Citation for published version:
Carley, M & Angioni, S 2017, 'Exact integration of surface and volume potentials', Journal of Engineering Mathematics, vol. 104, no. 1, pp. 93-106. https://doi.org/10.1007/s10665-016-9875-5

DOI:
10.1007/s10665-016-9875-5

Publication date:
2017

Document Version
Peer reviewed version

Journal of Engineering Mathematics manuscript No.
(will be inserted by the editor)

Exact integration of surface and volume potentials

Michael Carley · **Stefano Angioni**

Received: date / Accepted: date

Abstract A method for exact analytical integration of potentials from sources distributed on planar and volume elements is presented. The method is based on reduction of the surface integrals to a function similar to an incomplete elliptic integral, giving the integrals in closed form as functions of geometric properties of the surface or volume element. Explicit formulae and recursions are given for the integrals, allowing the evaluation of the potential for arbitrary polynomial sources. Volume integrals are derived from the surface integrals

The work described in this paper was partially funded by the Natural Environment Research Council under grant NE/K014951/1, "Susceptibility to Mass Extinctions: Ammonites as a Case Study".

M. Carley
Department of Mechanical Engineering, University of Bath, Bath BA2 7AY, UK
Tel.: +441-1225-386321
Fax: +44-1225-386928
E-mail: m.j.carley@bath.ac.uk

S. Angioni
Department of Mechanical Engineering, University of Bath, Bath BA2 7AY, UK
Tel.: +441-1225-386321
Fax: +44-1225-386928
E-mail: s.l.angioni@bath.ac.uk

using a simple coordinate transformation which gives the volume integral with little more effort than that required for the surface calculation.

Keywords Laplace equation · potential theory · boundary element method · integral equations · quadrature · elliptic integrals

1 Introduction

Many problems in mathematical physics can be dealt with using integral formulations. This makes the evaluation of potential integrals over surfaces and volumes one of the workhorses of numerical methods, and there is an extensive literature on numerical and analytical techniques for quadrature. Among other topics, this literature has concentrated on handling singularities in potential integrands, efficient coordinate transformations for numerical methods, and analytical approaches for elementary geometries, in particular planar elements such as triangles. The physical problems which can be handled by potential integral formulations cover a wide range of applications, including acoustics, fluid dynamics, electromagnetism, and optics, and are often accelerated using the Fast Multipole Method (FMM) applied to Boundary Element (BEM) or volume integral formulation. When acceleration methods are applied, the local interactions of elements must still be computed using quadratures, and this calculation can become a bottleneck in finding the solution.

To fix ideas, we consider integral formulations for the Laplace and Helmholtz problems,

$$\phi(\mathbf{x}) = \int_S \frac{\sigma(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}S(\mathbf{y}),$$
$$\phi(\mathbf{x}) = \int_S \sigma(\mathbf{y})\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}S(\mathbf{y}),$$

where potential ϕ at field point $\mathbf{x}$ is given by an integral over a surface S with source coordinates $\mathbf{y}$. A source term $\sigma(\mathbf{y})$ is prescribed on the surface and k is the wavenumber in the Helmholtz problem. If the source is distributed over a volume, the surface integration is replaced by a volume integration, but the principle remains the same. If the surface (volume) is discretized into planar (volume) elements such as triangles (tetrahedra), the core of a computational code becomes the evaluation of potential integrals on these elements. In the case of a Laplace problem, these integrals can be evaluated analytically using published methods [2, 3, 8–10], or the approach to be developed in this paper; for the Helmholtz problem, no such analytical calculation is possible, but the Laplace calculation can be interpreted as the integral of the leading-order singularity in the Helmholtz Green's function, which upon subtraction, leaves a remainder which can be integrated more easily using purely numerical methods, the so-called "singularity subtraction" approach. Our problem thus reduces to the case of evaluating integrals for the Laplace Green's function.

Numerous methods have been developed for this application, both numerical and analytical. The analytical formulations [2, 3, 8–10] differ principally in their approach to transforming the problem into a tractable form. A typical method [7, 8, 10] is to reduce surface integrals to line integrals on the boundary of a planar element, evaluating the line integrals analytically to yield a final result. Volume integrals on polyhedra are likewise performed by reducing the volume integral to surface integrals which can be evaluated using the plane element methods. These methods give rise to a number of special cases, such as that of a field point lying on an extension of an element edge, which must be handled explicitly in a code. Recently, one of us [3] has developed a formulation based on a coordinate transformation of a reference triangle which gives explicit formulae for the integrals on the surface, in terms of incomplete elliptic integrals whose arguments are functions of the element geometry.

It is our purpose in this paper to extend this existing formulation to the general case of planar elements with arbitrary monomial or polynomial source terms, writing the results explicitly in terms of incomplete generalized elliptic integrals, and to extend the method to integration over polyhedra, in particular tetrahedra. The evaluation of the elliptic-type integrals is performed using tabulated results and recursions, some of which seem not to have been published before, offering a conceptual framework which should allow the power of asymptotic and other methods to be brought to bear on the problem of evaluating potential integrals on general geometries, by treating it as a problem in developing expansions for a special function.

The extension of the method to volume integrals is dealt with by a very simple coordinate transformation which seems to have passed unnoticed in the literature, and allows integrals on polyhedra to be evaluated with very little extra effort over that of computing a surface integral. The net result is an exact analytical formulation for the potential integrals which arise in solving those fundamental physical problems which are governed by potential theory.

2 Analysis

The basic integral to be evaluated is taken on a triangle with one vertex at the origin and the field point on the z axis, Figure 1. This is a standard form into which a general triangle or other polygon can be decomposed. Figure 2 shows such a decomposition, with the general triangle broken into three partially overlapping triangles each with one vertex at the origin, which is taken as the projection of the field point onto the plane of the triangle. If the orientation of the subtriangles is properly taken into account when computing quantities, so that the signs of their areas respect the orientation of the vertices, the sum of integrals over the three elements will give the correct total for the original triangle by virtue of partial cancellation in the overlap regions. Routine coordinate transformations described in §2.3 allow a

