Stability of Higher-Order Discrete-Time Lur'e Systems*

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Abstract. We consider discrete-time Lur'e systems obtained by applying nonlinear feedback to a system of higher-order difference equations (ARMA models). The ARMA model relates the inputs and outputs of the linear system and does not involve any internal or state variables. A stability theory subsuming results of circle criterion type is developed, including criteria for input-to-output stability, a concept which is very much reminiscent of input-to-state stability.

Keywords. Absolute stability, ARMA models, circle criterion, complexified Aizerman conjecture, global asymptotic stability, input-to-output stability, input-output systems, Lur'e systems, polynomial matrices, stability in the large.

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1 Introduction

Lur'e systems in state-space form are a common and important class of nonlinear systems and there is a large body of work on the stability properties of these systems, see, for example, [4, 10, 11, 12, 17, 22, 24, 29, 30, 31, 37, 41]. In this paper, we consider forced discrete-time Lur'e systems defined by higher-order difference equations of the form

$$\mathbf{P}(\mathcal{L})y = \mathbf{Q}(\mathcal{L})u + \mathbf{Q}_{\mathrm{e}}(\mathcal{L})v, \quad u = f(y), \tag{1.1}$$

where $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{Q}_{\mathrm{e}}$ are polynomial matrices, $\mathcal{L}$ is the left-shift operator, u is an input used for feedback, v is an external input, y is the output and

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f is a nonlinearity. It is assumed that $\det \mathbf{P}(z) \not\equiv 0$ and that the rational matrices $\mathbf{P}^{-1}\mathbf{Q}$ and $\mathbf{P}^{-1}\mathbf{Q}_{\rm e}$ are proper. Under these conditions, the linear system

$$\mathbf{P}(\mathcal{L})y = \mathbf{Q}(\mathcal{L})u + \mathbf{Q}_{\rm e}(\mathcal{L})v = \big(\mathbf{Q}(\mathcal{L}), \mathbf{Q}_{\rm e}(\mathcal{L})\big)\begin{pmatrix} u \\ v \end{pmatrix} \tag{1.2}$$

is an input-output system in the sense of the behavioural approach to systems and control, see [27, Section 3.3] and [39]. The systematic investigation of models of the form (1.2) was started by Rosenbrock [28] and his work was further developed in algebraic systems theory, see, for example, [1]. In stochastic control, system (1.2) is also known as a so-called ARMA model [13]. Textbooks such as [20, 23, 27] contain detailed discussions of models of the form (1.2).

It is somewhat surprising that there seems to be hardly any literature on higher-order Lur'e systems. Exceptions include [2, 3, 26, 40] in which stability properties of certain unforced single-input single-output continuous-time higher-order Lur'e systems are studied. The main contribution of this paper consists of a number of stability criteria for systems of the form (1.1). The results obtained are reminiscent of the complexified Aizerman conjecture [15]-[17], nonlinear small-gain theorems and the circle criterion. In addition to stability concepts such as *stability in the large* and *global asymptotic stability*, which are relevant for system (1.1) without forcing ($v = 0$), we also consider *input-to-output stability*. The latter concept is similar in spirit to well-known state-space notions such as input-to-state stability [6, 33] and state-independent input-to-output stability [35] and should not be confused with the classical input-output concept of L^∞-stability due to Sandberg and Zames [7, 37].

The paper is organized as follows. In Section 2, we will present and discuss a number of preliminaries. Section 3 is devoted to the development of results relating to the behaviours and state-space realizations of linear input-output systems of the form (1.2). The key result (Theorem 3.2) in this context shows that, under the assumption of properness of $\mathbf{P}^{-1}\mathbf{Q}$ and $\mathbf{P}^{-1}\mathbf{Q}_{\rm e}$, there exists a state-space realization of (1.2) with the property that its full behaviour is isomorphic (in the vector space sense) to the behaviour of (1.2) (the isomorphism being induced by a certain "canonical" map). Moreover, stabilizability and controllability properties of the realization correspond nicely to natural conditions in terms of the polynomial matrices $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{Q}_{\rm e}$. We emphasize that the proof of Theorem 3.2, in which the so-called observer-form realization (see, for example, [20]) plays an important role, requires more than establishing the equality of transfer function matrices. In Section 4, we develop a stability theory for input-output Lur'e systems of the form (1.1) which is inspired by the complexified Aizerman conjecture and the classical circle criterion for state-space systems. Finally, the

Appendix contains the proofs of three auxiliary technical results.

Notation and terminology. Set $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers, respectively. We also define $\mathbb{E} := \{z \in \mathbb{C} : |z| > 1\}$, the exterior of the closed unit disc. The ring of polynomials with coefficients in $\mathbb{R}$ is denoted by $\mathbb{R}[z]$. For a polynomial matrix $\mathbf{M} \in \mathbb{R}[z]^{p\times m}$ given by $\mathbf{M}(z) = \sum_{j=0}^{k} M_j z^j$, where $M_j \in \mathbb{R}^{p\times m}$ with $M_k \neq 0$, we say that the degree of $\mathbf{M}$ is equal to k and write $\deg \mathbf{M} = k$. As usual, the degree of the zero matrix is defined to be equal to -1. The i-th row degree $r_i(\mathbf{M})$ of $\mathbf{M}$ is the degree of the polynomial row vector given by the i-th row of $\mathbf{M}$, or, equivalently, $r_i(\mathbf{M}) = \max_{1\leq j\leq m} \deg \mathbf{M}_{ij}$, where $\mathbf{M}_{ij} \in \mathbb{R}[z]$ is the polynomial in the i-th row and j-th column of $\mathbf{M}$. We say that a square polynomial matrix $\mathbf{M} \in \mathbb{R}[z]^{p\times p}$ is row reduced if $\deg\det \mathbf{M} = \sum_{i=1}^{p} r_i(\mathbf{M})$. Note that, by the Leibniz formula for determinants, we always have $\deg\det \mathbf{M} \leq \sum_{i=1}^{p} r_i(\mathbf{M})$.

It is convenient to state a simple lemma, the proof of which can be found in the Appendix. This lemma should be well known, but we were not able to find it in the literature.

Lemma 1.1. *Let $\mathbf{M} \in \mathbb{R}[z]^{p\times p}$ be row reduced and let $T \in \mathbb{R}^{p\times p}$ be invertible. Then $r_i(\mathbf{M}) = r_i(\mathbf{M}T)$ for all $i = 1, \ldots, p$. In particular, $\mathbf{M}T$ is row reduced.*

A square polynomial matrix $\mathbf{M} \in \mathbb{R}[z]^{p\times p}$ is said to be Schur if $\det \mathbf{M}(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| \geq 1$ and it is said to be unimodular if $\det \mathbf{M}(z) \equiv c$ for some non-zero constant c, or equivalently, if $\mathbf{M}$ has an inverse in $\mathbb{R}[z]^{p\times p}$.

Let $\mathbf{M} \in \mathbb{R}[z]^{p\times m}$ and $\mathbf{N} \in \mathbb{R}[z]^{p\times n}$. A polynomial matrix $\mathbf{L} \in \mathbb{R}[z]^{p\times q}$ is said to be a left divisor of $\mathbf{M}$ if there exists $\tilde{\mathbf{M}} \in \mathbb{R}[z]^{q\times m}$ such that $\mathbf{M} = \mathbf{L}\tilde{\mathbf{M}}$. We say that $\mathbf{L}$ is a common left divisor of $\mathbf{M}$ and $\mathbf{N}$ if $\mathbf{L}$ is a left divisor of both, $\mathbf{M}$ and $\mathbf{N}$. Furthermore, $\mathbf{L}$ is called a greatest common left divisor of $\mathbf{M}$ and $\mathbf{N}$ if every common left divisor of $\mathbf{M}$ and $\mathbf{N}$ is a left divisor of $\mathbf{L}$. It is well known that a greatest common left divisor $\mathbf{L} \in \mathbb{R}[z]^{p\times q}$ of $\mathbf{M}$ and $\mathbf{N}$ does exist, where

$$q := \max_{z\in\mathbb{C}} \operatorname{rk}\big(\mathbf{M}(z), \mathbf{N}(z)\big), \tag{1.3}$$

and $\mathbf{L}$ is unique up to right-multiplication by a unimodular matrix, that is, any other greatest common left divisor is of the form $\mathbf{L}\mathbf{U}$, where $\mathbf{U} \in \mathbb{R}[z]^{q\times q}$ is unimodular. Note that if $q = p$, where q is defined by (1.3), then $\mathbf{L}$ is square (of format $p \times p$) and $\det \mathbf{L}(z) \not\equiv 0$. If $\mathbf{L} \in \mathbb{R}[z]^{p\times q}$ is a greatest common left divisor of $\mathbf{M}$ and $\mathbf{N}$, then there exist $\tilde{\mathbf{M}} \in \mathbb{R}[z]^{m\times q}$ and $\tilde{\mathbf{N}} \in \mathbb{R}[z]^{n\times q}$ such that $\mathbf{M}\tilde{\mathbf{M}} + \mathbf{N}\tilde{\mathbf{N}} = \mathbf{L}$, and, moreover, for a given $z \in \mathbb{C}$, $\operatorname{rk}\big(\mathbf{M}(z), \mathbf{N}(z)\big) = p$ if, and only if, $\operatorname{rk} \mathbf{L}(z) = p$. The polynomial matrices $\mathbf{M}$ and $\mathbf{N}$ are said to be left-coprime if, for every greatest common left divisor $\mathbf{L} \in \mathbb{R}[z]^{p\times q}$, there exists $\tilde{\mathbf{L}} \in \mathbb{R}[z]^{q\times p}$ such that $\mathbf{L}\tilde{\mathbf{L}} = I_p$. It can be shown that $\mathbf{M}$ and $\mathbf{N}$ are left-coprime if, and only if, there exist $\tilde{\mathbf{M}} \in \mathbb{R}[z]^{m\times p}$ and $\tilde{\mathbf{N}} \in \mathbb{R}[z]^{n\times p}$ such that $\mathbf{M}\tilde{\mathbf{M}} + \mathbf{N}\tilde{\mathbf{N}} = I_p$, or, equivalently,

$$\operatorname{rk}\big(\mathbf{M}(z), \mathbf{N}(z)\big) = p \quad \forall\, z \in \mathbb{C}.$$

We refer to [5, 8, 20] for more details on polynomial matrices.

For $M \in \mathbb{C}^{p\times m}$, let M^* denote the Hermitian transposition of M (transposition, if M is real). For $K \in \mathbb{C}^{m\times p}$ and $r > 0$, we define the open ball in $\mathbb{C}^{m\times p}$ with centre K and radius r:

$$\mathbb{B}_{\mathbb{C}}(K, r) := \{M \in \mathbb{C}^{m\times p} : \|M - K\| < r\},$$

where the operator norm is induced by the 2-norms in $\mathbb{C}^p$ and $\mathbb{C}^m$.

The Hardy space of all bounded holomorphic functions $\mathbb{E} \to \mathbb{C}^{p\times m}$ is denoted by $H^\infty(\mathbb{C}^{p\times m})$, with norm given by

$$\|\mathbf{G}\|_{H^\infty} = \sup_{z\in\mathbb{E}} \|\mathbf{G}(z)\|\,.$$

The formal Z-transform associates with every $x \in (\mathbb{C}^n)^{\mathbb{N}_0}$ a formal power series in z^{-1}, namely $\hat{x}(z) := \sum_{j=0}^{\infty} z^{-j}x(j)$. If there exists $\rho > 0$ such that the power series converges for all complex z with $|z| > \rho$, then we omit the word "formal" and simply say that $\hat{x}$ is the Z-transform of x. If $\mathbf{G} \in H^\infty(\mathbb{C}^{p\times m})$, then there exists $G \in (\mathbb{C}^{p\times m})^{\mathbb{N}_0}$ such that $\mathbf{G}(z) = \sum_{j=0}^{\infty} z^{-j}G(j)$ and the power series converges for very $z \in \mathbb{E}$. In particular, $\mathbf{G}$ is the Z-transform of G and G is said to be the impulse response of $\mathbf{G}$.

The left-shift operator $\mathcal{L}\colon (\mathbb{R}^n)^{\mathbb{N}_0} \to (\mathbb{R}^n)^{\mathbb{N}_0}$ is defined by $(\mathcal{L}x)(t) = x(t + 1)$ for all $t \in \mathbb{N}_0$. Finally, we recall the definitions of certain classes of comparison functions.

$$\mathcal{K} := \{\alpha : [0,\infty) \to [0,\infty) : \alpha(0) = 0 \text{ and } \alpha \text{ is strictly increasing}\}\,,$$
$$\mathcal{K}_\infty := \Big\{\alpha \in \mathcal{K} : \lim_{s\to\infty} \alpha(s) = \infty\Big\}\,.$$

Finally, we denote by $\mathcal{KL}$ the set of functions $\beta : [0,\infty) \times \mathbb{N}_0 \to [0,\infty)$ with the following properties: $\beta(\cdot\,, t) \in \mathcal{K}$ for every $t \in \mathbb{N}_0$, and $\beta(s, \cdot\,)$ is non-increasing with $\lim_{t\to\infty} \beta(s,t) = 0$ for every $s \geq 0$. For more details on comparison functions, we refer the reader to [21].

