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Fundamental groups of toroidal compactifications

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Abstract

We compute the fundamental group of a toroidal compactification of a Hermitian locally symmetric space D/Γ, without assuming either that Γ is neat or that it is arithmetic. We also give bounds for the first Betti number.

Many important complex algebraic varieties can be described as locally symmetric varieties. Examples include modular curves $\mathbb{H}/\Gamma$, where $\mathbb{H}$ is the upper half-plane and $\Gamma < \mathrm{PSL}(2, \mathbb{Z})$; classifying spaces for Hodge structures or (in cases where a Torelli theorem holds) moduli spaces of polarised varieties, such as moduli of abelian varieties and of K3 surfaces; and special surfaces, such as Hilbert modular surfaces.

Locally symmetric varieties are in general non-compact, and we want to be able to compactify them and to study the geometry of the compactifications, especially the birational geometry, which does not depend on the choice of compactification. We work with toroidal compactifications as described in [AMRT].

Two basic birational invariants of a compact complex manifold X are the Kodaira dimension $\kappa(X)$ and the fundamental group $\pi_1(X)$. There is an extensive literature on computing Kodaira dimensions of specific locally symmetric varieties, which is usually very difficult.

Computing the fundamental group is easier, but there are some gaps in the literature which we aim to fill. We study the fundamental group of a toroidal compactification $(D/\Gamma)'_\Sigma$ of a non-compact, not necessarily arithmetic quotient D/Γ by a lattice Γ. In general this is not a manifold, but

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it is normal and can be chosen to have only quotient singularities. By [Ko, Sect 7] these do not affect the fundamental group.

The main result of the article is Theorem 4.3, describing $\pi_1((D/\Gamma)'_\Sigma)$ as a quotient of the lattice Γ.

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1 Background

In this section we explain the background to the problem and establish some terminology and notation.

A *symmetric space* of non-compact type is a quotient $D = G/K$ of a connected non-compact semisimple Lie group G, assumed to be the real points of a linear algebraic group defined over $\mathbb{Q}$, by a connected maximal compact subgroup K of G. If the centre of K is not discrete, then D carries a Hermitian structure and hence the structure of a complex manifold, in fact a Kähler manifold [He, Theorem VIII.6.1.]).

By a *lattice* in G we mean a discrete subgroup of G of finite covolume with respect to Haar measure. A lattice Γ is said to be *arithmetic* if $\Gamma \cap G(\mathbb{Z})$ is of finite index in both Γ and $G(\mathbb{Z})$. It is said to be *neat* if the subgroup of $\mathbb{C}^*$ generated by all eigenvalues of elements of Γ is torsion free.

A *locally symmetric variety* is the quotient of a Hermitian symmetric space D by a lattice $\Gamma < G$. If D/Γ is compact then Γ is said to be *cocompact* or *uniform*. Non-uniform lattices are very common, however, and it is this case that we are concerned with. By [BB] (for Γ arithmetic) and [Mok] these quotients are always algebraic varieties, not just complex analytic spaces.

Toroidal compactifications are constructed and described in detail in [AMRT], and it is shown in [Mok] that the construction applies to non-arithmetic lattices as well. In fact, for all $D = G/K$ except the complex ball $\mathbb{B}^n = \mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$, Margulis [Ma] showed that an irreducible lattice $\Gamma < G$ is necessarily arithmetic. An extensive reference on toroidal and other compactifications of locally symmetric spaces D/Γ is the monograph [BJ] of Borel and Ji.

The first results on the fundamental group of a smooth compactification of a locally symmetric space concerned Siegel modular 3-folds, where $G = \mathrm{Sp}(2,\mathbb{R})$. These are moduli spaces of abelian surfaces and some such cases were studied in [HK], in [Kn] and in [HSa]. More generally, the fundamental group of a toroidal compactification of an arbitrary Hermitian locally symmetric variety is studied in [Sa].

Write $G = G_1 \times \cdots \times G_s$ as a product of simple factors. We say that a parabolic proper subgroup $Q = Q_1 \times \cdots \times Q_s < G$ is *semimaximal* if each Q_j is either G_j or a maximal parabolic subgroup of G_j. Denote by

MPar$_\Gamma$ the set of Γ-rational semimaximal parabolic subgroups. It is shown in [Sa] that if $\Gamma < G$ is a neat arithmetic non-uniform lattice $\Gamma < G$, then a toroidal compactification $(D/\Gamma)'_\Sigma$ satisfies $\pi_1(D/\Gamma)'_\Sigma = \Gamma/\Upsilon$, where Υ is the subgroup of Γ generated by the centres of the unipotent radicals of all $Q \in \text{MPar}_\Gamma$. Moreover, in [GHS] it is shown that there is a surjective group homomorphism $\Gamma \to \pi_1((D/\Gamma)'_\Sigma)$, whose kernel contains all $\gamma \in \Gamma$ with a fixed point on D.

Let Λ be the subgroup of Γ generated by all $\gamma \in \Gamma \cap L_Q$ with $\gamma^k \in \Gamma \cap A_Q$ for some $k \in \mathbb{N}$ and some $Q \in \text{MPar}_\Gamma$ with split component A_Q and Levi subgroup $L_Q = A_Q \rtimes M_Q$: from this definition, Λ is normal in Γ.

Our main result, Theorem 4.3, is that $\pi_1((D/\Gamma)'_\Sigma) = \Gamma/\Lambda\Upsilon$ for an arbitrary (not necessarily arithmetic) non-uniform lattice $\Gamma < G$.

Here is a synopsis of the paper. Section 2 introduces some notation and terminology and describes the structure of Γ-rational parabolic subgroups, largely following [BJ]. Section 3 describes the toroidal compactifications $(D/\Gamma)'_\Sigma$ and their coverings $(D/\Gamma_o)'_\Sigma$ for normal subgroups (not necessarily lattices) $\Gamma_o \lhd \Gamma$ containing Υ. Section 4 comprises the main results of the article. We show in Proposition 4.4 that any element $\gamma\Upsilon \in \Gamma/\Upsilon$ with a fixed point on $(D/\Upsilon)'_\Sigma$ has a representative $\gamma \in \Gamma$ with $\gamma \in \Gamma \cap L_Q$ and $\gamma^k \in \Gamma \cap A_Q$ for some $Q \in \text{MPar}_\Gamma$ and $k \in \mathbb{N}$. This suffices to prove Theorem 4.3, and from that we deduce bounds on the first Betti numbers in Subsection 4.2.

2   Parabolic subgroups

We collect here some properties of Hermitian symmetric spaces $D = G/K$ of non-compact type and parabolic subgroups Q of G. For more details see [BJ, Chapter 1].

