Citation for published version:
Calderbank, DMJ, Slovak, J & Soucek, V 2018, 'Subriemannian metrics and the metrizability of parabolic geometries', ArXiv e-prints.

Publication date:
2018

Link to publication

SUBRIEMANNIAN METRICS AND THE METRIZABILITY OF PARABOLIC GEOMETRIES

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ABSTRACT. We present the linearized metrizability problem in the context of parabolic geometries and subriemannian geometry, generalizing the metrizability problem in projective geometry studied by R. Liouville in 1889. We give a general method for linearizability and a classification of all cases with irreducible defining distribution where this method applies. These tools lead to natural subriemannian metrics on generic distributions of interest in geometric control theory.

1. INTRODUCTION

Many areas of geometric analysis and control theory deal with distributions on smooth manifolds, i.e., smooth subbundles of the tangent bundle. Let $\mathcal{H} \leqslant TM$ be such a distribution of rank n on a smooth m-dimensional manifold M. A smooth curve $c\colon [a,b] \to M$ ($a \leqslant b \in \mathbb{R}$) is called *horizontal* if it is tangent to $\mathcal{H}$ at every point, i.e., for every $t \in [a,b]$, the tangent vector $\dot{c}(t)$ to c at $c(t) \in M$ belongs to $\mathcal{H}$. It is well known that, at least locally, any two points $x, y \in M$ can be connected by a horizontal curve c if and only if $\mathcal{H}$ is *bracket-generating* in the sense that any tangent vector can be obtained from iterated Lie brackets of sections of $\mathcal{H}$.

This paper is concerned with bracket-generating distributions arising in *parabolic geometries* [6], which are Cartan–Tanaka geometries modelled on homogeneous spaces G/P where G is a semisimple Lie group and $P \leqslant G$ a parabolic subgroup. On a manifold M equipped with such a parabolic geometry, each tangent space is modelled on the P-module $\mathfrak{g}/\mathfrak{p}$, and the socle $\mathfrak{h}$ of this P-module (the sum of its minimal nonzero P-submodules) induces a bracket-generating distribution $\mathcal{H}$ on M. Simple and well-known examples include projective geometry and (Levi-nondegenerate) hypersurface CR geometry: in the former case, $\mathfrak{g}/\mathfrak{p}$ is irreducible and so $\mathcal{H} = TM$, but in the latter case $\mathcal{H}$ is the corank one contact distribution of the hypersurface CR structure.

A more prototypical example for this paper is when $\mathcal{H} \leqslant TM$ is generic of rank n and corank $\frac{1}{2}n(n-1)$, i.e., $m = \frac{1}{2}n(n+1) = n + \frac{1}{2}n(n-1)$, and $[\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] = \Gamma(TM)$. In this case the Lie bracket on sections of $\mathcal{H}$ induces an isomorphism $\wedge^2\mathcal{H} \cong TM/\mathcal{H}$ and the distribution is said to be *free*. Any such manifold is a parabolic geometry where $G = SO(V)$ with $\dim V = 2n+1$ and P is the stabilizer of a maximal (n-dimensional) isotropic subspace U of V [9]. Then $\mathfrak{g}/\mathfrak{p}$ has socle $\mathfrak{h} \cong U^* \otimes (U^\perp/U)$ with quotient isomorphic to $\wedge^2\mathfrak{h}$, and $\mathfrak{h} \leqslant \mathfrak{g}/\mathfrak{p}$ induces the distribution $\mathcal{H} \leqslant TM$ on M.

While parabolic geometry is the main tool for the present work, our motivation is subriemannian geometry, which concerns the following notion [18].

Definition 1.1. Consider an m-dimensional manifold M with a given smooth distribution $\mathcal{H} \leqslant TM$ of constant rank n. A (pseudo-)Riemannian metric g on $\mathcal{H}$ is called a *horizontal* or *subriemannian metric on* M.

Horizontal metrics are important in both geometric analysis and control theory. Among the horizontal curves joining two points, it may be important to find those which are optimal

Date: March 29, 2018.

The authors thank the Czech Grant Agency, grant nr. P201/12/G028, for financial support.

in some sense, for example those of shortest length with respect to a horizontal metric. Horizontal metrics also allow for the definition of a hypo-elliptic sublaplacian [16], allowing methods of harmonic analysis to be applied. However, this raises the question: what is a good choice of horizontal metric?

For the distribution $\mathcal{H}$ on a parabolic geometry, there is a natural compatibility condition that can be imposed. Indeed, one of the key features of such a geometry is that it admits a canonical class of connections, called *Weyl connections*, which form an affine space modelled on the space of 1-forms.

Definition 1.2. A horizontal metric on the distribution $\mathcal{H} \leqslant TM$ induced from a parabolic geometry M is *compatible* if it is covariantly constant in horizontal directions with respect to some Weyl connection on M. We say M is (*locally*) *metrizable* if there exists (locally) a compatible horizontal metric.

The metrizability problem has been studied for several classes of parabolic geometry with $\mathcal{H} = TM$, in particular, the case of real projective. These examples exhibit several interesting features, which we seek to generalize to all parabolic geometries—in particular to those with $\mathcal{H} \neq TM$.

First, whereas the metrizability condition appears to be highly nonlinear, it linearizes when viewed as a condition on the inverse metric on $\mathcal{H}^*$ multiplied by a suitable power of the horizontal volume form. Secondly, this linear equation is highly overdetermined, with a finite dimensional solution space. Hence parabolic geometries admitting such horizontal metrics are rather special. This has been used to extract detailed information about the structure of the geometry [1, 3, 8, 10, 12, 17, 21].

If $\mathfrak{h}$ is the socle of $\mathfrak{g}/\mathfrak{p}$, it is not generally the case that $S^2\mathfrak{h}$ is irreducible—indeed $\mathfrak{h}$ itself need not be irreducible. In order to generalize the studied examples, we introduce a condition on P-submodules $B \leqslant S^2\mathfrak{h}$ containing nondegenerate elements, which we call the *algebraic linearization condition* (ALC). Our first main result (Theorem 1) justifies this terminology by showing that for parabolic geometries and P-submodules $B \leqslant S^2\mathfrak{h}$ satisfying the ALC, there is a bijection between compatible horizontal metrics and nondegenerate solutions of an overdetermined first order *linear* differential equation. (In fact, if $\mathfrak{h}$ is not irreducible we need a technical extra condition, which we call the *strong* ALC.)

Our second main result (Theorem 2) is a complete classification of all parabolic geometries and all P-submodules $B \leqslant S^2\mathfrak{h}$ such that $\mathfrak{h}$ is irreducible and B satisfies the ALC. The classification exhibits two nicely counterbalancing features. On the one hand, among parabolic geometries with irreducible socle, those admitting P-submodules $B \leqslant S^2\mathfrak{h}$ satisfying the ALC are rare. On the other hand, the list of examples is quite long: we state the classification using three tables containing 14 infinite families and 6 exceptional cases. Many of these examples invite further study (see e.g. [19]).

The structure of the paper is as follows. In section 2 we briefly outline the main notions and tools of parabolic geometry, referring to [6] for details, but concentrating on examples. We also establish the local metrizability of the homogeneous model. In section 3, we describe the linearization principle and prove Theorem 1. We give examples, and in particular show how explicit formulae can be obtained not only for the homogeneous model, but also for so-called *normal solutions*. Section 4 is devoted to the main classification result. We conclude by giving examples (Theorem 3) where the socle is not irreducible.

2. Background and motivating examples

We work throughout with real smooth manifolds M, real Lie groups P and real Lie algebras $\mathfrak{p}$ (e.g., we view $GL(n,\mathbb{C})$ as a real Lie group and $\mathfrak{gl}(n,\mathbb{C})$ as a real Lie algebra).

A (real or complex) *P-module* W is a finite dimensional (real or complex) vector space carrying a representation $\rho_W \colon P \to GL(W)$; W is then also a $\mathfrak{p}$-module, where $\mathfrak{p}$ is the Lie algebra of P, i.e., it carries a representation $\tilde{\rho}_W \colon \mathfrak{p} \to \mathfrak{gl}(W)$. We write $\xi \cdot w$ for $\tilde{\rho}_W(\xi)(w)$. The *nilpotent radical* of $\mathfrak{p}$ is the intersection $\mathfrak{n}$ of the kernels of all simple $\mathfrak{p}$-modules. It is an ideal in $\mathfrak{p}$ and the quotient $\mathfrak{p}_0 := \mathfrak{p}/\mathfrak{n}$ is reductive. We let $P_0 := P/\exp \mathfrak{n}$ be the corresponding quotient group with Lie algebra $\mathfrak{p}_0$. Any P-module W has a filtration

$$0 = W^{(0)} < W^{(1)} < \cdots < W^{(k)} = W \quad \text{with} \quad \mathfrak{n} \cdot W^{(j)} \leqslant W^{(j-1)} \quad \forall\, j \in \{1, \dots k\}, \tag{2.1}$$

by P-submodules, where $\mathfrak{n} \cdot W^{(j)}$ is the span of all $\xi \cdot w$ with $\xi \in \mathfrak{n}$ and $w \in W^{(j)}$. We let $gr(W) := \bigoplus_{1 \leqslant j \leqslant k} W^{(j)}/W^{(j-1)}$, which is a P_0-module.

2.1. Parabolic geometries and Weyl structures.

Let $P \leqslant G$ be a closed Lie subgroup of a Lie group G, whose Lie algebra $\mathfrak{p} \leqslant \mathfrak{g}$ has nilpotent radical $\mathfrak{n} \trianglelefteq \mathfrak{p}$.

Definition 2.1. A *Cartan geometry* of type G/P on a smooth manifold M is a principal P-bundle $\mathcal{G} \to M$ equipped with a P-equivariant 1-form $\theta \colon T\mathcal{G} \to \mathfrak{g}$ such that $\theta_p \colon T_p\mathcal{G} \to \mathfrak{g}$ is an isomorphism for all $p \in \mathcal{G}$, and $\theta(X_\xi) = \xi$ for all $\xi \in \mathfrak{p}$, where $\xi \mapsto X_\xi$ is the infinitesimal $\mathfrak{p}$ action on $\mathcal{G}$. The *homogeneous model* is the Cartan geometry $G \to G/P$ equipped with the Maurer–Cartan form of G.

Any P-module W induces a bundle $\mathcal{W} := \mathcal{G} \times_P W \to M$. A filtration (2.1) of W induces a bundle filtration $0 = \mathcal{W}^{(0)} < \mathcal{W}^{(1)} < \cdots < \mathcal{W}^{(k)} = \mathcal{W}$ with $gr(\mathcal{W}) := \bigoplus_{k \in \mathbb{N}} \mathcal{W}^{(k)}/\mathcal{W}^{(k+1)} \cong \mathcal{G}_0 \times_{P_0} gr(W)$ where $\mathcal{G}_0 := \mathcal{G}/\exp \mathfrak{n}$ is a principal P_0-bundle.

In particular, taking $W = \mathfrak{g}/\mathfrak{p}$, the projection of θ onto $\mathfrak{g}/\mathfrak{p}$ induces a bundle isomorphism $TM \to \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$. This P-module has an inductively defined filtration

$$0 = \mathfrak{h}^{(0)} < \mathfrak{h}^{(1)} < \cdots < \mathfrak{h}^{(k)} = \mathfrak{g}/\mathfrak{p}, \quad \text{where} \quad \mathfrak{h}^{(j)} := \{x \in \mathfrak{g}/\mathfrak{p} \mid \forall\, \xi \in \mathfrak{n},\ \xi \cdot x \in \mathfrak{h}^{(j-1)}\}.$$

In particular $\mathfrak{h} := \mathfrak{h}^{(1)}$ induces a distribution $\mathcal{H} \leqslant TM$ on M. We return to this in §2.3.

We specialize to the case that G is a semisimple Lie group and P is a parabolic subgroup of G, meaning that the nilpotent radical of $\mathfrak{p}$ is its Killing perp $\mathfrak{p}^\perp$ in $\mathfrak{g}$. Then Cartan geometries of type G/P are called *parabolic geometries* and have several distinctive features which we briefly explain and illustrate in the examples below (see [6] for further details).

First, the Killing form of $\mathfrak{g}$ induces a duality between $\mathfrak{p}^\perp$ and $\mathfrak{g}/\mathfrak{p}$, and hence on any parabolic geometry of type G/P, we have a natural isomorphism $\mathcal{G} \times_P \mathfrak{p}^\perp \cong T^*M$ dual to the isomorphism $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$.

Secondly, the principal P_0-bundle $\mathcal{G}_0$ has a distinguished family of principal connections called *Weyl connections.* To see this, it is convenient to fix a parabolic subalgebra $\mathfrak{p}^{\mathrm{op}}$ *opposite* to $\mathfrak{p}$ in the sense that $\mathfrak{g} = \mathfrak{p}^\perp \oplus \mathfrak{p}^{\mathrm{op}}$. This identifies P_0 with a subgroup of P, and induces a decomposition of P_0-modules

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}_0 \oplus \mathfrak{p}^\perp, \tag{2.2}$$

where $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{p}$ is the nilpotent radical of $\mathfrak{p}^{\mathrm{op}}$. A *Weyl structure* is a P_0-equivariant splitting $\iota \colon \mathcal{G}_0 \hookrightarrow \mathcal{G}$ of the projection $\mathcal{G} \to \mathcal{G}_0$ (i.e., a reduction of structure group of $\mathcal{G}$ to P_0); the corresponding Weyl connection is the $\mathfrak{p}_0$-component of $\iota^*\theta$. Weyl structures (or connections) form an affine space modelled on the space of 1-forms on M.

Summary. A manifold M with a parabolic geometry of type G/P comes equipped with: a filtration of the tangent bundle TM, a G_0 structure on $gr(TM)$, and a distinguished class of G_0-connections (the Weyl connections).

There are general results [5, 6] stating that these data are often sufficient to determine the parabolic geometry. Rather than explore this in generality, we turn to examples.

2.2. **Projective parabolic geometries.** We begin with some examples in which $\mathfrak{p}^\perp$ is abelian, hence the filtration of $\mathfrak{g}/\mathfrak{p}$ is trivial and (2.2) is a $\mathbb{Z}$-grading of $\mathfrak{g}$ as a Lie algebra, with $\mathfrak{p}_0$ in degree 0 and $\mathfrak{m}, \mathfrak{p}^\perp$ in degree ± 1 (also called a $|1|$-grading). There is thus a P_0-structure on TM and an algebraic bracket $[\![\cdot,\cdot]\!]$ on $TM \oplus \mathfrak{p}_0(M) \oplus T^*M$ where $\mathfrak{p}_0(M) \leqslant \mathfrak{gl}(TM)$ is the bundle induced by $\mathfrak{p}_0$. In this case a Weyl connection induces a P_0-connection ∇ on TM and any other Weyl connection is given (on vector fields Y, Z) by

$$\hat{\nabla}_Z Y = \nabla_Z Y + [\![[\![Z,\Upsilon]\!], Y]\!] = \nabla_Z Y + [\![Z,\Upsilon]\!] \cdot Y \tag{2.3}$$

for some 1-form Υ, and we write $\hat{\nabla} = \nabla + \Upsilon$ for short.

Projective geometry in dimension m may be viewed as a parabolic geometry of type G/P where $G = PGL(m+1,\mathbb{R})$ and P is the parabolic subgroup of block lower triangular matrices with blocks of sizes m and 1. Here $\mathfrak{m} = \mathbb{R}^m$, $\mathfrak{p}_0 = \mathfrak{gl}(m,\mathbb{R})$, and $\mathfrak{p}^\perp = \mathbb{R}^{m*}$, and the homogeneous model G/P is m-dimensional real projective space $\mathbb{R}P^m$.

