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A Regularity Theory for Energy Minimising Intrinsic Fractional Harmonic Maps

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Abstract

We define and develop an interior partial regularity theory for intrinsic energy minimising fractional harmonic maps from Euclidean space into smooth compact Riemannian manifolds for fractional powers strictly between zero and one. This constitutes a partial extension of the existing theory for intrinsic semi-harmonic maps, corresponding to the power one-half. Intrinsic fractional harmonic maps are critical points of an energy whose first variation is a Dirichlet to Neumann map for the harmonic map problem on a half-space with a Riemannian metric which can degenerate/become singular along the boundary, depending on the fractional power. We take advantage of the connection between fractional harmonic maps and free boundary problems for harmonic maps in order to develop a partial regularity theory for the fractional harmonic maps we consider. In particular, we prove partial regularity for minimising harmonic maps with (partially) free boundary data on half-spaces with the aforementioned metrics up to the boundary; fractional harmonic maps then inherit this regularity. As a by product of our methods, we establish a result which sheds new light on the monotonicity of the average energy of solutions of the degenerate linear elliptic equation related to fractional harmonic functions.

1 Introduction

Fractional harmonic maps from domains in Euclidean space into a smooth compact Riemannian manifold N have been studied by several individuals in recent years, particularly pertaining to their regularity. Our goal is to develop an interior regularity theory for a class of fractional harmonic maps for fractional powers in $(0,1)$. Our main result is a partial regularity theorem for *intrinsic* minimising fractional harmonic maps for powers between zero and one; for a precise statement see Theorem 3.0.1 of Section 3. Our theorem constitutes a generalisation, to powers other than $\frac{1}{2}$, of a regularity theory for intrinsic stationary fractional harmonic maps developed by Moser [29]. Our results regarding intrinsic fractional harmonic maps to powers other than $\frac{1}{2}$ are new; other authors consider their extrinsic counterparts.

Our main theorem generalises the regularity theory for harmonic mappings of Riemannian manifolds to the aforementioned fractional powers. In particular, intrinsic minimising fractional harmonic maps are smooth in the interior of their domain away from a singular set with Hausdorff dimension which depends on the dimension of the domain and the fractional power in question. Moreover, the Hausdorff dimension of the singular set is consistent with the scale-invariance of the problem and the dimensions of the singular set in the theories of harmonic maps and extrinsic fractional harmonic maps.

There are (at least) two types of fractional harmonic maps, namely intrinsic and extrinsic, and as a prelude to a discussion of these maps and the differences between them, we first consider the situation when N is replaced by $\mathbb{R}$. Let $\mathbb{R}^{m+1}_+ = \mathbb{R}^m \times (0,\infty)$ and $\beta \in (-1,1)$. Caffarelli and Silvestre [3] established that, for given boundary data $u : \mathbb{R}^m \to \mathbb{R}$, solutions $v : \mathbb{R}^{m+1}_+ \to \mathbb{R}$ of the Dirichlet problem:

$$\mathrm{div}(x_{m+1}^{\beta} \nabla v) = 0 \text{ in } \mathbb{R}^{m+1}_+ \quad \text{and} \quad v|_{\mathbb{R}^m} = u \tag{1.1}$$

satisfy $(-\Delta)^{\frac{1-\beta}{2}} u = \partial^{\beta}_{m+1} v := -(x^{\beta}_{m+1}\partial_{m+1} v)|_{\mathbb{R}^m}$, where $(-\Delta)^s$ is the fractional Laplace operator of order $s \in (0,1)$, defined via singular integral say. Observe that $\mathrm{div}(x^{\beta}_{m+1}\nabla v) = 0$ is the Euler-Lagrange equation for $E^{\beta}(v) := \frac{1}{2}\int_{\mathbb{R}^{m+1}_+} x^{\beta}_{m+1}|\nabla v|^2 \mathrm{d}x$ in a suitable Sobolev Space. One method of proof of Caffarelli and Silvestre's result is to show that

$$C||(-\Delta)^{\frac{s}{2}} u||^2_{L^2(\mathbb{R}^m)} = \inf\{E^{\beta}(v) : v|_{\mathbb{R}^m} = u\}$$

for some $C = C(m,s)$, where $s = \frac{1-\beta}{2}$. The first variation of the preceding functionals, respectively $u \mapsto (-\Delta)^{\frac{1-\beta}{2}} u$ and $u \mapsto \partial^{\beta}_{m+1} v$, must therefore coincide. We conclude that if we wish to study fractional harmonic functions (functions with $(-\Delta)^s u = 0$) then we may equivalently consider minimisers (or even critical points) of E^{β} with $v|_{\mathbb{R}^m} = u$ such that $\partial^{\beta}_{m+1} v = 0$. If instead we consider (1.1) for $u : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O} \subsetneq \mathbb{R}^m$ is open, then we still obtain a Dirichlet to Neumann map for the problem (1.1), but we may no longer identify this with a fractional Laplace operator.

The first to consider fractional harmonic maps between manifolds were Da Lio and Rivière [9]; they analysed the regularity of fractional $\frac{1}{2}$-harmonic maps into the round unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. These maps are critical points u of the energy $||(-\Delta)^{\frac{1}{4}} u||^2_{L^2(\mathbb{R})}$ in $\dot{H}^{\frac{1}{2}}$ under the constraint that u takes values in $\mathbb{S}^{n-1}$ almost everywhere. In other words, they considered critical points of the functional

$$L(u) = \inf\{E^0(v) : v|_{\mathbb{R}} = u, v \in \dot{W}^{1,2}(\mathbb{R}^2_+;\mathbb{R}^n), u(x) \in \mathbb{S}^{n-1} \text{ for a.e } x \in \mathbb{R}\}.$$

The motivation for studying such maps is rooted in conformal geometry and is linked to Rivière's result asserting the continuity of critical points of functionals with conformally invariant Lagrangian on two dimensional domains [33]; the functional L is invariant under the trace of conformal transformations of $\mathbb{R}^2_+$. Moreover, as Da Lio and Rivière suggested, it has been observed that $\frac{1}{2}$-harmonic maps appear in the asymptotic limit of a fractional Ginzburg-Landau equation [25].

Moser [29] observed that the definition of L is *extrinsic*, meaning that it depends on the choice of embedding of $\mathbb{S}^{n-1}$ into an ambient space. He proposed a modification of L to remove its dependence on the choice of embedding, making the new functional *intrinsic* so that it only depends on the geometry of the target manifold N. We assume henceforth that N is isometrically embedded in $\mathbb{R}^n$ for some n, which can always be achieved for smooth compact N by the theorem of Nash [30], and note that E^β is invariant under isometries of N. Define

$$I(u) = \inf\{E^0(v) : Tv = u, v \in \dot{W}^{1,2}(\mathbb{R}^{m+1}_+; \mathbb{R}^n), v(x) \in N \text{ for a.e } x \in \mathbb{R}^{m+1}_+\}.$$

Then I is intrinsic and its critical points are called intrinsic $\frac{1}{2}$-harmonic maps. When $m = 1$, this functional is conformally invariant in the same sense as L.

To facilitate our subsequent analysis and an illustration of the difference between L and I, we recall the notion of a harmonic mapping of Riemannian manifolds. Let (M, g) be a smooth Riemannian manifold. Let

$$W^{1,2}(M; N) := \{v \in W^{1,2}(M; \mathbb{R}^n) : v(x) \in N \text{ for almost every } x \in M\}.$$

If $x_1, \ldots, x_{m+1}$ are local coordinates in M, the energy density of $v \in W^{1,2}(M; N)$ is given by $e(v) = \sum_{i,j} g^{ij} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_j} \right\rangle$ where g^{ij} are the components the inverse of the matrix representing g and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^n$. The energy of v is

$$E_g(v) = \int_M e(v) \mathrm{d}vol_M.$$

In a coordinate chart, the Euler-Lagrange equations for E_g have the form

$$\Delta_g v + \sum_{i,j} g^{ij} A(v) \left(\frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_j} \right) = 0 \text{ in } M \tag{1.2}$$

where A is the second fundamental form of N and $\Delta_g v$ is the Laplace-Beltrami operator (with the same sign-convention as the Euclidean Laplacian). Critical points of E_g in $W^{1,2}(M; N)$ with respect to the dependent variable (weakly) satisfy (1.2) and are called *(weakly) harmonic maps with respect to g*. Now observe that any $v : \mathbb{R}^2_+ \to \mathbb{R}^n$ for which $L(u) = E^0(v)$ satisfies $\Delta v = 0$ in $\mathbb{R}^{m+1}_+$ (and is hence smooth there). In contrast, any $v : \mathbb{R}^{m+1}_+ \to N$ for which $I(u) = E^0(v)$ is a (weakly) harmonic map from $\mathbb{R}^{m+1}_+$ to N and satisfies

$$\Delta v + A(v)(\nabla v, \nabla v) = 0 \text{ in } \mathbb{R}^{m+1}_+ \tag{1.3}$$

where (in Euclidean coordinates) $A(v)(\nabla v, \nabla v) = \sum_i A(v) \left(\frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)$ and Δ is the usual Laplace operator on Euclidean space.

Extrinsic fractional harmonic maps into smooth target manifolds N, have been studied by Da Lio, Rivière and Schikorra [7, 8, 10, 35, 36] etc. for a range of fractional powers, as well as Millot and Sire [25] for the power $\frac{1}{2}$. The methods Da Lio, Rivière and Schikorra used to obtain regularity take advantage of compensation phenomena, namely improvements of the a-priori regularity in the Euler-Lagrange equations which are revealed using a well chosen gauge transformation, in analogy with those observed for harmonic maps. One of the earliest applications of this technique was used by Hélein to prove regularity for harmonic maps from Riemannian surfaces into spheres [21]. Millot and Sire [25] instead consider the extensions of fractional harmonic maps and exploit their connection with free boundary problems for harmonic maps, a method first applied by Moser in the intrinsic case for $\frac{1}{2}$-harmonic maps (when $\beta = 0$) [29], to obtain partial regularity. Sire et al have also recently released a pre-print which considers the regularity of extrinsic minimising fractional harmonic maps defined on the real line.

We will generalise the notion of intrinsic $\frac{1}{2}$-harmonic maps to fractional powers strictly between zero and one. In order to obtain a regularity result for these maps, we analyse their harmonic extensions, similarly to Moser [29]. Consider $u : \mathcal{O} \to N$ and its possible extensions $v : \mathbb{R}^{m+1}_{+} \to N$ with $v|_{\mathcal{O}} = u$. Moser observed that whenever I is differentiable at u, its first variation is the Dirichlet to Neumann map $\Lambda : u \mapsto -\partial^0_{m+1} v$ for the problem (1.1) but with (1.3) in place of the condition $\Delta v = 0$ in $\mathbb{R}^{m+1}_{+}$. Hence, intrinsic $\frac{1}{2}$-harmonic maps satisfy $\Lambda u = 0$, which may be interpreted as a zero Neumann condition on $\mathcal{O}$ for any v which is a harmonic extension of u in $\mathbb{R}^{m+1}_{+}$. Moser also re-formulated the Euler-Lagrange equations for critical points of I as $\Lambda u \perp N$ and then used the condition $\Lambda u \perp \Gamma$ to study critical points $u : \mathcal{O} \to N$ which are further constrained to lie in a smooth closed submanifold $\Gamma \subset N$. The condition $\Lambda u \perp \Gamma$ is equivalent to the Neumann data for v being orthogonal to Γ. Moreover, the conditions $\Lambda u \perp N$ and $\Lambda u \perp \Gamma$ respectively coincide with (partially) free ($v(\mathcal{O}) \subset N$) and constrained ($v(\mathcal{O}) \subset \Gamma$) boundary conditions for harmonic maps $v : \mathbb{R}^{m+1}_{+} \to N$ [18]. Consequently, interior regularity for intrinsic $\frac{1}{2}$-harmonic maps may be obtained from the regularity theory for free/constrained boundary problems for harmonic maps.

Without geometric constraints on the target manifold N (such as requiring N has non-positive sectional curvature), harmonic maps may be discontinuous everywhere on a domain of dimension at least 3 [32]. Harmonic maps with free/constrained boundary data also exhibit points of discontinuity on the boundary [12, 18, 19] unless further conditions on N are specified. Moser considered stationary $\frac{1}{2}$-harmonic maps $u : \mathcal{O} \to N$, in other words critical points of I with respect to inner and outer variations, as well as stationary $\frac{1}{2}$-harmonic maps constrained such that $u(\mathcal{O}) \subset \Gamma$. He observed that the harmonic extensions $v : \mathbb{R}^{m+1}_{+} \to N$ for such maps are (at least locally) stationary harmonic with respect to the free boundary condition. It therefore follows, using the regularity theory of Hélein for weakly harmonic maps and Bethuel [2] for stationary harmonic maps, that intrinsic $\frac{1}{2}$-harmonic maps are smooth when $m = 1$ and intrinsic stationary $\frac{1}{2}$-harmonic maps are smooth away from a set of vanishing Hausdorff dimension $m - 1$ when $m \geq 2$ respectively. Moser also observed that if u is constrained to lie in Γ then the regularity theory of Scheven [34] for stationary harmonic maps v with constrained boundary conditions $v(\mathcal{O}) \subset \Gamma$ implies the same regularity as for the unconstrained problem.

We will develop an interior regularity theory for energy minimising intrinsic fractional harmonic maps. In comparison to the regularity theories for stationary harmonic maps, the theory for minimising harmonic maps is relatively direct; it is possible to obtain regularity by constructing comparison maps to take advantage of the minimising property. Moreover, the estimates on the dimension of the singular set of minimisers is better than that obtained for stationary harmonic maps. Morrey first established the interior regularity of minimisers on two dimensional domains using comparison maps [26]. Schoen and Uhlenbeck later showed that on domains of dimension at least three, minimisers are regular away from a set of Hausdorff dimension at most the dimension of the domain minus three [37]; they also used manifold valued comparison maps, which are more difficult to construct on domains of dimension greater than two. Hardt and Lin [19] considered the regularity of minimising harmonic maps with free and constrained boundary data, using methods analogous to those of Schoen and Uhlenbeck. In the context considered here, i.e. for minimisers of E^β with respect to $v(\tilde{\mathcal{O}}) \subset N$ or $v(\tilde{\mathcal{O}}) \subset \Gamma$ for any open $\tilde{\mathcal{O}}$ with closure in $\mathcal{O}$, their results imply that $v|_{\mathcal{O}} = u$ is smooth away from a set of Hausdorff dimension at most $\dim\mathcal{O} - 3$ for the free boundary case and at most $\dim\mathcal{O} - 2$ for the constrained problem. They also give examples of minimisers with singular sets in $\mathcal{O}$ which have precisely the stated Hausdorff dimensions, thus showing that the dimension bounds for the singular set are optimal. Duzaar and Steffen [11, 12] simultaneously obtained the same partial regularity for the constrained problem. Baldes [1] and Gulliver and Jost [18] have considered the regularity of weakly harmonic maps with respect to (partially) free/constrained boundary data under geometric constraints on N and the maps in question. Such maps have harmonic extensions which minimise the Dirichlet energy, in particular, with respect to the (partially) free boundary condition $v(\tilde{\mathcal{O}}) \subset N$ for any open $\tilde{\mathcal{O}}$ with closure contained in $\mathcal{O}$.

There are two stages to proving partial regularity theories for stationary or minimising harmonic maps. First one proves partial Hölder continuity and then that (Hölder) continuity implies higher regularity. The latter fact was known when the first theories for minimising and stationary harmonic maps were developed. An exposition of the method used to show that continuous harmonic maps are smooth is given by Jost in [22]. Schoen showed that if a harmonic map is Hölder continuous, one can more readily deduce that this map is smooth [38]. The primary concern in regularity theories regarding harmonic maps to date has therefore been to establish their continuity. In the theory presented here, we are required to prove both continuity and higher regularity.

Our main theorem for fractional harmonic maps is obtained from two theorems regarding their extensions: an ε-regularity theorem and a partial regularity theorem, Theorems 4.0.1 and 4.0.2 in Section 4 respectively, which hold for the extensions up to $\mathcal{O}$. We do not provide an optimal estimate for the Hausdorff dimension of the singular set but we construct comparison maps which we expect can be used to establish the compactness of sequences of local minimisers of E^β and facilitate dimension reducing arguments of the singular set. It is indicated in Remark 3.3 of [11], which corresponds here to $\beta = 0$, that the construction of comparison maps which we generalise can be used to this end. We will explore this further in future work.

2 Preliminaries

The Euler-Lagrange equations for E^β at critical points $v : \mathbb{R}^{m+1}_+ \to N$ are semi-linear with leading order term $\mathrm{div}(x^\beta_{m+1}\nabla v)$. We will require Sobolev spaces suited to the analysis of solutions of such equations, as well as the associated linear equation $\mathrm{div}(x^\beta_{m+1}\nabla v) = 0$. For $\beta \neq 0$ the coefficient x^β_{m+1} degenerates or becomes singular on $\partial\mathbb{R}^{m+1}_+$ depending on the sign of β and the theory of uniformly elliptic second order partial differential equations does not apply on sets overlapping $\partial\mathbb{R}^{m+1}_+$. Viewing the coefficient x^β_{m+1} as a weighting (density) of the Lebesgue measure $\mathrm{d}x$, we may instead appeal to the theory of weighted second order degenerate elliptic equations. We recall and define the function spaces, and some of their analytical properties, necessary for our analysis and then record properties of solutions to $\mathrm{div}(x^\beta_{m+1}\nabla v) = 0$ which we will require for the study of solutions of the Euler-Lagrange equations of E^β.

For every $\beta \in (-1, 1)$, there exists $C > 0$ such that $\frac{1}{\int_B \mathrm{d}x}\int_B |x_{m+1}|^\beta \mathrm{d}x \frac{1}{\int_B \mathrm{d}x}\int_B |x_{m+1}|^{-\beta}\mathrm{d}x \leq C$ for every ball $B \subset \mathbb{R}^{m+1}$, where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}^{m+1}$. Hence $|x_{m+1}|^\beta$ is a weight of

Muckenhoupt class A_2, see [20] for an overview of these weights. Every such weight is canonically associated to corresponding Sobolev and Lebesgue spaces. Let $\Omega \subset \mathbb{R}^{m+1}$. Define

$$L^2_\beta(\Omega;\mathbb{R}^n) = \{f : \Omega \to \mathbb{R}^n : f \text{ is measurable}, \int_\Omega |f|^2 \, |x_{m+1}|^\beta \mathrm{d}x < \infty\}$$

which is a Hilbert space, see [6] Theorem 3.4.1, where the inner product of $f, g \in L^2_\beta(\Omega;\mathbb{R}^n)$ is given by $\langle f, g\rangle_{L^2_\beta(\Omega;\mathbb{R}^n)} = \int_\Omega \langle f, g\rangle \, |x_{m+1}|^\beta \mathrm{d}x$ where $\langle f, g\rangle$ is the inner product of f and g in $\mathbb{R}^n$. Define

$$W^{1,2}_\beta(\Omega;\mathbb{R}^n) = \{v : \Omega \to \mathbb{R}^n : v, \partial_i v \in L^p_\beta(\Omega;\mathbb{R}^n) \text{ for } i = 1, \ldots, m+1\}$$

where $\partial_i v$ denotes the weak partial derivative of v with respect to x_i. Proposition 2.1.2 of [40] guarantees that $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ is a Hilbert space the inner product $\langle v, w\rangle_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)} = \int_\Omega \langle v, w\rangle \, |x_{m+1}|^\beta \mathrm{d}x + \int_\Omega \langle \nabla u, \nabla v\rangle \, |x_{m+1}|^\beta \mathrm{d}x$ for $v, w \in W^{1,2}_\beta(\Omega;\mathbb{R}^n)$, where we write $\langle \nabla v, \nabla w\rangle = \sum_{i=1}^{m+1} \langle \partial_i v, \partial_i w\rangle$. We also define the Hilbert space $W^{1,2}_{\beta,0}(\Omega;\mathbb{R}^n)$ as the closure of $C^\infty_0(\Omega;\mathbb{R}^n)$ in $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ with respect to the norm induced by the inner product on $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$. When $\beta = 0$ we omit the subscript β from the preceding notation.

It is worth noting that, for every $\beta \in (-1,1)\backslash\{0\}$, approximation by smooth functions in $L^2_\beta(\Omega;\mathbb{R}^n)$ and $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ works in the same way as for the spaces $L^2(\Omega;\mathbb{R}^n)$ and $W^{1,2}(\Omega;\mathbb{R}^n)$. The details of this process are given in Theorem 2.1.4 and Corollary 2.1.5 in [40]. The spaces $W^{1,2}_\beta, W^{1,2}_{\beta,0}$ and L^2_β also have essentially the same analytical properties, such as completeness, reflexivity etc. as the unweighted spaces $W^{1,2}, W^{1,2}_0$ and L^2 respectively. Moreover, since $|x_{m+1}|^\beta$ is an A_2 weight, many inequalities that hold for the spaces $W^{1,p}$, such as the Poincarè inequality, have counterparts for functions in $W^{1,2}_\beta$ [20].

We will need to refer to the relationship between $W^{1,2}_\beta$ and $W^{1,p}$ and record it in the following Lemma.

Lemma 2.0.1. *Let $\beta \in (-1,1)$ and suppose $\Omega \subset \mathbb{R}^{m+1}_+$ is open and bounded.*

1. *If $\overline{\Omega} \subset \mathbb{R}^{m+1}_+$ then $W^{1,2}_\beta(\Omega;\mathbb{R}^n) = W^{1,2}(\Omega;\mathbb{R}^n)$.*
2. *If $\beta \leq 0$ then $W^{1,2}_\beta(\Omega;\mathbb{R}^n) \subset W^{1,2}(\Omega;\mathbb{R}^n)$.*
3. *If $\beta > 0$ then $W^{1,2}_\beta(\Omega;\mathbb{R}^n) \subset W^{1,p}(\Omega;\mathbb{R}^n)$ for every $1 \leq p < \frac{2}{1+\beta}$.*

Proof. Part 1 follows as in this case the norms on $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ and $W^{1,2}(\Omega;\mathbb{R}^n)$ are equivalent. Part 2 follows as in this case the $W^{1,2}(\Omega;\mathbb{R}^n)$ norm is dominated by a constant multiple (depending on Ω) of the $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ norm. Part 3 follows from Hölder's inequality. □

2.1 Weighted Homogeneous Sobolev Spaces

Consider the Dirichlet energies

$$E^\beta(v) = \frac{1}{2} \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \tag{2.1}$$

where $\beta \in (-1,1)$, which are well defined on

$$\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n) := \{ \ \phi \mid \phi = f|_{\mathbb{R}^{m+1}_+} \text{ for some } f \in C_0^\infty(\mathbb{R}^{m+1};\mathbb{R}^n)\}.$$

The energy E^β is naturally associated to the following Sobolev space.

Definition 2.1.1. Let $\beta \in (-1,1)$. The *Weighted Homogeneous Sobolev Space* $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ is the completion of $\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ with respect to the metric induced by the square root of E^β.

The elements of $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ are, strictly speaking, equivalence classes of Cauchy sequences and it will be necessary to have tangible representatives of these classes which may take values in N.

Lemma 2.1.1. *Let $m \in \mathbb{N}$ with $m \geq 2$ and Ω be an open bounded subset of $\mathbb{R}^{m+1}_+$. If $m = 2$ let $\beta \in (-3^{-1},1)$ and if $m \geq 3$ let $\beta \in (-1,1)$. Then there is a bounded linear operator I : $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n) \to W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ which satisfies $If = f|_\Omega$ for every $f \in \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$. Moreover,*

$$||Iv||_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)} \leq C||v||_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)} \tag{2.2}$$

for every $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ where C is a positive constant depending on m and Ω.

Proof. It suffices to establish that (2.2) holds for all $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$, as this space is dense in $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ and $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ is a Banach Space. Let $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$. By definition $||\nabla\phi||^2_{L^2_\beta(\Omega;\mathbb{R}^n)} \leq ||\phi||^2_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)}$ so we need to estimate $||\phi||^2_{L^2_\beta(\Omega;\mathbb{R}^n)}$ in terms of $||\phi||^2_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)}$.

Suppose $m \geq 3$. We have $\phi(\cdot, x_{m+1}) \in W^{1,2}(\mathbb{R}^m;\mathbb{R}^n)$ for every $x_{m+1} \in [0,\infty)$. Let ∇' denote the derivative of ϕ with respect to x' for $(x', x_{m+1}) \in \mathbb{R}^m \times [0,\infty)$. We apply Fubini's Theorem, together with Hölder's inequality, for conjugate exponents $\frac{m}{m-2}$ and $\frac{m}{2}$, and the Sobolev inequality in $\mathbb{R}^m$ to see that

$$\begin{aligned}
\int_\Omega x^\beta_{m+1}|\phi|^2\mathrm{d}x &= \int_a^b x^\beta_{m+1}\int_{l(x_{m+1})}|\phi|^2\mathrm{d}x'\mathrm{d}x_{m+1} \\
&\leq C\left(\mathrm{diam}(\Omega)\right)^2\int_a^b x^\beta_{m+1}\left(\int_{\mathbb{R}^m}|\phi|^{\frac{2m}{m-2}}\mathrm{d}x'\right)^{\frac{m-2}{m}}\mathrm{d}x_{m+1} \\
&\leq C\left(\mathrm{diam}(\Omega)\right)^2\int_a^b x^\beta_{m+1}\int_{\mathbb{R}^m}|\nabla'\phi|^2\mathrm{d}x'\mathrm{d}x_{m+1} \\
&= C\left(\mathrm{diam}(\Omega)\right)^2\int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1}|\nabla'\phi|^2\mathrm{d}x.
\end{aligned}$$

We conclude that $||\phi||^2_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)} \leq C||\phi||^2_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)}$.

The Sobolev Embedding Theorem as applied previously is no longer viable on planes of dimension 2. However, provided $\beta > -3^{-1}$, Corollary 2 in Section 2.1.7 of [24] implies that for every $\phi \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^n)$, and therefore by approximation every $\phi \in W^{1,2}_\beta(\mathbb{R}^3;\mathbb{R}^n)$, we have

$$\left(\int_{\mathbb{R}^3}|x_3|^{3\beta}|\phi|^6\mathrm{d}x\right)^{\frac{1}{3}} \leq C\int_{\mathbb{R}^3}|x_3|^\beta|\nabla\phi|^2\mathrm{d}x.$$

The even reflection with respect to $\partial\mathbb{R}^{m+1}_+$, denoted $\tilde{\phi}$, of a $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ is in $W^{1,2}_\beta(\mathbb{R}^3;\mathbb{R}^n)$ and hence, applying Hölder's inequality, we find

$$\int_\Omega x_3^\beta|\phi|^2\mathrm{d}x \leq |\Omega|^{\frac{2}{3}}\left(\int_{\mathbb{R}^3}|x_3|^{3\beta}|\tilde{\phi}|^6\mathrm{d}x\right)^{\frac{1}{3}} \leq C|\Omega|^{\frac{2}{3}}\int_{\mathbb{R}^3}|x_3|^\beta|\nabla\tilde{\phi}|^2\mathrm{d}x = 2C|\Omega|^{\frac{2}{3}}\int_{\mathbb{R}^3_+}x_3^\beta|\nabla\phi|^2\mathrm{d}x,$$

for every $\phi \in \mathcal{D}_+(\mathbb{R}^3_+;\mathbb{R}^n)$. We again conclude $||\phi||^2_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)} \leq C||\phi||^2_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)}$. □

Remark 2.1.1. We can further connect $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ with $W^{1,p}(\Omega;\mathbb{R}^n)$ for open bounded $\Omega \subset \mathbb{R}^{m+1}_+$ using Lemma 2.0.1.

2.2 Compactness of the Embedding $W^{1,2}_\beta \hookrightarrow L^2_\beta$

We will require an analogue of the compact embedding $W^{1,2} \hookrightarrow L^2$ in order to analyse re-scaled limits of bounded sequences of Sobolev functions. Away from the boundary, that is, for Ω with $\overline{\Omega} \subset \mathbb{R}^{m+1}_+$, the compactness of the inclusion $W^{1,2} \hookrightarrow L^2$ yields the compactness of the inclusion $W^{1,2}_\beta \hookrightarrow L^2_\beta$ in view of Lemma 2.0.1. We have not been able to find a proof of compactness near $\partial\mathbb{R}^{m+1}_+$ in the literature so present one for completeness. Let $B_r(y) = \{x \in \mathbb{R}^{m+1} : |x-y| < r\}$ and $Q_r(y) = \{x \in \mathbb{R}^{m+1} : |x_i - y_i| < r, i = 1,\ldots,m+1\}$ and define $B^+_r(y) = B_r(y) \cap \mathbb{R}^{m+1}_+$ and $Q^+_r(y) = Q_r(y) \cap \mathbb{R}^{m+1}_+$ for $y \in \partial\mathbb{R}^{m+1}_+$.

Lemma 2.2.1. *Let $r > 0$, $y \in \partial\mathbb{R}^{m+1}_+$ and suppose $(v_j)_{j\in\mathbb{N}}$ is a sequence in $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ with $\sup_j ||v_j||_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)} < \infty$ where Ω is either $Q^+_r(y)$ or $B^+_r(y)$. Then there exists a subsequence $(v_{j_k})_{k\in\mathbb{N}}$ and a $v \in W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ such that*

1. *$v_{j_k} \rightharpoonup v$ in $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$*
2. *$v_{j_k} \to v$ in $L^2_\beta(\Omega;\mathbb{R}^n)$*
3. *$\int_\Omega x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \leq \liminf_{k\to\infty}\int_\Omega x^\beta_{m+1}|\nabla v_{j_k}|^2\mathrm{d}x$.*

Proof. For $\beta = 0$, a proof can be found in [39] Section 1.3 Lemma 1. Otherwise, statement 1 follows from the weak sequential compactness of the unit ball in a Hilbert space and statement 3 follows from statement 2 and the lower semi-continuity of a Hilbert space norm. Hence, the main task is to prove statement 2. We may assume $r = 1$ and $y = 0$ since statement 2 is invariant under rescaling and translations with respect to x_i for $i = 1,\ldots,m$. We further assume that $\Omega = Q^+_1(0)$ since if $\Omega = B^+_1(0)$ and the result is true on $Q^+_1(0)$, we may compose with the, bi-Lipschitz piecewise C^1 with piecewise C^1 inverse map $Q^+_1(0) \to B^+_1(0) : x \mapsto \max_i|x_i||x|^{-1}x$ and deduce the statement on $B^+_1(0)$.

Suppose $(v_j)_{j\in\mathbb{N}}$ is a sequence with $v_j \in W^{1,2}_\beta(Q^+_1(0);\mathbb{R}^n)$ for every j and which satisfies

$$\sup_{j\in\mathbb{N}}||v_j||_{W^{1,2}_\beta(Q^+_1(0);\mathbb{R}^n)} \leq M \tag{2.3}$$

for some positive constant M. Relabelling if necessary, suppose $(v_j)_{j\in\mathbb{N}}$ is also the subsequence which satisfies $v_j \rightharpoonup v$ for $v \in W^{1,2}_\beta(Q^+_1(0);\mathbb{R}^n)$. Then

$$||v||_{W^{1,2}_\beta(Q^+_1(0);\mathbb{R}^n)} \leq M. \tag{2.4}$$

Let $Q' = (-1,1)^m$, define $Q_i = \{(x', x_{m+1}) \in Q_1^+(0) : (i+1)^{-1} < x_{m+1} \le 1\}$ for $i \in \mathbb{N}$ and let $\hat{Q}_i = Q_1^+(0)\backslash Q_i = Q' \times (0,(i+1)^{-1})$. In view of (2.3), we have $\sup_{j\in\mathbb{N}} ||v_j||_{W^{1,2}_\beta(Q_i;\mathbb{R}^n)} \le M$ for each $i \in \mathbb{N}$. Hence, using the compactness of the embedding $W^{1,2}_\beta(Q_1;\mathbb{R}^n) \hookrightarrow L^2_\beta(Q_1;\mathbb{R}^n)$, we find a $\tilde{v} \in W^{1,2}_\beta(Q_1;\mathbb{R}^n)$ and a subsequence, which we denote $(v_j)_{j\in\Lambda_1}$ for an infinite set $\Lambda_1 \subset \mathbb{N}$, which satisfies $v_j \rightharpoonup \tilde{v}$ in $W^{1,2}_\beta(Q_1;\mathbb{R}^n)$, $v_j \to \tilde{v}$ in $L^2_\beta(Q_1;\mathbb{R}^n)$ and almost everywhere as $j \to \infty$ with $j \in \Lambda_1$. Notice that $(v_j)_{j\in\Lambda_1}$ converges weakly to v in $W^{1,2}_\beta(Q_1;\mathbb{R}^n)$ because $(v_j)_{j\in\mathbb{N}}$ does and so, by the uniqueness of weak limits, we deduce $\tilde{v} = v$ in Q_1. Hence $v_j \to v$ in $L^2_\beta(Q_1;\mathbb{R}^n)$ and almost everywhere as $j \to \infty$ as well. Repeating this process inductively for every $i \in \mathbb{N}$, we obtain sequences $(v_j)_{j\in\Lambda_i}$ with $\Lambda_{i+1} \subset \Lambda_i$ such that $(v_j)_{j\in\Lambda_i}$ converges to v in $L^2_\beta(Q_i;\mathbb{R}^n)$ and almost everywhere in Q_i. Hence we can choose an increasing sequence of numbers $(k_i)_{i\in\mathbb{N}}$ with $k_i \in \Lambda_i$ such that

$$\int_{Q_i} x_{m+1}^\beta |v_k - v|^2 \mathrm{d}x < \frac{\int_{\frac{1}{i+1}}^{\frac{1}{i}} x_{m+1}^\beta \mathrm{d}x_{m+1}}{2^i} \le \frac{1}{i^{2+\beta}2^i} \tag{2.5}$$

for $k \ge k_i$. Then the sequence $(v_{k_i})_{i\in\mathbb{N}}$ converges to v almost everywhere in $Q_1^+(0)$ and in $L^2_\beta(Q_k;\mathbb{R}^n)$ for all $k \in \mathbb{N}$ as $i \to \infty$. Observe that

$$\int_{Q_1^+(0)} x_{m+1}^\beta |v_{k_i} - v|^2 \mathrm{d}x = \int_{\hat{Q}_i} x_{m+1}^\beta |v_{k_i} - v|^2 \mathrm{d}x + \int_{Q_i} x_{m+1}^\beta |v_{k_i} - v|^2 \mathrm{d}x. \tag{2.6}$$

By (2.5) we have $\int_{Q_i} x_{m+1}^\beta |v_{k_i} - v| \mathrm{d}x < \frac{1}{i^{2+\beta}2^i} \to 0$ as $i \to \infty$ so we consider the remaining term in (2.6). Using Chebychev's inequality combined with Fubini's Theorem, we may choose $c_i \in \big((i+1)^{-1}, i^{-1}\big)$ such that

$$\int_{Q'} |v_{k_i}(x', c_i) - v(x', c_i)|^2 \mathrm{d}x' \le \frac{1}{\left(\int_{\frac{1}{i+1}}^{\frac{1}{i}} x_{m+1}^\beta \mathrm{d}x_{m+1}\right)} \int_{\frac{1}{i+1}}^{\frac{1}{i}} \int_{Q'} x_{m+1}^\beta |v_{k_i} - v|^2 \mathrm{d}x. \tag{2.7}$$

Now for each $i \in \mathbb{N}$, we calculate

$$\begin{aligned} \int_{\hat{Q}_i} x_{m+1}^\beta |v_{k_i} - v|^2 \mathrm{d}x \le 4 \int_{\hat{Q}_i} x_{m+1}^\beta |v_{k_i} - v_{k_i}(x', c_i)|^2 \mathrm{d}x + 4 \int_{\hat{Q}_i} x_{m+1}^\beta |v - v(x', c_i)|^2 \mathrm{d}x \\ + 4 \int_{\hat{Q}_i} x_{m+1}^\beta |v_{k_i}(x', c_i) - v(x', c_i)|^2 \mathrm{d}x. \end{aligned} \tag{2.8}$$

We apply Hölder's inequality and (2.4) to see that

$$\begin{aligned} \int_{\hat{Q}_i} x_{m+1}^\beta |v(x', x_{m+1}) - v(x', c_i)|^2 \mathrm{d}x &= \int_{\hat{Q}_i} x_{m+1}^\beta \left| \int_{x_{m+1}}^{c_i} s^{-\frac{\beta}{2}} s^{\frac{\beta}{2}} \partial_{m+1} v(x', s) \mathrm{d}s \right|^2 \mathrm{d}x \\ &\le \frac{c_i^{1-\beta}}{1-\beta^2} c_i^{1+\beta} \int_0^{c_i} \int_{Q'} x_{m+1}^\beta \left|\partial_{m+1} v\right|^2 \mathrm{d}x \\ &\le \frac{1}{1-\beta^2} \frac{1}{i^2} M^2. \end{aligned} \tag{2.9}$$

The bound for the integral on the left hand side of (2.9) but with v replaced by v_{k_i} is identical. We apply (2.7) followed by (2.5) to see that

$$\int_{\hat{Q}_i} x_{m+1}^{\beta}|v_{k_i}(x',c_i)-v(x',c_i)|^2\mathrm{d}x \le \frac{\int_0^{\frac{1}{i+1}} x_{m+1}^{\beta}\mathrm{d}x_{m+1}}{\int_{\frac{1}{i+1}}^{\frac{1}{i}} x_{m+1}^{\beta}\mathrm{d}x_{m+1}} \int_{\frac{1}{i+1}}^{\frac{1}{i}}\int_{Q'} x_{m+1}^{\beta}|v_{k_i}-v|^2\mathrm{d}x$$
$$\le \frac{1}{(1+\beta)i^{1+\beta}2^i}. \tag{2.10}$$

Finally we combine (2.5), (2.6), (2.8), (2.9) and (2.10) and let $i\to\infty$ to conclude the proof. □

Remark 2.2.1. The method of proof of Lemma 2.2.1 is also valid for Ω of the form $\Omega=\mathcal{O}\times[0,r]$ for $r>0$ and $\mathcal{O}\subset\partial\mathbb{R}_+^{m+1}$.

