# A Novel Method of Polynomial Approximation for Parametric Problems in Power Systems

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***Abstract*—Many problems in power systems depend on parameters, i.e., system states are decided by the magnitude of parameters. Practically, these parameters could be stochastic variables or deterministic system control variables, e.g., generation outputs, nodal voltages, etc. Due to the nonlinearity of power systems, the analytical relation between system states and parameters cannot be obtained directly. To evaluate the influence of parameters on system states, the sampling method is very powerful but time-consuming. Another feasible approach is to use polynomial approximations, where system states are approximately expressed in the form of polynomials in terms of parameters. Galerkin method can be used to identify the approximate solution with high accuracy. In reality, however, it may encounter difficulties in solving high-dimensional equations if a large number of parameters are involved.**

**In this manuscript, an innovative method for resolving high-dimensional equations in power systems is proposed, where a sequence of decoupled equations are constructed to make polynomial expansions efficient to find. This new approach can provide a local approximation in the form of *Taylor expansion* at a given operation point. Its detailed process is introduced in the application to load flow problems. 6-bus system and IEEE 118-bus system are used to illustrate its effectiveness, where load flow are investigated. Results show that the proposed method provides approximation more efficiently than traditional Galerkin method, and 3-order polynomials can gain very accurate results.**

***Index Terms*—Parametric problems, polynomial approximation, perturbation method, Galerkin method, load flow problems.**

## I. Introduction

IN parametric problems of power systems, the influence of specific parameters on system states are evaluated to provide insights and recommendations to system planning and operation. For example, probabilistic load flow problems [1] are investigated to show how stochastic variables (i.e., parameters) influence the states of system operation, e.g., voltages, line flows, reliability, etc.

Normally, the problems are modeled as nonlinear equations in implicit formula, which implies that the states cannot be directly expressed in the form of analytical function of parameters. Thus, numerical approaches are typically employed, where sampling methods, e.g., Monte Carlo method [2], are the most straightforward approaches. Although simple and easy to implement, sampling methods are time-consuming and provide discrete rather than analytical continuous results. Hence, many other simplified methods are developed. For example, in load flow analysis, it is beneficial to use linear models which can be analytically handled, e.g., DC load flow models or linearized AC load flow models. However, they will inevitably introduce errors due to neglecting the nonlinearity.

To cope with the nonlinearity, polynomial approximation has been widely used, where polynomial functions are utilized to approximate the exact solutions of parametric problems. Then, the problem is transformed to solving unknown coefficients in predefined polynomials [3, 4]. Compared to sampling methods, the approximation provides an explicit formula to represent the relation governed by original models, based on which the influence of parameters on system states can be analytically assessed. Currently, polynomial approximation has been applied to many problems in power systems, e.g., stability analysis [5–8], load flow prediction [9, 10], power system modeling [11], probabilistic load flow [12–14], etc.

On the other hand, a large number of mathematical methods have been developed to find the results of polynomial approximation. In [6, 7], Taylor series are calculated straightforwardly by conducting corresponding derivative equations of each parameters, which however is technically difficult to implement in the cases beyond the third order. In [8], the polynomial approximation is obtained by least squares method based on sampling results. In [9–11], polynomial interpolation methods are used to determine polynomial coefficients by specific given operation points. Both least squares and interpolation methods are not easy to implement, if the number of parameters are very large. In order to increase computational efficiency, authors in [15] use particle swarm optimization method to find interpolation polynomial coefficients.

Many other novel methods for resolving parametric problems have attracted researchers worldwide. In [5, 13, 14], *collocation methods* are used. The idea is to use polynomials up to a certain degree as candidate solutions to find the solution which satisfies the given equations at some specific points, called *collocation points*. The main drawback of these methods is that the collocation points should be carefully designed for high accuracy and the stability of these methods are not guaranteed [16, 17]. In [12], *Galerkin method* is applied to stochastic analysis with generalized Polynomial Chaos (gPC), where the orthogonal projection makes the error minimized. Although this algorithm is very stable and accurate, the main difficulty in its application is that the decoupled high-dimensional Galerkin equations have to be solved at one go.

In this paper, a novel method for resolving polynomial approximations in power systems is proposed, which can provide a local approximation at the neighborhood of an assigned

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operation point. The algorithm is mainly based on *perturbation methods*, but derived in the form of generalized Galerkin method. By starting from an exact solution of parameters with specific values, the desired solution is expressed in terms of power series with respect to parameters, and the polynomial coefficients are determined in a decoupled approach. The main advantage of this method are twofold: i) the decoupled solution method can achieve high computational efficiency, especially in the cases with a large number of parameters; ii) the coefficients are sequentially found by substituting the previously determined coefficients into the associated expansions, so it is much easier to extend the approximation to higher orders.

The rest of this paper is organized as follows. Section II presents essential mathematical background. In section III, the detailed solution method is modeled and the application to the load flow problem is given in section IV. In section V, two cases are used to demonstrate the effectiveness of the approach, followed by the conclusion in the last section.

## II. Polynomial Approximation and Curse-of-Dimensionality

In this section, the concept of polynomial approximation and the main difficulty of its application to power systems are introduced.

### A. Concept of Polynomials Approximation

In this paper, it is assumed that the problem is governed by a set of equations:

$$A(\mathbf{u};\mathbf{p}) = 0, \tag{1}$$

where $A$ denotes the model, $\mathbf{u} = \{u_i\}_{i=1}^{M}$ is the state vector(for example, voltages) and $\mathbf{p} = \{p_i\}_{i=1}^{N_p}$ is the parameter vector (e.g., wind power output).