2 Preliminaries

Let Σ_{io} be the following subset of $\mathbb{R}[z]^{p\times p} \times \mathbb{R}[z]^{p\times m} \times \mathbb{R}[z]^{p\times m_{\mathrm{e}}}$: $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}) \in \Sigma_{\mathrm{io}}$ if, and only if, $\det \mathbf{P}(z) \not\equiv 0$ and both $\mathbf{P}^{-1}\mathbf{Q}$ and $\mathbf{P}^{-1}\mathbf{Q}_{\mathrm{e}}$ are proper (that is, the limits $\lim_{|z|\to\infty} \mathbf{P}^{-1}(z)\mathbf{Q}(z)$ and $\lim_{|z|\to\infty} \mathbf{P}^{-1}(z)\mathbf{Q}_{\mathrm{e}}(z)$ exist).

Let $(\mathbf{P},\mathbf{Q},\mathbf{Q}_\mathrm{e}) \in \Sigma_\mathrm{io}$ and set $k := \deg \mathbf{P}$, so that $\mathbf{P}(z) = \sum_{j=0}^k P_j z^j$ for suitable matrices $P_j \in \mathbb{R}^{p\times p}$, where $P_k \neq 0$. Note that, since $\mathbf{P}^{-1}\mathbf{Q}$ and $\mathbf{P}^{-1}\mathbf{Q}_\mathrm{e}$ are proper, $\deg \mathbf{Q} \leq k$ and $\deg \mathbf{Q}_\mathrm{e} \leq k$. Consequently, $\mathbf{Q}(z) = \sum_{j=0}^k Q_j z^j$ and and $\mathbf{Q}_\mathrm{e}(z) = \sum_{j=0}^k Q_{\mathrm{e}j} z^j$ for suitable matrices Q_j and $Q_{\mathrm{e}j}$. Obviously, if $\deg \mathbf{Q} < k$ or $\deg \mathbf{Q}_\mathrm{e} < k$, then $Q_k = 0$ or $Q_{\mathrm{e}k} = 0$, respectively. Furthermore, by properness of $\mathbf{P}^{-1}\mathbf{Q}$ and $\mathbf{P}^{-1}\mathbf{Q}_\mathrm{e}$, we have the following inequalities for the row degrees of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{Q}_\mathrm{e}$:

$$r_i(\mathbf{P}) \geq r_i(\mathbf{Q}) \quad \text{and} \quad r_i(\mathbf{P}) \geq r_i(\mathbf{Q}_\mathrm{e}) \qquad \forall\, i \in \{1,\dots,p\}.$$

With $(\mathbf{P},\mathbf{Q},\mathbf{Q}_\mathrm{e}) \in \Sigma_\mathrm{io}$, we associate the following input-output system

$$\mathbf{P}(\mathcal{L})y = \mathbf{Q}(\mathcal{L})u + \mathbf{Q}_\mathrm{e}(\mathcal{L})v = \big(\mathbf{Q}(\mathcal{L}), \mathbf{Q}_\mathrm{e}(\mathcal{L})\big)\begin{pmatrix} u \\ v \end{pmatrix} \tag{2.1}$$

or, equivalently,

$$\sum_{j=0}^k P_j y(t+j) = \sum_{j=0}^k Q_j u(t+j) + \sum_{j=0}^k Q_{\mathrm{e}j} v(t+j) \qquad \forall\, t \in \mathbb{N}_0, \tag{2.2}$$

where u is an input available for feedback, v is an external input and y is an output. The behaviour $\mathcal{B}(\mathbf{P},\mathbf{Q},\mathbf{Q}_\mathrm{e})$ of (2.1), or, equivalently, of (2.2), is defined by

$$\mathcal{B}(\mathbf{P},\mathbf{Q},\mathbf{Q}_\mathrm{e}) := \{(u,v,y) \in (\mathbb{R}^m)^{\mathbb{N}_0} \times (\mathbb{R}^{m_\mathrm{e}})^{\mathbb{N}_0} \times (\mathbb{R}^p)^{\mathbb{N}_0} : (u,v,y) \text{ satisfies (2.1)}\}.$$

It is obvious that $\mathcal{B}(\mathbf{P},\mathbf{Q},\mathbf{Q}_\mathrm{e})$ is a linear subspace of $(\mathbb{R}^m)^{\mathbb{N}_0} \times (\mathbb{R}^{m_\mathrm{e}})^{\mathbb{N}_0} \times (\mathbb{R}^p)^{\mathbb{N}_0}$.

Note that P_k is not assumed to be invertible and thus, for given $(u,v) \in (\mathbb{R}^m)^{\mathbb{N}_0} \times (\mathbb{R}^{m_\mathrm{e}})$, (2.2) does not always have a solution for all possible initial values $y(0),\dots,y(k-1) \in \mathbb{R}^p$. We illustrate this fact by a simple example.

Example 2.1. Let $p = 2$, $m = m_\mathrm{e} = 1$, $k = 1$,

$$\mathbf{P}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} 1 & -3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 1 & z-1 \end{pmatrix}$$

and $\mathbf{Q}(z) \equiv \mathbf{Q}_\mathrm{e}(z) \equiv (1,2)^T$. Then

$$\mathbf{P}^{-1}(z)\mathbf{Q}(z) = \mathbf{P}^{-1}(z)\mathbf{Q}_\mathrm{e}(z) = \frac{1}{z+2}\begin{pmatrix} z+5 \\ 1 \end{pmatrix}$$

and so, $\mathbf{P}^{-1}\mathbf{Q} = \mathbf{P}^{-1}\mathbf{Q}_\mathrm{e}$ is proper. Equation (2.2) takes the form

$$y_1(t) - 3y_2(t) = u(t) + v(t), \quad y_2(t+1) - y_2(t) + y_1(t) = 2(u(t) + v(t)),$$

and so, in particular, $y_1(0) = 3y_2(0) + u(0) + v(0)$, imposing a constraint on the initial vector $(y_1(0), y_2(0))^T$. ◊

For $n \in \mathbb{N}_0$, define

$$\Sigma_{\mathrm{ss}}^n := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m_\mathrm{e}} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m} \times \mathbb{R}^{p\times m_\mathrm{e}},$$

where, for $n = 0$, we set $\mathbb{R}^{n\times n} = \{0\}$, $\mathbb{R}^{n\times m} = \{0\}$ etc. The set of all state-space systems is then defined by

$$\Sigma_{\mathrm{ss}} := \bigcup_{n\in\mathbb{N}_0} \Sigma_{\mathrm{ss}}^n.$$

With $S := (A, B, B_\mathrm{e}, C, D, D_\mathrm{e}) \in \Sigma_{\mathrm{ss}}$, we associate the following controlled and observed linear state-space system

$$\mathcal{L}x = Ax + Bu + B_\mathrm{e}v, \quad y = Cx + Du + D_\mathrm{e}v. \tag{2.3}$$

The behaviour $\mathcal{B}(S)$ of (2.3) is the linear subspace of all $(u, v, x, y) \in (\mathbb{R}^m)^{\mathbb{N}_0} \times (\mathbb{R}^{m_\mathrm{e}})^{\mathbb{N}_0} \times (\mathbb{R}^n)^{\mathbb{N}_0} \times (\mathbb{R}^p)^{\mathbb{N}_0}$ which satisfy (2.3).

The transfer function of (2.3) (or of $S = (A, B, B_\mathrm{e}, C, D, D_\mathrm{e})$) is given by

$$C(zI - A)^{-1}(B, B_\mathrm{e}) + (D, D_\mathrm{e}) =: (\mathbf{G}(z), \mathbf{G}_\mathrm{e}(z)).$$

Let $K \in \mathbb{C}^{p\times m}$ and set $\mathbf{G}^K := \mathbf{G}(I - K\mathbf{G})^{-1}$. We define the set of all stabilizing complex output feedback gains for $\mathbf{G}$ by

$$\mathbb{S}_\mathbb{C}(\mathbf{G}) := \{K \in \mathbb{C}^{m\times p} : \mathbf{G}^K \in H^\infty(\mathbb{C}^{p\times m})\}.$$

Application of nonlinear feedback of the form $u = f(y)$ to the input-output system (2.1), where $f : \mathbb{R}^p \to \mathbb{R}^m$, leads to the following nonlinear higher-order difference equation

$$\mathbf{P}(\mathcal{L})y = \mathbf{Q}(\mathcal{L})(f \circ y) + \mathbf{Q}_\mathrm{e}(\mathcal{L})v, \tag{2.4}$$

or, equivalently,

$$\sum_{j=0}^{k} P_j y(t + j) = \sum_{j=0}^{k} Q_j f(y(t + j)) + \sum_{j=0}^{k} Q_{\mathrm{e}j} v(t + j) \qquad \forall\, t \in \mathbb{N}_0. \tag{2.5}$$

The behaviour $\mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}, f)$ of (2.4), or, equivalently, of (2.5), is defined by

$$\mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}, f) := \{(v, y) \in (\mathbb{R}^{m_\mathrm{e}})^{\mathbb{N}_0} \times (\mathbb{R}^p)^{\mathbb{N}_0} : (v, y) \text{ satisfies (2.4)}\}.$$

Occasionally, we will be interested in the unforced dynamics of (2.5) (that is, the dynamics of (2.5) for $v = 0$) and it is therefore convenient to define the corresponding behaviour $\mathcal{B}_0(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}, f)$ by

$$\mathcal{B}_0(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}, f) := \{y \in (\mathbb{R}^p)^{\mathbb{N}_0} : (0, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}, f)\}.$$

Similarly, applying the feedback $u = f(y)$ to the state-space system (2.3), yields the following closed-loop system

$$\mathcal{L}x = Ax + B(f \circ y) + B_{\mathrm{e}}v, \quad y = Cx + D(f \circ y) + D_{\mathrm{e}}v, \tag{2.6}$$

which will be denoted by $S^f := (A, B, B_{\mathrm{e}}, C, D, D_{\mathrm{e}}, f)$. The behaviour $\mathcal{B}(S^f)$ of (2.6) is the set of all $(v, x, y) \in (\mathbb{R}^{m_{\mathrm{e}}})^{\mathbb{N}_0} \times (\mathbb{R}^n)^{\mathbb{N}_0} \times (\mathbb{R}^p)^{\mathbb{N}_0}$ such that (v, x, y) satisfies (2.6). The behaviour $\mathcal{B}_0(S^f)$ of the unforced ($v = 0$) nonlinear state-space system is defined by

$$\mathcal{B}_0(S^f) = \{(x, y) \in (\mathbb{R}^n)^{\mathbb{N}_0} \times (\mathbb{R}^p)^{\mathbb{N}_0} : (0, x, y) \in \mathcal{B}(S^f)\}.$$

The following result, which will play an important role in Section 4, can be found in [31].[§]

Theorem 2.2. *Let $S = (A, B, B_{\mathrm{e}}, C, D, D_{\mathrm{e}}) \in \Sigma_{\mathrm{ss}}$, $f : \mathbb{R}^p \to \mathbb{R}^m$ be continuous, $K \in \mathbb{R}^{m \times p}$, $r > 0$ and set $\mathbf{G}(z) := C(zI - A)^{-1}B + D$. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subset \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ and consider the conditions:*

(H1) *(A, B, C) is stabilizable and detectable and $r \min_{|z|=1} \|\mathbf{G}^K(z)\| < 1$,*

(H2) *(A, B, C) is controllable and observable and $r\|\mathbf{G}^K(\infty)\| < 1$,*

(H3) *(A, B, C) is controllable and observable.*

The following statements hold.

