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Phase transition in a sequential assignment problem on graphs

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Abstract

We study the following sequential assignment problem on a finite graph $G = (V, E)$. Each edge $e \in E$ starts with an integer value $n_e \geq 0$, and we write $n = \sum_{e \in E} n_e$. At time t, $1 \leq t \leq n$, a uniformly random vertex $v \in V$ is generated, and one of the edges f incident with v must be selected. The value of f is then decreased by 1. There is a unit final reward if the configuration $(0, \ldots, 0)$ is reached. Our main result is that there is a *phase transition*: as $n \to \infty$, the expected reward under the optimal policy approaches a constant $c_G > 0$ when $(n_e/n : e \in E)$ converges to a point in the interior of a certain convex set $\mathcal{R}_G$, and goes to 0 exponentially when $(n_e/n : e \in E)$ is bounded away from $\mathcal{R}_G$. We also obtain estimates in the near-critical region, that is when $(n_e/n : e \in E)$ lies close to $\partial\mathcal{R}_G$. We supply quantitative error bounds in our arguments.

Keywords: phase transition, critical phenomenon, stochastic sequential assignment, Markov decision process, stochastic dynamic programming, discrete stochastic optimal control.

1 Introduction

Consider the following game (known in different versions [6], [11, Section 1.7]). Players start with a row of N empty boxes. In each of N rounds, a random digit is generated, and each player has to place it into one of the empty boxes they have. A player's score is the N digit number obtained after the last round. The game is a special case of *sequential stochastic assignment* introduced by Derman, Lieberman and Ross [3]. In sequential assignment, there are N jobs with given values $p_1 \leq \cdots \leq p_N$ that have to be assigned to N workers, as they appear in sequence. The i-th worker has ability X_i, where $X_1, \ldots, X_N$ are i.i.d. random variables from a given distribution F. The reward from assigning the job of value p_i to a worker with ability x is $p_i x$, and the overall reward of the assignment is the sum of the individual rewards. The game mentioned at the start is recovered when $p_i = 10^{i-1}$, and X_i is uniform in $\{0, \ldots, 9\}$.

The paper [3] showed that there is a strategy that maximizes the expected score independently of what $p_1, \ldots, p_N$ are. This strategy has the following form. There are numbers $-\infty = a_{0,n} \leq a_{1,n} \leq \cdots \leq a_{n-1,n} \leq a_{n,n} = \infty$, $n \geq 1$, that only depend on the distribution F, such that if there are n jobs remaining to be assigned, with values $p'_1 \leq \cdots \leq p'_n$, and the next worker has ability x with $a_{i-1,n} \leq x \leq a_{i,n}$, then the worker is assigned to the job with value p'_i.

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Albright and Derman [1] showed, using law of large numbers type arguments, that when F is absolutely continuous, one has $\lim_{n\to\infty} a_{qn,n} = F^{-1}(q)$, $0 < q < 1$, as $n \to \infty$. In particular, when the number n of jobs is large, a worker with ability x should be assigned to a job with rank approximately qn, where $F^{-1}(q) = x$. Note that when F is discrete, this way of determining the asymptotics breaks down: when x is an atom of F, the graph of F^{-1} has a horizontal piece at height x. For large finite n, the value of q where the profile $a_{qn,n}$ crosses height x can be expected to be somewhere in the corresponding interval of constancy of F^{-1}, and its precise location can be expected to be governed by large deviation effects.

In order to motivate the subject of our paper, consider the following modification of the game mentioned at the beginning. Suppose that each digit can take the values $1, \dots, k$, with equal probability. Also suppose that the goal of the player is to maximize the probability of achieving the maximum possible score, that is to reach the unique final assignment consisting of k contiguous intervals of equal digits. Let τ be the first time when all k numbers have occurred at least once. At time τ, the empty boxes form $k-1$ intervals of lengths $n_1, \dots, n_{k-1}$, where $n - \tau = \sum_{i=1}^{k-1} n_i$. The i-th interval has a box filled with i adjacent to it on the right, and a box filled with $i+1$ adjacent to it on the left. It is plausible that there exist numbers $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_{k-1} < \alpha_k = 1$, such that for large n, under the optimal strategy, $n_i/n \sim \alpha_{i+1} - \alpha_i$, $i = 1, \dots, k-1$. We will be interested in the following question. Suppose that an alternative position is imposed on the player, where the intervals have length $n'_i \sim (\beta_{i+1} - \beta_i)n'$, $i = 1, \dots, k-1$, where $0 = \beta_1 < \beta_2 < \cdots < \beta_{k-1} < \beta_k = 1$. What is the behaviour of the probability that the player can achieve the maximal score *from this position*?

We show that the above probability displays a sharp transition in the limit $n' \to \infty$. When the vector $(\beta_{i+1} - \beta_i : i = 1, \dots, k-1)$ lies in the interior of a certain convex set $\mathcal{R}_k$, the probability approaches a positive constant, whereas it goes to 0 exponentially when the vector is at a positive distance from $\mathcal{R}_k$.

More generally, we consider the above transition on a general finite graph $G = (V, E)$ with vertices labelled $1, \dots, k$. The starting position is a vector $(n_e : e \in E)$, and $n = \sum_{e\in E} n_e$. When a number $1 \le i \le k$ is rolled, one of the edges f incident with vertex i is selected by the player, and the value assigned to edge f is decreased by 1. We assign a final reward of 1 when the configuration $(0, \dots, 0)$ is reached, and refer to this as 'winning'. In the game described at the beginning, the graph is a path of length $k-1$.

We believe the study of this model is interesting for a number of reasons.

1. Questions of reachability have been studied in control theory for a long time [10, Sections 19,20]. In our model, the controllable set $\mathcal{R}_G$, that allows the player to reach the state $(0, \dots, 0)$ with uniformly positive probability, has a simple characterization, which however involves the graph structure in a non-trivial way; see Eqn. (2) and Lemma 4. As we show, choosing the right control is only essential near $\partial\mathcal{R}_G$. We believe our model, that is tractable on a general graph, is a useful example system to have in understanding the behaviour of discrete controlled systems with spatial structure near critical regions. Indeed, the main technical effort in this paper is getting estimates in the near critical region, that we do in Section 3.

2. In deriving the optimal strategy for sequential assignment, Derman, Lieberman and Ross [3] used Hardy's inequality, of which we have no analogue on graphs. Our proofs work

without knowledge of the optimal strategy, and only rely on martingale and Lyapunov function techniques, as well as an explicit relationship between $\mathcal{R}_G$ and available controls. Thus our arguments may be adaptable to other models. It may be that the transition phenomenon itself can be established with less effort, given more information on the optimal strategy (see for example Question 1 in Section 4). Nevertheless, we believe that the quantitative bounds we derive are of independent interest.

3. As the title of this paper suggests, we view the transition studied in this paper as an instance of a critical phenomenon.[1] While such transitions are ubiquitous in stochastic control, we found little in the literature that connects them with critical phenomena. We believe that such a point of view can be beneficial, and was indeed our original motivation for this study. Examples of works in the physics literature that address an interplay between controllability and network structure are [9, 7, 13].

4. Further problems that are important for applications can be studied in our model or suitable modifications thereof. For example, we see no obvious *distributed* control, where vertices would only have local information about the graph structure.

1.1 Definition of the model

Throughout $G = (V, E)$ will be a finite connected simple graph (without multiple edges or loops). We write $k = |V|$, and assume $|E| \geq 2$ (the case with one edge being trivial). We write $\deg_G(v)$ for the degree of $v \in V$, and $\deg_F(v)$ for the degree of v in the subgraph of G induced by the set of edges $F \subset E$.

The state at time $0 \leq t \leq n$ is an integer vector $\mathbf{N}(t) = (N_e(t) : e \in E)$, where the starting state is $\mathbf{N}(0) = \mathbf{n} = (n_e : e \in E)$. Usually we will use capitalized letters for random variables or random processes, and lowercase letters for their possible values. We write $n = \sum_{e \in E} n_e$. Let $V_1, \dots V_n \in V$ be an i.i.d. sequence of vertices with $\mathbf{P}[V_i = v] = \frac{1}{k}$, $v \in V$, $i = 1, \dots, n$. If the player allocates V_t to the edge e incident with V_t, the state is updated as

$$\mathbf{N}(t) = \mathbf{N}(t-1) - \mathbf{1}^e, \quad \text{where} \quad \mathbf{1}^e = (1^e_f : f \in E), \quad 1^e_f = \begin{cases} 1 & \text{if } f = e; \\ 0 & \text{if } f \neq e. \end{cases}$$

The gambler wins if $\mathbf{N}(n) = (0, \dots, 0) \in \mathbb{N}^E$, and looses otherwise. We denote by $p_G(\mathbf{n})$ the probability of winning under the optimal strategy, when the starting state is $\mathbf{n}$. This satisfies

$$p_G(\mathbf{n}) = \frac{1}{k} \sum_{v \in V} \max_{e \in E: e \sim v} p_G(\mathbf{n} - \mathbf{1}^e), \tag{1}$$

known as the *optimality equation* [12, Section I.1], where $e \sim v$ means that e is incident with v.

We introduce some notation needed to state our main theorem. We write $\mathcal{S}_G$ for the probability simplex in $\mathbb{R}^E$, that is, the set of non-negative vectors $\mathbf{x} \in \mathbb{R}^E$ such that $\sum_{e \in E} x_e = 1$.

[1] A reader unfamiliar with critical phenomena can find a good introduction in the short text [4]. We note that such familiarity is not required for understanding this paper.

We define

$$
\begin{aligned}
d(F) &= \left|\{v \in V : \deg_F(v) = \deg_G(v)\}\right|, \quad \emptyset \subset F \subset E; \\
\mathcal{R}_G &= \left\{ \mathbf{x} \in \mathcal{S}_G : \text{for all } \emptyset \subsetneq F \subsetneq E \text{ we have } \sum_{e \in F} x_e > \frac{1}{k} d(F) \right\}; \\
\mathcal{I}_G &= \left\{ \mathbf{x} \in \mathcal{S}_G : \text{there exists } \emptyset \subsetneq F \subsetneq E \text{ such that } \sum_{e \in F} x_e < \frac{1}{k} d(F) \right\}.
\end{aligned} \tag{2}
$$

The letters 'd', '$\mathcal{R}$' and '$\mathcal{I}$' are intended to evoke 'degree', 'reachable' and 'inaccessible', as we explain. For any non-empty set F of edges, $\frac{d(F)}{k}$ is the probability that the player receives a vertex that has full degree in F. Any such vertex *must* be allocated to one of the edges in F. For starting positions $\mathbf{n} = (n_e : e \in E)$ where the proportion of space $\sum_{e \in F} n_e/n$ available at the beginning is smaller than $d(F)/k$, the probability of winning goes to 0 (as $n \to \infty$). Therefore, from the region $\mathcal{I}_G$ the winning position is asymptotically inaccessible. On the other hand, as we show in Theorem 1, if $\mathbf{n} = n\,\mathbf{x}$ with $\mathbf{x} \in \mathcal{R}_G$, then the winning position is asymptotically reachable from $\mathbf{n}$. As we point out in Section 2.1, the set $\mathcal{R}_G$ arises as the region of controllability for a simple (deterministic) linear control system associated to the game. It can be verified that when G is a tree with k vertices ($k \geq 3$) $\mathcal{R}_G$ is a parallelepiped. As we will not need this fact, we omit the proof.

Remark. The arguments we present in this paper are also applicable to the slightly more general model when $V_1, \ldots, V_n$ are not uniformly distributed (but still i.i.d.). Suppose $\mathbf{P}[V_i = v] = p_v$ with a probability vector $\mathbf{p} = (p_v : v \in V)$ such that $p_v > 0$ for all $v \in V$. In this case $\mathcal{R}_G$ and $\mathcal{I}_G$ are replaced by

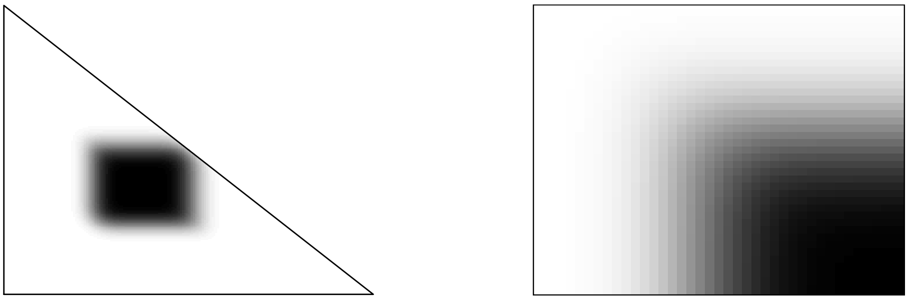
$$
\begin{aligned}
\mathcal{R}_{G,\mathbf{p}} &= \left\{ \mathbf{x} \in \mathcal{S}_G : \text{for all } \emptyset \subsetneq F \subsetneq E \text{ we have } \sum_{e \in F} x_e > \sum_{v : \deg_F(v) = \deg_G(v)} p_v \right\}; \\
\mathcal{I}_{G,\mathbf{p}} &= \left\{ \mathbf{x} \in \mathcal{S}_G : \text{there exists } \emptyset \subsetneq F \subsetneq E \text{ such that } \sum_{e \in F} x_e < \sum_{v : \deg_F(v) = \deg_G(v)} p_v \right\},
\end{aligned}
$$

As the required changes in the proofs are minor, but including them would burden the notation further, we state and prove the results only in the uniform case. All the essential difficulties are already present in the uniform model.

1.2 Main results

Theorems 1 and 2 below state our main results. Figure 1 illustrates these when G is a path of length three, that is $k = 4$.

Theorem 1. *Let G be a finite connected simple graph with $|E| \geq 2$.*
(i) If $\mathbf{x} \in \mathcal{I}_G$, and $\mathbf{n} = n\mathbf{x} + O(1)$, then $p_G(\mathbf{n}) \to 0$ exponentially fast, as $n \to \infty$, at a rate depending on $\mathbf{x}$. The rate of decay is bounded away from 0 on subsets bounded away from $\mathcal{R}_G$.
(ii) There exists a constant $c_G > 0$, such that if $\mathbf{x} \in \mathcal{R}_G$, and $\mathbf{n} = n\mathbf{x} + O(1)$, then $p_G(\mathbf{n}) \to c_G$, as $n \to \infty$.



