^{(k)}$ extended by zero into Ω_2 belongs to its domain). However it is not true. Functions $v^{(k)}$ satisfy the following equation

$$A_0 v^{(k)} = g_0^{(k)}\Theta_{\Omega_1}(x)\Theta_{Q_0}(y) + h^{(k)} - |Q_1|^{-1}\Theta_{Q_1}(y)\left\langle g_0^{(k)}\Theta_{\Omega_1}(x)\Theta_{Q_0}(y) + h^{(k)}\right\rangle_y.$$

Substituting the expression on the right hand side into $\|A_0 v^{(k)} - \lambda v^{(k)}\|_{\mathcal{H}_0}$ one finds that this entity does not converge to zero. Nevertheless, $v^{(k)}$ turns out to be a Weyl sequence for an operator $\widehat{A}_y$, see below, whose spectrum is contained in the essential spectrum of $\widehat{A}_0$,

$$\sigma(\widehat{A}_y) \subset \sigma_{\text{ess}}(\widehat{A}_0), \tag{2.49}$$

see [49]. We define a self-adjoint operator $\widehat{A}_y$ (cf. [48]) acting in $L^2(\Omega_1 \times Q_0)$ by

$$\widehat{A}_y v = -a_0 \Delta_y v = p, \quad p \in L^2(\mathbb{R}^n \times Q_0).$$

The domain of the operator, $\mathcal{D}(\widehat{A}_y) \subset L^2(\mathbb{R}^n, H_0^1(Q_0))$, is the set of all the solution of this equation. It is not difficult to see (by analysing (1.16)) that

$$\widehat{A}_y v^{(k)} = g_0^{(k)}\Theta_{\Omega_1}(x)\Theta_{Q_0}(y) + h^{(k)}, \tag{2.50}$$

i.e. $v^{(k)} \in \mathcal{D}(\widehat{A}_y)$. Combining (2.47), (2.48) and (2.50) we arrive at

$$\|(\widehat{A}_y - \lambda)v^{(k)}\|_{L^2(\mathbb{R}^n \times Q_0)} = \|g_0^{(k)}\Theta_{Q_0}(y) + h^{(k)} - \lambda v^{(k)}\|_{L^2(\mathbb{R}^n \times Q_0)} \to 0.$$

From (2.45) and (2.46) we conclude that other properties of Weyl sequence are fulfilled, and hence $\lambda \in \widehat{A}_y$. Hence $\lambda \in \sigma_{\rm ess}(\widehat{A}_0)$, see (2.49).

Now let (2.42) hold. Consider a sequence $w^{(j)} = u^{(k(j))}(1 - \eta_{i(j),R}) \in \mathcal{D}(\widehat{A}_0)$ (we remind that R is large enough to ensure $\Omega_2 \Subset B_R$). Then

$$\|w^{(j)}\|_{\widehat{\mathcal{H}}_0} \geq \|u^{(k(j))}(1 - \Theta_{B_{R+i(j)}})\|_{\mathcal{H}_0} \geq M,$$

i.e. (2.36) is satisfied for $w^{(j)}$. Since the sequence $1-\eta_{i(j),R}$ tends to 0 pointwise, (2.37) is valid. Analogously to (2.40) we derive

$$\|(\widehat{A}_0 - \lambda)w^{(j)}\|_{\widehat{\mathcal{H}}_0} = \|(A_0 - \lambda)w^{(j)}\|_{\mathcal{H}_0} \to 0,$$

yielding (2.38). Thus, we conclude that $\lambda \in \sigma_{\rm ess}(\widehat{A}_0)$, completing the proof of the theorem. □

Remark 2.3.2. Theorem 2.3.1 combined with [48] implies that $\sigma_{\rm ess}(A_0) = \{\lambda : \beta(\lambda) \geq 0\} \cup \sigma(A_y)$. Using the methods of [48] it is not hard to show further that $\sigma_{\rm ess}(A_0)$ contains no point spectrum (in particular, no embedded eigenvalues) except if λ is an eigenvalue of A_y corresponding to an eigenfunction with zero mean. It is natural to conjecture (cf. [48]) that, outside these eigenvalues, the spectrum is absolutely continuous and the “eigenfunctions of the continuous spectrum” are $u(x, y, \lambda) = u_0(x, \lambda)(1 + \lambda b(y, \lambda))$, where $u_0(x, \lambda)$ are solutions of the appropriate scattering problems:

$$\begin{gathered} \nabla \cdot A^{\rm hom}\nabla u_0 + \beta(\lambda)u_0 = 0,\ x \in \mathbb{R}^n\backslash\overline{\Omega}_2, \\ a_2\Delta u_0 + \lambda u_0 = 0,\ x \in \Omega_2 \end{gathered} \tag{2.51}$$

with the appropriate matching condition at $\partial\Omega_2$ and radiation condition at infinity. A detailed study of this as well as of the convergence of the related generalised eigenfunctions (cf. [48] for the defect-free case) is beyond the scope of the present study.

Summarising the main results of the chapter we conclude that Theorems 2.2.1 and 2.3.1 together with the results of [23, 29] (see the discussions at the end of

Section 2.2 and in the beginning of the present section) establish the validity of Theorem 1.1.1.

Part II

Spectral asymptotics in networks of thin domains

Chapter 3

Asymptotics of eigenfunctions and eigenvalues

This chapter is devoted to the construction of the asymptotics of the eigenvalue problem for the Laplacian in a thin curved domain Ω_h with Neumann boundary condition on one slanted end and Dirichlet condition elsewhere. We first state the problem and make a change of variables so as to pass to a problem in a fixed rectangle for a differential operator formally given by an asymptotic series. We then formally construct the asymptotics of the eigenvalues and eigenfunctions in Section 3.2 (outer problem), where the main order terms are functions of separated variables - transversal and longitudinal. In order to obtain proper boundary conditions for the functions of longitudinal variable we use the method of matched asymptotic expansions. Namely, we match the asymptotics of the outer problem with the asymptotics of the solution to the inner problem, which reveals the behaviour of the eigenfunctions of the problem in Ω_h in a small neighbourhood of the slanted end. The solution to the inner problem is described by means of scattering theory. In the last section of this chapter we provide the justification of the derived asymptotics and obtain relevant error bounds.

3.1 Problem formulation

We consider an eigenvalue problem for the Laplacian in a thin curved strip Ω_h with a slanted edge described as follows. Let Γ be a smooth curve in $\mathbb{R}^2$ with a natural parametrisation $\mathbf{r}(s) = (r_1(s), r_2(s))^T$, $s \in [0,1]$. The length of tangential vector $\mathbf{r}'(s)$ is one. The unit normal vector is given by $\mathbf{n}(s) = (-r_2'(s), r_1'(s))^T$. We assume that $\mathbf{r}(0) = 0$ and the curvature is zero in

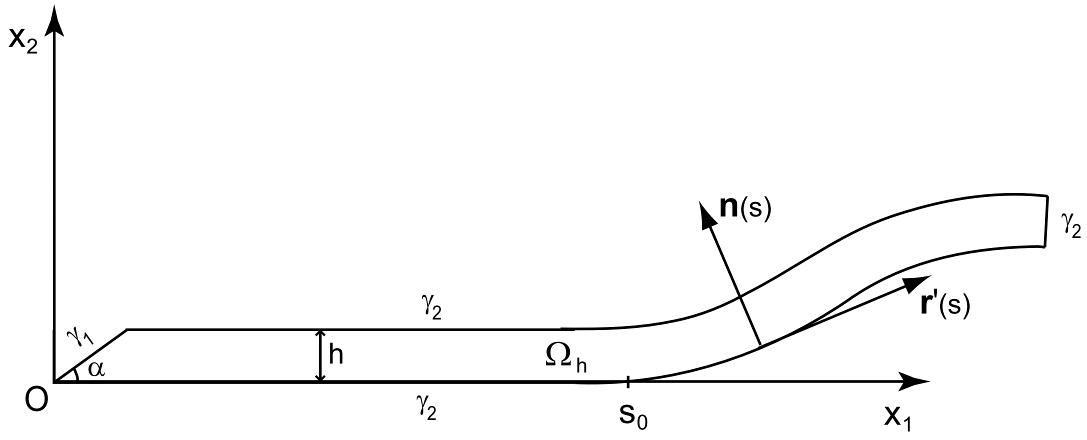

Figure 3-1: Thin curved strip

some neighbourhood of zero, say $s \in [0, s_0]$, $s_0 < 1$, for definiteness, and that $r_1(s) = s$, $r_2(s) = 0$, $s \in [0, s_0]$, i.e. Γ coincides with the positive part of x_1 axis in this neighbourhood. Let n be a normal coordinate along $\mathbf{n}(s)$. Then, for small enough $h > 0$ we define a thin curved strip Ω_h by

$$\Omega_h := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r_1(s) - nr_2'(s) \\ r_2(s) + nr_1'(s) \end{pmatrix} \,\middle|\, s \in (0,1), n \in (0,h), n < \tan(\alpha)\, s \right\}, \tag{3.1}$$

where $0 < h \ll 1$ is a small parameter and $0 < \alpha \leq \pi/2$ is some fixed angle describing the slant of the left edge, see Figure 1-1. We denote by γ_1 the part of the boundary of Ω_h that is described by the equation $n = \tan(\alpha)s$. Respectively $\gamma_2 = \partial\Omega_h \backslash \gamma_1$.

We study the following spectral problem:

$$\begin{aligned} -\Delta u_h = \lambda_h u_h, &\quad x \in \Omega_h, \\ \frac{\partial u}{\partial \nu} = 0, &\quad x \in \gamma_1, \\ u = 0, &\quad x \in \gamma_2, \end{aligned} \tag{3.2}$$

where ν is an exterior unit normal to the boundary of Ω_h. Denote the corresponding self-adjoint operator by A_h. We are interested in finding an asymptotic solution to the problem. The small parameter h describes the thickness of the domain Ω_h, i.e. the shape of the domain changes with h. We next aim at changing the variables so that the transformed spectral problem is in a fixed domain.

Let us rewrite the Laplacian in coordinates (s, n). First we write the partial derivatives with respect to s and n:

$$
\begin{aligned}
\frac{\partial}{\partial s} &= (r_1' - n r_2'')\frac{\partial}{\partial x_1} + (r_2' + n r_1'')\frac{\partial}{\partial x_2},\\
\frac{\partial}{\partial n} &= -r_2'\frac{\partial}{\partial x_1} + r_1'\frac{\partial}{\partial x_2}.
\end{aligned}
\tag{3.3}
$$

Since the curve's parametrisation is natural, the vector $\mathbf{r}''$ is normal to the curve (and hence parallel to $\mathbf{n}$). In this case the curvature is usually defined as the length of $\mathbf{r}''$. However in order to operate with the notation more conveniently we define the curvature with sign:

$$
\kappa = \mathbf{n}\cdot\mathbf{r}''. \tag{3.4}
$$

We assume that $\kappa \in C^2[0,1]$. Obviously, $\kappa\mathbf{n} = \mathbf{r}''$. This implies that $r_1'' = -\kappa r_2'$ and $r_2'' = \kappa r_1'$. Substituting the latter into (3.3) we arrive at

$$
\begin{pmatrix} \dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial n} \end{pmatrix} = \begin{pmatrix} A r_1' & A r_2' \\ -r_2' & r_1' \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \end{pmatrix},
$$

where $A = 1 - \kappa n$. We inverse matrix on the right to obtain

$$
\begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \end{pmatrix} = \frac{1}{A}\begin{pmatrix} r_1' & -A r_2' \\ r_2' & A r_1' \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial n} \end{pmatrix}.
$$

Therefore

$$
\begin{aligned}
\frac{\partial^2}{\partial x_1^2} = A^{-1} r_1' \left(\frac{r_1''}{A} + \frac{\kappa' n r_1'}{A^2} \right) \frac{\partial}{\partial s} + \frac{(r_1')^2}{A^2}\frac{\partial^2}{\partial s^2} - \\
- \frac{r_1' r_2''}{A}\frac{\partial}{\partial n} - \frac{\kappa r_1' r_2'}{A^2}\frac{\partial}{\partial s} - 2\frac{r_1' r_2'}{A}\frac{\partial^2}{\partial s \partial n} + (r_2')^2 \frac{\partial^2}{\partial n^2},
\end{aligned}
$$

and

$$\frac{\partial^2}{\partial x_2^2} = A^{-1} r_2' \left(\frac{r_2''}{A} + \frac{\kappa' n r_2'}{A^2} \right) \frac{\partial}{\partial s} + \frac{(r_2')^2}{A^2} \frac{\partial^2}{\partial s^2} +$$

$$+ \frac{r_1'' r_2'}{A} \frac{\partial}{\partial n} + \frac{\kappa r_1' r_2'}{A^2} \frac{\partial}{\partial s} + 2 \frac{r_1' r_2'}{A} \frac{\partial^2}{\partial s \partial n} + (r_2')^2 \frac{\partial^2}{\partial n^2}.$$

From the relations $\mathbf{r}' \cdot \mathbf{r}'' = 0$, $(\mathbf{r}')^2 = 1$ and (3.4) we derive

$$\Delta = \frac{\kappa' n}{A^3} \frac{\partial}{\partial s} + \frac{1}{A^2} \frac{\partial^2}{\partial s^2} - \frac{\kappa}{A} \frac{\partial}{\partial n} + \frac{\partial^2}{\partial n^2}.$$

We can rewrite this in a more convenient form

$$\Delta = (1 - \kappa n)^{-1} \frac{\partial}{\partial s} (1 - \kappa n)^{-1} \frac{\partial}{\partial s} + (1 - \kappa n)^{-1} \frac{\partial}{\partial n} (1 - \kappa n) \frac{\partial}{\partial n}. \tag{3.5}$$

We will seek an asymptotic solution of (3.2). To obtain an asymptotic approximation of the eigenvalue problem with respect to the small parameter h we introduce the following rescaling:

$$\eta = \frac{n}{h},$$

and consider an eigenvalue problem in the rectangular domain of variables (s, η),

$$D = (0, 1) \times (0, 1).$$

We use the Taylor's expansion $(1 - h\kappa\eta)^{-1} = 1 + h\kappa\eta + (h\kappa\eta)^2 + \ldots$ and $\partial/\partial n = h^{-1} \partial/\partial \eta$ to write a formal asymptotic expansion of the Laplacian:

$$\begin{aligned} -\Delta = -h^{-2} \frac{\partial^2}{\partial \eta^2} + h^{-1} \kappa \frac{\partial}{\partial \eta} + \left(\kappa^2 \eta \frac{\partial}{\partial \eta} - \frac{\partial^2}{\partial s^2} \right) + \\ + h \left(\kappa^3 \eta^2 \frac{\partial}{\partial \eta} - 2\kappa\eta \frac{\partial^2}{\partial s^2} - \kappa' \eta \frac{\partial}{\partial s} \right) + O(h^2) = -\Delta_h + O(h^2). \end{aligned} \tag{3.6}$$

3.2 Outer problem: asymptotic expansions

In this section we seek a formal asymptotic solution to the eigenvalue problem for the operator $-\Delta_h$ in a rectangular domain D:

$$-\Delta_h u_h = \lambda_h u_h, \quad (s, \eta) \in D, \tag{3.7}$$

satisfying Dirichlet boundary condition $u_h = 0$ on the part of ∂D corresponding to $\eta = 0$, $\eta = 1$ and $s = 1$. We do not specify any particular boundary condition at $s = 0$ at the moment. The problem of finding a correct boundary condition at this part of the boundary is one of the main goals of the present chapter and requires a considerable special attention.

Due to the structure of $-\Delta_h$ it is natural to seek the asymptotic solution to the spectral problem in the form of a standard regular asymptotic expansion:

$$u_h \approx u_0(s,\eta) + h\, u_1(s,\eta) + h^2\, u_2(s,\eta) + h^3\, u_3(s,\eta) + \dots \tag{3.8}$$

$$\lambda = \lambda_h \approx h^{-2}\lambda_{-2} + h^{-1}\lambda_{-1} + \lambda_0 + h\lambda_1 + \dots \tag{3.9}$$

We substitute (3.8), (3.9) into (3.7) and collect terms at the equal powers of h, obtaining a recurrent sequence of differential equations, as follows.

h^{-2} :

$$-\frac{\partial^2}{\partial\eta^2}u_0 = \lambda_{-2}u_0. \tag{3.10}$$

The variable s in this equation plays the role of a parameter. This, together with the boundary conditions, implies

$$\begin{aligned} u_0 &= \varphi_0(\eta)v_0(s), \\ \varphi_0 &= \sin(\pi\eta), \\ \lambda_{-2} &= \pi^2. \end{aligned} \tag{3.11}$$

Here v_0 is some function which will be defined at later stages. Notice that we restrict our attention to the eigenvalues λ_h corresponding to the first transversal mode π^2, the eigenvalues λ_h 'produced' by the transversal modes $n^2\pi^2, n = 2, 3, \dots$ are beyond the scope of the present work.

h^{-1}:

$$-\frac{\partial^2 u_1}{\partial\eta^2} - \lambda_{-2}u_1 = -\kappa\frac{\partial}{\partial\eta}u_0 + \lambda_{-1}u_0 = (-\varphi_0'\kappa + \lambda_{-1}\varphi_0)\; v_0. \tag{3.12}$$

As above, s is a parameter. This problem is solvable if and only if the right hand side is orthogonal to the eigenfunction φ_0. So we obtain

$$\int_0^1 (-\varphi_0'\kappa + \lambda_{-1}\varphi_0)\, v_0\, \varphi_0\, d\eta = -\frac{\kappa v_0}{2} \int_0^1 \frac{d}{d\eta} \sin^2(\pi\eta) d\eta +$$

$$+\lambda_{-1} v_0 \int_0^1 \sin^2(\pi\eta)\, d\eta = \lambda_{-1} v_0 \int_0^1 \sin^2(\pi\eta)\, d\eta = 0,$$

from which, assuming $v_0 \not\equiv 0$, it follows that

$$\lambda_{-1} = 0.$$

A general solution to (3.12) can be presented as a sum of the general solution of the homogeneous equation and some solution of the inhomogeneous equation. So,

$$u_1 = \varphi_1(\eta) v_1(s) + \varphi_0(\eta) w_0(s),$$

$$v_1 = \kappa v_0,$$

$$\varphi_1 = \frac{1}{2}\eta\varphi_0,$$

where w_0 is some function which will be defined at a later stage. We will obtain an equation for v_0 at the next step from the solvability condition.

h^0:

$$-\frac{\partial^2 u_2}{\partial\eta^2} - \lambda_{-2} u_2 = -\kappa \frac{\partial}{\partial\eta} u_1 - \left(\kappa^2 \eta \frac{\partial}{\partial\eta} - \frac{\partial^2}{\partial s^2}\right) u_0 + \lambda_0 u_0 =$$
$$= -\varphi_1'\kappa^2 v_0 - \varphi_0'\kappa w_0 - \eta\varphi_0'\kappa^2 v_0 + \varphi_0 v_0'' + \lambda_0 \varphi_0 v_0. \tag{3.13}$$

As earlier, the solvability condition for this equation consists in the orthogonality of the right hand side to φ_0 for any s,

$$\int_0^1 \left(-\varphi_1'\kappa^2 v_0 - \varphi_0'\kappa w_0 - \eta\varphi_0'\kappa^2 v_0 + \varphi_0 v_0'' + \lambda_0 \varphi_0 v_0\right) \varphi_0 d\eta = 0. \tag{3.14}$$