Polynomial approximation is to use *algebraic polynomials* to approximate the exact solution $\mathbf{u}^*(\mathbf{p})$ [18]. (Here, the upper script $*$ means the solution). The approximate solutions can be represented by:

$$\mathbf{u}_N^*(\mathbf{p}) = \sum_{i=1}^{N} \hat{\mathbf{c}}_i \Phi_i(\mathbf{p}). \tag{2}$$

where $\hat{\mathbf{c}}_i$ is the coefficient to be determined, $\Phi_i(\mathbf{p})$ is the predefined polynomial function in terms of $\mathbf{p}$, $N$ is the dimension of the approximation, and the superscript $\wedge$ means the coefficient is unknown.

Here, $\{\Phi_i\}_{i=1}^{N}$ can be interpreted as a set of basis in the form of polynomials, so the expansion of (2) is just the spectrum decomposition in the $N$-dimensional space spanned by $\{\Phi_i\}_{i=1}^{N}$. The main purpose of polynomial approximation is to find appropriate coefficients in (2) to approximate the exact solution.

### B. The Procedures of Galerkin Method

Galerkin method is one powerful tool to identify the unknown coefficients. The main procedures can be described concisely as follows [18]:

Step-1: A set of polynomial basis is chosen, and the approximate solution can be represented as (2). Here, $\Phi_i(\mathbf{p})$ is called *trial basis*.

Step-2: Substituting (2) into (1) produces a nonzero $\mathbf{R}$, called *residual*:

$$\mathbf{R} = A(\mathbf{u}_N^*(\mathbf{p});\mathbf{p}) = A(\sum_{i=1}^{N} \hat{\mathbf{c}}_i \Phi_i(\mathbf{p});\mathbf{p}). \tag{3}$$

Step-3: To identify the coefficients, Galerkin equations are formed by projecting the residual onto the *test basis* $\{\Gamma_k\}_{k=1}^{N}$:

$$\langle \mathbf{R}, \Gamma_k \rangle = 0, \quad k = 1, \cdots, N, \tag{4}$$

where the inner product $\langle \cdot, \cdot \rangle$ is defined in the manner of $\langle x, y \rangle = \int_{\mathcal{P}} xy dW(\mathbf{p})$. Here, $W(\mathbf{p})$ is the measure of $\mathbf{p}$ on $\mathcal{P}$. It should be noted that, in *traditional Galerkin method*, the test basis is the same as trial basis; otherwise, it is called *generalized Galerkin method*.

Step-4: The coefficients can be found by solving the Galerkin equations, and then the required approximate solution can be obtained by substituting them into (2).

### C. Curse-of-Dimensionality in Power Systems

From [3, 4], it is known that the number of unknown coefficients for each state is

$$N = \binom{N_p + N_d}{N_p}, \tag{5}$$

where $N_p$ is the number of parameters and $N_d$ is the degree of approximate polynomials.

It can be seen that $N$ grows very fast with $N_p$ and $N_d$, so that a large $N_p$ or $N_d$ may produce a tremendous number of coefficients to be determined, i.e., $M \times N$ ($M$ is the number of states). For example, in 118-bus system with 18 parameters, if $N_d = 3$, the number of coefficients to be determined is $234 \times \binom{18+3}{3} = 311220$.

In Galerkin method, the coupled *Galerkin equations* are formed to identify all the coefficients at one go, which may be infeasible in the cases with a large number of parameters. Thus, the uncoupled approaches are more attractive when polynomial approximation is applied to large-scale systems.

## III. The Proposed Methodology

The proposed method is started from a given operation point $\mathbf{p}_0$. By linear transformation, the parameters $\mathbf{p}$ can be expressed in terms of a perturbation about $\mathbf{p}_0$, $\mathbf{p} = \mathbf{p}_0 + \mathbf{p}'$. That means the approximate solution $\mathbf{u}_N^*(\mathbf{p})$ can also be expressed as $\mathbf{u}_N^*(\mathbf{p}')$. Therefore, for convenience, the parameters $\mathbf{p}$ refer to the perturbation about a given operation point in the following analysis.

### A. Choice of Trial Basis

In our approach, the power series of parameters are chosen as the polynomial basis in (2). For example, in the cases with one single parameter $p$, the solution could be expressed as

$$u_m^* = \sum_{i=1}^{N} \hat{c}_i^{(m)} p^{i-1}, \tag{6}$$

where $u_m^*$ denotes the $m$-th state in $\mathbf{u} = \{u_i\}_{i=1}^M$. Here the subscript $N$ for $u_m^*(p)$ is neglected for simplicity.

One conclusion can be obtained: in the expansion of multiplication of system states, the coefficient with $p$ of $(i-1)$-th order is the combination of the coefficients of each state with $p$ whose order is equal or less than $i-1$. For example, in the expansion of $u_m^* u_n^*$, the coefficient of $p^{i-1}$ is the combination of $\hat{c}_j^{(m)}$ and $\hat{c}_k^{(n)}$ with $j+k=i+1$. This is obvious, because if $j+k \neq i+1$, then $\left(\hat{c}_j^{(m)} p^{j-1}\right)\left(\hat{c}_k^{(n)} p^{k-1}\right) = \hat{c}_j^{(m)} \hat{c}_k^{(n)} p^{j+k-2}$ implies that the order of $p$ is $j+k-2 \neq i-1$. From $j \geq 1$ and $k \geq 1$, it can be derived that $1 \leq j \leq i$ and $1 \leq k \leq i$.

This property can provide a decoupled solution method, which will be explained later.

### B. Generalized Galerkin Approach and Decoupled Method

In traditional Galerkin method, the residual is projected onto the test basis, which is the same as trial basis, to form Galerkin equations. When the projection is taken onto a special test basis, the coefficients are decoupled in Galerkin equations. Then the coefficients can be determined in a decoupled approach.