(a) (b)

Figure 1: (a) Image of $p_G(m, 200-m-\ell, \ell)$ when G is a path of length three ($k = 4$) and $n = 200$. The limit of p_G is a positive constant in the rectangle $\frac{1}{4} < x = m/n, y = \ell/n < \frac{1}{2}$ (dark region), and goes to 0 when (x, y) is away from the rectangle (white region). The maximum of p_G is ≈ 0.2583299. (b) Detailed image of p_G near the corner of the critical region $0.15 \le m/n \le 0.35$, $0.4 \le \ell/n \le 0.6$.

In Section 3 we obtain bounds on the behaviour near $\partial\mathcal{R}_G$. These shows that the 'critical window' has width of order $\sqrt{n}$ around $n\partial\mathcal{R}_G$. Our bounds in particular imply the following upper bound on $p_G(\mathbf{n})$ in this region. Fix any $\delta > 0$, and let

$$\overline{M}_n = \overline{M}_n(\delta) = \max\left\{p_G(\mathbf{n}) : \mathbf{n}/n \in \mathcal{S}_G,\ \mathrm{dist}(\mathbf{n}/n, \partial\mathcal{R}_G) \le \delta\right\}.$$

Theorem 2. *For any $\delta > 0$ we have $\limsup_{n\to\infty} \overline{M}_n(\delta) \le c_G$.*

Combining Theorems 1 and 2 we obtain the following corollary.

Corollary 3. *The configuration $\mathbf{n}$ that maximizes $p_G(\mathbf{n})$ with n fixed, satisfies $p_G(\mathbf{n}) = c_G + o(1)$, as $n \to \infty$.*

Theorems 1 and 2 do not rule out the possibility that $p_G(\mathbf{n})$ is maximized near the critical surface, at a distance that is $o(n)$. But of course we expect that the location of the maximum, when rescaled by $1/n$, converges to a point in the interior of $\mathcal{R}_G$. It is also plausible that the location of this point can be characterized in terms of large deviation rates for events of the form 'the gambler runs out of space on the edges in F', that is:

$$\left\{ \sum_{v:\deg_F(v)=\deg_G(v)} \sum_{t=1}^{n} \mathbf{1}_{V_t=v} > \sum_{e\in F} n_e \right\}, \quad \emptyset \subsetneq F \subsetneq E.$$

We state an explicit conjecture for a path of length $k-1$, where this is easiest to formulate. Let

$$a_*(j;k) = \frac{\log\left(\frac{k-j-1}{k-j}\right)}{\log\left(\frac{j\,(k-j-1)}{(j+1)\,(k-j)}\right)}, \quad 1 \le j \le k-2 \qquad a_*(0;k) = 0 \qquad a_*(k-1;k) = 1.$$

Let $\mathbf{n}^{\max} = (n_j^{\max} : j = 1, \dots, k-1)$ denote a point in $n\,\mathcal{S}_G$ where $p_G(\mathbf{n})$ is maximized, $n \ge 1$.
Conjecture. Let $k \ge 3$. Then for $1 \le j \le k-2$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{j} n_j^{\max} = a_*(j;k).$$

The number $a_*(j;k)$ is obtained as the unique point $a \in \left(\frac{j}{k}, \frac{j+1}{k}\right)$, for which the 'cheaper' of the two large deviation events

$$\left\{\sum_{v=1}^{j}\sum_{t=1}^{n} \mathbf{1}_{V_t=v} > a\,n\right\} \quad \text{and} \quad \left\{\sum_{v=j+2}^{k}\sum_{t=1}^{n} \mathbf{1}_{V_t=v} > (1-a)\,n\right\}$$

is as 'expensive' as possible. (This number a can be obtained by equating the large deviation rates of the two events.) Each $a_*(j;k)$ marks out a linear submanifold of $\mathcal{S}_G$, and the location of the optimum is their intersection. We expect that a similar characterization holds for any connected graph G.

The structure of the paper is as follows. The proof of Theorem 1 is given in Section 2. We study the behaviour near $\partial\mathcal{R}_G$ in Section 3, and deduce Theorem 2. We stress however, that our analysis provides a much more refined picture than Theorem 2; see Propositions 10, 11 and 12, and their proof. The estimates in these propositions suggest Gaussian behaviour near $\partial\mathcal{R}_G$. We conclude with some further questions in Section 4.

2 Proof of the phase transition

The next section collects some preliminaries and useful notation.

2.1 Basic properties of $\mathcal{R}_G$

It will be convenient to have the version of $\mathcal{R}_G$ in which the inequalities are not strict:

$$\mathcal{K}_G = \left\{\mathbf{x} \in \mathcal{S}_G : \text{for all } F \subset E \text{ we have } \sum_{e\in F} x_e \ge \frac{1}{k} d(F)\right\}.$$

We denote by H_F the hyperplanes appearing in these inequalities:

$$H_F = \left\{\mathbf{x} \in \mathbb{R}^E : \sum_{e\in F} x_e = \frac{1}{k} d(F)\right\}, \emptyset \ne F \subset E.$$

In particular, $\mathcal{S}_G$, $\mathcal{R}_G$, $\mathcal{I}_G$ and $\mathcal{K}_G$ are all subsets of H_E.

Lemma 4.
(i) The sets $\mathcal{K}_G$ and $\mathcal{R}_G$ are convex with a non-empty interior relative to H_E.
(ii) $\mathcal{K}_G = \overline{\mathcal{R}_G}$ (the closure of $\mathcal{R}_G$ in H_E).

Proof. (i) As intersections of halfspaces with H_E, both $\mathcal{K}_G$ and $\mathcal{R}_G$ are convex. Also, since the halfspaces defining $\mathcal{R}_G$ (resp. $\mathcal{K}_G$) are open (resp. closed), $\mathcal{R}_G$ (resp. $\mathcal{K}_G$) is a relatively open (resp. closed) subset of H_E. The containment $\mathcal{R}_G \subset \mathcal{K}_G$ is immediate from the definitions. To show that $\mathcal{R}_G$ has non-empty interior, we check that the vector

$$\mathbf{x}^* = (x^*_e : e \in E), \quad x^*_e = \frac{1}{k} \sum_{\substack{v \in V \\ v \sim e}} \frac{1}{\deg(v)}, \quad e \in E, \tag{3}$$

belongs to $\mathcal{R}_G$. First, $\mathbf{x}^* \in H_E$ can be seen by summing the formula for x^*_e over $e \in E$ and exchanging the two sums. It is also immediate that $x^*_e > 0$, and therefore $\mathbf{x}^* \in \mathcal{S}_G$. Now fix any $\emptyset \subsetneq F \subsetneq E$. Since G is connected, there exists a vertex $v \in V$ such that $0 < \deg_F(v) < \deg_G(v)$. Therefore,

$$\begin{aligned}\sum_{e \in F} x^*_e &= \sum_{e \in F} \frac{1}{k} \sum_{\substack{v \in V \\ v \sim e}} \frac{1}{\deg(v)} = \frac{1}{k} \sum_{\substack{v \in V \\ \deg_F(v) = \deg_G(v)}} \sum_{\substack{e \in F \\ e \sim v}} \frac{1}{\deg_G(v)} + \frac{1}{k} \sum_{\substack{v \in V \\ \deg_F(v) < \deg_G(v)}} \sum_{\substack{e \in F \\ e \sim v}} \frac{1}{\deg_G(v)} \\ &> \frac{1}{k} \sum_{\substack{v \in V \\ \deg_F(v) = \deg_G(v)}} 1 = \frac{d(F)}{k}.\end{aligned}$$

This shows that $\mathbf{x}^* \in \mathcal{R}_G$, and since $\mathcal{R}_G$ is open in H_E, $\mathbf{X}^*$ is an interior point. The containment $\mathcal{R}_G \subset \mathcal{K}_G$ implies that $\mathbf{x}^*$ is also an interior point of $\mathcal{K}_G$.

(ii) Since $\mathcal{K}_G$ is closed, we have $\overline{\mathcal{R}_G} \subset \mathcal{K}_G$. Therefore, it is enough to show that $\mathcal{K}_G \setminus \mathcal{R}_G \subset \overline{\mathcal{R}_G}$. Let $\mathbf{x} \in \mathcal{K}_G \setminus \mathcal{R}_G$. Let $\mathbf{x}(t) = t\mathbf{x} + (1-t)\mathbf{x}^*$. Convexity of $\mathcal{K}_G$ implies that $\mathbf{x}(t) \in \mathcal{K}_G$ for all $0 \le t \le 1$. Moreover, since the expressions $\sum_{e \in F} x_e(t)$ are monotone linear functions of t, and $\sum_{e \in F} x_e(0) > d(F)/k$, and $\sum_{e \in F} x_e(1) \ge d(F)/k$, we must have the inequality $\sum_{e \in F} x_e(t) > \frac{1}{k} d(F)$ for all $0 \le t < 1$. This implies that $\mathbf{x}(t) \in \mathcal{R}_G$ for $0 \le t < 1$, and hence $\mathbf{x} \in \overline{\mathcal{R}_G}$, as required. ☐

The optimality equation implies that the optimal deterministic strategy is also optimal among randomized strategies. The next lemma states a connection between elements of $\mathcal{K}_G$ and possible moves in a randomized strategy. In its statement, we think of $q^{(v)}(e)$ as the probability of assigning vertex v to the edge e in such a move.

Lemma 5. *We have $\mathbf{x} \in \mathcal{K}_G$ if and only if there exists a collection $\{q^{(v)}(e) : v \in V, e \in E\}$ of non-negative numbers such that:*
(i) $\sum_{e \in E} q^{(v)}(e) = 1$ for all $v \in V$;
(ii) $q^{(v)}(e) = 0$ if e is not incident with v;
(iii) $\frac{1}{k} \sum_{v \in V} q^{(v)}(e) = x_e$ for all $e \in E$.

Proof. We deduce the statement from the Max-Flow-Min-Cut Theorem [2, Theorem III.1]. Define an auxilliary directed graph G' as follows. Replace each edge $\{v, w\}$ of G by two directed edges (v, u_e) and (w, u_e), introducing the new vertex u_e for each $e \in E$. Also add new vertices s and t. Add a directed edge (s, v) for each $v \in V$ and a directed edge (u_e, t) for each $e \in E$. Thus G' has $|V| + |E| + 2$ vertices and $2|E| + |V| + |E|$ edges.

Consider flows of strength 1 from s to t in G', where we assign capacity $1/k$ to each edge (s, v), $v \in V$, capacity 2 to each (v, u_e) and capacity x_e to each (u_e, t).

Suppose $q^{(v)}(e)$ satisfy (i)–(iii). Define a flow by letting $1/k$ flow on each (s, v), $q^{(v)}(e)/k$ flow on each (v, u_e), and x_e flow on each (u_e, t). This flow satisfies the capacity constraints, and it is a maximal flow, since $\{(s, v) : v \in V\}$ is a cut with value 1. Therefore any other other cut must have value at least 1. Given $\emptyset \subset F \subset E$, consider the cut

$$\{(s, v) : \deg_F(v) < \deg_G(v)\} \cup \{(u_e, t) : e \in F\}. \tag{4}$$

with value

$$\frac{k - d(F)}{k} + \sum_{e \in F} x_e = 1 - \frac{d(F)}{k} + \sum_{e \in F} x_e \geq 1.$$

This implies that $\mathbf{x} \in \mathcal{K}_G$.

For the converse, suppose that $\mathbf{x} \in \mathcal{K}_G$, and consider a maximal flow on G'. The conditions in the definition of $\mathcal{K}_G$ imply that all cuts of the form (4) have value ≥ 1, and the cut corresponding to $F = E$ has value 1. It is easy to check that any minimal cut is necessarily of this form, and therefore the maximal flow is 1. Letting $q^{(v)}(e)$ be k-times the amount flowing on (v, u_e) we obtain a collection satisfying (i)–(iii). □

Basic for Theorem 1 is the following computation. Suppose that our current state is $\mathbf{n} = n\mathbf{x}$, $\mathbf{x} \in \mathcal{S}_G$. Let $\{q^{(v)}(e)\}_{v \in V, e \in E}$ be a set of probabilities representing a randomized move (that is: $q_e^{(v)}$ is the probability that edge e will be used, conditional on the event that vertex v has been drawn). Let $\mathbf{N}' = (n-1)\mathbf{X}'$ be the random outcome of the move. Let $y_e = \frac{1}{k} \sum_{v \in V} q^{(v)}(e)$. We have

$$\mathbf{E}\mathbf{X}' = \frac{1}{n-1}\mathbf{E}\mathbf{N}' = \frac{1}{n-1}\left(\mathbf{n} - \sum_{e \in E} y_e \mathbf{1}^e\right) = \frac{n}{n-1}\mathbf{x} - \frac{1}{n-1}\mathbf{y} = \mathbf{x} + \frac{1}{n-1}(\mathbf{x} - \mathbf{y}). \tag{5}$$

If $\mathbf{x} \in \mathcal{R}_G$, then due to Lemma 5 it is possible to choose $\mathbf{y} \in \mathcal{K}_G$ in such a way that the average displacement points in any desired direction. On the other hand, if $\mathbf{x} \in \mathcal{I}_G$, convexity of $\mathcal{K}_G$ implies that the process will always move away from $\mathcal{R}_G$ on average.

The above observations are also reflected in the following deterministic controlled differential equation:

$$\frac{d\mathbf{x}}{dt} = \mathbf{x} - \mathbf{u}(t), \quad \text{where the control } \mathbf{u} \text{ satisfies } \mathbf{u}(t) \in \mathcal{K}_G \text{ for all } t \geq 0.$$

It is easy to see (for example using as Lyapunov function the distance from $H_E \cap H_F$ for suitable F) that:

(i) If $\mathbf{x}(0) \notin \mathcal{K}_G$, then for any control $\mathbf{u}$ we have $\mathbf{x}(t) \notin \mathcal{K}_G$ for all $t \geq 0$;

(ii) If $\mathbf{x}(0) \in \mathcal{R}_G$, then for any $\mathbf{x}' \in \mathcal{R}_G$ there exists a control $\mathbf{u}$ such that $\lim_{t \to \infty} \mathbf{x}(t) = \mathbf{x}'$.