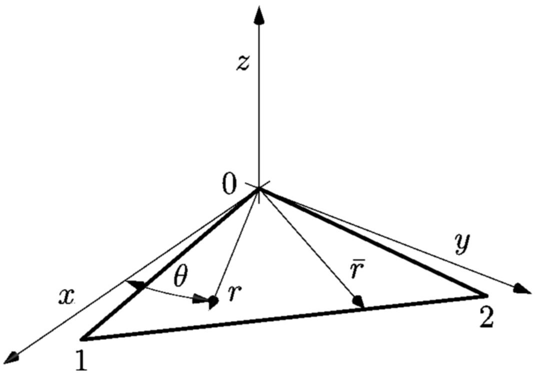

Fig. 1 Reference triangle for evaluation of potential integrals

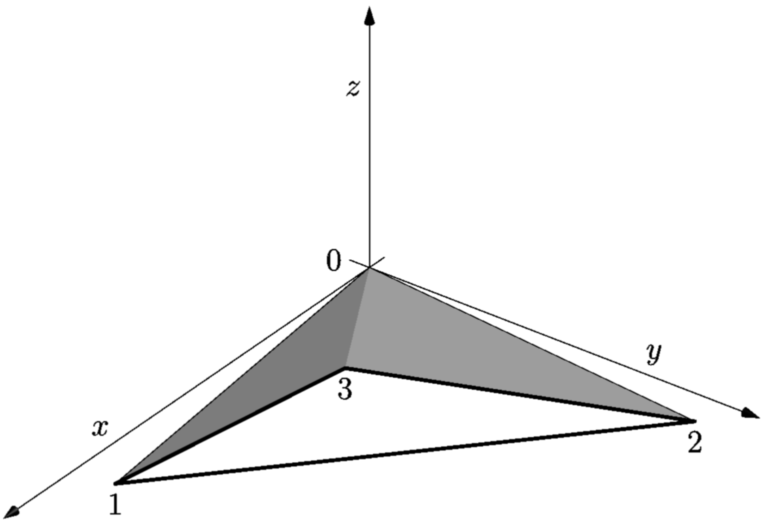

Fig. 2 Decomposition of general triangle 123 into oriented reference triangles 012 (positively oriented), 023, 031 (negatively oriented)

plane element to be transformed to the form of Figures 1 and 2. For now we consider only this reference case.

To proceed with the analysis, we adopt the notation of Figure 3. The reference triangle is rotated and defined by r_1 and r_2, the lengths of the two sides which meet at the origin, and

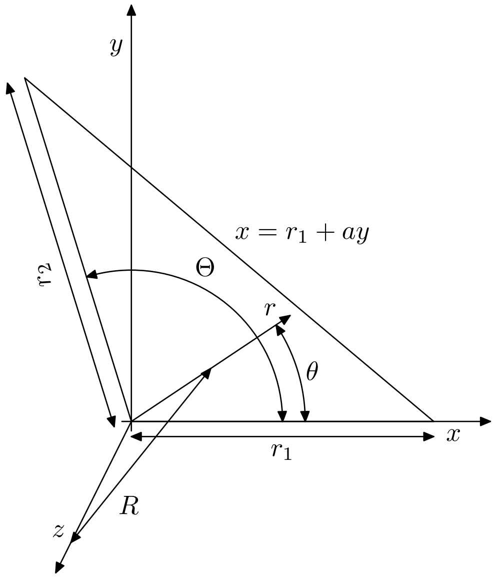

Fig. 3 Transformed reference triangle for evaluation of potential integrals

by Θ, the angle between them. The third side is the line $x = r_1 + ay$ with

$$a = \frac{r_2 \cos\Theta - r_1}{r_2 \sin\Theta}. \tag{1}$$

The general form of integral with arbitrary mononomial source term is then

$$I = \int_S x^n y^m R^\gamma \,\mathrm{d}S, \tag{2}$$
$$R^2 = x^2 + y^2 + z^2,$$

where S is element surface, the field point is $(0,0,z)$ and the element lies in the plane $z = 0$. For definiteness, the single-layer Laplace potential is given by Equation (2) with $\gamma = -1$. The double-layer potential is given by

$$\int_S x^n y^m \frac{\partial}{\partial n}\frac{1}{R}\,\mathrm{d}S = -\int_S x^n y^m \frac{\mathbf{n}.\mathbf{r}}{R^3}\,\mathrm{d}S,$$

where $\mathbf{n}$ is the surface unit normal and $\mathbf{r}$ is the source–field vector. The double-layer potential can thus be written in terms of Equation (2) with $\gamma = -3$.

Transforming Equation (2) to polar coordinates yields

$$I = \int_0^{\Theta} \sin^m \theta \cos^n \theta \int_0^{\bar{r}(\theta_1)} r^{n+m+1} R^{\gamma} \, \mathrm{d}r \, \mathrm{d}\theta, \tag{3}$$

$$R^2 = r^2 + z^2,$$

where

$$\bar{r}(\theta) = \frac{r_1}{(1+a^2)^{1/2}} \frac{1}{\cos(\theta+\phi)}, \; \phi = \tan^{-1} a.$$

The inner integral in Equation (3) can be evaluated analytically in terms of elementary functions leaving an integral which can also be evaluated analytically, but in terms of special functions. In order to evaluate these functions efficiently in a standard form, we rewrite the integrals in terms of arguments α and β derived from the geometric data. This reduces the integrals to previously studied functions which we describe as incomplete generalized elliptic integrals by analogy with generalized (complete) elliptic integrals [1,4, for examples of generalized complete elliptic integrals]. This allows the functions to be computed using formulae in standard tables.

After suitable manipulations, which are described in the following sections, I in equation (3) can be written as a linear combination of functions $I_{m,n}(\alpha, \theta)$ which are defined by

$$I_{m,n}^{(p)}(\alpha, \theta) = \int^{\theta} \sin^m \theta \cos^n \theta \Delta^p \, \mathrm{d}\theta_1, \tag{4}$$

$$\Delta(\theta) = (1 - \alpha^2 \sin^2 \theta)^{1/2},$$

$$\beta^2 = \frac{r_1^2 + z^2(1+a^2)}{1+a^2}, \; \alpha^2 = \frac{z^2}{\beta^2}, \; \alpha' = \left(1-\alpha^2\right)^{1/2}.$$

There are tabulated analytical expressions for the indefinite integral [5, 2.51, 2.52, 2.58] of Equation (4) so that for small values of m and n, $I_{m,n}^{(p)}$ can be evaluated directly, and recursion relations can be used to compute the integral for such other values of m, n and p as may be required. Our integration procedure is to write the integral over the reference triangle in terms of this function and then evaluate the function using tabulated expressions and recursion relations.

