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Logic programming: laxness and saturation[☆]

Ekaterina Komendantskaya*

Department of Computer Science, Heriot-Watt University, Edinburgh, UK

John Power*

Department of Computer Science, University of Bath, BA2 7AY, UK

Abstract

A propositional logic program P may be identified with a $P_f P_f$-coalgebra on the set of atomic propositions in the program. The corresponding $C(P_f P_f)$-coalgebra, where $C(P_f P_f)$ is the cofree comonad on $P_f P_f$, describes derivations by resolution. That correspondence has been developed to model first-order programs in two ways, with lax semantics and saturated semantics, based on locally ordered categories and right Kan extensions respectively. We unify the two approaches, exhibiting them as complementary rather than competing, reflecting the theorem-proving and proof-search aspects of logic programming. While maintaining that unity, we further refine lax semantics to give finitary models of logic programs with existential variables, and to develop a precise semantic relationship between variables in logic programming and worlds in local state.

Keywords: Logic programming, coalgebra, coinductive derivation tree, Lawvere theories, lax transformations, saturation

1. Introduction

Over recent years, there has been a surge of interest in category theoretic semantics of logic programming. Research has focused on two ideas: lax semantics, proposed by the current authors and collaborators [1], and saturated semantics, proposed by Bonchi and Zanasi [2]. Both ideas are based on coalgebra, agreeing on variable-free logic programs. Both ideas use subtle, well-established category theory, associated with locally ordered categories and with right Kan extensions respectively [3]. And both elegantly clarify and extend established logic programming constructs and traditions, for instance [4] and [5].

Until now, the two ideas have been presented as alternatives, competing with each other rather than complementing each other. A central thesis of this paper is that the competition is illusory, the two ideas being two views of a single, elegant body of theory, those views reflecting different but complementary aspects of logic programming, those aspects broadly corresponding with the notions of theorem proving and proof search. Such reconciliation has substantial consequences. In particular, it means that whenever one further refines one approach, as we shall do to the original lax approach in two substantial ways here, one should test whether the proposed refinement also applies to the other approach, and see what consequences it has from the latter perspective.

The category theoretic basis for both lax and saturated semantics is as follows. It has long been observed, e.g., in [6, 7], that logic programs induce coalgebras, allowing coalgebraic modelling of their operational semantics. Using the definition of logic program in Lloyd's book [8], given a set of atoms At, one can identify a variable-free logic program P built over At with a $P_f P_f$-coalgebra structure on At, where P_f is the finite powerset functor on Set: each atom is the head of finitely many clauses in P, and the body of each clause contains finitely many atoms. It was shown in [9] that if $C(P_f P_f)$ is the cofree comonad on $P_f P_f$, then, given a logic program P qua $P_f P_f$-coalgebra, the corresponding $C(P_f P_f)$-coalgebra structure characterises the and-or derivation trees generated by P, cf. [4]. That fact has formed the basis for our work on lax semantics [1, 10, 11, 12, 13] and for Bonchi and Zanasi's work on saturation semantics [14, 2].

In attempting to extend the analysis to arbitrary logic programs, both groups followed the tradition of [15, 6, 5, 16]: given a signature Σ of function symbols, let $\mathcal{L}_\Sigma$ denote the Lawvere theory generated by Σ, and, given a logic program P with function symbols in Σ, consider the functor category $[\mathcal{L}_\Sigma^{op}, Set]$, extending the set At of atoms in a variable-free logic program to the functor from $\mathcal{L}_\Sigma^{op}$ to Set sending a natural number n to the set $At(n)$ of atomic formulae with at most n variables generated by the function symbols in Σ and the predicate symbols in P. We all sought to model P by a $[\mathcal{L}_\Sigma^{op}, P_f P_f]$-coalgebra $p : At \longrightarrow P_f P_f At$ that, at n, takes an atomic formula $A(x_1, \ldots, x_n)$ with at most n variables, considers all substitutions of clauses in P into clauses with variables among $x_1, \ldots, x_n$ whose head agrees with $A(x_1, \ldots, x_n)$, and gives the set of sets of atomic formulae in antecedents, naturally extending the construction for variable-free logic programs. However, that idea is too simple for two reasons. We all dealt with the second problem in the same way, so we shall discuss it later, but the first problem is illustrated by the following example.

Example 1. *ListNat (for lists of natural numbers) denotes the logic program*
1. $\mathtt{nat(0)} \leftarrow$
2. $\mathtt{nat(s(x))} \leftarrow \mathtt{nat(x)}$
3. $\mathtt{list(nil)} \leftarrow$
4. $\mathtt{list(cons(x, y))} \leftarrow \mathtt{nat(x)}, \mathtt{list(y)}$

ListNat has nullary function symbols $\mathtt{0}$ *and* $\mathtt{nil}$*, a unary function symbol* $\mathtt{s}$*,*

and a binary function symbol cons. *So the signature Σ of ListNat contains four elements.*

There is a map in $\mathcal{L}_\Sigma$ of the form $0 \to 1$ that models the nullary function symbol 0. So, naturality of the map $p : At \longrightarrow P_f P_f At$ in $[\mathcal{L}_\Sigma^{op}, Set]$ would yield commutativity of the diagram

$$
\begin{array}{ccc}
At(1) & \xrightarrow{\;p_1\;} & P_f P_f At(1) \\
{\scriptstyle At(0)}\Big\downarrow & & \Big\downarrow{\scriptstyle P_f P_f At(0)} \\
At(0) & \xrightarrow[\;p_0\;]{} & P_f P_f At(0)
\end{array}
$$

But consider nat(x) $\in At(1)$: *there is no clause of the form* nat(x) $\leftarrow$ *in ListNat, so commutativity of the diagram would imply that there cannot be a clause in ListNat of the form* nat(0) $\leftarrow$ *either, but in fact there is one. Thus p is not a map in the functor category $[\mathcal{L}_\Sigma^{op}, Set]$.*

Problem 1. *As illustrated by Example 1, i.e., in ListNat, the natural construction of $p : At \longrightarrow P_f P_f At$ does not form a map, i.e., yield a natural transformation, in $[\mathcal{L}_\Sigma^{op}, Set]$.*

Proposed resolutions to Problem 1 diverged: at CALCO in 2011, we proposed lax transformations [17], then at CALCO 2013, Bonchi and Zanasi proposed saturation semantics [14]. First we shall describe our approach.

Our approach was to relax the naturality condition on p to a subset condition, following [18, 19, 20], so that, given a map in $\mathcal{L}_\Sigma$ of the form $f : n \to m$, the diagram

$$
\begin{array}{ccc}
At(m) & \xrightarrow{\;p_m\;} & P_f P_f At(m) \\
{\scriptstyle At(f)}\Big\downarrow & & \Big\downarrow{\scriptstyle P_f P_f At(f)} \\
At(n) & \xrightarrow[\;p_n\;]{} & P_f P_f At(n)
\end{array}
$$

need not commute, but rather the composite via $P_f P_f At(m)$ need only yield a subset of that via $At(n)$. So, for example, $p_1(\text{nat(x)})$ could be the empty set while $p_0(\text{nat(0)})$ could be non-empty in the semantics for ListNat as required. We extended *Set* to *Poset* in order to express such laxness, and we adopted established category theoretic research on laxness, notably that of [20], in order to prove that a cofree comonad exists and, on programs such as ListNat, behaves as we wish. This agrees with, and is indeed an instance of, He Jifeng and Tony Hoare's use of laxness to model data refinement [21, 22, 23, 24].

3

Bonchi and Zanasi's approach was to use saturation semantics [14, 2], following [6]. The key category theoretic result that supports it asserts that, regarding $ob(\mathcal{L}_\Sigma)$, equally $ob(\mathcal{L}_\Sigma)^{op}$, as a discrete category with inclusion functor $I : ob(\mathcal{L}_\Sigma) \longrightarrow \mathcal{L}_\Sigma$, the functor

$$[I, Set] : [\mathcal{L}_\Sigma^{op}, Set] \longrightarrow [ob(\mathcal{L}_\Sigma)^{op}, Set]$$

that sends $H : \mathcal{L}_\Sigma^{op} \longrightarrow Set$ to the composite $HI : ob(\mathcal{L}_\Sigma)^{op} \longrightarrow Set$ has a right adjoint, given by right Kan extension. The data for $p : At \longrightarrow P_f P_f At$, although not forming a map in $[\mathcal{L}_\Sigma^{op}, Set]$, may be seen as a map in $[ob(\mathcal{L}_\Sigma)^{op}, Set]$. So, by the adjointness, the data for p corresponds to a map $\bar{p} : At \longrightarrow R(P_f P_f AtI)$ in $[\mathcal{L}_\Sigma^{op}, Set]$, thus to a coalgebra on At in $[\mathcal{L}_\Sigma^{op}, Set]$, where $R(P_f P_f AtI)$ is the right Kan extension of $P_f P_f AtI$ along the inclusion I. The right Kan extension is defined by

$$R(P_f P_f AtI)(n) = \prod_{m \in \mathcal{L}_\Sigma} (P_f P_f At(m))^{\mathcal{L}_\Sigma(m,n)}$$

and the function

$$\bar{p}(n) : At(n) \longrightarrow \prod_{m \in \mathcal{L}_\Sigma} (P_f P_f At(m))^{\mathcal{L}_\Sigma(m,n)}$$

takes an atomic formula $A(x_1, \ldots, x_n)$, and, for every substitution for $x_1, \ldots, x_n$ generated by the signature Σ, gives the set of sets of atomic formulae in the tails of clauses with head $A(t_1, \ldots, t_n)$, where the t_i's are determined by the substitution. By construction, $\bar{p}$ is natural, but one quantifies over all possible substitutions for $x_1, \ldots, x_n$ in order to obtain that naturality, and one ignores the laxness of p.

As we shall show in Section 5, the two approaches can be unified. If one replaces

$$[I, Set] : [\mathcal{L}_\Sigma^{op}, Set] \longrightarrow [ob(\mathcal{L}_\Sigma)^{op}, Set]$$

by the inclusion

$$[\mathcal{L}_\Sigma^{op}, Poset] \longrightarrow Lax(\mathcal{L}_\Sigma^{op}, Poset)$$

$[\mathcal{L}_\Sigma^{op}, Set]$ being a full subcategory of $[\mathcal{L}_\Sigma^{op}, Poset]$, one obtains exactly Bonchi and Zanasi's correspondence between p and $\bar{p}$, with exactly the same formula, starting from lax transformations as we proposed. Thus, from a category theoretic perspective, saturation can be seen as complementary to laxness rather than as an alternative to it. This provides a robustness test for future refinements to models of logic programming: a refinement of one view of category theoretic semantics can be tested by its effect on the other. We now turn to such refinements.

Recently, we have refined lax semantics in two substantial ways, the first of which was the focus of the workshop paper [25] that this paper extends, with the second being to start to build a precise relationship with the semantics for local variables [26], which is new here. For the first, a central contribution of

lax semantics has been the inspiration it provided towards the development of an efficient logic programming algorithm [1, 10, 11, 12, 13]. That development drew our attention to the semantic significance of *existential* variables: such variables do not appear in ListNat, and they are not needed for a considerable body of logic programming, but they do appear in logic programs such as the following, which is a leading example in Sterling and Shapiro's book [27].

Example 2. *GC (for graph connectivity) denotes the logic program*
1. $\mathtt{connected}(\mathtt{x}, \mathtt{x}) \leftarrow$
2. $\mathtt{connected}(\mathtt{x}, \mathtt{y}) \leftarrow \mathtt{edge}(\mathtt{x}, \mathtt{z}), \mathtt{connected}(\mathtt{z}, \mathtt{y})$

There is a variable z in the tail of the second clause of GC that does not appear in its head, whereas no such variable appears in ListNat. Such a variable is called an existential variable, the presence of which challenges the algorithmic significance of lax semantics. In describing the putative coalgebra $p : At \longrightarrow P_f P_f At$ just before Example 1, we referred to *all* substitutions of clauses in P into clauses with variables among $x_1, \ldots, x_n$ whose head agrees with $A(x_1, \ldots, x_n)$. If there are no existential variables, that amounts to term-matching, which is algorithmically efficient; but if existential variables do appear, the mere presence of a unary function symbol generates an infinity of such substitutions, creating algorithmic difficulty, which, when first introducing lax semantics, we avoided modelling by replacing the outer instance of P_f by P_c, thus allowing for countably many choices.