One can easily check that

$$\begin{aligned} \int_0^1 \varphi_0' \varphi_0 d\eta &= 0, \\ \int_0^1 \eta \varphi_0' \varphi_0 d\eta &= -\frac{1}{2} \int_0^1 \varphi_0^2 d\eta \\ \int_0^1 \varphi_1' \varphi_0 d\eta &= \frac{1}{4} \int_0^1 \varphi_0^2 d\eta. \end{aligned} \tag{3.15}$$

Hence from (3.14) and (3.15) we obtain the equation for v_0:

$$-v_0'' - \frac{1}{4}\kappa^2 v_0 = \lambda_0 v_0, \quad s \in (0,1). \tag{3.16}$$

From the setting of the original problem it follows naturally that the Dirichlet boundary condition has to be prescribed at the right end of the interval,

$$v_0(1) = 0,$$

but the condition at the left end is still to be determined. Substituting equation (3.16) back into (3.13) we transform it into

$$-\frac{\partial^2 u_2}{\partial \eta^2} - \lambda_{-2} u_2 = \left(-\frac{3}{4}\varphi_0 - \frac{3}{2}\eta\varphi_0'\right)\kappa^2 v_0 - \varphi_0' \kappa w_0. \tag{3.17}$$

A solution to this equation is given by the formula

$$\begin{aligned} u_2 &= \varphi_2 v_2 + \varphi_1 w_1 + \varphi_0 z_0, \\ v_2 &= \kappa^2 v_0, \\ \varphi_2 &= \frac{3}{8}\eta^2 \varphi_0, \\ w_1 &= \kappa w_0, \end{aligned} \tag{3.18}$$

where $z_0(s)$ is an arbitrary function. For the purposes of the present chapter we choose it to be identically zero,

$$z_0 \equiv 0, \ \ s \in [0,1].$$

In the next step we obtain an equation for w_0.

h^1:

$$-\frac{\partial^2 u_3}{\partial \eta^2} - \lambda_{-2} u_3 = -\kappa \frac{\partial u_2}{\partial \eta} - \kappa^2 \eta \frac{\partial u_1}{\partial \eta} + \frac{\partial^2 u_1}{\partial s^2} - \\ -\kappa^3 \eta^2 \frac{\partial u_0}{\partial \eta} + 2\kappa\eta \frac{\partial^2 u_0}{\partial s^2} + \kappa' \eta \frac{\partial u_0}{\partial s} + \lambda_0 u_1 + \lambda_1 u_0. \tag{3.19}$$

As usual, the right hand side must be orthogonal to φ_0. So we multiply the right hand side by φ_0, integrate over the interval $[0, 1]$ with respect to η and work out all the terms separately. (Note that $\int_0^1 \eta \varphi_0^2 d\eta = \frac{1}{2} \int_0^1 \varphi_0^2 d\eta$ and $\int_0^1 \eta^2 \varphi_0' \varphi_0 d\eta = -\frac{1}{2} \int_0^1 \varphi_0^2 d\eta$.) As a result,

$$\begin{aligned}
-\int_0^1 \kappa \frac{\partial u_2}{\partial \eta} \varphi_0 d\eta &= -\left(\frac{3}{16}\kappa^3 v_0 + \frac{1}{4}\kappa^2 w_0\right) \int_0^1 \varphi_0^2 d\eta, \\
-\int_0^1 \kappa^2 \eta \frac{\partial u_1}{\partial \eta} \varphi_0 d\eta &= \frac{1}{2}\kappa^2 w_0 \int_0^1 \varphi_0^2 d\eta, \\
\int_0^1 \frac{\partial^2 u_1}{\partial s^2} \varphi_0 d\eta &= \left(\frac{1}{4}\left(\kappa'' v_0 + 2\kappa' v_0' + \kappa v_0''\right) + w_0''\right) \int_0^1 \varphi_0^2 d\eta, \\
-\int_0^1 \kappa^3 n^2 \frac{\partial u_0}{\partial \eta} \varphi_0 d\eta &= \frac{1}{2}\kappa^3 v_0 \int_0^1 \varphi_0^2 d\eta, \\
\int_0^1 2\kappa n \frac{\partial^2 u_0}{\partial s^2} \varphi_0 d\eta &= \kappa v_0'' \int_0^1 \varphi_0^2 d\eta, \\
\int_0^1 \kappa' n \frac{\partial u_0}{\partial s} \varphi_0 d\eta &= \frac{1}{2}\kappa' v_0' \int_0^1 \varphi_0^2 d\eta, \\
\int_0^1 \lambda_0 u_1 \varphi_0 d\eta &= \left(\frac{1}{4}\lambda_0 \kappa v_0 + \lambda_0 w_0\right) \int_0^1 \varphi_0^2 d\eta, \\
\int_0^1 \lambda_1 u_0 \varphi_0 d\eta &= \lambda_1 v_0 \int_0^1 \varphi_0^2 d\eta.
\end{aligned}$$

We take the sum of the expressions on the right hand side of the above, equate it to zero and use equation (3.16) to eliminate the second derivatives of v_0. Thus the solvability condition gives us the following equation for w_0:

$$-w_0'' - \frac{1}{4}\kappa^2 w_0 - \lambda_0 w_0 = \lambda_1 v_0 - \lambda_0 \kappa v_0 + \frac{1}{4}\kappa'' v_0 + \kappa' v_0', \ s \in (0,1). \tag{3.20}$$

The solution to (3.19) (if it exists) is not unique. We fix a particular one by imposing the orthogonality condition:

$$\int_0^1 u_3(s,\eta)\varphi_0(\eta)d\eta \equiv 0. \tag{3.21}$$

So, in summary, the formal asymptotic approximation to the solution of (3.2) is given by

$$\begin{aligned} u_h^{(3)} &= \sum_{i=0}^{3} h^i u_i(s,\eta), \\ \lambda_h^{(3)} &= h^{-2}\lambda_{-2} + \lambda_0 + h\lambda_1. \end{aligned} \tag{3.22}$$

The eigenelements λ_0 and v_0 (as well as λ_1 and w_0) are not yet defined, since the boundary condition at the left end of the interval $(0,1)$ is unclear. In order to obtain proper boundary conditions on v_0 and w_0 we need to match the asymptotics (3.22) with the asymptotics of a solution of the *inner* problem, i.e. a solution near the origin which satisfy Neumann boundary condition on γ_1.

3.3 Inner problem and scattering matrix

In order to obtain proper boundary conditions for functions v_0 we need to consider the behaviour of the solution of (3.2) in the neighbourhood of the origin. We will use the method of matched asymptotic expansions, adjusting expansion (3.22) to the asymptotic expansion of the solution to problem (3.2) near the origin. From now on we assume that $\lambda_0 \neq 0$. By the assumptions of this chapter the domain Ω_h in the neighbourhood of the origin coincides with a straight strip of the width h slanted at the origin. Then we can introduce a stretched variable $y = h^{-1}x$ (hence $\Delta_x = h^{-2}\Delta_y$) and consider an 'inner' eigenvalue problem in a semi-infinite cylinder (see Figure 3-2)

$$\Pi_\alpha := \{y \,|\, y_1 > 0, y_2 \in (0,1), y_2 < \tan(\alpha)\, y_1\}, \tag{3.23}$$

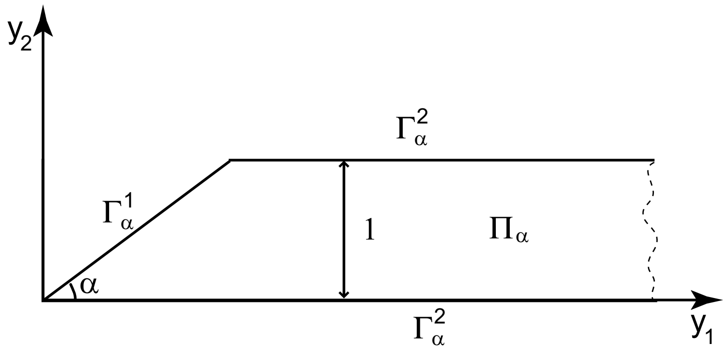

Figure 3-2: Semi-infinite cylinder

$$
\begin{aligned}
-\Delta_y g(y) &= k_h^2\, g(y), \quad y \in \Pi_\alpha, \\
\frac{\partial g}{\partial \nu} &= 0, \quad y \in \Gamma_\alpha^1, \\
g &= 0, \quad y \in \Gamma_\alpha^2,
\end{aligned} \tag{3.24}
$$

(ν is the exterior normal in the y-coordinates) with

$$
k_h^2 = (\pi^2 + h^2\lambda_0 + h^3\lambda_1), \tag{3.25}
$$

via (3.9) and (3.11). Here Γ_α^1 is the slanted part of the boundary of Π_α (i.e. corresponding to $y_2 = \tan(\alpha)\, y_1$) and $\Gamma_\alpha^2 = \partial\Pi_\alpha \backslash \Gamma_\alpha^1$. Denote $\mu_h = \sqrt{\lambda_0 + h\lambda_1}$. A solution to problem (3.24) depends obviously on the angle α. We therefore use index α in our notation wherever necessary.

In this section we will make some use of the reasoning and results of [66]. In general, equation (3.24) does not have a nontrivial solution from $L^2(\Pi_\alpha)$. Nevertheless, there always exists a solution that is given us a sum of the *Floquet waves* and of some function decaying exponentially at infinity. Its structure depends on the values of α and k_h. The term 'Floquet waves' is used here for the solutions of the eigenvalue problem akin to (3.24) where the domain Π_α is replaced by the infinite strip $0 < y_2 < 1$. These solutions are given by the following formula

$$
\psi^{\pm}(y) = \exp(\pm i h \mu_h y_1) \sin(\pi y_2) \ \text{if} \ \mu_h \neq 0. \tag{3.26}
$$

Formally setting $h = 0$ we have $k_0 = \pi^2$. We call this *threshold* case. The Floquet waves then are the following

$$
\psi_0(y) = \sin(\pi y_2), \quad \psi_1(y) = y_1 \sin(\pi y_2). \tag{3.27}
$$

When λ_0 is positive (*super-threshold* case) the Floquet waves are oscillating waves of constant amplitude. They have clear physical meaning. Namely

$$g^+ := \exp(-ih\mu_h\, y_1)\sin(\pi y_2), \tag{3.28}$$

is called the incoming wave (as travelling from plus infinity), and

$$g^- := \exp(ih\mu_h\, y_1)\sin(\pi y_2), \tag{3.29}$$

is called the outgoing wave (as travelling to plus infinity). When λ_0 is negative (*sub-threshold* case) the Floquet waves are exponentially growing and exponentially decaying functions. In this case there is no similar intuitive classification. Nevertheless, for some technical reasons (see [66] for some explanations), it is convenient to call the following combinations of Floquet waves the incoming and the outgoing waves respectively:

$$\begin{aligned} g^+ &:= \frac{1}{\sqrt{2}}\left[\exp(ih\mu_h y_1) - i\exp(-ih\mu_h y_1)\right]\sin(\pi y_2), \\ g^- &:= \frac{1}{\sqrt{2}}\left[\exp(ih\mu_h y_1) + i\exp(-ih\mu_h y_1)\right]\sin(\pi y_2). \end{aligned} \tag{3.30}$$

(Notice that the normalising coefficient $\frac{1}{\sqrt{2}}$ is introduced to make the amplitude of the waves the same as in (3.28), (3.29).)

Problem (3.24) is solvable in weighted spaces, see e.g. [66], and the solution could be written as a linear combination of incoming and outgoing waves g^+ and g^- and some exponentially decaying function z,

$$g = g^+ + Sg^- + z. \tag{3.31}$$

The function z decays exponentially in the following sense. Let $H^2_\beta(\Pi_\alpha)$ be a completion in the norm $\|e^{\beta y_1}u\|_{H^2(\Pi_\alpha)}$ of the set of functions in $C^\infty(\overline{\Pi}_\alpha)$ with compact supports vanishing in the neighbourhood of Γ^2_α. Then we require $z \in H^2_\beta(\Pi_\alpha)$ for some positive β. Finally, S is a unitary scattering matrix (in the present case it is simply a complex number depending on h and μ_h, $|S| = 1$), whose asymptotic behaviour as $h \to 0$ determines in fact the boundary conditions for equations (3.16), (3.20). One can write the asymptotics of S in terms of the scattering matrix $\overset{\circ}{s}$ relevant to Floquet waves (3.27) on the threshold, as follows.

Solutions $\sin(\pi y_2)$ and $y_1 \sin(\pi y_2)$ correspond to the spectral parameter $k_0^2 = \lambda_{-2} = \pi^2$, i.e. by formally setting $h = 0$ in (3.25). The incoming and outgoing waves in this case will be defined as

$$\begin{aligned} \overset{\circ}{g}{}^{+} &:= \frac{1}{\sqrt{2}}(1 - iy_1)\sin(\pi y_2), \\ \overset{\circ}{g}{}^{-} &:= \frac{1}{\sqrt{2}}(1 + iy_1)\sin(\pi y_2). \end{aligned} \tag{3.32}$$

We assume that there does not exist a solution of (3.24) with $k_0^2 = \pi^2$ belonging to $L^2(\Pi_\alpha)$. Then the solution to the problem in a weighted space can be presented in the form

$$\overset{\circ}{g} = \overset{\circ}{g}{}^{+} + \overset{\circ}{s}\overset{\circ}{g}{}^{-} + z, \tag{3.33}$$

where $\overset{\circ}{s}$, $|\overset{\circ}{s}| = 1$, is a scattering matrix and $z \in H^2_\beta(\Pi_\alpha)$ for some $\beta > 0$ (this z is obviously different from the one in (3.31)).

The asymptotics of S can then be written in terms of $\overset{\circ}{s}$ as follows. For $\overset{\circ}{s} \neq 1$ one can obtain an explicit formula for the second term of the asymptotics of S, see [66]. Namely, when λ_0 is negative

$$S = i - h2\mu_h \frac{1 + \overset{\circ}{s}}{1 - \overset{\circ}{s}} + O(h^2) = i - h2\sqrt{\lambda_0}\frac{1 + \overset{\circ}{s}}{1 - \overset{\circ}{s}} + O(h^2); \tag{3.34}$$

and when λ_0 is positive

$$S = -1 + h2\mu_h \frac{1 + \overset{\circ}{s}}{1 - \overset{\circ}{s}} + O(h^2) = -1 + h2\sqrt{\lambda_0}\frac{1 + \overset{\circ}{s}}{1 - \overset{\circ}{s}} + O(h^2). \tag{3.35}$$

(In our notation $\sqrt{\lambda_0} = i\sqrt{|\lambda_0|}$ for $\lambda_0 < 0$.)

It is obvious, that in the case $\overset{\circ}{s} = 1$ (the critical case) formulas (3.34), (3.35) are unsuitable. We will derive an asymptotics of the scattering matrix

$$S = \sum_{m=0}^{\infty} h^m S_m \tag{3.36}$$

when $\overset{\circ}{s} = 1$ also following the general reasoning from [66]. (Notice in passing that the following derivation is easily adopted to the simpler non-critical cases yielding (3.34) and (3.35).) The special solution of (3.24) is given by (3.31), (3.30). We seek the asymptotics of the solution in the form

$$g \approx g^+ + \sum_{m=0}^{\infty} h^m S_m g^- \quad \text{for} \quad y_1 > |\log h|, \tag{3.37}$$

$$g \approx \sum_{m=0}^{\infty} h^m V_m(y) \quad \text{for} \quad y_1 < 2|\log h|. \tag{3.38}$$

Matching these two expansions in the intermediate region $|\log h| < y_1 < 2|\log h|$ we can find (3.36).

Substituting (3.38) into (3.24) we obtain the following sequence of boundary value problems

$$\begin{gathered} -(\Delta + k_0^2)V_m = 0, \;\; y \in \Pi_\alpha, \;\; m = 0, 1, \\ -(\Delta + k_0^2)V_m = \mu_h^2 V_{m-2}, \;\; y \in \Pi_\alpha, \;\; m = 2, 3, \dots, \\ \frac{\partial V_m}{\partial \nu} = 0, \quad y \in \Gamma_\alpha^1, \\ V_m = 0, \quad y \in \Gamma_\alpha^2, \;\; m = 0, 1, \dots. \end{gathered} \tag{3.39}$$

For $m = 0, 1$ the solution is given by

$$V_m = A_m \overset{\circ}{g},$$

where $\overset{\circ}{g}$ is from (3.33). Then for $m = 2$ we obtain

$$-(\Delta + k_0^2)V_2 = A_0 \mu_h^2 (\overset{\circ}{g}{}^+ + \overset{\circ}{g}{}^- + z)$$

(recall that $\overset{\circ}{s} = 1$). A solution to this problem exists and has the form

$$V_2 = -\frac{1}{2} A_0 \mu_h^2 y_1^2 (\overset{\circ}{g}{}^+ + \overset{\circ}{g}{}^-) + \widetilde{V}_2. \tag{3.40}$$

The function $\widetilde{V}_2$ solves the following boundary value problem

$$\begin{gathered} -(\Delta + k_0^2)\widetilde{V}_2 = z, \\ \frac{\partial \widetilde{V}_2}{\partial \nu} = \frac{\partial}{\partial \nu}\Big(\frac{1}{2} A_0 \mu_h^2 y_1^2 (\overset{\circ}{g}{}^+ + \overset{\circ}{g}{}^-)\Big), \quad y \in \Gamma_\alpha^1, \\ \widetilde{V}_2 = 0, \quad y \in \Gamma_\alpha^2. \end{gathered}$$

$\widetilde{V}_2$ is given by

$$\widetilde{V}_2 = A_2 \overset{\circ}{g} + B \overline{\overset{\circ}{g}} + \widetilde{z} \tag{3.41}$$

where A_2 and B are some constants and $\widetilde{z} \in H^2_\beta$, see [66].