Assume that, there exsits a set of test basis $\{\Gamma_k\}_{k=1}^N$ which satisfies

1) When $1 \leq k \leq N-1$,

$$\begin{cases} \int_{\mathcal{P}} p^i \Gamma_k dW(p) = 1, & 0 \leq i < k \\ \int_{\mathcal{P}} p^i \Gamma_k dW(p) = 0, & i \geq k \end{cases} \tag{7}$$

2) When $k = N$,

$$\int_{\mathcal{P}} p^i \Gamma_k dW(p) = 1, \quad i \geq 0 \tag{8}$$

When forming Galerkin equations, the residual is projected onto $\{\Gamma_k\}_{k=1}^N$, and then the parameter $p$ is eliminated.

(1) For $\langle \mathbf{R}, \Gamma_1 \rangle = 0$, the coefficients with $p$ whose orders are greater than 0 will be eliminated, and only coefficients with 0-th order $p$ will be preserved in $\langle \mathbf{R}, \Gamma_1 \rangle = 0$;

(2) For $\langle \mathbf{R}, \Gamma_2 \rangle = 0$, the coefficients with $p$ whose orders are greater than 1 will be eliminated, and only coefficients with 0-th order and first-order $p$ will be preserved in $\langle \mathbf{R}, \Gamma_2 \rangle = 0$;

$\vdots$

(N-1) For $\langle \mathbf{R}, \Gamma_{N-1} \rangle = 0$, the coefficients with $p$ whose orders are greater than $N-2$ will be eliminated, and only coefficients with $p$ whose order are less than $N-1$ will be preserved in $\langle \mathbf{R}, \Gamma_{N-1} \rangle = 0$;

(N) For $\langle \mathbf{R}, \Gamma_N \rangle = 0$, all the coefficients are preserved.

Hence, the decoupled solution can be set up as follows. The coefficients with regard to $p^0$ can be found by solving $\langle \mathbf{R}, \Gamma_1 \rangle = 0$ firstly, and then the coefficients with regard to $p^1$ can be determined by solving $\langle \mathbf{R}, \Gamma_2 \rangle = 0$, where the coefficients with regard to $p^0$ are known. This process can be repeated until all the coefficients are found, where the previously determined coefficients are used at each step of this approach. It is similar to the back-substitution process of solving linear equations by using the *Gaussian elimination* approach.

TABLE I
AN EXAMPLE OF TEST BASIS WHEN $N_d = 2$

| $k$ | $\Gamma_k$ |
|---|---|
| 1 | $\frac{1}{128}\left(225 - 1050p^2 + 945p^4\right)$ |
| 2 | $\frac{1}{128}\left(225 + 1200p - 1050p^2 - 1680p^3 + 945p^4\right)$ |
| 3 | $\frac{1}{16}\left(15 - 60p - 210p^2 + 140p^3 + 315p^4\right)$ |

One necessary condition of the proposed method is the existence of the test basis $\{\Gamma_k\}_{k=1}^N$, which should satisfy (7) and (8). The proof of existence is shown in Appendix A. For example, in the second order systems when $N_d = 2$, the maximum order of monomials is $2 \times N_d = 4$. Thus, the test basis should satisfy 5 conditions from (7) and (8), and TABLE I gives the example of test basis.

Additionally, this approach can be interpreted as a kind of perturbation method. It starts from solving coefficients with regard to $p^0$, where $p$ is set to 0 in Galerkin equations, which is just the solution at a given operation point with the perturbation being zero. Then the following coefficients are identified from $p^1$ to powers of $p$ at higher levels. Although the result is a local approximation, this approach gains high computational efficiency from the decoupled approach. It is very easy to extend the approximation to higher orders, because the determination of lower order coefficients are not influenced by the solution of higher order ones.

## IV. Implementation of Parametric Load Flow

### A. Load Flow Formulation

The proposed approach is applied to load flow model in rectangular form, where the voltage $V$ is given by $V = e + jf$. Recall the load flow equations:

$$\begin{cases} \sum\limits_{k=1}^{N_E} \left[G_{ik}\left(e_i e_k + f_i f_k\right) + B_{ik}\left(f_i e_k - e_i f_k\right)\right] - P_i = 0 \\ \qquad\qquad \forall i \notin \textit{slack-bus} \\ \sum\limits_{k=1}^{N_E} \left[G_{ik}\left(f_i e_k - e_i f_k\right) - B_{ik}\left(f_i f_k + e_i e_k\right)\right] - Q_i = 0 \\ \qquad\qquad \forall i \in \textit{PQ-bus} \\ e_i^2 + f_i^2 - V_i^2 = 0, \qquad \forall i \in \textit{PV-bus} \end{cases} \tag{9}$$

where $N_E$ is the total bus number, $G_{ik}$ and $B_{ik}$ are the entries of nodal admittance matrix. In our analysis, $e_i$, $f_i (i \notin$ *slack-bus*) are assumed to be system states, and parameters are other measurable variables that affect system voltages, e.g., generation outputs, etc.