Bonchi and Zanasi, in [14, 2], followed the lead of lax semantics in using P_c rather than P_f in order to account for existential variables, but one needs a careful study to see that. In saturated semantics, countability arises in two ways: applying the right Kan extension R yields countability as there may be countably many substitutions for variables; and using P_c rather than P_f also yields countability. In the absence of existential variables, Bonchi and Zanasi could have applied saturation to the map $p : At \longrightarrow P_f P_f At$, with the right Kan extension generating the countability required for saturation. However, in the presence of existential variables, there is no such map into $P_f P_f At$ to which to apply saturation. So for saturated semantics, our analysis of existential variables makes for a subtle difference, clarifying where countability is required.

That is the second of the two problems mentioned just before Example 1. More succinctly, it may be expressed as follows:

Problem 2. *As illustrated by Example 2, an arbitrary logic program does not generate a map of the form $p : At \longrightarrow P_f P_f At$. Previous work addressed that by an artificial use of countability.*

We have long sought a solution to Problem 2. We finally found and presented such a resolution in the workshop paper [25] that this paper extends. We both refine it a little more, as explained later, and give more detail here.

The conceptual key to the resolution was to isolate and give finitary lax semantics to the notion of *coinductive tree* [28, 1]. Coinductive trees arise from

term-matching resolution [28, 1], which is a variant of SLD-resolution. Term-matching captures the theorem proving aspect of logic programming, which is distinct from, but complementary with, its problem solving aspect, which is captured by SLD-resolution [12, 11]. The difference is that in the term-matching approach, one only substitutes in a goal after having exhausted all possible term-matching, whereas in SLD-resolution, one uses full unification at any time. We called the derivation trees arising from term-matching coinductive trees in order to mark their connection with coalgebraic logic programming, which we also developed.

Syntactically, one can observe the difference between lax semantics and saturation semantics in that lax semantics models coinductive trees, which are finitely branching, whereas saturation involves infinitely many possible substitutions, leading Bonchi and Zanasi to model different kinds of trees, their focus being on proof search rather than on theorem proving.

Chronologically, we introduced lax semantics in 2011 as above [17]; lax semantics inspired us to investigate term-matching and to introduce the notion of coinductive tree [28]; because of the possibility of existential variables, our lax semantics for coinductive trees, despite inspiring the notion, was potentially infinitary [1]; so we have now refined lax semantics to ensure finitariness of the semantics for coinductive trees, even in the presence of existential variables [25], introducing it in the workshop paper that this paper extends. We further refine lax semantics here to start to build a precise relationship with the semantics of local variables [26], which we plan to develop further in future. We regard it as positive that lax semantics brings to the fore, in semantic terms, the significance of existential variables, and allows a precise semantic relationship between the role of variables in logic programming and local variables as they arise in programming more generally.

The semantics we give in this paper is subtly different to that in [25]. Here, we disambiguate the role of P_f in our modelling of existential variables: in Section 6, we consider $\int At$, then apply $P_f P_f$ to it, whereas we mixed a construction for $\int$ with P_f in [25], but that does not quite match the modelling of local state. We also give far more detail throughout this paper: in giving examples, in explaining the relationship with Bonchi and Zanasi's saturated semantics, in proofs, and in developing the relationship with local state.

The paper is organised as follows. In Section 2, we set logic programming terminology, explain the relationship between term-rewriting and SLD-resolution, and introduce the notion of coinductive tree. In Section 3, we give semantics for variable-free logic programs. This semantics could equally be seen as lax semantics or saturated semantics, as they agree in the absence of variables. In Section 4, we model coinductive trees for logic programs without existential variables and explain the difficulty in modelling coinductive trees for arbitrary logic programs. In Section 5, we recall saturation semantics and make precise the relationship between it and lax semantics. We devote Section 6 of the paper to refining lax semantics, while maintaining the relationship with saturation semantics, to model the coinductive trees generated by logic programs with existential variables, and in Section 7, we start to build a precise relationship with

6

the semantics of local state [26].

2. Theorem proving in logic programming

A *signature* Σ consists of a set $\mathcal{F}$ of function symbols $f, g, \ldots$ each equipped with an arity. Nullary (0-ary) function symbols are constants. For any set *Var* of variables, the set $Ter(\Sigma)$ of terms over Σ is defined inductively as usual:

- $x \in Ter(\Sigma)$ for every $x \in Var$.

- If f is an n-ary function symbol ($n \geq 0$) and $t_1, \ldots, t_n \in Ter(\Sigma)$, then $f(t_1, \ldots, t_n) \in Ter(\Sigma)$.

A *substitution* over Σ is a (total) function $\sigma : Var \to Ter(\Sigma)$. Substitutions are extended from variables to terms as usual: if $t \in Ter(\Sigma)$ and σ is a substitution, then the *application* $\sigma(t)$ is a result of applying σ to all variables in t. A substitution σ is a *unifier* for t, u if $\sigma(t) = \sigma(u)$, and is a *matcher* for t against u if $\sigma(t) = u$. A substitution σ is a *most general unifier* (*mgu*) for t and u if it is a unifier for t and u and is more general than any other such unifier, i.e., all unifiers factor through any most general unifier. A *most general matcher* (*mgm*) σ for t against u is defined analogously.

In line with logic programming (LP) tradition [8], we consider a set $\mathcal{P}$ of predicate symbols each equipped with an arity. It is possible to define logic programs over terms only, in line with the term-rewriting (TRS) tradition [29], as in [11], but we will follow the usual LP tradition here. That gives us the following inductive definitions of the sets of atomic formulae, Horn clauses and logic programs (we also include the definition of terms for convenience).

Definition 1.
$$\text{Terms } Ter \ ::= \ Var \mid \mathcal{F}(Ter, ..., Ter)$$
$$\text{Atomic formulae (or atoms) } At \ ::= \ \mathcal{P}(Ter, ..., Ter)$$
$$\text{(Horn) clauses } HC \ ::= \ At \leftarrow At, ..., At$$
$$\text{Logic programs } Prog \ ::= HC, ..., HC$$

In what follows, we will use letters A, B, C, D, possibly with subscripts, to refer to elements of *At*.

Given a logic program P, we may ask whether a given atom is logically entailed by P. E.g., given the program ListNat we may ask whether `list(cons(0,nil))` is entailed by ListNat. The following rule, which is a restricted form of SLD-resolution, provides a semi-decision procedure to derive the entailment.

Definition 2 (Term-matching (TM) Resolution). *Given a program P and an atomic formula A, we say P entails A, written as $P \vdash A$, if there is a derivation of $P \vdash A$ from an empty goal using the following rules:*

$$\frac{}{P \vdash [\,]} \qquad \frac{P \vdash \sigma A_1 \quad \cdots \quad P \vdash \sigma A_n}{P \vdash \sigma A} \ \ if \ (A \leftarrow A_1, \ldots, A_n) \in P$$

In contrast, the SLD-resolution rule could be presented in the following form:

$$B_1, \ldots, B_j, \ldots, B_n \rightsquigarrow_P \sigma B_1, \ldots, \sigma A_1, \ldots, \sigma A_n, \ldots, \sigma B_n$$

if $(A \leftarrow A_1, \ldots, A_n) \in P$, and σ is the mgu of A and B_j. The derivation for A succeeds when $A \rightsquigarrow_P [\,]$; we use $\rightsquigarrow_P^*$ to denote several steps of SLD-resolution.

At first sight, the difference between TM-resolution and SLD-resolution may seem only to be notational. Indeed, both $ListNat \vdash \mathtt{list(cons(0,nil))}$ and $\mathtt{list(cons(0,nil))} \rightsquigarrow_{ListNat}^* [\,]$ by the above rules (see also Figure 1). However, $ListNat \nvdash \mathtt{list(cons(x,y))}$ whereas $\mathtt{list(cons(x,y))} \rightsquigarrow_{ListNat}^* [\,]$. And, even more mysteriously, $GC \nvdash \mathtt{connected(x,y)}$ while $\mathtt{connected(x,y)} \rightsquigarrow_{GC} [\,]$.

In fact, TM-resolution reflects the *theorem proving* aspect of LP: the rules of Definition 2 can be used to semi-decide whether a given term t is entailed by P. In contrast, SLD-resolution reflects the *problem solving* aspect of LP: using the SLD-resolution rule, one asks whether, for a given t, a substitution σ can be found such that $P \vdash \sigma(t)$. There is a subtle but important difference between these two aspects of proof search.

For example, when considering the successful derivation $\mathtt{list(cons(x,y))}$ $\rightsquigarrow_{ListNat}^* [\,]$, we assume that $\mathtt{list(cons(x,y))}$ holds only relative to a computed substitution, e.g. $\mathtt{x} \mapsto \mathtt{0}$, $\mathtt{y} \mapsto \mathtt{nil}$. Of course this distinction is natural from the point of view of theorem proving: $\mathtt{list(cons(x,y))}$ is not a "theorem" in this generality, but its special case, $\mathtt{list(cons(0,nil))}$, is. Thus, $ListNat \vdash \mathtt{list(cons(0,nil))}$ but $ListNat \nvdash \mathtt{list(cons(x,y))}$ (see also Figure 1). Similarly, $\mathtt{connected(x,y)} \rightsquigarrow_{GC} [\,]$ should be read as: $\mathtt{connected(x,y)}$ holds relative to the computed substitution $\mathtt{y} \mapsto \mathtt{x}$.

According to the soundness and completeness theorems for SLD-resolution [8], the derivation $\rightsquigarrow$ has *existential* meaning, i.e. when $\mathtt{list(cons(x,y))} \rightsquigarrow_{ListNat}^*$ $[\,]$, the successful goal $\mathtt{list(cons(x,y))}$ is not meant to be read as universally quantified over $\mathtt{x}$ and $\mathtt{y}$. In contrast, TM-resolution proves a universal statement. So $GC \vdash \mathtt{connected(x,x)}$ reads as: $\mathtt{connected(x,x)}$ is entailed by GC for any $\mathtt{x}$.

Much of our recent work has been devoted to formal understanding of the relation between the theorem proving and problem solving aspects of LP [11, 12]. The type-theoretic semantics of TM-resolution, given by "Horn clauses as types, λ-terms as proofs" is given in [12, 13].

Definition 2 gives rise to derivation trees. E.g. the derivation (or, equivalently, the proof) for $ListNat \vdash \mathtt{list(cons(0,nil))}$ can be represented by the following derivation tree:
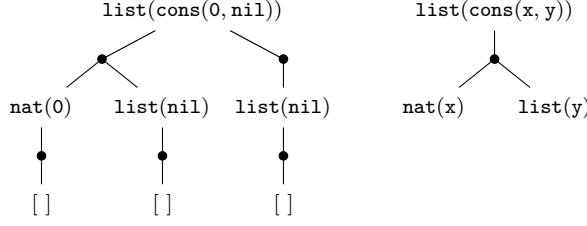
```
            list(cons(0, nil))
               ╱        ╲
         nat(0)          list(nil)
            |               |
           [ ]             [ ]
```

8
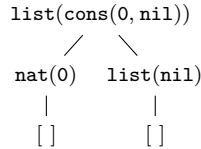
Figure 1: **Left:** The coinductive tree for list(cons(0, nil)) and the extended program $ListNat^+$. **Right:** The coinductive tree for list(cons(x, y)) and $ListNat^+$. Each •-node marks a clause whose head matches the antecedent of the •-node in the tree.
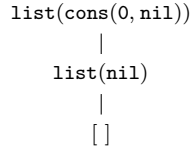
In general, given a term t and a program P, more than one derivation for $P \vdash t$ is possible. For example, if we add a fifth clause to the program $ListNat$:

$$5.\ \text{list}(\text{cons}(0, x)) \leftarrow \text{list}(x)$$

then yet another, alternative, proof is possible for the extended program:

$$ListNat^+ \vdash \text{list}(\text{cons}(0, \text{nil}))$$

via Clause 5:

To reflect the choice of derivation strategies at every stage of the derivation, we introduce a new kind of node called an *or-node*, which we depict by a •-node, e.g., as in Figure 1.