2.1   Langlands decomposition of a parabolic subgroup

Any parabolic subgroup Q of G has a *Langlands decomposition* [BJ, Equation (I.1.10)]

$$Q = N_Q A_Q M_Q$$

where N_Q is the *unipotent radical* of Q. We write $L_Q = A_Q M_Q$, the *Levi subgroup* of Q, and $R_Q = N_Q \rtimes A_Q$, the *solvable radical* of Q. The subgroup A_Q is called the *split component* of Q, and M_Q is a semisimple complement of R_Q. All these groups are uniquely defined once we choose a maximal compact subgroup K of G.

We denote by U_Q the centre of the unipotent radical N_Q of Q. Since N_Q is a 2-step nilpotent group, i.e. $[[N_Q, N_Q], N_Q] = 0$, we have $U_Q = [N_Q, N_Q]$, the commutator subgroup. We may identify U_Q with its Lie algebra $\mathfrak{u}_Q \cong \mathbb{R}^m$, for $m = \dim_\mathbb{R} U_Q$. The quotient $V_Q = N_Q/U_Q$ is also an abelian group,

naturally isomorphic to $\mathbb{C}^n$ [BJ, (III.7.9)] and $N_Q = U_Q \rtimes V_Q$ is a semi-direct product of U_Q and V_Q.

The semi-simple complement M_Q of the solvable radical R_Q of Q is a product $M_Q = G'_{Q,l} \times G_{Q,h}$ of semisimple groups $G'_{Q,l}$, $G_{Q,h}$ of noncompact type [BJ, (III.7.8)].

This gives us the *refined Langlands decomposition*

$$Q = [(U_Q \rtimes V_Q) \rtimes A_Q] \rtimes (G'_{Q,l} \times G_{Q,h}) \tag{1}$$

of an arbitrary parabolic subgroup Q of G. Note also that $G = QK$.

The group $A_Q \cong (\mathbb{R}^s_{>0}, .)$ is an $\mathbb{R}$-split torus of G of dimension $s \leq r$, where $r = \mathrm{rk}_{\mathbb{R}} G$ is the real rank of G.

The symmetric space D has an embedding in a space $\check{D}$, the compact dual, on which G acts. The topological boundary of D then decomposes into complex analytic boundary components corresponding to parabolic subgroups Q: namely, Q is the normaliser of the boundary component $F(Q)$. See [BJ, Proposition I.5.28] or [AMRT, Proposition III.3.9.] for details.

If $F(P) \subseteq \overline{F(Q)}$ then $U(P) \supseteq U(Q)$ by [AMRT, Theorem III.4.8(i)].

2.2 Horospherical decomposition

For any parabolic subgroup we have $G = QK$ [BJ, (I.1.20)], so Q acts transitively on D; moreover, $Q \cap K = M_Q \cap K$. As a result, the refined Langlands decomposition (1) of Q induces the *refined horospherical decomposition*

$$D = U_Q \times V_Q \times A_Q \times D'_{Q,l} \times D_{Q,h} \tag{2}$$

of D with $D'_{Q,l} = G'_{Q,l}/G'_{Q,l} \cap K$ and $D_{Q,h} = G_{Q,h}/G_{Q,h} \cap K$ [BJ, Lemma III.7.9]). The equality in (2) is a real analytic diffeomorphism. The factors $D_{Q,h} \cong F(Q)$ and $D'_{Q,l}$ are respectively Hermitian and Riemannian symmetric spaces of noncompact type.

The parabolic group $Q = (N_Q \rtimes A_Q) \rtimes (G'_{Q,l} \times G_{Q,h})$ acts on the Hermitian symmetric space $D = N_Q \times A_Q \times D'_{Q,l} \times D_{Q,h}$ by the rule

$$(n_0, a_0, g'_0, g_0)(n, a, z', z) = ((a_0, g'_0, g_0)^{-1} n_0 (a_0, g'_0, g_0)n, a_0 a, g'_0 z', g_0 z) \tag{3}$$

for $(n_0, a_0, g'_0, g_0) \in (N_Q \rtimes A_Q) \rtimes (G'_{Q,l} \times G_{Q,h})$ and $(n, a, z', z) \in N_Q \times A_Q \times D'_{Q,l} \times D_{Q,h}$ (cf. [BJ, Equation (I.1.11)]).

We need more detail about this action. Since $U_Q = [N_Q, N_Q]$, we see that U_Q is a normal subgroup of Q, and the action of $A_Q \rtimes (G'_{Q,l} \times G_{Q,h})$ on N_Q and U_Q descends to $V_Q \cong N_Q/U_Q$. If $\alpha_0 = (a_0, g'_0, g_0) \in A_Q \rtimes (G'_{Q,l} \times G_{Q,h})$ and $n = (u, v) \in U_Q \rtimes V_Q$ then the first term in the right-hand side of (3) is given by

$$\alpha_0^{-1}(u_0, v_0)\alpha_0(u, v) = (\alpha_0^{-1} u_0 \alpha_0 + u, \alpha_0^{-1} v_0 \alpha_0 + v).$$

Altogether, the Q-action on D is given by

$$(u_0, v_0, \alpha_0)(u, v, \zeta', \zeta) = (\alpha_0^{-1} u_0 \alpha_0 + u, \alpha_0^{-1} v_0 \alpha_0 + v, a_0 a, g'_0 \zeta', g_0 \zeta). \tag{4}$$

2.3 Siegel domains

In [P-S] Pyatetskii-Shapiro realises the Hermitian symmetric spaces $D = G/K$ of noncompact type as Siegel domains of third kind. These are families of open cones, parametrised by products of complex Euclidean spaces and Hermitian symmetric spaces of noncompact type.

In the refined horospherical decomposition (2), the A_Q-orbit

$$C_Q = A_Q D'_{Q,l} = \{(a, \zeta') \mid a \in A_Q, \zeta' \in D'_{Q,l}\} \tag{5}$$

of the Riemannian symmetric space $D'_{Q,l}$ is an open, strongly convex cone in $U_Q \cong \mathbb{R}^m$ [BJ, Lemma III.7.7]. Note that the reductive group $G_{Q,l} = A_Q \rtimes G'_{Q,l}$ acts transitively on C_Q since $C_Q = G_{Q,l}/G_{Q,l} \cap K = G_{Q,l}/G'_{Q,l} \cap K$.

We embed C_Q in the complexification $U_Q \otimes_{\mathbb{R}} \mathbb{C} \cong (\mathbb{C}^m, +)$ of U_Q as a subset $iC_Q \subset iU_Q$ with pure imaginary components. Combining with (2), one obtains a real analytic diffeomorphism of D onto the product

$$(U_Q + iC_Q) \times V_Q \times D_{Q,h}. \tag{6}$$

which will be called the Siegel domain realisation of D associated with Q. See [AMRT] for the relation between (6) and the classical Siegel domain presentation of D.

