The Annals of Applied Probability
2015, Vol. 25, No. 5, 2416–2461
DOI: 10.1214/14-AAP1052

SPACE–TIME PERCOLATION AND DETECTION BY MOBILE NODES

BY ALEXANDRE STAUFFER[1]

University of Bath

Consider the model where nodes are initially distributed as a Poisson point process with intensity λ over $\mathbb{R}^d$ and are moving in continuous time according to independent Brownian motions. We assume that nodes are capable of detecting all points within distance r of their location and study the problem of determining the first time at which a target particle, which is initially placed at the origin of $\mathbb{R}^d$, is detected by at least one node. We consider the case where the target particle can move according to any continuous function and can adapt its motion based on the location of the nodes. We show that there exists a sufficiently large value of λ so that the target will eventually be detected almost surely. This means that the target cannot evade detection even if it has full information about the past, present and future locations of the nodes. Also, this establishes a phase transition for λ since, for small enough λ, with positive probability the target can avoid detection forever. A key ingredient of our proof is to use fractal percolation and multi-scale analysis to show that cells with a small density of nodes do not percolate in space and time.

1. Introduction. Let Π_0 be a Poisson point process over $\mathbb{R}^d$ of intensity $\lambda > 0$. We refer to the points of Π_0 as *nodes*, and let each node of Π_0 move as an independent Brownian motion. Define Π_s to be the point process obtained after the nodes of Π_0 have moved for time s. More formally, for each $x \in \Pi_0$, let $(\zeta_x(s))_{s\geq 0}$ be a standard Brownian motion and define $\Pi_s = \{x + \zeta_x(s) : x \in \Pi_0\}$. It is well known that Brownian motion is a measure-preserving transformation of Poisson point processes [31], Proposition 1.3, which gives that, for any fixed s, Π_s is also distributed as a Poisson point process. However, Π_s and Π_0 are *not* independent, and it is this feature that makes this model challenging to analyze.

We consider a fixed constant r so that, at any time, a node is able to detect all points inside the ball of radius r centered at its location. Then, letting $B(x, r)$ stand for the ball of radius r centered at x, we have that

$$\text{at time } s, \text{ the nodes of } \Pi_s \text{ detect the region } \bigcup_{x\in\Pi_s} B(x, r). \tag{1}$$

Received March 2012; revised July 2014.

[1]This work was done while the author was at Microsoft Research, Redmond WA, USA.

MSC2010 subject classifications. Primary 82C43; secondary 60G55, 60J65, 60K35, 82C21.

Key words and phrases. Poisson point process, Brownian motion, fractal percolation, multi-scale analysis.

The region in (1) is related to the random geometric graph model [20, 25], where (nonmobile) nodes are given by a Poisson point process and edges are added between pairs of nodes whose distance is at most r.

This model is closely related to a model of mobile graphs introduced by van den Berg, Meester and White [31]. This and other similar models of mobile graphs have been considered as natural models for mobile wireless networks [7, 8, 11, 18, 28].

Detection time. Consider an additional target particle u that is located at the origin of $\mathbb{R}^d$ at time 0. Let u move according to a continuous function $g(s)$, and define the *detection time* T_{det} as the first time at which a node is within distance r from u. More formally, we have

$$T_{\mathrm{det}} = \inf\Bigl\{t \geq 0 : g(t) \in \bigcup_{x\in\Pi_t} B(x,r)\Bigr\}.$$

Detection is a fundamental problem in wireless networks and it appears, for example, in the contexts of area surveillance and disaster recovery, where mobile sensors are randomly deployed to explore a region which, due to some natural disaster, is unsafe for humans [32].

We consider the case of a target u that wants to evade detection and can adapt its motion according to the position of the nodes of Π_0. A fundamental question is whether there exists a phase transition on λ so that, for sufficiently large λ, the target has no way to avoid detection almost surely as $t \to \infty$. More formally, define a *trajectory* $h : \mathbb{R}_+ \to \mathbb{R}^d$ as a continuous function such that $h(0)$ is the origin of $\mathbb{R}^d$. Then we say that h is *not detected* from time 0 to t if, for each $s \in [0,t]$, all nodes of Π_s are at distance larger than r from $h(s)$. The existence of such h implies that, if the target chooses its location at time s to be $g(s) = h(s)$, then it avoids detection up to time t. We then define

$$\rho_t(\lambda) = \mathbf{P}\bigl(\exists \text{ a trajectory that is not detected by } (\Pi_s)_s \text{ from time 0 to } t\bigr)$$

and, since $\rho_t(\lambda)$ is nonincreasing with t and nonnegative, the limit exists and we let

$$\rho = \rho(\lambda) = \lim_{t\to\infty} \rho_t.$$

Let $V_t = \mathbb{R}^d \setminus \bigcup_{x\in\Pi_t} B(x,r)$ be the subset of $\mathbb{R}^d$ that is not detected by the nodes of Π_t, which is usually referred to as the *vacant* region. Well-known results on random geometric graphs and mobile graphs [20, 31] give that there exists a critical value λ_{c} so that, if $\lambda < \lambda_{\mathrm{c}}$, then V_t contains an infinite connected component (also called the infinite vacant cluster) at all times. Therefore, since time is continuous and the target is allowed to move with arbitrary speed, if $\lambda < \lambda_{\mathrm{c}}$, the target can avoid detection if the origin belongs to the infinite component of V_0, an event that occurs with positive probability when $\lambda > \lambda_{\mathrm{c}}$. This gives that $\rho(\lambda) > 0$

for all $\lambda < \lambda_c$. Our Theorem 1.1 below establishes that, if λ is larger than some value λ_t, where t stands for trajectory, then $\rho = 0$, which means that the target cannot avoid detection almost surely even if it is able to foresee the locations of all nodes at *all* times (including future times). Since ρ is monotone in λ, this gives a phase transition on the value of λ for the existence of a trajectory that is not detected by the nodes.

THEOREM 1.1. *In dimensions $d \geq 2$, there exists a value $\lambda_t = \lambda_t(d) \in [\lambda_c, \infty)$ such that $\rho = 0$ for all $\lambda > \lambda_t$, and $\rho > 0$ for all $\lambda < \lambda_t$. Furthermore, there exist $\lambda_t' \geq \lambda_t$, an explicit positive constant $c = c(d)$ and a positive C independent of t such that, for all large enough t and all $\lambda > \lambda_t'$,*

$$\rho_t(\lambda) \leq \begin{cases} \exp\left(-C\dfrac{t}{(\log t)^c}\right), & \text{for } d = 2, \\ \exp(-Ct), & \text{for } d \geq 3. \end{cases} \tag{2}$$

REMARK 1.1. (i) For $d = 1$, we have that $\rho = 0$ for all $\lambda > 0$. This holds since, for any two nodes $v_1, v_2 \in \Pi_0$, after a finite time, v_1 and v_2 will meet almost surely. Therefore, by considering v_1 and v_2 such that u is between them at time 0, we have that u will be detected in finite time almost surely.

(ii) Note that $\rho < 1$ for all $\lambda > 0$ since, with constant probability, the origin of $\mathbb{R}^d$ is detected at time 0.

(iii) Based on results by Kesidis, Konstantopoulos and Phoha [14, 17] (see also the discussion in [26]), we have the following lower bound for $\rho_t(\lambda)$, which is obtained by considering a nonmobile target:

$$\rho_t(\lambda) \geq \begin{cases} \exp(-C'\sqrt{t}), & \text{for } d = 1, \\ \exp\left(-C'\dfrac{t}{\log t}\right), & \text{for } d = 2, \\ \exp(-C't), & \text{for } d \geq 3, \end{cases}$$

for all large enough t and where $C' > 0$ does not depend on t. Therefore, comparing this lower bound with the tail bound in (2) for $\lambda > \lambda_t'$ reveals that, disregarding logarithmic factors for $d = 2$ and constant factors for $d \geq 3$, a target that is able to choose its motion strategically in response to the past, present and even future positions of the nodes and is also capable of moving with arbitrary speed cannot do much better in terms of avoiding detection than a target that does not move at all.

We believe that there exists a regime for λ so that V_t contains no infinite component at every t, but the target is still able to avoid detection; that is, $\lambda_t > \lambda_c$. If this is true, when $\lambda \in (\lambda_c, \lambda_t)$, we have that the target is completely surrounded by nodes at all times; that is, at any time, the positions at which the target can be at

that time form a bounded set. However, this set may not collapse to the empty set as time proceeds, and thus the target may be able to avoid detection with positive probability. Also, we believe that $\lambda_t = \lambda_t'$; that is, as soon as λ is large enough so that $\rho = 0$, then the exponential tail bounds in (2) should hold. We formalize these ideas in the following conjecture.

CONJECTURE 1. *For all dimensions $d \geq 2$, we have $\lambda_c < \lambda_t$ and $\lambda_t = \lambda_t'$.*

Space–time percolation. In order to prove Theorem 1.1, which is the main motivation of this paper, we develop a framework that gives a more general result regarding space–time percolation of increasing events. We consider a tessellation of $\mathbb{R}^d$ into cubes of side length ℓ, and a tessellation of the time interval $[0, t]$ into subintervals of length β. Let $i \in \mathbb{Z}^d$ be an index for the cubes of the tessellation and $\tau \in \mathbb{Z}_+$ be an index for the time intervals. Let $\varepsilon \in (0, 1)$ be fixed and let $E(i, \tau)$ be the indicator random variable for the event that $\Pi_{\tau\beta}$ contains at least $(1 - \varepsilon)\lambda\ell^d$ nodes in the cube indexed by i.

The pairs (i, τ) can be seen as a tessellation of the *space–time* region $\mathbb{R}^{d+1}$ and $E(i, \tau)$ is a process over this region. We refer to the space–time subregion indexed by (i, τ) as a *cell*. For this process, we say that a cell (i, τ) is *adjacent* to a cell (i', τ') if $\|i - i'\|_\infty \leq 1$ and $|\tau - \tau'| \leq 1$. We say that a cell (i, τ) is *bad* if $E(i, \tau) = 0$ and, in this case, define $K(i, \tau)$ as the set of bad cells from which there exists a path of adjacent bad cells to (i, τ); if $E(i, \tau) = 1$ we let $K(i, \tau) = \varnothing$. $K(i, \tau)$ is usually referred to as the *bad cluster* of (i, τ). Our Theorem 1.2 below establishes an upper bound for the bad cluster of the origin $(0, 0)$, which implies that the bad cells do not percolate in space and time. We note that the theorem below is carried out for all dimensions $d \geq 1$, unlike Theorem 1.1 for which the case $d = 1$ is trivial [see Remark 1.1(i)].