This intuition is made precise in the following definition of a *coinductive tree*, which first appeared in [17, 1] and was refined in [11] under the name of a rewriting tree. Over a succession of papers, we have made minor modifications and a few minor corrections to the precise formulation of the notion of coinductive tree, but the idea and application has remained the same.

Definition 3 (Coinductive tree). *Let P be a logic program and A be an atomic formula. The* coinductive tree *for A is the possibly infinite tree T satisfying the following properties.*

- *the root of T is labelled by A*

- *Each node in T is either an and-node or an or-node*

- *Each or-node is labelled by* •

9

- *Each and-node is labelled by an atom*

- *For every and-node A' occurring in T, if there is a clause C_i in P of the form $B_i \leftarrow B_1^i, \ldots, B_{n_i}^i$, such that there is an mgm θ of B_i against A', then A' has an or-node as a child, and that or-node has children given by and-nodes $\theta(B_j^i), \ldots, \theta(B_k^i)$, where $\{B_j, \ldots, B_k\} \subseteq \{B_1, \ldots, B_{n_i}\}$ and $B_j, \ldots, B_k$ is the maximal such set for which $\theta(B_j^i), \ldots, \theta(B_k^i)$ are distinct.*

Note the use of mgms (rather than mgus) in the last item. There may exist clauses with empty antecedents: some such exist in Figure 1. An *or-node*, thus a •-node with a single child labelled [ ], represents such a clause.

Coinductive trees provide a convenient model for proofs by TM-resolution. Note that coinductive trees are necessarily finitely branching, logic programs being inherently finite.

Let us make one final observation on TM-resolution. Generally, given a program P and an atom t, one can prove that

$$t \rightsquigarrow_P^* [\,] \text{ with computed substitution } \sigma \text{ if and only if } P \vdash \sigma t.$$

This simple fact may leave the impression that proofs (and correspondingly coinductive trees) for TM-resolution are in some sense fragments of reductions by SLD-resolution. Compare, for example, the right-hand tree of Figure 1 before substitution with the larger left-hand tree obtained after the substitution. In this case, we could emulate the problem solving aspect of SLD-resolution by using coinductive trees and allowing the application of substitutions within coinductive trees, as was proposed in [28, 11, 12]. That works perfectly for programs such as ListNat, but not for existential programs: although there is a one step SLD-derivation for $\texttt{connected}(\texttt{x}, \texttt{y}) \rightsquigarrow_{GC} [\,]$ (with $\texttt{y} \mapsto \texttt{x}$), there is no TM-resolution proof for $\texttt{connected}(\texttt{x}, \texttt{y})$, as the derivation diverges and gives rise to the following infinite coinductive tree:
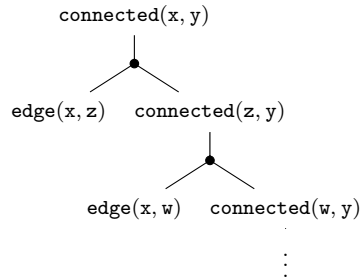
The above tree is not a fragment of the derivation $\texttt{connected}(\texttt{x}, \texttt{y}) \rightsquigarrow_{GC} [\,]$, moreover, it requires more (infinitely many) variables. Thus, the operational semantics of TM-resolution and SLD-resolution can be very different for existential programs, in regard both to termination and to the number of variables involved.

This issue is largely orthogonal to that of non-termination. Consider the non-terminating (but not existential) program Bad:

    bad(x) ← bad(x)

For Bad, the operational behaviours of TM-resolution and SLD-resolution are similar: in both cases, derivations do not terminate, and both require only finitely many variables. Moreover, such programs can be analysed using similar coinductive methods in TM- and SLD-resolution [13, 30].

The problems caused by existential variables are known in the literature on theorem proving and term-rewriting [29]. In TRS [29], existential variables are not allowed to appear in rewriting rules, and in type inference based on term rewriting or TM-resolution, the restriction to non-existential programs is common [31].

So theorem-proving, in contrast to problem-solving, is modelled by term-matching; term-matching gives rise to coinductive trees; and as explained in the introduction and, in more detail, later, coinductive trees give rise to laxness. So in this paper, we use laxness to model coinductive trees, and thereby theorem-proving in LP, and we relate our semantics with Bonchi and Zanasi's saturated semantics, which we believe primarily models the problem-solving aspect of logic programming.

Categorical semantics for existential programs, which are known to be challenging for theorem proving, is a central contribution of Section 6 and of this paper.

3. Semantics for variable-free logic programs

In this section, we recall and develop the work of [9], in regard to variable-free logic programs, i.e., we take $Var = \emptyset$ in Definition 1. Variable-free logic programs are operationally equivalent to propositional logic programs, as substitutions play no role in derivations. In this (propositional) setting, coinductive trees resemble the and-or derivation trees known in the LP literature [4], and this semantics appears as the ground case of both lax semantics [1] and saturated semantics [2].

Proposition 1. *For any set At, there is a bijection between the set of variable-free logic programs over the set of atoms At and the set of $P_f P_f$-coalgebra structures on At, where P_f is the finite powerset functor on Set.*

Theorem 1. *Let $C(P_f P_f)$ denote the cofree comonad on $P_f P_f$. Then, given a logic program P over At, equivalently $p : At \longrightarrow P_f P_f(At)$, the corresponding $C(P_f P_f)$-coalgebra $\overline{p} : At \longrightarrow C(P_f P_f)(At)$ sends an atom A to the coinductive tree for A.*

PROOF. Applying the work of [32] to this setting, the cofree comonad is in general determined as follows: $C(P_f P_f)(\text{At})$ is the limit of the diagram

$$\ldots \longrightarrow \text{At} \times P_f P_f(\text{At} \times P_f P_f(\text{At})) \longrightarrow \text{At} \times P_f P_f(\text{At}) \longrightarrow \text{At}$$

11

with maps determined by the projection $\pi_0 : At \times P_f P_f(At) \longrightarrow At$, with applications of the functor $At \times P_f P_f(-)$ to it.

Putting $At_0 = At$ and $At_{n+1} = At \times P_f P_f At_n$, and defining the cone

$$
\begin{aligned}
p_0 &= id : At \longrightarrow At(= At_0) \\
p_{n+1} &= \langle id, P_f P_f(p_n) \circ p \rangle : At \longrightarrow At \times P_f P_f At_n(= At_{n+1})
\end{aligned}
$$

the limiting property of the diagram determines the coalgebra $\overline{p} : At \longrightarrow C(P_f P_f)(At)$. The image $\overline{p}(A)$ of an atom A is given by an element of the limit, equivalently a map from 1 into the limit, equivalently a cone of the diagram over 1.

To give the latter is equivalent to giving an element A_0 of At, specifically $p_0(A) = A$, together with an element A_1 of $At \times P_f P_f(At)$, specifically $p_1(A) = (A, p_0(A)) = (A, p(A))$, together with an element A_2 of $At \times P_f P_f(At \times P_f P_f(At))$, etcetera. The definition of the coinductive tree for A is inherently coinductive, matching the definition of the limit, and with the first step agreeing with the definition of p. Thus it follows by coinduction that $\overline{p}(A)$ can be identified with the coinductive tree for A.

Example 3. *Let At consist of atoms* A, B, C *and* D*. Let P denote the logic program*

$$
\begin{aligned}
\text{A} &\leftarrow \text{B}, \text{C} \\
\text{A} &\leftarrow \text{B}, \text{D} \\
\text{D} &\leftarrow \text{A}, \text{C}
\end{aligned}
$$

So $p(\text{A}) = \{\{\text{B}, \text{C}\}, \{\text{B}, \text{D}\}\}$, $p(\text{B}) = p(\text{C}) = \emptyset$, and $p(\text{D}) = \{\{\text{A}, \text{C}\}\}$.

Then, as depicted in Figure 2, $p_0(\text{A}) = \text{A}$, which is the root of the coinductive tree for A.

Then $p_1(\text{A}) = (\text{A}, p(\text{A})) = (\text{A}, \{\{\text{B}, \text{C}\}, \{\text{B}, \text{D}\}\})$, which consists of the same information as in the first three levels of the coinductive tree for A*, i.e., the root* A*, two or-nodes, and below each of the two or-nodes, nodes given by each atom in each antecedent of each clause with head* A *in the logic program P: nodes marked* B *and* C *lie below the first or-node, and nodes marked* B *and* D *lie below the second or-node, exactly as $p_1(\text{A})$ describes.*

Continuing, note that $p_1(\text{D}) = (\text{D}, p(\text{D})) = (\text{D}, \{\{\text{A}, \text{C}\}\})$. So

$$
\begin{aligned}
p_2(\text{A}) &= (\text{A}, P_f P_f(p_1)(p(\text{A}))) \\
&= (\text{A}, P_f P_f(p_1)(\{\{\text{B}, \text{C}\}, \{\text{B}, \text{D}\}\})) \\
&= (\text{A}, \{\{(\text{B}, \emptyset), (\text{C}, \emptyset)\}, \{(\text{B}, \emptyset), (\text{D}, \{\{\text{A}, \text{C}\}\})\}\})
\end{aligned}
$$

which is the same information as that in the first five levels of the coinductive tree for A*: $p_1(\text{A})$ provides the first three levels of $p_2(\text{A})$ because $p_2(\text{A})$ must map to $p_1(\text{A})$ in the cone; in the coinductive tree, there are two and-nodes at level 5, labelled by* A *and* C*. As there are no clauses with head* B *or* C*, no or-nodes lie*

12

below the first three of the and-nodes at level 3. However, there is one or-node lying below D, *it branches into and-nodes labelled by* A *and* C, *which is exactly as* $p_2(A)$ *tells us.*
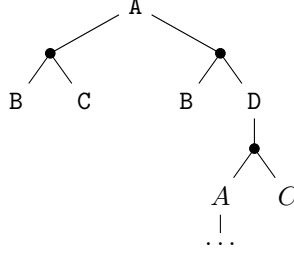

Figure 2: The coinductive tree for A and the program P from Example 3.

4. Lax semantics for logic programs

We now lift the restriction on $Var = \emptyset$ in Definition 1 and consider first-order terms and atoms in full generality.

There are several equivalent ways in which to describe the Lawvere theory generated by a signature. So, for precision, in this paper, we define the *Lawvere theory $\mathcal{L}_\Sigma$ generated by* a signature Σ as follows: $\mathrm{ob}(\mathcal{L}_\Sigma)$ is the set of natural numbers. For each natural number n, let $x_1, \ldots, x_n$ be a specified list of distinct variables. Define $\mathcal{L}_\Sigma(n, m)$ to be the set of m-tuples $(t_1, \ldots, t_m)$ of terms generated by the function symbols in Σ and variables $x_1, \ldots, x_n$. Define composition in $\mathcal{L}_\Sigma$ by substitution.

One can readily check that these constructions satisfy the axioms for a category, with $\mathcal{L}_\Sigma$ having strictly associative finite products given by the sum of natural numbers. The terminal object of $\mathcal{L}_\Sigma$ is the natural number 0. There is a canonical identity-on-objects functor from Nat^{op} to $\mathcal{L}_\Sigma$, just as there is for any Lawvere theory, and it strictly preserves finite products.

Example 4. *Consider ListNat. The constants* 0 *and* nil *are maps from* 0 *to* 1 *in* $\mathcal{L}_\Sigma$, s *is modelled by a map from* 1 *to* 1, *and* cons *is modelled by a map from* 2 *to* 1. *The term* s(0) *is the map from* 0 *to* 1 *given by the composite of the maps modelling* s *and* 0.

Given an arbitrary logic program P with signature Σ, we can extend the set At of atoms for a variable-free logic program to the functor $At : \mathcal{L}_\Sigma^{op} \to Set$ that sends a natural number n to the set of all atomic formulae, with variables among $x_1, \ldots, x_n$, generated by the function symbols in Σ and by the predicate symbols in P. A map $f : n \to m$ in $\mathcal{L}_\Sigma$ is sent to the function $At(f) : At(m) \to At(n)$ that sends an atomic formula $A(x_1, \ldots, x_m)$ to

$A(f_1(x_1, \ldots, x_n)/x_1, \ldots, f_m(x_1, \ldots, x_n)/x_m)$, i.e., $At(f)$ is defined by substitution.

