



*Citation for published version:*

Dereudre, D & Penrose, M 2018, 'On the critical threshold for continuum AB percolation', *Journal of Applied Probability*, pp. 1228-1237. <https://doi.org/10.1017/jpr.2018.81>

*DOI:*

[10.1017/jpr.2018.81](https://doi.org/10.1017/jpr.2018.81)

*Publication date:*

2018

*Document Version*

Peer reviewed version

[Link to publication](#)

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# On the critical threshold for continuum AB percolation

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September 17, 2018

## Abstract

Consider a bipartite random geometric graph on the union of two independent homogeneous Poisson point processes in  $d$ -space, with distance parameter  $r$  and intensities  $\lambda, \mu$ . For any  $\lambda > 0$  we consider the percolation threshold  $\mu_c(\lambda)$  associated to the parameter  $\mu$ . Denoting by  $\lambda_c$  the percolation threshold for the standard Poisson Boolean model with radii  $r$ , we show the lower bound  $\mu_c(\lambda) \geq c \log(c/(\lambda - \lambda_c))$  for any  $\lambda > \lambda_c$  with  $c > 0$  a fixed constant. In particular, there is no phase transition in  $\mu$  at the critical value of  $\lambda$ , that is,  $\mu_c(\lambda_c) = \infty$ .

*Keywords:* continuum percolation, bipartite geometric graph, connectivity threshold, enhancement.

*AMS classifications:* primary 60D05, secondary 60K35, 82B43.

## 1 Introduction and statement of results

In the continuum AB percolation model, particles of two types A and B are scattered randomly in Euclidean space as two independent Poisson processes, and edges are added between particles of opposite type that are sufficiently close together. This provides a continuum analogue to lattice AB percolation which is discussed in e.g. [4]. The model was introduced by Iyer and Yogeshwaran [5], where motivation is discussed in detail; the main motivation comes from wireless communications networks with two types of transmitter. As discussed in [9], a complementary (but distinct)

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continuum percolation model with two types of particle is the *secrecy random graph* [12, 10].

To describe continuum AB percolation more precisely, we make some definitions. Let  $d \in \mathbb{N}$  with  $d \geq 2$ . Given any two locally finite sets  $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d$ , and given  $r > 0$ , let  $G(\mathcal{X}, \mathcal{Y}, r)$  be the bipartite graph with vertex sets  $\mathcal{X}$  and  $\mathcal{Y}$ , and with an undirected edge  $\{X, Y\}$  included for each  $X \in \mathcal{X}$  and  $Y \in \mathcal{Y}$  with  $|X - Y| \leq r$ , where  $|\cdot|$  is the Euclidean norm in  $\mathbb{R}^d$ . Also, let  $G(\mathcal{X}, r)$  be the graph with vertex set  $\mathcal{X}$  and with an undirected edge  $\{X, X'\}$  included for each  $X, X' \in \mathcal{X}$  with  $|X - X'| \leq r$ .

For  $\lambda, \mu > 0$  let  $\mathcal{P}_\lambda, \mathcal{Q}_\mu$  be independent homogeneous Poisson point processes in  $\mathbb{R}^d$  of intensity  $\lambda, \mu$  respectively, where we view each point process as a random subset of  $\mathbb{R}^d$ . We are here concerned with the bipartite graph  $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r)$ .

Let  $\mathcal{I}$  be the class of graphs having at least one infinite component. By a version of the Kolmogorov zero-one law, given parameters  $r, \lambda, \mu$  (and  $d$ ), we have  $\mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}] \in \{0, 1\}$ . Provided  $r, \lambda$ , and  $\mu$  are sufficiently large, we have  $\mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}] = 1$ ; see [5], or [9]. Set

$$\mu_c(r, \lambda) := \inf\{\mu : \mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}] = 1\},$$

with the infimum of the empty set interpreted as  $+\infty$ . Also, for the more standard one-type continuum percolation graph  $G(\mathcal{P}_\lambda, r)$ , define

$$\lambda_c(r) := \inf\{\lambda : \mathbb{P}[G(\mathcal{P}_\lambda, r) \in \mathcal{I}] = 1\}.$$

Thus  $\lambda_c(r)$  is the critical Poisson intensity for percolation of the occupied region for a spherical Poisson Boolean model (see [8]) where the balls are all of radius  $r/2$ . In the notation of [8] it would be called  $\lambda_c(r/2)$ .

By scaling (see Proposition 2.11 of [8])  $\lambda_c(2r) = r^{-d}\lambda_c(2)$ . The value of  $\lambda_c(2)$  is not known analytically, but is well known to be finite for  $d \geq 2$  [4, 8], and explicit bounds are provided in [8]. Simulation studies indicate that  $1 - e^{-\pi\lambda_c(2)} \approx 0.67635$  for  $d = 2$  [11] and  $1 - e^{-(4\pi/3)\lambda_c(2)} \approx 0.28957$  for  $d = 3$  [7].

Obviously if  $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}$  then also  $G(\mathcal{P}_\lambda, 2r) \in \mathcal{I}$ , and hence  $\mu_c(r, \lambda) = \infty$  for  $\lambda < \lambda_c(2r)$ . In [5, 9], it is proved that  $\mu_c(r, \lambda) < \infty$  for  $\lambda > \lambda_c(2r)$ . Indeed, with  $\pi_d$  denoting the volume of the unit radius ball in  $d$  dimensions we have from [9] that

$$\limsup_{\delta \downarrow 0} \left( \frac{\mu_c(r, \lambda_c + \delta)}{\delta^{-2d} |\log \delta|} \right) \leq \left( \frac{4\lambda_c(2r)^2}{r} \right)^d d^{3d} (d+1) \pi_d. \quad (1.1)$$

It is also indicated in [9] how, for any given  $\lambda > \lambda_c(2r)$ , one can compute an explicit upper bound for  $\mu_c(r, \lambda)$ .