(1) *If* (H1) *or* (H2) *holds and*

$$\|f(\xi) - K\xi\| \le r\,\|\xi\| \quad \forall\, \xi \in \mathbb{R}^p,$$

then there exists $\kappa \ge 1$ such that, for all $(x, y) \in \mathcal{B}_0(S^f)$,

$$\|x(t)\| + \|y(t)\| \le \kappa\,\|x(0)\| \quad \forall\, t \in \mathbb{N}_0. \tag{2.7}$$

(2) *If* (H1) *or* (H2) *holds and*

$$\|f(\xi) - K\xi\| < r\,\|\xi\| \quad \forall\, \xi \in \mathbb{R}^p,\ \xi \ne 0,$$

then there exists $\kappa \ge 1$ such that, for all $(x, y) \in \mathcal{B}_0(S^f)$, (2.7) *holds, $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$.*

(3) *If* (H1) *or* (H3) *holds and there exists $\alpha \in \mathcal{K}_\infty$ such that*

$$\|f(\xi) - K\xi\| \le r\,\|\xi\| - \alpha(\|\xi\|) \qquad \forall\, \xi \in \mathbb{R}^p,$$

[§] Non-zero feedthrough is not considered in [31], that is, the theory developed in [31] applies to (2.6) with $D = 0$ and $D_{\mathrm{e}} = 0$. However, the extension to the non-zero feedthrough case is not difficult. In the special case wherein $B_{\mathrm{e}} = B$ and $D_{\mathrm{e}} = D$, the non-zero feedthrough case is covered in [29].

then there exist $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}_\infty$ *such that, for all* $(v, x, y) \in \mathcal{B}(S^f)$*,*

$$\|x(t)\| + \|y(t)\| \leq \beta(\|x(0)\|, t) + \gamma\big(\|v\|_t\big) \quad \forall\, t \in \mathbb{N}_0, \tag{2.8}$$

where $\|v\|_t := \sup\{\|v(s)\| : s = 0, \dots, t\}$*.*

Note that statements (1) and (2) of the above theorem relate to the Lur'e system (2.6) with $v = 0$ and imply that the unforced feedback system is stable in the large and globally asymptotically stable, respectively, whilst statement (3) addresses the issue of input-to-state stability (ISS). In particular, (2.8) shows that the forced nonlinear feedback system (2.6) is ISS. The concept of ISS, for a general controlled nonlinear system, appears first in [32] published in 1989. The theory of ISS which has been subsequently developed, provides a natural stability framework for nonlinear systems with inputs, merging, in a sense, Lyapunov and input-output approaches to stability (the latter initiated by Sandberg and Zames in the 1960s, see [7, 37]). We refer the reader to [6, 33] for overviews of ISS theory.

It is not difficult to show that if

$$\mathbb{B}_{\mathbb{C}}(K, r) \subset \mathbb{S}_{\mathbb{C}}(\mathbf{G}), \tag{2.9}$$

then $r\|\mathbf{G}^K\|_{H^\infty} \leq 1$, see [31, Lemma 6]. We conclude that the condition $r \min_{|z|=1} \|\mathbf{G}^K(z)\| < 1$ is violated if, and only if, $\|\mathbf{G}^K(e^{i\omega})\| = \|\mathbf{G}^K\|_{H^\infty} = 1/r$ for all $\omega \in [0, 2\pi)$. Consequently, if (2.9) holds, $m = p$ ("square" case) and $\det \mathbf{G}(z) \not\equiv 0$, then $r \min_{|z|=1} \|\mathbf{G}^K(z)\| < 1$ if, and only if, $\underline{\sigma}\big(\mathbf{G}^{-1}(e^{i\omega}) - K\big) \not\equiv r$, where $\underline{\sigma}$ denotes the smallest singular value. If $m = p = 1$, the latter condition means that the inverse Nyquist plot $\{1/\mathbf{G}(e^{i\omega}) : \omega \in [0, 2\pi)\}$ is *not* equal to the circle of radius r centred at K.

Similarly, if (2.9) holds, the condition $r\|\mathbf{G}^K(\infty)\| < 1$ is violated if, and only if, $r\|\mathbf{G}^K(\infty)\| = 1$. Therefore, if $r \min_{|z|=1} \|\mathbf{G}^K(z)\| < 1$, then $r\|\mathbf{G}^K(\infty)\| < 1$, because otherwise, $r\|\mathbf{G}^K(\infty)\| = 1$ and so, by the maximum principle [14, Theorem 3.13.1], $\|\mathbf{G}^K(z)\| \equiv 1/r$. Moreover, in the case wherein $m = p = 1$, if $r\|\mathbf{G}^K(\infty)\| = 1$, then $\mathbf{G}^K$ is constant and so is $\mathbf{G}$.

Next we provide an example which shows that if $r\|\mathbf{G}^K(\infty)\| = 1$, then the conclusion of statement (1) of Theorem 2.2 does not necessarily hold despite the fact that all other hypotheses are satisfied.

Example 2.3. Consider (2.6) with

$$A = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad B = C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad B_{\mathrm{e}} = 0, \quad D_{\mathrm{e}} = 0,$$

and $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(\xi) = \begin{pmatrix} 0 \\ (1/2)\xi_2 \sin \xi_2 \end{pmatrix}, \quad \forall\, \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{R}^2.$$

Note that (A, B, C) is controllable and observable. Moreover,

$$\mathbf{G}(z) = C(zI - A)^{-1}B + D = \begin{pmatrix} z/(z^2 - 1/4) & 0 \\ 0 & 2 \end{pmatrix},$$

and so $\|\mathbf{G}\|_{H^\infty} = \|\mathbf{G}(\infty)\| = \|D\| = 2$. With $K = 0$ and $r = 1/2$ all assumptions of statement (1) of Theorem 2.2 are satisfied with the exception of $r\|\mathbf{G}^K(\infty)\| < 1$. We show that the conclusion of statement (1) does not hold. The behaviour $\mathcal{B}_0(S^f)$ consists of all $(x, y) \in (\mathbb{R}^2)^{\mathbb{N}_0} \times (\mathbb{R}^2)^{\mathbb{N}_0}$ satisfying $x(t) = A^t x(0)$, $y_1(t) = x_1(t)$ and $y_2(t) = y_2(t) \sin y_2(t)$, where x_1, x_2, y_1 and y_2 denote the components of x and y. Whilst x is always bounded, $\mathcal{B}_0(S^f)$ contains unbounded trajectories (choose $y_2(t) = (4t + 1)\pi/2$ for all $t \in \mathbb{N}_0$), and so the conclusion of statement (1) of Theorem 2.2 does not hold. $\diamondsuit$

3 Linear higher-order input-output systems: behaviour and state-space realization

In the following, let $\star$ denote convolution. The first result of this section provides a simple characterization of the behaviour of an input-output system.

Proposition 3.1. *Let $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}) \in \Sigma_\mathrm{io}$ and let G and G_e be the impulse responses of $\mathbf{P}^{-1}\mathbf{Q}$ and $\mathbf{P}^{-1}\mathbf{Q}_\mathrm{e}$, respectively. The following statements hold.*

(1) $\dim \ker \mathbf{P}(\mathcal{L}) = \deg \det \mathbf{P}$.

(2) $(u, v, G \star u + G_\mathrm{e} \star v) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e})$ *for all* $(u, v) \in (\mathbb{R}^m)^{\mathbb{N}_0} \times (\mathbb{R}^{m_\mathrm{e}})^{\mathbb{N}_0}$.

(3) $(u, v, y) \in (\mathbb{R}^m)^{\mathbb{N}_0} \times (\mathbb{R}^{m_\mathrm{e}})^{\mathbb{N}_0} \times (\mathbb{R}^p)^{\mathbb{N}_0}$ *is in* $\mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e})$ *if, and only if,*

$$y - G \star u - G_\mathrm{e} \star v \in \ker \mathbf{P}(\mathcal{L}).$$

A proof of Proposition 3.1 can be found in the Appendix. Note that, by part (3) of Proposition 3.1,

$$\mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}) = \mathcal{B}_{00}(\mathbf{P}) + \{(u, v, G \star u + G_\mathrm{e} \star v) : (u, v) \in (\mathbb{R}^m)^{\mathbb{N}_0} \times (\mathbb{R}^{m_\mathrm{e}})^{\mathbb{N}_0}\},$$

where

$$\mathcal{B}_{00}(\mathbf{P}) := \{0\} \times \{0\} \times \ker \mathbf{P}(\mathcal{L}) \subset \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}).$$

A state-space system $(A, B, B_\mathrm{e}, C, D, D_\mathrm{e}) \in \Sigma_\mathrm{ss}$ is said to be a realization of $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}) \in \Sigma_\mathrm{io}$ if

$$\mathbf{P}^{-1}(z)(\mathbf{Q}(z), \mathbf{Q}_\mathrm{e}(z)) = C(zI - A)^{-1}(B, B_\mathrm{e}) + (D, D_\mathrm{e}).$$

We say that the dimension of the realization is n if A has format $n \times n$. If $\mathbf{P}$ is a constant matrix, then, by properness of $\mathbf{P}^{-1}(\mathbf{Q}, \mathbf{Q}_\mathrm{e})$, it follows that the polynomial matrices $\mathbf{Q}$ and $\mathbf{Q}_e$ are constant, in which case, $(\mathbf{P},\mathbf{Q},\mathbf{Q}_e)$ has the 0-dimensional realization $(0,0,0,0,D,D_e)$, where D and D_e are the values of the constant functions $\mathbf{P}^{-1}\mathbf{Q}$ and $\mathbf{P}^{-1}\mathbf{Q}_e$, respectively. A realization of minimal dimension is said to be a minimal realization.

For a sequence $w \in (\mathbb{R}^q)^{\mathbb{N}_0}$ and $l \in \mathbb{N}$, it is convenient to define

$$w^l = \big(w(0)^*, w(1)^*, \dots, w(l-1)^*\big)^* \in \mathbb{R}^{lq},$$

where * denotes transposition. The following theorem is the main result of this section.

Theorem 3.2. *Let* $(\mathbf{P},\mathbf{Q},\mathbf{Q}_e) \in \Sigma_{io}$, *define*

$$D := \lim_{|z|\to\infty} \mathbf{P}^{-1}(z)\mathbf{Q}(z), \quad D_e := \lim_{|z|\to\infty} \mathbf{P}^{-1}(z)\mathbf{Q}_e(z) \tag{3.1}$$

and set $n := \deg\det\mathbf{P}$. *Then there exists an* n*-dimensional realization* $S := (A, B, B_e, C, D, D_e)$ *of* $(\mathbf{P},\mathbf{Q},\mathbf{Q}_e)$ *such that* (C, A) *is observable,* $(u, v, y) \in \mathcal{B}(\mathbf{P},\mathbf{Q},\mathbf{Q}_e)$ *for every* $(u, v, x, y) \in \mathcal{B}(S)$, *the map*

$$\Lambda : \mathcal{B}(S) \to \mathcal{B}(\mathbf{P},\mathbf{Q},\mathbf{Q}_e),\ (u, v, x, y) \mapsto (u, v, y)$$

is a vector space isomorphism and there exists $\lambda > 0$ *such that*

$$\|x(0)\| \le \lambda\big(\|u^k\| + \|v^k\| + \|y^k\|\big) \quad \forall\,(u, v, x, y) \in \mathcal{B}(S), \tag{3.2}$$

where $k = \deg\mathbf{P}$. *Moreover, the following statements hold.*

(1) *Under the additional assumption that there exists* $L \in \mathbb{C}^{m\times p}$ *such that* $\mathbf{P}(I + DL) - \mathbf{Q}L$ *is Schur, the pair* (A, B) *is stabilizable.*

(2) *Under the additional assumption that* $\mathbf{P}$ *and* $\mathbf{Q}$ *are left coprime, the pair* (A, B) *is controllable and* S *is a minimal realization of* $(\mathbf{P},\mathbf{Q},\mathbf{Q}_e)$.

Before we prove Theorem 3.2, we provide some commentary on the assumption in statement (1). This hypothesis postulates the existence of a matrix L such that $\mathbf{P}(I + DL) - \mathbf{Q}L$ is Schur. The following proposition provides a sufficient condition for the existence of such a matrix L.

Proposition 3.3. *Let* $(\mathbf{P},\mathbf{Q}) \in \mathbb{R}[z]^{p\times p} \times \mathbb{R}[z]^{p\times m}$ *such that* $\det\mathbf{P}(z) \not\equiv 0$ *and* $\mathbf{G} := \mathbf{P}^{-1}\mathbf{Q}$ *is proper, set*

$$D := \lim_{|z|\to\infty} \mathbf{P}^{-1}(z)\mathbf{Q}(z),$$

and let $K \in \mathbb{C}^{m\times p}$. *The following statements are equivalent.*

(1) $K \in \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ *and the rank condition*

$$\mathrm{rk}\big(\mathbf{P}(z), \mathbf{Q}(z)\big) = p \quad \forall\, z \in \mathbb{C} \text{ with } |z| \ge 1 \tag{3.3}$$

is satisfied.

(2) *The polynomial matrix* $\mathbf{P} - \mathbf{Q}K$ *is Schur and* $I - DK$ *is invertible.*

In particular, if statement (1) *holds, then* $I - DK$ *is invertible and the matrix* $L := K(I - DK)^{-1}$ *is such that* $\mathbf{P}(I + DL) - \mathbf{Q}L$ *is Schur.*

Note that the condition (3.3) is necessary and sufficient for the greatest common left divisor of $\mathbf{P}$ and $\mathbf{Q}$ to be Schur. A proof of Proposition 3.3 can be found in the Appendix.

