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# Travelling waves for a Frenkel-Kontorova chain

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## Abstract

In this article, the Frenkel-Kontorova model for dislocation dynamics is considered, where the on-site potential consists of quadratic wells joined by small arcs, which can be spinodal (concave) as commonly assumed in physics. The existence of heteroclinic waves —making a transition from one well of the on-site potential to another— is proved by means of a Schauder fixed point argument. The setting developed here is general enough to treat such a Frenkel-Kontorova chain with smooth ( $C^2$ ) on-site potential. It is shown that the method can also establish the existence of two-transition waves for a piecewise quadratic on-site potential.

Mathematics Subject Classification: 37K60, 34C37, 58F03, 70H05

## 1 Introduction

In 1938, Frenkel and Kontorova proposed a fundamental model of dislocation dynamics [8]. The model is given by Newton's equation of motion for a chain of atoms,

$$mu_k'' = \beta(u_{k+1} - 2u_k + u_{k-1}) - 2\pi \frac{\tilde{\alpha}}{\gamma} g' \left( \frac{2\pi}{\gamma} u_k \right) \quad (1)$$

with some constants  $\tilde{\alpha}$ ,  $\beta$ ,  $\gamma$ , describing the displacement  $u_k$  of atom  $k \in \mathbb{Z}$  in a one-dimensional chain. The nonlinearity is the derivative of an on-site potential describing the interaction with atoms above and below the chain of atoms considered. The periodicity of the nonlinearity thus reflects the periodic nature of a crystalline lattice. The Frenkel-Kontorova chain is a fundamental model of dislocation dynamics, describing how an imperfection (dislocation) travels through a crystalline lattice; see for example the survey [3]. The simplest motion that may exist is that of a travelling wave,  $u_j(t) = u(j - ct)$  with wave speed  $c$ . This *ansatz* transforms (1), after rescaling, into

$$c^2 u'' - \Delta_D u + g'(u) = 0 \quad (2)$$

on  $\mathbb{R}$ , where  $\Delta_D$  is the discrete Laplacian,

$$\Delta_D u(x) := u(x+1) - 2u(x) + u(x-1).$$

In the original paper [8], the on-site potential  $g$  is sinusoidal. A dislocation corresponds to a heteroclinic wave in the sense that the states near  $-\infty$  are in one well of  $g$  and the states near  $+\infty$  are in another well, which we here take to be a neighbouring one; see Figure 1 for a plot of an approximate solution, where the dislocation is in travelling wave coordinates positioned at the origin; the oscillations in the left half plane take place in one well of the on-site potential, and the solutions in the positive half plane take place in the neighbouring well. It thus suffices to consider potentials  $g$  with two wells only, say

$$g(u) = \frac{1}{2}\alpha u^2 - \alpha\psi(u), \quad (3)$$

which leads to the reformulation of (2) as

$$c^2 u'' - \Delta_D u + \alpha u - \alpha\psi'(u) = 0 \quad (4)$$

(in the setting considered here, it is easy to show that a solution to (4) with a two-well on-site potential is also a solution to the same equation with a suitable periodic continuation of the on-site potential).

The study of equation (4) has a long history in mechanics. Initial work by Frank and van der Merwe [7] resorted to the analysis of the continuum approximation of (1), the sine-Gordon equation, is analysed. This simplifies the analysis significantly, but already Schrödinger [18] pointed out the difference between PDE limits and underlying lattice equations. Indeed, the analysis of (1) has proved to be very hard. To the best of our knowledge, all existing analytic results, spanning more than 50 years, rely on the assumption that  $g$  is piecewise quadratic; then the force  $g'$  in (1) is piecewise linear and Fourier methods can be applied. We refer the reader to papers by Atkinson and Cabrera [2], Earmme [5]) and extensive work by Truskinovsky and collaborators, covering the so-called Fermi-Pasta-Ulam-Tsingou chain with piecewise quadratic interaction [22] and the Frenkel-Kontorova model [14]. Kresse and Truskinovsky have studied the case of an on-site potential with different moduli (second derivatives at the minima) [15]. We also mention important contributions by Slepian, for example [21, 20]. Flytzanis, Crowley and Celli [6] apply Fourier techniques to a problem where the potential consists of three parabolas, the middle one being concave.

A difficulty with a piecewise quadratic on-site potential is that it is not smooth, but has a cusp. One would expect the physical potential to be smooth, but the use of Fourier methods automatically leads to singularities in the force. Does the presence or absence of this singularity affect the existence of waves? The answer is neither obvious from physical view, since the singularity can be seen as an additional force, albeit only acting very locally; mathematically the persistence of solutions would mean that one can perturb from a singular situation to a regular one; we are not aware of well-developed tools to address this question.

In this article, we give an affirmative answer to this question, by showing that a solution to (4) exists for suitable choices of parameters, for nonlinearities which

are suitable mollified versions of the sign function,  $\alpha\psi'(u) \approx \alpha \operatorname{sgn}(u)$ . Note that the unperturbed case  $\alpha\psi'(u) = \alpha \operatorname{sgn}(u)$  corresponds to the piecewise quadratic force (3). The main technical restriction is that the admissible perturbations are small, in a well-defined sense. This is a technical requirement stemming from the use of a fixed point argument employed in the proof. It may be possible to apply continuity methods to show that the solutions we found persist for a wider range of perturbations; but this is a problem not studied in this article.

In terms of the equation, (4) combines a number of difficulties. It combines a differential operator (the second derivative) with a difference operator ( $\Delta_D$ ). See, e.g., [10] for the subject of such functional equations. Here the equation is looking ‘forward’,  $u(x+1)$ , and ‘backward’,  $u(x-1)$ . The theory of such advance-delay equations is still not very well developed, though there are very remarkable results, employing tools ranging from variational techniques to centre manifold/normal form analysis, for example [9, 12, 4]. The non-monotonicity of  $g'$  finally is the core difficulty of the problem.

In terms of the methods, we have to deviate completely from any previous work, as we no longer can rely on Fourier arguments. Instead we develop a perturbative approach capable of starting from a degenerate situation. Two of us have with M. Herrmann and K. Matthies recently developed a different approach for such a perturbation from the so-called Fermi-Pasta-Tsingou chain with smooth nonconvex interaction potential, showing the existence of heteroclinic waves for cases where the potential has a small spinodal (concave) region [11]. As the method used here, the approach relies on a perturbation argument, but then proceeds differently by relying on the Banach fixed point theorem, following a careful analysis of an integral equation describing the travelling wave equation. The framework developed in the present article is relatively flexible and allows potentially the analysis of a range of problems in the setting of (at least) the Frenkel-Kontorova chain. To give an example, we study in Section 4 the problem with a piecewise quadratic on-site potential,  $\psi'(u) = \operatorname{sgn}(u)$ , and establish what is to our knowledge the first proof of solutions exhibiting two transitions between the wells of the on-site potential. It can be regarded as a simplified version of the shadowing lemma [1].

We close the introduction by relating the result here to a few results in the literature. There are few rigorous results for nonconvex interaction potentials available, in particular for heteroclinic solutions as we will study. A very remarkable existence result for such solutions is that of Iooss and Kirchgässner [12]; there a general theory for small solutions is developed. Here we are interested in (large) heteroclinic solutions that stay asymptotically for  $x \rightarrow -\infty$  in one well of a nonconvex on-site potential  $g$  and for  $x \rightarrow \infty$  in another well. For the particular choice  $\alpha\psi'(u) = \alpha \operatorname{sgn}(u)$ , the existence of such travelling waves has been established for suitable parameters with an argument based on Fourier estimates [13]. Here we show that this result holds true in greater generality, in particular for on-site potentials where the concave part is not degenerate as it is assumed in [13]. We work in a nonlinear setting where the Fourier methods of [13] are not applicable. The existence of heteroclinic travelling waves for the Frenkel-Kontorova problem (1) has been open since 1939 (for coherent spatially

localised temporally periodic solutions, existence was established in the seminal paper by MacKay and Aubry [16]; see also [17]).

## 2 Setup and main result

The central argument we are going to employ is a Schauder fixed point theorem. This is possibly surprising, as equation (4) is defined on the whole real line and therefore there is *a priori* no reason to expect compactness properties for (4). We now sketch the setting in which the Schauder theorem applies.

We start by considering the linear part of (4). The linear operator

$$u \rightarrow Lu = c^2 u'' - \Delta_D u + \alpha u \quad (5)$$

has in Fourier space the representation

$$-c^2 \zeta^2 + 2(1 - \cos \zeta) + \alpha = -c^2 \zeta^2 + 4 \sin^2(\zeta/2) + \alpha =: D(\zeta), \quad (6)$$

where  $D$  is the *dispersion function*. Obviously, for the sound speed,  $c = 1$ , the dispersion relation  $D$  has exactly two nonzero roots  $\pm k_0$ , where

$$k_0 := \frac{\pi}{2} \quad (7)$$

if

$$\alpha = c^2 \left(\frac{\pi}{2}\right)^2 - 2, \quad (8)$$

and furthermore  $D'(\zeta) = -2c^2 \zeta + 2 \sin \zeta$  vanishes only at  $\zeta = 0$ . We will work in a parameter regime where  $c$  is marginally subsonic; we keep  $k_0$  fixed by (7) and  $\alpha$  given by (8). Then  $c$  is the only free parameter in the dispersion relation. Since we seek to find heteroclinic solutions, we will focus on subsonic waves, that is,  $c \leq 1$ .

By continuity, the dispersion function will have exactly two real roots near  $\pm k_0$  for ‘almost sonic’ subsonic speeds  $c$ .

Our main theorem can be considered as perturbation result, allowing for a family of smooth, even potentials  $\psi_\epsilon \in C^2(\mathbb{R})$  which are ‘close’ to the special case  $\psi'(u) = \text{sgn}(u)$  considered in [13]. More specifically, we will assume in our main Theorem 2.2 below that  $\psi'_\epsilon(x) = \text{sgn}(x)$  for  $|x| \geq \epsilon$  and  $|\psi''_\epsilon(x)| \leq 2\epsilon^{-1}$  for  $|x| < \epsilon$ .