In these coordinates, the action of Q (with notation as in (4)) is given by

$$\begin{aligned}
(u_0, v_0, a_0, g'_0, g_0)&(u + i(a, \zeta'), v, \zeta) \\
&= ((\alpha_0^{-1} u_0 \alpha_0 + u) + i(a_0 a, g'_0 \zeta'), \alpha_0^{-1} v_0 \alpha_0 + v, g_0 \zeta)
\end{aligned} \tag{7}$$

where $(u + i(a, \zeta'), v, \zeta) \in (U_Q + iC_Q) \times V_Q \times D_{Q,h}$.

3 Toroidal compactifications

We recall briefly enough detail on toroidal compactification for our immediate purposes: for full details we refer to [AMRT].

3.1 Admissible fans and collections

Suppose that $Q \in \mathrm{MPar}_\Gamma$: then $\Upsilon_Q = \Gamma \cap U_Q \cong \mathbb{Z}^m$ is a lattice in $U_Q \cong \mathbb{R}^m$. We say that a closed polyhedral cone $\sigma \subset U_Q$ is Υ_Q-rational if $\sigma = \mathbb{R}_{\geq 0} u_1 + \cdots + \mathbb{R}_{\geq 0} u_s$ for some $u_i \in \Upsilon_Q$.

A *fan* (see [Fu]) $\Sigma(Q)$ is a collection of closed polyhedral cones in U_Q such that any face of a cone in $\Sigma(Q)$ is also in $\Sigma(Q)$ and any two cones in $\Sigma(Q)$ intersect in a common face. It is Υ_Q-*rational* if all cones in $\Sigma(Q)$ are Υ_Q-rational.

The fan $\Sigma(Q)$ in U_Q is said to be Γ-*admissible* if it is Υ_Q-rational, it decomposes C_Q (that is, $C_Q \subseteq \bigcup_{\sigma \in \Sigma(Q)} \sigma$) and $\Gamma_{Q,l} = \Gamma \cap G_{Q,l}$ acts on $\Sigma(Q)$ with only finitely many orbits.

The lattice Γ acts on MPar_Γ by conjugation. We say that a family $\Sigma = \{\Sigma(Q)\}_{Q \in \mathrm{MPar}_\Gamma}$ of Γ-admissible fans $\Sigma(Q)$ is a Γ-*admissible family* if:

(i) $\gamma\Sigma(Q) = \Sigma(Q^\gamma)$ for all $\gamma \in \Gamma$, $Q \in \mathrm{MPar}_\Gamma$ and

(ii) $\Sigma(Q) = \{\sigma \cap U_Q \mid \sigma \in \Sigma(P)\}$ whenever $F(P) \subseteq \overline{F(Q)}$, for $P, Q \in \mathrm{MPar}_\Gamma$.

3.2 Partial compactification at a cusp

For $Q \in \mathrm{MPar}_\Gamma$, the quotient $\mathbb{T}(Q) = (U_Q \otimes_\mathbb{R} \mathbb{C})/\Upsilon_Q \cong (\mathbb{C}^*)^m$ is an algebraic torus over $\mathbb{C}$.

A Γ-admissible fan $\Sigma(Q)$ determines a toric variety $X_{\Sigma(Q)}$ that includes $\mathbb{T}(Q)$ as a dense Zariski-open subset. More precisely, $X_{\Sigma(Q)}$ is the disjoint union of $\mathbb{T}(Q)$ with the quotients $\mathbb{T}(Q)_\sigma$ of $\mathbb{T}(Q)$ by the complex algebraic tori $\mathrm{Span}_\mathbb{C}(\sigma)/(\mathrm{Span}_\mathbb{C}(\sigma) \cap \Upsilon_Q)$, generated by the cones $\sigma \in \Sigma(Q)$.

Bearing in mind that $(U_Q + iC_Q)/\Upsilon_Q$ is an open subset of $\mathbb{T}(Q)$, we take the closure $\overline{(U_Q + iC_Q)/\Upsilon_Q}$ of $(U_Q + iC_Q)/\Upsilon_Q$ in $X_{\Sigma(Q)}$ and define $Y_{\Sigma(Q)}$ as the interior of $\overline{(U_Q + iC_Q)/\Upsilon_Q}$ in $X_{\Sigma(Q)}$.

The Siegel domain presentation (6) of D associated with Q provides a real analytic diffeomorphism

$$D/\Upsilon_Q = (U_Q + iC_Q)/\Upsilon_Q \times V_Q \times D_{Q,h},$$

and the Γ-admissible fan $\Sigma(Q)$ defines a partial compactification

$$(D/\Upsilon_Q)_{\Sigma(Q)} = Z_{\Sigma(Q)} = Y_{\Sigma(Q)} \times V_Q \times D_{Q,h}. \tag{8}$$

By subdividing $\Sigma(Q)$ we may, and henceforth do, assume that $X_{\Sigma(Q)}$ and $Y_{\Sigma(Q)}$ are smooth: see [Fu].

To describe the Q-action on $Z_{\Sigma(Q)}$, consider the Υ_Q-covering map

$$\epsilon_Q \colon D = (U_Q + iC_Q) \times V_Q \times D_{Q,h} \longrightarrow D/\Upsilon_Q = (U_Q + iC_Q)/\Upsilon_Q \times V_Q \times D_{Q,h},$$

given in the notation of (5) and (7) by

$$\epsilon_Q(u + i(a, \zeta'), v, \zeta) = \big(\mathbf{e}_Q(u + i(a, \zeta')), v, \zeta\big),$$

where $\mathbf{e}_Q \colon U_Q \otimes \mathbb{C} \to \mathbb{T}(Q)$ is the canonical map with kernel Υ_Q. If we identify Υ_Q with $\mathbb{Z}^m$ then we can identify $\mathbf{e}_Q$ with exponentiation, i.e. $\mathbf{e}_Q(z_1, \ldots, z_m) = (e^{2\pi i z_1}, \ldots, e^{2\pi i z_m})$ for $(z_1, \ldots, z_m) \in \mathbb{C}^m$. According to (7), the action of $(u_0, v_0, \alpha_0) \in Q$ with $u_0 \in U_Q$, $v_0 \in V_Q$, $\alpha_0 = (a_0, g'_0, g_0) \in A_Q \rtimes (G'_{Q,l} \times G_{Q,h})$ on $D/\Upsilon_Q = \epsilon_Q(D)$ is by the rule

$$\begin{aligned}
(u_0, v_0, &a_0, g'_0, g_0) \big(\mathbf{e}_Q(u + i(a, \zeta')), v, \zeta\big) \\
&= \big(\mathbf{e}_Q((\alpha_0^{-1} u_0 \alpha_0) + u + i(a_0 a, g'_0 \zeta')), \alpha_0^{-1} v_0 \alpha_0 + v, g_0 \zeta\big)
\end{aligned} \tag{9}$$

with $\alpha_0 = (a_0, g'_0, g_0)$. This Q-action extends by continuity to $Z_{\Sigma(Q)}$.