On a parabolic geometry of this type, the G_0-structure carries no information as $G_0 \cong GL(m,\mathbb{R})$, but two Weyl connections ∇ and $\hat{\nabla} = \nabla + \Upsilon$ are related (on vector fields Y, Z) by

$$\hat{\nabla}_Z Y = \nabla_Z Y + [\![[\![Z,\Upsilon]\!], Y]\!] = \nabla_Z Y + \Upsilon(Z)Y + \Upsilon(Y)Z. \tag{2.4}$$

Using abstract indices we may write this as

$$\hat{\nabla}_a Y^b = \nabla_a Y^b + \Upsilon_a Y^b + \Upsilon_c Y^c \delta^b_a.$$

Thus the connections ∇ and $\hat{\nabla}$ have the same torsion and the same geodesics (as unparametrized curves). Setting the torsion to zero, we have that the Weyl connections form a projective class $[\nabla]$.

(*Almost*) *c-projective geometry* is a complex analogue of projective geometry [3, 14, 15, 23] with $G = PGL(m+1,\mathbb{C})$ and $P \leqslant G$ block lower triangular as in the projective case, so the homogeneous model G/P is complex projective space $\mathbb{C}P^m$ viewed as a real homogeneous space. A parabolic geometry of this type on a $2m$-manifold M is given by an almost complex structure $J \in \mathfrak{gl}(TM)$ and a class $[\nabla]$ of connections preserving J which differ by

$$\hat{\nabla}_a Y^b = \nabla_a Y^b + \Upsilon_a Y^b - \Upsilon_c J^c_a J^b_d Y^d + \Upsilon_c Y^c \delta^b_a - \Upsilon_c J^c_d Y^d J^b_a,$$

This can be obtained from the real projective formula by substituting $(1,0)$-forms $\Upsilon - iJ\Upsilon$ and $(1,0)$ vectors into (2.4).

(*Almost*) *grassmannian geometries* are generalizations of real projective geometry with $G = PGL(m+k,\mathbb{R})$ and P block lower triangular with blocks of size m and k. The homogeneous model G/P is the grassmannian of k-planes in $\mathbb{R}^{m+k}$. On a parabolic geometry of this type, the G_0-structure is given by an identification of the tangent space with the tensor product of two auxiliary vector bundles E^* and F of ranks k and m (with $\wedge^k E^* \simeq \wedge^m F$). In abstract index notation, we write $e_{A'}$ for a section of E and f_A for a section of F, hence $Y^{A'}_A$ for a vector field and $\eta^B_{A'}$ for a one-form.

The Weyl connections are tensor products of connections on E^* and F with fixed torsion, and the freedom in their choice is (cf. [6, p. 514])

$$\hat{\nabla}^A_{A'} Y^{B'}_B = \nabla^A_{A'} Y^{B'}_B + \delta^{B'}_{A'} \Upsilon^A_{C'} Y^{C'}_B + \delta^A_B \Upsilon^C_{A'} Y^{B'}_C. \tag{2.5}$$

When $m = 2\ell$ and $k = 2$ there is an interesting related geometry obtained by replacing $PGL(2\ell+2,\mathbb{R})$ by another real form of $PGL(2\ell+2,\mathbb{C})$, namely $PGL(\ell+1,\mathbb{H})$. The homogeneous model is then quaternionic projective space $\mathbb{H}P^\ell$, and a parabolic geometry of this type is an (almost) quaternionic manifold [15].

2.3. **Parabolic geometries on filtered manifolds.** We now turn to the examples of greater interest to us, in which $\mathcal{H}$ is a proper subbundle of TM. In fact, in these examples, the geometry is often entirely determined by the distribution $\mathcal{H}$, as we now discuss.

Given a smooth manifold M of dimension m, equipped with a distribution $\mathcal{H} = \mathcal{H}^{(1)} \leqslant TM$ of rank n, the Lie bracket of sections of $\mathcal{H}$ (as vector fields) defines a bundle map $\wedge^2\mathcal{H} \to TM/\mathcal{H}$ called the *Levi form* of $\mathcal{H}$. If we assume the image of the Levi form has constant rank, it defines a subbundle $\mathcal{H}^{(2)} \leqslant TM$ with $\mathcal{H}^{(2)}/\mathcal{H}$ equal to the image. We thus inductively define $\mathcal{H}^{(j)} \leqslant \mathcal{H}^{(j+1)} \leqslant TM$ such that $\mathcal{H}^{(j+1)}/\mathcal{H}^{(j)}$ is the image of the Lie bracket $\mathcal{H} \otimes \mathcal{H}^{(j)} \to TM/\mathcal{H}^{(j)}$. If we further assume $\mathcal{H}$ is *bracket-generating* i.e., $\mathcal{H}^{(k)} = TM$ for some $k \in \mathbb{N}$, then we obtain a filtration

$$0 = \mathcal{H}^{(0)} < \mathcal{H}^{(1)} < \cdots < \mathcal{H}^{(k)} = TM$$

such that the Lie bracket of sections of $\mathcal{H}^{(i)}$ and $\mathcal{H}^{(j)}$ is a section of $\mathcal{H}^{(i+j)}$. The associated graded vector bundle $gr(TM)$ is, at each $x \in M$, a graded Lie algebra $\mathfrak{g}_x$ called the *symbol algebra* of $\mathcal{H}$ at x. We finally assume that the Lie algebras $\mathfrak{g}_x$ are all isomorphic to the same nilpotent radical $\mathfrak{m}$ of a fixed parabolic subalgebra $\mathfrak{p}^{\mathrm{op}}$ in a semisimple Lie algebra $\mathfrak{g}$. In many cases $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{p}^{\mathrm{op}}$, where $\mathfrak{p}$ and $\mathfrak{p}^{\mathrm{op}}$ are opposite in $\mathfrak{g}$, is the full algebra of automorphisms of $\mathfrak{m}$ (as a graded Lie algebra), and, as discussed in [5, 6], this suffices to equip M with a parabolic geometry of type G/P.

The decomposition (2.2) of $\mathfrak{g}$ is no longer $|1|$-graded and this complicates the description of Weyl connections considerably. However, if we work only with *horizontal* (or *partial*) *connections*, i.e., restrict the Weyl connections to covariant derivatives in $\mathcal{H}$ directions only, then the theory is as simple as in the $|1|$-graded case: the Lie bracket between $\mathfrak{m}$ and $\mathfrak{p}^{\perp}$ in $\mathfrak{g}$ induces a Lie bracket between $\mathfrak{h} \leqslant \mathfrak{m}$ and $\mathfrak{p}^{\perp}/[\mathfrak{p}^{\perp}, \mathfrak{p}^{\perp}] \cong \mathfrak{h}^*$ with values in $\mathfrak{p}_0$, and hence an algebraic bracket $[\![\cdot,\cdot]\!] \colon \mathcal{H} \otimes \mathcal{H}^* \to \mathfrak{p}_0(M)$. Any two Weyl connections ∇ and $\hat{\nabla}$ are related by

$$\hat{\nabla}_Z v = \nabla_Z v + [\![Z, \Upsilon]\!] \cdot v$$

where Υ is a section of $\mathcal{H}^*$, Z is a section of $\mathcal{H}$, and v is a section of $\mathcal{G}_0 \times_{P_0} V$ for any G_0-module V. We write $\hat{\nabla}|_{\mathcal{H}} = \nabla|_{\mathcal{H}} + \Upsilon$ for short.

Free distributions are parabolic geometries with $G = SO(n+1, n)$ and P block lower triangular with blocks of sizes n, 1, n, where the inner product is defined on the standard basis $e_0, e_1 \ldots e_{2n}$ by $\langle e_i, e_{n+1+i}\rangle = \langle e_n, e_n\rangle = \langle e_{n+1+i}, e_i\rangle = 1$ for $0 \leqslant i \leqslant n-1$ and all other inner products zero, see [9]. The homogeneous model G/P is the grassmannian of maximal isotropic subspaces of $\mathbb{R}^{2n+1}$. Elements of the Lie algebra $\mathfrak{g} = \mathfrak{so}(n+1, 1)$ have the form

$$\begin{pmatrix} -A^{\mathrm{T}} & -\xi^{\mathrm{T}} & B \\ -\gamma^{\mathrm{T}} & 0 & \xi \\ C & \gamma & A \end{pmatrix}$$

where $B^{\mathrm{T}} = -B$ and $C^{\mathrm{T}} = -C$. Here $A \in \mathfrak{gl}(n, \mathbb{R}) \cong \mathfrak{p}_0$, $\xi \in \mathbb{R}^n \cong \mathfrak{h}$, $\gamma \in \mathbb{R}^{n*} \cong \mathfrak{h}^*$, $B \in \wedge^2\mathbb{R}^n \cong \wedge^2\mathfrak{h}$ and $C \in \wedge^2\mathbb{R}^{n*} \cong \wedge^2\mathfrak{h}^*$.

A parabolic geometry of this type on a manifold M of dimension $\frac{1}{2}n(n+1)$ may be determined by a distribution $\mathcal{H}$ of rank n whose Levi form $\wedge^2\mathcal{H} \to TM/\mathcal{H}$ is an isomorphism, hence the term "free distribution". The P_0-structure is no additional data, and Weyl connections may be determined as P_0-connections ∇ such that for any sections Y, Z of $\mathcal{H}$, the projection of $\nabla_Z Y - \nabla_Y Z$ onto $TM/\mathcal{H} \cong \wedge^2\mathcal{H}$ is $X \wedge Y$. If $\hat{\nabla}|_{\mathcal{H}} = \nabla|_{\mathcal{H}} + \Upsilon$ we then compute that

$$\hat{\nabla}_Z Y = \nabla_Z Y + \Upsilon(Y)Z. \tag{2.6}$$

Free CR or quaternionic CR distributions are obtained by replacing $\mathfrak{so}(n+1,n)$ with $\mathfrak{g} = \mathfrak{su}(n+1,n)$ or $\mathfrak{sp}(n+1,n)$, again with (complex or quaternionic) blocks of sizes n, 1, n, and $\mathfrak{p}$ being block lower triangular [20]. Elements of $\mathfrak{g}$ now have the form

$$\begin{pmatrix} -A^\dagger & -\xi^\dagger & B \\ -\gamma^\dagger & \mu & \xi \\ C & \gamma & A \end{pmatrix}$$

where † denotes the (complex or quaternionic) hermitian conjugate, $B^\dagger = -B$, $C^\dagger = -C$ and $\overline{\mu} = -\mu$. We may thus compute, using matrix commutators

$$\big[[\xi - \xi^\dagger, \gamma - \gamma^\dagger], \eta - \eta^\dagger\big] = \big(\xi\gamma\eta + \eta(\gamma\xi - \xi^\dagger\gamma^\dagger)\big) - \big(\xi\gamma\eta + \eta(\gamma\xi - \xi^\dagger\gamma^\dagger)\big)^\dagger.$$

Note that the order here is important in the quaternionic case.

A parabolic geometry of this type has a complex or quaternionic rank n distribution $\mathcal{H}$ for which the Levi form is complex or quaternionic skew hermitian, inducing an isomorphism of $TM/\mathcal{H}$ with such forms on $\mathcal{H}$. If $\hat{\nabla}|_{\mathcal{H}} = \nabla|_{\mathcal{H}} + \Upsilon$ we then have, on sections Y, Z of $\mathcal{H}$,

$$\hat{\nabla}_Z Y = \nabla_Z Y + Z\,\Upsilon(Y) + Y\,(\Upsilon(Z) - \overline{\Upsilon(Z)}).$$

2.4. **First BGG operators, local metrizability of the homogeneous model, and normal solutions.** Let $\mathcal{G} \to M$, θ be a Cartan geometry of type G/P. The extension of $\mathcal{G}$ by the left action of P on G is a principal G-bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_P G$ with G-connection $\tilde{\theta}\colon \tilde{\mathcal{G}} \to \mathfrak{g}$, and (by construction) a reduction $\mathcal{G} \subseteq \tilde{\mathcal{G}}$ of structure group to P, and this provides an alternative description of the Cartan geometry. It follows that for any G-module V, there is a canonical induced linear connection on $\mathcal{V} = \mathcal{G} \times_P V \cong \tilde{\mathcal{G}} \times_G V$. These bundles are called *tractor bundles* and their sections *tractors*.

In the parabolic case, the *BGG machinery* of [7, 2] provides a sequence of invariant linear differential operators between bundles induced by P-modules associated to V. The first such operator is defined on the bundle $\mathcal{G} \times_P V/(\mathfrak{p}^\perp \boldsymbol{\cdot} V) \cong \mathcal{V}/(T^*M \boldsymbol{\cdot} \mathcal{V})$ and is overdetermined.

When $M = G/P$ is the homogeneous model, the kernel of this *first BGG operator* is in bijection with the space of parallel sections of the tractor bundle $\mathcal{V}$, and the solutions have an explicit polynomial expression in normal coordinates. In more detail, fix an opposite parabolic subalgebra $\mathfrak{p}^{\mathrm{op}}$ to $\mathfrak{p} \leqslant \mathfrak{g}$, inducing a decomposition (2.2). Then $\exp\mathfrak{m} \leqslant G$ is a unipotent subgroup of G which determines a reduction $\mathcal{G}_0 \cong P_0 \exp\mathfrak{m} \leqslant G$ of the homogeneous model $G \to G/P$ to the structure group P_0 over the image M of $\exp\mathfrak{m}$ in G/P, hence a Weyl connection over M, the *normal flat Weyl connection*.

Now if $\mathcal{V} = \mathcal{G}_0 \times_{P_0} V$ is induced by a P_0-module V, the Weyl covariant derivative of sections can be defined as the differentiation of P_0-equivariant V-valued functions on $\mathcal{G}_0$ in the direction of the constant vector fields with respect to the Weyl connection, and the subgroup $\exp\mathfrak{m}$ is tangent to all such constant vector fields. Thus any constant coordinate function $f : \exp\mathfrak{m} \to V$, with $f(x) = f_0$ for all $x \in \exp\mathfrak{m}$, defines a covariantly constant section with values in $\mathcal{V}$. In particular, choosing any nondegenerate symmetric 2-form g in $S^2\mathfrak{m}^*$, the metric defined by the constant g in the normal flat coordinates is covariantly constant with respect to the normal flat Weyl connection. Thus the homogeneous model G/P is locally metrizable. By [4], such explicit formulae also apply on general curved geometries to the so called *normal solutions*, which are those induced by parallel sections of the corresponding tractor bundle. We discuss this further in §3.5.

3. Metrizability and the linearization principle

3.1. **First order operators.** In [22], the second and third authors developed a theory of invariant first order linear operators for parabolic geometries, generalizing work of Fegan [11] in the conformal case (cf. [13, Appendix B]).