$$R_i^P = \sum_{k=1}^{N_E}\left[G_{ik}\left(\sum_{j=1}^{N}\hat{c}_j^{(e_i)}\Phi_j\sum_{j=1}^{N}\hat{c}_j^{(e_k)}\Phi_j+\sum_{j=1}^{N}\hat{c}_j^{(f_i)}\Phi_j\sum_{j=1}^{N}\hat{c}_j^{(f_k)}\Phi_j\right)+B_{ik}\left(\sum_{j=1}^{N}\hat{c}_j^{(f_i)}\Phi_j\sum_{j=1}^{N}\hat{c}_j^{(e_k)}\Phi_j-\sum_{j=1}^{N}\hat{c}_j^{(e_i)}\Phi_j\sum_{j=1}^{N}\hat{c}_j^{(f_k)}\Phi_j\right)\right]-\sum_{j=1}^{N}c_j^{(P_i)}\Phi_j, \tag{10a}$$

$$R_i^Q = \sum_{k=1}^{N_E}\left[G_{ik}\left(\sum_{j=1}^{N}\hat{c}_j^{(f_i)}\Phi_j\sum_{j=1}^{N}\hat{c}_j^{(e_k)}\Phi_j-\sum_{j=1}^{N}\hat{c}_j^{(e_i)}\Phi_j\sum_{j=1}^{N}\hat{c}_j^{(f_k)}\Phi_j\right)-B_{ik}\left(\sum_{j=1}^{N}\hat{c}_j^{(f_i)}\Phi_j\sum_{j=1}^{N}\hat{c}_j^{(f_k)}\Phi_j+\sum_{j=1}^{N}\hat{c}_j^{(e_i)}\Phi_j\sum_{j=1}^{N}\hat{c}_j^{(e_k)}\Phi_j\right)\right]-\sum_{j=1}^{N}c_j^{(Q_i)}\Phi_j, \tag{10b}$$

$$R_i^V = \left(\sum_{j=1}^{N}\hat{c}_j^{(e_i)}\Phi_j\right)^2+\left(\sum_{j=1}^{N}\hat{c}_j^{(f_i)}\Phi_j\right)^2-\left(\sum_{j=1}^{N}c_j^{(V_i)}\Phi_j\right)^2. \tag{10c}$$

### B. Polynomial Approximation of Parametric Load Flow

The trial basis is assumed to be $\{\Phi_i\}$, so $P_i$, $Q_i$, and $V_i$ can be represented as $\sum_{j=1}^{N}c_j^{(P_i)}\Phi_i$, $\sum_{j=1}^{N}c_j^{(Q_i)}\Phi_i$, and $\sum_{j=1}^{N}c_j^{(V_i)}\Phi_i$, respectively. The coefficients $c_j^{(P_i)}$, $c_j^{(Q_i)}$, and $c_j^{(V_i)}$ can be obtained directly by evaluating $\langle P_i,\Phi_j\rangle$, $\langle Q_i,\Phi_j\rangle$, and $\langle V_i,\Phi_j\rangle$, respectively. Then, the residual can be obtained by substituting the approximate solution into (9), given as (10).

The determination of coefficients starts from the zeroth-order at a given operation point. As illustrated in subsection III-B, only coefficients of zeroth-order are contained in the Galerkin equations:

$$\sum_{k=1}^{N_E}\left[G_{ik}\left(\hat{c}_1^{(e_i)}\hat{c}_1^{(e_k)}+\hat{c}_1^{(f_i)}\hat{c}_1^{(f_k)}\right)+B_{ik}\left(\hat{c}_1^{(f_i)}\hat{c}_1^{(e_k)}-\hat{c}_1^{(e_i)}\hat{c}_1^{(f_k)}\right)\right]-c_1^{(P_i)}=0,\qquad \forall i\notin \textit{slack-bus} \tag{10a}$$

$$\sum_{k=1}^{N_E}\left[G_{ik}\left(\hat{c}_1^{(f_i)}\hat{c}_1^{(e_k)}-\hat{c}_1^{(e_i)}\hat{c}_1^{(f_k)}\right)-B_{ik}\left(\hat{c}_1^{(f_i)}\hat{c}_1^{(f_k)}+\hat{c}_1^{(e_i)}\hat{c}_1^{(e_k)}\right)\right]-c_1^{(Q_i)}=0,\qquad \forall i\in \textit{PQ-bus} \tag{10b}$$

$$\hat{c}_1^{(e_i)}\hat{c}_1^{(e_i)}+\hat{c}_1^{(f_i)}\hat{c}_1^{(f_i)}-c_1^{(V_i)}c_1^{(V_i)}=0,\ i\in \textit{PV-bus}, \tag{10c}$$

where $c_1^{(P_i)}$, $c_1^{(Q_i)}$, and $c_1^{(V_i)}$ are the known zeroth-order coefficients of the parametric expressions of $P_i$, $Q_i$, and $V_i$, respectively.

Thus, the zeroth-order coefficients can be determined by solving (10), where the number of equations is equal to the original load flow model. Note that, by taking projection onto $\Gamma_1$, the Galerkin equation is transformed into the traditional load flow equations.

After knowing the zeroth-order coefficients, one can find the first-order coefficients by solving $\langle\mathbf{R},\Gamma_2\rangle=0$, which only contains the zeroth and first-order coefficients. The Galerkin equations are formed as:

$$\sum_{k=1}^{N_E}\left[G_{ik}\left(c_1^{(e_i)}\hat{c}_2^{(e_k)}+\hat{c}_2^{(e_i)}c_1^{(e_k)}+c_1^{(f_i)}\hat{c}_2^{(f_k)}+\hat{c}_2^{(f_i)}c_1^{(f_k)}\right)+B_{ik}\left(c_1^{(f_i)}\hat{c}_2^{(e_k)}+\hat{c}_2^{(f_i)}c_1^{(e_k)}-c_1^{(e_i)}\hat{c}_2^{(f_k)}-\hat{c}_2^{(e_i)}c_1^{(f_k)}\right)\right]-c_2^{(P_i)}=0, \tag{11a}$$