As explained in the Introduction and in [9], we cannot model a logic program by a natural transformation of the form $p : At \longrightarrow P_f P_f At$ as naturality breaks down, e.g., in ListNat. So, in [17, 1], we relaxed naturality to lax naturality. In order to define it, we extended $At : \mathcal{L}_\Sigma^{op} \to Set$ to have codomain $Poset$ by composing At with the inclusion of Set into $Poset$. Mildly overloading notation, we denote the composite by $At : \mathcal{L}_\Sigma^{op} \to Poset$.

Definition 4. *Given functors $H, K : \mathcal{L}_\Sigma^{op} \longrightarrow Poset$, a lax transformation from H to K is the assignment to each object n of $\mathcal{L}_\Sigma$, of an order-preserving function $\alpha_n : Hn \longrightarrow Kn$ such that for each map $f : n \longrightarrow m$ in $\mathcal{L}_\Sigma$, one has $(Kf)(\alpha_m) \leq (\alpha_n)(Hf)$, pictured as follows:*

$$
\begin{array}{ccc}
Hm & \xrightarrow{\ \alpha_m\ } & Km \\
\Big\downarrow{\scriptstyle Hf} & \geq & \Big\downarrow{\scriptstyle Kf} \\
Hn & \xrightarrow[\ \alpha_n\ ]{} & Kn
\end{array}
$$

Functors and lax transformations, with pointwise composition, form a locally ordered category denoted by $Lax(\mathcal{L}_\Sigma^{op}, Poset)$. Such categories and generalisations have been studied extensively, e.g., in [18, 19, 20, 23].

Definition 5. *Define $P_f : Poset \longrightarrow Poset$ by letting $P_f(P)$ be the partial order given by the set of finite subsets of P, with $A \leq B$ if for all $a \in A$, there exists $b \in B$ for which $a \leq b$ in P, with behaviour on maps given by image. Define P_c similarly but with countability replacing finiteness.*

We are not interested in arbitrary posets in modelling logic programming, only those that arise, albeit inductively, by taking subsets of a set qua discrete poset. So we gloss over the fact that, for an arbitrary poset P, Definition 5 may yield factoring, with the underlying set of $P_f(P)$ being a quotient of the set of subsets of P. The potential difficulty is that if Definition 5 makes $A \leq B$ and $B \leq A$, then as P_f is defined in terms of posets, it follows that A is put equal to B, which would be inconvenient, but in fact, it does not substantially affect us as we start with discrete posets rather than arbitrary ones.

Example 5. *Modelling Example 1, ListNat generates a lax transformation of the form $p : At \longrightarrow P_f P_f At$ as follows: $At(n)$ is the set of atomic formulae in ListNat with at most n variables.*

For example, $At(0)$ consists of `nat(0)`, `nat(nil)`, `list(0)`, `list(nil)`, `nat(s(0))`, `nat(s(nil))`, `list(s(0))`, `list(s(nil))`, `nat(cons(0,0))`, `nat(cons(0,nil))`, `nat(cons(nil,0))`, `nat(cons(nil,nil))`, *etcetera.*

Similarly, $At(1)$ includes all atomic formulae containing at most one (specified) variable x, thus all the elements of $At(0)$ together with nat(x), list(x), nat(s(x)), list(s(x)), nat(cons(0,x)), nat(cons(x,0)), nat(cons(x,x)), *etcetera.*

The function $p_n : At(n) \longrightarrow P_f P_f At(n)$ sends each element of $At(n)$, i.e., each atom $A(x_1, \ldots, x_n)$ with variables among $x_1, \ldots, x_n$, to the set of sets of atoms in the antecedent of each unifying substituted instance of a clause in P with head for which a unifying substitution agrees with $A(x_1, \ldots, x_n)$.

Taking $n = 0$, nat(0) $\in At(0)$ *is the head of one clause, and there is no other clause for which a unifying substitution will make its head agree with* nat(0). *The clause with head* nat(0) *has the empty set of atoms as its tail, so $p_0(\text{nat}(0)) = \{\emptyset\}$.*

Taking $n = 1$, list(cons(x,0)) $\in At(1)$ *is the head of one clause given by a unifying substitution applied to the final clause of ListNat, and accordingly $p_1(\text{list}(\text{cons}(x,0))) = \{\{\text{nat}(x), \text{list}(0)\}\}$.*

The family of functions p_n satisfy the inequality required to form a lax transformation because if $A(x_1, \cdots, x_n)$ is the head of a substituted instance of a clause in P, then so is $A(t_1, \cdots, t_n)$ for any substitutions t_i for x_i. The family does not satisfy the strict requirement of naturality as explained in the Introduction.

We now analyse the relationship between a lax transformation $p : At \longrightarrow P_f P_f At$ and $\overline{p} : At \longrightarrow C(P_f P_f)At$, the corresponding coalgebra for the cofree comonad $C(P_f P_f)$ on $P_f P_f$.

We recall the central abstract result of [17], the notion of an "oplax" map of coalgebras being required to match that of lax transformation. Notation of the form H-*coalg* refers to coalgebras for an endofunctor H, while notation of the form C-*Coalg* refers to coalgebras for a comonad C. The subscript *oplax* refers to oplax maps and, given an endofunctor E on *Poset*, the notation $Lax(\mathcal{L}_\Sigma^{op}, E)$ denotes the endofunctor on $Lax(\mathcal{L}_\Sigma^{op}, Poset)$ given by post-composition with E; similarly for a comonad.

Theorem 2. *[17] For any locally ordered endofunctor E on Poset, if $C(E)$ is the cofree comonad on E, then there is a canonical isomorphism*

$$Lax(\mathcal{L}_\Sigma^{op}, E)\text{-}coalg_{oplax} \simeq Lax(\mathcal{L}_\Sigma^{op}, C(E))\text{-}Coalg_{oplax}$$

Theorem 2 tells us that for any endofunctor E on *Poset*, the relationship between E-coalgebras and $C(E)$-coalgebras extends pointwise from *Poset* to $Lax(\mathcal{L}_\Sigma^{op}, Poset)$ providing one matches lax natural transformations by oplax maps of coalgebras. It follows that, given an endofunctor E on *Poset* with cofree comonad $C(E)$, the cofree comonad for the endofunctor on $Lax(\mathcal{L}_\Sigma^{op}, Poset)$ sending $H : \mathcal{L}_\Sigma^{op} \longrightarrow Poset$ to the composite $EH : \mathcal{L}_\Sigma^{op} \longrightarrow Poset$ sends H to the composite $C(E)H$. Taking the example $E = P_f P_f$ allows us to conclude the following.

Corollary 1. *[17] $Lax(\mathcal{L}_\Sigma^{op}, C(P_f P_f))$ is the cofree comonad on $Lax(\mathcal{L}_\Sigma^{op}, P_f P_f)$.*

Corollary 1 means that there is a natural bijection between lax transformations

$$p : At \longrightarrow P_f P_f At$$

and lax transformations

$$\overline{p} : At \longrightarrow C(P_f P_f)At$$

subject to the two conditions required of a coalgebra of a comonad given pointwise, thus by applying the construction of Theorem 1 pointwise. So it is the abstract result we need in order to characterise the coinductive trees generated by logic programs with no existential variables, extending Theorem 1. As explained in the Introduction, an existential variable in a logic program is a variable that appears in the tail of a clause but not in its head. We explain the situation in detail in Example 6, but for now, just note that ListNat does not have existential variables, so the following result applies directly to it.

Theorem 3. *Let $C(P_f P_f)$ denote the cofree comonad on the endofunctor $P_f P_f$ on Poset. Then, given a logic program P with no existential variables on At, defining $p_n(A(x_1, \ldots, x_n))$ to be the set of sets of atoms in each antecedent of each unifying substituted instance of a clause in P with head for which a unifying substitution agrees with $A(x_1, \ldots, x_n)$, the corresponding $Lax(\mathcal{L}_\Sigma^{op}, C(P_f P_f))$-coalgebra $\overline{p} : At \longrightarrow C(P_f P_f)At$ sends an atom $A(x_1, \ldots, x_n)$ to the coinductive tree for $A(x_1, \ldots, x_n)$.*

PROOF. The absence of existential variables ensures that any variable that appears in the antecedent of a clause must also appear in its head. So every atom in every antecedent of every unifying substituted instance of a clause in P with head for which a unifying substitution agrees with $A(x_1, \ldots, x_n)$ actually lies in $At(n)$. Moreover, there are only finitely many sets of sets of such atoms. So the construction of each p_n is well-defined, i.e., the image of $A(x_1, \ldots, x_n)$ lies in $P_f P_f At(n)$. The collection of maps given by p_n for each object n of $\mathcal{L}_\Sigma$ forms a lax transformation from At to $P_f P_f At$: the laxness condition holds because substitution preserves the truth of a clause, i.e., if one makes a substitution into both the head and tail of a clause that is true, the substituted instance of the clause is also true.

By Corollary 1, $\overline{p}$ is determined pointwise. So, to construct it, we may fix n and follow the proof of Theorem 1, consistently replacing At by $At(n)$. To complete the proof, observe that the construction of p from a logic program P matches the construction of the coinductive tree for an atom $A(x_1, \ldots, x_n)$ if P has no existential variables. So following the proof of Theorem 1 completes this proof.

Example 6. *Attempting to model Example 2, that of graph connectedness, GC, by mimicking the modelling of ListNat in Example 5, i.e., defining the function $p_n : At(n) \longrightarrow P_f P_f At(n)$ by sending each element of $At(n)$, i.e., each atom $A(x_1, \ldots, x_n)$ with variables among $x_1, \ldots, x_n$, to the set of sets of atoms in the*

antecedent of each unifying substituted instance of a clause in P with head for which a unifying substitution agrees with $A(x_1, \ldots, x_n)$, *fails.*

Consider the clause

$$\texttt{connected}(\texttt{x}, \texttt{y}) \leftarrow \texttt{edge}(\texttt{x}, \texttt{z}), \texttt{connected}(\texttt{z}, \texttt{y})$$

Modulo possible renaming of variables, the head of the clause, i.e., the atom $\texttt{connected}(\texttt{x}, \texttt{y})$, *lies in* $At(2)$ *as it has two variables. There is trivially only one substituted instance of a clause in GC with head for which a unifying substitution agrees with* $\texttt{connected}(\texttt{x}, \texttt{y})$, *and the singleton set consisting of the set of atoms in its antecedent is* $\{\{\texttt{edge}(\texttt{x}, \texttt{z}), \texttt{connected}(\texttt{z}, \texttt{y})\}\}$, *which does not lie in* $P_f P_f At(2)$ *as it has three variables appear in it rather than two. See Section 2 for a picture of the coinductive tree for* $\texttt{connected}(\texttt{x}, \texttt{y})$.

We dealt with that inelegantly in [17]: in order to force $p_2(\texttt{connected}(\texttt{x}, \texttt{y}))$ *to lie in* $P_f P_f At(2)$ *and model GC in any reasonable sense, we allowed substitutions for z in* $\{\{\texttt{edge}(\texttt{x}, \texttt{z}), \texttt{connected}(\texttt{z}, \texttt{y})\}\}$ *by any term on* x, y *on the basis that there is no unifying such, so we had better allow all possibilities. So, rather than modelling the clause directly, recalling that* $At(2) \subseteq At(3) \subseteq At(4)$, *etcetera, modulo renaming of variables, we put*

$$p_2(\texttt{connected}(\texttt{x}, \texttt{y})) \;=\; \{\{\texttt{edge}(\texttt{x}, \texttt{x}), \texttt{connected}(\texttt{x}, \texttt{y})\}, \{\texttt{edge}(\texttt{x}, \texttt{y}), \texttt{connected}(\texttt{y}, \texttt{y})\}\}$$

$$p_3(\texttt{connected}(\texttt{x}, \texttt{y})) \;=\; \{\{\texttt{edge}(\texttt{x}, \texttt{x}), \texttt{connected}(\texttt{x}, \texttt{y})\}, \{\texttt{edge}(\texttt{x}, \texttt{y}), \texttt{connected}(\texttt{y}, \texttt{y})\},$$
$$\{\texttt{edge}(\texttt{x}, \texttt{z}), \texttt{connected}(\texttt{z}, \texttt{y})\}\}$$

$$p_4(\texttt{connected}(\texttt{x}, \texttt{y})) \;=\; \{\{\texttt{edge}(\texttt{x}, \texttt{x}), \texttt{connected}(\texttt{x}, \texttt{y})\}, \{\texttt{edge}(\texttt{x}, \texttt{y}), \texttt{connected}(\texttt{y}, \texttt{y})\},$$
$$\{\texttt{edge}(\texttt{x}, \texttt{z}), \texttt{connected}(\texttt{z}, \texttt{y})\}, \{\texttt{edge}(\texttt{x}, \texttt{w}), \texttt{connected}(\texttt{w}, \texttt{y})\}\}$$

etcetera: for p_2, *as only two variables* x *and* y *appear in any element of* $P_f P_f At(2)$, *we allowed substitution by either* x *or* y *for* z; *for* p_3, *a third variable may appear in an element of* $P_f P_f At(3)$, *allowing an additional possible subsitution; for* p_4, *a fourth variable may appear, etcetera.*

Countability arises if a unary symbol s *is added to GC, as in that case, for* p_2, *not only did we allow* x *and* y *to be substituted for* z, *but we also allowed* $s^n(x)$ *and* $s^n(y)$ *for any* $n > 0$, *and to do that, we replaced* $P_f P_f$ *by* $P_c P_f$, *allowing for the countably many possible substitutions.*

Those were inelegant decisions, but they allowed us to give some kind of model of all logic programs. We shall revisit this in Section 6.