As mentioned in [9], it is of interest to give complementary *lower* bounds for  $\mu_c(r, \lambda)$ . In this note, we make some progress in this direction by showing that there exists  $c > 0$  (depending on  $d$ ) such that for all  $\delta > 0$  we have

$$\mu_c\left(\frac{1}{2}, \lambda_c(1) + \delta\right) \geq c \log(c/\delta). \quad (1.2)$$

We make no attempt here to give an explicit numerical value for  $c$  as a function of  $d$ . In principle it should be possible to do this by keeping track of the constants at each step of the proof, but it would undoubtedly give extremely large values for  $c$ .

By scaling arguments, the previous lower bound is true for any radius  $r > 0$  (after changing the constant  $c$ ), so in particular,

$$\lim_{\delta \downarrow 0} \mu_c(r, \lambda_c(2r) + \delta) = +\infty. \quad (1.3)$$

Immediately we obtain

$$\mu_c(r, \lambda_c(2r)) = +\infty. \quad (1.4)$$

We note from (1.1) that if  $\lambda > \lambda_c(2r)$ , we can find finite  $\mu$  such that  $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}$  almost surely. If we were able to prove this under the weaker hypothesis that  $G(\mathcal{P}_\lambda, 2r) \in \mathcal{I}$  almost surely, then combining this with (1.4) we would have shown that in fact  $G(\mathcal{P}_{\lambda_c}, 2r) \notin \mathcal{I}$  almost surely, which would solve the classic open problem of proving non-percolation at the critical point (in any dimension) for this continuum percolation model.

We shall prove (1.2) in the next section. Our strategy of proof goes as follows. We deem all A-particles having no B-particle nearby to be *useless*, since they cannot be used in any percolating AB cluster. Given  $\mu$ , we use a version of the technique of *enhancement* to show that there exists a value of  $\lambda$  such that  $\mathcal{P}_\lambda$  is supercritical for A-percolation (with distance parameter  $2r$ ) but becomes subcritical after removal of all the useless particles (a thinning process with only local dependence). We shall give an outline of the enhancement argument in the next section. The technique of enhancement has previously been applied to one-type continuum percolation in [2, 3], and further discussion of enhancement can be found there.

## 2 Proof of the lower bound

This section is devoted to proving the following theorem.

**Theorem 2.1.** *There exists  $c > 0$  (depending on  $d$ ) such that for all  $\delta > 0$ , we have (1.2). In particular,*

$$\lim_{\delta \downarrow 0} \mu_c\left(\frac{1}{2}, \lambda_c(1) + \delta\right) = +\infty. \quad (2.1)$$

Fix  $\mu > 0$ . To prove (2.1), we need to show that there exists  $\lambda > \lambda_c(1)$  such that  $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, 1/2) \notin \mathcal{I}$  almost surely. To obtain (1.2) we need a suitable quantitative lower bound for this  $\lambda$  (or rather, for  $\lambda - \lambda_c(1)$ ) in terms of  $\mu$ .

We fix some  $\lambda_0 > \lambda_c(1)$ . Given a realization of  $(\mathcal{P}_{\lambda_0}, \mathcal{Q}_\mu)$ , let us say that a point  $x \in \mathbb{R}^d$  is *useless* if no point of  $\mathcal{Q}_\mu$  lies within distance  $1/2$  of  $x$ . Otherwise, let us say  $x$  is *useful*. We shall apply these notions mainly (but not always) in the case where  $x$  is itself a point of  $\mathcal{P}_{\lambda_0}$ .

Given also  $p, q \in [0, 1]$  let  $\mathcal{P}_{\lambda_0, p, q}$  be a thinned version of  $\mathcal{P}_{\lambda_0}$  where each useful point is independently retained with probability  $p$  and each useless point is independently retained with probability  $q$ . In particular  $\mathcal{P}_{\lambda_0, p, p}$  has the same distribution as  $\mathcal{P}_{\lambda_0 p}$ .

For  $R > 0$  let  $B_R$  denote the Euclidean ball of radius  $R$  centred at the origin. Let  $\theta(p, q)$  be the probability that there exists an infinite component of  $G(\mathcal{P}_{\lambda_0, p, q}, 1)$  that includes at least one vertex in  $B_1$ , and, for  $n \in \mathbb{N}$ , let  $\theta_n(p, q)$  be the probability that there exists a component of  $G(\mathcal{P}_{\lambda_0, p, q}, 1)$  that includes at least one vertex in  $B_1$  and at least one vertex outside  $B_n$ . Then for all  $p, q$  we have  $\theta(p, q) = \lim_{n \rightarrow \infty} \theta_n(p, q)$ .

Before giving further details, we now sketch out our enhancement strategy. By a continuum version of Russo's formula (Lemma 2.1 below), the partial derivative  $\partial\theta_n/\partial p$  (respectively  $\partial\theta_n/\partial q$ ) can be expressed as an integral over  $x \in B_n$  of the probability that  $x$  is useful and pivotal (respectively, useless and pivotal), where we say  $x$  is *pivotal*, loosely speaking, if event  $A_{x, n, p, q}$  (defined in Lemma 2.1) occurs.