Let us introduce some further notation. Throughout we write $\|\mathbf{w}\|_1 = \sum_{e\in E} |w_e|$ and $|\mathbf{w}| = \sqrt{\sum_{e\in E} |w_e|^2}$ for any vector $\mathbf{w} = (w_e : e \in E) \in \mathbb{R}^E$. For $\mathbf{w} \in \mathbb{R}^E$ and $A \subset \mathbb{R}^E$ we write $\mathrm{dist}(\mathbf{w}, A) = \inf_{\mathbf{y}\in A} |\mathbf{w} - \mathbf{y}|$. We will write $\langle \cdot, \cdot \rangle$ for the Euclidean scalar product.

For each $\emptyset \subsetneq F \subsetneq E$ we fix a point $\mathbf{z}^F \in \mathcal{K}_G$ such that $\sum_{e\in F} z_e^F = \frac{d(F)}{k}$. Let $\mathbf{u}^F$ be the unit vector of the form

$$u_e^F = \begin{cases} a_F & \text{if } e \in F; \\ -b_F & \text{if } e \in E \setminus F, \end{cases}$$

with $a_F, b_F > 0$, and such that $\sum_{e\in E} u_e^F = 0$. For all $\mathbf{w} \in \mathcal{K}_G$ we have $\langle \mathbf{w} - \mathbf{z}^F, \mathbf{u}^F \rangle \geq 0$. We will often use linear functions of the form:

$$L^{F,n}(\mathbf{n}) = \langle \mathbf{n} - n\mathbf{z}^F, \mathbf{u}^F \rangle = \sum_{e\in E} (n_e - n z_e^F) u_e^F.$$

The last expression can be rewritten as follows:

$$\begin{aligned}\sum_{e\in E} (n_e - n z_e^F) u_e^F &= a_F \sum_{e\in F} n_e + (-b_F) \left(n - \sum_{e\in F} n_e \right) - n a_F \sum_{e\in F} z_e^F - n(-b_F) \left(1 - \sum_{e\in F} z_e^F \right) \\ &= (a_F + b_F) \sum_{e\in F} n_e - n b_F - n(a_F + b_F) \sum_{e\in F} z_e^F + n b_F = (a_F + b_F) \left(\sum_{e\in F} n_e - n\frac{d(F)}{k} \right).\end{aligned}$$

We define $\kappa = \kappa(G) = \min\{(a_F + b_F) : \emptyset \subsetneq F \subsetneq E\} > 0$. We will need the following lemma.

Lemma 6. *There exist constants $b = b(G) > 0$ and $B = B(G)$ such that for all $\mathbf{w} \in \mathcal{K}_G$ we have*

$$b\,\mathrm{dist}(\mathbf{w}, \partial\mathcal{R}_G) \leq \min_{\emptyset\subsetneq F\subsetneq E} \left\{ \sum_{e\in F} w_e - \frac{d(F)}{k} \right\} \leq B\,\mathrm{dist}(\mathbf{w}, \partial\mathcal{R}_G). \tag{6}$$

We also have

$$\frac{1}{2} L^{F,n}(n\mathbf{w}) \leq n \left(\sum_{e\in F} w_e - \frac{d(F)}{k} \right) \leq \frac{1}{\kappa} L^{F,n}(n\mathbf{w}), \quad n \geq 1. \tag{7}$$

Proof. The proof of Lemma 4(ii) showed that $\mathcal{K}_G \setminus \mathcal{R}_G = \partial\mathcal{R}_G$. Therefore, if $\mathbf{w} \in \mathcal{K}_G \setminus \mathcal{R}_G$ then $\sum_{e\in F} w_e = d(F)/k$ for some $\emptyset \subsetneq F \subsetneq E$, and $\mathrm{dist}(\mathbf{w}, \partial\mathcal{R}_G) = 0$. In particular, the first statement of the lemma holds when $\mathbf{w} \in \mathcal{K}_G \setminus \mathcal{R}_G$. Henceforth assume that $\mathbf{w} \in \mathcal{R}_G$. Then since $\partial\mathcal{R}_G = \cup_{\emptyset\subsetneq F\subsetneq E} H_F \cap \mathcal{K}_G$, we have

$$\mathrm{dist}(\mathbf{w}, \partial\mathcal{R}_G) = \min_{\emptyset\subsetneq F\subsetneq E} \mathrm{dist}(\mathbf{w}, \mathcal{K}_G \cap H_F) \geq \min_{\emptyset\subsetneq F\subsetneq E} \mathrm{dist}(\mathbf{w}, H_E \cap H_F). \tag{8}$$

We claim that the last inequality is in fact an equality. Let F be a set for which the minimum in the right hand side of (8) is attained. Let $\mathbf{w}_0$ be the orthogonal projection of $\mathbf{w}$ onto $H_E \cap H_F$ in the linear space H_0. If the line segment $\mathbf{w}\,\mathbf{w}_0$ had any interior point $\mathbf{w}_1$ belonging to any other $H_{F'}$, then this would contradict the minimality of F'. Therefore, the entire line segment $\mathbf{w}\,\mathbf{w}_0$, apart from $\mathbf{w}_0$, belongs to $\mathcal{R}_G$, with $\mathbf{w}_0 \in \partial\mathcal{R}_G$. Hence $\mathrm{dist}(\mathbf{w}, H_E \cap H_F) = \mathrm{dist}(\mathbf{w}, \mathbf{w}_0) \geq$

$\mathrm{dist}(\mathbf{w}, \partial\mathcal{R}_G)$. This proves our claim. Since $\mathbf{w} \in H_E$, there exists a constant B_0, that only depends on $\min\{\text{angle between } H_E \text{ and } H_F : \emptyset \subsetneq F \subsetneq E\}$, such that

$$\mathrm{dist}(\mathbf{w}, H_F) \le \mathrm{dist}(\mathbf{w}, H_E \cap H_F) \le B_0 \, \mathrm{dist}(\mathbf{w}, H_F).$$

This implies the first statement of the lemma, since $\mathrm{dist}(\mathbf{w}, H_F) = |F|^{-1/2}\left(\sum_{e \in F} w_e - \frac{d(F)}{k}\right)$. The second statement of the lemma follows from the definition of $\kappa(G)$, and the fact that $a_F, b_F \le 1$ (since $\mathbf{u}^F$ is a unit vector). □

Recall that we write $\mathbf{n} = n\mathbf{x}$ for the starting state. Given a randomized strategy, we write $\mathbf{X}(t) = \frac{1}{n-t}\mathbf{N}(t)$. Note that we allow the processes $\mathbf{N}(t)$, $\mathbf{X}(t)$, etc. to have negative entries, and once this happens, we have $\mathbf{X}(t) \notin \mathcal{S}_G$ for all further times. We write $\mathbf{Y}(t-1)$ for the vector of edge weights that our strategy prescribes for round t, and $E(t) \in E$ for the random edge selected in round t according to this strategy. We write

$$\mathcal{F}_t = \sigma\left(\mathbf{N}(s),\, \mathbf{Y}(s) : 0 \le s \le t\right)$$

for the filtration of the process.

2.2 Steering

In the following proposition we show that if n is large enough, then starting from any state in $\mathcal{R}_G$ that is bounded away from the boundary, there is a strategy that steers the process close to any other such point in $\mathcal{R}_G$.

Proposition 7. *Given $\delta > 0$, there exist $c_1 = c_1(G, \delta) > 0$, $\lambda_1 = \lambda_1(G, \delta) > 0$, $n_0 = n_0(G, \delta)$, $K_1 = K_1(G, \delta)$ and $C_1 = C_1(G, \delta)$ such that the following holds. Let n and n_1 be any positive integers such that $n \ge (1 + K_1)n_1$ and $n_1 \ge n_0$. Suppose that $\mathbf{n} = n\mathbf{x}$ with $\mathrm{dist}(\mathbf{x}, \partial\mathcal{R}_G) \ge \delta$. Suppose also that $\mathbf{z} \in \mathcal{R}_G$ with $\mathrm{dist}(\mathbf{z}, \partial\mathcal{R}_G) \ge \delta$, with $n_1\mathbf{z}$ having integer coordinates. There exists a randomized strategy starting from state $\mathbf{n}$ such that under this strategy we have:*

$$\mathbf{P}[\mathbf{N}(n - n_1) = n_1\mathbf{z}] \ge c_1; \tag{9}$$

and for all $q \ge 1$ we have

$$\mathbf{P}\left[|\mathbf{N}(n - n_1) - n_1\mathbf{z}| > q\right] \le C_1 \exp(-\lambda_1 q). \tag{10}$$

The strategy will be defined in three stages: in the first stage we reduce $|\mathbf{N}(t) - (n-t)\mathbf{z}|$ to $O(1)$; in the second stage we keep it within $O(1)$ until time $n - n_1 - O(1)$; and we use the last $O(1)$ steps to attempt to hit $n_1\mathbf{z}$ exactly. The first two of these steps are the content of the next two lemmas. After proving the lemmas we assemble them to prove Proposition 7.

Lemma 8. *Given $\delta > 0$ there exists $K_2 = K_2(G, \delta)$, $d_0 = d_0(\delta)$, $\lambda_2 = \lambda_2(G, \delta) > 0$ and $C_2 = C_2(G)$ such that for any $\mathbf{x}, \mathbf{z}$ with $\mathrm{dist}(\mathbf{x}, \partial\mathcal{R}_G), \mathrm{dist}(\mathbf{z}, \partial\mathcal{R}_G) \ge \delta$ the following holds. For any n, n' with $n \ge K_2 n'$ and n' large enough there is a randomized strategy starting from state $\mathbf{n} = n\mathbf{x}$ such that the stopping time*

$$\tau_{d_0} = \inf\{t \ge 0 : |\mathbf{N}(t) - (n-t)\mathbf{z}| \le d_0\}$$

satisfies

$$\mathbf{P}[\tau_{d_0} > n - n'] \leq C_2 \exp(-\lambda_2 n'). \tag{11}$$

Proof. The value of $d_0 > 0$ will be chosen in course of the proof. We are also going to use a small parameter $0 < \varepsilon_0 < \delta/4$, chosen later. The first step of the proof is to reach an ε_0-neighbourhood of $\mathbf{z}$.

Let $\mathbf{y}$ be the point where the halfline starting at $\mathbf{z}$ and passing through $\mathbf{x}$ intersects $\partial\mathcal{R}_G$. Let $\mathbf{u}$ denote the unit vector with the same direction as $\mathbf{x} - \mathbf{z}$. In the first step, we use the following strategy: given the current state $\mathbf{N}(t) = (n-t)\mathbf{X}(t)$, we select $\mathbf{Y}(t) \in \partial\mathcal{R}_G$ such that $\mathbf{Y}(t) - \mathbf{X}(t)$ is a positive multiple of $\mathbf{u}$. In particular, $\mathbf{Y}(0) = \mathbf{y}$. We employ this strategy until the stopping time $\tau(1)$ defined by

$$\tau(1) = \inf\{t \geq 0 : |\mathbf{X}(t) - \mathbf{z}| \leq \varepsilon_0\}.$$

Let us write $\mathbf{X}^{\mathrm{ort}}(t)$ for the component of the vector $\mathbf{X}(t) - \mathbf{z}$ orthogonal to $\mathbf{u}$. Let

$$S(t) = \langle \mathbf{N}(t) - (n-t)\mathbf{z}, \mathbf{u} \rangle. \tag{12}$$

Since

$$\mathbf{N}(t+1) = (\mathbf{N}(t) - \mathbf{Y}(t)) + \left(\mathbf{Y}(t) - \mathbf{1}^{E(t+1)}\right),$$

and the second term has mean $\mathbf{0}$ given $\mathcal{F}_t$, we have

$$\mathbf{E}[S(t+1) \,|\, \mathcal{F}_t] = S(t) - \langle \mathbf{Y}(t) - \mathbf{z}, \mathbf{u} \rangle. \tag{13}$$

Since $\mathbf{x}$ and $\mathbf{z}$ are bounded away from $\partial\mathcal{R}_G$, there exist $\mu = \mu(G,\delta) > 1$ and $\varepsilon_0 = \varepsilon_0(G,\delta) > 0$ such that as long as $|\mathbf{X}^{\mathrm{ort}}(t)| \leq \frac{\varepsilon_0}{2}$, we have

$$\langle \mathbf{Y}(t) - \mathbf{z}, \mathbf{u} \rangle \geq \mu |\mathbf{x} - \mathbf{z}|. \tag{14}$$

This implies that $S'(t) = S(t) + t\mu|\mathbf{x} - \mathbf{z}|$ is a supermartingale as long as $|\mathbf{X}^{\mathrm{ort}}(t)| \leq \varepsilon_0/2$. On the other hand, due to the calculation in (5), $\mathbf{X}^{\mathrm{ort}}(t)$ is a martingale.