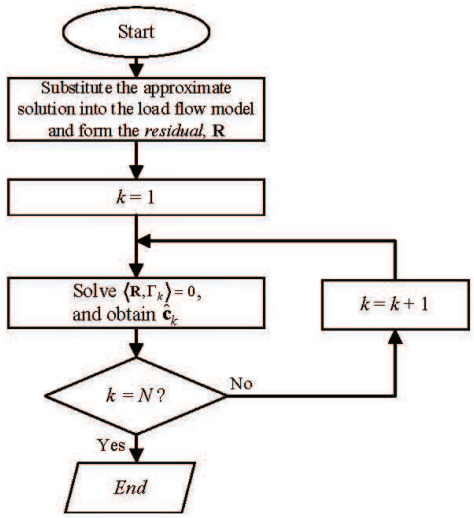


Fig. 1. Flowchart of the solution of approximation

$$\sum_{k=1}^{N_E}\left[G_{ik}\left(c_1^{(f_i)}\hat{c}_2^{(e_k)}+\hat{c}_2^{(f_i)}c_1^{(e_k)}-c_1^{(e_i)}\hat{c}_2^{(f_k)}-\hat{c}_2^{(e_i)}c_1^{(f_k)}\right)-B_{ik}\left(c_1^{(f_i)}\hat{c}_2^{(f_k)}+\hat{c}_2^{(f_i)}c_1^{(f_k)}+c_1^{(e_i)}\hat{c}_2^{(e_k)}+\hat{c}_2^{(e_i)}c_1^{(e_k)}\right)\right]-c_2^{(Q_i)}=0, \tag{11b}$$

$$c_1^{(e_i)}\hat{c}_2^{(e_i)}+c_1^{(f_i)}\hat{c}_2^{(f_i)}-c_1^{(V_i)}c_2^{(V_i)}=0, \tag{11c}$$

where $c_2^{(P_i)}$, $c_2^{(Q_i)}$, and $c_2^{(V_i)}$ are the known first-order coefficients of the parametric expressions of $P_i$, $Q_i$, and $V_i$, respectively.

Here, the previous determined zeroth-order coefficients are used. The following coefficients can be determined by solving $\langle\mathbf{R},\Gamma_k\rangle=0,(3\le k\le N)$ successively. The main process can be summarized as Fig. 1.

### C. Extension of Parametric Load Flow

In the previous subsection, the process of solving the approximation of system voltages is discussed. If one want to analyze some other states, e.g., system netloss, line flow,

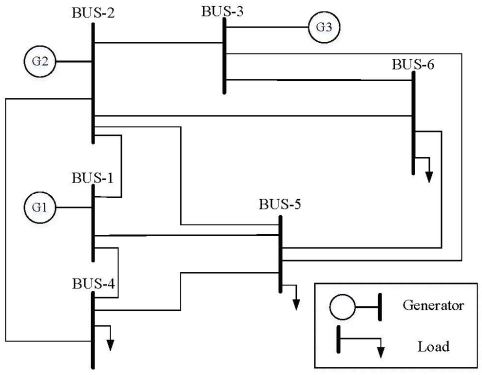


Fig. 2. 6-bus system

etc., additional equations with respect to these states should be involved. For example, if the system netloss is considered, the additional equation is $\sum_{i\in\mathcal{G}} P_i - \sum_{i\in\mathcal{L}} P_i - P_{loss} = 0$, whose residual, $R_{loss}$, can be expressed as

$$R_{loss} = \sum_{j=1}^{N} \left( \sum_{i\in\mathcal{G}} c_j^{(P_i)} - \sum_{i\in\mathcal{L}} c_j^{(P_i)} - \hat{c}_j^{(P_{loss})} \right) \Phi_j, \quad (12)$$

where $\mathcal{G}$ is the generation set, $\mathcal{L}$ is the load set, $\hat{c}_j^{(P_{loss})}$ is the coefficients of netloss to be identified.

Similarly, other system states can be considered when additional equations with respect to them are involved in the approximation.

## V. Case Studies

In this section, two cases are studied to show the effectiveness of the proposed method. Firstly, a 6-bus system with a small number of parameters is used to illustrate the accuracy. Secondly, the IEEE 118-bus system with a large number of parameters are analyzed to verify the computational efficiency. All calculations are performed on the platform of *Wolfram Mathematica* 10.0 with a CPU of *Intel i7-4710MQ* 2.5 *GHz*.

### A. Case 1: 6-bus system

As shown in Fig. 2, the 6-bus system in [19] is used. It is assumed that the capacity of the three generators is 100 MW. The Newton-Raphson method is used to solve the nonlinear equations, where the iteration threshold is set to $1 \times 10^{-10}$. In our analysis, the active power output of generators at bus-2 ($P_{G2}$) and bus-3 ($P_{G3}$) are regarded as parameters, while bus-1 is the swing bus.

*1) Results Presentation:* The approximate solution of $e_3$, $f_3$ and netloss, are given in TABLE II when $N_d = 3$. As seen, explicit expressions with respect to $P_{G2}$ and $P_{G3}$ can be obtained, and Fig. 3 depicts the system netloss, $P_{loss}$.

The results of states can be obtained directly by substituting the value of $P_{G2}$ and $P_{G3}$ into the expressions, rather than evaluating the whole load flow equations. This is very efficient if we want to evaluate the states quickly for a new set of parameters.