2.3 Energy Decay for a Linear Neumann-type Problem

When examining the limit of re-scaled sequences of Sobolev functions, as in the proof of Lemma 4.9.1, we will obtain a weak solution of the Neumann-type problem

$$\operatorname{div}(x_{m+1}^{\beta}\nabla v)=0 \text{ in } B_R^+(x_0) \quad\text{and}\quad x_{m+1}^{\beta}\partial_{m+1}v=0 \text{ in } \partial B_R^+(x_0)\cap\partial\mathbb{R}_+^{m+1}, \tag{2.11}$$

for some $R>0$ and $x_0\in\partial\mathbb{R}_+^{m+1}$. The rate of decay of the re-scaled energy of such solutions on concentric half-balls centred at x_0 will play a role in the proof of the aforementioned lemma and we estimate this decay here.

Lemma 2.3.1. *Let $\beta\in(-1,1)$ and suppose $v\in W_\beta^{1,2}(B_R^+(x_0);\mathbb{R}^n)$ satisfies*

$$\int_{B_R^+(x_0)} x_{m+1}^{\beta}\langle\nabla v,\nabla\psi\rangle\mathrm{d}x=0 \tag{2.12}$$

for every $\psi\in C_0^\infty(B_R(x_0);\mathbb{R}^n)$. There exists a $\gamma=\gamma(m,\beta)\in(0,1)$ and a positive constant C such that

$$\left(\frac{r}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{r}{2}}(x_0)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x\le Cr^{2\gamma} \tag{2.13}$$

for every $r\le\frac{R}{2}$.

Proof. The even reflection of v in $\partial\mathbb{R}_+^{m+1}$, which we don't relabel, belongs to $W_\beta^{1,2}(B_R(x_0);\mathbb{R}^n)$ and satisfies

$$\int_{B_R(x_0)} |x_{m+1}|^{\beta}\langle\nabla v,\nabla\psi\rangle\mathrm{d}x=0 \tag{2.14}$$

for every $\psi\in C_0^\infty(B_R(x_0);\mathbb{R}^n)$. A result of Fabes et al, see [14] Theorem 2.3.12, implies the local Hölder continuity of v in $B_R(x_0)$. In particular, there exists a constant C such that

$$|v(x)-v(y)|\le C|x-y|^{\gamma} \tag{2.15}$$

for some $\gamma\in(0,1)$ and every $x,y\in B_{\frac{R}{2}}(x_0)$.

By approximation, (2.14) holds for every $\psi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$. Let $\eta \in C_0^\infty(B_r(x_0);[0,1])$ be a cutoff function with $\eta \equiv 1$ in $B_{\frac{r}{2}}(x_0)$ and $|\nabla\eta| \leq \frac{C}{r}$ for a fixed positive $C \geq 2$. We observe that $\phi = \eta^2(v-\lambda)$ is an admissible test function for every $\lambda \in \mathbb{R}^n$. Testing (2.14) against ϕ and using Young's inequality, $ab \leq \delta\frac{a^2}{2} + \frac{b^2}{2\delta}$ for $a, b \geq 0$ and $\delta > 0$, we see that

$$\int_{B_r(x_0)} |x_{m+1}|^\beta \eta^2 |\nabla v|^2 \mathrm{d}x \leq \delta \int_{B_r(x_0)} |x_{m+1}|^\beta \eta^2 |\nabla v|^2 \mathrm{d}x + \frac{C}{\delta}\int_{B_r(x_0)} |x_{m+1}|^\beta |\nabla \eta|^2 |v-\lambda|^2 \mathrm{d}x.$$

We choose $\delta = \frac{1}{2}$, set $\lambda = v(x_0)$, recall $|\nabla\eta| \leq \frac{C}{r}$ and apply (2.15) to see that

$$\int_{B_r(x_0)} |x_{m+1}|^\beta \eta^2 |\nabla v|^2 \mathrm{d}x \leq C r^{1-m-\beta+\gamma} \tag{2.16}$$

for another positive $C > 0$, independent of $r \leq \frac{R}{2}$. Multiplying (2.16) by $(2^{-1}r)^{1-m-\beta}$ concludes the proof. □

2.4 Boundary Monotonicity Formula for the Average Energy of Solutions to the Degenerate Linear Equation

In order to utilise a version of the method of harmonic replacement, see Lemma 4.12.1 in Section 4.12, we need to know how the average energy of solutions to $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ behaves on concentric balls. When these balls are centred on $\partial\mathbb{R}^{m+1}_+$, we have the following.

Theorem 2.4.1. *Let $B_R(x_0) \subset \mathbb{R}^{m+1}$ with $(x_0)_{m+1} = 0$ and $R \leq 1$. Suppose $v \in W^{1,2}_\beta(B_R(x_0);\mathbb{R}^n)$ is a weak solution of div$(|x_{m+1}|^\beta \nabla v) = 0$ in $B_R(x_0)$. If $\beta \in (-1, 0]$ or if $\beta \in (0,1)$ and v is symmetric with respect to $\partial\mathbb{R}^{m+1}_+$ in $B_R(x_0)$ then*

$$s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x \leq r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x$$

for every $0 < s \leq r \leq R$.

Remark 2.4.1. Theorem 2.4.1 is a particular case of Theorem 2.6 of [4] where the assumptions are slightly different: suppose v is a weak solution of $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_1(0)$ with $v \in W^{1,2}_\beta(B_1(0);\mathbb{R}^n)$ and $v \in L^2(B_1^+(0);\mathbb{R}^n)$, then the conclusion of Theorem 2.4.1 holds. This statement is actually false in general for $\beta \in (0,1)$ as is illustrated by considering the function

$$\tilde{v}(x', x_{m+1}) = \begin{cases} \frac{1}{1-\beta} x_{m+1}^{1-\beta} & \text{if } x_{m+1} \geq 0 \\ -\frac{1}{1-\beta}(-x_{m+1})^{1-\beta} & \text{if } x_{m+1} < 0 \end{cases}$$

for $\beta \in (-1, 1)$.

The remainder of this section is predominantly devoted to a proof of Theorem 2.4.1. We also establish a similar monotonicity formula for balls $B_R(x_0)$ with $(x_0)_{m+1} \geq \theta R$ for $\theta \geq 2$ and give explicit dependence on θ. Our method is essentially that used to show the well known monotonicity results for the average energy of sub-harmonic functions, which is based on the observation $\Delta|\nabla v|^2 \geq 0$ if $\Delta v = 0$. When $\beta = 0$ our result reduces to the usual growth formula for the average energy of sub-harmonic functions.

Define $v^* := |x_{m+1}|^\beta \partial_{m+1} v$. This function will be integral to our proof of Theorem 2.4.1 because, as we will see in more detail later, it satisfies $\operatorname{div}(|x_{m+1}|^{-\beta}\nabla v^*) = 0$ when $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$. This fact that if $\operatorname{div}(x_{m+1}^\beta \nabla v) = 0$ in $\mathbb{R}^{m+1}_+$ then $\operatorname{div}(x_{m+1}^{-\beta}\nabla v^*) = 0$ was first observed by Caffarelli and Silvestre [3].

First, we consider the regularity of solutions of $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$. The following results have essentially been obtained in, for example, [5]. We give a proof regardless, as an illustration that the method of difference quotients works essentially unchanged in the directions $x_1, \ldots, x_m$, which we will take advantage of when considering the regularity of manifold valued minimisers of E^β.

Let $i = 1, \ldots, m$, $\Omega \subset \mathbb{R}^{m+1}$ and let $h \in \mathbb{R}$. Define the difference quotient of $v : \Omega \to \mathbb{R}^n$ by $\Delta_i^h v(x) = h^{-1}(v(x+he_i) - v(x))$ where e_i denotes the i-th basis vector in $\mathbb{R}^{m+1}$ and $\operatorname{dist}(x, \partial\Omega) < |h|$.

Lemma 2.4.1. *Let $\Omega \subset \mathbb{R}^{m+1}$ be open and $v \in W^{1,2}_\beta(\Omega; \mathbb{R}^n)$. Then for any $i = 1, \ldots, m$ we have $\Delta_i^h v \in L^2_\beta(K; \mathbb{R}^n)$ for any compact $K \subset \Omega$, provided $|h| <$ dist$(K, \partial\Omega)$. In particular,*

$$\int_K |x_{m+1}|^\beta |\Delta_i^h v|^2 \mathrm{d}x \le \int_\Omega |x_{m+1}|^\beta \left|\partial_i v\right|^2 \mathrm{d}x.$$

Proof. This proof follows the proof of Lemma 7.23 in [16]. We assume that $h \ge 0$, the argument for negative h is analogous. Let $v \in C^1(\Omega; \mathbb{R}^n) \cap W^{1,2}_\beta(\Omega; \mathbb{R}^n)$. Using the notation $K_h = \{x \in \mathbb{R}^{m+1} : \operatorname{dist}(x, K) \le h\}$ and noting $K_h \subset \Omega$, by Fubini's Theorem and the compactness of K, for h with $|h| < \operatorname{dist}(K, \partial\Omega)$ and any $i = 1, \ldots, m$, we calculate

$$\begin{aligned}
\int_K |x_{m+1}|^\beta |\Delta_i^h v|^2 \mathrm{d}x &\le \int_K |x_{m+1}|^\beta \frac{1}{h} \int_0^h \left|\partial_i v(x + te_i)\right|^2 \mathrm{d}t\mathrm{d}x \\
&\le \frac{1}{h} \int_0^h \int_{K_h} |x_{m+1}|^\beta \left|\partial_i v\right|^2 \mathrm{d}x\mathrm{d}t \\
&\le \int_\Omega |x_{m+1}|^\beta \left|\partial_i v\right|^2 \mathrm{d}x.
\end{aligned}$$

We deduce the result for $v \in W^{1,2}_\beta(\Omega; \mathbb{R}^n)$ by approximation. □

Next we prove a criterion for the existence of weak derivatives in the directions x_i for $i = 1, \ldots, m$.

Lemma 2.4.2. *Suppose $\Omega \subset \mathbb{R}^{m+1}$ is open and bounded and let $v \in L^2_\beta(\Omega; \mathbb{R}^n)$. For any $i = 1, \ldots, m$, suppose there exist constants $M > 0$ and $\tilde{h} > 0$ such that*

$$\int_K |x_{m+1}|^\beta |\Delta_i^h v|^2 \mathrm{d}x \le M$$

for every $h \ne 0$ with $|h| < \tilde{h}$ and compact $K \subset \Omega$ with $\operatorname{dist}(K, \partial\Omega) > |h|$. Then the weak derivative $\partial_i v$ exists in Ω and satisfies

$$\int_\Omega |x_{m+1}|^\beta \left|\partial_i v\right|^2 \mathrm{d}x \le M.$$

Proof. First choose a sequence $(h_k)_{k\in\mathbb{N}}$ with $h_k \to 0$. We discard h_k with $|h_k| \ge \tilde{h}$ and re-index to $k \in \mathbb{N}$. Define $v_i^{h_k}(x) = \Delta_i^{h_k} v(x)$ when $x \in \Omega$ and $\operatorname{dist}(x, \partial\Omega) \ge 2|h_k|$ and $v_i^{h_k} = 0$ otherwise.

It follows that $\{v_i^{h_k}\}_{k\in\mathbb{N}}$ is a bounded sequence in $L^2_\beta(\Omega;\mathbb{R}^n)$. Hence there is a subsequence, which we index again by $k \in \mathbb{N}$, such that $h_k \to 0$ and $v_i^{h_k} \to \tilde{v}_i$ weakly in $L^2_\beta(\Omega;\mathbb{R}^n)$. Furthermore, this convergence, together with the weak lower semi-continuity of a Hilbert space norm, guarantees that $\int_\Omega x_{m+1}^\beta |\tilde{v}_i|^2 \,\mathrm{d}x \leq M$. Note that Hölder's inequality implies $L^2_\beta(\Omega;\mathbb{R}^n) \subset L^p(\Omega;\mathbb{R}^n)$ for some $p \in (1,2]$ depending on β. Thus each linear functional on $L^p(\Omega;\mathbb{R}^n)$ restricts to a linear functional on $L^2_\beta(\Omega;\mathbb{R}^n)$ and $v_i^{h_k}$ converges to $\tilde{v}_i$ weakly in $L^p(\Omega;\mathbb{R}^n)$. As in the proof of Lemma 7.24 in [16], it follows that $\tilde{v}_i$ is the weak derivative $\partial_i v$. □

We now establish the regularity properties of solutions of $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$ needed for the proof of Theorem 2.4.1.

Lemma 2.4.3. *Let $v \in W^{1,2}_\beta(B_R(x_0);\mathbb{R}^n)$ and suppose v is a weak solution of $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_R(x_0)$. For every $r < R$ and $i = 1,\ldots,m$ it follows that $\partial_i v \in W^{1,2}_\beta(B_r(x_0);\mathbb{R}^n)$, $\partial_i v$ is a weak solution of $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_r(x_0)$ and $\partial_i v$ is locally Hölder continuous in $B_R(x_0)$. In addition, $v^* \in W^{1,2}_{-\beta}(B_r(x_0);\mathbb{R}^n)$ and v^* is a weak solution of $\operatorname{div}(|x_{m+1}|^{-\beta} \nabla v^*) = 0$ in $B_r(x_0)$ and v^* is locally Hölder continuous in $B_R(x_0)$.*

Proof. Elliptic regularity theory shows that v is smooth in $B_R(x_0)\backslash\partial\mathbb{R}^{m+1}_+$ [16]. Observe that v satisfies

$$\int_{B_R(x_0)} |x_{m+1}|^\beta \langle \nabla v, \nabla \phi \rangle \,\mathrm{d}x \tag{2.17}$$

for every $\phi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$ by approximation. Let $r < R$ and choose $\eta \in C_0^\infty(B_R(x_0))$ with $\eta \equiv 1$ in $B_r(x_0)$, $\eta \equiv 0$ in $B_R(x_0)\backslash B_{r+\frac{R-r}{2}}(x_0)$, $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq \frac{C}{R-r}$. Let $\Delta_i^h v$ be the difference quotient of v for some $i = 1,\ldots,m$ and suppose $|h| < \frac{R-r}{4}$. Then $\phi = -\Delta_i^{-h}(\eta^2 \Delta_i^h v) \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$ is an admissable test function for (2.17) and an application of Young's inequality, $ab \leq \delta\frac{a^2}{2} + \delta^{-1}\frac{a^2}{2}$ for $a, b \geq 0$ and $\delta > 0$, together with an integration by parts and Lemma 2.4.1 implies

$$\begin{aligned}\int_{B_R(x_0)} \eta^2 |x_{m+1}|^\beta |\nabla\Delta_i^h v|^2 \mathrm{d}x &= -\int_{B_R(x_0)} 2\eta |x_{m+1}|^\beta \left\langle \nabla\Delta_i^h v \cdot \nabla\eta, \Delta_i^h v\right\rangle \mathrm{d}x \\ &\leq \frac{C}{R-r}\delta \int_{B_R(x_0)} \eta^2 |x_{m+1}|^\beta |\nabla\Delta_i^h v|^2 \mathrm{d}x \\ &\quad + \frac{C}{R-r}\delta^{-1} \int_{B_R(x_0)} |x_{m+1}|^\beta |\partial_i v|^2 \mathrm{d}x.\end{aligned}$$

Since $\eta \equiv 1$ in $B_r(x_0)$, choosing $\delta = \frac{R-r}{2C}$ we deduce that

$$\int_{B_r(x_0)} |x_{m+1}|^\beta |\nabla\Delta_i^h v|^2 \mathrm{d}x \leq \frac{C}{(R-r)^2} \int_{B_R(x_0)} |x_{m+1}|^\beta |\partial_i v|^2 \mathrm{d}x.$$

The right hand side above is independent of h and thus Lemma 2.4.2 implies the weak derivative $\nabla\partial_i v$ exists and is in $L^2_\beta(B_r(x_0);\mathbb{R}^{(m+1)n})$. Hence $\partial_i v \in W^{1,2}_\beta(B_r(x_0);\mathbb{R}^n)$ for every $r < R$. We integrate by parts in (2.17) to see that $\partial_i v$ is a weak solution of $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_r(x_0)$ for every $r < R$. It follows from [14] Theorem 2.3.12 that each $\partial_i v$ is locally Hölder continuous in $B_R(x_0)$. We inductively deduce that for any multi-index α' with $\alpha'_{m+1} = 0$, $D^{\alpha'} v \in W^{1,2}_\beta(B_r(x_0);\mathbb{R}^n)$ is a weak solution of $\operatorname{div}(|x_{m+1}|^{\beta}\nabla D^{\alpha'}v) = 0$ in $B_r(x_0)$ for every $r < R$ and $D^{\alpha'}v$ is locally Hölder continuous in $B_R(x_0)$.

Since $\partial_i v \in W^{1,2}_{\beta}(B_r(x_0);\mathbb{R}^n)$ for every $r < R$ and $i = 1,\ldots,m$, we have

$$\int_{B_r(x_0)} |x_{m+1}|^{-\beta}|\partial_i v^*|^2 \mathrm{d}x = \int_{B_r(x_0)} |x_{m+1}|^{\beta}|\partial_i\partial_{m+1}v|^2 \mathrm{d}x < \infty. \tag{2.18}$$

We also have $\Delta' v \in W^{1,2}_{\beta}(B_r(x_0);\mathbb{R}^n)$, where Δ' is the Laplace operator with respect to the variables $x_1,\ldots,x_m$. Furthermore, as v solves $\operatorname{div}(|x_{m+1}|^{\beta}\nabla v) = 0$ classically in $B_R(x_0)\backslash\partial\mathbb{R}^{m+1}_+$, we have $|x_{m+1}|^{-\beta}\partial_{m+1}v^* = -\Delta' v \in W^{1,2}_{\beta}(B_r(x_0);\mathbb{R}^n)$ for every $r < R$. Hence

$$\int_{B_r(x_0)} |x_{m+1}|^{-\beta}|\partial_{m+1}v^*|^2 \mathrm{d}x = \int_{B_r(x_0)} |x_{m+1}|^{\beta}|\Delta' v|^2 \mathrm{d}x < \infty. \tag{2.19}$$

Together, (2.18) and (2.19) imply $\int_{B_r(x_0)} |x_{m+1}|^{-\beta}|\nabla v^*|^2 \mathrm{d}x < \infty$ for every $r < R$. Moreover, we have

$$\int_{B_r(x_0)} |x_{m+1}|^{-\beta}|v^*|^2 \mathrm{d}x = \int_{B_r(x_0)} |x_{m+1}|^{\beta}|\partial_{m+1}v|^2 \mathrm{d}x < \infty, \tag{2.20}$$

since $v \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$. We can directly verify that $\partial_i v^*$, $i = 1,\ldots,m$, are the weak derivatives of v^* and omit the details. Now consider $\partial_{m+1}v^*$. Let ∇' denote the gradient operator with respect to the variables $x_1,\ldots,x_m$ and let $\psi \in C^{\infty}_0(B_R(x_0);\mathbb{R}^n)$. Since $|x_{m+1}|^{-\beta}\partial_{m+1}v^* = -\Delta' v \in W^{1,2}_{\beta}(B_r(x_0);\mathbb{R}^n)$ and v is a weak solution of $\operatorname{div}(|x_{m+1}|^{\beta}\nabla v) = 0$, we see that

$$\begin{aligned}\int_{B_R(x_0)} \langle \partial_{m+1}v^*, \psi\rangle \,\mathrm{d}x &= -\int_{B_R(x_0)} |x_{m+1}|^{\beta} \langle \Delta' v, \psi\rangle \,\mathrm{d}x \\ &= \int_{B_R(x_0)} |x_{m+1}|^{\beta} \langle \nabla' v, \nabla'\psi\rangle \,\mathrm{d}x \\ &= -\int_{B_R(x_0)} \langle v^*, \partial_{m+1}\psi\rangle \,\mathrm{d}x.\end{aligned}$$

It follows that $v^* \in W^{1,2}_{-\beta}(B_r(x_0);\mathbb{R}^n)$ for every $r < R$. In a similar manner we calculate

$$\begin{aligned}\int_{B_R(x_0)} |x_{m+1}|^{-\beta} \langle \nabla v^*, \nabla\psi\rangle \,\mathrm{d}x &= \int_{B_R(x_0)} |x_{m+1}|^{-\beta} \langle \partial_{m+1}v^*, \partial_{m+1}\psi\rangle \,\mathrm{d}x \\ &\quad + \int_{B_R(x_0)} |x_{m+1}|^{-\beta} \langle \nabla' v^*, \nabla'\psi\rangle \,\mathrm{d}x \\ &= -\int_{B_R(x_0)} \langle \Delta' v, \partial_{m+1}\psi\rangle \,\mathrm{d}x + \int_{B_R(x_0)} \langle \Delta' v, \partial_{m+1}\psi\rangle \,\mathrm{d}x \\ &= 0.\end{aligned}$$

Hence v^* is a weak solution of $\operatorname{div}(|x_{m+1}|^{\beta}\nabla v^*) = 0$ in $B_r(x_0)$ for every $r < R$. It follows from [14] Theorem 2.3.12 that v^* is locally Hölder continuous in $B_R(x_0)$. □

Corollary 2.4.1. *Suppose $v \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ and assume v weakly satisifes div$(|x_{m+1}|^{\beta}\nabla v) = 0$ in $B_R(x_0)$. Then the derivatives $D^{\alpha'}v$, where α' is a multi-index with $(\alpha')_{m+1} = 0$, are elements* *of* $W^{1,2}_\beta(B_r(x_0);\mathbb{R}^n)$ *and weak solutions of* $div(|x_{m+1}|^\beta\nabla v) = 0$ *in* $B_r(x_0)$ *for every* $r < R$ *and are locally Hölder continuous in* $B_R(x_0)$*. Furthermore, the functions* $(D^{\alpha'}v)^* := |x_{m+1}|^\beta\partial_{m+1}D^{\alpha'}v$ *are elements of* $W^{1,2}_{-\beta}(B_r(x_0);\mathbb{R}^n)$ *and weak solutions of* $div(|x_{m+1}|^{-\beta}\nabla(D^{\alpha'}v)^*) = 0$ *in* $B_r(x_0)$ *for every* $r < R$ *and are locally Hölder continuous in* $B_R(x_0)$ *.*

Proof. This follows from an immediate and direct application of Lemma 2.4.3. □

Next we record a condition on integral of the normal derivative of Sobolev functions which implies the type of monotonicity we want to establish in Theorem 2.4.1.

Lemma 2.4.4. *Suppose that* $v \in W^{1,2}_\beta(B_R(x_0))$*, where* $x_0 \in \partial\mathbb{R}^{m+1}_+$*, and that*

$$\int_{\partial B_\rho(x_0)} \nu \cdot |x_{m+1}|^\beta \nabla v \mathrm{d}S(x) \geq 0$$

for almost every $\rho \in (0, R)$*, where* ν *is the outward pointing unit normal on* $\partial B_\rho(x_0)$*, then*

$$s^{-(1+m+\beta)}\int_{B_s(x_0)} |x_{m+1}|^\beta v\mathrm{d}x \leq r^{-(1+m+\beta)}\int_{B_r(x_0)} |x_{m+1}|^\beta v\mathrm{d}x \tag{2.21}$$

for every $0 < s \leq r \leq R$*.*

Proof. This proof follows the proof of Theorem 2.1 in Section 2 of [16] and the proof of Proposition 2.2 in Section III of [15]. For almost every $0 < s \leq r < R$, using Fubini's Theorem, we see that

$$\begin{aligned} &r^{-(m+\beta)}\int_{\partial B_r(x_0)} |x_{m+1}|^\beta v\mathrm{d}S(x) - s^{-(m+\beta)}\int_{\partial B_s(x_0)} |x_{m+1}|^\beta v\mathrm{d}S(x) \\ &= \int_{\partial B_1(0)} |\omega_{m+1}|^\beta \int_s^r \frac{\partial}{\partial t} v(t\omega + x_0)\mathrm{d}t\mathrm{d}\omega \\ &= \int_s^r t^{-(m+\beta)}\int_{\partial B_t(x_0)} |x_{m+1}|^\beta \nu\cdot\nabla v\mathrm{d}S(x)\mathrm{d}t \\ &\geq 0. \end{aligned} \tag{2.22}$$

Define the absolutely continuous function $f(r) = \int_{B_r(x_0)} |x_{m+1}|^\beta v\mathrm{d}x$ for $0 \leq r \leq R$. Using (2.22) we calculate

$$f(r) = \int_0^r f'(\rho)\mathrm{d}\rho = \int_0^r \rho^{m+\beta}\rho^{-(m+\beta)}f'(\rho)\mathrm{d}\rho \leq \int_0^r \rho^{m+\beta}r^{-(m+\beta)}f'(r)\mathrm{d}\rho = \frac{r}{1+m+\beta}f'(r)$$

for $0 < r < R$. It follows that $(r^{-(1+m+\beta)}f(r))' \geq 0$ and integrating between $s \leq r \leq R$ completes the proof. □

Proof of Theorem 2.4.1. Note that v is smooth in $B_R(x_0)\backslash\partial\mathbb{R}^{m+1}_+$. Hence $\mathrm{div}(|x_{m+1}|^\beta\nabla\partial_i v) = 0$ and $\mathrm{div}(|x_{m+1}|^{-\beta}\nabla v^*) = 0$ classically in this set. Furthermore, we have

$$\begin{aligned} 0 = \partial_{m+1}\mathrm{div}(|x_{m+1}|^\beta\nabla v) &= \mathrm{div}(|x_{m+1}|^\beta\nabla\partial_{m+1}v) + \mathrm{sgn}(x_{m+1})\frac{\beta}{|x_{m+1}|}\mathrm{div}(|x_{m+1}|^\beta\nabla v) \\ &\quad - \beta|x_{m+1}|^{\beta-2}\partial_{m+1}v \\ &= \mathrm{div}(|x_{m+1}|^\beta\nabla\partial_{m+1}v) - \beta|x_{m+1}|^{\beta-2}\partial_{m+1}v, \end{aligned}$$

so that $\operatorname{div}(|x_{m+1}|^\beta \nabla \partial_{m+1} v) = \beta |x_{m+1}|^{\beta-2}\partial_{m+1} v$ in $B_R(x_0)\backslash \partial\mathbb{R}^{m+1}_+$. Hence on $B_R(x_0)\backslash \partial\mathbb{R}^{m+1}_+$ we have

$$\operatorname{div}(|x_{m+1}|^\beta \nabla |\partial_i v|^2) = 2|x_{m+1}|^\beta |\nabla \partial_i v|^2 + 2\langle \partial_i v, \operatorname{div}(|x_{m+1}|^\beta \nabla \partial_i v)\rangle \geq 0,$$

$$\operatorname{div}(|x_{m+1}|^{-\beta} \nabla |v^*|^2) = 2|x_{m+1}|^\beta |\nabla v^*|^2 + 2\langle v^*, \operatorname{div}(|x_{m+1}|^{-\beta} \nabla v^*)\rangle \geq 0,$$

and, when $\beta \in (0,1)$,

$$\operatorname{div}(|x_{m+1}|^\beta \nabla |\partial_{m+1} v|^2) = 2|x_{m+1}|^\beta |\nabla \partial_{m+1} v|^2 + 2\beta |x_{m+1}|^{\beta-2} |\partial_{m+1} v|^2 \geq 0,$$

classically where $i = 1, \ldots, m$.

Fix $R > r > \varepsilon > 0$ and let $B^\varepsilon_r(x_0) = B_r(x_0) \cap \{x \in \mathbb{R}^{m+1} : |x_{m+1}| \geq \varepsilon\}$. Using the divergence theorem, we calculate

$$\begin{aligned} 0 \leq & \int_{\partial B_r(x_0)} \mathbb{1}_{\overline{B^\varepsilon_r(x_0)}} |x_{m+1}|^\beta \nu \cdot \nabla |\partial_i v|^2 \mathrm{d}S(x) \\ & - \int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^\beta e_{m+1} \cdot (\nabla |\partial_i v|^2 (x', \varepsilon) - \nabla |\partial_i v|^2 (x', -\varepsilon)) \mathrm{d}x', \end{aligned} \tag{2.23}$$

$$\begin{aligned} 0 \leq & \int_{\partial B_r(x_0)} \mathbb{1}_{\overline{B^\varepsilon_r(x_0)}} |x_{m+1}|^{-\beta} \nu \cdot \nabla |v^*|^2 \mathrm{d}S(x) \\ & - \int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^{-\beta} e_{m+1} \cdot (\nabla |v^*|^2 (x', \varepsilon) - \nabla |v^*|^2 (x', -\varepsilon)) \mathrm{d}x' \end{aligned} \tag{2.24}$$

and, when $\beta \in (0,1)$,

$$\begin{aligned} 0 \leq & \int_{\partial B_r(x_0)} \mathbb{1}_{\overline{B^\varepsilon_r(x_0)}} |x_{m+1}|^\beta \nu \cdot \nabla |\partial_{m+1} v|^2 \mathrm{d}S(x) \\ & - \int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^\beta e_{m+1} \cdot (\nabla |\partial_{m+1} v|^2 (x', \varepsilon) - \nabla |\partial_{m+1} v|^2 (x', -\varepsilon)) \mathrm{d}x' \end{aligned} \tag{2.25}$$

where $\mathbb{1}_{\overline{B^\varepsilon_r(x_0)}}$ is the indicator function of $\overline{B^\varepsilon_r(x_0)}$, ν is the outward unit normal on $\partial B_r(x_0)$, $\mathrm{d}S$ is the Lebesgue measure on $\partial B_r(x_0)$ and $B^m_s(x_0) = \{x \in \mathbb{R}^m : |x - x_0| < s\}$.

We consider the terms on the right hand side of (2.23)-(2.25) separately with a view to taking the limit as $\varepsilon \to 0^+$. Lemma 2.4.3 and Corollary 2.4.1 imply that $\partial_i v, \partial_j \partial_i v \in W^{1,2}_\beta(B_r(x_0); \mathbb{R}^n)$ and are locally Hölder continuous in $B_R(x_0)$ for $i, j = 1, \ldots, m$. The lemma and corollary further imply that $v^*, (\partial_i v)^* \in W^{1,2}_{-\beta}(B_r(x_0); \mathbb{R}^n)$, where $i = 1, \ldots, m$, are locally Hölder continuous in $B_R(x_0)$ and hence uniformly continuous in $\overline{B_r(x_0)}$. We can therefore check that $|\partial_i v|^2 \in W^{1,2}_\beta(B_r(x_0))$, integrating over $B^\varepsilon_r(x_0)$ and letting $\varepsilon \to 0^+$ for the $m+1$-th derivative. It follows that

$$\begin{aligned} ||x_{m+1}|^\beta \nu \cdot \nabla |\partial_i v|^2| & \leq 2 \left|\langle \nu_{m+1} (\partial_i v)^*, \partial_i v\rangle\right| + 2 \sum_{j=1}^m \left| |x_{m+1}|^\beta \langle \nu_i \partial_j \partial_i v, \partial_i v\rangle \right| \\ & \leq C(1 + |x_{m+1}|^\beta), \end{aligned} \tag{2.26}$$

where C is a positive constant that may depend on r but is independent of ε. Furthermore, since $\partial_i v$ and $(\partial_i v)^*$ are uniformly continuous in $\overline{B_r(x_0)}$, we see that

$$\mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} e_{m+1}\varepsilon^\beta \cdot (\nabla|\partial_i v|^2(x',\varepsilon) - \nabla|\partial_i v|^2(x',-\varepsilon)) \to 0 \text{ uniformly as } \varepsilon \to 0^+ \tag{2.27}$$

for $(x',0) \in \overline{B_r(x_0)}$.

Since each $\partial_j\partial_i v \in W^{1,2}_\beta(B_r(x_0);\mathbb{R}^n)$ is locally Hölder continuous in $B_R(x_0)$ for $i,j = 1,\ldots,m$ and $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$ classically in $B_R(x_0)\backslash\partial\mathbb{R}^{m+1}_+$, we have $|x_{m+1}|^{-\beta}\partial_{m+1}v^* = -\Delta' v \in W^{1,2}_\beta(B_r(x_0);\mathbb{R}^n)$ is uniformly continuous in $\overline{B_r(x_0)}$ for every $r < R$. We can hence check $|v^*|^2 \in W^{1,2}_{-\beta}(B_r(x_0))$. We also have

$$\begin{aligned} ||x_{m+1}|^{-\beta}\nu\cdot\nabla|v^*|^2| &\le 2\left||x_{m+1}|^{-\beta}\langle\nu_{m+1}\partial_{m+1}v^*, v^*\rangle\right| + 2\sum_{i=1}^m \left||x_{m+1}|^{-\beta}\langle\nu_i\partial_i v^*, v^*\rangle\right| \\ &\le C(1+|x_{m+1}|^{-\beta}) \end{aligned} \tag{2.28}$$

and

$$\mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} e_{m+1}\varepsilon^{-\beta} \cdot (\nabla|v^*|^2(x',\varepsilon) - \nabla|v^*|^2(x',-\varepsilon)) \to 0 \text{ uniformly as } \varepsilon \to 0^+ \tag{2.29}$$

for $(x',0) \in \overline{B_r(x_0)}$.

In order to derive similar conclusions to (2.26) and (2.27) for the constituent integrands of (2.25), we assume that $\beta \in (0,1)$ and that v is *symmetric* with respect to $\partial\mathbb{R}^{m+1}_+$, namely $v(x',x_{m+1}) = v(x',-x_{m+1})$ for every $(x',x_{m+1}) \in B_R(x_0)$. The symmetry of v implies v^* must be odd with respect to $\partial\mathbb{R}^{m+1}_+$, that is $v^*(x',x_{m+1}) = -v^*(x',-x_{m+1})$ for every $(x',x_{m+1}) \in B_R(x_0)$ and hence, as it is also continuous in $B_R(x_0)$ we have $v^*(x',0) = 0$ for every $(x',0) \in B_R(x_0)$.

Fix $(x',0) \in B_R(x_0)$ and note that $(x',0) \in B_r(x_0)$ for some $r < R$ and choose h with $|h|$ sufficiently small as to ensure $(x',h) \in B_r(x_0)$. We see that

$$\begin{aligned} |h|^{-1}|v(x',h) - v(x',0)| &= |\partial_{m+1}v(x',x_{m+1})| \\ &= |x_{m+1}|^{-\beta}|v^*(x',x_{m+1}) - 0| \\ &= |x_{m+1}|^{-\beta}|\partial_{m+1}v^*(x',\xi)||x_{m+1}| \\ &\le |x_{m+1}|||\xi|^{-\beta}\partial_{m+1}v^*(x',\xi)| \\ &\le C|h| \to 0 \text{ as } h \to 0, \end{aligned}$$

where x_{m+1} with $|x_{m+1}| \in (0,|h|)$ and ξ with $|\xi| \in (0,|x_{m+1}|)$ are chosen such that the Mean Value Theorem holds. Thus we see that $\partial_{m+1}v(x',0) = 0$ classically for $(x',0) \in B_R(x_0)$. Analogous calculations to those on the right hand side above show that $\partial_{m+1}v$ is continuous at $(x',0)$ and hence continuous in $B_R(x_0)$. We also have

$$|x_{m+1}|^\beta \partial^2_{m+1}v = \partial_{m+1}v^* - \beta x^{-1}_{m+1}v^*$$

for $(x',x_{m+1}) \in \overline{B_r(x_0)}\backslash\partial\mathbb{R}^{m+1}_+$ and hence

$$\begin{aligned} ||x_{m+1}|^\beta \partial^2_{m+1}v(x',x_{m+1})| &\le |\partial_{m+1}v^*| + \beta|x^{-1}_{m+1}v^*| \\ &\le C + \beta|\partial_{m+1}v^*(x',\xi)| \\ &\le C, \end{aligned}$$

in $\overline{B_r(x_0)}\backslash\partial\mathbb{R}^{m+1}_+$, where ξ is chosen with $|\xi| \in (0, |x_{m+1}|)$ such that the Mean Value Theorem holds. It follows that $|x_{m+1}|^\beta \partial^2_{m+1} v$ is essentially bounded in $\overline{B_r(x_0)}$ and $\partial B_r(x_0)$.