THEOREM 1.2. *Let $\varepsilon > 0$, $\ell > 0$ and $\beta > 0$ be fixed. Let Q_t be the space–time region $(-t, t)^d \times [0, t/2)$. Then there exist a positive constant $c = c(d)$, a positive value C, and values $\lambda_0, t_0 > 0$ such that, for all $\lambda > \lambda_0$ and $t > t_0$, we have*

$$\mathbf{P}(\exists \text{ a cell of } K(0,0) \text{ not contained in } Q_t) \leq \begin{cases} \exp\left(-C\dfrac{\sqrt{t}}{(\log t)^c}\right), & \text{for } d = 1, \\ \exp\left(-C\dfrac{t}{(\log t)^c}\right), & \text{for } d = 2, \\ \exp(-Ct), & \text{for } d \geq 3. \end{cases}$$

In Section 3, we prove a more general form of Theorem 1.2 (which we state in Theorem 3.1). In this general form, E is not restricted to be the event defined above, but can be taken to be the indicator random variable of *any increasing* event that depends only on a *bounded* neighborhood of cells and whose marginal probability is large enough. Under these assumptions, we prove that the cells (i, τ)

for which $E(i,\tau)=1$ percolate in space and time, which implies that the cells for which $E(i,\tau)=0$ do not percolate. We prove Theorem 1.1 from this result by defining E with respect to a specific event that implies that if the target enters a cell (i,τ) for which $E(i,\tau)=1$ then the target is detected. Since the statement of Theorem 3.1 requires some extra notation, we defer it to Section 3.

Proof overview. The proof of Theorem 1.2 proceeds via a multi-scale argument inspired by fractal percolation. We will consider tessellations of the space–time region $\mathbb{R}^{d+1}$ into cells of various scale. We start with very large cells so that, by standard arguments, we can show that, with sufficiently large probability, all such cells contain a sufficiently high density of nodes at the beginning of their time interval. Then we consider only the cells that were seen to be sufficiently dense and partition them into smaller cells; the cells that were observed not to be dense are simply disregarded, similarly to a fractal percolation process. We use that the large cells were dense to infer that, with sufficiently large probability, the smaller cells also contain a high density of nodes. We then repeat this procedure until we obtain cells of side length ℓ, for which the density requirement translates to the event $E(i,\tau)=1$. We show that, despite the procedure of removing nondense cells described above, the cells of side length ℓ that are observed to be dense percolate in space and time. The proof of this result requires a delicate construction that allows us to control dependencies among cells of various scales. The details are given in Section 3.

With this framework of space–time percolation, the proof of Theorem 1.1 follows rather easily from Theorem 1.2. The intuition is that, whenever $E(i,\tau)=1$, the cube i contains sufficiently many nodes that can prevent the target to cut through the cube i during the interval $[\tau\beta,(\tau+1)\beta]$. Indeed, by setting the parameters ℓ and β small enough, we can ensure that the target can only avoid detection if it never enters a cell for which $E(i,\tau)=1$. However, since the cells for which $E(i,\tau)=0$ do not percolate in space and time by Theorem 1.2, we are assured that the target must be detected. The details are given in Section 4.

Related work. The detection of a target that moves independently of the nodes of $(\Pi_s)_s$ is by now well understood. Kesidis, Konstantopoulous and Phoha [14, 17] observed that, for the case $g\equiv 0$ (i.e., u does not move), a very precise asymptotic expression for $\mathbf{P}(T_{\mathrm{det}}\geq t)$ can be obtained using ideas from stochastic geometry [30]. This was later extended by Peres et al. [26] and Peres and Sousi [27] for the case when g is any function independent of the nodes of $(\Pi_s)_s$. They establish the interesting fact that the best strategy for a target that moves independently of the nodes and wants to avoid detection is to stay put and not to move. A result of similar flavor was proved for continuous-time random walks on the square lattice by Drewitz et al. [10] and Moreau et al. [22]. The detection problem has also being analyzed in different models of static and mobile networks [1, 9, 19].

Multi-scale arguments as developed in our proof of Theorem 1.2 are not uncommon. The most related references are the papers by Kesten and Sidoravicius [15, 16] for the study of the spread of infection in a moving population, the work of Peres et al. [26] for the so-called percolation time of mobile geometric graphs, and a paper by Chatterjee et al. [3] for the gravitational allocation of Poisson point processes. However, the techniques in these papers are tailored to specific problems. Our Theorem 1.2 (or, more precisely, the detailed version of Theorem 3.1) provides a flexible multi-scale analysis that establish space–time percolation of increasing events. We believe this technique can be extended to handle other problems as well.

Organization of the paper. The remainder of the paper is organized as follows. In Section 2, we generalize a coupling argument developed in [26, 29] for the mixing of moving nodes, which we will need in our main proofs. Then, in Section 3, we give the precise statement and proof of Theorem 1.2. Finally, we apply this result in Section 4, where we prove Theorem 1.1.

2. Mixing of mobile graphs. In this section, we extend a coupling argument developed by Sinclair and Stauffer [29] (see also [26]). In a high-level description, the result in [29] establishes that, if a point process contains sufficiently many nodes in each cell of a suitable tessellation, then after the nodes have moved as *independent Brownian motions* for some time interval ϱ that depends on the size of the cells of the tessellation, the nodes will contain an independent Poisson point process with high probability. This argument gives a way to handle dependencies on mobile geometric graphs, but is not enough in our proof of Theorem 1.2. The reason is that, when nodes are moving as independent Brownian motions, it may be the case that a node moves atypically far away during the time interval ϱ. Therefore, a node may affect a large region in space during ϱ, causing large dependencies among the cells of the tessellation. In order to better control dependencies, we only consider nodes that do not move very far away during the interval ϱ. These nodes are then moving as independent Brownian motions *conditioned* on not moving very far away during ϱ. We carry out a more careful analysis of the coupling argument in [29] in order to derive a corresponding result to this more general setting for the motion of the nodes.

We note that, very recently, Benjamini and Stauffer [2] employed similar techniques to show that, for a particular point process, after the nodes have moved for some time, they will be *contained* in an independent Poisson point process.

Now we describe the setting. Fix $\ell > 0$ and consider the cube $Q_\ell = [-\ell/2, \ell/2]^d$. Consider a node that, at time 0, is located at some arbitrary position $x \in Q_\ell$. We assume that the position of this node at time Δ is distributed according to a translation-invariant function f_Δ, which means that the probability density function for this node to move from x to some arbitrary position $y \in \mathbb{R}^d$ after time Δ

is $f_\Delta(y-x)$. Let $M > \ell$. We say that a subdensity function $g:\mathbb{R}^d \to \mathbb{R}_+$ is (ε,ℓ)-*indistinguishable from* f_Δ *over the cube* Q_M if

$$g(y) \le f_\Delta(y-x) \quad \text{for all } x \in Q_\ell \text{ and } y \in Q_M, \text{ and also } \int_{Q_{M-\ell}} g(y)\,dy \ge 1-\varepsilon. \tag{3}$$

[Recall that a function h is called a *subdensity* function if $h(x) \ge 0$ for all $x \in \mathbb{R}^d$ and $\int_{\mathbb{R}^d} h(x)\,dx \le 1$.] We will apply (3) in the following context. Consider two nodes u and v that are initially in arbitrary positions inside $Q_{\ell/2}$; so their distance vector is in Q_ℓ and is represented by x in (3). Assume that, at time 0, u is at position a_0 and that, at time Δ, the position of u is a_Δ, where $a_\Delta - a_0$ is distributed according to the density function f_Δ. Similarly, assume that v is at position b_0 at time 0 and that b_Δ is v's position at time Δ, where $b_\Delta - b_0$ is distributed according to the density function given by a renormalization of g. Suppose we are given a_0 and b_0, and we want to sample b_Δ and a_Δ in a coupled way. Then, for any b_Δ such that $b_\Delta - b_0 \in Q_M$, we have from (3) that $g(b_\Delta - b_0) \le f_\Delta(b_\Delta - a_0)$. Thus, we can obtain a coupling so that $a_\Delta = b_\Delta$ whenever $b_\Delta - b_0 \in Q_M$. Also, since the integral of g over $Q_{M-\ell}$ is at least $1-\varepsilon$, we obtain that this coupling gives $\mathbf{P}(a_\Delta = b_\Delta) \ge 1-\varepsilon$. In words, an indistinguishable function g is such that a motion according to g inside Q_M is very similar to a motion according to f_Δ under any perturbation of the starting point that is within Q_ℓ.

We now state Proposition 2.1, which is the most general version of our extension to [29], Proposition 4.1. Later, in Proposition 2.3, we give a special case of Proposition 2.1, which we will use in our proofs for space–time percolation. In [29], a proof is carried out by constructing a (ε,ℓ)-indistinguishable function for the special case when f_Δ is a (standard) Brownian motion run for time Δ. Below, for any two sets $A, A' \subset \mathbb{R}^d$, we define $A + A'$ to be the set $\{x + x' : \text{for all } x \in A \text{ and } x' \in A'\}$.