We sketch the situation for this degenerate potential briefly. For  $|\lambda| < 1$  and  $\theta \in [0, 2\pi)$ , trivially  $1 + \lambda \sin(k_0 \cdot + \theta)$  is a solution to (4) on  $[1, \infty)$  and  $-1 + \lambda \sin(k_0 \cdot - \theta)$  is a solution on  $(-\infty, -1]$ . The question is whether these two solution segments can be glued together to form a heteroclinic solution, traversing from one well of the on-site potential  $g$  to another.

The answer is affirmative for the degenerate potential discussed in this paragraph, as shown in [13] (recalled in Theorem 2.1 below). This solution  $u \in H_{\text{loc}}^2(\mathbb{R})$  is odd,  $u(x) = -u(-x)$ , and heteroclinic in the sense that

$$\lim_{x \rightarrow \pm\infty} [u(x) \mp 1 - \lambda \sin(k_0 x \pm \theta)] = 0$$

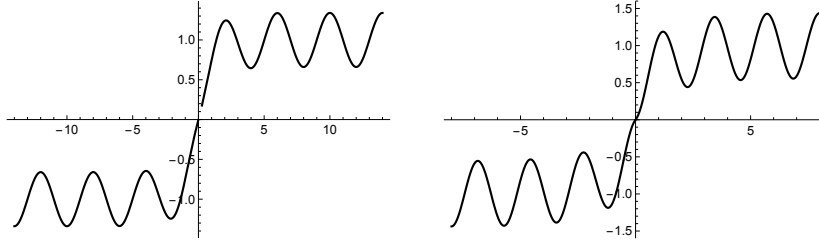


Figure 1: Left: A plot of  $u_{\text{pa}}$  for  $k_0 = \frac{\pi}{2}$  and  $c^2 = 0.9$ , which is in the parameter regime studied in this article. Right: The analogous plot for  $k_0 = \frac{7}{8}\pi$  and  $c^2 = 0.8$ .

for some  $\lambda$  and  $\theta$ , and  $\alpha$  given by (8). This solution is well approximated by the explicit function

$$u_{\text{pa}}(x) := \text{sgn}(x) \left[ A \left( 1 - e^{-\beta|x|} \right) + B (1 - \cos(k_0 x)) \right], \quad (9)$$

with

$$A = \frac{c^2 k_0^2 - \alpha}{c^2 (\beta^2 + k_0^2)} \quad \text{and} \quad B = \frac{\alpha + \beta^2 c^2}{c^2 (\beta^2 + k_0^2)} \quad (10)$$

and

$$\beta^2 = \frac{\alpha}{c^2} \cdot \frac{k_0 \sin(k_0)}{2 - 2 \cos(k_0) - k_0 \sin(k_0)} = \frac{\alpha}{c^2} \cdot \frac{k_0}{2 - k_0}.$$

Plots of  $u_{\text{pa}}$  for two different sets of parameter values are shown in Figure 1.

The argument in [13] and this paper uses an idea developed by Schwetlick and Zimmer for a Fermi-Pasta-Ulam chain with nonconvex interaction potential, and no on-site potential [19]. This idea is to represent the solution  $u$  as  $u = u_{\text{p}} - r$  with explicitly given  $u_{\text{p}}$ ; then the analysis is reduced to a careful investigation of the Fourier representation of  $r$ . Here, we will argue similarly and consider a “profile” function  $u_{\text{p}} \in H_{\text{loc}}^2(\mathbb{R})$ . By profile function we mean that the function  $c^2 u_{\text{p}}'' - \Delta_D u_{\text{p}} + \alpha u_{\text{p}} - \alpha \text{sgn}(u_{\text{p}})$  satisfies

$$(1 + x^2)(c^2 u_{\text{p}}'' - \Delta_D u_{\text{p}} + \alpha u_{\text{p}} - \alpha \text{sgn}(u_{\text{p}})) \in L^2(\mathbb{R}) \quad (11)$$

$$\int_{\mathbb{R}} [c^2 u_{\text{p}}'' - \Delta_D u_{\text{p}} + \alpha u_{\text{p}} - \alpha \text{sgn}(u_{\text{p}})] \sin(k_0 \cdot) dx = 0. \quad (12)$$

The former condition implies  $c^2 u_{\text{p}}'' - \Delta_D u_{\text{p}} + \alpha u_{\text{p}} - \alpha \text{sgn}(u_{\text{p}}) \in L^1(\mathbb{R})$ , so the latter condition is well posed. In addition, we demand the function should be odd,  $\text{sgn}(u_{\text{p}}(x)) = \text{sgn}(x)$  on  $\mathbb{R}$ , vanishes at  $x = 0$ , satisfy  $u_{\text{p}}'(0) > 0$  and  $\liminf_{|x| \rightarrow \infty} |u_{\text{p}}(x)| > 0$ , so that equation (16) below holds. It is somewhat tedious but not difficult to find such a  $u_{\text{p}}$ . In this paper, we will use as profile  $u_{\text{p}}$  a solution to (4) with the special force  $\psi'(x) = \text{sgn}(x)$ , obtained in [13] and recalled in Theorem 2.1 below. The advantage is that for this choice of  $u_{\text{p}}$ , the verification of essential conditions such as (11), (12) and (C2) below is relatively straightforward.

**Theorem 2.1** ([13, Theorem 4.1]). *Let  $\psi'(x) = \text{sgn}(x)$ . Let  $c$  be such that  $c^2 \in [0.83, 1]$ . Let  $k_0$  be given by (7) and  $\alpha$  be given by (8). Then (4) has a solution  $u_p = u_{pa} - r$  with  $u_{pa}$  given by (9) with*

$$\sqrt{\frac{\pi}{2}} |r(z)| \leq \begin{cases} 0.257 & \text{for } c^2 \in [0.9, 1], \\ 0.339 & \text{for } c^2 \in [0.83, 0.9], \end{cases}$$

and

$$\sqrt{\frac{\pi}{2}} |r'(z)| \leq \begin{cases} 0.43 & \text{for } c^2 \in [0.9, 1], \\ 0.34 & \text{for } c^2 \in [0.83, 0.9]. \end{cases}$$

Furthermore,  $r \in H_{\text{odd}}^2(\mathbb{R})$ ,  $\text{sgn}(u_p(x)) = \text{sgn}(x)$  on  $\mathbb{R}$ ,  $u_p(0) = 0$ ,  $u_p'(0) > 0$  and  $\liminf_{|x| \rightarrow \infty} |u_p(x)| > 0$ .

The last claim follows from [13, p. 1147, bottom], while the conditions at the origin are direct consequences of [13, (25), (30) and (32)]; the sign condition is [13, (9), proof of Theorem 4.1] and  $r \in H_{\text{odd}}^2(\mathbb{R})$  is a consequence of [13, Corollary 3.2] and the symmetric construction.

**Remark** For lower wave speeds, the theorem above is not applicable, and one thus would have to find a profile  $u_p$  satisfying the assumptions of this paper, in particular (11), (12) and (C2). Since we want to focus on the method to prove existence, which takes a suitable  $u_p$  as mere input, we do not pursue this question here.

We are left with having to find  $r \in H_{\text{odd,loc}}^2(\mathbb{R})$  (that is,  $r \in H_{\text{loc}}^2(\mathbb{R})$  and  $r(-x) = -r(x)$ ) such that  $u_p - r$  is a solution:

$$c^2(u_p - r)'' - \Delta_D(u_p - r) + \alpha(u_p - r) - \alpha\psi'(u_p - r) = 0,$$

and hence for  $r$

$$c^2r'' - \Delta_D r + \alpha r = c^2u_p'' - \Delta_D u_p + \alpha u_p - \alpha\psi'(u_p - r),$$

which is an equation of the form

$$c^2r'' - \Delta_D r + \alpha r = Q,$$

or  $Lr = Q$  with nonlinear  $Q$ . This is not a standard perturbation problem, because of two important interrelated features. Firstly, the solution  $u$  given by Theorem 2.1 is not isolated. Namely,  $u_p(x_0 + \cdot) + \gamma_s \sin(k_0 \cdot) + \gamma_c \cos(k_0 \cdot)$  is another solution, for all  $\gamma_s$  and  $\gamma_c$  close enough to 0 and all  $x_0 \in \mathbb{R}$ . Secondly, the linear operator  $L$  is not invertible, in the sense that  $\sin(k_0 \cdot)$  and  $\cos(k_0 \cdot)$  are in its kernel. Although  $\sin(k_0 \cdot)$  and  $\cos(k_0 \cdot)$  do not decay to 0, their existence creates difficulties even when working with  $r$  decaying to 0; a difficulty is that in finite balls in  $L^2$  as one uses in fixed point arguments, the corrector  $r$  could take the shape of a kernel function restricted to a finite domain, which could

lead to an approximate solution on the boundary of the ball which is different from the intended solution. For simplicity, we shall in the following theorem assume that  $\psi_\epsilon$  is even, so that we can consider only odd solutions and reduce partially in this way the degeneracy of the problem.

To apply perturbative techniques, we first need to recast the problem (see Subsection 3.1). Then we will employ Schauder's fixed point theorem, for the following reason. As Schauder's fixed point theorem does not require assumptions on the derivative of the nonlinear operator or a Lipschitz condition, it is in principle easier to deal with than the Implicit Function Theorem or the Banach fixed point theorem, for example, provided that some compactness is available. As a consequence, it allows in principle to obtain existence results under less restrictive hypotheses. Although we are far away from any satisfying general result, see the hypotheses of Theorem 3.5, which involve explicit constants. Note that Theorem 3.5 is formulated in terms of a rather general potential  $\Psi$  that is later in this work (in Section 3.4) related to the particular  $\psi$  under consideration.

The main result can be stated as follows.

**Theorem 2.2.** *For  $\epsilon > 0$  small enough, let the even function  $\psi = \psi_\epsilon \in C^2(\mathbb{R})$  be such that  $\psi'_\epsilon(x) = \text{sgn}(x)$  for  $|x| \geq \epsilon$  and  $|\psi''_\epsilon(x)| \leq 2\epsilon^{-1}$  for  $|x| < \epsilon$ . Let  $k_0$  be given by (7),  $\alpha$  be given by (8). Then there exists a range of subsonic velocities  $c$  close to 1 such that for these velocities, there exists a heteroclinic solution to (4).*

We remark that one of the conditions imposed on closeness of  $c$  to 1 is  $c^2 \in [0.83, 1]$  as only in this case we can build on the existence result Theorem 2.1. The existence result Theorem 2.1 provides a one-transition wave which we expect to persist under small perturbations only.