The preceding discussion implies $|\partial_{m+1} v|^2 \in W^{1,2}_\beta(B_r(x_0))$ and

$$||x_{m+1}|^\beta \nu \cdot \nabla |\partial_{m+1} v|^2| \le C \tag{2.30}$$

on $\overline{B_r(x_0)}\backslash\partial\mathbb{R}^{m+1}_+$. Furthermore, using the symmetry of v, we see that

$$\mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} e_{m+1}\varepsilon^\beta \cdot (\nabla|\partial_{m+1}v|^2(x', \varepsilon) - \nabla|\partial_{m+1}v|^2(x', -\varepsilon)) \to 0 \text{ uniformly as } \varepsilon \to 0^+ \tag{2.31}$$

for $(x', 0) \in \overline{B_r(x_0)}$.

Using Lebesgue's Dominated Convergence Theorem, we combine (2.23) with (2.26) and (2.27), (2.24) with (2.28) and (2.29) and (2.25) with (2.30) and (2.31) to see that

$$0 \le \int_{\partial B_r(x_0)} |x_{m+1}|^\beta \nu \cdot \nabla |\partial_i v|^2 \mathrm{d}S(x), \quad 0 \le \int_{\partial B_r(x_0)} |x_{m+1}|^{-\beta} \nu \cdot \nabla |v^*|^2 \mathrm{d}S(x)$$

and, when $\beta \in (0, 1)$ and v is symmetric with respect to $\partial\mathbb{R}^{m+1}_+$,

$$0 \le \int_{\partial B_r(x_0)} |x_{m+1}|^\beta \nu \cdot \nabla |\partial_{m+1} v|^2 \mathrm{d}S(x)$$

respectively. Noting that $|x_{m+1}|^{-\beta}|v^*|^2 = |x_{m+1}|^\beta |\partial_{m+1}v|^2$, we apply Lemma 2.4.4, to see that

$$s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\nabla' v|^2 \mathrm{d}x \le r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\nabla' v|^2 \mathrm{d}x, \tag{2.32}$$

$$s^{-(1+m-\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\partial_{m+1} v|^2 \mathrm{d}x \le r^{-(1+m-\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\partial_{m+1} v|^2 \mathrm{d}x \tag{2.33}$$

and, when $\beta \in (0, 1)$ and v is symmetric with respect to $\partial\mathbb{R}^{m+1}_+$,

$$s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\partial_{m+1} v|^2 \mathrm{d}x \le r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\partial_{m+1} v|^2 \mathrm{d}x \tag{2.34}$$

for every $0 < s \le r < R$. We apply Lebesgue's dominated convergence, sending $r \to R^-$ and combine (2.32), (2.33) and (2.34) conclude the proof to see the theorem holds for $0 < s \le r \le R$. $\square$

Remark 2.4.2. We observe from (2.33) and (2.34) that without the symmetry condition on v, the monotonicity of the average energy of $|\partial_{m+1}v|^2$ is better than required if $\beta \in (-1, 0]$ and worse than required if $\beta \in (0, 1)$; the symmetry of v with respect to $\partial_{m+1}v$ resolves this issue by implying continuity of $\partial_{m+1}v$ on $\partial\mathbb{R}^{m+1}_+$ which, as Lemma 2.4.1 shows, cannot be expected in general.

2.5 Interior Monotonicity Formula

We need a counterpart to Theorem 2.4.1 for balls in the interior of $\mathbb{R}^{m+1}_+$. Since the weight x^β_{m+1} is not scale invariant on a (Euclidean) ball with $B_\rho(y)$ with $y_{m+1} \ge 2\rho$, we only expect monotonicity of the average energy (with respect to the Lebesgue measure) of solutions to $\mathrm{div}(x^\beta_{m+1}\nabla v) = 0$ up to a correcting factor. As the radius tends to zero, the coefficients of uniform ellpiticity for the preceding equation tend to constants. Accordingly, the correcting factor becomes smaller with the radius. General monotonicity-type formulas for linear uniformly elliptic equations are available in [15] for example. We perform the following calculations in order to determine how the correcting factor behaves explicitly as the radius decays geometrically. We first establish monotonicity on the boundary of concentric balls.

Lemma 2.5.1. *Let $\beta \in (-1,1)$, $B_R(y) \subset \mathbb{R}^{m+1}_+$ with $\overline{B_R(y)} \subset \mathbb{R}^{m+1}_+$ and suppose $v \in C^2(B_R(y);\mathbb{R}^n)$ satisfies $div(x_{m+1}^{\beta}\nabla|v|^2) \geq 0$ classically in $B_R(y)$. Then, letting $\mathrm{d}S$ denote Lebesgue surface measure, for $0 < s \leq r < R$ we have*

$$s^{-m}\frac{1}{(y_{m+1} - sgn(\beta)s)^{\beta}}\int_{\partial B_s(y)} x_{m+1}^{\beta}|v|^2\mathrm{d}S(x) \leq r^{-m}\frac{1}{(y_{m+1} - sgn(\beta)r)^{\beta}}\int_{\partial B_r(y)} x_{m+1}^{\beta}|v|^2\mathrm{d}S(x).$$

Proof. Let $\rho < R$. We calculate $0 \leq \int_{\partial B_\rho(y)} x_{m+1}^{\beta}\nu\cdot\nabla|v|^2\mathrm{d}S(x)$, where ν is the unit outward normal on $\partial B_\rho(y)$. Now using variables $\rho = |x-y|$ and $\omega = \frac{x-y}{\rho}$ we have

$$\begin{aligned}
0 &\leq \rho^m\int_{\mathbb{S}^m}(\rho\omega_{m+1} + y_{m+1})^{\beta}\frac{\partial}{\partial\rho}(|v(\rho\omega+y)|^2)\mathrm{d}\omega \\
&= \rho^m\int_{\mathbb{S}^m}\frac{\partial}{\partial\rho}\left((\rho\omega_{m+1} + y_{m+1})^{\beta}|v(\rho\omega+y)|^2\right)\mathrm{d}\omega \\
&\quad - \rho^m\int_{\mathbb{S}^m}\beta\omega_{m+1}(\rho\omega_{m+1} + y_{m+1})^{\beta-1}|v(\rho\omega+y)|^2\mathrm{d}\omega \\
&= \rho^m\frac{\partial}{\partial\rho}\left(\rho^{-m}\int_{\partial B_\rho(y)} x_{m+1}^{\beta}|v|^2\mathrm{d}S(x)\right) - \int_{\partial B_\rho(y)}\frac{\beta}{\rho}x_{m+1}^{\beta-1}(x_{m+1} - y_{m+1})|v|^2\mathrm{d}S(x).
\end{aligned}$$

We define $f(\rho) = \rho^{-m}\int_{\partial B_\rho(y)} x_{m+1}^{\beta}|v|^2\mathrm{d}S(x)$ and divide by ρ^m to see that

$$\begin{aligned}
0 &\leq f'(\rho) - \frac{\beta}{\rho}f(\rho) + \frac{y_{m+1}\beta}{\rho^{m+1}}\int_{\partial B_\rho(y)} x_{m+1}^{\beta-1}|v|^2\mathrm{d}S(x) \\
&\leq f'(\rho) - \frac{\beta}{\rho}f(\rho) + \frac{y_{m+1}\beta}{\rho(y_{m+1} - \mathrm{sgn}(\beta)\rho)}f(\rho) \\
&= f'(\rho) + \left(\frac{|\beta|}{y_{m+1} - \mathrm{sgn}(\beta)\rho}\right)f(\rho).
\end{aligned}$$

Hence $0 \leq \left((y_{m+1} - \mathrm{sgn}(\beta)\rho)^{-\beta}f(\rho)\right)'$ and integrating between $s < r$ concludes the proof. □

With this lemma in hand we can establish the following counterpart to Theorem 2.4.1.

Lemma 2.5.2. *Let $B_R(y) \subset \mathbb{R}^{m+1}$ with $y_{m+1} \geq \theta R$ for $\theta \geq 2$. Suppose $v \in C^2(B_R(y);\mathbb{R}^n) \cap W^{1,2}_{\beta}(B_R(y);\mathbb{R}^n)$ satisfies $div(x_{m+1}^{\beta}\nabla v) = 0$ in $B_R(y)$. Then there exists $C = C(m)$ such that*

$$\left(\frac{R}{2}\right)^{-(m+1)}\int_{B_{\frac{R}{2}}(y)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x \leq (1 + \frac{C}{\theta-1})R^{-(m+1)}\int_{B_R(y)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x.$$

Proof. Let $g(r) := \int_{B_r(y)} x_{m+1}^\beta |f|^2 \mathrm{d}x$ where f satisfies the assumptions of Lemma 2.5.1 and $r \in (0,R)$. Then we have

$$\begin{aligned}
g(r) &= \int_0^r \int_{\partial B_\rho(y)} x_{m+1}^\beta |f|^2 \mathrm{d}S(x)\mathrm{d}\rho \\
&= \int_0^r (y_{m+1} - \mathrm{sgn}(\beta)\rho)^\beta \rho^m (y_{m+1} - \mathrm{sgn}(\beta)\rho)^{-\beta} \rho^{-m} \int_{\partial B_\rho(y)} x_{m+1}^\beta |f|^2 \mathrm{d}S(x)\mathrm{d}\rho \\
&\le \int_0^r (y_{m+1} - \mathrm{sgn}(\beta)\rho)^\beta \rho^m \mathrm{d}\rho (y_{m+1} - \mathrm{sgn}(\beta)r)^{-\beta} r^{-m} \int_{\partial B_r(y)} x_{m+1}^\beta |f|^2 \mathrm{d}S(x) \\
&\le y_{m+1}^\beta \frac{r}{m+1} (y_{m+1} - \mathrm{sgn}(\beta)r)^{-\beta} \int_{\partial B_r(y)} x_{m+1}^\beta |f|^2 \mathrm{d}S(x).
\end{aligned} \tag{2.35}$$

Hence

$$\begin{aligned}
0 &\le g'(r) - g(r) y_{m+1}^{-\beta} \frac{m+1}{r} (y_{m+1} - \mathrm{sgn}(\beta)r)^\beta \\
&= g'(r) - g(r) \frac{m+1}{r} (1 - \mathrm{sgn}(\beta) \frac{r}{y_{m+1}})^\beta \\
&= g'(r) - g(r) \frac{m+1}{r} \left(1 + \beta(1 - \mathrm{sgn}(\beta) \frac{s}{y_{m+1}})^{\beta-1} \mathrm{sgn}(\beta) \frac{-r}{y_{m+1}} \right) \\
&= g'(r) - g(r) \frac{m+1}{r} + g(r) \frac{m+1}{r} |\beta| (1 - \mathrm{sgn}(\beta) \frac{s}{y_{m+1}})^{\beta-1} \frac{r}{y_{m+1}},
\end{aligned}$$

where $s \in (0,r)$ is such that the Mean Value Theorem holds for the function $r \mapsto (1 - \mathrm{sgn}(\beta)\frac{r}{y_{m+1}})^\beta$. Now recall that $y_{m+1} \ge \theta R \ge \theta r$ for $\theta \ge 2$. We hence find

$$0 \le g'(r) - g(r) \frac{m+1}{r} + g(r) \frac{m+1}{r} |\beta| \frac{1}{\theta - 1}.$$

It follows that $0 \le (r^{-(m+1)} r^{\frac{|\beta|(m+1)}{\theta-1}} g(r))'$ and consequently, if $\frac{R}{2} \le r < R$, we have

$$\begin{aligned}
\left(\frac{R}{2}\right)^{-(m+1)} g(\frac{R}{2}) &\le 2^{\frac{|\beta|(m+1)}{\theta-1}} r^{-(m+1)} g(r) \\
&= (1 + \frac{|\beta|(m+1)}{\theta-1} \xi^{\frac{|\beta|(m+1)}{\theta-1} - 1}) r^{-(m+1)} g(r) \\
&\le (1 + \frac{|\beta|(m+1)}{\theta-1} 2^{|\beta|(m+1)}) r^{-(m+1)} g(r)
\end{aligned} \tag{2.36}$$

where $\xi \in (1,2)$ is such that the Mean Value theorem holds for the function $t \mapsto t^{\frac{|\beta|(m+1)}{\theta-1}}$. If $\mathrm{div}(x_{m+1}^\beta \nabla v) = 0$ then, as observed in the proof of Theorem 2.4.1, we have $\mathrm{div}(x_{m+1}^\beta \nabla |\partial_i v|^2) \ge 0$ for $i = 1, \dots, m$ and $\mathrm{div}(x_{m+1}^{-\beta} \nabla |x_{m+1}^\beta \partial_{m+1} v|^2) \ge 0$. We apply (2.36) with $f = \partial_i v$ for $i = 1, \dots, m$, and with $f = x_{m+1}^\beta \partial_{m+1} v$ and $-\beta$ in place of β and combine the results, letting $r \to R^-$ to conclude the proof. □

2.6 Solutions of the Linear Degenerate Dirichlet Problem

We require further results regarding the following Dirichlet problem: solve

$$\operatorname{div}(|x_{m+1}|^{\beta}\nabla v) = 0 \text{ in } B_R(x_0) \qquad \text{and} \qquad v = \phi \text{ on } \partial B_R(x_0) \tag{2.37}$$

for a given ϕ, where $x_0 \in \mathbb{R}^{m+1}$, in order to apply a version of the method of harmonic replacement in the proof of Lemma 4.12.1. A weak solution of (2.37) is a $v \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ which weakly satisfies $\operatorname{div}(|x_{m+1}|^{\beta}\nabla v) = 0$ in $B_R(x_0)$ with $v - \phi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$. We collect the results we require, which can be found in [20], in the form of a lemma.

Lemma 2.6.1. *Suppose $\phi \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$. Then there exists a $v \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ which is a weak solution of the Dirichlet problem* (2.37)*. Any such solution is unique and continuous in $B_R(x_0)$, if $\phi \in C(\overline{B_R(x_0)};\mathbb{R}^n)$ then $v(x) \to \phi(z)$ as $x \to z$ for $z \in \partial B_R(x_0)$ and the weak maximum principle*

$$\max_{\overline{B_R(x_0)}} v = \max_{\partial B_R(x_0)} v = \max_{\partial B_R(x_0)} \phi$$

and weak minimum principle

$$\min_{\overline{B_R(x_0)}} v = \min_{\partial B_R(x_0)} v = \min_{\partial B_R(x_0)} \phi$$

both hold, where we take the maximum and minimum component-wise. If $w \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ also satisfies $w - \phi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$ then

$$\int_{B_R(x_0)} |x_{m+1}|^{\beta}|\nabla v|^2 \mathrm{d}x \le \int_{B_R(x_0)} |x_{m+1}|^{\beta}|\nabla w|^2 \mathrm{d}x.$$

Proof. Since $|x_{m+1}|^{\beta}$ is of Muckenhoupt class A_2 it follows from 1.6 of [20] that $|x_{m+1}|^{\beta}$ is a 2-admissible weight so we may apply the theory of [20]. Aside from the minimising property of v, the assertions of the lemma are consequences of Theorem 3.70, Corollary 6.32, the strong maximum principle 6.5 and lastly 3.17 in [20]. If v is a weak solution of (2.37) for a given ϕ and $w \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ with $w - \phi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$ then $w - v \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$. Hence, by approximation, we have

$$\int_{B_R(x_0)} |x_{m+1}|^{\beta}\langle \nabla v, \nabla(w - v)\rangle \mathrm{d}x = 0,$$

so that

$$\int_{B_R(x_0)} |x_{m+1}|^{\beta}|\nabla w|^2 \mathrm{d}x = \int_{B_R(x_0)} |x_{m+1}|^{\beta}|\nabla v|^2 \mathrm{d}x + \int_{B_R(x_0)} |x_{m+1}|^{\beta}|\nabla (w-v)|^2 \mathrm{d}x$$

which concludes the proof. □

The uniqueness of solutions to the Dirichlet problem (2.37) implies that solutions with boundary data which are symmetric with respect to $\partial\mathbb{R}^{m+1}_{+}$ are themselves symmetric. More precisely, we have the following.

Lemma 2.6.2. *Suppose $v, \phi \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ and v is a weak solution of the Dirichlet problem* (2.37) *with ϕ as boundary data. Let $\phi \in C(\overline{B_R(x_0)};\mathbb{R}^n)$ and suppose $\phi(x', x_{m+1}) = \phi(x', -x_{m+1})$ for every $(x', x_{m+1}) \in \overline{B_R(x_0)}$. Then $v(x', x_{m+1}) = v(x', -x_{m+1})$ for every $(x', x_{m+1}) \in B_R(x_0)$.*

Proof. The continuity of ϕ in $\overline{B_R(x_0)}$, combined with an application of Lemma 2.6.1, implies that v and, consequently, $\tilde{v}(x', x_{m+1}) := v(x', -x_{m+1})$ are continuous in $\overline{B_R(x_0)}$. We observe that $\tilde{v} \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ weakly satisfies $\operatorname{div}(|x_{m+1}|^{\beta}\nabla\tilde{v}) = 0$ in $B_R(x_0)$ and $\tilde{v}|_{\partial B_R(x_0)} = \phi|_{\partial B_R(x_0)}$ so that $\tilde{v} - \phi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$. Hence v and $\tilde{v}$ solve the same Dirichlet problem; solutions to this problem are unique by Lemma 2.6.1 and thus $\tilde{v} = v$. □

3 Intrinsic Fractional Harmonic Maps

We assume, translating N if necessary, that $0 \in N$. For technical reasons, when $m \geq 3$ we let $\beta \in (-1, 1)$ and when $m = 2$ we let $\beta \in (-3^{-1}, 1)$. Then Lemma 2.1.1 may be applied and we can define

$$\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N) = \{v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+;\mathbb{R}^n) : v(x) \in N \text{ for almost every } x \in \mathbb{R}^{m+1}_+\}.$$

Let $\mathcal{O} \subset \partial\mathbb{R}^{m+1}_+$ be open and such that a continuous linear trace operator with respect to $\mathcal{O}$ exists. We can, for example, obtain such a trace operator $T : \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+;\mathbb{R}^n) \to L^p(\mathcal{O};\mathbb{R}^n)$, where $p = p(\beta) \in (1, 2]$, by combining Lemmata 2.1.1 and 2.0.1 with [13] Section 4.3 Theorem 1 whenever $\mathcal{O}$ is contained in the boundary of a Lipschitz $\Omega \subset \mathbb{R}^{m+1}_+$. Define

$$I^{\beta}(u) = \inf\{E^{\beta}(v) : v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N), Tv = u\}$$

for $u \in T(\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N))$. The functional I^{β} serves as an intrinsic energy for u; it does not depend on the choice of embedding of N into Euclidean space. Moreover, I^{β} coincides with the fractional Sobolev semi-norm $||u||_{\dot{H}^{\frac{1-\beta}{2}}(\mathbb{R}^m;\mathbb{R}^n)}$ when $N = \mathbb{R}^n$ and $\mathcal{O} = \mathbb{R}^m$.

For every $u \in T(\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N))$, an application of the direct method of the calculus of variations shows that there exists $v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$ such that $I^{\beta}(u) = E^{\beta}(v)$. For a given u, such a v is referred to henceforth as a *minimal harmonic map*. Any minimal harmonic map v is weakly harmonic in $\mathbb{R}^{m+1}_+$ with respect to the metric represented in Euclidean coordinates by $x^{\alpha}_{m+1}\delta_{ij}$, where $\beta = \frac{\alpha(m-1)}{2}$; the Dirichlet energy on $\mathbb{R}^{m+1}_+$ for this metric is precisely E^{β}. Observe that when $m = 1$, for every $\alpha \in \mathbb{R}$ the Dirichlet energy density in $\mathbb{R}^{m+1}_+$ corresponding to the metric $x^{\alpha}_{m+1}\delta_{ij}$ satisfies $e(v) = |\nabla v|^2$. Hence, if we take the point of view that $\mathbb{R}^{m+1}_+$ is a Riemannian manifold with metric $x^{\alpha}_{m+1}\delta_{ij}$ and try to define I^{β} as above when $m = 1$, we would have $I^{\beta} \equiv I^0$. Every critical point of I^0 is smooth when $m = 1$ by the theory of Moser [29] and so we only consider the case $m \geq 2$ henceforth. Since a minimal harmonic map v is weakly harmonic in $\mathbb{R}^{m+1}_+$, it satisfies

$$\int_{\mathbb{R}^{m+1}_+} x^{\beta}_{m+1} \left(\langle\psi, A(v)(\nabla v, \nabla v)\rangle - \langle\nabla v, \nabla\psi\rangle\right) \mathrm{d}x = 0 \tag{3.1}$$

for every $\phi \in C^{\infty}_0(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$, where A is the second fundamental form of N and

$$\langle\nabla v, \nabla\psi\rangle = \sum_{i=1}^{m+1} \langle\partial_i v, \partial_i\psi\rangle \quad \text{and} \quad A(v)\left(\nabla v, \nabla v\right) = \sum_{i=1}^{m+1} A(v)\left(\partial_i v, \partial_i v\right).$$

Formally, if v is sufficiently regular in $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$, we calculate

$$\int_{\mathbb{R}^{m+1}_+} x_{m+1}^{\beta} \left(\langle \psi, A(v)(\nabla v, \nabla v)\rangle - \langle \nabla v, \nabla \psi\rangle\right) \mathrm{d}x = \int_{\mathcal{O}} \left\langle (x_{m+1}^{\beta}\partial_{m+1} v)(x', 0), \phi(x')\right\rangle \mathrm{d}x',$$

for every $\phi \in C_0^{\infty}(\mathcal{O}; \mathbb{R}^n)$ and any $\psi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ with $\psi(x', 0) = \phi(x')$, where $\mathrm{d}x'$ is the Lebsegue measure on $\mathbb{R}^m$. In general, the integral in 3.1 defines a distribution on $\mathcal{O}$ given by

$$\partial_{m+1}^{\beta} v(\phi) := \int_{\mathbb{R}^{m+1}_+} x_{m+1}^{\beta} \left(\langle \psi, A(v)(\nabla v, \nabla v)\rangle - \langle \nabla v, \nabla \psi\rangle\right) \mathrm{d}x$$

for $\phi \in C_0^{\infty}(\mathcal{O}; \mathbb{R}^n)$. This observation allows us, analogously to [29] Proposition 1.1, to identify a superdifferential for I^{β}. Recall that since N is compact, Theorem 1 in Section 2.12.3 of [39] gives a tubular neighbourhood of N, which has the form $U_{\delta}(N) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, N) < \delta\}$ for a $\delta = \delta(N) > 0$, and a smooth map $\pi_N : U_{\delta}(N) \to N$ such that $|\pi_N(y) - y| = \mathrm{dist}(y, N)$ for every $y \in U_{\delta}(N)$. Using the same method of proof as [29] Proposition 1.1 we deduce the following.

Lemma 3.0.1. *Let $u \in T(\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N))$ and $v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N)$ be a minimal harmonic map with $Tv = u$. Then for $\phi \in C_0^{\infty}(\mathcal{O}; \mathbb{R}^n)$,*

$$I^{\beta}(\pi_N(u + t\phi)) \leq I^{\beta}(u) - t\partial_{m+1}^{\beta} v(\phi) + o(|t|) \tag{3.2}$$

as $t \to 0$.

It follows from this proposition that if $\frac{\partial}{\partial t}\big|_{t=0} I^{\beta}(\pi_N(u+t\phi))$ exists then it is equal to $-\partial_{m+1}^{\beta} v(\phi)$ where v is any minimal harmonic map with $Tv = u$; this indicates a candidate for the first variation of I^{β}.

Definition 3.0.1. Let $\beta \in (-1, 1)$ and $\mathcal{D}_{\beta}$ be the collection of all $u \in T(\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N))$ such that there exists a distribution $\lambda_{\beta} \in (C_0^{\infty}(\Omega; \mathbb{R}^n))^*$ with $\lambda_{\beta} = -\partial_{m+1}^{\beta} v$ for every minimal harmonic map $v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$. Then we may define a map $\Lambda_{\beta} : \mathcal{D}_{\beta} \to (C_0^{\infty}(\Omega; \mathbb{R}^n))^* : u \mapsto \lambda_{\beta} = \Lambda_{\beta} u$.

In [29] Theorem 1.1 Moser showed that Λ_0 is the first variation of I^0. The method of proof of Moser's theorem, applied with the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}^{m+1}$ replaced by $x_{m+1}^{\beta}\mathrm{d}x$, yields the following.

Lemma 3.0.2. *If $u \in \mathcal{D}_{\beta}$, then*

$$\frac{\partial}{\partial t}\bigg|_{t=0} I^{\beta}(\pi_N(u + t\phi)) = \Lambda_{\beta} u(\phi)$$

for all $\phi \in C_0^{\infty}(\mathcal{O}; \mathbb{R}^n)$. If $u \notin \mathcal{D}_{\beta}$, then there exists $\phi \in C_0^{\infty}(\mathcal{O}; \mathbb{R}^n)$ such that the function $t \mapsto I^{\beta}(\pi_N(u + t\phi))$ is not differentiable at 0.

Consequently, we may define intrinsic fractional harmonic maps as follows.

Definition 3.0.2. Let $\beta \in (-1, 1)$ and $u \in T(\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N))$. If $\Lambda_{\beta} u = 0$ then we say that u is an intrinsic $\frac{1-\beta}{2}$-harmonic map.

As discussed in the introduction, intrinsic $\frac{1-\beta}{2}$-harmonic maps are the boundary values of free boundary harmonic maps from $\mathbb{R}^{m+1}_+$ to N. In general, such maps may have singularities in $\mathcal{O}$ and we may not expect regularity in general. We consider a smaller class of fractional harmonic maps which minimise I^β in order to obtain partial regularity in $\mathcal{O}$.

Definition 3.0.3. We say that $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N))$ minimises I^β if for every compact $K \subset \mathcal{O}$ and every $\tilde{u} \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N))$ with $u|_{\mathcal{O}\setminus K} = \tilde{u}|_{\mathcal{O}\setminus K}$ we have $I^\beta(u) \leq I^\beta(\tilde{u})$. A minimiser of I^β will be called an intrinsic minimising $\frac{1-\beta}{2}$-harmonic map. For convenience, as we consider no other kind of fractional harmonic map, we drop the prefixes intrinsic and minimising. Any $\frac{1-\beta}{2}$-harmonic map will also be broadly referred to as a fractional harmonic map.

This definition allows us to consider interior regularity for a class of critical points of I^β without explicitly specifying boundary conditions. For example, minimisers of I^β with respect to Dirichlet or free boundary conditions satisfy the definition. In order to deduce regularity results for u, we analyse their minimal harmonic extensions in more detail. To this end, we make the following definition.

Definition 3.0.4. Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$. We say that v is E^β minimising, or energy minimising, in $\mathbb{R}^{m+1}_+$ relative to $\mathcal{O} \subset \partial\mathbb{R}^{m+1}_+$, if for every compact $K \subset \mathbb{R}^{m+1}$ with $K \cap \partial\mathbb{R}^{m+1}_+ \subset \mathcal{O}$ and for every $w \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ with $v|_{\mathbb{R}^{m+1}_+\setminus K} = w|_{\mathbb{R}^{m+1}_+\setminus K}$ we have $E^\beta(v) \leq E^\beta(w)$.

Minimisers of I^β and minimisers of E^β relative to $\mathcal{O}$ are connected as follows.

Lemma 3.0.3. *Suppose $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N))$ minimises I^β in the sense of Definition 3.0.3 and fix a minimal harmonic map $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$. Then v is a minimiser of E^β relative to $\mathcal{O}$.*

Proof. Let $K \subset \mathbb{R}^{m+1}$ be compact such that the compact set $K_m := K \cap \partial\mathbb{R}^{m+1}_+ \subset \mathcal{O}$ and suppose that $w \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ satisfies $v|_{\mathbb{R}^{m+1}_+\setminus K} = w|_{\mathbb{R}^{m+1}_+\setminus K}$. Define $\tilde{u} = Tw$ and let $\tilde{v}$ be a minimal harmonic map with $T\tilde{v} = \tilde{u}$. Since $\mathcal{O}$ is open in $\partial\mathbb{R}^{m+1}_+$ and $K_m \subset \mathcal{O}$ is compact we have $\mathrm{dist}^m(K_m; \partial\mathcal{O}) > 0$, where dist^m is the distance in $\mathbb{R}^m \times \{0\}$. We can therefore choose an open set $\tilde{\mathcal{O}} \subset \mathcal{O}$ with $K_m \subset \tilde{\mathcal{O}} \subset \overline{\tilde{\mathcal{O}}} \subset \mathcal{O}$. Since K_m is closed and $\tilde{\mathcal{O}}$ is open we have $\mathrm{dist}^m(K_m; \partial\tilde{\mathcal{O}}) > 0$ as well. It follows that $\mathrm{dist}(\mathcal{O}\setminus\tilde{\mathcal{O}}; K) := \kappa > 0$, where dist is the Euclidean distance in $\overline{\mathbb{R}^{m+1}_+}$. The continuity of the trace operator yields

$$\int_{\mathcal{O}\setminus\tilde{\mathcal{O}}} |u - \tilde{u}|^p \mathrm{d}x = \int_{\mathcal{O}\setminus\tilde{\mathcal{O}}} |T(v-w)|^p \mathrm{d}x \leq C||v-w||^p_{W^{1,p}((\mathcal{O}\setminus\tilde{\mathcal{O}})\times(0,\kappa);\mathbb{R}^n)} = 0,$$

since $v = w$ in $\mathcal{O}\setminus\overline{\tilde{\mathcal{O}}} \times (0,\kappa)$. Since v and $\tilde{v}$ are minimal harmonic maps and u is a minimiser of I^β, we have

$$E^\beta(v) = I^\beta(u) \leq I^\beta(\tilde{u}) = E^\beta(\tilde{v}) \leq E^\beta(w)$$

as required. □

As a consequence of the preceding lemma, we can consider the regularity of minimisers of E^β relative to $\mathcal{O}$ on relatively open balls (in the Euclidean topology) centred on $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$ in order to prove regularity of fractional harmonic maps. Our main results, stated and proved in Section

4, constitute an ε-regularity and a corresponding parital regularity theorem for minimisers of E^β relative to $\mathcal{O}$ which translate into the following partial regularity result for fractional harmonic maps.

Theorem 3.0.1. *Suppose $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N))$ minimises I^β. Then there exists a $\gamma(m, N, \beta) \in (0,1)$ and a relatively closed set $\Sigma \subset \mathcal{O}$ with $\mathcal{H}^{m+\beta-1}(\Sigma) = 0$ such that $u \in C^\infty(\mathcal{O}\backslash\Sigma; N)$.*

Proof. Fix a minimal harmonic map $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$. As u is a minimiser of I^β, Theorem 3.0.3 implies v is a minimiser of E^β relative to $\mathcal{O}$. An application of Theorem **??** implies the result. □

4 Partial Regularity of Minimisers of E^β relative to $\mathcal{O}$

The main result of the paper is the following ε-regularity theorem for minimisers of E^β relative to $\mathcal{O}$. To state the theorem and subsequent results we will need the following notation. For a set $\Omega \subset \mathbb{R}^{m+1}_+$ we will sometimes split the boundary $\partial\Omega$ into the (possibly empty) sets $\partial^+\Omega = \partial\Omega \cap \mathbb{R}^{m+1}_+$ and $\partial^0\Omega = \partial\Omega \cap \partial\mathbb{R}^{m+1}_+$. Let $x_0 \in \partial\mathbb{R}^{m+1}_+$ and recall the notation $B^+_R(x_0) = \{x \in \mathbb{R}^{m+1}_+ : |x - x_0| < R\}$.

Theorem 4.0.1. *If $m \geq 3$, let $\beta \in (-1,1)$ and if $m = 2$ let $\beta \in (-3^{-1}, 1)$. Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ be a minimiser of E^β relative to $\mathcal{O}$ and let $B^+_R(x_0)$ be a half-ball with $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$. There exists $\varepsilon = \varepsilon(m, N, \beta)$ such that the following holds. If $R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \leq \varepsilon$ then there is a $\theta = \theta(m, N, \beta) \in (0,1)$ and a $\gamma = \gamma(m, N, \beta) \in (0,1)$ such that $v \in C^{0,\gamma}(\overline{B^+_{\theta R}(x_0)}; N)$. Furthermore, for every $l \in \mathbb{N}$ there is a $\theta = \theta(m, N, \beta, l) \in (0,1)$ and a $\gamma = \gamma(m, N, \beta, l) \in (0,1)$ such that $D^{\alpha'}v \in C^{0,\gamma}(\overline{B^+_{\theta R}(x_0)}; \mathbb{R}^n)$ for every $\alpha' \in \mathbb{N}^{m+1}_0$ with $|\alpha'| \leq l$ and $\alpha'_{m+1} = 0$.*

Remark 4.0.1. Henceforth, we assume the conditions on m and β from Theorem 4.0.1. We have restricted to considering α' with $\alpha'_{m+1} = 0$ as (partial) regularity of these derivatives up to the boundary will yield the desired regularity for fractional harmonic maps stated in Theorem 3.0.1. The main purpose of the theorem is to provide regularity estimates which are uniform up to $\mathcal{O}$; such estimates do not follow from known theory.

Theorem 4.0.1, combined with the partial regularity theory for harmonic maps yields the following. We use the notation $\mathcal{H}^t$ to denote the t-dimensional Hausdorff measure, with respect to the Euclidean metric on $\mathbb{R}^{m+1}$, for $t \geq 0$.

Theorem 4.0.2. *Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ be a minimiser of E^β relative to $\mathcal{O}$. There exists sets $\Sigma_{int} \subset \mathbb{R}^{m+1}_+$ and $\Sigma_{bdry} \subset \mathcal{O}$ such that the following holds. The set Σ_{int} is relatively closed in $\mathbb{R}^{m+1}_+$ and has Hausdorff dimension at most $m-2$. The set Σ_{bdry} is relatively closed in $\mathcal{O}$ and $\mathcal{H}^{m+\beta-1}(\Sigma_{bdry}) = 0$. The set $\Sigma := \Sigma_{int} \cup \Sigma_{bdry}$ is relatively closed in $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$ and $\mathcal{H}^{m+\beta-1}(\Sigma) = 0$. Furthermore, we have $v \in C^\infty(\mathbb{R}^{m+1}_+\backslash\Sigma_{int}; N)$, $v \in C^{0,1}_{loc}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; N)$ and for every multi-index $\alpha' \in \mathbb{N}^{m+1}$ with $\alpha'_{m+1} = 0$ we have $D^{\alpha'}v \in C^{0,1}_{loc}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; \mathbb{R}^n)$ and $\nabla D^{\alpha'}v \in L^\infty_{loc}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; \mathbb{R}^{(m+1)n})$. Finally, for every $\alpha' \in \mathbb{N}^{m+1}_0$ with $\alpha'_{m+1} = 0$, we have $x^\beta_{m+1}\partial_{m+1}D^{\alpha'}v \in C^{0,\gamma}_{loc}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; \mathbb{R}^{(m+1)n})$ for some $\gamma = \gamma(m, N, \beta, \alpha') \in (0,1)$.*

Remark 4.0.2. The existence and properties of Σ_{int} follows from the theory of Schoen and Uhlenbeck [37].

4.1 Euler-Lagrange and Stationary Equations for Minimisers

Minimisers of E^β relative to $\mathcal{O}$ are critical points of E^β with respect to outer and inner variations, including those which vary their boundary data in $\mathcal{O}$. As a consequence, they satisfy two systems of partial differential equations which we describe presently.

Let $\psi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$, as defined in Section 2.1, with $\psi(\cdot,0) \in C_0^\infty(\mathcal{O};\mathbb{R}^n)$. For sufficiently small t we define an outer variation of v by $v_t = \pi_N(v + t\psi) \in N$, where π_N is the nearest point projection onto N. Critical points of E^β with respect to variations of the form v_t satisfy

$$\int_{\mathbb{R}^{m+1}_+} x_{m+1}^\beta \left(\langle \nabla v, \nabla\psi\rangle - \langle \psi, A(v)\,(\nabla v, \nabla v)\rangle\right) \mathrm{d}x = 0. \tag{4.1}$$

Note that there is a Neumann-type boundary condition implicit in (4.1). In particular, if v is sufficiently smooth in $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$ we have

$$x_{m+1}^\beta \partial_{m+1} v = 0 \text{ in } \mathcal{O}. \tag{4.2}$$

A (weakly) harmonic map satisfying (4.1) is said to be *(weakly) harmonic with respect to the Neumann type boundary condition* (4.2).