Let  $R > 0$ . Suppose, in the event that  $x$  is useful and pivotal, that we resample  $\mathcal{P}_{\lambda, p, q}$  inside the translated ball  $B_R(x) := x + B_R$  (leaving the Poisson process unchanged outside this ball). In Lemma 2.2 we shall show there exists a fixed  $R > 0$  such that with (conditional) probability bounded away from zero (uniformly in  $x$  and  $n$ ) the point  $x$  is useless and pivotal in the resampled process, given that  $x$  is useful and pivotal in the original process. Therefore  $(\partial\theta_n/\partial q)/(\partial\theta_n/\partial p)$  is bounded away from zero, uniformly in  $n$ . Using this and setting  $p_c := \lambda_c/\lambda_0$ , we can show that for some sufficiently small  $\varepsilon > 0$  (depending on our initial choice of  $\mu$ ), if we change  $(p, q)$  from  $(p_c - \varepsilon, p_c - \varepsilon)$  to  $(p_c + \varepsilon, 0)$  the 'bad' effect of reducing  $q$  to zero outweighs the 'good' effect of increasing  $p$  by  $2\varepsilon$ , so that  $\theta(p_c + \varepsilon, 0) \leq \theta(p_c - \varepsilon, p_c - \varepsilon) = 0$ , so removing useless vertices of  $\mathcal{P}_{\lambda_c + \varepsilon\lambda_0}$  makes it subcritical.

**Lemma 2.1.** *Given  $x \in \mathbb{R}^d$ , denote by  $A_{x, n, p, q}$  the event that  $G(\mathcal{P}_{\lambda_0, p, q} \cup \{x\}, 1)$  contains a path including vertices both in  $B_1$  and in  $B_n^c$ , but  $G(\mathcal{P}_{\lambda_0, p, q}, 1)$  does not, and let  $F_x$  be the event that  $x$  is useful.*

*Then for any  $n \geq 1$  and  $p, q \in (0, 1)$ ,*

$$\frac{\partial\theta_n(p, q)}{\partial p} = \int_{B_{n+1}} \mathbb{P}[A_{x, n, p, q} \cap F_x] \lambda_0 dx$$

and

$$\frac{\partial \theta_n(p, q)}{\partial q} = \int_{B_{n+1}} \mathbb{P}[A_{x, n, p, q} \cap F_x^c] \lambda_0 dx,$$

*Proof.* Adapting the proof of Lemma 1 in [2], we prove only the first identity since the second is obtained in exactly the same way. We denote by  $\mathcal{F}$  the  $\sigma$ -algebra generated by  $(\mathcal{P}_{\lambda_0}, \mathcal{Q}_\mu)$ . In particular  $\mathcal{F}$  does not contain any information on the thinning procedures for the useful and useless vertices. We denote by  $A_n$  the event that there exists a path in  $G(\mathcal{P}_{\lambda_0, p, q}, 1)$  from  $B_1$  to outside  $B_n$ . Let us note that the distribution of  $\mathcal{P}_{\lambda_0, p, q}$ , given the  $\sigma$ -algebra  $\mathcal{F}$ , consists of a collection of independent Bernoulli variables which indicate whether the vertices are retained or removed by the thinning procedure. Then, applying the standard coupling of Bernoulli variables and Russo's formula (also attributed to Margulis but in fact dating back at least to [1, eqn (5.2)]), we obtain for any  $h \in (0, 1 - p]$  that

$$0 \leq \mathbb{P}(\mathcal{P}_{\lambda_0, p+h, q} \in A_n | \mathcal{F}) - \mathbb{P}(\mathcal{P}_{\lambda_0, p, q} \in A_n | \mathcal{F}) \leq h \#(\mathcal{P}_{\lambda_0} \cap B_{n+1})$$

and

$$\lim_{h \downarrow 0} \frac{1}{h} \left( \mathbb{P}(\mathcal{P}_{\lambda_0, p+h, q} \in A_n | \mathcal{F}) - \mathbb{P}(\mathcal{P}_{\lambda_0, p, q} \in A_n | \mathcal{F}) \right) = \mathbb{E} [N_{n, p, q} | \mathcal{F}],$$

where  $N_{n, p, q}$  is the number of useful vertices in  $\mathcal{P}_{\lambda_0}$  that are pivotal for the occurrence of  $A_n$  (with the  $(p, q)$ -thinning applied to all of the vertices of  $\mathcal{P}_{\lambda_0}$  except for the one being counted as pivotal). Recall that a vertex  $x$  in a configuration  $\mathcal{X}$  is said to be pivotal for an increasing event  $\mathcal{A}$  if  $\mathcal{X}$  belongs to  $\mathcal{A}$  whereas  $\mathcal{X} \setminus \{x\}$  does not.