Let us derive a formula for the coefficient B. We need this because, as we will see later, B enters in the formula for the first order ($O(h)$) term in the asymptotics of S. In order to obtain the formula we apply integration by parts to the following integral (bar over a symbol denotes its complex conjugate).

$$\begin{aligned} 0 = \int\limits_{\Pi_{\alpha,R}} (\Delta + k_0^2)\, \overset{\circ}{g}\, \overline{V}_2 dy = \int\limits_{\Pi_{\alpha,R}} \overset{\circ}{g}\, (\Delta + k_0^2) \overline{V}_2 dy + \\ + \int\limits_{\partial\Pi_{\alpha,R}} \frac{\partial}{\partial \nu} \overset{\circ}{g}\, \overline{V}_2 dS - \int\limits_{\partial\Pi_{\alpha,R}} \overset{\circ}{g}\, \frac{\partial}{\partial \nu} \overline{V}_2 dS = \\ = -A_0 \mu_h^2 \int\limits_{\Pi_{\alpha,R}} |\overset{\circ}{g}|^2 dy + \int\limits_{\partial\Pi_{\alpha,R}} \frac{\partial}{\partial \nu} \overset{\circ}{g}\, \overline{V}_2 dS - \int\limits_{\partial\Pi_{\alpha,R}} \overset{\circ}{g}\, \frac{\partial}{\partial \nu} \overline{V}_2 dS, \end{aligned} \tag{3.42}$$

where $\Pi_{\alpha,R}$ denotes the part of Π_α satisfying condition $y_1 < R$. It becomes clear from the last formula why we integrate over the bounded domain: the reason is that function $\overset{\circ}{g}$ does not belong to $L^2(\Pi_\alpha)$. Due to the boundary conditions and asymptotic behaviour of $\overset{\circ}{g}$ and V_2 the second term on the right hand side of the latter converges to zero as $R \to \infty$. As for the last term, we derive via (3.32), (3.33), (3.40) and (3.41) the following

$$\begin{aligned} \int\limits_{\partial\Pi_{\alpha,R}} \overset{\circ}{g} \frac{\partial}{\partial \nu} \overline{V}_2 dS = \int\limits_{y_1 = R} \overset{\circ}{g} \frac{\partial}{\partial y_1} \overline{V}_2 dy_2 = \\ = - \int\limits_{y_1 = R} \sin(\pi y_2)(2 A_0 \mu_h^2 y_1 \sin(\pi y_2) + iB \sin(\pi y_2)) dy_2 + o(1) = \\ = -A_0 \mu_h^2 R - \frac{i}{2} B + o(1). \end{aligned} \tag{3.43}$$

From (3.42) and (3.43) we obtain

$$B = -i 2 A_0 \mu_h^2 \lim_{R \to \infty} \left[\int\limits_{\Pi_{\alpha,R}} |\overset{\circ}{g}|^2 dy - R \right]. \tag{3.44}$$

Notice that the limit in the latter formula indeed exists. This is easy to see from the following observation:

$$R = R \int\limits_{y_1=R} |\overset{\circ}{g}^{+} + \overset{\circ}{g}^{-}|^2 dy_2 = \int\limits_{\Pi_{\alpha,R}} |\overset{\circ}{g}^{+} + \overset{\circ}{g}^{-}|^2 dy + const.$$

Let us denote

$$\sigma = \lim_{R\to\infty} \left[\int\limits_{\Pi_{\alpha,R}} |\overset{\circ}{g}|^2 dy - R \right]. \tag{3.45}$$

Then

$$B = -i2A_0\mu_h^2\sigma.$$

Let us write the asymptotics for the Floquet waves as $h \to 0$ and $y_1 \sim |\log h|$:

$$\exp(\pm ih\mu_h y_1) = 1 \pm ih\mu_h y_1 - \frac{1}{2}h^2\mu_h^2 y_1^2 + O(h^3|\log h|^3).$$

Notice that

$$\frac{1}{2}(\overset{\circ}{g}^{+} + \overset{\circ}{g}^{-}) = \frac{1}{\sqrt{2}}\sin(\pi y_2),$$

$$\frac{1}{2}(\overset{\circ}{g}^{-} - \overset{\circ}{g}^{+}) = i\frac{1}{\sqrt{2}}y_1\sin(\pi y_2).$$

First we consider case of negative λ_0 . Then we can write asymptotics (3.37) as follows,

$$\begin{gathered} g = \sigma_-(\overset{\circ}{g}^{+} + \overset{\circ}{g}^{-}) + S_0\sigma_+(\overset{\circ}{g}^{+} + \overset{\circ}{g}^{-}) + h[\sigma_+\mu_h(\overset{\circ}{g}^{-} - \overset{\circ}{g}^{+}) + \\ + S_0\sigma_-\mu_h(\overset{\circ}{g}^{-} - \overset{\circ}{g}^{+}) + S_1\sigma_+(\overset{\circ}{g}^{+} + \overset{\circ}{g}^{-})] + h^2[-\frac{1}{2}\sigma_-\mu_h^2 y_1^2(\overset{\circ}{g}^{+} + \overset{\circ}{g}^{-}) - \\ -\frac{1}{2}S_0\sigma_+\mu_h^2 y_1^2(\overset{\circ}{g}^{+} + \overset{\circ}{g}^{-}) + S_1\sigma_-\mu_h(\overset{\circ}{g}^{-} - \overset{\circ}{g}^{+}) + S_2\sigma_+(\overset{\circ}{g}^{+} + \overset{\circ}{g}^{-})] + \\ + O(h^3|\log h|^3), \end{gathered} \tag{3.46}$$

where we denote

$$\sigma_\pm = \frac{1 \pm i}{2}.$$

Now we derive the first two terms of asymptotics (3.36). Matching expansions (3.37) and (3.38) in the intermediate region $|\log h| < y_1 < 2|\log h|$ we first equate main terms of V_0 and term of order one in (3.46). From this we obtain

$$A_0 = \sigma_+(-i + S_0).$$

Equating the terms of order h and collecting the coefficients at $\overset{\circ}{g}^+$ and $\overset{\circ}{g}^-$ we respectively have two equations

$$A_1 = \sigma_+(-\mu_h + i\mu_h S_0 + S_1)$$

and

$$A_1 = \sigma_+(\mu_h - i\mu_h S_0 + S_1).$$

Hence it follows that

$$A_1 = S_1,$$

$$S_0 = -i,$$

$$A_0 = 2\sigma_-.$$

Equating terms of order h^2 we obtain

$$A_2 = \sigma_+(i\mu_h S_1 + S_2)$$

$$A_2 + B = \sigma_+(-i\mu_h S_1 + S_2).$$

Then we arrive at

$$S_1 = \frac{\sigma_+}{\mu_h} B = -i2\mu_h\sigma.$$

Notice that since μ_h is purely imaginary the first order corrector S_1 is real. So, the asymptotics of S in case when $\overset{\circ}{s} = 1$ and λ_0 is negative is given by

$$S = -i - hi2\mu_h\sigma + O(h^2) = -i - hi2\sqrt{\lambda_0}\sigma + O(h^2). \tag{3.47}$$

Analogously we obtain the asymptotics for the case of positive λ_0. Now, the asymptotics of g is given by

$$\begin{aligned} g = \frac{1}{\sqrt{2}}(1 + S_0)(\overset{\circ}{g}^+ + \overset{\circ}{g}^-) + h[-\frac{1}{\sqrt{2}}\mu_h(\overset{\circ}{g}^- - \overset{\circ}{g}^+) + \frac{1}{\sqrt{2}}S_0\mu_h(\overset{\circ}{g}^- - \overset{\circ}{g}^+) + \\ + \frac{1}{\sqrt{2}}S_1(\overset{\circ}{g}^+ + \overset{\circ}{g}^-)] + h^2[-\frac{1}{2\sqrt{2}}(1 + S_0)\mu_h^2 y_1^2(\overset{\circ}{g}^+ + \overset{\circ}{g}^-) + \\ + \frac{1}{\sqrt{2}}S_1\mu_h(\overset{\circ}{g}^- - \overset{\circ}{g}^+) + \frac{1}{\sqrt{2}}S_2(\overset{\circ}{g}^+ + \overset{\circ}{g}^-)] + O(h^3|\log h|^3). \end{aligned} \tag{3.48}$$

Equating terms of the same order in (3.38) and (3.48) we derive sequentially

$$A_0 = \frac{1}{\sqrt{2}}(1 + S_0),$$

$$A_1 = \frac{1}{\sqrt{2}}(\mu_h - S_0\mu_h + S_1) = \frac{1}{\sqrt{2}}(-\mu_h + S_0\mu_h + S_1),$$

hence

$$S_0 = 1,$$

$$A_2 = \frac{1}{\sqrt{2}}(-\mu_h S_1 + S_2),$$

$$A_2 + B = \frac{1}{\sqrt{2}}(\mu_h S_1 + S_2),$$

hence

$$S_1 = -i2\mu_h\sigma.$$

In this case μ_h is real and S_1 is purely imaginary. Finally we have

$$S = 1 - hi2\mu_h\sigma + O(h^2) = 1 - hi2\sqrt{\lambda_0}\sigma + O(h^2). \tag{3.49}$$

Remark 3.3.1. The scattering matrix S depends on the choice of a coordinate system, i.e. on the position of the domain Π_α in a coordinate system. In particular, the formulas for S in this section are valid only for Π_α positioned as described in (3.23).

3.4 Matching of asymptotics and limit boundary conditions

In this section we will derive proper boundary condition for the function v_0 at the left end of the interval $[0, 1]$. In order to do this we need to match the outer asymptotic solution (3.22) to problem (3.7) and asymptotics of the solution to inner problem (3.24) given by (3.31), (3.36). The matching will be made in some intermediate region lying near $s = 0$. Accomplishing this we will eliminate the uncertainty about the approximate solution $u_h^{(3)}$ in (3.22).

Note that due to the straight shape of Ω_h when $s < s_0$ the coordinates are related by the formula $(s, \eta) = (hy_1, y_2)$. In this section we mostly use the coordinates (s, η), so we must rewrite formulas from the previous section, in particular the Floquet waves (3.26) read

$$\psi^{\pm} = \exp(\pm i\mu_h s)\sin(\pi\eta) = \exp(\pm i\mu_h s)\varphi_0(\eta).$$

We carry out the matching of the asymptotic expansions in the region $s \in (h^{1/3}, 2h^{1/3})$. For such s the curvature κ is identically zero. Then it is easy to see that equation (3.19) locally becomes homogeneous, and due to condition (3.21) we have $u_3 = 0$. Also obviously we have $u_1 = \varphi_0 w_0$ and $u_2 = 0$. Thus the approximate solution to (3.2) for $s < s_0$ simplifies to

$$u_h^{(3)} = \varphi_0(\eta)(v_0(s) + hw_0(s)), \tag{3.50}$$

where v_0 and w_0 locally satisfy differential equations

$$-v_0'' = \lambda_0 v_0 \tag{3.51}$$

and

$$-w_0'' - \lambda_0 w_0 = \lambda_1 v_0,$$

cf. (3.16) and (3.20). Then v_0 is a linear combination of exponents,

$$v_0 = C_1 \exp(i\sqrt{\lambda_0}s) + C_2 \exp(-i\sqrt{\lambda_0}s), \tag{3.52}$$

and w_0, consequently, can be presented in the form

$$\begin{gathered} w_0 = C_1 \exp(i\sqrt{\lambda_0}s)\frac{i\lambda_1}{2\sqrt{\lambda_0}}s - C_2 \exp(-i\sqrt{\lambda_0}s)\frac{i\lambda_1}{2\sqrt{\lambda_0}}s+ \\ +C_3 \exp(i\sqrt{\lambda_0}s) + C_4 \exp(-i\sqrt{\lambda_0}s). \end{gathered} \tag{3.53}$$

Let us write an asymptotics of the Floquet wave as $h \to 0$. Notice that

$$\mu_h = \sqrt{\lambda_0} + h\frac{\lambda_1}{2\sqrt{\lambda_0}} + O(h^2).$$

Then

$$\begin{gathered} \psi^{\pm} = \exp(\pm i\mu_h s)\sin(\pi\eta) = \\ = \exp(\pm i\sqrt{\lambda_0}s)\exp\left(\pm h\frac{i\lambda_1}{2\sqrt{\lambda_0}}s + O(h^2)\right)\varphi_0(\eta) = \\ = \exp(\pm i\sqrt{\lambda_0}s)\left(1 \pm h\frac{i\lambda_1}{2\sqrt{\lambda_0}}s\right)\varphi_0(\eta) + O(h^2). \end{gathered} \tag{3.54}$$

Consider first the non-critical case and negative λ_0. Let us denote

$$\sigma = -i\,\frac{1+\overset{\circ}{s}}{1-\overset{\circ}{s}}. \tag{3.55}$$

Notice that we use the same notation for the different objects, see (3.45) and (3.55). We do this because these objects play a similar role in the formulas for the asymptotics of the scattering matrix. Also there should not be any confusion since formula (3.45) is only used for the critical case $\overset{\circ}{s} = 1$ and (3.55) is valid for the non-critical case $\overset{\circ}{s} \neq 1$. Notice further that straightforward calculation shows that σ is real,

$$\sigma = \frac{\mathrm{Im}\overset{\circ}{s}}{1-\mathrm{Re}\overset{\circ}{s}}.$$

In view of (3.54), employing (3.34) we obtain the following asymptotics for the solution of (3.24):

$$\begin{gathered}
g = \frac{1}{\sqrt{2}}\big(\,[\exp(i\mu_h s) - i\exp(-i\mu_h s)] + \\
+S\,[\exp(i\mu_h s) + i\exp(-i\mu_h s)]\,\big)\varphi_0(\eta) + z = \\
= \frac{1+i}{\sqrt{2}}\exp(i\sqrt{\lambda_0}s)\varphi_0(\eta) - \frac{1+i}{\sqrt{2}}\exp(-i\sqrt{\lambda_0}s)\varphi_0(\eta) + \\
+\frac{h}{\sqrt{2}}\left((1+i)\frac{i\lambda_1}{2\sqrt{\lambda_0}}s - i2\sqrt{\lambda_0}\,\sigma\right)\exp(i\sqrt{\lambda_0}s)\varphi_0(\eta) + \\
+\frac{h}{\sqrt{2}}\left((1+i)\frac{i\lambda_1}{2\sqrt{\lambda_0}}s + 2\sqrt{\lambda_0}\,\sigma\right)\exp(-i\sqrt{\lambda_0}s)\varphi_0(\eta) + \\
+O(h^2) + z.
\end{gathered} \tag{3.56}$$

In formulas (3.54) and (3.56) notation $O(h^2)$ stands for the remainder that is an infinitely smooth and uniformly bounded with respect to h function times h^2, i.e. this remainder together with all its derivatives is bounded by Ch^2 in the L^∞-norm. (Constant C is independent of h although may depend on the order of the derivative). Another remainder is exponentially decaying: $z = z(y) \in H^1_\beta(\Pi_\alpha)$. Since $z = z(h^{-1}s, \eta)$, it must be a rapidly decaying function even for relatively small values of s. However it depends on h as a parameter, since z enters formula (3.31) for the solution of eigenvalue problem (3.24) where the eigenvalue depends on h. Fortunately, $z(y)$ is bounded in $H^1_\beta(\Pi_\alpha)$ uniformly in h, see

[67]. This means that the H^1-norm of $z(h^{-1}s, \eta)$ in the region $s \in (h^{1/3}, 2h^{1/3})$ is exponentially small ($\sim \exp(-h^{-2/3}\beta)$) uniformly with respect to h.

We match asymptotics of $u_h^{(3)}$ and Mg, where M is arbitrary constant. This gives us relations between coefficients in (3.52) and (3.53). Matching the main terms of the asymptotics we derive from (3.53) and (3.56) that

$$C_2 = -C_1 = -\frac{1+i}{\sqrt{2}}M. \tag{3.57}$$

It is obvious from (3.52) and (3.57) that $v_0(0) = 0$. Thus, v_0 and λ_0 is a solution to eigenvalue problem (3.16) with Dirichlet boundary conditions

$$v_0(0) = v_0(1) = 0. \tag{3.58}$$

We assume also that v_0 is normalised,

$$\int_0^1 |v_0|^2 ds = 1. \tag{3.59}$$

This condition fixes some precise value of the coefficients C_1 and M. Matching $w_0\varphi_0$ with the coefficient next to h in the asymptotics of Mg we determine the coefficients in (3.53). We see that C_3 and C_4 must be the following

$$\begin{aligned} C_3 &= -i\sqrt{2\lambda_0}\,\sigma\, M = -(1+i)\sqrt{\lambda_0}\,\sigma\, C_1, \\ C_4 &= \sqrt{2\lambda_0}\,\sigma\, M = (1-i)\sqrt{\lambda_0}\,\sigma\, C_1. \end{aligned} \tag{3.60}$$

It follows from (3.53) and the latter that w_0 must satisfy the following heterogeneous the Dirichlet condition at the point $s = 0$:

$$w_0(0) = C_3 + C_4 = -2\,i\,\sqrt{\lambda_0}\,\sigma\, C_1. \tag{3.61}$$

In order that the function $u_1 = \varphi_1 v_1 + \varphi_0 v_0$ comply with Dirichlet condition on the right end of the strip Ω_h we must set

$$w_0(1) = 0. \tag{3.62}$$

Now we are going to demonstrate that there is a unique choice of λ_1 (which was not defined yet) such that there exists a solution of (3.20) satisfying boundary conditions (3.61), (3.62). Indeed, it is well known that the aforementioned problem has a solution if and only if the right hand side of (3.20) satisfies the following solvability condition:

$$\int_0^1 \left(\lambda_1 v_0 - \lambda_0 \kappa v_0 + \frac{1}{4}\kappa'' v_0 + \kappa' v_0'\right) \overline{v}_0 ds = -w_0(0)\overline{v}_0'(0). \tag{3.63}$$

From (3.52) and (3.57) we have $v_0'(0) = 2i\sqrt{\lambda_0}C_1$. Then from (3.61) we obtain that

$$-w_0\overline{v}_0'(0) = 4|\lambda_0||C_1|^2\sigma = |v_0'(0)|^2\sigma,$$

since

$$|v_0'(0)|^2 = 4|\lambda_0||C_1|^2.$$

Notice also that in the neighbourhood of zero where $\kappa = 0$ the eigenfunction v_0 has a form $C_1\left[\exp(-\sqrt{|\lambda_0|}s) - \exp(\sqrt{|\lambda_0|}s)\right]$. It follow from the theory of ordinary differential equations that $v_0 = C_1 f(s)$, $s \in [0,1]$, where $f(s)$ is a real valued function. In this case $v_0'\overline{v}_0 = \frac{1}{2}(|v_0|^2)'$ and one can apply integration by parts as follows,

$$\int_0^1 \kappa' v_0' \overline{v}_0 ds = -\frac{1}{2}\int_0^1 \kappa'' |v_0|^2 ds.$$

Thus solvability condition (3.63) is fulfilled if λ_1 is given by:

$$\lambda_1 = \left[\int_0^1 \left(\lambda_0\kappa + \frac{1}{4}\kappa''\right)|v_0|^2 ds + |v_0'(0)|^2\sigma\right]\|v_0\|^{-2}, \tag{3.64}$$

where $\|v_0\|^2 = \int_0^1 |v_0|^2 ds$.

Therefore, problem (3.20), (3.61), (3.62) has a solution (which actually is not unique). We need to chose a solution that satisfies (3.53), (3.57) and (3.60) (so that $u_h^{(3)}$ would match with the inner solution Mg). Let us show that this is possible. We fix some arbitrary solution $\widetilde{w}_0$ of (3.20), (3.61), (3.62). In the neighbourhood of zero it has a form

$$\begin{gathered}\widetilde{w}_0 = C_1 \exp(i\sqrt{\lambda_0}s)\frac{i\lambda_1}{2\sqrt{\lambda_0}}s - C_2\exp(-i\sqrt{\lambda_0}s)\frac{i\lambda_1}{2\sqrt{\lambda_0}}s + \\ +\widetilde{C}_3\exp(i\sqrt{\lambda_0}s) + \widetilde{C}_4\exp(-i\sqrt{\lambda_0}s),\end{gathered} \tag{3.65}$$

where the coefficients C_1 and C_2 are as in (3.57), but $\widetilde{C}_3$ and $\widetilde{C}_4$ may differ from C_3 and C_4, cf. (3.51)–(3.53). From the boundary conditions for $\widetilde{w}_0$ we have

$$\widetilde{w}_0(0) = \widetilde{C}_3 + \widetilde{C}_4 = -2i\sqrt{\lambda_0}\sigma C_1. \tag{3.66}$$

Notice that function $w_0 = \widetilde{w}_0 + mv_0$, where m is arbitrary constant, is also a solution of the concerned problem. In the neighbourhood of zero it can be written as

$$w_0 = \widetilde{w}_0 + mv_0 = C_1 \exp(i\sqrt{\lambda_0}s)\frac{i\lambda_1}{2\sqrt{\lambda_0}}s - C_2 \exp(-i\sqrt{\lambda_0}s)\frac{i\lambda_1}{2\sqrt{\lambda_0}}s +$$
$$+(\widetilde{C}_3 + mC_1)\exp(i\sqrt{\lambda_0}s) + (\widetilde{C}_4 + mC_2)\exp(-i\sqrt{\lambda_0}s).$$

We can choose m such that $\widetilde{C}_3 + mC_1 = -(1+i)\sqrt{\lambda_0}\sigma C_1$. Then from (3.57), (3.66) we obtain that $\widetilde{C}_4 + mC_2 = \widetilde{C}_4 - mC_1 = (1-i)\sqrt{\lambda_0}\sigma C_1$. Hence for such choice of m the solution

$$w_0 = \widetilde{w}_0 + mv_0$$

behaves in neighbourhood of zero as described by (3.53), (3.57) and (3.60).