Define $\Psi_t(x) = x + t\phi(x)$ for $x \in \overline{\mathbb{R}^{m+1}_+}$, where $\psi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^{m+1})$ is such that $\psi(\cdot,0) \in C_0^\infty(\mathcal{O};\partial\mathbb{R}^{m+1}_+)$ and $|t|$ is small enough to make Φ_t into a diffeomorphism of $\overline{\mathbb{R}^{m+1}_+}$ with $\Phi_t(\mathcal{O}) \subset \mathcal{O}$. We say $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N)$ is a critical point of the Dirichlet energy corresponding to inner variations $v_t := v \circ \Phi_t$, or variations of the independent variable, if v satisfies

$$\int_{\mathbb{R}^{m+1}_+} \sum_{i=1}^{m+1}\sum_{k=1}^{m+1} x_{m+1}^\beta \left(2\left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_k}\right\rangle - \delta_{ik}|\nabla v|^2\right)\frac{\partial \phi_k}{\partial x_i}\mathrm{d}x = \int_{\mathbb{R}^{m+1}_+} \beta x_{m+1}^{\beta-1}\phi_{m+1}|\nabla v|^2 \mathrm{d}x \tag{4.3}$$

for every ψ as above. A weakly harmonic map with respect to the Neumann type boundary condition (4.2) which satisfies (4.3) for every $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^{m+1})$ with $\phi(\cdot,0) \in C_0^\infty(\mathcal{O};\partial\mathbb{R}^{m+1}_+)$ is called *weakly stationary harmonic, or stationary harmonic, with respect to the Neumann type boundary condition* (4.2).

4.2 Energy Monotonicity

Stationary harmonic maps satisfy a monotonicity formula for an appropriately scaled version of the energy over balls with closure in $\mathbb{R}^{m+1}_+$. This property was proved by Schoen and Uhlenbeck for energy minimisers, see [37] Proposition 2.4, and Price, see the remark after Theorem 1 in [31], for stationary harmonic maps.

As a consequence of 4.3, we show that stationary harmonic maps with respect to the Neumann-type boundary condition 4.2 satisfy a similar monotonicity formula on half-balls $B^+_\rho(y)$ with centre y in $\mathcal{O}$ and which satisfy $\overline{\partial^0 B^+_\rho(y)} = \overline{B^m(y)} \subset \mathcal{O}$. Moreover, we state a version of the formula for balls with closure contained in $\mathbb{R}^{m+1}_+$, giving an explicit expression for the factors that the constants involved depend upon.

Lemma 4.2.1. *Suppose $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N)$ is a weakly stationary harmonic map with respect to the Neumann-type boundary condition 4.2. Suppose y in $\mathcal{O}$ and consider $B^+_R(y)$ with $\overline{\partial^0 B^+_R(y)} \subset \mathcal{O}$.*

Then

$$r^{1-m-\beta}\int_{B_r^+(y)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x - s^{1-m-\beta}\int_{B_s^+(y)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x$$
$$= \; 2\int_{B_r^+(y)\setminus B_s^+(y)} x_{m+1}^{\beta}\frac{|(x-y)\cdot\nabla v|^2}{|x-y|^{m+1+\beta}}\mathrm{d}x$$

whenever $0 \leq s \leq r \leq R$ *and therefore* $\rho \mapsto \rho^{1-m-\beta}\int_{B_\rho^+(y)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x$ *is a non-decreasing function of* ρ *for* $0 < \rho \leq R$.

Proof. The proof is analogous to that of the monotonicity formula for stationary harmonic maps. We follow [39] Section 2.4 and [28] Lemma 3.3; we test (4.3) with $\phi(x) = (x-y)\eta(x)$, where $\eta \in C_0^\infty(B_\rho(y))$, which yields

$$(m-1+\beta)\int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta}\,|\nabla v|^2\,\eta\mathrm{d}x + \int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta}(x-y)\cdot\nabla\eta\,|\nabla v|^2\,\mathrm{d}x$$
$$= 2\int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta}\,\langle (x-y)\cdot\nabla v, \nabla\eta\cdot\nabla v\rangle\,\mathrm{d}x. \tag{4.4}$$

Let $\chi \in C_0^\infty(\mathbb{R};[0,1])$ with $\chi(s) \equiv 1$ for $s \geq 1$ and $\chi(s) \equiv 0$ for $s \leq \frac{1}{2}$. The smooth functions defined by $\eta_j(x) = \chi(j(\rho - |x-y|))$ are admissible choices for η in (4.4) and $\{\eta_j\}_{j\in\mathbb{N}}$ converges pointwise to the indicator function of $B_\rho^+(y)$. We substitute η_j for η in (4.4) and take the limit as $j \to \infty$, using Lebesgue's Dominated Convergence and Differentiation Theorems, to see that

$$(m-1+\beta)\int_{B_\rho^+(y)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x - \rho\int_{\partial^+ B_\rho^+(y)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}S(x)$$
$$= -\frac{2}{\rho}\int_{\partial^+ B_\rho^+(y)} x_{m+1}^{\beta}\,|(x-y)\cdot\nabla v|^2\,\mathrm{d}S(x)$$

for almost every $\rho > 0$, where $\mathrm{d}S$ is the Lebesgue measure on $\partial B_\rho(y)$. Multiplying the above by the factor $-\rho^{-(\beta+m)}$ and bearing in mind that $\frac{\mathrm{d}}{\mathrm{d}\rho}\int_{B_\rho^+(y)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x = \int_{\partial^+ B_\rho^+(y)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}S(x)$ for almost all $\rho > 0$, we find

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^{1-m-\beta}\int_{B_\rho^+(y)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x\right) = 2\int_{\partial^+ B_\rho^+(y)} x_{m+1}^{\beta}\frac{|(x-y)\cdot\nabla v|^2}{|x-y|^{m+1+\beta}}\mathrm{d}S(x)$$

for almost every $\rho > 0$. Integrating between $0 < s < r$ concludes the proof. □

Remark 4.2.1. A consequence of Lemma 4.2.1 is that we can define the *density function*

$$\Theta_v^{\beta}(y) = \lim_{\rho\to 0^+}\rho^{1-m-\beta}\int_{B_\rho^+(y)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x$$

for every $y \in \mathcal{O}$, analogously to Definition 1 in Section 2.5 of [39]. Using Lemma 4.2.1 we deduce Θ_v^{β} is upper semi-continuous in $\mathcal{O}$ for any map v which is weakly stationary harmonic with respect to the Neumann-type boundary condition (4.2).

The following version of the energy monotonicity formula is due to Grosse-Brauckmann, [17] Theorem 1. We do not give a proof, but remark that the explicit form of the constant in the forthcoming formula can be determined using the method of proof of Lemma 4.2.1.

Lemma 4.2.2. *Suppose $v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N)$ is a weakly stationary harmonic critical point of E^β. Fix a ball $B_{\rho_0}(y)$ with $\overline{B_{\rho_0}(y)} \subset \mathbb{R}^{m+1}_+$ for some $\rho_0 > 0$ and suppose r and s satisfy $0 < s < r < \rho_0$. Then*

$$\begin{aligned} & e^{rC|\beta|} r^{1-m} \int_{B_r(y)} x_{m+1}^{\beta} |\nabla v|^2 \,\mathrm{d}x \; - \; e^{sC|\beta|} s^{1-m} \int_{B_s(y)} x_{m+1}^{\beta} |\nabla v|^2 \,\mathrm{d}x \\ & \geq 2 \int_{B_r(y) \setminus B_s(y)} x_{m+1}^{\beta} e^{|x-y|C|\beta|} \frac{|(x-y)\cdot \nabla v|^2}{|x-y|^{m+1}} \mathrm{d}x \end{aligned} \tag{4.5}$$

and therefore, for $0 < \rho < \rho_0$, $e^{\rho C|\beta|} \rho^{1-m} \int_{B_\rho(y)} x_{m+1}^{\beta} |\nabla v|^2 \,\mathrm{d}x$ is a non-decreasing function of ρ where $C = (y_{m+1} - \rho_0)^{-1} = (dist(B_{\rho_0}(y), \mathbb{R}^m \times \{0\}))^{-1}$.

4.3 A Modified Lemma of Morrey

In order to prove Theorem 4.11.1, in analogy with the regularity theory of harmonic maps, we will show that the re-scaled, scale-invariant energies in the monotonicity formulas in Section 4.2 decay slightly faster than implied by the Lemmata as the radius decreases. This will permit the application of a well-known lemma of Morrey, see [28] Lemma 2.1 for example, which is used to derive Hölder continuity from sufficiently fast energy decay. We will reduce the hypothesis of this lemma to similar hypothesis for the re-scaled energies from the monotonicity formula. To this end, we introduce a class of ball with closure in $\mathbb{R}^{m+1}_+$ on which the metrics $x^{\alpha}_{m+1}\delta_{ij}$, discussed in Section 3 and corresponding weights x^{β}_{m+1} are uniformly equivalent to the Euclidean metric and 1 respectively. We also introduce classes of balls and half-balls contained in the interior of a given larger half-ball $B^+_R(x_0)$ for $R > 0$ and $x_0 \in \partial\mathbb{R}^{m+1}_+$.

Define

$$\mathcal{B} = \{B_\rho(y) \subset \mathbb{R}^{m+1}_+ : y_{m+1} \geq 2\rho\} \qquad \text{and} \qquad \mathcal{B}_\theta = \{B_\rho(y) \subset \mathbb{R}^{m+1}_+ : y_{m+1} \geq \theta\rho\}$$

for $\theta \geq 2$. Then $\mathcal{B}_\theta \subset \mathcal{B}$ and $\mathcal{B}_2 = \mathcal{B}$. We further define

$$\mathcal{B}_\theta(x_0, R, r) = \{B_\rho(y) \subset B^+_R(x_0) : y_{m+1} \geq \theta\rho, y \in B^+_r(x_0)\},$$

omitting the subscript θ in the case $\theta = 2$, and let

$$\mathcal{B}^+(x_0, R, r) = \{B^+_\rho(y) \subset B^+_R(x_0) : y_{m+1} = 0, |x_0 - y| < r, \rho \leq r\}.$$

Observe that, on any $B_\rho(y) \in \mathcal{B}$, we can choose constants c, C, c_0 and C_0 independently of β such that for every $x \in B_\rho(y)$ and $\beta \in (-1, 1)$ we have

$$c y^{\beta}_{m+1} \leq x^{\beta}_{m+1} \leq C y^{\beta}_{m+1} \qquad \text{and} \qquad c_0 \leq \frac{\sup_{B_\rho(y)} x^{\beta}_{m+1}}{\inf_{B_\rho(y)} x^{\beta}_{m+1}} \leq C_0. \tag{4.6}$$

Lemma 4.3.1. *Let $\gamma > 0$, $x_0 \in \partial\mathbb{R}^{m+1}_+$, $a > 0$, $\theta_1 \geq 2$ and $\theta_2 \leq \frac{1}{2}$. Define $\theta = \frac{\theta_2}{2\theta_1}$. Then there exists a constant $C_0 = C_0(m, \gamma, \theta_1, \beta)$ such that if $v \in W^{1,2}_\beta(B^+_R(x_0); \mathbb{R}^n)$ with*

$$r^{-2m} \int_B |x_{m+1}|^{-\beta} \mathrm{d}x \int_B |\nabla v|^2 |x_{m+1}|^\beta \mathrm{d}x \leq ar^\gamma \tag{4.7}$$

for every $B = B^+_r(y) \in \mathcal{B}^+(x_0, R, \theta_2 R)$ and every $B = B_r(y) \in \mathcal{B}_{\theta_1}(x_0, R, \theta_2 R)$, then for almost every $x_1, x_2 \in B^+_{\theta R}(x_0)$,

$$|v(x_1) - v(x_2)| \leq C_0 a^{\frac{1}{2}} |x_1 - x_2|^{\frac{\gamma}{2}}.$$

Proof. Let $B_r(y) \subset B_{2\theta R}(x_0)$ with $y_{m+1} \geq 0$, $y \in B_{\theta R}(x_0)$ and $r \leq \theta R$, for a $\gamma \in (0, 1)$, an $a > 0$ and θ as specified. Such a $B_r(y)$ must satisfy either $B_r(y) \in \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$ or $B_r(y) \notin \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$. We consider these cases in turn and we work with the even reflection of v with respect to $\partial\mathbb{R}^{m+1}_+$, which we do not relabel and which is in $W^{1,2}_\beta(B_R(x_0); \mathbb{R}^n)$.

Suppose $B_r(y) \in \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$ with $r \leq \theta R$. Then $B_r(y) \in \mathcal{B}_{\theta_1}(x_0, R, \theta_2 R)$ and an application of Hölder's inequality and the assumptions of the lemma yields

$$r^{-m} \int_{B_r(y)} |\nabla v| \mathrm{d}x \leq \left(r^{-2m} \int_{B_r(y)} |x_{m+1}|^{-\beta} \mathrm{d}x \int_{B_r(y)} |\nabla v|^2 |x_{m+1}|^\beta \mathrm{d}x \right)^{\frac{1}{2}} \leq a^{\frac{1}{2}} r^{\frac{\gamma}{2}}. \tag{4.8}$$

Now suppose $B_r(y) \notin \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$ and $r \leq \theta R$. In this case, since $B_r(y) \subset B_{2\theta R}(x_0)$ and $y \in B_{\theta R}(x_0)$ by assumption, we must have $y_{m+1} < \theta_1 r$. Hence $y_{m+1} - r < \zeta r$, where $\zeta \geq 1$ is such that $\theta_1 = \zeta + 1$, and thus $B_r(y) \subset B_{(2+\zeta)r}(y_0)$ where $y_0 = y - (0, y_{m+1})$. We observe that $B^+_{(2+\zeta)r}(y_0) \in \mathcal{B}^+(x_0, R, \theta_2 R)$. Therefore, defining $s = (2 + \zeta)r$ and using the assumptions of the lemma, the symmetry of v and applying Hölder's inequality, we find

$$\begin{aligned} r^{-m} \int_{B_r(y)} |\nabla v| \mathrm{d}x &\leq r^{-m} \int_{B_s(y_0)} |\nabla v| \mathrm{d}x \\ &= 2r^{-m} \int_{B^+_s(y_0)} |\nabla v| \mathrm{d}x \\ &\leq 2(2+\zeta)^m \left(s^{-2m} \int_{B^+_s(y_0)} |x_{m+1}|^{-\beta} \mathrm{d}x \int_{B^+_s(y_0)} |\nabla v|^2 |x_{m+1}|^\beta \mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq 2(2+\zeta)^{m+\frac{\gamma}{2}} a^{\frac{1}{2}} r^{\frac{\gamma}{2}}. \end{aligned} \tag{4.9}$$

Since v is even with respect to $\partial\mathbb{R}^{m+1}_+$, we deduce that either (4.8) or (4.9) holds on any $B_r(y) \subset B_{2\theta R}(x_0)$ with $y \in B_{\theta R}(x_0)$ and $r \leq \theta R$. Hence we have established that the hypothesis of the decay lemma of Morrey hold on $B_{2\theta R}(x_0)$, see [28] Lemma 2.1. An application of this lemma concludes the proof. □

4.4 Interior Estimates for Hölder Continuity

Using the regularity theory of Schoen and Uhlenbeck [37] and Schoen [38], we show that minimisers $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ of E^β relative to $\mathcal{O}$ essentially satisfy (4.7) in Lemma 4.3.1, provided the scale-invariant energy $R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x$ is sufficiently small. To this end, we show that that the preceding scale-invariant energy $B_R^+(x_0)$ controls the scale-invariant Euclidean energy on a class of ball with closure in $B_R^+(x_0)$. We also recall the relevant theory from [37] and [38] Sections 1, 2 and 3, stating the results in our context with slightly different notation. Our goal is to prove the following.

Lemma 4.4.1. *Suppose $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1};N)$ is a minimiser of E^β relative to $\mathcal{O}$. Let $B_R^+(x_0)$ be a half-ball with $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$. There exists an $\varepsilon_0 = \varepsilon_0(m,N) > 0$, a $\theta = \theta(m,N) \geq 2$ and a positive $C = C(m,N)$ such that if*

$$R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta \left|\nabla v\right|^2 \mathrm{d}x \leq \varepsilon_0,$$

then

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \mathrm{d}x \leq C \left(\frac{\rho}{r}\right)^\gamma r^{1-m} \int_{B_r(y)} |\nabla v|^2 \mathrm{d}x \tag{4.10}$$

on every $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$ for $0 < \rho \leq r$ and a $\gamma = \gamma(m,N) \in (0,1)$.

To establish the preceding lemma, we observe the following relationship between the scale-invariant energy on $B_R^+(x_0)$ and the scale-invariant Euclidean energy on a class of ball with closure in $B_R^+(x_0)$.

Lemma 4.4.2. *Suppose $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1};N)$ is a weakly stationary harmonic map with respect to the Neumann type boundary condition (4.2). Let $B_R^+(x_0)$ be a half-ball with $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$ and suppose $B_\rho(y) \in \mathcal{B}(x_0, R, \frac{R}{3})$. Then there is a constant $C = C(m)$ such that*

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \mathrm{d}x \leq C R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x. \tag{4.11}$$

Proof. Notice that any ball $B_\rho(y) \in \mathcal{B}$ satisfies $B_\rho(y) \subset B_{\frac{y_{m+1}}{2}}(y)$ so we can choose the scaling factor $e^{\frac{2|\beta|\rho}{y_{m+1}}}$ in Lemma 4.2.2. Furthermore, $e^{\frac{2|\beta|\rho}{y_{m+1}}} \leq e$ since $y_{m+1} \geq 2\rho$ and $\beta \in (-1,1)$. Hence, using (4.6) and applying Lemma 4.2.2, we find

$$\begin{aligned} \rho^{1-m} \int_{B_\rho(y)} \left|\nabla v\right|^2 \mathrm{d}x &\leq y_{m+1}^{-\beta} e^{\frac{2|\beta|\rho}{y_{m+1}}} \rho^{1-m} \int_{B_\rho(y)} x_{m+1}^\beta \left|\nabla v\right|^2 \mathrm{d}x \\ &\leq C \left(\frac{y_{m+1}}{2}\right)^{1-m-\beta} \int_{B_{\frac{y_{m+1}}{2}}(y)} x_{m+1}^\beta \left|\nabla v\right|^2 \mathrm{d}x. \end{aligned} \tag{4.12}$$

Let $y = (y_1, \ldots, y_{m+1})$ and $y^+ = (y_1, \ldots, y_m, 0)$. Note that $B_{\frac{y_{m+1}}{2}}(y) \subset B^+_{\frac{3y_{m+1}}{2}}(y^+)$ and, since $B_\rho(y) \in \mathcal{B}\left(x_0, R, \frac{R}{3}\right)$, we have $y \in B^+_{\frac{R}{3}}(x_0)$ and $B^+_{\frac{3y_{m+1}}{2}}(y^+) \subset B^+_{\frac{R}{2}}(y^+) \subset B_R^+(x_0)$. Using these facts we apply Lemma 4.2.1 to see that

$$\begin{aligned}\left(\frac{y_{m+1}}{2}\right)^{1-m-\beta}\int_{B_{\frac{y_{m+1}}{2}}(y)} x_{m+1}^{\beta}\left|\nabla v\right|^2 \mathrm{d}x &\le C\left(\frac{3y_{m+1}}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{3y_{m+1}}{2}}(y^+)} x_{m+1}^{\beta}\left|\nabla v\right|^2 \mathrm{d}x\\ &\le C\left(\frac{R}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{R}{2}}(y^+)} x_{m+1}^{\beta}\left|\nabla v\right|^2 \mathrm{d}x\\ &\le CR^{1-m-\beta}\int_{B^+_R(x_0)} x_{m+1}^{\beta}\left|\nabla v\right|^2 \mathrm{d}x \qquad (4.13)\end{aligned}$$

where $C = C(m)$. The combination of (4.12) and (4.13) yields (4.11). □

The maps considered in [37] and [38] belong to

$$W^{1,2}(\Omega; N) = \{v \in W^{1,2}(\Omega; \mathbb{R}^n) : v(x) \in N \text{ for almost every } x \in B_r(y)\},$$

for open $\Omega \subset \mathbb{R}^{m+1}_+$. Consider the compact Riemannian manifold $\overline{B_1(0)}$ with metric $\tilde{g}$. Recall from the introduction that the Dirichlet energy functional on $\overline{B_1(0)}$ is given by

$$E_{\tilde{g}}(v) = \int_{B_1(0)} |\nabla v|^2_{\tilde{g}} \sqrt{\det(\tilde{g})}\mathrm{d}x.$$

A minimiser of $E_{\tilde{g}}$ with fixed boundary data is defined as follows.

Definition 4.4.1. [[37] Section 1] Any $v \in W^{1,2}(B_1(0); N)$ is an $E_{\tilde{g}}$ *minimising map* if it satisfies $E_{\tilde{g}}(v) \le E_{\tilde{g}}(w)$ for any $w \in W^{1,2}(B_1(0); N)$ with $v - w \in W^{1,2}_0(\overline{B_1(0)}; \mathbb{R}^n)$.

The metric $\tilde{g}$ is assumed to be of class C^2 on $\overline{B_1(0)}$. For $\Lambda > 0$ denote by $\mathscr{E}_\Lambda$ the class of functionals $E_{\tilde{g}}$ on $\overline{B_1(0)}$ with metric $\tilde{g}$ such that $\tilde{g}_{ij}(0) = \delta_{ij}$ and

$$\sum_{i,j,k} |\partial_k \tilde{g}_{ij}| \le \Lambda.$$

If v is $E_{\tilde{g}}$-minimising with $E_{\tilde{g}} \in \mathscr{E}_\Lambda$ then we say $v \in \mathscr{H}_\Lambda$.

Schoen and Uhlenbeck [37] proved their ε-regularity theorem for minimisers of functionals of the form $\tilde{E}_{\tilde{g}} + F$, where F gives rise to terms in the Euler-Lagrange equations which are lower order than those coming from the energy. We state the result of their theorem with $F = 0$.

Lemma 4.4.3 (Theorem 3.1 in [37])**.** *There exists $\varepsilon = \varepsilon(m, N) > 0$ such that if $v \in \mathscr{H}_\Lambda$, $\Lambda \le \varepsilon$ and $\int_{B_1(0)} |\nabla v|^2 \mathrm{d}x \le \varepsilon$, then v is Hölder continuous in $B_{\frac{1}{2}}(0)$ and*

$$|v(x_1) - v(x_2)| \le C|x_1 - x_2|^\gamma$$

for constants $C = C(m, N)$ and $\gamma = \gamma(m, N) \in (0, 1)$ and every $x_1, x_2 \in B_{\frac{1}{2}}(0)$.

It is well known that continuous weakly harmonic maps are smooth, see [22] for example. It is more readily shown that Hölder continuous harmonic maps are smooth; this is the content of the following lemma, proved by Schoen in [38].

Lemma 4.4.4 (Lemma 3.1 of [38])**.** *Consider a ball $B_r(y) \subset \mathbb{R}^{m+1}_+$ and suppose $v \in W^{1,2}(B_r(y); N)$ is a weakly harmonic map which is Hölder continuous on $B_r(y)$. Then v is smooth on $B_r(y)$.*

The final lemma we will need is from [38] as follows.

Lemma 4.4.5 (Theorem 2.2 of [38])**.** *Let $v \in C^2(B_r(0); N)$ and $\tilde{g}$ be a Riemannian metric on $B_r(0)$. Suppose v is harmonic with respect to $\tilde{g}$ in $B_r(0)$ and $\tilde{g}$ satisfies $|\partial_k \tilde{g}_{ij}| \le \Lambda r^{-1}$ for $i, j, k = 1, \dots, m+1$ and $\Lambda^{-1}(\delta_{ij}) \le (\tilde{g}_{ij}) \le \Lambda(\delta_{ij})$ in the sense of tensors, where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ otherwise. Then there exists an $\varepsilon = \varepsilon(\Lambda, m, N)$ such that if*

$$r^{1-m} \int_{B_r(0)} |\nabla v|^2_{\tilde{g}} \left(det(\tilde{g})\right)^{\frac{1}{2}} \mathrm{d}x \le \varepsilon$$

then

$$\sup_{B_{\frac{r}{2}}(0)} |\nabla v|^2_{\tilde{g}} \le C r^{-(1+m)} \int_{B_r(0)} |\nabla v|^2_{\tilde{g}} \left(det(\tilde{g})\right)^{\frac{1}{2}} \mathrm{d}x \tag{4.14}$$

for a constant $C = C(\Lambda, m, N)$.

Proof of Lemma 4.4.1. Suppose $R^{1-m-\beta} \int_{B^+_{\frac{R}{3}}(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \le \varepsilon_0$ for an $\varepsilon_0 > 0$ to be chosen small and let ε be the number from Lemma 4.4.3.

Recall the metric g given in Euclidean coordinates by $g := x^\alpha_{m+1}\delta_{ij}$ and define $\hat{g}$ on $\overline{B_1(0)}$ by

$$\hat{g}_{ij}(x) = \delta_{ij}\left(1 + r y^{-1}_{m+1} x_{m+1}\right)^\alpha = y^{-\alpha}_{m+1} g_{ij}(rx + y). \tag{4.15}$$

The energy corresponding to $\hat{g}$ is

$$E_{\hat{g}}(\hat{v}) = \frac{1}{2} \int_{B_1(0)} \left(1 + r y^{-1}_{m+1} x_{m+1}\right)^\beta |\nabla \hat{v}|^2 \mathrm{d}x$$

for maps $\hat{v} \in W^{1,2}(B_1(0); \mathbb{R}^n)$. Since $B_r(y) \in B_\theta(x_0, R, \frac{R}{3})$ for a $\theta \ge 2$ to be chosen, we have $B_r(y) \in \mathcal{B}$. Hence, using (4.6) and noting that $\beta = \alpha\left(\frac{m-1}{2}\right) \in (-1, 1)$, we find constants c, C depending only on m such that

$$c \le \left(1 + r y^{-1}_{m+1} x_{m+1}\right)^\beta \le C \qquad \text{and} \qquad c \le \hat{g}_{ij}(x) \le C. \tag{4.16}$$

We note that $\partial_k \hat{g} = 0$ for $k \ne m+1$ and, again using (4.6), we calculate

$$\left|\partial_{m+1}\left(1 + r y^{-1}_{m+1} x_{m+1}\right)^\alpha\right| = r y^{-1}_{m+1} |\alpha| \left|\left(1 + r y^{-1}_{m+1} x_{m+1}\right)^{\alpha-1}\right| \le C r y^{-1}_{m+1}$$

where C is chosen independently of α. Hence, if we set $\theta = \theta(m, N) \ge \max\left\{2, (m+1)C\varepsilon^{-1}\right\}$ then we conclude that

$$\sum_{i,j,k} |\partial_k \tilde{g}_{ij}| = \sum_{i=1}^{m+1} |\partial_{m+1} \tilde{g}_{ii}| \le \varepsilon. \tag{4.17}$$

We assume the preceding choice of θ henceforth so that (4.17) holds on any $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$.

Define $v_{r,y}(x) = v(rx + y)$ for $x \in B_1(0)$. Lemmata 2.0.1 and 2.1.1 imply $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N) \hookrightarrow W^{1,2}(B_r(y); N)$ for every $B_r(y)$ with $\overline{B_r(y)} \subset \mathbb{R}^{m+1}_+$, regardless of $\beta \in (-1, 1)$. A change of variables then yields $v_{r,y} \in W^{1,2}(B_1(0); N)$. Furthermore, since v is a minimiser of E^β relative to $\mathcal{O}$ we readily calculate that $v_{r,y}$ is a minimiser of $E_{\hat{g}}$ in the sense of Definition 4.4.1 on $B_1(0)$, that is, $v_{r,y}$ minimises $E_{\hat{g}}$ among all maps in $W^{1,2}(B_1(0); N)$ with the same boundary values as $v_{r,y}$.

Our considerations so far imply that if $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$, for our preceding choice of θ, then $v_{r,y} \in \mathscr{H}_\varepsilon$. Let C be the constant from Lemma 4.4.2 and suppose that $\varepsilon_0 \leq C^{-1}\varepsilon$. An application of this lemma, combined with a change of variables, yields

$$\int_{B_1(0)} |\nabla v_{r,y}|^2 \,\mathrm{d}x = r^{1-m} \int_{B_r(y)} |\nabla v|^2 \,\mathrm{d}x \leq CR^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \,\mathrm{d}x \leq C\varepsilon_0 \leq \varepsilon. \quad (4.18)$$

This holds for every $B_r(y) \in \mathcal{B}_\theta\left(x_0, R, \frac{R}{3}\right)$. We may therefore apply Lemma 4.4.3 to $v_{r,y}$ to deduce that it is Hölder continuous in $B_{\frac{1}{2}}(0)$. Re-scaling implies v is Hölder continuous in every $B_{\frac{r}{2}}(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$, or equivalently v is Hölder continuous in every $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$.

Since v is weakly harmonic in $\mathbb{R}^{m+1}_+$ with respect to the metric represented by $x^\alpha_{m+1}\delta_{ij}$ (where $\beta = \alpha\frac{m-1}{2}$), it is weakly harmonic with respect to $x^\alpha_{m+1}\delta_{ij}$ on every $B_r(y)$ with $\overline{B_r(y)} \subset \mathbb{R}^{m+1}_+$. Thus it follows from Lemma 4.4.4 that v is smooth in each $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$, which holds if and only if each $v_{r,y}$ corresponding to such a $B_r(y)$ is smooth in $B_1(0)$. We further deduce that $v_{r,y}$ is harmonic in $B_1(0)$ with respect to $\hat{g}$ using the chain rule. Moreover, it follows from (4.16) and (4.17) that $\hat{g}$ satisfies the assumptions required of the metric in Lemma 4.4.5. We combine (4.16) with (4.18) to see that

$$\int_{B_1(0)} \left(1 + ry^{-1}_{m+1}x_{m+1}\right)^\beta |\nabla v_{r,y}|^2 \mathrm{d}x \leq C \int_{B_1(0)} |\nabla v_{r,y}|^2 \mathrm{d}x \leq C\varepsilon_0. \quad (4.19)$$

In addition to our previous stipulation for ε_0, we further require that $\varepsilon_0 \leq \frac{\varepsilon_1}{C}$ where ε_1 is the number from Lemma 4.4.5. We apply this lemma, recalling $\beta = \alpha(\frac{m-1}{2})$ and $B_r(y) \in \mathcal{B}$, to see that

$$\begin{aligned} r^2 \sup_{B_{\frac{r}{2}}(y)} \left(x_{m+1}y^{-1}_{m+1}\right)^\beta |\nabla v|^2 &= \sup_{B_{\frac{1}{2}}(0)} \left(1 + ry^{-1}_{m+1}x_{m+1}\right)^\beta |\nabla v_{r,y}|^2 \\ &\leq C \sup_{B_{\frac{1}{2}}(0)} \left(1 + ry^{-1}_{m+1}x_{m+1}\right)^{-\alpha} |\nabla v_{r,y}|^2 \\ &\leq C \int_{B_1(0)} \left(1 + ry^{-1}_{m+1}x_{m+1}\right)^\beta |\nabla v_{r,y}|^2 \mathrm{d}x \\ &= r^{1-m} \int_{B_r(y)} \left(x_{m+1}y^{-1}_{m+1}\right)^\beta |\nabla v|^2 \mathrm{d}x. \end{aligned}$$

As a result, for any $\sigma \in (0, \frac{1}{2}]$ we have

$$(\sigma r)^{1-m} \int_{B_{\sigma r}(y)} \left(x_{m+1}y^{-1}_{m+1}\right)^\beta |\nabla v|^2 \mathrm{d}x \leq C\sigma^2 r^{1-m} \int_{B_r(y)} \left(x_{m+1}y^{-1}_{m+1}\right)^\beta |\nabla v|^2 \mathrm{d}x.$$

Choose σ such that $\sigma^2 \leq \frac{1}{2C}$ and let $\rho \leq r$. Then

$$e^{\frac{|\beta|\sigma r}{y_{m+1}-r}} (\sigma r)^{1-m} \int_{B_{\sigma r}(y)} \left(x_{m+1}y^{-1}_{m+1}\right)^\beta |\nabla v|^2 \mathrm{d}x \leq \frac{1}{2} e^{\frac{|\beta| r}{y_{m+1}-r}} r^{1-m} \int_{B_{\sigma r}(y)} \left(x_{m+1}y^{-1}_{m+1}\right)^\beta |\nabla v|^2 \mathrm{d}x$$

on every $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$. The map $\rho \mapsto e^{\frac{|\beta|\rho}{y_{m+1}-r}} \rho^{1-m} \int_{B_\rho(y)} \left(x_{m+1} y_{m+1}^{-1}\right)^\beta |\nabla v|^2 \mathrm{d}x$ is non-decreasing in ρ for $\rho \leq r$ as a result of Lemma 4.2.2. Lemma 8.23 of [16] hence implies

$$e^{\frac{|\beta|\rho}{y_{m+1}-r}} \rho^{1-m} \int_{B_\rho(y)} \left(x_{m+1} y_{m+1}^{-1}\right)^\beta |\nabla v|^2 \mathrm{d}x \leq C \left(\frac{\rho}{r}\right)^\gamma e^{\frac{|\beta| r}{y_{m+1}-r}} r^{1-m} \int_{B_r(y)} \left(x_{m+1} y_{m+1}^{-1}\right)^\beta |\nabla v|^2 \mathrm{d}x$$

for every $\rho \leq r$ on every $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$. It follows that

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \mathrm{d}x \leq C \left(\frac{\rho}{r}\right)^\gamma r^{1-m} \int_{B_r(y)} |\nabla v|^2 \mathrm{d}x$$

on every $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$ for $0 < \rho \leq r$. This concludes the proof. □

4.5 A Modified Lemma of Luckhaus

Here we begin our construction of comparison maps. We prove a partial extension (to the particular case of our degenerate/singular metrics $x_{m+1}^\alpha \delta_{ij}$) of a lemma of Luckhaus, Lemma 3 in [23], as presented in Lemma 1 Section 2.6 of [39].

Let $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ denote the round m dimensional unit sphere, centred at the origin and equipped with the metric induced by the Euclidean metric on $\mathbb{R}^{m+1}$. Define $\mathbb{S}^m_+ = \mathbb{S}^m \cap \mathbb{R}^{m+1}_+$ with the metric induced from $\mathbb{S}^m$. We let ω denote a point in $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ or $\mathbb{S}^m_+ \subset \mathbb{R}^{m+1}_+$and write $\mathrm{d}\omega$ for the volume element corresponding to the induced metric. Recall the notation $\partial^+\Omega = \partial\Omega \cap \mathbb{R}^{m+1}_+$ for $\Omega \subset \mathbb{R}^{m+1}$ and $Q_r(y) = \{x \in \mathbb{R}^{m+1} : |x_i - y_i| < r, i = 1, \ldots, m+1\}$ for $y \in \mathbb{R}^{m+1}$. We also write $Q_r^+(y) = Q_r(y) \cap \mathbb{R}^{m+1}_+$ for $y \in \partial\mathbb{R}^{m+1}_+$.

In order to state the modified Luckhaus lemma precisely we introduce the notion of a Sobolev space for functions whose domain is either $\mathbb{S}^m$ or $\mathbb{S}^m_+$.

Definition 4.5.1. Let $\varepsilon > 0$ and $\rho > 0$. Suppose $S = \rho\mathbb{S}^m$ and $V_\varepsilon = B_{\rho+\varepsilon}(0) \backslash B_{\rho-\varepsilon}(0)$ or $S = \rho\mathbb{S}^m_+$ and $V_\varepsilon = B^+_{\rho+\varepsilon}(0) \backslash B^+_{\rho-\varepsilon}(0)$. An element $v \in L^2_\beta(S; \mathbb{R}^n)$ is said to be in $W^{1,2}_\beta(S; \mathbb{R}^n)$ if the map $v(\rho\frac{x}{|x|}) \in W^{1,2}_\beta(V_\varepsilon; \mathbb{R}^n)$ for some $\varepsilon > 0$. An element $v \in L^2_\beta(S \times [a, b]; \mathbb{R}^n)$, with $a < b$ real numbers, is said to be in $W^{1,2}_\beta(S \times [a, b]; \mathbb{R}^n)$ if the map $v(\rho\frac{x}{|x|}, s) \in W^{1,2}_\beta(V_\varepsilon \times [a, b]; \mathbb{R}^n)$ for some $\varepsilon > 0$. If $N \subset \mathbb{R}^n$ is compact, we say v is in $W^{1,2}_\beta(S; N)$ or $W^{1,2}_\beta(S \times [a, b]; N)$ if v is in $W^{1,2}_\beta(S; \mathbb{R}^n)$ or $W^{1,2}_\beta(S \times [a, b]; \mathbb{R}^n)$ respectively and $v(x) \in N$ for almost every $x \in S$.

We now state our version of the Luckhaus lemma.