By the dominated convergence theorem we obtain

$$\begin{aligned} \frac{\partial^+ \theta_n(p, q)}{\partial p} &= \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E} \left[ \mathbb{P}(\mathcal{P}_{\lambda_0, p+h, q} \in A_n | \mathcal{F}) - \mathbb{P}(\mathcal{P}_{\lambda_0, p, q} \in A_n | \mathcal{F}) \right] \\ &= \mathbb{E} \left[ \lim_{h \downarrow 0} \frac{1}{h} \left[ \mathbb{P}(\mathcal{P}_{\lambda_0, p+h, q} \in A_n | \mathcal{F}) - \mathbb{P}(\mathcal{P}_{\lambda_0, p, q} \in A_n | \mathcal{F}) \right] \right] \\ &= \mathbb{E} [\mathbb{E} [N_{n, p, q} | \mathcal{F}]] = \mathbb{E} [N_{n, p, q}]. \end{aligned}$$

By the Mecke formula (see [6]) it follows that

$$\begin{aligned} \mathbb{E} [N_{n, p, q}] &= \mathbb{E} \left[ \sum_{x \in \mathcal{P}_{\lambda_0}} \mathbf{1}_{\{x \text{ is useful and pivotal for } A_n\}} \right] \\ &= \int_{B_{n+1}} \mathbb{P}[A_{x, n, p, q} \cap F_x] \lambda_0 dx. \end{aligned}$$

One may argue similarly for the left derivative. The lemma is proved.  $\square$

Now we want to apply enhancement arguments as in [2, 3]. For this we need to control the ratio between  $\frac{\partial\theta_n(p,q)}{\partial q}$  and  $\frac{\partial\theta_n(p,q)}{\partial p}$ . The crucial lemma is given here.

**Lemma 2.2.** *Given  $\alpha \in (0, 1)$  and  $\lambda_0 > 0$ , there exists a constant  $c > 0$  such that for any  $p \geq \alpha$ ,  $q \geq \alpha$ ,  $\mu > 0$ ,  $n \geq c^{-1}$  and  $x \in B_{n+1}$ ,*

$$\frac{\mathbb{P}[A_{x,n,p,q} \cap F_x^c]}{\mathbb{P}[A_{x,n,p,q} \cap F_x]} \geq ce^{-\mu/c}. \quad (2.2)$$

The proof, based on geometrical arguments, is given at the end of the section.

*Proof of Theorem 2.1.* Set  $\lambda_c := \lambda_c(1)$  and  $p_c := \lambda_c/\lambda_0$ . Choose  $\alpha := p_c/2$ . Then by Lemmas 2.1 and 2.2, there exists  $c > 0$  such that for any  $p, q \in (\alpha, 1)$ , any  $\mu > 0$  and any  $n \geq c^{-1}$ , we have

$$\frac{\partial\theta_n(p, q)}{\partial q} \geq (ce^{-\mu/c}) \frac{\partial\theta_n(p, q)}{\partial p}. \quad (2.3)$$

Given  $\mu > 0$ , let  $\delta > 0$  be small enough so that  $p_c + \delta < 1$  and  $p_c - \delta(1 + (2/c)e^{\mu/c}) \geq \alpha$ , that is,  $\delta(1 + (2/c)e^{\mu/c}) \leq p_c/2$ . There exists  $c' > 0$  such that the choice

$$\delta = c'e^{-\mu/c'} \quad (2.4)$$

is suitable, for all  $\mu > 0$ . By the finite-increments formula on the segment  $[(p_c - \delta, p_c - \delta); (p_c + \delta, p_c - \delta(1 + \frac{2}{c}e^{\mu/c}))]$  and inequality (2.3),

$$\theta_n(p_c + \delta, p_c - \delta(1 + \frac{2}{c}e^{\mu/c})) \leq \theta_n(p_c - \delta, p_c - \delta).$$

By passing to the limit  $n \rightarrow +\infty$  and noting that  $\theta(p_c - \delta, p_c - \delta) = 0$  we obtain

$$\theta(p_c + \delta, 0) \leq \theta(p_c + \delta, p_c - \delta(1 + \frac{2}{c}e^{\mu/c})) \leq \theta(p_c - \delta, p_c - \delta) = 0,$$

so  $G(\mathcal{P}_{\lambda_0, p_c + \delta, 0}, 1) \notin \mathcal{I}$  almost surely and hence  $G(\mathcal{P}_{\lambda_c + \delta\lambda_0}, \mathcal{Q}_\mu, \frac{1}{2}) \notin \mathcal{I}$  almost surely.

In conclusion, setting  $\lambda := \lambda_c + \delta\lambda_0$  with  $\delta$  given by (2.4), we have  $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, \frac{1}{2}) \notin \mathcal{I}$  almost surely. Hence

$$\mu_c(\frac{1}{2}, \lambda) \geq \mu = c' \log \left( \frac{\lambda_0 c'}{\lambda - \lambda_c} \right),$$

so the theorem is proved.  $\square$

It remains now to show Lemma 2.2. Let us first give two geometrical lemmas. We denote by  $B_k(x)$  the translated ball  $x + B_k$  and for each point  $a \in \mathbb{R}^d$  and subset  $S \subset \mathbb{R}^d$ , we denote by  $d(a, S)$  the Euclidean distance from  $a$  to  $S$ .