PROPOSITION 2.1. *Let $S \supset S'$ be two bounded regions of $\mathbb{R}^d$ and define $R = \sup\{k > 0 : S' + Q_k \subseteq S\}$. Consider any partition of S into sets $S_1, S_2, \ldots, S_m$ that we call cells such that the diameter of each cell is at most γ for some fixed $\gamma > 0$. Let Φ_0 be an arbitrary point process at time 0 such that, for each $i = 1, 2, \ldots, m$, Φ_0 contains at least $\beta\,\mathrm{vol}(S_i)$ nodes in S_i for some $\beta > 0$. Let Φ_Δ be the point process obtained at time Δ from Φ_0 after the nodes have moved independently according to a translation-invariant density function f_Δ. Fix $\varepsilon \in (0,1)$. If there exists a translation-invariant subdensity function $g:\mathbb{R}^d \to \mathbb{R}_+$ that is $(\varepsilon/2, 2\gamma)$-indistinguishable from f_Δ over the cube $Q_{R+2\gamma}$, then we can couple the nodes of Φ_Δ with those of a Poisson point process Ξ that is independent of Φ_0 and has intensity $(1-\varepsilon)\beta$ so that the nodes of Ξ are a subset of the nodes of Φ_Δ inside S' with probability at least*

$$1 - \sum_{i=1}^{m} \exp\bigl(-c\varepsilon^2 \beta\,\mathrm{vol}(S_i)\bigr) \qquad \textit{for some positive constant } c = c(d).$$

PROOF. We will construct Ξ via three Poisson point processes: Ξ_0, Ξ_0' and Ξ_Δ'. We start by defining Ξ_0 as a Poisson point process over S with intensity $(1-\varepsilon/2)\beta$. Then, for each $i=1,2,\dots,m$, Ξ_0 has fewer nodes than Φ_0 in S_i if Ξ_0 has less than $\beta\,\mathrm{vol}(S_i)$ nodes in that cell, which by a standard Chernoff bound (cf. Lemma A.1) occurs with probability larger than $1-\exp(-\frac{\delta^2(1-\varepsilon/2)\beta\,\mathrm{vol}(S_i)}{2}(1-\delta/3))$ for δ such that $(1+\delta)(1-\varepsilon/2)=1$. Note that $\delta\in(\varepsilon/2,1)$, so the probability above can be bounded below by $1-\exp(-c_1\varepsilon^2\beta\,\mathrm{vol}(S_i))$ for some universal constant c_1. Let $\{\Xi_0\preceq\Phi_0\}$ be the event that, for every $i=1,2,\dots,m$, Ξ_0 has fewer nodes than Φ_0 in every cell. Using the union bound we obtain

$$\mathbf{P}(\Xi_0\preceq\Phi_0)\geq 1-\sum_{i=1}^{m}\exp\bigl(-c_1\varepsilon^2\beta\,\mathrm{vol}(S_i)\bigr). \tag{4}$$

If $\{\Xi_0\preceq\Phi_0\}$ holds, then we can pair each node $u\in\Xi_0$ to a unique node of $v\in\Phi_0$ in the same cell. We call u the corresponding pair of v (and vice versa). We will now show that we can couple the motion of the nodes in Ξ_0 with the motion of their corresponding pairs in Φ_0 so that the probability that an arbitrary pair is at the same location at time Δ is sufficiently large.

To describe the coupling, let v' be a node of Ξ_0 located at $y'\in S$, and let v be the corresponding pair of v' in Φ_0. Let y be the location of v in S, and note that since v and v' belong to the same cell we have $\|y-y'\|_2\leq\gamma$. Since g is $(\varepsilon/2,2\gamma)$-indistinguishable from f_Δ, we assume that the motion of v' from time 0 to Δ is such that its density function is a renormalization of g. Then, for any point z such that $z-y'\in Q_R$, we have

$$g\bigl(z-y'\bigr)=g\bigl(z-y-\bigl(y'-y\bigr)\bigr)\leq f_\Delta(z-y),$$

since $y'-y\in Q_{2\gamma}$ and $z-y=(z-y')-(y-y')\in Q_{R+2\gamma}$. Then we have that the function g is smaller than the densities for the motions of v and v' to the location z for all z such that $z-y'\in Q_R$ and $y-y'\in Q_{2\gamma}$.

Define $\tilde g(x)=g(x)\mathbf{1}(x\in Q_R)$ and

$$\psi=\int_{Q_R}\tilde g(x)\,dx=\int_{Q_R}g(x)\,dx\geq 1-\frac{\varepsilon}{2}. \tag{5}$$

Hence, with probability ψ we can use the density function $\frac{\tilde g(x)}{\psi}$ to sample a single location $y'+x$ for the position of both v and v' at time Δ. Then the second Poisson point process in the construction of Ξ, which we denote by Ξ_0', is defined as a Poisson point process of intensity $\psi(1-\varepsilon/2)\beta$ obtained by *thinning* Ξ_0 (i.e., deleting each node of Ξ_0 with probability $1-\psi$). At this step, we have used the fact that the function $\tilde g(x)$ is oblivious of the location of v and, consequently, is independent of the point process Φ_0.

Then the third Poisson point process Ξ_Δ' is obtained from Ξ_0' after the nodes have moved according to the density function $\frac{\tilde g(z)}{\psi}$. Thus, since g is translation

invariant, we have that Ξ'_Δ is a Poisson point process and, due to the coupling described above, the nodes of Ξ'_Δ are a subset of the nodes of Φ_Δ and are independent of the nodes of Φ_0, where Φ_Δ is obtained by letting the nodes of Φ_0 move from time 0 to time Δ according to the density function f_Δ.

Note that Ξ'_Δ is a *nonhomogeneous* Poisson point process. It remains to show that the intensity of Ξ'_Δ is strictly larger than $(1-\varepsilon)\beta$ in S' so that Ξ can be obtained from Ξ'_Δ via thinning; since Ξ'_Δ is independent of Φ_0, so is Ξ.

For $z \in \mathbb{R}^d$, let $\mu(z)$ be the intensity of Ξ'_Δ. Since Ξ'_0 has no node outside S, we obtain, for any $z \in S'$,

$$\mu(z) \geq \psi(1-\varepsilon/2)\beta \int_{z+Q_R} \frac{\tilde{g}(z-x)}{\psi}\,dx = (1-\varepsilon/2)\beta \int_{Q_R} \tilde{g}(x)\,dx,$$

where the inequality follows since $z + Q_R \subset S$ for all $z \in S'$. Using (5), we have $\mu(z) \geq (1-\varepsilon/2)(1-\varepsilon/2)\beta \geq (1-\varepsilon)\beta$, which is the intensity of Ξ. Therefore, when $\{\Xi_0 \preceq \Phi_0\}$ holds, which occurs with probability given by (4), the nodes of Ξ are a subset of the nodes of Φ_Δ, which completes the proof of Proposition 2.1. □

Now we illustrate an application of Proposition 2.1 by giving an example that we will use later in our proofs. Consider a node that, at time 0, is located at an arbitrary position $x \in Q_\ell$. Let this node move for time Δ according to a Brownian motion and, for each $z > 0$, define $F_\Delta(z)$ to be the event that this node never leaves the cube $x + Q_z$ during the interval $[0, \Delta]$. Let $M \geq \ell$ and, for $y = (y_1, y_2, \ldots, y_d) \in Q_{2M}$, define $\tilde{f}_\Delta(y)$ to be the probability density function for the location of this node at position $x + y$ at time Δ conditioned on $F_\Delta(3M)$. It follows from the reflection principle of Brownian motion that

$$\begin{aligned}
&\tilde{f}_\Delta(y)\mathbf{P}\big(F_\Delta(3M)\big) \\
&\quad \geq \prod_{i=1}^{d}\bigg(\frac{1}{\sqrt{2\pi\Delta}}\exp\bigg(-\frac{y_i^2}{2\Delta}\bigg) - \frac{1}{\sqrt{2\pi\Delta}}\exp\bigg(-\frac{(3M-y_i)^2}{2\Delta}\bigg) \\
&\qquad\qquad - \frac{1}{\sqrt{2\pi\Delta}}\exp\bigg(-\frac{(3M+y_i)^2}{2\Delta}\bigg)\bigg) \\
&\quad = \frac{1}{(2\pi\Delta)^{d/2}}\exp\bigg(-\frac{\|y\|_2^2}{2\Delta}\bigg) \\
&\qquad \times \prod_{i=1}^{d}\bigg[1 - \exp\bigg(-\frac{9M^2}{2\Delta}\bigg)\bigg(\exp\bigg(\frac{6My_i}{2\Delta}\bigg) + \exp\bigg(-\frac{6My_i}{2\Delta}\bigg)\bigg)\bigg] \\
&\quad \geq \frac{1}{(2\pi\Delta)^{d/2}}\exp\bigg(-\frac{\|y\|_2^2}{2\Delta}\bigg)\bigg(1 - 2d\exp\bigg(-\frac{3M^2}{2\Delta}\bigg)\bigg),
\end{aligned} \tag{6}$$

where the last step follows since the function $e^a + e^{-a}$ is increasing in $|a|$ and $y_i \in [-M, M]$ for all $y \in Q_{2M}$.

Our goal is to apply Proposition 2.1 with $f_\Delta = \tilde{f}_\Delta$. In order to do this, we will use the technical lemma below, which constructs a subdensity function that is indistinguishable from $\tilde{f}_\Delta$.

LEMMA 2.2. *Let $\xi \in (0,1)$ and $m > 0$. Let*

$$\Delta \geq \frac{d^3 m^2}{\xi^2} \quad \text{and} \quad M \geq \sqrt{8\Delta \log(8d\xi^{-1})}.$$

For $z \in Q_M$, define

$$g(z) = \frac{1}{(2\pi\Delta)^{d/2}} \exp\biggl(-\frac{(\|z\|_2 + m\sqrt{d}/2)^2}{2\Delta}\biggr)\biggl(1 - 2d\exp\biggl(-\frac{3M^2}{2\Delta}\biggr)\biggr),$$

and, for $z \notin Q_M$, set $g(z) = 0$. Then, g is (ξ, m)-indistinguishable from $\tilde{f}_\Delta$ over Q_M, where $\tilde{f}_\Delta$ is the probability density function for the location of a Brownian motion at time Δ given that it never leaves the cube Q_{3M} during the interval $[0, \Delta]$.