2.1 Potential from plane triangles

As an example corresponding to previously published work, we give explicit expressions for the potential from a triangular element with constant source term, $m = n = 0$, for which

$$I = \int_0^{\Theta} \int_0^{\overline{r}(\theta)} R^{\gamma} r \, \mathrm{d}r \mathrm{d}\theta. \tag{5}$$

Under a change of variables from r to R:

$$I = \int_0^{\Theta} \int_{|z|}^{\overline{R}(\theta)} R^{\gamma+1} \mathrm{d}R \, \mathrm{d}\theta, \tag{6}$$

$$= \frac{1}{\gamma+2} \int_0^{\Theta} \overline{R}^{\gamma+2} - |z|^{\gamma+2} \, \mathrm{d}\theta, \tag{7}$$

$$\text{with} \overline{R}(\theta) = [\overline{r}(\theta)^2 + z^2]^{1/2},$$

which can be evaluated with the change of variable $\theta + \phi \to \theta$,

$$I = \frac{1}{\gamma+2} \left[\beta^{\gamma+2} \int_{\phi}^{\Theta+\phi} \frac{\Delta^{\gamma+2}}{\cos^{\gamma+2}\theta} \, \mathrm{d}\theta - |z|^{\gamma+2} \Theta \right], \tag{8}$$

which can then be written in terms of the basic integrals

$$I = \frac{1}{\gamma+2} \left\{ \beta^{\gamma+2} \left[I_{0,-\gamma-2}^{(\gamma+2)}(\alpha, \Theta+\phi) - I_{0,-\gamma-2}^{(\gamma+2)}(\alpha, \phi) \right] - |z|^{\gamma+2} \Theta \right\}. \tag{9}$$

Standard references [5] give explicit expressions for $I_{m,n}^{(p)}$ for the most important values of m, n, and p, summarized in Table 1, and recursion relations can be used to generate the function for other values of the parameters, as required.

For the general case with a monomial source term, closed form expressions can be derived with the aid of formulae for the inner integral [5, 2.260], using the expression

$$r^{2q} = \left(R^2 - z^2\right)^q,$$

which can be expanded into a polynomial in R. Thus, for $n+m$ even, and writing $n+m = 2q$ for compactness,

$$I = \int_0^{\Theta} \sin^m\theta \cos^n\theta \int_0^{\bar{r}(\theta)} r^{n+m+1} R^{\gamma}\,\mathrm{d}r\,\mathrm{d}\theta, \tag{10}$$

$$\begin{aligned}
&= \sum_{k=0}^{q} \binom{q}{k} \frac{(-z^2)^k}{2q-2k+\gamma+2} \times \\
&\left\{ \left[I^{(2q-2k+\gamma+2)}_{m,n+2k-2q-\gamma-2}(\alpha, \Theta+\phi) - I^{(2q-2k+\gamma+2)}_{m,n+2k-2q-\gamma-2}(\alpha,\phi) \right] \beta^{2q-2k+\gamma+2} \right. \\
&\left. - \left[I^{(0)}_{m,n}(\alpha,\Theta+\phi) - I^{(0)}_{m,n}(\alpha,\phi) \right] z^{2q-2k+\gamma+2} \right\}.
\end{aligned} \tag{11}$$

For the case of $n+m$ odd, we write $m+n+1 = 2q$ so that

$$\begin{aligned}
I &= \int_0^{\Theta} \sin^m\theta \cos^n\theta \int_0^{\bar{r}} R^{\gamma} r^{m+n+1}\,\mathrm{d}r\,\mathrm{d}\theta, \\
&= \sum_{s=0}^{q} \binom{q}{s} (-z^2)^{q-s} \int_0^{\Theta} \sin^m\theta\cos^n\theta \int_0^{\bar{r}} R^{\gamma+2s}\,\mathrm{d}r\,\mathrm{d}\theta.
\end{aligned}$$

From tabulated integrals [5, 2.260.3],

$$\begin{aligned}
\int_0^{\bar{r}} R^{2p+1}\,\mathrm{d}r = \bar{r}\bar{R}^{2p+1} + \sum_{k=0}^{p-1} \frac{(2p+1)\ldots(2p-2k+1)}{p(p-1)\ldots(p-k)} \left(\frac{z^2}{2}\right)^{k+1} \bar{r}\bar{R}^{2p-2k-1} \\
+ \frac{(2p+1)!!}{(p+1)!} \left(\frac{z^2}{2}\right)^{p+1} \log\frac{\bar{R}+\bar{r}}{|z|},\ p \geq -1,
\end{aligned}$$

where $(\cdot)!!$ denotes the double factorial, and

$$\begin{aligned}
&\int_{\phi}^{\phi+\Theta} \sin^m\theta\cos^n\theta \int_0^{\bar{r}} R^{2p+1}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \beta^{2p+2}\alpha' I^{(2p+1)}_{m,n-2p-2}(\alpha,\phi+\Theta) - I^{(2p+1)}_{m,n-2p-2}(\alpha,\phi) \\
&- \sum_{k=0}^{p-1} \frac{(2p+1)\ldots(2p-2k+1)}{p(p-1)\ldots(p-k)} \left(\frac{z^2}{2}\right)^{k+1} \beta^{2p-2k}\alpha' \left[I^{(2p-2k-1)}_{m,n-2p+2k}(\alpha,\phi+\Theta) - I^{(2p-2k-1)}_{m,n-2p+2k}(\alpha,\phi) \right] \\
&+ \frac{(2p+1)!!}{(p+1)!} \left(\frac{z^2}{2}\right)^{p+1} \int_{\phi}^{\phi+\Theta} \sin^m\theta\cos^n\theta \log\frac{\Delta+\alpha'}{\Delta-\alpha'}\,\mathrm{d}\theta,\ p \geq -1.
\end{aligned}$$

This leaves the final term to be evaluated using integration by parts:

$$\int^{\theta} \sin^m\theta\cos^n\theta\log\frac{\Delta+\alpha'}{\Delta-\alpha'}\,\mathrm{d}\theta$$
$$=\sum_{q=0}^{m'}\binom{m'}{q}\frac{(-1)^q}{n+2q+1}\left[-\log\frac{\Delta+\alpha'}{\Delta-\alpha'}\cos^{n+2q+1}+2\alpha' I^{(-1)}_{1,n+2q}(\alpha,\theta)\right],\quad m=2m';\quad (12a)$$
$$=\sum_{q=0}^{n'}\binom{n'}{q}\frac{(-1)^q}{m+2q+1}\left[-\log\frac{\Delta+\alpha'}{\Delta-\alpha'}\sin^{m+2q+1}-2\alpha' I^{(-1)}_{m+2q+2,-1}(\alpha,\theta)\right],\quad n=2n'. \quad (12b)$$

For $p<-1$, the resulting integral is [5, 2.263.4]

$$\int^{\theta}\sin^m\theta\cos^n\theta\int_0^{\bar{r}}R^{2p+1}\,\mathrm{d}r\,\mathrm{d}\theta$$
$$=\frac{1}{2p'-1}\frac{\beta\alpha'}{z^2}\left[\frac{1}{\beta^{2p'-1}}I^{(1-2p')}_{m,n+2p'-2}(\alpha,\theta)\right.$$
$$\left.+\sum_{k=1}^{p'-1}\frac{1}{\beta^{2p'-2k-1}}\left(\frac{2}{z^2}\right)^k\frac{(p'-1)(p'-2)\dots(p'-k)}{(2p'-3)(2p'-5)\dots(2p'-2k-1)}I^{(1+2k-2p')}_{m,n+2p'-2k-2}(\alpha,\theta)\right],\ p'=-p-1.$$

Using the results of this section, the potential integral of Equation (2) on a reference triangle can be evaluated for arbitrary n, m, and odd γ in terms of the generalized incomplete elliptic integral defined in Equation (4).