TABLE II
Results of 6-bus system when $N_d = 3$

| | approximate solutions |
|---|---|
| $e_3$ | $1.0418 + 0.0237P_{G2} + 0.0455P_{G3} - 0.0057P_{G2}^2$ $-0.0195P_{G3}^2 - 0.0205P_{G2}P_{G3} + 0.0002P_{G2}^3$ $-0.0010P_{G3}^3 + 0.0014P_{G2}^2P_{G3} + 0.0014P_{G2}P_{G3}^2$ |
| $f_3$ | $-0.2444 + 0.1015P_{G2} + 0.1961P_{G3} - 0.0030P_{G2}^2$ $+0.0002P_{G3}^2 - 0.0044P_{G2}P_{G3} + 0.0002P_{G2}^3$ $-0.0002P_{G3}^3 - 0.0003P_{G2}^2P_{G3} + 0.0003P_{G2}P_{G3}^2$ |
| $netloss$ | $0.1926 - 0.1430P_{G2} - 0.1787P_{G3} + 0.0517P_{G2}^2$ $+0.0754P_{G3}^2 + 0.0930P_{G2}P_{G3} - 0.0021P_{G2}^3$ $-0.0041P_{G3}^3 - 0.0060P_{G2}^2P_{G3} - 0.0060P_{G2}P_{G3}^2$ |

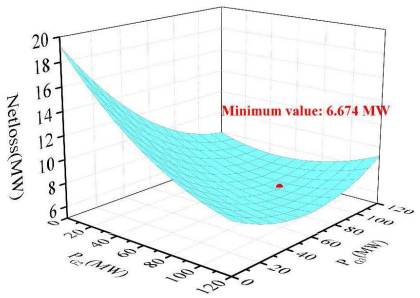


Fig. 3. Approximate solution of netloss

The results can provide additional insights of the dependence of states on parameters. For example, the minimization of system netloss can be obtained directly by evaluating $\frac{\partial P_{loss}}{\partial P_{G2}} = \frac{\partial P_{loss}}{\partial P_{G3}} = 0$, where $P_{G2} = 78.91$ MW and $P_{G3} = 83.16$ MW.

TABLE III
RMSE of all results when $N_d = 1, 2, 3$

| | $N_d = 1$ | $N_d = 2$ | $N_d = 3$ |
|---|---|---|---|
| $e_2$ | $2.12 \times 10^{-3}$ | $1.18 \times 10^{-4}$ | $1.78 \times 10^{-5}$ |
| $e_3$ | $4.19 \times 10^{-3}$ | $1.82 \times 10^{-4}$ | $3.01 \times 10^{-5}$ |
| $e_4$ | $1.44 \times 10^{-3}$ | $7.97 \times 10^{-5}$ | $1.17 \times 10^{-5}$ |
| $e_5$ | $2.23 \times 10^{-3}$ | $1.09 \times 10^{-4}$ | $1.50 \times 10^{-5}$ |
| $e_6$ | $3.19 \times 10^{-3}$ | $1.45 \times 10^{-4}$ | $2.03 \times 10^{-5}$ |
| $f_2$ | $1.22 \times 10^{-3}$ | $7.26 \times 10^{-5}$ | $1.53 \times 10^{-5}$ |
| $f_3$ | $1.26 \times 10^{-3}$ | $2.46 \times 10^{-5}$ | $4.68 \times 10^{-5}$ |
| $f_4$ | $8.47 \times 10^{-4}$ | $4.11 \times 10^{-5}$ | $7.38 \times 10^{-6}$ |
| $f_5$ | $8.50 \times 10^{-4}$ | $1.38 \times 10^{-5}$ | $8.20 \times 10^{-6}$ |
| $f_6$ | $9.81 \times 10^{-4}$ | $1.30 \times 10^{-5}$ | $2.49 \times 10^{-5}$ |
| Average | $1.83 \times 10^{-3}$ | $7.99 \times 10^{-5}$ | $1.98 \times 10^{-5}$ |

*2) Accuracy:* For benchmarking, the results from sampling method and traditional Galerkin method are considered respectively. In sampling method, $P_{G2}$ and $P_{G3}$ are sampled with equal intervals for 21 times from the lower boundary to the upper boundary, where the calculation is performed repeatedly just as the traditional load flow. From [20], it is known that the traditional Galerkin method using orthogonal basis can provide global optimal approximation, where the basis is chosen from

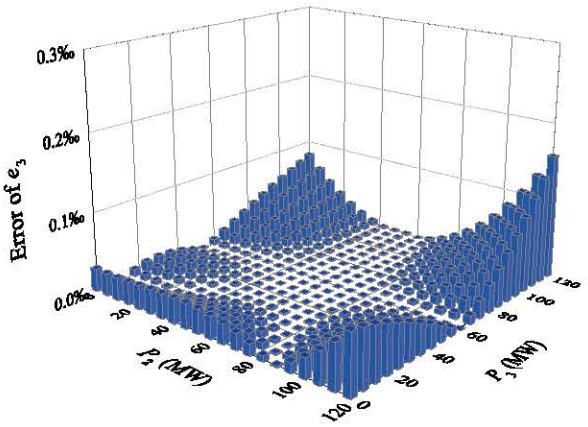


Fig. 4. Absolute error of $e_3$ by proposed method

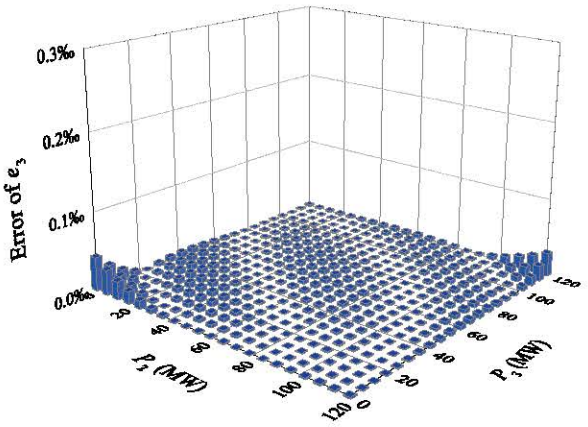


Fig. 5. Absolute error of $e_3$ by traditional Galerkin method

*Legendre Polynomials* in our analysis.