We first fix some notation. The Killing form of $\mathfrak{g}$ induces a nondegenerate invariant scalar product on $\mathfrak{p}_0 = \mathfrak{p}/\mathfrak{p}^\perp$, such that the decomposition into the semisimple part $\mathfrak{p}_0^{ss} = [\mathfrak{p}_0, \mathfrak{p}_0]$ and the centre $\mathfrak{z}(\mathfrak{p}_0)$ is orthogonal. Thus any Cartan subalgebra of $\mathfrak{p}_0 \otimes \mathbb{C}$ has an orthogonal decomposition $\mathfrak{t} = \mathfrak{t}' \oplus \mathfrak{t}_0$, where $\mathfrak{t}'$ is a Cartan subalgebra of $\mathfrak{p}_0^{ss} \otimes \mathbb{C}$ and $\mathfrak{t}_0 = \mathfrak{z}(\mathfrak{p}_0) \otimes \mathbb{C}$. Further, $\mathfrak{t}^* = \mathfrak{t}'^* \oplus \mathfrak{t}_0^*$ is the dual decomposition, hence is orthogonal with respect to the induced scalar product on $\mathfrak{t}^*$. We write the corresponding decomposition of a weight $\lambda \in \mathfrak{t}^*$ as $\lambda = \lambda' + \lambda^0$. Let Σ_0 be the set of simple roots α of $\mathfrak{g}$ whose root space $\mathfrak{g}_\alpha$ is in $\mathfrak{h}^* \otimes \mathbb{C}$. The remaining simple roots have root spaces in $\mathfrak{p}_0 \otimes \mathbb{C}$, and hence belong to $\mathfrak{t}'^*$ (i.e., they vanish on $\mathfrak{t}_0$). Hence α^0, for $\alpha \in \Sigma_0$, form a basis for $\mathfrak{t}_0^*$ (dual to the basis of $\mathfrak{t}_0$ formed by the fundamental coweights which belong to $\mathfrak{t}_0$).

Let V_λ be an irreducible complex $\mathfrak{p}_0$-module with highest weight $\lambda = \lambda' + \lambda^0 \in \mathfrak{t}^*$, let $\alpha = \alpha' + \alpha^0 \in \Sigma_0$, and let $\mu = \mu' + \mu^0$ be the highest weight of a component V_μ in the tensor product $V_\lambda \otimes V_\alpha$. The key observation from [22, Theorem 4.4] is that there is a first order invariant operator between sections of the bundles induced by V_λ and V_μ if and only if the scalar expression

$$c_{\lambda,\mu,\alpha} = \tfrac{1}{2}\big((\mu - \lambda, \mu + \lambda + 2\rho') - (\alpha, \alpha + 2\rho')\big)$$

vanishes, where $\rho' \in \mathfrak{t}'^*$ is half the sum of the positive roots of $\mathfrak{p}_0$. We split this expression into contributions from $\mathfrak{t}'^*$ and $\mathfrak{t}_0^*$ using the fact that $\mu^0 = \lambda^0 + \alpha^0$. Thus

$$\begin{aligned} c_{\lambda,\mu,\alpha} &= c_{\lambda',\mu',\alpha'} + \tfrac{1}{2}\big((\alpha_0, 2\lambda^0 + \alpha^0) - (\alpha^0, \alpha^0)\big) \\ &= c_{\lambda',\mu',\alpha'} + (\lambda^0, \alpha^0). \end{aligned} \tag{3.1}$$

If we fix λ', α and μ', this decomposition provides one (real) linear equation on the central weight λ^0. This establishes the existence of many first order operators [22]. Here we exploit (3.1) in a more specific way.

Proposition 3.1. *Let $\lambda' \in \mathfrak{t}'^*$ be the highest weight of a $\mathfrak{p}_0^{ss}$-module, and for each $\alpha \in \Sigma_0$, let $\mu'_\alpha \in \mathfrak{t}'^*$ be the highest weight of an irreducible component of $V_{\alpha'} \otimes V_{\lambda'}$. Then there is a unique central weight $\lambda^0 \in \mathfrak{t}_0^*$ such that for all $\alpha \in \Sigma_0$, there is an invariant linear first order operator between sections of the bundles induced by V_λ and V_{μ_α}, where $\lambda = \lambda' + \lambda^0$ and $\mu_\alpha = \mu'_\alpha + \lambda^0 + \alpha^0$.*

A particular case of this result arises when $\mu'_\alpha = \lambda' + \alpha'$ so that $\mu_\alpha = \lambda + \alpha$ and V_{μ_α} is the Cartan product of V_λ and V_α. In this case, the unique λ^0 is such that $(\lambda, \alpha) = 0$ for all $\alpha \in \Sigma_0$, so that λ is a dominant weight for $\mathfrak{g}$ and the first order system is the first BGG operator on the bundle induced by V_λ.

3.2. **The algebraic linearization condition.** Let $(\mathcal{G} \to M, \theta)$ be a parabolic geometry of type (G, P) and let $\mathfrak{h}$ be the socle of the $\mathfrak{p}$-module $\mathfrak{g}/\mathfrak{p}$, whose central weights form a basis of $\mathfrak{z}(\mathfrak{p}_0)^*$. As we have seen, $\mathcal{G} \times_P \mathfrak{h} \subseteq \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} \cong TM$ defines a (bracket generating) "horizontal" distribution $\mathcal{H} \subseteq TM$. Our aim is to construct compatible subriemannian (or pseudo-riemannian) metrics, i.e., pseudo-riemannian metrics g on $\mathcal{H}$ for which there exists a horizontal metric Weyl connection (a Weyl connection ∇ with $\nabla_Z g = 0$ for all horizontal vector fields Z).

Let $c\colon \mathfrak{h}^* \otimes S^2\mathfrak{h} \to \mathfrak{h}$ be the natural contraction. We then posit the following.

Definition 3.2. A nontrivial $\mathfrak{p}_0$-submodule $B \leqslant S^2\mathfrak{h}$ satisfies the *algebraic linearization condition* (ALC) if and only if B has nondegenerate elements, and there exist $\mathfrak{p}_0$-submodules $\mathfrak{h}_i \leqslant \mathfrak{h}$ and $B_i \leqslant S^2\mathfrak{h}_i$ ($i \in \{1, \dots r\}$) with $\mathfrak{h} = \bigoplus_{i=1}^r \mathfrak{h}_i$ and $B = \bigoplus_{i=1}^r B_i$ such that for each $i \in \{1, \dots r\}$, B_i is irreducible, and for any $\alpha \in \Sigma_0$ and any irreducible component W of $B_i \otimes \mathbb{C}$, $(V_\alpha \otimes W) \cap (\ker c \otimes \mathbb{C})$ is irreducible or zero.

Remark 3.3. Note that $\eta \in B$ is nondegenerate if and only if the same is true for each component $\eta_i \in B_i$. The restrictions $b_i\colon \mathfrak{h}^* \otimes B_i \to \mathfrak{h}_i$ of c are then surjective, and so we may write $\mathfrak{h}^*\otimes B_i = \ker b_i \oplus \zeta_i(\mathfrak{h}_i)$ where $\zeta_i\colon \mathfrak{h}_i \to \mathfrak{h}^*\otimes B_i$ is a $\mathfrak{p}_0$-invariant map with $b_i\circ\zeta_i = id_{\mathfrak{h}_i}$. Since B_i is irreducible, it must lie in a single weight space of $\mathfrak{t}_0$, with weight $-\alpha^0-\beta^0$ where $\alpha,\beta\in\Sigma_0$; hence it is in the image of $\mathfrak{h}_\alpha\otimes\mathfrak{h}_\beta \to S^2\mathfrak{h}\otimes\mathbb{C}$ for the corresponding weight spaces and so $\mathfrak{h}_i\otimes\mathbb{C}$ has at most two irreducible components.

3.3. **The linearization principle.** Suppose first for simplicity that $B\leqslant S^2\mathfrak{h}$ is absolutely irreducible and satisfies the ALC (so $\mathfrak{h}$ has at most two irreducible components) and let $\pi = id_{\mathfrak{h}^*\otimes B} - \zeta\circ b$ be the projection onto $\ker(b\colon \mathfrak{h}^*\otimes B\to\mathfrak{h})$. The linearization method constructs a (pseudo-riemannian) metric on $\mathcal{H}$, i.e., a nondegenerate section g of $S^2\mathcal{H}^*$ from a weighted inverse metric, i.e., a section η of $S^2\mathcal{H}\otimes\mathcal{L}$ for some line bundle $\mathcal{L}$. For this we suppose η is a section of $\mathcal{B}\otimes\mathcal{L}$, where $\mathcal{B}=\mathcal{G}\times_P B$ and $\mathcal{L}$ is a line bundle induced by a weight of $\mathfrak{z}(\mathfrak{p}_0)$. We write b,ζ,π also for the induced bundle homomorphisms (tensored by the identity on $\mathcal{L}$) and choose $\mathcal{L}$ so that there is an invariant first order linear operator $\mathcal{D}$ from $\Gamma(\mathcal{B}\otimes\mathcal{L})$ to $\Gamma(\ker b)$ with $\mathcal{D}=\pi\circ\nabla|_{\mathcal{H}}$ for any Weyl structure ∇. If $\dim B=1$, then $\ker b=0$, so $\mathcal{D}$ is the zero operator, and we take $\mathcal{L}$ to be trivial. Otherwise $\mathcal{L},\mathcal{D}$ are determined by Proposition 3.1. Due to the ALC, $\ker b$ is then a sum of Cartan products of summands of $\mathfrak{h}^*$ and B, and the operator $\mathcal{D}$ is the first BGG operator.

Solutions η of the linear differential equation $\mathcal{D}\eta=0$ are characterized by the fact that for some (hence any) Weyl structure ∇, there is a section X^∇ of $\mathcal{H}\otimes\mathcal{L}$ such that

$$\nabla|_{\mathcal{H}}\eta = \zeta(X^\nabla).$$

Now suppose $\hat\nabla|_{\mathcal{H}} = \nabla|_{\mathcal{H}} + \Upsilon$ with Υ in $\mathcal{H}^*$. Then for any $Z\in\Gamma\mathcal{H}$, $\hat\nabla_Z\eta = \nabla_Z\eta + [\![Z,\Upsilon]\!]\bullet\eta$, and $[\![\cdot,\Upsilon]\!]\bullet\eta$ is in the image of ζ by the invariance of $\mathcal{D}$. Hence by Schur's lemma and §3.1 (i.e., [22]):

$$[\![\cdot,\Upsilon]\!]\bullet\eta = (\zeta\circ b)([\![\cdot,\Upsilon]\!]\bullet\eta) = (\zeta\circ b)\Big(\textstyle\sum_{\alpha\in\Sigma_0}\ell_\alpha\Upsilon_\alpha\otimes\eta\Big)$$

for nonzero scalars ℓ_α, where $\Upsilon=\sum_{\alpha\in\Sigma_0}\Upsilon_\alpha$ with $\Upsilon_\alpha\in V_\alpha\subseteq\mathfrak{h}^*\otimes\mathbb{C}$. If we define $\sharp_\eta(\Upsilon) = \sum_{\alpha\in\Sigma_0}\ell_\alpha b(\Upsilon_\alpha\otimes\eta)$, we deduce that

$$\hat\nabla|_{\mathcal{H}}\eta = \nabla|_{\mathcal{H}}\eta + \zeta(\sharp_\eta(\Upsilon)).$$

Now if η is a nondegenerate solution of $\mathcal{D}\eta=0$, with $\nabla|_{\mathcal{H}}\eta=\zeta(X^\nabla)$ for some Weyl connection ∇ and $X^\nabla\in\Gamma(\mathcal{H}\otimes\mathcal{L})$, we may take $\Upsilon=-\sharp_\eta^{-1}(X^\nabla)$ to obtain

$$\hat\nabla|_{\mathcal{H}}\eta = \zeta(X^\nabla)+\zeta(\sharp_\eta(\Upsilon)) = 0.$$

Hence η is (inverse to) a horizontal compatible metric, up to the shift of the weight via the line bundle $\mathcal{L}$. Finally, the nondegenerate weighted metric η allows us to build a nonvanishing section σ of the line bundle $\wedge^m\mathcal{H}\otimes\mathcal{L}^{m/2}$, where $m=\dim\mathfrak{h}$, with $\hat\nabla|_{\mathcal{H}}\sigma=0$. This line bundle cannot be trivial because the central weight of $\mathcal{B}\otimes\mathcal{L}$ is not zero. If $\mathfrak{h}$ is absolutely irreducible, then $\wedge^m\mathcal{H}\otimes\mathcal{L}^{m/2}\cong\mathcal{L}^k$ for some nonzero k, and then $\psi=(\sigma^{-1/k}\eta)^{-1}$ is a section of $\mathcal{B}^*$ with $\hat\nabla|_{\mathcal{H}}\psi=0$. Otherwise, we need to assume the central weights of $\wedge^m\mathcal{H}$ and $\mathcal{L}$ are linearly dependent. The most natural way to achieve this is to suppose that the simple roots α,β with $\mathfrak{h}\otimes\mathbb{C}=\mathfrak{h}_\alpha\oplus\mathfrak{h}_\beta$ are related by an automorphism of the Dynkin diagram of $\mathfrak{g}$.

Definition 3.4. A $\mathfrak{p}_0$-submodule $B\leqslant S^2\mathfrak{h}$ satisfies the *strong algebraic linearization condition* (strong ALC) if and only if B satisfies the ALC with respect to $\mathfrak{p}_0$-submodules $\mathfrak{h}_i\leqslant\mathfrak{h}$ such that whenever $\mathfrak{h}_i\otimes\mathbb{C}=\mathfrak{h}_\alpha\oplus\mathfrak{h}_\beta$ for $\alpha,\beta\in\Sigma_0$, there is an automorphism of the Dynkin diagram of $\mathfrak{g}$ preserving Σ_0 and interchanging α and β.

With this definition, the linearization method yields the following result.

Theorem 1. *Let $B\leqslant S^2\mathfrak{h}$ satisfy the strong ALC with respect to $B=\bigoplus_{i=1}^r B_i$ and $\mathfrak{h}=\bigoplus_{i=1}^r\mathfrak{h}_i$. Then for all $i\in\{1,\dots r\}$ there are induced line bundles $\mathcal{L}_i$ and invariant first* *order linear operators $\mathcal{D}_i$ acting on sections of $\mathcal{B}_i \otimes \mathcal{L}_i$ such that there is a bijection between nondegenerate solutions $\eta_i : i \in \{1, \dots r\}$ of the equations $\mathcal{D}_i(\eta_i) = 0$, and nondegenerate sections ψ of $\mathcal{B}^*$ with $\nabla|_{\mathcal{H}}\psi = 0$ for some Weyl connection ∇.*

Proof. Define b_i, ζ_i as in Remark 3.3 so that $\mathfrak{h}^* \otimes B_i = \ker b_i \oplus \zeta_i(\mathfrak{h}_i)$, let $\pi_i = id_{\mathfrak{h}^*\otimes B_i} - \zeta_i \circ b_i$ be the projection onto $\ker b_i$, and let $\Sigma_0^i = \{\alpha \in \Sigma_0 : V_\alpha \subseteq \mathfrak{h}_i^* \otimes \mathbb{C}\}$. We apply the same ideas as in the absolutely irreducible case to each irreducible component $V_{\lambda'}$ of $B_i \otimes \mathbb{C}$. If $\dim V_{\lambda'} \geqslant 2$ then the ALC implies that $(V_\alpha \otimes V_{\lambda'}) \cap (\ker b_i \otimes \mathbb{C})$ is irreducible for all $\alpha \in \Sigma_0$, hence Proposition 3.1 provides a unique λ^0 so that there is an invariant first order operator between sections of the bundles induced by V_λ and $\ker b_i \otimes \mathbb{C}$. If instead, $\dim V_{\lambda'} = 1$, then $V_\alpha \otimes V_{\lambda'}$ is irreducible, and is contained in $\ker b_i \otimes \mathbb{C}$ unless $\alpha \in \Sigma_0^i$. We thus supplement (3.1) by the equations $(\lambda^0, \alpha^0) = 0$ when $\alpha \in \Sigma_0^i$.