Lemma 4.5.1. *Let $m \geq 2$ and $\beta \in (-1, 1)$. Let N be a compact subset of $\mathbb{R}^n$ and suppose $u, v \in W^{1,2}_\beta(\mathbb{S}^m_+; N)$. Then for all $\varepsilon \in (0, 1)$ there is a $w \in W^{1,2}_\beta(\mathbb{S}^m_+ \times [0, \varepsilon]; \mathbb{R}^n)$ such that w agrees with u on $\mathbb{S}^m_+ \times \{0\}$ and v on $\mathbb{S}^m_+ \times \{\varepsilon\}$ in the sense of traces and which satisfies the following. Let $\overline{D}$ be the gradient on $\mathbb{S}^m_+ \times [0, \varepsilon]$ and D the gradient on $\mathbb{S}^m_+$. Then $w = w(\omega, s)$ satisfies*

$$\begin{aligned} &\int_{\mathbb{S}^m_+ \times [0,\varepsilon]} \omega_{m+1}^\beta |\overline{D}w|^2 \mathrm{d}\omega \mathrm{d}s \\ &\leq C_1 \varepsilon \int_{\mathbb{S}^m_+} \omega_{m+1}^\beta \left(|Du|^2 + |Dv|^2\right) \mathrm{d}\omega + \frac{C_1}{\varepsilon} \int_{\mathbb{S}^m_+} \omega_{m+1}^\beta |u - v|^2 \mathrm{d}\omega \end{aligned} \tag{4.20}$$

where $C_1 = C_1(m, \beta)$. Furthermore, w satisfies

$$\begin{aligned} &dist^2(w(\omega,s),N) \\ &\leq \frac{C_2}{\varepsilon^{m+\frac{\beta}{2}+\frac{|\beta|}{2}}}\left(\int_{\mathbb{S}^m_+}\omega_{m+1}^{\beta}\left(|Du|^2+|Dv|^2\right)\mathrm{d}\omega\right)^{\frac{1}{q}}\left(\int_{\mathbb{S}^m_+}\omega_{m+1}^{\beta}|u-v|^2\mathrm{d}\omega\right)^{1-\frac{1}{q}} \\ &\quad+\frac{C_2}{\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}}\int_{\mathbb{S}^m_+}\omega_{m+1}^{\beta}|u-v|^2\mathrm{d}\omega \end{aligned} \tag{4.21}$$

for almost every $(\omega, s) \in \mathbb{S}^m_+ \times [0, \varepsilon]$ where $C_2 = C_2(m, \beta)$ and q satisfies the following. If $\beta \in (-1, 0]$ then (4.21) *holds for $q = 2$. If $\beta \in (0, 1)$, for any $p \in (1, \frac{2}{1+\beta})$, there exists $q \in \{2, p\}$ such that* (4.21) *holds.*

Our proof of Lemma 4.5.1 follows the proof, given in Section 2.12.2 of [39], of Lemma 1 in Section 2.6 of [39].

4.6 Absolute Continuity Properties of Functions in $W^{1,2}_\beta$

We recall the discussion in [39] Section 2.12.1 regarding the absolute continuity properties of $W^{1,p}$ functions, which are inherited by $W^{1,2}_\beta$ functions in view of Lemma 2.0.1. Let $\mathcal{H}^t$ denote the t-dimensional Hausdorff measure with respect to the Euclidean metric. Consider a rectangle $Q \subset \overline{\mathbb{R}^{m+1}_+}$ of the form $Q = [a_1, b_1] \times \ldots \times [a_{m+1}, b_{m+1}]$ where $a_i < b_i$. Suppose $v \in W^{1,2}_\beta(Q; \mathbb{R}^n)$ with $\beta \in (-1, 1)$. It follows from Lemma 2.0.1 that if $a_{m+1} > 0$ then $Q \subset \mathbb{R}^{m+1}_+$ and $v|_Q \in W^{1,2}(Q; \mathbb{R}^n)$. Lemma 2.0.1 also implies that if $a_{m+1} = 0$ then $v|_Q \in W^{1,p}(Q; \mathbb{R}^n)$ for $p = p(\beta)$. Hence, by Lemma 3.1.1 and Theorem 3.1.8 in [27], if $a_{m+1} \geq 0$, we may infer the existence of a representative $\hat{v}$ of v such that, for each $i = 1, \ldots, m+1$, $\hat{v}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{m+1})$ is an absolutely continuous function of x_i for almost all fixed values of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+1}$ with respect to the m dimensional Hausdorff measure $\mathcal{H}^m$ on $[a_1, b_1] \times \ldots \times [a_{i-1}, b_{i-1}] \times [a_{i+1}, b_{i+1}] \times \ldots \times [a_{m+1}, b_{m+1}]$. The classical partial derivatives $\frac{\partial \hat{v}}{\partial x_i}$ agree almost everywhere with the weak derivatives $\frac{\partial v}{\partial x_i}$. Furthermore, for any closed subset N of $\mathbb{R}^n$, if $v(x) \in N$ for almost every x then it is possible to choose $\hat{v}(x) \in N$ for every $x \in \mathbb{R}^{m+1}_+$.

4.7 Proof of Lemma 4.5.1

Proof of Lemma 4.5.1. We follow the proof, given in Section 2.12.2 of [39], of Lemma 1 in Section 2.6 of [39]. Throughout, C denotes a constant only depending on m and β.

Suppose $u, v \in W^{1,2}_\beta(\mathbb{S}^m_+; N)$. We reflect u and v evenly in $\mathbb{R}^m \times \{0\}$, without relabelling, to get $u, v \in W^{1,2}_\beta(\mathbb{S}^m; N)$ and choose extensions of u and v to $\mathbb{R}^{m+1} \setminus \{0\}$ which are homogeneous of degree zero with respect to the origin. Then we choose representatives of these extensions which satisfy the absolute continuity properties described in Section 4.6 on $\overline{Q_1(0)}$. We will denote the representatives of the extensions of u and v by $\hat{u}$ and $\hat{v}$ respectively. Then $\hat{u}(\rho\omega) = \hat{u}(\omega)$, $\hat{v}(\rho\omega) = \hat{v}(\omega)$ for almost every $\rho > 0$ and $\omega \in \mathbb{S}^m$. Moreover, we have the identity $\nabla \hat{u} = |x|^{-1} \nabla \hat{u}(\omega(x)) = Du(\omega(x))$ for $\omega(x) = |x|^{-1} x$, where D is the gradient on $\mathbb{S}^m_+$ and ∇ is the gradient on $\mathbb{R}^{m+1}_+$. We therefore calculate

$$\int_{Q_1^+(0)} x_{m+1}^{\beta}\left(|\nabla\hat{u}|^2+|\nabla\hat{v}|^2\right)\mathrm{d}x \leq C\int_{\mathbb{S}^m_+}\omega_{m+1}^{\beta}\left(|Du|^2+|Dv|^2\right)\mathrm{d}\omega \tag{4.22}$$

and

$$\int_{Q_1^+(0)} x_{m+1}^{\beta}|\hat{u}-\hat{v}|^2\mathrm{d}x \le C\int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta}|u-v|^2\mathrm{d}\omega. \tag{4.23}$$

Let $\varepsilon \in (0,\frac{1}{8})$ and define the closed rectangles $Q_{i,\varepsilon} = [i_1\varepsilon,(i_1+1)\varepsilon]\times\ldots\times[i_{m+1}\varepsilon,(i_{m+1}+1)\varepsilon]$ for $i=(i_1,\ldots,i_{m+1})\in\mathbb{Z}^{m+1}$. Fix $\varepsilon\in(0,\frac{1}{8})$ arbitrarily henceforth. Let F^l denote any l-dimensional face of a $Q_{i,\varepsilon}$. We define

$$\mathcal{Q} = \{Q_{i,\varepsilon} : i\in\mathbb{Z}^{m+1}, Q_{i,\varepsilon}\subset\overline{Q_{\frac{1}{2}}(0)}\} \text{ and } \mathcal{F}_i^l = \left\{F^l \text{ faces of } Q_{i,\varepsilon}\right\}.$$

In addition, we write $x+\mathcal{F}_i^l$ to denote the collection of the translations of all faces in $\mathcal{F}_i^l$ by $x\in\mathbb{R}^{m+1}$.

Consider a non-negative, measurable function $f:\overline{Q_1(0)}\to\mathbb{R}$ which is even with respect to the hyperplane $\partial\mathbb{R}^{m+1}_+$. Invoking [39] Section 2.12.2 estimate (3), which is a consequence of Chebychev's inequality and Fubini's theorem, we see that for every $K\ge 1$ there exists a set $P\subset Q_{0,\varepsilon}$ of measure $|P|\le\frac{C\varepsilon^{m+1}}{K}$, with $C=C(m)$, such that for all $y\in Q_{0,\varepsilon}\backslash P$ and $l\in\{0,\ldots,m+1\}$ we have

$$\varepsilon^{m+1-l}\sum_{\{i:Q_{i,\varepsilon}\in\mathcal{Q}\}}\sum_{y+\mathcal{F}_i^l}\int_{F^l} f\mathrm{d}\mathcal{H}^l \le K\int_{Q_1(0)} f\mathrm{d}x = 2K\int_{Q_1^+(0)} f\mathrm{d}x. \tag{4.24}$$

Since we chose $\hat{u}$ and $\hat{v}$ with the absolute continuity properties described in Section 4.6 on $\overline{Q_1(0)}$ it follows that for almost every $x\in Q_{0,\varepsilon}$, with respect to the $m+1$-dimensional Lebesgue measure, all of the functions $\hat{u},\hat{v},\nabla\hat{u},\nabla\hat{v}$ are $\mathcal{H}^l$ almost everywhere defined on each of the l-dimensional faces of $x+Q_{i,\varepsilon}$ for $Q_{i,\varepsilon}\in\mathcal{Q}$ and $l=1,\ldots,m+1$. Moreover, the gradients of $\hat{u}$ and $\hat{v}$ on any l-dimensional face of $x+Q_{i,\varepsilon}$ coincide $\mathcal{H}^l$ almost everywhere with the tangential parts of $\nabla\hat{u}$ and $\nabla\hat{v}$ respectively. Thus we may choose $x=a\in Q_{0,\varepsilon}$ such that these properties hold and, provided we choose K (depending on m) sufficiently large in (4.24), such that $a_{m+1}\ge\frac{\varepsilon}{2}$ and such that we may apply (4.24) simultaneously for $f(x)=|x_{m+1}|^{\beta}\tilde{f}(x)$ with $\tilde{f}(x)=|\hat{u}(x)-\hat{v}(x)|^2$ and $\tilde{f}(x)=|\nabla\hat{u}(x)|^2+|\nabla\hat{v}(x)|^2$ (where ∇ is the gradient on $\mathbb{R}^{m+1}$). In particular, after discarding the integrals in (4.24) taken over any cube faces which do not intersect $\mathbb{R}^{m+1}_+$, we have

$$\varepsilon^{m+1-l}\sum_{\left\{i:\substack{Q_{i,\varepsilon}\in\mathcal{Q}\\ i_{m+1}\ge -1}\right\}}\sum_{a+\mathcal{F}_i^l}\int_{F^l\cap\mathbb{R}^{m+1}_+} x_{m+1}^{\beta}\tilde{f}\mathrm{d}\mathcal{H}^l \le C\int_{Q_1^+(0)} x_{m+1}^{\beta}\tilde{f}\mathrm{d}x. \tag{4.25}$$

Now we begin the construction of w by defining a map on the one dimensional faces of every $Q\times[0,\varepsilon]$ where $Q:=(a+Q_{i,\varepsilon})\cap\mathbb{R}^{m+1}_+$ with $Q_{i,\varepsilon}\in\mathcal{Q}$ and $i_{m+1}\ge -1$. Let E_j denote a one dimensional face of Q parallel to the jth coordinate axis for $j=1,\ldots,m+1$. Define $w(x,0)=\hat{u}(x)$ on $Q\times\{0\}$ and $w(x,\varepsilon)=\hat{v}(x)$ on $Q\times\{\varepsilon\}$ and let $w(x,s)=(1-\frac{s}{\varepsilon})\hat{u}(x)+\frac{s}{\varepsilon}\hat{v}(x)$ for $x\in E_j$ and $s\in[0,\varepsilon]$. Since $\hat{u}(\mathbb{R}^{m+1}_+)\subset N$ by definition, it follows that

$$\mathrm{dist}^2(w(x,s),N)\le \max_{j=1,\ldots,m+1}\sup_{E_j}|\hat{u}-\hat{v}|^2 \tag{4.26}$$

for x in the 1-dimensional edges of Q and $s\in[0,\varepsilon]$.

We now estimate $\sup_{E_j}|\hat{u}-\hat{v}|^2$ using the Sobolev embedding theorem for $W^{1,2}_{\beta}$ along the line segments E_j, which one can deduce analogously to the case for $W^{1,2}$ functions. Note that in our construction so far, we have discarded any edges $E_j \subset \partial\mathbb{R}^{m+1}_+$. If E_j, $j = 1, \dots, m+1$, lies in a $Q = (a + Q_{i,\varepsilon}) \cap \mathbb{R}^{m+1}_+$ with $i_{m+1} \geq -1$, with the exception of the case $i_{m+1} = -1$, $j = m+1$ and $\beta \in (0,1)$, we calculate

$$\sup_{E_j} |\hat{u} - \hat{v}|^2 \leq \frac{C}{\varepsilon^{\frac{\beta}{2} + \frac{|\beta|}{2}}} \left(\int_{E_j} x^\beta_{m+1} \left(|\partial_j \hat{u}|^2 + |\partial_j \hat{v}|^2 \right) \mathrm{d}x_j \right)^{\frac{1}{2}} \left(\int_{E_j} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 \mathrm{d}x_j \right)^{\frac{1}{2}} + \frac{C}{\varepsilon^{1 + \frac{\beta}{2} + \frac{|\beta|}{2}}} \int_{E_j} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 \mathrm{d}x_j. \tag{4.27}$$

If $\beta > 0$, $i_{m+1} = -1$ and $j = m+1$ then we calculate

$$\sup_{E_{m+1}} |\hat{u} - \hat{v}|^2 \leq C \left(\int_{E_{m+1}} x^\beta_{m+1} (|\partial_{m+1} \hat{u}|^2 + |\partial_{m+1} \hat{v}|^2) \mathrm{d}x_{m+1} \right)^{\frac{1}{p}} \left(\int_{E_{m+1}} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 \mathrm{d}x_{m+1} \right)^{1 - \frac{1}{p}} + C\varepsilon^{-(1+\beta)} \int_{E_{m+1}} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 \mathrm{d}x_{m+1}, \tag{4.28}$$

for any $p \in (1, \frac{2}{1+\beta})$. The combination of (4.26), (4.27) and (4.28) with (4.25), applied with $l = 1$, yields

$$\mathrm{dist}^2(w(x,s), N) \leq \frac{C}{\varepsilon^{m + \frac{\beta}{2} + \frac{|\beta|}{2}}} \left(\int_{Q^+_1(0)} x^\beta_{m+1} \left(|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2 \right) \mathrm{d}x \right)^{\frac{1}{q}} \left(\int_{Q^+_1(0)} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 \mathrm{d}x \right)^{1 - \frac{1}{q}} + \frac{C}{\varepsilon^{m + 1 + \frac{\beta}{2} + \frac{|\beta|}{2}}} \int_{Q^+_1(0)} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 \mathrm{d}x, \tag{4.29}$$

where $q \in \{p, 2\}$, for p fixed as above, depends on β.

Next we bound the gradient of w on the product of the 1-dimensional edges of Q with $[0, \varepsilon]$. Let $\overline{\nabla}$ denote the gradient on $E_j \times [0, \varepsilon]$. Recall that $\hat{u}, \hat{v}$ are defined so that the tangential parts of their gradients $\nabla \hat{u}, \nabla \hat{v}$ on $\mathbb{R}^{m+1}_+$ coincide $\mathcal{H}^1$ almost everywhere with their gradients on the edges E_j. It follows that

$$\sup_{s \in [0,\varepsilon]} |\overline{\nabla} w(x,s)|^2 \leq 8 \left(|\nabla \hat{u}(x)|^2 + |\nabla \hat{v}(x)|^2 \right) + \frac{2}{\varepsilon^2} |\hat{u}(x) - \hat{v}(x)|^2,$$

for x in any edge E_j, $j = 1, \dots, m+1$, of Q. Integrating over $E_j \times [0, \varepsilon]$ with respect to $x^\beta_{m+1} \mathrm{d}x_j \mathrm{d}s$ yields

$$\int_{E_j \times [0,\varepsilon]} x^\beta_{m+1} |\overline{\nabla} w|^2 \mathrm{d}x_j \mathrm{d}s \leq 8\varepsilon \int_{E_j} x^\beta_{m+1} (|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2) \mathrm{d}x_j + \frac{2}{\varepsilon} \int_{E_j} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 \mathrm{d}x_j. \tag{4.30}$$

Consider again $Q \times [0, \varepsilon]$ for $Q = (a + Q_{i,\varepsilon}) \cap \mathbb{R}^{m+1}_+$ with $Q_{i,\varepsilon} \in \mathcal{Q}$ and $i_{m+1} \geq -1$. Recall that we are excluding cube faces in $\partial\mathbb{R}^{m+1}_+$ from our construction. We use a slightly different procedure to extend w to higher dimensions depending on whether $i_{m+1} = -1$ or $i_{m+1} \geq 0$. Accordingly we introduce some temporary notation for two classes of F^l that we consider. Let $F^l_\perp$ denote any l-dimensional face of any Q with no edges in the $m+1$ direction and let F^l_{m+1} denote any face of any such Q with edges in the $m+1$ direction. Suppose that $l \geq 2$ and w is already defined with L^2 gradient on every $F^l_\perp$ and $F^{l-1}_\perp \times [0,\varepsilon]$, square $x^\beta_{m+1}\mathrm{d}\mathcal{H}^l$-integrable gradient on every F^l_{m+1} and square $x^\beta_{m+1}\mathrm{d}\mathcal{H}^{l-1}\mathrm{d}s$-integrable $F^{l-1}_{m+1} \times [0,\varepsilon]$. Furthermore, suppose that $w(x,0) = \hat{u}(x)$ and $w(x,\varepsilon) = \hat{v}(x)$ for $x \in F^l_\perp$ or $x \in F^l_{m+1}$. These assumptions imply that w is defined $\mathcal{H}^l$ almost everywhere on all the l-dimensional faces of Q for $l \geq 2$. Since $\partial(F^l_\perp \times [0,\varepsilon])$ and $\partial^+(F^l_{m+1} \times [0,\varepsilon])$ are the union of such l-dimensional faces, w is defined $\mathcal{H}^l$ almost everywhere on these sets. If Q is such that $i_{m+1} \geq 0$ then we do not distinguish between $F^l_\perp$ and F^l_{m+1} and extend w to each $F^l_\perp \times [0,\varepsilon]$ and $F^l_{m+1} \times [0,\varepsilon]$ by homogeneous extension of degree zero with respect to $(y, \frac{\varepsilon}{2})$, where y is the centre point of $F^l_\perp$ or F^l_{m+1}. If $i_{m+1} = -1$ then we extend w into $F^l_\perp \times [0,\varepsilon]$ using the same method. We extend w homogeneously of degree 0 from $\partial^+(F^l_{m+1} \times [0,\varepsilon])$ into $F^l_{m+1} \times [0,\varepsilon]$ with respect to the point $\left(y^+, \frac{\varepsilon}{2}\right)$, where y is the centre point of F^l_{m+1} and $y^+ = y - (0, y_{m+1})$.

Now let F^l denote any l-dimensional face of any Q again. Since the tangential parts of the gradients $\nabla\hat{u}, \nabla\hat{v}$ on $\mathbb{R}^{m+1}$ coincide with the gradients of $\hat{u}$ and $\hat{v}$ on F^l for $\mathcal{H}^l$ almost every $x \in F^l$, using the fact that $\hat{u}$ and $\hat{v}$ are homogeneous of degree zero, we calculate

$$\int_{F^l \times [0,\varepsilon]} x^\beta_{m+1} |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^l \mathrm{d}s \leq C\varepsilon \int_{F^l} x^\beta_{m+1} (|\nabla\hat{u}|^2 + |\nabla\hat{v}|^2) \mathrm{d}\mathcal{H}^l + C\varepsilon \sum_{a+\mathcal{F}^{l-1}_i} \int_{F^{l-1} \times [0,\varepsilon]} x^\beta_{m+1} |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^{l-1} \mathrm{d}s, \tag{4.31}$$

where $\overline{\nabla}$ is the gradient on $F^l \times [0,\varepsilon]$. From (4.31), we inductively deduce that for any $l \in \{2, \ldots, m+1\}$ we can extend w to each $F^l \times [0,\varepsilon]$ in $Q \times [0,\varepsilon]$ so that w has an L^2 or $x^\beta_{m+1}\mathrm{d}\mathcal{H}^l\mathrm{d}s$-integrable gradient $\overline{\nabla} w$ on these faces. Moreover, $\overline{\nabla} w$ satisfies

$$\int_{F^l \times [0,\varepsilon]} x^\beta_{m+1} |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^l \mathrm{d}s \leq C\varepsilon^{l-1} \sum_{a+\mathcal{F}^1_i} \int_{F^1 \times [0,\varepsilon]} x^\beta_{m+1} |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^1 \mathrm{d}s + C \sum_{j=1}^{l} \varepsilon^{l-j+1} \sum_{a+\mathcal{F}^j_i} \int_{F^j} x^\beta_{m+1} (|\nabla\hat{u}|^2 + |\nabla\hat{v}|^2) \mathrm{d}\mathcal{H}^j. \tag{4.32}$$

So far, we have constructed a map $w = w^{i,\varepsilon}$ on each cube and rectangle $Q = (a + Q_{i,\varepsilon}) \cap \mathbb{R}^{m+1}_+$ such that $Q_{i,\varepsilon} \in \mathcal{Q}$ with $i_{m+1} \geq -1$. It follows from the construction that $w^{(i,\varepsilon)} = w^{(j,\varepsilon)}$ $\mathcal{H}^{l+1}$-almost everywhere on common faces $F^l \times [0,\varepsilon]$ of $(a + Q_{i,\varepsilon}) \cap \mathbb{R}^{m+1}_+$ and $(a + Q_{j,\varepsilon}) \cap \mathbb{R}^{m+1}_+$. Furthermore, for $0 < \varepsilon < \frac{1}{8}$ it follows that

$$Q^+_{\frac{1}{4}}(0) \subset \bigcup_{\left\{i : \substack{Q_{i,\varepsilon} \in \mathcal{Q} \\ i_{m+1} \geq -1}\right\}} a + Q_{i,\varepsilon}.$$

We may therefore define $w \in W^{1,2}_\beta(Q^+_{\frac{1}{4}}(0) \times [0,\varepsilon]; \mathbb{R}^n)$ by $w|_{(a+Q_{i,\varepsilon}) \cap \mathbb{R}^{m+1}_+}(x,s) = w^{(i,\varepsilon)}(x,s)$ for $s \in [0,\varepsilon]$. Since w is homogeneous of degree 0 on any l-dimensional face of any $Q \times [0,\varepsilon]$ with $l \geq 3$, our inductive procedure preserves (4.29) for all (x,s) in $Q^+_{\frac{1}{4}}(0) \times [0,\varepsilon]$, with the possible exception of a set P of m-dimensional Hausdorff measure 0. It follows from (4.29) that for $(x,s) \in$

$(Q^+_{\frac{1}{4}}(0)\times[0,\varepsilon])\backslash P$ we have

$$\begin{aligned}\mathrm{dist}^2(w(x,s),N) \le \frac{C}{\varepsilon^{m+\frac{\beta}{2}+\frac{|\beta|}{2}}}\left(\int_{Q^+_1(0)} x^\beta_{m+1}\left(|\nabla\hat{u}|^2+|\nabla\hat{v}|^2\right)\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{Q^+_1(0)} x^\beta_{m+1}|\hat{u}-\hat{v}|^2\mathrm{d}x\right)^{1-\frac{1}{q}} \\ + \frac{C}{\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}}\int_{Q^+_1(0)} x^\beta_{m+1}|\hat{u}-\hat{v}|^2\mathrm{d}x \qquad (4.33)\end{aligned}$$

where $q\in\{2,p\}$ for some fixed $p\in(1,\frac{2}{1+\beta})$. Moreover, we combine (4.30), (4.32) and (4.25) to see that

$$\int_{Q^+_{\frac{1}{4}}(0)\times[0,\varepsilon]} x^\beta_{m+1}|\overline{\nabla}w|^2\mathrm{d}x\mathrm{d}s \le C\varepsilon\int_{Q^+_1(0)} x^\beta_{m+1}\left(|\nabla\hat{u}|^2+|\nabla\hat{v}|^2\right)\mathrm{d}x + \frac{C}{\varepsilon}\int_{Q^+_1(0)} x^\beta_{m+1}|\hat{u}-\hat{v}|^2\mathrm{d}x. \qquad (4.34)$$

The definition of w as required now follows from combining (4.33) and (4.34) with (4.22) and (4.23). The absolute continuity properties, described in Section 4.6, of w, viewed as a function defined on a rectangle in polar coordinates, guarantee that for almost every $\rho\in[\frac{1}{8},\frac{1}{4}]$, w has square $x^\beta_{m+1}\mathrm{d}\mathcal{H}^m\mathrm{d}s$-integrable gradient $\partial^+B^+_\rho(0)\times[0,\varepsilon]$ which coincides $\mathcal{H}^m\mathrm{d}s$ almost everywhere with the tangential part of $\overline{\nabla}w$. Using Fubini's theorem and Chebychev's inequality, applied to the map $\rho\mapsto\int_{\partial^+B^+_\rho(0)\times[0,\varepsilon]} x^\beta_{m+1}|\overline{\nabla}w|^2\mathrm{d}\mathcal{H}^m\mathrm{d}s$, we may therefore choose $\rho\in[\frac{1}{8},\frac{1}{4}]$ such that w has square $x^\beta_{m+1}\mathrm{d}\mathcal{H}^m\mathrm{d}s$-integrable gradient on $\partial^+B^+_\rho(0)\times[0,\varepsilon]$ and satisfies

$$\int_{\partial^+B^+_\rho(0)\times[0,\varepsilon]} x^\beta_{m+1}|\overline{\nabla}w|^2\mathrm{d}\mathcal{H}^m\mathrm{d}s \le C\int_{B^+_{\frac{1}{4}}(0)\times[0,\varepsilon]} x^\beta_{m+1}|\overline{\nabla}w|^2\mathrm{d}\mathcal{H}^{m+1}\mathrm{d}s.$$

We define $\tilde{w}$ on $\mathbb{S}^m_+\times[0,\varepsilon]$ by $\tilde{w}(\omega,s)=w(\rho\omega,s)$ and observe that this map has the required properties. □

4.8 Comparison Maps

With Lemma 4.5.1 in hand, we may now construct comparison maps for $W^{1,2}_\beta$ functions which have values in N and are defined on half-balls centred in $\partial\mathbb{R}^{m+1}_+$, provided the re-scaled energy is sufficiently small. We use the notation $\overline{v}_{B^+_\rho(y),\beta}=\left(\int_{B^+_\rho(y)} x^\beta_{m+1}\mathrm{d}x\right)^{-1}\int_{B^+_\rho(y)} x^\beta_{m+1}v\mathrm{d}x$.

Lemma 4.8.1. *There exists a $\delta_0=\delta_0(m,N,\beta)>0$ such that the following holds. Let $\varepsilon\in(0,1)$ and $v\in W^{1,2}_\beta(B^+_\rho(y);N)$ with $\rho^{1-m-\beta}\int_{B^+_\rho(y)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x\le\delta_0^2\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}$. Then there is a $\sigma\in(\frac{3\rho}{4},\rho)$ such that we can find a map $w_\varepsilon\in W^{1,2}_\beta(B^+_\rho(y);N)$ which agrees with v in $B^+_\rho(y)\backslash B^+_\sigma(y)$ and such that*

$$\begin{aligned}&\sigma^{1-m-\beta}\int_{B^+_\sigma(y)} x^\beta_{m+1}|\nabla w|^2\mathrm{d}x \\ &\le C\varepsilon\rho^{1-m-\beta}\int_{B^+_\rho(y)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x + \frac{C}{\varepsilon}\rho^{-(1+m+\beta)}\int_{B^+_\rho(y)} x^\beta_{m+1}|v-\overline{v}_{B^+_\rho(y),\beta}|^2\mathrm{d}x \qquad (4.35)\end{aligned}$$

for a constant $C=C(m,\beta)$.

Proof. We follow the proof of Corollary 1 in Section 2.7 of [39]. Throughout, C denotes a constant which depends on m and possibly β and we only distinguish different C when necessary. We also assume, without loss of generality, that $\varepsilon \leq \frac{1}{2}$.

Let $\delta_0 > 0$ to be chosen as required and suppose the assumptions of the lemma hold for δ_0. As a consequence of Poincaré Inequality for the A_2 weights $|x_{m+1}|^\beta$ [20], we have

$$\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}x \leq C\rho^{1-m-\beta} \int_{B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \tag{4.36}$$

As discussed in Section 4.6, we may work with a representative of v, which we don't relabel, such that $v(B_\rho^+(y)) \subset N$. It follows that

$$\operatorname{dist}^2(\overline{v}_{B_\rho^+(y),\beta}, N) \leq |v(x) - \overline{v}_{B_\rho^+(y),\beta}|^2 \tag{4.37}$$

for every $x \in B_\rho^+(y)$. Integrating (4.37) over $B_\rho^+(y)$ with respect to $x_{m+1}^{\frac{\beta}{2}+\frac{|\beta|}{2}} \mathrm{d}x$ and dividing by $\int_{B_\rho^+(y)} x_{m+1}^{\frac{\beta}{2}+\frac{|\beta|}{2}} \mathrm{d}x$ we see that

$$\operatorname{dist}^2(\overline{v}_{B_\rho^+(y),\beta}, N) \leq C\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}x. \tag{4.38}$$

Combining (4.38) with (4.36) we find

$$\operatorname{dist}^2(\overline{v}_{B_\rho^+(y),\beta}, N) \leq C\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}x \leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}.$$

Hence, we may choose $\lambda \in N$ such that

$$|\lambda - \overline{v}_{B_\rho^+(y),\beta}|^2 \leq C\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}x \leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \tag{4.39}$$

Using Chebychev's inequality, we choose a $C > 0$ such that there exists $\sigma \in (\frac{3\rho}{4}, \rho)$ such that $\hat{v} \in W_\beta^{1,2}(\mathbb{S}_+^m; N)$, where $\hat{v}(\omega) = v(\sigma\omega + y)$, and such that

$$\begin{aligned} \int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |D\hat{v}|^2 \mathrm{d}\omega &\leq C\sigma^{2-m-\beta} \int_{\partial^+ B_\sigma^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}S(x) \\ &\leq C\rho^{1-m-\beta} \int_{B_\rho^+(y)\setminus B_{\frac{\rho}{2}}^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \\ &\leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}} \end{aligned} \tag{4.40}$$

where D is the gradient on $\mathbb{S}_+^m$, and

$$\begin{aligned} \int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |\hat{v} - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}\omega &\leq \sigma^{-m-\beta} \int_{\partial^+ B_\sigma^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}S(x) \\ &\leq C\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)\setminus B_{\frac{\rho}{2}}^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}x \\ &\leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \end{aligned} \tag{4.41}$$

We may therefore apply Lemma 4.5.1 to $\hat{v} \in W^{1,2}_\beta(\mathbb{S}^m_+; N)$ and λ. This yields a $w_0 : \mathbb{S}^m_+ \times [0,\varepsilon] \to \mathbb{R}^n$ with $w_0 = \hat{v}$ on $\mathbb{S}^m_+ \times \{0\}$ and $w_0 = \lambda$ on $\mathbb{S}^m_+ \times \{\varepsilon\}$ in the sense of traces. Furthermore, (4.20) yields

$$\int_{\mathbb{S}^m_+ \times [0,\varepsilon]} \omega^\beta_{m+1} |\overline{D} w_0|^2 \mathrm{d}\omega \mathrm{d}s \le C\varepsilon \int_{\mathbb{S}^m_+} \omega^\beta_{m+1} |D\hat{v}|^2 \mathrm{d}\omega + \frac{C}{\varepsilon} \int_{\mathbb{S}^m_+} \omega^\beta_{m+1} |\hat{v} - \lambda|^2 \mathrm{d}\omega, \tag{4.42}$$

where $\overline{D}$ is the gradient on $\mathbb{S}^m_+ \times [0,\varepsilon]$ and D is the gradient on $\mathbb{S}^m_+$. In addition, (4.21) implies that

$$\begin{aligned} \mathrm{dist}^2(w_0(\omega,s), N) \le{}& \frac{C}{\varepsilon^{m+\frac{\beta}{2}+\frac{|\beta|}{2}}} \left(\int_{\mathbb{S}^m_+} \omega^\beta_{m+1} |D\hat{v}|^2 \mathrm{d}\omega \right)^{\frac{1}{q}} \left(\int_{\mathbb{S}^m_+} \omega^\beta_{m+1} |\hat{v} - \lambda|^2 \mathrm{d}\omega \right)^{1-\frac{1}{q}} \\ &+ \frac{C}{\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}} \int_{\mathbb{S}^m_+} \omega^\beta_{m+1} |\hat{v} - \lambda|^2 \mathrm{d}\omega \end{aligned} \tag{4.43}$$

for every $(\omega, s) \in \mathbb{S}^m_+ \times [0,\varepsilon]$, where $q \in (1,2]$ depends on β. Henceforth we assume that $\delta_0 \le 1$. Using (4.39) and (4.41) we deduce that

$$\begin{aligned} \int_{\mathbb{S}^m_+} \omega^\beta_{m+1} |\hat{v} - \lambda|^2 \mathrm{d}\omega &\le 2 \int_{\mathbb{S}^m_+} \omega^\beta_{m+1} |\hat{v} - \overline{v}_{B^+_\rho(y),\beta}|^2 \mathrm{d}\omega + 2 \int_{\mathbb{S}^m_+} \omega^\beta_{m+1} |\overline{v}_{B^+_\rho(y),\beta} - \lambda|^2 \mathrm{d}\omega \\ &\le C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \end{aligned} \tag{4.44}$$

The combination of (4.43) with (4.40) and (4.44) yields

$$\mathrm{dist}(w_0(x,s), N) \le C\delta_0 \tag{4.45}$$

for every $(\omega, s) \in \mathbb{S}^m_+ \times [0,\varepsilon]$ and for $q \in (1,2]$ depending on β.

Choose δ_0, depending on N, m, β, such that $C\delta_0 \le \hat{\alpha}$ where C is the constant in (4.45) and $\hat{\alpha} > 0$ is sufficiently small to guarantee that the nearest point projection π_N onto N exists and has bounded derivatives in $N_{\hat{\alpha}} = \{x \in \mathbb{R}^n : \mathrm{dist}(x, N) \le \hat{\alpha}\}$. It then follows from (4.45) that we may apply π_N to w_0. Let $\omega \in \mathbb{S}^m_+$ satisfy $\omega = \omega(x) = \frac{x-y}{|x-y|}$, $r = |x-y|$ and define $w \in W^{1,2}_\beta(B^+_\rho(y); N)$ by

$$w(x) = w(y + r\omega(x)) = \begin{cases} v(y + r\omega(x)) & r \in (\sigma, \rho) \\ \pi_N(w_0(\omega(x), (1 - \frac{r}{\sigma}))) & r \in [(1-\varepsilon)\sigma, \sigma] \\ \lambda & r \in (0, (1-\varepsilon)\sigma). \end{cases}$$

Note that w agrees with v in $B^+_\rho(y) \backslash B^+_\sigma(y)$. We then readily calculate that w satisfies (4.35) as required. □

4.9 Control of the Mean Squared Oscillation

The Euler-Lagrange equations of E^β satisfy the structural conditions $|\mathrm{div}(x^\beta_{m+1} \nabla v)| \le c_0 x^\beta_{m+1} |\nabla v|^2$, together with the Neumann condition (4.2). For functions satisfying such conditions, if the re-scaled energy is sufficiently small it is possible to control their mean squared oscillation using the energy as follows.

Lemma 4.9.1. *For every $\delta > 0$ and every $c_0 > 0$ there exist two constants $\varepsilon = \varepsilon(m, n, \delta, c_0) > 0$ and $\theta = \theta(m, n, \delta, c_0) \in (0, \frac{1}{4}]$ such that the following holds. Let $x_0 \in \partial\mathbb{R}^{m+1}_+$, $R > 0$ and $B^+_R(x_0) \subset$*

$\mathbb{R}^{m+1}_+$. *Suppose* $v \in W^{1,2}_\beta(B^+_R(x_0);\mathbb{R}^n)$ *satisfies*

$$\left|\int_{B^+_R(x_0)} x^\beta_{m+1}\,\langle\nabla v,\nabla\phi\rangle\,\mathrm{d}x\right| \le c_0\int_{B^+_R(x_0)} x^\beta_{m+1}\,|\phi|\,|\nabla v|^2\,\mathrm{d}x$$

for every $\phi \in C^\infty_0(B_R(x_0);\mathbb{R}^n)$. *If*

$$R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \le \varepsilon$$

then

$$(\theta R)^{-(1+m+\beta)}\int_{B^+_{\theta R}(x_0)} x^\beta_{m+1}|v-\overline{v}_{B^+_{\theta R}(x_0),\beta}|^2\mathrm{d}x \le \delta R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}\,|\nabla v|^2\,\mathrm{d}x.$$

Proof. The proof of the lemma is based on a blow-up procedure analogous to that of the proof of Lemma 3.5 in [28] for example.