3.3 The gluing maps

For $P, Q \in \mathrm{MPar}_\Gamma$ with $F(P) \subseteq \overline{F(Q)}$, we are going to describe explicitly the holomorphic map $\mu_P^Q \colon Z_{\Sigma(Q)} \longrightarrow Z_{\Sigma(P)}$ of [AMRT, Lemma III.5.4].

According to [AMRT, Theorem III.4.8], U_Q is an $\mathbb{R}$-linear subspace of U_P. Therefore $\Upsilon_Q < \Upsilon_P$ and the identity map

$$\mathrm{id}_D \colon D = (U_Q + iC_Q) \times V_Q \times D_{Q,h} \longrightarrow D = (U_P + iC_P) \times V_P \times D_{P,h}$$

induces a holomorphic covering

$$\mu_P^Q \colon D/\Upsilon_Q = (U_Q + iC_Q)/\Upsilon_Q \times V_Q \times D_{Q,h} \longrightarrow D/\Upsilon_P$$

given by $\mu_P^Q(\Upsilon_Q x) = \Upsilon_P x$.

The inclusions $U_Q \otimes \mathbb{C} \subset U_P \otimes \mathbb{C}$ and $\Upsilon_Q < \Upsilon_P$ induce a homomorphism $\mu_{P,1}^Q \colon \mathbb{T}(Q) \longrightarrow \mathbb{T}(P)$, which extends to $\mu_{P,1}^Q \colon X_{\Sigma(Q)} \longrightarrow X_{\Sigma(P)}$, mapping $Y_{\Sigma(Q)}$ into $Y_{\Sigma(P)} \subset \overline{(U_P + iC_P)/\Upsilon_P}$. In this way, one obtains a holomorphic gluing map

$$\mu_P^Q \colon Z_{\Sigma(Q)} = Y_{\Sigma(Q)} \times V_Q \times D_{Q,h} \longrightarrow Y_{\Sigma(P)} \times V_P \times D_{P,h} = Z_{\Sigma(P)},$$

given by

$$\mu_P^Q \left(\lim_{t \to \infty} (y_t, v, z) \right) = \lim_{t \to \infty} \mu_P^Q(y_t, v, z) = \lim_{t \to \infty} (y_t + \Upsilon_P/\Upsilon_Q, v, z)$$

where $y_t \in (U_Q + iC_Q)/\Upsilon_Q$ for $t \in \mathbb{R}$ tends to some point $\lim_{t \to \infty} y_t \in Y_{\Sigma(Q)}$. From this definition, μ_Q^Q is the identity on $Z_{\Sigma(Q)} = (D/\Upsilon_Q)_{\Sigma(Q)}$.

3.4 Toroidal compactifications and coverings

We recall the construction of a toroidal compactification $(D/\Gamma)'_\Sigma$ of a locally symmetric variety D/Γ, associated with a Γ-admissible family $\Sigma = \{\Sigma(Q)\}_{Q \in \mathrm{MPar}_\Gamma}$ of fans $\Sigma(Q)$ in U_Q. In the notation of subsection 3.2, consider the disjoint union $\coprod_{Q \in \mathrm{MPar}_\Gamma} Z_{\Sigma(Q)}$.

We denote by Υ the subgroup of Γ generated by Υ_Q for all $Q \in \mathrm{MPar}_\Gamma$. Suppose that Γ_o is a normal subgroup of Γ containing Υ. Its action on D induces an equivalence relation $\sim_{\Gamma_o}$ on $\coprod_{Q \in \mathrm{MPar}_\Gamma} Z_{\Sigma(Q)}$, as in the proof of [Sa, Theorem 2.1]: for $\Gamma_o = \Gamma$ it is described in [AMRT, III.5]. Let $z_1 \in Z_{\Sigma(Q_1)}$ and $z_2 \in Z_{\Sigma(Q_2)}$: then $z_1 \sim_{\Gamma_o} z_2$ if there exist $\gamma \in \Gamma_o$, $Q \in \mathrm{MPar}_\Gamma$ and $z \in Z_{\Sigma(Q)}$, such that $\overline{F(Q)} \supseteq F(Q_1)$, $\overline{F(Q)} \supseteq F(Q_2^\gamma)$, $\mu_{Q_1}^Q(z) = z_1$ and $\mu_{Q_2^\gamma}^Q(z) = \gamma z_2$.

Then we put

$$(D/\Gamma_o)'_\Sigma = \left(\coprod_{Q \in \mathrm{MPar}_\Gamma} Z_{\Sigma(Q)} \right) / \sim_{\Gamma_o}.$$

In [Sa] this is used to construct the (Γ/Υ)-Galois covering $(D/\Upsilon)'_\Sigma$ of $(D/\Gamma)'_\Sigma$ and show that $(D/\Upsilon)'_\Sigma$ is a simply connected complex analytic space. Notice that Γ_o is not required to be a lattice, and that $(D/\Upsilon)'_\Sigma$ is not compact.

In the proof of [Sa, Theorem 1.5] it is shown that $Z_{\Sigma(Q)}$, which are diffeomorphic to $Y_{\Sigma(Q)} \times V_Q \times D_{Q,h}$ for all $Q \in \mathrm{MPar}_\Gamma$, are simply connected. Further, the proof of [Sa, Theorem 2.1] establishes that the natural coverings $D/\Upsilon_Q \to D/\Upsilon$ extend to open holomorphic maps $\pi^U_{\Sigma(Q)} \colon Z_{\Sigma(Q)} \to (D/\Upsilon)'_\Sigma$, which are biholomorphic onto their images.

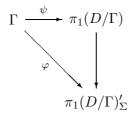# 4 The fundamental group and first Betti number

4.1 The fundamental group

We begin by stating two theorems that summarise the results from [Sa] and [GHS] on the fundamental group of a toroidal compactification $(D/\Gamma)'_\Sigma$ of a quotient of $D = G/K$ by an arithmetic lattice $\Gamma < G$.