Since B_i is irreducible, $B_i \otimes \mathbb{C}$ is either irreducible or has two irreducible components with conjugate weights. Now the system of equations (3.1) and $(\lambda^0, \alpha^0) = 0$ that we impose to find λ^0 are conjugation invariant. Hence in either case, we obtain a line bundle $\mathcal{L}_i$ and an invariant first order linear operator $\mathcal{D}_i := \pi_i \circ \nabla|_{\mathcal{H}}$ on $\mathcal{B}_i \otimes \mathcal{L}_i$, so that any section η_i satisfies $\mathcal{D}_i(\eta_i) = 0$ if and only if

$$\nabla|_{\mathcal{H}}\eta_i = \zeta_i(X_i^\nabla)$$

for a suitable section X_i^∇ of $\mathcal{H}_i \otimes \mathcal{L}_i$. Given such sections η_i, let $\eta = \sum_{i=1}^r \eta_i$. By construction, the operator $b_i \circ \nabla|_{\mathcal{H}}$ is not invariant on $\mathcal{B}_i \otimes \mathcal{L}_i$. Hence by Schur's Lemma, there are *nonzero* scalars ℓ_α such that

$$[\![\cdot, \Upsilon]\!] \bullet \eta = (\zeta \circ b)([\![\cdot, \Upsilon]\!] \bullet \eta) = (\zeta \circ b)\Big(\textstyle\sum_{i=1}^r \sum_{\alpha\in\Sigma_0^i} \ell_\alpha \Upsilon_\alpha \otimes \eta_i\Big) = \Big(\textstyle\sum_{\alpha\in\Sigma_0} \ell_\alpha \Upsilon_\alpha \otimes \eta\Big),$$

where $\Upsilon = \sum_{\alpha\in\Sigma_0} \Upsilon_\alpha$ as before. As before, we define $\sharp_\eta(\Upsilon) = \sum_{\alpha\in\Sigma_0} \ell_\alpha b(\Upsilon_\alpha \otimes \eta)$, so that if $\hat\nabla|_{\mathcal{H}} = \nabla|_{\mathcal{H}} + \Upsilon$ then

$$\hat\nabla|_{\mathcal{H}}\eta = \nabla|_{\mathcal{H}}\eta + \zeta(\sharp_\eta(\Upsilon)).$$

If η is a nondegenerate then $\sharp_\eta$ is invertible, and so if $\mathcal{D}(\eta) = 0$, i.e., $\mathcal{D}_i(\eta_i) = 0$ for all i, then we may set $\Upsilon := -\sharp_\eta^{-1}(X^\nabla)$, where $X^\nabla = \sum_{i=1}^r X_i^\nabla$ to obtain $\hat\nabla|_{\mathcal{H}}\eta = 0$.

Finally, taking volume forms of η_i on $\mathcal{H}_i$ for each i, we obtain nonvanishing sections σ_i of $\wedge^{m_i}\mathcal{H}_i \otimes \mathcal{L}_i^{m_i/2}$ with $\hat\nabla|_{\mathcal{H}}\sigma_i = 0$. The weights of the σ_i are linearly independent, and the strong ALC ensures that the central weights of the $\mathcal{L}_i$ are linear combinations of the central weights of $\wedge^{m_j}\mathcal{H}_j$, so for every i, we can solve the linear system $\eta_i \otimes \bigotimes_j \sigma_j^{a_{ij}} \in \mathcal{B}_i$, and hence, inverting each component, obtain the section ψ of $\mathcal{B}^*$ as required. Since the system is invertible, η can be obtained from ψ and its volume forms on each $\mathcal{H}_i$. ☐

If only the ALC is assumed, then the proof yields, in place of horizontally parallel metrics on $\mathcal{H}$, horizontally parallel conformal structures on each $\mathcal{H}_i$ and horizontally parallel sections of some line bundles.

3.4. **Example: projective geometry.** Let us illustrate the metrizability procedure by showing how the well-known example of projective geometry [8, 10, 17, 21] fits into the general method. Here $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R}) = \mathfrak{h} \oplus \mathfrak{gl}(\mathfrak{h}) \oplus \mathfrak{h}^*$ and $S^2\mathfrak{h}$ is irreducible. Since $\mathfrak{h}^* \otimes S^2\mathfrak{h} \cong \mathfrak{h} \oplus (\mathfrak{h}^* \otimes_0 S^2\mathfrak{h})$, where the second summand is the trace-free part (the Cartan product), $B = S^2\mathfrak{h}$ satisfies the ALC. The class of covariant derivatives defining the projective structure depends on an arbitrary 1-form Υ_a and two of them are related by (2.4). Hence on a section φ of $\mathcal{B} = S^2TM$, we have

$$[\![Z, \Upsilon]\!] \bullet \varphi = 2\Upsilon(Z)\varphi + Z \otimes \varphi(\Upsilon, \cdot) + \varphi(\Upsilon, \cdot) \otimes Z$$

for any vector field Z and 1-form Υ. If we twist by the line bundle $\mathcal{L}$ induced by the P_0-module L with highest weight $-2\omega_1$, then for $\eta \in \Gamma(\mathcal{B} \otimes \mathcal{L})$ and $\hat\nabla = \nabla + \Upsilon$, we have

$$\hat\nabla_Z\eta = \nabla_Z\eta + b(\Upsilon \otimes \eta) \odot Z$$

where $X \odot Z = X \otimes Z + Z \otimes X$ and $b(\Upsilon \otimes \eta) = \eta(\Upsilon, \cdot)$ is the natural contraction. In abstract indices this contraction of $\Upsilon_c \eta^{ab}$ is $\Upsilon_a \eta^{ab}$ and hence

$$\hat{\nabla}_c \eta^{ab} = \nabla_c \eta^{ab} + \delta^a_c \Upsilon_d \eta^{bd} + \delta^b_c \Upsilon_d \eta^{ad}.$$

We thus have an invariant first order operator acting on η (a first BGG operator) whose solutions satisfy

$$\nabla_Z \eta = \langle Z, \zeta(X^\nabla) \rangle = \tfrac{1}{n+1} X^\nabla \odot Z.$$

for some section X^∇ of $TM \otimes \mathcal{L}$. (Here $\zeta(X) = \frac{1}{n+1} X \odot id$, or in abstract indices, $\zeta(X^a) = \frac{1}{n+1}\left(X^a \delta^b_c + X^b \delta^a_c\right)$ so that $b(\zeta(X)) = X$.) Evidently η is parallel for $\hat{\nabla}$ provided $b(\Upsilon \otimes \eta) = -\frac{1}{n+1} X^\nabla$, which we can solve for Υ if η is nondegenerate. Direct computation shows that $\det(\eta)$ is a section of $\mathcal{L}^{-2}$. So $g^{ab} := \det(\eta)\eta^{ab}$ is a nondegenerate section of S^2TM and its inverse is parallel with respect to $\hat{\nabla}$. In terms of the general theory herein, if

$$[\![\cdot, \Upsilon]\!] \bullet \eta = \ell\, (\zeta \circ b)(\Upsilon \otimes \eta),$$

then

$$b(\Upsilon \otimes \eta) \odot Z = [\![Z, \Upsilon]\!] \bullet \eta = \ell \,\langle Z, (\zeta \circ b)(\Upsilon \otimes \eta) \rangle = \frac{\ell}{n+1} b(\Upsilon \otimes \eta) \odot Z,$$

and so $\ell = n + 1$. Hence $\sharp_\eta(\Upsilon) = (n + 1)b(\Upsilon \otimes \eta)$ and the solution is $\Upsilon = -\sharp_\eta^{-1}(X_\nabla)$.

3.5. **The metric tractor bundle.** As we have seen in §2.4, the homogeneous model G/P is always locally metrizable, and solutions in the kernel of a given first BGG operator are induced by parallel sections of a corresponding *metric* tractor bundle. In general, if M has nontrivial curvature, not all solutions to a linearized metrizability problem will correspond to such parallel sections: as discussed in §2.4, those that do are called *normal solutions* and exhibit special features. In particular, as shown in [4], they are always of a simple polynomial forms in normal coordinates, exactly as in the homogeneous model. Thus the explicit formulae from the homogeneous case form an ansatz for solutions in general.

Let us discuss this in the case of free distributions from §2.3. Here $\mathfrak{g} = \mathfrak{so}(n + 1, n) = \wedge^2\mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{gl}(\mathfrak{h}) \oplus \mathfrak{h}^* \oplus \wedge^2\mathfrak{h}^*$, $\mathcal{B} = S^2\mathfrak{h}$ is irreducible and satisfies the ALC, just as in the case of projective geometry. In this case, however, there is no need to twist by a line bundle, since by (2.6), we already have

$$\hat{\nabla}_Z \eta = \nabla_Z \eta + b(\Upsilon \otimes \eta) \odot Z$$

for any sections Z of $\mathcal{H}$ and η of $S^2\mathfrak{h}$, where $\hat{\nabla}|_{\mathcal{H}} = \nabla|_{\mathcal{H}} + \Upsilon$ and b is the natural contraction. The solution of the linearized metrizability problem then proceeds exactly as in the projective case, so we now consider the form of the normal solutions.

The *standard tractor bundle* is the bundle associated to the defining representation V of $G = SO(n + 1, n)$. Explicitly, using the matrix description in §2.3, we may write elements of V as column vectors

$$v = \begin{pmatrix} \lambda^a \\ \tau \\ \ell_a \end{pmatrix}$$

on which the action of the nilpotent radical $\mathfrak{m}$ of $\mathfrak{p}^{\mathrm{op}}$ is given by

$$\boldsymbol{x} \bullet v = \begin{pmatrix} 0 & x^a & y^{ab} \\ 0 & 0 & -x^a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda^b \\ \tau \\ \ell_b \end{pmatrix} = \begin{pmatrix} x^a\tau + y^{ab}\ell_b \\ -x^b\ell_b \\ 0 \end{pmatrix}.$$

The metric tractor bundle in this example is associated to the symmetric tracefree square $S^2_0 V$ of V. Elements of the symmetric square S^2V are given by

$$\Phi = \begin{pmatrix} \nu^{ab} \\ \sigma^b \\ \kappa \mid \psi^c_b \\ \xi_b \\ \tau_{bc} \end{pmatrix},$$

where ν^{ab} and τ_{bc} are symmetric, and such and element is in $S^2_0 V$ if $\kappa = -\psi^c_c$. Our convention is such that $\Phi = v \odot \tilde{v}$ has components

$$\nu^{ab} = \lambda^a \tilde{\lambda}^b + \tilde{\lambda}^a \lambda^b; \quad \sigma^b = \lambda^b \tilde{\tau} + \tau \tilde{\lambda}^b; \quad \kappa = \tau\tilde{\tau}; \quad \psi^c_b = \ell_b \tilde{\lambda}^c + \lambda^c \tilde{\ell}_b;$$
$$\xi_b = \ell_b \tilde{\tau} + \tau \tilde{\ell}_b; \quad \tau_{bc} = \ell_a \tilde{\ell}_b + \tilde{\ell}_a \ell_b.$$

The action of the nilpotent radical on the symmetric square is given by

$$\boldsymbol{x} \bullet \Phi := \begin{pmatrix} 0 & x^a & y^{ab} \\ 0 & 0 & -x^a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \nu^{ab} \\ \sigma^b \\ \kappa \mid \psi^c_b \\ \xi_b \\ \tau_{bc} \end{pmatrix} = \begin{pmatrix} x^{(a}\sigma^{b)} - y^{c(a}\psi^{b)}_c \\ x^c \psi^b_c + y^{bc}\xi_c - x^b \kappa \\ -x^b \xi_b \mid x^c \xi_b \\ -x^a \tau_{ab} \\ 0 \end{pmatrix},$$

where $x^{(a}\sigma^{b)} = x^a \otimes \sigma^b + \sigma^b \otimes x^a$. The iterated action is therefore given by

$$\boldsymbol{x} \bullet \boldsymbol{x} \bullet \Phi = \begin{pmatrix} x^c x^{(a}\psi^{b)}_c + 2x^{(a}y^{b)c}\xi_c - x^a x^b \kappa \\ 2x^c x^b \xi_c - y^{bc} x^a \tau_{ac} \\ x^b x^a \tau_{ab} \mid -x^c x^a \tau_{ab} \\ 0 \\ 0 \end{pmatrix},$$

$$\boldsymbol{x} \bullet \boldsymbol{x} \bullet \boldsymbol{x} \bullet \Phi = \begin{pmatrix} 4x^a x^b x^c \xi_c - 2x^{(a}y^{b)c}x^d \tau_{dc} \\ -2x^b x^a x^c \tau_{ac} \\ 0 \mid 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{x} \bullet \boldsymbol{x} \bullet \boldsymbol{x} \bullet \boldsymbol{x} \bullet \Phi = \begin{pmatrix} -4x^a x^b x^c x^d \tau_{cd} \\ 0 \\ 0 \mid 0 \\ 0 \\ 0 \end{pmatrix}$$

with all further iterates zero. The normal solution is the projection onto $S^2\mathcal{H}$ of $\exp(\boldsymbol{x}) \cdot \Phi$, which is given by

$$\eta^{ab}(x,y) = \nu^{ab} + x^{(a}\sigma^{b)} - y^{c(a}\psi^{b)}_c + \tfrac{1}{2}x^c x^{(a}\psi^{b)}_c + x^{(a}y^{b)c}\xi_c + \tfrac{1}{2}x^a x^b \psi^c_c$$
$$+ \tfrac{2}{3}x^a x^b x^c \xi_c - \tfrac{1}{3}x^{(a}y^{b)c}x^d \tau_{dc} - \tfrac{1}{6}x^a x^b x^c x^d \tau_{cd}.$$

4. Classification of metric parabolic geometries with irreducible $\mathfrak{h}$

We have seen that the linearizability problem of the existence of compatible subriemannian metrics on parabolic geometries reduces to a purely algebraic question related to the number of components in certain tensor products of the $\mathfrak{p}_0$-modules $\mathfrak{h}$ and its dual $\mathfrak{h}^*$. In fact, we are only interested in the actions of the semisimple part of $\mathfrak{p}_0 = \mathfrak{p}/\mathfrak{p}^\perp$.

In this section, we classify all cases of the ALC where the defining distribution of the parabolic geometry corresponding to $\mathfrak{h}$ is irreducible. This is the case with all $|1|$-graded geometries, but many $|2|$-graded and some more general geometries are involved too. In order to keep the story short, while still providing a complete and simple picture, we use the schematic description of the chosen type of parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ by crosses on the Dynkin diagram for $\mathfrak{g}$ and we write weights of $\mathfrak{p}$-modules as linear combinations of the fundamental weights for $\mathfrak{g}$, depicted as the nonzero coefficients over the nodes of the diagrams, ignoring those over the crossed nodes (see e.g. [6, §3.2] for these conventions). This exactly provides the complete information on the representation of the semisimple part of $\mathfrak{p}_0$ in the case of complex algebras and we always add further information on specific real forms of them. Actually for practical reasons (and in accordance with common practice), we rather write the weights of the dual $\mathfrak{p}_0$-modules over the Dynkin diagram. Moreover, the displayed diagrams and weights always correspond to the complexified versions and thus we have to keep in mind their meaning for particular real forms.

The classification is given in the following theorem. In the proof we also describe the geometric properties of the metrics in any admissible component B, mostly in terms of special structure related to the given parabolic geometry. The classification in Table 1 was also obtained in [19].

Theorem 2. *Let $\mathfrak{p}$ be a parabolic subalgebra in a real simple Lie algebra $\mathfrak{g}$ and let B be a $\mathfrak{p}$-submodule of $S^2\mathfrak{h}$, with $\mathfrak{h} \cong (\mathfrak{p}^\perp/[\mathfrak{p}^\perp, \mathfrak{p}^\perp])^*$ irreducible. Then B satisfies the ALC and admits nondegenerate elements if and only if one of the following holds*:

- $\mathfrak{g}$ *is complex and the complexification of* $(\mathfrak{p}, B)$ *appears in Table* 1;
- $(\mathfrak{g}, \mathfrak{p}, B)$ *appears as a real form in Table* 2 *or* 3;
- $(\mathfrak{g}, \mathfrak{p}, B)$ *is* (*the underlying real Lie algebra of*) *the complexification of a triple appearing in Table* 2.