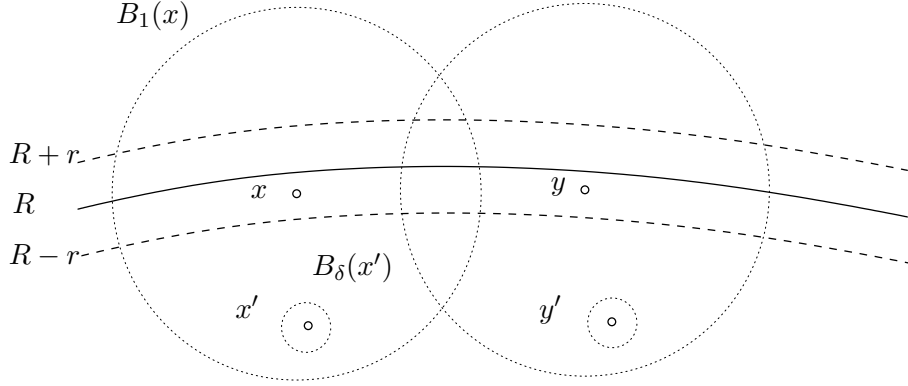


Figure 1: An example of configuration  $x, y, x'$  and  $y'$  in Lemma 2.3.

**Lemma 2.3.** *There exists  $K > 10$  such that for all  $R > K$  and all  $r, \delta \in (0, 1/K)$ , we have: for all  $x, y \in B_{R+r}(0) \setminus B_{R-r}(0)$  with  $|x - y| > 1$  there exist  $x', y' \in B(0, R - 1/2)$  satisfying*

- *i)  $|x - x'| \leq 1 - \delta$ ;*
- *ii)  $|y - y'| \leq 1 - \delta$ ;*
- *iii)  $|x' - y'| \geq 1 + 2\delta$ ;*
- *iv)  $d(x', B_R(0)^c \setminus B_1(x)) \geq 1 + \delta$ ;*
- *v)  $d(y', B_R(0)^c \setminus B_1(y)) \geq 1 + \delta$ .*

*Proof.* Given  $x, y \in B_{R+r}(0) \setminus B_{R-r}(0)$  with  $|x - y| > 1$ , set  $x' := x - (1 - 3\delta)x/|x|$ , and

$$y' := \begin{cases} y - (1 - 3\delta)x/|x| + 2\delta(y - x)/|y - x| & \text{if } |y - x| < 3 \\ y - (1 - 3\delta)y/|y| & \text{otherwise.} \end{cases}$$

It is clear that for  $R \geq 99$  and  $r$  and  $\delta$  small enough  $x', y' \in B(0, R - 1/2)$ . Moreover item *i)* holds trivially, as do items *ii)* and *iii)* by the triangle inequality. Finally, provided  $R$  is large enough and  $r, \delta$  small enough, the distances  $d(x', B_R(0)^c \setminus B_1(x))$  and  $d(y', B_R(0)^c \setminus B_1(y))$  are bounded below by  $\sqrt{1.9}$ . Therefore choosing  $R$  large enough and  $r, \delta$  small enough, items *iv)* and *v)* are satisfied.  $\square$

**Lemma 2.4.** *Let  $r > 0$ . Then there exists  $K' > 10$  such that for any  $R > K'$  and any  $\delta \in (0, 1/K')$  we have: for all  $x \in B_R(0) \setminus B_{R-r}(0)$  and  $y \in B_{R-r}(0)$  with  $|x - y| > 1$  and  $d(y, B_{R+r}(0) \setminus B_1(x)) \leq 1$ , there exist  $x', y' \in B(0, R - 1/2)$  satisfying*



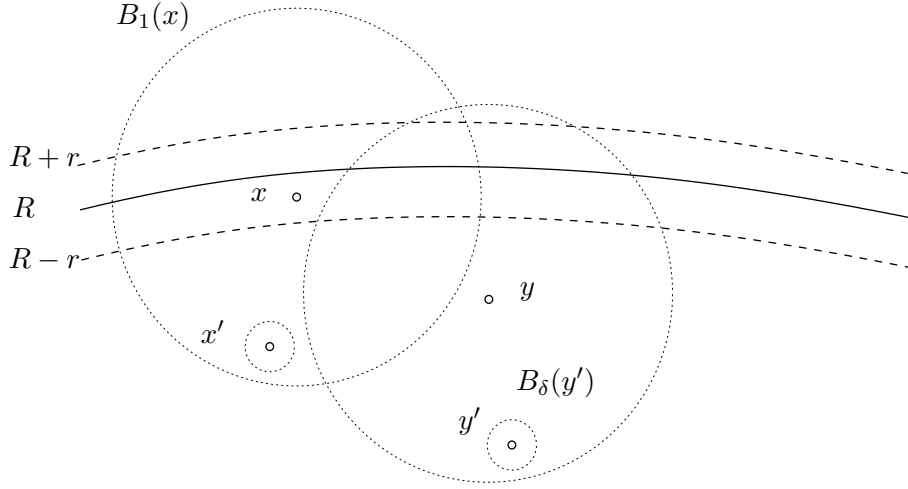


Figure 2: An example of configuration  $x, y, x'$  and  $y'$  in Lemma 2.4.

- *i)*  $|x - x'| \leq 1 - \delta$ ;
- *ii)*  $|y - y'| \leq 1 - \delta$ ;
- *iii)*  $|x' - y'| \geq 1 + 2\delta$ ;
- *iv)*  $|x' - y| \geq 1 + \delta$ ;
- *v)*  $|y' - x| \geq 1 + \delta$ ;
- *vi)*  $d(y', B_R(0)^c \setminus B_1(y)) \geq 1 + \delta$ ;
- *vii)*  $d(x', B_{R+r}(0)^c \setminus B_1(x)) \geq 1 + \delta$ .