Proof of Theorem 3.2. It is well-known that there exists a unimodular $\mathbf{U} \in \mathbb{R}[z]^{p\times p}$ such that $\mathbf{UP}$ is row reduced and $\deg \mathbf{UP} \le \deg \mathbf{P}$, see [25, Theorem 1]. Moreover, it is clear that $\mathbf{U}$ can be chosen such that

$$r_1(\mathbf{UP}) \ge r_2(\mathbf{UP}) \ge \ldots \ge r_p(\mathbf{UP}).$$

Trivially,

$$\mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}) = \mathcal{B}(\mathbf{UP}, \mathbf{UQ}, \mathbf{UQ}_{\mathrm{e}})$$

and, moreover, a sextuple $(A, B, B_{\mathrm{e}}, C, D, D_{\mathrm{e}}) \in \Sigma_{\mathrm{ss}}$ is a realization of $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}})$ if, and only if, it is a realization of $(\mathbf{UP}, \mathbf{UQ}, \mathbf{UQ}_{\mathrm{e}})$. Therefore, without loss of generality, we may assume that $\mathbf{P}$ is row reduced, that is, $n = \deg\det \mathbf{P} = \sum_{i=1}^{p} r_i(\mathbf{P})$, and moreover,

$$r_1(\mathbf{P}) \ge r_2(\mathbf{P}) \ge \ldots \ge r_p(\mathbf{P}). \tag{3.4}$$

Let q be the number of rows with $r_i(\mathbf{P}) \ge 1$. Obviously, $0 \le q \le p$. First we deal with the (not very interesting) case wherein $q = 0$. Noting that $r_i(\mathbf{P}) = -1$ is not possible since $\det \mathbf{P}(z) \not\equiv 0$, it follows that, in the case $q = 0$, $r_i(\mathbf{P}) = 0$ for all $i = 1, \ldots, p$. Consequently, $\mathbf{P}(z) \equiv P_0$ and, by properness of $\mathbf{P}^{-1}\mathbf{Q}$ and $\mathbf{P}^{-1}\mathbf{Q}_{\mathrm{e}}$, $\mathbf{Q}(z) \equiv Q_0$ and $\mathbf{Q}_{\mathrm{e}}(z) \equiv Q_{\mathrm{e}\,0}$. Therefore, $\deg\det \mathbf{P} = 0$, and, trivially, the 0-dimensional state space system $(0, 0, 0, 0, D, D_{\mathrm{e}})$ is a realization of $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}})$ which has all the required properties.

Let us now assume that $q \ge 1$. Then, by (3.4),

$$r_1(\mathbf{P}) \ge r_2(\mathbf{P}) \ge \ldots \ge r_q(\mathbf{P}) \ge 1,$$

and

$$r_i(\mathbf{P}) = 0 \quad \forall\, i \in \{q + 1, \ldots, p\}. \tag{3.5}$$

Consequently, there exists an invertible matrix $T \in \mathbb{R}^{p\times p}$ such that $\mathbf{P}(z)T$ is of the form

$$\mathbf{P}(z)T = \begin{pmatrix} \mathbf{P}_0(z) & \mathbf{P}_1(z) \\ 0 & I \end{pmatrix}, \tag{3.6}$$

where $\mathbf{P}_0$ is polynomial matrix of format $q \times q$.

We claim that $\mathbf{P}_0$ is row reduced and $r_i(\mathbf{P}_0) \geq 1$ for all $i = 1, \ldots, q$. To this end, note that, by row reducedness of $\mathbf{P}$ and Lemma 1.1, $r_i(\mathbf{P}T) = r_i(\mathbf{P})$ for all $i = 1, \ldots, p$. In particular,

$$r_i(\mathbf{P}T) = r_i(\mathbf{P}) \geq 1 \quad \forall\, i \in \{1, \ldots, q\}. \tag{3.7}$$

and

$$\sum_{i=1}^{q} r_i(\mathbf{P}T) = \sum_{i=1}^{p} r_i(\mathbf{P}T) = \sum_{i=1}^{p} r_i(\mathbf{P}) = \deg\det \mathbf{P} = n. \tag{3.8}$$

Moreover, by (3.6),

$$r_i(\mathbf{P}_0) \leq r_i(\mathbf{P}T) \quad \forall\, i \in \{1, \ldots, q\}. \tag{3.9}$$

On the other hand,

$$\sum_{i=1}^{q} r_i(\mathbf{P}_0) \geq \deg\det \mathbf{P}_0 = \deg\det(\mathbf{P}T) = \deg\det \mathbf{P} = n.$$

Together with (3.8) and (3.9) this shows that

$$r_i(\mathbf{P}_0) = r_i(\mathbf{P}T) = r_i(\mathbf{P}) \quad \forall\, i \in \{1, \ldots, q\}. \tag{3.10}$$

Therefore, by (3.8), $\sum_{i=1}^{q} r_i(\mathbf{P}_0) = n = \deg\det \mathbf{P}_0$, showing that $\mathbf{P}_0$ is row reduced. Furthermore, by (3.7), $r_i(\mathbf{P}_0) \geq 1$ for all $i = 1, \ldots, q$.

Setting $\mathbf{R} := \mathbf{Q} - \mathbf{P}D$ and $\mathbf{R}_{\mathrm{e}} := \mathbf{Q}_{\mathrm{e}} - \mathbf{P}D_{\mathrm{e}}$, we have that $\mathbf{P}^{-1}\mathbf{R} = \mathbf{P}^{-1}\mathbf{Q} - D$ and $\mathbf{P}^{-1}\mathbf{R}_{\mathrm{e}} = \mathbf{P}^{-1}\mathbf{Q}_{\mathrm{e}} - D_{\mathrm{e}}$ and thus, $\mathbf{P}^{-1}\mathbf{R}$ and $\mathbf{P}^{-1}\mathbf{R}_{\mathrm{e}}$ are strictly proper. Together with (3.5) this implies that $\mathbf{R}$ and $\mathbf{R}_{\mathrm{e}}$ are of the form

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_0 \\ 0 \end{pmatrix}, \quad \mathbf{R}_{\mathrm{e}} = \begin{pmatrix} \mathbf{R}_{\mathrm{e}0} \\ 0 \end{pmatrix}, \quad \text{where } \mathbf{R}_0 \in \mathbb{R}[z]^{q \times m},\ \mathbf{R}_{\mathrm{e}0} \in \mathbb{R}[z]^{q \times m_{\mathrm{e}}}.$$

Consequently, by (3.6),

$$T^{-1}\mathbf{P}^{-1}(\mathbf{R}, \mathbf{R}_{\mathrm{e}}) = \begin{pmatrix} \mathbf{P}_0^{-1} & -\mathbf{P}_0^{-1}\mathbf{P}_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} \mathbf{R}_0 & \mathbf{R}_{\mathrm{e}0} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_0^{-1}\mathbf{R}_0 & \mathbf{P}_0^{-1}\mathbf{R}_{\mathrm{e}0} \\ 0 & 0 \end{pmatrix}. \tag{3.11}$$

Let $(A, B, B_{\mathrm{e}}, C_0, 0, 0)$ be the observer-form realization of $(\mathbf{P}_0, \mathbf{R}_0, \mathbf{R}_{\mathrm{e}0})$, see [20, pp. 413-417]. This realization has dimension n, (C_0, A) is observable and the following identity holds

$$\mathbf{\Psi}_0(z)\big[zI - A, (B, B_{\mathrm{e}})\big] = \big[\mathbf{P}_0(z)C_0, (\mathbf{R}_0(z), \mathbf{R}_{\mathrm{e}0}(z))\big]. \tag{3.12}$$

Here $\mathbf{\Psi}_0 \in \mathbb{R}[z]^{q \times n}$ is the block diagonal polynomial matrix defined by

$$\mathbf{\Psi}_0(z) := \operatorname{blockdiag}_{1 \leq i \leq q}\big(z^{\rho_i - 1}, z^{\rho_i - 2}, \ldots, z, 1\big),$$

where $\rho_i := r_i(\mathbf{P}_0) = r_i(\mathbf{P}T) = r_i(\mathbf{P})$ for all $i = 1, \ldots, q$. Note that $\sum_{i=1}^q \rho_i = \deg\det \mathbf{P}_0 = n$. Defining

$$\tilde{C} := \begin{pmatrix} C_0 \\ 0 \end{pmatrix} \in \mathbb{R}^{p\times n}, \;\; \tilde{D} := T^{-1}D, \;\; \tilde{D}_{\mathrm{e}} := T^{-1}D_{\mathrm{e}}, \;\; \mathbf{\Psi} := \begin{pmatrix} \mathbf{\Psi}_0 \\ 0 \end{pmatrix},$$

and setting $\tilde{S} := (A, B, B_{\mathrm{e}}, \tilde{C}, \tilde{D}, \tilde{D}_{\mathrm{e}})$, we conclude that $\tilde{S}$ is a realization of $(\mathbf{P}T, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}})$ (as follows from (3.11)), $\tilde{S}$ has dimension n, $(\tilde{C}, A)$ is observable, and, by (3.12),

$$\mathbf{\Psi}(z)\big[zI - A, (B, B_{\mathrm{e}})\big] = \big[\mathbf{P}(z)T\tilde{C}, (\mathbf{R}(z), \mathbf{R}_{\mathrm{e}}(z))\big]. \tag{3.13}$$

Setting $C := T\tilde{C}$, it is clear that $S := (A, B, B_{\mathrm{e}}, C, D, D_{\mathrm{e}})$ is a n-dimensional realization of $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}})$, (C, A) is observable, and, by (3.13),

$$\mathbf{\Psi}(z)\big[zI - A, (B, B_{\mathrm{e}})\big] = \big[\mathbf{P}(z)C, (\mathbf{R}(z), \mathbf{R}_{\mathrm{e}}(z))\big]. \tag{3.14}$$

To show

$$(u, v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}) \quad \forall\, (u, v, x, y) \in \mathcal{B}(S), \tag{3.15}$$

we note that, if $(u, v, x, y) \in \mathcal{B}(S)$, then $y = y_{x(0)} + G \star u + G_{\mathrm{e}} \star v$, where G and G_{e} are the impulse responses of $\mathbf{P}^{-1}\mathbf{Q}$ and $\mathbf{P}^{-1}\mathbf{Q}_{\mathrm{e}}$, respectively, and, for $\xi \in \mathbb{R}^n$, the function y_ξ is defined by $y_\xi(t) = CA^t\xi$ for all $t \in \mathbb{N}_0$. Invoking Proposition 3.1, we see that (3.15) is equivalent to

$$y_{x(0)} = y - G \star u + G_{\mathrm{e}} \star v \in \ker \mathbf{P}(\mathcal{L}) \quad \forall\, (u, v, x, y) \in \mathcal{B}(S). \tag{3.16}$$

To establish (3.16), we will show that

$$\ker \mathbf{P}(\mathcal{L}) = \{y_\xi : \xi \in \mathbb{R}^n\}. \tag{3.17}$$

By Proposition 3.1, the dimension of $\ker \mathbf{P}(\mathcal{L})$ is equal to n. Therefore, (3.17) will follow from the inclusion

$$\ker \mathbf{P}(\mathcal{L}) \subset \{y_\xi : \xi \in \mathbb{R}^n\}. \tag{3.18}$$

To prove that (3.18) holds, let $w \in \ker \mathbf{P}(\mathcal{L})$ and $\xi \in \mathbb{R}^n$. Application of the Z-transform to the identity $\mathbf{P}(\mathcal{L})w = 0$ yields

$$\mathbf{P}(z)\hat{w}(z) = \sum_{j=1}^{k} P_j \sum_{i=0}^{j-1} z^{j-i} w(i) = \sum_{l=1}^{k} h^l(w) z^l,$$

where $h^l : \ker \mathbf{P}(\mathcal{L}) \to \mathbb{R}^p$ is the linear map given by

$$h^l(w) = \sum_{j=l}^{k} P_j w(j-l), \quad 1 \le l \le k. \tag{3.19}$$

Consequently,

$$\hat{w}(z) = \mathbf{P}^{-1}(z)\sum_{l=1}^{k} h^l(w)z^l \quad \text{and} \quad \hat{y}_\xi(z) = zC(zI - A)^{-1}\xi. \tag{3.20}$$

It follows from (3.20) that $w = y_\xi$ if, and only if, $z\mathbf{P}(z)C(zI - A)^{-1}\xi = \sum_{l=1}^{k} h^l(w)z^l$. By (3.14), $\mathbf{\Psi}(z) = \mathbf{P}(z)C(zI - A)^{-1}$ and so, $w = y_\xi$ if, and only if, $z\mathbf{\Psi}(z)\xi = \sum_{l=1}^{k} h^l(w)z^l$, or, equivalently,

$$\begin{pmatrix} \operatorname{blockdiag}_i\big(z^{\rho_i}, z^{\rho_i-1}, \ldots, z^2, z\big) \\ 0 \end{pmatrix} \xi = \sum_{l=1}^{k} h^l(w)z^l. \tag{3.21}$$

Let $e_1, \ldots, e_p$ be the canonical basis of $\mathbb{R}^p$. Using that, by (3.10), $r_i(\mathbf{P}) = r_i(\mathbf{P}_0) = \rho_i$ for all $i = 1, \ldots, q$ and, by (3.5), $r_i(\mathbf{P}) = 0$ for all $i = q+1, \ldots, p$, it follows that

$$e_i^* P_j = 0 \quad \text{for all } i = 1, \ldots, q \text{ and } j \text{ such that } \rho_i < j \le k.$$

and

$$e_i^* P_j = 0 \quad \text{for all } i = q+1, \ldots, p \text{ and } j = 1, \ldots, k.$$

Consequently, letting h_i^l denote the i-th component of h^l, we obtain

$$h_i^l = 0 \quad \text{for all } i = 1, \ldots, q \text{ and } l \text{ such that } \rho_i < l \le k \tag{3.22}$$

and

$$h_i^l = 0 \quad \text{for all } i = q+1, \ldots, p \text{ and } l = 1, \ldots, k. \tag{3.23}$$

Setting $\sigma_1 := 0$ and

$$\sigma_i := \sum_{j=1}^{i-1} \rho_j, \quad i = 2, \ldots, q,$$

the i-th component of the vector on the left-hand side of (3.21) is given by $\sum_{l=1}^{\rho_i} z^{\rho_i+1-l}\xi_{\sigma_i+l}$. Invoking (3.22) and (3.23), it now follows that, for every $w \in \ker \mathbf{P}(\mathcal{L})$, (3.21) has a unique solution $\xi = \xi(w)$ in $\mathbb{R}^n$ which is given by

$$\xi_{\sigma_i+l} = \xi_{\sigma_i+l}(w) = h_i^{\rho_i+1-l}(w), \quad 1 \le l \le \rho_i,\ 1 \le i \le q. \tag{3.24}$$

We have now established (3.18). Hence, as has already been pointed out, (3.15) follows.