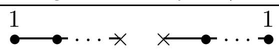
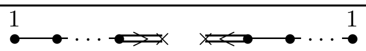
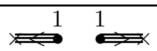
Case	*Diagram* Δ_ℓ *for* $\mathfrak{p}, B$	*Real simple* $\mathfrak{g}$	*Growth*
A_ℓ^h		$\mathfrak{sl}(\ell+1,\mathbb{C})$ $\ell \geqslant 2$	2ℓ
B_ℓ^h		$\mathfrak{so}(2\ell+1,\mathbb{C})$ $\ell \geqslant 2$	$2k, 2k+k(k-1)$
G_2^h		$G_2^{\mathbb{C}}$	$4, 6, 10$

TABLE 1. Complex geometries with hermitian B

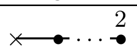
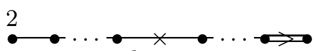
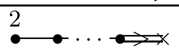
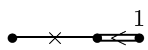
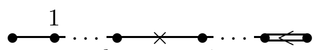
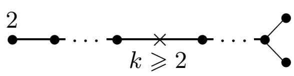
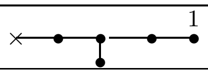
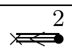
Case	*Diagram* Δ_ℓ *for* $\mathfrak{p}, B$	*Real simple* $\mathfrak{g}$	*Growth*
$A_\ell^{1,1}$		$\mathfrak{sl}(\ell+1,\mathbb{R})$ $\ell \geqslant 2$	ℓ
$A_\ell^{1,2}$		$\mathfrak{sl}(\ell+1,\mathbb{R})$, $\mathfrak{sl}(p+1,\mathbb{H})$ $\ell = 2p+1$, $p \geqslant 2$	$4p$
$B_\ell^{1,k}$	 $k \geqslant 2$	$\mathfrak{so}(p,q)$, $k \leqslant p \leqslant q$ $p+q = 2\ell+1$	$d = k(2\ell-2k+1)$, $n = d + \frac{1}{2}k(k-1)$
$B_\ell^{1,\ell}$		$\mathfrak{so}(\ell,\ell+1)$ $\ell \geqslant 2$	$k, k+\frac{1}{2}k(k-1)$
$C_4^{1,2}$		$\mathfrak{sp}(8,\mathbb{R})$ $\mathfrak{sp}(2,2)$ $\mathfrak{sp}(1,3)$	$8, 11$
$C_\ell^{1,k}$	 $k = 2j \geqslant 4$	$\mathfrak{sp}(2\ell,\mathbb{R})$ $\mathfrak{sp}(p,q)$ $\ell = p+q$, $k \leqslant p \leqslant q$	$d = k(2\ell-2k)$, $n = d + \frac{1}{2}k(k+1)$
$D_\ell^{1,k}$	 $k \geqslant 2$	$\mathfrak{so}(p,q)$ $\mathfrak{so}^*(2\ell)$ $2\ell = p+q$ $k = 2j$ $k \leqslant p \leqslant q$ $k \leqslant \ell-2$	$d = k(2\ell-2k)$, $n = d + \frac{1}{2}k(k-1)$
$E_6^{1,1}$		$E_{6(6)}$, $E_{6(-26)}$	16
$G_2^{1,1}$		$G_{2(2)}$	$2, 3, 5$

TABLE 2. Real geometries with absolutely irreducible $\mathfrak{h}$

Outline of Proof. In the gradings of the complex algebras $\mathfrak{g}$ corresponding to parabolic geometries, the number of irreducible components of $\mathfrak{h}^*$ is equal to the number of crosses in

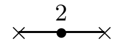
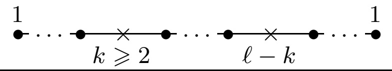
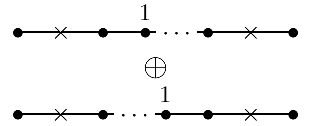
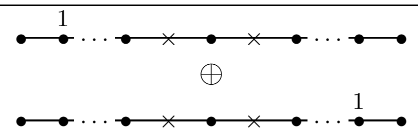
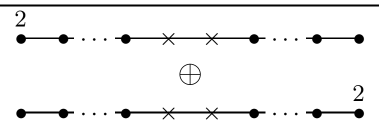
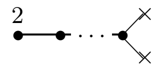
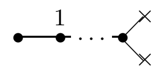
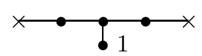
Case	*Diagram Δ_ℓ for $\mathfrak{p}, B$*	*Real simple $\mathfrak{g}$*	*Growth*
$A_3^{2,1}$		$\mathfrak{su}(1,3),\ \mathfrak{su}(2,2)$	$4, 5$
$A_\ell^{2,k}$		$\mathfrak{su}(p,q),\ k \leqslant p \leqslant q$ $\ell = p+q-1 \geqslant 4$	$d = 2k(\ell - 2k + 1)$, $n = d + k^2$
$A_\ell^{2,h}$		$\mathfrak{su}(p,q),\ 2 \leqslant p \leqslant q$ $\ell = p+q-1 \geqslant 6$	$4(\ell-3), 4(\ell-2)$
$A_{2k+1}^{2,s}$		$\mathfrak{su}(k, k+2)$, $\mathfrak{su}(k+1, k+1)$ $\ell = 2k+1 \geqslant 7$	$4k, 4k + k^2$
$A_{2k}^{2,s}$		$\mathfrak{su}(k, k+1)$ $\ell = 2k \geqslant 4$	$2k, 2k + k^2$
$D_\ell^{2,s}$		$\mathfrak{so}(\ell-1, \ell+1)$ $\mathfrak{so}^*(2\ell),\ \ell = 2j+1$	$d = 2(\ell-1)$, $d + \frac{1}{2}(\ell-1)(\ell-2)$
$D_\ell^{2,h}$		$\mathfrak{so}(\ell-1, \ell+1)$ $\mathfrak{so}^*(2\ell),\ \ell = 2j+1$	$d = 2(\ell-1)$, $d + \frac{1}{2}(\ell-1)(\ell-2)$
$E_6^{2,h}$		$E_{6(2)}$	$16, 24$

TABLE 3. Real geometries with $\mathfrak{h}$ not absolutely irreducible

the Dynkin diagram describing the chosen parabolic subalgebra. However, in the real forms of $\mathfrak{g}$, there might be complex or quaternionic components giving rise to two components in the complexification. These two complex components have to be either conjugate (in the complex case) or isomorphic (in the quaternionic case).

The latter observation reduces our quest to diagrams with two crosses placed in a symmetric way. Indeed, more than two crosses cannot result in one component, while asymmetric positions of the crosses inevitably yield two complex components which are neither conjugate nor isomorphic. Moreover, having two components in the complexified $\mathfrak{h}$, we may ignore the symmetric products of the individual parts in $S^2\mathfrak{h}$, because there cannot be any nondegenerate metrics there.

We first dispense with the case that $\mathfrak{g}$ is complex but B is not, so that $B \otimes \mathbb{C}$ is irreducible in $\mathfrak{g} \otimes \mathbb{C} \cong \mathfrak{g} \oplus \mathfrak{g}$ and the diagram for $(\mathfrak{p}, B)$ is invariant under the automorphism exchanging the two components of the Dynkin diagram. Thus $B \otimes \mathbb{C} = \mathfrak{h}_\alpha \otimes \mathfrak{h}_\beta$ where $\mathfrak{h} \otimes \mathbb{C} = \mathfrak{h}_\alpha \oplus \mathfrak{h}_\beta$. Now the ALC is satisfied provided $\mathfrak{h}_\alpha \otimes \mathfrak{h}_\alpha^*$ (and hence also $\mathfrak{h}_\beta \otimes \mathfrak{h}_\beta^*$) has precisely two irreducible components as a representation of a component of $\mathfrak{p}_0 \otimes \mathbb{C}$. Only the (dual) defining representations in type A have this property, and so $\mathfrak{g}$ must have type A, B or G, where the nodes crossed in $\mathfrak{g} \otimes \mathbb{C}$ are end nodes corresponding to short simple roots. The possibilities are listed in Table 1, covering the following three cases:

Case 1 (A_ℓ^h)**.** The c-projective geometries may be equipped with distinguished hermitian metrics.

Case 2 (B_ℓ^h)**.** The almost complex version of a free distribution of rank k, may be equipped with distinguished hermitian metrics.

Case 3 (G_2^h)**.** The almost complex version of the $(2,3,5)$-distributions may be equipped with distinguished hermitian metrics.

We analyse the remaining real cases with irreducible $\mathfrak{h}$ by the Dynkin type of $\mathfrak{g}$ in the following sections. □

4.1. **Proof of Theorem 2 when $\mathfrak{g}$ has type A_ℓ.** The case $\ell = 1$ is trivial, so we assume $\ell \geqslant 2$, and first consider the case of a single crossed node. If the crossed node is one of the ends of the Dynkin diagram, the only real $\mathfrak{g}$ is the split form, $\mathfrak{h}$ and $S^2\mathfrak{h}$ are irreducible, and $B = S^2\mathfrak{h}$ satisfies the ALC: when $\ell = 2$,

$$B \simeq \times\!-\!\overset{2}{\bullet} \qquad \mathfrak{h}^* \otimes B \simeq \times\!-\!\overset{3}{\bullet} \oplus \times\!-\!\overset{1}{\bullet}$$

and when $\ell \geqslant 3$,

$$B \simeq \times\!-\!\bullet \cdots \overset{2}{\bullet} \qquad \mathfrak{h}^* \otimes B \simeq \times\!-\!\overset{1}{\bullet} \cdots \overset{2}{\bullet} \oplus \times\!-\!\bullet \cdots \overset{1}{\bullet}.$$

These examples can be summarized in the following statement.

Case 4 ($A_\ell^{1,1}$)**.** Here $\mathfrak{g} = \mathfrak{sl}(\ell+1, \mathbb{R})$, $\ell \geqslant 2$, $\mathfrak{h} \cong \mathbb{R}^\ell$ and $B = S^2\mathfrak{h}$. This is the most classical case of projective structures on ℓ-dimensional manifolds M, and nondegenerate sections of $\mathcal{B}$ are inverse to arbitrary pseudo-Riemannian metrics on M.

Suppose next that the cross is adjacent to one end of the diagram, with $\ell \geqslant 3$. We then have $S^2\mathfrak{h} = B \oplus B'$, where

$$\mathfrak{h} \simeq \overset{1}{\bullet}\!-\!\times\!-\!\bullet \cdots \overset{1}{\bullet} \qquad \mathfrak{h}^* \simeq \overset{1}{\bullet}\!-\!\times\!-\!\overset{1}{\bullet} \cdots \bullet$$

$$B \simeq \bullet\!-\!\times\!-\!\bullet \cdots \overset{1}{\bullet}\!-\!\bullet \;\; (\ell \geqslant 4) \qquad B' \simeq \overset{2}{\bullet}\!-\!\times\!-\!\bullet \cdots \overset{2}{\bullet}$$

and B is trivial for $\ell = 3$ (when $\mathfrak{h} \cong \mathfrak{h}^*$). The tensor product $\mathfrak{h}^* \otimes B'$ decomposes into four irreducible components, except for the real form $\mathfrak{su}(2,2)$ when $\ell = 3$, in which case there are only three components. In any case, B' does not satisfy the ALC.

In order for B to have nondegenerate elements, ℓ must be odd, and for $\ell = 2p + 1 \geqslant 5$, $\mathfrak{h}^* \otimes B \simeq \overset{1}{\bullet}\!-\!\times\!-\!\overset{1}{\bullet}\!-\!\bullet \cdots \overset{1}{\bullet}\!-\!\bullet \oplus \overset{1}{\bullet}\!-\!\times\!-\!\bullet \cdots \overset{1}{\bullet}$; thus the ALC holds for B.

Case 5 ($A_\ell^{1,2}$)**.** For each $\ell = 2p + 1 \geqslant 5$, there are two real forms. When $\mathfrak{g} \simeq \mathfrak{sl}(2p+2, \mathbb{R})$, the geometries are the almost grassmannian structures on manifolds M of dimension $4p$, modelled on the grassmannian of 2-planes in $\mathbb{R}^{2p}$. The tangent bundle TM is identified with a tensor product $E \otimes F$, where *rank* $E = 2$, *rank* $F = 2p$, and the nondegenerate metrics in $\mathcal{B}$ are tensor products of area forms on E and symplectic forms on F. When $\mathfrak{g} \simeq \mathfrak{sl}(p, \mathbb{H})$, the geometries are almost quaternionic geometries, where TM is a quaternionic vector bundle, and the nondegenerate metrics in $\mathcal{B}$ are the (real parts of) quaternionic hermitian forms.

When the cross is further from the ends of the diagram, we have $S^2\mathfrak{h} = B \oplus B'$ with

$$B \simeq \bullet\!-\!\overset{1}{\bullet} \cdots \bullet\!-\!\times\!-\!\bullet \cdots \overset{1}{\bullet}\!-\!\bullet \qquad B' \simeq \overset{2}{\bullet} \cdots \bullet\!-\!\times\!-\!\bullet \cdots \overset{2}{\bullet}.$$

and there are too many components in both $\mathfrak{h}^* \otimes B$ and $\mathfrak{h}^* \otimes B'$ to satisfy the ALC.

We now turn to cases with two crossed nodes, related by the diagram automorphism of A_ℓ. First suppose the crossed nodes are the endpoints. In order to have nontrivial B we must have $\ell \geqslant 3$, in which case $S^2\mathfrak{h} = B \oplus B' \oplus B''$ where

$$\mathfrak{h} \simeq \times\!-\!\overset{1}{\bullet} \cdots \bullet\!-\!\times \oplus \times\!-\!\bullet \cdots \overset{1}{\bullet}\!-\!\times \simeq \mathfrak{h}^*$$

$$B \simeq \times\!-\!\overset{2}{\bullet}\!-\!\times \text{ or } \times\!-\!\overset{1}{\bullet}\!-\!\bullet \cdots \bullet\!-\!\overset{1}{\bullet}\!-\!\times \qquad B' \simeq \times\!-\!\overset{2}{\bullet} \cdots \bullet\!-\!\times \oplus \times\!-\!\bullet \cdots \overset{2}{\bullet}\!-\!\times$$

and B'' is trivial. Clearly $\mathfrak{h}^* \otimes B'$ has too many irreducible components to satisfy the ALC, no matter which real form we consider.

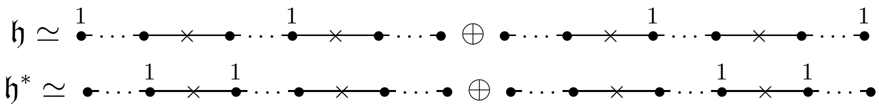
It remains to consider B, first in the case $\ell = 3$, where the possible real forms (with $\mathfrak{h}$ irreducible) are $\mathfrak{su}(2,2)$ and $\mathfrak{su}(1,3)$. Then

$$\mathfrak{h}^* \otimes B \simeq \big(\times\!-\!\overset{3}{\bullet}\!-\!\times \oplus \times\!-\!\overset{3}{\bullet}\!-\!\times \big) \oplus \big(\times\!-\!\overset{1}{\bullet}\!-\!\times \oplus \times\!-\!\overset{1}{\bullet}\!-\!\times \big)$$

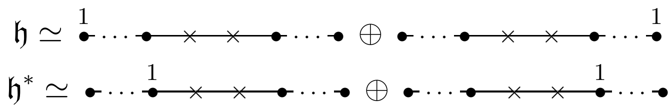
and the ALC is satisfied, since these are complexifications of two complex components for the real form in question. However, for $\ell \geqslant 4$, we find that the product $\mathfrak{h}^* \otimes B$ leads to complexifications with three complex components, so the ALC is not satisfied.

