



Citation for published version:

Cox, A & Källblad, S 2017, 'Model-independent bounds for Asian options: A dynamic programming approach', SIAM Journal on Control and Optimization, vol. 55, no. 6, pp. 3409–3436. <https://doi.org/10.1137/16M1087527>

DOI:

[10.1137/16M1087527](https://doi.org/10.1137/16M1087527)

Publication date:

2017

Document Version

Publisher's PDF, also known as Version of record

[Link to publication](#)

© 2017 SIAM. The final publication is available at epubs.siam.org via <https://doi.org/10.1137/16M1087527>.

University of Bath

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

MODEL-INDEPENDENT BOUNDS FOR ASIAN OPTIONS: A DYNAMIC PROGRAMMING APPROACH*

ALEXANDER M. G. COX[†] AND SIGRID KÄLLBLAD[‡]

Abstract. We consider the problem of finding model-independent bounds on the price of an Asian option, when the call prices at the maturity date of the option are known. Our methods differ from most approaches to model-independent pricing in that we consider the problem as a dynamic programming problem, where the controlled process is the conditional distribution of the asset at the maturity date. By formulating the problem in this manner, we are able to determine the model-independent price through a PDE formulation. Notably, this approach does not require specific constraints on the payoff function (e.g. convexity), and would appear to generalize to many related problems.

Key words. model-independent pricing, dynamic programming, measure-valued martingale

AMS subject classifications. 91G20, 93E20, 49L20, 60G48

DOI. 10.1137/16M1087527

1. Introduction. Since the seminal paper of Hobson [37], there has been substantial interest in questions of the following form: Given an asset with price $(S_t)_{t \in [0, T]}$, a derivative whose payoff X_T depends on the path of the asset, and the prices of call options at maturity time T , find a static portfolio of calls and a dynamic trading strategy in the asset which superhedges the derivative at time T , under (essentially) *any* model for the asset. The class of models considered are usually very large (for example, all models with continuous paths), and so the resulting price is usually called the model-independent superhedging price.

The problem of finding the model-independent superhedging price is closely related to the problem of identifying the largest model-based price: specifically, in a classical setting, one would expect the prices of all options to be given as the expected value under some risk-neutral measure,¹ and by specifying the call prices at time 0, the distribution of S_T under this risk-neutral measure is determined. It is therefore natural to conjecture that the model-independent superhedging price is equal to $\sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}[X_T]$, where the supremum is taken over all probability measures \mathbb{Q} such that $(S_t)_{t \in [0, T]}$ is a martingale and S_T has the distribution determined by the call options. Recently, a number of papers, starting with Beiglböck, Henry-Labordère, and Penkner [10] in discrete time, and followed up by Dolinsky and Soner [24] in continuous time (see also [1, 7, 9, 11, 25, 40]), have made this result explicit under a variety of technical conditions. Note that in this formulation it is very natural to consider the supremum

*Received by the editors August 2, 2016; accepted for publication (in revised form) May 11, 2017; published electronically November 2, 2017.

<http://www.siam.org/journals/sicon/55-6/M108752.html>

Funding: This project began while the first author was visiting École Polytechnique. The work of the second author was supported by the ERC 321111 Rofirm, the ANR Isotace, the Chairs Financial Risks (Risk Foundation, sponsored by Société Générale), and Finance and Sustainable Development (IEF sponsored by EDF and CA).

[†]Department of Mathematical Sciences, University of Bath, Bath, UK (a.m.g.cox@bath.ac.uk, <http://www.maths.bath.ac.uk/~mapamgc/>).

[‡]CMAP, École Polytechnique, Paris, France; and Vienna University of Technology, Vienna, Austria (sigrid.kaellblad@tuwien.ac.at).

¹For ease of presentation, we largely assume that the interest rate is zero; this has no substantial effect on our main results.

over the set of probability measures as a primal problem, and the infimum over the class of superhedging strategies as the corresponding dual problem.

An alternative approach to these problems, following Hobson [37], is to use a time-change argument to reformulate the primal problem in terms of an optimization over solutions to the Skorokhod embedding problem; that is, to argue that, up to an unknown time change τ_t , the martingale $S_t = B_{\tau_t}$ is a time change of a Brownian motion. For a number of important quantities (maximum, quadratic variation, local time, etc.) the values of these quantities for the asset price up to time T and for the Brownian motion up to (the stopping time) τ_T agree. It can often then be argued that the choice of a model for S_t with a given law and the choice of a stopping time τ_T are equivalent provided B_{τ_T} has the required distribution (and satisfies an integrability constraint). The latter problem is well known as the Skorokhod embedding problem (SEP). A common approach to solving the model-independent superhedging problem is then to consider the corresponding SEP. If an optimal solution to this problem can be found, then it is often possible to guess the correct solution to the corresponding dual problem, and interpret this in terms of a superhedging strategy. This approach has been used in, e.g., [14, 16, 17, 18, 20, 19, 34, 38, 36, 42]; see also the survey article of Hobson [35].

Of note in all of the known optimal solutions to the SEP is that some underlying structure is required on the form of the option payoff. For example, if we write $M_t := \sup_{u \leq t} B_u$ for the maximum process, then the optimal constructions are known to maximize $\mathbb{E}[F(M_\tau)]$ over solutions to the SEP, provided that F is monotonic. To the best of our knowledge, the optimal construction when F is not monotonic is not known. Similarly, in the case of variance options or the local time, the function F must be concave/convex in order to have a known optimal solution. In Beiglböck, Cox, and Huesmann [8], this behavior was explained in terms of a natural convexity property which holds when a path-swapping operation is performed. It follows from this operation that many constructions of solutions to the SEP are optimal when the payoff to be optimized has such a convexity property. However, without the corresponding convexity, a “nice” description of the optimal solution seems impossible. One of the key strengths of the results described in this paper is that our methods are *not* constrained by such a convexity assumption on the payoff, and therefore will work for very general payoffs.

The main results in this paper concern the case where the option described above is an Asian option, that is, $X_T = F(A_T)$, where $A_t = \int_0^t S_u du$ (we omit the usual scaling factor $\frac{1}{t}$ for notational ease), and we consider the primal version of the problem, that is, we look to maximize $\mathbb{E}[F(A_T)]$ over all price processes $(S_u)_{u \in [0, T]}$ which are martingales, and which satisfy a constraint on the terminal law $S_T \sim \mu$. Notably, the Asian example already falls outside the case of payoffs which can easily be handled by SEP methods, since the whole time change $(\tau_u)_{u \in [0, T]}$ and not just the final time τ_T is already important in determining the value of A_T . However, in the case where the function F is convex, the optimal model is still easily determined: essentially, the asset will jump to its terminal distribution immediately, and the manner in which this is done (the ‘embedding component’) turns out to be irrelevant. This result was first given in Stebegg [45], which, to the best of our knowledge, is the first paper to characterize optimality in a setting where the SEP approach fails, or more generally to consider a problem of this form in continuous time without using the SEP approach. The standing assumptions in [45] are slightly different from ours — essentially, [45] allows a slightly more general setup (general starting measures;

discrete and continuously monitored payoffs are included) at the cost of considering only convex payoff functions (see also section 4.1). At a heuristic level, this restriction to convex functions in [45] appears comparable to the convexity constraint described above for the SEP in determining the “simple” set of optimizers. We also observe that there is a long history of considering model-independent bounds for the prices of Asian options (e.g., [23, 4, 3, 15, 22, 31]), although we note that, in contrast to the case considered in this paper, existing results tend to consider discretely monitored Asian options, often when call options on the underlying asset are traded at some or all intermediate maturities.

The novelty of our approach relates to the manner in which we formulate the problem as a dynamic programming problem. In particular, we include the conditional law of the final value of the asset price in the formulation of our problem. The condition that the process is a martingale with this conditional law is then formulated in terms of the behavior of the conditional law. Specifically, we require the conditional law ξ_t to be a *measure-valued martingale*, by which we mean that $(\xi_t(A))_{t \in [0, T]}$ is required to be a martingale for any (Borel) set A . We will show in section 2 that this condition is equivalent to the original formulation. In particular, by requiring $\xi_0 = \mu$ and requiring ξ_T to be singular, we enforce the condition that the terminal law of $S_t = \int x \xi_t(dx)$ is μ . The concept of a measure-valued martingale is classical (see, e.g., Dawson [21] and Horowitz [39]; in this literature, the rather confusing terminology “martingale measure” is also common), and has appeared in the context of the SEP in Eldan [29]. A key result for our purposes is that we are able to show that our value function is continuous in ξ , where the space of measures is equipped with the Wasserstein topology. This allows us to approximate ξ by atomic measures, which enables us to reduce the whole problem to a finite-dimensional problem, at which point classical methods can be used (section 3). We note that, in this discrete formulation, our problem could be compared to (a special case of) the problems considered in Žitković [46], El Karoui and Tan [28], and Bouchard and Nutz [12], although we prove our results via more direct, classical methods. We also remark that Galichon, Henry-Labordère, and Touzi [32] have also previously used a stochastic control approach to solve a similar problem, but in a rather different manner to the approach of this paper. In section 4 we are able to use these results to provide concrete solutions to certain problems.

We believe that the methods and ideas we describe in this paper can be applied far beyond the case of Asian options. However, the Asian option setting does provide us with a useful structure which we are able to exploit in the construction and formulation of our problem. In particular, it is easy to show that “small” changes in the conditional terminal law result in small changes in the value function for the problem, the increase in the average dA_t is easy to write in terms of the current conditional law, and also our underlying problem is not strongly affected by jumps in the process: the value functions for the problem where the path is assumed to be continuous and for the problem where the path is assumed to be càdlàg are identical (although optimizers may exist in the càdlàg formulation and not in the continuous formulation). In section 5 we discuss further extensions.

2. Problem formulation using measure-valued martingales. Consider the following problem: We have an asset $(S_t)_{t \in [0, T]}$ in a market with a riskless bank account and a time horizon T , and we wish to find a model-independent superhedge of an option which pays the holder $F(A_T)$, where $\frac{1}{T}A_T = \frac{1}{T} \int_0^T S_t dt$ is the running

average.² We will consider the problem where the law of the underlying asset at maturity S_T is given at time 0, and we consider the primal optimization problem, that is, to find the law of the process which maximizes $\mathbb{E}[F(A_T)]$ subject to $S_T \sim \mu$. Here, we consider the case where the interest rate $\rho = 0$, although the extension to constant interest rates is straightforward.

Our basic approach is to consider the problem as a dynamic programming problem where the current state includes the conditional distribution of the process at maturity. Specifically, we assume $S_T \in \mathbb{R}_+$, and with $\mathcal{M}(\mathbb{R}_+)$ the set of Borel measures on \mathbb{R}_+ , we write

$$(1) \quad \mathcal{P}^1 := \left\{ \mu \in \mathcal{M}(\mathbb{R}_+) : \mu(\mathbb{R}_+) = 1, \int |x| \mu(dx) < \infty \right\}.$$

Our aim is to set the problem up as a dynamic programming problem. We suppose that the problem evolves on an artificial time horizon $r \geq 0$ on which a measure-valued process $(\xi_r)_{r \geq 0}, \xi_r \in \mathcal{P}^1$ evolves. We let $(T_r)_{r \geq 0}$ be an increasing process in $[0, T]$. Our interpretation of this quantity is that T_r represents the “real” time at the artificial time r . Roughly, the slower T_r increases, the higher “volatility” we see in the real-time scale. We set the problem up in this way since we wish to allow a substantial change in the r timescale to happen instantaneously in real time, which may correspond to jumps in the asset price. However, we wish to maintain a “continuous” evolution of the measure-valued process over its natural timescale (we do not wish to deal with jumps in the measure-valued process). The choice of the increasing processes (T_r) will form part of the control of the problem—specifically, we optimize over $\lambda_r \in [0, 1]$ and define

$$(2) \quad T_r = \int_0^r \lambda_s ds.$$

The second part of the control will be the choice of the measure-valued process ξ . This process will determine the conditional distribution of the asset (S_t) . Specifically, the initial value is $\xi_0 = \mu$, where μ is the terminal law of the asset at time 0, and over time we suppose that (ξ_r) evolves in a manner that ensures that (S_t) remains a martingale.

DEFINITION 2.1. *We say that an adapted process $(\xi_r)_{r \geq 0}$ with $\xi_r \in \mathcal{P}^1$ is a measure-valued martingale if, for any $f \in C_b(\mathbb{R}_+)$, $\xi.(f) := \int f(x) \xi.(dx)$ is a martingale.*

Note trivially that if $f \in C_b(\mathbb{R}_+)$, then the martingale $\xi.(f)$ is uniformly integrable with well-defined limit $\xi_\infty(f)$ (in particular, ξ_∞ is a measure; see [39, Proposition 2.1]).

Remark 2.2. An adapted process (ξ_r) with $\xi_r \in \mathcal{P}^1$ is a measure-valued martingale if and only if $\xi.(A)$ is a martingale for any $A \in \mathcal{B}(\mathbb{R})$. Indeed, the indicator function over an interval of \mathbb{R} may be approximated by continuous functions, and an application of the monotone class theorem yields that the claim holds for any $A \in \mathcal{B}(\mathbb{R})$ (see Lemma 2.12 for a similar argument). Conversely, any $f \in C_b(\mathbb{R}_+)$ may be approximated from below by simple functions. In fact, by the same argument, we see that $\xi.(f)$ is a martingale for any (nonnegative) measurable function.

²We use the slightly unconventional notation $A_T = \int_0^T S_t dt$ to avoid an unnecessary number of terms of the form $\frac{1}{T}$ in all our calculations; it is clear that this is just a scaling factor and can be removed.

We will think of measure-valued martingales as processes (ξ_r) evolving in time. In order to discuss their continuity, we make the following definition.

DEFINITION 2.3. *We say that a measure-valued martingale is continuous if, for any 1-Lipschitz function f , $\xi.(f) = \int f(x)\xi.(dx)$ is a.s. continuous.*

It immediately follows that $X. = \int x\xi.(dx)$ is a continuous process whenever (ξ_r) is continuous; conversely, whenever (ξ_r) is continuous (a.s.) in the sense of weak convergence of measures, and its mean is also continuous, it is continuous in the above sense. Specifically, Definition 2.3 is equivalent to requiring (almost sure) continuity of $r \mapsto \xi_r$ in the topology of \mathcal{W}_1 , the first Wasserstein metric, by the duality of the Wasserstein distance [5, Theorem 6.1.1].

Remark 2.4. Our underlying probability spaces will generally be assumed to satisfy the usual conditions. Under this assumption, of course, for every $f \in C_b(\mathbb{R}_+)$, the martingale $\xi.(f)$ has a càdlàg version. More pertinently, we can choose a version of any given measure-valued martingale (ξ_r) such that $\xi.(f)$ is càdlàg for every 1-Lipschitz function f . Indeed, naturally $\eta_r := (1 \vee x)\xi_r(dx)$, $r \geq 0$, also satisfies Definition 2.1 (with exception for the fact that it need not lie in \mathcal{P}^1). Hence, we may apply Theorem 2.5 in [39] to deduce the existence of a version of (η_r) , and thus of (ξ_r) , such that $\eta.(f)$ is càdlàg for any bounded Borel function f ; the latter implies that $\xi.(f)$ is càdlàg for any 1-Lipschitz function f . In what follows, we will assume that we always take this càdlàg (in the sense of Definition 2.3) version of (ξ_r) .