Using (3.19) and (3.24), we conclude that there exists $\lambda_1 > 0$ such that

$$\|\xi\| = \|\xi(w)\| \le \lambda_1 \|w^k\| \quad \forall\, w \in \ker \mathbf{P}(\mathcal{L}). \tag{3.25}$$

Appealing to (3.16), we see that there exists $\lambda_2 > 0$ (depending only on $\mathbf{G}$ and $\mathbf{G}_\mathrm{e}$) such that

$$\|y_{x(0)}^k\| \le \lambda_2\big(\|y^k\| + \|u^k\| + \|v^k\|\big) \quad \forall\, (u, v, x, y) \in \mathcal{B}(S),$$

it follows from (3.25) that,

$$\|x(0)\| \leq \lambda_1 \|y^k_{x(0)}\| \leq \lambda_1\lambda_2\big(\|u^k\| + \|v^k\| + \|y^k\|\big) \quad \forall\,(u,v,x,y) \in \mathcal{B}(S).$$

Consequently, (3.2) holds with $\lambda := \lambda_1\lambda_2$.

As for the map Λ, it is clear that Λ is linear and it follows from the observability of the pair (C,A) that Λ is injective. To show surjectivity of Λ, let $(u,v,y) \in \mathcal{B}(\mathbf{P},\mathbf{Q},\mathbf{Q}_\mathrm{e})$. Then, by Proposition 3.1, $w := y - G \star u - G_\mathrm{e} \star v$ is in $\ker \mathbf{P}(\mathcal{L})$ and thus, by (3.17), there exists $\xi \in \mathbb{R}^n$ such that $w(t) = CA^t\xi$ for all $t \in \mathbb{N}_0$. Let $x \in (\mathbb{R}^n)^{\mathbb{N}_0}$ be the solution of the initial-value problem

$$\mathcal{L}x = Ax + Bu + B_\mathrm{e}v, \quad x(0) = \xi.$$

The corresponding output of S is then $w + G \star u + G_\mathrm{e} \star v = y$, and thus, $(u,v,x,y) \in \mathcal{B}(S)$, showing that Λ is surjective. We have now established that Λ is an isomorphism.

We proceed to prove statement (1). To this end, note that, by (3.14),

$$\mathbf{\Psi}(z)\big(zI - A, B\big) = \big(\mathbf{P}(z)C, \mathbf{R}(z)\big),$$

Multiplying this identity from the right by the $(n{+}m){\times}n$-matrix $(I, -C^*L^*)^*$ gives

$$\mathbf{\Psi}(z)\big(zI - A - BLC\big) = (\mathbf{P}(z) - \mathbf{R}(z)L)C. \tag{3.26}$$

Let $z \in \mathbb{C}$ with $|z| \geq 1$ and $\zeta \in \mathbb{C}^n$ and consider the equation

$$(zI - A - BLC)\zeta = 0. \tag{3.27}$$

Obviously, to establish stabilizability of (A,B), it is sufficient to show that $\zeta = 0$. It follows from (3.26) and (3.27) that $(\mathbf{P}(z) - \mathbf{R}(z)L)C\zeta = 0$. Since, by hypothesis, $\mathbf{P} - \mathbf{R}L$ is Schur, and thus, $\mathbf{P}(z) - \mathbf{R}(z)L$ is invertible, we conclude that $C\zeta = 0$. Together with (3.27) this implies

$$\begin{pmatrix} zI - A \\ C \end{pmatrix} \zeta = 0.$$

Since (C,A) is observable, it follows, via the Hautus criterion for observability, that $\zeta = 0$, implying the stabilizability of (A,B).

To prove statement (2), we note that, by left coprimeness of $\mathbf{P}$ and $\mathbf{Q}$, the polynomial matrices $\mathbf{P}$ and $(\mathbf{Q},\mathbf{Q}_\mathrm{e})$ are also left coprime. Consequently, the McMillan degrees of $\mathbf{P}^{-1}\mathbf{Q}$ and $\mathbf{P}^{-1}(\mathbf{Q},\mathbf{Q}_\mathrm{e})$ are equal to n. Therefore, the realizations (A,B,C,D) and $(A,B,B_\mathrm{e},C,D,D_\mathrm{e})$ of $\mathbf{P}^{-1}\mathbf{Q}$ and $\mathbf{P}^{-1}(\mathbf{Q},\mathbf{Q}_\mathrm{e})$, respectively, are minimal. In particular, the pair (A,B) is controllable. □

Whilst Theorem 3.2 has some overlap with [38, Theorem 5.1], we emphasize that, for our purposes, Theorem 3.2 is more appropriate than [38, Theorem 5.1]. In particular, it contains information relevant in Section 4 which is not included in [38, Theorem 5.1], for example, inequality (3.2) and statements (1) and (2).

4 Higher-order input-output Lur'e systems

We will now apply the results from Sections 2 and 3 to obtain stability criteria for input-output Lur'e systems of the form

$$\mathbf{P}(\mathcal{L})y = \mathbf{Q}(\mathcal{L})(f \circ y) + \mathbf{Q}_{\mathrm{e}}(\mathcal{L})v, \tag{4.1}$$

where $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}) \in \Sigma_{\mathrm{io}}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$ is continuous.

The following proposition relates the behaviour $\mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}, f)$ of the nonlinear input-output system (4.1) to the behaviour $\mathcal{B}(S^f)$ of the nonlinear state-space system

$$\mathcal{L}x = Ax + B(f \circ y) + B_{\mathrm{e}}v, \quad y = Cx + D(f \circ y) + D_{\mathrm{e}}v, \tag{4.2}$$

where $S = (A, B, B_{\mathrm{e}}, C, D, D_{\mathrm{e}})$ is the realization of $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}})$ having the properties guaranteed by Theorem 3.2 and $S^f := (A, B, B_{\mathrm{e}}, C, D, D_{\mathrm{e}}, f)$.

Proposition 4.1. *Let* $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}) \in \Sigma_{\mathrm{io}}$*, let* $f : \mathbb{R}^p \to \mathbb{R}^m$ *be a nonlinearity and let* S *be the realization of* $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}})$ *guaranteed to exist by Theorem 3.2. Then* $(v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}, f)$ *for every* $(v, x, y) \in \mathcal{B}(S^f)$*, the map*

$$\Lambda^f : \mathcal{B}(S^f) \to \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}, f),\ (v, x, y) \mapsto (v, y)$$

is a bijection and there exists $\lambda > 0$ *such that*

$$\|x(0)\| \leq \lambda\big(\|v^k\| + \|y^k\| + \|(f \circ y)^k\|\big) \quad \forall\, (v, x, y) \in \mathcal{B}(S^f), \tag{4.3}$$

where $k = \deg \mathbf{P}$*. In particular, if* f *is linearly bounded, that is,*

$$\sup_{\xi \in \mathbb{R}^p,\, \xi \neq 0} \frac{\|f(\xi)\|}{\|\xi\|} < \infty,$$

then there exists $\lambda^f > 0$ *such that*

$$\|x(0)\| \leq \lambda^f \big(\|v^k\| + \|y^k\|\big) \quad \forall\, (v, x, y) \in \mathcal{B}(S^f).$$

Proof. Set $n := \deg \det \mathbf{P}$ and write $S = (A, B, B_{\mathrm{e}}, C, D, D_{\mathrm{e}})$, where D and D_{e} are given by (3.1). By Theorem 3.2, the realization S is n-dimensional, the pair (C, A) is observable, the map

$$\Lambda : \mathcal{B}(S) \to \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}),\ (u, v, x, y) \mapsto (u, v, y)$$

is a vector space isomorphism and there exists $\lambda > 0$ such that (3.2) holds.

Let $(v, x, y) \in \mathcal{B}(S^f)$. Then $(f \circ y, v, x, y) \in \mathcal{B}(S)$ and thus, $(f \circ y, v, y) = \Lambda(f \circ y, v, x, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}})$. As a consequence, $(v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}, f)$, showing that Λ^f maps $\mathcal{B}(S^f)$ into $\mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}, f)$. Injectivity of Λ^f follows from observability of (C, A). To prove surjectivity of Λ^f, let $(v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}, f)$. Then $(f \circ y, v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e})$, and so, by surjectivity of Λ, there exists $x \in (\mathbb{R}^n)^{\mathbb{N}_0}$ such that $(f \circ y, v, x, y) \in \mathcal{B}(S)$. Therefore, $(v, x, y) \in \mathcal{B}(S^f)$ and so, $\Lambda^f(v, x, y) = (v, y)$, establishing surjectivity of Λ^f.

Finally, if $(v, x, y) \in \mathcal{B}(S^f)$, then $(f \circ y, v, x, y) \in \mathcal{B}(S)$ and therefore, by (3.2),

$$\|x(0)\| \leq \lambda\big(\|v^k\| + \|y^k\| + \|(f \circ y)^k\|\big),$$

completing the proof. □

We are now in the position to state and prove the main stability result of this paper.

Theorem 4.2. *Let* $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}) \in \Sigma_\mathrm{io}$, $f : \mathbb{R}^p \to \mathbb{R}^m$ *be continuous,* $r > 0$, $K \in \mathbb{R}^{m \times p}$ *and set* $\mathbf{G} = \mathbf{P}^{-1}\mathbf{Q}$. *Assume that* $\mathbb{B}_\mathbb{C}(K, r) \subset \mathbb{S}_\mathbb{C}(\mathbf{G})$ *and consider the conditions:*

(H1′) *Rank condition* (3.3) *holds and* $r \min_{|z|=1} \|\mathbf{G}^K(z)\| < 1$,

(H2′) $\mathbf{P}$ *and* $\mathbf{Q}$ *are left coprime and* $r\|\mathbf{G}^K(\infty)\| < 1$,

(H3′) $\mathbf{P}$ *and* $\mathbf{Q}$ *are left coprime.*

The following statements hold.

(1) *If* (H1′) *or* (H2′) *holds and*

$$\|f(\xi) - K\xi\| \leq r\,\|\xi\| \quad \forall\, \xi \in \mathbb{R}^p, \tag{4.4}$$

then there exists $\kappa \geq 1$ *such that, for all* $y \in \mathcal{B}_0(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}, f)$,

$$\|y(t)\| \leq \kappa \|y^k\| \quad \forall\, t \in \mathbb{N}_0, \tag{4.5}$$

where $k = \deg \mathbf{P}$.

(2) *If* (H1′) *or* (H2′) *is satisfied and*

$$\|f(\xi) - K\xi\| < r\,\|\xi\| \quad \forall\, \xi \in \mathbb{R}^p,\ \xi \neq 0, \tag{4.6}$$

then there exists $\kappa \geq 1$ *such that, for all* $y \in \mathcal{B}_0(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}, f)$, (4.5) *holds and* $y(t) \to 0$ *as* $t \to \infty$.

(3) *If* (H1′) *or* (H3′) *holds and there exists* $\alpha \in \mathcal{K}_\infty$ *such that*

$$\|f(\xi) - K\xi\| \leq r\,\|\xi\| - \alpha(\|\xi\|) \qquad \forall\, \xi \in \mathbb{R}^p, \tag{4.7}$$

then there exist $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such that for all* $(v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_\mathrm{e}, f)$

$$\|y(t)\| \leq \beta(\|y^k\|, t) + \gamma(\|v\|_t) \quad \forall\, t \in \mathbb{N}_0, \tag{4.8}$$

where $k = \deg \mathbf{P}$ *and* $\|v\|_t := \sup\{\|v(s)\| : s = 0, 1, \ldots, t\}$.

Note that if (3.3) does not hold, then the greatest common left divisor of $\mathbf{P}$ and $\mathbf{Q}$ is not Schur, implying that, for any linear f, the feedback system (4.1) is not stable in the sense of statements (2) or (3).