Theorem 2.2 is proved in the next section.

## 3 Proof of Theorem 2.2

### 3.1 Preliminaries

We now turn to the proof of Theorem 2.2. We seek a solution to (4),

$$c^2 u'' - \Delta_D u + \alpha u - \alpha \psi'(u) = 0.$$

By assumption,  $\psi \in C^2(\mathbb{R})$  is even and for its derivative it holds that  $\psi' = \text{sgn}$  outside a bounded set. We split the solution  $u$  sought to (4) as

$$u = u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r, \tag{13}$$

where the profile function  $u_p \in H_{\text{loc}}^2(\mathbb{R})$  is odd and satisfies properties (11) and (12). Further,  $\gamma \in \mathbb{R}$  is assumed to be sufficiently close to 0, and  $u_o \in H_{\text{loc}}^2(\mathbb{R})$  is an odd function such that for each  $l = 0, 1, 2$ ,

$$(1 + x^2) \frac{d^l}{dx^l} (u_o(x) - \text{sgn}(x) \cos(k_0 x)) \in L^2(\mathbb{R} \setminus [-1, 1]). \tag{14}$$



For example, one can choose  $u_o$  to agree with  $\text{sgn}(x) \cos(k_0 x)$  outside a bounded interval. It is not hard to give an explicit representation for  $u_o$ , whereas  $u_p$  is the solution given by Theorem 2.1; the task is then to find the corrector  $r \in H_{\text{odd}}^2(\mathbb{R})$  such that  $u$  as in (13) solves (4). The periodic term  $\gamma \sin(k_0 \cdot)$  is separated from  $u_p$  for mere convenience; obviously this term could be added to  $u_p$  and then  $\tilde{u}_p := u_p + \gamma \sin(k_0 \cdot)$  satisfies (11) and (12) if  $\text{sgn}(\tilde{u}_p) = \text{sgn}(u_p)$ , and could replace  $u_p$ .

With this notation, we can now restate Theorem 2.2 in a more detailed form we are going to establish.

**Theorem 3.1.** *For  $\epsilon > 0$ , let the even function  $\psi = \psi_\epsilon \in C^2(\mathbb{R})$  be such that  $\psi'_\epsilon(x) = \text{sgn}(x)$  for  $|x| \geq \epsilon$  and  $|\psi''_\epsilon(x)| \leq 2\epsilon^{-1}$  for  $|x| < \epsilon$ . Let  $k_0$  be given by (7),  $\alpha$  be given by (8). Then there exists a range of subsonic velocities  $c$  close to 1 such that a heteroclinic solution to (4) exists, in the following sense. Let the odd function  $u_p \in H_{\text{loc}}^2(\mathbb{R})$  be the solution to the equation  $c^2 u'' - \Delta_D u + \alpha u - \alpha \text{sgn}(u) = 0$  of Theorem 2.1, and let the odd function  $u_o \in H_{\text{loc}}^2(\mathbb{R})$  satisfy (14).*

*Then for all  $|\gamma|$  and  $\rho > 0$  small enough, there exists  $\epsilon_0 > 0$  satisfying the following property. For every  $\epsilon \in (0, \epsilon_0)$ , there exists  $r \in H_{\text{odd}}^2(\mathbb{R})$  and  $\beta \in \mathbb{R}$  such that  $\|r\|_{H^2(\mathbb{R})} < \rho$  and  $u := u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r$  is a solution to (4),*

$$\begin{aligned} & c^2(u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r)'' - \Delta_D(u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r) \\ & + \alpha(u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r) - \alpha \psi'_\epsilon(u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r) = 0. \end{aligned} \quad (15)$$

Theorem 2.2 follows immediately once Theorem 3.1 is established, and the rest of the article is devoted to the proof of Theorem 3.1.

We start the proof by considering the linear operator  $L$  of (5) with  $\alpha$  as in (8) and  $c$  being slightly subsonic. Specifically, we first study the equation  $Lr = Q$  under the hypothesis  $\int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = 0$ , with  $k_0 = \pi/2$ . Roughly speaking, in the equation  $Lr = Q$ , the right-hand side is replaced by a new expression  $Q$  depending on  $u_o$  and a real parameter  $\beta$  chosen so that  $\int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = 0$ .

**Lemma 3.2.** *Let  $u_p$  be the solution to the special case  $\psi'(x) = \text{sgn}(x)$  recalled in Theorem 2.1. There exists  $\rho > 0$  such that, for all  $r$  in the ball  $\overline{B(0, \rho)} \subset H_{\text{odd}}^2(\mathbb{R})$ ,  $\text{sgn}(u_p(x) - r(x)) = \text{sgn}(x)$  on  $\mathbb{R}$ .*

*Proof.* Recall the Sobolev estimates

$$\begin{aligned} \|r\|_{L^\infty(\mathbb{R})} & \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1+k^2} (1+k^2) |\widehat{r}| dk \\ & \leq \frac{1}{\sqrt{2\pi}} \sqrt{\int_{\mathbb{R}} \frac{1}{(1+k^2)^2} dk} \|r\|_{H^2(\mathbb{R})} = \frac{1}{2} \|r\|_{H^2(\mathbb{R})} \end{aligned}$$

and

$$\begin{aligned} \|r'\|_{L^\infty(\mathbb{R})} & \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{|k|}{1+k^2} (1+k^2) |\widehat{r}| dk \\ & \leq \frac{1}{\sqrt{2\pi}} \sqrt{\int_{\mathbb{R}} \frac{k^2}{(1+k^2)^2} dk} \|r\|_{H^2(\mathbb{R})} = \frac{1}{2} \|r\|_{H^2(\mathbb{R})}. \end{aligned}$$

By symmetry, it suffices to consider positive  $x$ . Hence it suffices to choose  $\rho_0 > 0$  such that there is a point  $x_0 \in (0, 1)$  such that

$$u_p(x) > \rho_0/2 \text{ for } x > x_0 \text{ and } u'_p(x) > \rho_0/2 \text{ for every } x \in [0, x_0]. \quad (16)$$

Since  $u_p$  satisfies this property for some  $\rho_0$ , so the claim follows for any  $\rho \in (0, \rho_0)$ .  $\square$

Throughout this article, we will assume  $\rho \in (0, \rho_0)$ . We also assume that  $\epsilon < \rho_0/6$ , so that  $\psi'(s) = \text{sgn}(s)$  for all  $|s| \geq \rho_0/6$ .

If we add the requirement on  $\beta$ ,  $\gamma$  and  $r$  that the condition

$$|\beta u_o(x) + \gamma \sin(k_0 x) - r(x)| \leq \frac{2}{3} |u_p(x)| \quad (17)$$

is fulfilled for all  $x \in \mathbb{R}$ , then solving (4) with the *ansatz* (13) is equivalent to finding a solution of a problem with a relaxed nonlinearity  $\Psi(u, x)$

$$c^2 u'' - \Delta_D u + \alpha u - \alpha \partial_1 \Psi(u, x) = 0 \text{ with } u = u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r, \quad (18)$$

where  $\Psi: \mathbb{R}^2 \rightarrow \mathbb{R}$  is the specific embedding of the original interaction force

$$\begin{cases} \Psi(u, x) = \psi(u) & \text{for } |x| < 1, \\ \Psi(u, x) = \text{sgn}(x)u & \text{for } |x| \geq 1. \end{cases} \quad (19)$$

A suitable constant could be added to  $\text{sgn}(x)u$  in the definition of  $\Psi$ , but such a constant is irrelevant, as only  $\partial_1 \Psi$  matters in what follows. We prove the existence of a solution using Schauder's fixed point theorem. Note that the *ansatz* (17) for  $u$  means that if a solution is found the transition between wells is found well within the interval  $x \in (-1, 1)$ , so that  $\partial_1 \Psi$  will coincide with the original  $\psi'$  for all  $x \in \mathbb{R}$ . Indeed, by (16) and (17), we get for all  $|x| > x_0$  that

$$|u(x)| = |u_p(x) + \beta u_o(x) + \gamma \sin(k_0 x) - r(x)| \geq \frac{1}{3} |u_p(x)| > \frac{\rho_0}{6} > \epsilon$$

and

$$\psi'(u(x)) = \text{sgn}(x) = \partial_1 \Psi(u(x), x).$$

### 3.2 Application of Schauder's fixed point theorem

In this section, we prove the existence of a solution of a slightly relaxed problem, Equation (26) below, under fairly abstract assumptions, notably (C1), (C2) in Theorem 3.5 below. The following sections then establish that the original problem can be cast in the setting studied here.

Specifically, consider the following modification of (15) for  $r \in H^2_{\text{odd}}(\mathbb{R})$  and  $\beta \in \mathbb{R}$ , and recall  $\psi'(u(x)) = \partial_1 \Psi(u(x), x)$  for the function  $u$  we have in mind,

$$\begin{aligned} & c^2 (u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r)'' - \Delta_D (u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r) \\ & + \alpha (u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r) - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, x) = 0; \end{aligned} \quad (20)$$

here the new ingredient is a function  $\xi \in C^1(\mathbb{R})$  with

$$\|\xi\|_{L^\infty(\mathbb{R})} + \|\xi'\|_{L^\infty(\mathbb{R})} < \infty.$$

Thus, in a first step, we replaced  $\beta$  by  $\xi(\beta)$  in the nonlinear term. As  $\xi$  and  $\xi'$  are assumed to be bounded, the function  $\xi$  allows us to control the nonlinear term without restrictions on the size of  $\beta$ . In a second step, we shall assume that  $\xi$  is the identity near 0 and show that the relevant values of  $\beta$  are sufficiently close to 0, so that  $\xi(\beta) = \beta$  for these values of  $\beta$ .