2.2 Evaluation of basic function

The core of our procedure for evaluating the integral of Equation (2) is the computation of the function defined in Equation (4). For small values of m, n, and p, it is probably most convenient to use tabulated expressions hard-coded into the computational technique. In this section, we give explicit expressions for the most commonly encountered values of the parameters, and recursion relations which allow the efficient computation of $I^{(p)}_{m,n}$ for arbitrary values of the parameters.

The basic formulae and recursions are tabulated in the literature [5, 2.51, 2.52, 2.58] and a small number of additional results are readily derived. The basic recursion relations which are needed are, for m and n at fixed p [5, 2.581],

$$I_{m,n}^{(p)}(\alpha,\theta) = \frac{1}{(m+n+p)\alpha^2}\Big[\sin^{m-3}\theta\cos^{n+1}\theta\Delta^{p+2} \\ + [(m+n-2)+(m+p-1)\alpha^2]I_{m-2,n}^{(p)}(\alpha,\theta) - (m-3)I_{m-4,n}^{(p)}(\alpha,\theta)\Big] \quad (13a)$$

$$I_{m,n}^{(p)}(\alpha,\theta) = \frac{1}{(m+n+p)\alpha^2}\Big[\sin^{m+1}\theta\cos^{n-3}\theta\Delta^{p+2} \\ + [(n+p-1)\alpha^2 - (m+n-2)\alpha'^2]I_{m,n-2}^{(p)}(\alpha,\theta) - (n-3)\alpha^2 I_{m,n-4}^{(p)}(\alpha,\theta)\Big]. \quad (13b)$$

For recursions on p, corresponding to changing γ in Equation (2), integration by parts gives a recursion for p decreasing, and expanding $\Delta^2 = 1-\alpha^2\sin^2\theta$ gives a formula for p increasing:

$$I_{m,n}^{(p)}(\alpha,\theta) = \frac{1}{(p+2)\alpha^2}\Big[-\sin^{m-1}\theta\cos^{n-1}\theta\Delta^{p+2} \\ + (m-1)I_{m-2,n}^{(p+2)}(\alpha,\theta) - (n-1)I_{m,n-2}^{(p+2)}(\alpha,\theta)\Big], \quad (14a)$$

$$I_{m,n}^{(p)}(\alpha,\theta) = I_{m,n}^{(p-2)}(\alpha,\theta) - \alpha^2 I_{m+2,n}^{(p-2)}(\alpha,\theta), \quad (14b)$$

$$I_{m,n}^{(p)}(\alpha,\theta) = \alpha'^2 I_{m,n}^{(p-2)}(\alpha,\theta) + \alpha^2 I_{m,n+2}^{(p-2)}(\alpha,\theta). \quad (14c)$$

Table 1 gives explicit formulae for $I_{m,n}^{(p)}$ in the cases $p = \pm 1$ as well as for $p = 0$ which corresponds to $\alpha = 0$, when the field point lies in the element plane. To compute $I_{m,n}^{(p)}$ for general values of the parameters, the entries of Table 1 can be precomputed and recursion relations allow the table to be extended to such values of the parameters as may be required.

2.3 Integration on general polygons

The previous two sections give a method for the evaluation of potential integrals with arbitrary monomial source terms. Figure 4 shows an arbitrary polygon composed of N vertices given as $(r_i\cos\psi_i, r_i\sin\psi_i)$, $i = 0,\dots,N-1$, with $(r_N,\psi_N) = (r_0,\psi_0)$ when the polygon