Let $t_1 = \frac{1+\mu}{2\mu} n$. Due to the choice of μ and ε_0, we have the inclusions

$$\begin{aligned}
\{\tau(1) > t_1\} \subset \{&|\mathbf{X}^{\mathrm{ort}}(s)| > \varepsilon_0/2 \text{ for some } 0 \leq s \leq t_1\} \\
&\cup \{S(s) > \mu(n-s)|\mathbf{x} - \mathbf{z}| \text{ for some } 0 \leq s \leq t_1\} \cup \{S(t_1) \geq 1\} \\
\subset &\left\{\max_{0 \leq s \leq t_1} |\mathbf{X}^{\mathrm{ort}}(s)| > \varepsilon_0/2\right\} \cup \left\{\max_{0 \leq s \leq t_1} S'(s) - S'(0) > (\mu - 1)n|\mathbf{x} - \mathbf{z}|\right\} \\
&\cup \left\{\max_{0 \leq s \leq t_1} S'(s) - S'(0) > \frac{\mu - 1}{2} n|\mathbf{x} - \mathbf{z}|\right\}.
\end{aligned} \tag{15}$$

The inclusions (15) imply

$$\mathbf{P}[\tau(1) > t_1] \leq \mathbf{P}\left[\max_{0 \leq s \leq t_1} S'(s) - S'(0) > \frac{\mu - 1}{2} n|\mathbf{x} - \mathbf{z}|\right] + \mathbf{P}\left[\max_{0 \leq s \leq t_1} |\mathbf{X}^{\mathrm{ort}}(s)| > \varepsilon_0/2\right]. \tag{16}$$

Since $S'(t)$ has increments bounded by $(1+\mu)\sqrt{2}$, while $|\mathbf{X}^{\mathrm{ort}}(t+1) - \mathbf{X}^{\mathrm{ort}}(t)| \le \sqrt{2}/(n-t-1)$, we can apply the Azuma-Hoeffding inequality (see [14, Exercise E14.2] or [5, Theorem 12.2(3)]) to $\{S'(t)\}_{t\ge 0}$ as well as to the projection of $\{\mathbf{X}^{\mathrm{ort}}(t)\}_{t\ge 0}$ to each coordinate direction. This yields

$$\begin{aligned}\mathbf{P}[\tau(1) > t_1] &\le \exp\left(-\frac{(\mu-1)^2}{8}\frac{n^2|\mathbf{x}-\mathbf{z}|^2}{t_1\,2\,(1+\mu)^2}\right) + 2|E|\exp\left(-\frac{1}{8}\frac{\varepsilon_0^2}{t_1|E|\sum_{s=1}^{t_1}\frac{2}{(n-s)^2}}\right) \\ &\le C'\exp(-\lambda' n)\end{aligned} \tag{17}$$

for some $\lambda' = \lambda'(\mu, \varepsilon_0) > 0$ and $C' = C'(G)$.

For the second step we condition on the point $\mathbf{n}_1 = n_1\mathbf{x}_1 = \mathbf{N}(\tau(1))$, such that $n - n_1 \le t_1$ and $|\mathbf{x}_1 - \mathbf{z}| \le \varepsilon_0 < \delta/4$. For ease of notation, we re-parametrize time for this step so that $\mathbf{N}(0) = \mathbf{n}_1$. We choose $\mathbf{Y}(t)$ to be the point where the halfline starting at $\mathbf{z}$ and passing through $\mathbf{X}(t)$ intersects $\partial\mathcal{R}_G$. Let us write $\mathbf{u}(t)$ for the unit vector with the same direction as $\mathbf{X}(t) - \mathbf{z}$. Decompose $\mathbf{X}(t+1) - \mathbf{z} = X'(t+1)\mathbf{u}(t) + \mathbf{X}''(t+1)$, where $\langle \mathbf{X}''(t+1), \mathbf{u}(t)\rangle = 0$. As long as $|\mathbf{N}(t) - (n-t)\mathbf{z}| \ge d_0$, we have

$$\begin{aligned}|\mathbf{N}(t+1) - (n-t-1)\mathbf{z}| &= \sqrt{\langle \mathbf{N}(t+1) - (n-t-1)\mathbf{z}, \mathbf{u}(t)\rangle^2 + (n-t-1)^2|\mathbf{X}''(t+1)|^2} \\ &\le \sqrt{\langle \mathbf{N}(t+1) - (n-t-1)\mathbf{z}, \mathbf{u}(t)\rangle^2 + 2} \\ &\le \langle \mathbf{N}(t+1) - (n-t-1)\mathbf{z}, \mathbf{u}(t)\rangle + \frac{2}{d_0 - \sqrt{2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{E}\big(|\mathbf{N}(t+1) - (n-t-1)\mathbf{z}| \,\big|\, \mathcal{F}_t\big) &\le \mathbf{E}\big(\langle \mathbf{N}(t+1) - (n-t-1)\mathbf{z}, \mathbf{u}(t)\rangle \,\big|\, \mathcal{F}_t\big) + \frac{2}{d_0 - \sqrt{2}} \\ &= \langle \mathbf{N}(t) - (n-t)\mathbf{z}, \mathbf{u}(t)\rangle - \langle \mathbf{Y}(t) - \mathbf{z}, \mathbf{u}(t)\rangle + \frac{2}{d_0 - \sqrt{2}} \\ &\le |\mathbf{N}(t) - (n-t)\mathbf{z}| - \delta + \frac{2}{d_0 - \sqrt{2}}.\end{aligned}$$

Hence if we require that $d_0 \ge \sqrt{2} + \frac{4}{\delta}$, then

$$D(t) = |\mathbf{N}(t) - (n-t)\mathbf{z}| + \frac{\delta}{2}t, \quad t \ge 0,$$

is a supermartingale until τ_{d_0}. Since the increments of $D(t)$ are bounded by $2 + \frac{\delta}{2} < 3$, and $\varepsilon_0 < \frac{\delta}{4}$, it follows with $t_2 = \frac{3}{4}n_1$ that

$$\mathbf{P}\left[\tau_{d_0} > t_2\right] \le \mathbf{P}\left[\max_{0\le s\le t_2}(D(s) - D(0)) > \frac{\delta}{8}n_1\right] \le \exp\left(-\frac{\delta^2 t_2^2}{64\cdot 3^2\, t_2}\right) \le \exp(-\lambda'' n_1)$$

with some $\lambda'' = \lambda''(\delta) > 0$.

Putting the two parts together, the statement follows if we choose $K_2 = \frac{8\mu}{\mu-1}$. □

Lemma 9. *Given $\delta > 0$ there exist $\lambda_3 = \lambda_3(\delta) > 0$ and $C_3 = C_3(\delta)$ such that such that for all $n' \geq n'' \geq 0$ and all $\mathbf{w}, \mathbf{z} \in \mathcal{K}_G$ with $\operatorname{dist}(\mathbf{z}, \partial\mathcal{R}_G) \geq \delta$, $|n'\mathbf{w} - n'\mathbf{z}| \leq d_0(\delta)$ the following holds. There exists a randomized strategy starting in state $\mathbf{n}' = n'\mathbf{w}$ such that for all $q \geq 1$ we have*

$$\mathbf{P}\left[|\mathbf{N}(n' - n'') - n''\mathbf{z}| > q\right] \leq C_3 \exp(-\lambda_3 q). \tag{18}$$

Proof. When $|\mathbf{N}(t) - (n' - t)\mathbf{z}| < d_0$, let us apply an arbitrary move, otherwise, let us follow the strategy used in the second part of Lemma 8. We saw in the proof of Lemma 8 that

$$D(t) = |\mathbf{N}(t) - (n - t)\mathbf{z}| + \frac{\delta}{2} \sum_{0 \leq s < t} I[|\mathbf{N}(s) - (n - s)\mathbf{z}| \geq d_0]$$

is a supermartingale on any time interval $s \in [t_1, t_2)$ on which $|\mathbf{N}(s) - (n - s)\mathbf{z}| \geq d_0$. Assume the event

$$F(q) = \left\{|\mathbf{N}(n' - n'') - n''\mathbf{z}| > 4q\right\},$$

and suppose $q > d_0$. When $n' - n'' < q$, the event $F(q)$ is impossible, because $|\mathbf{N}(0) - n'\mathbf{z}| \leq d_0 < q$ and the increments of $|\mathbf{N}(t) - (n - t)\mathbf{z}|$ are bounded by 2. Hence we may assume that $\ell_{\max} := \lfloor (n' - n'')/q \rfloor \geq 1$. Since $D(0) \leq d_0 < q$, the inequalities

$$|\mathbf{N}(n' - n'' - \ell q) - (n'' + \ell q)\mathbf{z}| > 4q, \quad \ell = 0, \ldots, \ell_{\max}, \tag{19}$$

cannot all simultaneously be satisfied. Summing over the smallest ℓ for which (19) fails, we have

$$\begin{aligned} \mathbf{P}[F(q)] &\leq \sum_{1 \leq \ell \leq \ell_{\max}} \mathbf{P}\left[D(n' - n'') - D(n' - n'' - \ell q) > \frac{\delta}{2} q\ell\right] \\ &\leq \sum_{\ell \geq 1} \exp\left(-\frac{1}{8} \frac{\delta^2 q^2 \ell^2}{3^2 q\ell}\right) \leq C_3 \exp(-\lambda_3 q). \end{aligned} \tag{20}$$

Adjusting the constant C_3, if necessary, we have the statement for all $q > 0$. This completes the proof. ☐

Remark. Note that the above strategy does not require the coordinates to stay positive. This will become important in Section 3.3.

Proof of Proposition 7. Observe that if there is no point $\mathbf{w}$ such that $\operatorname{dist}(\mathbf{w}, \partial\mathcal{R}_G) \geq \delta$, then the statement of the Proposition holds vacuously. Henceforth assume that δ is small enough so that the set above is non-empty. We choose $q_0 \geq 2$ so that for the event $F(q)$ introduced in the proof of Lemma 9 we have $\mathbf{P}[F(q_0/4)] \leq \frac{1}{2}$. Let M be the smallest integer such that

$$M \geq \left(\min\left\{w_e : e \in E,\ \mathbf{w} \in \mathcal{R}_G,\ \operatorname{dist}(\mathbf{w}, \partial\mathcal{R}_G) \geq \delta\right\}\right)^{-1},$$

which is finite by our assumption on δ. We choose K_1 and n_0 such that $n \geq K_1 n_1$ and $n_1 \geq n_0$ imply $n \geq K_2(n_1 + Mq_0)$, where K_2 is the constant from Lemma 8. Following the strategies in Lemmas 8 and 9 over the time interval $[n, n - n_1 - Mq_0]$ we have

$$\mathbf{P}\left[|\mathbf{N}(n - n_1 - Mq_0) - (n_1 + Mq_0)| \leq q_0\right] \geq \frac{1}{2} - C_2 \exp(-\lambda_2 n_1) \geq \frac{1}{4}, \tag{21}$$

if n_0 is large enough. On the event in (21) we have

$$\begin{aligned}
&N_e(n-n_1-Mq_0)-n_1z_e\\
&\quad\geq (Mq_0)z_e-|N_e(n-n_1-Mq_0)-(n_1+Mq_0)z_e|\\
&\quad\geq q_0-q_0=0, \quad e\in E.
\end{aligned}$$

Therefore, $\mathbf{N}(n-n_1-Mq_0)\geq n_1\mathbf{z}$ componentwise, and there is a strictly positive probability $c_1=c_1(G,\delta)>0$ that $n_1\mathbf{z}$ can be hit exactly from the state $\mathbf{N}(n-n_1-Mq_0)$. This proves (9) of the Proposition. Since the form of the bound (18) is not affected by taking Mq_0 extra steps, statement (10) follows from the estimates (11) and (18) of Lemmas 8 and 9. □

2.3 Proof of the Main Theorem

In this section we complete the proof of Theorem 1.

Proof of Theorem 1(i). Fix $\mathbf{x}\in\mathcal{I}_G$, and let $\emptyset\subsetneq F\subsetneq E$ be a set such that $\sum_{e\in F}x_e<\frac{d(F)}{k}$. Then for some $\varepsilon=\varepsilon(G,\mathbf{x})>0$ and sufficiently large n we have $\frac{1}{n}\sum_{e\in F}N_e(0)<\frac{d(F)}{k}-\varepsilon$. Let

$$Y_t=\begin{cases}1 & \text{if } V_t=v \text{ and } \deg_F(v)=\deg_G(v);\\ 0 & \text{otherwise.}\end{cases}$$

Since any v with $\deg_F(v)=\deg_G(v)$ must be assigned to one of the edges in F, we have

$$p_G(\mathbf{n})\leq\mathbf{P}\left[\sum_{t=1}^n Y_t\leq\sum_{e\in F}N_e(0)\right]\leq\mathbf{P}\left[\frac{1}{n}\sum_{t=1}^n Y_t<\frac{d(F)}{k}-\varepsilon\right]\leq\exp\left(-n\frac{\varepsilon^2}{4}\right).$$

by Bernstein's inequality; see [5, Theorem 2.2(1)]. The rate of decay is bounded away from 0 as long as $\mathbf{x}$ is bounded away from $\partial\mathcal{R}_G$. □

Proof of Theorem 1(ii). We show that for any fixed $\delta>0$ we have

$$\lim_{n\to\infty}M_n=\lim_{n\to\infty}m_n=\alpha, \tag{22}$$

where

$$\begin{aligned}
m_n=m_n(\delta)&=\min\left\{p_G(\mathbf{n}):\sum_{e\in E}n_e=n,\ \mathrm{dist}(\mathbf{n}/n,\partial\mathcal{R}_G)\geq\delta\right\}, \quad n\geq 1;\\
M_n=M_n(\delta)&=\max\left\{p_G(\mathbf{n}):\sum_{e\in E}n_e=n,\ \mathrm{dist}(\mathbf{n}/n,\partial\mathcal{R}_G)\geq\delta\right\}, \quad n\geq 1;\\
\alpha=\alpha(\delta)&=\liminf_{n\to\infty}m_n(\delta).
\end{aligned}$$

We consider $n'\geq n_0$, $n\geq K_1n'$ and $\mathbf{n}=n\mathbf{x}$ such that $m_n=p_G(\mathbf{n})$. We apply Proposition 7 with $\mathbf{z}=\mathbf{n}'/n'$, where $\mathbf{n}'$ is chosen so that $M_{n'}=p_G(\mathbf{n}')$.