The assumptions in this Subsection are as follows (they will be used in the abstract Theorem 3.5 below). We recall  $k_0$  is given by (7),  $\alpha$  is given by (8), and  $c$  is close to 1. We have seen that then the dispersion function in (6) has exactly two simple roots  $\pm k_0$ . Furthermore, for the linear operator given in (5),  $L \sin(k_0 \cdot) = 0$ . Let  $\Psi: \mathbb{R}^2 \rightarrow \mathbb{R}$  be of class  $C^2$  with respect to the first variable,  $\Psi$ ,  $\partial_1 \Psi$  and  $\partial_{11}^2 \Psi$  be measurable with respect to the second variable,  $\partial_1 \Psi$  be odd and

$$|\partial_{11}^2 \Psi(s, x)| \leq \frac{\mu}{(1+x^2)^{3/2}} \quad (21)$$

for some constant  $\mu > 0$ . Note that

$$(1+x^2) \frac{1}{(1+x^2)^{3/2}} \in L^2(\mathbb{R}).$$

The size of  $\mu$  does not matter in what follows (in particular, it is not assumed to be small).

We recall that the parameter  $\gamma$  is real-valued, and that  $u_p$  is a given odd function in  $H_{\text{loc}}^2(\mathbb{R})$  satisfying

$$\sup_{\beta \in \mathbb{R}} \left\| (1+x^2)^{3/2} \left( c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 x), x) \right) \right\|_{L^\infty(\mathbb{R})} < \infty. \quad (22)$$

The odd function  $u_o \in H_{\text{loc}}^2(\mathbb{R})$  satisfies (14). Thus, since  $L \cos(k_0 \cdot) = 0$ ,

$$(1+x^2) L u_o = (1+x^2) (c^2 u_o'' - \Delta_D u_o + \alpha u_o) \in L_{\text{odd}}^2(\mathbb{R}).$$

It follows that the map

$$(r, \beta) \rightarrow \Gamma(r, \beta) = (1+x^2) \left( c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 x) - r, x) \right) \in L_{\text{odd}}^2(\mathbb{R})$$

is well-defined on  $H_{\text{odd}}^2(\mathbb{R}) \times \mathbb{R}$  and of class  $C^1$ . Note that it can be written as

$$\begin{aligned} \Gamma(r, \beta) &= (1+x^2) \left( c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 x), x) \right) \\ &\quad + \alpha (1+x^2) \int_0^1 \partial_{11}^2 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 x) - sr, x) r ds, \quad (23) \end{aligned}$$

which is the sum of two terms in  $L_{\text{odd}}^2(\mathbb{R})$  (see (21) and (22)).

**Lemma 3.3.** *The map  $\Gamma: H_{\text{odd}}^2(\mathbb{R}) \times \mathbb{R} \rightarrow L_{\text{odd}}^2(\mathbb{R})$  is compact.*

*Proof.* Let  $\{(r_n, \beta_n)\} \subset H_{\text{odd}}^2(\mathbb{R}) \times \mathbb{R}$  be a bounded sequence. We verify that  $\{\Gamma(r_n, \beta_n)\}$  has a Cauchy subsequence in  $L_{\text{odd}}^2(\mathbb{R})$ . Let  $\varepsilon > 0$ .

Since  $\xi$  is continuous on  $\mathbb{R}$ , the sequence  $\{\xi_n\} := \{\xi(\beta_n)\}$  is bounded. Taking a convergent subsequence  $\{\xi_{n_k}\}$ , equation (22) and the dominated convergence theorem ensure that the first term of  $\Gamma(r_{n_k}, \beta_{n_k})$  converges as  $k \rightarrow \infty$ . Hence, for  $k, l$  large enough,

$$\left\| (1+x^2) \left( c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta_{n_k}) u_o + \gamma \sin(k_0 x), x) \right) - (1+x^2) \left( c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta_{n_l}) u_o + \gamma \sin(k_0 x), x) \right) \right\|_{L^2(\mathbb{R})} < \frac{\varepsilon}{2}.$$

To deal with the second term, we split  $\mathbb{R}$  in two parts, namely  $I_\varepsilon := [-x_\varepsilon, x_\varepsilon]$  and its complement in  $\mathbb{R}$ , where  $x_\varepsilon > 0$  is large. The motivation for this split is that many Sobolev embeddings are compact on an bounded interval, whereas the second term can be assumed as small as needed when restricted to the complement of  $I_\varepsilon$ . More precisely, given  $\varepsilon > 0$ , choose  $x_\varepsilon$  large enough so that for all  $k$

$$\alpha \left\| (1+x^2) \int_0^1 \partial_{11}^2 \Psi(u_p + \xi_{n_k} u_o + \gamma \sin(k_0 x) - s r_{n_k}, x) r_{n_k} ds \right\|_{L^2(\mathbb{R} \setminus I_\varepsilon)} < \frac{\varepsilon}{8}$$

(see (21)). Using the compact embedding  $H^2(-x_\varepsilon, x_\varepsilon) \subset C[-x_\varepsilon, x_\varepsilon]$ , by taking a further subsequence if necessary, we can assume that  $\{r_{n_k}\}$  converges in  $C[-x_\varepsilon, x_\varepsilon]$ . It follows, again from the dominated convergence theorem, that

$$\alpha(1+x^2) \int_0^1 \partial_{11}^2 \Psi(u_p + \xi_{n_k} u_o + \gamma \sin(k_0 x) - s r_{n_k}, x) r_{n_k} ds$$

converges in  $L^2(-x_\varepsilon, x_\varepsilon)$ . Hence, for  $k, l$  large enough,

$$\left\| \alpha(1+x^2) \int_0^1 \partial_{11}^2 \Psi(u_p + \xi_{n_k} u_o + \gamma \sin(k_0 x) - s r_{n_k}, x) r_{n_k} ds - \alpha(1+x^2) \int_0^1 \partial_{11}^2 \Psi(u_p + \xi_{n_l} u_o + \gamma \sin(k_0 x) - s r_{n_l}, x) r_{n_l} ds \right\|_{L^2(\mathbb{R})} < \varepsilon/2.$$

Thus  $\{\Gamma(r_{n_k}, \beta_{n_k})\}$  is a Cauchy subsequence.  $\square$

By (33) of Proposition A.2 in the Appendix,

$$\int_{\mathbb{R}} (c^2 u_o'' - \Delta_D u_o + \alpha u_o) \sin(k_0 \cdot) dx = -2c^2 k_0 + 2 < 0$$

if  $c > k_0^{-1/2}$ . Assume that, for all  $r$  in some subset of  $H_{\text{odd}}^2(\mathbb{R})$  and all  $\beta \in \mathbb{R}$ ,

$$\begin{aligned} & \left| \int_{\mathbb{R}} \alpha \partial_{11}^2 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \xi'(\beta) u_o \sin(k_0 \cdot) dx \right| \\ & \leq C \left| \int_{\mathbb{R}} (c^2 u_o'' - \Delta_D u_o + \alpha u_o) \sin(k_0 \cdot) dx \right| = C 2(c^2 k_0 - 1) \end{aligned}$$

for some constant  $C \in [0, 1)$ . Then for fixed  $r$  in the given subset, the equation

$$\int_{\mathbb{R}} \left( c^2 (u_p + \beta u_o + \gamma \sin(k_0 \cdot))'' - \Delta_D (u_p + \beta u_o + \gamma \sin(k_0 \cdot)) + \alpha (u_p + \beta u_o + \gamma \sin(k_0 \cdot)) - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 \cdot) - r, x) \right) \sin(k_0 \cdot) dx = 0$$

can uniquely be solved for  $\beta$  as a  $C^1$ -function of  $r$ ,  $\beta = \beta(r)$ , thanks to Banach's fixed point theorem and the implicit function theorem.

**Lemma 3.4.** *The map  $r \rightarrow \beta(r)$  is bounded on bounded sets.*

*Proof.* The splitting (23) and the hypothesis that  $\xi$  is bounded show an additional property, namely that the map  $(r, \beta) \rightarrow \Gamma(r, \beta)$  is bounded on every set on which the  $r$ -component is bounded. As a consequence, by definition of  $\beta = \beta(r)$ ,

$$2(c^2 k_0 - 1)\beta = -\beta \int_{\mathbb{R}} (c^2 u_o - \Delta_D u_o + \alpha u_o) \sin(k_0 x) dx = \int_{\mathbb{R}} [c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 \cdot) - r, \cdot)] \sin(k_0 x) dx$$

and

$$\beta = \frac{\int_{\mathbb{R}} [c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 \cdot) - r, \cdot)] \sin(k_0 x) dx}{2(c^2 k_0 - 1)}. \quad (24)$$

The map  $r \rightarrow \beta(r) = \frac{1}{2}(c^2 k_0 - 1)^{-1} \int_{\mathbb{R}} \Gamma(r, \beta(r))(1 + x^2)^{-1} \sin(k_0 x) dx$  is thus bounded on bounded sets.  $\square$

Hence the problem can be written as  $c^2 r'' - \Delta_D r + \alpha r = Q$ , with

$$\begin{aligned} Q &= c^2 (u_p + \beta(r) u_o + \gamma \sin(k_0 \cdot))'' - \Delta_D (u_p + \beta(r) u_o + \gamma \sin(k_0 \cdot)) \\ &\quad + \alpha (u_p + \beta(r) u_o + \gamma \sin(k_0 \cdot)) - \alpha \partial_1 \Psi(u_p + \xi(\beta(r)) u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \\ &= \beta(r) (c^2 u_o'' - \Delta_D u_o + \alpha u_o) + (1 + x^2)^{-1} \Gamma(r, \beta(r)) \in L^2_{\text{odd}}(\mathbb{R}) \end{aligned} \quad (25)$$

and  $\int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = 0$  by definition of  $\beta(r)$ . On the other hand, if  $Q \in L^2(\mathbb{R})$  is odd with

$$(1 + x^2)Q \in L^2(\mathbb{R}) \quad \text{and} \quad \int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = 0,$$

and by Proposition A.1 in Appendix A, there exists a unique odd  $r = L^{-1}Q \in H^2(\mathbb{R})$  such that  $Lr = c^2 r'' - \Delta_D r + \alpha r = Q$ . Moreover

$$\|L^{-1}Q\|_{H^2(\mathbb{R})} = \|r\|_{H^2(\mathbb{R})} \leq \{C_1 + ((4 + \alpha)C_1 + 1)/c^2\} \|(1 + x^2)Q\|_{L^2(\mathbb{R})}$$

for some constant  $C_1 > 0$ .