m	n	$I^{(-1)}_{m,n}(\alpha,\theta)$	$I^{(0)}_{m,n}(\alpha,\theta)$	$I^{(1)}_{m,n}(\alpha,\theta)$
-1	-1	$\frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}+\frac{1}{2\alpha'}\ln\frac{\Delta+\alpha'}{\Delta-\alpha'}$	$\ln\tan\theta$	$\frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}+\frac{\alpha'}{2}\ln\frac{\Delta+\alpha'}{\Delta-\alpha'}$
0	-1	$-\frac{1}{2\alpha'}\ln\frac{\Delta-\alpha' S}{\Delta+\alpha' S}$	$\ln\left(\frac{1+S}{1-S}\right)^{1/2}$	$\frac{\alpha'}{2}\ln\frac{\Delta+\alpha' S}{\Delta-\alpha' S}+\alpha\sin^{-1}(\alpha S)$
1	-1	$\frac{1}{2\alpha'}\ln\frac{\Delta+\alpha'}{\Delta+\alpha'}$	$-\ln C$	$-\Delta+\frac{\alpha'}{2}\ln\frac{\Delta+\alpha'}{\Delta-\alpha'}$
2	-1	$\frac{1}{2\alpha'}\ln\frac{\Delta+\alpha' S}{\Delta-\alpha' S}-\frac{1}{\alpha}\sin^{-1}(\alpha S)$	$-S+\ln\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$	$-\frac{\Delta S}{2}+\frac{2\alpha^2-1}{2\alpha}\sin^{-1}(\alpha S)$
				$+\frac{\alpha'}{2}\ln\frac{\Delta+\alpha' S}{\Delta-\alpha' S}$
3	-1	$\frac{\Delta}{\alpha^2}+\frac{1}{2\alpha'}\ln\frac{\Delta+\alpha'}{\Delta-\alpha'}$	$\frac{C^2}{2}-\ln C$	$-\frac{\alpha^2S^2+3\alpha^2-1}{3\alpha^2}\Delta+\frac{\alpha'}{2}\ln\frac{\Delta+\alpha'}{\Delta-\alpha'}$
-1	0	$-\frac{1}{2}\ln\frac{\Delta+C}{\Delta-C}$	$\ln\tan\frac{\theta}{2}$	$-\frac{1}{2}\ln\frac{\Delta+C}{\Delta-C}+\alpha\ln\alpha(\alpha C+\Delta)$
-1	1	$\frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}$	$\ln S$	$\Delta+\frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}$
-1	2	$-\frac{1}{2}\ln\frac{\Delta+C}{\Delta-C}+\frac{1}{\alpha}\ln(\alpha C+\Delta)$	$C+\ln\tan\frac{\theta}{2}$	$\frac{\Delta C}{2}+\frac{\alpha^2+1}{2\alpha}\ln(\alpha C+\Delta)$
				$-\frac{1}{2}\ln\frac{\Delta+C}{\Delta-C}$
-1	3	$\frac{\Delta}{\alpha^2}-\frac{1}{2}\ln\frac{1+\Delta}{1-\Delta}$	$\frac{C^2}{2}+\ln S$	$-\frac{\alpha^2S^2-3\alpha^2-1}{3\alpha^2}\Delta+\frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}$
0	0	F	θ	E
0	1	$\frac{1}{\alpha}\tan^{-1}\frac{\alpha S}{\Delta}$	S	$\frac{\Delta S}{2}+\frac{1}{2\alpha}\sin^{-1}(\alpha S)$
1	0	$-\frac{1}{\alpha}\ln(\alpha C+\Delta)$	$-C$	$-\frac{\Delta C}{2}-\frac{\alpha'^2}{2\alpha}\ln(\alpha C+\Delta)$
0	2	$\frac{E}{\alpha^2}-\frac{\alpha'^2}{\alpha^2}F$	$\frac{SC}{2}+\frac{\theta}{2}$	$\frac{\Delta}{3}SC$
1	1	$-\frac{\Delta}{\alpha^2}$	$\frac{S^2}{2}$	$-\frac{\Delta^3}{3\alpha^2}$
2	0	$\frac{F-E}{\alpha^2}$	$-\frac{SC}{2}+\frac{\theta}{2}$	$-\frac{\Delta}{3}SC+\frac{\alpha'^2}{3\alpha^2}D+\frac{2\alpha^2-1}{3\alpha^2}E$
0	3	$\frac{\Delta S}{2\alpha^2}+\frac{2\alpha^2-1}{2\alpha^3}\sin^{-1}(\alpha S)$	$S-\frac{S^3}{3}$	$\frac{2\alpha^2C^2+2\alpha^2+1}{8\alpha^2}\Delta S$
				$+\frac{4\alpha^2-1}{8\alpha^3}\sin^{-1}(\alpha S)$
1	2	$-\frac{\Delta C}{2\alpha^2}+\frac{\alpha'^2}{2\alpha^3}\ln(\alpha C+\Delta)$	$-\frac{C^3}{3}$	$-\frac{2\alpha^2C^2+\alpha'^2}{8\alpha^2}\Delta C+\frac{\alpha'^4}{8\alpha^3}\ln(\alpha C+\Delta)$
2	1	$-\frac{\Delta S}{2\alpha^2}+\frac{\sin^{-1}(\alpha S)}{2\alpha^3}$	$\frac{S^3}{3}$	$\frac{2\alpha^2S^2-1}{8\alpha^2}\Delta S+\frac{1}{8\alpha^3}\sin^{-1}(\alpha S)$
3	0	$\frac{\Delta C}{2\alpha^2}-\frac{\alpha^2+1}{2\alpha^3}\ln(\alpha C+\Delta)$	$\frac{C^3}{3}-C$	$-\frac{2\alpha^2S^2+3\alpha^2-1}{8\alpha^2}\Delta C$
				$+\frac{3\alpha^4-2\alpha^2-1}{8\alpha^3}\ln(\alpha C+\Delta)$
0	4	$\frac{\Delta SC}{3\alpha^2}+\frac{4\alpha^2-2}{3\alpha^4}E$	$\frac{3SC}{8}+\frac{SC^3}{4}$	$\frac{3\alpha^2C^2+3\alpha^2+1}{15\alpha^2}\Delta SC$
		$+\frac{3\alpha^4-5\alpha^2+2}{3\alpha^4}F$	$+\frac{3\theta}{8}$	$+\frac{2\alpha'^2(\alpha'^2-2\alpha^2}{15\alpha^4}F+\frac{3\alpha^4+7\alpha^2-2}{15\alpha^4}E$
1	3	$-\frac{\Delta}{3\alpha^4}(\alpha^2C^2-2\alpha'^2)$	$-\frac{C^4}{4}$	$-\frac{3\alpha^4S^4-\alpha^2(5\alpha^2+1)S^2+5\alpha^2-2}{15\alpha^4}\Delta$
2	2	$-\frac{\Delta SC}{3\alpha^2}+\frac{2-\alpha^2}{3\alpha^4}E+$	$-\frac{CS}{8}+\frac{CS^3}{4}$	$-\frac{3\alpha^2C^2-2\alpha^2+1}{15\alpha^2}\Delta SC-\frac{\alpha'^2\alpha^2}{15\alpha^4}F$
		$\frac{2\alpha^2-2}{3\alpha^4}F$	$+\frac{\theta}{8}$	$+\frac{2(\alpha^4-\alpha^2+1)}{15\alpha^4}E$
3	1	$-\frac{\Delta}{3\alpha^4}(2+\alpha^2S^2)$	$\frac{S^4}{4}$	$\frac{3\alpha^4S^4-\alpha^2S^2-2}{15\alpha^4}\Delta$
4	0	$-\frac{\Delta SC}{3\alpha^2}+\frac{2+\alpha^2}{3\alpha^4}F$	$-\frac{3SC}{8}-\frac{S^3C}{4}$	$-\frac{3\alpha^2S^2+4\alpha^2-1}{15\alpha^2}\Delta SC$
		$-\frac{2(1+\alpha^2)}{3\alpha^4}E$	$+\frac{3\theta}{8}$	$-\frac{2(2\alpha^4-\alpha^2-1)}{15\alpha^4}F+\frac{8\alpha^4-3\alpha^2-2}{15\alpha^4}E$

Table 1 Indefinite integral $I^{(p)}_{m,n}(\alpha,\theta)$: $S=\sin\theta$; $C=\cos\theta$; $E=E(\theta,\alpha)$ and $F=F(\theta,\alpha)$ incomplete elliptic integrals

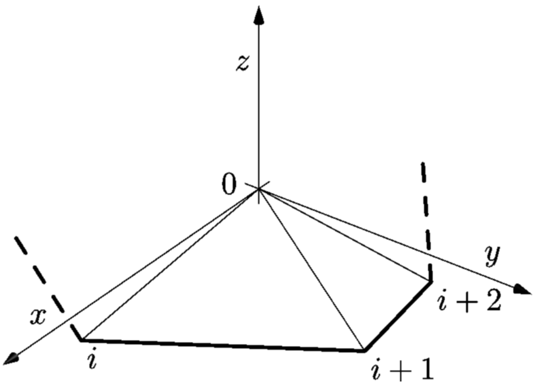

Fig. 4 General polygon in plane $z = 0$

does not enclose the origin and $(r_0, \psi_0 + 2\pi)$ when the origin lies inside the polygon. The potential integral can be evaluated on this surface by summing contributions over triangles formed by the origin and successive pairs of vertices. As noted above, if the vertex orientations are respected, certain triangles will make a negative contribution to the sum and the resulting total will be correct for the surface bounded by the polygon.