Theorem 3 models the coinductive trees generated by ListNat as the latter has no existential variables, but for GC, as explained in Example 6, the natural construction of p did *not* model the clause

$$\texttt{connected}(\texttt{x}, \texttt{y}) \leftarrow \texttt{edge}(\texttt{x}, \texttt{z}), \texttt{connected}(\texttt{z}, \texttt{y})$$

directly, and so its extension *a fortiori* could *not* model the coinductive trees generated by $\texttt{connected}(\texttt{x}, \texttt{y})$.

For arbitrary logic programs, the way we defined $\overline{p}(A(x_1, \ldots, x_n))$ in earlier papers such as [1] was in terms of a variant of the coinductive tree generated by $A(x_1, \ldots, x_n)$ in two key ways:

17

1. coinductive trees allow new variables to be introduced as one passes down the tree, e.g., with

$$\texttt{connected}(\texttt{x},\texttt{y}) \leftarrow \texttt{edge}(\texttt{x},\texttt{z}), \texttt{connected}(\texttt{z},\texttt{y})$$

appearing in it. However, because of the presence of the variable $\texttt{z}$, the set $\{\{\texttt{edge}(\texttt{x},\texttt{z}), \texttt{connected}(\texttt{z},\texttt{y})\}\}$ does not appear in $P_f P_f At(2)$. In previous papers, we made clumsy adaptations of the natural model in order to model the clause.

2. coinductive trees are finitely branching, as one expects in logic programming, but in previous papers, we allowed $\overline{p}(A(x_1, \ldots, x_n))$ to be infinitely branching in order to model the countably many possible applications of a unary function symbol.

5. Saturated semantics for logic programs

Bonchi and Zanasi's saturated semantics approach to modelling logic programming in [14] was to consider $P_f P_f$ as we did in [17], sending At to $P_f P_f At$, but to ignore the inherent laxness, replacing $Lax(\mathcal{L}_\Sigma^{op}, Poset)$ by $[ob(\mathcal{L}_\Sigma), Set]$, where $ob(\mathcal{L}_\Sigma)$ is the set of objects of $\mathcal{L}_\Sigma$ treated as a discrete category, i.e., as a category containing only identity maps. Their central construction may be seen in a more axiomatic setting as follows.

For any small category C, let $ob(C)$ denote the discrete subcategory with the same objects as C, with inclusion $I : ob(C) \longrightarrow C$. Then the functor

$$[I, Set] : [C, Set] \longrightarrow [ob(C), Set]$$

has a right adjoint given by right Kan extension, and that remains true when one extends from Set to any complete category, and it all enriches, e.g., over $Poset$ [3]. As $ob(C)$ has no non-trivial arrows, the right Kan extension is a product, given by

$$(ran_I H)(c) = \prod_{d \in C} Hd^{C(c,d)}$$

By the Yoneda lemma, to give a natural transformation from K to $(ran_I H)(-)$ is equivalent to giving a natural, or equivalently in this setting, a "not necessarily natural", transformation from KI to H. Taking $C = \mathcal{L}_\Sigma^{op}$ gives exactly Bonchi and Zanasi's formulation of saturated semantics [14].

It was the fact of the existence of the right adjoint, rather than its characterisation as a right Kan extension, that enabled Bonchi and Zanasi's constructions of saturation and desaturation, but the description as a right Kan extension informed their syntactic analysis.

Note for later that products in $Poset$ are given pointwise, so agree with products in Set. So if we replace Set by $Poset$ here, and if C is an ordinary category without any non-trivial $Poset$-enrichment, the right Kan extension

would yield the same set as above, with an order on it determined by that on H.

In order to unify saturated semantics with lax semantics, we need to rephrase Bonchi and Zanasi's formulation a little. Upon close inspection, one can see that, in their semantics, they only used objects of $[ob(\mathcal{L}_\Sigma)^{op}, Set]$, equivalently $[ob(\mathcal{L}_\Sigma), Set]$, of the form HI for some $H : \mathcal{L}_\Sigma^{op} \longrightarrow Set$ [14]. That allows us, while making no substantive change to their body of work, to reformulate it a little, in axiomatic terms, as follows.

Let $[C, Set]_d$ denote the category of functors from C to Set and "not necessarily natural" transformations between them, i.e., a map from H to K consists of, for all $c \in C$, a function $\alpha_c : Hc \longrightarrow Kc$, without demanding a naturality condition. The functor $[I, Set] : [C, Set] \longrightarrow [ob(C), Set]$ factors through the inclusion of $[C, Set]$ into $[C, Set]_d$ as follows:

$$[C, Set] \xrightarrow{J_{Set}} [C, Set]_d \xrightarrow{J'} [ob(C), Set]$$

The functor $J' : [C, Set]_d \longrightarrow [ob(C), Set]$ sends a functor $H : C \longrightarrow Set$ to the functor $HI : ob(C) \longrightarrow Set$. The composite $[I, Set] = J'J_{Set}$ has a right adjoint given by right Kan extension, and J' is fully faithful. By elementary category theory [33], it follows that $J_{Set} : [C, Set] \longrightarrow [C, Set]_d$ has a right adjoint that sends $H : C \longrightarrow Set$ to the right Kan extension of $J'(H) = HI$ along I.

Thus one can rephrase Bonchi and Zanasi's work to assert that the central mathematical fact that supports saturated semantics is that the inclusion

$$[\mathcal{L}_\Sigma^{op}, Set] \longrightarrow [\mathcal{L}_\Sigma^{op}, Set]_d$$

has a right adjoint that sends a functor $H : \mathcal{L}_\Sigma^{op} \longrightarrow Set$ to the right Kan extension $ran_I HI$ of the composite $HI : ob(\mathcal{L}_\Sigma)^{op} \longrightarrow Set$ along the inclusion $I : ob(\mathcal{L}_\Sigma) \longrightarrow \mathcal{L}_\Sigma$.

We can now unify lax semantics with saturated semantics by developing a precise body of theory that relates the inclusion

$$J_{Set} : [C, Set] \longrightarrow [C, Set]_d$$

which has a right adjoint that sends $H : C \longrightarrow Set$ to $ran_I HI$, with the inclusion

$$J : [C, Poset] \longrightarrow Lax(C, Poset)$$

which also has a right adjoint, that right adjoint being given by a restriction of the right Kan extension $ran_I HI$ of the composite $HI : ob(C) \longrightarrow Poset$ along the inclusion $I : ob(C) \longrightarrow C$.

The existence of the right adjoint follows from Theorem 3.13 of [19], but we give an independent proof here and a description of it in terms of right Kan extensions in order to show that Bonchi and Zanasi's explicit constructions of saturation and desaturation apply equally in this setting.

Consider the inclusions

$$[C, Poset] \longrightarrow Lax(C, Poset) \longrightarrow [C, Poset]_d$$

Exactly as for Set as above, for any small category C, the inclusion

$$J_{Poset} : [C, Poset] \longrightarrow [C, Poset]_d$$

has a right adjoint sending $H : C \longrightarrow Poset$ to $ran_I HI$. In order to give a right adjoint to $J : [C, Poset] \longrightarrow Lax(C, Poset)$, we will restrict $ran_I HI$ to a subfunctor $R(H)$ in $[C, Poset]$ so that to give a natural transformation from K into the restriction $R(H)$ of $ran_I HI$ is equivalent to giving a map from K to H in $Lax(C, Poset)$, i.e., a map in $[C, Poset]_d$ that satisfies the condition that, for all $f : c \longrightarrow d$, one has $Hf.\alpha_c \leq \alpha_d.Kf$. This can be done by defining $R(H)$ to be an *inserter*, which is a particularly useful kind of limit that applies to locally ordered categories and is a particular kind of generalisation of the notion of equaliser.

Definition 6. *[19] Given parallel maps $f, g : X \longrightarrow Y$ in a locally ordered category D, an* inserter *from f to g is an object $Ins(f, g)$ of D together with a map $i : Ins(f, g) \longrightarrow X$ such that $fi \leq gi$ and is universal such, i.e., for any object Z and map $z : Z \longrightarrow X$ for which $fz \leq gz$, there is a unique map $k : Z \longrightarrow Ins(f, g)$ such that $ik = z$. Moreover, for any such z and z' for which $z \leq z'$, then $k \leq k'$, where k and k' are induced by z and z' respectively.*

An inserter is a form of limit. Taking D to be $Poset$, the poset $Ins(f, g)$ is given by the full sub-poset of X determined by $\{x \in X | f(x) \leq g(x)\}$. Being limits, inserters in functor categories are determined pointwise.

The inserter we require is subtle, in the spirit of the work of [19] and [3]. Given a functor $H : C \longrightarrow Poset$, we define a parallel pair of maps in $[C, Poset]$, i.e., natural transformations, δ_1 and δ_2. Their domains and codomains, which must be functors from C to $Poset$, are given as follows:

$$\delta_1, \delta_2 : \prod_{d \in C} Hd^{C(-,d)} \longrightarrow \prod_{d,d' \in C} Hd'^{C(-,d) \times C(d,d')}$$

i.e., the domain of both δ_1 and δ_2 is the functor from C to $Poset$ that sends c to $\prod_{d \in C} Hd^{C(c,d)}$, and similarly for the codomain.

Using the property of products, to give such natural transformations is equivalent, via Currying, to giving, for each c, d and d' in C, maps of the form

$$(\delta_1)_{(d,d')}c, (\delta_2)_{(d,d')}c : C(c,d) \times C(d,d') \times \prod_{d \in C} Hd^{C(c,d)} \longrightarrow Hd'$$

natural in c. We define the two maps as follows:

1. the (d, d')-component of $\delta_1 c$ is determined by composing

$$\circ_C \times id : C(c,d) \times C(d,d') \times \prod_{d \in C} Hd^{C(c,d)} \longrightarrow C(c,d') \times \prod_{d \in C} Hd^{C(c,d)}$$

with the evaluation of the product at d', i.e., taking the d'-th component of the product and applying evaluation to $C(c, d') \times Hd^{C(c,d')}$

2. the (d, d')-component of $\delta_2 c$ is determined by evaluating the product at d

$$C(c, d) \times C(d, d') \times \prod_{d \in C} Hd^{C(c,d)} \longrightarrow C(d, d') \times Hd$$

then composing with

$$C(d, d') \times Hd \xrightarrow{H \times id} Hd'^{Hd} \times Hd \xrightarrow{eval} Hd'$$

Theorem 4. *The right adjoint R to the inclusion $J : [C, Poset] \longrightarrow Lax(C, Poset)$ sends $H : C \longrightarrow Poset$ in $Lax(C, Poset)$ to the inserter in $[C, Poset]$ from δ_1 to δ_2*

$$\delta_1, \delta_2 : (ran_I HI)(-) = \prod_{d \in C} Hd^{C(-,d)} \longrightarrow \prod_{d,d' \in C} Hd'^{C(-,d) \times C(d,d')}$$

Although the statement of the theorem is complex, the proof is routine, along similar lines to proofs in [3]. One simply needs to check that δ_1 and δ_2 are natural, which they routinely are, and that the inserter satisfies the universal property we seek, which it does by construction.