PROOF. Note that, for any $x \in Q_m$, the triangle inequality gives that $\|z - x\|_2 \leq \|x\|_2 + \|z\|_2 \leq \frac{m\sqrt{d}}{2} + \|z\|_2$. Thus, for all $x \in Q_m$ and $z \in Q_M$, we have that $z - x \in Q_{M+m} \subseteq Q_{2M}$ and, from (6), we obtain that $g(z) \leq \tilde{f}_\Delta(z - x)$ as required by the first condition in (3). Now, let $\rho = m\sqrt{d}/2$ and

$$\upsilon(x) = \frac{1}{\sqrt{2\pi\Delta}} \exp\biggl(-\frac{(|x| + \rho)^2}{2\Delta}\biggr),$$

for $x \in \mathbb{R}$. Note that $\sum_{i=1}^d (|z_i| + \rho)^2 = \|z\|_2^2 + 2\rho\|z\|_1 + d\rho^2 \geq (\|z\|_2 + \rho)^2$, so

$$\biggl(1 - 2d\exp\biggl(-\frac{3M^2}{2\Delta}\biggr)\biggr)\prod_{i=1}^d \upsilon(z_i) \leq g(z) \qquad \text{for } z = (z_1, \ldots, z_d) \in Q_M. \tag{7}$$

Next, observe that

$$\begin{aligned}\int_{-\infty}^{\infty} \upsilon(x)\,dx &= 1 - \int_{-\rho}^{\rho} \frac{1}{\sqrt{2\pi\Delta}} \exp\biggl(-\frac{y^2}{2\Delta}\biggr)\,dy \\ &\geq 1 - \frac{2\rho}{\sqrt{2\pi\Delta}} \geq 1 - \frac{\rho}{\sqrt{\Delta}} \geq 1 - \frac{\xi}{2d},\end{aligned}$$

where the last step follows from $\Delta \geq \frac{d^3 m^2}{\xi^2} = \frac{4d^2\rho^2}{\xi^2}$. Now, since $\frac{M-m}{2} + \rho \geq \frac{M}{2} \geq \sqrt{\Delta}$, we apply the Gaussian tail bound (cf. Lemma A.2) to obtain

$$\begin{aligned}\int_{(M-m)/2}^{\infty} \upsilon(x)\,dx &\leq \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\Delta}}{(M-m)/2 + \rho} \exp\biggl(-\frac{((M-m)/2 + \rho)^2}{2\Delta}\biggr) \\ &\leq \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\Delta}}{M/2} \exp\biggl(-\frac{M^2}{8\Delta}\biggr) \leq \frac{\xi}{8d},\end{aligned}$$

for any $\xi \in (0,1)$ since $M \geq \sqrt{8\Delta \log(8d\xi^{-1})}$. Thus, $\int_{-(M-m)/2}^{(M-m)/2} \upsilon(x)\,dx \geq 1 - \frac{\xi}{2d} - 2\frac{\xi}{8d} = 1 - \frac{3\xi}{4d}$. We can now deduce from (7) that

$$\begin{aligned}\int_{Q_{M-m}} g(z)\,dz &\geq \left(1 - 2d\exp\left(-\frac{3M^2}{2\Delta}\right)\right)\int_{Q_{M-m}} \prod_{i=1}^{d} \upsilon(z_i)\,dz \\ &\geq \left(1 - 2d\exp\left(-\frac{3M^2}{2\Delta}\right)\right)\left(1 - \frac{3\xi}{4d}\right)^d \\ &\geq \left(1 - 2d\left(\frac{\xi}{8d}\right)^{12}\right)\left(1 - \frac{3\xi}{4}\right) \\ &\geq \left(1 - \frac{\xi}{4}\right)\left(1 - \frac{3\xi}{4}\right) \geq 1 - \frac{\xi}{4} - \frac{3\xi}{4} = 1 - \xi. \qquad \square\end{aligned}$$

The next proposition is a special case of Proposition 2.1 that we will use in our proofs.

PROPOSITION 2.3. *Fix $K > \ell > 0$ and consider the cube Q_K tessellated into subcubes of side length ℓ. Let Φ_0 be an arbitrary point process at time 0 that contains at least $\beta\ell^d$ nodes in each subcube for some $\beta > 0$. For any $z > 0$, let $\Phi_\Delta(z)$ be the point process obtained by letting the nodes of Φ_0 move for time Δ according to independent Brownian motions that are conditioned on being inside Q_z throughout the interval $[0,\Delta]$. Fix $\varepsilon \in (0,1)$. There are constants c_1, c_2, c_3 depending only on d such that, if $\Delta \geq \frac{c_1\ell^2}{\varepsilon^2}$ and $K' \leq K - c_2\sqrt{\Delta \log(16d\varepsilon^{-1})} > 0$, we can couple the nodes of $\Phi_\Delta(3(K - K' + 2\sqrt{d}\ell))$ with those of a Poisson point process Ξ that is independent of Φ_0 and has intensity $(1-\varepsilon)\beta$ so that the nodes of Ξ are a subset of the nodes of $\Phi_\Delta(3(K - K' + 2\sqrt{d}\ell))$ inside the cube $Q_{K'}$ with probability at least*

$$1 - \frac{K^d}{\ell^d}\exp(-c_3\varepsilon^2\beta\ell^d).$$

PROOF. Denote by $\tilde{f}_\Delta(y)$ the probability density function for the location of a Brownian motion at time Δ conditioned on the motion never leaving $Q_{3(K-K'+2\sqrt{d}\ell)}$ during the whole of $[0,\Delta]$. Now, if c_1 and c_2 are large enough with respect to d, we can apply Lemma 2.2 with $\xi = \varepsilon/2$, $M = K - K' + 2\sqrt{d}\ell$ and $m = 2\sqrt{d}\ell$, which gives that g is $(\varepsilon/2, 2\sqrt{d}\ell)$-indistinguishable from $\tilde{f}_\Delta$ over $Q_{K-K'+2\sqrt{d}\ell}$. Thus, we apply Proposition 2.1 with $S = Q_K$ and $S' = Q_{K'}$. This gives that $R = K - K'$ and $\gamma = \ell\sqrt{d}$. Thus, we have that a Poisson point process Ξ with intensity $(1-\varepsilon)\beta$ over $Q_{K'}$ can be coupled with $\Phi_\Delta(3(K - K' + 2\sqrt{d}\ell))$ so that Ξ is a subset of $\Phi_\Delta(3(K - K' + 2\sqrt{d}\ell))$ with probability at least

$$1 - \frac{K^d}{\ell^d}\exp(-c_3\varepsilon^2\beta\ell^d) \qquad \text{for some positive constant } c_3 = c_3(d). \qquad \square$$

3. Space–time percolation. In this section, we develop the main technical result of this paper, which we stated in a simplified form in Theorem 1.2. In order to state this theorem in full generality, we need to introduce some notation, which will extend the setting introduced in the part *space–time percolation* in Section 1.

We tessellate $\mathbb{R}^d$ into cubes of side length ℓ. We index the cubes by integer vectors $i \in \mathbb{Z}^d$ such that

$$\text{cube } i = (i_1, i_2, \ldots, i_d) \text{ corresponds to the region } \prod_{j=1}^{d} [i_j \ell, (i_j + 1)\ell] \subset \mathbb{R}^d.$$

Let $t > 0$, and tessellate the time interval $[0, t]$ into subintervals of length β. We index the subintervals by $\tau \in \mathbb{Z}_+$, where

$$\text{subinterval } \tau \text{ represents the time interval } [\tau\beta, (\tau + 1)\beta] \subset \mathbb{R}.$$

We call each pair (i, τ) of the space–time tessellation a *cell*, and define the region of a cell as

$$R_1(i, \tau) = \prod_{j=1}^{d} [i_j \ell, (i_j + 1)\ell] \times [\tau\beta, (\tau + 1)\beta].$$

The space–time tessellation defined above will be sufficient when we apply in Section 4 the technique developed in this section to the detection problem. However, in previous works, being able to handle overlapping cells turned out to be very useful; for example, in the study of the spread of infection [15] and in the estimation of the length of the shortest path between nodes [13]. For this reason, we generalize our framework slightly by introducing bigger, overlapping cells. Consider an integer parameter $\eta \geq 1$ and, for each cube $i = (i_1, i_2, \ldots, i_d)$ and time interval τ,

$$\text{define the } \textit{super cube } i \text{ as } \prod_{j=1}^{d} [i_j \ell, (i_j + \eta)\ell] \quad \text{and}$$

$$\text{the } \textit{super interval } \tau \text{ as } [\tau\beta, (\tau + \eta)\beta].$$

Then define the *super cell* (i, τ) as the Cartesian product of the super cube i and the super interval τ.

For a sequence of point processes $(\Pi_s)_{s \geq 0}$, we say that

> an event E is *increasing* for $(\Pi_s)_{s \geq 0}$ if the fact that E holds for $(\Pi_s)_{s \geq 0}$ implies that it holds for all $(\Pi'_s)_{s \geq 0}$ for which $\Pi'_s \supseteq \Pi_s$ for all $s \geq 0$.

We also say that an event E is *restricted* to a region $X \subset \mathbb{R}^d$ and a time interval $[t_0, t_1]$ if it is measurable with respect to the σ-field generated by the nodes that are inside X at time t_0 and their positions from time t_0 to t_1. For an increasing event

E that is restricted to a region X and a time interval $[t_0, t_1]$, we have the following definition:

(8) ν_E is called the *probability associated to* E if, for any $\mu \geq 0$ and region $X' \subset \mathbb{R}^d$, $\nu_E(\mu, X')$ is the probability that E happens given that, at time t_0, the nodes in X are given by a Poisson point process of intensity μ and their motions from t_0 to t_1 are independent Brownian motions conditioned to have displacement inside X' from t_0 to t_1.

In other words, if $(x_t)_{t\geq 0}$ are the locations of one such node, then $x_{t_0} \in X$ and, for each $s \in [t_0, t_1]$, we have $x_s - x_{t_0} \in X'$.

We say that:

two distinct cells (i, τ) and (i', τ') are *adjacent* if $\|i - i'\|_\infty \leq 1$ and $|\tau - \tau'| \leq 1$.