Case 6 ($A_3^{2,1}$)**.** Here $\mathfrak{g}$ is $\mathfrak{su}(2,2)$ or $\mathfrak{su}(1,3)$, and M has a CR structure, i.e., a contact distribution $\mathcal{H}$ equipped with a complex structure. The Levi form induces the class of trivial parallel hermitian metrics (the Weyl connections corresponding to the contact forms leave parallel both the complex structure and the symplectic form, thus also the associated metric, and the metrizability problem is trivial as in the conformal case). However, we now see that there may also be interesting compatible subriemannian metrics on $\mathcal{H} \leqslant TM$ which are hermitian and tracefree with respect to the Levi form.

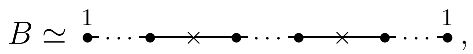
Now suppose the crosses are not placed at the ends, say the left one at the k-th position, $2 \leqslant k$. Thus we consider the real forms $\mathfrak{su}(p,q)$ with $k \leqslant p \leqslant q$. We have



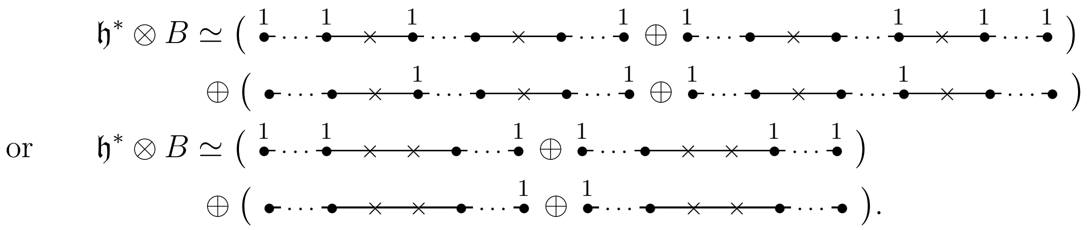
for $\ell > 2k$ and



for $\ell = 2k$. In particular, we have $S^2\mathfrak{h} \supset B$ where

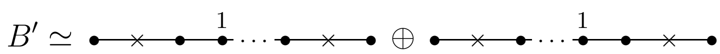


which admits nondegenerate metrics and satisfies the ALC, with



Case 7 ($A_\ell^{2,k}$)**.** Here $\mathfrak{g} \simeq \mathfrak{su}(p,q)$ with nodes k and $\ell+1-k$ crossed, where $2 \leqslant k \leqslant p \leqslant q, p+q = \ell+1$. In these geometries, $\mathcal{H} \cong E \otimes F$, where E is a complex vector bundle of rank k, and the rank $(\ell - 2k + 1)$ complex vector bundle F comes with a hermitian form of signature $(p-k, q-k)$. The corank of $\mathcal{H} \leqslant TM$ is k^2, and the metrics on $\mathcal{H}$ are the products of hermitian metrics on E with the given ones on F. When $\ell = 2k$ (i.e., F has rank 1), $\mathfrak{g} = \mathfrak{su}(k, k+1)$ with the nodes $k, k+1$ are crossed. These are the free CR geometries with complex structure on $\mathcal{H}$ studied in [20] (where it is also explained how complex structure arises on $\mathcal{H}$).

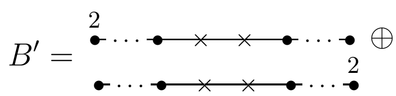
The remaining components of $S^2\mathfrak{h}$ do not satisfy the ALC, except in special cases $k = 2$, $2k = \ell$ and $2k+1 = \ell$. In particular, when $k = 2$,



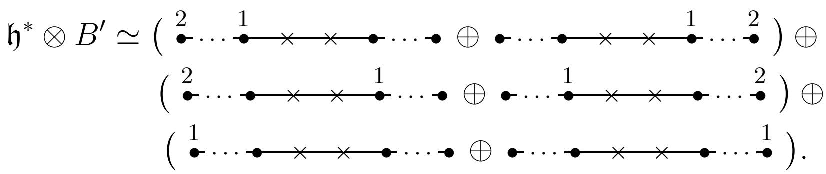
satisfies the ALC (and is nontrivial for $\ell \geqslant 6$).

Case 8 ($A_\ell^{2,h}$)**.** Here $\mathfrak{g} \simeq \mathfrak{su}(p,q)$ with nodes 2 and $\ell - 1$ crossed, where $2 \leqslant p \leqslant q$ and $\ell = p+q-1 \geqslant 6$. In this geometry, $\mathcal{H} \cong E \otimes F$, where E is a complex vector bundle of rank 2, and F is a complex vector bundle of rank $\ell - 3$. The corank of $\mathcal{H} \leqslant TM$ is 4. The eligible metrics are the complex symmetric bilinear forms of the form of tensor product of two exterior forms.

When $2k = \ell$, we obtain $S^2\mathfrak{h} = B \oplus B'$ where



which admits nondegenerate metrics, and satisfies the ALC, with



Case 9 ($A^{2,s}_{2k}$)**.** This case is again the free CR geometry, with $\mathfrak{g} = \mathfrak{su}(k, k+1)$, but the eligible metrics are the complex bilinear metrics on $\mathcal{H}$.

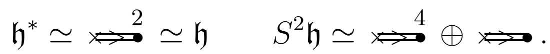
Similarly, when $\ell = 2k+1$ with the k-th and $(k+2)$-nd nodes crossed,

satisfies the ALC.

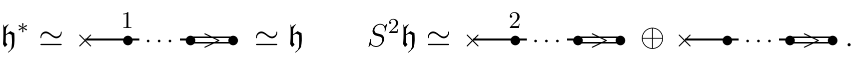
Case 10 ($A^{2,s}_{2k+1}$)**.** Here $\ell = 2k+1$, $\mathfrak{g}$ is $\mathfrak{su}(k, k+2)$, or $\mathfrak{su}(k+1, k+1)$, with nodes k and $k+2$ crossed. In this geometry, $\mathcal{H} \cong E \otimes F$, where E is a complex vector bundle of rank k, and F is a complex vector bundle of rank 2. The codimension of $\mathcal{H} \leqslant TM$ is k^2. The eligible metrics are the complex symmetric bilinear forms of the form of tensor product of two exterior forms.

We have now exhausted all possibilities, completing the proof in type A.

4.2. **Proof of Theorem 2 when $\mathfrak{g}$ has type B_ℓ.** In the type B case, there are no complex or quaternionic modules to consider, so the irreducible cases have one cross only. The unique grading of length one is odd dimensional conformal geometry. In dimension three we then have

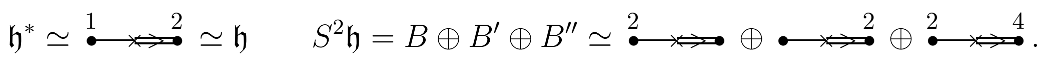


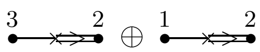
The trivial representation in $S^2\mathfrak{h}$ corresponds to the trivial case of metrics in the conformal class, which are excluded from our classification, and choosing B to be the other component leads to three components in $B \otimes \mathfrak{h}^*$, so the ALC fails. Similarly, for conformal geometries of dimensions $2\ell - 1 \geqslant 5$ we obtain

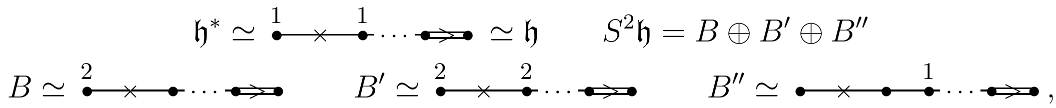


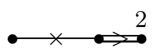
As before, the trivial summand is excluded, and the other component fails the ALC.

We turn now to Lie contact geometries, with the second node crossed. For B_3,

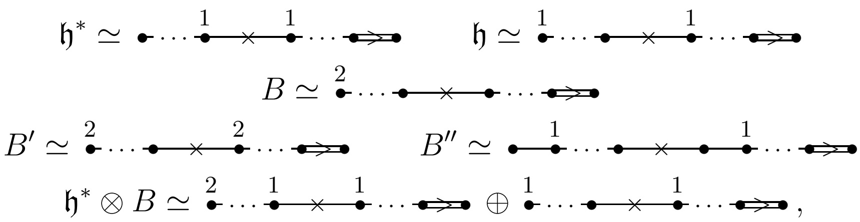


Here, $B \otimes \mathfrak{h}^* =$  and satisfies the ALC. The other choices lead to too many components. For B_ℓ with $\ell \geqslant 4$, we have instead

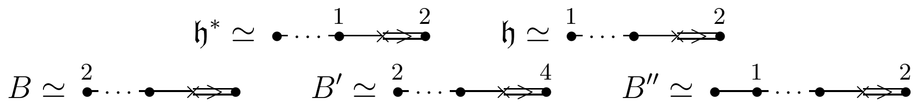


except that when $\ell = 4$, $B'' =$ . Now we check that $B' \otimes \mathfrak{h}^*$ has six components, $B'' \otimes \mathfrak{h}^*$ has three components, but the ALC is again satisfied by B. Lie contact geometries exist for $\mathfrak{g} = \mathfrak{so}(p, q)$ with $2 \leqslant p \leqslant q$; $\mathfrak{h}$ is the tensor product of defining representations $\mathbb{R}^2$ of $\mathfrak{sl}(2, \mathbb{R})$ and $\mathbb{R}^{p+q-4}$ of $\mathfrak{so}(p-2, q-2)$, and B is the tensor product of a symmetric form on $\mathbb{R}^2$ and the defining inner product of signature $(p-2, q-2)$ on $\mathbb{R}^{p+q-4}$. See [6, §4.2.5] for more details on these geometries.

Next we consider B_ℓ with the cross on k-th position, $3 \leqslant k \leqslant \ell - 1$; the outcome is quite similar to the Lie contact case. For $k \neq \ell - 1$, $S^2\mathfrak{h} = B \oplus B' \oplus B''$, where



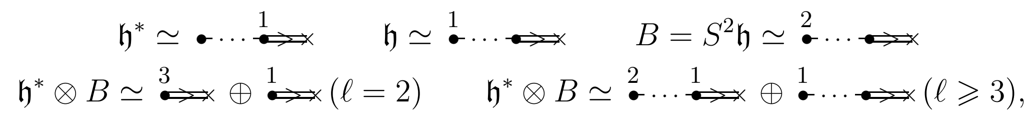
so B satisfies the ALC, but B' and B'' do not. If $k = \ell - 1$, $S^2\mathfrak{h} = B \oplus B' \oplus B''$ with



and again, B satisfies the ALC, but B' and B'' do not. These $|2|$-graded geometries are modelled on the flag variety of isotropic k-planes and exist for the real forms $\mathfrak{so}(p, q)$ with $k \leqslant p \leqslant q$. We have $\mathfrak{h} \cong \mathbb{R}^k \otimes \mathbb{R}^{p+q-k}$ and B corresponds to the tensor product of a symmetric form on $\mathbb{R}^k$ with the defining inner product on $\mathbb{R}^{p+q-k}$.

Case 11 ($B_\ell^{1,k}$)**.** Here $\mathfrak{g} \simeq \mathfrak{so}(p, q)$ with $k \leqslant p \leqslant q$ and $p + q = 2\ell + 1$, and the geometries come equipped with the identification of the horizontal distribution $\mathcal{H} \leqslant TM$ with the tensor product $E \otimes F$, where E has rank k and F carries a metric of signature $(p-k, q-k)$. The corank of $\mathcal{H} \leqslant TM$ is $\frac{1}{2}k(k-1)$. The metrics in B are the tensor products of symmetric nondegenerate forms on E and the given metric on F.

Finally, we arrive at the cross at the very end. For B_ℓ with $\ell \geqslant 2$, we have

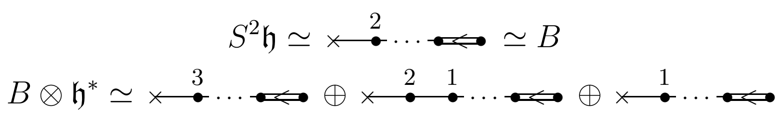


and the ALC is satisfied.

Case 12 ($B_\ell^{1,\ell}$)**.** Here $\mathfrak{g}$ is the split form $\mathfrak{so}(\ell, \ell + 1)$. The geometries are the well known free distributions, cf. [9], with rank ℓ horizontal distribution $\mathcal{H} \leqslant TM$ of corank $\frac{1}{2}\ell(\ell - 1)$. The metrics in B are all nondegenerate metrics on $\mathcal{H}$.

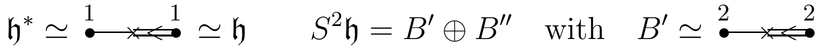
4.3. **Proof of Theorem 2 when $\mathfrak{g}$ has type C_ℓ.** As with type B_ℓ, we only have to consider cases with a single crossed node. We begin with the first node crossed, corresponding to the well known contact projective structures, with

we have discussed the lowest dimension three already as the B_2 case, which coincides with the free distribution of rank two. For $\ell \geqslant 3$, the picture changes since

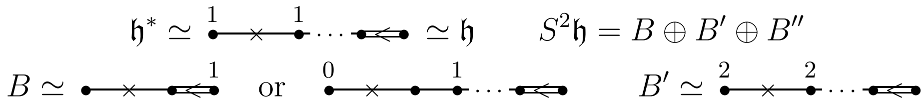


and thus the ALC fails.

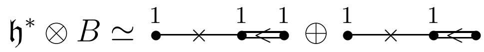
Moving on to the second node, we obtain another well known family of examples: the quaternionic contact geometries (for $\mathfrak{g} \cong \mathfrak{sp}(p, \ell - p)$, $1 \leqslant p \leqslant \ell/2$) or their split analogues (for $\mathfrak{g} \cong \mathfrak{sp}(2\ell, \mathbb{R})$)—see [6, §4.3.3]. For $\ell = 3$, we have



and B'' trivial, while for $\ell \geqslant 4$, we have



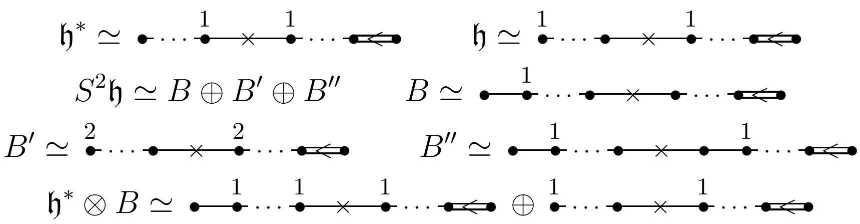
and B'' trivial. Since $\mathfrak{h}^* \otimes B'$ decomposes into four components, there are only nontrivial possibilities for $\ell \geqslant 4$. For $\ell = 4$,



and so the ALC holds for B, but for $\ell \geqslant 5$, $\mathfrak{h}^* \otimes B$ has three irreducible components, and the ALC is not satisfied.