The problem (20) studied in this Subsection can be written as

$$r = L^{-1}Q = L^{-1}\left(c^2(u_p + \beta u_o + \gamma \sin(k_0 \cdot))'' - \Delta_D(u_p + \beta u_o + \gamma \sin(k_0 \cdot)) + \alpha(u_p + \beta u_o + \gamma \sin(k_0 \cdot)) - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, x)\right) \quad (26)$$

with  $\beta = \beta(r)$ .

**Theorem 3.5.** *Let  $\xi$  be in  $C^1(\mathbb{R})$  with  $\|\xi\|_{L^\infty(\mathbb{R})} + \|\xi'\|_{L^\infty(\mathbb{R})} < \infty$ . Let  $k_0$  be as in (7) and  $\alpha$  given by (8). Let  $\Psi: \mathbb{R}^2 \rightarrow \mathbb{R}$  be of class  $C^2$  with respect to the first variable, let  $\Psi$ ,  $\partial_1 \Psi$  and  $\partial_{11}^2 \Psi$  be measurable with respect to the second variable, and  $\partial_1 \Psi$  be odd. Assume that the hypotheses (14), (21) and (22) hold. Suppose that there exists an open ball  $B(0, \rho) \subset H_{\text{odd}}^2(\mathbb{R})$  such that*

$$\sup_{r \in \overline{B(0, \rho)}, \beta \in \mathbb{R}} \left| \int_{\mathbb{R}} \alpha \partial_{11}^2 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \xi'(\beta)u_o \sin(k_0 \cdot) dx \right| < 2(c^2 k_0 - 1) \quad (C1)$$

and

$$\sup_{r \in \overline{B(0, \rho)}} \|(1 + x^2)Q(r)\|_{L^2(\mathbb{R})} < \{C_1 + ((4 + \alpha)C_1 + 1)/c^2\}^{-1} \rho, \quad (C2)$$

here the notation  $Q = Q(r)$  insists on the fact that  $Q$  defined in (25) depends on  $r$ . Then there exists a solution  $r \in B(0, \rho)$  to (26).

*Proof.* For all  $r \in \overline{B(0, \rho)}$ ,  $Q = Q(r)$  is well defined with values in

$$Z = \left\{ f \in L_{\text{odd}}^2(\mathbb{R}) : (1 + x^2)f \in L^2(\mathbb{R}), \int_{\mathbb{R}} f(x) \sin(k_0 x) dx = 0 \right\}$$

and completely continuous in  $r$  (that is, continuous and compact; see (25), Lemma 3.3 and Lemma 3.4). The map  $r \rightarrow L^{-1}Q(r)$  sends  $\overline{B(0, \rho)}$  into  $B(0, \rho)$  and is completely continuous. The Schauder fixed point theorem gives a solution  $r \in \overline{B(0, \rho)}$  to the equation  $r = L^{-1}Q(r)$ , and in fact  $r \in B(0, \rho)$ .  $\square$

### 3.3 On the verification of condition (C2)

In this section, we establish one condition, (C2') below, for the verification of condition (C2) in Theorem 3.5. This simpler condition will then be shown in Subsection 3.4 to hold under the assumptions of Theorem 3.1.

By the formula (24) for  $\beta = \beta(r)$  and

$$Q = (c^2 u_o'' - \Delta_D u_o + \alpha u_o) \beta + c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, \cdot),$$

one has

$$\begin{aligned} & \|(1+x^2)Q\|_{L^2(\mathbb{R})} \leq \|(1+x^2)(c^2u_o'' - \Delta_D u_o + \alpha u_o)\|_{L^2(\mathbb{R})} \\ & \times \left| \frac{\int_{\mathbb{R}} \left\{ c^2u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \right\} \sin(k_0 x) dx}{2(c^2k_0 - 1)} \right| \\ & + \|(1+x^2)(c^2u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, \cdot))\|_{L^2(\mathbb{R})}. \end{aligned}$$

Hence condition (C2) is ensured by the following condition

$$\begin{aligned} & \sup_{r \in B(0, \rho)} \left\{ \|(1+x^2)(c^2u_o'' - \Delta_D u_o + \alpha u_o)\|_{L^2(\mathbb{R})} \frac{\|(1+x^2)^{-1} \sin(k_0 \cdot)\|_{L^2(\mathbb{R})}}{2(c^2k_0 - 1)} + 1 \right\} \\ & \times \|(1+x^2)(c^2u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, \cdot))\|_{L^2(\mathbb{R})} \\ & < \frac{1}{C_1 + ((4 + \alpha)C_1 + 1)/c^2} \rho, \end{aligned}$$

which in turn is ensured by the condition

$$\begin{aligned} & \sup_{r \in B(0, \rho)} \left\| (1+x^2)(c^2u_p'' - \Delta_D u_p + \alpha u_p \right. \\ & \quad \left. - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, \cdot)) \right\|_{L^2(\mathbb{R})} \\ & < \frac{\{C_1 + ((4 + \alpha)C_1 + 1)/c^2\}^{-1} \rho}{\|(1+x^2)(c^2u_o'' - \Delta_D u_o + \alpha u_o)\|_{L^2(\mathbb{R})} \sqrt{\pi/8}(c^2k_0 - 1)^{-1} + 1}. \end{aligned}$$

If  $u_p$  is a particular solution to the ‘‘unperturbed’’ equation  $c^2u_p'' - \Delta_D u_p + \alpha u_p - \alpha S(u_p, \cdot) = 0$  for some function  $S$ , if

$$\begin{aligned} & \|(1+x^2)(\alpha S(u_p, \cdot) - \alpha \partial_1 \Psi(u_p, \cdot))\|_{L^2(\mathbb{R})} \\ & + \sup_{r \in B(0, \rho)} \left\| (1+x^2) \left( \alpha \partial_1 \Psi(u_p, \cdot) \right. \right. \\ & \quad \left. \left. - \alpha \partial_1 \Psi(u_p + \xi(\beta(r))u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \right) \right\|_{L^2(\mathbb{R})} \\ & < \frac{(C_1 + ((4 + \alpha)C_1 + 1)/c^2)^{-1} \rho}{\|(1+x^2)(c^2u_o'' - \Delta_D u_o + \alpha u_o)\|_{L^2(\mathbb{R})} \sqrt{\pi/8}(c^2k_0 - 1)^{-1} + 1} \quad (\text{C2}') \end{aligned}$$

and if the condition (C1) holds true, then the ‘‘perturbed’’ problem, in which  $S$  is replaced by  $\partial_1 \Psi$  and the parameter  $\gamma$  can be chosen in  $\mathbb{R}$ , has a solution  $r \in B(0, \rho)$ .

### 3.4 Verification of the conditions in Theorem 3.5

In this section, we prove Theorem 3.1. We have to show that the assumptions made there imply those of Theorem 3.5, and show that  $\xi$  can be chosen to be the identity in the region of interest.

We make the same assumptions on  $k_0$ ,  $\alpha$ ,  $u_o$  and  $u_p$  as in Theorem 3.1. In particular, the chosen  $u_p$  is such that  $u'_p(0) > 0$ ,

$$\int_{\mathbb{R}} (c^2 u''_p - \Delta_D u_p + \alpha u_p - \alpha \operatorname{sgn}(u_p)) \sin(k_0 \cdot) dx = 0,$$

and

$$\left\| (1+x^2)^{3/2} (c^2 u''_p - \Delta_D u_p + \alpha u_p - \alpha \operatorname{sgn}(u_p)) \right\|_{L^\infty(\mathbb{R})} < \infty$$

(indeed  $c^2 u''_p - \Delta_D u_p + \alpha u_p - \alpha \operatorname{sgn}(u_p) = 0$ ). Let  $\rho_0 > 0$  satisfy (16); then  $|u_p(x)| > \rho_0/2$  for all  $|x| \geq 1$ .

**Lemma 3.6.** *In the setting of this subsection,  $\xi$  can be chosen such that the solution given by Theorem 3.5 solves (15).*

*Proof.* In Equation (20), we choose  $\xi$  such that it is the identity function in a neighbourhood of  $\beta = 0$  and

$$\|\xi\|_{L^\infty(\mathbb{R})} |u_o(x)| \leq \frac{1}{3} |u_p(x)| \text{ for all } x \in \mathbb{R}.$$

If  $|\gamma|$  and  $\|r\|_{H^2(\mathbb{R})}$  are small enough, then for every  $x \in \mathbb{R}$

$$|u_p(x) + \xi(\beta)u_o(x) + \gamma \sin(k_0 x) - r(x)| \geq \frac{1}{3} |u_p(x)| \quad (27)$$

and thus

$$\partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, x) = \psi'(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r).$$

Hence, we will obtain the solution  $u = u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r$  to

$$c^2 u'' - \Delta_D u + \alpha u - \alpha \psi'(u) = 0$$

if, in addition,  $\xi(\beta) = \beta$ . □

**Lemma 3.7.** *Under the assumptions of Theorem 3.1, assumption (21) of Theorem 3.5 holds.*

*Proof.* This is immediate because, by (19),  $\partial_{11}^2 \Psi(s, x) = 0$  if  $|x| \geq 1$ ,  $\partial_{11}^2 \Psi(s, x) = \psi''(s) = 0$  if  $|x| < 1$  and  $|s| \geq \epsilon$ , and  $|\partial_{11}^2 \Psi(s, x)| = |\psi''(s)| \leq 2\epsilon^{-1}$  if  $|x| < 1$  and  $|s| < \epsilon$ . □