For each $(i, \tau) \in \mathbb{Z}^{d+1}$, let $E(i, \tau)$ be the indicator random variable for an increasing event restricted to the super cube i and the super interval τ. We say that a cell (i, τ) is *bad* if $E(i, \tau) = 0$, and in this case, we define the *bad cluster* $K(i, \tau)$ of (i, τ) as the following set of cells:

$$K(i,\tau) = \big\{(i',\tau') \in \mathbb{Z}^{d+1} : E(i',\tau') = 0 \text{ and } \exists \text{ a path of adjacent bad cells from } (i,\tau) \text{ to } (i',\tau')\big\}. \tag{9}$$

If $E(i, \tau) = 1$, we let $K(i, \tau) = \varnothing$. Finally, define

$$\mathcal{R}_1^t = \big\{(i,\tau) \in \mathbb{Z}^{d+1} : R_1(i,\tau) \subset (-t, t) \times [0, t)\big\}. \tag{10}$$

Our main technical result, Theorem 3.1 below, establishes a bound on the bad cluster of the origin.

THEOREM 3.1. *For each $(i, \tau) \in \mathbb{Z}^{d+1}$, let $E(i, \tau)$ be the indicator random variable for an increasing event that is restricted to the super cube i and the super interval τ, and let ν_E be the probability associated to E as defined in (8). Fix a constant $\varepsilon \in (0, 1)$, an integer $\eta \geq 1$ and the ratio $\beta/\ell^2 > 0$. Fix also w such that*

$$w \geq \sqrt{18\eta \frac{\beta}{\ell^2} \log\Big(\frac{8d}{\varepsilon}\Big)}.$$

Then there exist positive numbers α_0 and t_0 so that if

$$\alpha = \min\Big\{\varepsilon^2 \lambda \ell^d, \log\Big(\frac{1}{1 - \nu_E((1-\varepsilon)\lambda, Q_{w\ell})}\Big)\Big\} \geq \alpha_0,$$

we have a positive constant $c = c(d)$ and a positive value C independent of t such that, for all $t \geq t_0$,

$$\mathbf{P}\big(K(0,0) \nsubseteq \mathcal{R}_1^t\big) \leq \begin{cases} \exp\Big(-C\dfrac{\sqrt{t}}{(\log t)^c}\Big), & \text{for } d = 1, \\ \exp\Big(-C\dfrac{t}{(\log t)^c}\Big), & \text{for } d = 2, \\ \exp(-Ct), & \text{for } d \geq 3. \end{cases}$$

REMARK 3.1. (i) Theorem 1.2 is a special case of Theorem 3.1 with $\eta = 1$ and $E(i,\tau)$ corresponding to the event that the cell i has at least $(1-\varepsilon)\lambda\ell^d$ nodes at time $\tau\beta$. Note that, in this case, w can be taken to be any positive constant since ν_E does not depend on w.

(ii) In Theorem 3.1, C may depend on all the parameters (including λ), but does not depend on t.

(iii) Note that, in Theorem 3.1, we first fix the values of ε, η and β/ℓ^2. Given these values, we fix w, and only then we set λ and ℓ so that the condition on α is satisfied. Note that setting ℓ in this last step is equivalent to setting β, since the ratio β/ℓ^2 is fixed. It is important that w does not depend on λ or ℓ. In typical applications of Theorem 3.1, w is set to be sufficiently large so that $\nu_E((1-\varepsilon)\lambda, Q_{w\ell})$ can be made arbitrarily close to 1 by setting λ or ℓ large enough after having fixed w. This is usually the case since, if w is large enough with respect to η and β/ℓ^2, we have that the probability that a standard Brownian motion stays inside $Q_{w\ell}$ during $[0, \eta\beta]$ is at least $1 - \exp(-\frac{cw^2\ell^2}{\eta\beta})$ for some positive constant c. Then we have that most of the nodes of the Poisson point process of intensity $(1-\varepsilon)\lambda$ in the definition of ν_E stay inside the cube of side length $w\ell$ that is centered at their initial position.

(iv) As observed in [14, 17], if we define $E(i,\tau) = 0$ when there is no node in the cube i at time $\tau\beta$, then $\mathbf{P}(K(0,0) \nsubseteq \mathcal{R}_1^t)$ is at least $\exp(-c\sqrt{t})$ for $d = 1$, $\exp(-c\frac{t}{\log t})$ for $d = 2$ and $\exp(-ct)$ for $d \geq 3$, where c does not depend on t. This lower bound is achieved by the event that $E(0,\tau) = 0$ for all $\tau = 0, 1, \ldots, \lceil t/\beta \rceil$. Hence, with respect to t, the exponents in the bound of Theorem 3.1 are tight up to logarithmic factors for $d = 1, 2$ and up to constants for $d \geq 3$.

In order to prove Theorem 3.1, we will do a multi-scale analysis, where, at each scale, we tessellate $\mathbb{R}^{d+1}$ into cells of a given size. We then analyze the density of nodes in these cells using the framework of fractal percolation. The proof of Theorem 3.1 is rather lengthy and, for this reason, we split the proof in many parts. We start by introducing the multi-scale tessellation in Section 3.1. Then, in Section 3.2, we define the fractal percolation process we will analyze. In Section 3.3, we present a high-level sketch of the proof using the notation established in the previous two sections, and also discuss the relation with standard fractal percolation. In Section 3.4, we define the support of a cell and prove some geometric

properties of the support. In Section 3.5, we use the definition of the support of a cell to introduce a new notion of path of cells, and also prove some lemmas about the paths. Finally, in Section 3.6, we put these results together to give the proof of Theorem 3.1.

3.1. *Multi-scale tessellation of space and time.* We start with the tessellation of space. Let m be a sufficiently large integer. For each scale $k \geq 1$, tessellate $\mathbb{R}^d$ into cubes of length ℓ_k such that

$$\ell_1 = \ell \quad \text{and} \quad \ell_k = mk^3\ell_{k-1} = m^{k-1}(k!)^3\ell. \tag{11}$$

We also set $\ell_0 = \ell/m$.

REMARK 3.2. We let m be sufficiently large with respect to ε, η, w and the ratio β/ℓ^2; however, m does not depend on λ or ℓ. Hence, after having m fixed, we can make α arbitrarily large by setting either λ or ℓ large enough.

It will be useful to refer to Figure 1 in the discussion below. We index the cubes by integer vectors $i \in \mathbb{Z}^d$ and denote them by $S_k(i)$. Therefore, for $i = (i_1, i_2, \ldots, i_d)$, we have

$$S_k(i) = \prod_{j=1}^{d} [i_j\ell_k, (i_j + 1)\ell_k].$$

So, $S_1(i)$ is exactly the cube i introduced in the beginning of Section 3. Note that $S_k(i)$ is the union of $(mk^3)^d$ cubes of scale $k-1$; in particular, a cube of scale

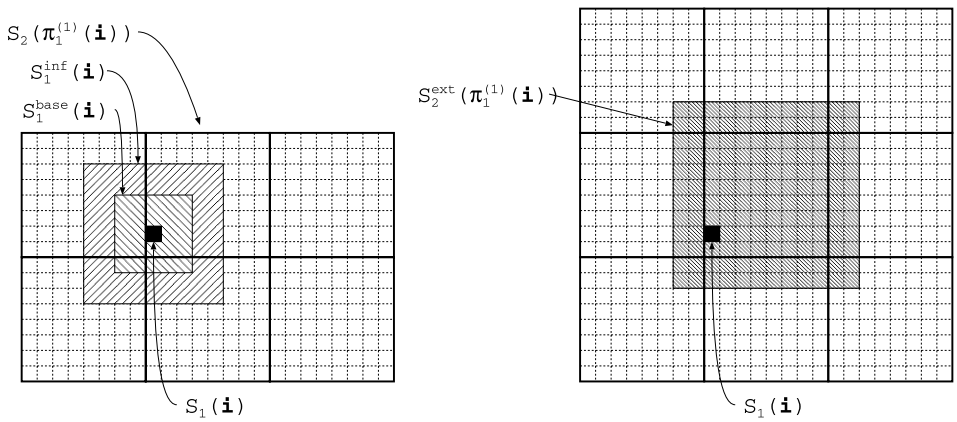

FIG. 1. *Illustration of the tessellation of $\mathbb{R}^d$. Different scales are represented by the thickness of the lines; for example, $S_2(\pi_1^{(1)}(i))$ is the square with thick borders that contains $S_1(i)$, which is the black square. Note that $S_1(i)$ is at the same position in both the left and the right pictures above, illustrating that $S_1^{\text{base}}(i) \subset S_2^{\text{ext}}(\pi_1^{(1)}(i))$ as given in (13).*

smaller than k is contained inside a unique cube of scale k. With this, for each k, $j \geq 0$ and $i \in \mathbb{Z}^d$, we can define

$$\pi_k^{(j)}(i) = i' \quad \text{iff} \quad S_k(i) \subseteq S_{k+j}(i').$$

Thus, $\pi_k^{(j)}$ defines a hierarchy over cubes since, for all $j' \leq j$, $\pi_k^{(j)}(i) = \pi_{k+j'}^{(j-j')}(\pi_k^{(j')}(i))$. Using this notion, for $i, i' \in \mathbb{Z}^d$ and $k \geq 0$, we say that $(k+1, i')$ is the *parent* of (k, i) if $\pi_k^{(1)}(i) = i'$; in this case, we also say that (k, i) is a *child* of $(k+1, i')$. We define the set of descendants of (k, i) as (k, i) and the union of the descendants of the children of (k, i), or only (k, i) in case (k, i) has no child.

We now introduce a new variable n that satisfies

$$n^d = \frac{m}{7\eta}. \tag{12}$$

The variable m must be large enough so that $n > 1$. We also assume that m is specified in a way that makes n an integer.