To show the accuracy, *Root Mean Square Error* (RMSE) is used here. As seen, the RMSE of $e$ and $f$ at all buses are given in TABLE III with $N_d$ increasing from 1 to 3. When $N_d$ becomes larger, the RMSE of every state decreases very fast. The average value of RMSEs of all states decreases from $1.83 \times 10^{-3}$ when $N_d = 1$ to $1.98 \times 10^{-5}$ when $N_d = 3$ exponentially, indicating that the approximate solutions converge to the exact solutions at a very fast speed. When $N_d = 3$, the approximate results at bus-3 have the maximum RMSE, which are $3.01 \times 10^{-5}$ for $e_3$ and $4.68 \times 10^{-5}$ for $f_3$ respectively.

The absolute error of $e_3$ is selected to show the error distribution where $N_d = 3$, as shown in Fig. 4 and Fig. 5. It can be seen that the result by our method is larger than that by traditional Galerkin method. In our method, the biggest error is 0.16‰, while the maximum error is only about 0.04‰ in traditional Galerkin method. However, from practical view, the approximation by our method can provide very accurate results by increasing $N_d$.

*3) Trend of coefficients:* The convergence property of coefficients when $N_d$ increases is investigated. For facilitating analysis, we consider the case with a single parameter $P_{G2}$, and $P_{G3}$ is set to 60 MW. The absolute value of coefficients of $e_3$ is shown on a semi-log plot in Fig. 6 when $N_d = 20$. It is seen that the coefficients converge exponentially. In the

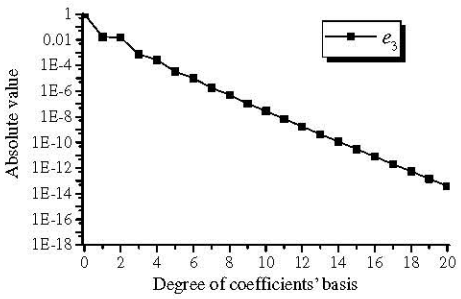


Fig. 6. Convergence of coefficients

TABLE IV
TIME CONSUMPTION COMPARISON OF TRADITIONAL GALERKIN METHOD AND PROPOSED METHOD

| $N_d$ | | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $N$ | | 3 | 6 | 10 | 15 | 21 |
| Time (s) | TGM | 0.234 | 2.688 | 14.157 | 62.766 | 227.078 |
| | PM | 0.016 | 0.037 | 0.061 | 0.104 | 0.187 |
| Equation dimension | TGM | 30 | 60 | 100 | 150 | 210 |
| | PM | | | 10 | | |

proposed method, the coefficients are identified from 0th-order successively, such that the approximation can be extended to higher orders. If the coefficients are small enough, it is reasonable to say the corresponding monomial terms have little influence on the results. In our method, the calculation can be terminated if the coefficients are small enough, i.e., the absolute value of coefficients are less than $10^{-4}$.

*4) Computational efficiency:* The efficiency of the proposed method is demonstrated from two aspects: the computational time; the maximum memory. To show the improvement on numerical effort, the time consumption and maximum memory used by traditional Galerkin method (TGM) and proposed method (PM) is compared. $N_d$ is increased from 1 to 5, and the results are shown in TABLE IV and Fig. 7.

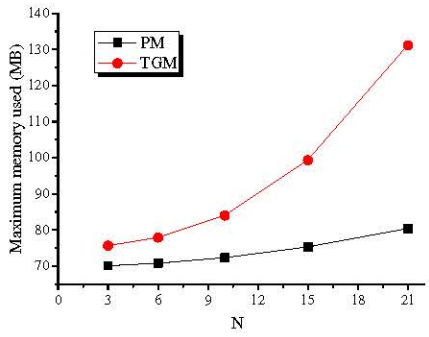


Fig. 7. Comparison of memory used between TGM and PM

From the table, it can be seen that the computational time of TGM is much larger than that of PM. In TGM, the time has increased about 970 times when $N_d$ increases from 1 to 5, while the time in PM only increases about 12 times. As seen in Fig. 7, in the TGM, the consumed memory

increases exponentially with growing $N$, but in our approach the occupied memory grows much slower. For example, when the order is 5, the consumed maximum memory increases around 7 times of that increased in our approach. The proposed method enjoys much better efficiency than TGM from both two aspects. Particularly, when $N_d$ becomes even larger, the computation will be easier to conduct by using the proposed method but the TGM might fail.

The reason is that the TGM solves an $(M \times N)$-dimensional equation at one go, while our new method only needs to solve $M$-dimensional equations for $N$ times. For example, when $N_d = 5$, TGM needs to solve a 210-dimensional equation, but our method only needs to solve 10-dimensional equations for 21 times.

### B. *IEEE* 118-*bus system*

To show the effectiveness of proposed method in large scaled system, IEEE 118-bus system is utilized here [21]. The generators, whose active power output are nonzero, are regarded as parameters, shown in TABLE V, and 3rd-order polynomial is used as approximation.

TABLE V
PARAMETERS IN IEEE 118-BUS SYSTEM

| Parameters numbers | Node number |
|---|---|
| 18 | 10, 12, 25, 26, 31, 46, 49, 54, 59, 61, 65, 66, 80, 87, 89, 100, 103, 111 |

From (5), we know that the number of coefficients to be identified is $234 \times \frac{(18+3)!}{18!3!} = 311220$. In traditional Galerkin method, it is very hard to solve such high-dimensional equations. If there are more parameters, the equations will have even higher dimension, causing that the solution of coefficients may fail. In the proposed method, the coefficients are found by solving the decoupled Galerkin equations successively. Thus, the difficulty in solving high-dimensional Galerkin equations at one go has been avoided. The results by the proposed method are shown in TABLE VI, and the calculation in this case by traditional Galerkin method fails.