Theorem 4.1 *[Sa, Corollary 1.6, Theorem 2.1] Let $D = G/K$ be a Hermitian symmetric space and let Γ be a non-uniform arithmetic lattice in G. Then the fundamental group $\pi_1((D/\Gamma)'_\Sigma)$ of a toroidal compactification $(D/\Gamma)'_\Sigma$ of D/Γ is a quotient group of Γ/Υ. In particular, if Γ is a neat arithmetic non-uniform lattice then $\pi_1((D/\Gamma)'_\Sigma) = \Gamma/\Upsilon$.*

The above is a more carefully stated version of the results in [Sa]. Recall that Υ is the group generated by Υ_Q for all $Q \in \mathrm{MPar}_\Gamma$. In a similar style, we define Λ to be the subgroup of Γ generated by all $\gamma \in \Gamma \cap L_Q$ such that $\gamma^k \in A_Q$ for some $k \in \mathbb{N}$ and some $Q \in \mathrm{MPar}_\Gamma$.

Theorem 4.2 *[GHS, Lemma 5.2, Proposition 5.3] Under the conditions of Theorem 4.1, there is a commutative diagram*

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\ \psi\ } & \pi_1(D/\Gamma) \\
& {}_{\varphi}\searrow & \downarrow \\
& & \pi_1(D/\Gamma)'_\Sigma
\end{array}
$$

of surjective group homomorphisms, such that $\ker\varphi$ and $\ker\psi$ contain all $\gamma \in \Gamma$ with a fixed point on D.

The following is the main result of the present paper. We use the notation from Subsection 2.1.

Theorem 4.3 *Let $D = G/K$ be a Hermitian symmetric space and let Γ be a non-uniform lattice in G. Then for any Γ-admissible family Σ, the toroidal compactification $(D/\Gamma)'_\Sigma$ has fundamental group*

$$\pi_1((D/\Gamma)'_\Sigma) = \Gamma/\Lambda\Upsilon.$$

Proof. According to [Sa], $(D/\Upsilon)'_\Sigma$ is a path connected simply connected locally compact topological space and Γ/Υ acts properly discontinuously on $(D/\Upsilon)'_\Sigma$ by homeomorphisms. More precisely, $\gamma\Upsilon \colon (\Upsilon q) \mapsto \Upsilon\gamma q$ defines a Γ/Υ-action on D/Υ, which extends continuously to $(D/\Upsilon)'_\Sigma$. The quotient space $(D/\Upsilon)'_\Sigma/(\Gamma/\Upsilon) = (D/\Gamma)'_\Sigma$ is the toroidal compactification of D/Γ associated with Σ. By a theorem of Armstrong [Ar]

$$\pi_1((D/\Gamma)'_\Sigma) = (\Gamma/\Upsilon)/(\Gamma/\Upsilon)_o^{\mathrm{Fix}}$$

where $(\Gamma/\Upsilon)_o^{\mathrm{Fix}}$ is the subgroup of Γ/Υ generated by elements $\gamma\Upsilon$ with a fixed point on $(D/\Upsilon)'_\Sigma$.

Theorem 4.3 therefore follows from Proposition 4.4, which establishes that $(\Gamma/\Upsilon)_o^{\mathrm{Fix}} = \Lambda\Upsilon/\Upsilon$. $\quad\square$

In order to describe the action of Γ/Υ on $(D/\Upsilon)'_\Sigma$ note that the Γ-action on MPar_Γ by conjugation determines holomorphic maps $\gamma \colon Z_{\Sigma(Q)} \to Z_{\Sigma(Q^\gamma)}$ for all $\gamma \in \Gamma$ and $Q \in \mathrm{MPar}_\Gamma$. Any $\gamma \in \Gamma$ transforms the $\sim_\Upsilon$-equivalence class of $z \in Z_{\Sigma(Q)}$ into the $\sim_\Upsilon$-equivalence class of γz, giving a biholomorphic map $\gamma \colon (D/\Upsilon)'_\Sigma \longrightarrow (D/\Upsilon)'_\Sigma$. By definition of $\sim_\Upsilon$, all $\gamma \in \Upsilon$ act trivially on $(D/\Upsilon)'_\Sigma$ and the Γ-action on $(D/\Upsilon)'_\Sigma$ reduces to a (Γ/Υ)-action

$$(\Gamma/\Upsilon) \times (D/\Upsilon)'_\Sigma \longrightarrow (D/\Upsilon)'_\Sigma,$$

given by

$$(\gamma\Upsilon)\pi_{\Sigma(Q)}^U(z) = \pi_{\Sigma(Q^\gamma)}^U(\gamma z) \tag{10}$$

for $\gamma\Upsilon \in \Gamma/\Upsilon$ and $z \in Z_{\Sigma(Q)}$.

Proposition 4.4 *In the notations from Theorem 4.3, a coset $\gamma_0\Upsilon \in \Gamma/\Upsilon$ has a fixed point on $(D/\Upsilon)'_\Sigma$ if and only if for some $k \in \mathbb{N}$ and some $Q \in \mathrm{MPar}_\Gamma$, there is a representative $\gamma \in \Gamma \cap L_Q$ of $\gamma_0\Upsilon = \gamma\Upsilon$ with $\gamma^k \in \Gamma \cap A_Q$. Hence the subgroup $(\Gamma/\Upsilon)_o^{\mathrm{Fix}}$ of Γ/Υ satisfies $(\Gamma/\Upsilon)_o^{\mathrm{Fix}} = \Lambda\Upsilon/\Upsilon$.*

Proof. We first prove the "only if" part of the statement.

We claim that if $\gamma_0\Upsilon \in \Gamma/\Upsilon$ has a fixed point on

$$(D/\Upsilon)'_\Sigma = \left(\coprod_{P \in \mathrm{MPar}_\Gamma} Z_{\Sigma(P)} \right) / \sim_\Upsilon$$

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then there exist $\gamma_1 \in \gamma_0 \Upsilon$ and $y \in Z_{\Sigma(Q)}$ for some $Q \in \mathrm{MPar}_\Gamma$, such that $\gamma_1 y = y$. That is, if a coset of Υ has a fixed point mod Υ then some representative of that coset has a fixed point "on the nose".

To prove this, notice that if $z_0 \sim_\Upsilon \gamma_0 z_0$ for some $z_0 \in Z_{\Sigma(P)}$, then there exist $Q_1 \in \mathrm{MPar}_\Gamma$, $z_1 \in Z_{\Sigma(Q_1)}$ and $u_1 \in \Upsilon$ such that $F(P) \subseteq \overline{F(Q_1)}$ and $\mu_P^{Q_1}(z_1) = z_0$, and $F(P^{u_1 \gamma_0}) \subseteq \overline{F(Q_1)}$ and $\mu_{P^{u_1 \gamma_0}}^{Q_1}(z_1) = u_1 \gamma_0 z_0$.