*Proof.* Let  $x \in B_R \setminus B_{R-r}$  and  $y \in B_{R-r}$  with  $|x - y| > 1$  and  $d(y, B_{R+r} \setminus B_1(x)) \leq 1$ . Without loss of generality, we assume that  $|x - y| < 3$ ; otherwise, an accurate construction as in the previous lemma is possible. Let  $\varepsilon := 10^{-3}$  be fixed, and note that  $\sin(\pi/3 - 3\varepsilon) > 1/2$ . Note also that provided  $R$  is large enough, the vector  $y - x$  is at an angle at most  $\pi/3 + \varepsilon$  with the hyperplane tangent to  $B_{|x|}$  at  $x$ .

Let  $u$  and  $v$  be unit vectors in the vector space generated by  $x$  and  $y$  such that the angle between  $y - x$  and  $u$  is equal to  $\pi/3 + \varepsilon$ , the angle between  $y - x$  and  $v$  is equal to  $\pi/3 + 2\varepsilon$ , and such that both  $u$  and  $v$  have negative scalar product with  $x$ . (The choice of  $u$  and of  $v$  is unique provided that  $R$  is large enough.) Then we set  $x' = x + (1 - 3\delta)v$  and  $y' = y + (1 - 3\delta)u$ .

It is clear that for  $\delta$  small enough  $x', y' \in B(0, R - 1/2)$ . Moreover items *i)* and *ii)* are trivially true, and *iii)* holds provided  $\delta$  is small enough. The quadrilateral

$x, y, y', x'$  is almost a parallelogram; the opposite edges  $(y, y')$  and  $(x, x')$  are the same length, and at an angle of  $\varepsilon$  to each other. The angle between  $y - x$  and  $x' - x$  is equal to  $\pi/3 + 2\varepsilon$ , and all sides of this quadrilateral are of length at least  $1 - 3\delta$ . Provided  $\delta$  is small enough, the norms of the diagonals  $|y' - x|$  and  $|y - x'|$  exceed  $1 + \delta$ ; that is, items *iv*) and *v*) hold. Also, for  $R$  sufficiently large, the angle between  $y - y'$  and  $y$  is smaller than  $\pi/6 + 2\varepsilon$ . Moreover  $y \in B_{R-r}(0)$  so  $d(y', B_R(0)^c \setminus B_1(y)) \geq 1 + r/2$ . Item *vi*) follows for  $\delta$  small enough. The proof of the last item *vii*) is similar.  $\square$

*Proof of Lemma 2.2.* Assume now that parameters  $R > 10$ ,  $r > 0$  and  $\delta > 0$  are chosen such that all items in Lemmas 2.3 and 2.4 are satisfied. Assume also that  $R \in \mathbb{N}$  and that  $\delta < R/99$ . Noting that

$$\frac{\mathbb{P}[A_{x,n,p,q} \cap F_x^c]}{\mathbb{P}[A_{x,n,p,q} \cap F_x]} = \frac{\mathbb{P}[F_x^c | A_{x,n,p,q}]}{\mathbb{P}[F_x | A_{x,n,p,q}]} \geq \mathbb{P}[F_x^c | A_{x,n,p,q}],$$

we have to find a lower bound for  $\mathbb{P}[F_x^c | A_{x,n,p,q}]$  of the type  $ce^{-\mu/c}$ . Assume for now that  $B_R(x) \subset B_n \setminus B_1$ . (We shall consider the other cases at the end.)

Consider creating the Poisson processes  $\mathcal{P}_{\lambda_0}$ ,  $\mathcal{Q}_\mu$  and  $\mathcal{P}_{\lambda_0,p,q}$  in stages, as follows (this is similar to arguments seen in [2] and [3]).

First we generate the points of  $\mathcal{P}_{\lambda_0}$  outside  $B_R(x)$ , the points of  $\mathcal{Q}_\mu$  outside  $B_{1/2}(x)$  and the retained points  $\mathcal{P}_{\lambda_0,p,q}$  outside  $B_R(x)$ .

At this stage, we let  $V$  be the set of vertices of  $\mathcal{P}_{\lambda_0,p,q}$  created so far which are connected by a path (in  $G(\mathcal{P}_{\lambda_0,p,q} \setminus B_R(x), 1)$ ) to  $B_1$ , and let  $T$  be the set of vertices of  $\mathcal{P}_{\lambda_0,p,q}$  created so far which are connected by a path to  $B_n^c$ . Let  $V_1$  and  $T_1$  denote the 1-neighbourhood of  $V$ ,  $T$  respectively. Let  $E_1$  be the event that  $T \cap V_1 = \emptyset$  and  $(T_1 \setminus V_1) \cap B_R \neq \emptyset$  and  $(V_1 \setminus T_1) \cap B_R \neq \emptyset$ . If  $A_{x,n,p,q}$  is realized,  $x$  is pivotal and therefore  $E_1$  occurs.

Now, assuming  $E_1$  occurs, build up the point process of retained vertices  $\mathcal{P}_{\lambda_0,p,q}$  inwards into  $B_R(x) \cap (V_1 \Delta T_1)$  from the boundary of the ball  $B_R(x)$ , until the appearance of the first new vertex (here,  $\Delta$  denotes symmetric set difference). Denote this new vertex  $X$ . Let  $E_2$  be the event that such a vertex exists, that is,  $E_2 := \{\mathcal{P}_{\lambda_0,p,q} \cap B_R(x) \cap (V_1 \Delta T_1) \neq \emptyset\}$ . If  $A_{x,n,p,q}$  is realized, then  $E_2$  must occur.