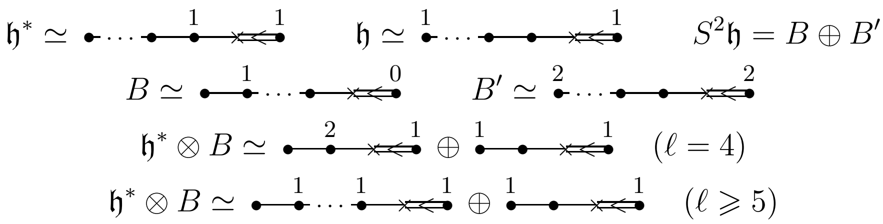
Case 13 ($C_4^{1,2}$)**.** Here the possible real Lie algebras are $\mathfrak{sp}(8,\mathbb{R})$, $\mathfrak{sp}(2,2)$, or $\mathfrak{sp}(1,3)$, with the second node crossed. In the first case, the geometries come equipped with the identification of the horizontal distribution $\mathcal{H} \leqslant TM$ with the tensor product $E \otimes F$, where E is rank 2 and the rank 4 vector bundle F comes with a symplectic form. The eligible metrics in B are the tensor products of a area form on E and the given symplectic form on F. In the quaternionic cases, $\mathcal{H}$ is quaternionic and the eligible metrics in B are quaternionic hermitian forms.

Let us next suppose that the k-th node is crossed for $3 \leqslant k \leqslant \ell - 2$. Then



and so B satisfies the ALC, but the other components do not. The relevant metrics are again tensor products of an exterior form on the rank k auxiliary bundle E and the given symplectic form on F (where the horizontal distribution is identified with $E \otimes F$). These geometries are available for the split form $\mathfrak{sp}(2\ell,\mathbb{R})$ and, if k is even then also for the real forms $\mathfrak{sp}(p,q)$, $k \leqslant p < q$.

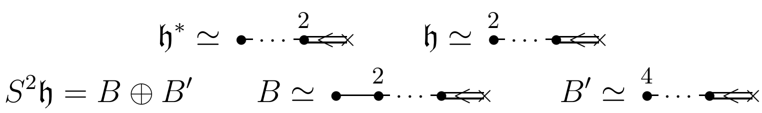
The case with the cross at the last but one node is very similar. Here



and so B satisfies the ALC (while B does not).

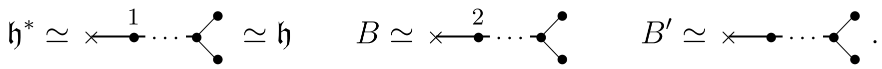
Case 14 ($C_\ell^{1,k}$)**.** With the k-th node crossed for $3 \leqslant k \leqslant \ell - 1$, the possible real Lie algebras are $\mathfrak{sp}(2n,\mathbb{R})$, and if k is even, then also $\mathfrak{sp}(p,q)$, $k \leqslant p \leqslant q$. In the split case, the horizontal distribution is a tensor product $\mathcal{H} \simeq E \otimes F$ with E of rank k and F symplectic of rank $2\ell - 2k$, the eligible metrics are tensor products of antisymmetric forms on E and the given symplectic form on F. In the quaternionic cases, $\mathcal{H}$ comes with a quaternionic structure, and the eligible metrics are quaternionic hermitian forms.

Finally, we consider the cross at the last node of C_ℓ with $\ell \geq 3$ ($\ell = 2$ is equivalent to the B_2 case with the first node crossed). In this case



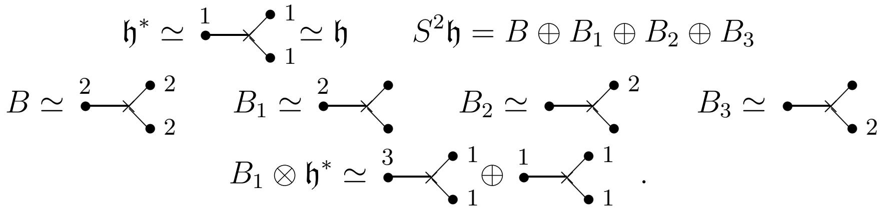
and both B and B' have too many components in their tensor products with $\mathfrak{h}^*$ to satisfy the ALC.

4.4. **Proof of Theorem 2 when $\mathfrak{g}$ has type D_ℓ.** We first consider the cases with one cross on D_ℓ, $\ell \geqslant 4$, starting with the the first node, i.e., the even dimensional conformal geometries, where $S^2\mathfrak{h} = B \oplus B'$ with

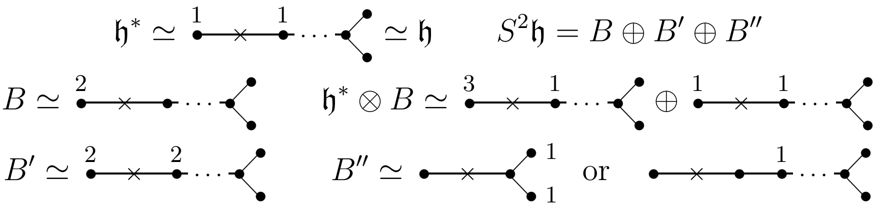


As in the odd dimensional case (type B_ℓ), B does not satisfy the ALC, and the trivial summand B' yields metrics in the conformal class, which we exclude.

We turn now to the Lie contact case, with the second node crossed. For $\ell = 4$,

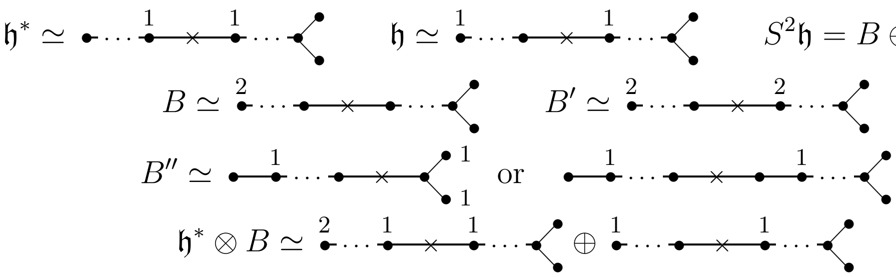


While $\mathfrak{h}^* \otimes B$ has too many components, B_1 satisfies the ALC, as do B_2 and B_3 by symmetry. The metrics are tensor products of two area forms and a symmetric form on $\mathfrak{h} \cong \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$. The geometries exist for the real forms $\mathfrak{so}(4,4)$, $\mathfrak{so}(3,5)$ and the quaternionic $\mathfrak{so}^*(8) \simeq \mathfrak{so}(2,6)$. Similarly, for $\ell \geqslant 5$, we have

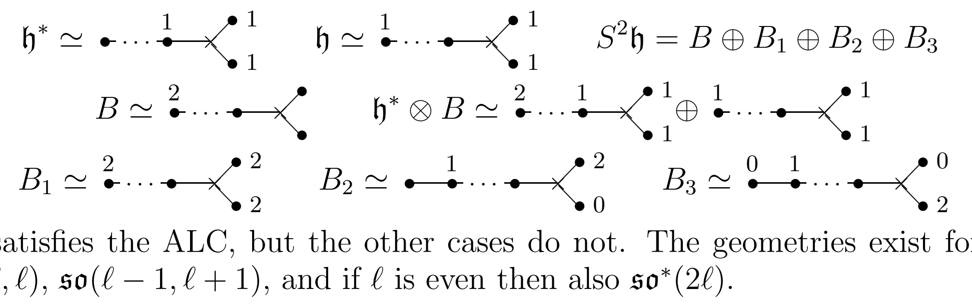


where B satisfies the ALC, but B' and B'' do not. In addition to the real forms $\mathfrak{so}(p,q)$, $2 \leqslant p \leqslant q$, $p+q = 2\ell$, which are analogous to the Lie contact geometries of type B_ℓ, the real form $\mathfrak{so}^*(2\ell)$ is also possible.

As with type B_ℓ, the cases where the k-th node is crossed, with $3 \leqslant k \leqslant \ell - 2$ behave in a similar way. If $k \leqslant \ell - 3$ then



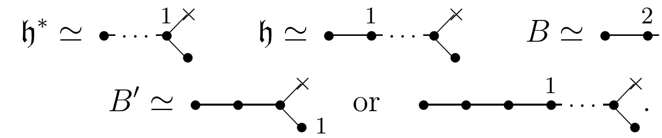
so that B satisfies the ALC, while B' and B'' do not. The geometries exist for real forms $\mathfrak{so}(p,q)$, $k \leqslant p \leqslant q$, $p+q = 2\ell$, and if k is even, then also for $\mathfrak{so}^*(2\ell)$. If $k = \ell - 2$, the computation differs slightly, but the outcome is similar:



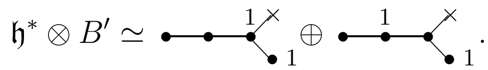
where B satisfies the ALC, but the other cases do not. The geometries exist for the real forms $\mathfrak{so}(\ell,\ell)$, $\mathfrak{so}(\ell-1,\ell+1)$, and if ℓ is even then also $\mathfrak{so}^*(2\ell)$.

Case 15 ($D_\ell^{1,k}$)**.** Here $\mathfrak{g} \simeq \mathfrak{so}(p,q)$ with $2 \leqslant k \leqslant p \leqslant q$ and $p+q = 2n$, the geometries come equipped with the identification of the horizontal distribution $\mathcal{H} \leqslant TM$ with the tensor product $E \otimes F$, where E has rank k and F carries a metric of signature $(p-k, q-k)$. The corank of $\mathcal{H} \leqslant TM$ is $\frac{1}{2}k(k-1)$. The metrics in B are the tensor products of symmetric nondegenerate forms on E and the given metric on F. When $\mathfrak{g} \simeq \mathfrak{so}^*(2n)$ and k is even, the geometries come with the identification of the horizontal distribution $\mathcal{H}$ with the tensor product of a quaternionic rank k bundle E and a quaternionic rank $n-2k$ bundle F equipped with a quaternionic skew-hermitian form. The metrics in B are quaternionic hermitian forms.

The remaining case with one cross is the so called spinorial geometry with the cross on one of the nodes in the fork. The case of D_4 coincides with the 6-dimensional conformal Riemannian geometry. For $\ell \geqslant 5$, we have $S^2\mathfrak{h} = B \oplus B'$ with

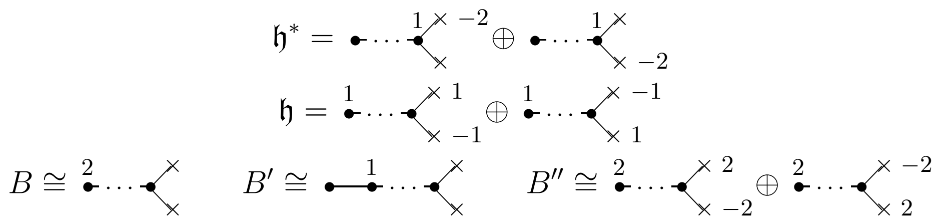


Now $\mathfrak{h}^* \otimes B$ has three summands, as does $\mathfrak{h}^* \otimes B'$, except for $\ell = 5$, when

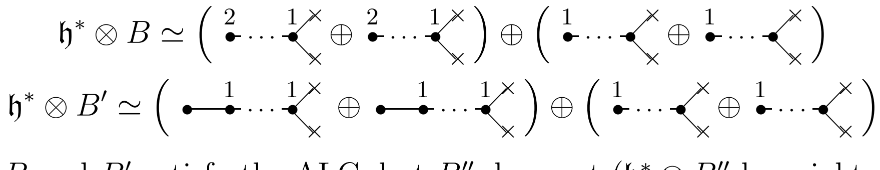


Here, in the complex setting, $\mathfrak{h} \cong \wedge^2\mathbb{C}^5$, and $B \cong \mathbb{C}^{5*} \cong \wedge^4\mathbb{C}^5 \leqslant S^2\wedge^2\mathbb{C}^5$, where $\alpha \in B \cong \mathbb{C}^{5*}$ determines a metric g_α on $\mathfrak{h}^* \cong \wedge^2\mathbb{C}^{5*}$ by $g_\alpha(\xi,\eta) = \alpha \wedge \xi \wedge \eta \in \wedge^5\mathbb{C}^{5*} \cong \mathbb{C}$. Such a metric is never nondegenerate, so this case is excluded.

We next consider D_ℓ cases with two crossed nodes. For $\mathfrak{h}$ to be irreducible, the semisimple part of the Levi factor $\mathfrak{p}/\mathfrak{p}^\perp$ must be simple. Indeed, working in the complex setting, a direct check reveals that breaking the Dynkin diagram by two crosses into more than one part always leads to non-isomorphic representations for the two components of $\mathfrak{h}$. Furthermore, the only way to obtain isomorphic components is to take the two spinorial nodes of the D_ℓ diagram. The only real forms compatible with this geometry are the split form $\mathfrak{so}(\ell,\ell)$, the quasi-split form $\mathfrak{so}(\ell+1,\ell-1)$ and the quaternionic form $\mathfrak{so}^*(2\ell)$ with $\ell = 2p+1$ odd. In the split case, $\mathfrak{h}$ is not irreducible, so this does not fit into our classification. In the quasi-split case, $\mathfrak{h} \cong \mathbb{R}^{\ell-1} \otimes_{\mathbb{R}} \mathbb{C}$ is complex, while in the quaternionic case, $\mathfrak{h}$ is quaternionic. In $S^2\mathfrak{h} = B \oplus B' \oplus B''$, with



where we denote the nonzero weights over the crossed nodes for clarity. Observe that

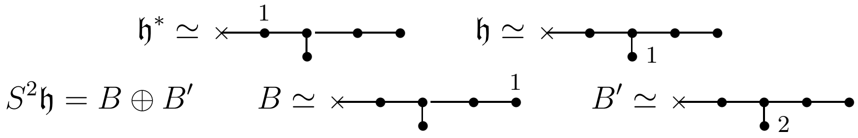


so that both B and B' satisfy the ALC, but B'' does not ($\mathfrak{h}^* \otimes B''$ has eight components).

Case 16 ($D_\ell^{2,s}$)**.** When $\mathfrak{g} = \mathfrak{so}(\ell-1,\ell+1)$, the horizontal distribution $\mathcal{H} \leqslant TM$ is a complex vector bundle of complex rank $\ell - 1$, and the metrics in B are complex bilinear. When $\mathfrak{g} = \mathfrak{so}^*(2\ell)$, with $\ell = 2p+1$ odd, the horizontal distribution $\mathcal{H} \leqslant TM$ has a quaternionic structure of quaternionic rank p and the metrics in B are quaternionic skew-hermitian.

Case 17 ($D_\ell^{2,h}$)**.** This case involves the same geometries as in the previous case, with $\ell = 2p+1$ odd, except that when $\mathfrak{g} = \mathfrak{so}(\ell-1, \ell+1)$, the metrics in B' are hermitian, while for $\mathfrak{g} = \mathfrak{so}^*(2\ell)$, the metrics in B are quaternionic hermitian.

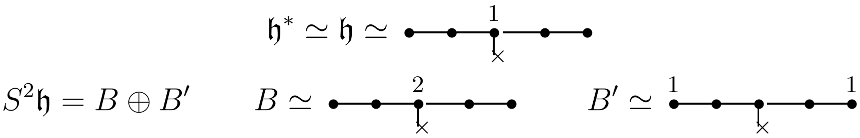
4.5. **Proof of Theorem 2 when $\mathfrak{g}$ has exceptional type.** The first case we consider is the Lie algebra E_6. Let us consider possibilities for parabolic subalgebras with one crossed node. The first possibility is



The product $\mathfrak{h}^* \otimes B$ decomposes (as the product of a spinor and defining representation of SO(10)) into the sum of two P-modules, hence the ALC is satisfied.