**Lemma 3.8.** *Under the assumptions of Theorem 3.1, the assumptions (22), (C1) and (C2') hold.*

*Proof.* We first establish the claim for (C1). Let us recall that  $\psi$  is such that  $|\psi''(s)| \leq 2\epsilon^{-1}$  for  $|s| < \epsilon$  and  $\psi''(s) = 0$  otherwise, where  $\epsilon > 0$ . If  $\epsilon$  is small enough and  $|x| = 6\epsilon/u'_p(0)$ , then

$$|u_p(x)| = u'_p(0) |x| (1 + o(x)) \geq \frac{1}{2} u'_p(0) |x| \geq 3\epsilon$$



and thus  $|u_p(x)| \geq 3\epsilon$  for all  $|x| \geq 6\epsilon/u'_p(0)$  if  $\epsilon$  is small enough. Hence

$$\psi''(u_p(x) + \xi(\beta)u_o(x) + \gamma \sin(k_0x) - r(x)) = 0$$

for all  $|x| \geq 6\epsilon/u'_p(0)$  if  $|\gamma|$ ,  $\|r\|_{H^2(\mathbb{R})}$  and  $\epsilon$  are small enough (see (27)). Therefore

$$\begin{aligned} & \left| \int_{\mathbb{R}} \alpha \psi''(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r) \xi'(\beta)u_o \sin(k_0 \cdot) dx \right| \\ & \leq \int_{-6\epsilon/u'_p(0)}^{6\epsilon/u'_p(0)} \alpha 2\epsilon^{-1} |\xi'(\beta)u_o \sin(k_0 \cdot)| dx \\ & \leq \alpha 2\epsilon^{-1} \|\xi'(\beta)u_o\|_{L^\infty(\mathbb{R})} \int_{-6\epsilon/u'_p(0)}^{6\epsilon/u'_p(0)} |k_0 x| dx \\ & \leq \alpha 2\epsilon^{-1} \|\xi'(\beta)u_o\|_{L^\infty(\mathbb{R})} k_0 (6\epsilon/u'_p(0))^2 \rightarrow 0 \end{aligned}$$

as  $\epsilon \rightarrow 0$ , uniformly in  $\beta \in \mathbb{R}$  and  $r \in \overline{B(0, \rho)}$  if  $|\gamma|$  and  $\rho > 0$  are small enough. Hence (C1) holds true. Assumption (22) can be verified similarly.

We now show that (C2') is satisfied. We choose for  $u_p$  the solution of the degenerate problem  $c^2 u'' - \Delta_D u + \alpha u - \alpha \text{sgn}(u) = 0$ , see Theorem 2.1, and choose  $\epsilon > 0$  small enough so that

$$\begin{aligned} & \left\| (1+x^2)(\alpha \text{sgn}(u_p) - \alpha \partial_1 \Psi(u_p, \cdot)) \right\|_{L^2(\mathbb{R})} \\ & < \frac{(C_1 + ((4+\alpha)C_1 + 1)/c^2)^{-1} \rho}{2 \left( \|(1+x^2)(c^2 u''_o - \Delta_D u_o + \alpha u_o)\|_{L^2(\mathbb{R})} \sqrt{\pi/8} (c^2 k_0 - 1)^{-1} + 1 \right)}. \end{aligned}$$

Then observe that, for all  $r \in \overline{B(0, \rho)}$ ,

$$\begin{aligned} & \left\| (1+x^2)(\alpha \partial_1 \Psi(u_p, \cdot) - \alpha \partial_1 \Psi(u_p + \xi(\beta(r))u_o + \gamma \sin(k_0 \cdot) - r, \cdot)) \right\|_{L^2(\mathbb{R})} \\ & \leq \left\| (1+x^2) \alpha \sup_{\lambda \in [0,1]} |\partial_{11}^2 \Psi(u_p + \lambda \xi(\beta(r))u_o + \lambda \gamma \sin(k_0 \cdot) - \lambda r, \cdot)| \right. \\ & \quad \times \left. \left\| \xi(\beta(r))u_o + \gamma \sin(k_0 \cdot) - r \right\|_{L^2(\mathbb{R})} \right\|. \end{aligned}$$

Arguing as above,

$$\begin{aligned} & \left\| (1+x^2)(\alpha \partial_1 \Psi(u_p, \cdot) - \alpha \partial_1 \Psi(u_p + \xi(\beta(r))u_o + \gamma \sin(k_0 \cdot) - r, \cdot)) \right\|_{L^2(\mathbb{R})} \\ & \leq \alpha 2\epsilon^{-1} \left\| (1+x^2)(\xi(\beta(r))u_o + \gamma \sin(k_0 \cdot) - r) \right\|_{L^2([-6\epsilon/u'_p(0), 6\epsilon/u'_p(0)])} \rightarrow 0 \end{aligned}$$

as  $\epsilon \rightarrow 0$ , uniformly in  $r \in \overline{B(0, \rho)}$  if  $|\gamma|$  and  $\rho > 0$  are small enough.  $\square$

### 3.5 Proof of Theorem 3.1

By Theorem 3.5, there exists  $r \in H_{\text{odd}}^2(\mathbb{R})$  such that  $\|r\|_{H^2(\mathbb{R})} < \rho$  and

$$\begin{aligned} & c^2(u_p + \beta(r)u_o + \gamma \sin(k_0 \cdot) - r)'' - \Delta_D(u_p + \beta(r)u_o + \gamma \sin(k_0 \cdot) - r) \\ & + \alpha(u_p + \beta(r)u_o + \gamma \sin(k_0 \cdot) - r) - \alpha \psi'(u_p + \xi(\beta(r))u_o + \gamma \sin(k_0 \cdot) - r) = 0. \end{aligned} \quad (28)$$

We also get that  $\beta(r)$  belongs to the neighbourhood of 0 on which  $\xi$  is the identity if  $|\gamma|, \rho, \epsilon$  are small enough. Indeed, by (24),

$$\begin{aligned} |\beta(r)| & \leq \frac{\left| \int_{\mathbb{R}} \left\{ \alpha \operatorname{sgn}(u_p) - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \right\} \sin(k_0 x) dx \right|}{2(c^2 k_0 - 1)} \\ & \leq \frac{1}{2(c^2 k_0 - 1)} \int_{-6\epsilon/u_p'(0)}^{6\epsilon/u_p'(0)} \alpha \left( 1 + \frac{2}{\epsilon} |u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r| \right) k_0 |x| dx \\ & = O(1) \int_{-6\epsilon/u_p'(0)}^{6\epsilon/u_p'(0)} |x| dx = O(\epsilon^2). \end{aligned}$$

Since  $\xi$  is the identity near the origin, and there is a solution to (28), this solution is also a solution to (15).  $\square$

Since Theorem 2.2 is an immediate consequence of Theorem 3.1, the main claim of the paper is proved.

## 4 Two-transition solutions

To demonstrate the flexibility of our method, we show in this section how it can be used to obtain two-transition solutions, that is, travelling waves starting in one well of the on-site potential, making a transition to another well before returning to the first well. Again we focus on the on-site potential to be taken as a piecewise quadratic,  $\psi'(x) = \operatorname{sgn}(x)$ , as in [13]. Also, we consider the same velocity regime  $c^2 \in [0.83, 1]$  as in that paper. However, in contrast to the perturbation of the potential studied in Sections 2 and 3, we now keep the special potential, but prove the existence of solutions representing two transitions between the two wells. We construct the solution similarly as in (13) for the case of a single transition, where the odd profile function  $u_p$  will be replaced by an even profile function  $v_p$ , and similarly the odd function  $u_o$  will be replaced by an even function  $u_e$ . That is, we use a decomposition of the form

$$u(x) = v_p(x) + \beta_e u_e(x) + \tilde{\gamma} \cos(k_0 x) - \tilde{r}(x). \quad (29)$$

Here  $v_p$  is the primary profile,  $\beta_e$  a small coefficient scaling the contribution from  $u_e$ ,  $\tilde{\gamma}$  a coefficient to be chosen later, and  $\tilde{r}$  a (small) remainder.

We first turn the attention to  $v_p$ .

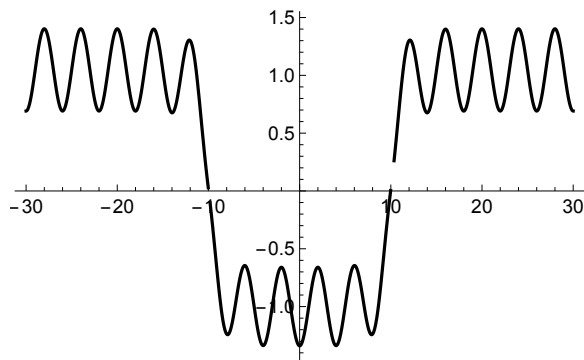


Figure 2: An approximation to the two-transition profile  $v_p$ .

**Lemma 4.1.** *Let  $x_0 \in (\pi/k_0)\mathbb{Z} = 2\mathbb{Z}$  be positive. Then there exist an even profile  $v_p \in H_{\text{loc}}^2(\mathbb{R})$  such that  $v_p$  vanishes exactly at the two points  $\pm x_0$ . Furthermore,*

$$\|(1+x^2)(Lv_p - \alpha \text{sgn}(v_p))\|_{L^2(\mathbb{R})} \rightarrow 0 \quad (30)$$

as  $x_0 \rightarrow \infty$ .

*Proof.* The odd solution  $x \rightarrow u_{\text{pa}}(x) - r(x)$  in [13] (see (9) and (10)) converges in  $H^2(z-2, z+2)$  as  $|z| \rightarrow \infty$  to the function

$$\text{sgn}(x) \left( A + B - B \cos(k_0 x) \right), \text{ where } A + B = 1 \text{ and } B = \frac{c^2 k_0^2 - 2}{c^2 k_0^2 - k_0},$$

where the expression for  $B$  makes use of (7) and (8). More precisely,

$$\left\| u_{\text{pa}} - r - \text{sgn}(\cdot) \left( A + B - B \cos(k_0 \cdot) \right) \right\|_{H^2(z-2, z+2)} \rightarrow 0$$

as  $|z| \rightarrow \infty$ .

It is straightforward to see that  $-u_{\text{pa}} + r$  is also a single-transition solution to the solution to the problem with piecewise quadratic on-site potential studied. We now introduce a two-transition profile  $v_p$  by combining these two single-transition solutions. Namely, for positive  $x_0 \in 2\mathbb{Z}$ , we define  $v_p$  as

$$v_p(x) := \left( \frac{1}{2} + \lambda(x) \right) (u_{\text{pa}}(x - x_0) - r(x - x_0)) \\ - \left( \frac{1}{2} - \lambda(x) \right) (u_{\text{pa}}(x + x_0) - r(x + x_0)),$$

where the step function  $\lambda \in C^\infty(\mathbb{R}, \mathbb{R})$  is odd and non-decreasing with  $\lambda(x) := -1/2$  for  $x \leq -1$  and  $\lambda(x) := 1/2$  for  $x \geq 1$ ; see Figure 2 for an approximation of  $v_p$ .