TABLE VI
RESULTS OF THE IEEE 118-BUS SYSTEM

| Order | 0th | 1st | 2nd | 3rd | Total |
|---|---|---|---|---|---|
| Number of coefficients | 234 | 4212 | 40014 | 266760 | 311220 |
| Time consumed (s) | 4.14 | 3.125 | 54.562 | 453.266 | 515.093 |

It can be seen that the coefficients number of first-order is 18 times of zeroth-order, but the time for solving the first-order coefficients is less than that for zeroth-order. The reason is that the load flow model is a second-order system, so the Galerkin equation for zeroth-order shown as (10) is nonlinear, while the equation for first-order given as (11) is linear.

Fig. 8 shows the trend of coefficients with respect to different polynomial orders. As seen, the coefficients have

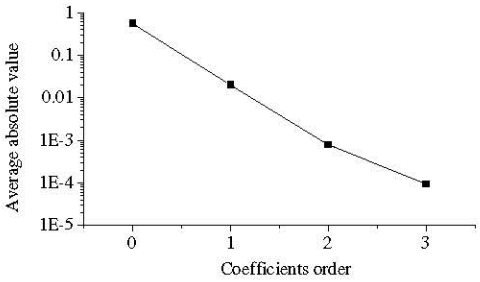


Fig. 8. Trend of coefficients in 118-bus system

an exponential convergence trend as well, when polynomial order increases. The average absolute value of the third-order coefficients is only $9.49 \times 10^{-5}$, which is very small from a practical view.

## VI. CONCLUSION

In parametric problems, polynomial approximation is one powerful tool to cope with the nonlinearity. In traditional Galerkin method, the practical problems with large number of parameters and states may involve coupled and high-dimensional equations, causing the computation to be very difficult and sometimes even to fail. A new methodology for this type of problems in power systems is introduced in this paper, which is derived in the form of generalized Galerkin method. In our approach, we choose the power series as trial basis functions and then the Galerkin equations are decoupled by projecting the residual onto a set of special test basis. Thus, the equations can be solved in a low-dimensional manner successively.

It is admitted that the approximate solution by our method is not as accurate as that by traditional Galerkin method, but the accuracy can be improved if the polynomial order is increased. The decoupled solution process enables the approximation to be extended to higher order cases easily. Case studies show that the proposed method gains better efficiency especially in large-scaled systems.

## APPENDIX A
## PROOF OF THE EXISTENCE OF TEST BASIS

For the test basis satisfying (7) and (8), we try to proof the $\Gamma_k$ does always exist. It is assumed that, for any continuous arbitrary parameter $p$, one can find a neighborhood $[a, b]$ where $p$ can be turned into $p' \in [-1, 1]$ by a linear transformation. Thus, the parameter domain is assumed as $[-1, 1]$.

In parametric problem, the maximum polynomial order of governing equations is assumed to be $N_{MP}$, so the maximum polynomial order of the residual will be $N_{MP} \times N_d$, denoted as $N_{\mathcal{G}}$. To implement the calculation, the condition in (7) and (8), $N_{\mathcal{G}} + 1$ equations should be satisfied for each $k$, so a polynomial

$$\Gamma_k = \sum_{j=0}^{N_{\mathcal{G}}} \hat{\beta}_{k,j} p^j \tag{A.1}$$

with $N_{\mathcal{G}} + 1$ unknown coefficients $\hat{\beta}_{k,i}$ can be uniquely determined for every $\Gamma_k \in \{\Gamma_k\}_{k=1}^{N}$.

Substitute (A.1) into (7) and (8), then for $1 \leq k < N$,

$$\begin{cases} \sum_{j=0}^{N_{\mathcal{G}}} \hat{\beta}_{k,j} \int_{-1}^{1} p^{i+j} dp &= 1, \quad 0 \leq i < k \\ \sum_{j=0}^{N_{\mathcal{G}}} \hat{\beta}_{k,j} \int_{-1}^{1} p^{i+j} dp &= 0, \quad k \leq i \leq N_{\mathcal{G}} \end{cases} \tag{A.2}$$

and for $k = N$

$$\sum_{j=0}^{N_{\mathcal{G}}} \hat{\beta}_{k,j} \int_{-1}^{1} p^{i+j} dp = 1, \quad 0 \leq i \leq N_{\mathcal{G}} \tag{A.3}$$

Rewrite the left side of (A.2) and (A.3) in matrix form:

$$\begin{pmatrix} \int_{-1}^{1} p^{0+0} dp & \cdots & \int_{-1}^{1} p^{0+N_{\mathcal{G}}} dp \\ \vdots & \ddots & \vdots \\ \int_{-1}^{1} p^{N_{\mathcal{G}}+0} dp & \cdots & \int_{-1}^{1} p^{N_{\mathcal{G}}+N_{\mathcal{G}}} dp \end{pmatrix} \begin{pmatrix} \hat{\beta}_{k,0} \\ \vdots \\ \hat{\beta}_{k,N_{\mathcal{G}}} \end{pmatrix} \tag{A.4}$$

For simplicity, we use $\mathbf{H}\hat{\mathbf{b}}_k$ to represent (A.4). Here $\mathbf{H}$ is a *Hankel matrix* with dimension $(N_{\mathcal{G}}+1) \times (N_{\mathcal{G}}+1)$, and $\hat{\mathbf{b}}_k$ is the vector $(\hat{\beta}_{k,0}, \cdots, \hat{\beta}_{k,N_{\mathcal{G}}})^T$.

If $\hat{\mathbf{b}}_k$ is solvable, the matrix $\mathbf{H}$ should be invertible. From [22], it is known that this matrix is always definite positive, such that the test basis in the form of (A.1) does always exist.

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