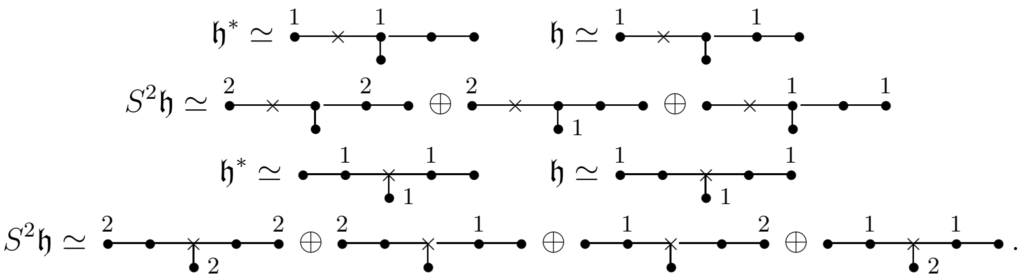
Case 18 ($E_6^{1,1}$)**.** This is the |1|-graded geometry for E_6 for which the allowed real forms are the split form $E_{6(6)}$, or $E_{6(-26)}$, and $\mathcal{H} = TM$ carries the structure of basic spinor representation S^+ of $\mathfrak{so}(5,5)$, or $\mathfrak{so}(1,9)$ respectively. The P-module B corresponding to the eligible metrics is the defining representation of $\mathfrak{so}(5,5)$ or $\mathfrak{so}(1,9)$.

Consider next the adjoint variety, with the node on the short leg crossed. We have

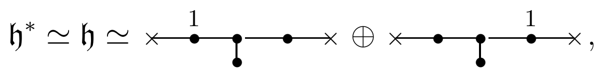


and find that both $\mathfrak{h}^* \otimes B$ and $h^* \otimes B'$ have four components.

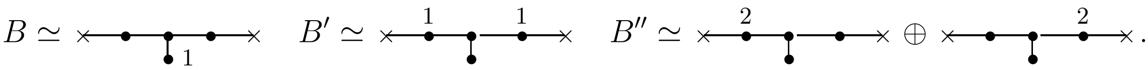
In the remaining two cases with one crossed node, the semisimple part of $\mathfrak{p}/\mathfrak{p}^\perp$ is not simple, and it is easy to see that the ALC cannot be satisfied:



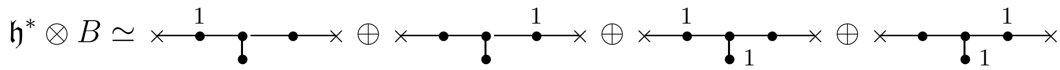
The only case with two crosses for which $\mathfrak{h}$ could be irreducible is



and indeed, $\mathfrak{h}$ is irreducible for the quasi-split real form $E_{6(2)}$. For this real form, the nontrivial irreducible summands in $S^2\mathfrak{h}$ are



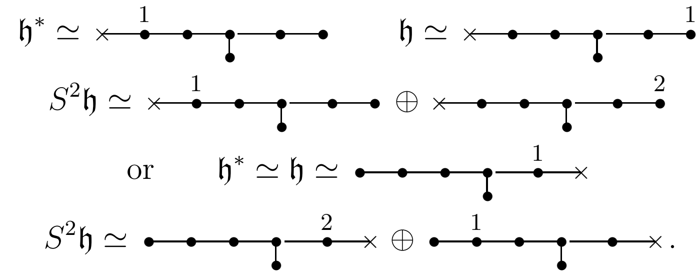
The products $\mathfrak{h}^* \otimes B'$ and $\mathfrak{h}^* \otimes B''$ have too many components but



and so the ALC is satisfied.

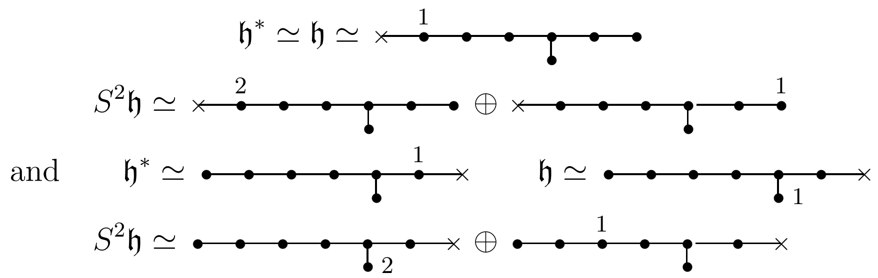
Case 19 ($E_6^{2,h}$)**.** This is a |2|-graded geometry for the quasi-split Lie algebra $E_{6(2)}$. The horizontal distribution $\mathcal{H}$ carries the structure of the spinor representation S of $\mathfrak{so}(3,5)$, while the eligible metrics are induced by the defining representation of $\mathfrak{so}(3,5)$.

For E_7 and its real forms, irreducibility of $\mathfrak{h}$ implies that only one node may be crossed, and a similar analysis to the E_6 type shows that the cases with the best chance to satisfy the ALC are those with cross over the first or last node, where



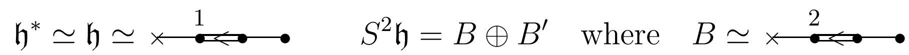
It is easy to see that none of these cases satisfy the ALC.

Similarly, for E_8, even the most promising candidates



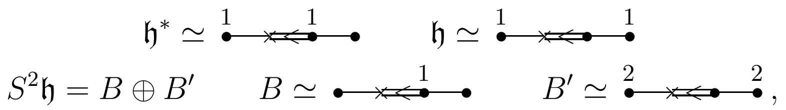
fail the ALC. Again there can be no cases with more than one cross.

For F_4, the only non-split possibility is



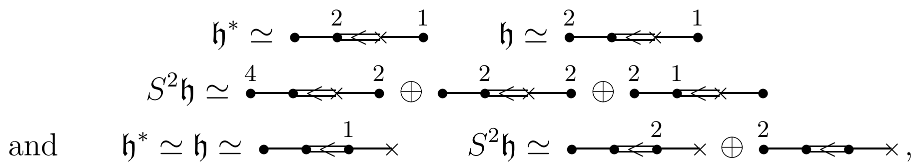
and B' is trivial. However, B does not satisfy the ALC.

For the split form, all cases can have only one crossed node. When



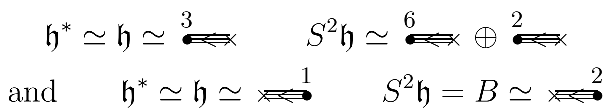
the elements of B are all degenerate, whereas $\mathfrak{h}^* \otimes B'$ does not satisfy the ALC.

In the remaining two possibilities for the crossed node,

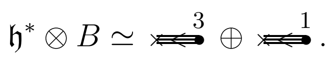


the ALC fails in all cases.

Finally, for G_2, only the split case is possible, with one crossed node.



and only the last of these satisfies the ALC, with

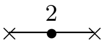


Case 20 ($G_2^{1,1}$)**.** The real Lie algebra is the split form of G_2 and the geometry is given by Cartan's famous $(2,3,5)$ distribution. Hence the horizontal distribution has rank 2 and the P-module B corresponding to the eligible metrics is the second symmetric power of the defining representations of $\mathfrak{sl}(2,\mathbb{R})$.

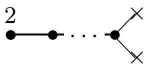
5. Examples of reducible cases

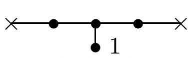
We now discuss a few cases of geometries with reducible $\mathcal{H}$, where the linearized metrizability procedure works. Actually, we have seen several such examples already, when dealing with real forms with irreducible, but not absolutely irreducible $\mathfrak{h}$ in Theorem 2. We list some of those with irreducible B in the following result.

Theorem 3. *The following real parabolic geometries with the Lie algebra $\mathfrak{g}$ and choice of B satisfy the ALC and the linearized metrizability procedure works.*

(i) $B \simeq$ , $\mathfrak{g} \simeq \mathfrak{sl}(4,\mathbb{R})$. *These are Lagrangian contact structures in dimension 5, where a decomposition $\mathcal{H} = E \oplus F$ of the contact subbundle into a direct sum of two Lagrangian subbundles is given. The metrics in B are the split signature metrics with both E and F isotropic.*

(ii) $B \simeq$ , $\mathfrak{g} \simeq \mathfrak{sl}(n+1,\mathbb{R})$, *$n$ even, the first cross at the k-th root ($2k < n$), crosses at symmetric places. These geometries come with $\mathcal{H}$ identified with the sum of two vector bundles of the form $(E \otimes F^*) \oplus (F \otimes G^*)$, where E and G are real vector bundles of rank k, and F is a real vector bundle of rank $n - 2k + 1$. The metrics are the split signature ones, in the subbundle $E \otimes G^* \leqslant E \otimes F^* \otimes F \otimes G^*$.*

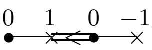
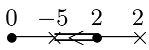
(iii) $B \simeq$ , *the real Lie algebra is $\mathfrak{so}(p,p)$, $2p = n$. The horizontal distribution $\mathcal{H} \leqslant TM$ is the sum of two rank $p-1$ bundles E and F coming from the defining representations of $\mathfrak{sl}(p-1,\mathbb{R})$ with different weights, and B stays for general split metrics on $E \oplus F$.*

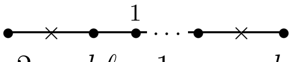
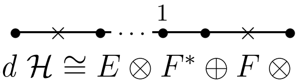
(iv) $B \simeq$ , *the real Lie algebra is the split form of type E. The geometry is $|2|$-graded, and the horizontal subspace $\mathcal{H} \leqslant TM$ corresponds to the direct sum of two of the three isomorphic defining representations of $\mathfrak{so}(4,4)$. The eligible metrics are the generic tracefree split ones and the P-module B corresponds to the third defining representation $\mathbb{R}^8$, up to the weight.*

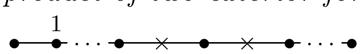
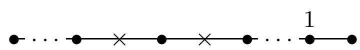
Proof. All cases were already treated for different real forms in the previous section, except for the very last case. The computation presented there showed that the ALC is satisfied but the subbundle $\mathcal{H}$ is not irreducible, but a sum of two subbundles. At the same time, the strong ALC holds, thus the linearized metrizability procedure works as required. $\square$

Our final result illustrates the possibility of finding examples with reducible B, including one in which a trivial one-dimensional component occurs.

Theorem 4. *The following real parabolic geometries with the Lie algebra $\mathfrak{g}$ and choice of B satisfy the ALC.*

(i) $B \simeq$  $\oplus$ , *the real Lie algebra is the split form of type F ($|6|$-graded). The horizontal distribution $\mathcal{H} \leqslant TM$ is built of two rank 2 bundles E and F coming from the defining $\mathfrak{sl}(2,\mathbb{R})$ representations of the different components in $\mathfrak{p}_0$. The first component $\mathcal{H}_1$ is a tensor product $E \otimes F$ with appropriate weight, while F stays for the other component $\mathcal{H}_2$ with another weight. The eligible metrics are the sums of the metrics in $\Lambda^2 E \otimes \Lambda^2 F \leqslant S^2\mathcal{H}_1$, and the metrics in $S^2\mathcal{H}_2$.*

(ii) $B \simeq$  $\oplus$ . *In this case $\mathfrak{g} \simeq \mathfrak{sl}(\ell+1), 5 \leqslant \ell$, with nodes 2 and $\ell - 1$ crossed, and $\mathcal{H} \cong E \otimes F^* \oplus F \otimes G$, where E is a real vector bundle of rank 2, and F is a real vector bundle of rank $\ell - 3$. The corank of $\mathcal{H} \leqslant TM$ is 4. The eligible metrics are sums of the symmetric bilinear forms on $\mathcal{H}_1$ and $\mathcal{H}_2$, both of the form of tensor product of two exterior forms.*

(iii) $B \simeq$  $\oplus$ . *Similarly to the previous case, $\mathfrak{g} \simeq \mathfrak{sl}(2k)$, $4 \leqslant k$, with nodes crossed at symmetric positions, and $\mathcal{H} \cong E \otimes F^* \oplus F \otimes G$,* *where E and G are real vector bundles of rank $k-1$, while F is a real vector bundle of rank 2. The corank of $\mathcal{H} \leqslant TM$ is $(k-1)^2$. The eligible metrics are sums of the symmetric bilinear forms on $\mathcal{H}_1$ and $\mathcal{H}_2$, both of the form of tensor product of two exterior forms.*

(iv) $B \simeq \overset{2}{\bullet}\cdots\bullet - \times - \times - \bullet \cdots \bullet \oplus \bullet \cdots \bullet - \times - \times - \bullet \cdots \overset{2}{\bullet}$. *Here $\mathfrak{g} = \mathfrak{sl}(2k+1)$, the horizontal distribution is the sum of two vector bundles of the same rank k, corresponding to the defining representations of the two semisimple components in $\mathfrak{p}_0$. The metrics are sums of metrics on these two parts of $\mathcal{H}$.*

Proof. (i) Since the strong ALC cannot hold in the case of split forms of the algebras, we work with the complete weights. The form of $\mathfrak{h}$ is seen from the Cartan matrix of type F, while the sum and difference of the second and last lines in the inverse Cartan matrix (which corresponds to the crossed nodes in the Dynkin diagram) provide the coefficients (4 8 11 6) and (2 4 5 2) expressing two generating weights in the centre of $\mathfrak{p}_0$. With their help, we find

$$\mathfrak{h}^* = \overset{1}{\bullet} - \overset{-2}{\times} \Leftarrow \overset{1}{\bullet} - \overset{0}{\times}, \oplus \overset{0}{\bullet} - \overset{0}{\times} \Leftarrow \overset{1}{\bullet} - \overset{-2}{\times}.$$

$$\mathfrak{h} = \overset{1}{\bullet} - \overset{-1}{\times} \Leftarrow \overset{1}{\bullet} - \overset{-1}{\times} \oplus \overset{0}{\bullet} - \overset{-2}{\times} \Leftarrow \overset{1}{\bullet} - \overset{1}{\times}.$$

The part of interest in $S^2\mathfrak{h}$ is

$$B_1 \oplus B_2 = \overset{0}{\bullet} - \overset{1}{\times} \Leftarrow \overset{0}{\bullet} - \overset{-1}{\times} \oplus \overset{0}{\bullet} - \overset{-5}{\times} \Leftarrow \overset{2}{\bullet} - \overset{2}{\times}.$$

Now, B_1 is trivial, while

$$B_2 \otimes \mathfrak{h}^* \simeq \overset{1}{\bullet} - \times \Leftarrow \overset{3}{\bullet} - \times \oplus \overset{1}{\bullet} - \times \Leftarrow \overset{1}{\bullet} - \times \oplus \bullet - \times \Leftarrow \overset{3}{\bullet} - \times \oplus \bullet - \times \Leftarrow \overset{1}{\bullet} - \times.$$

Hence the kernel of b does not exceed the allowed number of components and the ALC holds. Finally,

$$\Lambda^4\mathfrak{h}_1 = \overset{0}{\bullet} - \overset{2}{\times} \Leftarrow \overset{0}{\bullet} - \overset{-2}{\times} \qquad \Lambda^2\mathfrak{h}_2 = \overset{0}{\bullet} - \overset{-2}{\times} \Leftarrow \overset{0}{\bullet} - \overset{-3}{\times}$$

so that the weight of $\mathcal{L}$ can be expressed in terms of them and thus the linearized metrizability procedure can be completed.

(ii)–(iv) All the other cases have been already discussed as the complex versions of some cases in the previous section. The only remaining bit of the proof is the check that the top exterior forms on the individual components provide linearly independent weights and thus may be used to rescale the metrics properly. This can be done exactly as in case (i). $\square$

Remark 5.1. Actually, the arguments in the cases (iii) and (iv) above work also in any of the situations where the crosses are either apart by one or next to each other, i.e., without assuming they are placed symmetrically, except if the adjacent crosses appear right at the ends of the diagram. In the latter case of the so called paths geometries, one of the top degree forms on $\mathfrak{h}_i$ has trivial weight zero and thus the linearized metrizability procedure fails at the stage when we change the weight of the solutions in order to get genuine metrics.

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