We emphasize that the support of a measure-valued martingale can only ever decrease: if $\xi_{r_0}(A) = 0$, then $\xi_r(A) = 0$ for all $r \geq r_0$. In the particularly nice case that ξ_{r_0} is an atomic measure, then for all $r \geq r_0$ the measure will also be atomic, and supported on the same set of points. In particular, the spatial distribution of such a measure will not change, only the weights attributed to each point. Since the weight associated to each point is a martingale and is constrained to lie in the interval $[0, 1]$, it follows that in the limit as $r \rightarrow \infty$, the weight assigned to each point must converge to a limit; commonly, this limit will be assumed to be either 0 or 1, and this motivates the following definitions. Consider the set of singular measures on \mathbb{R}_+ , $\mathcal{P}^s = \{\mu \in \mathcal{M}(\mathbb{R}_+) : \mu = \delta_y, y \in \mathbb{R}_+\}$; then we have the following.

DEFINITION 2.5. *We say that a measure-valued martingale (ξ_r) is terminating if $\xi_r \rightarrow \xi_\infty \in \mathcal{P}^s$ a.s. as $r \rightarrow \infty$, where the convergence is in the sense of weak convergence of measures. It is finitely terminating if $\tau_s := \inf\{r \geq 0 : \xi_r \in \mathcal{P}^s\}$ is a.s. finite.*

LEMMA 2.6. *Suppose (ξ_r) is a terminating measure-valued martingale with $\xi_0 = \mu$. Then $X. = \int x\xi.(dx)$ is a nonnegative uniformly integrable (UI) martingale with $X_\infty \sim \mu$.*

Proof. The martingale property follows from Remark 2.2. In particular, for $y \in \mathbb{R}_+$, we have that

$$\mathbb{E}[(y - X_\infty)_+] = \mathbb{E}\left[\int (y - x)_+ \xi_\infty(dx)\right] = \int (y - x)_+ \mu(dx).$$

Since $\mathbb{E}[(y - X_\infty)_+]$ characterizes the law of X_∞ uniquely, $X_\infty \sim \mu$. As $X_0 = \int x \mu(dx) < \infty$, it also follows that $\mathbb{E}[|X_r|] = \mathbb{E}[X_r] < \infty$. Finally, we observe that $X_r = \mathbb{E}[X_\infty | \mathcal{F}_r]$ and $X_\infty \sim \mu \in \mathcal{P}^1$ imply that X is a UI martingale. \square

COROLLARY 2.7. *If (ξ_r) is a terminating measure-valued martingale with $\xi_0 = \mu$, then for every 1-Lipschitz function f , $X^f := \xi.(f) = \int f(x)\xi.(dx)$ is a UI martingale with $X_0^f = \int f d\mu$ and $X_\infty^f \sim f(\mu)$.*

Having introduced these concepts, we will take the second control in our problem to be the choice of a process (ξ_r) subject to the constraint that (ξ_r) is a terminating measure-valued martingale with $\xi_0 = \mu$.

Observe that, once we have chosen a pair of processes (λ_r) and (ξ_r) , the idea is to let the “asset price” at time T_r be given by $\int x \xi_r(dx)$. More precisely, since the process T_r is nondecreasing, there exists a right-continuous inverse $T_t^{-1} = \inf\{r > 0 : T_r > t\}$; note that there can be only countably many jumps in T_t^{-1} . We then introduce $T_t^{-1,*} = \inf\{r > 0 : T_r \wedge T > t\}$ and define the càdlàg process

$$(3) \quad S_t = \int x \xi_{T_t^{-1,*}}(dx), \quad t \leq T,$$

noting that $S_T = \int x \xi_\infty(dx)$. The average process is then given by

$$(4) \quad A_t = \int_0^t S_s ds = \int_0^t \int x \xi_{T_s^{-1,*}}(dx) ds.$$

Then the main problem we wish to solve is as follows.

Problem 2.8 (Basic optimization problem). Given an integrable probability measure μ on \mathbb{R}_+ and a function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, we want to find a probability space $(\Omega, \mathcal{H}, (\mathcal{H}_t)_{t \in [0, T]}, \mathbb{P})$ which satisfies the usual conditions and a càdlàg UI martingale $(S_t)_{t \in [0, T]}$ on this space with $S_T \sim \mu$ which maximizes $\mathbb{E}[F(A_T)]$ over the class of such probability spaces and processes.

Remark 2.9. Since we do not require \mathcal{H}_0 to be trivial, S_0 need not be a constant. However, for the Asian option, it holds for any probability space and càdlàg martingale S_t , as given in Problem 2.8, that one may construct a sequence of càdlàg martingales (S^n) such that $S_0^n = s_0 \in \mathbb{R}$, $S_T^n \sim \mu$, and

$$(5) \quad \lim_{n \rightarrow \infty} \mathbb{E}[F(A_T^n)] = \mathbb{E}[F(A_T)].$$

Hence, the value of Problem 2.8 remains the same under the additional assumption that $S_0 = s_0$, and for any optimizer to the former problem an approximately optimal sequence may be constructed for the latter; we refer the reader to Lemma 5.1 in Stebegg [45] for a precise argument (see also Assumption 3.9 in Guo, Tan, and Touzi [33] and the proof of Lemma 4.1 in Dolinsky and Soner [25] for related arguments). We argue in the proof of Lemma 2.11 below that the value of Problem 2.8 remains the same if we restrict to martingales which are piecewise constant over arbitrary but finite partitions. Hence, a similar argument yields that the value of the problem also remains the same if we restrict to continuous martingales.

To formalize this remark, and since we generally wish to work with probability spaces satisfying the usual conditions, we extend our framework slightly: given a complete probability space with a right-continuous filtration $(\mathcal{G}_t)_{t \geq 0}$, we can always extend the filtration to $(-\varepsilon, \infty)$ for some $\varepsilon > 0$ by taking \mathcal{G}_t to be the (completion of the) trivial σ -algebra for $t < 0$. Similarly, a càdlàg process Z_t on $[0, \infty)$ can be extended to a càdlàg process on $(-\varepsilon, \infty)$ by setting Z_t to be some constant value for $t < 0$. Since this is constant we may write Z_{0-} for this value without confusion. Similarly, to avoid the excessive use of ε 's, we write $(\mathcal{G}_t)_{t \in [0-, \infty)}$ to denote a filtration extended in this manner. All other terminology (e.g., martingales) are then to be understood in the obvious way.

Our first claim is that Problem 2.8 is equivalent to the following formulation.

Problem 2.10 (Measure-valued martingale formulation). Given an integrable probability measure μ on \mathbb{R}_+ and a function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, we want to find a probability space $(\Omega, \mathcal{G}, (\mathcal{G}_r)_{r \in [0-, \infty)}, \mathbb{P})$ satisfying the usual conditions, a progressively measurable process $\lambda_r \in [0, 1]$ such that $\int_0^T \lambda_s ds \rightarrow \infty$ a.s. as $r \rightarrow \infty$, and a finitely terminating measure-valued $(\mathcal{G}_r)_{r \in [0-, \infty)}$ -martingale $(\xi_r)_{r \in [0-, \infty)}$ with $\xi_{0-} = \mu$, which maximizes $\mathbb{E}[F(A_T)]$ with A_T given by (4).

LEMMA 2.11. *Problems 2.8 and 2.10 are equivalent, in the sense that the values coincide, and if there exists an optimizer in Problem 2.8, then we can construct an optimizer for Problem 2.10 and vice versa; if the supremum for the problem can only be approximated, then equivalent approximating sequences can be found.*

Moreover, if F is continuous, then the value of the problem remains the same if we restrict Problem 2.10 to probability spaces and processes such that the filtration \mathcal{G}_r is the usual augmentation of the natural filtration of a $(\mathcal{G}_r)_{r \geq 0}$ -Brownian motion and $(\xi_r)_{r \geq 0}$ is continuous in the sense of Definition 2.3.

As a consequence, if the restricted measure-valued martingale problem admits a solution, then a corresponding optimizer may be constructed also for the basic optimization problem. Before proving this result, we give an auxiliary lemma.

LEMMA 2.12. *Suppose $(X_r)_{r \in [0, \infty]}$ is a martingale on $(\Omega, \mathcal{G}, (\mathcal{G}_r)_{r \in [0, \infty)}, \mathbb{P})$ such that $\mathbb{E}[|X_\infty|] < \infty$. Then there exists a terminating measure-valued martingale $(\xi_r)_{r \in [0, \infty]}$ such that $X_r = \int x \xi_r(dx)$ a.s. for all $r \in [0, \infty]$.*

Proof. Define the \mathcal{G}_∞ -measurable random measure $\xi_\infty(dx) := \delta_{X_\infty}(dx)$. Then $\xi_\infty \in \mathcal{P}^1$ a.s. Further, let \mathcal{A} be a countable Boolean algebra generating $\mathcal{B}(\mathbb{R})$ and define the \mathcal{G}_r -measurable set function ξ_r by

$$(6) \quad \xi_r(A) := \mathbb{E}[\xi_\infty(A) | \mathcal{G}_r], \quad A \in \mathcal{A}.$$

Since ξ_∞ is countably additive a.s., then so is ξ_r . Indeed, for $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, such that $\cup A_n \in \mathcal{A}$ and $A_i \cap A_j = \emptyset$, $i \neq j$,

$$\xi_r(\cup A_n) = \mathbb{E}[\xi_\infty(\cup A_n) | \mathcal{G}_r] = \sum_{n=1}^\infty \mathbb{E}[\xi_\infty(A_n) | \mathcal{G}_r] = \sum_{n=1}^\infty \xi_r(A_n).$$

Since ξ_r is also finite, it follows that (6) uniquely defines a \mathcal{G}_r -measurable measure on $\mathcal{B}(\mathbb{R})$, up to a null set, on which we arbitrarily take $\xi_t = \delta_0$. Next, let $\mathcal{O} := \{A \in \mathcal{B}(\mathbb{R}) : \xi_r(A) \text{ is a martingale on } [0, \infty]\}$. Since, for any $r \in [0, \infty]$, ξ_r is a measure and is thus continuous from below, it follows that \mathcal{O} is a monotone class. Indeed, for $A_n \in \mathcal{O}$, $n \in \mathbb{N}$, with $A_n \subset A_{n+1} \subset \dots$,

$$\mathbb{E}[\xi_\infty(\cup A_n) | \mathcal{G}_r] = \lim_{n \rightarrow \infty} \mathbb{E}[\xi_\infty(A_n) | \mathcal{G}_r] = \lim_{n \rightarrow \infty} \xi_r(A_n) = \xi_r(\cup A_n).$$

Since $\mathcal{A} \subset \mathcal{O}$, we have by the monotone class theorem that $\xi_r(A)$ is a martingale for all $A \in \mathcal{B}(\mathbb{R})$. Since $\mathbb{E}[\int |x| \xi_\infty(dx)] = \mathbb{E}[|X_\infty|] < \infty$, this yields in particular that $\xi_r \in \mathcal{P}^1$ for $r \in [0, \infty]$. According to Remark 2.2, (ξ_r) is thus a measure-valued martingale. It is terminating by definition. It therefore follows directly from Lemma 2.6 that $\int x \xi_r(dx) = X_r$ a.s. for $r \in [0, \infty]$. \square

Remark 2.13. The above result can be partially found in [39, Theorem 1.3], on taking $(\xi_r)_{r \in [0, \infty]}$ as the optional projection of the random measure δ_{X_∞} .

Proof of Lemma 2.11. We show that every candidate solution to Problem 2.8 gives rise to a candidate solution to Problem 2.10, and *vice versa*. The claim about optimizers follows.

We first suppose that we have a solution to Problem 2.8. By Monroe [43, Theorem 11], there exists a probability space $(\Omega, \mathcal{G}', (\mathcal{G}'_s)_{s \in [0, \infty)}, \mathbb{P})$, a (\mathcal{G}'_s) -Brownian motion (W_s) with $W_0 = \int x \mu(dx)$, and a right-continuous (\mathcal{G}'_s) -time change $(\tau_t)_{t \in [0, T]}$ such that (S_t) and (W_{τ_t}) are equal in law, τ_T is a.s. finite, and $W_{\cdot \wedge \tau_T}$ is a UI martingale. We then define ξ'_s to be the law of W_{τ_T} conditional on \mathcal{G}'_s . That is, we apply Lemma 2.12 to the process $W_{\cdot \wedge \tau_T}$ to obtain a terminating measure-valued martingale $(\xi'_s)_{s \in [0, \infty]}$ such that $\int x \xi'_s(dx) = W_{s \wedge \tau_T}$, $s \geq 0$, a.s.; those properties of ξ'_s are preserved by defining $\xi'_{0-} = \mu$.

We now need to construct a measurable process λ_r giving rise to a time change T_r via (2) such that the process (S_t) given by (3) is the required process. Note that by construction, $(S_t)_{t \geq 0}$ and $(\int x \xi'_{\tau_t}(dx))_{t \geq 0} = (W_{\tau_t})_{t \geq 0}$ are equal in law, and therefore they both give rise to the same value of $\mathbb{E}[F(A_T)]$. We will now modify the time change and deduce that this gives rise to the correct process. Specifically, we recall that τ_T is finite a.s. Let

$$(7) \quad T_t^{-1} = \tau_{t \wedge T} + t$$

and, in turn, define $T_r := \sup\{t \geq 0 : T_t^{-1} \leq r\}$, $r \geq 0$. From (7) we immediately see that T_t^{-1} is strictly increasing with $T_t^{-1} - T_s^{-1} \geq t - s$ for $t > s$, so that T_r is nondecreasing and 1-Lipschitz. In particular, $T_{\tau_T + T + r} = T + r$ for $r \geq 0$ so that T_t^{-1} given by (7) is indeed the right-continuous inverse of T_r . Further, with $R_r = r - T_r \wedge T$, $r \geq 0$, it follows that $R_{T_t^{-1}}(\omega) = \tau_t(\omega)$, $t < T$, and thus, defining $\xi_r := \xi'_{R_r}$, $r \geq 0$, we have that $(S_t)_{t < T}$ and $(\int x \xi_{T_t^{-1}}(dx))_{t < T}$ are equal in law. Therefore, let (\mathcal{G}_r) be the (right-continuous) time-changed filtration given by $\mathcal{G}_r = \mathcal{G}'_{R_r}$, $r \geq 0$. Then ξ_r is a finitely terminating measure-valued (\mathcal{G}_r) -martingale; indeed, $\xi_r \in \mathcal{P}^s$ for $r \geq \tau_T + T$. Further, $T_t^{-1} \in \mathcal{G}'_{\tau_t} = \mathcal{G}'_{R(T_t^{-1})} = \mathcal{G}_{T_t^{-1}}$ and thus $T_r \in \mathcal{G}_r$. Recalling the properties of T_r , we also deduce that there exists a process $\lambda_r \in [0, 1]$ which is \mathcal{G}_r -measurable and such that $T_r = \int_0^r \lambda_s ds$. Hence (possibly by taking a modification), λ_r can be assumed to be progressively measurable, and it is immediate that $T_r \rightarrow \infty$ as $r \rightarrow \infty$.

To see the converse, suppose we are given a solution to Problem 2.10. From Lemma 2.6 it follows immediately that $S = \int x \xi_{T^{-1},*}(dx)$ is the required process.