We define some larger cubes based on $S_k(i)$. For $k \geq 0$, define the *base* $S_k^{\text{base}}(i)$ of $S_k(i)$ and the *area of influence* $S_k^{\text{inf}}(i)$ of $S_k(i)$ by

$$S_k^{\text{base}}(i) = \bigcup_{i' : \|i - i'\|_\infty \leq \eta m n (k+1)^3} S_k(i')$$

and

$$S_k^{\text{inf}}(i) = \bigcup_{i' : \|i - i'\|_\infty \leq 2\eta m n (k+1)^3} S_k(i').$$

We also define the *extended* cube $S_k^{\text{ext}}(i)$ by

$$S_k^{\text{ext}}(i) = \bigcup_{i' : \pi_{k-1}^{(1)}(i') = i} S_{k-1}^{\text{base}}(i').$$

Note that $S_k^{\text{ext}}(i)$ is the union of the bases of the children of (k, i), which are the $(k-1)$-cubes contained in $S_k(i)$. It is easy to see that $S_k(i) \subset S_k^{\text{base}}(i) \subset S_k^{\text{inf}}(i)$ and

$$S_{k+1}^{\text{ext}}(\pi_k^{(1)}(i)) = \bigcup_{i' : \pi_k^{(1)}(i') = \pi_k^{(1)}(i)} S_k^{\text{base}}(i') \supset S_k^{\text{base}}(i). \tag{13}$$

REMARK 3.3. One important property obtained from these definitions is that an extended cube of scale 1 has side length $\ell + 2\eta m n \ell_0 = (1 + 2\eta n)\ell$. Therefore, for any $i \in \mathbb{Z}^d$, the extended cube $S_1^{\text{ext}}(i)$ contains the super cell i defined in the beginning of Section 3.

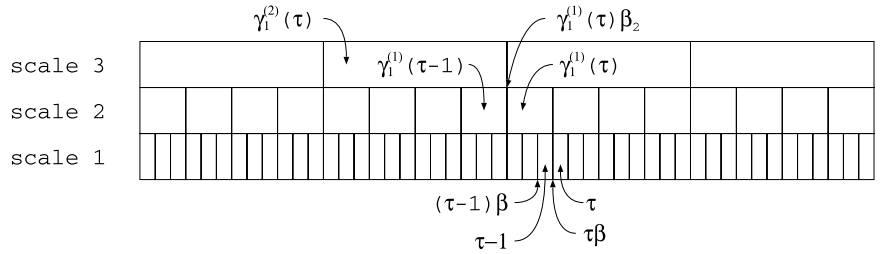

FIG. 2. *Time scale. The horizontal axis represents time and the vertical axis represents the scale. Note that* $\gamma_1^{(1)}(\tau) = \gamma_1^{(1)}(\tau+1) = \gamma_1^{(1)}(\tau+2)$.

Now, we define a multi-scale tessellation of time. In this discussion, it will be useful to refer to Figure 2. We define

$$\varepsilon_1 = \varepsilon \quad \text{and} \quad \varepsilon_k = \varepsilon_{k-1} - \frac{\varepsilon}{k^2} \qquad \text{for all } k \geq 1.$$

There will be no scale 0 for time, but we have defined $\varepsilon_0 = 2\varepsilon$ for consistency. Now we define

$$\beta_k = C_{\text{mix}} \frac{\ell_{k-1}^2}{(\varepsilon_{k-1} - \varepsilon_k)^2} = C_{\text{mix}} \frac{\ell_{k-1}^2 k^4}{\varepsilon^2} \qquad \text{for all } k \geq 1, \tag{14}$$

where $C_{\text{mix}} \geq 4c_1$ and c_1 is the constant in Proposition 2.3 that depends on d only. Note that, for $k = 1$, we have

$$\beta = \beta_1 = C_{\text{mix}} \frac{(\ell/m)^2}{\varepsilon^2}. \tag{15}$$

Hence, given β/ℓ^2 and ε, we can set m sufficiently large so that $C_{\text{mix}} \geq 4c_1$. Also, note that

$$\frac{\beta_{k+1}}{\beta_k} = \frac{\ell_k^2 (k+1)^4}{\ell_{k-1}^2 k^4} = m^2 k^2 (k+1)^4 \qquad \text{for all } k \geq 1. \tag{16}$$

Now, for scale $k \geq 1$, we tessellate time into intervals of length β_k. We index the time intervals by $\tau \in \mathbb{Z}$ and denote them by $T_k(\tau)$, where

$$T_k(\tau) = \big[\tau\beta_k, (\tau+1)\beta_k\big].$$

We allow time to be negative, and note that β_{k+1}/β_k is always an integer by (16), which gives that

(17) a time interval of scale k is contained inside a unique time interval of scale $k+1$.

At scale 1, we will consider the time intervals that intersect $[0, t]$.

We also introduce a hierarchy over time, but it is important to emphasize that this time hierarchy is conceptually different than the spatial hierarchy induced by π. For all k and τ, let $\gamma_k^{(0)}(\tau) = \tau$, and, for $j \geq 1$, define

$$\gamma_k^{(j)}(\tau) = \tau' \qquad \text{if } \gamma_k^{(j-1)}(\tau)\beta_{k+j-1} \in T_{k+j}(\tau' + 1).$$

In the spatial tessellation, we constructed the hierarchy π based on whether a large cell contains a small cell. For the time tessellation, if $\tau' = \gamma_k^{(1)}(\tau)$, then the interval at scale $k+1$ that contains $T_k(\tau)$ [which is uniquely defined as noted in (17)] is $T_{k+1}(\tau' + 1)$. In other words, as we move from scale $k+1$ to k, this hierarchy shifts time forward. The reason for this definition is that this shift in time will be used to allow nodes to mix inside their cells using the techniques of Section 2. This will become clearer in Section 3.5.

Now note that γ indeed establishes a hierarchy over time since, for any $j' \leq j$, we have $\gamma_k^{(j)} = \gamma_{k+j'}^{(j-j')}(\gamma_k^{(j')})$. Thus, for $\tau, \tau' \in \mathbb{Z}$ and $k \geq 1$, we say that $(k+1, \tau')$ is the *parent* of (k, τ), if $\gamma_k^{(1)}(\tau) = \tau'$; in this case we also say that (k, τ) is a *child* of $(k+1, \tau')$. We also define the set of descendants of (k, τ) as (k, τ) and the union of the descendants of the children of (k, τ), or only (k, τ) in case (k, τ) has no child.

Now, for any $i \in \mathbb{Z}^d$, $k \geq 1$, $\tau \in \mathbb{Z}$, we define the space–time parallelogram

$$R_k(i, \tau) = S_k(i) \times T_k(\tau),$$

and note that these parallelograms induce a tessellation over space and time. From now on, we reserve the word *cube* to refer to the spatial region defined by $S_k(i)$, *interval* to refer to the time region defined by $T_k(\tau)$ and *cell* to refer to the space–time region $R_k(i, \tau)$.

For $k \geq 1$, define $\mathcal{S}_k$ to be the set of indices $i \in \mathbb{Z}^d$ given by

$$\mathcal{S}_k = \{i \in \mathbb{Z}^d : S_k(i) \text{ intersects } [-t, t]^d\}.$$

Equivalently, we can see $\mathcal{S}_k$ as the set of cubes of scale k that have a descendant of scale 1 intersecting $[-t, t]^d$. Similarly, we define $\mathcal{T}_k$ as the set of indices $\tau \in \mathbb{Z}$ for time intervals of scale k that have a descendant at scale 1 intersecting $[0, t]$. More formally, we have

$$\mathcal{T}_k = \{\tau \in \mathbb{Z} : \exists \tau' \text{ s.t. } \gamma_1^{(k-1)}(\tau') = \tau \text{ and } T_1(\tau') \text{ intersects } [0, t]\}.$$

Note that an interval in $\mathcal{T}_k$ with $k \geq 2$ may not intersect $[0, t]$. Using these definitions, we set

$$\mathcal{R}_k = \mathcal{S}_k \times \mathcal{T}_k.$$

Note that $\mathcal{R}_1 \supset \mathcal{R}_1^t$, where $\mathcal{R}_1^t$ is defined in (10). Also, for any cell $(i, \tau) \in \mathcal{R}_1^t$, the cells that are adjacent to (i, τ) belong to $\mathcal{R}_1$. Now define

$$\mathcal{R} = \{(k, i, \tau) : 1 \leq k \leq \kappa \text{ and } (i, \tau) \in \mathcal{R}_k\};$$

that is, $\mathcal{R}$ is set of tuples of the form (k, i, τ) giving the index set of all cells of all scales we consider. Later in the proof of Theorem 3.1 we will set $\kappa = c\frac{\log t}{\log\log t}$ for some sufficiently large constant c, but we define the tessellation now for all scales. We extend the hierarchy of space and time to the cells of $\mathcal{R}$. Then, for $(k, i, \tau) \in \mathcal{R}$, we define the *descendants* of (k, i, τ) as the cells (k', i', τ') so that (k', i') is a descendant of (k, i) and (k', τ') is a descendant of (k, τ). As usual, we say that (k, i, τ) is an *ancestor* of (k', i', τ') if (k', i', τ') is a descendant of (k, i, τ).

3.2. *A fractal percolation process.* Here, we define the percolation process we will analyze. For $k \geq 1$, define $S_k(i)$ to be *k-dense* if each cube of scale $k-1$ that is contained in $S_k(i)$ has at least $(1-\varepsilon_k)\lambda\ell_{k-1}^d$ nodes. For $(k, i, \tau) \in \mathcal{R}$, let $D_k(i, \tau)$ be the indicator random variable such that

$$D_k(i, \tau) = 1 \quad \text{iff} \quad S_k(i) \text{ is } k\text{-dense at time } \tau\beta_k.$$

Now, take a node $v \in \Pi_0$ and let $(x_t)_t$ be the locations of v in $\mathbb{R}^d$. For a time interval $[t_0, t_1] \subset \mathbb{R}$ and a region $X \subset \mathbb{R}^d$, we say that:

$$\text{the } \textit{displacement} \text{ of } v \text{ throughout } [t_0, t_1] \text{ is in } X \text{ if } \bigcup_{s=t_0}^{t_1} (x_s - x_{t_0}) \subset X.$$

In other words, v never leaves the region $x_{t_0} + X$ during the interval $[t_0, t_1]$.

We need to introduce a more restrictive notion of density. For this, we define the indicator random variable $D_k^{\text{ext}}(i, \tau)$ for each $(k, i, \tau) \in \mathcal{R}$ so that

> $D_k^{\text{ext}}(i, \tau) = 1$ iff, at time $\tau\beta_k$, all cubes of scale $k-1$ contained in $S_k^{\text{ext}}(i)$ have at least $(1-\varepsilon_k)\lambda\ell_{k-1}^d$ nodes whose displacement throughout $[\tau\beta_k, (\tau+2)\beta_k]$ is in $Q_{\eta mnk^3\ell_{k-1}}$,

where Q_z denotes the cube of side length z given by $[-z/2, z/2]^d$. Clearly, $D_k^{\text{ext}}(i, \tau) \leq D_k(i, \tau)$ for all $(k, i, \tau) \in \mathcal{R}$.