Resuming the above we conclude that such chosen solutions v_0 and w_0 match with the asymptotics of inner solution Mg up to the term of order h^2 and we have the following relation in the region $s \in (h^{1/3}, 2h^{1/3})$:

$$Mg - u_h^{(3)} = h^2 f_1 + z, \tag{3.67}$$

where

$$|f_1|, \left|\frac{\partial}{\partial s} f_1\right| \le CM,$$
$$\|z\|_{H^1((h^{1/3},2h^{1/3})\times(0,1))} \le CM(h^m) \text{ for any } m. \tag{3.68}$$

For positive λ_0 in the non-critical case we have

$$g = \big(\exp(-i\mu_h s) + S\exp(i\mu_h s)\big)\varphi_0(\eta) + z =$$
$$= \left(-1 + hi2\sqrt{\lambda_0}\sigma - h\frac{i\lambda_1}{2\sqrt{\lambda_0}}s\right)\exp(i\sqrt{\lambda_0}s)\varphi_0(\eta) +$$
$$+ \left(1 - h\frac{i\lambda_1}{2\sqrt{\lambda_0}}s\right)\exp(-i\sqrt{\lambda_0}s)\varphi_0(\eta) + O(h^2) + z.$$

In this case we set in (3.52) and (3.53)

$$
\begin{aligned}
C_2 &= -C_1 = M,\\
C_3 &= -2\,i\,\sqrt{\lambda_0}\,\sigma\,C_1,\\
C_4 &= 0.
\end{aligned}
\tag{3.69}
$$

We choose v_0 being a solution of (3.16), (3.58) satisfying (3.59) and (3.52), (3.69). Analogously to the above one can show that for λ_1 given by (3.64) there exists a solution of (3.20), (3.61), (3.62) satisfying (3.53), (3.69). It is easy to see then that (3.67) holds true.

In the critical case $\overset{\circ}{s} = 1$ we similarly obtain the coefficients C_i, $i = 1, 2, 3, 4$. For the case of negative λ_0 we have

$$
\begin{aligned}
C_1 &= C_2 = \frac{1-i}{\sqrt{2}} M,\\
C_3 &= (1-i)\sqrt{\lambda_0}\sigma C_1,\\
C_4 &= (1+i)\sqrt{\lambda_0}\sigma C_1,
\end{aligned}
\tag{3.70}
$$

and if λ_0 is positive, then

$$
\begin{aligned}
C_1 &= C_2 = M,\\
C_3 &= -2\,i\,\sqrt{\lambda_0}\sigma C_1,\\
C_4 &= 0.
\end{aligned}
\tag{3.71}
$$

These formulas imply that v_0 must satisfy Neumann boundary condition at zero. Thus v_0 is a normalised solution of (3.16) subject to boundary conditions

$$
v_0'(0) = v_0(1) = 0. \tag{3.72}
$$

In this case the solvability condition for the equation for w_0:

$$
\int_0^1 \left(\lambda_1 v_0 - \lambda_0 \kappa v_0 + \frac{1}{4}\kappa'' v_0 + \kappa' v_0'\right) \overline{v}_0 ds = w_0'(0)\overline{v}_0(0). \tag{3.73}
$$

involves the value of w_0' at zero. So we impose the following boundary conditions

$$
w_0'(0) = 2\lambda_0\sigma C_1, \quad w_0(1) = 0,
$$

from which the first one is implied by (3.53) and (3.70) (or (3.71)). Then

$$w_0'(0)\overline{v}_0(0) = 4\lambda_0|C_1|^2\sigma = |v_0(0)|^2\lambda_0\sigma. \tag{3.74}$$

Consequently from (3.73) we obtain that

$$\lambda_1 = \left[\int_0^1 \left(\lambda_0\kappa + \frac{1}{4}\kappa''\right)|v_0|^2 ds + |v_0(0)|^2\lambda_0\sigma\right]\|v_0\|^{-2}. \tag{3.75}$$

Then one can show that there exits a solution of (3.20), (3.74) which satisfies (3.53) and (3.70) (or (3.71)) and we still have (3.67).

Thus w_0 is fully defined as a solution of (3.20) with Dirichlet boundary condition at the right end of the interval and satisfying condition (3.53) with an appropriate coefficients near the left end. For such w_0 the solvability condition for equation (3.19) is fulfilled, hence there exists a solution $u_3 \in C^\infty(D)$ such that $u(s,0) = u(s,1) \equiv 0$ and

$$\int_0^1 u_3(s,\eta)\varphi_0(\eta)d\eta \equiv 0.$$

3.5 Error bounds and justification of the asymptotics

In this section we justify the asymptotics obtained earlier. In order to do this we first need to construct a function satisfying the boundary conditions in (3.2) such that after the substitution into equation (3.2) we get asymptotically small (of order $h^{3/2}$) error on its right hand side. It is well known that the operator A_h (as elliptic and defined in bounded domain) has a discrete spectrum with the only accumulation point at infinity. Let $\lambda_{1,h} \le \lambda_{2,h} \le \dots$ be all the eigenvalues of A_h repeating accordingly to there multiplicity and $u_{i,h}$, $i \in \mathbb{N}$ be the corresponding orthonormalised eigenfunctions. Let us introduce a smooth cut-off function

$$\chi(s) = \begin{cases} 1, & s \le 1, \\ 0, & s \ge 2. \end{cases} \tag{3.76}$$

We formulate the main results of the present section in the following theorem.

Theorem 3.5.1. *Let* $\lambda_h^{(3)}$ *be given by (3.22) and* $\lambda_0 \neq 0$. *Then there exists* $h_0 > 0$ *and a constant* C *independent of* h *such that for any* $0 < h \le h_0$ *there exist an eigenvalue* $\lambda_{i,h}$ *of the operator* A_h *such that*

$$|\lambda_{i,h} - \lambda_h^{(3)}| \le Ch^{3/2}. \tag{3.77}$$

Moreover, a function

$$u_h^{appr} = \chi(sh^{-1/3})Mg + (1 - \chi(sh^{-1/3}))(u_0 + hu_1),$$

where g *is a solution of scattering problem (3.24) and* M *is a constant such that (3.67) is satisfied, approximates eigenfunctions of* A_h *in the following sense: for any* $d > 0$ *and any* $0 < h \le h_0$ *there exist coefficients* $c_i(h)$ *such that*

$$\left\| u_h^{appr} - \sum_{|\lambda_{i,h} - \lambda_h^{(3)}| \le d} c_i(h) u_{i,h} \right\|_{L^2(\Omega_h)} \le Cd^{-1}h^2. \tag{3.78}$$

Remark 3.5.2. The error estimate in (3.78) is somewhat deceptive. The fact is that the norm of u_h^{appr} is not of order one. Indeed, roughly speaking the main term of the asymptotics u_h^{appr} is $v_0(s)\sin(h^{-1}\pi n)$, where $\int_0^1 v_0^2 ds = 1$. It is clear then that the norm of $v_0(s)\sin(h^{-1}\pi n)$ in u_h^{appr} is of order $h^{1/2}$. One can consider normalised u_h^{appr}, for which the error estimate (3.78) holds with the right hand side equal $Cd^{-1}h^{3/2}$ (which is of the same order as the estimate for eigenvalues). But in this case the main term of u_h^{appr} is of order $h^{-1/2}$ in L^∞-norm. This seems to us to be improper in some way, so we prefer to normalise v_0 rather than u_h^{appr}.

Proof. We will first mention the regularity properties of the functions g and $u_h^{(3)}$. Obviously $\varphi_0 = \sin(\pi\eta) \in C^\infty([0,1])$. From the general theory of ordinary differential equations we know that v_0 and consequently w_0 belong to $C^\infty([0,1])$. Then obviously $u_0, u_1, u_2 \in C^\infty(\overline{D})$ as elementary combinations of C^∞ functions. The third term of the asymptotics $u_3 \in C^\infty(\overline{D})$ as a solution of ordinary differential equation with respect to η (3.19), where s plays the role of a parameter and the right hand side belongs to $C^\infty(\overline{D})$. Furthermore, in the coordinates (s,η) the operator $-\Delta$ is presented in the form

$$-\Delta = -\Delta_h + h^2 L_h,$$

where L_h is a second order differential operator with smooth bounded coefficients, cf. (3.5), (3.6). Due to equations (3.10), (3.12), (3.13), (3.19) the approximation $u_h^{(3)}$ solves the following equation

$$-\Delta u_h^{(3)} = \lambda_h^{(3)} u_h^{(3)} + h^2 f_2 \text{ in } D, \tag{3.79}$$

where

$$f_2(s,\eta) = L_h u_h^{(3)} + \left(\kappa^3 \eta^2 \frac{\partial}{\partial \eta} - 2\kappa\eta \frac{\partial^2}{\partial s^2} - \kappa' \eta \frac{\partial}{\partial s} \right) u_1 +$$
$$+ \left(\kappa^2 \eta \frac{\partial}{\partial \eta} - \frac{\partial^2}{\partial s^2} \right) u_2 + \kappa \frac{\partial}{\partial \eta} u_3 -$$
$$-h^2 \lambda_0 (u_2 + h u_3) - h^2 \lambda_1 (u_1 + h u_2 + h^2 u_3).$$

It is smooth and hence

$$|f_2| \leq C \tag{3.80}$$

uniformly in h.

In order to justify the asymptotics, the approximation to the actual solution of (3.2) must satisfy boundary conditions imposed in (3.2). To this end we will slightly modify the function $u_h^{(3)}$. Namely, since the functions u_i, $i = 0, 1, 2$, comply with the proper boundary conditions on γ_2, and u_3 vanishes everywhere on γ_2 except the part corresponding to $s = 1$, we need only to multiply u_3 by the appropriate cut-off function. Consider function

$$\widehat{u}_h^{(3)} = \sum_{i=0}^{2} h^i u_i + h^3 u_3 \chi((s-1)h^{-\beta} + 2),$$

Where β is some positive number. Obviously this function satisfies Dirichlet boundary condition on the whole of γ_2. We can rewrite it as

$$\widehat{u}_h^{(3)} = u_h^{(3)} + h^3 u_3 \big[\chi((s-1)h^{-\beta} + 2) - 1 \big].$$

Let us derive the equation for the latter. We will drop the argument of the function $\chi((s-1)h^{-\beta}+2)$ to shorten the notation hoping that this will not lead to any confusion. From (3.79) we obtain

$$-\Delta \widehat{u}_h^{(3)} = -\Delta u_h^{(3)} - h^3 \Delta \left[u_3 (\chi - 1) \right] = \lambda_h^{(3)} \widehat{u}_h^{(3)} + h^2 f_2 + h^3 f_3 \text{ in } D, \tag{3.81}$$

where

$$f_3 = -\Delta\left[u_3(\chi - 1)\right] - \lambda_h^{(3)} u_3(\chi - 1).$$

Due to formula (3.6) we have

$$\begin{aligned} f_3 &= (-\Delta u_3 - \lambda_h^{(3)} u_3)(\chi - 1) + \widetilde{f}_3 \frac{\partial}{\partial s}\chi + \widehat{f}_3 \frac{\partial^2}{\partial s^2}\chi = \\ &= (-\Delta u_3 - \lambda_h^{(3)} u_3)(\chi - 1) + h^{-\beta}\widetilde{f}_3 \chi' + h^{-2\beta}\widehat{f}_3 \chi''. \end{aligned}$$

The components of the latter formula are bounded as follows.

$$|\widetilde{f}_3|,\ |\widehat{f}_3| \leq C,$$

uniformly with respect to h. Since $u_3 \in C^\infty(D)$ and bounded together with its derivatives uniformly in h, and Δ in the coordinates (s, η) is an operator of second order with smooth coefficients of order h^{-2}, we have

$$|\Delta u_3| \leq h^{-2} C,$$

uniformly with respect to h. It is important that support of the functions $\chi - 1$, χ' and χ'' defined on the interval $[0, 1]$ is small, namely, $\sup(\chi - 1) = \sup(\chi') = \sup(\chi'') = [1 - h^\beta, 1]$. Then we arrive at

$$\begin{aligned} &|f_3| \leq (h^{-2} + h^{-2\beta}) C, \\ &\sup(f_3) = [1 - h^\beta, 1]. \end{aligned} \tag{3.82}$$

In order to comply with the Neumann boundary condition on γ^1 we replace $\widehat{u}_h^{(3)}(s, h\eta)$ by $g(hs, h\eta)$ in the small neighbourhood of the origin, where $g(y_1, y_2)$ is the matching inner solution (3.31). In this neighbourhood $\kappa \equiv 0$ and due to (3.24) we have

$$-\Delta g = \lambda_h^{(3)} g, \tag{3.83}$$

and g satisfies conditions

$$\begin{aligned} &\frac{\partial g}{\partial \nu} = 0 \text{ on } \gamma^1, \\ &g = 0 \text{ on } \gamma^2. \end{aligned}$$

Since on the interval $s \in [h^{1/3}, 2h^{1/3}]$ we have relation (3.67), we match there the outer and inner solutions $\widehat{u}_h^{(3)}$ and Mg.

We choose the following function as an approximate solution to the eigenvalue problem (3.2),

$$\widehat{u}_h^{appr}(s,\eta) = \chi(sh^{-1/3})Mg(hs,\eta) + (1-\chi(sh^{-1/3}))\widehat{u}_h^{(3)}(s,\eta).$$

Notice that in the region $s \in [h^{1/3}, 2h^{1/3}]$, since the curvature κ is zero and the corresponding part of the domain Ω_h is a strip parallel to the axis x, the operator simply has form

$$-\Delta = -\frac{\partial^2}{\partial s^2} - h^{-2}\frac{\partial^2}{\partial \eta^2}.$$

Then we derive via (3.83), (3.81) and (3.67)

$$\begin{aligned} -\Delta\widehat{u}_h^{appr} &= -\chi M\Delta g - (1-\chi)\Delta\widehat{u}_h^{(3)} - 2\frac{\partial}{\partial s}\chi\frac{\partial}{\partial s}(Mg - \widehat{u}_h^{(3)}) - \\ -\frac{\partial^2}{\partial s^2}\chi(Mg - \widehat{u}_h^{(3)}) &= \lambda_h^{(3)} Mg\chi + (\lambda_h^{(3)}\widehat{u}_h^{(3)} + h^2 f_2 + h^3 f_3)(1-\chi) - \\ -h^{5/3}2\chi'\frac{\partial}{\partial s}f_1 - h^{4/3}\chi'' f_1 &- h^{-1/3}2\chi'\frac{\partial}{\partial s}z - h^{-2/3}\chi'' z = \\ &= \lambda_h^{(3)}\widehat{u}_h^{appr} + (h^2 f_2 + h^3 f_3)(1-\chi) + h^{4/3} f_4 + \widetilde{z}, \end{aligned} \tag{3.84}$$

where function

$$f_4 = h^{1/3}2\chi'\frac{\partial}{\partial s}f_1 - \chi'' f_1$$

is bounded due (3.68) and has a small support:

$$|f_4| \le C,$$
$$\sup(f_4(s,\eta)) = [h^{1/3}, 2h^{1/3}] \times [0,1],$$

and function

$$\begin{aligned} \widetilde{z} &= -h^{-1/3}2\chi'\frac{\partial}{\partial s}z - h^{-2/3}\chi'' z, \\ \|\widetilde{z}\|_{L^2(\Omega_h)} &= O(h^m) \text{ for any } m \end{aligned} \tag{3.85}$$

due to (3.68).

Now it is easy to estimate in $L^2(\Omega_h)$ (in variables (x,y)) the discrepancy

$-\Delta\widehat{u}_h^{appr} - \lambda_h^{(3)}\widehat{u}_h^{appr}$ via (3.84), (3.80), (3.82) and (3.84)

$$\| -\Delta\widehat{u}_h^{appr} - \lambda_h^{(3)}\widehat{u}_h^{appr}\|_{L^2(\Omega_h)} =$$

$$= \|(h^2 f_2 + h^3 f_3)(1-\chi) + h^{4/3} f_4 + \widetilde{z}\|_{L^2(\Omega_h)} \leq$$

$$\leq h^2\|C\|_{L^2(\Omega_h)} + h^3(h^{-2} + h^{-2\beta})\|C\|_{L^2(\Omega_h\cap\{s\in[1-h^\beta,1]\})} +$$

$$+ h^{4/3}\|C\|_{L^2(\Omega_h\cap\{s\in[h^{1/3},2h^{1/3}]\})} \leq$$

$$\leq (h^{5/2} + h^{(3+\beta)/2} + h^{(7-3\beta)/2} + h^2)C.$$

Choosing $\beta = 1$ we obtain

$$\| -\Delta\widehat{u}_h^{appr} - \lambda_h^{(3)}\widehat{u}_h^{appr}\|_{L^2(\Omega_h)} \leq Ch^2. \tag{3.86}$$

It is well known that the set $u_{i,h}$, $i \in \mathbb{N}$ of the orthonormalised eigenfunctions of A_h forms a basis in $L^2(\Omega_h)$. Then $\widehat{u}_h^{appr}$ can be written in the form

$$\widehat{u}_h^{appr} = \sum_{i=1}^{\infty} c_i u_{i,h}. \tag{3.87}$$

The main term of $\widehat{u}_h^{appr}$ is $v_0(s)\sin h^{-1}\pi n$, v_0 is normalised, and other terms are of order $O(h)$ or have relatively small support. Then one can easily check that

$$\|\widehat{u}_h^{appr}\|^2_{L^2(\Omega_h)} = \sum_{i=1}^{\infty} c_i^2 = \frac{1}{2}h + o(h). \tag{3.88}$$

Substituting (3.87) into (3.86) we obtain

$$\sum_{i=1}^{\infty} c_i^2(\lambda_{i,h} - \lambda_h^{(3)})^2 \leq Ch^4.$$

Then from (3.88) follows that

$$\min_i |\lambda_{i,h} - \lambda_h^{(3)}| \leq Ch^{3/2}, \tag{3.89}$$

which prove the validity of (4.44).