It remains to argue the second part of the lemma. Indeed, in general, the time change granted by Monroe [43] may not necessarily be measurable with respect to the Brownian filtration. However, for any probability space $(\Omega, \mathcal{H}, (\mathcal{H}_t)_{t \in [0, T]}, \mathbb{P})$ and càdlàg martingale $(S_t)_{t \in [0, T]}$, we may define a sequence $(S_t^n)_{t \in [0, T]}$ by

$$S_t^n = S_{[nt/T]T/n}, \quad n \geq 1.$$

Then the (S_t^n) are still martingales with $S_T \sim \mu$. Further, since F is continuous, $F(A_T^n)$ converges a.s. to $F(A_T)$, and an application of Fatou's lemma gives that $\mathbb{E}[F(A_T)] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[F(A_T^n)]$. In consequence, the value of Problem 2.8 remains the same if restricting to martingales which are piecewise constant over arbitrary but finite partitions. Since any discrete martingale may be embedded in a Brownian motion with stopping times measurable with respect to the Brownian filtration (cf., e.g., [26]), it follows that we may restrict to Brownian filtrations (\mathcal{G}_r) in Problem 2.10. By the martingale representation theorem, any (\mathcal{G}_r) -martingale is continuous. In consequence, recalling Remark 2.4, the (ξ_r) defined above can be assumed to be

continuous in the sense of Definition 2.3. The fact that the resulting measure-valued martingale is finitely terminating, and that the first time the integral of λ_s reaches T is finite also follows immediately from this embedding procedure. \square

Remark 2.14. We note that the embedding of a process $(S_t)_{t \in [0,1]}$ into the pair of a continuous measure-valued martingale $(\xi_r)_{r \in [0,\infty]}$ and time change $(\lambda_r)_{r \in [0,\infty]}$ is not unique. In particular, choosing $T_t^{-1} := 1 - e^{-\tau t} + \frac{t}{T}$ (cf. (7)) renders $T_T^{-1} \leq 2$ a.s., and the problem might be viewed as evolving on the *finite* timescale $r \in [0, 2]$. In Lemma 3.4 below, we will consider yet another scaling which yields a specific relation between the evolution of the ξ and the λ .

Remark 2.15. We observe in fact that, from the proof of the lemma, if $\lambda_r = 1$ for $r \in [u, v)$ for some interval $[u, v)$, then $\xi_r = \xi_u$ for all $r \in [u, v)$. In particular, $\lambda = 1$ corresponds to a constant ξ , and thus (S_t) is constant on $t \in [T_u, T_v)$.

3. The dynamic programming problem.

3.1. Problem formulation and continuity. We want to write the optimization problem as a “Markovian” optimization problem and address our original problem by embedding it into a family of conditional ones: we suppose that at time r , we have “real” time $T_r = t$, current law $\xi_r = \xi \in \mathcal{P}^1$, and running average $A_{T_r} = a$, and we wish to find

$$(8) \quad U(r, t, \xi, a) = \sup \mathbb{E}[F(A_T) | T_r = t, \xi_r = \xi, A_{T_r} = a],$$

where the supremum is taken over all time-change determining processes $(\lambda_u)_{u \in [r,\infty)}$ and measure-valued martingales $(\xi_u)_{u \in [r,\infty)}$ satisfying the conditions of Problem 2.10; by Lemma 2.11 it follows that $U(0, 0, \mu, 0)$ will be the value of the Asian option under the optimal model. As shown in Theorem A.3 in Appendix A, such a dynamic treatment can be formalized at the general level of Problem 2.10, and the corresponding dynamic programming principle does hold. However, to get a more explicit characterization of the function U , we will show that one can reduce the problem to one that is easier to handle by the use of classical methods. To this end, we first argue the continuity of the problem with respect to its marginal constraint.

At this stage, we directly define the function in (8) to equal the value of Problem 2.10 when the law to be embedded is given by ξ , the horizon by $T - t$, and the payoff function by $F(a + \cdot)$. Then we have the following lemma.

LEMMA 3.1. *Suppose F is a nonnegative Lipschitz function. The function $U : \mathbb{R}_+ \times [0, T] \times \mathcal{P}^1 \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is continuous (here the topology on \mathcal{P}^1 is the topology derived from the Wasserstein-1 metric) and independent of r .*

Proof. We begin by observing that continuity in all the variables except ξ follows immediately from the continuity properties of F : any small change in a will result in a direct shift in the final value of A_T , while small changes in t can be handled by computing the average of the same model over the modified time horizon. In addition, the independence of the problem on the value of the “measure-valued” timescale r is immediate.

We consider finally the continuity in the measure ξ . Consider a given probability space $(\Omega, \mathcal{G}, (\mathcal{G}_s)_{s \geq r}, \mathbb{P})$ and a measure-valued martingale $(\xi_s)_{s \geq r}$. Recall that \mathcal{W}_1 is the Wasserstein-1 metric space, and write $d_{\mathcal{W}_1}$ for the metric on this space. Let $\xi' \in \mathcal{P}^1$. We will first show that if $d_{\mathcal{W}_1}(\xi_r, \xi') < \varepsilon$, then there is a measure-valued martingale $(\xi'_s)_{s \geq r}$ such that $\xi'_r = \xi'$ and $\mathbb{E}[|\int x \xi_s(dx) - \int x \xi'_s(dx)| | \mathcal{G}_r] < \varepsilon$ for all $s \in [r, \infty)$. Recall that $d_{\mathcal{W}_1}(\xi_r, \xi') < \varepsilon$ implies that there exists a transport

plan $\Gamma \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}_+)$ such that $\xi'(dy) = \Gamma(\mathbb{R}_+, dy)$, $\xi_r(dx) = \Gamma(dx, \mathbb{R}_+)$, and $\int \int |x - y| \Gamma(dx, dy) < \varepsilon$. First, by disintegration (e.g., [5, Theorem 5.3.1]) there exists a Borel family of probability measures $m(x, dy)$ such that $\Gamma(dx, dy) = \xi_r(dx) m(x, dy)$.

Now define the process

$$\xi'_s(dy) := \int \xi_s(dx) m(x, dy), \quad s \geq r.$$

Then $\xi'_s \in \mathcal{P}^1$ and $\xi'_r = \xi'$. Further, for any $A \in \mathcal{B}(\mathbb{R})$, since $m(\cdot, A)$ is measurable,

$$\mathbb{E}[\xi'_u(A) | \mathcal{G}_s] = \mathbb{E}\left[\int m(x, A) \xi_u(dx) | \mathcal{G}_s\right] = \int m(x, A) \xi_s(dx) = \xi'_s(A), \quad s \leq u,$$

and thus ξ'_s , $s \geq r$, is a measure-valued martingale. Next, note that

$$\begin{aligned} \left| \int x \xi_s(dx) - \int x \xi'_s(dx) \right| &= \left| \int \int (x - y) \xi_s(dx) m(x, dy) \right| \\ &\leq \int \int |x - y| \xi_s(dx) m(x, dy). \end{aligned}$$

Hence

$$\mathbb{E}\left[\left| \int x \xi_s(dx) - \int x \xi'_s(dx) \right| \middle| \mathcal{G}_r\right] \leq \int \int |x - y| \Gamma(dx, dy).$$

By the definition of the metric on \mathcal{W}_1 , since $d_{\mathcal{W}_1}(\xi_r, \xi'_r) < \varepsilon$, we can find a transport plan Γ with the desired marginals and $\int \int |x - y| \Gamma(dx, dy) < \varepsilon$. Fix some process $(\lambda_s)_{s \geq r}$, and write $A_t^{\xi, \lambda}$ for the average process corresponding to the measure-valued process ξ and the time-change process λ , conditional on \mathcal{F}_r . Recalling (4), we have

$$\begin{aligned} \mathbb{E}\left[\left| A_T^{\xi, \lambda} - A_T^{\xi', \lambda} \right| \middle| \mathcal{G}_r\right] &= \mathbb{E}\left[\left| \int_{T_r}^T \int x \xi_{T_s^{-1}}(dx) ds - \int_{T_r}^T \int x \xi'_{T_s^{-1}}(dx) ds \right| \middle| \mathcal{G}_r\right] \\ &\leq \mathbb{E}\left[\left| \int_{T_r}^T \int x \xi_{T_s^{-1}}(dx) - \int x \xi'_{T_s^{-1}}(dx) \right| ds \middle| \mathcal{G}_r\right] \\ &\leq \varepsilon(T - T_r). \end{aligned}$$

Now fix $\varepsilon' > 0$ and consider $\xi, \xi' \in \mathcal{P}^1$ such that $d_{\mathcal{W}_1}(\xi, \xi') < \varepsilon'/(2T\zeta)$, where ζ is the Lipschitz constant of F . Then there exists $(\xi_s, \lambda_s)_{s \geq r}$ such that $\xi_r = \xi$ and $U(r, t, \xi, a) \leq \mathbb{E}[F(A_T^{\xi, \lambda}) | \mathcal{G}_r] + \varepsilon'/2$. Using the estimate above, and by the Lipschitz property of F , we can moreover find $(\xi'_s)_{s \geq r}$ such that $\xi'_r = \xi'$ and $\mathbb{E}[|F(A_T^{\xi, \lambda}) - F(A_T^{\xi', \lambda})| | \mathcal{G}_r] \leq \varepsilon'/2$. It follows that

$$U(r, t, \xi, a) \leq \mathbb{E}\left[F\left(A_T^{\xi, \lambda}\right) \middle| \mathcal{G}_r\right] + \varepsilon'/2 \leq \mathbb{E}\left[F\left(A_T^{\xi', \lambda}\right) \middle| \mathcal{G}_r\right] + \varepsilon' \leq U(r, t, \xi', a) + \varepsilon'.$$

By symmetry, $|U(r, t, \xi', a) - U(r, t, \xi, a)| \leq \varepsilon'$. Finally, we note that joint continuity follows as a simple adaptation of this argument combined with the arguments for the other parameters. \square

Since the function $U(r, t, \xi, a)$ is independent of the parameter r , we will often write $U(t, \xi, a)$ where there is no confusion.

Remark 3.2. Continuity of the primal problem as a function of μ was proved by alternative methods in Dolinsky and Soner [25, Theorem 4.1]. As demonstrated in Guo, Tan, and Touzi [33, Proposition 4.3], upper semicontinuity can be proved by yet an alternative method. We now recall their argument in the present context. To this end, consider the space of all càdlàg paths on $[0, 1]$ and let the filtration be that generated by the canonical process (S_t) . Problem 2.8 can then be formulated as maximizing $\mathbb{E}[F(A_T)]$ over martingale measures satisfying the constraint $S_T \sim^{\mathbb{P}} \mu$. Given a sequence of probability measures (μ_n) on \mathbb{R}_+ converging in $d_{\mathcal{W}_1}$ to μ , let (\mathbb{P}_n) be a sequence of martingale measures such that $S_T \sim^{\mathbb{P}_n} \mu_n$ and

$$(9) \quad \lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{P}_n} [F(A_T)] = \limsup_{n \rightarrow \infty} U(0, \mu_n, 0).$$

According to Jakubowski [41], there exists a subsequence $(\mathbb{P}_{n_k})_{k \geq 1}$ which is weakly convergent with respect to the so-called S -topology on the set of càdlàg paths. Let \mathbb{P}_0 be the limiting measure. According to Guo, Tan, and Touzi [33], \mathbb{P}_0 is then a martingale measure and $S_T \sim^{\mathbb{P}_0} \mu$. Since the mapping $\omega \mapsto A_T(\omega)$ is S -continuous (see Corollary 2.11 in [41]), it follows that

$$U(0, \mu, 0) \geq \mathbb{E}^{\mathbb{P}_0} [F(A_T)] \geq \lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{P}_{n_k}} [F(A_T)],$$

which, combined with (9), yields the upper semicontinuity.

3.2. Reduction to a finite-dimensional problem. Our aim now is to provide a more concrete description of the function U . However, because the function U is continuous in ξ , we can restrict ourselves to a nicer class of problems. Specifically, we can approximate our object of primary interest $U(t, \xi, a)$ by a sequence $U(t, \xi^N, a)$, where ξ^N can be chosen to have nice properties. For our purposes, a natural simplifying assumption is to assume that the measures ξ^N are atomic measures. In this case, as we shall see, the problem becomes much more tractable via classical methods.

To do this, we let $\mathbb{X}_N = \{x_0, x_1, \dots, x_N\}$, where $0 \leq x_0 < x_1 < \dots < x_N$, and write $\mathcal{P}^1(\mathbb{X}_N) = \mathcal{P}^1 \cap \mathcal{M}(\mathbb{X}_N)$ and $\mathcal{P}^s(\mathbb{X}_N) = \mathcal{P}^s \cap \mathcal{M}(\mathbb{X}_N)$. Observe that if (ξ_r) is a terminating measure-valued martingale and $\xi_0 \in \mathcal{P}^1(\mathbb{X}_N)$, then $\xi_r \in \mathcal{P}^1(\mathbb{X}_N)$ a.s. for all $r \geq 0$ and $\xi_\infty = \delta_{x_i}$ for some $x_i \in \mathbb{X}_N$. Further, write $\alpha \subseteq \{0, 1, \dots, N\}$, $\mathbb{X}_\alpha = \{x_i : i \in \alpha\}$, and $\mathcal{P}^1(\mathbb{X}_\alpha), \mathcal{P}^s(\mathbb{X}_\alpha)$, etc., as above. In particular, $\mathbb{X}_N = \mathbb{X}_{\{0, 1, \dots, N\}}$.

We then have the following characterization.

LEMMA 3.3. *Suppose $\mu \in \mathcal{P}^1(\mathbb{X}_N)$. Then (ξ_r) is a measure-valued martingale with $\xi_0 = \mu$ if and only if $\xi_r^n := \xi_r(\{x_n\})$ is a nonnegative martingale for each n and $\sum_{i=0}^N \xi_r^i = 1$. Moreover, (ξ_r) is terminating if and only if $\xi_\infty^n = 0$ for all but one $n \in \{0, 1, 2, \dots, N\}$ a.s., and (ξ_r) is continuous if and only if ξ_r^n is continuous for each n .*

It is clear that there are similar statements where $\mathcal{P}^1(\mathbb{X}_N)$ is replaced by $\mathcal{P}^1(\mathbb{X}_\alpha)$.

Then we consider the further consequence of Lemma 2.11: By the martingale representation theorem, working on the probability space granted by Lemma 2.11, we can assume that the dynamics of (ξ_r) are given by a controlled Brownian motion in a recursive formulation. For fixed $N \geq 1$, we suppose that (ξ_r) solves the SDE

$$(10) \quad d\xi_r^n = w_r^n dW_r$$

for (W_r) a standard Brownian motion, and where $\mathbf{w}_r = (w_r^1, \dots, w_r^N) \in \mathbb{R}^N$, $w_r^0 = -\sum_{n=1}^N w_r^n$, and $\xi_r^n \in \{0, 1\}$ implies $w_r^n = 0$, $n \in \{0, \dots, N\}$; that is, as soon as one of the atoms disappears, it can never be resurrected.

Next, we show that (ξ_r) and (λ_r) may be chosen so that a specific relation holds between \mathbf{w}_r and λ_r .

LEMMA 3.4. *Let $\mu \in \mathcal{P}^1(\mathbb{X}_N)$ and consider a martingale $(S_t)_{t \in [0, T]}$ represented via (3) by processes $(\lambda_r, \xi_r)_{r \in [0, \infty)}$ given on a probability space $(\Omega, \mathcal{G}, (\mathcal{G}_r)_{r \in [0, \infty)}, \mathbb{P})$ such that $\lambda_r \in [0, 1]$ is a progressively measurable process and (ξ_r) is a measure-valued martingale with $\xi_0 = \mu$. Suppose further that (\mathcal{G}_r) is the natural filtration of a Brownian motion (W_r) , $\inf\{r \geq 0 : \int_0^r \lambda_s ds = T\} < \infty$ a.s., and (ξ_r) is continuous and finitely terminating. Then, without loss of generality (w.l.o.g.), we may assume that*

$$(11) \quad \|\mathbf{w}_u\|^2 + \lambda_u = 1 - \mathbf{1}_{\{\xi_u \in \mathcal{P}^s\}} \mathbf{1}_{\{T_u = T\}}.$$

That is, we can always choose a multiple $(\Omega, \mathcal{G}, (\mathcal{G}_r), \mathbb{P}, (\lambda_r, \xi_r))$ which represents (S_t) via (3), and which satisfies the above properties as well as (11).