Obviously  $v_p$  is even, piecewise  $C^2$ , and satisfies  $Lv_p - \alpha \operatorname{sgn}(v_p) = 0$  on  $\mathbb{R} \setminus [-2, 2]$ . To show (30), we thus only have to show that  $\|Lv_p - \alpha \operatorname{sgn}(v_p)\|_{L^2(-2,2)}$  tends to 0 as  $x_0 \rightarrow \infty$  with  $x_0 \in 2\mathbb{Z}$ . We first deal with  $x_0 \in 4\mathbb{Z}$ . For  $x \in (-2, 2)$ , we find that as  $x_0 \rightarrow \infty$

$$\begin{aligned}
v_p(x) &\rightarrow \left(\frac{1}{2} + \lambda(x)\right) \operatorname{sgn}(x - x_0) \{1 - B \cos(k_0(x - x_0))\} \\
&\quad - \left(\frac{1}{2} - \lambda(x)\right) \operatorname{sgn}(x + x_0) \{1 - B \cos(k_0(x + x_0))\} \\
&= - \left(\frac{1}{2} + \lambda(x)\right) \{1 - B \cos(k_0(x - x_0))\} \\
&\quad - \left(\frac{1}{2} - \lambda(x)\right) \{1 - B \cos(k_0(x + x_0))\} \\
&= -1 + B \cdot \left\{ \left(\frac{1}{2} + \lambda(x)\right) \cos(k_0(x - x_0)) \right. \\
&\quad \left. + \left(\frac{1}{2} - \lambda(x)\right) \cos(k_0(x + x_0)) \right\} \\
&= -1 + B \cdot \{\cos(k_0x) \cos(k_0x_0) + 2\lambda(x) \sin(k_0x) \sin(k_0x_0)\} \\
&= -1 + B \cdot \cos(k_0x) \cos(k_0x_0) =: v_p^\infty(x),
\end{aligned}$$

as  $\sin(k_0x_0) = 0$  and  $\cos(k_0x_0) = 1$  is independent of  $x_0 \in 4\mathbb{Z}$ .

On  $(-2, 2)$ , this limit function  $v_p^\infty$  solves  $Lv_p^\infty - \alpha \operatorname{sgn}(v_p^\infty) = 0$ , since  $\cos(k_0x_0) = 1$  and  $B = \frac{c^2 k_0^2 - 2}{c^2 k_0^2 - k_0} = 1 - \frac{2 - k_0}{c^2 k_0^2 - k_0} < 1$  gives

$$v_p^\infty(x) = -1 + B \cdot \cos(k_0x) \cos(k_0x_0) < 0$$

for all  $x \in (-2, 2)$ . Hence

$$Lv_p^\infty - \alpha \operatorname{sgn}(v_p^\infty) = B \cos(k_0x_0) L \cos(k_0 \cdot) = 0.$$

As a consequence,  $\|Lv_p - \alpha \operatorname{sgn}(v_p)\|_{L^2(-2,2)} \rightarrow 0$  as  $x_0 \in 4\mathbb{Z}$  tends to  $\infty$ .

The same argument works for  $x_0 \rightarrow \infty$  with  $x_0 \in 2\mathbb{Z} \setminus 4\mathbb{Z}$ , but this time  $\cos(k_0x_0) = -1$ .  $\square$

Let us now turn to the even function  $u_e$ . For example, one can choose  $u_e$  to agree with  $\operatorname{sgn}(x) \sin(k_0x)$  outside a fixed bounded interval. The essential property used is that such a function will satisfy the condition in Proposition A.2 in Appendix A.

For any choice of the parameter  $\beta_e \in \mathbb{R}$  and any  $\tilde{r} \in H_e^2(\mathbb{R})$ , we can choose the remaining parameter  $\tilde{\gamma}$  to ensure that  $u$  of (29) inherits the two zeros  $\pm x_0$  from  $v_p$ . That is, we set

$$\tilde{\gamma} := \{\tilde{r}(x_0) - \beta_e u_e(x_0)\} \cos(k_0x_0)^{-1},$$

where we note that  $\cos(k_0 x_0) = \pm 1$  for  $x_0 \in 2\mathbb{Z}$ .

To motivate the definition of  $\tilde{r}$ , let us assume for the moment that  $\pm x_0$  are the only zeros of  $u$ . In other words, let us assume for now that the sign condition

$$\operatorname{sgn}(v_p + \beta_e u_e + \tilde{\gamma} \cos(k_0 \cdot) - \tilde{r}) = \operatorname{sgn}(v_p) \quad (31)$$

holds. In analogy to (15) as an equation for the remainder  $r$  in Section 3, we now consider the equation

$$L\tilde{r} = \beta_e L u_e + L v_p - \alpha \operatorname{sgn}(v_p) \quad (32)$$

for  $\tilde{r} \in H_e^2(\mathbb{R})$ , where the subscript  $e$  stands for even functions. Note that if (32) has a solution  $\tilde{r}$ , then the function  $u$ , with the decomposition (29) will be a solution to (4) provided the sign condition (31) holds.

The solvability of (32) is addressed in the following lemma.

**Lemma 4.2.** *Define*

$$\beta_e := \frac{1}{2(c^2 k_0 - 1)} \int_{\mathbb{R}} [-L v_p + \alpha \operatorname{sgn}(v_p)] \cos(k_0 \cdot) dx.$$

*Then equation (32) has an even solution  $\tilde{r} \in H_e^2(\mathbb{R})$ . In particular, we have the estimate*

$$\|\tilde{r}\|_{H^2(\mathbb{R})} \leq C \left( |\beta_e| + \|(1+x^2)(L v_p - \alpha \operatorname{sgn}(v_p))\|_{L^2(\mathbb{R})} \right).$$

*Proof.* By the choice of  $\beta_e$  and Proposition A.2,

$$\int_{\mathbb{R}} \left( \beta_e L u_e + L v_p - \alpha \operatorname{sgn}(v_p) \right) \cos(k_0 \cdot) dx = 0.$$

The expression  $L^{-1}Q$  given by Proposition A.1 in Appendix A can be applied to the right hand side of (32),

$$Q := \beta_e L u_e + L v_p - \alpha \operatorname{sgn}(v_p),$$

because  $(1+x^2)Q \in L^2(\mathbb{R})$  and  $\int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = \int_{\mathbb{R}} Q(x) \cos(k_0 x) dx = 0$ . Hence

$$\tilde{r} := L^{-1}(\beta_e L u_e + L v_p - \alpha \operatorname{sgn}(v_p))$$

is well-defined. It is immediate that  $\tilde{r}$  is even.  $\square$

**Theorem 4.3.** *Under the assumptions of Theorem 2.1 (in particular, for a piecewise quadratic on-site potential,  $\psi'(x) = \operatorname{sgn}(x)$ ), there exists a family of even solutions*

$$u = v_p + \beta_e u_e + \tilde{\gamma} \cos(k_0 \cdot) - \tilde{r}$$

*to (4), parametrised by the choice of sufficiently large  $x_0 \in 2\mathbb{Z}$  in Lemma 4.1.*

*Each of these solutions making two transitions between the wells of the on-site potential, located at  $-x_0$  and  $+x_0$ , so that they remain in the well around  $-1$  only on a large but finite interval  $(-x_0, x_0)$ .*

*Proof.* Lemma 4.1 provides  $v_p$ . Further,  $u_e$  is as discussed above. In addition, Lemma 4.2 defines  $\beta_e$  and  $\tilde{r}$ .

As

$$\beta_e = \frac{1}{2(c^2 k_0 - 1)} \int_{\mathbb{R}} [-Lv_p + \alpha \operatorname{sgn}(v_p)] \cos(k_0 \cdot) dx,$$

we obtain by estimate (30)

$$|\beta_e| \leq C \left\| (1+x^2)(Lv_p - \alpha \operatorname{sgn}(v_p)) \right\|_{L^2(\mathbb{R})} \cdot \left\| \frac{\cos(k_0 x)}{1+x^2} \right\|_{L^2(\mathbb{R})} \rightarrow 0$$

for a sequence of points  $x_0 \in 2\mathbb{Z}$  with  $x_0 \rightarrow \infty$ .

It remains to verify the sign condition (31) for  $u$ , i.e., to show that  $\pm x_0$  are the only roots of

$$u = v_p + \beta_e u_e + \tilde{\gamma} \cos(k_0 \cdot) - \tilde{r}.$$

Recall that the choice

$$\tilde{\gamma} = \{\tilde{r}(x_0) - \beta_e u_e(x_0)\} \cos(k_0 x_0)^{-1}$$

was made so that  $u$  vanishes at  $\pm x_0$ . The bounded embedding  $H^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$  and Lemma 4.2 show that  $\tilde{r}(x_0)$  is small. Moreover, smallness of  $\beta_e$  and  $\tilde{r}(x_0)$  imply that  $\tilde{\gamma}$  is small itself.

As  $v_p$  changes sign at precisely  $\pm x_0$ , we now use that the derivative  $v_p'(\pm x_0)$  is bounded below independently of large  $x_0$ . Thus, uniform smallness of the additional term  $\beta_e u_e + \tilde{\gamma} \cos(k_0 \cdot) - \tilde{r}$  and its derivative establishes the sign condition for all sufficiently large  $x_0$ .  $\square$

## A Appendix

We first state an auxiliary statement for the equation  $Lr = Q$ . This result is used after Lemma 3.4 and in the proof of Lemma 4.2.