Corollary 2. *Let R be the right adjoint to the inclusion $J : [C, Poset] \longrightarrow Lax(C, Poset)$. Then, for any functor $H : C \longrightarrow Poset$ in $Lax(C, Poset)$, the functor $R(H) : C \longrightarrow Poset$ is a subfunctor in $[C, Poset]$ of the right Kan extension, $ran_I HI$, of HI along the inclusion $I : ob(C) \longrightarrow C$.*

The inclusion of $R(H)$ into $ran_I HI$ is given by the canonical map from the inserter that defines $R(H)$ in the statement of Theorem 4 into $ran_I HI$, the domain of δ_1 and δ_2.

Bonchi and Zanasi's saturation and desaturation constructions remain exactly the same: the saturation of $p : At \longrightarrow P_f P_f At$ is a natural transformation $\overline{p} : At \longrightarrow ran_I P_f P_f At I$ that factors through $Ins(\delta_1, \delta_2)$ without any change whatsoever to its construction, that being so because of the fact of p being lax.

With this result in hand, it is routine to work systematically through Bonchi and Zanasi's papers, using their saturation and desaturation constructions exactly as they had them, without discarding the inherent laxness that logic programming, cf data refinement, possesses.

So this unifies lax semantics, which flows from, and may be seen as an instance of, Tony Hoare's semantics for data refinement [21, 22, 23], with saturated semantics and its more denotational flavour [6].

6. Lax semantics for logic programs refined: existential variables

In Section 4, following [17], we gave lax semantics for logic programs without existential variables, such as ListNat. In particular, we modelled the coinductive trees they generate. Restriction to non-existential examples such as ListNat is

common for implementational reasons [1, 11, 12, 13], so Section 4 allowed the modelling of coinductive trees for a natural class of logic programs.

Nevertheless, we would like to model coinductive trees generated by logic programming in full generality, including examples such as that of GC. We need to refine the lax semantics of Section 4 in order to do so, and, having just unified lax semantics with saturated semantics in Section 5, we would like to retain that unity in making such a refinement. So that is what we do in this section.

We initially proposed such a refinement in the workshop paper [25] that this paper extends, but since the workshop, we have found a further refinement that strengthens the relationship with the modelling of local state [26]. So our constructions here are a little different to those in [25].

In order to model coinductive trees, it follows from Example 6 that the endofunctor $Lax(\mathcal{L}_\Sigma^{op}, P_f P_f)$ on $Lax(\mathcal{L}_\Sigma^{op}, Poset)$ that sends At to $P_f P_f At$, needs to be refined as $\{\{\text{edge}(\text{x}, \text{z}), \text{connected}(\text{z}, \text{y})\}\}$ is not an element of $P_f P_f At(2)$ as it involves three variables x, y and z. In general, we need to allow the image of p_n to lie in the set given by applying $P_f P_f$ to a superset of $At(n)$, one that includes $At(m)$ for all $m \geq n$. In Example 6, that would allow the image of p_2 to lie in $P_f P_f At(3)$ rather than in $P_f P_f At(2)$, thus allowing us to map $\text{connected}(\text{x}, \text{y})$ to $\{\{\text{edge}(\text{x}, \text{z}), \text{connected}(\text{z}, \text{y})\}\}$ as desired.

However, we do not want to double-count: there are six injections of 2 into 3, inducing six inclusions $At(2) \subseteq At(3)$, and one only wants to count each atom in $At(2)$ once. So we refine $P_f P_f At(n)$ to become $P_f P_f(\int At(n))$, where $\int At$ is defined as follows.

Letting Inj denote the category of natural numbers and injections, for any Lawvere theory L, there is a canonical identity-on-objects functor $J : Inj^{op} \longrightarrow L$.

Definition 7. *We define $\int At(n)$ to be the colimit of the composite functor*

$$n/Inj \xrightarrow{\ cod\ } Inj \xrightarrow{\ J\ } \mathcal{L}_\Sigma^{op} \xrightarrow{\ At\ } Poset$$

This functor sends an injection $j : n \longrightarrow m$ to $At(m)$, with the j-th component of the colimiting cocone being of the form $\rho_j : At(m) \longrightarrow \int At(n)$.

The colimiting property is precisely the condition required to ensure no double-counting (see [33] or, for the enriched version, [3] of this construction in a general setting).

It is not routine to extend the construction of $\int At(n)$ for each n to give a functor $\int At : \mathcal{L}_\Sigma^{op} \longrightarrow Poset$. To do so, we mimic the construction on arrows used to define the monad for local state in [26]. We first used this idea in [25] and we refine our use of it in this paper to make for a closer technical relationship with the semantics of local state in [26]: we do not fully understand the relationship yet, but there seems considerable potential based on the work here to make precise comparison between the role of variables in logic programming with that of worlds in modelling local state.

Following [26], we extend $\int At(n)$ to a functor $\int At : \mathcal{L}_\Sigma^{op} \longrightarrow Poset$ by sending a map $f : n \longrightarrow n'$ in $\mathcal{L}_\Sigma$ to the order-preserving function

$$\int At(f) : \int At(n') \longrightarrow \int At(n)$$

determined by the colimiting property of $\int At(n')$ as follows.

For each $j' : n' \longrightarrow l \in n'/Inj$, there are a unique natural number k and at least one isomorphism between l and $n' + k$, such that composite of j' with the isomorphism is the canonical injection from n' to $n' + k$. So, in calculating with the colimit, we may identify j', via composition with a chosen such isomorphism, with the canonical injection from n' to $n' + k$. This induces a cocone

$$At(l) \cong At(n' + k) \xrightarrow{At(f + k)} At(n + k) \xrightarrow{\rho_j} \int At(n)$$

where $j : n \longrightarrow n + k$ is the canonical injection of n into $n + k$, thus determining $\int At(f)$. It is routine to check that this assignment makes $\int At$ functorial with respect to maps in $\mathcal{L}_\Sigma^{op}$.

There is nothing specific about the functor $At : \mathcal{L}_\Sigma^{op} \longrightarrow Poset$ in the above construction. The construction generalises without fuss from defining the functor $\int At : \mathcal{L}_\Sigma^{op} \longrightarrow Poset$ to the definition of $\int H : \mathcal{L}_\Sigma^{op} \longrightarrow Poset$ for any functor $H : \mathcal{L}_\Sigma^{op} \longrightarrow Poset$.

In order to make each map $\alpha : H \Rightarrow K$ generate a map $\int \alpha : \int H \Rightarrow \int K$, we need to specify in exactly what category we treat H as an object. $Lax(\mathcal{L}_\Sigma^{op}, Poset)$ is not possible because $\int H(n)$ is defined to be a (strict rather than lax) colimit, so a (strict) cocone into $\int K(n)$ is required in order to generate the map $\int \alpha(n) : \int H(n) \rightarrow \int K(n)$. We would have that if we had strict naturality of α with respect to injections, but that is not true of an arbitrary map α in $Lax(\mathcal{L}_\Sigma^{op}, Poset)$. So we refine $Lax(\mathcal{L}_\Sigma^{op}, Poset)$ by restricting its maps in order to do that.

We accordingly restrict the maps of $Lax(\mathcal{L}_\Sigma^{op}, Poset)$ to allow only those lax transformations $\alpha : H \Rightarrow K$ that are strict with respect to maps in Inj, i.e., those α such that for any injection $i : n \longrightarrow m$, the diagram

$$
\begin{array}{ccc}
Hn & \xrightarrow{\alpha_n} & Kn \\
{\scriptstyle Hi}\downarrow & & \downarrow{\scriptstyle Ki} \\
Hm & \xrightarrow[\alpha_m]{} & Km
\end{array}
$$

commutes. A restriction of this nature is standard in 2-category theory, e.g., in [19], as one typically needs to distinguish between lax and strict maps, with strict commutativity only in regard to the latter.

Summarising this discussion yields the following:

Definition 8. *Let $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$ denote the category with objects given by functors from $\mathcal{L}_\Sigma^{op}$ to $Poset$, maps given by lax transformations that strictly respect injections, and composition given pointwise.*

Proposition 2. *cf [26] Let $J : Inj^{op} \longrightarrow \mathcal{L}_\Sigma$ be the canonical inclusion. Define*

$$\int : Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset) \longrightarrow Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$$

on objects as above. Given $\alpha : H \Rightarrow K$ in $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$, define $\int \alpha(n)$ by the fact that $j \in n/Inj$ is coherently isomorphic to the canonical inclusion $j : n \longrightarrow n + k$ for a unique natural number k, and applying the definition of $\int H(n)$ as a colimit to the cocone given by composing

$$\alpha_{n+k} : H(m) = H(n + k) \longrightarrow K(n + k) = K(m)$$

with the canonical map $K(m) \longrightarrow \int K(n)$ exhibiting $\int K(n)$ as a colimit. Then $\int(-)$ is an endofunctor on $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$.

The proof is routine, albeit after lengthy calculation involving colimits.

We can now model an arbitrary logic program by a map $p : At \longrightarrow P_f P_f \int At$ in $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$, modelling ListNat as we did in Example 5 but now modelling the clauses of GC directly rather than using the awkward substitution instances of Example 6.

Example 7. *Except for the restriction of $Lax(\mathcal{L}_\Sigma^{op}, Poset)$ to $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$, ListNat is modelled in exactly the same way here as it was in Example 5, the reason being that no clause in ListNat has a variable in the tail that does not already appear in the head. We need only observe that, although p is not strictly natural in general, it does strictly respect injections. For example, if one views* $\mathtt{list(cons(x,0))}$ *as an element of $At(2)$, its image under p_2 agrees with its image under p_1.*

Example 8. *In contrast to Example 6, using $P_f P_f \int$, we can emulate the construction of Examples 5 and 7 for ListNat to model GC.*

Modulo possible renaming of variables, $\mathtt{connected(x,y)}$ *is an element of $At(2)$. The function p_2 sends it to the element* $\{\{\mathtt{edge(x,z)}, \mathtt{connected(z,y)}\}\}$ *of $(P_f P_f \int At)(2)$. This is possible by taking $n = 2$ and $m = 3$ in Definition 7. In contrast,* $\{\{\mathtt{edge(x,z)}, \mathtt{connected(z,y)}\}\}$ *is not an element of $P_f P_f At(2)$, hence the failure of Example 6.*

The behaviour of $P_f P_f \int At$ on maps ensures that the lax transformation p strictly respects injections. For example, if $\mathtt{connected(x,y)}$ *is seen as an element of $At(3)$, the additional variable is treated as a fresh variable w, so does not affect the image of* $\mathtt{connected(x,y)}$ *under p_3.*

For the proof of the centrepiece theorem of this section, we need briefly to analyse Definition 7, which defines $\int At(n)$. We do not assert that the colimit

defining $\int At(n)$ is ω-directed, but it can be built from an ω-directed colimit as follows.

For any injection $j : n \longrightarrow l$, there are a unique k and at least one isomorphism between l and $n+k$ such that the composite of j with the isomorphism is the canonical injection from n to $n + k$. There may be more than one such isomorphism: for instance, if $n = 2$ and $l = 4$, there are two. So the colimit $\int At(n)$ is given by first taking the ω-directed colimit $colim_{m \geq n} At(m)$ determined by the canonical injections, then by quotienting $colim_{m \geq n} At(m)$ by structural permutations. This extends routinely to arbitrary functors.

Lemma 1. *The construction $\int$ preserves monomorphisms in $[Inj, Poset]$.*

PROOF. Monomorphisms in $[Inj, Poset]$ are given pointwise. An ω-directed colimit is, a fortiori, a filtered colimit, so, by Gabriel-Ulmer, it commutes with finite limits, so respects monomorphisms. So, given a monomorphism $\iota : H \Rightarrow K$, for any n, the induced map from $colim_{m \geq n} H(m)$ to $colim_{m \geq n} K(m)$ is a monomorphism. $\int H(n)$ is the quotient of $colim_{m \geq n} H(m)$ under structural permutations; likewise for $\int K(n)$. Given two elements of $colim_{m \geq n} H(m)$ that are sent by ι to equivalent elements of $colim_{m \geq n} K(m)$, without loss of generality, a permutation θ in Inj generates the equivalence. Naturality of ι, together with injectivity of ι_m, implies that applying $H(\theta)$ to the original two elements of $colim_{m \geq n} H(m)$ also renders them equivalent, thus yielding injectivity of $\int \iota(n)$.