Proof. Suppose $(\Omega, \mathcal{G}, (\mathcal{G}_r), \mathbb{P}, (\lambda_r, \xi_r))$ satisfies the assumptions of the lemma (except for (11)). We aim to construct a time change such that the time-changed filtration $(\bar{\mathcal{G}}_u)$ and the time-changed processes $(\bar{\lambda}_u, \bar{\xi}_u)_{u \geq 0}$ satisfy the assertions. To this end, recall that T_r is given by (2) and let $\tau := \inf\{r : \xi_r \in \mathcal{P}^s \text{ and } T_r = T\}$; since ξ_r is finitely terminating, τ is finite a.s. Let $\phi : \Omega \times [0, \infty) \rightarrow \mathbb{R}_+$ be given by $\phi(u) = \int_0^u \eta_s^2 ds$ for some positive adapted process η_u such that $\phi(\infty) \geq \tau$. Then ϕ is continuous and increasing in u , and its inverse ϕ^{-1} is well defined and continuous on $[0, \tau]$. We define

$$(12) \quad \bar{\xi}_u := \xi_{\phi(u)} \quad \text{and} \quad \bar{T}_u := T_{\phi(u)}$$

and let $(\bar{\mathcal{G}}_u)_{u \in [0, \infty)}$ be the time-changed filtration with $\bar{\mathcal{G}}_u = \mathcal{G}_{\phi(u)}$. Note that $\bar{\xi}_u$ is a continuous measure-valued $(\bar{\mathcal{G}}_u)$ -martingale. Moreover, (12) implies that $\bar{T}_t^{-1} = \phi^{-1}(T_t^{-1})$, $t < T$ (recall that $\bar{T}_T^{-1} = \infty$). Hence, S_t is given by (3) defined with respect to $\bar{\xi}_u$ and \bar{T}_u . It remains to argue that η_u can be chosen such that

$$(13) \quad \|\bar{\mathbf{w}}_u\|^2 + \bar{\lambda}_u = 1 - \mathbf{1}_{\{\bar{\xi}_u \in \mathcal{P}^s\}} \mathbf{1}_{\{\bar{T}_u = T\}}.$$

First, note that $\phi^{-1}(\tau) = \inf\{u : \bar{\xi}_u \in \mathcal{P}^s \text{ and } \bar{T}_u = T\}$. By the martingale representation theorem, we know that ξ_r is given by (10) for some process $(\mathbf{w}_r) \in \mathbb{R}^N$. Since there is a $(\bar{\mathcal{G}}_u)$ -Brownian motion (B_u) such that $dW_{\phi(u)} = \eta_u dB_u$, it follows from (12) that

$$(14) \quad d\bar{\xi}_u^n = w_{\phi(u)}^n \eta_u dB_u \quad \text{and} \quad \bar{\lambda}_u = \eta_u^2 \lambda_{\phi(u)}.$$

Let $\bar{\mathbf{w}}_u = \mathbf{w}_{\phi(u)} \eta_u$. Then (13) holds for $u \leq \phi^{-1}(\tau)$ if η satisfies

$$\eta_u^2 = \frac{1}{\|\mathbf{w}_{\phi(u)}\|^2 + \lambda_{\phi(u)}}, \quad u \leq \phi^{-1}(\tau).$$

We therefore proceed by defining ϕ^{-1} via

$$d\phi^{-1}(r) := (\|\mathbf{w}_r\|^2 + \lambda_r) dr, \quad r \leq \tau.$$

It follows from the construction of $(\xi_r)_{r \in [0, \infty)}$ and $(\lambda_r)_{r \in [0, \infty)}$ that for $r \leq \tau$, $\|\mathbf{w}_r\| = 0$ if and only if $\lambda_r = 1$ (see Remark 2.15, and note that $\|\mathbf{w}\| = 0$ if and only if

$\|w^0, \mathbf{w}\| = 0$). In consequence, ϕ^{-1} is well defined, continuous, and strictly increasing on $[0, \tau]$. In particular,

$$\phi^{-1}(\tau) = \int_0^\tau (\|\mathbf{w}_r\|^2 + \lambda_r) dr = T + \int_0^\tau \|\mathbf{w}_r\|^2 dr,$$

and we observe that, as the quadratic variation process of a bounded martingale (in \mathbb{R}^N), $\int_0^\tau \|\mathbf{w}_r\|^2 dr$ is a.s. finite. It follows that ϕ_u and η_u are well defined for $u \leq \phi^{-1}(\tau)$, and that ϕ_u and ϕ_r^{-1} are adapted with respect to the respective filtrations $(\bar{\mathcal{G}}_u)$ and (\mathcal{G}_r) . In consequence, $\bar{\xi}_u$ and $\bar{\lambda}_u$ are well defined via (12) for $u \leq \phi^{-1}(\tau)$ and, according to (14), $\bar{\lambda}_u = \eta_u^2 \lambda_{\phi(u)} \in [0, 1]$, and can therefore be assumed to be progressively measurable (possibly after taking a modification). For $u > \phi^{-1}(\tau)$, we take $\mathbf{w}_u = \mathbf{0} = \lambda_u$. \square

Remark 3.5. When embedding piecewise constant martingales, as was done in the second part of the proof of Lemma 2.11, it follows that $\lambda_r \in \{0, 1\}$. However, the solution to Problem 2.10 (which in general is not unique), and thus to the basic optimization problem, may be attained for more general processes $\lambda_u \in [0, 1]$; cf. the nonconvex example considered in section 4.2. Hence, we do not further restrict the set of λ 's even though the value of the problem would remain unaffected.

Given the above dynamics of the stochastic factors, we note that $U(r, t, \xi, a)$ in (8) is now well defined as the value function corresponding to a class of dynamic control problems. In particular, w.l.o.g., we may optimize over controlled processes defined on a fixed reference probability space; see, e.g., [30]. The following result is now an immediate consequence of the lemma above. Let $\mathcal{A}_{u_0}^N = \{(\mathbf{w}_u)_{u \geq u_0} : \mathbf{w}_u \text{ is progressively measurable, } \mathbf{w}_u = (w_u^1, \dots, w_u^N) \in \mathbb{R}^N, \text{ and } \|\mathbf{w}_u\| \leq 1\}$ be the set of admissible controls.

COROLLARY 3.6. *For each $\mathbf{w} \in \mathcal{A}_{u_0}^N$, define $(\lambda_u)_{u \geq u_0}$ by (11) and $(\xi_u)_{u \geq u_0}$ by (10) with $w^0 = -\sum_{i=1}^N w^i$. Then, for $\mu \in \mathcal{P}(\mathbb{X}_N)$, Problem 2.10 in its restricted form (cf. Lemma 2.11) is equivalent to finding a process $\mathbf{w} \in \mathcal{A}_{u_0}^N$ such that $\xi_u^n \in \{0, 1\}$ implies $w_s^n = 0$, $s \geq u$, for $n \in \{0, \dots, N\}$, and such that \mathbf{w} maximizes $\mathbb{E}[F(A_T)]$ over the class of such processes where*

$$(15) \quad dA_{T_u} = (x_0, \dots, x_N) \cdot (\xi_u^0, \dots, \xi_u^N) \lambda_u T du.$$

Moreover, for all $\varepsilon > 0$, there exists $u^* = u^*(\varepsilon) > 0$ such that, for any μ, t, a ,

$$\inf_{\mathbf{w} \in \mathcal{A}_{u_0}} \mathbb{P}(\xi_{u_0+u^*} \in \mathcal{P}^s(\mathbb{X}_N) | \xi_{u_0} = \mu, T_{u_0} = t, A_{T_{u_0}} = a) > 1 - \varepsilon.$$

Proof. The only part of the first half of the corollary that does not follow immediately from the previous result is the part where the process $(\xi_u)_{u \geq 0}$, which results from a given choice $\mathbf{w} \in \mathcal{A}_{u_0}$, is terminating, and this follows once we show the second half of the result.

To see the second half of the result, note that it is sufficient to show that there is a similar bound for the first time that $\xi_u^n = 0$ for some $n \in \{0, 1, \dots, N\}$. Consider the process at time $u^* \gg u_0$, and suppose that the measure ξ has not already become singular at this time, so we have

$$\int_{u_0}^{u^*} (\|\mathbf{w}_u\| + \lambda_u) du = u^* - u \implies \int_{u_0}^{u^*} \|\mathbf{w}_u\| du \geq u^* - u - T.$$

In particular, we can ensure that $\max_k \{ \int_{u_0}^{u^*} (w_u^k)^2 du \}$ is arbitrarily large by choosing u^* sufficiently large. Let k^* be the maximizing component; it follows immediately, from the fact that $\xi_u^{k^*}$ is a $[0, 1]$ -valued martingale with quadratic variation process $\int_{u_0}^{u^*} (w_u^{k^*})^2 du$, that with high probability at least one component must have become zero or one. \square

Between Lemma 3.1 and Corollary 3.6, we have shown that Problem (8) is equivalent to choosing the variable \mathbf{w} in the problem above. Moreover, according to Theorem A.3 below, we know that the dynamic programming principle holds for our problem.³ Hence, we can break the problem up into a sequence of independent problems by considering the process only up to the first time that one of the current atoms becomes zero. Specifically, for $\xi \in \mathcal{P}(\mathbb{X}_N)$, we can consider the problem

$$(16) \quad \begin{aligned} \tilde{V}_N(u, t, \xi, a) = & \sup_{\mathbf{w} \in \mathcal{A}_u^N} \mathbb{E} \left[\tilde{V}_{N-1}(\sigma, T_\sigma, \xi_\sigma, A_{T_\sigma}) \mathbf{1}_{\{T_\sigma < T\}} \right. \\ & \left. + F(A_{T_\sigma}) \mathbf{1}_{\{T_\sigma = T\}} \mid A_{T_u} = a, \xi_u = \xi, T_u = t \right], \end{aligned}$$

where $\sigma = \inf\{s \geq u : \xi_s^n \notin (0, 1) \text{ some } n \in \{0, \dots, N\} \text{ or } T_s = T\}$. We also have the boundary conditions $\tilde{V}_0(u, t, \xi, a) = F(a + (1 - t)x)$, where $\xi = \delta_x$. Here, the function $\tilde{V}_k(u, t, \xi, a)$ is determined for $\xi \in \mathcal{P}^1(\mathbb{X}_\alpha)$ with $|\alpha| = k + 1$. Specifically, for $\xi \in \mathcal{P}^1(\mathbb{X}_\alpha)$ with $|\alpha| = k + 1$,

$$\begin{aligned} \tilde{V}_k(u, t, \xi, a) = & \sup_{\mathbf{w} \in \mathcal{A}_u^k} \mathbb{E} \left[\tilde{V}_{k-1}(\sigma, T_\sigma, \xi_\sigma, A_{T_\sigma}) \mathbf{1}_{\{T_\sigma < T\}} \right. \\ & \left. + F(A_{T_\sigma}) \mathbf{1}_{\{T_\sigma = T\}} \mid A_{T_u} = a, \xi_u = \xi, T_u = t \right], \end{aligned}$$

where we set $\sigma = \inf\{s \geq u : \xi_s \in \mathcal{P}^1(\mathbb{X}_\alpha)$ some $\alpha, |\alpha| \leq k$ or $T_r = T\}$. Observe in particular that each $\xi \in \mathcal{P}^1(\mathbb{X}_N)$ determines a unique set α such that $\xi \in \mathcal{P}^1(\mathbb{X}_{\alpha'})$ implies $\alpha' \supset \alpha$. Write $\alpha(\xi)$ for this unique subset, and we observe that we have the consistency conditions

$$\tilde{V}_{|\alpha(\xi)|-1}(u, t, \xi, a) = \tilde{V}_k(u, t, \xi, a) \quad \text{for all } k \geq |\alpha(\xi)| - 1.$$

Finally, fix ξ with $|\alpha(\xi)| = k + 1$. We can make the identification between the probability measure $\xi = \sum_{j=0}^k \xi^{i_j} \delta_{x_{i_j}}$ (on \mathbb{X}_α), where i_0, i_1, \dots, i_k are the ordered elements of α and the vector $\boldsymbol{\xi}^\alpha = (\xi^{i_1}, \dots, \xi^{i_k}) \in \Delta^k := \{\mathbf{z} \in \mathbb{R}_+^k : \sum z_i \leq 1\}$. Specifically, $\xi^{i_0} = 1 - \mathbf{1} \cdot \boldsymbol{\xi}^\alpha$. With this identification, we define

$$V_\alpha(u, t, \boldsymbol{\xi}^\alpha, a) = \tilde{V}_k(u, t, \xi, a).$$

We write $\mathbf{x}^\alpha = (x_{i_0}, x_{i_1}, \dots, x_{i_k})$ and $\mathbb{S}^k = \{\mathbf{z} \in \mathbb{R}^k : \|\mathbf{z}\| = 1\}$ for the unit sphere in \mathbb{R}^k . Finally, note that for $|\alpha| = 1$, $V_\alpha(t, a) = \tilde{V}_0(t, \xi, a) = F(a + (T - t)x_{i_0})$, and we then define $(1 - \mathbf{1} \cdot \boldsymbol{\xi}^\alpha, \boldsymbol{\xi}^\alpha) := \xi^{i_0} = 1$. We also use the conventions $\mathbb{S}^0 := \emptyset$ and $\sup \emptyset := -\infty$.

³In Theorem A.3, the dynamic programming principle is proved for the original weak formulation given in Problem 2.10 (cf. Lemma A.1). Notably, by use of similar arguments to those used in the proof of Lemma 2.11, this implies that the dynamic programming principle holds also within the fixed Brownian probability space provided by Lemma 2.11 which we consider here. Alternatively, this might also be seen from the fact that Theorem A.3 and Lemma 2.11 immediately yield the “difficult” direction of the dynamic programming principle for the fixed Brownian setting (cf. “ \geq ” in (16)); the reverse inequality follows by standard arguments.

THEOREM 3.7. *Suppose $F(a)$ is continuous and nonnegative; fix $\alpha \subseteq \{0, \dots, N\}$ with $|\alpha| \geq 1$, and write $k = |\alpha| - 1$. The function $V_\alpha : \mathbb{R}_+ \times [0, T] \times \Delta^k \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is independent of u and is the unique nonnegative viscosity solution bounded by $F(a + (T - t)x_N)$ to*

$$(17) \quad \max \left\{ \frac{\partial V_\alpha}{\partial t} + \mathbf{x}^\alpha \cdot (1 - \mathbf{1} \cdot \boldsymbol{\xi}^\alpha, \boldsymbol{\xi}^\alpha) \frac{\partial V_\alpha}{\partial a}, \sup_{\mathbf{w} \in \mathbb{S}^k} [\text{tr}(\mathbf{w}\mathbf{w}^T D_{\boldsymbol{\xi}}^2 V_\alpha)] \right\} = 0$$

for $\boldsymbol{\xi}^\alpha \in (\Delta^k)^\circ$ and $t < T$ with the boundary conditions

$$(18) \quad \begin{aligned} V_\alpha(u, T, \boldsymbol{\xi}^\alpha, a) &= F(a), \\ V_\alpha(u, t, \boldsymbol{\xi}^{\alpha'}, a) &= V_{\alpha'}(u, t, \boldsymbol{\xi}^{\alpha'}, a), \end{aligned}$$

where the second equation is taken when $\boldsymbol{\xi}^\alpha \in \partial\Delta^k$. Here α' is the subset of α corresponding to nonzero entries of $(1 - \mathbf{1} \cdot \boldsymbol{\xi}^\alpha, \boldsymbol{\xi}^\alpha)$, and $\boldsymbol{\xi}^{\alpha'}$ is the vector identifying the corresponding probability measure.