Let us denote by f_h the discrepancy $-\Delta\widehat{u}_h^{appr} - \lambda_h^{(3)}\widehat{u}_h^{appr}$. Then $f_h = \sum_{i=1}^{\infty} b_i u_{i,h}$, where $\sum_{i=1}^{\infty} b_i^2 \leq Ch^4$. We can assume that $\lambda_h^{(3)} \neq \lambda_{i,h}$ (the case $\lambda_h^{(3)} = \lambda_{i,h}$ is trivial). Then $c_i = (\lambda_{i,h} - \lambda_h^{(3)})^{-1} b_i$. Let us represent $\widehat{u}_h^{appr}$ as a sum of two functions:

$$\widehat{u}_h^{appr} = \sum_{|\lambda_{i,h}-\lambda_h^{(3)}|\leq d} \frac{b_i}{\lambda_{i,h}-\lambda_h^{(3)}} u_{i,h} + \sum_{|\lambda_{i,h}-\lambda_h^{(3)}|>d} \frac{b_i}{\lambda_{i,h}-\lambda_h^{(3)}} u_{i,h} =$$

$$= \psi_1 + \psi_2.$$

It easily follows from the above that

$$\|\widehat{u}_h^{appr} - \psi_1\|_{L^2(\Omega_h)} = \|\psi_2\|_{L^2(\Omega_h)} \leq Cd^{-1}h^2. \tag{3.90}$$

The asymptotics $\widehat{u}_h^{appr}$ includes high order terms in its formula. They are required for the justification of (4.44), but not necessary for the approximation of eigenfunctions of A_h. Consider the function $u_h^{appr} = \chi(sh^{-1/3})Mg + (1 - \chi(sh^{-1/3}))(u_0 + hu_1)$. It is easy to see that it differs from $\widehat{u}_h^{appr}$ by a term of order $h^{5/2}$ in $L^2(\Omega_h)$ norm. Hence the second part of the statement of theorem follows. □

Chapter 4

Models on graphs

In this chapter we obtain crucial results concerning the structure of the limiting problem on the graph. We show that in the non-critical case the more accurate model on the graph involves not Dirichlet, but 'almost Dirichlet' boundary conditions at the vertices, namely, the condition of the type

$$V(0) + hTV'(0) = 0 \tag{4.1}$$

at each vertex. Here T is a $d \times d$ matrix, where d is the number of all edges incident to the vertex, $V(0)$ is the vector $(v_1(0), \ldots, v_d(0))^t$ of the values at the vertex that the function v attains along the edges, and $V'(0) = (v_1'(0), \ldots, v_d'(0))^t$ is the vector of the values at the vertex of the derivatives taken along the edges taken in outgoing directions. This is important for the following reason. If one imposes Dirichlet boundary conditions at the vertices, the limiting problem on the graph splits into a number of disjoint problems on its edges, whereas boundary conditions (4.1) insure that the interaction between different edges exists although it is weak. The boundary conditions at the vertices for the limiting problem are fully defined by the scattering matrix at the threshold (the first transversal eigenvalue), which in turn is determined only by the geometry of the junction (in our simplified model by the angle of the slant). In the last section of the chapter we provide an explicit example for the case of zero-curvature, $\kappa \equiv 0$.

4.1 Limiting operator on graph

Let us consider the differential operator

$$\widehat{L}_h = L_0 + hL_1 = -\frac{d^2}{ds^2} - \frac{1}{4}\kappa^2 + h\left(-\kappa'\frac{d}{ds} - \frac{1}{4}\kappa''\right)$$

acting on the interval $(0,1)$. It follows from (3.16) and (3.20) that

$$\begin{aligned}\widehat{L}_h(v_0 + hw_0) = \lambda_0 v_0 + h(\lambda_0 w_0 + \lambda_1 v_0 - \lambda_0 \kappa v_0) + h^2 L_1 w_0 = \\ = (\lambda_0 + h\lambda_1)(1 - h\kappa)(v_0 + hw_0) + \\ + h^2 L_1 w_0 - h^2(\lambda_1 w_0 - \lambda_1 \kappa v_0 - \lambda_0 \kappa w_0) + h^3 \lambda_1 \kappa w_0.\end{aligned} \tag{4.2}$$

Notice that $v_0,\, w_0 \in C^\infty([0,1])$ (see the discussion in the beginning of the proof of Theorem 3.5.1) and $\kappa \in C^2([0,1])$ by the assumptions of the present chapter. Notice also that

$$e^{-h\kappa} = 1 - h\kappa + O(h^2),$$

where the last term must be understood in terms of the norm $L^\infty(0,1)$. Then we obtain from (4.2)

$$\widehat{L}_h(v_0 + hw_0) = (\lambda_0 + h\lambda_1)e^{-h\kappa}(v_0 + hw_0) + O(h^2), \tag{4.3}$$

where the last term is understood in the norm $L^\infty(0,1)$. The operator $\widehat{L}_h$ cannot be symmetric, however we can slightly change it to obtain a symmetric operator. Indeed,

$$\frac{d}{ds}\left(e^{h\kappa}\frac{d}{ds}\right) = e^{h\kappa}\left(\frac{d^2}{ds^2} + h\kappa'\frac{d}{ds}\right).$$

Then multiplying (4.3) by $e^{h\kappa}$ we obtain

$$L_h(v_0 + hw_0) = (\lambda_0 + h\lambda_1)(v_0 + hw_0) + O(h^2), \tag{4.4}$$

where

$$L_h = e^{h\kappa}\widehat{L}_h = -\frac{d}{ds}\left(e^{h\kappa}\frac{d}{ds}\right) - \frac{1}{4}\, e^{h\kappa}(\kappa^2 + h\kappa''). \tag{4.5}$$

In view of (4.4) it is natural to try to approximate the function

$$\widetilde{v}_h = v_0 + hw_0$$

by some eigenfunction of the operator L_h with appropriate boundary conditions, which we will derive first. From (3.52), (3.53) we have

$$\begin{aligned} \widetilde{v}_h(0) &= C_1 + C_2 + h(C_3 + C_4), \\ \widetilde{v}_h{}'(0) &= (C_1 - C_2)\, i\sqrt{\lambda_0} + h\left((C_1 - C_2)\frac{i\lambda_1}{2\sqrt{\lambda_0}} + (C_3 - C_4)\, i\sqrt{\lambda_0}\right). \end{aligned} \tag{4.6}$$

Then for the non-critical case, $\lambda_0 < 0$, via (3.57) and(3.60) we obtain

$$\begin{aligned} \widetilde{v}_h(0) &= -h2i\sqrt{\lambda_0}\sigma C_1, \\ \widetilde{v}_h{}'(0) &= 2\, i\sqrt{\lambda_0}C_1 + h\left(\frac{i\lambda_1}{\sqrt{\lambda_0}} - 2\, i\lambda_0\sigma\right) C_1. \end{aligned}$$

Hence

$$\widetilde{v}_h(0) + h\sigma\widetilde{v}_h{}'(0) = h^2\left(\frac{i\lambda_1}{\sqrt{\lambda_0}} - 2\, i\lambda_0\sigma\right)\sigma C_1.$$

(On the right end we obviously have $\widetilde{v}_h(1) = 0$.) This suggests that we need to consider the self-adjoint operator L_h acting in $L^2(0,1)$, given by (4.42) with boundary conditions

$$\begin{aligned} v(0) + h\,\sigma\, v'(0) &= 0, \\ v(1) &= 0. \end{aligned} \tag{4.7}$$

Ideally, we would like to describe the asymptotic solution to spectral problem (3.2) in terms of the eigenvalues and eigenfunctions of L_h.

We slightly modify the function $\widetilde{v}_h$ so that it would satisfy boundary conditions (4.7). Consider the function

$$\widehat{v}_h = \widetilde{v}_h + h^2 N(e^{i\sqrt{\lambda_0}s} + e^{-i\sqrt{\lambda_0}s})\,\chi(4s), \tag{4.8}$$

where

$$N = -\left(\frac{i\lambda_1}{2\sqrt{\lambda_0}} - i\lambda_0\sigma\right)\sigma C_1,$$

and χ is from (3.76). Clearly, $\widehat{v}_h$ satisfies (4.7). Moreover, $\widehat{v}_h$ is approximate solution to the eigenvalue problem,

$$L_h\widehat{v}_h = (\lambda_0 + h\lambda_1)\widehat{v}_h + O(h^2). \tag{4.9}$$

The last term is understood in the norm $L^\infty(0,1)$ and, hence, it is $O(h^2)$ in

$L^2(0,1)$. Consider the spectral problem for the operator L_h,

$$L_h v_h^{(k)} = \mu_h^{(k)} v_h^{(k)},$$
$$v_h^{(k)}(0) + h\sigma \frac{d}{dx} v_h^{(k)}(0) = 0, \;\; v_h^{(k)}(1) = 0.$$

Applying absolutely the same reasoning as in the previous section when estimates (3.89), (3.90) have been derived, we conclude that

$$\begin{aligned} &\min_k |\mu_h^{(k)} - (\lambda_0 + h\lambda_1)| \leq Ch^2, \\ &\| \sum_{k \in K_d} c_k(h) v_h^{(k)} - \widehat{v}_h \|_{L^2(0,1)} \leq Cd^{-1}h^2, \end{aligned} \tag{4.10}$$

where $\mu_h^{(k)}$ and $v_h^{(k)}$ are the eigenvalues and corresponding eigenfunctions of L_h, C is an h-independent constant, and summation is taken over the set of indices K_d such that $|\mu_h^{(k)} - (\lambda_0 + h\lambda_1)| \leq d$.

The case of the positive λ_0 is analogous, and one can easily obtain (4.10) for function (4.8) with N given by

$$N = -\left(\frac{i\lambda_1}{2\sqrt{\lambda_0}} + \lambda_0 \sigma \right) \sigma C_1.$$

Estimates (4.10) show that all the asymptotics $\lambda_0 + h\lambda_1$ and $v_0 + hw_0$ can be approximated by the eigenvalues and eigenfunctions of L_h. Moreover, an almost converse statement is valid. But in order to prove this we need to obtain more precise information about the spectrum of L_h. This is the main purpose of the next section.

4.2 Spectrum of the limiting operator L_h

We are interested in the behaviour of the spectrum of L_h as $h \to 0$, in particular, in its relation to the spectrum of the operator L_0,

$$L_0 v = -v'' - \frac{1}{4}\kappa^2 v,$$
$$D(L_0) = H_0^1 \cap H^2 = H_0^1(0,1) \cap H^2(0,1).$$

Remark 4.2.1. Notice that if $\sigma = 0$ (i.e. $\overset{\circ}{s} = -1$), boundary conditions (4.7) are purely Dirichlet. In this case the operator L_h is a regular perturbation of the operator L_0. Then the spectrum of L_h converges to the spectrum of L_0. So we assume in the following that $\sigma \neq 0$.

We consider first the operator $L_{h,0}$ defined by the same differential operation as L_0,

$$L_{h,0}\, v = -v'' - \frac{1}{4}\kappa^2 v,$$

with the domain $D(L_{h,0})$ that consists of all $v \in H^2$ satisfying boundary conditions (4.7).

The operators L_0, $L_{h,0}$, L_h are self-adjoint and their spectra are discrete with the only limiting point at infinity. Integration by parts yields that the following bilinear forms correspond to the operators:

$$\begin{aligned}
\Lambda_0(v,w) &= \int_0^1 v'w'ds - \int_0^1 \frac{1}{4}\kappa^2 vwds,\ \ v,w \in H_0^1 = H_0^1(0,1),\\
\Lambda_{h,0}(v,w) &= \int_0^1 v'w'ds - \int_0^1 \frac{1}{4}\,\kappa^2 vwds - \frac{1}{h\sigma}v(0)w(0),\ \ v,w \in H^1_{(0)},\\
\Lambda_h(v,w) &= \int_0^1 e^{h\kappa}v'w'ds - \int_0^1 \frac{1}{4}\,e^{h\kappa}(\kappa^2 + h\kappa'')vwds - \frac{1}{h\sigma}v(0)w(0),\ \ v,w \in H^1_{(0)}
\end{aligned} \tag{4.11}$$

respectively, where $H^1_{(0)}$ is the set of functions from H^1 vanishing at 1, $v(1) = 0$. We use the minimax definition for eigenvalues of an operator L with a bilinear form Λ:

$$\mu^{(k)} := \inf_{\dim W = k} \sup_{v \in W} \frac{\Lambda(v,v)}{\|v\|^2},\ \ k = 1, 2, \ldots, \tag{4.12}$$

where W denotes a subspace of the domain of the bilinear form, and $\|v\|^2$ denotes $\int_0^1 v^2 ds$ for short.

Lemma 4.2.2. *The eigenvalues of* $L_{h,0}$ *and* L_0 *alternate:*

$$\mu_{h,0}^{(k)} < \mu_0^{(k)} < \mu_{h,0}^{(k+1)},\ k = 1, 2, \ldots.$$

If $\sigma < 0$, *then*

$$\lim_{h\to 0} \mu_{h,0}^{(k)} = \mu_0^{(k)},\ k = 1, 2, \ldots.$$

If $\sigma > 0$ *, then*

$$\mu_{h,0}^{(1)} = -\frac{1}{(h\sigma)^2} + O\left(h^m\right), \; \forall m > 0,$$
$$\lim_{h\to 0} \mu_{h,0}^{(k+1)} = \mu_0^{(k)}, \; k = 1, 2, \ldots, \tag{4.13}$$

and the first eigenfunction of $L_{h,0}$ *can be approximated by the exponentially decaying function* f_h *:*

$$\|v_{h,0}^{(1)} - f_h\| = O(h^m), \; \forall m > 0, \tag{4.14}$$

where f_h *is given by*

$$f_h = \frac{\exp(-(h\sigma)^{-1}s)\chi(2s/s_0)}{\|\exp\left(-(h\sigma)^{-1}s\right)\chi(2s/s_0)\|}. \tag{4.15}$$

Proof. Since for an arbitrary function $v \in H_0^1$ we have $\Lambda_0(v,v) = \Lambda_{h,0}(v,v)$, it follows from the minimax principle that

$$\mu_{h,0}^{(k)} \le \mu_0^{(k)}.$$

Let us consider the spectrum of the self-adjoint operator L^γ corresponding to the bilinear form

$$\Lambda^\gamma(v,w) = \int_0^1 v'w'ds - \int_0^1 \frac{1}{4}\kappa^2 vwds + \gamma v(0)w(0)$$

defined on $H_{(0)}^1$, where the parameter $\gamma \in \mathbb{R}^n$. Denote its eigenvalues by $\mu^{(k)}(\gamma)$. We have $\mu^{(k)}(\gamma) = \mu_{h,0}^{(k)}$ (and the equality of the corresponding eigenfunctions) provided that $\gamma = -\frac{1}{h\sigma}$. Since for any fixed v the bilinear form $\Lambda^\gamma(v,v)$ is a continuous non-decreasing function of γ, each eigenvalue $\mu^{(k)}(\gamma)$ is a continuous non-decreasing function of γ as well. Let us fix some $\mu \in \mathbb{R}$. It follows from the theory of ordinary differential equations that if $v \in H_{(0)}^1$, $v \not\equiv 0$, is some solution of the equation

$$-v'' - \frac{1}{4}\kappa^2 v = \mu v, \tag{4.16}$$

then any other solution of (4.16) from $H_{(0)}^1$ is given by Cv, where C is constant. For any $\mu \in \mathbb{R}$ there exists a solution (4.16) from $H_{(0)}^1$, therefore μ is an eigenvalue of either L^γ (for some particular value of γ) or L_0. These observations imply several important consequences. Firstly, each eigenvalue $\mu^{(k)}(\gamma)$ is a continuous strictly increasing function of γ ; secondly,

$$\mu^{(k)}(\gamma_1) < \mu_0^{(k)} < \mu^{(k+1)}(\gamma_2), \ \forall \gamma_1, \gamma_2 \in \mathbb{R}, \ k = 1, 2, \ldots ; \tag{4.17}$$

and, thirdly,

$$\lim_{\gamma \to +\infty} \mu^{(k)}(\gamma) = \mu_0^{(k)} = \lim_{\gamma \to -\infty} \mu^{(k+1)}(\gamma), \, k = 1, 2, \ldots .$$

The statements of the lemma follow immediately except the one concerning $\mu_{h,0}^{(1)}$ and $v_{h,0}^{(1)}$ when $\sigma > 0$.

The function f_h belongs to the domain of $L_{h,0}$ and satisfies the following equation:

$$L_{h,0} f_h = -f_h'' = -(h\sigma)^{-2} f_h + O\big((h\sigma)^{-3/2} \exp(-(h\sigma)^{-1} s_0/2)\big). \tag{4.18}$$

Then it is easy to show in the way absolutely analogous to the proof of error bounds (3.89), (3.90) the validity of the first equality in (4.13) and asymptotics (4.14). □

Remark 4.2.3. Notice that if $\sigma > 0$, then equally to the case of the operator $L_{h,0}$ we have the following asymptotics for the first eigenvalue and the corresponding eigenfunction of L_h :

$$\begin{aligned} L_h f_h = -f_h'' = -(h\sigma)^{-2} f_h + O\left(h^m\right), \\ \mu_h^{(1)} = -\frac{1}{(h\sigma)^2} + O\left(h^m\right), \\ \|v_h^{(1)} - f_h\| = O(h^m), \, \forall m > 0. \end{aligned} \tag{4.19}$$

The next lemma establishes asymptotic proximity of the eigenvalues of L_h and $L_{h,0}$. In turn this will provide the desired result on the convergence of the eigenvalues of L_h to the eigenvalues of L_0 .

Lemma 4.2.4. *The eigenvalues of the operators L_h and $L_{h,0}$ are asymptotically close:*

$$\lim_{h \to 0} |\mu_{h,0}^{(k)} - \mu_h^{(k)}| = 0, \, k = 1, 2, \ldots . \tag{4.20}$$

Proof. Let us consider the difference between the bilinear forms Λ_h and $\Lambda_{h,0}$:

$$|\Lambda_h(v,v) - \Lambda_{h,0}(v,v)| =$$
$$= \left| \int_0^1 (e^{h\kappa} - 1)(v')^2 ds - \int_0^1 \frac{1}{4} (e^{h\kappa}(\kappa^2 + h\kappa'') - \kappa^2) v^2 ds \right| .$$

It follows immediately from (4.11) that if $\sigma < 0$ then

$$|\Lambda_h(v,v) - \Lambda_{h,0}(v,v)| \le h\, C(\Lambda_{h,0}(v,v) + C\|v\|^2), \tag{4.21}$$

where C is some constant independent of h. These estimates allow us to derive the statement of the lemma using the minimax definition of the eigenvalues. Indeed, from (4.21) we obtain

$$\Lambda_h(v,v) \le \Lambda_{h,0}(v,v) + |\Lambda_h(v,v) - \Lambda_{h,0}(v,v)| \le$$
$$\le \Lambda_{h,0}(v,v) + h\, C(\Lambda_{h,0}(v,v) + C\|v\|^2),$$
$$\Lambda_{h,0}(v,v) \le \Lambda_h(v,v) + |\Lambda_h(v,v) - \Lambda_{h,0}(v,v)| \le$$
$$\le \Lambda_h(v,v) + h\, C(\Lambda_{h,0}(v,v) + C\|v\|^2).$$

Then it follows from definition (4.12) that

$$\mu_{h,0}^{(k)} - h\, C(\mu_{h,0}^{(k)} + C) \le \mu_h^{(k)} \le \mu_{h,0}^{(k)} + h\, C(\mu_{h,0}^{(k)} + C).$$

The latter implies (4.20) for the case $\sigma < 0$.

If $\sigma > 0$ the matter is more complicated. In this case we need to employ the following lemma, whose proof we provide later.

Lemma 4.2.5. *Let v be the following linear combination of the eigenfunction of either L_h:*

$$v = \sum_{i=2}^{k} a_i v_h^{(i)},$$

or $L_{h,0}$:

$$v = \sum_{i=2}^{k} a_i v_{h,0}^{(i)}.$$

Then for such v inequality (4.21) is valid.