Theorem 5. *The functor $P_f P_f \int : Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset) \longrightarrow Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$ induces a cofree comonad $C(P_f P_f \int)$ on $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$. Moreover, given a logic program P qua $P_f P_f \int$-coalgebra $p : At \longrightarrow P_f P_f \int At$, the corresponding $C(P_f P_f \int)$-coalgebra $\overline{p} : At \longrightarrow C(P_f P_f \int)(At)$ sends an atom $A(x_1, \ldots, x_n) \in At(n)$ to the coinductive tree for $A(x_1, \ldots, x_n)$.*

PROOF. Proposition 2 defined $\int$ as an endofunctor on $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$. It is routine to verify, by inspection, that the construction of $\int$ also yields an endofunctor on $[Inj, Poset]$, giving a restriction of $P_f P_f \int$ to $[Inj, Poset]$.

If one restricts $P_f P_f \int$ to $[Inj, Poset]$, there is a cofree comonad on it for general reasons, $[Inj, Poset]$ being locally finitely presentable and $P_f P_f \int$ being an accessible functor [32]. The cofree comonad may be constructed as follows.

By Lemma 1, $\int$ preserves monomorphisms in $[Inj, Poset]$. Post-composition with P_f does too (up to equivalence). Thus $[Inj, Poset]$ is accessible and $P_f P_f \int$ is accessible and preserves monomorphisms (up to equivalence). So we may apply Corollary 3.3 of [32] to obtain a description of the cofree comonad on the restriction of $P_f P_f \int$ to $[Inj, Poset]$ in terms of limits.

We cannot directly apply the same argument to $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$ as the category $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$ does not have arbitrary limits (and so is not locally finitely presentable), cf, the analysis of categories of maps of algebras in 2-categories in [19, Section 2]. However, the specific limit that describes the cofree comonad on the restriction of $P_f P_f \int$ to $[Inj, Poset]$ does extend to $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$, with that extension describing the cofree comonad on $P_f P_f \int$. So, if we check that the limiting properties of that specific pointwise

limit allow us to define functoriality with respect to arbitrary maps in $\mathcal{L}_\Sigma$, i.e., for every functor $H : \mathcal{L}_\Sigma \longrightarrow Poset$, they allow us to define the functor $P_f P_f \int H : \mathcal{L}_\Sigma \longrightarrow Poset$, and then in turn, they allow us to define functoriality with respect to arbitrary maps in $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$, together with a small amount of coherence, the fact of its describing the cofree comonad on the restriction, that yielding the data and axioms for the comultiplication and counit, gives us the result we seek.

In detail, the cofree comonad $C(P_f P_f \int)$ on $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$ is given pointwise by the same limit as in the proof of Theorem 1, similarly to Theorem 3. In particular, replacing At by $At(n)$ and replacing $P_f P_f$ by $P_f P_f \int$ in the diagram in the proof of Theorem 1, one has

$$\ldots \longrightarrow At(n) \times (P_f P_f \int)(At \times P_f P_f \int At)(n) \longrightarrow At(n) \times (P_f P_f \int At)(n) \longrightarrow At(n)$$

with maps determined by the projection $\pi_0 : At(-) \times (P_f P_f \int) At(-) \longrightarrow At(-)$, with the endofunctor $At(n) \times P_f P_f \int (-)$ applied to it. For any n, $C(P_f P_f \int)(At)(n)$ is the limit, potentially transfinite, of the diagram [32, Corollary 3.3].

Products in the category $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$ are given pointwise, with pointwise projections. Moreover, those projections are strictly natural, as explained in Proposition 2.1 and Remark 2.9 of [19]. Because the projections are strictly natural, and because $\int$ and P_f send strictly natural transformations to strictly natural transformations, the limiting property applied to the above directed cone, of which $C(P_f P_f \int)(At)(n)$ is the limit, determines, for each map $f : n \longrightarrow m$ in $\mathcal{L}_\Sigma$, a map of posets from $C(P_f P_f \int)(At)(m)$ to $C(P_f P_f \int)(At)(n)$. The unicity property of the limit ensures that this construction makes $C(P_f P_f \int)(At)$ into a functor from $\mathcal{L}_\Sigma^{op}$ to $Poset$.

The construction and proof apply equally to an arbitrary functor $H : \mathcal{L}_\Sigma^{op} \longrightarrow Poset$ in $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$ as they do to At, yielding the construction of $C(P_f P_f \int)(H)$ in general.

A map in $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$ is a lax transformation $\alpha : H \Rightarrow K$ that is strictly natural with respect to injections. The data for any such map, i.e., the collection of poset maps $\alpha(n)$, is fully determined by its restriction to a map in $[Inj, Poset]$. The limiting property of $C(P_f P_f \int)(K)(n)$ as explained above yields the data, i.e., the components, for $C(P_f P_f \int)(\alpha) : C(P_f P_f \int)(H) \Rightarrow C(P_f P_f \int)(K)$. Moreover, the limiting property ensures that this data satisfies the properties required of an endofunctor on $[Inj, Poset]$, i.e., it respects composition and identities.

We next need to see that $C(P_f P_f \int)(\alpha)$ is lax natural with respect to arbitrary maps in $\mathcal{L}_\Sigma$, i.e., for any map $f : n \longrightarrow m$ in $\mathcal{L}_\Sigma$, that there is an

inequality

$$C(P_f P_f \textstyle\int)(H)(m) \xrightarrow{\;C(P_f P_f \int)(\alpha)(m)\;} C(P_f P_f \textstyle\int)(K)(m)$$

$$C(P_f P_f \textstyle\int)(H)(f) \Big\downarrow \qquad\qquad \geq \qquad\qquad \Big\downarrow C(P_f P_f \textstyle\int)(K)(f)$$

$$C(P_f P_f \textstyle\int)(H)(n) \xrightarrow[\;C(P_f P_f \int)(\alpha)(n)\;]{} C(P_f P_f \textstyle\int)(K)(n)$$

i.e., for every element of the poset $C(P_f P_f \int)(H)(m)$, the diagram yields an inequality as indicated in the poset $C(P_f P_f \int)(K)(n)$.

That follows from the two-dimensional property of limits in *Poset*, cf the proof of [19, Proposition 2.1]. Given any small diagram $F : D \longrightarrow Poset$ with limit L and limiting cone $\pi : L \Rightarrow F$, for any cone $c : C \Rightarrow F$, the *one-dimensional property* of the limit L asserts the existence of a unique intermediary map $\gamma : C \longrightarrow L$. The *two-dimensional property* asserts that, given any two cones c and c' with the same vertex C, if $c_x \leq c'_x$ for each x in D, it follows that $\gamma \leq \gamma'$.

Here, $C(P_f P_f \int)(K)(n)$ is given by the limit of the potentially transfinite diagram

$$\ldots \longrightarrow K(n) \times (P_f P_f \textstyle\int)(K \times P_f P_f \textstyle\int K)(n) \longrightarrow K(n) \times (P_f P_f \textstyle\int K)(n) \longrightarrow K(n)$$

So $F : D \longrightarrow Poset$ is the diagram. We denote Fx by $K_x n$. By induction, together with Proposition 2, K_x is functorial in n. The limit L of the diagram F is the poset $C(P_f P_f \int)(K)(n)$. All four vertices of the potential inequality above may be viewed as limits in the same way. We take C to be $C(P_f P_f \int)(H)(m)$ and the cones c and c' to be the precomposition of γ and γ' respectively with the limiting cone π for $C(P_f P_f \int)(K)(n)$, with γ and γ' given by the top-right and bottom-left legs of the potential inequality respectively. In order to deduce lax naturality of $C(P_f P_f \int)(\alpha)$ from that of α, it suffices, by the two-dimensional property, to show that for any $x \in D$, one has $\pi_x \gamma \leq \pi_x \gamma'$.

Not only are the four vertices in the potential inequality given by limits, but also, necessarily, the four legs are determined by the one-dimensional property of limits. So $\pi_x \gamma$ is given by composing $\pi_x : C(P_f P_f \int)(H)(m) \longrightarrow H_x m$ with the map from $H_x m$ to $K_x n$ that generates γ. We denote the latter map by γ_x. The same applies to γ'. So it suffices for us to show that, for each $x \in D$, one has $\gamma_x \leq \gamma'_x$, with the order on the products $K_x n$ taken pointwise. We do that by potentially transfinite induction.

For the base case, $K_x n = K(n)$, $H_x m = H(m)$, and γ_x and γ'_x are the top-right and bottom-left legs of the following inequality, which holds by lax

naturality of α:

$$
\begin{array}{ccc}
H(m) & \xrightarrow{\;\alpha(m)\;} & K(m) \\
{\scriptstyle H(f)}\big\downarrow & \geq & \big\downarrow{\scriptstyle K(f)} \\
H(n) & \xrightarrow[\alpha(n)]{} & K(n)
\end{array}
$$

Thus for $x = 0$, α is a lax transformation from H_x to K_x that we denote by α_x.

For a successor ordinal, we may assume that $\alpha_x : H_x \Rightarrow K_x$ is lax natural, and we need to see that we have an inequality in the following diagram:

$$
\begin{array}{ccc}
H(m) \times (P_f P_f \int H_x)(m) & \xrightarrow{\;\alpha_{x+1}(m)\;} & K(m) \times (P_f P_f \int K_x)(m) \\
{\scriptstyle H(f) \times (P_f P_f \int H_x)(f)}\big\downarrow & \geq & \big\downarrow{\scriptstyle K(f) \times (P_f P_f \int K_x)(f)} \\
H(n) \times (P_f P_f \int H_x)(n) & \xrightarrow[\alpha_{x+1}(n)]{} & K(n) \times (P_f P_f \int K_x)(n)
\end{array}
$$

where $\alpha_{x+1} = \alpha \times P_f P_f \int \alpha_x$. By Proposition 2, $\int \alpha_x$ is lax natural as $\int$ is functorial; composition with P_f is also functorial; and so, α_{x+1}, being a product of two lax natural transformations, is itself lax natural, yielding the inequality.

For a limit ordinal, we have the inequality we seek by a simple application of the two-dimensional property applied to the limit of the partial diagram for $C(P_f P_f \int)(K)(n)$ up to stage x.

Thus we have defined $C(P_f P_f \int)$ as an endofunctor on $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$. The counit and comultiplication of $C(P_f P_f \int)$ are also constructed by restricting to $[Inj, Poset]$, using the classical result for the strict setting [32], then using the two-dimensional property of the limit to verify coherence in regard to arbitrary maps in $\mathcal{L}_\Sigma$.

The construction of $\overline{p}$ is given pointwise, with it following from its coinductive construction that it yields the coinductive trees as required: because of our construction of $\int At$ to take the place At in Theorem 3, the image of p lies in $P_f P_f \int At$.

The lax naturality in respect to general maps $f : m \longrightarrow n$ means that a substitution applied to an atom $A(x_1, \ldots, x_n) \in At(n)$, i.e., application of the function $At(f)$ to $A(x_1, \ldots, x_n)$, followed by application of $\overline{p}$, i.e., taking the coinductive tree for the substituted atom, or application of the function $(C(P_f P_f \int)At)f)$ to the coinductive tree for $A(x_1, \ldots, x_n)$ potentially yield different trees: the former substitutes into $A(x_1, \ldots, x_n)$, then takes its coinductive tree, while the latter applies a substitution to each node of the coinductive tree for $A(x_1, \ldots, x_n)$, then prunes to remove redundant branches.