Proof. We work by induction. Suppose the problem has been solved for $k' < k$ to give continuous value functions, independent of time. The case where $k = 0$ is trivial. The first step is to approximate by a problem with a finite time horizon. To this end, we fix a sequence $K \nearrow \infty$ and consider the following two problems: For given (u, t, ξ, a) with $\xi \in \mathcal{P}^1(\mathbb{X}_\alpha)$ and $|\alpha| = k + 1$, we set $\sigma^K = \sigma \wedge (K + 1)$ and define the functions $\tilde{V}_k^{\overline{K}}$ and \tilde{V}_k^K by

$$\begin{aligned} \tilde{V}_k^{\overline{K}}(u, t, \xi, a) &= \sup_{\mathbf{w} \in \mathcal{A}_u^{k, K}} \mathbb{E} \left[(F(a + (T - t)x_N)(\sigma^K - K)_+) \vee \left(F(A_{T_{\sigma^K}}) \mathbf{1}_{\{T_{\sigma^K} = T\}} \right. \right. \\ &\quad \left. \left. + \tilde{V}_{k-1}(\sigma^K, T_{\sigma^K}, \xi_{\sigma^K}, A_{T_{\sigma^K}}) \mathbf{1}_{\{T_{\sigma^K} < T\}} \right) \mid T_u = t, \xi_u = \xi, A_{T_u} = a \right] \end{aligned}$$

and

$$\begin{aligned} \tilde{V}_k^K(u, t, \xi, a) &= \sup_{\mathbf{w} \in \mathcal{A}_u^{k, K}} \mathbb{E} \left[(F(a + (T - t)x_N)(K + 1 - \sigma^K)_+) \wedge \left(F(A_{T_{\sigma^K}}) \mathbf{1}_{\{T_{\sigma^K} = T\}} \right. \right. \\ &\quad \left. \left. + \tilde{V}_{k-1}(\sigma^K, T_{\sigma^K}, \xi_{\sigma^K}, A_{T_{\sigma^K}}) \mathbf{1}_{\{T_{\sigma^K} < T\}} \right) \mid T_u = t, \xi_u = \xi, A_{T_u} = a \right], \end{aligned}$$

where $\mathcal{A}_u^{k, K} = \{(\mathbf{w}_s)_{s \in [u, K+1]} : (\mathbf{w}_s) \text{ is progressively measurable with } \mathbf{w}_s \in \mathbb{R}^k \text{ and } \|\mathbf{w}\| \leq 1\}$ and, as previously, $\sigma = \inf\{s \geq u : \xi_s^n \notin (0, 1) \text{ some } n \in \{0, \dots, k\} \text{ or } T_s = T\}$. With the same identification as above, we define $V_\alpha^{\overline{K}}$ and V_α^K by

$$V_\alpha^{\overline{K}}(u, t, \boldsymbol{\xi}^\alpha, a) = \tilde{V}_{|\alpha(\xi)|-1}^{\overline{K}}(u, t, \xi, a) \quad \text{and} \quad V_\alpha^K(u, t, \boldsymbol{\xi}^\alpha, a) = \tilde{V}_{|\alpha(\xi)|-1}^K(u, t, \xi, a).$$

Recall that the dynamics of the involved factors is governed by (2), (10), and (15), with (λ_s) given by (11). Note that, w.l.o.g., we may write $\mathcal{A}_u^{k, K} = \{(\lambda_s, \mathbf{w}_s)_{s \in [u, K+1]} : (\lambda_s, \mathbf{w}_s) \text{ is progressively measurable with } \mathbf{w}_s \in \mathbb{S}^k \text{ and } \lambda_s \in [0, 1]\}$. It follows from [30, Corollary V.3.1] that on the domain $[0, K + 1] \times [0, T] \times \mathbb{R}^k \times \mathbb{R}$, $V_\alpha^{\overline{K}}$ and V_α^K are both viscosity solutions to

$$(19) \quad \frac{\partial V_\alpha}{\partial u} - \sup_{\substack{\mathbf{w} \in \mathbb{S}^k \\ \lambda \in [0, 1]}} \left[\frac{1}{2} (1 - \lambda) \text{tr}(\mathbf{w}\mathbf{w}^T D_{\boldsymbol{\xi}}^2 V_\alpha) + \lambda \left(\frac{\partial V_\alpha}{\partial t} + \mathbf{x}^\alpha \cdot (1 - \mathbf{1} \cdot \boldsymbol{\xi}^\alpha, \boldsymbol{\xi}^\alpha) \frac{\partial V_\alpha}{\partial a} \right) \right] = 0$$

equipped with the boundary conditions

$$(20) \quad \begin{cases} V_\alpha(u, T, \boldsymbol{\xi}^\alpha, a) = F(a), \\ V_\alpha(u, t, \boldsymbol{\xi}^{\alpha'}, a) = V_{\alpha'}(u, t, \boldsymbol{\xi}^{\alpha'}, a) \end{cases}$$

for $u < K$, and either increasing to $F(a + (T - t)x_N)$ for $u \in [K, K + 1]$ in the first case, or decreasing to 0 in $[K, K + 1]$ in the second case. Specifically, in both cases, we have a viscosity equation with controls in a compact set and continuous boundary data on a compact domain; it follows that both equations have unique, continuous viscosity solutions ([30, Corollaries V.3.1 and V.8.1]), and the viscosity solutions to both equations correspond to the given value functions of the above optimal control problems. In particular, we see immediately that $V_\alpha^K(u, t, \boldsymbol{\xi}, a) \geq V_\alpha(u, t, \boldsymbol{\xi}, a) \geq V_\alpha^K(u, t, \boldsymbol{\xi}, a)$ for $u \leq K + 1$. Moreover, from Lemma 3.1, identifying U and V_α , we know that the function V_α is continuous, and moreover, from Corollary 3.6, we know that $V_\alpha^K(u, t, \boldsymbol{\xi}, a)$ will decrease pointwise to $V_\alpha(u, t, \boldsymbol{\xi}, a)$ as $K \rightarrow \infty$, and $V_\alpha^K(u, t, \boldsymbol{\xi}, a)$ will increase pointwise to the same limit. We conclude that V_α is a viscosity solution to the given equation (see, e.g., Barles and Souganidis [6]).

Now suppose that W is another viscosity solution to the same equation, also nonnegative and bounded by $F(a + (T - t)x_N)$. By the comparison principle, for any K , $V_\alpha^K(u, t, \boldsymbol{\xi}, a) \geq W(u, t, \boldsymbol{\xi}, a) \geq V_\alpha^K(u, t, \boldsymbol{\xi}, a)$ for $u \leq K$. Hence $V_\alpha \geq W \geq V_\alpha$; that is, $W = V_\alpha$. Finally, we observe that the solution V_α is independent of u by Lemma 3.1, so $\frac{\partial V_\alpha}{\partial u} = 0$, and optimizing over λ immediately gives the equivalent formulation. \square

Remark 3.8. We note that some obvious generalizations of this setup can easily be handled. For example, consider Asian options with nonconstant weighting, so $A_T = \int_0^T f(t)S_t dt$ for some (possibly signed) continuous function $f : [0, T] \rightarrow \mathbb{R}$. A simple modification to the arguments above gives the same result with the corresponding PDE:

$$\max \left\{ \frac{\partial V_\alpha}{\partial t} + f(t)\mathbf{x}^\alpha \cdot (1 - \mathbf{1} \cdot \boldsymbol{\xi}^\alpha, \boldsymbol{\xi}^\alpha) \frac{\partial V_\alpha}{\partial a}, \sup_{\mathbf{w} \in \mathbb{S}^k} [\text{tr}(\mathbf{w}\mathbf{w}^T D_\xi^2 V_\alpha)] \right\} = 0.$$

4. Examples and superhedging. In this section we consider some simple cases where explicit solutions to the viscosity equations in Theorem 3.7 can be given. We also give some arguments regarding the construction of superhedging strategies. A number of the results in this section can be compared to the recent work of Stebegg [45], but we emphasize that our results extend beyond the case where F is convex, and we will consider such an example below.

4.1. Convex payoff functions.

LEMMA 4.1. *Suppose the function F is convex and Lipschitz. Then, for all $\xi \in \mathcal{P}^1(\mathbb{R}_+)$,*

$$U(t, \xi, a) = \int F(a + (T - t)x) \xi(dx).$$

Moreover, an optimal model is given by

$$\begin{aligned} S_{0-} &= \int x \xi(dx), \\ S_t &= S_T, \quad t \geq 0, \end{aligned}$$

where $S_T \sim \xi$.

In terms of the class of models considered in Corollary 3.6, this corresponds to a model which takes $\lambda_u = 0$ until the measure $\xi_u \in \mathcal{P}^s$, and then setting $\lambda_u = 1$ until $T_u = T$.

Proof. By continuity, we are only required to check that (17) holds for atomic ξ . However, if we write $\bar{\xi} = \int x \xi(dx)$, then

$$\begin{aligned} \frac{\partial U}{\partial t} + \bar{\xi} \frac{\partial U}{\partial a} &= \int F'(a + (T - t)x) (\bar{\xi} - x) \xi(dx) \\ &\leq \int F'(a + (T - t)\bar{\xi}) (\bar{\xi} - x) \xi(dx) = 0. \end{aligned}$$

Moreover, if $t = T$ or $\xi \in \mathcal{P}^s$, then we have equality.

Recalling the notation of Theorem 3.7, we have

$$U(t, \xi, a) = \sum_{j=0}^{|\alpha(\xi)|-1} F(a + (T - t)x_{i_j}) \xi^{i_j},$$

and computing the second derivatives, we have $D_{\bar{\xi}}^2 U = 0$. Hence (17) holds. □

In this convex case, we are easily able to provide a martingale inequality interpretation of this result. Indeed, this has already appeared in [45]. Since this will help our intuition, we provide an alternative approach to [45], which will enable us to gain insight into the optimal strategies for the nonconvex case. We restrict first to the case where $F(a) = (a - K)_+$ for some $K > 0$, and we write $Y_t = A_t + (T - t)S_t$. We suppose also that (S_t) is a continuous semimartingale (although a pathwise analogue of this argument is possible, where S_t is assumed just to have continuous paths). From the definition of local time, we get

$$(A_T - K)_+ = (Y_T - K)_+ = (Y_0 - K)_+ + M_T + L_T^{Y,K},$$

where M_T is a local martingale and $L_T^{Y,K}$ is the local time of Y at the level K . It follows from the definition of Y that we have

$$L_T^{Y,K} = \int_0^T (T - t) dL_t^{S,K_t}, \quad \text{where } K_t = \frac{K - A_t}{T - t},$$

so L^{S,K_t} is the local time of the asset price along the curve K_t . That is, we have a local time contribution coming from the crossing of the curve K_t by the asset price. However, for a given distribution of S_T , the local time at each value of x is fixed and decreases as $|x - S_0|$ increases. We now argue that $L_T^{Y,K}$ is maximized by trying to accumulate all the local time on the curve K_t as close as possible to time zero; that is, all crossings of $S_t = K_t$ should happen as close to time zero as possible. This happens because if $S_t \neq K_t$, then $|S_t - K_t|$ is increasing, and there will be less local time which can later be accumulated at K_t , since the process needs to accumulate the local time at a (relatively) more distant point. In addition, the factor $(T - t)$ which appears in the integral only makes the weight of local time accumulated at later times smaller.

It follows (and again, this can be made rigorous) that the optimal model should make all crossings of K_t necessary to embed in a short time interval. After this time, it is irrelevant how the process behaves, so long as it either remains above or below K_t .

Remark 4.2. The cases where there is a positive interest rate can be handled similarly (the process $Y_t = A_t + \frac{S_t}{\rho}(e^{\rho(T-t)} - 1)$ should be used instead). In addition, by adding constraints, one can extend to general convex functions, with the model which crosses each relevant curve K_t corresponding to a convexity point of F immediately being the optimal choice.

4.2. A nonconvex example. In this section, we provide a solution to the problem for a nonconvex example. Specifically, we use the intuition from the convex case established above to try to find a solution to the problem for a payoff function of the form

$$(21) \quad F(A_T) = (A_T - K_1)_+ - (A_T - K_2)_+, \quad K_1 < K_2.$$

Given certain additional assumptions on the measure we wish to embed, we will then verify that an optimal model may be determined through the use of Theorem 3.7. We observe that the results of this paper simply verify the existence of an optimal model. Given the existence of an optimal model, the existence of a superhedging strategy follows from general results (e.g., Dolinsky and Soner [24]).

The intuition established above suggests that we wish to gain the benefit of the convexity at K_1 immediately, while leaving the concavity at K_2 until as late as possible. However there is a trade-off, since the process may sacrifice some of the convexity at K_1 by waiting at K_2 . To specify this, note that since the payoff is constant for $A_T \geq K_2$, it must be suboptimal to have positive support on events for which $Y_t > K_2$, $t \in (0, T]$. Indeed, the payoff will not be improved by this, but the martingale property of Y implies that more mass must then be put on events yielding an average strictly less than K_2 . In consequence, at least for some values of K_1, K_2 , it is natural to conjecture that at time 0, S will either run to K_2 or to some lower level; at the lower level, the paths will behave as indicated by Lemma 4.1.

For a measure μ with continuous support, we therefore define the level η by

$$(22) \quad \eta := \inf \left\{ y \in \mathbb{R} : \frac{\int_{[y, \infty)} x \mu(dx)}{\mu([y, \infty))} \geq K_2 \right\}.$$

We then expect to accumulate all mass above $x = \eta$ into a branch of the underlying asset, taking the value $S_t = K_2$, $t \in (0, T)$ and embedding $\mathbf{1}_{x \geq \eta} \mu(dx)$ at $t = T$. As for the mass to be embedded on $[0, \eta)$, we expect the same optimal behavior as detected for the convex case in section 4.1. Put differently, at $u = T_0^{-1}$, with probability $\mu([\eta, \infty))$ we expect the measure-valued martingale ξ_u to take the value $\frac{\mathbf{1}_{x \geq \eta} \mu(dx)}{\mu([\eta, \infty))}$ and stay constant until T_T^{-1} , and with probability $\mu([0, \eta))$ we expect to recover the structure of Lemma 4.1.