Notice that f_h is almost orthogonal to the eigenfunctions of L_h and $L_{h,0}$

(clearly except for the first ones):

$$\begin{aligned}\left|\big(f_h, v_h^{(k)}\big)\right| &= \left|\big(f_h - v_h^{(1)}, v_h^{(k)}\big)\right| = O(h^m)\|v_h^{(k)}\|, \ \forall m>0, \ k\geq 2,\\ \left|\big(f_h, v_{h,0}^{(k)}\big)\right| &= \left|\big(f_h - v_{h,0}^{(1)}, v_{h,0}^{(k)}\big)\right| = O(h^m)\|v_{h,0}^{(k)}\|, \ \forall m>0, \ k\geq 2,\end{aligned} \tag{4.22}$$

due to (4.19). Here $(\bullet,\bullet)$ denotes the scalar product in $L^2(0,1)$. Then via (4.19), (4.18) and (4.22) one has

$$\begin{aligned}\Lambda_h(f_h, v) &= (L_h f_h, v) = O(h^m)\|v\|, \ \forall m>0,\\ \Lambda_{h,0}(f_h, v) &= (L_{h,0} f_h, v) = O(h^m)\|v\|, \ \forall m>0,\end{aligned} \tag{4.23}$$

where v is as in Lemma 4.2.5. Notice also the following obvious observation:

$$\begin{aligned}\Lambda_h(f_h, f_h) &= -\frac{1}{h^2\sigma^2} + O(h^m) = \mu_h^{(1)} + O(h^m),\\ \Lambda_{h,0}(f_h, f_h) &= -\frac{1}{h^2\sigma^2} + O(h^m) = \mu_{h,0}^{(1)} + O(h^m), \ \forall m>0.\end{aligned} \tag{4.24}$$

Then from (4.23) and (4.24) we derive

$$\begin{aligned}\Lambda_h(a_1 f_h + v, a_1 f_h + v) &= a_1^2\Lambda_h(f_h, f_h) + 2a_1\Lambda_h(f_h, v) + \Lambda_h(v,v) \leq\\ &\leq \Lambda_h(v,v) + O(h^m)\|a_1 f_h + v\|^2, \ \forall m>0,\end{aligned} \tag{4.25}$$

and analogously

$$\Lambda_{h,0}(a_1 f_h + v, a_1 f_h + v) \leq \Lambda_{h,0}(v,v) + O(h^m)\|a_1 f_h + v\|^2, \ \forall m>0.$$

Let W_k be a subspace of $H^1_{(0)}$ spanned by the functions $f_h, v_{h,0}^{(2)}, \dots, v_{h,0}^{(k)}$ and let W_{k-1} be a subspace of $H^1_{(0)}$ spanned by the functions $v_{h,0}^{(2)}, \dots, v_{h,0}^{(k)}$. Clearly $\dim W_k = k$ since f_h is 'very close' to $v_{h,0}^{(1)}$, see (4.14). Let $v \in W_k$. Then the function v can be presented in the form $v = a_1 f_h + \widetilde{v}$, where $\widetilde{v} \in W_{k-1}$, and $\|v\|^2 = \|a_1 f_h\|^2 + \|\widetilde{v}\|^2 + O(h^m)\|a_1 f_h\|\,\|\widetilde{v}\|, \ \forall m > 0$, due to (4.22). Hence we obtain from (4.25) that

$$\frac{\Lambda_h(v,v)}{\|v\|^2} \leq \frac{\Lambda_h(\widetilde{v},\widetilde{v})}{\|v\|^2} + O(h^m) \leq \frac{\Lambda_h(\widetilde{v},\widetilde{v})}{\|\widetilde{v}\|^2} + O(h^m). \tag{4.26}$$

That is for any $v \in W_k$ there exists $\widetilde{v} \in W_{k-1}$ such that (4.26) holds. (We assumed in the above that $v \neq a_1 f_h$, otherwise the reasoning is trivial.) Then via (4.12), (4.21) and (4.26) we derive for $k \geq 2$ that

$$\mu_h^{(k)} \leq \sup_{v \in W_k} \frac{\Lambda_h(v,v)}{\|v\|^2} \leq \sup_{v \in W_{k-1}} \frac{\Lambda_h(v,v)}{\|v\|^2} + O(h^m) \leq$$

$$\leq \sup_{v \in W_{k-1}} \frac{\Lambda_{h,0}(v,v) + h\,C(\Lambda_{h,0}(v,v) + C\|v\|^2)}{\|v\|^2} + O(h^m) = \tag{4.27}$$

$$= \mu_{h,0}^{(k)}(1 + hC) + O(h).$$

Analogously, let W_k be a subspace of $H^1_{(0)}$ spanned by the functions $f_h, v_h^{(2)}, \ldots, v_h^{(k)}$ and W_{k-1} be a subspace of $H^1_{(0)}$ spanned by the functions $v_h^{(2)}, \ldots, v_h^{(k)}$. Then for $k \geq 2$

$$\mu_{h,0}^{(k)} \leq \sup_{v \in W_k} \frac{\Lambda_{h,0}(v,v)}{\|v\|^2} \leq \sup_{v \in W_{k-1}} \frac{\Lambda_{h,0}(v,v)}{\|v\|^2} + O(h^m) \leq$$

$$\leq \sup_{v \in W_{k-1}} \frac{\Lambda_h(v,v) + h\,C(\Lambda_{h,0}(v,v) + C\|v\|^2)}{\|v\|^2} + O(h^m) = \tag{4.28}$$

$$= \mu_h^{(k)} + hC\mu_{h,0}^{(k)} + O(h).$$

Estimates (4.27) and (4.28) imply the statement of the lemma. □

Proof of Lemma 4.2.5. In order to proof the validity of (4.21) we need to estimate somehow the term $-\frac{1}{h\sigma}v^2(0)$ for the eigenfunctions $v_h^{(k)}$ and $v_{h,0}^{(k)}$, $k = 2, 3, \ldots$. The following reasoning is equally valid for both $v_h^{(k)}$ and $v_{h,0}^{(k)}$, so we give it only for the eigenfunctions of L_h. Since $\kappa(s) = 0$ for $s \in [0, s_0]$,

$$-\frac{d^2}{ds^2} v_h^{(k)} = \mu_h^{(k)} v_h^{(k)}, \;\; s \in [0, s_0]. \tag{4.29}$$

Let $\mu_h^{(k)}$ be positive. Then

$$v_h^{(k)} = a_1 \sin\left(\sqrt{\mu_h^{(k)}}\, s\right) + a_2 \cos\left(\sqrt{\mu_h^{(k)}}\, s\right), \;\; s \in [0, s_0]. \tag{4.30}$$

From boundary conditions (4.7) it follows that

$$a_2 = -h\,\sigma\sqrt{\mu_h^{(k)}}\, a_1. \tag{4.31}$$

The coefficient a_1 is determined by the norm $\|v_h^{(k)}\|$. Indeed,

$$\|v_h^{(k)}\|^2 \geq a_1^2 \int_0^{s_0} \left[\sin^2\left(\sqrt{\mu_h^{(k)}}\, s\right) + h^2\sigma^2\mu_h^{(k)} \cos^2\left(\sqrt{\mu_h^{(k)}}\, s\right) - \right.$$

$$\left. -2h\,\sigma\sqrt{\mu_h^{(k)}} \sin\left(\sqrt{\mu_h^{(k)}}\, s\right) \cos\left(\sqrt{\mu_h^{(k)}}\, s\right)\right] ds =$$

$$= a_1^2 \left[\frac{s_0}{2}\left(1 + h^2\sigma^2\mu_h^{(k)}\right) + \frac{-1 + h^2\sigma^2\mu_h^{(k)}}{4\sqrt{\mu_h^{(k)}}} \sin\left(2\sqrt{\mu_h^{(k)}}\, s_0\right) + \right. \tag{4.32}$$

$$\left. + \frac{h\,\sigma}{2}\left(\cos\left(2\sqrt{\mu_h^{(k)}}\, s_0\right) - 1\right)\right].$$

If the eigenvalue $\mu_h^{(k)}$ is bounded away from zero uniformly with respect to h, $\mu_h^{(k)} > C > 0$, we conclude from (4.32) that

$$\|v_h^{(k)}\|^2 \geq a_1^2 C,$$

for come $C > 0$, and hence

$$|a_1| \leq C\|v_h^{(k)}\|, \tag{4.33}$$

where C independent of h. If for some sequence $h \to 0$ we have $|\mu_h^{(k)}| \to 0$, then by applying Taylor expansion to the right hand side of (4.32) we arrive at

$$\|v_h^{(k)}\|^2 \geq a_1^2 \left[\frac{1}{3}\mu_h^{(k)} s_0^3 + O\left(h\mu_h^{(k)} + \left(\mu_h^{(k)}\right)^2\right)\right].$$

Therefore,

$$|a_1| \leq \frac{C}{\sqrt{\mu_h^{(k)}}}\|v_h^{(k)}\|, \tag{4.34}$$

for small enough h and some constant C independent of h.

At this stage we need to use the following proposition whose proof we provide after the proof of the lemma.

Proposition 4.2.6. *Each eigenvalue $\mu_h^{(k)}$, $k = 2, 3, \dots$ of L_h is bounded uniformly with respect to h, i.e. for any $k = 2, 3, \dots$ there exists a constant C such that*

$$|\mu_h^{(k)}| \leq C. \tag{4.35}$$

We obtain from (4.30), (4.31) and either (4.34) or (4.33) and Proposition 4.2.6

that

$$|v_h^{(k)}(0)| \leq Ch\|v_h^{(k)}\|, \; k = 2, 3, \ldots. \tag{4.36}$$

The case of negative $\mu_h^{(k)}$ is very similar. In this instance

$$v_h^{(k)} = a_1 \exp\left(\sqrt{|\mu_h^{(k)}|}\, s\right) + a_2 \exp\left(-\sqrt{|\mu_h^{(k)}|}\, s\right), \; s \in [0, s_0]. \tag{4.37}$$

From boundary conditions (4.7) it follows that

$$a_1 + a_2 = -h\,\sigma\sqrt{|\mu_h^{(k)}|}\,(a_1 - a_2). \tag{4.38}$$

Denote $b_1 = (a_1 - a_2)/2$ and $b_2 = (a_1 + a_2)/2$. Then

$$\begin{aligned}
\|v_h^{(k)}\|^2 \geq b_1^2 \int\limits_0^{s_0} \Bigg[& \left(1 - h\sqrt{|\mu_h^{(k)}|\sigma}\right)^2 \exp\left(2\sqrt{|\mu_h^{(k)}|}\, s\right) + \\
+ \left(1 + h\sqrt{|\mu_h^{(k)}|\sigma}\right)^2 & \exp\left(-2\sqrt{|\mu_h^{(k)}|}\, s\right) - 2\left(1 - h^2\mu_h^{(k)}\sigma^2\right)\Bigg] ds = \\
= b_1^2 \Bigg[\frac{1}{2\sqrt{|\mu_h^{(k)}|}} & \left(1 - h\sqrt{|\mu_h^{(k)}|\sigma}\right)^2 \exp\left(2\sqrt{|\mu_h^{(k)}|}\, s\right)\Bigg|_0^{s_0} + \\
- \frac{1}{2\sqrt{|\mu_h^{(k)}|}} & \left(1 + h\sqrt{|\mu_h^{(k)}|\sigma}\right)^2 \exp\left(-2\sqrt{|\mu_h^{(k)}|}\, s\right)\Bigg|_0^{s_0} - \\
& -2\left(1 - h^2\mu_h^{(k)}\sigma^2\right) s_0 \Bigg].
\end{aligned} \tag{4.39}$$

If the eigenvalue $\mu_h^{(k)}$ is bounded away from zero uniformly with respect to h, $|\mu_h^{(k)}| > C > 0$, we arrive at

$$|b_1| \leq C\|v_h^{(k)}\|,$$

where C independent of h. Then (4.36) follows from (4.37), (4.38) and Proposition 4.2.6.

If for some sequence $h \to 0$ we have $|\mu_h^{(k)}| \to 0$, then by applying Taylor expansion to the right hand side of (4.39) we arrive at

$$\|v_h^{(k)}\|^2 \geq b_1^2 \left[\frac{4}{3}\left|\mu_h^{(k)}\right| s_0^3 + O\left(h\mu_h^{(k)} + \left(\mu_h^{(k)}\right)^2\right)\right].$$

Therefore,

$$|b_1| \le \frac{C}{\sqrt{|\mu_h^{(k)}|}} \|v_h^{(k)}\|,$$

for small enough h and some constant C independent of h, and (4.36) follows from (4.37), (4.38).

The case $\mu_h^{(k)} = 0$ is trivial. For $s \in [0, s_0]$ we have

$$v_h^{(k)} = a_1 s - h\sigma a_1.$$

Then

$$\|v_h^{(k)}\|^2 \ge \int_0^{s_0} (a_1 s - h\sigma a_1)^2 ds \ge C a_1^2,$$

and (4.36) follows.

Analogously for the eigenfunctions of $L_{h,0}$ we have

$$|v_{h,0}^{(k)}(0)| \le Ch\|v_{h,0}^{(k)}\|, \ k = 2, 3, \dots. \tag{4.40}$$

Remark 4.2.7. Constants in (4.36), (4.40) are uniform in h, however may depend on k.

The crucial estimates (4.36), (4.40) are valid not only for the eigenfunctions of L_h and $L_{h,0}$ but also for their finite linear combinations. Indeed, let

$$v = \sum_{i=2}^{k} a_i v_h^{(i)}. \tag{4.41}$$

Then

$$v^2(0) \le C \sum_{i=2}^{k} a_i^2 \big(v_h^{(i)}(0)\big)^2 \le Ch^2 \sum_{i=2}^{k} a_i^2 \|v_h^{(i)}\|^2 = Ch^2\|v\|^2,$$

and analogously for $v = \sum_{i=2}^{k} a_i v_{h,0}^{(i)}$. From the latter estimate it follows, that in the case of positive σ inequalities (4.21) are valid for such linear combinations of the eigenfunctions. □

Proof of Proposition 4.2.6. Let us introduce two self-adjoint operators $\mathcal{L}_h$, $\mathcal{L}_{h,\gamma}$

via their bilinear forms:

$$\Psi_h(v,w) := \int\limits_0^1 e^{h\kappa} v'w'ds - \int\limits_0^1 \frac{1}{4}\, e^{h\kappa}(\kappa^2 + h\kappa'')vwds,$$

$$\Psi_{h,\gamma}(v,w) := \int\limits_0^1 e^{h\kappa} v'w'ds - \int\limits_0^1 \frac{1}{4}\, e^{h\kappa}(\kappa^2 + h\kappa'')vwds + \gamma v(0)w(0),$$

with the domains

$$D(\Psi_h) = H_0^1,$$
$$D(\Psi_{h,\gamma}) = H_{(0)}^1.$$

The corresponding eigenvalue problems for these operators are given by the equation

$$-\frac{d}{ds}\left(e^{h\kappa}\frac{d}{ds}\right)v - \frac{1}{4}\, e^{h\kappa}(\kappa^2 + h\kappa'')v = \omega_h v,\ s \in (0,1), \tag{4.42}$$

accompanied by Dirichlet boundary conditions for $\mathcal{L}_h$ or the conditions

$$\gamma v(0) - v'(0) = 0,$$
$$v(1) = 0,$$

for $\mathcal{L}_{h,\gamma}$. Analogously to the case of the operators L_γ and L_0 one can show that for every fixed h

$$\omega_h^{(k)}(\gamma) < \omega_h^{(k)} < \omega_h^{(k+1)}(\gamma),\ \ \gamma \in \mathbb{R},\ \ k = 1, 2, \ldots,$$

where $\omega_h^{(k)}$ and $\omega_h^{(k)}(\gamma)$ are the eigenvalues of $\mathcal{L}_h$ and $\mathcal{L}_{h,\gamma}$ arranged in non-decreasing order. In particular, when $\gamma = -\frac{1}{h\sigma}$ (then $\omega_h^{(k)}(\gamma) = \mu_h^{(k)}$), we have

$$\mu_h^{(k)} < \omega_h^{(k)} < \mu_h^{(k+1)},\ \ k = 1, 2, \ldots.$$

The operator $\mathcal{L}_h$ is a regular perturbation of L_0, hence $\omega_h^{(k)} \to \mu_0^{(k)}$. Therefore for each $k \geq 2$ the eigenvalue $\mu_h^{(k)}$ is bounded uniformly with respect to h. $\square$

Combining Lemmas 4.2.2 and 4.2.4 and Remark 4.2.1 we obtain the description of the asymptotic behaviour of the spectrum of L_h.

Theorem 4.2.8. *Let $\sigma \leq 0$, then*

$$\lim_{h\to 0} \mu_h^{(k)} = \mu_0^{(k)},\ k = 1, 2, \ldots.$$

Let $\sigma > 0$, *then*

$$\mu_h^{(1)} = -\frac{1}{(h\sigma)^2} + O\left(h^m\right), \forall m > 0,$$
$$\lim_{h\to 0} \mu_h^{(k+1)} = \mu_0^{(k)},\ k = 1, 2, \ldots.$$

4.3 Approximation of the problem in Ω_h by the limiting problem on graph

Theorem 4.2.8 together with (4.10) imply the following

Theorem 4.3.1. *Let* $\mu_h^{(k)}$ *be the* k*-th eigenvalue of the operator* L_h *such that* $\lim_{h\to 0} \mu_h^{(k)} \neq 0$, $k = 1, 2, \ldots$ *if* $\sigma < 0$, $k = 2, 3, \ldots$ *if* $\sigma > 0$. *Then for* $\lambda_0^{(k)}$ *-* k*-th eigenvalue of problem (3.16), (3.58), and* $\lambda_1^{(k)}$, *given by (3.64) with* $\lambda_0 = \lambda_0^{(k)}$, *we have*

$$|\mu_h^{(k)} - (\lambda_0^{(k)} + h\lambda_1^{(k)})| \leq Ch^2,\ k = 1, 2, \ldots \text{ if } \sigma < 0,$$
$$|\mu_h^{(k)} - (\lambda_0^{(k-1)} + h\lambda_1^{(k-1)})| \leq Ch^2,\ k = 2, 3, \ldots \text{ if } \sigma > 0.$$

Let $v_0^{(k)}$ *be the eigenfunction corresponding to* $\lambda_0^{(k)}$, *and* $w_0^{(k)}$ *be the solution of (3.20) satisfying the conditions described in Section 3.4. Then*

$$\begin{aligned} \|v_h^{(k)} - (v_0^{(k)} + hw_0^{(k)})\|_{L^2(0,1)} \leq Ch^2,\ k = 1, 2, \ldots \text{ if } \sigma < 0, \\ \|v_h^{(k)} - (v_0^{(k-1)} + hw_0^{(k-1)})\|_{L^2(0,1)} \leq Ch^2,\ k = 2, 3, \ldots \text{ if } \sigma > 0. \end{aligned} \tag{4.43}$$

Proof. The first part of the statement of the theorem follows immediately from Theorem 4.2.8 and the first inequality in (4.10) (notice that $\lambda_0^{(k)} = \mu_0^{(k)}$). Theorem 4.2.8 implies also that for any $\lambda_0^{(k)}$ there exists constant d such that for small enough h the d-neighbourhood of $\lambda_0^{(k)} + h\lambda_1^{(k)}$ contains exactly one eigenvalue $\mu_h^{(k)}$. Then the second part of the statement follows from (4.10) and a simple observation that the function $\widetilde{v}_h^{(k)} = v_0^{(k)} + hw_0^{(k)}$ differs from $\widehat{v}_h^{(k)}$ only by the term of order h^2. □

Remark 4.3.2. The limiting operator L_h corresponds to the non-critical case. The critical case is different from the non-critical one. The reason is that in the critical case the limiting operator on the graph must include a spectral parameter in the boundary condition. Indeed, for example if λ_0 is negative, via (4.6) and (3.70) we have

$$-h\lambda_0\sigma\widetilde{v}_h(0) + \widetilde{v}_h{}'(0) = -h^2 2\lambda_0^{3/2}\sigma^2 C_1.$$

The study of this problem goes beyond the scope of the present thesis.