Writing the integral over the polygon as a sum over the subtriangles,

$$I = \sum_{i=0}^{N-1} I_i,$$

where the integral for the ith subtriangle is

$$I_i = \int_{\psi_i}^{\psi_{i+1}} \sin^m \psi \cos^n \psi \int_0^{\bar{r}} r^{n+m+1} R^\gamma \, \mathrm{d}r \, \mathrm{d}\psi. \tag{15}$$

Writing $\psi = \psi_i + \theta$ and expanding the trigonometric functions gives

$$I_i = \sum_{q=0}^{m} \sum_{s=0}^{n} \binom{m}{q} \binom{n}{s} (-1)^s \cos^{n+q-s} \psi_i \sin^{m+n-q-s} \psi_i \times$$
$$\int_0^{\psi_{i+1}-\psi_i} \sin^{q+s} \theta \cos^{n+m-q-s} \theta \int_0^{\bar{r}} r^{n+m+1} R^\gamma \, \mathrm{d}r \, \mathrm{d}\theta. \tag{16}$$

The integrals can then be evaluated using the methods given above to yield the transformed result for Equation (15).

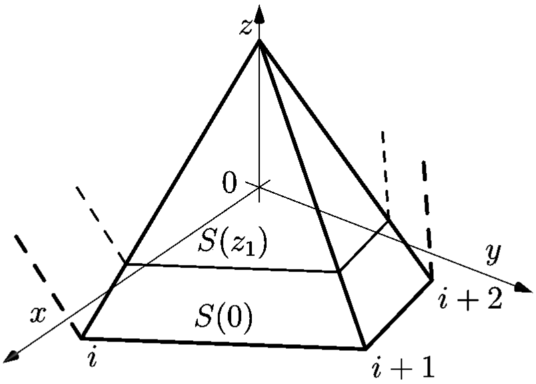

Fig. 5 Integration on general pyramid

2.4 Volume integrals

To evaluate potential integrals from a volume source, we perform a decomposition similar to that in the case of a polygon, and compute integrals on a pyramid with one vertex at the field point. As we shall see, there is a remarkably simple coordinate transformation which gives the volume integral for practically no greater effort than evaluating the integral on the base of the polyhedron. Our general case is for integration of a monomial over a polyhedron,

$$I_V = \int_V x_1^n y_1^m z_1^p R^\gamma \,\mathrm{d}V, \tag{17}$$

with V the polyhedron volume.

Analogously to the plane element case, we decompose the element into reference pyramids each with one vertex at the field point, Figure 5, so that

$$I_V = \int_V x_1^n y_1^m z_1^p R^\gamma \,\mathrm{d}V, \tag{18}$$
$$R^2 = x_1^2 + y_1^2 + (z - z_1)^2,$$

which can be rewritten

$$I_V = \int_0^z z_1^p \int_{S(z_1)} x_1^n y_1^m R^\gamma \,\mathrm{d}S(z_1)\,\mathrm{d}z_1, \tag{19}$$

where $S(z_1)$ is the element of area at z_1. The pyramid cross-section is of constant shape so that coordinates on a cross-section can be written as coordinates on the pyramid base scaled on $(1-z_1/z)$. Applying the coordinate transformation, Equation (19) becomes

$$I_V = \int_0^z z_1^p \left(1-\frac{z_1}{z}\right)^{n+m+\gamma+2} \mathrm{d}z_1 \int_{A(0)} x_1^n y_1^m R^\gamma \,\mathrm{d}S(0), \tag{20}$$

yielding

$$I_V = z^{p+1}\frac{p!q!}{(p+q+1)!}I, \tag{21}$$

where I is the integral over the base polygon from §2.3 and $q = n+m+\gamma+2$. This approach appears to be novel, although a similar argument was implicitly used in a purely numerical scheme for evaluation of singular and near-singular integrals on polyhedra [6, Figure 12]. The method has the advantage that the volume integration requires little more computational effort than a routine surface integration over each face of the polyhedron (four in the case of a general tetrahedron) given that the scaling factor in Equation (21) is trivially simple to evaluate. This conclusion holds no matter what the method used to evaluate the surface integral on the reference pyramid base, the only requirement being that the tetrahedron orientation be respected in computing contributions from elements generated by decomposition of general geometries.

2.5 Integration on polygons in arbitrary orientations

The method developed so far has depended on elements lying in a reference position in the plane $z=0$. To correctly integrate on geometries with general monomial source terms, we require some means of transforming results from the reference geometry to the problem coordinate system. We do so using the notation given in Figure 6. A point $\mathbf{x}$ in the global

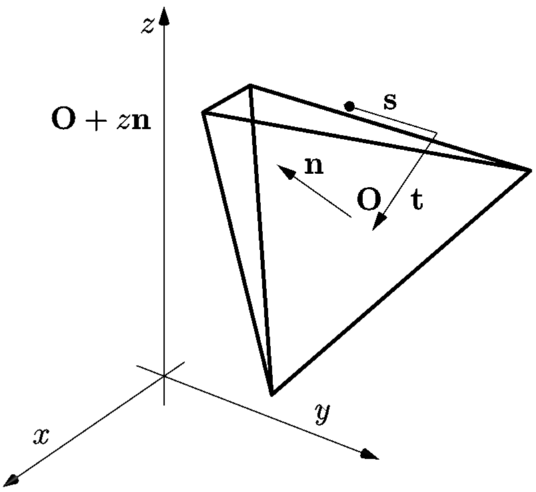

Fig. 6 Local and global coordinate systems

coordinate system is written

$$\mathbf{x} = \mathbf{O} + a\mathbf{s} + b\mathbf{t} + c\mathbf{n}, \tag{22}$$

with **O** an origin for the local coordinate system, **s** and **t** unit vectors in the reference plane, and $\mathbf{n} = \mathbf{s} \times \mathbf{t}$. For integration on a pyramid, the origin **O** is taken as the projection of the pyramid vertex onto the plane of the base. Such a coordinate system is readily established for any element using standard geometric operations.

Then, for example,

$$x^m = \sum_{i=0}^{m} \sum_{j=0}^{i} \sum_{k=0}^{m-i} \binom{m}{i} \binom{i}{j} \binom{m-i}{k} s_x^j t_x^{i-j} n_x^k O_x^{m-i-k} a^j b^{i-j} c^k, \tag{23}$$

where s_x is the x–component of **s**, etc. Thus, integrals over a polyhedron in general position with source term x_1^m become triple sums over the integrals in reference position. For general source terms $x_1^{m_x} y_1^{m_y} z_1^{m_z}$, the same procedure yields a nine-fold sum and the integral for the general pyramid can be written as a sum of integrals evaluated in reference position:

$$
\begin{aligned}
&\int_V x_1^{m_x} y_1^{m_y} z_1^{m_z} R^\gamma \,\mathrm{d}V = \\
&\sum_{i_x=0}^{m_x} \sum_{j_x=0}^{i_x} \sum_{k_x=0}^{m_x-i_x} \sum_{i_y=0}^{m_y} \sum_{j_y=0}^{i_y} \sum_{k_y=0}^{m_y-i_y} \sum_{i_z=0}^{m_z} \sum_{j_z=0}^{i_z} \sum_{k_z=0}^{m_z-i_z} \\
&\quad \binom{m_x}{i_x}\binom{i_x}{j_x}\binom{m_x-i_x}{k_x}\binom{m_y}{i_y}\binom{i_y}{j_y}\binom{m_y-i_y}{k_y}\binom{m_z}{i_z}\binom{i_z}{j_z}\binom{m_z-i_z}{k_z} \\
&\qquad \times s_x^{j_x} t_x^{i_x-j_x} n_x^{k_x} O_x^{m_x-i_x-k_x} s_y^{j_y} t_y^{i_y-j_y} n_y^{k_y} O_y^{m_y-i_y-k_y} s_z^{j_z} t_z^{i_z-j_z} n_z^{k_z} O_z^{m_z-i_z-k_z} \\
&\qquad \times \int_{V_r} R^\gamma a^{j_x+j_y+j_z} b^{i_x+i_y+i_z-j_x-j_y-j_z} c^{k_x+k_y+k_z} \,\mathrm{d}V_r, \qquad (24)
\end{aligned}
$$

with V_r the signed volume corresponding to the pyramid in the reference position with axes **s**, **t**, and **n** with integrals evaluated using the method of §2.4. With the reference integrals precomputed using the recursion methods, the sum of Equation (24) is a matter of arithmetic and can be quickly evaluated for a series of source terms on an arbitrary pyramid.

2.6 Algorithms

We summarize the analysis in a number of basic algorithms. First, the evaluation of the potential from monomial sources up to order $|m+n| \leq N$ on a polygon in the plane $z=0$:

1. generate the list of polar coordinates for the vertices (r_i, ψ_i);
2. for each triangle formed by the origin and adjacent vertices:
   (a) compute $I_{m,n}^{(p)}$, $p=\pm 1$, $m+n=-1,\ldots,4$ using §2.2;
   (b) rotate and sum $I_{m,n}^{(p)}$ to form integrals over the triangle, using Equation (16).

For a pyramid in reference position:

1. compute integrals over the base polygon using the previous algorithm;
2. generate scaling factors using Equation (21).

3. return integrals as integrals on the base polygon multiplied by scaling factors from Equation (21).

For a tetrahedron in general orientation:

1. for each tetrahedron formed by the field point and three vertices of the original tetrahedron:
   (a) transform to the coordinate system of Figure 6;
   (b) compute integrals for a pyramid in reference position, respecting the sign of the orientation of the base (i.e. do not take absolute values of quantities);
   (c) use Equation (24) to compute the contribution to the total integral;
2. sum the four contributions to find the total integral over the original tetrahedron.

3 Numerical testing

Our initial test for the integration method was evaluation of the constant source potential from a reference triangle using the method of Newman [8] and the technique of this paper. Newman's approach is also analytical, so that the results from our method should be the same to within machine precision, and a comparison of timings will give an indication of the effect of requiring computation of special functions in the calculation.

The geometry of the test triangle is shown in Figure 7. The potential was calculated at 64 evenly spaced points $(0,0,z)$, $0 \leq z \leq 1$ and the difference between the Newman method calculation and ours checked. This was never greater than 5×10^{-15}, essentially machine precision, confirming the correctness of our approach. The computation time for the new integration method was, however, eight times greater than that for the Newman approach. We believe that this can be reduced by code optimization in the new technique, and possibly by introducing well-chosen approximations for the incomplete elliptic integrals, but

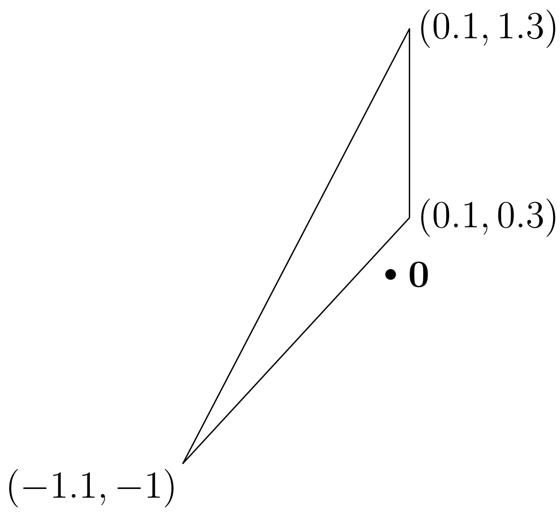

Fig. 7 Geometry for triangle integration test

our method will inevitably require more time than Newman's because of its use of special functions and the need to decompose the triangular element into subtriangles, which is not required in Newman's method. On the other hand, our implementation is designed to integrate higher-order terms with less extra work, by reuse of lower order results in the recursion.

As a more extensive test case for the integration method, we employ a reference tetrahedron shown in Figure 8. Since the method relies on integration over the faces of the tetrahedron, this is also a test for the surface integration technique. The choice of field points is governed by the requirement to check for the cases of points strictly outside the tetrahedron, on an edge or an extension of an edge, at or approaching an interior point of a face, and strictly within the volume, at varying distances from the vertices, edges and faces. Field points are chosen on lines (x, y_i, z_i), $-2 \leq x \leq 2$, with $(y_1, z_1) = (0,0)$, and $(y_2, z_2) = (0.1.0.1)$. The integral Equation (17) is computed for $0 \leq |m+n+p| \leq 4$ corresponding to integration of shape functions up to fourth order on a tetrahedral element. Values of $\gamma = -1, 1, 3$ are chosen to check for evaluation of a Laplace potential and for the first three odd order terms in the ex-

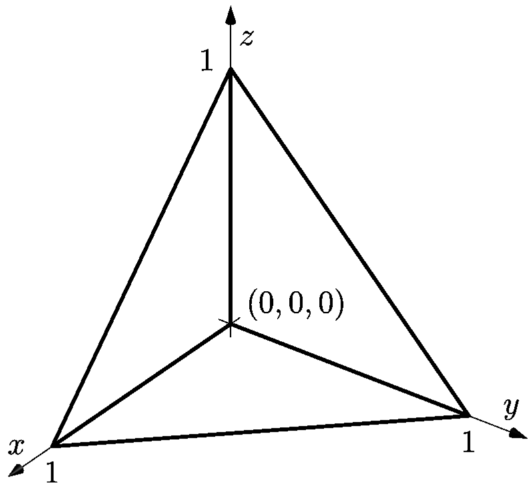

Fig. 8 Tetrahedron for test integrals

pansion of the Helmholtz Green's function, as would be needed in a singularity-subtraction method. For brevity representative results are presented for varying orders of source term: other combinations of m, n, and p give similar behaviour.