Observe that the statement of the lemma is invariant under rescaling and translation by any point in $\partial\mathbb{R}^{m+1}_+$; henceforth we assume $R=1$ and $x_0=0$. Suppose, for a contradiction, that there exist $\delta>0$ and $c_0>0$ such that the claim is false. Then for any $\theta\in(0,\frac{1}{4}]$ there is a sequence of maps $(v_k)_{k\in\mathbb{N}}$, with $v_k\in W^{1,2}_\beta(B^+_1(0);\mathbb{R}^n)$ for every k, such that

$$\left|\int_{B^+_1(0)} x^\beta_{m+1}\,\langle\nabla v_k,\nabla\phi\rangle\,\mathrm{d}x\right| \le c_0\int_{B^+_1(0)} x^\beta_{m+1}|\phi||\nabla v_k|^2\mathrm{d}x \tag{4.46}$$

for every $\phi\in C^\infty_0(B_1(0);\mathbb{R}^n)$ and

$$\int_{B^+_1(0)} x^\beta_{m+1}|\nabla v_k|^2\mathrm{d}x := \varepsilon_k \to 0 \text{ as } k\to\infty$$

but

$$\theta^{-(1+m+\beta)}\int_{B^+_\theta(0)} x^\beta_{m+1}|v_k-\overline{(v_k)}_{B^+_\theta(0),\beta}|^2\mathrm{d}x > \delta\int_{B^+_1(0)} x^\beta_{m+1}|\nabla v_k|^2\mathrm{d}x = \delta\varepsilon_k. \tag{4.47}$$

Consider the normalised sequence $(w_k)_{k\in\mathbb{N}}$ defined by $w_k=\varepsilon_k^{-\frac{1}{2}}(v_k-\overline{(v_k)}_{B^+_\theta(0),\beta})$. Then $\nabla w_k = \varepsilon_k^{-\frac{1}{2}}\nabla v_k$ and thus

$$\int_{B^+_1(0)} x^\beta_{m+1}|\nabla w_k|^2\mathrm{d}x = 1 \quad\text{and}\quad \overline{(w_k)}_{B^+_\theta(0),\beta}=0. \tag{4.48}$$

Furthermore, we deduce from (4.47) that

$$\theta^{-(1+m+\beta)}\int_{B^+_\theta(0)} x^\beta_{m+1}\,|w_k|^2\,\mathrm{d}x > \delta. \tag{4.49}$$

Using (4.48) and the Poincaré inequality for A_2 weights we deduce that $(w_k)_{k\in\mathbb{N}}$ is bounded $W^{1,2}_\beta(B^+_1(0);\mathbb{R}^n)$. Hence, the Compactness Lemma, Lemma 2.2.1, yields a subsequence $(w_{k_j})_{j\in\mathbb{N}}$ which converges weakly in $W^{1,2}_\beta(B^+_1(0);\mathbb{R}^n)$ and strongly in $L^2_\beta(B^+_1(0);\mathbb{R}^n)$ to a $w\in W^{1,2}_\beta(B^+_1(0);\mathbb{R}^n)$.

In view of (4.46) and (4.48) we calculate

$$\left| \int_{B_1^+(0)} x_{m+1}^{\beta} \langle \nabla w_k, \nabla \phi \rangle \, \mathrm{d}x \right| \leq c_0 ||\phi||_{L^\infty(B_1^+(0);\mathbb{R}^n)} \varepsilon_k^{\frac{1}{2}}$$

for every $\phi \in C_0^\infty(B_1(0);\mathbb{R}^n)$. Since $w_{k_j} \rightharpoonup w$ in $W_\beta^{1,2}(B_1^+(0);\mathbb{R}^n)$, it follows that

$$\left| \int_{B_1^+(0)} x_{m+1}^{\beta} \langle \nabla w, \nabla \phi \rangle \mathrm{d}x \right| = \lim_{j\to\infty} \left| \int_{B_1^+(0)} x_{m+1}^{\beta} \langle \nabla w_{k_j}, \nabla \phi \rangle \mathrm{d}x \right| \leq c_0 ||\phi||_{L^\infty(B_1^+(0);\mathbb{R}^n)} \lim_{j\to\infty} \varepsilon_{k_j}^{\frac{1}{2}} = 0$$

for every $\phi \in C_0^\infty(B_1(0);\mathbb{R}^n)$. Hence w is a weak solution of the linear Neumann-type problem (2.11) and, in particular, satisfies (2.12) from Lemma 2.3.1 in $B_1^+(0)$.

We also conclude, using the Compactness Lemma to take limits in (4.48) and (4.49), that

$$\int_{B_1^+(0)} x_{m+1}^{\beta} |\nabla w|^2 \, \mathrm{d}x \leq 1 \quad \text{and} \quad \overline{w}_{B_\theta^+(0),\beta} = 0$$

and

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^{\beta} |w|^2 \, \mathrm{d}x \geq \delta \tag{4.50}$$

respectively. Now, since $\overline{w}_{B_\theta^+(0),\beta} = 0$, the Poincaré inequality yields

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^{\beta} |w|^2 \mathrm{d}x \leq C\theta^{1-m-\beta} \int_{B_\theta^+(0)} x_{m+1}^{\beta} |\nabla w|^2 \mathrm{d}x. \tag{4.51}$$

We apply Lemma 2.3.1 to w with $\theta \leq \frac{1}{4}$ (so that $2\theta \leq \frac{1}{2}$). This gives a positive constant C (independent of θ) and a $\gamma \in (0,1)$ such that

$$\theta^{1-m-\beta} \int_{B_\theta^+(0)} x_{m+1}^{\beta} |\nabla w|^2 \mathrm{d}x \leq C(2\theta)^{2\gamma}. \tag{4.52}$$

Combining (4.51) and (4.52) we see that

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^{\beta} |w|^2 \mathrm{d}x \leq C(2\theta)^{2\gamma}. \tag{4.53}$$

This holds for all fixed $\theta \in (0, \frac{1}{4}]$ and we choose $\theta < \frac{1}{2} \left(\frac{\delta}{C}\right)^{\frac{1}{2\gamma}}$ so that (4.53) contradicts (4.50). □

Remark 4.9.1. We could have used Lemma 2.4.1 in place of Lemma 2.3.1 to the same effect. In using the latter lemma, we observe that Hölder continuity of solutions to the linear Neumann-type problem (2.11) is sufficient to obtain energy decay and consequently Hölder continuity of minimisers of E^β relative to $\mathcal{O}$; we do not need higher regularity for the linear problem at this point.

4.10 Energy Decay

We combine our construction of comparison maps in Section 4.8, with the improved control of the mean squared deviation obtained in Section 4.9 in order to show that the re-scaled energy decays faster than implied by the boundary monotonicity formula, Lemma 4.2.1.

Lemma 4.10.1. *Let $v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N)$ be a minimiser of E^{β} relative to $\mathcal{O}$. Suppose $B^+_R(x_0)$ is a half-ball with $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$. There exist $\varepsilon_0 = \varepsilon_0(m, N, \beta) > 0$ and $\theta_0 = \theta_0(m, N, \beta) \in (0, \frac{1}{4})$ such that if*

$$R^{1-m-\beta} \int_{B^+_R(x_0)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x \leq \varepsilon_0,$$

then

$$(\theta_0 r)^{1-m-\beta} \int_{B^+_{\theta_0 r}(y)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x \leq \frac{1}{2} r^{1-m-\beta} \int_{B^+_r(y)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x,$$

for every $B^+_r(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$.

Proof. Let $B^+_{\rho}(y) \subset B^+_r(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$. Then $\rho \leq r \leq \frac{R}{2}$, $y \in \partial \mathbb{R}^{m+1}_+$, $|x_0 - y| < \frac{R}{2}$ and $y \in \mathcal{O}$. Suppose v satisfies $R^{1-m-\beta} \int_{B^+_R(x_0)} |\nabla v|^2 \mathrm{d}x \leq \varepsilon_0$ for $\varepsilon_0 > 0$ to be chosen. Then for any $\rho \in (0, r]$ the monotonicity formula, Lemma 4.2.1, yields

$$\begin{aligned} \rho^{1-m-\beta} \int_{B^+_{\rho}(y)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x &\leq r^{1-m-\beta} \int_{B^+_r(y)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x \\ &\leq \left(\frac{R}{2}\right)^{1-m-\beta} \int_{B^+_{\frac{R}{2}}(y)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x \\ &\leq C \varepsilon_0. \end{aligned} \tag{4.54}$$

We apply Lemma 4.8.1 on $B^+_{\rho}(y) \subset B^+_r(y)$, with $\rho \leq r$ to be chosen later. This gives a δ_0 such that for any $\varepsilon \in (0, 1)$, if

$$\rho^{1-m-\beta} \int_{B^+_{\rho}(y)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x \leq \delta_0^2 \varepsilon^{1+m+\frac{\beta}{2}+\frac{|\beta|}{2}} \tag{4.55}$$

then there is a $\sigma \in (\frac{3\rho}{4}, \rho)$ such that we can find a $w_{\varepsilon} \in W^{1,2}_{\beta}(B^+_{\rho}(y); N)$ which agrees with v in $B^+_{\rho}(y) \backslash B^+_{\sigma}(y)$ and satisfies

$$\begin{aligned} &\sigma^{1-m-\beta} \int_{B^+_{\sigma}(y)} x^{\beta}_{m+1} |\nabla w|^2 \mathrm{d}x \\ &\leq C \varepsilon \rho^{1-m-\beta} \int_{B^+_{\rho}(y)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x + \frac{1}{\varepsilon} C \rho^{-(1+m+\beta)} \int_{B^+_{\rho}(y)} x^{\beta}_{m+1} |v - \overline{v}_{B^+_{\rho}(y), \beta}|^2 \mathrm{d}x. \end{aligned} \tag{4.56}$$

Assuming (4.55) and consequently (4.56) hold, we make use of the comparison property of w. Since $v = w$ in $B^+_{\rho}(y) \backslash B^+_{\sigma}(y)$ we may extend w to an element of $\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N)$ by requiring $w = v$ on $\mathbb{R}^{m+1}_+ \backslash B^+_{\rho}(y)$. As v is a minimiser of E^{β} relative to $\mathcal{O}$, we deduce that $\int_{B^+_{\sigma}(y)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x \leq \int_{B^+_{\sigma}(y)} x^{\beta}_{m+1} |\nabla w|^2 \mathrm{d}x$. Combining this fact with the monotonicity formula, Lemma 4.2.1, and (4.56) gives

$$\left(\frac{3\rho}{4}\right)^{1-m-\beta}\int_{B^+_{\frac{3\rho}{4}}(y)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x$$
$$\leq C\varepsilon\rho^{1-m-\beta}\int_{B^+_\rho(y)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x + \frac{1}{\varepsilon}C\rho^{-(1+m+\beta)}\int_{B^+_\rho(y)} x_{m+1}^{\beta}|v-\overline{v}_{B^+_\rho(y),\beta}|^2\mathrm{d}x. \tag{4.57}$$

Fix $\varepsilon = \min\{\frac{1}{4}, \frac{1}{4C}\}$, where C is the constant in (4.57) and let $\varepsilon_0 \leq \frac{1}{C}\delta_0^2\varepsilon^{1+m+\frac{\beta}{2}+\frac{|\beta|}{2}}$ where C is the constant from (4.54). It follows from (4.54) that (4.55) is satisfied and hence, substituting this ε into (4.57), we have

$$\left(\frac{3\rho}{4}\right)^{1-m-\beta}\int_{B^+_{\frac{3\rho}{4}}(y)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x$$
$$\leq \frac{1}{4}\rho^{1-m-\beta}\int_{B^+_\rho(y)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x + \hat{C}\rho^{-(1+m+\beta)}\int_{B^+_\rho(y)} x_{m+1}^{\beta}|v-\overline{v}_{B^+_\rho(y),\beta}|^2\mathrm{d}x \tag{4.58}$$

for a constant $\hat{C}$ and any $\rho \leq r \leq \frac{R}{2}$. Observe that

$$\left|\int_{B^+_r(y)} x_{m+1}^{\beta}\langle\nabla v, \nabla\phi\rangle\,\mathrm{d}x\right| = \left|\int_{B^+_r(y)} x_{m+1}^{\beta}\langle\phi, A(v)(\nabla v, \nabla v)\rangle\,\mathrm{d}x\right| \leq c_0\int_{B^+_r(y)} x_{m+1}^{\beta}|\phi||\nabla v|^2\mathrm{d}x,$$

for every $\phi \in C_0^\infty(B_r(y);\mathbb{R}^n)$ on every $B^+_r(y) \subset B^+_R(x_0)$ where $c_0 = c_0(m,N)$. Hence, we may apply Lemma 4.9.1 for δ, c_0 as above to obtain a corresponding $\varepsilon_1 > 0$ and $\theta_1 \in (0, \frac{1}{4}]$ such that if $r^{1-m-\beta}\int_{B^+_r(y)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x \leq \varepsilon_1$ then

$$(\theta_1 r)^{-(1+m+\beta)}\int_{B^+_{\theta_1 r}(y)} x_{m+1}^{\beta}|v-\overline{v}_{B^+_{\theta_1 r}(y),\beta}|^2\mathrm{d}x \leq \frac{1}{4\hat{C}}r^{1-m-\beta}\int_{B^+_r(y)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x. \tag{4.59}$$

Now choose $\varepsilon_0 = \frac{1}{C}\min\{\delta_0^2\varepsilon^{1+m+\frac{\beta}{2}+\frac{|\beta|}{2}}, \varepsilon_1\}$ where C is the constant from (4.54). It follows that (4.58) and (4.59) hold on any $B^+_\rho(y) \subset B^+_r(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$. Thus we may apply (4.58) with $\rho = \theta_1 r$. In turn, assuming this choice of ρ, we combine (4.58) with the monotonicity formula and (4.59) to see that

$$\left(\frac{3\theta_1 r}{4}\right)^{1-m-\beta}\int_{B^+_{\frac{3\theta_1 r}{4}}(y)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x \leq \frac{1}{2}r^{1-m-\beta}\int_{B^+_r(y)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x.$$

Hence the lemma is proved with the above choice of ε_0 and $\theta_0 = \frac{3\theta_1}{4}$. □

4.11 ε-regularity as far as Hölder Continuity

The culmination of the results in this section so far lead to the following ε-regularity theorem for minimisers of E^β relative to $\mathcal{O}$ which establishes the first part of Theorem 4.0.1.

Theorem 4.11.1. *If $m \geq 3$, let $\beta \in (-1,1)$ and if $m=2$ let $\beta \in (-3^{-1},1)$. Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N)$ be a minimiser of E^β relative to $\mathcal{O}$. Suppose $B^+_R(x_0)$ satisfies $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$. There exists an $\varepsilon = \varepsilon(m,N,\beta) > 0$ and a $\theta = \theta(m,N,\beta) \in (0,1)$ such that if*

$$R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \leq \varepsilon,$$

then $v \in C^{0,\gamma}(\overline{B^+_{\theta R}(x_0)};N)$ for some $\gamma = \gamma(m,N,\beta) \in (0,1)$. In particular,

$$|v(x_1)-v(x_2)| \leq C\left(R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}}\left(\frac{|x_1-x_2|}{R}\right)^\gamma \tag{4.60}$$

for every $x_1, x_2 \in B^+_{\theta R}(x_0)$ and a constant $C = C(m,N,\beta)$.

Proof. Throughout the proof we adopt the convention that all constants depend only on m, N and β unless stated otherwise. We reinforce this dependence where appropriate.

Let v be a minimiser of E^β relative to $\mathcal{O}$ with $R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \leq \varepsilon$ for $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$, where ε_0 is the number from Lemma 4.10.1 and ε_1 is the number from lemma 4.4.1.

Observe that the function $\tilde{r} \mapsto \tilde{r}^{1-m-\beta}\int_{B^+_{\tilde{r}}(z)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x$ is non-decreasing on $(0, \frac{R}{2}]$ by the monotonicity formula, Lemma 4.2.1. Furthermore, the choice of ε allows us to apply Lemma 4.10.1. We apply this lemma, together with Lemma 8.23 of [16] to deduce that on every $B^+_{\tilde{r}}(z) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$ we have

$$\begin{aligned}\tilde{r}^{1-m-\beta}\int_{B^+_{\tilde{r}}(z)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x &\leq C\left(2\frac{\tilde{r}}{R}\right)^{\gamma_0}\left(\frac{R}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{R}{2}}(z)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \\ &\leq C\left(\frac{\tilde{r}}{R}\right)^{\gamma_0} R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \end{aligned} \tag{4.61}$$

for a constant C and a $\gamma_0 \in (0,1)$ which depend on m, N, β and θ_0, and hence only on m, N, β.

Our choice of ε also permits the application of Lemma 4.4.1; this lemma implies that for any $B_r(y) \in \mathcal{B}_{\theta_1}(x_0, R, \frac{R}{3})$, with $\theta_1 \geq 2$ given by the lemma, and any $0 < \rho \leq r$ we have

$$\rho^{1-m}\int_{B_\rho(y)} |\nabla v|^2 \,\mathrm{d}x \leq C\left(\frac{\rho}{r}\right)^{\gamma_1} r^{1-m}\int_{B_r(y)} |\nabla v|^2 \,\mathrm{d}x \tag{4.62}$$

for some $\gamma_1 \in (0,1)$. Since $\theta_1 \geq 2$, for any $B_r(y) \in \mathcal{B}_{\theta_1}(x_0, R, \frac{R}{3})$ we have the inclusions

$$B_r(y) \subset B_{\frac{y_{m+1}}{\theta_1}}(y) \subset B^+_{\left(\frac{\theta_1+1}{\theta_1}\right)y_{m+1}}(y^+) \subset B^+_{\frac{3y_{m+1}}{2}}(y^+) \in \mathcal{B}^+\left(x_0, R, \frac{R}{2}\right), \tag{4.63}$$

where $y^+ = y - (0, y_{m+1})$. It follows, applying (4.6), that

$$\left(\frac{y_{m+1}}{\theta_1}\right)^{1-m}\int_{B_{\frac{y_{m+1}}{\theta_1}}(y)} |\nabla v|^2 \,\mathrm{d}x \leq C\left(\frac{(\theta_1+1)y_{m+1}}{\theta_1}\right)^{1-m-\beta}\int_{B^+_{\left(\frac{\theta_1+1}{\theta_1}\right)y_{m+1}}(y^+)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x, \tag{4.64}$$

where C depends on m, N, β and θ_1 and thus only on m, N, β. We combine (4.62), applied with $r = \frac{y_{m+1}}{\theta_1}$, with (4.63), (4.64) and (4.61), applied on $B^+_{\tilde{r}}(z)$ with $\tilde{r} = \left(\frac{\theta_1+1}{\theta_1}\right) y_{m+1}$ and $z = y^+$. It follows, after defining $\hat{\gamma} = \min\{\gamma_0, \gamma_1\}$, that

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \,\mathrm{d}x \leq C \left(\frac{\rho}{R}\right)^{\hat{\gamma}} R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x. \tag{4.65}$$

This holds for any $B_\rho(y) \in \mathcal{B}_{\theta_1}(x_0, R, \frac{R}{3})$. Since (4.61) holds on every $B^+_{\tilde{r}}(z) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$ for γ_0, it holds on every $B^+_{\tilde{r}}(z) \in \mathcal{B}^+(x_0, R, \frac{R}{3})$ for $\hat{\gamma}$. We deduce from (4.61) and (4.65) that the hypothesis of Lemma 4.3.1 are satisfied; applying this lemma concludes the proof. □

Remark 4.11.1. Once we know that a minimiser of E^β relative to $\mathcal{O}$ is Hölder continuous in some $B^+_R(x_0)$ with $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$, known theory for harmonic maps, see Lemma 4.4.4 which is from [38], implies that v is smooth in $B^+_R(x_0)$. However, this theory does not imply v is smooth up to $\partial^0 B^+_R(x_0)$; henceforth our goal is essentially to prove this fact.

4.12 An L^∞ Bound for the Gradient

The first step in our proof of a higher partial regularity of minimising fractional harmonic maps consists of establishing an L^∞ bound for the gradients to solutions of systems of semi-linear equations with growth conditions satisfied by minimisers v of E^β relative to $\mathcal{O}$ and their derivatives $D^{\alpha'} v$ where $\alpha' \in \mathbb{N}_0^{m+1}$ is a multi-index with $\alpha'_{m+1} = 0$. The method of proof is that of harmonic replacement; compare the growth of the average Dirichlet energy of solutions ot the semi-linear equations with that of solutions to the linearised system. We follow [38] for example. The monotonicity formulas established in Theorem 2.4.1 and Lemma 2.5.2 are a key ingredient of the proof.

We will use the notation $|\Omega|_\beta = \int_\Omega |x_{m+1}|^\beta \mathrm{d}x$ and $|\Omega| = \int_\Omega \mathrm{d}x$ for $\Omega \subset \mathbb{R}^{m+1}$.

Lemma 4.12.1. *Suppose* $v \in W^{1,2}_\beta(B^+_R(x_0); \mathbb{R}^n) \cap C^{0,\gamma}(\overline{B^+_R(x_0)}; \mathbb{R}^n)$ *where* $B^+_R(x_0)$ *is a half-ball with* $R \leq 1$ *and* $\gamma \in (0,1)$*. Suppose* v *satisfies*

$$\int_{B^+_R(x_0)} x^\beta_{m+1} \langle \nabla v, \nabla \psi \rangle \,\mathrm{d}x = \int_{B^+_R(x_0)} x^\beta_{m+1} \langle \psi, G(x, \nabla v) \rangle \,\mathrm{d}x$$

for every $\psi \in C^\infty_0(B_R(x_0); \mathbb{R}^n)$*, where* $G : \mathbb{R}^m \times \mathbb{R}^{(m+1)n}$ *is measurable and* $|G(x,q)| \leq C_1 |q|^2 + C_2$ *for a positive* $C_1 \leq C^*$ *and non-negative* $C_2 \leq C^*$ *for some* $C^* > 0$*. Then* $\nabla v \in L^\infty(B^+_{\frac{R}{3}}(x_0); \mathbb{R}^{(m+1)n})$ *and, in particular, we have*

$$||\nabla v||^2_{L^\infty(B^+_{\frac{R}{3}}(x_0); \mathbb{R}^{(m+1)n})} \leq C_3 \frac{1}{|B^+_R(x_0)|_\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x + C_4 \tilde{C}_2$$

where $C_3 = C_3(m, N, \beta, C^*)$*,* $C_4 = C_4(m, N, \beta, C^*)$ *and* $\tilde{C}_2 = C_2^{\frac{1}{2}} + C_2$*. In particular, if* $C_2 = 0$ *then then* $\tilde{C}_2 = 0$*.*

Proof. Without relabelling, we reflect v evenly across the hyperplane $\partial \mathbb{R}^{m+1}_+$. It follows that $v \in C^{0,\gamma}(\overline{B_R(x_0)}; \mathbb{R}^n) \cap W^{1,2}_\beta(B_R(x_0); \mathbb{R}^n)$ is a weak solution of $\mathrm{div}(|x_{m+1}|^\beta \nabla v) + |x_{m+1}|^\beta \tilde{G} = 0$ in $B_R(x_0)$, where $\tilde{G}$ is measurable and $|\tilde{G}(x,q)| \leq C_1 |q|^2 + C_2$. We will derive estimates on classes of

$B_\rho(y)$ with $y \in B_{\frac{R}{2}}(x_0)$ and $\rho \leq \frac{R}{2}$. We focus initially on an estimate for the average energy on $B_{\frac{\rho}{2}}(y)$ in terms of that on $B_\rho(y)$. Since v is even with respect to $\partial\mathbb{R}^{m+1}_+$ we only need to consider $B_\rho(y)$ with $y_{m+1} \geq 0$. We consider two cases, $y_{m+1} = 0$ and $y_{m+1} > 0$.

Suppose $B_\rho(y)$ is such that $B^+_\rho(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$. An application of Minkowski's inequality, for maps in $L^2_\beta(B_{\frac{\rho}{2}}(y); \mathbb{R}^{n(m+1)})$, yields

$$\left(\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta}\int_{B_{\frac{\rho}{2}}(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq \left(\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta}\int_{B_{\frac{\rho}{2}}(y)} |x_{m+1}|^\beta \left|\nabla w\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\frac{C}{|B_\rho(y)|_\beta}\int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla(v-w)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{4.66}$$

for any $w \in W^{1,2}_\beta(B_\rho(y); \mathbb{R}^n)$. Let $w \in W^{1,2}_\beta(B_\rho(y); \mathbb{R}^n)$ be the weak solution of $\operatorname{div}(|x_{m+1}|^\beta \nabla w) = 0$ in $B_\rho(y)$ with $w = v$ on $\partial B_\rho(y)$, given by Lemma 2.6.1. Then w is smooth in $B_\rho(y)\backslash\partial\mathbb{R}^{m+1}_+$ and continuous in $\overline{B_\rho(y)}$. Furthermore, since v is symmetric with respect to $\partial\mathbb{R}^{m+1}_+$, it follows from Lemma 2.6.2 that w is symmetric with respect to $\partial\mathbb{R}^{m+1}_+$ and, crucially, we are now free to apply Theorem 2.4.1 to w for every $\beta \in (-1, 1)$.

As $w - v \in C(\overline{B_\rho(y)}; \mathbb{R}^n) \cap W^{1,2}_{\beta,0}(B_\rho(y); \mathbb{R}^n)$ and v satisfies $\operatorname{div}(|x_{m+1}|^\beta \nabla v) + |x_{m+1}|^\beta \tilde{G} = 0$ and w satisfies $\operatorname{div}(|x_{m+1}|^\beta \nabla w) = 0$ weakly in $B_\rho(y)$, we calculate

$$\begin{aligned}\int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla(v-w)|^2 \mathrm{d}x &= \int_{B_\rho(y)} |x_{m+1}|^\beta \left\langle v - w, \tilde{G}\right\rangle \mathrm{d}x \\ &\leq C_1 \sup_{B_\rho(y)} |v-w| \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x \\ &\quad + C_2 \sup_{B_\rho(y)} |v-w| \int_{B_\rho(y)} |x_{m+1}|^\beta \mathrm{d}x. \end{aligned} \tag{4.67}$$

The Hölder continuity of v, together with the weak maximum and minimum principles given by Lemma 2.6.1 imply

$$\sup_{B_\rho(y)} |v - w| \leq C\rho^\gamma. \tag{4.68}$$

Next we use the monotonicity and minimising properties of w to scale its averaged energy. An application of Theorem 2.4.1, followed by an application of Lemma 2.6.1 yields

$$\begin{aligned}\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta}\int_{B_{\frac{\rho}{2}}(y)} |x_{m+1}|^\beta |\nabla w|^2 \mathrm{d}x &\leq \frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla w|^2 \mathrm{d}x \\ &\leq \frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x. \end{aligned} \tag{4.69}$$

Combining (4.66) - (4.69) we see that

$$\left(\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta}\int_{B_{\frac{\rho}{2}}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\right)^{\frac{1}{2}} \leq \left(\frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\right)^{\frac{1}{2}} + C\left(C_1\rho^\gamma\frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x + C_2\rho^\gamma\right)^{\frac{1}{2}}.$$

Define $\tilde{C}_1 = (C_1+C_2)^{\frac{1}{2}} + C_1$ and $\tilde{C}_2 = C_2^{\frac{1}{2}} + C_2$. We square both sides of the preceding inequality, using Young's inequality ($ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ for $a, b \geq 0$) and the fact that $\gamma \in (0,1)$ and $\rho \leq R \leq 1$ to see that

$$\begin{aligned}
&\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta}\int_{B_{\frac{\rho}{2}}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x \\
&\leq \frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x \\
&\quad + C\left(C_1\rho^\gamma\left(\frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\right)^2 + C_2\rho^\gamma\frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad + C\left(C_1\rho^\gamma\frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x + C_2\rho^\gamma\right) \\
&\leq \frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x \\
&\quad + C\left((C_1+C_2)\rho^\gamma\left(\frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\right)^2 + C_2\rho^\gamma\right)^{\frac{1}{2}} \\
&\quad + CC_1\rho^\gamma\frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x + CC_2\rho^\gamma \\
&\leq (1 + C\tilde{C}_1\rho^{\frac{\gamma}{2}})\frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x + C\tilde{C}_2\rho^{\frac{\gamma}{2}}.
\end{aligned} \tag{4.70}$$

This holds on every $B_\rho(y)$ with $B_\rho^+(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$.

Now we iterate this estimate on concentric balls. Consider $B_{\frac{R}{2}}(y)$ with $B^+_{\frac{R}{2}}(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$. Let $\rho_k = 2^{-k}\frac{R}{2}$ for $k \in \mathbb{N}_0$. First notice that $\rho_k^{\frac{\gamma}{2}} \leq 2^{-\frac{k\gamma}{2}}$. Hence

$$\prod_{j=0}^\infty\left(1 + C\tilde{C}_1\rho_j^{\frac{\gamma}{2}}\right) \leq \prod_{j=0}^\infty\left(1 + C((C^*)^{\frac{1}{2}} + C^*)2^{-\frac{j\gamma}{2}}\right) \leq \tilde{C} < \infty$$

where $\tilde{C}$ depends on m, N, β and C^*. It follows from (4.70) that, for every $k \geq 1$, we have

$$\begin{aligned}\frac{1}{|B_{\rho_k}(y)|_\beta}\int_{B_{\rho_k}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x &\leq \prod_{j=1}^{k}\left(1+C\tilde{C}_1\rho_{k-j}^{\frac{\gamma}{2}}\right)\frac{1}{|B_{\frac{R}{2}}(y)|_\beta}\int_{B_{\frac{R}{2}}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\\ &\quad + C\tilde{C}_2\sum_{j=1}^{k}\rho_{k-j}^{\frac{\gamma}{2}}\prod_{l=1}^{j-1}\left(1+C\tilde{C}_1\rho_{k-l}^{\frac{\gamma}{2}}\right)\\ &\leq C\tilde{C}\frac{1}{|B_R(x_0)|_\beta}\int_{B_R(x_0)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x + C\tilde{C}_2\tilde{C}\sum_{j=1}^{k}2^{\frac{\gamma}{2}(j-k)}\\ &\leq C\tilde{C}\frac{1}{|B_R^+(x_0)|_\beta}\int_{B_R^+(x_0)}x_{m+1}^\beta|\nabla v|^2\mathrm{d}x + C\tilde{C}_2\tilde{C}. \end{aligned}\tag{4.71}$$

Now we consider estimates on $B_\rho(y)$ with $B_{\frac{y_{m+1}}{2}}(y) \in \mathcal{B}(x_0, R, \frac{R}{3})$; on this class of ball we have

$$B_\rho(y) \subset B_{\frac{y_{m+1}}{2}}(y) \subset B^+_{\frac{3}{2}y_{m+1}}(y^+) \subset B^+_{\frac{R}{2}}(y^+) \in \mathcal{B}^+(x_0, R, \frac{R}{2}), \tag{4.72}$$

where $y^+ = y - (y', y_{m+1})$. Let $w \in W^{1,2}_\beta(B_\rho(y); \mathbb{R}^n)$ be the weak solution of $\mathrm{div}(|x_{m+1}|^\beta \nabla w) = 0$ in $B_\rho(y)$ with $w = v$ on $\partial B_\rho(y)$, given by Lemma 2.6.1 and suppose $\theta \geq 2$ is such that $y_{m+1} \geq \theta\rho$. Then Lemma 2.5.2 yields

$$\frac{1}{|B_{\frac{\rho}{2}}(y)|}\int_{B_{\frac{\rho}{2}}(y)}x_{m+1}^\beta|\nabla w|^2\mathrm{d}x \leq (1+\frac{C}{\theta-1})\frac{1}{|B_\rho(y)|}\int_{B_\rho(y)}x_{m+1}^\beta|\nabla w|^2\mathrm{d}x.$$

Hence, repeating (4.66)-(4.69) but with $|B_{\frac{\rho}{2}}(y)|_\beta$ replaced by $|B_{\frac{\rho}{2}}(y)|$, we find

$$\begin{aligned}\left(\frac{1}{|B_{\frac{\rho}{2}}(y)|}\int_{B_{\frac{\rho}{2}}(y)}x_{m+1}^\beta|\nabla v|^2\mathrm{d}x\right)^{\frac{1}{2}} &\leq \left((1+\frac{C}{\theta-1})\frac{1}{|B_\rho(y)|}\int_{B_\rho(y)}x_{m+1}^\beta|\nabla v|^2\mathrm{d}x\right)^{\frac{1}{2}}\\ &\quad + C\left(C_1\rho^\gamma\frac{1}{|B_\rho(y)|}\int_{B_\rho(y)}x_{m+1}^\beta|\nabla v|^2\mathrm{d}x + C_2\rho^\gamma y_{m+1}^\beta\right)^{\frac{1}{2}}.\end{aligned}$$

We square both sides of this inequality analogously to (4.70), noting that $\theta \geq 2$ so $\frac{1}{\theta-1} \leq 1$, $\gamma \in (0,1)$ and $\rho \leq R \leq 1$, to see that

$$\begin{aligned}&\frac{1}{|B_{\frac{\rho}{2}}(y)|}\int_{B_{\frac{\rho}{2}}(y)}x_{m+1}^\beta|\nabla v|^2\mathrm{d}x\\ &\leq (1+C\tilde{C}_1\rho^{\frac{\gamma}{2}}+\frac{C}{\theta-1})\frac{1}{|B_\rho(y)|}\int_{B_\rho(y)}x_{m+1}^\beta|\nabla v|^2\mathrm{d}x + C\tilde{C}_2\rho^{\frac{\gamma}{2}}y_{m+1}^\beta.\end{aligned}\tag{4.73}$$

This holds on every $B_\rho(y)$ with $B_{\frac{y_{m+1}}{2}}(y) \in \mathcal{B}(x_0, R, \frac{R}{3})$ and $y_{m+1} \geq \theta\rho$.

We iterate this estimate on concentric balls. Consider $B_{\frac{y_{m+1}}{2}}(y) \in \mathcal{B}(x_0, R, \frac{R}{3})$. Let $\rho_k = 2^{-k}\frac{y_{m+1}}{2}$ for $k \in \mathbb{N}_0$ and observe that $y_{m+1} \geq 2^{k+1}\rho_k$ and $\rho_k^{\frac{\gamma}{2}} \leq 2^{-\frac{k\gamma}{2}}$. Observe

$$\prod_{j=0}^{\infty}\left(1+C\tilde{C}_1\rho_j^{\frac{\gamma}{2}}+\frac{C}{2^{j+1}-1}\right) \leq \prod_{j=0}^{\infty}\left(1+C((C^*)^{\frac{1}{2}}+C^*)2^{-\frac{j\gamma}{2}}+\frac{C}{2^{j+1}-1}\right) \leq \hat{C} < \infty$$

where $\hat{C}$ depends on m, N, β and C^*. It follows from (4.73) that, for every $k \geq 1$, we have

$$\begin{aligned}
&\frac{1}{|B_{\rho_k}(y)|}\int_{B_{\rho_k}(y)} x_{m+1}^{\beta}|\nabla v|^2 \mathrm{d}x \\
&\leq \prod_{j=1}^{k}\left(1 + C\tilde{C}_1\rho_{k-j}^{\frac{\gamma}{2}} + \frac{C}{2^{k-j+1}-1}\right)\frac{1}{|B_{\frac{y_{m+1}}{2}}(y)|}\int_{B_{\frac{y_{m+1}}{2}}(y)} x_{m+1}^{\beta}|\nabla v|^2 \mathrm{d}x \\
&\quad + C\tilde{C}_2 y_{m+1}^{\beta}\sum_{j=1}^{k}\rho_{k-j}^{\frac{\gamma}{2}}\prod_{l=1}^{j-1}\left(1 + C\tilde{C}_1\rho_{k-l}^{\frac{\gamma}{2}} + \frac{C}{2^{k-l+1}-1}\right) \\
&\leq \hat{C}\frac{1}{|B_{\frac{y_{m+1}}{2}}(y)|}\int_{B_{\frac{y_{m+1}}{2}}(y)} x_{m+1}^{\beta}|\nabla v|^2 \mathrm{d}x + C\tilde{C}_2\hat{C}y_{m+1}^{\beta}.
\end{aligned} \tag{4.74}$$

Now fix $y \in B^+_{\frac{R}{3}}(x_0)$ which implies $B_{\frac{y_{m+1}}{2}}(y) \in \mathcal{B}(x_0, R, \frac{R}{3})$. We divide (4.74) by y_{m+1}^{β}, let $y^+ = y - (0, y_{m+1})$ and combine (4.71), (4.72) and (4.74) to see that

$$\begin{aligned}
\frac{y_{m+1}^{-\beta}}{|B_{\rho_k}(y)|}\int_{B_{\rho_k}(y)} x_{m+1}^{\beta}|\nabla v|^2 \mathrm{d}x &\leq C\hat{C}\frac{y_{m+1}^{-\beta}}{|B_{\frac{y_{m+1}}{2}}(y)|}\int_{B_{\frac{y_{m+1}}{2}}(y)} x_{m+1}^{\beta}|\nabla v|^2 \mathrm{d}x + C\tilde{C}_2\hat{C} \\
&\leq C\hat{C}\frac{1}{|B_{\frac{3y_{m+1}}{2}}(y^+)|_{\beta}}\int_{B_{\frac{3y_{m+1}}{2}}(y^+)} |x_{m+1}|^{\beta}|\nabla v|^2 \mathrm{d}x + C\tilde{C}_2\hat{C} \\
&\leq C\tilde{C}\hat{C}\frac{1}{|B^+_R(x_0)|_{\beta}}\int_{B^+_R(x_0)} x_{m+1}^{\beta}|\nabla v|^2 \mathrm{d}x + C(\hat{C} + \hat{C}\tilde{C})\tilde{C}_2.
\end{aligned}$$

An application of Lebesgue's differentiation theorem concludes the proof. □

Remark 4.12.1. A consequence of the preceding lemma is that Hölder continuous weak solutions of $\operatorname{div}(x_{m+1}^{\beta}\nabla v) + x_{m+1}^{\beta}G = 0$ in $B^+_R(x_0)$ and $x_{m+1}^{\beta}\partial_{m+1}v = 0$ in $\partial^0 B^+_R(x_0)$, with G satisfying the assumptions of the lemma on $B^+_R(x_0)$, are actually Lipschitz continuous on $B^+_{\frac{R}{3}}(x_0)$.