Now we can prove that the limiting problem on the graph gives the correct asymptotics of the problem in the thin domain Ω. Indeed, it is easy to see via (4.43) that

$$\|(u_0 + hu_1) - (\varphi_0 + h\kappa\varphi_1)v_h\|_{L^2(\Omega_h)} \leq Ch^{5/2}.$$

Then the assertion follows from Theorem 3.5.1 and Theorem 4.3.1 immediately:

Theorem 4.3.3. *Let $\mu_h^{(k)}$ be as in Theorem 4.3.1 and $v_h^{(k)}$ be the corresponding normalised eigenfunction. Then there exists $h_0 > 0$ and a constant C independent of h such that for any $0 < h \leq h_0$ there exist an eigenvalue $\lambda_{i,h}$ of the operator A_h such that*

$$|\lambda_{i,h} - (h^{-2}\lambda_{-2} + \mu_h^{(k)})| \leq Ch^{3/2}. \tag{4.44}$$

Moreover, the function $(\varphi_0 + h\kappa\varphi_1)v_h^{(k)}$ approximates eigenfunctions $u_{i,h}$ of the operator A_h outside a small neighbourhood of the origin. Namely, let Θ_h be a characteristic function of the set $(0, 2h^{1/3})^2$, then for any $d > 0$ and any $0 < h \leq h_0$ there exist coefficients $c_i(h)$ such that

$$\left\| \left(u_h^{appr} - \sum_{|\lambda_{i,h} - (h^{-2}\lambda_{-2} + \mu_h)| \leq d} c_i(h) u_{i,h} \right) (1 - \Theta_h) \right\|_{L^2(\Omega_h)} \leq Cd^{-1}h^2.$$

Remark 4.3.4. In the last two chapters we never essentially used the assumption about the boundary conditions imposed on the slanted end γ_1 of the domain Ω_h. All the arguments and results obtained are equally valid for the eigenvalue problem

$$\begin{aligned} -\Delta u_h &= \lambda_h u_h, \quad x \in \Omega_h, \\ u &= 0, \quad x \in \partial\Omega_h, \end{aligned} \tag{4.45}$$

except that in this setting the critical case is not possible. Let u_h^N be the eigenfunction of problem (3.2) and u_h^D be the eigenfunction of problem (4.45). Then a symmetric extension of u_h^N into $\widehat{\Omega}_h$ (see Introduction and Figure 0-4 in particular), and antisymmetric extension of u_h^D into $\widehat{\Omega}_h$ are eigenfunctions of the problem

$$\begin{aligned} -\Delta u_h &= \lambda_h u_h, \quad x \in \widehat{\Omega}_h, \\ u &= 0, \quad x \in \partial\widehat{\Omega}_h. \end{aligned} \tag{4.46}$$

Denote by $v_h^N(s)$ and $v_h^D(s)$ the eigenfunction of the limiting operators for problems (3.2) and (4.45) respectively. They satisfy the following boundary conditions at the vertex:

$$v_h^N(0) + h\,\sigma^N \frac{d}{ds} v^N(0) = 0,$$
$$v_h^D(0) + h\,\sigma^D \frac{d}{ds} v^D(0) = 0.$$

(In particular $\sigma^N = \sigma$ as is in (3.55), and σ^D is given by the same formula with $\overset{\circ}{s}$ corresponding to the Dirichlet case.)

The limiting problem for (4.46) is posed on the graph consisting of two edges adjacent in the single vertex. Let $V(s) = (v_1(s), v_2(s))^t$ be the vector of representatives of an eigenfunction of the limiting problem along the edges of the graph. Then either

$$V(s) = (v_h^N(s), v_h^N(s))^t,$$

or

$$V(s) = (v_h^D(s), -v_h^D(s))^t.$$

It is easy to see then that the following boundary conditions a the vertex must be imposed:

$$V(0) + hTV'(0) = 0,$$

where

$$T = \frac{1}{2}\begin{pmatrix} \sigma^N + \sigma^D & \sigma^N - \sigma^D \\ \sigma^N - \sigma^D & \sigma^N + \sigma^D \end{pmatrix}.$$

4.4 Explicit example for zero-curvature case

Let $\kappa \equiv 0$. Then the eigenvalue problem for the operator L_h takes the form

$$-\frac{d^2}{ds^2} v_h^{(k)} = \mu_h^{(k)} v_h^{(k)}, \ s \in (0,1),$$
$$v_h^{(k)}(0) + h\sigma \frac{d}{ds} v_h^{(k)}(0) = 0,$$
$$v_h^{(k)}(1) = 0.$$

The eigenvalues $\mu_h^{(k)}$ (except the first eigenvalue in the case $\sigma > 0$ which we do not consider here) converge to the eigenvalues of the problem

$$-\frac{d^2}{ds^2} v_0^{(k)} = \mu_0^{(k)} v_0^{(k)}, \; s \in (0,1),$$
$$v_0^{(k)}(0) = v_0^{(k)}(1) = 0,$$

as stated in Theorem 4.2.8. The solutions to the latter are elementary:

$$\mu_0^{(k)} = k^2\pi^2, \; k = 1, 2, \ldots,$$
$$v_0^{(k)} = \sin(k\pi s).$$

It is easy to see that the eigenfunctions of L_h are given by

$$v_h^{(k)} = \sin\left(\sqrt{\mu_h^{(k)}}(1-s)\right),$$

where $\mu_h^{(k)}$ can be found from the boundary condition. Namely, the eigenvalues $\mu_h^{(k)}$ are all the solutions of the equation

$$\tan\left(\sqrt{\mu_h^{(k)}}\right) = h\sigma\sqrt{\mu_h^{(k)}}.$$

Asymptotically one has

$$\mu_h^{(k)} = k^2\pi^2 + h\,2\,\sigma\,k^2\pi^2 + O(h^2), \; k = 1, 2, \ldots, \; \sigma < 0,$$
$$\mu_h^{(k+1)} = k^2\pi^2 + h\,2\,\sigma\,k^2\pi^2 + O(h^2), \; k = 1, 2, \ldots, \; \sigma > 0.$$

These perfectly agree with the results of Chapter 3 that give

$$v_0 = c v_h^{(k)}$$
$$\lambda_0 = k^2\pi^2,$$
$$\lambda_1 = 2\sigma k^2\pi^2,$$

for some $k = 1, 2, \ldots,$ and some coefficient c.

From Theorem 4.2.8 it follows that for any $k = 1, 2, \ldots,$ there exist an eigenvalue λ_h of the operator A_h in Ω_h such that

$$\lambda_h = h^{-2}\pi^2 + k^2\pi^2 + h\,2\,\sigma\,k^2\pi^2 + O(h^{3/2}).$$

Chapter 5

Localisation effects and the bottom of the spectrum

In the above two chapters we studied the eigenvalues of problem (3.2) generated by the first transversal eigenvalue π^2. Now we will address the behaviour of the bottom of the spectrum of A_h. The first eigenvalues of A_h correspond to the so-called bound states, i.e. eigenvalues of the operator A_α (see (3.24)) with the corresponding eigenfunctions localised near the end of the semi-infinite strip and exponentially decaying at infinity. The related eigenfunctions of A_h demonstrate the same behaviour: they are confined to the slanted end of Ω_h and decay at the rate of order $\exp(-h^{-1} l\, x_1)$, where $l > 0$ is some fixed number. Thus, our purpose is to study the point spectrum of the operator A_α lying below its essential spectrum $[\pi^2, \infty)$. In the first section we develop some methods for estimation of the number of bound states. The second section is devoted to the proof of the monotonicity of the first bound state as a function of the angle of the slant.

5.1 Boundary localisation

Theorem 5.1.1. *There exists at least one eigenvalue λ of the operator A_α such that $\lambda < \pi^2$, the corresponding eigenfunction $u(y)$ decays exponentially at infinity ($u(y) \sim \exp(-l\, y_1)$ as $y_1 \to \infty$, where l is some positive number). Then there exists $h_0 > 0$ and a constant C independent of h such that for any $0 < h \le h_0$ there exist an eigenvalue λ_h of the operator A_h such that*

$$|\lambda_h - h^{-2}\lambda| \le C e^{-lh^{-1}}.$$

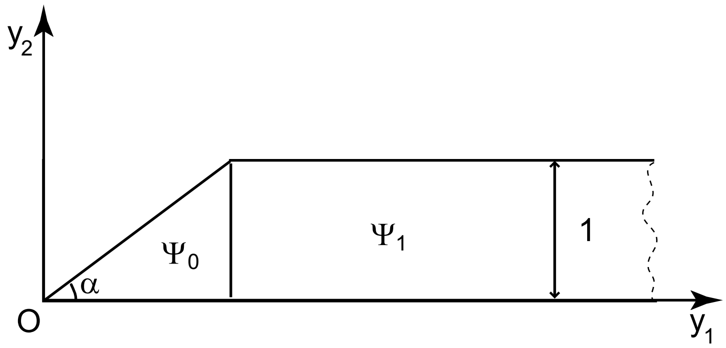

Figure 5-1: Semi-infinite strip

Moreover, a function

$$u_h^{appr} = \chi(2x_1/s_0)u(h^{-1}x)$$

approximates eigenfunctions of A_h *in the following sense: for any* $d > 0$ *and any* $0 < h \leq h_0$ *there exist coefficients* $c_i(h)$ *such that*

$$\left\| u_h^{appr} - \sum_{|\lambda_{i,h} - h^{-2}\lambda| \leq d} c_i(h) u_{i,h} \right\|_{L^2(\Omega_h)} \leq C d^{-1} e^{-lh^{-1}}.$$

Proof. It is well known, see e.g. [69], that there exists at least one eigenvalue of A_α below its essential spectrum and the corresponding eigenfunction decays exponentially at infinity. The rest of the proof is analogous to the proof of Theorem 3.5.1 although is much simpler. One only needs to notice that the discrepancy $-\Delta_x u_h^{appr} - h^{-2}\lambda u_h^{appr}$ is of order $e^{-lh^{-1}}$. □

In view of Theorem 5.1.1, it is important to describe somehow the discrete spectrum of A_α. Our aim is to study the relation between the value of angle α and the behaviour of the discrete spectrum of A_α. For this purpose we will use methods from [62, Chapter IV]. Let

$$\begin{aligned} \Psi_0 &:= \{(y_1, y_2) \,|\, 0 < y_1 < \cot(\alpha), 0 < y_2 < \tan(\alpha)\, y_1\}, \\ \Psi_1 &:= \{(y_1, y_2) \,|\, y_1 > \cot(\alpha), y_2 \in (0, 1)\}. \end{aligned} \tag{5.1}$$

Denote by T_0 a self-adjoint operator for the negative Laplacian $-\Delta$ in Ψ_0 with Dirichlet boundary condition on $y_2 = 0$ and Neumann boundary condition on the rest of the boundary of Ψ_0. Analogously, we denote by T_1 a self-adjoint operator for $-\Delta$ in Ψ_1 with Dirichlet boundary condition on $y_2 = 0$, $y_2 = \pi$ and Neumann boundary condition on the rest of the boundary of Ψ_1. Denote by

D_0 and D_1 spaces of functions from $H^1(\Psi_0)$ and $H^1(\Psi_1)$ vanishing on $y_2 = 0$ and $y_2 = 0$, $y_2 = \pi$ respectively. It is well known that $\sigma(T_1) = [\lambda_{-2}, \infty)$. The following lemma provides a tool of estimating the number of eigenvalues of A_α from above (cf. [62]).

Lemma 5.1.2. *The number of eigenvalues of the operator A_α lying below its essential spectrum can be estimated from above by the number of eigenvalues of T_0 lying below $\sigma_{\mathrm{ess}}(A_\alpha)$.*

Proof. Let $\gamma_1 \le \gamma_2 \le \ldots \le \gamma_k < \pi^2$ and $\omega_1 \le \omega_2 \le \ldots \le \omega_n < \pi^2$ be all eigenvalues of A_α and T_0 respectively and $v_1, \ldots, v_k$, $\psi_1, \ldots, \psi_n$ be the corresponding orthonormalised eigenfunctions. Our aim is to prove that

$$k \le n. \tag{5.2}$$

Assume the contrary, i.e. that $k > n$. Then there exists a linear combination of eigenfunctions of A_α

$$v = \sum_{i=1}^{k} c_i v_i,$$

such that

$$\int_{\Psi_0} v\,\psi_j\,dx = 0,\ j = 1, \ldots, n.$$

For each eigenfunction of A_α we have

$$\|\nabla v_i\|^2_{L^2(\Pi_\alpha)} = \gamma_i \|v_i\|^2_{L^2(\Pi_\alpha)} = \gamma_i.$$

Then we obtain the following inequality for v:

$$\begin{gathered}\|\nabla v\|^2_{L^2(\Pi_\alpha)} = \sum_{i=1}^{k} c_i^2 \|\nabla v_i\|^2_{L^2(\Pi_\alpha)} = \\ = \sum_{i=1}^{k} c_i^2 \gamma_i \|v_i\|^2_{L^2(\Pi_\alpha)} \le \gamma_k \|v\|^2_{L^2(\Pi_\alpha)} < \pi^2 \|v\|^2_{L^2(\Pi_\alpha)}.\end{gathered} \tag{5.3}$$

The lower bound of the spectrum of T_1 is π^2, so by the variational principle

$$\|\nabla v\|^2_{L^2(\Psi_1)} \ge \pi^2 \|v\|^2_{L^2(\Psi_1)}.$$

Deducting the last inequality from (5.3) we obtain

$$\|\nabla v\|^2_{L^2(\Psi_0)} < \pi^2 \|v\|^2_{L^2(\Psi_0)}.$$

Then, since $v \in D_0$, one should have

$$\inf_{\substack{D_0 \ni \psi \perp \psi_j \\ j=1,\dots,n}} \frac{\|\nabla \psi\|^2_{L^2(\Psi_0)}}{\|\psi\|^2_{L^2(\Psi_0)}} \le \frac{\|\nabla v\|^2_{L^2(\Psi_0)}}{\|v\|^2_{L^2(\Psi_0)}} < \pi^2,$$

which is impossible because there are only n eigenvalues of T_0 less than π^2. We arrive at a contradiction. Hence $k \le n$. □

Theorem 5.1.3. *If $\alpha = \pi/4$ the operator A_α has exactly one eigenvalue lying below its essential spectrum.*

Proof. Let α be equal to $\pi/4$, so that Ψ_0 is an isosceles right-angled triangle. Let ψ be a solution to the eigenvalue problem

$$\begin{aligned} -\Delta\psi = \omega\psi,\ & y \in \Psi_0, \\ \psi = 0,\ & y_2 = 0; \\ \frac{\partial \psi}{\partial \nu} = 0,\ & y_1 = 1 \text{ or } y_2 = y_1. \end{aligned} \tag{5.4}$$

We can extend ψ to the unit square $(0,1)^2$ symmetrically reflecting it with respect to the line $y_1 = y_2$, i.e. by the formula

$$\text{for any } (y_1, y_2) \in (0,1)^2 \backslash \overline{\Pi}_0,\ \ \psi(y_1,y_2) = \psi(y_2,y_1),\ \ (y_2,y_1) \in \Psi_0.$$

Then, by the symmetry principal for the Laplacian, the extended function, which we still denote by ψ, is a solution to the eigenvalue problem

$$\begin{aligned} -\Delta\psi = \omega\psi,\ & y \in (0,1)^2, \\ \psi = 0,\ & y_1 = 0 \text{ or } y_2 = 0, \\ \frac{\partial \psi}{\partial \nu} = 0,\ & y_1 = 1 \text{ or } y_2 = 1. \end{aligned} \tag{5.5}$$

So we can seek the solution to (5.4) amongst the eigenfunctions of the operator given by (5.5). By separation of variables, all the eigenfunctions and correspond-

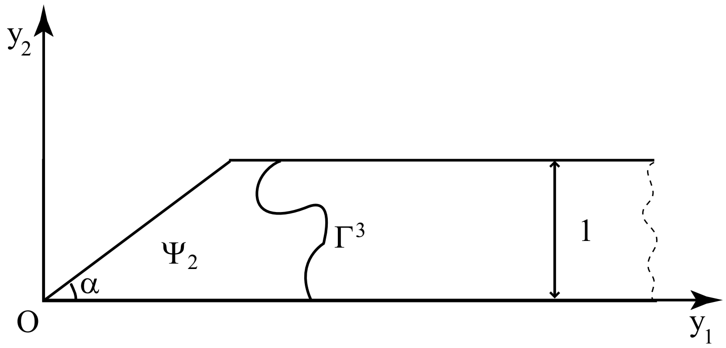

Figure 5-2: Semi-infinite cylinder

ing eigenvalues of (5.5) are the following:

$$\psi_{n,m} = \sin\left(\pi(2n-1)y_1/2\right)\sin\left(\pi(2m-1)y_2/2\right),$$

$$\omega_{n,m} = \frac{\pi^2}{4}\left((2n-1)^2+(2m-1)^2\right),\ n,m\in\mathbb{N}.$$

It is easy to see that $\omega_1 = \omega_{1,1} = \frac{\pi^2}{2}$ and $\psi_1 = \psi_{1,1}$ are eigenvalue and eigenfunction of (5.4). The next eigenvalue of the problem ω_2 is greater that π^2 (since $\omega_2 \geq \omega_{1,2} = \omega_{2,1} = \frac{5}{4}\pi^2$). Then the theorem follows from Theorem 5.1.1 and Lemma 5.1.2. □

Now, when we have a method of estimating the number of eigenvalues A_α from above, we would like to know how to estimate it from below. This can be done in somewhat similar way. Let Ψ_2 be a bounded domain contained between the lines $y_2 = 0$, $y_2 = 1$, $y_2 = \tan(\alpha)\, y_1$ and some smooth simple curve Γ^3 lying entirely in $\overline{\Pi}_\alpha$ such that its one end belongs to $\{y_1 \geq 0,\ y_2 = 0\}$, another belongs to $\{y_1 \geq \cot(\alpha),\ y_2 = 1\}$ and it has no other common points with $\partial\Pi_\alpha$, see Figure 5-2. Let T_2 be an operator defined by $-\Delta$ in Ψ_2 with Dirichlet boundary conditions imposed on $\partial\Psi_2 \cap \Gamma^2_\alpha \cup \Gamma^3$ and Neumann conditions on the rest of the boundary (i.e. on Γ^1_α). Denote by D_2 the space of functions from $H^1(\Psi_2)$ vanishing on $\partial\Psi_2 \cap \Gamma^2_\alpha \cup \Gamma^3$. Then the following assertion is true (cf. [62]).