As a reference method, we employ the technique of Khayat and Wilton [6] who develop an approach which handles singularities and near-singularities in the integrand using a coordinate transformation which renders the distance term analytic in the transformed coordinates on a triangle, combined with a Gaussian quadrature in the z direction, an idea based on the constant cross-section argument which we employ in §2.4, though the implications for scaling of the integrand are not explicitly recognized or used to simplify the calculation as in Equation (21). In this calculation we use quadrature rules with $25^3 = 15625$ nodes. Computation time is not reported since this is constant for an analytical method and would not be a meaningful comparison of the algorithms.

Figure 9 shows the difference ε between analytical and numerical results for selected values of the parameters. In each case, it is clear that the difference between the two methods is of the order of machine precision outside the tetrahedron volume, where no singularities

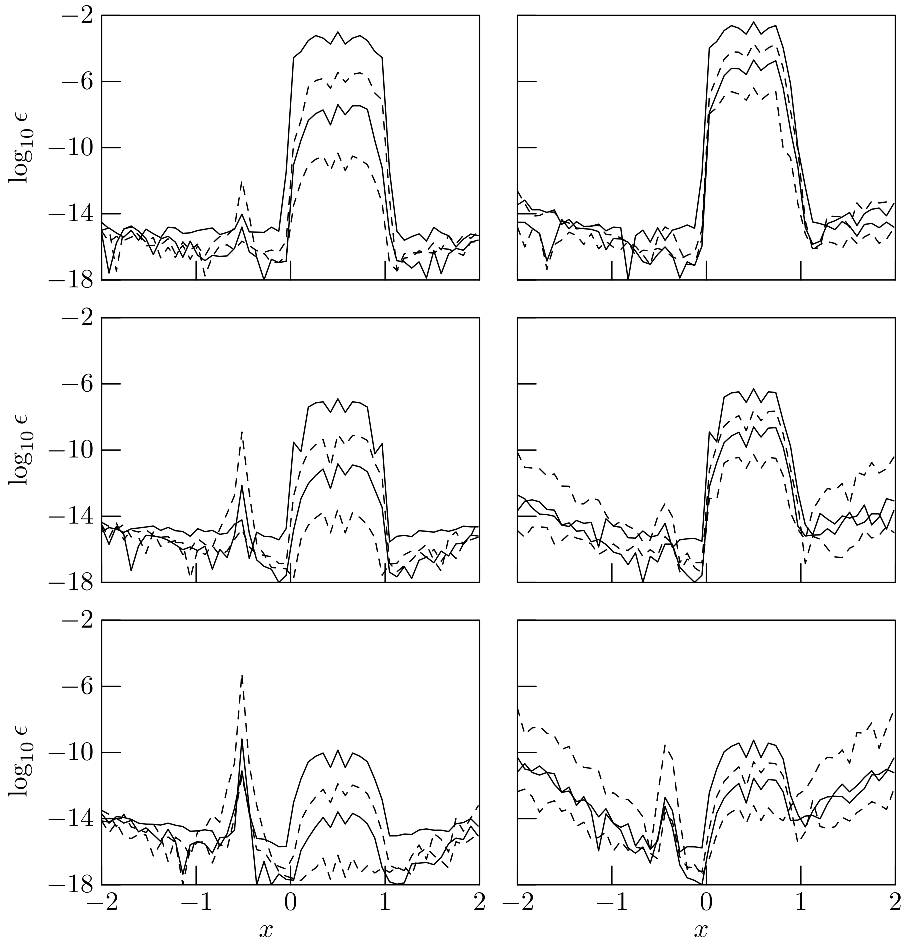

Fig. 9 Difference ε between analytical and Khayat-Wilton computation of potential integrals on reference tetrahedron: top to bottom, $\gamma = -1, 1, 3$; left-hand column, $(y,z) = (0,0)$; right-hand column $(y,z) = (0.1, 0.1)$. Data are plotted for $(n,m,p) = (0,0,0), (1,1,0), (1,1,1), (2,0,2)$.

are encountered, but that there is a larger difference within the volume, where singularities are encountered in the integrand. This remains true for the $\gamma = 1, 3$ cases where the derivatives of the integrand can be singular, affecting the smoothness of the integrand, with a corresponding effect on the quality of the integration.

4 Conclusions

A method for the exact evaluation of potentials generated by arbitrary monomial source terms on planar and polyhedral elements has been presented. The method reduces integrals on triangular elements to incomplete generalized elliptic integrals, which can be evaluated using tabulated formulae and recursions. Volume integrations are performed exactly using a trivially simple transformation which gives integrals over a tetrahedral volume at the cost of four analytically-evaluated surface integrals, rather than requiring explicit handling of the volume integration proper, as in wholly-numerical methods. The approach as presented can be used directly in production codes and, through its link to elliptic integrals, opens a possibility for asymptotic and other analysis of the basic functions required with the potential for further gains in efficiency.

References

1. Björkberg, J., Kristensson, G.: Electromagnetic scattering by a perfectly conducting elliptic disk. Canadian Journal of Physics **65**, 723–734 (1987)
2. Carini, A., Salvadori, A.: Analytical integrations in 3D BEM: preliminaries. Computational Mechanics **28**, 177–185 (2002). DOI 10.1007/S00466-001-0278-7
3. Carley, M.: Analytical formulae for potential integrals on triangles. ASME Journal of Applied Mechanics **80**(4) (2013). DOI 10.1115/1.4007853
4. Epstein, L.F., Hubbell, J.H.: Evaluation of a generalized elliptic-type integral. Journal of Research of the National Bureau of Standards B **67**(1), 1–17 (1963)
5. Gradshteyn, I., Ryzhik, I.M.: Table of integrals, series, and products, 5th edn. Academic, London (1980)
6. Khayat, M.A., Wilton, D.R.: Numerical evaluation of singular and near-singular potential integrals. IEEE Transactions on Antennas and Propagation **53**(10), 3180–3190 (2005). DOI 10.1109/TAP.2005.856342
7. Marshall, J.S., Grant, J.R., Gossler, A.A., Huyer, S.A.: Vorticity transport on a Lagrangian tetrahedral mesh. Journal of Computational Physics **161**(1), 85–113 (2000). DOI 10.1006/jcph.2000.6490

8. Newman, J.N.: Distributions of sources and normal dipoles over a quadrilateral panel. Journal of Engineering Mathematics **20**, 113–126 (1986). DOI 10.1007/BF00042771
9. Salvadori, A.: Analytical integrations in 3D BEM for elliptic problems: Evaluation and implementation. International Journal for Numerical Methods in Engineering **84**, 505–542 (2010). DOI 10.1002/nme.2906
10. Suh, J.C.: The evaluation of the Biot–Savart integral. Journal of Engineering Mathematics **37**, 375–395 (2000). DOI 10.1023/A:1004666000020