If $F(Q_1) = F(P)$, then $Q_1 = P$ and in (10) we have $\mu_{P^{u_1 \gamma_0}}^{Q_1} = \mu_P^{Q_1} = \mathrm{id}_{Z_{\Sigma(P)}}$ and $z_0 = u_1 \gamma_0 z_0$. Since Υ is a normal subgroup of Γ, we may take $\gamma_1 = u_1 \gamma_0 \in \Upsilon \gamma_0 = \gamma_0 \Upsilon$. In particular, this shows that the claim is true if $F(P)$ is of maximal dimension.

Now we conclude the proof of the claim by induction on $\mathrm{codim}\, F(P)$: suppose that the claim holds for all $P' \in \mathrm{MPar}_\Gamma$ with $\dim F(P') > \dim F(P)$, and take Q_1 as above. If $F(Q_1) = F(P)$ we are done. If not, then $z_0 \sim_\Upsilon z_1$ and $\gamma_0 z_0 \sim_\Upsilon \gamma_0 z_1$, because $\mu_{P^{\gamma_0}}^{Q_1^{\gamma_0}}(\gamma_0 z_1) = \gamma_0 z_0$. On the other hand, $\gamma_0 z_0 \sim_\Upsilon z_1$, so $z_1 \sim_\Upsilon \gamma_0 z_1$ because $\sim_\Upsilon$ is an equivalence relation. Thus $\gamma_0 \Upsilon$ has the fixed point $z_1 \in Z_{\Sigma(Q_1)}$ so the claim follows by taking $P' = Q_1$.

Suppose then that $\gamma_1 \in \Gamma$ has a fixed point $y \in Z_{\Sigma(Q)}$ for some $Q \in \mathrm{MPar}_\Gamma$. Then $y = \gamma_1 y \in Z_{\Sigma(Q^{\gamma_1})}$ implies that $Q^{\gamma_1} = Q$; but the parabolic subgroup Q of G coincides with its normaliser in G, so $\gamma_1 \in Q$. We may therefore use the Langlands decomposition of Q and write

$$\gamma_1 = (u_1, v_1, a_1, g_1', g_1) \in [(U_Q \rtimes V_Q) \rtimes A_Q] \rtimes (G_{Q,l}' \times G_{Q,h}).$$

As above we take

$$\alpha_1 = (a_1, g_1', g_1) \in L_Q = A_Q \rtimes (G_{Q,l}' \times G_{Q,h}).$$

Any element of $X_{\Sigma(Q)}$ may be written as a limit of elements of $\mathbb{T}(Q)$, as $\lim_{t \to \infty} (\mathbf{e}_Q(u_t + ix_t))$ and if the element is in $Y_{\Sigma(Q)}$ then we may take $x_t \in C_Q$. So

$$y = (\lim_{t \to \infty} \mathbf{e}_Q(u_t + ix_t), v, \zeta) \in Y_{\Sigma(Q)} \times V_Q \times D_{Q,h}.$$

Then by (9) and the continuity of the Q-action on $(D/\Upsilon_Q)_{\Sigma(Q)}$

$$\gamma_1 y = (\lim_{t \to \infty} \mathbf{e}_Q((\alpha_1^{-1} u_1 \alpha_1) + u_t + i(a_1, g_1')x_t), \alpha_1^{-1} v_1 \alpha_1 + v, g_1 \zeta) = y$$

Comparing the V_Q coordinates gives $\alpha_1^{-1} v_1 \alpha_1 = 0 \in V_Q \cong \mathbb{C}^n$ and hence $v_1 = 0$. From the last component we get $g_1 \zeta = \zeta$.

From the first component we get the equation in $X_{\Sigma(Q)}$

$$\lim_{t \to \infty} \mathbf{e}_Q(\alpha_1^{-1} u_1 \alpha_1 + u_t + i(a_1, g_1')x_t) = \lim_{t \to \infty} \mathbf{e}_Q(u_t + ix_t)$$

and this holds exactly when $\alpha_1^{-1} u_1 \alpha_1 \in \Upsilon_Q = \ker \mathbf{e}_Q$ and $\lim\limits_{t\to\infty} i(a_1, g_1') x_t = \lim\limits_{t\to\infty} i x_t$. Therefore we may take $\gamma' = (-\alpha_1^{-1} u_1 \alpha_1, 0, \mathrm{id})$ and $\gamma = \gamma_1 \gamma' \in \gamma_0 \Upsilon$, and we compute

$$
\begin{aligned}
\gamma &= (u_1, 0, \alpha_1)(-\alpha_1^{-1} u_1 \alpha_1, 0, \mathrm{id}) \\
&= (\alpha_1^{-1} u_1 \alpha_1 - \alpha_1^{-1} u_1 \alpha_1, \alpha_1^{-1} 0 \alpha_1 + 0, \alpha_1) \\
&= (0, 0, \alpha_1) \in \Gamma \cap L_Q.
\end{aligned}
$$

The remaining assertion of the "only if" part is that γ is torsion mod A_Q.

If γ has a fixed point y and $y \in D/\Upsilon_Q$ then γ belongs to the compact stabiliser of y in the isometry group G of D and hence γ is torsion. If $y \in Z_{\Sigma(Q)} \setminus (D/\Upsilon_Q)$ we need to look at g_1 and g_1'. For g_1 what we need is immediate: it is in the stabiliser of $\zeta \in D_{Q,h}$ and isotropy groups in symmetric spaces are always torsion, so g_1 is of finite order: by replacing γ with a power we may assume that g_1 is the identity.

For the Riemannian part g_1' a little more work is needed. The element γ has a fixed point $y' = \lim\limits_{t\to\infty} \mathbf{e}_Q(u_t + i x_t) \in X_{\Sigma(Q)} \setminus \mathbb{T}(Q)$, so $y' \in \mathbb{T}(Q)_\sigma$ for some unique $\sigma \in \Sigma(Q)$. Therefore γ preserves σ. If we assume, as we may do, that Q has been chosen so as to maximise $\dim F(Q)$ (see [AMRT, Lemma III.5.5]), then $\sigma \cap C_Q \neq \emptyset$ (remember that C_Q is an open cone but σ is closed): this follows from [AMRT, Theorem III.4.8(ii)].

Since γ preserves σ, it permutes the top-dimensional cones of which σ is a face: there are finitely many of these as long as $\sigma \cap C_Q \neq \emptyset$. Therefore some power of γ preserves a top-dimensional cone, so we may as well assume that σ is top-dimensional. The action of γ is thus determined by its action on $\sigma = \sum_{i=1}^q \mathbb{R}_{\geq 0} u_i$.

Thus γ permutes the rays $\mathbb{R}_{\geq 0} u_i$ (it may not fix them pointwise) and therefore some power, in fact $\gamma^{q!}$, fixes all the rays, so we may as well assume that γ fixes all the rays. In particular it fixes a rational basis of U_Q up to scalars. Now consider the real subgroup of Q that fixes that basis up to scalars. Its identity component is a torus, and because the u_i are defined over $\mathbb{Q}$ it is $\mathbb{R}$-split (in fact $\mathbb{Q}$-split) and therefore it is contained in the maximal $\mathbb{R}$-split torus in Q, which is A_Q. So some power of γ is in A_Q, as required.