Statements (1) and (2) of the above theorem relate to the Lur'e system (4.1) with $v = 0$ and imply that the unforced feedback system is stable in a sense which is reminiscent of stability in the large and global asymptotic stability in the state-space context, respectively. The stability property in statement (3) corresponds very naturally to the ISS concept which applies to the state-space Lur'e systems (4.2). In the following, we will say that the input-output Lur'e system (4.1) is *input-to-output stable* (IOS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that (4.8) holds for all $(v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\rm e}, f)$.

Theorem 4.2 is reminiscent of the complex Aizerman conjecture [15, 16, 17, 30, 31] in the sense that the assumption of stability for all linear feedback gains in the *complex* ball $\mathbb{B}_{\mathbb{C}}(K, r)$ guarantees stability of the nonlinear Lur'e system for every nonlinearity f satisfying the "nonlinear" ball condition (4.4), (4.6) or (4.7). For a counterexample to the classical real Aizerman conjecture in discrete-time see [4, 12].

We present a simple example which shows that there exist input-output Lur'e system of the form (4.1) which are globally asymptotically stable in the sense of statement (2) of Theorem 4.2 but which are not IOS.

Example 4.3. Consider (4.1) with $\mathbf{P}(z) = z$, $\mathbf{Q}(z) = \mathbf{Q}_{\rm e}(z) = 1$, $K = 0$ and

$$f(\xi) = \begin{cases} \xi + 1 & \xi < -2, \\ \xi/2 & |\xi| \leq 2, \\ \xi - 1 & \xi > 2. \end{cases}$$

Then $\mathbf{G}(z) = 1/z$, $\mathbb{B}_{\mathbb{C}}(0, 1) \subset \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, (H2$'$) (and, a fortiori, (H3$'$)) holds and

$$|f(\xi)| = |\xi| - \varphi(|\xi|) \quad \forall\, \xi \in \mathbb{R}, \tag{4.9}$$

where

$$\varphi(s) = \begin{cases} s/2 & 0 \leq \xi \leq 2, \\ 1 & \xi > 2. \end{cases} \tag{4.10}$$

In particular, $|f(\xi)| < |\xi|$ for all $\xi \neq 0$, and so the conclusions of statement (2) of Theorem 4.2 hold. However, for $v(t) \equiv 2$ and $y(t) = 3 + t$, the pair (v, y) is in $\mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\rm e}, f)$, showing that (4.1) is not IOS. Note that it follows from (4.9) and (4.10) that there does not exist $\alpha \in \mathcal{K}_\infty$ such that (4.7) is satisfied. ◊

If (4.1) is IOS, then, trivially, (4.1) is bounded-input bounded-output (BIBO) stable in the sense that, for all $(v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\rm e}, f)$, if v is bounded, then y is bounded. The next example shows that an IOS Lur'e system of the form (4.1) is not necessarily BIBO stable with finite gain in the sense of "classical" input-output theory [7, 37].

Example 4.4. Consider the same system as in Example 4.3 (that is, $\mathbf{P}(z) = z$, $\mathbf{Q}(z) = \mathbf{Q}_{\rm e}(z) = 1$ and $K = 0$), but now with nonlinearity f given by

$$f(\xi) = \begin{cases} \xi + \pi/2 - 1 + \sqrt{|\xi| - \pi/2} & \xi < -\pi/2, \\ \sin \xi & |\xi| \le \pi/2, \\ \xi - \pi/2 + 1 - \sqrt{\xi - \pi/2} & \xi > \pi/2. \end{cases}$$

Defining $\alpha \in \mathcal{K}_\infty$ by

$$\alpha(s) = \begin{cases} s - \sin s & 0 \le s \le \pi/2, \\ \pi/2 - 1 + \sqrt{s - \pi/2} & s > \pi/2, \end{cases}$$

we have that

$$|f(\xi)| = |\xi| - \alpha(|\xi|) \quad \forall \, \xi \in \mathbb{R}.$$

Furthermore, $\mathbb{B}_{\mathbb{C}}(0,1) \subset \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ (where $\mathbf{G}(z) = 1/z$) and (H3$'$) is satisfied and thus, it follows from statement (3) of Theorem 4.2 that the Lur'e system is IOS.

For $\rho > 0$, let $\theta_\rho \in \mathbb{R}^{\mathbb{N}_0}$ denote the constant function with value ρ and let $y_\rho \in \mathbb{R}^{\mathbb{N}_0}$ be the unique solution of the initial-value problem

$$\mathcal{L}y = f \circ y + \theta_\rho, \quad y(0) = 0.$$

Obviously, $(\theta_\rho, y_\rho) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\rm e}, f)$ for every $\rho > 0$ and it is not difficult to show, by induction on t, that

$$\lim_{\rho \to \infty} \frac{y_\rho(t)}{\rho} = t = \lim_{\rho \to 0} \frac{y_\rho(t)}{\rho} \quad \forall \, t = 1, 2, 3, \ldots,$$

implying that the "linear" BIBO gain (also referred to as l^∞-gain) of the Lur'e system (in the sense of [37, Section 6.2]) is infinite. $\diamondsuit$

Proof of Theorem 4.2. Invoking Proposition 4.1, we conclude that there exists a realization $S := (A, B, B_{\rm e}, C, D, D_{\rm e})$ of $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\rm e})$ such that S has dimension equal to $\deg \det \mathbf{P}$, (C, A) is observable and the map

$$\Lambda^f : \mathcal{B}(S^f) \to \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\rm e}, f), \ (v, x, y) \mapsto (v, y)$$

is a bijection. Moreover, any of the conditions (4.4), (4.6) or (4.7) implies that f is linearly bounded, and so (again by Proposition 4.1) there exists a constant $\lambda^f > 0$ such that

$$\|x(0)\| \le \lambda^f \left(\|v^k\| + \|y^k\| \right) \quad \forall \, (v, x, y) \in \mathcal{B}(S^f). \tag{4.11}$$

If (H2$'$) or (H3$'$) holds, then, by Theorem 3.2, (A, B) is controllable. If (H1$'$) is satisfied, then, by Proposition 3.3, $I - DK$ is invertible and the matrix $L := K(I - DK)^{-1}$ is such that $\mathbf{P}(I + DL) - \mathbf{Q}L$ is Schur. Consequently, by Theorem 3.2, (A, B) is stabilizable. Furthermore, $C(zI - A)^{-1}B + D =$

$\mathbf{G}(z)$, and we conclude that if (H1′), (H2′) or (H3′) holds, then (H1), (H2) or (H3) holds, respectively, in the context of the linear state-space system $S = (A, B, B_e, C, D, D_e)$. Therefore, we may apply Theorem 2.2 to the nonlinear state-space system S^f.

Statements (1) and (2) now follow immediately from Theorem 2.2, the bijectivity of the map Λ^f and (4.11). To prove statement (3), we note that Theorem 2.2 guarantees the existence of comparison functions $\beta_0 \in \mathcal{KL}$ and $\gamma_0 \in \mathcal{K}$ such that

$$\|x(t)\| + \|y(t)\| \leq \beta_0(\|x(0)\|, t) + \gamma_0(\|v\|_t) \quad \forall\, t \in \mathbb{N}_0,\ \forall\,(v, x, y) \in \mathcal{B}(S^f).$$

Invoking (4.11) and the bijectivity of Λ^f, we obtain

$$\|y(t)\| \leq \beta_0(\lambda^f(\|y^k\|+\|v^k\|), t)+\gamma_0(\|v\|_t) \quad \forall\, t \in \mathbb{N}_0,\ \forall\,(v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_e, f).$$

Since $\beta_0 \in \mathcal{KL}$, the following estimate holds

$$\beta_0\big(\lambda^f(\|y^k\| + \|v^k\|), t\big) \leq \beta_0\big(2\lambda^f\|y^k\|, t\big) + \beta_0\big(2\lambda^f\|v^k\|, 0\big).$$

Together with

$$k\|v\|_t^2 \geq \sum_{j=0}^{k-1} \|v(j)\|^2 = \|v^k\|^2 \quad \forall\, t \geq k,$$

this leads to

$$\|y(t)\| \leq \beta_1(\|y^k\|, t) + \gamma(\|v\|_t) \quad \forall\, t \geq k,\ \forall\,(v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_e, f), \quad (4.12)$$

where $\beta_1(s, t) := \beta_0(2\lambda^f s, t)$ and $\gamma(s) := \beta_0(2\lambda^f\sqrt{k}s, 0)+\gamma_0(s)$. It is obvious that β_1 and γ are in $\mathcal{KL}$ and $\mathcal{K}$, respectively. Finally, defining $\beta : [0, \infty) \times \mathbb{N}_0 \to [0, \infty)$ by

$$\beta(s, t) = \begin{cases} s + \beta_1(s, k-1) & t = 0, \ldots, k-1 \\ \beta_1(s, t) & t = k, k+1, \ldots \end{cases}$$

it is clear that $\beta \in \mathcal{KL}$ and (4.12) implies that

$$\|y(t)\| \leq \beta(\|y^k\|, t) + \gamma(\|v\|_t) \quad \forall\, t \in \mathbb{N}_0,\ \forall\,(v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_e, f),$$

completing the proof of statement (3). □

In the corollary below, Theorem 4.2 is expressed in form of a "nonlinear small-gain" result.

Corollary 4.5. *Let* $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_e) \in \Sigma_{io}$*,* $f : \mathbb{R}^p \to \mathbb{R}^m$ *be continuous and* $K \in \mathbb{S}_{\mathbb{C}}(\mathbf{G})$*, where* $\mathbf{G} := \mathbf{P}^{-1}\mathbf{Q}$*. The following statements hold.*

(1) *If* (H1$'$) *or* (H2$'$) *is satisfied and*

$$\|\mathbf{G}^K\|_{H^\infty}\frac{\|f(\xi)-K\xi\|}{\|\xi\|}\leq 1 \quad \forall\,\xi\in\mathbb{R}^p,\,\xi\neq 0, \tag{4.13}$$

then there exists $\kappa\geq 1$ *such that* (4.5) *holds for all* $y\in\mathcal{B}_0(\mathbf{P},\mathbf{Q},\mathbf{Q}_{\rm e},f)$*.*

(2) *If* (H1$'$) *or* (H2$'$) *is satisfied and*

$$\|\mathbf{G}^K\|_{H^\infty}\frac{\|f(\xi)-K\xi\|}{\|\xi\|}< 1 \quad \forall\,\xi\in\mathbb{R}^p,\,\xi\neq 0, \tag{4.14}$$

then there exists $\kappa\geq 1$ *such that, for all* $y\in\mathcal{B}_0(\mathbf{P},\mathbf{Q},\mathbf{Q}_{\rm e},f)$*,* (4.5) *holds and* $y(t)\to 0$ *as* $t\to\infty$*.*

(3) *If* (H1$'$) *or* (H3$'$) *is satisfied and there exists* $\alpha\in\mathcal{K}_\infty$ *such that*

$$\|\mathbf{G}^K\|_{H^\infty}\frac{\|f(\xi)-K\xi\|}{\|\xi\|}\leq 1-\frac{\alpha(\|\xi\|)}{\|\xi\|} \quad \forall\,\xi\in\mathbb{R}^p,\,\xi\neq 0, \tag{4.15}$$

then the input-output Lur'e system (4.1) *is IOS.*

Note that (4.13)-(4.15) are not small-gain conditions in the sense of classical input-output theory of feedback systems (as presented, for example, in [7, 37]): whilst the RHS of, for example, (4.15) is smaller than 1 for all $\xi\neq 0$, it is in general not uniformly bounded away from 1. Indeed, it is possible that the RHS of (4.15) is converging to 1 as $\|\xi\|\to 0$ or $\|\xi\|\to\infty$. Therefore, rather than comparing Corollary 4.5 with classical small-gain theorems [7, 37], it is more appropriate to view it in the context of "modern" nonlinear ISS small-gain results, see for example [18, 19, 34]. However, we emphasize that Corollary 4.5 is not a special case of the general nonlinear small-gain theorems derived in [18, 19, 34].

Proof of Corollary 4.5. Setting $r=1/\|\mathbf{G}^K\|_{H^\infty}$, we have $\mathbb{B}_\mathbb{C}(K,r)\subset\mathbb{S}_\mathbb{C}(\mathbf{G})$ [31, Lemma 6]. The claim now follows from Theorem 4.2. □.

In the following, for a Hermitian matrix M, we will write $M\succeq 0$ ($M\succ 0$) if M is positive semi-definite (positive definite). Recall that a square rational matrix $\mathbf{H}$ is said to be positive real, if for every complex number $z\in\mathbb{E}$ which is not a pole of $\mathbf{H}$, we have that $\operatorname{Re}\mathbf{H}(z):=\big(\mathbf{H}(z)+\mathbf{H}^*(z)\big)/2\succeq 0$.

The following corollary can be considered is an extension of the well-known circle criterion to input-output Lur'e systems of the form (4.1). It shows that conditions very similar to those of the circle criterion for unforced state-space systems [10, 11] guarantee certain stability properties of (4.1), including IOS.