REMARK 3.4. An important property of this definition is that, when $D_k^{\text{ext}}(i, \tau) = 1$, if $(k-1, i', \tau')$ is a child of (k, i, τ), then we know that there are enough nodes in $S_{k-1}^{\text{base}}(i')$ at time $\tau\beta_k$ and these nodes never leave the cube $S_{k-1}^{\text{inf}}(i')$ during the interval $[\tau\beta_k, \tau'\beta_{k-1}]$. This will allow us to apply the mixing technique of Section 2 to show that if $D_k^{\text{ext}}(i, \tau) = 1$ then $D_{k-1}^{\text{ext}}(i', \tau')$ is likely to be 1.

Now, for $k \leq \kappa - 1$, let

> $D_k^{\text{base}}(i, \tau) = 1$ iff, at time $\gamma_k^{(1)}(\tau)\beta_{k+1}$, all cubes of scale k inside $S_k^{\text{base}}(i)$ contain at least $(1-\varepsilon_{k+1})\lambda\ell_k^d$ nodes whose displacement throughout $[\gamma_k^{(1)}(\tau)\beta_{k+1}, \tau\beta_k]$ is in $Q_{\eta mn(k+1)^3\ell_k}$.

Note that if $D_{k+1}^{\mathrm{ext}}(\pi_k^{(1)}(i), \gamma_k^{(1)}(\tau)) = 1$ then $D_k^{\mathrm{base}}(i,\tau) = 1$. This gives that

$$D_k^{\mathrm{base}}(i,\tau) \geq D_{k+1}^{\mathrm{ext}}\big(\pi_k^{(1)}(i), \gamma_k^{(1)}(\tau)\big) \qquad \text{for all } (k,i,\tau) \in \mathcal{R}. \tag{18}$$

For scale κ, we define

$$A_\kappa(i,\tau) = D_\kappa^{\mathrm{ext}}(i,\tau),$$

and, for k satisfying $2 \leq k \leq \kappa - 1$, we set

$$A_k(i,\tau) = \max\big\{D_k^{\mathrm{ext}}(i,\tau), 1 - D_k^{\mathrm{base}}(i,\tau)\big\}.$$

Finally, for scale 1 we set

$$A_1(i,\tau) = \max\big\{E(i,\tau), 1 - D_1^{\mathrm{base}}(i,\tau)\big\}$$

and define

$$A(i,\tau) = \prod_{k=1}^{\kappa} A_k\big(\pi_1^{(k-1)}(i), \gamma_1^{(k-1)}(\tau)\big). \tag{19}$$

We will analyze the random variables $A(i,\tau)$ instead of $E(i,\tau)$. We say that a cell $(i,\tau) \in \mathcal{R}_1$ has a *bad ancestry* if $A(i,\tau) = 0$, and in this case we define

$$\begin{aligned} K'(i,\tau) = \big\{&\big(i',\tau'\big) \in \mathbb{Z}^{d+1} : A\big(i',\tau'\big) = 0 \text{ and } \exists \text{ a path of adjacent cells} \\ &\text{from } (i,\tau) \text{ to } \big(i',\tau'\big) \text{ where each cell of the path has a bad ancestry}\big\}. \end{aligned}$$

If $A(i,\tau) = 1$ then $K'(i,\tau) = \varnothing$. In words, $K'(i,\tau)$ is the set of cells of scale 1 with bad ancestry that can be reached from (i,τ) via a path of cells of scale 1 with bad ancestry. The lemma below shows that we can bound $K(i,\tau)$ by $K'(i,\tau)$.

LEMMA 3.2. *For each cell $(i,\tau) \in \mathbb{Z}^{d+1}$ of scale 1, we have that $E(i,\tau) \geq A(i,\tau)$. This implies that $K(i,\tau) \subseteq K'(i,\tau)$.*

PROOF. We first fix $(i,\tau) \in \mathbb{Z}^{d+1}$. Then, for $k = 1$, define $X_1 = E(i,\tau)$ and, for $k \geq 2$, define $X_k = D_k^{\mathrm{ext}}(\pi_1^{(k-1)}(i), \gamma_1^{(k-1)}(\tau))$. Also, let $Y_k = D_k^{\mathrm{base}}(\pi_1^{(k-1)}(i), \gamma_1^{(k-1)}(\tau))$. Therefore, by the definition of A in (19), we have

$$A(i,\tau) = \left(\prod_{k=1}^{\kappa-1} \max\{X_k, 1 - Y_k\}\right) X_\kappa. \tag{20}$$

By (18), we have that $Y_k \geq X_{k+1}$ for all k. Thus, for any $k \leq \kappa - 1$, we have

$$\max\{X_k, 1 - Y_k\} X_{k+1} \leq \max\{X_k, 1 - X_{k+1}\} X_{k+1} = X_k X_{k+1}.$$

Therefore, applying the inequality above repetitively in (20) we have

$$A(i,\tau) \leq \left(\prod_{k=1}^{\kappa-2} \max\{X_k, 1 - Y_k\}\right) X_{\kappa-1} X_\kappa \leq \prod_{k=1}^{\kappa} X_k \leq X_1 = E(i,\tau). \qquad \square$$

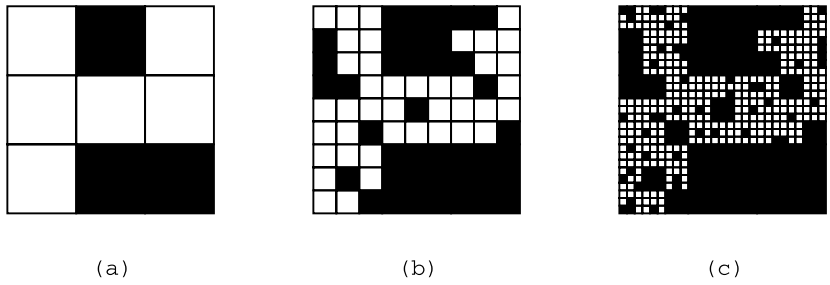

FIG. 3. *Illustration of a fractal percolation process with* $\mu = 3$ *and its* 1*-cubes* (a), 2*-cubes* (b) *and* 3*-cubes* (c). *Black squares represent closed cubes and white squares represent open cubes.*

3.3. *High-level overview.* In this section, we give some intuition for the definitions of Sections 3.1 and 3.2, and give a high-level overview of the proof of Theorem 3.1.

First, we give the definition of standard fractal percolation. Take the cube $[0, 1]^d$ and partition it into μ^d identical subcubes of side length $1/\mu$, where $\mu > 1$ is an integer. We refer to such subcubes as 1-cubes, and declare each 1-cube to be *open* independently with probability $p \in (0, 1)$; otherwise, declare it to be *closed*. Now repeat this process independently for each open 1-cube, splitting it into μ^d identical subcubes of size $1/\mu^2$ that we call 2-cubes and declaring each 2-cube to be open independently with probability p. The 1-cubes that are closed are not partitioned again, and all the region spanned by these cubes is considered to be closed (see Figure 3). We continue this procedure until we obtain z-cubes of side length $1/\mu^z$. Many results in this area [4–6, 12, 21, 24] study properties of the set of open z-cubes as $z \to \infty$.

Now, we discuss the intuition behind our definitions and the connection with fractal percolation. We start at scale κ. We tessellate space and time into very large cells. These are the cells indexed by the tuples in $\mathcal{R}_\kappa$, and each cell corresponds to a cube in space and a time interval. Then, for each cell $(i, \tau) \in \mathcal{R}_\kappa$, we check whether the cell contains sufficiently many nodes at the beginning of its time interval; that is, we check whether $A_\kappa(i, \tau) = D_\kappa^{\text{ext}}(i, \tau) = 1$. If $A_\kappa(i, \tau) = 1$, we do a finer tessellation of the cell in both space and time. In terms of fractal percolation, this corresponds to the event that a large cube is open and then is subdivided into smaller cubes. On the other hand, if $A_\kappa(i, \tau) = 0$, we skip that cell and tessellate it no further, similarly to what happens to cubes that are closed in a fractal percolation process. We iterate this procedure until we obtain cells of volume $\ell^d \beta$ (i.e., cells of scale 1). The main reason for employing this idea instead of analyzing the events $D_k(i, \tau)$ directly is that the $D_k(i, \tau)$ are highly dependent.

In the analysis, we start with the random variables $A_k(i, \tau)$ of scale $k = \kappa$, where cells are so large that we can easily obtain that $A_\kappa(i, \tau) = 1$ for all $(i, \tau) \in R_\kappa$. Then we move from scale $k + 1$ to scale k. Let $(i, \tau) \in \mathcal{R}_k$. In order to analyze $A_k(i, \tau)$, we need to observe $A_{k+1}(i', \tau')$ such that $\pi_k^{(1)}(i) = i'$ and $\gamma_k^{(1)}(\tau) = \tau'$;

that is, $(k+1, i', \tau')$ is the parent of (k, i, τ) with respect to the hierarchies π and γ. If $A_{k+1}(i', \tau') = 0$, then we do not need to observe $A_k(i, \tau)$ since we will not do the finer tessellation of $R_{k+1}(i', \tau')$ that produces the cell (k, i, τ). In this case, we will consider all the descendants at scale 1 of the cell $(k+1, i', \tau')$ as "bad," and hence we will not need to observe any other descendant of $(k+1, i', \tau')$ such as (k, i, τ). On the other hand, if $A_{k+1}(i', \tau') = 1$, we know that there is a sufficiently large density of nodes in the region $S_k^{\text{base}}(i) \subset S_{k+1}^{\text{ext}}(i')$ that surrounds $S_k(i)$ at time $\tau'\beta_{k+1}$. Then, by allowing these nodes to move from $\tau'\beta_{k+1}$ to $\tau\beta_k$, we obtain that many of these nodes move inside $S_k(i)$, giving that the probability that $A_k(i, \tau) = 0$, which corresponds to the event $D_k^{\text{base}}(i, \tau) = 1$ and $D_k^{\text{ext}}(i, \tau) = 0$, is small. We then apply this reasoning for all $(k, i, \tau) \in \mathcal{R}$. The key fact is that a dense cell at scale k makes the children of this cell likely to be dense.