Lemma 5.1.4. *The number of eigenvalues of the operator A_α lying below its essential spectrum can be estimated from below by the number of eigenvalues of T_2 lying below $\sigma_{\rm ess}(A_\alpha)$.*

Proof. Denote all the eigenvalues lying below $\sigma_{\rm ess}(A_\alpha)$ and the corresponding eigenfunctions of T_2 by μ_i and φ_i, $i = 1,\ldots,m$, respectively, $\mu_1 \leq \ldots \leq \mu_m <$

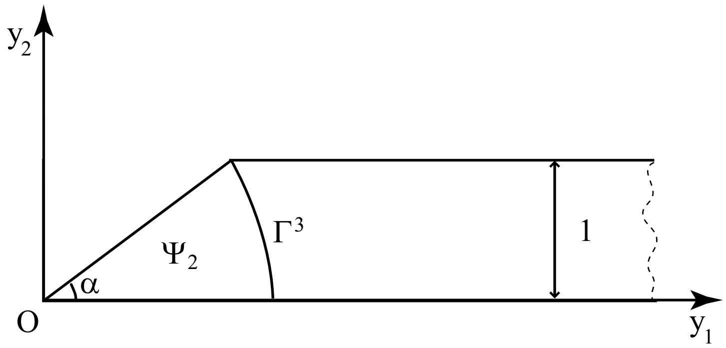

Figure 5-3: Semi-infinite cylinder

π^2. Let k be the number of eigenvalues of A_α as in Lemma 5.1.2. Assume to the contrary that $m > k$. Then there exists a linear combination of eigenfunctions of T_2,

$$\varphi = \sum_{i=1}^{m} c_i \varphi_i,$$

orthogonal to all $v_i,\ i = 1, \dots, k$. Notice that due to the boundary conditions for φ we can extend in by zero to the rest of Π_α, so that the extension $\varphi \in H_0^1(\Pi_\alpha, \Gamma_\alpha^2)$, where $H_0^1(\Pi_\alpha, \Gamma_\alpha^2)$ denotes the space of functions from $H^1(\Pi_\alpha)$ vanishing on Γ_α^2, see (3.24). Then we have

$$\inf_{\substack{H_0^1(\Pi_\alpha, \Gamma_\alpha^2) \ni v \perp v_j \\ j=1,\dots,k}} \frac{\|\nabla v\|^2_{L^2(\Pi_\alpha)}}{\|v\|^2_{L^2(\Pi_\alpha 0)}} \leq \frac{\|\nabla \varphi\|^2_{L^2(\Pi_\alpha)}}{\|\varphi\|^2_{L^2(\Pi_\alpha)}} \leq \mu_m < \pi^2,$$

which means existence of $k+1$ eigenvalues of A_α below its essential spectrum. We obtain a contradiction. □

Remark 5.1.5. The argument of Lemmas 5.1.2 and 5.1.4 is known as Dirichlet-Neumann bracketing.

Suppose now that Ψ_2 is a sector given in the polar coordinates by

$$\Psi_2 := \left\{ (\rho, \theta) | 0 < \rho < (\sin(\alpha))^{-1},\ 0 < \theta < \alpha \right\},$$

see Figure 5-3. Let us consider an eigenvalue problem

$$\begin{gathered} -\Delta \varphi = \mu \varphi,\ y \in \Psi_2, \\ \varphi = 0,\ \theta = 0 \text{ or } \rho = (\sin(\alpha))^{-1}, \\ \frac{\partial \varphi}{\partial \theta} = 0,\ \theta = \alpha. \end{gathered} \tag{5.6}$$

The Laplace operator in the polar coordinates is given by the formula

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}.$$

We can separate the variables in the equation, let $\varphi = u(\rho)v(\theta)$. Then we obtain the following equations for v and u:

$$-v'' = \nu^2 v,$$

$$v(0) = 0,\ v'(\alpha) = 0;$$

$$\begin{gathered} u'' + \frac{1}{\rho}u' + \left(\mu - \frac{\nu^2}{\rho^2}\right) u = 0, \\ u(0) = u\left((\sin(\alpha))^{-1}\right) = 0. \end{gathered} \tag{5.7}$$

Obviously, the sequence of eigenfunctions and eigenvalues satisfying the first equation is given by

$$v_i = \sin\left(\frac{\pi}{2\alpha}(2i-1)\theta\right),\ i = 1, 2, \dots,$$

$$\nu_i = \frac{\pi}{2\alpha}(2i-1).$$

Making the substitution $r = \sqrt{\mu}\rho$, $u(\rho) = \widetilde{u}(r)$ in the second equation one arrives at Bessel equation

$$\widetilde{u}'' + \frac{1}{r}\widetilde{u}' + \left(1 - \frac{\nu_i^2}{r^2}\right)\widetilde{u} = 0.$$

The solution to this equation is given by Bessel function

$$J_{\nu_i}(r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu_i + k + 1)} \left(\frac{r}{2}\right)^{\nu_i + 2k}.$$

Denote by $j_{\nu_i,l},\ l, j = 1, 2, \dots$ zeros of Bessel function J_{ν_i}. In order to satisfy boundary conditions in (5.7) we need to make a substitution $r = \sqrt{\mu_{i,l}}\rho$ where

$$\mu_{i,l} = \sin^2(\alpha)\, j_{\nu_i,l}^2,\ i, l = 1, 2, \dots. \tag{5.8}$$

Then the function

$$u_{i,l} = J_{\nu_i}(\sqrt{\mu_{i,l}}\rho)$$

Solves problem (5.7) with $\mu = \mu_{i,l}$. Then the solution to eigenvalue problem (5.6) is given by the sequence of eigenvalues (5.8) and corresponding eigenfunctions

$$\psi_{i,l} = \sin\left(\frac{\pi}{2\alpha}(2i-1)\theta\right) J_{\nu_i}(\sqrt{\mu_{i,l}}\rho),\ i,l = 1,2,\dots.$$

Thus we have proved the following

Theorem 5.1.6. *The number of eigenvalues of A_α lying below its essential spectrum is greater than or equal to the number of those eigenvalues*

$$\mu_{i,l} = \sin^2(\alpha)\, j^2_{\nu_i,l},\ i,l = 1,2,\dots.$$

of problem (5.6) which are less than π^2, where $j_{\nu_i,1} < j_{\nu_i,2} < \dots$ are zeros of Bessel function J_{ν_i} and

$$\nu_i = \frac{\pi}{2\alpha}(2i-1).$$

Notice that for the first zero of Bessel function J_ν the following inequality is valid:

$$j_{\nu,1} > \sqrt{\nu(\nu+2)},$$

see e.g. [81]. Hence we have

$$\mu_{i,1} > \sin^2(\alpha)\left(\frac{(2i-1)^2}{4\alpha^2}\pi^2 + \frac{2i-1}{\alpha}\pi\right).$$

The function $\sin(\alpha)/\alpha$ is a strictly decreasing function on the interval $\alpha \in (0,\pi/2)$. Then it is easy to see by direct calculations that

$$\mu_{i,1} > \pi^2,\ i \geq 2.$$

On the other hand the asymptotic formula for zeros of Bessel function J_ν for large values of ν,

$$j_{\nu,l} = \nu + o(\nu),\ l \in \mathbb{N},$$

see e.g. [51, 65], ensures that for small enough $\alpha > 0$ there are arbitrary many eigenvalues of problem (5.6) lying below π^2. More precisely we have

$$\mu_{1,l} = \frac{\pi^2}{4} + o(1),\ l \in \mathbb{N},$$

as $\alpha \to 0$. Then the following assertion follows from Theorem 5.1.6.

Theorem 5.1.7. *For small enough* $\alpha > 0$ *there are arbitrary many eigenvalues of* A_α *lying below its essential spectrum.*

5.2 Monotonicity of the first eigenvalue with respect to the angle α

It is clear that the first eigenvalue of problem (3.24) is simple. It is interesting question how it depends on the value of the angle α. Bearing in mind the results of the previous section it is natural to conjecture that the first eigenvalue decays as α gets smaller.

Theorem 5.2.1. *Let* λ_α *be the first eigenvalue of* A_α. *Then* λ_α *is a strictly increasing function of the argument* $\alpha \in (0, \pi/2)$.

Proof. First let us consider eigenvalue problem (3.24) in the sequence of domains Π_{α_i}, $i = 1, \ldots, n$, corresponding to angles α_i, where $0 < \alpha_1 < \ldots < \alpha_n \leq \pi/2$, $\alpha_i = \alpha_1 + (i-1)\Delta\alpha$, $\Delta\alpha > 0$, $i = 2, \ldots, n$, and n is such that $\alpha_{n+1} > \pi/2$. Denote by $\lambda_{\alpha_i} < \pi^2$ and φ_{α_i}, first eigenvalues and respective eigenfunctions of the corresponding problems. A position of a domain on the coordinate plane is insignificant, therefore for the sake of simplified and more illustrative narration we shift the domains Π_{α_i} by $\cot(\alpha_1) - \cot(\alpha_i)$ in positive direction of axis x so as the top vertices of Π_{α_1} and Π_{α_i} to coincide (in the point C), see Figure 5-4. Points B and D on the Figure 5-4 are located such that triangles Δ_{ABC} and Δ_{ADC} are equal with sides $AB = AD$ and $BC = CD$. We denote the domains enclosed within Δ_{ABC} and Δ_{ADC} by Σ_1 and Σ_2 respectively.

The following equality is true:

$$\|\nabla \varphi_{\alpha_2}\|^2_{L^2(\Pi_{\alpha_2})} = \lambda_{\alpha_2} \|\varphi_{\alpha_2}\|^2_{L^2(\Pi_{\alpha_2})}. \tag{5.9}$$

By the variational properties of eigenvalues

$$\lambda_{\alpha_1} = \inf_{f \in H^1_0(\Pi_{\alpha_1}; \Gamma^2_{\alpha_1})} \frac{\|\nabla f\|^2_{L^2(\Pi_{\alpha_1})}}{\|f\|^2_{L^2(\Pi_{\alpha_1})}}, \tag{5.10}$$

Let us extend the eigenfunction φ_{α_2} into Σ_1 by symmetric reflection against the line AC and by zero into Δ_{OAB}. The extension, which we denote by $\widetilde{\varphi}_{\alpha_2}$, belongs to $H^1_0(\Pi_{\alpha_1}; \Gamma^2_{\alpha_1})$ due to the boundary conditions imposed on φ_{α_2}. Thus

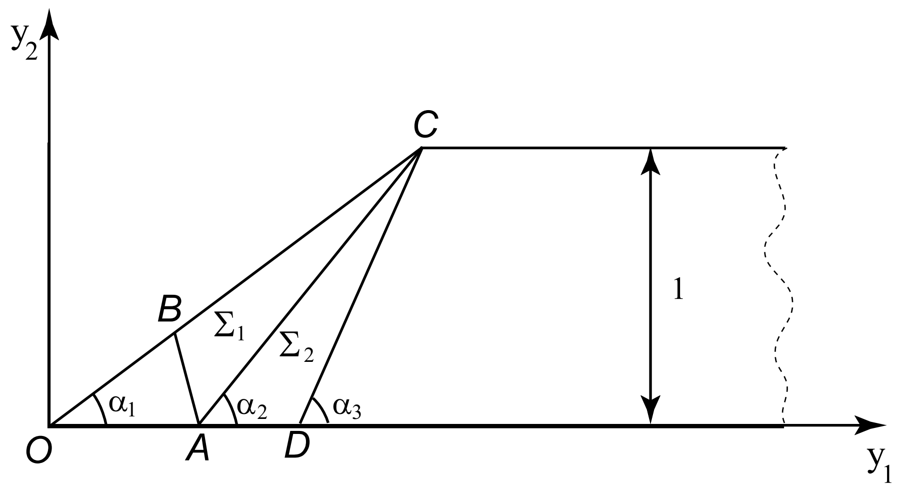

Figure 5-4:

we have

$$\lambda_{\alpha_1} \leq \frac{\|\nabla \widetilde{\varphi}_{\alpha_2}\|^2_{L^2(\Pi_{\alpha_1})}}{\|\widetilde{\varphi}_{\alpha_2}\|^2_{L^2(\Pi_{\alpha_1})}}. \tag{5.11}$$

Notice that from the symmetry of $\widetilde{\varphi}_{\alpha_2}$ against the line AC follows that $\|\nabla \widetilde{\varphi}_{\alpha_2}\|_{L^2(\Sigma_1)} = \|\nabla \varphi_{\alpha_2}\|_{L^2(\Sigma_2)}$ and $\|\widetilde{\varphi}_{\alpha_2}\|_{L^2(\Sigma_1)} = \|\varphi_{\alpha_2}\|_{L^2(\Sigma_2)}$. Then we can write (5.11) as

$$\|\nabla \varphi_{\alpha_2}\|^2_{L^2(\Pi_{\alpha_2})} + \|\nabla \varphi_{\alpha_2}\|^2_{L^2(\Sigma_2)} \geq \lambda_{\alpha_1}\left(\|\varphi_{\alpha_2}\|^2_{L^2(\Pi_{\alpha_2})} + \|\varphi_{\alpha_2}\|^2_{L^2(\Sigma_2)}\right). \tag{5.12}$$

Deducting (5.9) from (5.12) we obtain

$$\|\nabla \varphi_{\alpha_2}\|^2_{L^2(\Sigma_2)} \geq (\lambda_{\alpha_1} - \lambda_{\alpha_2})\|\varphi_{\alpha_2}\|^2_{L^2(\Pi_{\alpha_2})} + \lambda_{\alpha_1}\|\varphi_{\alpha_2}\|^2_{L^2(\Sigma_2)}.$$

Now we deduct the latter from (5.9):

$$\begin{aligned} \|\nabla \varphi_{\alpha_2}\|^2_{L^2(\Pi_{\alpha_3})} &= \|\nabla \varphi_{\alpha_2}\|^2_{L^2(\Pi_{\alpha_2})} - \|\nabla \varphi_{\alpha_2}\|^2_{L^2(\Sigma_2)} < \\ &< (2\lambda_{\alpha_2} - \lambda_{\alpha_1})\|\varphi_{\alpha_2}\|^2_{L^2(\Pi_{\alpha_3})} + 2(\lambda_{\alpha_2} - \lambda_{\alpha_1})\|\varphi_{\alpha_2}\|^2_{L^2(\Sigma_2)}. \end{aligned} \tag{5.13}$$

Assume that $\lambda_{\alpha_1} \geq \lambda_{\alpha_2}$ and denote $\Delta\lambda = \lambda_{\alpha_1} - \lambda_{\alpha_2} \geq 0$. Since $\varphi_{\alpha_2} \in H^1_0(\Pi_{\alpha_3}; \Gamma^2_{\alpha_3})$ we conclude from (5.13) that

$$\lambda_{\alpha_3} = \inf_{f \in H^1_0(\Pi_{\alpha_3}; \Gamma^2_{\alpha_3})} \frac{\|\nabla f\|^2_{L^2(\Pi_{\alpha_3})}}{\|f\|^2_{L^2(\Pi_{\alpha_3})}} \leq \frac{\|\nabla \varphi_{\alpha_2}\|^2_{L^2(\Pi_{\alpha_3})}}{\|\varphi_{\alpha_2}\|^2_{L^2(\Pi_{\alpha_3})}} \leq \lambda_{\alpha_2} - \Delta\lambda.$$

Then we obtain $\lambda_{\alpha_2} - \lambda_{\alpha_3} \geq \Delta\lambda = \lambda_{\alpha_1} - \lambda_{\alpha_2}$. Reasoning by induction we derive that

$$\lambda_{\alpha_n} \leq \lambda_{\alpha_1} - (n-1)\Delta\lambda < \pi^2. \tag{5.14}$$

Now let us prove the monotonicity property of λ_α. We aim to show that the first eigenvalue λ_α of (3.24) is strictly increasing function of the argument $\alpha \in (0, \pi/2)$. Reasoning by contradiction we assume that there exist α' and α'' from $(0, \pi/2)$ such that $\alpha' < \alpha'' < \pi/2$ and $\lambda_{\alpha'} \geq \lambda_{\alpha''}$. It is obvious that in this case one can choose $\alpha \in [\alpha', \alpha'']$ such that for any $\delta > 0$ there exists $0 < \Delta\alpha < \delta$ satisfying $\lambda_\alpha \geq \lambda_{\alpha+\Delta\alpha}$ Then by (5.14) we have

$$\lambda_\gamma < \lambda_\alpha < \pi^2, \tag{5.15}$$

where the angle $\gamma = \alpha + n\Delta\alpha$ for some n such that $\gamma \leq \pi/2$, $|\gamma - \pi/2| < \Delta\alpha$. So γ can be chosen arbitrary close to $\pi/2$.

On the other hand the eigenvalue λ_γ must be close to the bottom of the essential spectrum of problem (3.24) when the angle γ is close to $\pi/2$. Indeed, one can show this using Poincaré inequality. Let us introduce a new coordinates obtained from (y_1, y_2) by rotation on the angle $-(\pi/2 - \gamma)$:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} \cos(\pi/2-\gamma) & -\sin(\pi/2-\gamma) \\ \sin(\pi/2-\gamma) & \cos(\pi/2-\gamma) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

so that the part of the boundary Γ_γ^1 lies on the positive part of the axis y_2'. We denote by Π_γ' the domain Π_γ in the new coordinates:

$$\Pi_\gamma' = \left\{ (y_1', y_2') \,\middle|\, y_1' > 0, y_1' \tan\left(\frac{\pi}{2} - \gamma\right) < y_2' < y_1' \tan\left(\frac{\pi}{2} - \gamma\right) + \frac{1}{\sin(\gamma)} \right\}.$$

For a function $\varphi \in H_0^1(0, 1/\sin(\gamma))$ we have well known Poincaré inequality with an explicit constant,

$$\int_0^{1/\sin(\gamma)} (\varphi(y_2'))^2 \, dy_2' \leq (\pi \sin(\gamma))^{-2} \int_0^{1/\sin(\gamma)} \left(\frac{d}{dy_2'} \varphi(y_2') \right)^2 dy_2'.$$

Due to the properties of rotation the modulus of the gradient of function remains the same and the Jacobian equals 1. So we derive for $\varphi \in H_0^1(\Pi_\gamma, \Gamma_\gamma^2)$

$$\int\limits_{\Pi_\gamma} \varphi^2 dxdy = \int\limits_{\Pi'_\gamma} \varphi^2 dy'_1 dy'_2 \leq (\pi \sin(\gamma))^{-2} \int\limits_{\Pi'_\gamma} \left(\frac{d}{dy'_2} \varphi \right)^2 dy'_1 dy'_2 \leq$$

$$\leq (\pi \sin(\gamma))^{-2} \int\limits_{\Pi'_\gamma} |\nabla' \varphi|^2 \, dy'_1 dy'_2 = (\pi \sin(\gamma))^{-2} \int\limits_{\Pi_\gamma} |\nabla \varphi|^2 \, dxdy.$$

Thus, as γ tends to $\pi/2$ the first eigenvalue

$$\lambda_\gamma = \inf_{\varphi \in H_0^1(\Pi_\gamma, \Gamma_\gamma^2)} \frac{\|\nabla \varphi\|^2_{L^2(\Pi_\gamma)}}{\|\varphi\|^2_{L^2(\Pi_\gamma)}} \geq (\pi \sin(\gamma))^2$$

tends to π^2, which contradicts to (5.15). This proves the theorem. $\square$

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