Corollary 4.6. *Let* $(\mathbf{P},\mathbf{Q},\mathbf{Q}_{\rm e})\in\Sigma_{\rm io}$*,* $f:\mathbb{R}^p\to\mathbb{R}^m$ *be continuous,* $K_1,K_2\in\mathbb{R}^{m\times p}$ *and set* $\mathbf{G}=\mathbf{P}^{-1}\mathbf{Q}$*. Assume that the rational matrix* $\mathbf{H}:=(I-K_2\mathbf{G})(I-K_1\mathbf{G})^{-1}$ *is positive real and consider the following conditions:*

(H1″) *Rank condition* (3.3) *holds and* $\operatorname{Re}\mathbf{H}(e^{i\theta}) \succ 0$ *for some* $\theta \in [0, 2\pi)$,

(H2″) $\mathbf{P}$ *and* $\mathbf{Q}$ *are left coprime and* $\operatorname{Re}\mathbf{H}(\infty) \succ 0$,

(H3″) $\mathbf{P}$ *and* $\mathbf{Q}$ *are left coprime.*

The following statements hold.

(1) *If* (H1″) *or* (H2″) *is satisfied,* $\ker(K_1 - K_2) = \{0\}$ *and*

$$\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \leq 0 \quad \forall\, \xi \in \mathbb{R}^p, \tag{4.16}$$

then there exists $\kappa \geq 1$ *such that* (4.5) *holds for all* $y \in \mathcal{B}_0(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}, f)$.

(2) *If* (H1″) *or* (H2″) *is satisfied and*

$$\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle < 0 \quad \forall\, \xi \in \mathbb{R}^p,\ \xi \neq 0, \tag{4.17}$$

then there exists $\kappa \geq 1$ *such that, for all* $y \in \mathcal{B}_0(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}, f)$, (4.5) *holds and* $y(t) \to 0$ *as* $t \to \infty$.

(3) *If* (H1″) *or* (H3″) *holds and there exists* $\alpha \in \mathcal{K}_\infty$ *such that*

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \leq -\alpha(\|\xi\|)\|\xi\| \quad \forall\, \xi \in \mathbb{R}^p, \tag{4.18}$$

then the input-output Lur'e system (4.1) *is IOS.*

Assume that $\mathbf{H}$ is positive real and, for any $\zeta \in \mathbb{C}^m$, define $h_\zeta : \mathbb{E} \to \mathbb{R}$ by $h_\zeta(z) = \operatorname{Re}\langle \mathbf{H}(z)\zeta, \zeta\rangle = \langle \operatorname{Re}\mathbf{H}(z)\zeta, \zeta\rangle$. Then $h_\zeta(z) \geq 0$ for all $z \in \mathbb{E}$, and, since $\mathbf{H}$ is holomorphic on $\mathbb{E}$, the function h_ζ is harmonic. Consequently, by the maximum principle for harmonic functions (applied to $-h_\zeta$), if the condition $\operatorname{Re}\mathbf{H}(\infty) \succ 0$ is violated, then there does not exist $\theta \in [0, 2\pi)$ such that $\operatorname{Re}\mathbf{H}(e^{i\theta}) \succ 0$. Consequently, if $\mathbf{H}$ is positive real and $\operatorname{Re}\mathbf{H}(e^{i\theta}) \succ 0$ for some $\theta \in [0, 2\pi)$, then $\operatorname{Re}\mathbf{H}(\infty) \succ 0$.

Proof of Corollary 4.6. We provide a proof of statement (3) only. Statements (1) and (2) can be proved by similar means. To prove statement (3), we proceed in several steps.

Step 1. Setting

$$M := \frac{1}{2}(K_1 + K_2), \quad N := \frac{1}{2}(K_1 - K_2), \tag{4.19}$$

it follows that

$$\operatorname{Re}\,\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle = \|f(\xi) - M\xi\|^2 - \|N\xi\|^2 \quad \forall\, \xi \in \mathbb{F}^p \tag{4.20}$$

Thus, by (4.18), $\ker N = \{0\}$. Hence, N^*N is invertible, and $N^\sharp := (N^*N)^{-1}N^* \in \mathbb{F}^{p\times m}$ is a left-inverse of N.

Define a new nonlinearity $\tilde{f} : \mathbb{R}^m \to \mathbb{R}^m$ by

$$\tilde{f}(\xi) = f(N^\sharp \xi) - K_1 N^\sharp \xi \quad \forall\, \xi \in \mathbb{R}^m. \tag{4.21}$$

The sector condition (4.18) together with arguments identical to those used in the proof of [31, Corollary 16] can be invoked to show that there exists a $\mathcal{K}_\infty$-function φ such that

$$\|\tilde{f}(\xi) + NN^\sharp \xi\| \le \|\xi\| - \varphi(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^m. \tag{4.22}$$

Step 2. Setting $\tilde{\mathbf{G}} := N\mathbf{G}^{K_1} = N\mathbf{G}(I - K_1\mathbf{G})^{-1}$, a routine calculation shows that

$$\mathbf{H} = (I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1} = I + 2N\mathbf{G}^{K_1} = I + 2\tilde{\mathbf{G}}. \tag{4.23}$$

Consequently, invoking the hypothesis that $\mathbf{H}$ is positive real, we conclude that

$$\|\tilde{\mathbf{G}}(I + \tilde{\mathbf{G}})^{-1}\|_{H^\infty} = \|(I - \mathbf{H})(I + \mathbf{H})^{-1}\|_{H^\infty} \le 1.$$

The identity

$$\tilde{\mathbf{G}}^{(-NN^\sharp)} = \tilde{\mathbf{G}}(I + NN^\sharp\tilde{\mathbf{G}})^{-1} = \tilde{\mathbf{G}}(I + N\mathbf{G}^{K_1})^{-1} = \tilde{\mathbf{G}}(I + \tilde{\mathbf{G}})^{-1}, \tag{4.24}$$

shows that $\|\tilde{\mathbf{G}}^{(-NN^\sharp)}\|_{H^\infty} \le 1$. Consequently, by [31, Lemma 6],

$$\mathbb{B}_{\mathbb{C}}(-NN^\sharp, 1) \subset \mathbb{S}_{\mathbb{C}}(\tilde{\mathbf{G}}). \tag{4.25}$$

Step 3. Let $S = (A, B, B_{\rm e}, C, D, D_{\rm e})$ be the realization of $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\rm e})$ guaranteed to exist by Theorem 3.2. In particular, (C, A) is observable. Invoking Proposition 4.1, we conclude that there exists $\lambda^f > 0$ such that

$$\|x(0)\| \le \lambda^f \big(\|v^k\| + \|y^k\|\big) \quad \forall\, (v, x, y) \in \mathcal{B}(S^f). \tag{4.26}$$

Since (H1$''$) or (H3$''$) holds, it follows from Theorem 3.2 and Proposition 3.3 that (A, B) is stabilizable or controllable, respectively.

Let us consider the state-space system $\tilde{S} = (\tilde{A}, \tilde{B}, \tilde{B}_{\rm e}, \tilde{C}, \tilde{D}, \tilde{D}_{\rm e})$, where

$$\tilde{A} := A + B(I - K_1D)^{-1}K_1C, \;\; \tilde{B} := B(I - K_1D)^{-1}, \;\; \tilde{B}_{\rm e} := \tilde{B}K_1D_{\rm e} + B_{\rm e},$$
$$\tilde{C} := N(I - DK_1)^{-1}C, \;\; \tilde{D} := N(I - DK_1)^{-1}D, \;\; \tilde{D}_{\rm e} := N(I - DK_1)^{-1}D_{\rm e}.$$

Obviously, for these definitions to be well-posed, we need to show that $I - K_1D$ is invertible (in which case $I - DK_1$ is also invertible). To this end, we observe that, by (4.23) and the positive realness of $\mathbf{H}$, the rational matrix $\tilde{\mathbf{G}}$ is proper. Consequently, $K_1N^\sharp\tilde{\mathbf{G}} = K_1\mathbf{G}(I - K_1\mathbf{G})^{-1} = (I - K_1\mathbf{G})^{-1} - I$, is proper, implying the properness of $(I - K_1\mathbf{G})^{-1}$, which in turn shows that $I - K_1D = I - K_1\mathbf{G}(\infty)$ is invertible.

As is well known (see, for example, [36, Remark 7.1.4]), $(\tilde{A}, \tilde{B}, N^\sharp C, N^\sharp D)$ is a realization of $\mathbf{G}^{K_1} = \mathbf{G}(I - K_1\mathbf{G})^{-1}$ and therefore, $\tilde{\mathbf{G}} = N\mathbf{G}^{K_1}$ is the transfer function of the state-space system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. It is clear that observability of (C, A) implies that of $(\tilde{C}, \tilde{A})$ and, if (A, B) is controllable or stabilizable, then $(\tilde{A}, \tilde{B})$ is controllable or stabilizable, respectively. Furthermore, if there exists $\theta \in [0, 2\pi)$ such that $\operatorname{Re}\mathbf{H}(e^{i\theta})$ is positive definite, then, since $\tilde{\mathbf{G}}(I + \tilde{\mathbf{G}})^{-1} = (I - \mathbf{H})(I + \mathbf{H})^{-1}$, it follows from (4.24) that

$$\|\tilde{\mathbf{G}}^{(-NN^\sharp)}(e^{i\theta})\| = \|\big(I - \mathbf{H}(e^{i\theta})\big)\big((I + \mathbf{H}(e^{i\theta})\big)^{-1}\| < 1,$$

whence

$$\min_{|z|=1} \|\tilde{\mathbf{G}}^{(-NN^\sharp)}(z)\| < 1.$$

We now conclude that (H1) or (H3) holds in the context given by the linear system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, the feedback gain $K = -NN^\sharp$ and the radius $r = 1$.

Step 4. By the previous three steps, Theorem 2.2 applies to the Lur'e system $\tilde{S}^{\tilde{f}} = (\tilde{A}, \tilde{B}, \tilde{B}_{\mathrm{e}}, \tilde{C}, \tilde{D}, \tilde{D}_{\mathrm{e}}, \tilde{f})$ defined by $\tilde{S}$ and $\tilde{f}$. Therefore, there exist $\tilde{\beta} \in \mathcal{KL}$ and $\tilde{\gamma} \in \mathcal{K}_\infty$ such that, for all $(v, x, y) \in \mathcal{B}(\tilde{S}^{\tilde{f}})$,

$$\|x(t)\| + \|y(t)\| \leq \tilde{\beta}(\|x(0)\|, t) + \tilde{\gamma}(\|v\|_t) \quad \forall\, t \in \mathbb{N}_0. \tag{4.27}$$

Finally, let $(v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}, f)$ be arbitrary. By Proposition 4.1 there exists unique $x \in (\mathbb{R}^n)_0^{\mathbb{N}}$ such that $(x, v, y) \in \mathcal{B}(S^f)$. A routine calculation shows that $(x, v, Ny) \in \mathcal{B}(\tilde{S}^{\tilde{f}})$ and consequently, by (4.27),

$$\|x(t)\| + \|Ny(t)\| \leq \tilde{\beta}(\|x(0)\|, t) + \tilde{\gamma}(\|v\|_t) \quad \forall\, t \in \mathbb{N}_0.$$

Now $y = N^\sharp(Ny)$, and so

$$\|y(t)\| \leq \|N^\sharp\|\big(\tilde{\beta}(\|x(0)\|, t) + \tilde{\gamma}(\|v\|_t)\big) \quad \forall\, t \in \mathbb{N}_0.$$

This, together with (4.26) and the argument used towards end of the proof of Theorem 4.2 shows that the input-output Lur'e system (4.1) is IOS, completing the proof of statement (3). □

Finally, we present an IOS result for input-output Lur'e systems of the form (4.1) which is the natural analogue to the "classical" circle criterion for unforced state-space Lur'e systems [10, 11]. To this end, we recall that a square rational matrix $\mathbf{H}$ is said to be strictly positive real, if there exists $\rho \in (0, 1)$ such that the rational matrix function $z \mapsto \mathbf{H}(\rho z)$ is positive real.