To specify this, we restrict to a certain class of measures μ . Specifically, we consider the problem at time $t \in [0, T]$ with current average $A_t = a$ when we take $\mathbf{x}^\alpha = (-1, 0, 1)$, so $|\alpha| = 3$, and consider the terminal distribution⁴

$$(23) \quad \mu = (1 - \beta - \gamma)\delta_{-1} + \beta\delta_0 + \gamma\delta_1, \quad \beta, \gamma \in (0, 1).$$

That is, $\boldsymbol{\xi}^\alpha = (\beta, \gamma)$, and we write $V(t, a; \beta, \gamma) = V_\alpha(t, a; \boldsymbol{\xi}^\alpha)$. Further, we let $K_1 \in (-1, 1)$ and $K_2 \in (0, 1)$. Suppose now that $a + \frac{\gamma}{\gamma + \beta}(T - t) < K_2 \leq a + (T - t)$. That is, the expected averages considering the mass at both $x = 0$ and $x = 1$, and at $x = 1$ only, are, respectively, smaller and greater than K_2 . Following the reasoning above, at $u = T_t^{-1}$, we then expect to have split ξ_u into the two measures

$$(24) \quad \xi^1 = \frac{\bar{\eta}\delta_0 + \gamma\delta_1}{\bar{\eta} + \gamma} \quad \text{and} \quad \xi^2 = \frac{(1 - \gamma - \beta)\delta_{-1} + (\beta - \bar{\eta})\delta_0}{1 - \gamma - \bar{\eta}},$$

⁴Here we consider a μ with negative support although our main results are formulated for measures on \mathbb{R}_+ ; this is only to simplify the explicit formulae given in this section and notably does not affect the applicability of Theorem 3.7, which can be applied here by means of a simple shift.

where (cf. (22)) $\bar{\eta}$ is given by

$$\bar{\eta} = \sup \left\{ y : \frac{\gamma}{\gamma + y} \geq \frac{K_2 - a}{T - t} \right\} = \gamma \left(\frac{T - t}{K_2 - a} - 1 \right).$$

If $a < K_1$, this yields $V(t, a; \beta, \gamma) = (\gamma + \bar{\eta})(K_2 - K_1)$. However, if $a - (T - t) < K_1 \leq a$, the result for the convex case guides us to further split the measure ξ^2 into δ_{-1} and δ_0 , and it follows that $V(t, a; \beta, \gamma) = (\gamma + \bar{\eta})(K_2 - K_1) + (\beta - \bar{\eta})(a - K_1)$. Similar considerations for the other cases guide us to define the following candidate value function:

(25)

$$V(t, a; \beta, \gamma) := \begin{cases} K_2 - K_1, & \text{(i) } K_2 \leq a^{-101}, \\ (2\gamma + \beta - 1)(T - t) + a - K_1, & \text{(ii) } K_1 \leq a^{-1}, a^{-101} < K_2, \\ \frac{2\gamma + \beta}{1 + \frac{K_2 - a}{T - t}} (K_2 - K_1), & \text{(iii) } a^{-1} < K_1, a^{-101} < K_2 \leq a^{01}, \\ \gamma(T - t) - (\gamma + \beta)(K_1 - a), & \text{(iv) } a^{-1} < K_1 \leq a^0, a^{01} < K_2, \\ \gamma \frac{T - t}{K_2 - a} (K_2 - K_1), & \text{(v) } a^0 < K_1, a^{01} < K_2 \leq a^1, \\ \gamma(T - t - (K_1 - a)), & \text{(vi) } a^0 < K_1, K_1 \leq a^1 < K_2, \\ 0, & \text{(vii) } a^1 < K_1, \end{cases}$$

where we used the notation $a^N = a + s^N(T - t)$ with $s^i = i, i \in \{-1, 0, 1\}$, $s^{01} = \frac{\gamma}{\gamma + \beta}$ (with the convention $\frac{\gamma}{\gamma + \beta} = K_2$ when $\gamma + \beta = 0$), and $s^{-101} = s = 2\gamma + \beta - 1$, i.e., the expected average taking the mass at various atoms into account. The function is depicted in Figure 1, together with a candidate sample path.

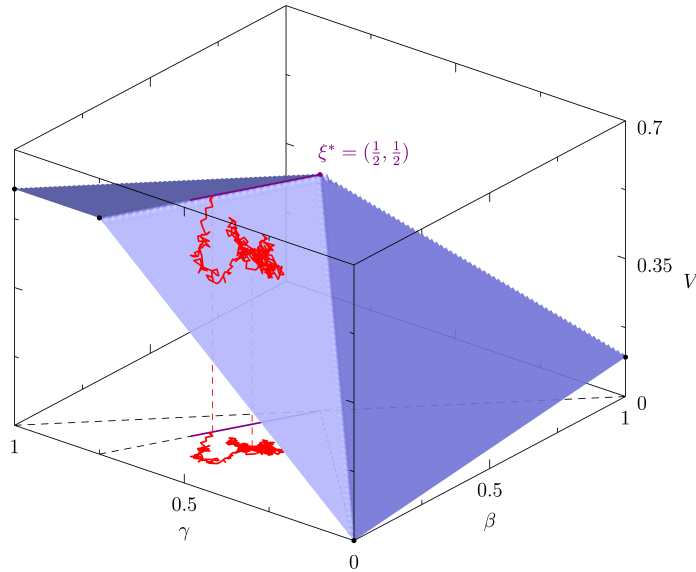


FIG. 1. The value function plotted as a function of β, γ . Here $t = a = 0, T = 1, K_1 = -0.1$, and $K_2 = 0.5$. Also shown (in red) is a possible path of (ξ_r) starting from $(\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})$. The planar regions in the diagram correspond to the regions (i), (iii), and (iv) given in (25). The process starts in region (iii), and runs until it hits the boundary of this region and region (i). The continuing path is then shown running along the boundary of regions (i) and (iii). In an optimal model, this behavior happens at time 0, although note that there are many possible solutions; for example, the movement along the boundary between regions (i) and (iii) could happen at any time between $t = 0$ and $t = T$. On reaching the point ξ^* , the process is unable to move any further before the time $t = T$ without becoming suboptimal.

Example 4.3. Observe that in sending $K_2 \rightarrow \infty$, $V(t, a; \beta, \gamma)$ reduces to the value-function for the (convex) payoff function $F(A_T) = (A_T - K_1)_+$ given in section 4.1 with μ given by (23). Alternatively, let $\beta = 0$, $K_1 = 0$, and $K_2 = \frac{1}{2}$. Then, $V(t, a; \gamma)$ reduces in the following way:

$$(26) \quad V(t, a; \gamma) = \begin{cases} \frac{1}{2}, & 2\gamma - 1 > \frac{1/2-a}{T-t}, \\ \gamma \frac{1}{1 + \frac{1/2-a}{T-t}}, & 2\gamma - 1 \leq \frac{1/2-a}{T-t}. \end{cases}$$

Note that for $t = a = 0$ and $2\gamma - 1 \leq 1/2$, $V_t + sV_a = 0$ if and only if $\gamma = 0$ or $\gamma = 3/4$. Therefore the optimal model (S_t) will jump to either -1 or $1/2$ at time $t = 0$, and then stay constant until time $t = T$ where μ is embedded.

It can be verified by elementary calculations that $V(t, a; \beta, \gamma)$ is continuous. The next result verifies that it is a (bounded) solution to (17) with $F(a)$ and μ given, respectively, by (21) and (23); its proof is deferred to Appendix B. In consequence, according to Theorem 3.7, $V(t, a; \beta, \gamma)$ is indeed the associated value function.

PROPOSITION 4.4. *The function $V(t, a; \beta, \gamma)$ defined in (25) is the unique non-negative viscosity solution bounded by $K_2 - K_1$ to the equation (17) equipped with the boundary condition (18). In particular, it is the value function associated with the payoff (21) and μ given by (23).*

We now discuss the optimal control associated with the value function (25) and the corresponding solution to the basic optimization problem, Problem 2.8. Indeed, recall Lemma 2.11, which says that if Problem 2.10 admits an optimal solution, this solution will correspond to a solution of Problem 2.8. Naturally, the solution coincides with the conjectured optimal model used to deduce the form of $V(t, a; \beta, \gamma)$. However, our aim below is to illustrate how it may be deduced directly from the value function and, in consequence, from the dynamic programming equation (19)–(20), and to show that it is nonunique and also nontrivial, since it necessarily has a jump at $t = T$ as well as $t = 0$. We let $T = 1$ and split the behavior into three parts.

- (I) *Real time is kept constant while the measure-valued martingale evolves ($\lambda_u = 0$ and (β_u, γ_u) diffuses): (S_t) jumps to certain points at time $t = 0$.*

Depending on the parameters of the problem, the starting point $(0, 0; \beta, \gamma)$ lies in one of the regions (i), (iii), (iv), or (v). It follows from the dynamic programming principle equation (17) that the model can evolve in real time only if $V_t + sV_a = 0$. According to (32) (see the proof of Proposition 4.4), while $V_t + sV_a = 0$ for all (β, γ) in regions (i), (ii), and (vii), it holds for the remaining regions that $V_t + sV_a = 0$ if and only if

$$(27) \quad \begin{cases} 2\gamma + \beta - 1 = \frac{K_2 - a}{T - t} & \text{or } (\beta, \gamma) = (0, 0) & \text{in (iii),} \\ \gamma + \beta = 1 & \text{or } (\beta, \gamma) = (0, 0) & \text{in (iv),} \\ 2\gamma + \beta - 1 = \frac{K_2 - a}{T - t} & \text{or } \gamma = 0 & \text{in (v),} \\ \gamma = 0 & \text{or } (\beta, \gamma) = (0, 1) & \text{in (vi),} \end{cases}$$

In consequence, if starting in region (i), one may immediately evolve in (real) time. However, if starting in regions (iii)–(v), (real) time must be kept constant while (β_u, γ_u) are allowed to diffuse until the above boundaries are reached; that is, $\lambda_u = 0$ until the measure-valued martingale $\xi = (\beta, \gamma)$ satisfies (27). If at the line $\frac{\gamma}{\gamma + \beta} = \frac{K_2 - a}{T - t}$, since $V(0, 0; \beta_u, \gamma_u)$ is a martingale, diffusion will take place only in the direction of that line until either of its boundary points are reached. Effectively, the associated price process (S_t) thus jumps to certain points at time $t = 0$.

(II) *Progress in real time only* ($\lambda_u = 1$): (S_t) is kept constant.

Once the measure-valued martingale satisfies (27), (real) time might start to evolve ($\lambda_u > 0$). In particular, one might let $\lambda_u = 1$, which implies that (S_t) is kept constant. On a case by case basis, it can be verified that once at a point where $V_t + sV_a = 0$, this remains the case.

(III) *When $T_u = T$, the measure-valued martingale $\xi. = (\beta., \gamma.)$ terminates: (S_t) jumps and embeds μ at $t = T$.*

Real time may run until $T_u = T$. Thereafter, $\lambda_u = 0$ and (β, γ) diffuses until ξ_u terminates, that is, until $\xi. = (\beta., \gamma.)$ reaches $(0, 0)$, $(0, 1)$, or $(1, 0)$. As expected, $V(T, a(T); \beta, \gamma)$ stays constant during this process as it is independent of γ and β . This step corresponds to S embedding μ via a jump at $t = T$.

The evolution in time and measure described in (II) and (III) could, in part, happen simultaneously or in reverse order. This implies that the optimal model is not unique. For example, having reached the line $\gamma + \beta = 1$ in region (iv), one might let (β_u, γ_u) continue to diffuse until reaching either of the points $(1 - \frac{K_2 - A_u}{T - u}, \frac{K_2 - A_u}{T - u})$ or $(1, 0)$, before letting (real) time evolve. This corresponds to the behavior used to deduce $V(t, a; \beta, \gamma)$, i.e., (S_t) jumps to one of the values $-1, 0$, or K_2 at time $t = 0$. Alternatively, by letting $\lambda_u \in (0, 1)$ and supposing that $A_t \geq K_1$, one may let time and measure evolve simultaneously, which corresponds to $Y_u = A_u + (T - u)S_u$, $u \in (t, 1)$ being either constantly equal to -1 or moving (as a continuous martingale) between the values 0 and K_2 . Observe that this behavior may result in a different distribution of A_T in comparison with the case where all the diffusion happens immediately. Similar behavior can be observed in the regions (i), (ii), and (v), although the distribution of A_T then remains the same.

While the optimal model is not unique, we note that it has certain characteristics: the model necessarily has a jump at both $t = 0$ and $t = T$. Indeed, there is a certain amount of mass which is “locked in” and cannot be embedded until the terminal time $t = T$. This is to ensure that $S_t = K_2$, $t \in (0, T)$, with a certain probability (e.g., in regions (iii) and (v)). On the other hand, sending $K_2 \rightarrow \infty$ and thus isolating the behavior at the convex kink K_1 , we see that the mass terminating above/below K_1 must already at time $t = 0$ be accumulated above/below K_1 . Although affected by the presence of K_2 , this feature is present also for the general case (e.g., in regions (iv) and (vi)).

5. Conclusions and further work. In this paper we have considered the model-independent pricing problem for Asian options using a novel approach based on measure-valued martingales. While this paper concentrated on the case of Asian options, the main ideas should generalize to other cases, and may provide insights beyond the existing literature. Moreover, there are a number of natural questions which arise from our work.

- Is it possible to generalize the results in this paper to the case of a general starting law? Financially, this has the interpretation of pricing a forward starting option at time 0, where $0 < T_0 < T_1$, the option pays the holder the amount $F(\int_{T_0}^{T_1} S_u du)$ at time T_1 , and the price of call options are known at times T_0 and T_1 . Write λ for the implied law of S_{T_0} and μ for the implied law of S_{T_1} . It follows immediately from the results of this paper that the problem is equivalent to finding a function $m : \mathbb{R} \rightarrow \mathcal{P}^1, x \mapsto m_x$ which maximizes

$\int U(0, m_x, 0) \lambda(dx)$ over all functions m such that $\int m_x(A) \lambda(dx) = \mu(A)$ for all Borel sets A and $x = \int y m_x(dy)$. However, it would be interesting to have a dynamic formulation of the problem, similar to the simple case.

- The PDE (17) is formulated for the case of atomic measures. Is there a similar formulation that holds when ξ is only assumed to be measure-valued?
- What is the corresponding formulation for (17) in the case of (say) options on variance?
- Do the methods described above extend to related problems in higher dimensions? If the formulation is given in terms of a measure μ on \mathbb{R}^d , one might hope that a very similar approach would be possible. Is this also true of (the financially more meaningful) case where $S_t \in \mathbb{R}^d$ and the marginal distribution of each component of S_T is specified?

Appendix A. A formal dynamic programming principle. In this section, we formally derive the dynamic programming principle for the pricing problem in its weak form given in Definition 2.10. We choose to follow closely the setup in [46]; see also, however, [27] and [44] for similar arguments.

We denote by \mathbb{D} the set of càdlàg paths on $[0, \infty)$ taking values in $E := \mathcal{P}(\mathbb{R}) \times [0, T] \times \mathbb{R}$, where we equip $\mathcal{P}(\mathbb{R})$ with the topology induced by the \mathcal{W}_1 -metric and E with the product topology; in particular, this renders E a Polish space, and using the Skorokhod topology on \mathbb{D} makes it a Polish space too. For $x, x' \in E$, we write $d(x, x') := \mathcal{W}_1(\xi, \xi') \vee |t - t'| \vee |a - a'|$. A generic path in \mathbb{D} is denoted by ω and we use $X = (\xi, T, A)$ for the coordinate process: $X_r(\omega) = (\xi_r, T_r, A_r)(\omega) = \omega(r)$.

The set of all probability measures on $\mathcal{B}(\mathbb{D})$ is denoted by \mathfrak{P} . A map $\nu : E \times \mathcal{B}(\mathbb{D}) \rightarrow [0, 1]$ is called a (universally) measurable kernel if (i) $\nu(x, \cdot) \in \mathfrak{P}$ for all $x \in E$, and (ii) $E \ni x \rightarrow \nu(x, A)$ is (universally) measurable for all $A \in \mathcal{B}(\mathbb{D})$; recall that the universal σ -algebra is the intersection of the completions of the Borel σ -algebra over all probability measures on the space, and that universally measurable functions are integrable with respect to any such probability measure. We write ν_x for the probability measure $\nu(x, \cdot)$ and interpret ν as a (universally) measurable map $E \rightarrow \mathfrak{P}$.