4.13 Existence of Higher Order Derivatives

The existence of higher order derivatives of minimisers of E^{β} relative to $\mathcal{O}$ in directions tangential to $\partial\mathbb{R}^{m+1}_+$ follows using the usual method of difference quotients.

Lemma 4.13.1. *Fix $l \in \mathbb{N}_0$. Suppose v is a minimiser of E^{β} relative to $\mathcal{O}$ and let $B^+_R(x_0)$ be a half-ball with $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$. Suppose further that for every multi-index $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| \leq l$ we have $D^{\alpha'}v \in C^{0,\gamma}(\overline{B^+_R(x_0)}; \mathbb{R}^n) \cap W^{1,2}_{\beta}(B^+_R(x_0); \mathbb{R}^n)$ for some $\gamma \in (0,1)$ and $\nabla D^{\alpha'}v \in L^{\infty}(B^+_R(x_0); \mathbb{R}^{(m+1)n})$. Then for $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$ and $i = 1, \ldots, m$, the weak derivative $\nabla\partial_i D^{\alpha'}v$ exists and $\nabla\partial_i D^{\alpha'}v \in L^2_{\beta}(B^+_{\frac{R}{2}}(x_0); \mathbb{R}^{(m+1)n})$.*

Proof. Without relabelling, we extend A to a smooth section of $T^*\mathbb{R}^n \otimes T^*\mathbb{R}^n \otimes T\mathbb{R}^n$. Fix $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$. Since v is a minimiser of E^{β} relative to $\mathcal{O}$, the regularity assumptions on v and $D^{\alpha'}v$ imply we may integrate by parts l times in (4.1); for any $\phi \in C_0^\infty(B_R(x_0);\mathbb{R}^n)$ we have

$$\int_{B_R^+(x_0)} x_{m+1}^\beta \langle \nabla D^{\alpha'} v, \nabla\phi\rangle \mathrm{d}x = \int_{B_R^+(x_0)} x_{m+1}^\beta \langle \phi, D^{\alpha'}(A(v)(\nabla v, \nabla v))\rangle \mathrm{d}x. \tag{4.75}$$

Let $\eta \in C_0^\infty(B_{\frac{3R}{4}}(x_0))$ be a smooth cutoff function such that $\eta \equiv 1$ in $B_{\frac{R}{2}}(x_0)$, $1 \geq \eta \geq 0$ in $B_{\frac{3R}{4}}(x_0)\backslash B_{\frac{R}{2}}(x_0)$ and $|\nabla\eta| \leq \frac{C}{R}$. Furthermore, let $\Delta_i^h D^{\alpha'}v = h^{-1}(D^{\alpha'}v(x+he_i) - D^{\alpha'}v(x))$ be the difference quotient of $D^{\alpha'}v$ and assume $|h| < \frac{R}{4}$. Observe that, by approximation, $w = -\Delta_i^{-h}(\eta^2\Delta_i^h D^{\alpha'}v)$ is an admissible test function for (4.75). We substitute w into (4.75) and apply 'integration by parts' for difference quotients to see that

$$\begin{aligned}\int_{B_R^+(x_0)} \eta^2 x_{m+1}^\beta |\Delta_i^h \nabla D^{\alpha'} v|^2 \mathrm{d}x = &\int_{B^+_{\frac{3R}{4}}(x_0)} \eta^2 x_{m+1}^\beta \langle \Delta_i^h D^{\alpha'} v, \Delta_i^h D^{\alpha'}(A(v)(\nabla v, \nabla v))\rangle \mathrm{d}x \\ &- \int_{B^+_{\frac{3R}{4}}(x_0)} 2\eta x_{m+1}^\beta \langle \Delta_i^h \nabla D^{\alpha'} v\cdot\nabla\eta, \Delta_i^h D^{\alpha'} v\rangle \mathrm{d}x.\end{aligned} \tag{4.76}$$

We now use Young's inequality, $ab \leq \frac{a^2}{\delta 2} + \delta\frac{b^2}{2}$ for $a, b \geq 0$ and $\delta > 0$, to move all of the terms involving $\Delta_i^h \nabla D^{\alpha'}v$ on the right hand side of (4.76) to the left hand side. We calculate

$$\begin{aligned}-\int_{B^+_{\frac{3R}{4}}(x_0)} 2\eta x_{m+1}^\beta \langle \Delta_i^h \nabla D^{\alpha'} v\cdot\nabla\eta, \Delta_i^h D^{\alpha'} v\rangle \mathrm{d}x \leq &\, C\delta \int_{B_R^+(x_0)} \eta^2 x_{m+1}^\beta |\Delta_i^h \nabla D^{\alpha'} v|^2 \mathrm{d}x \\ &+ \frac{C}{\delta}\int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla\eta|^2 |\Delta_i^h D^{\alpha'} v|^2 \mathrm{d}x.\end{aligned} \tag{4.77}$$

We need to estimate the term involving $\Delta_i^h D^{\alpha'}(A(v)(\nabla v, \nabla v))$ in a similar fashion. An application of the Mean Value Theorem, noting we are working on $B^+_{\frac{3R}{4}}(x_0)$ and $|h| < \frac{R}{4}$, implies

$$|\Delta_i^h D^{\alpha'}(A(v)(\nabla v, \nabla v))| \leq C_1 |\Delta_i^h \nabla D^{\alpha'} v| + C_2$$

where C_1, C_2 depend on m, N, β and $||\nabla D^{\tilde{\alpha}'} v||_{L^\infty(B_R^+(x_0);\mathbb{R}^{(m+1)n})}$ where $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $|\tilde{\alpha}'| \leq |\alpha'|$ and $\tilde{\alpha}'_{m+1} = 0$. Hence, using Young's inequality again, we deduce

$$\begin{aligned}&\int_{B^+_{\frac{3R}{4}}(x_0)} \eta^2 x_{m+1}^\beta \langle \Delta_i^h D^{\alpha'} v, \Delta_i^h D^{\alpha'}(A(v)(\nabla v, \nabla v))\rangle \mathrm{d}x \\ &\leq \delta \int_{B^+_{\frac{3R}{4}}(x_0)} \eta^2 x_{m+1}^\beta |\Delta_i^h \nabla D^{\alpha'} v|^2 \mathrm{d}x + C\frac{C_3}{\delta}\int_{B^+_{\frac{3R}{4}}(x_0)} x_{m+1}^\beta \mathrm{d}x,\end{aligned} \tag{4.78}$$

where C_3 depends on m, N, β and $||\nabla D^{\tilde{\alpha}'} v||_{L^\infty(B_R^+(x_0);\mathbb{R}^{(m+1)n})}$ where $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $|\tilde{\alpha}'| \leq |\alpha'|$ and $\tilde{\alpha}'_{m+1} = 0$. Choosing δ sufficiently small in (4.77) and (4.78) we combine these inequalities with (4.76). Since $|\nabla\eta| \leq \frac{C}{R}$ and $\eta \equiv 1$ in $B^+_{\frac{R}{2}}(x_0)$, we see that

$$\int_{B^+_{\frac{R}{2}}(x_0)} x_{m+1}^\beta |\Delta_i^h \nabla D^{\alpha'} v|^2 \mathrm{d}x \leq C_4(R^{-2}+1)\int_{B_R^+(x_0)} x_{m+1}^\beta \mathrm{d}x.$$

where C_4 depends on m, N, β and $||\nabla D^{\tilde{\alpha}'}v||_{L^\infty(B_R^+(x_0);\mathbb{R}^{(m+1)n})}$ where $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $|\tilde{\alpha}'| \leq |\alpha'|$ and $\tilde{\alpha}'_{m+1} = 0$. This bound is independent of h with $|h| < \frac{R}{4}$. Hence by Lemma 2.4.2 we conclude that the weak derivative $\nabla\partial_i D^{\alpha'}v$ exists and satisfies the above inequality with $\nabla\partial_i D^{\alpha'}v$ in place of $\Delta_i^h \nabla D^{\alpha'}v$. This concludes the proof. □

4.14 Caccioppoli-Type Inequality

Here we show that the derivatives of minimisers of E^β relative to $\mathcal{O}$ in directions tangential to $\partial\mathbb{R}_+^{m+1}$ all satisfy essentially the same Caccioppoli-type inequality.

Lemma 4.14.1. *Fix $l \in \mathbb{N}_0$, let $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ be a minimiser of E^β relative to $\mathcal{O}$ and let $B_R^+(x_0)$ be a half-ball with $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$. Suppose that for every multi-index $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| \leq l$ we have $D^{\alpha'}v \in C^{0,\gamma}(\overline{B_R^+(x_0)}; \mathbb{R}^n) \cap W_\beta^{1,2}(B_R^+(x_0); \mathbb{R}^n)$ for some $\gamma \in (0,1)$ and $\nabla D^{\alpha'}v \in L^\infty(B_R^+(x_0); \mathbb{R}^{(m+1)n})$. Suppose further that for $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$ and some $i \in \{1, \ldots, m\}$, we have $\nabla\partial_i D^{\alpha'}v \in L_\beta^2(B_R^+(x_0); \mathbb{R}^{(m+1)n})$. Let $B_\rho(y) \subset B_R(x_0)$ with $y_{m+1} \geq 0$. For each α' with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$ there are constants $C = C(m, N, \beta)$ and C_1, C_2 which depend on m, N, β and are comprised of polynomial functions, with no constant terms, of $||\nabla D^{\tilde{\alpha}'}v||_{L^\infty(B_\rho(y)\cap\mathbb{R}_+^{m+1};\mathbb{R}^{(m+1)n})}$, where $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $|\tilde{\alpha}'| \leq |\alpha'| = l$ and $\tilde{\alpha}'_{m+1} = 0$, such that*

$$\int_{B_{\frac{\rho}{2}}(y)\cap\mathbb{R}_+^{m+1}} x_{m+1}^\beta |\nabla\partial_i D^{\alpha'}v|^2 \mathrm{d}x \leq C\left(C_1 + \frac{1}{\rho^2}\right)\int_{B_\rho(y)\cap\mathbb{R}_+^{m+1}} x_{m+1}^\beta |\partial_i D^{\alpha'}v - \lambda|^2 \mathrm{d}x + C_2 \int_{B_\rho(y)\cap\mathbb{R}_+^{m+1}} x_{m+1}^\beta \mathrm{d}x \tag{4.79}$$

for any $\lambda \in \mathbb{R}^n$.

Proof. Fix α' with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$. Integrating by parts $l+1$ times in (4.1) shows that for every $\psi \in C_0^\infty(B_R(x_0); \mathbb{R}^n)$, we have

$$\int_{B_R^+(x_0)} x_{m+1}^\beta \langle \nabla\partial_i D^{\alpha'}v, \nabla\psi\rangle \mathrm{d}x = \int_{B_R^+(x_0)} x_{m+1}^\beta \langle \psi, \partial_i D^{\alpha'}(A(v)(\nabla v, \nabla v))\rangle \mathrm{d}x$$

Now, by approximation, we may choose $\psi = \eta^2(\partial_i D^{\alpha'}v - \lambda)$ where $\lambda \in \mathbb{R}^n$ is a constant vector and $\eta \in C_0^\infty(B_\rho(y))$ is a cutoff function with $\eta \equiv 1$ in $B_{\frac{\rho}{2}}(y)$, $0 \leq \eta \leq 1$, and $|\nabla\eta| \leq \frac{C}{\rho}$. We calculate

$$\begin{aligned}\int_{B_\rho(y)\cap\mathbb{R}_+^{m+1}} x_{m+1}^\beta \eta^2 |\nabla\partial_i D^{\alpha'}v|^2 \mathrm{d}x \leq{}& \tilde{C}_1 \int_{B_\rho(y)\cap\mathbb{R}_+^{m+1}} x_{m+1}^\beta \eta^2 |\partial_i D^{\alpha'}v - \lambda||\nabla\partial_i D^{\alpha'}v| \mathrm{d}x \\ &+ \tilde{C}_2 \int_{B_\rho(y)\cap\mathbb{R}_+^{m+1}} x_{m+1}^\beta \eta^2 |\partial_i D^{\alpha'}v - \lambda| \mathrm{d}x \\ &+ C \int_{B_\rho(y)\cap\mathbb{R}_+^{m+1}} x_{m+1}^\beta \eta |\nabla\eta||\partial_i D^{\alpha'}v - \lambda||\nabla\partial_i D^{\alpha'}v| \mathrm{d}x,\end{aligned} \tag{4.80}$$

where $C = C(m,N,\beta)$ and $\tilde{C}_1, \tilde{C}_2$ which depend on m, N, β and are comprised of polynomial functions, with no constant terms, of $||\nabla D^{\tilde{\alpha}'}v||_{L^\infty(B_\rho(y)\cap\mathbb{R}^{m+1}_+;\mathbb{R}^{(m+1)n})}$ where $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $|\tilde{\alpha}'| \leq l$ and $\tilde{\alpha}'_{m+1} = 0$. We apply Young's inequalitiy, $ab \leq \frac{\delta a^2}{2} + \frac{b^2}{\delta 2}$ for $a, b \geq 0$ and $\delta > 0$, to each term on the right hand side of (4.80). We first apply this inequality to the term corresponding to C, choosing δ sufficiently small depending on C and hence only on m, N, β and recalling $|\nabla\eta| \leq \frac{C}{\rho}$ to see that

$$\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x^\beta_{m+1}\eta^2|\nabla\partial_i D^{\alpha'}v|^2\mathrm{d}x \leq \tilde{C}_1\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x^\beta_{m+1}\eta^2|\partial_i D^{\alpha'}v-\lambda||\nabla\partial_i D^{\alpha'}v|\mathrm{d}x$$
$$+\tilde{C}_2\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x^\beta_{m+1}\eta^2|\partial_i D^{\alpha'}v-\lambda|\mathrm{d}x$$
$$+\frac{C}{\rho^2}\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x^\beta_{m+1}|\partial_i D^{\alpha'}v-\lambda|^2\mathrm{d}x \qquad (4.81)$$

where $\tilde{C}_1, \tilde{C}_2$ are possibly different from before but have the same structure and dependence as the constants in (4.80). Applying Young's inequality again, now to the terms in (4.81) corresponding to $\tilde{C}_1, \tilde{C}_2$ concludes the proof. □

4.15 Control of the Mean Squared Oscillation of the Derivatives on the Boundary

We prove an analogue of Lemma 4.9.1 for the derivatives of minimisers of E^β relative to $\mathcal{O}$.

Lemma 4.15.1. *Fix $l \in \mathbb{N}_0$. For every $\delta > 0$ there exist numbers $\varepsilon > 0$, $\tau \in (0,1)$ and $\theta \in (0, \frac{1}{4}]$ such that the following holds. Suppose $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ is a minimiser of E^β relative to $\mathcal{O}$ with $D^{\alpha'}v \in C^{0,\gamma}(\overline{B^+_R(x_0)}; \mathbb{R}^n) \cap W^{1,2}_\beta(B^+_R(x_0); \mathbb{R}^n)$ for a $\gamma \in (0,1)$ and $\nabla D^{\alpha'}v \in L^\infty(B^+_R(x_0); \mathbb{R}^{(m+1)n})$ for every multi-index $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| \leq l$, where $B^+_R(x_0) \subset \mathbb{R}^{m+1}_+$ satisfies $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$. Suppose further that for $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$ and some $i \in \{1, \ldots, m\}$, we have $\nabla\partial_i D^{\alpha'}v \in L^2_\beta(B^+_R(x_0); \mathbb{R}^{(m+1)n})$. If $|\alpha'| = l$ and $\alpha'_{m+1} = 0$ and*

$$R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \leq \varepsilon^2,$$

then, for every $B^+_r(y) \in \mathcal{B}^+(x_0, R, \tau R)$, either

$$r^{1-m-\beta}\int_{B^+_r(y)} x^\beta_{m+1}|\nabla\partial_i D^{\alpha'}v|^2\mathrm{d}x \leq \delta\left(\frac{r}{R^{|\alpha'|+2}}\right)^2 \qquad (4.82)$$

or

$$(\theta r)^{-(1+m+\beta)}\int_{B^+_{\theta r}(y)} x^\beta_{m+1}|\partial_i D^{\alpha'}v - \overline{\partial_i D^{\alpha'}v}_{B^+_{\theta r}(y),\beta}|^2\mathrm{d}x \leq \delta r^{1-m-\beta}\int_{B^+_r(y)} x^\beta_{m+1}|\nabla\partial_i D^{\alpha'}v|^2\mathrm{d}x. \qquad (4.83)$$

Proof. We use a blow-up argument, analogous in spirit to the argument we used in the proof of Lemma 4.9. First we note that the statement of the lemma is invariant under rescaling and translation by any point in $\partial\mathbb{R}^{m+1}_+$. In particular, suppose the lemma holds for minimisers of E^β relative to $\tilde{\mathcal{O}}$ whenever $\overline{\partial^0 B_1^+(0)} \subset \tilde{\mathcal{O}}$. If the hypothesis of the lemma hold for minimisers of E^β relative to $\mathcal{O}$ and $B_R^+(x_0)$ satisfies $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$, then applying the lemma to $v_R := v(R \cdot + x_0)$ yields the conclusion of the lemma on $B_R^+(x_0)$.

We now prove the lemma when $R = 1$, $x_0 = 0$ and v is a minimiser of E^β relative to $\mathcal{O}$ and $\overline{\partial^0 B_1^+(0)} \subset \mathcal{O}$. Suppose the statement is false. Then there exists $\delta > 0$ such that, for any fixed $\theta \in (0, \frac{1}{4}]$, we may find a sequence $(v_k)_{k\in\mathbb{N}}$ of minimisers of E^β relative to $\mathcal{O}$ such that the following holds. Each v_k satisfies $D^{\alpha'} v_k \in C^{0,\gamma}(\overline{B_1^+(0)}; \mathbb{R}^n) \cap W^{1,2}_\beta(B_1^+(0); \mathbb{R}^n)$ for a $\gamma \in (0,1)$ and $\nabla D^{\alpha'} v_k \in L^\infty(B_1^+(0); \mathbb{R}^{(m+1)n})$ for every multi-index $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| \leq l$. For $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$ and a fixed $i = 1, \ldots, m$, each v_k further satisfies $\nabla \partial_i D^{\alpha'} v_k \in L^2_\beta(B_1^+(0); \mathbb{R}^{(m+1)n})$. Moreover, the v_k satisfy

$$\int_{B_1^+(0)} x_{m+1}^\beta \, |\nabla v_k|^2 \, \mathrm{d}x := \varepsilon_k^2 \to 0,$$

and, furthermore, there exists a sequence of numbers $0 < \tau_k \to 0^+$, half-balls $B^+_{r_k}(y_k) \in \mathcal{B}^+(0, 1, \tau_k)$, and numbers $0 < r_k \leq \tau_k \to 0^+$ such that

$$r_k^{1-m-\beta} \int_{B^+_{r_k}(y_k)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v_k|^2 \mathrm{d}x > \delta r_k^2 \tag{4.84}$$

and

$$\begin{aligned}(\theta r_k)^{-(1+m+\beta)} &\int_{B^+_{\theta r_k}(y_k)} x_{m+1}^\beta |\partial_i D^{\alpha'} v_k - \overline{(\partial_i D^{\alpha'} v_k)}_{B^+_{\theta r_k}(y_k),\beta}|^2 \mathrm{d}x \\ > \delta r_k^{1-m-\beta} &\int_{B^+_{r_k}(y_k)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v_k|^2 \mathrm{d}x \end{aligned} \tag{4.85}$$

for α' with $|\alpha'| = l$ and $\alpha'_{m+1} = 0$.

Since each v_k is a minimiser of E^β relative to $\mathcal{O}$ and $v_k \in W^{1,2}_\beta(B_1^+(0); \mathbb{R}^n) \cap C^{0,\gamma}(\overline{B_1^+(0)}; \mathbb{R}^n)$ we deduce from Lemma 4.12.1 that each v_k satisfies

$$||\nabla v_k||^2_{L^\infty(B^+_{\frac{1}{3}}(0);\mathbb{R}^{(m+1)n})} \leq C \int_{B_1^+(0)} x_{m+1}^\beta |\nabla v_k|^2 \mathrm{d}x \leq C\varepsilon_k^2 \to 0. \tag{4.86}$$

The assumptions of the lemma guarantee that we may apply Lemma 4.14.1 on $B^+_{\frac{1}{6}}(0)$ with $\lambda = 0$. We do so and conclude that there are constants $C = C(m, N, \beta)$ and C_1, C_2, which depend on m, N, β and are comprised of a polynomial function, no constant terms, of $||\nabla v_k||_{L^\infty(B^+_{\frac{1}{3}}(0);\mathbb{R}^{(m+1)n})}$ and consequently satisfy $C_1, C_2 \to 0$ as $k \to \infty$, such that for α' with $|\alpha'| = 1$ and $\alpha'_{m+1} = 0$

$$\begin{aligned}\int_{B^+_{\frac{1}{6}}(0)} x_{m+1}^\beta |\nabla D^{\alpha'} v_k|^2 \mathrm{d}x &\leq C\,(C_1 + 1) \int_{B^+_{\frac{1}{3}}(0)} x_{m+1}^\beta |D^{\alpha'} v_k|^2 \mathrm{d}x + C_2 \\ &\leq C\,(C_1+1)\, ||\nabla v_k||^2_{L^\infty(B^+_{\frac{1}{3}}(0);\mathbb{R}^{(m+1)n})} + C_2 \to 0 \end{aligned} \tag{4.87}$$

as $k \to \infty$. If $l \geq 1$, integrating by parts in (4.1), we see that when $|\alpha'| = 1$ and $\alpha'_{m+1} = 0$, $D^{\alpha'} v_k$ satisfies

$$\int_{B^+_{\frac{1}{6}}(0)} x^{\beta}_{m+1} \langle \nabla D^{\alpha'} v_k, \nabla \psi \rangle \mathrm{d}x = \int_{B^+_{\frac{1}{6}}(0)} x^{\beta}_{m+1} \langle \psi, G \rangle \mathrm{d}x$$

for every $\psi \in C^{\infty}_0(B_{\frac{1}{6}}(0); \mathbb{R}^n)$ where, by Young's inequality, $|G(x,q)| \leq C_3 |q|^2 + C_4$ for constants $C_3 = C_3(m, N, \beta)$ and C_4 which depends on m, N, β and is comprised of a polynomial function, with no constant term, of $||\nabla v_k||_{L^{\infty}(B^+_{\frac{1}{3}}(0); \mathbb{R}^{(m+1)n})}$ and hence $C_4 \to 0$ as $k \to \infty$. Now recall that by assumption $D^{\alpha'} v_k \in C^{0,\gamma}(\overline{B^+_{\frac{1}{6}}(0)}; \mathbb{R}^{(m+1)n}) \cap W^{1,2}_{\beta}(B^+_{\frac{1}{6}}(0); \mathbb{R}^{(m+1)n})$ for each $\alpha' \in \mathbb{N}^{m+1}$ with $|\alpha'| = 1$ and $\alpha'_{m+1} = 0$. Hence applying Lemma 4.12.1 again in conjunction with (4.86) and (4.87), we deduce that there exist positive constants $\tilde{C}_3 = \tilde{C}_3(m, N, \beta)$ and $\tilde{C}_4$ which depends on m, N, β and k with $\tilde{C}_4 \to 0$ as $k \to \infty$ such that

$$||\nabla D^{\alpha'} v_k||^2_{L^{\infty}(B^+_{\frac{1}{3}\frac{1}{6}}(0); \mathbb{R}^{(m+1)n})} \leq \tilde{C}_3 \frac{1}{|B^+_{\frac{1}{6}}(0)|_{\beta}} \int_{B^+_{\frac{1}{6}}(0)} x^{\beta}_{m+1} |\nabla D^{\alpha'} v_k|^2 \mathrm{d}x + \tilde{C}_4 \to 0$$

as $k \to \infty$. Repeating the preceding process for $D^{\alpha'} v_k$ with $|\alpha'| = 2$, then $|\alpha'| = 3, \ldots, |\alpha'| = l$, we see that

$$||\nabla D^{\alpha'} v_k||^2_{L^{\infty}(B^+_{\frac{1}{3}6^{-|\alpha'|}}(0); \mathbb{R}^{(m+1)n})} \to 0 \tag{4.88}$$

as $k \to \infty$ for every α' with $|\alpha'| \leq l$ and $\alpha'_{m+1} = 0$.

Now fix α' with $|\alpha'| = l$ and $\alpha'_{m+1} = 0$. Discarding as may v_k as necessary and re-indexing the resulting sequence we may assume that $2\tau_k \leq 6^{-(l+1)}$ so that $B^+_{r_k}(y_k) \subset B^+_{2r_k}(y_k) \subset B^+_{2\tau_k}(y_k) \in \mathcal{B}^+(0, \frac{1}{3}6^{-l}, 6^{-(l+1)})$ and, in particular, $B^+_{2\tau_k}(y_k) \subset B^+_{\frac{1}{3}6^{-l}}(0)$. Define

$$r_k^{1-m-\beta} \int_{B^+_{r_k}(y_k)} x^{\beta}_{m+1} |\nabla \partial_i D^{\alpha'} v_k|^2 \mathrm{d}x := \tilde{\varepsilon}^2_k.$$

Note that it is possible to show, combining Lemma 4.14.1 with $\lambda = 0$ and (4.88), that $\tilde{\varepsilon}^2_k \to 0$, but this is not required in what follows.

We see from (4.84) and (4.85) that

$$\tilde{\varepsilon}^2_k > \delta r^2_k \tag{4.89}$$

and

$$(\theta r_k)^{-(1+m+\beta)} \int_{B^+_{\theta r_k}(y_k)} x^{\beta}_{m+1} |\partial_i D^{\alpha'} v_k - \overline{(\partial_i D^{\alpha'} v_k)}_{B^+_{\theta r_k}(y_k), \beta}|^2 \mathrm{d}x > \delta \tilde{\varepsilon}^2_k. \tag{4.90}$$

Define

$$w_k = \frac{\partial_i D^{\alpha'} v_k(r_k x + y_k) - \overline{(\partial_i D^{\alpha'} v_k)}_{B^+_{\theta r_k}(y_k), \beta}}{\tilde{\varepsilon}_k}$$

Then

$$\nabla w_k(x) = \frac{r_k}{\tilde{\varepsilon}_k} \nabla \partial_i D^{\alpha'} v_k(r_k x + y_k). \tag{4.91}$$

Hence, using the change of variables $x \mapsto r_k x + y_k$, we find

$$\int_{B_1^+(0)} x_{m+1}^\beta \left|\nabla w_k\right|^2 \mathrm{d}x = 1 \quad \text{and} \quad \overline{(w_k)}_{B_\theta^+(0),\beta} = 0. \tag{4.92}$$

Furthermore, after changing variables again, we deduce from (4.90) that

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta \left|w_k\right|^2 \mathrm{d}x > \delta. \tag{4.93}$$

The combination of (4.92) and the Poincaré inequality for A_2 weights implies $(w_k)_{k\in\mathbb{N}}$ is bounded in $W_\beta^{1,2}(B_1^+(0);\mathbb{R}^n)$. The the Rellich Compactness lemma, Lemma 2.2.1, therefore yields a subsequence $(w_{k_j})_{j\in\mathbb{N}}$ which converges weakly in $W_\beta^{1,2}(B_1^+(0);\mathbb{R}^n)$ and strongly in $L_\beta^2(B_1^+(0);\mathbb{R}^n)$ to a $w \in W_\beta^{1,2}(B_1^+(0);\mathbb{R}^n)$.

Now we show that w is a weak solution of the Neumann-type problem (2.11) in $B_1^+(0)$. Let $\phi \in C_0^\infty(B_1(0);\mathbb{R}^n)$ and define $\tilde{\phi} \in C_0^\infty(B_{r_k}(y_k);\mathbb{R}^n)$ by $\tilde{\phi}(z) = \phi\left(\frac{z-y_k}{r_k}\right)$. We observe that $r_k\nabla\tilde{\phi}(z) = \nabla\phi(x)$ where $x \in B_1(0)$ and $z \in B_{r_k}(y_k)$ satisfy $z = r_k x + y_k$. Hence, using the change of variables $x \mapsto r_k x + y_k$ and (4.91), we find

$$\begin{aligned}
\int_{B_1^+(0)} x_{m+1}^\beta \left\langle \nabla w_k, \nabla\phi\right\rangle \mathrm{d}x &= \frac{r_k}{\tilde{\varepsilon}_k}\int_{B_1^+(0)} x_{m+1}^\beta \langle \nabla\partial_i D^{\alpha'} v_k(r_k x + y_k), \nabla\phi(x)\rangle \mathrm{d}x \\
&= \frac{r_k^{-m-\beta}}{\tilde{\varepsilon}_k}\int_{B_{r_k}^+(y_k)} z_{m+1}^\beta \langle \nabla\partial_i D^{\alpha'} v_k, \nabla\phi\left(\frac{z-y_k}{r_k}\right)\rangle \mathrm{d}z \\
&= \frac{r_k^{1-m-\beta}}{\tilde{\varepsilon}_k}\int_{B_{r_k}^+(y_k)} z_{m+1}^\beta \langle \nabla\partial_i D^{\alpha'} v_k, \nabla\tilde{\phi}\rangle \mathrm{d}z.
\end{aligned} \tag{4.94}$$

As $\tilde{\phi} \in C_0^\infty(B_{r_k}(y_k);\mathbb{R}^n)$, v_k is a minimiser of E^β relative to $\mathcal{O}$ and, in view of (4.89), we have $\frac{r_k^2}{\tilde{\varepsilon}_k^2} < \frac{1}{\delta}$, it follows that

$$\begin{aligned}
\left|\int_{B_{r_k}^+(y_k)} z_{m+1}^\beta \langle \nabla\partial_i D^{\alpha'} v_k, \nabla\tilde{\phi}\rangle \mathrm{d}z\right| &\le C\int_{B_{r_k}^+(y_k)} z_{m+1}^\beta (|\nabla\partial_i D^{\alpha'} v_k| + 1)\mathrm{d}z \\
&= C r_k^{1+m+\beta}\int_{B_1^+(0)} x_{m+1}^\beta (|\nabla\partial_i D^{\alpha'} v_k(r_k x + y_k)| + 1)\mathrm{d}x \\
&= C\tilde{\varepsilon}_k r_k^{m+\beta}\int_{B_1^+(0)} x_{m+1}^\beta (|\nabla w_k| + \frac{r_k}{\tilde{\varepsilon}_k})\mathrm{d}x \\
&\le C\tilde{\varepsilon}_k r_k^{m+\beta}\int_{B_1^+(0)} x_{m+1}^\beta (|\nabla w_k| + \delta^{-\frac{1}{2}})\mathrm{d}x,
\end{aligned} \tag{4.95}$$

where C depends on m, N, β, $||\tilde{\phi}||_{L^\infty(B_{r_k}^+(y_k);\mathbb{R}^n)} = ||\phi||_{L^\infty(B_1^+(0);\mathbb{R}^n)}$ and is comprised of a polynomial, with no constant terms, of $||\nabla D^{\tilde{\alpha}'} v_k||_{L^\infty(B^+_{\frac{1}{3}6^{-l}}(0);\mathbb{R}^n)}$ where $|\tilde{\alpha}'| \le l = |\alpha'|$ and $\tilde{\alpha}'_{m+1} = 0$ and is therefore independent of k in view of (4.88).

We combine (4.92) and (4.95) to see that, for any $\phi \in C_0^\infty(B_1(0); \mathbb{R}^n)$, the weak convergence of w_{k_j} to w in $W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n)$ yields

$$\begin{aligned}\left|\int_{B_1^+(0)} x_{m+1}^\beta \langle \nabla w, \nabla \phi \rangle \mathrm{d}x\right| &= \lim_{j\to\infty} \left|\int_{B_1^+(0)} x_{m+1}^\beta \langle \nabla w_{k_j}, \nabla \phi \rangle \mathrm{d}x\right| \\ &\le C \lim_{j\to\infty} r_{k_j} \int_{B_1^+(0)} x_{m+1}^\beta (|\nabla w_{k_j}| + \delta^{-\frac{1}{2}}) \mathrm{d}x \\ &= 0\end{aligned}$$

since $r_{k_j} \to 0$. Hence w is a weak solution of (2.11) in $B_1^+(0)$.

Using the Rellich Compactness Lemma, Lemma 2.2.1, we take limits in (4.92) and (4.93) to see that

$$\int_{B_1^+(0)} x_{m+1}^\beta |\nabla w|^2 \, \mathrm{d}x \le 1 \quad \text{and} \quad \overline{w}_{B_\theta^+(0),\beta} = 0 \tag{4.96}$$

and

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 \, \mathrm{d}x \ge \delta \tag{4.97}$$

respectively. Now, in view of (4.96), the Poincaré inequality for A_2 weights yields

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 \mathrm{d}x \le C\theta^{1-m-\beta} \int_{B_\theta^+(0)} x_{m+1}^\beta |\nabla w|^2 \mathrm{d}x. \tag{4.98}$$

Lastly, since w is a weak solution of (2.11) we may apply Corollary 2.3.1 to w with $\theta \le \frac{1}{4}$ (so that $2\theta \le \frac{1}{2}$). This gives a positive constant C (independent of θ) and a $\gamma \in (0,1)$ such that

$$\theta^{1-m-\beta} \int_{B_\theta^+(0)} x_{m+1}^\beta |\nabla w|^2 \mathrm{d}x \le C(2\theta)^{2\gamma}. \tag{4.99}$$

Combining (4.98) and (4.99) we see that

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 \mathrm{d}x \le C(2\theta)^{2\gamma}. \tag{4.100}$$

This holds for all fixed $\theta \in (0, \frac{1}{4}]$ and we choose $\theta < 2^{-1} \left(\frac{\delta}{C}\right)^{\frac{1}{2\gamma}}$ so that (4.100) contradicts (4.97). Hence the lemma is proved. □

4.16 Control of the Mean Squared Oscillation of the Derivatives in the Interior

We need a counterpart to Lemma 4.15.1 which holds on a class of balls with closure contained in the interior of $\mathbb{R}_+^{m+1}$.

Lemma 4.16.1. *Fix $l \in \mathbb{N}_0$. For every $\delta > 0$ there exist numbers $\varepsilon > 0$, $\tau \in (0,1)$ and $\theta \in (0, \frac{1}{4}]$ such that the following holds. Suppose $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ is a minimiser of E^β relative to $\mathcal{O}$ with $D^{\alpha'} v \in C^{0,\gamma}(\overline{B_R^+(x_0)}; \mathbb{R}^n) \cap W_\beta^{1,2}(B_R^+(x_0); \mathbb{R}^n)$ for a $\gamma \in (0,1)$ and $\nabla D^{\alpha'} v \in L^\infty(B_R^+(x_0); \mathbb{R}^{(m+1)n})$*

for every multi-index $\alpha' \in \mathbb{N}_0^{m+1}$ *with* $\alpha'_{m+1} = 0$ *and* $|\alpha'| \leq l$*, where* $B_R^+(x_0) \subset \mathbb{R}_+^{m+1}$ *satisfies* $R \leq 1$ *and* $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$*. Suppose further that for* $\alpha' \in \mathbb{N}_0^{m+1}$ *with* $\alpha'_{m+1} = 0$ *and* $|\alpha'| = l$ *and some* $i \in \{1, \dots, m\}$*, we have* $\nabla \partial_i D^{\alpha'} v \in L^2_\beta(B_R^+(x_0); \mathbb{R}^{(m+1)n})$*. If* $|\alpha'| = l$ *and* $\alpha'_{m+1} = 0$ *and*

$$R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \leq \varepsilon^2,$$

then, for every $B_r(y) \in \mathcal{B}_4(x_0, R, \tau R)$*, either*

$$r^{1-m} \int_{B_r(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq \delta \left(\frac{r}{R^{|\alpha'|+2}} \right)^2 \tag{4.101}$$

or

$$(\theta r)^{-(1+m)} \int_{B_{\theta r}(y)} |\partial_i D^{\alpha'} v - \overline{\partial_i D^{\alpha'} v}_{B_{\theta r}(y)}|^2 \mathrm{d}x \leq \delta r^{1-m} \int_{B_r(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x. \tag{4.102}$$

Proof. The method of proof is similar to the proof of Lemma 4.15.1. We observe that the lemma is invariant under scaling and translation with respect to x_0 in $\partial \mathbb{R}_+^{m+1}$ in the same way as Lemma 4.15.1. Hence we assume $R = 1$, $x_0 = 0$, v is a minimiser of E^β relative to $\mathcal{O}$ and $\overline{\partial^0 B_1^+(0)} \subset \mathcal{O}$.