Let $\varphi(\mathbf{r})$ denote the probability that with the strategy described in Proposition 7 the state at time $n - n'$ is $n'\mathbf{z} + \mathbf{r}$, where $\sum_{e \in E} r_e = 0$. Due to Proposition 7, we have $\varphi(0) \geq c_1$. Therefore, we can write

$$\begin{aligned} m_n = p_G(\mathbf{n}) &\geq \sum_{\mathbf{r}: \sum_{e \in E} r_e = 0} \varphi(\mathbf{r})\, p_G(n'\mathbf{z} + \mathbf{r}) \geq c_1 p_G(n'\mathbf{z}) + \sum_{\substack{\mathbf{r} \neq \mathbf{0}: \\ \sum_{e \in E} r_e = 0}} \varphi(\mathbf{r})\, p_G(n'\mathbf{z} + \mathbf{r}) \\ &\geq c_1 (M_{n'} - m_{n'}) + \sum_{\mathbf{r}: \sum_{e \in E} r_e = 0} \varphi(\mathbf{r})\, m_{n'} \\ &\geq c_1 (M_{n'} - m_{n'}) + m_{n'} - C \exp(-\lambda n') \end{aligned}$$

with some $\lambda > 0$ and C depending on δ and $\lambda_1, \lambda_2, \lambda_3$. Rearranging gives

$$M_{n'} - m_{n'} \leq \frac{1}{c_1}(m_n - m_{n'}) + \frac{C}{c_1} \exp(-\lambda n'). \tag{23}$$

Since $n \geq Kn'$ was arbitrary, taking $\liminf_{n \to \infty}$ yields

$$M_{n'} - m_{n'} \leq \frac{1}{c_1}(\alpha - m_{n'}) + \frac{C}{c_1} \exp(-\lambda n'). \tag{24}$$

Taking $\limsup_{n' \to \infty}$ in (24) yields $M_{n'} - m_{n'} \to 0$. Taking $\liminf_{n' \to \infty}$ in (24) yields

$$0 \leq \liminf_{n' \to \infty} (M_{n'} - m_{n'}) \leq \frac{1}{c_1}(\alpha - \limsup_{n' \to \infty} m_{n'}) \leq 0.$$

This shows that $\lim_{n' \to \infty} m_{n'} = \alpha$, and the proof of (22) is complete.

The limit does not depend on δ, since for $0 < \delta_1 < \delta_2$ we have

$$m_n(\delta_1) \leq m_n(\delta_2) \leq M_n(\delta_2) \leq M_n(\delta_2),$$

and hence $\alpha(\delta_1) = \alpha(\delta_2) = c_G$.

We conclude the proof by noting that $c_G > 0$. This is because Proposition 7 implies that the process can be steered close to the point $n_0\mathbf{x}^*$ for a sufficiently large n_0 with positive probability, and from here there is a strictly positive probability of winning. □

Remark. Since the left hand side of (24) is non-negative, we can rearrange to get

$$m_{n'} \leq \alpha + C \exp(-\lambda n'), \quad n' \geq n_0.$$

We do not have a corresponding exponential lower bound on the speed at which the limit α is approached. See Question 1 in Section 4.

3 Upper bounds in the critical region

In this section we obtain estimates in the critical region. This requires distinguishing a few cases that we state as separate propositions in the next section, and use them to prove Theorem 2. The proofs of the three propositions are given in Sections 3.2, 3.3 and 3.4, respectively.

3.1 Statements of upper bounds in three subregions

We define the sets of configurations

$$\begin{aligned}\mathcal{B}^I_G(n;A) &= \left\{\mathbf{n}\in n\mathcal{S}_G:\ \text{for some } \emptyset \subsetneq F \subsetneq E \text{ we have } L^{F,n}(\mathbf{n}) \le -A\sqrt{n}\right\}\\ \mathcal{B}^{II}_G(n;A) &= \left\{\mathbf{n}\in n\mathcal{S}_G:\ \text{for all } F \text{ with } 0<d(F)<k \text{ we have } L^{F,n}(\mathbf{n}) \ge A\sqrt{n}\right\}\\ \mathcal{B}^{III}_G(n;A) &= \left\{\mathbf{n}\in n\mathcal{S}_G:\ -A\sqrt{n} < \min_{F:0<d(F)<k} L^{F,n}(\mathbf{n}) < A\sqrt{n}\right\}.\end{aligned} \tag{25}$$

Proposition 10. *For all $A > 0$ we have*

$$\limsup_{n\to\infty} \max\{p_G(\mathbf{n}) : \mathbf{n}\in\mathcal{B}^I_G(n;A)\} \le \exp\left(-\frac{A^2}{8}\right).$$

In particular, the lim sup *is at most c_G, if $A \ge \sqrt{8\log(1/c_G)}$.*

Proposition 11. *There exist constants $C_4 = C_4(G)$ and $\lambda_4 = \lambda_4(G) > 0$ such that for all $A \ge 1$ we have*

$$\limsup_{n\to\infty} \max\{p_G(\mathbf{n}) : \mathbf{n}\in\mathcal{B}^{II}_G(n;A)\} \le c_G + C_4\exp(-\lambda_4 A^2). \tag{26}$$

Proposition 12. *There exists $A_0 = A_0(G)$ such that for all $A \ge A_0$ we have*

$$\limsup_{n\to\infty} \max\{p_G(\mathbf{n}) : \mathbf{n}\in\mathcal{B}^{III}_G(n;A)\} \le c_G + C_4\exp(-\lambda_4 A^2).$$

Proof of Theorem 2 assuming Propositions 10, 11, 12. Given $\varepsilon > 0$, choose A sufficiently large so that each of the upper bounds in Propositions 10, 11 and 12 is at most $c_G + \varepsilon$. Since with this fixed choice of A the sets $\mathcal{B}^I_G$, $\mathcal{B}^{II}_G$ and $\mathcal{B}^{III}_G$ cover all possibilities, the statement follows. □

3.2 Upper bound for $\mathcal{B}^I_G$

Proof of Proposition 10. We may fix the set F in the definition of $\mathcal{B}^I_G(n;A)$ and argue separately for each such set. Let us fix $\delta > 0$. Due to Theorem 1(i), we may restrict to $\mathbf{n}$ such that

$$-\delta n < L^{F,n}(\mathbf{n}) \le -A\sqrt{n}.$$

Let us follow the optimal strategy starting in configuration $\mathbf{n}$. The process $S(t) = L^{F,n-t}(\mathbf{N}(t))$ is a supermartingale due to

$$\mathbf{E}[S(t+1)\,|\,\mathcal{F}_t] = S(t) - \langle \mathbf{Y}(t) - \mathbf{z}^F, \mathbf{u}^F\rangle \le S(t). \tag{27}$$

Consider the stopping time

$$\tau = \left(\lfloor n - c\sqrt{n}\rfloor + 1\right) \wedge \inf\{t \ge 0 : S(t) < -\delta(n-t)\},$$

where $c = \frac{A}{2\delta}$. Then we have

$$\begin{aligned}\mathbf{P}[\tau > n - c\sqrt{n}] &\le \mathbf{P}\left[\max_{0\le t\le \lfloor n-c\sqrt{n}\rfloor} S(t) - S(0) > (A-\delta c)\sqrt{n}\right]\\ &\le \exp\left(-\frac{1}{2}\frac{(A-\delta c)^2\, n}{\lfloor n - c\sqrt{n}\rfloor}\right) \le \exp\left(-\frac{A^2}{8}\right).\end{aligned}$$

Due to the optimality equation, $p_G(\mathbf{N}(t))$ is a bounded martingale. Hence by optional stopping we have

$$p_G(\mathbf{n}) = \mathbf{E}[p_G(\mathbf{N}(\tau));\, \tau \le n - c\sqrt{n},\, S(\tau) < -\delta(n-\tau)] + \mathbf{E}[p_G(\mathbf{N}(\tau));\, \tau > n - c\sqrt{n}], \quad (28)$$

The first term in the right hand side of (28) is at most

$$\max\left\{p_G(\mathbf{n}') : \|\mathbf{n}'\|_1 \ge c\sqrt{n},\, L^{F,n'}(\mathbf{n}') < -\delta n'\right\},$$

which goes to 0, as $n \to \infty$, due to Theorem 1(i). The second term in the right hand side of (28) is at most $\mathbf{P}[\tau > n - c\sqrt{n}] \le \exp(-\frac{A^2}{8}) < c_G$, due to our choice of A. This completes the proof of the Proposition. □

3.3 Upper bound for $\mathcal{B}_G^{II}$

We start with two propositions that strengthen Proposition 7, and will be used in the proof of Proposition 11. In the first, we give a lower bound on the probability that the process can be steered away from the boundary, if at least order $\sqrt{n}$ away.

Proposition 13. *There exist $\lambda_5 = \lambda_5(G) > 0$, $\gamma = \gamma(G) > 0$, $c_5 = c_5(G)$, $C_5 = C_5(G)$ and $n_0' = n_0'(G)$ such that for all $A \ge 1$ the following holds. Let n, n' satisfy $n^\gamma \ge n' \ge n_0'$, and let $\mathbf{n} = n\mathbf{x}$ be a configuration such that*

$$\sum_{e \in F} x_e \ge \frac{1}{k} d(F) + \frac{A}{\sqrt{n}}, \quad \text{for all } \emptyset \subsetneq F \subsetneq E. \quad (29)$$

There exists a randomized strategy starting from $\mathbf{n}$ such that for the stopping time

$$\tau = \inf\{t \ge 0 : \mathrm{dist}(\mathbf{X}(t), \partial\mathcal{R}_G) \ge c_5\}$$

we have

$$\mathbf{P}[\tau > n - n'] \le C_5 \exp(-\lambda_5 A^2).$$

Proof. Let $\mathbf{y}$ be the point where the halfline starting at $\mathbf{x}^*$ and passing through $\mathbf{x}$ intersects $\partial\mathcal{R}_G$. Write $d = |\mathbf{x} - \mathbf{y}|$, and note that $d \ge \frac{A}{B}\frac{1}{\sqrt{n}}$, due to Lemma 6. Let r be the smallest integer such that $(3/2)^r d \ge \frac{1}{2}|\mathbf{x}^* - \mathbf{y}|$. We fix a small number $\eta > 0$ such that $\frac{1}{2} - \eta > \frac{4}{9}$. Then it is straightforward to check that the choice of r ensures that there exists $0 < \gamma = \gamma(G) < 1$ such that $(\frac{1}{2} - \eta)^r n \ge n^\gamma$, if $n \ge n_0$ for some $n_0 = n_0(G)$.

Consider the sequence of points $\mathbf{x} = \mathbf{y}(0), \mathbf{y}(1), \dots, \mathbf{y}(r)$ defined by

$$\mathbf{y}(i) = \mathbf{y} + (3/2)^i(\mathbf{x} - \mathbf{y}), \quad i = 0, 1, \dots, r.$$

The following statement can be proved in essentially the same way as Lemma 8. For $\varepsilon > 0$ sufficiently small, there exists $\lambda = \lambda(G, \eta, \varepsilon) > 0$ such that given any point $\mathbf{w} \in \mathcal{R}_G$ with $|\mathbf{w} - \mathbf{y}(i)| < \varepsilon(3/2)^i d$ and any n such that $(\frac{1}{2} - \eta)n \ge n_0$ the following holds. There exists a randomized strategy starting in state $n\mathbf{w}$ such that for the stopping time

$$\tau(i) = \inf\{t \ge 0 : |\mathbf{X}(t) - \mathbf{y}(i+1)| < \varepsilon(3/2)^{i+1} d\}$$

we have

$$\mathbf{P}\left[\tau(i) > \left(\frac{1}{2}+\eta\right)n\right] \leq \exp\left(-\lambda(3/2)^{2i}A^2\right).$$

Summing the upper bounds on $\tau(0), \tau(1), \ldots, \tau(r-1)$ we obtain that there is a randomized strategy starting from state $\mathbf{n}$ such that for the stopping time

$$\tau' = \inf\{t \geq 0 : |\mathbf{X}(t) - \mathbf{y}(r)| < \varepsilon(3/2)^r d\}$$

we have

$$\mathbf{P}[\tau' > n - n^\gamma] \leq C\exp(-\lambda A^2).$$

Due to the choice of r, and for a sufficiently small ε, the point $\mathbf{X}(\tau')$ is at least a fixed positive distance c_5 from $\partial\mathcal{R}_G$, and hence $\tau \leq \tau'$. This completes the proof. □

The next proposition extends the result of Proposition 7 to the case when the target state is anywhere in $\mathcal{K}_G$.

Proposition 14. *Given $\delta > 0$, there exists $\lambda_6 = \lambda_6(G) > 0$, $C_6 = C_6(G)$, $c_6 = c_6(G) > 0$, $K_6 = K_6(G, \delta)$ and $n_6 = n_6(G, \delta)$ such that for any $n_1 \geq K_6 n'$, $n' \geq n_6$ and configurations $\mathbf{n}_1 = n_1\mathbf{x}$, $\mathbf{x} \in \mathcal{R}_G$, $\operatorname{dist}(\mathbf{x}, \partial\mathcal{R}_G) \geq \delta$ and $\mathbf{n}' = n'\mathbf{z}$, $\mathbf{z} \in \mathcal{K}_G$ the following holds. There exists a randomized strategy starting in state $\mathbf{n}_1$ such that*

$$\mathbf{P}\left[\mathbf{N}(n_1 - n') = \mathbf{n}'\right] \geq c_6, \tag{30}$$

and

$$\mathbf{P}\left[|\mathbf{N}(n_1 - n') - \mathbf{n}'| > q\right] \leq C_6\exp(-\lambda_6 q), \quad q > 0. \tag{31}$$

Proof. We consider the following intermediate point:

$$\mathbf{x}'' = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{x}' \qquad \text{and} \qquad \mathbf{n}'' = n'\mathbf{x} + \mathbf{n}' + O(1),$$

where the $O(1)$ term guarantees that $\mathbf{n}''$ has integer coordinates. Observe that $\operatorname{dist}(\mathbf{x}'', \partial\mathcal{R}_G)$ is at least a positive constant. Due to Proposition 7 we can steer the process from $\mathbf{n}_1$ to a $(\delta/4)$-neighbourhood of $\mathbf{x}''$ with probability at least $1 - C_1\exp(-\lambda_1 n')$, provided $K_6 \geq 2K_1(G, \delta)$. Let us call the point reached this way $(2n')\mathbf{y}''$. Since

$$\mathbf{y}'' = \mathbf{x}'' + (\mathbf{y}'' - \mathbf{x}'') = \frac{1}{2}(\mathbf{x} - 2(\mathbf{y}'' - \mathbf{x}'')) + \frac{1}{2}\mathbf{x}',$$

and $|2(\mathbf{y}'' - \mathbf{x}'')| < \frac{\delta}{2}$, the point $\mathbf{w} = \mathbf{x} - 2(\mathbf{y}'' - \mathbf{x}'')$ satisfies $\operatorname{dist}(\mathbf{w}, \partial\mathcal{R}_G) \geq \frac{\delta}{2}$.