A Borel-measurable map from \mathbb{D} to $[0, \infty)$ is called a random time. For any random time τ , we define the shift-operator θ_τ on \mathbb{D} by $X_r(\theta_\tau(\omega)) = X_{\tau(\omega)+r}(\omega)$. Further, for a random time τ and any two paths $\omega, \omega' \in \mathbb{D}$ such that $X_\tau(\omega) = X_0(\omega')$, the concatenation $\omega *_\tau \omega'$ is an element of \mathbb{D} specified by

$$X_t(\omega *_\tau \omega') = \mathbf{1}_{\{t < \tau(\omega)\}} X_t(\omega) + \mathbf{1}_{\{t \geq \tau(\omega)\}} X_{t-\tau(\omega)}(\omega').$$

For a random time τ , a probability measure $\mu \in \mathfrak{P}$, and a universally measurable kernel ν , we then define the concatenation $\mu *_\tau \nu$ as the probability measure in \mathfrak{P} given by

$$(\mu *_\tau \nu)(A) = \iint \mathbf{1}_A(\omega *_\tau \omega') \nu_{X_\tau(\omega)}(d\omega') \mu(d\omega), \quad A \in \mathcal{B}(\mathbb{D}).$$

We let $\mathbb{F}^0 = \{\mathcal{F}_r^0\}_{r \in [0, \infty)}$ denote the raw filtration generated by the coordinate process X , and let $\mathbb{F} = \{\mathcal{F}_r\}_{r \in [0, \infty)}$ be its right-continuous hull, i.e., $\mathcal{F}_r = \bigcap_{s > r} \mathcal{F}_s^0$ for $r \geq 0$. For $x = (\xi, t, a) \in E$, we denote by \mathcal{P}_x the set of probability measures in \mathfrak{P} for which

- (i) $X_0 = x$ a.s.;
- (ii) ξ_r is a measure-valued \mathbb{F} -martingale;
- (iii) T_r is nondecreasing with $\lim_{r \rightarrow \infty} T_r = \infty$ a.s.;

(iv) $A_r = a + \int_0^{r \wedge \tau_0} \bar{\xi}_{u-} dT_u$ a.s., where $\bar{\xi} = \int x \xi.(dx)$ and $\tau_0 = \inf\{r : T_r \geq T\}$. Finally, we note that, according to Lemma 3.12 in [46], there exists a measurable functional $\bar{X} = (\bar{\xi}, \bar{T}, \bar{A}) : \mathbb{D} \rightarrow E$ such that $\bar{X}(\omega) = \lim_{t \rightarrow \infty} X_t(\omega)$ whenever the limit exists and $\bar{X}(\theta_t(\omega)) = \bar{X}(\omega)$ for all $t \geq 0$. We let $G(\omega) := F(\bar{A}(\omega))$; for any $\mu \in \mathcal{P}_x$, $x \in E$, we then have that $G = \lim_{t \rightarrow \infty} F(A_t)$ a.s. We define the problem as

$$(28) \quad v(x) = \sup_{\mu \in \mathcal{P}_x} \mathbb{E}^\mu [G].$$

LEMMA A.1. *The value function defined in (28) coincides with the value function as defined in Lemma 3.1. In particular, $x \mapsto v(x)$ is continuous.*

Proof. Let $(\Omega, \mathcal{G}, (\mathcal{G}_r), \mathbb{P}, (\xi_r), (\lambda_r))$ be a multiple as specified in Problem 2.10. W.l.o.g., let $x = (\mu, 0, 0)$. (Note that in Problem 2.10 it was assumed only that $\xi_{0-} = x$; by considering a time transformation $t \mapsto \frac{(t-\varepsilon)_+}{T-\varepsilon} T$, this difference can be seen to be irrelevant.) Since any martingale is a martingale in (the right-continuous hull of) its own filtration, it follows that any such multiple induces on the canonical space \mathbb{D} a measure $\mu \in \mathcal{P}_x$. Conversely, any probability measure $\mu \in \mathcal{P}_x$, together with the space $(\mathbb{D}, \mathcal{B}(\mathbb{D}), \mathbb{F}^\mu)$ and the canonical process (ξ, T) , produces such a multiple. Indeed, the martingale property is preserved by passing to the augmented filtration \mathbb{F}^μ , and the fact that one may, w.l.o.g., assume that T is absolutely continuous a.s. follows as in the proof of Lemma 2.11. Moreover, since $\tau_0 < \infty$ a.s., for any pair (T_r, ξ_r) with ξ a measure-valued martingale, one may construct a terminating measure-valued martingale $\tilde{\xi}_r$ such that $(T_r, \tilde{\xi}_r)$ yields the same value of the payoff. The continuity is then an immediate consequence of Lemma 3.1. \square

Remark A.2. Let $\tilde{\mathbb{F}}^0$ be the filtration generated by $(T, \bar{\xi})$, where $\bar{\xi} = \int x \xi.(dx)$, and let $\tilde{\mathbb{F}}$ be its right-continuous hull. Further, let $\tilde{\mathcal{P}}_x$ denote the set of measures in \mathfrak{P} which satisfy properties (i) to (iv) above, with the difference that ξ is only assumed to be a measure-valued $\tilde{\mathbb{F}}$ -martingale. We then have that

$$(29) \quad v(x) = \sup_{\mu \in \tilde{\mathcal{P}}_x} \mathbb{E}^\mu [G].$$

Indeed, this follows from the proof of Lemma 2.11, where the constructed measure-valued martingales are indeed adapted to the filtration generated by T and $\bar{\xi}$.

We are now ready to state the dynamic programming principle. For simplicity, we provide it here for bounded payoff functions. We denote by \mathcal{T} the set of finite \mathbb{F} -stopping times.

THEOREM A.3. *Let $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be bounded and Lipschitz. Then, for all $x \in E$ and $\tau \in \mathcal{T}$,*

$$v(x) = \sup_{\mu \in \mathcal{P}_x} \mathbb{E} [v(X_\tau)].$$

Proof. Given $\varepsilon > 0$, $x \in E$, and $\tau \in \mathcal{T}$, take $\mu \in \mathcal{P}_x$ such that $v(x) - \varepsilon \leq \mathbb{E}^\mu [G]$. Let ν_x be the regular conditional probability distribution of θ_τ under μ given $X_\tau = x$; since \mathbb{D} is Polish it exists $\mu \circ X_\tau^{-1}$ -a.s. Recall that, for any $f \in C_b(\mathbb{R}_+)$, $\xi.(f)$ is a bounded μ -martingale. By use of the same argument as given in the proof of Proposition 3.11 in [46],⁵ we may then conclude that $\xi.(f)$ is a ν_x -martingale for

⁵The argument given in [46] is for the case where τ is bounded, but it can readily be generalized; see also [27].

$\mu \circ X_\tau^{-1}$ -almost all $x \in E$. It follows that $\nu_x \in \mathcal{P}_x$ for $\mu \circ X_\tau^{-1}$ -almost all $x \in E$. As argued in the proof of Proposition 2.5 in [46], we may further pick a universally measurable version of ν_x such that $\nu_x \in \mathcal{P}_x$ for all $x \in E$. Now, note that $G(\omega') = G(\omega *_\tau \omega')$ for all $\omega, \omega' \in \mathbb{D}$ with $X_\tau(\omega) = X_0(\omega')$, and thus

$$\iint G(\omega') \nu_{X_\tau(\omega)}(d\omega') \mu(d\omega) = \iint G(\omega *_\tau \omega') \nu_{X_\tau(\omega)}(d\omega') \mu(d\omega).$$

By use of the properties of the regular conditional probability distribution, we thus obtain the following line of equalities:

$$\mathbb{E}^\mu[G] = \mathbb{E}^\mu[G \circ \theta_\tau] = \mathbb{E}^{\mu *_\tau \nu}[G] = \mathbb{E}^\mu[g(X_\tau)],$$

where $g(x) = \mathbb{E}^{\nu_x}[G] = \int G(\omega') \nu_x(d\omega')$. Hence, $v(x) - \varepsilon \leq \mathbb{E}^\mu[v(X_\tau)]$ for some $\mu \in \mathcal{P}_x$, and since ε was chosen arbitrarily we obtain $v(x) \leq \sup_{\mu \in \mathcal{P}_x} \mathbb{E}[v(X_\tau)]$.

In order to argue the reverse inequality, for any $\varepsilon > 0$, we first argue the existence of a measurable kernel ν with $\nu_x \in \mathcal{P}_x$ and $\mathbb{E}^{\nu_x}[G] \geq v(x) - \varepsilon$ for each $x \in E$. To this end, we define a mapping $E \times \mathbb{D} \ni (x, \bar{\omega}) \mapsto \omega^{x, \bar{\omega}} \in \mathbb{D}$ such that, for each $x = (\xi, t, a) \in E$, the mapping $\bar{\omega} = (\bar{\xi}, \bar{t}, \bar{a}) \mapsto \alpha^x(\bar{\omega}) := \omega^{x, \bar{\omega}} = (\xi, t, a)$ modifies the path $\bar{\omega}$ as follows:

$$(30) \quad \begin{cases} \xi_r(dy) &= \int \bar{\xi}_r(dx) m^\xi(x, dy), \\ t_r &= \bar{t}_r + t - \bar{t}_0, \\ a_r &= a + \int_0^{r \wedge \tau_0} \int x \xi_{u-}(dx) dt_u, \end{cases}$$

where the family $m^\xi(\cdot, dy)$ satisfies $\mathcal{W}_1(\bar{\xi}_0, \xi) = \iint |x - y| \bar{\xi}_0(dx) m^\xi(x, dy)$, and $\tau_0 = \inf\{r : t_r = T\}$. Then $(x, \bar{\omega}) \mapsto \omega^{x, \bar{\omega}}$ is $\mathcal{B}(E) \times \mathcal{B}(\mathbb{D})$ measurable. Hence, for any $\bar{\mathbb{P}} \in \cup_{x \in E} \mathcal{P}_x$, defining $\bar{\nu}_x := \bar{\mathbb{P}} \circ (\alpha^x)^{-1}$, $x \in E$, yields a measurable kernel $\bar{\nu}$ with $\bar{\nu}_x \in \mathcal{P}_x$. Indeed, the martingale property of ξ under $\bar{\nu}_x$ follows as in the proof of Lemma 3.1. Further, from (30) we have that

$$\begin{aligned} |\bar{a}_\infty - a_\infty| &\leq \int_{\bar{t}_0}^T \left| \int x \bar{\xi}_{\bar{t}_s-1}(dx) - \int x \xi_{\bar{t}_s-1}(dx) \right| ds \\ &\quad + \int_{T-(t-\bar{t}_0)}^T \left(\int x \bar{\xi}_{\bar{t}_s-1}(dx) + \int x \xi_{\bar{t}_s-1}(dx) \right) ds + |\bar{a}_0 - a|. \end{aligned}$$

Proceeding as in the proof of Lemma 3.1, for any $\bar{\mathbb{P}} \in \mathcal{P}_{\bar{x}}$ with $\bar{x} = (\bar{\xi}, \bar{t}, \bar{a})$, we then have that

$$\mathbb{E}^{\bar{\mathbb{P}}}[|A_\infty(\bar{\omega}) - A_\infty(\alpha^x \circ \bar{\omega})|] \leq T \mathcal{W}_1(\bar{\xi}, \xi) + (t - \bar{t}) \int x \bar{\xi}(dx) \vee \int x \xi(dx) + |\bar{a} - a|,$$

and with $\bar{\nu}_x = \bar{\mathbb{P}} \circ (\alpha^x)^{-1}$, the Lipschitz property of F thus yields $|\mathbb{E}^{\bar{\mathbb{P}}}[G] - \mathbb{E}^{\bar{\nu}_x}[G]| \leq \delta^{\bar{\xi}}(d(\bar{x}, x))$ for some modulus of continuity $\delta^{\bar{\xi}}$. Now, let $\varepsilon > 0$, and let $\{x^n\}_{n \in \mathbb{N}}$ be a countable dense subset of E . For each n , let $\mathbb{P}_n \in \mathcal{P}_{x^n}$ such that $\mathbb{E}^{\mathbb{P}_n}[G] \geq v(x^n) - \frac{\varepsilon}{3}$. Further, for each x^n , let r_n be such that for all $x \in B^n := \{x \in E : d(x, x^n) \leq r_n\}$, it holds that $v(x^n) \geq v(x) - \frac{\varepsilon}{3}$ and $|\mathbb{E}^{\mathbb{P}_n}[G] - \mathbb{E}^{\nu_x^n}[G]| \leq \frac{\varepsilon}{3}$ with $\nu_x^n := \mathbb{P}_n \circ (\alpha^x)^{-1}$; the existence of such r_n , $n \in \mathbb{N}$, follows from the above and Lemma 3.1. We then define the measurable kernel $(\nu_x)_{x \in E}$ by

$$(31) \quad \nu_x := \sum_{n \in \mathbb{N}} \mathbf{1}_{C^n}(x) \mathbb{P}_n \circ (\alpha^x)^{-1}, \quad \text{where } C^n = B^n \setminus \bigcup_{k=1}^{n-1} B^k.$$

By construction, for $x \in C^n$, $n \in \mathbb{N}$, we then have that

$$\mathbb{E}^{\nu_x}[G] \geq \mathbb{E}^{\mathbb{P}^n}[G] - \frac{1}{3}\varepsilon \geq v(x^n) - \frac{2}{3}\varepsilon \geq v(x) - \varepsilon.$$

Hence, ν is a measurable kernel with $\nu_x \in \mathcal{P}_x$ and $\mathbb{E}^{\nu_x}[G] \geq v(x) - \varepsilon$ for $x \in E$.

To conclude, we take $x_0 \in E$, $\mu \in \mathcal{P}_{x_0}$, $\tau \in \mathcal{T}$, and ν as constructed in (31). Since $\xi.(f)$ is a bounded μ -martingale for any $f \in C_b(\mathbb{R}_+)$, we may use the same arguments as in the proof of Proposition 3.10 in [46] to deduce that $\xi.(f)$ is also a $\mu *_{\tau} \nu$ -martingale. We may thus conclude that $\mu *_{\tau} \nu \in \mathcal{P}_{x_0}$. Letting $g(x) = \mathbb{E}^{\nu_x}[G]$ and noticing that g is measurable, we thus obtain

$$v(x_0) \geq \mathbb{E}^{\mu *_{\tau} \nu}[G] = \mathbb{E}^{\mu}[g(X_{\tau})] \geq \mathbb{E}^{\mu}[v(X_{\tau})] - \varepsilon.$$

Since ε and $\mu \in \mathcal{P}_{x_0}$ were both chosen arbitrarily, we obtain $v(x) \geq \sup_{\mu \in \mathcal{P}_x} \mathbb{E}[v(X_{\tau})]$ and conclude. \square

The above proof exploits the continuity properties of our problem in order to construct an approximately optimal measurable kernel; see [2] and [13] for similar approaches.

Appendix B. Proof of Proposition 4.4.