For the converse (the "if" part), suppose that $\gamma \Upsilon \in \Gamma/\Upsilon$ with $\gamma \in \Gamma \cap L_Q$ and $\gamma^k = a \in A_Q$ for some $Q \in \mathrm{MPar}_\Gamma$. Since $\gamma \in Q$ it preserves the cone $C = C_Q$. By the Brouwer fixed point theorem, γ preserves a ray ρ' in $\overline{C}_Q$.

We claim that there exists a boundary component $F(P)$, for some $P \in \mathrm{MPar}_\Gamma$, fixed by γ such that γ preserves a ray $\rho = \mathbb{R}_{>0} u_\rho$ in the interior of C_P. This is trivial if $\dim C_Q = 1$. We shall proceed by induction on $\dim C_Q$.

The ray ρ' is preserved by a, with eigenvalue λ say, and ρ' belongs to a unique real boundary component C' of C, since $\overline{C}$ is the disjoint union of its real boundary components by [AMRT, Proposition II.3.1]. Let H_λ be the

λ-eigenspace of a in U_Q. Then $H_\lambda \cap \overline{C}$ is a boundary component of C and contains ρ', so $H_\lambda \cap \overline{C} = C'$. Thus the normaliser of C' in $\mathrm{Aut}(C)$ is rational because it is the normaliser of the rational linear subspace H_λ. Therefore by [AMRT, Corollary II.3.22], C' is a rational boundary component. But γ preserves C', and $\dim C' < \dim C$.

We may write $\gamma = (a_1, g_0', g_0)$ for some $a_1 \in A_Q$, because $L_Q = A_Q \rtimes (G_{Q,l}' \times G_{Q,h})$. Then $g_0 \in G_{Q,h}$ is of finite order and has a fixed point $\zeta \in D_{Q,h}$, and we may assume that γ preserves $\rho = \mathbb{R}_{>0} u_\rho$ in C_Q.

Writing $\mathbf{e}_Q(u + i\infty_\rho)$ for $\lim_{t \to \infty} \mathbf{e}_Q(u + i(tu_\rho)) \in X_{\Sigma(Q)}$, the point

$$z = (\mathbf{e}_Q(0 + i\infty_\rho), 0, \zeta) \in Z_{\Sigma(Q)} = Y_{\Sigma(Q)} \times V_Q \times D_{Q,h}$$

is fixed by $\gamma = (0_m, 0_n, a_1, g_0', g_0) \in ((U_Q \rtimes V_Q) \rtimes A_Q) \times (G_{Q,l}' \times G_{Q,h})$, since, writing $\alpha_0 = (a_1, g_0', g_0) \in L_Q = A_Q \rtimes (G_{Q,l}' \times G_{Q,h})$, we have

$$\gamma z = (\mathbf{e}_Q(\alpha_0^{-1} 0_m \alpha_0 + ia_1\infty_\rho), \alpha_0^{-1} 0_n \alpha_0, g_0\zeta)$$
$$= (\mathbf{e}_Q(0 + i\infty_\rho), 0, \zeta) = z.$$

Therefore $\gamma\Upsilon$ fixes the image of z in $(D/\Upsilon)'_\Sigma$ and $\Lambda\Upsilon/\Upsilon \subseteq (\Gamma/\Upsilon)_o^{\mathrm{Fix}}$. That concludes the proof of Proposition 4.4. $\square$

4.2 The first Betti number

We can use Theorem 4.3 to give bounds on the first Betti number of the toroidal compactifications.

Corollary 4.5 *Suppose that $D = G/K$ is a Hermitian symmetric space of non-compact type without 1-dimensional factors and Γ is a non-uniform lattice of G. Let r be the real rank of G and h be the number of Γ-conjugacy classes of Γ-rational semimaximal parabolic subgroups of G. Then*

$$\mathrm{rk}_\mathbb{Z} H_1(D/\Gamma, \mathbb{Z}) - hr \leq \mathrm{rk}_\mathbb{Z} H_1((D/\Gamma)'_\Sigma, \mathbb{Z}) \leq \mathrm{rk}_\mathbb{Z} H_1(D/\Gamma, \mathbb{Z}). \quad (11)$$

If Γ is neat then

$$\mathrm{rk}_\mathbb{Z} H_1((D/\Gamma)'_\Sigma, \mathbb{Z}) = \mathrm{rk}_\mathbb{Z} H_1(D/\Gamma, \mathbb{Z}).$$

Proof. For an arbitrary group $\mathfrak{G}$ we denote by $\mathrm{ab}\,\mathfrak{G}$ its abelianisation $\mathrm{ab}(\mathfrak{G}) = \mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]$. If S is a complex analytic space then $H_1(S, \mathbb{Z}) = \mathrm{ab}\,\pi_1(S)$.

If $\mathfrak{H}$ is a normal subgroup of $\mathfrak{G}$ then $[\mathfrak{G}/\mathfrak{H}, \mathfrak{G}/\mathfrak{H}] = [\mathfrak{G}, \mathfrak{G}]\mathfrak{H}/\mathfrak{H}$ and

$$\mathrm{ab}(\mathfrak{G}/\mathfrak{H}) = (\mathfrak{G}/\mathfrak{H})/([\mathfrak{G}, \mathfrak{G}]\mathfrak{H}/\mathfrak{H}) \cong \mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]\mathfrak{H} = \mathfrak{G}/\mathfrak{H}[\mathfrak{G}, \mathfrak{G}].$$

Therefore by Theorem 4.3

$$H_1((D/\Gamma)'_\Sigma, \mathbb{Z}) \cong \mathrm{ab}(\Gamma/\Upsilon\Lambda) \cong \Gamma/\Lambda\Upsilon[\Gamma, \Gamma].$$

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On the other hand, D is a path connected, simply connected locally compact space with a properly discontinuous action of Γ by homeomorphisms. Let Φ be the subgroup of Γ generated by the elements $\gamma \in \Gamma$ with a fixed point on D. By [Ar], the fundamental group of D/Γ is $\pi_1(D/\Gamma) = \Gamma/\Phi$. Therefore $H_1(D/\Gamma, \mathbb{Z}) \cong \mathrm{ab}(\Gamma/\Phi) \cong \Gamma/\Phi[\Gamma,\Gamma]$ and

$$H_1((D/\Gamma)'_\Sigma, \mathbb{Z}) \cong (\Gamma/\Phi[\Gamma,\Gamma])/(\Lambda\Upsilon[\Gamma,\Gamma]/\Phi[\Gamma,\Gamma]) \cong H_1(D/\Gamma, \mathbb{Z})/F$$

for the abelian group

$$F = \Lambda\Upsilon[\Gamma,\Gamma]/\Phi[\Gamma,\Gamma] < \Gamma/\Phi[\Gamma,\Gamma] \cong H_1(D/\Gamma, \mathbb{Z}).$$