Now consider the steps of the strategy of Lemma 9 for the starting state $n'\mathbf{w}$ and target state $0\mathbf{w}$ over the time interval $[0, n' - Mq_0]$, where $M \geq (\min\{w_e : e \in E\})^{-1}$, and q_0 is chosen so that $F(q_0/4) \geq \frac{1}{2}$. Let $\widetilde{\mathbf{N}}(t)$, $t \geq 0$ denote this process. If the coordinates do stay positive until time $n' - Mq_0$, there is a strictly positive probability of hitting state $\mathbf{0}$. When $\mathbf{0}$ is not hit exactly, we have the bound

$$\mathbf{P}[|\widetilde{\mathbf{N}}(n')| > q] = \mathbf{P}[|\widetilde{\mathbf{N}}(n') - \mathbf{0}| > q] \leq C_2\exp(-\lambda_2 q).$$

If we now apply exactly the same moves to the configuration $(2n')\mathbf{y}''$, we obtain that the process $\mathbf{N}(t) = \mathbf{n}' + \widetilde{\mathbf{N}}(t)$ hits $\mathbf{n}' = n'\mathbf{x}'$ with positive probability, and satisfies the bound in (31). □

Since the proof of Proposition 11 is quite long, we first give a brief outline. Suppose we can select configurations $\mathbf{n}$ and $\mathbf{n}(\ell), \dots, \mathbf{n}(1)$ in such a way that:
(a) $\mathbf{n}/n$ is bounded away from $\partial\mathcal{R}_G$, so that we have $p_G(\mathbf{n}) \le c_G + \varepsilon$;
(b) $\mathbf{n}(\ell), \dots, \mathbf{n}(1)$ are in the respective sets $\mathcal{B}_G^{II}$ with each $p_G(\mathbf{n}(i))$ close to the lim sup in (26);
(c) We can steer the process as follows: $\mathbf{n} \to \mathbf{n}(\ell) \to \mathbf{n}(\ell-1) \to \cdots \to \mathbf{n}(1)$;
(d) In each steering step we hit the target exactly with probability bounded away from 0.
If ℓ is large, step (d) ensures that $p_G(\mathbf{n})$ cannot be much smaller than the smallest of the $p_G(\mathbf{n}(i))$'s, and the claim will follow. The crux of the proof is parts (c)–(d), which rely on Propositions 13 and 14. The argument is somewhat delicate, since the $\mathbf{n}(i)$'s now can be arbitrarily close to $\partial\mathcal{R}_G$; recall the definition of $\mathcal{B}_G^{II}$ in (25). Therefore, Propositions 13 and 14 will be applied on a suitable subgraph that omits some edges.

We carry out the plan (a)–(d). We start with some preliminaries. The first step is to subdivide $\mathcal{B}_G^{II}$ according to which part of $\partial\mathcal{R}_G$ is close. Given $\mathbf{n} \in \mathcal{B}_G^{II}$, let

$$\mathcal{G} = \mathcal{G}(\mathbf{n}; G, A) = \left\{ F \subset E : L^{F,n}(\mathbf{n}) < \frac{\kappa A}{2^{|E|+1}}\sqrt{n} \right\} \qquad \text{and} \qquad \overline{F} = \cup\mathcal{G},$$

where κ is the constant from Lemma 6. It may so happen that $\overline{F} = \emptyset$, in which case the arguments we have to make are similar to and simpler than when $\overline{F} \ne \emptyset$. We will not spell out such arguments. Note that $F \in \mathcal{G}$ implies $d(F) = 0$, since $\mathbf{n} \in \mathcal{B}_G^{II}$. Hence we have

$$\sum_{e \in \overline{F}} n_e \le \sum_{F \in \mathcal{G}} \sum_{e \in F} n_e \le \sum_{F \in \mathcal{G}} \frac{1}{2\kappa} L^{F,n}(\mathbf{n}) < \frac{1}{2} A\sqrt{n}. \tag{32}$$

This implies $d(\overline{F}) = 0$, for n large enough. Note that any F with $d(F) = 0$ that is not contained entirely inside $\overline{F}$ satisfies

$$\sum_{e \in F} n_e \ge \frac{1}{2} L^{F,n}(\mathbf{n}) \ge \frac{\kappa A}{2^{|E|+2}}\sqrt{n}.$$

Let us abreviate $\kappa_0 = \kappa/2^{|E|+2}$. In the remainder of this section, we are going to fix a possible value F_0 of $\overline{F}$, and argue separately for each F_0. With this in mind we make the following definitions. For any F_0 such that $d(F_0) = 0$, let

$$\begin{aligned}
\mathcal{B}_G^{II}(n; A, F_0) &= \left\{ \mathbf{n} \in \mathcal{B}_G^{II}(n; A) : \begin{array}{l} \sum_{e \in F_0} n_e < \frac{1}{2} A\sqrt{n}, \text{ and for all } F \text{ not contained} \\ \text{in } F_0 \text{ we have } \sum_{e \in F} n_e - \frac{n}{k} d(F) \ge \kappa_0 A\sqrt{n} \end{array} \right\} \\
M_n(F_0) &= \max\left\{ p_G(\mathbf{n}) : \mathbf{n} \in \mathcal{B}_G^{II}(n; A, F_0) \right\} \\
\beta &= \limsup_{n \to \infty} M_n(F_0).
\end{aligned} \tag{33}$$

Our task is to show that $\beta \le c_G + C\exp(-\lambda A^2)$ for each F_0 such that $\mathcal{B}_G^{II}(n; A, F_0)$ is non-empty.

We will need to work on subgraphs of the form $G^H = (V, E^H)$, where $E^H = E \setminus H$, $H \subset F_0$. We write $\mathbf{n}^H$ for the restriction of $\mathbf{n}$ to G^H, that is: $\mathbf{n}^H = (n_e : e \in E^H)$. When no confusion can arise, we will write $n^H = \sum_{e \in E^H} n_e$.

Lemma 15. *If $\mathcal{B}_G^{II}(n; A, F_0)$ is non-empty, then for any $H \subset F_0$ the graph G^H is connected.*

Proof. It is enough to consider $H = F_0$. Should G^{F_0} not be connected, we could write $E = E_1 \cup F_0 \cup E_2$ as a disjoint union, where E_1 and E_2 are non-empty and do not share any vertex. Then we have $0 < d(E_1 \cup F_0), d(E_2 \cup F_0) < k$ and $d(E_1 \cup F_0) + d(E_2 \cup F_0) \geq k$. Therefore, if $\mathbf{n} \in \mathcal{B}_G^{II}(n; A, F_0)$, we have

$$\begin{aligned}\sum_{e \in E} n_e &= \sum_{e \in E_1 \cup F_0} n_e + \sum_{e \in E_2 \cup F_0} n_e - \sum_{e \in F_0} n_e \\ &\geq \frac{n}{k} d(E_1 \cup F_0) + \frac{1}{2} A\sqrt{n} + \frac{n}{k} d(E_2 \cup F_0) + \frac{1}{2} A\sqrt{n} - \frac{1}{2} A\sqrt{n} \\ &\geq n + \frac{1}{2} A\sqrt{n} > n,\end{aligned}$$

a contradiction. □

Lemma 16. *Let $H \subset F_0$ and $\mathbf{n} \in \mathcal{B}_G^{II}(n; A, F_0)$.*
(i) We have $\mathbf{n}^H / n^H \in \mathcal{K}_{G^H}$.
(ii) Suppose in addition that $n_e \geq cA\sqrt{n}$ for all $e \in F_0 \setminus H$, with some $c > 0$. Then $\mathbf{n}^H$ satisfies the assumption on the starting state of Proposition 13, with A replaced by $\min\{cA, \kappa_0 A\}$.

Proof. Both statements will be proved by the same computations. Let $\emptyset \subsetneq F \subsetneq (E \setminus H)$. Since $d(H) \leq d(F_0) = 0$, we have $d(F \cup H; G) = d(F; G^H)$. When this common value is ≥ 1, we have

$$\begin{aligned}\sum_{e \in F} n_e &\geq \sum_{e \in F \cup H} n_e - \frac{1}{2} A\sqrt{n} \geq \frac{n}{k} d(F \cup H; G) + A\sqrt{n} - \frac{1}{2} A\sqrt{n} \\ &\geq \frac{n^H}{k} d(F; G^H) + \frac{1}{2} A\sqrt{n^H} \geq \frac{n^H}{k} d(F; G^H).\end{aligned} \tag{34}$$

This already suffices for part (i). When $d(F \cup H; G) = d(F; G^H) = 0$ and F is not a subset of F_0, we have

$$\sum_{e \in F} n_e \geq \kappa_0 A\sqrt{n} \geq \kappa_0 A\sqrt{n^H}. \tag{35}$$

When $\emptyset \subsetneq F \subset F_0 \setminus H$, under the assumption made in part (ii) we have

$$\sum_{e \in F} n_e \geq cA\sqrt{n} \geq cA\sqrt{n^H}. \tag{36}$$

The three cases (34), (35) and (36) complete the proof of part (ii). □

The main technical difficulty in the proof of Proposition 11 is that we have no control over how small $n_e(i)$ can get for $e \in F_0$, and therefore these coordinates *must* be hit exactly at each stage. We can do this, if the difference $n_e(i+1) - n_e(i) \geq 0$ is sufficiently small so that we have enough opportunity to play these edges (once the exact value is achieved, we can ignore any such edge, since $d(F_0) = 0$. The configurations introduced next will help us overcome this technical difficulty.

Let $\mathbf{x}^{*,F_0}$ denote the configuration introduced in (3), with the graph G replaced by G^{F_0}. Given $\delta > 0$ and $H \subsetneq F_0$, let

$$\mathbf{y}^{*,F_0}(\delta; H) = (1-\delta)\mathbf{x}^{*,F_0} + \delta \frac{1}{|F_0 \setminus H|} \sum_{e \in F_0 \setminus H} \mathbf{1}^e,$$

where all vectors are regarded as being in $\mathbb{R}^{E^H}$. Let $\mathbf{n}^{*,F_0}(H) = n\mathbf{y}^{*,F_0}(\delta; H) + O(1)$.

Lemma 17.
(i) We have $\mathbf{x}^{*,F_0} \in \mathcal{K}_{G^H}$.
(ii) For all sufficiently small $\delta > 0$ *we have* $\mathbf{y}^{*,F_0}(\delta; H) \in \mathcal{R}_{G^H}$ *and* $\mathrm{dist}(\mathbf{y}^{*,F_0}(\delta; H), \partial\mathcal{R}_{G^H}) \geq \delta(B|F_0 \setminus H|)^{-1}$.
(iii) There exists $c_7(G) > 0$ *such that for all sufficiently small* $\delta > 0$ *and all* $\emptyset \subsetneq F \subsetneq E^{F_0}$ *we have*

$$\left(\sum_{e \in E^{F_0}} n_e^{*,F_0}(H) \right)^{-1} \sum_{e \in F} n_e^{*,F_0}(H) \geq \frac{d(F; G^{F_0})}{k} + c_7.$$

Proof. (i) Let $\emptyset \subsetneq F \subsetneq E^H$. We first consider the case when $F \not\subset F_0 \setminus H$ and $E \setminus F_0 \not\subset F$. Then we have

$$\sum_{e \in F} x_e^{*,F_0} = \sum_{e \in F \setminus F_0} x_e^{*,F_0} > \frac{d(F \setminus F_0; G^{F_0})}{k} = \frac{d(F \cup (F_0 \setminus H); G^H)}{k} \geq \frac{d(F; G^H)}{k}. \tag{37}$$

When $F \not\subset F_0 \setminus H$ and $E \setminus F_0 \subset F$, we have instead

$$\sum_{e \in F} x_e^{*,F_0} = \sum_{e \in F \setminus F_0} x_e^{*,F_0} = 1 > \frac{d(F; G^H)}{k}. \tag{38}$$

If $\emptyset \subsetneq F \subset F_0 \setminus H$, we have

$$\sum_{e \in F} x_e^{*,F_0} = 0 = \frac{d(F; G^H)}{k}. \tag{39}$$

This completes the proof of part (i).

(ii) If δ is sufficiently small, the inequalities (37) and (38), with $\mathbf{x}^{*,F_0}$ replaced by $\mathbf{y}^{*,F_0}(\delta; H)$, remain strict. Also, Eqn. (39) becomes a strict inequality. The lower bound on the distance follows from Lemma 6.

(iii) This follows from (37), since the normalization factor in the front is $[n(1-O(\delta))]^{-1}$. $\square$

Proof of Proposition 11. Given $\varepsilon > 0$, we select a subsequence along which $M_n(F_0) > \beta - \varepsilon$. For each n in the subsequence, select $\mathbf{n} \in \mathcal{B}_G^{II}(n, F_0)$ such that $p_G(\mathbf{n}) > \beta - \varepsilon$. By passing to a further subsequence, we may assume that for each $e \in F_0$ the coordinates n_e are nondecreasing along the subsequence.