Assuming that  $E_2$  occurs, we suppose  $X \in T_1$  (the other case,  $X \in V_1$ , can be treated in the same way). We now have to distinguish several cases.

*Case 1:*  $X \in B_R(x) \setminus B_{R-r}(x)$ , which means that  $X$  is close to the boundary of  $B_R(x)$ . Now we have also two sub-cases to consider:

*Case 1.1:* there exists  $Y$  in  $V \cap B_{R+r}(x) \setminus B_R(x)$ . Note that  $X$  and  $Y$  are exactly in position for applying Lemma 2.3. By that result, there exist  $x_1$  and  $y_1$  in  $B_{R-1/2}(x)$  such that any point in  $B_\delta(x_1)$  (respectively  $B_\delta(y_1)$ ) can be the next point for the path from  $B_n$  (respectively  $B_1$ ) going to  $x$ . These points are now sufficiently far

from the boundary of  $B_R(x)$  to create two paths  $(x_i)_{1 \leq i \leq R+2}$  and  $(y_i)_{1 \leq i \leq R+2}$  of  $R+2$  points (deterministic, given what has been revealed so far) from  $T$  to  $x$  and  $V$  to  $x$ , where each step size in each path is at most  $1 - 2\delta$  (that is,  $|x_{i+1} - x_i| < 1 - 2\delta$  and  $|y_{i+1} - y_i| < 1 - 2\delta$  for each  $i$ , and  $|x - x_{R+2}| \leq 1 - 2\delta$  and  $|x - y_{R+2}| \leq 1 - 2\delta$ ), and such that these paths remain a distance greater than  $1 + 2\delta$  from each other; see Figure 3.

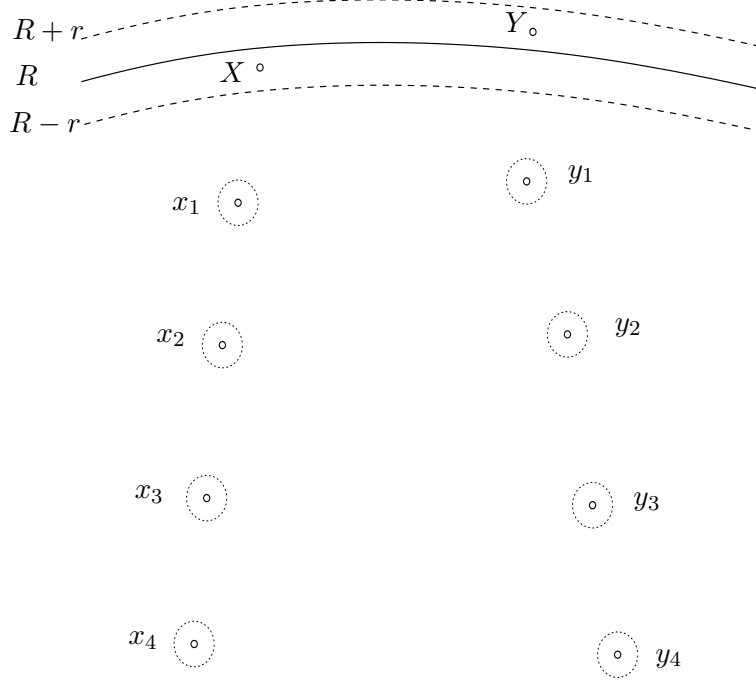


Figure 3: Example of paths  $(x_i)_{1 \leq i \leq R+2}$  and  $(y_i)_{1 \leq i \leq R+2}$ .

If each ball  $B_\delta(x_i)$  or  $B_\delta(y_i)$ ,  $1 \leq i \leq R+2$ , contains a single (retained) point, and there are no other points of  $\mathcal{P}_{\lambda_0} \cap B_R(x)$  besides those already considered, and there are no points of  $\mathcal{Q}_\mu$  inside  $B_{1/2}(x)$ , then  $A_{x,n,p,q} \cap F_x^c$  occurs. Recall that  $p$  and  $q$  are at least  $\alpha$ . Then the probability that all of the above occur is clearly bounded from below by  $ce^{-\pi_d \mu / 2^d}$  for a suitable constant  $c > 0$  (depending on  $\lambda_0, R, r, \delta$  and  $\alpha$ ), where  $\pi_d$  is the volume of the unit ball in  $\mathbb{R}^d$ .

*Case 1.2:* the set  $V \cap B_{R+r}(x) \setminus B_R(x)$  is empty. In this case we continue to build inwards the point process of retained vertices  $\mathcal{P}_{\lambda_0,p,q}$ , but only in  $V_1 \setminus T_1$ . Continue until the next new vertex appears, denoted  $Y$  and then stop. Let  $E_3$  be the event that such a new vertex  $Y$  does indeed appear. If this occurs, then two sub-cases are possible:

*Case 1.2.1:* the point  $Y$  is close to the boundary (i.e.  $Y \in B_R(x) \setminus B_{R-r}(x)$ ).

Then  $X$  and  $Y$  are exactly in position for applying Lemma 2.3 as in Case 1.1 and we conclude this case in the same way.