Suppose the statement is false. Then there exists $\delta > 0$ such that, for any fixed $\theta \in (0, \frac{1}{4}]$, we may find a sequence $(v_k)_{k \in \mathbb{N}}$ of minimisers of E^β relative to $\mathcal{O}$ such that the following holds. Each v_k satisfies $D^{\alpha'} v_k \in C^{0,\gamma}(\overline{B_1^+(0)}; \mathbb{R}^n) \cap W^{1,2}_\beta(B_1^+(0); \mathbb{R}^n)$ for a $\gamma \in (0,1)$ and $\nabla D^{\alpha'} v_k \in L^\infty(B_1^+(0); \mathbb{R}^{(m+1)n})$ for every multi-index $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| \leq l$. For $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$ and a fixed $i = 1, \dots, m$, each v_k further satisfies $\nabla \partial_i D^{\alpha'} v_k \in L^2_\beta(B_1^+(0); \mathbb{R}^{(m+1)n})$. The v_k also satisfy

$$\int_{B_1^+(0)} x_{m+1}^\beta \left| \nabla v_k \right|^2 \mathrm{d}x := \varepsilon_k^2 \to 0.$$

There furthermore exists a sequence of numbers $0 < \tau_k \to 0$, balls $B_{r_k}(y_k) \in \mathcal{B}_4(0, 1, \tau_k)$, and numbers $0 < r_k \leq \tau_k \to 0$ such that

$$r_k^{1-m} \int_{B_{r_k}(y_k)} |\nabla \partial_i D^{\alpha'} v_k|^2 \mathrm{d}x > \delta r_k^2 \tag{4.103}$$

and

$$\begin{aligned} &(\theta r_k)^{-(1+m)} \int_{B_{\theta r_k}(y_k)} |\partial_i D^{\alpha'} v_k - \overline{(\partial_i D^{\alpha'} v_k)}_{B_{\theta r_k}(y_k)}|^2 \mathrm{d}x \\ &> \delta r_k^{1-m} \int_{B_{r_k}(y_k)} |\nabla \partial_i D^{\alpha'} v_k|^2 \mathrm{d}x \end{aligned} \tag{4.104}$$

for α' with $|\alpha'| = l$ and $\alpha'_{m+1} = 0$.

Since the assumptions of the lemma are the same as the assumptions of Lemma 4.15.1 we still have (4.88), namely, for every α' with $|\alpha'| \leq l$ and $\alpha'_{m+1} = 0$

$$||\nabla D^{\alpha'} v_k||^2_{L^\infty(B^+_{\frac{1}{3}6^{-|\alpha'|}}(0); \mathbb{R}^{(m+1)n})} \to 0. \tag{4.105}$$

Define

$$r_k^{1-m}\int_{B_{r_k}(y_k)}|\nabla\partial_i D^{\alpha'}v_k|^2\mathrm{d}x := \tilde{\varepsilon}_k^2.$$

Fix α' with $|\alpha'| = l$ and $\alpha'_{m+1} = 0$. Discarding as many k as necessary, and re-indexing the resulting sequence to $k \in \mathbb{N}$, we may assume $2\tau_k \leq 6^{-(l+1)}$ so that $B_{r_k}(y_k) \in \mathcal{B}_4(0, \frac{1}{3}6^{-l}, 6^{-(l+1)})$ and, in particular, $B_{r_k}(y_k) \subset B^+_{\frac{1}{3}6^{-l}}(0)$. Note that similarly to in the proof of Lemma 4.15.1, using Lemma 4.14.1 with $\lambda = 0$ and (4.105) we can show $\tilde{\varepsilon}_k^2 \to 0$, but this is not used in what follows. Now consider the normalised sequence

$$w_k(x) = \frac{\partial_i D^{\alpha'}v_k(r_k x + y_k) - \overline{(\partial_i D^{\alpha'}v_k)}_{B_{\theta r_k}(y_k)}}{\tilde{\varepsilon}_k}.$$

We have

$$\nabla w_k(x) = \frac{r_k}{\tilde{\varepsilon}_k}\nabla\partial_i D^{\alpha'}v_k\left(r_k x + y_k\right). \tag{4.106}$$

Hence, using the change of variables $x \mapsto r_k x + y_k$, we find

$$\int_{B_1(0)}|\nabla w_k|^2\,\mathrm{d}x = 1 \quad\text{and}\quad \overline{(w_k)}_{B_\theta(0)} = 0 \tag{4.107}$$

and, also using (4.104),

$$\theta^{-(1+m)}\int_{B_\theta(0)}|w_k|^2\,\mathrm{d}x > \delta. \tag{4.108}$$

As a result of (4.107) and the Poincaré inequality, we observe $(w_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,2}(B_1(0);\mathbb{R}^n)$. The Rellich Compactness lemma, [39] Section 1.3 Lemma 1, thus yields a subsequence $(w_{k_j})_{j\in\mathbb{N}}$ which converges weakly in $W^{1,2}(B_1(0);\mathbb{R}^n)$ and strongly in $L^2(B_1(0);\mathbb{R}^n)$ to some $w \in W^{1,2}(B_1(0);\mathbb{R}^n)$.

Define $f_k(x) = \left(1 + (y_k)^{-1}_{m+1}r_k x_{m+1}\right)^\beta$ for each $k \in \mathbb{N}$. Observe that $a_{k_j} = (y_{k_j})^{-1}_{m+1}r_{k_j} \in [0, 4^{-1}]$ for every j, since each $B_{r_k}(y_k) \in \mathcal{B}_4(0, 1, \tau_k)$. Thus there is a subsequence, which we also index with k_j, which converges to $a \in [0, 4^{-1}]$. Furthermore, $(f_{k_j})_{j\in\mathbb{N}}$ is uniformly bounded and equicontinuous so, by the Arzelà-Ascoli theorem, there is a uniformly convergent subsequence which we again index by k_j. Since $f_{k_j}(x) \to f(x) = (1 + a x_{m+1})^\beta$ pointwise, we must also have $f_{k_j} \to f$ uniformly.

Now, for $\phi \in C_0^\infty(B_1(0);\mathbb{R}^n)$, similar calculations to those in the proof of Lemma 4.15.1 yield

$$\left|\int_{B_1(0)} f_k\left\langle\nabla w_k, \nabla\phi\right\rangle\mathrm{d}x\right| \leq C||\phi||_{L^\infty(B_1(0);\mathbb{R}^n)}r_k\int_{B_1(0)}|\nabla w_k| + \delta^{-\frac{1}{2}}\mathrm{d}x \to 0 \tag{4.109}$$

as $k \to \infty$. Furthermore, as w_{k_j} converges weakly to w in $W^{1,2}(B_1(0);\mathbb{R}^n)$ and $f_{k_j} \to f$ uniformly, we conclude that

$$\int_{B_1(0)} f\langle\nabla w, \nabla\phi\rangle\mathrm{d}x = \lim_{j\to\infty}\int_{B_1(0)} f_{k_j}\langle\nabla w_{k_j}, \nabla\phi\rangle\mathrm{d}x = 0. \tag{4.110}$$

Hence w is a weak solution of $\operatorname{div}((1+ax_{m+1})^{\beta}\nabla w) = 0$ in $B_1(0)$. By linear elliptic regularity theory, w is smooth in $B_1(0)$. We also conclude by taking limits in (4.107) and (4.108) that

$$\int_{B_1(0)} |\nabla w|^2 \, \mathrm{d}x \leq 1 \quad \text{and} \quad \overline{w}_{B_\theta(0)} = 0 \quad \text{and} \quad \theta^{-(1+m)} \int_{B_\theta(0)} |w|^2 \, \mathrm{d}x \geq \delta \tag{4.111}$$

respectively using the Rellich Compactness Lemma. Since $|\nabla w|^2$ satisfies a mean value inequality, namely $\sup_{B_\theta(0)} |\nabla w|^2 \leq C(m,\beta) \int_{B_1(0)} |\nabla w|^2 \mathrm{d}x$ as shown in theorem 2.1 in section III of [15], we apply the Poincaré inequality and (4.111) to see that

$$\theta^{-(1+m)} \int_{B_\theta(0)} |w|^2 \mathrm{d}x \leq \theta^{1-m} \int_{B_\theta(0)} |\nabla w|^2 \mathrm{d}x \leq C\theta^2 \int_{B_1(0)} |\nabla w|^2 \mathrm{d}x \leq C\theta^2. \tag{4.112}$$

This holds for all fixed $\theta \in (0, \frac{1}{2}]$ and we choose $\theta < \left(\frac{\delta}{C}\right)^{\frac{1}{2}}$ so that (4.112) contradicts the last statement of (4.111). ☐

4.17 Higher Order ε-Regularity

With the preceding theory in hand, we are now in a position to prove our main ε-regularity theorem.

Proof of Theorem 4.0.1. Observe that the hypothesis of the theorem are invariant with respect to the rescaling $x \mapsto Rx + x_0$. Thus we will assume $R = 1, x_0 = 0$ and $\overline{\partial^0 B_1^+(0)} \subset \mathcal{O}$.

We use proof by strong induction. We choose ε to be the number from Theorem 4.11.1. Then the combination of Theorem 4.11.1, Lemma 4.12.1 and Lemma 4.13.1 yield a $\hat{\theta} = \hat{\theta}(m, N, \beta) \leq \frac{1}{2}$ and a $\hat{\gamma} \in (0,1)$ such that $v \in C^{0,\hat{\gamma}}(\overline{B_{\hat{\theta}}^+(0)}; \mathbb{R}^n) \cap W_\beta^{1,2}(B_{\hat{\theta}}^+(0); \mathbb{R}^n)$, $\nabla v \in L^\infty(B_{\hat{\theta}}^+(0); \mathbb{R}^{(m+1)n})$ and $\partial_i v \in W_\beta^{1,2}(B_{\hat{\theta}}^+(0); \mathbb{R}^n)$ for $i = 1, \ldots, m$. Now fix $l \in \mathbb{N}_0$. The induction hypothesis is that there exists $\tilde{\theta} = \tilde{\theta}(m, N, \beta, l) \leq \frac{1}{2}$ and a $\tilde{\gamma} = \tilde{\gamma}(m, N, \beta, l) \in (0,1)$ such that the following holds. For $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| \leq l$, we have $D^{\alpha'} v \in C^{0,\tilde{\gamma}}(\overline{B_{\tilde{\theta}}^+(0)}; \mathbb{R}^n) \cap W_\beta^{1,2}(B_{\tilde{\theta}}^+(0); \mathbb{R}^n)$ and $\nabla D^{\alpha'} v \in L^\infty(B_{\tilde{\theta}}^+(0); \mathbb{R}^{(m+1)n})$. Furthermore, when $|\alpha'| = l$ and $\alpha'_{m+1} = 0$ we suppose $\nabla \partial_i D^{\alpha'} v \in L_\beta^2(B_{\tilde{\theta}}^+(0); \mathbb{R}^n)$ for $i = 1, \ldots, m$. We have already observed that this is true when $l = 0$. The inductive step will be to show the preceding statement holds, possibly for a different $\tilde{\theta}$ and $\tilde{\gamma}$, for $D^{\alpha'} v$ with $|\alpha'| \leq l+1$ and $\alpha'_{m+1} = 0$. We fix α' with $|\alpha'| = l \geq 0$ and $\alpha'_{m+1} = 0$ henceforth.

Applying Lemma 4.14.1, we see that

$$\begin{aligned}\int_{B_{\frac{\rho}{2}}(y) \cap \mathbb{R}_+^{m+1}} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq C_0 \left(C_1 + \frac{1}{\rho^2}\right) \int_{B_\rho(y) \cap \mathbb{R}_+^{m+1}} x_{m+1}^\beta |\partial_i D^{\alpha'} v - \lambda|^2 \mathrm{d}x \\ + C_2 \int_{B_\rho(y) \cap \mathbb{R}_+^{m+1}} x_{m+1}^\beta \mathrm{d}x\end{aligned} \tag{4.113}$$

for any $B_\rho(y) \subset B_{\tilde{\theta}}(0)$ with $y_{m+1} \geq 0$ and $i = 1, \ldots, m$, where $C_0 = C_0(m, N, \beta)$ and C_1, C_2 depend on m, N, β and are polynomial functions, with no constant terms, of $||\nabla D^{\tilde{\alpha}'} v||_{L^\infty(B_\rho(y) \cap \mathbb{R}_+^{m+1}; \mathbb{R}^{(m+1)n})}$ where $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $|\tilde{\alpha}'| \leq l$ and $\tilde{\alpha}'_{m+1} = 0$. We apply Lemmata 4.15.1 and 4.16.1, with

$\delta = \frac{1}{2}\frac{1}{2^{m+2}C_0}$, to respectively obtain numbers $\varepsilon_1 > 0$, $\tau_1 \in (0,1)$ and $\theta_1 \in (0,\frac{1}{4}]$ and $\varepsilon_2 > 0$, $\tau_2 \in (0,1)$ and $\theta_2 \in (0,\frac{1}{4}]$, depending only on δ and hence only on m, N, β, such that if $\tilde{R} \leq \tilde{\theta}$ and

$$\tilde{R}^{1-m-\beta}\int_{B^+_{\tilde{R}}(0)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \leq \min\{\varepsilon_1^2, \varepsilon_2^2\} \tag{4.114}$$

then either (4.82) or (4.83) holds for every $B^+_{r_1}(y) \in \mathcal{B}^+(0,\tilde{R},\tau_1\tilde{R})$ and either (4.101) or (4.102) holds for every $B_{r_2}(y) \in \mathcal{B}_4(0,\tilde{R},\tau_2\tilde{R})$. It follows from the proof of Theorem 4.11.1, bearing in mind $R = 1$ and $x_0 = 0$, that for every $B^+_r(y) \in \mathcal{B}^+(0,1,\frac{1}{2})$ we have

$$r^{1-m-\beta}\int_{B^+_r(y)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \leq Cr^{\tilde{\gamma}}$$

for some $\tilde{\gamma} \in (0,1)$ and some constant $C = C(m,N,\beta)$. In particular, this holds for $y = 0$ and $r \leq \frac{1}{2}$. Hence if $\tilde{R} = \tilde{R}(m,N,\beta,l) = (\min\{\frac{\varepsilon_1^2}{C}, \frac{\varepsilon_2^2}{C}, \left(\frac{\tilde{\theta}}{2}\right)^{\tilde{\gamma}}\})^{\frac{1}{\tilde{\gamma}}}$ then (4.114) holds on $B^+_{\tilde{R}}(0)$. We have assumed $\tilde{R} \leq \frac{\tilde{\theta}}{2}$ so that we may later apply (4.113) with impunity on any ball or half-ball in $B^+_{\tilde{R}}(0)$.

First we show that (4.7) essentially holds for $\nabla\partial_i D^{\alpha'}v$ on every $B^+_{r_1}(y) \in \mathcal{B}^+(0,\tilde{R},\tau_1\tilde{R})$. We know that (4.82) or (4.83) holds on $B^+_{r_1}(y)$. We apply (4.113) with $\lambda = \overline{\partial_i D^{\alpha'}v}_{B^+_{\theta_1 r_1}(y),\beta}$, noting that $|\lambda| \leq ||\nabla D^{\alpha'}v||_{L^\infty(B^+_{\theta_1 r_1}(y);\mathbb{R}^{(m+1)n})}$, to see that

$$\begin{aligned}
&\left(\frac{\theta_1 r_1}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{\theta_1 r_1}{2}}(y)} x^\beta_{m+1}|\nabla\partial_i D^{\alpha'}v|^2 \mathrm{d}x \\
&\leq C_0 2^{m+\beta-1}(\theta_1 r_1)^{-(1+m+\beta)}\int_{B^+_{\theta_1 r_1}(y)} x^\beta_{m+1}|\partial_i D^{\alpha'}v - \overline{\partial_i D^{\alpha'}v}_{B^+_{\theta_1 r_1}(y),\beta}|^2 \mathrm{d}x + Cr_1^2.
\end{aligned} \tag{4.115}$$

Hence, regardless of which of (4.82) or (4.83) holds (bearing in mind our choice of δ above), we have

$$(\sigma_1 r_1)^{1-m-\beta}\int_{B^+_{\sigma_1 r_1}(y)} x^\beta_{m+1}|\nabla\partial_i D^{\alpha'}v|^2 \mathrm{d}x \leq \frac{1}{2}r_1{}^{1-m-\beta}\int_{B^+_{r_1}(y)} x^\beta_{m+1}|\nabla\partial_i D^{\alpha'}v|^2 \mathrm{d}x + Cr_1^2, \tag{4.116}$$

where $\sigma_1 = \frac{\theta_1}{2}$ and C depends on $||\nabla D^{\tilde{\alpha}'}v||_{L^\infty(B^+_{\tilde{\theta}}(0);\mathbb{R}^n)}$ where $|\tilde{\alpha}'| \leq l$ with $\tilde{\alpha}'_{m+1} = 0$ and, moreover, may depend on $\tilde{R}, \theta_1, m, N$ and β and hence only on m, N, β, l as $\tilde{R} = \tilde{R}(m,N,\beta,l)$ and $\theta_1 = \theta_1(m,N,\beta)$. This holds for any $B^+_{r_1}(y) \in \mathcal{B}^+(0,\tilde{R},\tau_1\tilde{R})$. We may apply (4.116) with r_1 replaced by $\sigma_1^k r_1$ for every $k \in \mathbb{N}$ and iterate to see that

$$\begin{aligned}
&(\sigma_1^k r_1)^{1-m-\beta}\int_{B^+_{\sigma_1^k r_1}(y)} x^\beta_{m+1}|\nabla\partial_i D^{\alpha'}v|^2 \mathrm{d}x \\
&\leq \frac{1}{2^k}r_1{}^{1-m-\beta}\int_{B^+_{r_1}(y)} x^\beta_{m+1}|\nabla\partial_i D^{\alpha'}v|^2 \mathrm{d}x + C\sum_{j=0}^{k-1} 2^{-j}(\sigma_1^{k-1-j}r_1)^2 \\
&\leq \frac{1}{2^k}\left(r_1{}^{1-m-\beta}\int_{B^+_{r_1}(y)} x^\beta_{m+1}|\nabla\partial_i D^{\alpha'}v|^2 \mathrm{d}x + Cr_1^2\right).
\end{aligned} \tag{4.117}$$

Setting $\gamma_1 = -\frac{\ln 2}{\ln \sigma_1} \in (0,1)$ we conclude that

$$
\begin{aligned}
&r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \\
&\leq \sigma_1^{1-m-\beta-\gamma_1} \left(\frac{r}{r_1}\right)^{\gamma_1} \left({r_1}^{1-m-\beta} \int_{B_{r_1}^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C r_1^2 \right)
\end{aligned}
\tag{4.118}
$$

for any $r \leq r_1$, where C depends on the same factors as the constant in (4.116). This holds for any $B_{r_1}^+(y) \in \mathcal{B}^+(0, \tilde{R}, \tau_1 \tilde{R})$.

We want a similar estimate for ${r_2}^{1-m} \int_{B_{r_2}(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x$ on balls $B_{r_2}(y) \in \mathcal{B}_4(0, \tilde{R}, \tau_2 \tilde{R})$. We calculate the constants in (4.6) from Section 4.3 explicitly. Then our choice of δ and a similar argument which lead to (4.118) yields the existence of a $\gamma_2 = \gamma_2(m, N, \beta) \in (0,1)$ such that for any $B_{r_2}(y) \in \mathcal{B}_4(0, \tilde{R}, \tau_2 \tilde{R})$ and any $r \leq r_2$ we have

$$
r^{1-m} \int_{B_r(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq \sigma_2^{1-m-\gamma_2} \left(\frac{r}{r_2}\right)^{\gamma_2} \left({r_2}^{1-m} \int_{B_{r_2}(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C r_2^2 \right), \tag{4.119}
$$

where $\sigma_2 = \frac{\theta_2}{2}$ and C depends on the same factors as in (4.118). We now use (4.118) and (4.119) to show the hypothesis (4.7) from Lemma 4.3.1 is satisfied.

Define $\tau = \min\{\frac{\tau_1}{2}, \tau_2\} < \frac{1}{2}$, $\gamma = \min\{\gamma_1, \gamma_2\}$. We apply (4.118) with $r_1 = \tau \tilde{R}$. It follows that for every $B_r^+(y) \in \mathcal{B}^+(0, \tilde{R}, \tau \tilde{R})$ we have $B_{\tau \tilde{R}}^+(y) \in \mathcal{B}^+(0, \tilde{R}, \tau_1 \tilde{R})$ and hence

$$
\begin{aligned}
&r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \\
&\leq C \left(\frac{r}{\tau \tilde{R}}\right)^{\gamma} \left((\tau \tilde{R})^{1-m-\beta} \int_{B_{\tau \tilde{R}}^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C (\tau \tilde{R})^2 \right).
\end{aligned}
\tag{4.120}
$$

Furthermore, applying (4.113) with $\lambda = 0$ and $\rho = 2\tau \tilde{R}$ implies that

$$
(\tau \tilde{R})^{1-m-\beta} \int_{B_{\tau \tilde{R}}^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq C(1 + (\tau \tilde{R})^2) \leq C, \tag{4.121}
$$

where C depends on m, N, β, l and $||\nabla D^{\tilde{\alpha}'} v||_{L^\infty(B_{\tilde{\theta}}^+(0); \mathbb{R}^n)}$ where $|\tilde{\alpha}'| \leq l$ and $\tilde{\alpha}'_{m+1} = 0$. We combine (4.120) and (4.121) to see that for every $B_r^+(y) \in \mathcal{B}^+(0, \tilde{R}, \tau \tilde{R})$ we have

$$
r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq C \left(\frac{r}{\tau \tilde{R}}\right)^{\gamma} \leq C_3 r^\gamma, \tag{4.122}
$$

where C_3 depends on m, N, β, l and $||\nabla D^{\tilde{\alpha}'} v||_{L^\infty(B_{\tilde{\theta}}^+(0); \mathbb{R}^n)}$ where $|\tilde{\alpha}'| \leq l$ and $\tilde{\alpha}'_{m+1} = 0$.

Now let $B_r(y) \in \mathcal{B}_4(0, \tilde{R}, \frac{2\tau}{3} \tilde{R})$. Then $B_r(y) \subset B_{\frac{y_{m+1}}{4}}(y) \subset B^+_{\frac{3y_{m+1}}{2}}(y^+) \subset B^+_{\tau \tilde{R}}(y^+) \in \mathcal{B}^+(0, \tilde{R}, \tau \tilde{R})$, where $y^+ = y - (0, y_{m+1})$. Recalling again (4.6) from Section 4.3, we note that

$$
\left(\frac{y_{m+1}}{4}\right)^{1-m} \int_{B_{\frac{y_{m+1}}{4}}(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq C \left(\frac{3y_{m+1}}{2}\right)^{1-m-\beta} \int_{B^+_{\frac{3y_{m+1}}{2}}(y^+)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x. \tag{4.123}
$$

Since $y_{m+1} \leq \tau \tilde{R} \leq 1$, applying (4.119) on $B_{\frac{y_{m+1}}{4}}(y) \in \mathcal{B}_4(0, \tilde{R}, \tau_2 \tilde{R})$, using (4.123), and then applying (4.122) gives

$$\begin{aligned}
&r^{1-m} \int_{B_r(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \\
&\leq C \left(\frac{4r}{y_{m+1}}\right)^{\gamma} \left(\left(\frac{y_{m+1}}{4}\right)^{1-m} \int_{B_{\frac{y_{m+1}}{4}}(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C y_{m+1}^2 \right) \\
&\leq C \left(\frac{r}{y_{m+1}}\right)^{\gamma} \left(C \left(\frac{3y_{m+1}}{2}\right)^{1-m-\beta} \int_{B^+_{\frac{3y_{m+1}}{2}}(y^+)} x_{m+1}^{\beta} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C y_{m+1}^2 \right) \\
&\leq C \left(\frac{r}{y_{m+1}}\right)^{\gamma} \left(C_3 \left(\frac{3y_{m+1}}{2}\right)^{\gamma} + C y_{m+1}^2 \right) \\
&\leq C_4 r^{\gamma} \qquad (4.124)
\end{aligned}$$

where C_4 depends on m, N, β, l and $||\nabla D^{\tilde{\alpha}'} v||_{L^\infty(B^+_{\hat{\theta}}(0);\mathbb{R}^n)}$ where $|\tilde{\alpha}'| \leq l$ and $\tilde{\alpha}'_{m+1} = 0$. Together, (4.122) and (4.124) imply (4.7) from Lemma 4.3.1 holds for $B^+_r(y) \in \mathcal{B}^+(0, \tilde{R}, \frac{2\tau}{3}\tilde{R})$ and $B_r(y) \in \mathcal{B}_4(0, \tilde{R}, \frac{2\tau}{3}\tilde{R})$. Applying this lemma shows that $\partial_i D^{\alpha'} v \in C^{0,\hat{\gamma}}(\overline{B^+_{\hat{\theta}}(0)}; \mathbb{R}^n)$ for some $\hat{\theta}, \hat{\gamma} \in (0,1)$ depending on m, N, β, l. Hence, recalling that by the inductive hypothesis $\partial_i D^{\alpha'} v \in W^{1,2}_{\beta}(B^+_{\hat{\theta}(0)}; \mathbb{R}^n)$, we may apply Lemma 4.12.1 and Lemma 4.13.1 to respectively imply $\nabla \partial_i D^{\alpha'} v \in L^\infty(B^+_{\frac{\hat{\theta}}{3}}(0); \mathbb{R}^{(m+1)n})$ and $\nabla \partial_j \partial_i D^{\alpha'} v \in L^2_\beta(B^+_{\frac{\hat{\theta}}{6}}(0); \mathbb{R}^n)$ for $i, j = 1, \ldots, m$. This completes the inductive step and therefore the proof. □

Remark 4.17.1. A consequence of the proof is that the number ε in Theorem 4.0.1 can be taken to be the number from Theorem 4.11.1.

Theorem 4.0.1 yields an improvement to Theorem 4.11.1 for minimisers of E^β relative to $\mathcal{O}$. We are now in a position to prove our partial regularity theorem.

Proof of Theorem 4.0.2. Aspects of the proof closely follow the proof of Theorem 3.2 in [28]. First, it follows from the theory of Schoen and Uhlenbeck [37] that there exists a set $\Sigma_{\text{int}} \subset \mathbb{R}^{m+1}_+$, with Hausdorff dimension at most $m-2$, such that v is smooth in a neighbourhood of any point in Σ_{int}.

Define

$$\Sigma_{\text{bdry}} = \{y \in \mathcal{O} : \Theta^\beta_v(y) \geq \varepsilon\}$$

where ε is the number given by the Theorem 4.11.1 and $\Theta^\beta_v(y)$ is the density function defined in Remark 4.2.1. The upper semi-continuouty of Θ^β_v was established in Remark 4.2.1 which, when combined with the definition of Σ_{bdry}, shows that Σ_{bdry} is relatively closed in $\mathcal{O}$.

We write Σ_{bdry} as a countable union of compact sets of the form $K \cap \Sigma_{\text{bdry}}$, where $K \subset \mathcal{O}$ is compact, and let $\Sigma' \subset \Sigma_{\text{bdry}}$ be such a set. Fix $\delta > 0$ and cover Σ' by a collection of balls $B^m_{r_i}(x_i) \subset \mathcal{O}$ with $\overline{B^m_{r_i}(x_i)} \subset \mathcal{O}$ with $x_i \in \Sigma'$ and $0 < r_i \leq \delta$. The compactness of Σ', combined with Vitali's covering theorem yields a finite subcollection of balls, $B^m_{r_1}(x_1), \ldots, B^m_{r_I}(x_I)$ for some $I \in \mathbb{N}$, of any such cover of Σ', which satisfies $B_{r_i}(x_i) \cap B_{r_j}(x_j) = \emptyset$ for $i \neq j, 1 \leq i, j \leq I$ and $\Sigma' \subset$

$\bigcup_{i=1}^{I} B_{5r_i}(x_i)$. Using the boundary energy monotonicitly formula, Lemma 4.2.1, we see that

$$\sum_{i=1}^{I}(10r_i)^{m+\beta-1} \leq \frac{10^{m+\beta-1}}{\varepsilon}\sum_{i=1}^{I}\int_{B^+_{r_i}(x_i)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \leq \frac{10^{m+\beta-1}}{\varepsilon}\int_{\mathcal{O}\times[0,\delta]} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x.$$

We send $\delta \to 0^+$ and use Lebesgue's Dominated Convergence Theorem to see that $\mathcal{H}^{m-1+\beta}(\Sigma') = 0$ and hence $\mathcal{H}^{m-1+\beta}(\Sigma_{\mathrm{bdry}}) = 0$.

Let $x_0 \in (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$. If $x_0 \in \mathbb{R}^{m+1}_+$ then $x_0 \in \mathbb{R}^{m+1}_+\backslash\Sigma_{\mathrm{int}}$ and v is smooth in an open ball centred at x_0 and contained in $\mathbb{R}^{m+1}_+\backslash\Sigma_{\mathrm{int}} \subset (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$. If $x_0 \in \mathcal{O}$ then $x_0 \in \mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$ and $\Theta^\beta_v(x_0) < \varepsilon$ which, combined with the fact that $\mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$ is open in $\mathcal{O}$, implies there exists an $R > 0$ such that $R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \leq \varepsilon$, $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$. Consequently, Theorem 4.11.1 implies that there are $\theta, \gamma \in (0,1)$ such that $v \in C^{0,\gamma}(\overline{B^+_{\theta R}(x_0)}; N)$. Furthermore, we deduce from (4.61) in the proof of Theorem 4.11.1 that

$$r^{1-m-\beta}\int_{B^+_r(z)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \leq C\left(\frac{r}{R}\right)^\gamma \varepsilon$$

on every $B^+_r(z) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$ which shows that $\Theta^\beta_v(z) = 0$ for every $z \in \partial^0 B^+_{\frac{R}{2}}(x_0)$. Now setting $\sigma = \min\{\theta, \frac{1}{2}\}$ we see that $\Theta^\beta_v(z) = 0$ for $z \in \partial^0 B^+_{\sigma R}(x_0)$ which implies $\overline{\partial^0 B^+_{\sigma R}(x_0)} \subset \mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$. Furthermore, v is a Hölder continuous weakly harmonic map in any $B_r(y)$ with $\overline{B_r(y)} \subset B^+_{\sigma R}(x_0)$. We apply Lemma 4.4.4 to see that v is smooth in $B^+_{\sigma R}(x_0)$ and conclude that $B^+_{\sigma R}(x_0) \subset \mathbb{R}^{m+1}_+\backslash\Sigma_{\mathrm{int}}$. Consequently, we have $B^+_{\sigma R}(x_0) \cup \partial^0 B^+_{\sigma R}(x_0) \subset (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$. Note that $B^+_{\sigma R}(x_0) \cup \partial^0 B^+_{\sigma R}(x_0)$ is an open ball centred at x_0 in the (Euclidean) topology of $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$. Hence Σ is relatively closed in $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$. As the Hausdorff dimension of Σ_{int} is at most $m-2 < m-1+\beta$ and $\mathcal{H}^{m-1+\beta}(\Sigma_{\mathrm{bdry}}) = 0$, we deduce that $\mathcal{H}^{m+\beta-1}(\Sigma) = 0$. We also conclude $v \in C^{0,\gamma}_{loc}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; N)$.

Consider $x_0 \in (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$ with $x_0 \in \mathcal{O}$. Then, as above, we observe there is an $R > 0$ such that $R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \leq \varepsilon$, $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$. In view of Remark 4.17.1, Theorem 4.0.1 implies that for every $l \in \mathbb{N}_0$ there exist $\theta, \gamma \in (0,1)$ such that for every $\alpha' \in \mathbb{N}^{m+1}_0$ with $|\alpha'| \leq l$ and $\alpha'_{m+1} = 0$ we have $D^{\alpha'}v \in C^{0,\gamma}(\overline{B^+_{\theta R}(x_0)}; \mathbb{R}^n)$. However, we also know that $(\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$ is open in $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$. Hence there exists $\tilde{R} > 0$ such that $B^+_{\tilde{R}}(x_0) \cup \partial^0 B^+_{\tilde{R}}(x_0) \subset (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$. Setting $r = \min\{\theta R, \tilde{R}\}$ we conclude $D^{\alpha'}v \in C^{0,\gamma}(\overline{B^+_r(x_0)}; \mathbb{R}^n)$ and $B^+_r(x_0) \cup \partial^0 B^+_r(x_0) \subset (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$. We iteratively apply Lemmata 4.12.1 and 4.13.1 to see that for every $\alpha' \in \mathbb{N}^{m+1}_0$ with $|\alpha'| \leq l$ and $\alpha'_{m+1} = 0$ we have $\nabla D^{\alpha'}v \in L^\infty(B^+_{\tilde{r}}(x_0); \mathbb{R}^{(m+1)n})$ and $D^{\alpha'}v \in W^{1,2}_\beta(B^+_{\tilde{r}}(x_0); \mathbb{R}^n)$ for some $\tilde{r} \leq r$. It follows that $B^+_{\tilde{r}}(x_0) \cup \partial^0 B^+_{\tilde{r}}(x_0) \subset (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$. Hence $v \in C^{0,1}_{loc}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; N)$ and for every multi-index $\alpha' \in \mathbb{N}^{m+1}$ with $\alpha'_{m+1} = 0$ we have $D^{\alpha'}v \in C^{0,1}_{loc}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; \mathbb{R}^n)$ and $\nabla D^{\alpha'}v \in L^\infty_{loc}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; \mathbb{R}^{(m+1)n})$.

Lastly, for x_0 as in the preceding paragraph, fix $\alpha' \in \mathbb{N}^{m+1}_0$ with $\alpha'_{m+1} = 0$ and $\tilde{r} \leq 1$ such that $B^+_{\tilde{r}}(x_0) \cup \partial^0 B^+_{\tilde{r}}(x_0) \subset (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$. Making $\tilde{r}$ smaller if necessary, we may assume that $\nabla D^{\tilde{\alpha}'}v \in L^\infty(B^+_{\tilde{r}}(x_0); \mathbb{R}^{(m+1)n})$ and $D^{\tilde{\alpha}'}v \in C^{0,1}(\overline{B^+_{\tilde{r}}(x_0)}; \mathbb{R}^n)$ for every $\tilde{\alpha}' \in \mathbb{N}^{m+1}_0$ with $\tilde{\alpha}'_{m+1} = 0$ and $|\tilde{\alpha'}| \leq |\alpha'| + 2$. We also observe that since $v \in C^{0,1}(\overline{B^+_{\tilde{r}}(x_0)}; N)$ is a Hölder continuous harmonic map, it is smooth in $B^+_{\tilde{r}}(x_0)$ by Lemma 4.4.4 and so we have $x^{-\beta}_{m+1}\partial_{m+1}(x^\beta_{m+1}\partial_{m+1}D^{\alpha'}v) = -(\Delta' D^{\alpha'}v + D^{\alpha'}(A(v)(\nabla v, \nabla v)))$ classically in $B^+_{\tilde{r}}(x_0)$, where Δ' is the Laplacian with respect to x_i, $i = 1, \ldots, m$. Hence, $x_{m+1}^{-\beta} \partial_{m+1}(x_{m+1}^{\beta} \partial_{m+1} D^{\alpha'} v)$ is bounded in $B_{\tilde{r}}^{+}(x_0)$. Hence, for every $B_\rho^+(y) \in \mathcal{B}^+(x_0, \frac{\tilde{r}}{3}, \frac{\tilde{r}}{6})$ we calculate

$$\rho^{1-m+\beta} \int_{B_\rho^+(y)} x_{m+1}^{-\beta} |\nabla (x_{m+1}^{\beta} \partial_{m+1} D^{\alpha'} v)|^2 \mathrm{d}x \leq C \rho^{2+2\beta}.$$

Moreover, for every $B_\rho(y) \in \mathcal{B}(x_0, \frac{\tilde{r}}{3}, \frac{\tilde{r}}{6})$ we calculate

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla (x_{m+1}^{\beta} \partial_{m+1} D^{\alpha'} v)|^2 \mathrm{d}x \leq C \rho^{2-2|\beta|}.$$

An application of Lemma 4.3.1 concludes the proof. □

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