We now choose $\mathbf{n}(1), \dots, \mathbf{n}(\ell)$ and $\mathbf{n}$. Let $n(1) < \cdots < n(\ell)$ and let $\mathbf{n}(i) \in \mathcal{B}_G^{II}(n(i); F_0)$, $i = 1, \dots, \ell$, be a sequence of points such that:
(i) $n(i+1) \geq 2(2K_6 n(i))^{1/\gamma}$, $i = 1, \dots, \ell - 1$;

(ii) $n_e(i+1) \geq n_e(i)$, for all $e \in F_0$, $i = 1, \dots, \ell - 1$;
(iii) $p_G(\mathbf{n}(i)) \geq \beta - \varepsilon$, $i = 1, \dots, \ell$.
We further define $\mathbf{n}$ in the following way. Let $n = 2K_6 n(\ell)$, where K_6 is the constant of Proposition 14, and let $\mathbf{n} = K_6\, n(\ell)\, \mathbf{y}^{*,F_0}(\delta_1; \emptyset) + K_6\, \mathbf{n}(\ell) + O(1)$ for a small $\delta_1 > 0$ for which the conclusions of Lemma 17(ii)–(iii) hold. We will need that for all $e \in F_0$ we have

$$n_e \leq K_6\, n(\ell)\, \frac{\delta_1}{|F_0|} + K_6\, \frac{1}{2}\, A\, \sqrt{n(\ell)} + O(1) < 2\delta_1 K_6\, n(\ell) = \delta_1 n, \tag{40}$$

if $n(\ell)$ is large enough. Also note that an application of Theorem 1(ii) yields $p_G(\mathbf{n}) < c_G + \varepsilon$.

We now define the strategy to steer from $\mathbf{n}$ towards $\mathbf{n}(\ell)$. We first employ a strategy that plays an edge $e \in F_0$ with $N_e(t) > n_e(\ell)$, whenever that is possible, but never plays an edge $e \in F_0$ with $N_e(t) = n_e(\ell)$. We stop the first time t when for all $e \in F_0$ we have $N_e(t) = n_e(\ell)$. Such a strategy exists, since $d(F_0) = 0$. Since we start with $N_e(0) - n_e(\ell) \leq \delta_1 n$ (recall (40)), if δ_1 is sufficiently small, there is probability $\geq 1 - \exp(-\lambda n)$ that we stop before time $C\delta n$ for some $C = C(G)$ and $\lambda > 0$. Moreover, the value on every edge is decreased by an amount at most $C\delta n$, and therefore it follows from Lemma 17(iii) that the configuration $\mathbf{n}'$ reached has the property that $(\mathbf{n}')^{F_0}$ is bounded away from $\partial \mathcal{R}_{G^{F_0}}$.

We can now apply Proposition 14 to $(\mathbf{n}')^{F_0}$ and $(\mathbf{n}(\ell))^{F_0}$ on the connected graph G^{F_0}. We can implement the moves given by the strategy in that proposition as a strategy on G, because $d(F_0) = 0$. Let $\varphi_\ell(\mathbf{r}(\ell))$ denote the probability that at time $n(\ell)$ we reach state $\mathbf{n}(\ell) + \mathbf{r}(\ell)$. Let us write $c_\ell = \varphi_\ell(\mathbf{0})$ for the probability that $\mathbf{n}(\ell)$ was hit exactly. Note that since we applied the strategy on G^{F_0}, we have $r_e(\ell) = 0$ for all $e \in F_0$. This restriction will be implicit in our notation. Proposition 14 implies

$$\begin{aligned} c_G + \varepsilon \geq p_G(\mathbf{n}) &\geq c_\ell p_G(\mathbf{n}(\ell)) + \sum_{\mathbf{r}(\ell) \neq \mathbf{0}} \varphi_\ell(\mathbf{r}(\ell))\, p_G(\mathbf{n}(\ell) + \mathbf{r}(\ell)) \\ &\geq c_\ell(\beta - \varepsilon) + \sum_{0 < |\mathbf{r}(\ell)| < \nu A \sqrt{n(\ell)}} \varphi_\ell(\mathbf{r}(\ell))\, p_G(\mathbf{n}(\ell) + \mathbf{r}(\ell)). \end{aligned} \tag{41}$$

with any $\nu > 0$. The value of ν will be chosen in what follows.

We now inductively define the strategy that steers from $\mathbf{n}(i+1) + \mathbf{r}(i+1)$ towards $\mathbf{n}(i)$, for $i = \ell - 1, \ell - 2, \dots, 1$. We assume $|\mathbf{r}(i+1)| < \nu A \sqrt{n(i+1)}$. Let

$$H = \{e \in F_0 : n_e(i+1) < \delta_2 A \sqrt{n_{i+1}}\},$$

where $\delta_2 > 0$ will be chosen in a moment. We will first reduce the edges in H to their target value $n_e(i)$. Then we use Proposition 13 and Proposition 7 in G^H to reach a target where the edges $e \in F_0 \setminus H$ do not have much excess compared to $n_e(i)$, so that these can be reduced to $n_e(i)$ as well. Following this, we use Proposition 14 in G^{F_0} to hit $\mathbf{n}(i)$.

The first part of the strategy is to reduce the value on each edge $e \in H$, whenever that is possible, until it equals $n_e(i)$, and in such a way that no edge in $F_0 \setminus H$ is used. We stop the first time t when $N_e(t) = n_e(i)$ for all $e \in H$. Since $d(F_0) = 0$, such strategy exists. The goal is achieved before time $C\delta_2 A \sqrt{n(i+1)}$ with probability $\geq 1 - \exp(-\lambda \sqrt{n(i+1)})$, if δ_2 is sufficiently small. Moreover, the value of every $e \in E \setminus F_0$ is decreased by no more than $C\delta_2 A \sqrt{n(i+1)}$. Let $\mathbf{n}'(i+1)$ denote the configuration reached.

Lemma 18. *If δ_2 and ν are sufficiently small, the restriction of the configuration $\mathbf{n}'(i+1)$ to G^H satisfies the assumption on the starting state of Proposition 13 with A replaced by $\min\{\frac{1}{2}\kappa_0 A, \delta_2 A\}$.*

Proof. The proof is similar to the proof of Lemma 16. Let $\emptyset \subsetneq F \subsetneq E \setminus H$. If $d(F \cup H; G) \geq 1$, we have

$$\begin{aligned}\sum_{e\in F} n'_e(i+1) &= \sum_{e\in F\cup H} n'_e(i+1) - \sum_{e\in H} n_e(i) \geq \sum_{e\in F\cup H} n'_e(i+1) - \sum_{e\in H}(n_e(i+1) + r_e(i+1)) \\ &\geq \sum_{e\in F\cup H}(n_e(i+1) + r_e(i+1)) - (C + |H|)\delta_2 A\sqrt{n(i+1)} \\ &\geq \sum_{e\in F\cup H} n_e(i+1) - \sqrt{|E|}|\mathbf{r}(i+1)| - (C+|H|)\delta_2 A\sqrt{n(i+1)} \\ &\geq \frac{n(i+1)}{k} d(F\cup H; G) + A\sqrt{n(i+1)} - (\sqrt{|E|}\nu + (C+|H|)\delta_2)A\sqrt{n(i+1)} \\ &\geq \frac{n'(i+1)}{k} d(F; G^H) + (1 - C'\nu + C''\delta_2)A\sqrt{n'(i+1)}.\end{aligned} \tag{42}$$

Hence we will require that $1 - C'\nu - C''\delta_2 \geq \frac{1}{2}$, say.

When $d(F \cup H; G) = 0$ and F is not a subset of F_0, we have

$$\begin{aligned}\sum_{e\in F} n'_e(i+1) &\geq \sum_{e\in F}(n_e(i+1) + r_e(i+1)) - C\delta_2 A\sqrt{n(i+1)} \\ &\geq \sum_{e\in F} n_e(i+1) - (\sqrt{|E|}\nu + C\delta_2)A\sqrt{n(i+1)} \\ &\geq (\kappa_0 - \sqrt{|E|}\nu - C\delta_2)A\sqrt{n(i+1)} \\ &\geq \frac{1}{2}\kappa_0 A\sqrt{n'(i+1)},\end{aligned} \tag{43}$$

if ν and δ_2 are small enough.

Finally, if $\emptyset \subsetneq F \subset F_0 \setminus H$, we have

$$\sum_{e\in F} n'_e(i+1) = \sum_{e\in F} n_e(i+1) \geq \sum_{e\in F} \delta_2 A\sqrt{n(i+1)} \geq \delta_2 A\sqrt{n'(i+1)}. \tag{44}$$

The cases (42), (43) and (44) complete the proof. ◻

We need one more auxilliary configuration. Let $n''(i) = 2K_6 n(i)$, where K_6 is the constant from Proposition 14, and let

$$\mathbf{n}''(i) = K_6 n(i)\mathbf{y}^{*,F_0}(\delta_1; H) + (K_6 - 1)\frac{n(i)}{(n(i))^H}(\mathbf{n}(i))^H + \mathbf{n}(i) + O(1).$$

Due to Lemma 17(ii), $\mathbf{n}''(i)/n''(i) \in \mathcal{R}_G$ and $(\mathbf{n}''(i))^H/(n''(i))^H$ is at least distance $c\delta_1$ away from $\partial\mathcal{R}_{G^H}$. Therefore, we can apply Proposition 7 on the graph G^H to steer the process from $(\mathbf{n}'(i+1))^H$ to a δ_3 neighbourhood of $(\mathbf{n}''(i))^H$, which succeeds with probability at least $1-C_1\exp(-\lambda_1\delta_3 n(i))$. Moreover, due to Lemma 17(iii), the configuration $\mathbf{n}''(i)+\mathbf{s}$ reached this way satisfies

$$(2K_6 n(i))^{-1}\sum_{e\in F}(n''_e(i)+s_e) \geq \frac{d(F;G^{F_0})}{k} + c'_7, \quad \emptyset \subsetneq F \subsetneq E^{F_0}. \tag{45}$$

Also, for $e \in F_0 \setminus H$ we have

$$\begin{aligned}(n''_e(i)+s_e) - n_e(i) &\geq K_6 n(i) y_e^{*,F_0}(\delta_1;H) - \sqrt{|E|}|\mathbf{s}| - \frac{1}{2}A\sqrt{n(i)} \\ &\geq K_6 n(i)\frac{\delta_1}{|F_0|} - 2K_6 n(i)\sqrt{|E|}\delta_3 - \frac{1}{2}A\sqrt{n(i)} \geq 0,\end{aligned}$$

if $\delta_3 < \delta_1(4|F_0|\sqrt{|E|})^{-1}$ and $n(i)$ is large enough. On the other hand:

$$\begin{aligned}n''_e(i)+s_e &\leq K_6 n(i)\delta_1 + \sqrt{|E|}|\mathbf{s}| + K_6\frac{1}{2}A\sqrt{n(i)}(1+O(n(i)^{-1/2})) \\ &\leq K_6 n(i)\delta_1 + 2K_6 n(i)\sqrt{|E|}\delta_3 \leq 2K_6 n(i)\delta_1,\end{aligned}$$

if $n(i)$ is large enough.

If δ_1 is sufficiently small, we can now employ a strategy starting from state $\mathbf{n}''(i)+\mathbf{s}$, that reduces the values on all $e \in F_0 \setminus H$, whenever that is possible, until they all equal $n_e(i)$, but never uses an edge in H. This only changes the values on $e \in E^{F_0}$ by at most $2C\delta_1 K_6 n(i)$, and succeeds with probability at least $1-\exp(-\lambda 2K_6 n(i))$. Let $\mathbf{n}'''(i)$ denote the configuration reached. It follows from (45) that $(\mathbf{n}''')^{F_0}$ is bounded away from $\partial\mathcal{R}_{G^{F_0}}$.

Finally, we can apply Proposition 14 on the graph G^{F_0} with starting state $(\mathbf{n}'''(i))^{F_0}$ and target state $(\mathbf{n}(i))^{F_0}$. Let $\varphi_i(\mathbf{r}(i))$ denote the probability that at time $n(i)$ we reach state $\mathbf{n}(i)+\mathbf{r}(i)$. Let us write $c_i = \varphi_i(\mathbf{0})$ for the probability that $\mathbf{n}(i)$ is hit exactly. This gives the following inductive bound:

$$\begin{aligned}p_G(\mathbf{n}(i+1)+\mathbf{r}(i+1)) &\geq c_i p_G(\mathbf{n}(i)) + \sum_{\mathbf{r}(i)\neq\mathbf{0}} \varphi_i(\mathbf{r}(i))\, p_G(\mathbf{n}(i)+\mathbf{r}(i)) \\ &\geq c_i(\beta-\varepsilon) + \sum_{0<|\mathbf{r}(i)|<\nu A\sqrt{n_i}} \varphi_i(\mathbf{r}(i))\, p_G(\mathbf{n}(i)+\mathbf{r}(i)).\end{aligned} \tag{46}$$

Combining (41) and (46), Proposition 14 yields

$$\begin{aligned}c_G + \varepsilon \geq (\beta-\varepsilon)\,&[c_\ell + (1-c_\ell)c_{\ell-1} + \cdots + (1-c_\ell)\cdots(1-c_2)c_1] \\ &- C\ell\exp(-\lambda A^2) - C\exp(-\lambda\nu A\sqrt{n_1}).\end{aligned}$$

Since each $c_j \geq c > 0$, we extract a factor arbitrarily close to $\beta-\varepsilon$. Letting $\varepsilon \downarrow 0$ shows that $c_G \geq \beta(1-e^{-c\ell}) - C\ell\exp(-\lambda A^2)$. Choosing ℓ of order A^2 completes the proof. ∎

3.4 Upper bound for $\mathcal{B}_G^{III}$

In the proof of Proposition 12 we are going to need the following lemma about supermartingales. It is a close variant of [8, Propositions 17.19 and 17.20] and hence we omit the proof.