Proof. Differentiating $V(t, a; \beta, \gamma)$ within the respective regions, we obtain that $V_t(t, a; \beta, \gamma) + sV_a(t, a; \beta, \gamma) = 0$ in regions (i), (ii), and (vii), and that

$$(32) \quad V_t + sV_a = \begin{cases} \frac{(K_2 - K_1)(2\gamma + \beta)}{(1 + \frac{K_2 - a}{T - t})^2} \left(-\frac{K_2 - a}{(T - t)^2} + \frac{2\gamma + \beta - 1}{T - t} \right) & \text{in (iii),} \\ -\gamma + (2\gamma + \beta - 1)(\gamma + \beta) & \text{in (iv),} \\ -\gamma \frac{K_2 - K_1}{(K_2 - a)^2} \left(K_2 - a - (2\gamma + \beta - 1)(T - t) \right) & \text{in (v),} \\ 2\gamma \left(\gamma + \frac{\beta}{2} - 1 \right) & \text{in (vi),} \end{cases}$$

where $s = \mathbf{x}^{\alpha} \cdot (1 - \beta - \gamma, \beta, \gamma) = 2\gamma + \beta - 1$. Using that $2\gamma + \beta - 1 \leq \frac{K_2 - a}{T - t}$ in regions (iii) and (v), and that $\gamma + \beta \leq 1$ implies $(2\gamma + \beta - 1) \frac{\gamma + \beta}{\gamma} \leq 1$ for $\gamma > 0$, it follows from (32) that $V_t + sV_a \leq 0$ within the respective regions. In consequence, with $V_{\mathbf{v}}$ denoting the directional derivative in the direction of \mathbf{v} , it holds on $\{\gamma + \beta \leq 1\}$ that

$$(33) \quad V_{\mathbf{v}} \leq 0, \quad \mathbf{v} = (1, 2\gamma + \beta - 1, 0, 0).$$

Recall that $V(t, a; \xi)$ is a viscosity super- (resp., sub-) solution to (17) if, for each $\varphi \in \mathcal{C}^{1,1,2}$ and at each point $(\bar{t}, \bar{a}, \bar{\xi})$ minimizing (resp., maximizing) $V - \varphi$,

$$(34) \quad \max \left\{ \varphi_t + (2\bar{\gamma} + \bar{\beta} - 1)\varphi_a, \frac{1}{2} \sup_{\mathbf{w} \in \mathbb{S}^2} [\text{tr}(\mathbf{w}\mathbf{w}^T D_{\bar{\xi}}^2 \varphi)] \right\} \leq 0 \quad (\text{resp.}, \geq 0).$$

We first argue that V is a subsolution. To this end, let $\varphi \in \mathcal{C}^{1,1,2,2}$ and $\bar{\mathbf{z}} = (\bar{t}, \bar{a}, \bar{\beta}, \bar{\gamma})$ such that $\bar{\mathbf{z}}$ maximizes $V - \varphi$. Note that there exists $\bar{\mathbf{w}} \in \mathbb{S}^2$ such that the directional derivative at $\bar{\mathbf{z}}$ along $\bar{\mathbf{w}}$ (keeping a and t constant) satisfies $V_{\bar{\mathbf{w}\bar{\mathbf{w}}}} = 0$; if $\bar{\mathbf{z}} \in \{2\gamma + \beta - 1 = \frac{K_2 - a}{T - t}\}$ or $\bar{\mathbf{z}} \in \{\frac{\gamma}{\gamma + \beta} = \frac{K_2 - a}{T - t}\}$, let $\bar{\mathbf{w}}$ be in the direction of that line. Since $(V - \varphi)_{\bar{\mathbf{w}\bar{\mathbf{w}}}} \leq 0$, it follows that $\text{tr}(\bar{\mathbf{w}}\bar{\mathbf{w}}^T D_{\bar{\xi}}^2 \varphi) \geq 0$ (note that $\text{tr}(\mathbf{w}\mathbf{w}^T D_{\xi}^2 \varphi) = \mathbf{w}^T D_{\xi}^2 \varphi \mathbf{w} = \varphi_{\mathbf{w}\mathbf{w}}$). In consequence,

$$(35) \quad \sup_{\mathbf{w} \in \mathbb{S}^2} [\text{tr}(\mathbf{w}\mathbf{w}^T D_{\bar{\xi}}^2 \varphi)] \geq 0.$$

In order to show that V is a supersolution, let $\varphi \in \mathcal{C}^{1,1,2,2}$ and let $\bar{\mathbf{z}} = (\bar{t}, \bar{a}, \bar{\beta}, \bar{\gamma})$ be a minimizer to $V - \varphi$. Due to the concavity of V as a function of β and γ and the differentiability of φ , $\bar{\mathbf{z}}$ must lie strictly within one of the regions given in (32). Hence, for all $\mathbf{w} \in \mathbb{S}^2$, $\text{tr}(\mathbf{w}\mathbf{w}^T D_{\bar{\xi}}^2(V - \varphi)) \geq 0$ and $\text{tr}(\mathbf{w}\mathbf{w}^T D_{\bar{\xi}}^2 V) = 0$. In consequence,

$$(36) \quad \sup_{\mathbf{w} \in \mathbb{S}^2} [\text{tr}(\mathbf{w}\mathbf{w}^T D_{\bar{\xi}}^2 \varphi)] \leq 0.$$

Let now $\mathbf{v} = (1, 2\bar{\gamma} + \bar{\beta} - 1, 0, 0)$. Since $\bar{\mathbf{z}}$ minimizes $V - \varphi$, it follows that $\varphi_{\mathbf{v}} \leq V_{\mathbf{v}}$. According to (33), we thus obtain

$$\varphi_t + (2\bar{\gamma} + \bar{\beta} - 1) \varphi_a \leq V_{\mathbf{v}} \leq 0,$$

which when combined with (36) renders (34).

It remains to argue the boundary conditions (18). Note that for $t = T$, the only possible regions are (i), (ii), and (vii) (for $a \geq K_2$, $a \in [K_1, K_2)$, and $a < K_1$) for which $V(t, a; \beta, \gamma)$ is given, respectively, by $K_2 - K_1$, $a - K_1$, and 0. Hence, the terminal condition is satisfied. Next, note that for $\gamma = 0$ and $\beta = 0$ or $\gamma + \beta = 1$, the problem reduces, respectively, to the convex case and the case presented in Example 4.3. This verifies the second boundary condition and we conclude. \square

Acknowledgments. We are grateful to the referees and editors for their helpful comments and suggestions.

REFERENCES

- [1] B. ACCIAIO, M. BEIGLBÖCK, F. PENKNER, AND W. SCHACHERMAYER, *A model-free version of the fundamental theorem of asset pricing and the super-replication theorem*, Math. Finance, 26 (2016), pp. 233–251, doi:10.1111/mafi.12060.
- [2] A. AKSAMIT, Z. HOU, AND J. OBLÓJ, *Robust Framework for Quantifying the Value of Information in Pricing and Hedging*, preprint, arXiv:1605.02539, 2016.
- [3] H. ALBRECHER, J. DHAENE, M. GOOVAERTS, AND W. SCHOUTENS, *Static hedging of Asian options under Lévy models*, J. Derivatives, 12 (2005), pp. 63–72, doi:10.3905/jod.2005.479381.
- [4] H. ALBRECHER, P. A. MAYER, AND W. SCHOUTENS *General lower bounds for arithmetic Asian option prices*, Appl. Math. Finance, 15 (2008), pp. 123–149, doi:10.1080/13527260701356633.
- [5] L. AMBROSIO, N. GIGLI, AND G. SAVARÉ, *Gradient Flows: In Metric Spaces and in the Space of Probability Measures*, 2nd ed., Lectures in Mathematics ETH Zurich, Birkhäuser, Basel, 2008.
- [6] G. BARLES AND P. E. SOUGANIDIS, *Convergence of approximation schemes for fully nonlinear second order equations*, Asymptot. Anal., 4 (1991), pp. 271–283, doi:10.3233/ASY-1991-4305.
- [7] E. BAYRAKTAR AND Z. ZHOU, *On arbitrage and duality under model uncertainty and portfolio constraints*, Math. Finance, 27 (2017), pp. 988–1012, doi:10.1111/mafi.12104.
- [8] M. BEIGLBÖCK, A. M. G. COX, AND M. HUESMANN, *Optimal transport and Skorokhod embedding*, Invent. Math., 208 (2017), pp. 327–400, doi:10.1007/s00222-016-0692-2.
- [9] M. BEIGLBÖCK, A. M. G. COX, M. HUESMANN, N. PERKOWSKI, AND D. J. PRÖMEL, *Pathwise Super-replication via Vovk's Outer Measure*, preprint, arXiv:1504.03644, 2015.
- [10] M. BEIGLBÖCK, P. HENRY-LABORDÈRE, AND F. PENKNER, *Model-independent bounds for option prices: A mass transport approach*, Finance Stoch., 17 (2013), pp. 477–501, doi:10.1007/s00780-013-0205-8.
- [11] S. BIAGINI, B. BOUCHARD, C. KARDARAS, AND M. NUTZ, *Robust fundamental theorem for continuous processes*, Math. Finance, 27 (2017), pp. 963–987, doi:10.1111/mafi.12110.
- [12] B. BOUCHARD AND M. NUTZ, *Stochastic target games and dynamic programming via regularized viscosity solutions*, Math. Oper. Res., 41 (2015), pp. 109–124, doi:10.1287/moor.2015.0718.
- [13] B. BOUCHARD AND N. TOUZI, *Weak dynamic programming principle for viscosity solutions*, SIAM J. Control Optim., 49 (2011), pp. 948–962, doi:10.1137/090752328.
- [14] H. BROWN, D. HOBSON, AND L. C. G. ROGERS, *The maximum maximum of a martingale constrained by an intermediate law*, Probab. Theory Related Fields, 119 (2001), pp. 558–578, doi:10.1007/PL00008771.

- [15] X. CHEN, G. DEELSTRA, J. DHAENE, AND M. VANMAELE, *Static super-replicating strategies for a class of exotic options*, Insurance Math. Econom., 42 (2008), pp. 1067–1085, doi:10.1016/j.insmatheco.2008.01.002.
- [16] A. M. G. COX, DAVID HOBSON, AND JAN OBLÓJ, *Pathwise inequalities for local time: Applications to Skorokhod embeddings and optimal stopping*, Ann. Appl. Probab., 18 (2008), pp. 1870–1896, doi:10.1214/07-AAP507.
- [17] A. M. G. COX AND JAN OBLÓJ, *Robust hedging of double touch barrier options*, SIAM J. Financial Math., 2 (2011), pp. 141–182, doi:10.1137/090777487.
- [18] A. M. G. COX, J. OBLÓJ, AND N. TOUZI, *The Root Solution to the Multi-marginal Embedding Problem: An Optimal Stopping and Time-reversal Approach*, preprint, arXiv:1505.03169, 2015.
- [19] A. M. G. COX AND J. WANG, *Root's barrier: Construction, optimality and applications to variance options*, Ann. Appl. Probab., 23 (2013), pp. 859–894, doi:10.1214/12-AAP857.
- [20] A. M. G. COX AND J. WANG, *Optimal Robust Bounds for Variance Options*, preprint, arXiv:1308.4363, 2013.
- [21] D. DAWSON, *Measure-valued Markov processes*, in Ecole d'Eté de Probabilités de Saint-Flour XXI - 1991, Lecture Notes in Mathematics 1541, P.-L. Hennequin, ed., Springer, Berlin, 1993, pp. 1–260, doi:10.1007/BFb0084190.
- [22] G. DEELSTRA, I. DIALLO, AND M. VANMAELE, *Bounds for Asian basket options*, J. Comput. Appl. Math., 218 (2008), pp. 215–228, doi:10.1016/j.com.2006.12.017.
- [23] J. DHAENE, M. DENUIT, M. J. GOOVAERTS, R. KAAS, AND D. VYNCKE, *The concept of comonotonicity in actuarial science and finance: Applications*, Insurance Math. Econom., 31 (2002), pp. 133–161, doi:10.1016/S0167-6687(02)00135-X.
- [24] Y. DOLINSKY AND H. M. SONER, *Martingale optimal transport and robust hedging in continuous time*, Probab. Theory Related Fields, 160 (2013), pp. 391–427, doi:10.1007/s00440-013-0531-y.
- [25] Y. DOLINSKY AND H. M. SONER, *Martingale optimal transport in the Skorokhod space*, Stochastic Process. Appl., 125 (2015), pp. 3893–3931, doi:10.1016/j.spa.2015.05.009.
- [26] L. E. DUBINS, *On a theorem of Skorohod*, Ann. Math. Statist., 39 (1968), pp. 2094–2097.
- [27] N. EL KAROUI AND X. TAN, *Capacities, Measurable Selection and Dynamic Programming Part I: Abstract Framework*, preprint, arXiv:1310.3363.
- [28] N. EL KAROUI AND X. TAN, *Capacities, Measurable Selection and Dynamic Programming Part II: Application in Stochastic Control Problems*, preprint, arXiv:1310.3364, 2013.
- [29] R. EL DAN, *Skorokhod embeddings via stochastic flows on the space of Gaussian measures*, Ann. Inst. Henri Poincaré Probab. Stat., 52 (2016), pp. 1259–1280, doi:10.1214/15-AIHP682.
- [30] W. H. FLEMING AND H. M. SONER, *Controlled Markov Processes and Viscosity Solutions*, 2nd ed., Stochastic Modelling and Appl. Probability 25, Springer, New York, 2006.
- [31] M. FORDE AND A. JACQUIER, *Robust approximations for pricing Asian options and volatility swaps under stochastic volatility*, Appl. Math. Finance, 17 (2010), pp. 241–259, doi:10.1080/13504860903335348.
- [32] A. GALICHON, P. HENRY-LABORDÈRE, AND N. TOUZI, *A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options*, Ann. Appl. Probab., 24 (2014), pp. 312–336, doi:10.1214/13-AAP925.
- [33] G. GUO, X. TAN, AND N. TOUZI, *Tightness and duality of martingale transport on the Skorokhod space*, Stochastic Process. Appl., 127 (2017), pp. 927–956, doi:10.1016/j.spa.2016.07.005.
- [34] P. HENRY-LABORDÈRE, J. OBLÓJ, P. SPOIDA, AND N. TOUZI, *The maximum maximum of a martingale with given \mathbf{n} marginals*, Ann. Appl. Probab., 26 (2016), pp. 1–44, doi:10.1214/14-AAP1084.
- [35] D. HOBSON, *The Skorokhod embedding problem and model-independent bounds for option prices*, Paris-Princeton Lectures on Mathematical Finance 2010, Lecture Notes in Mathematics 2003, J.-M. Morel, F. Takens, and B. Teissier, eds., Springer, Berlin, pp. 267–318, doi:10.1007/978-3-642-14660-2_4, 2011.
- [36] D. HOBSON AND M. KLIMMEK, *Maximizing functionals of the maximum in the Skorokhod embedding problem and an application to variance swaps*, Ann. Appl. Probab., 23 (2013), pp. 2020–2052, doi:10.1214/12-AAP893.
- [37] D. G. HOBSON, *Robust hedging of the lookback option*, Finance Stoch., 2 (1998), pp. 329–347, doi:10.1007/s007800050044.
- [38] D. G. HOBSON AND J. L. PEDERSEN, *The minimum maximum of a continuous martingale with given initial and terminal laws*, Ann. Probab., 30 (2002), pp. 978–999, doi:10.1214/aop/1023481014.
- [39] J. HOROWITZ, *Measure-valued random processes*, Z. Wahrscheinlichkeit., 70 (1985), pp. 213–236, doi:10.1007/BF02451429.

- [40] Z. HOU AND J. OBLÓJ, *On Robust Pricing–Hedging Duality in Continuous Time*, preprint, arXiv:1503.02822, 2015.
- [41] A. JAKUBOWSKI, *A non-Skorohod topology on the Skorohod space*, *Electron. J. Probab.*, 2 (1997), doi:10.1214/EJP.v2-18.
- [42] S. KÄLLBLAD, X. TAN, AND N. TOUZI, *Optimal Skorokhod embedding given full marginals and Azéma-Yor peacocks*, *Ann. Appl. Probab.*, 27 (2017), pp. 686–719, doi:10.1214/16-AAP1191.
- [43] I. MONROE, *On embedding right continuous martingales in Brownian motion*, *Ann. Math. Statist.*, 43 (1972), pp. 1293–1311, doi:10.1214/aoms/1177692480.
- [44] M. NUTZ AND R. VAN HANDEL, *Constructing sublinear expectations on path space*, *Stochastic Process. Appl.*, 123 (2013), pp. 3100–3121, doi:10.1016/j.spa.2013.03.022.
- [45] F. STEBEGG, *Model-Independent Pricing of Asian Options via Optimal Martingale Transport*, preprint, arXiv:1412.1429.
- [46] G. ŽITKOVIĆ, *Dynamic programming for controlled markov families: Abstractly and over martingale measures*, *SIAM J. Control Optim.*, 52 (2014), pp. 1597–1621, doi:10.1137/130926481.