*Case 1.2.2:* the point  $Y$  is in  $B_{R-r}(x)$ . Then  $X$  and  $Y$  are exactly in position for applying Lemma 2.4. So there exist  $x_1$  and  $y_1$  in  $B_{R-1/2}(x)$  such that any points in  $B_\delta(x_1)$  (respectively  $B_\delta(y_1)$ ) can be the next point for the path from  $B_n$  (respectively  $B_1$ ) going to  $x$ . These points are now sufficiently far from the boundary of  $B_R(x)$  to create two paths  $(x_i)_{1 \leq i \leq R+2}$  and  $(y_i)_{1 \leq i \leq R+2}$  as in Case 1.1.

*Case 2:*  $X \in B_{R-r}(x)$ , which means that  $X$  is far from the boundary of  $B_R(x)$ . Then we continue to build inwards the point process of retained vertices  $\mathcal{P}_{\lambda_0, p, q}$ , but only in  $V_1 \setminus T_1$ , as in Case 1.2. Continue until the next new vertex appears, denoted  $Y$ ; let  $E_4$  be the event that such a vertex  $Y$  does appear, and assume  $E_4$  occurs. Now  $X$  and  $Y$  are sufficiently far from the boundary (both in  $B_{R-r}(x)$ ) in order to build  $(x_i)_{1 \leq i \leq R+2}$  and  $(y_i)_{1 \leq i \leq R+2}$  as in Case 1.1.

Let  $E$  be the event that events  $E_1, E_2$ , and (if in Case 1.2)  $E_3$ , and (if in Case 2)  $E_4$  (along with corresponding events for the case when  $X \in V_1$ ) all occur. If  $A_{x, n, p, q}$  occurs, then  $E$  must occur, and therefore

$$\mathbb{P}[F_x^c | A_{x, n, p, q}] = \frac{\mathbb{P}[F_x^c \cap A_{x, n, p, q} \cap E]}{\mathbb{P}[A_{x, n, p, q}]} \geq \mathbb{P}[F_x^c \cap A_{x, n, p, q} | E].$$

Thus, the conclusion of all these cases is that there exists a constant  $c > 0$  such that

$$\mathbb{P}[F_x^c | A_{x, n, p, q}] \geq ce^{-\mu/c}.$$

Recall that we have been assuming  $B_R(x) \subset B_n \setminus B_1$ . A similar proof can be derived in the other two cases, namely the case with  $B_R(x) \cap B_1 \neq \emptyset$  and the case with  $B_R(x) \cap B_n^c \neq \emptyset$ . We may assume  $n \geq 99R$ . When  $B_R(x) \cap B_1 \neq \emptyset$ , consider a path  $(x_i)_{1 \leq i \leq 2R+2}$  from outside  $B_{2R}(x)$  to  $x$  and a path  $(y_i)_{1 \leq i \leq R+2}$  inside  $B_{2R}(x)$  joining  $x$  to  $B_1$ . When  $B_R(x) \cap B_n^c \neq \emptyset$ , consider a path  $(x_i)_{1 \leq i \leq 2R+2}$  from outside  $B_{2R}(x)$  to  $x$  and a path  $(y_i)_{1 \leq i \leq R+2}$  inside  $B_{2R}(x)$  joining  $x$  to  $B_n^c$ . For brevity we omit the details of these cases here (which are similar to the corresponding cases treated in [2]). Lemma 2.2 is proved.  $\square$

*Acknowledgement:* This work was supported in part by the Labex CEMPI (ANR-11-LABX-0007-01), the CNRS GdR 3477 GeoSto and the ANR project PPP (ANR-16-CE40-0016).

We thank the anonymous referees for some very helpful comments.

## References

- [1] Esary, J. D. and Proschan, F. (1963). Coherent structures of non-identical components. *Technometrics* **5**, 191–209.

- [2] Franceschetti, M., Penrose, M. D., and Rosoman, T. (2010). Strict inequalities of critical probabilities on Gilbert’s continuum percolation graph. arXiv:1004.1596
- [3] Franceschetti, M., Penrose, M. D., and Rosoman, T. (2011). Strict inequalities of critical values in continuum percolation *J. Statist. Phys.* **142**, 460–486.
- [4] Grimmett, G. (1999). *Percolation*. Second edition. Springer-Verlag, Berlin.
- [5] Iyer, S. K. and Yogeshwaran, D. (2012). Percolation and connectivity in AB random geometric graphs. *Adv. in Appl. Probab.* **44**, 21-41.
- [6] Last, G. and Penrose, M. (2018). *Lectures on the Poisson Process*. Cambridge University Press, Cambridge.
- [7] Lorenz, C. D. and Ziff, R. M. (2000). Precise determination of the critical percolation threshold for the three dimensional “Swiss cheese” model using a growth algorithm. *J. Chem. Phys.* **114**, 3659-3661.
- [8] Meester, R. and Roy, R. (1996). *Continuum Percolation*. Cambridge University Press, Cambridge.
- [9] Penrose, M.D. (2014) Continuum AB percolation and AB random geometric graphs. *J. Appl. Probab.* **51A**, 333-344.
- [10] Pinto, P.C. and Win, Z. (2012) Percolation and connectivity in the intrinsically secure communications graph. *IEEE Trans. Inform. Theory* **58**, 1716-1730.
- [11] Quintanilla, J. A. and Ziff, R. M. (2007). Asymmetry of percolation thresholds of fully penetrable disks with two different radii. *Phys. Rev. E* **76**, 051115 [6 pages].
- [12] Sarkar, A. and Haenggi, M. (2013) Percolation in the Secrecy Graph. *Discrete Appl. Math.* **161**, 2120-2132.