In particular,

$$\mathrm{rk}_\mathbb{Z} H_1(D/\Gamma, \mathbb{Z}) = \mathrm{rk}_\mathbb{Z} H_1((D/\Gamma)'_\Sigma, \mathbb{Z}) + \mathrm{rk}_\mathbb{Z} F.$$

To verify (11), it suffices to show that $F_o = \Phi\Upsilon[\Gamma,\Gamma]/\Phi[\Gamma,\Gamma]$ is a finite subgroup of F and that $\mathrm{rk}_\mathbb{Z}(F) = \mathrm{rk}_\mathbb{Z}(F/F_o) \le hr$.

We check the rank condition first. For any $Q \in \mathrm{MPar}_\Gamma$ we define Λ_Q to be the subgroup of $\Gamma \cap L_Q$ generated by all $\gamma \in \Gamma \cap L_Q$ such that $\gamma^k \in A_Q$ for some $k \in \mathbb{N}$ (i.e. the elements that are torsion mod A_Q). We define Λ_Q^A to be the subgroup of $\Gamma \cap A_Q$ of elements that arise in this way: that is, Λ_Q^A is the group generated by $\{\gamma^k \in \Gamma \cap A_Q \mid \gamma \in \Lambda_Q, \ k \in \mathbb{N}\}$.

Consider the finitely generated abelian group $\mathfrak{L}_Q = \Lambda_Q\Upsilon[\Gamma,\Gamma]/\Phi\Upsilon[\Gamma,\Gamma]$ and the subgroup $\mathfrak{A}_Q = \Lambda_Q^A\Upsilon[\Gamma,\Gamma]/\Phi\Upsilon[\Gamma,\Gamma]$. These both have the same rank, because the quotient $\mathfrak{L}_Q/\mathfrak{A}_Q$ is abelian and generated by torsion elements so it is finite.

Choose representatives $Q_1, \ldots, Q_h$ for each Γ-conjugacy class in MPar_Γ. Then F/F_o is generated by the $\mathfrak{L}_{Q_i}$, and therefore

$$\mathrm{rk}_\mathbb{Z}(F) = \mathrm{rk}_\mathbb{Z}(F/F_o) \le \sum_{i=1}^h \mathrm{rk}_\mathbb{Z}(\mathfrak{L}_{Q_i}) = \sum_{i=1}^h \mathrm{rk}_\mathbb{Z}(\mathfrak{A}_{Q_i}).$$

However, $\mathfrak{A}_Q$ is a discrete subgroup of A_Q, so $\mathrm{rk}_\mathbb{Z}\mathfrak{A}_Q < \mathrm{rk}_\mathbb{R} A_Q \le r = \mathrm{rk}_\mathbb{R}(G)$, and this gives the bound on $\mathrm{rk}_\mathbb{Z}(F)$.

It remains to show that F_o is finite. It is certainly finitely generated, because it is generated by all $\Upsilon_Q[\Gamma,\Gamma]\Phi/[\Gamma,\Gamma]\Phi$; but each Υ_Q is finitely generated and $\Upsilon_{Q^\gamma} = \Upsilon_Q$ so we need only the Υ_{Q_i}.

Since F_o is abelian, it is now enough to show that any element of F_o is of finite order. For $Q \in \mathrm{MPar}_\Gamma$, consider the group $\mathbf{N}_Q = \Gamma \cap N_Q$: we have

$$[\mathbf{N}_Q, \mathbf{N}_Q] \le \Gamma \cap [N_Q, N_Q] = \Gamma \cap U_Q = \Upsilon_Q.$$

It suffices to prove that $\mathrm{Span}_\mathbb{R}[\mathbf{N}_Q, \mathbf{N}_Q] = U_Q$, because then $[\mathbf{N}_Q, \mathbf{N}_Q]$ is of finite index in Υ_Q so $\Upsilon_Q/[\mathbf{N}_Q, \mathbf{N}_Q]$ is finite, of exponent k_Q say. Then

$(\Upsilon_Q)^{k_Q} \leq [\mathbf{N}_Q, \mathbf{N}_Q] \leq [\Gamma, \Gamma]$ and any element of $\Upsilon_Q \Phi[\Gamma, \Gamma]/\Phi[\Gamma, \Gamma]$ is of order dividing k_Q.

To prove $\mathrm{Span}_{\mathbb{R}}[\mathbf{N}_Q, \mathbf{N}_Q] = U_Q$, note that the group N_Q is 2-step nilpotent, so $[\mathbf{N}_Q, \mathbf{N}_Q] \subset U_Q$ and hence $\mathrm{Span}_{\mathbb{R}}[\mathbf{N}_Q, \mathbf{N}_Q] \subseteq U_Q$. For the other inclusion, let $\beta_1, \dots, \beta_{m+2n} \in \mathfrak{n}_Q$ be such that $b_j = \exp(\beta_j) \in N_Q$ generate the lattice $\mathbf{N}_Q$. Then $\mathfrak{n}_Q = \mathrm{Span}_{\mathbb{R}}(\beta_1, \dots, \beta_{m+2n})$ and

$$\mathfrak{u}_Q = [\mathfrak{n}_Q, \mathfrak{n}_Q] = \mathrm{Span}_{\mathbb{R}}(\{[\beta_i, \beta_j]\}).$$

Here $[\beta_i, \beta_j]$ is the Lie bracket, but U_Q is isomorphic to $\mathfrak{u}_Q$ via exp, so that $\exp[\beta_i, \beta_j] = [\exp(\beta_i), \exp(\beta_j)] = [b_i, b_j]$. Thus $U_Q = \mathrm{Span}_{\mathbb{R}}(\{[b_i, b_j]\}) \subseteq \mathrm{Span}_{\mathbb{R}}[\mathbf{N}_Q, \mathbf{N}_Q]$, as required.

For neat Γ, it is shown in [Sa] that $(\Gamma/\Upsilon)_o^{\mathrm{Fix}}$ is trivial. Combining this with $(\Gamma/\Upsilon)_o^{\mathrm{Fix}} = \Lambda\Upsilon/\Upsilon$ from Lemma 4.4, one concludes that $\Lambda \subseteq \Upsilon$. Therefore $F = F_o$ and $\mathrm{rk}_{\mathbb{Z}}(F) = 0$. □

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