**Proposition A.1.** *Let  $Q \in L^2(\mathbb{R})$  with  $(1+x^2)Q \in L^2(\mathbb{R})$ . We assume that either  $Q$  is odd and satisfies*

$$\int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = 0,$$

*or that  $Q$  is even and satisfies*

$$\int_{\mathbb{R}} Q(x) \cos(k_0 x) dx = 0.$$

*Then, for all  $c$  near enough to 1, there exists a unique function  $r \in H_{odd}^2(\mathbb{R})$  such that  $Lr = c^2 r'' - \Delta_D r + \alpha r = Q$ . Moreover*

$$\|r\|_{H^2(\mathbb{R})} := \|(1+k^2)\hat{r}\|_{L^2(\mathbb{R})} \leq \{C_1 + ((4+\alpha)C_1 + 1)/c^2\} \|(1+x^2)Q\|_{L^2(\mathbb{R})}$$

*for some constant  $C_1 > 0$  (independent of  $c$  near 1).*

*Proof.* The assumptions imply that  $\widehat{Q} \in H^2(\mathbb{R}, \mathbb{C})$ ,  $\widehat{Q}(\pm k_0) = 0$  and that there exists a unique  $r \in H_{\text{odd}}^2(\mathbb{R})$  such that  $c^2 r'' - \Delta_D r + \alpha r = Q$ , namely

$$\widehat{r}(k) = \frac{\widehat{Q}(k)}{D(k)}, \text{ for } k \in \mathbb{R}.$$

If  $Q$  is odd and real-valued,  $i\widehat{Q}$  is odd and real-valued, therefore so are  $i\widehat{r}$  and  $r$ . Analogously, when  $Q$  is even, and real-valued, then  $\widehat{Q}$ ,  $\widehat{r}$  and  $r$  are even and real-valued. Moreover, the derivative of the Fourier transform of  $Q$  satisfies

$$\begin{aligned} \left\| \widehat{Q}' \right\|_{L^\infty(\mathbb{R})} &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{|x|}{1+x^2} (1+x^2) |Q(x)| dx \\ &\leq \frac{1}{\sqrt{2\pi}} \left( \int_{\mathbb{R}} \frac{x^2}{(1+x^2)^2} dx \right)^{1/2} \|(1+x^2)Q\|_{L^2(\mathbb{R})} = \frac{1}{2} \|(1+x^2)Q\|_{L^2(\mathbb{R})}, \end{aligned}$$

(note that  $(1/2) \arctan x - (1/2)x/(1+x^2)$  is a primitive of  $x^2(1+x^2)^{-2}$ ).

Consider for a while  $c = 1$ . For  $|k| \in [k_0/2, 3k_0/2] \setminus \{k_0\}$ , one gets by Cauchy's mean value theorem applied to the real-valued functions  $i\widehat{Q}$  and  $D$

$$\left| \frac{\widehat{Q}(k)}{D(k)} \right| \leq \sup_{|s| \in [k_0/2, 3k_0/2] \setminus \{k_0\}} \left| \frac{\widehat{Q}'(s)}{D'(s)} \right| \leq |D'(k_0/2)|^{-1} \frac{1}{2} \|(1+x^2)Q\|_{L^2(\mathbb{R})}.$$

For  $|k| \notin [k_0/2, 3k_0/2]$ , one gets  $|D(k)| \geq \min\{|D(k_0/2)|, |D(3k_0/2)|\}$ . Hence

$$\begin{aligned} \int_{\mathbb{R}} \left| \frac{\widehat{Q}(k)}{D(k)} \right|^2 dk &\leq \max\{|D(k_0/2)|^{-2}, |D(3k_0/2)|^{-2}\} \int_{|k| \notin [k_0/2, 3k_0/2]} \left| \widehat{Q}(k) \right|^2 dk \\ &\quad + 2k_0 |D'(k_0/2)|^{-2} \frac{1}{4} \|(1+x^2)Q\|_{L^2(\mathbb{R})}^2 \\ &\leq \left( \max\{|D(k_0/2)|^{-2}, |D(3k_0/2)|^{-2}\} + \frac{1}{2} k_0 |D'(k_0/2)|^{-2} \right) \|(1+x^2)Q\|_{L^2(\mathbb{R})}^2 \\ &= C_1^2 \|(1+x^2)Q\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

This estimate remains valid for all  $c$  close to 1 if we first increase slightly  $C_1$ . As a consequence

$$c^2 \|r''\|_{L^2(\mathbb{R})} \leq (4+\alpha) \|r\|_{L^2(\mathbb{R})} + \|Q\|_{L^2(\mathbb{R})} \leq ((4+\alpha)C_1+1) \|(1+x^2)Q\|_{L^2(\mathbb{R})}$$

and

$$\begin{aligned} \|r\|_{H^2(\mathbb{R})} &= \|(1+k^2)\widehat{r}\|_{L^2(\mathbb{R})} \leq \|r\|_{L^2(\mathbb{R})} + \|r''\|_{L^2(\mathbb{R})} \\ &\leq \{C_1 + ((4+\alpha)C_1+1)/c^2\} \|(1+x^2)Q\|_{L^2(\mathbb{R})}. \end{aligned}$$

□

The following proposition establishes orthogonality relations and estimates for the Fourier mode associated with  $k_0$  for  $L$  applied to even and odd functions. The estimate (33) is used just after the compactness proof (Lemma 3.3).

**Proposition A.2.** *Consider the odd function  $u_o \in H_{\text{loc}}^2(\mathbb{R})$  satisfying (14). In addition, let  $u_e \in H_{\text{loc}}^2(\mathbb{R})$  be an even function such that*

$$(1+x^2) \frac{d^l}{dx^l} (u_e(x) - \text{sgn}(x) \sin(k_0 x)) \in L^2(\mathbb{R} \setminus [-1, 1])$$

for  $l = 0, 1, 2$ , analogously to (14). If  $c > k_0^{-1/2}$ , then

$$\begin{aligned} \int_{\mathbb{R}} \sin(k_0 \cdot) (c^2 u_o'' - \Delta_D u_o + \alpha u_o) dx &= -2c^2 k_0 + 2 < 0, \\ \int_{\mathbb{R}} \cos(k_0 \cdot) (c^2 u_e'' - \Delta_D u_e + \alpha u_e) dx &= 2c^2 k_0 - 2 > 0, \\ \int_{\mathbb{R}} \cos(k_0 \cdot) (c^2 u_o'' - \Delta_D u_o + \alpha u_o) dx &= 0 \end{aligned} \tag{33}$$

and

$$\int_{\mathbb{R}} \sin(k_0 \cdot) (c^2 u_e'' - \Delta_D u_e + \alpha u_e) dx = 0.$$

*Proof.* The two last integrals vanish because the integrands are odd functions of  $x$ . For the first integral, two integrations by parts and the identity  $L \sin(k_0 \cdot) = 0$



give

$$\begin{aligned}
& \lim_{z \rightarrow \infty} \int_{-z}^z \sin(k_0 \cdot) (c^2 u_o'' - \Delta_D u_o + \alpha u_o) dx \\
&= \lim_{z \rightarrow \infty} \int_{-z}^z \left[ c^2 \frac{d^2}{dx^2} \sin(k_0 \cdot) - \Delta_D \sin(k_0 \cdot) + \alpha \sin(k_0 \cdot) \right] u_o dx \\
&\quad + \lim_{z \rightarrow \infty} c^2 [\sin(k_0 z) u_o'(z) - k_0 \cos(k_0 z) u_o(z) \\
&\quad\quad - \sin(-k_0 z) u_o'(-z) + k_0 \cos(-k_0 z) u_o(-z)] \\
&\quad - \lim_{z \rightarrow \infty} \left( \int_{-z+1}^{z+1} - \int_{-z}^z \right) \sin(k_0(x-1)) u_o(x) dx \\
&\quad - \lim_{z \rightarrow \infty} \left( \int_{-z-1}^{z-1} - \int_{-z}^z \right) \sin(k_0(x+1)) u_o(x) dx \\
&\stackrel{(14)}{=} \lim_{z \rightarrow \infty} c^2 (-k_0 \sin^2(k_0 z) - k_0 \cos^2(k_0 z) - k_0 \sin^2(-k_0 z) - k_0 \cos^2(-k_0 z)) \\
&\quad - \lim_{z \rightarrow \infty} \int_z^{z+1} \sin(k_0(x-1)) \cos(k_0 x) dx - \lim_{z \rightarrow \infty} \int_{-z}^{-z+1} \sin(k_0(x-1)) \cos(k_0 x) dx \\
&\quad + \lim_{z \rightarrow \infty} \int_{-z-1}^{-z} \sin(k_0(x+1)) \cos(k_0 x) dx + \lim_{z \rightarrow \infty} \int_{z-1}^z \sin(k_0(x+1)) \cos(k_0 x) dx \\
&= -2c^2 k_0 + \lim_{z \rightarrow \infty} \int_{z-1}^{z+1} \cos^2(k_0 x) dx + \lim_{z \rightarrow \infty} \int_{-z-1}^{-z+1} \cos^2(k_0 x) dx \\
&= -2c^2 k_0 + 2 < 0.
\end{aligned}$$

Analogously,

$$\begin{aligned}
& \int_{\mathbb{R}} \cos(k_0 \cdot) (c^2 u_e'' - \Delta_D u_e + \alpha u_e) dx = \int_{\mathbb{R}} \sin(k_0 \cdot + k_0) (c^2 u_e'' - \Delta_D u_e + \alpha u_e) dx \\
&= \lim_{z \rightarrow \infty} c^2 (-k_0 \sin(k_0 z + k_0) \sin(k_0 z - k_0) - k_0 \cos(k_0 z + k_0) \cos(k_0 z - k_0) \\
&\quad - k_0 \sin(-k_0 z + k_0) \sin(-k_0 z - k_0) - k_0 \cos(-k_0 z + k_0) \cos(-k_0 z - k_0)) \\
&\quad - \lim_{z \rightarrow \infty} \int_z^{z+1} \sin(k_0(x-1) + k_0) \cos(k_0 x - k_0) dx \\
&\quad - \lim_{z \rightarrow \infty} \int_{-z}^{-z+1} \sin(k_0(x-1) + k_0) \cos(k_0 x - k_0) dx \\
&\quad + \lim_{z \rightarrow \infty} \int_{-z-1}^{-z} \sin(k_0(x+1) + k_0) \cos(k_0 x - k_0) dx \\
&\quad + \lim_{z \rightarrow \infty} \int_{z-1}^z \sin(k_0(x+1) + k_0) \cos(k_0 x - k_0) dx \\
&= 2c^2 k_0 - 2 > 0.
\end{aligned}$$

□

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