Lemma 19. *Let $Z(t)$ be a non-negative supermartingale with respect to $\mathcal{F}_t$, and τ a stopping time with respect to $\mathcal{F}_t$. Suppose that*
(i) $Z(0) = k \geq 1$;
(ii) $|Z(t+1) - Z(t)| \leq B$;
(iii) there exist constants $\sigma^2 > 0$ and $b > 0$ such that almost surely on the event $\{\tau > t\}$, either $\mathsf{Var}(Z(t+1) \,|\, \mathcal{F}_t) \geq \sigma^2$ or $\mathsf{Var}(Z(t+1) \,|\, \mathcal{F}_t) = 0$ and $\mathbf{E}[Z(t+1) = Z(t) \,|\, \mathcal{F}_t] \leq -b$. Then there exists $u_1 = u_1(B, b, \sigma)$ and $C = C(b, \sigma)$ such that if $u \geq u_1$ then

$$\mathbf{P}[\tau > u] \leq C \frac{k}{\sqrt{u}}.$$

Proof of Proposition 12. Given $\varepsilon > 0$ choose $A_0(\varepsilon)$ large enough so that the conclusions of Propositions 10 and 11 are satisfied for all $A \geq A_0$. Under the optimal strategy, we consider the process

$$Z(t) = \min\{L^{F,n-t}(\mathbf{N}(t)) : F, \, 0 < d(F) < k\}, \tag{47}$$

which is a supermartingale, because the $L^{F,n-t}$ are. Since the increments of $L^{F,n}$ are bounded, condition (ii) of Lemma 19 is satisfied. We show that $Z(t)$ satisfies the condition (iii) of Lemma 19 as well. Let F be the set contributing the minimum in (47). Since $d(F) > 0$, there exists an edge $e \in F$ such that N_e gets updated with probability at least $1/k$. On this event we have

$$L^{F,n-t-1}(\mathbf{N}(t+1)) - L^{F,n-t}(\mathbf{N}(t)) = -\langle \mathbf{1}^e - \mathbf{z}^F, \mathbf{u}^F \rangle =: -b(e; F) < 0,$$

since $d(F) < k$. Therefore, if $\mathsf{Var}(Z(t+1) \,|\, \mathcal{F}_t) = 0$, we have $\mathbf{E}[Z(t+1) - Z(t) \,|\, \mathcal{F}_t] \leq -b(e; F)$. On the other hand, since there are only finitely many possible shifts in the values of the $L^{F,n-t}$, and only finitely many possible vectors $\mathbf{Y}(t)$ (recall that there exists a deterministic optimal strategy), if $\mathsf{Var}(Z(t+1) \,|\, \mathcal{F}_t)$ is non-zero, then it is bounded below by some $\sigma^2 = \sigma^2(G) > 0$.

We will choose a small $a > 0$, and subdivide $\mathcal{B}_G^{III}(n; A)$ into the slices:

$$\mathcal{B}_G^{III}(n; a, k) = \left\{\mathbf{n} \in n\mathcal{S}_G : \min\left\{L^{F,n}(\mathbf{n}) : F, \, 0 < d(F) < k\right\} \in [ak\sqrt{n}, a(k+1)\sqrt{n})\right\},$$
$$a > 0, \; -k_{\max} - 2 \leq k \leq k_{\max} + 1,$$

where $k_{\max} = \lceil A/a \rceil$. Let $\mathbf{n} \in \mathcal{B}_G^{III}(n; a, k)$. The idea of the proof is to run the martingale $p_G(\mathbf{N}(t))$ until $Z(t)$ moves well into one of the neighbouring slices, and use optional stopping to get an inequality relating the maximum of $p_G(\mathbf{n})$ over $\mathcal{B}_G^{III}(n; a, k)$ to the maxima over $\mathcal{B}_G^{III}(n'; a, k-1)$ and $\mathcal{B}_G^{III}(n'; a, k+1)$, with $\frac{1}{4}n \leq n' < n$. The parameter a will be chosen small so that we can apply Lemma 19 to the stopping rule. We will need to handle $k \geq 1$, $k = 0, -1$ and $k \leq -2$ separately. It will be convenient to introduce the following notation:

$$\begin{aligned} M_n(k) &= \max\left\{p_G(\mathbf{n}) : \mathbf{n} \in \mathcal{B}_G^{III}(n; a, k)\right\} \\ \overline{M}_n(k) &= \sup_{m \geq n} M_m(k) \\ \beta(k) &= \limsup_{n \to \infty} M_n(k) = \lim_{n \to \infty} \overline{M}_n(k). \end{aligned}$$

Case $1 \le k \le k_{\max}$. We define the stopping time

$$\tau_k = \sqrt{a}n\left(\frac{1}{k}-\frac{1}{4k^2}\right) \wedge \inf\left\{t \ge 0 : Z(t) < \left(k-\frac{1}{2}\right)a\sqrt{n-t}\right\}$$
$$\wedge \inf\left\{t \ge 0 : Z(t) \ge \left(k+\frac{3}{2}\right)a\sqrt{n-t}\right\},$$

It is straightforward to check that whenever $\tau_k < n(\frac{1}{k}-\frac{1}{4k^2})$, the value of $Z(\tau_k)$ is such that $\mathbf{N}(\tau_k)$ is either in the slice $\mathcal{B}_G^{III}(n-\tau_k;a,k-1)$ or in the slice $\mathcal{B}_G^{III}(n-\tau_k;a,k+1)$. An application of Lemma 19 to $Z(t)-(k-1)a\sqrt{n}$ yields

$$\mathbf{P}\left[\tau_k \ge \sqrt{a}n\left(\frac{1}{k}-\frac{1}{4k^2}\right)\right] \le C\frac{2a\sqrt{n}}{a^{1/4}\sqrt{n}\sqrt{\frac{1}{k}-\frac{1}{4k^2}}} \le C\frac{4a^{3/4}}{\sqrt{\frac{a}{2A}\left(4-\frac{a}{2A}\right)}} = \frac{4C}{\sqrt{\frac{1}{A}\left(\frac{2}{\sqrt{a}}-\frac{\sqrt{a}}{2A}\right)}}. \quad (48)$$

By optional stopping, we have

$$\begin{aligned} p_G(\mathbf{n}) &= \mathbf{E}[p_G(\mathbf{N}(\tau_k))] \\ &\le \mathbf{P}[Z(\tau_k) < ka\sqrt{n-\tau_k}]\overline{M}_{n/4}(k-1) + \mathbf{P}[Z(\tau_k) \ge (k+1)a\sqrt{n-\tau_k}]\overline{M}_{n/4}(k+1) \quad (49)\\ &\quad + \mathbf{P}[Z(\tau_k) \in [ka\sqrt{n-\tau_k},(k+1)a\sqrt{n-\tau_k})]\overline{M}_{n/4}(k). \end{aligned}$$

Note that due to our choice of a in (48) the probability in the third term of (49) is at most $C(A)\sqrt{a}$. Maximizing $p_G(\mathbf{n})$ over its slice yields

$$M_n(k) \le c_n(k)\overline{M}_{n/4}(k-1) + d_n(k)\overline{M}_{n/4}(k) + e_n(k)\overline{M}_{n/4}(k+1), \quad 1 \le k \le k_{\max}, \quad (50)$$

where $d_n(k) \le C(A)\sqrt{a}$. By stopping the supermartingale $Z'(t) = Z(t)-(k-1)a\sqrt{n}$ at τ_k we have

$$2a\sqrt{n} \ge Z'(0) \ge \mathbf{E}[Z'(\tau_k); Z'(\tau_k) \ge \frac{5}{2}a\sqrt{n-\tau_k}] \ge \frac{5}{2}a\sqrt{n}\sqrt{1-\sqrt{a}}\, e_n(k). \quad (51)$$

When a is sufficiently small, the inequalties (51) and $d_n(k) \le C(A)\sqrt{a}$ imply that $c_n(k) \ge \frac{1}{6}$.

Case $k=-1,0$. We define

$$\tau_k = \frac{3}{4}an \wedge \inf\left\{t \ge 0 : Z(t) < \left(k-\frac{1}{2}\right)a\sqrt{n-t}\right\} \wedge \inf\left\{t \ge 0 : Z(t) \ge \left(k+\frac{3}{2}\right)a\sqrt{n-t}\right\}.$$

We now have

$$\mathbf{P}\left[\tau_k \ge \frac{3}{4}an\right] \le C\frac{2a\sqrt{n}}{\sqrt{\frac{3}{4}an}} = \frac{2\sqrt{a}C}{\sqrt{3/4}}. \quad (52)$$

Analogously to (50) we obtain

$$M_n(k) \le c_n(k)\overline{M}_{n/4}(k-1) + d_n(k)\overline{M}_{n/4}(k) + e_n(k)\overline{M}_{n/4}(k+1), \quad k=-1,0. \quad (53)$$

By an argument similar to the one for the previous case, for a sufficiently small we have $c_n(k) \ge \frac{1}{4}$.

Case $-k_{\max} - 1 \le k \le -2$. This time we define

$$\tau_k = n\sqrt{a}\left(\frac{1}{1-k} - \frac{1}{4(1-k)^2}\right) \wedge \inf\left\{t \ge 0 : Z(t) < \left(k - \frac{1}{2}\right) a\sqrt{n-t}\right\}$$
$$\wedge \inf\left\{t \ge 0 : Z(t) \ge \left(k + \frac{3}{2}\right) a\sqrt{n}\right\}.$$

Then with the same choice of a as in the case $k \ge 1$ we have

$$\mathbf{P}\left[\tau_k > n\left(\frac{1}{1-k} - \frac{1}{4(1-k)^2}\right)\right] \le C\frac{4a^{3/4}}{\sqrt{\frac{a}{2A}\left(4 - \frac{a}{2A}\right)}} \le C(A)\sqrt{a}.$$

This yields the relation

$$M_n(k) \le c_n(k)\overline{M}_{n/4}(k-1) + d_n(k)\overline{M}_{n/4}(k) + e_n(k)\overline{M}_{n/4}(k+1), \quad -k_{\max} - 1 \le k \le -2, \tag{54}$$

where $c_n(k) \ge \frac{1}{4}$ for sufficiently small a.

We select a subsequence of n along which $c_n(k), d_n(k), e_n(k)$ all converge to some limits $c(k), d(k), e(k)$, as well as all $M_n(k)$ converge to $\beta(k)$. Then we get

$$\beta(k) \le c(k)\beta(k-1) + d(k)\beta(k) + e(k)\beta(k+1), \tag{55}$$

Due to Proposition 10 we have $\beta(-k_{\max} - 2) \le \varepsilon$ and $\beta(k_{\max} + 1) \le c_G + \varepsilon$. It is easy to deduce from the relation (55) and $c(k) \ge \frac{1}{4} > 0$ that if $\beta(k) \ge \beta(k+1)$ then also $\beta(k-1) \ge \beta(k)$. Hence the maximum in the variable k occurs at the right endpoint and $\beta(k) \le c_G + \varepsilon$ for all $-k_{\max} - 2 \le k < k_{\max} + 1$. This completes the proof of the Proposition. □

4 Further Questions

Question 1. It is plausible that the limit c_G is reached at an exponential rate everywhere in $\mathcal{R}_G$. If one could show that $p_G(\mathbf{n})$ is maximized in the interior of $\mathcal{R}_G$, then this would follow rather easily from (24). Can one describe the asymptotic behaviour of the optimal strategy?

Question 2. The estimates in Section 3 strongly suggest Gaussian behaviour near $\partial\mathcal{R}_G$. Can one make this more precise?

Question 3. It is plausible that under the optimal strategy, the games starting from $\mathbf{n}, \mathbf{n}' \in n\mathcal{R}_G$ (and with the same sequence of vertices drawn) couple with high probability. This may provide an alternative approach to the rather technical arguments of Theorem 1(ii) and Proposition 11.

Question 4. We describe a possible definition of an "order parameter", in analogy with statistical physics models. Let $0 \le \alpha \le 1$, and suppose that the player has to give up proportion α of her/his moves to an adversary, at which times the move is chosen by the adversary. Let $p_{G,\alpha}(\mathbf{n})$ denote the probability of winning in such a game. Let

$$\theta(\mathbf{x}) = \inf\{0 \le \alpha \le 1 : \lim_{n\to\infty} p_{G,\alpha}(n\mathbf{x}) = 0\}.$$

The methods of Theorem 1 show that $\theta(\mathbf{x}) > 0$ in $\mathcal{R}_G$ and $\theta(\mathbf{x}) = 0$ in $\mathcal{I}_G$. Can one analyze θ, or a suitable alternative?

References

[1] C. Albright and C. Derman, Asymptotic optimal policies for the stochastic sequential assignment problem, Management Sci. **19** (1972), 46–51.

[2] B. Bollobás, *Modern graph theory*, Graduate Texts in Mathematics, **184**, Springer, New York, 1998.

[3] C. Derman, G. J. Lieberman and S. M. Ross, A sequential stochastic assignment problem, Management Sci. **18** (1972), 349–355.

[4] G. Grimmett, *Probability on graphs*, Institute of Mathematical Statistics Textbooks, **1**, Cambridge Univ. Press, Cambridge, 2010.

[5] G. R. Grimmett and D. R. Stirzaker, *Probability and random processes*, second edition, Oxford Univ. Press, New York, 1992.

[6] A. A. Járai, The dice and numbers game, To appear in The Mathematical Gazette, 2016.

[7] T. Jia, Y-Y. Liu, E. Csóka, M. Pósfai, J-J. Slotine, A-L. Barabási, Emergence of bimodality in controlling complex networks, Nature Communications, **4** (2013), Article No. 2002, pp. 4, `doi:10.1038/ncomms3002`

[8] D. A. Levin, Y. Peres and E. L. Wilmer, *Markov chains and mixing times*, Amer. Math. Soc., Providence, RI, 2009.

[9] T. Nepusz and T. Vicsek, Controlling edge dynamics in complex networks, Nature Physics, **8** (2012), 568–573.

[10] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, *The mathematical theory of optimal processes*, Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt, Interscience Publishers John Wiley & Sons, Inc. New York, 1962.

[11] M. L. Puterman, *Markov decision processes: discrete stochastic dynamic programming*, Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, Wiley, New York, 1994.

[12] S. Ross, *Introduction to stochastic dynamic programming*, Probability and Mathematical Statistics, Academic Press, New York, 1983.

[13] J. Sun and A. E. Motter, Controllability Transition and Nonlocality in Network Control, Phys. Rev. Lett. **110** (2013), pp. 5, 208701.

[14] D. Williams, *Probability with martingales*, Cambridge Mathematical Textbooks, Cambridge Univ. Press, Cambridge, 1991.