Corollary 4.7. *Let* $(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_{\mathrm{e}}) \in \Sigma_{\mathrm{io}}$, $f : \mathbb{R}^p \to \mathbb{R}^m$ *be continuous,* $K_1, K_2 \in \mathbb{R}^{m\times p}$ *and set* $\mathbf{G} = \mathbf{P}^{-1}\mathbf{Q}$. *If the rational matrix* $\mathbf{H} := (I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ *is strictly positive real,* $\ker(K_1 - K_2) = \{0\}$, *the nonlinearity satisfies*

$$\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \leq 0 \quad \forall\, \xi \in \mathbb{R}^p, \tag{4.28}$$

and (H1″) *or* (H3″) *holds, then the input-output Lur'e system* (4.1) *is IOS.*

Proof. Set $L := K_2 - K_1$, let $\lambda \geq 0$ and define

$$\mathbf{H}_\lambda := \big(I - (K_2 + \lambda L)\mathbf{G}\big)\big(I - (K_1 - \lambda L)\mathbf{G}\big)^{-1}.$$

By hypothesis, $\mathbf{H}_0$ is strictly positive real. We claim that there exists $\hat{\lambda} > 0$ such that $\mathbf{H}_\lambda$ is strictly positive real for all $\lambda \in [0, \hat{\lambda}]$. To this end, note that

$$\mathbf{H}_\lambda = I - (1 + 2\lambda)L\mathbf{G}\big(I - (K_1 - \lambda L)\mathbf{G}\big)^{-1}. \tag{4.29}$$

Since $\mathbf{H}_0$ is strictly positive real, it follows from a well-known result (see, for example, [11, Theorem 13.31]) that $\mathbf{H}_0 \in H^\infty(\mathbb{C}^{m\times m})$ and, furthermore, there exists $\delta > 0$ such that

$$\operatorname{Re}\mathbf{H}_0(e^{i\theta}) - \delta I \succ 0 \quad \forall\, \theta \in [0, 2\pi). \tag{4.30}$$

Since $\ker L = \{0\}$, the matrix L is left-invertible, and it follows from (4.29) (with $\lambda = 0$) that $\mathbf{G}(I - K_1\mathbf{G})^{-1} \in H^\infty(\mathbb{C}^{p\times m})$. Consequently, there exists $\tilde{\lambda} > 0$ such that $\mathbf{G}\big(I - (K_1 - \lambda L)\mathbf{G}\big)^{-1} \in H^\infty(\mathbb{C}^{p\times m})$ for all $\lambda \in [0, \tilde{\lambda}]$ and the map

$$[0, \tilde{\lambda}] \to H^\infty(\mathbb{C}^{m\times m}),\ \lambda \mapsto \mathbf{H}_\lambda$$

is continuous. Invoking (4.30), we conclude that there exists $\hat{\lambda} \in (0, \tilde{\lambda}]$ such that, for each $\lambda \in [0, \hat{\lambda}]$, $\operatorname{Re}\mathbf{H}_\lambda(e^{i\theta}) - (\delta/2)I \succ 0$ for all $\theta \in [0, 2\pi)$. It follows (see [11, Theorem 13.31]) that, for all $\lambda \in [0, \hat{\lambda}]$, $\mathbf{H}_\lambda$ is strictly positive real and, a fortiori, positive real.

The claim will follow from statement (3) of Corollary 4.6, provided we can show that, for $\lambda \in (0, \hat{\lambda}]$, there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\langle f(\xi) - (K_1 - \lambda L)\xi, f(\xi) - (K_2 + \lambda L)\xi\rangle \leq -\alpha(\|\xi\|)\|\xi\| \quad \forall\, \xi \in \mathbb{R}^p. \tag{4.31}$$

Invoking (4.28), a straightforward calculation shows that

$$\langle f(\xi) - (K_1 - \lambda L)\xi, f(\xi) - (K_2 + \lambda L)\xi\rangle \leq -\lambda(\lambda + 1)\|L\xi\|^2 \quad \forall\, \xi \in \mathbb{R}^p.$$

By left-invertibility of L, there exists $\mu > 0$ such that $\|L\xi\| \geq \mu\|\xi\|$ for all $\xi \in \mathbb{R}^p$, and so,

$$\langle f(\xi) - (K_1 - \lambda L)\xi, f(\xi) - (K_2 + \lambda L)\xi\rangle \leq -\mu^2\lambda(\lambda + 1)\|\xi\|^2 \quad \forall\, \xi \in \mathbb{R}^p,$$

showing that (4.31) holds with $\alpha(s) = \mu^2\lambda(\lambda + 1)s$. ☐

5 Appendix

Proof of Lemma 1.1. Set $\mathbf{N} := \mathbf{M}T$ and denote the entries of $\mathbf{M}$, $\mathbf{N}$ and T by $\mathbf{M}_{ij}$, $\mathbf{N}_{ij}$ and T_{ij}, respectively. Obviously, $\deg\det\mathbf{M} = \deg\det\mathbf{N}$ and, since $\mathbf{M}$ is row reduced

$$\sum_{i=1}^{p} r_i(\mathbf{M}) = \deg\det\mathbf{M} \leq \sum_{i=1}^{p} r_i(\mathbf{N}). \tag{5.1}$$

Furthermore,

$$\mathbf{N}_{ij}(z) = \sum_{k=1}^{p} \mathbf{M}_{ik}(z) T_{kj},$$

showing that $r_i(\mathbf{N}) \leq r_i(\mathbf{M})$ for all $i = 1, \ldots, p$. Together with (5.1) this implies that $r_i(\mathbf{N}) = r_i(\mathbf{M})$ for all $i = 1, \ldots, p$. □

We proceed to prove Proposition 3.1. To this end, it is useful to recall that the right-shift operator $\mathcal{R} : (\mathbb{R}^q)^{\mathbb{N}_0} \to (\mathbb{R}^q)^{\mathbb{N}_0}$ is defined by

$$(\mathcal{R}g)(t) = \begin{cases} 0 & t = 0, \\ g(t-1) & t = 1, 2, \ldots \end{cases}$$

Proof of Proposition 3.1. Statement (1) is a standard result and can be found, for example, in [9, Theorem S1.8].

To prove statement (2), let $(u, v) \in (\mathbb{R}^m)^{\mathbb{N}_0} \times (\mathbb{R}^{m_\mathrm{e}})^{\mathbb{N}_0}$ and set $y := G \star u$ and $y_\mathrm{e} := G_\mathrm{e} \star v$. Furthermore, we define

$$u_\mathrm{o} = \mathcal{R}^k u, \;\; v_\mathrm{o} = \mathcal{R}^k v, \;\; y_\mathrm{o} = \mathcal{R}^k y, \;\; y_\mathrm{eo} = \mathcal{R}^k y_\mathrm{e},$$

where $k = \deg \mathbf{P}$. Then, since $\mathcal{R}^k(G \star u) = G \star (\mathcal{R}^k u)$ and $\mathcal{R}^k(G_\mathrm{e} \star v) = G_\mathrm{e} \star (\mathcal{R}^k v)$

$$y_\mathrm{o} = \mathcal{R}^k(G \star u) = G \star u_\mathrm{o}, \quad y_\mathrm{eo} = \mathcal{R}^k(G_\mathrm{e} \star v) = G_\mathrm{e} \star v_\mathrm{o},$$

and, taking formal Z-transforms (denoted by superscript $\hat{}$), we obtain

$$\mathbf{P}\hat{y}_\mathrm{o} = \mathbf{Q}\hat{u}_\mathrm{o}, \quad \mathbf{P}\hat{y}_\mathrm{eo} = \mathbf{Q}_\mathrm{e}\hat{v}_\mathrm{o}. \tag{5.2}$$

Since $u_\mathrm{o}(t) = 0$, $v_\mathrm{o}(t) = 0$, $y_\mathrm{o}(t) = 0$ and $y_\mathrm{eo}(t) = 0$ for $t = 0, \ldots, k-1$, we have that

$$(\widehat{\mathbf{P}(\mathcal{L})y_\mathrm{o}})(z) = \mathbf{P}(z)\hat{y}_\mathrm{o}(z), \quad (\widehat{\mathbf{Q}(\mathcal{L})u_\mathrm{o}})(z) = \mathbf{Q}(z)\hat{u}_\mathrm{o}(z),$$

and

$$(\widehat{\mathbf{P}(\mathcal{L})y_\mathrm{eo}})(z) = \mathbf{P}(z)\hat{y}_\mathrm{eo}(z), \quad (\widehat{\mathbf{Q}_\mathrm{e}(\mathcal{L})v_\mathrm{o}})(z) = \mathbf{Q}_\mathrm{e}(z)\hat{v}_\mathrm{o}(z).$$

Together with (5.2) this implies

$$\mathbf{P}(\mathcal{L})(y_\mathrm{o} + y_\mathrm{eo}) = \mathbf{Q}(\mathcal{L})u_\mathrm{o} + \mathbf{Q}_\mathrm{e}(\mathcal{L})v_\mathrm{o}. \tag{5.3}$$

Since $\mathcal{L}$ commutes with $\mathbf{P}(\mathcal{L})$, $\mathbf{Q}(\mathcal{L})$ and $\mathbf{Q}_\mathrm{e}(\mathcal{L})$ and

$$\mathcal{L}^k u_\mathrm{o} = u, \;\; \mathcal{L}^k v_\mathrm{o} = v, \;\; \mathcal{L}^k y_\mathrm{o} = y, \;\; \mathcal{L}^k y_\mathrm{eo} = y_\mathrm{e},$$

it follows from (5.3) that

$$\mathbf{P}(\mathcal{L})(G \star u + G_\mathrm{e} \star v) = \mathbf{P}(\mathcal{L})(y + y_\mathrm{e}) = \mathbf{Q}(\mathcal{L})u + \mathbf{Q}_\mathrm{e}(\mathcal{L})v,$$

showing that $(u, v, G \star u + G_e \star v) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_e)$.

To prove statement (3), let $(u, v, y) \in (\mathbb{R}^m)^{\mathbb{N}_0} \times (\mathbb{R}^{m_e})^{\mathbb{N}_0} \times (\mathbb{R}^p)^{\mathbb{N}_0}$ and set $w := G \star u + G_e \star v$. By statement (2), $(u, v, w) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_e)$. Therefore, if $(u, v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_e)$, then $\mathbf{P}(\mathcal{L})y = \mathbf{P}(\mathcal{L})w$, showing that $y - w \in \ker \mathbf{P}(\mathcal{L})$. Conversely, if $y - w \in \ker \mathbf{P}(\mathcal{L})$. Then

$$\mathbf{P}(\mathcal{L})y = \mathbf{P}(\mathcal{L})(y - w) + \mathbf{P}(\mathcal{L})w = \mathbf{P}(\mathcal{L})w = \mathbf{Q}(\mathcal{L})u + \mathbf{Q}_e(\mathcal{L})v,$$

showing that $(u, v, y) \in \mathcal{B}(\mathbf{P}, \mathbf{Q}, \mathbf{Q}_e)$. □

Proof of Proposition 3.3. Assume that statement (1) holds. By hypothesis, $\mathbf{G}^K \in H^\infty(\mathbb{C}^{p\times m})$, and so $\lim_{|z|\to\infty} \mathbf{G}^K(z) =: \mathbf{G}^K(\infty) \in \mathbb{C}^{p\times m}$ exists. Since $\mathbf{G}^K(I - K\mathbf{G}) = \mathbf{G}$, we conclude that

$$\mathbf{G}^K(\infty)(I - KD) = D.$$

Let $\zeta \in \mathbb{C}^m$ and assume that $(I - KD)\zeta = 0$. Then, $D\zeta = 0$, and thus $\zeta = 0$, showing that $I - KD$ is invertible, which in turn is equivalent to the invertibility of $I - DK$. Let $\mathbf{L}$ be the greatest common left divisor of $\mathbf{P}$ and $\mathbf{Q}$. Invoking (3.3), it follows that $\mathbf{L}$ is Schur. Moreover, $\mathbf{P} = \mathbf{L}\mathbf{P}_0$ and $\mathbf{Q} = \mathbf{L}\mathbf{Q}_0$, where $\mathbf{P}_0$ and $\mathbf{Q}_0$ are left-coprime polynomial matrices. Obviously, $\mathbf{P}_0^{-1}\mathbf{Q}_0 = \mathbf{P}^{-1}\mathbf{Q} = \mathbf{G}$ and thus

$$(\mathbf{P}_0 - \mathbf{Q}_0K)^{-1}\mathbf{Q}_0 = (I - \mathbf{P}_0^{-1}\mathbf{Q}_0K)^{-1}\mathbf{P}_0^{-1}\mathbf{Q}_0 = \mathbf{G}^K \in H^\infty(\mathbb{C}^{p\times m}).$$

Left coprimeness of $\mathbf{P}_0$ and $\mathbf{Q}_0$ implies left coprimeness of $\mathbf{P}_0 - \mathbf{Q}_0K$ and $\mathbf{Q}_0$ and it follows that $\mathbf{P}_0 - \mathbf{Q}_0K$ is Schur. Now $\mathbf{L}$ is Schur and thus, $\mathbf{P} - \mathbf{Q}K = \mathbf{L}(\mathbf{P}_0 - \mathbf{Q}_0K)$ is Schur. Together with the invertibility of $I - DK$ this shows that statement (2) holds.

Conversely, assume that statement (2) is true. Since $\mathbf{P} - \mathbf{Q}K$ is Schur, it is clear that the rank condition (3.3) is satisfied and that

$$\mathbf{G}^K = \mathbf{G}(I - K\mathbf{G})^{-1} = (I - \mathbf{G}K)^{-1}\mathbf{G} = (\mathbf{P} - \mathbf{Q}K)^{-1}\mathbf{Q}$$

does not have any poles in $\mathbb{E}$. Finally, by the invertibility of $I - DK$, $\mathbf{G}^K$ does not have a pole at ∞ either and consequently, $\mathbf{G}^K \in H^\infty(\mathbb{C}^{p\times m})$, showing that statement (1) holds. □

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