Example 9. *Extending Example 8, consider* $\mathtt{connected(x,y)} \in At(2)$. *In expressing GC as a map* $p : At \longrightarrow P_f P_f \int At$ *in Example 8, we put*

$$p_2(\mathtt{connected(x,y)}) = \{\{\mathtt{edge(x,z)}, \mathtt{connected(z,y)}\}\}$$

Accordingly, $\overline{p}_2(\mathtt{connected(x,y)})$ *is the coinductive tree for* $\mathtt{connected(x,y)}$, *thus the infinite tree generated by repeated application of the same clause modulo renaming of variables.*

If we substitute x for y in the coinductive tree, i.e., apply the function $(C(P_f P_f \int)At)(x,x)$ *to it (see the definition of* L_Σ *at the start of Section 4 and observe that* (x,x) *is a 2-tuple of terms, thus an arrow from 1 to 2), we obtain the same tree but with y systematically replaced by x. However, if we substitute x for y in* $\mathtt{connected(x,y)}$, *i.e., apply the function* $At(x,x)$ *to it, we obtain* $\mathtt{connected(x,x)} \in At(1)$, *whose coinductive tree has additional branching as the first clause of GC, i.e.,* $\mathtt{connected(x,x)} \leftarrow$ *may also be applied.*

In contrast to this, we have strict naturality with respect to injections: for example, an injection $i : 2 \longrightarrow 3$ *yields the function* $At(i) : At(2) \longrightarrow At(3)$ *that, modulo renaming of variables, sends* $\mathtt{connected(x,y)} \in At(2)$ *to itself seen as an element of* $At(3)$, *and the coinductive tree for* $\mathtt{connected(x,y)}$ *is accordingly also sent by* $(C(P_f P_f \int)At)(i)$ *to itself seen as an element of* $(C(P_f P_f \int)At)(3)$.

Example 9 illustrates why, although the condition of strict naturality with respect to injections holds for $P_f P_f \int$, it does not hold for $Lax(\mathcal{L}_\Sigma^{op}, P_f P_f)$ in Example 6 as we did not model the clause

$$\mathtt{connected(x,y)} \leftarrow \mathtt{edge(x,z)}, \mathtt{connected(z,y)}$$

directly there, but rather modelled all substitution instances into all available variables.

Turning to the relationship between lax semantics and saturated semantics given in Section 5, we need to refine our construction of the right adjoint to the inclusion

$$[\mathcal{L}_\Sigma^{op}, Poset] \longrightarrow Lax(\mathcal{L}_\Sigma^{op}, Poset)$$

to give a construction of a right adjoint to the inclusion

$$[\mathcal{L}_\Sigma^{op}, Poset] \longrightarrow Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$$

As was the case in Section 5, such a right adjoint exists for general reasons as an example of the main result of [19]. An explicit construction of it arises by emulating the construction of Theorem 4. In the statement of Theorem 4, putting $C = \mathcal{L}_\Sigma^{op}$, we described a parallel pair of maps in $[\mathcal{L}_\Sigma^{op}, Poset]$ and constructed their inserter, the inserter being exactly the universal property corresponding to the laxness of the maps in $Lax(\mathcal{L}_\Sigma^{op}, Poset)$. Here, we use the same technique but with equaliser replacing inserter, to account for the equalities in

$Lax_{Inj}(C, Poset)$. Thus we take an equaliser of two variants of δ_1 and δ_2 seen as maps in $[\mathcal{L}_\Sigma^{op}, Poset]$ with domain $Ins(\delta_1, \delta_2)$

Again, one may use the same constructions of saturation and desaturation as before, i.e., use a right Kan extension. However, now, instead of applying the right Kan extension to $P_c P_f$ as in [2], one may apply it to $P_f P_f \int At$. This still generates countability as application of the right Kan extension generates countability, but the saturation construction is now applied to an inherently finitary map, $p : At \longrightarrow P_f P_f \int At$, rather than to a map with countability already built into it. So our account of existential variables allows a refinement, albeit a subtle one, to saturated semantics, in addition to refining lax semantics.

7. Semantics for variables in logic programs: local variables

The relationship between the semantics of logic programming we propose here and that of local state is yet to be explored fully, and we leave the bulk of it to future work. However, as explained in Section 6, the definition of $\int$ was informed by the semantics for local state in [26], and we have preliminary results that strengthen the relationship.

Proposition 3. *The endofunctor $\int(-)$ on $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$ canonically supports the structure of a monad, with unit $\eta_H : H \Rightarrow \int H$ defined, at n, by the id_n component $\rho_{id_n} : Hn \longrightarrow \int Hn$ of the colimiting cocone, and with multiplication $\mu_H : \int\int H \Longrightarrow \int H$ defined, at n, by observing that if $m = n + k$ and $p = m + l$, then $p = n + (k+l)$ with canonical injections j_k, j_l and j_{k+l} coherent with each other, and applying the doubly indexed colimiting property of $\int\int H$ to $\rho_{j_{k+l}} : H(p) \longrightarrow \int H(n)$,*

This bears direct comparison with the monad for local state in the case where one has only one value, as studied by Stark [34]. The setting is a little different. Stark does not consider maps in $\mathcal{L}_\Sigma$ or laxness, and his base category is Set rather than $Poset$. However, if one restricts our definition of $\int$ and the other data for the monad of Proposition 3 to $[Inj, Set]$, one obtains Stark's construction.

The monad for local state in [26] also extends Stark's construction but in a different direction: for local state, neither Inj nor Set is extended, but state, which is defined by a functor into Set is interpolated into the definition of the functor $\int$, which restricts to $[Inj, Set]$. That interpolation of state is closely related to our application of $P_f P_f$ to $\int$: just as the former gives rise to a monad for local state on $[Inj, Set]$, the latter bears the ingredients for a monad as follows.

Proposition 4. *For any endofunctor P on Set, here is a canonical distributive law*

$$\int P(-) \longrightarrow P \int (-)$$

of the endofunctor $P \circ -$ over the monad $\int$ on $[Inj, Set]$.

The canonicity of the distributive law arises as $\int$ is defined pointwise as a colimit, and the distributive law is the canonical comparison map determined by applying P pointwise to the colimiting cone defining $\int$.

The functor $P_f P_f$ does not quite satisfy the axioms for a monad [35] (see also [2]), but variants of $P_f P_f$, in particular $P_f M_f$, where M_f is the finite multiset monad on Set, do [35] (also see [2]). Putting $P = P_f M_f$, the distributive law of Proposition 4 respects the monad structure of $P_f M_f$, yielding a canonical monad structure on the composite $P_f M_f \int$.

The full implications of that are yet to be investigated, but, trying to emulate the analysis of local state in [26], we believe we have a natural set of operations and equations that generate the monad $P_f M_f \int$. That encourages us considerably towards the possibility of seeing the semantics for logic programming, both lax and saturated, as an example of a general semantics of local effects. We have not yet fully understood the significance of the specific combination of operations and equations generating the monad, but we are currently investigating it.

8. Conclusions and Further Work

Let P_f be the covariant finite powerset functor on Set. Then, to give a variable-free logic program P is equivalent to giving a $P_f P_f$-coalgebra structure $p : At \longrightarrow P_f P_f At$ on the set At of atoms in the program. Now let $C(P_f P_f)$ be the cofree comonad on $P_f P_f$. Then, the $C(P_f P_f)$-coalgebra $\overline{p} : At \longrightarrow C(P_f P_f)At$ corresponding to p sends an atom to the coinductive tree it generates. This fact is the basis for both our lax semantics and Bonchi and Zanasi's saturated semantics for logic programming.

Two problems arise when, following standard category theoretic practice, one tries to extend this semantics to model logic programs in general by extending from Set to $[\mathcal{L}_\Sigma^{op}, Set]$, where $\mathcal{L}_\Sigma$ is the free Lawevere theory generated by a signature Σ. The first is that the natural construction $p : At \longrightarrow P_f P_f At$ does not form a natural transformation, so is not a map in $[\mathcal{L}_\Sigma^{op}, Set]$.

Two resolutions were proposed to that: lax semantics [1], which we have been developing in the tradition of semantics for data refinement [21], and saturated semantics [2], which Bonchi and Zanasi have adopted. In this paper, we have shown that the two resolutions are complementary rather than competing, the first modelling the theorem-proving aspect of logic programming, while the latter models proof search.

In modelling theorem-proving, lax semantics led us to identify and develop the notion of coinductive tree. To express the semantics, we extended $[\mathcal{L}_\Sigma^{op}, Set]$ to $Lax(\mathcal{L}_\Sigma^{op}, Poset)$, the category of strict functors and lax transformations between them. We followed standard semantic practice in extending P_f from Set to $Poset$ and we postcomposed the functor $At : \mathcal{L}_\Sigma^{op} \longrightarrow Poset$ by $P_f P_f$. Bonchi and Zanasi also postcomposed At by $P_f P_f$, but then saturated. We showed that their saturation and desaturation constructions are generated exactly by starting from $Lax(\mathcal{L}_\Sigma^{op}, Poset)$ rather than from $[ob(\mathcal{L}_\Sigma)^{op}, Set]$ as they did, thus

unifying the underlying mathematics of the two developments, supporting their computational coherence.

The second problem mentioned above relates to existential variables, those being variables that appear in the antecedent of a clause but not in its head. The problem of existential clauses is well-known in the literature on theorem proving and within communities that use term-rewriting, TM-resolution or their variants. In TRS [29], existential variables are not allowed to appear in rewriting rules, and in type inference, the restriction to non-existential programs is common [31]. In logic programming, the problem of handling existential variables when constructing proofs with TM-resolution marks the boundary between its theorem-proving and problem-solving aspects.

Existential variables are not present in many logic programs, but they do occasionally occur in important examples, such as those developed by Sterling and Shapiro [27]. The problem for us was that, in the presence of existential variables, the natural model $p : At \longrightarrow P_f P_f At$ of a logic program might escape its codomain, i.e., $p_n(A)$ might not lie in $P_f P_f At(n)$ because of the new variables. On one hand, we want to model them, but on the other, the fact of the difficulty for us means that we have semantically identified the concept of existential variable, which is positive.

In this paper, we have resolved the problem by refining $Lax(\mathcal{L}_\Sigma^{op}, Poset)$ to $Lax_{Inj}(\mathcal{L}_\Sigma^{op}, Poset)$, insisting upon strict naturality for injections, and by refining the construction $P_f P_f At$ to $P_f P_f \int At$, thus allowing for additional variables in the tail of a clause in a logic program. That has allowed us to model coinductive trees for arbitrary logic programs, in particular those including existential variables. We have also considered the effect of such refinement on saturated semantics, where the right Kan extension generates countability but use of P_c is no longer required.

In order to refine $P_f P_f(-)$, we followed a technique developed in the semantics of local state [26]. That alerted us to the relationship between variables in logic programming with the use of worlds in modelling local state. So, as ongoing work, we are now relating our semantics for logic programming with that for local state. For the future, we shall continue to develop that, with the hope of being able to locate our semantics of logic programming within a general semantics for local effects.

Beyond that, a question that we have not considered semantically at all yet but which our applied investigations are encouraging is that of modelling recursion. There are fundamentally two kinds of recursion that arise in logic programming as there may be recursion in terms and recursion in proofs. For example, $\texttt{stream}(\texttt{scons}(x, y)) \leftarrow \texttt{stream}(y)$ is a standard (co-)recursive definition of infinite streams in logic programming literature. More abstractly, the following program P_1: $\texttt{p}(\texttt{f}(x)) \leftarrow \texttt{p}(x)$ defines an infinite data structure p with constructor f. For such cases, proofs given by coinductive trees will be finite. An infinite sequence of (finite) coinductive trees will be needed to approximate the intended operational semantics of such a program, as we discuss in detail in [1, 36]. In contrast, there are programs like P_2: $\texttt{p}(x) \leftarrow \texttt{p}(x)$ or P_3: $\texttt{p}(x) \leftarrow \texttt{p}(\texttt{f}(x))$ that are also recursive, but additionally their proofs as given by

coinductive trees will have an infinite size.

In [37, 38] programs like P_1 were called productive, for producing (infinite) data, and programs like P_2 and P_3 – non-productive, for recursing without producing substitutions. The productive case amounts to a loop in the Lawvere theory $\mathcal{L}_\Sigma$ [39], while the non-productive case amounts to repetition within a coinductive tree, possibly modulo a substitution. This paper gives a close analysis of trees. That should set the scene for investigation of recursion, as it seems likely to yield more general kinds of graph that arise by identifying loops in $\mathcal{L}_\Sigma$ and by equating nodes in trees.

The lax semantics we presented here has recently inspired investigations into the importance of TM-resolution (as modelled by the coinductive trees) in programming languages. In particular, TM-resolution is used in type class inference in Haskell [31]. In [13, 40] we showed applications of nonterminating TM-resolution in Haskell type classes. We plan to continue looking for applications of this work in programming language design beyond logic programming.

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