We now give the intuition behind the different types of cubes. Let $(k, i, \tau) \in \mathcal{R}$ and assume that $(k+1, i', \tau')$ is the parent of (k, i, τ). We consider the extended cube $S_{k+1}^{\text{ext}}(i')$ instead of just $S_{k+1}(i')$ to assure that, when $D_{k+1}^{\text{ext}}(i', \tau') = 1$, then there is a large density of nodes around $S_k(i)$ at time $\tau'\beta_{k+1}$ even if $S_k(i)$ lies near the boundary of $S_{k+1}(i')$; this happens since $\{D_{k+1}^{\text{ext}}(i', \tau') = 1\}$ guarantees that there are sufficiently many nodes in $S_k^{\text{base}}(i) \subset S_{k+1}^{\text{ext}}(i')$. We then let the nodes move for time $\tau\beta_k - \tau'\beta_{k+1} \geq \beta_{k+1}$, thereby allowing them to mix in $S_k^{\text{base}}(i)$ and move inside $S_k(i)$. While these nodes move in the interval $[\tau'\beta_{k+1}, \tau\beta_k]$, they never leave the *area of influence* $S_k^{\text{inf}}(i)$. This allows us to argue that cells that are sufficiently far apart in space are "roughly independent" since we only observe nodes that stay inside the area of influence of their cells.

Now, we give a brief sketch of the proof. We want to give an upper bound for the probability that $K(0,0)$ is not contained in $\mathcal{R}_1^t$. When $K(0,0) \nsubseteq \mathcal{R}_1^t$, then there exists a *very large* path of adjacent bad cells of scale 1. A natural strategy is to consider any fixed path of adjacent cells from the cell $(0,0)$ to a cell outside $\mathcal{R}_1^t$ and show that the probability that all cells in this path are bad is exponentially small, and then take the union bound over all such paths. However, this strategy seems challenging due to the dependencies among the events that the cells of a given path are bad. We use two ideas to solve this problem: path of cells of varying scales and well separated cells.

We start with cells of scale κ, which are so large that we can show that, with very large probability, $A_\kappa(i, \tau) = 1$ for all $(i, \tau) \in \mathcal{R}_\kappa$. Therefore, if a cell (i, τ) of scale 1 has $A(i, \tau) = 0$, we know that there exists an ancestor (k', i', τ') of $(1, i, \tau)$ such that (k', i', τ') is bad but its parent is good [i.e., $A_{k'}(i', \tau') = 0$]. With this, we have that if a path of bad cells of scale 1 exists, then there is a path of bad cells of varying scales. This path must contain sufficiently many cells because it must connect the cell $(0,0)$ to a cell outside $\mathcal{R}_1^t$. We take any fixed path of cells of varying scale and show that, given that this path contains sufficiently many cells, we can obtain a subset of the cells of the path so that these cells are "well separated" in space and time. This implies that the $A_k(i, \tau)$ are "roughly" independent for the well

separated cells. Using this, we can show that the probability that all cells in this subset are bad is very small. Then, by applying the union bound over all sets of well separated cells that can be obtained from a path of cells of varying scales, we establish Theorem 3.1.

3.4. *The support of a cell.* We define the *time of influence* $T_k^{\inf}(\tau)$ of (k,τ) as

$$T_1^{\inf}(\tau) = \bigl[\gamma_1^{(1)}(\tau)\beta_2, \bigl(\tau + \max\{\eta, 2\}\bigr)\beta_1\bigr]$$

and

$$T_k^{\inf}(\tau) = \bigl[\gamma_k^{(1)}(\tau)\beta_{k+1}, (\tau+2)\beta_k\bigr] \qquad \text{for } k \geq 2$$

and set the *region of influence* as

$$R_k^{\inf}(i,\tau) = S_k^{\inf}(i) \times T_k^{\inf}(\tau).$$

Intuitively, the event $\{A_k(i,\tau) = 1\}$ depends on the motion of nodes during a subinterval of $T_k^{\inf}(\tau)$ and these nodes never leave the region $S_k^{\inf}(i)$. Thus, we will be able to argue later that two cells with disjoint regions of influence are "roughly independent."

We assume that m is sufficiently large with respect to η so that $\max\{\eta,2\}\beta \leq \beta_2 = 16m^2\beta$, which gives that

$$T_k^{\inf}(\tau) \subseteq T_{k+1}\bigl(\gamma_k^{(1)}(\tau)\bigr) \cup T_{k+1}\bigl(\gamma_k^{(1)}(\tau)+1\bigr) \cup T_{k+1}\bigl(\gamma_k^{(1)}(\tau)+2\bigr). \tag{21}$$

We define the *time support* $T_k^{\sup}(\tau)$ of (k,τ) as

$$\begin{aligned} T_k^{\sup}(\tau) &= \bigcup_{i=0}^{8} T_{k+1}\bigl(\gamma_k^{(1)}(\tau) - 3 + i\bigr) \\ &= \bigl[\bigl(\gamma_k^{(1)}(\tau) - 3\bigr)\beta_{k+1}, \bigl(\gamma_k^{(1)}(\tau) + 6\bigr)\beta_{k+1}\bigr], \end{aligned} \tag{22}$$

and note that, by (21),

$$T_k^{\inf}(\tau) \subset T_k^{\sup}(\tau).$$

We also define the *spatial support* $S_k^{\sup}(i)$ of (k,i) as

$$S_k^{\sup}(i) = \bigcup_{i' :\, \|i' - \pi_k^{(1)}(i)\|_\infty \leq m} S_{k+1}(i'), \tag{23}$$

and, for any $(k,i,\tau) \in \mathcal{R}$, we define

$$R_k^{\sup}(i,\tau) = S_k^{\sup}(i) \times T_k^{\sup}(\tau). \tag{24}$$

The main idea behind the definition of the support is that, in order to control dependencies, we will not only need to consider path of cells of varying scales, but we will also need to restrict our attention to paths of cells that are sufficiently far

apart in both space and time; we will define later two cells to be far apart if the support of one cell does not contain the support of the other. We now prove some useful geometric properties of the support.

The lemma below gives that if the regions of influence of two cells intersect, then the region of influence of the cell of smaller scale is contained in the support of the other cell. In other words, the support of a given cell contains all the cells of smaller scale whose region of influence intersects the region of influence of the given cell.

LEMMA 3.3. *For any sufficiently large m the following is true. For any $(k,i,\tau),(k',i',\tau')\in\mathcal{R}$ with $k\geq k'$, if $R^{\inf}_{k'}(i',\tau')\not\subseteq R^{\sup}_{k}(i,\tau)$ then $R^{\inf}_{k'}(i',\tau')\cap R^{\inf}_{k}(i,\tau)=\varnothing$.*

PROOF. Note that, if $R^{\inf}_{k'}(i',\tau')\not\subseteq R^{\sup}_{k}(i,\tau)$, then either $T^{\inf}_{k'}(\tau')\not\subseteq T^{\sup}_{k}(\tau)$ or $S^{\inf}_{k'}(i')\not\subseteq S^{\sup}_{k}(i)$. We first assume that $T^{\inf}_{k'}(\tau')\not\subseteq T^{\sup}_{k}(\tau)$ and show that this implies

$$T^{\inf}_{k'}(\tau')\cap T^{\inf}_{k}(\tau)=\varnothing,$$

which yields $R^{\inf}_{k'}(i',\tau')\cap R^{\inf}_{k}(i,\tau)=\varnothing$.

Note that the interval $T^{\inf}_{k'}(\tau')$ has length at most $3\beta_{k'+1}$ by (21). Then, since $T^{\inf}_{k'}(\tau')\not\subseteq T^{\sup}_{k}(\tau)$,

$$T^{\inf}_{k'}(\tau')\cap\big[(\gamma^{(1)}_{k}(\tau)-3)\beta_{k+1}+3\beta_{k'+1},(\gamma^{(1)}_{k}(\tau)+6)\beta_{k+1}-3\beta_{k'+1}\big]=\varnothing. \tag{25}$$

Using the fact that $\beta_{k'}\leq\beta_k$, we obtain

$$\begin{aligned}&\big[(\gamma^{(1)}_{k}(\tau)-3)\beta_{k+1}+3\beta_{k'+1},(\gamma^{(1)}_{k}(\tau)+6)\beta_{k+1}-3\beta_{k'+1}\big]\\&\quad\supseteq\big[\gamma^{(1)}_{k}(\tau)\beta_{k+1},(\gamma^{(1)}_{k}(\tau)+3)\beta_{k+1}\big]\\&\quad=T_{k+1}\big(\gamma^{(1)}_{k}(\tau)\big)\cup T_{k+1}\big(\gamma^{(1)}_{k}(\tau)+1\big)\cup T_{k+1}\big(\gamma^{(1)}_{k}(\tau)+2\big)\\&\quad\supseteq T^{\inf}_{k}(\tau),\end{aligned}$$

where the last step follows from (21). This, together with (25), implies that $T^{\inf}_{k'}(\tau')$ does not intersect $T^{\inf}_{k}(\tau)$.

Now, we turn to the case $S^{\inf}_{k'}(i')\not\subseteq S^{\sup}_{k}(i)$, for which we want to show

$$S^{\inf}_{k'}(i')\cap S^{\inf}_{k}(i)=\varnothing.$$

Let $x_1,x_2,\dots,x_d$ be defined so that $S_k(i)=\prod_{j=1}^{d}[x_j,x_j+\ell_k]$. Then we can write

$$S^{\inf}_{k}(i)=\prod_{j=1}^{d}[x_j-2\eta mn(k+1)^3\ell_k,x_j+\ell_k+2\eta mn(k+1)^3\ell_k]. \tag{26}$$

Now, let $y_1, y_2, \ldots, y_d$ be defined so that $S_k^{\sup}(i) = \prod_{j=1}^d [y_j, y_j + (2m+1)\ell_{k+1}]$. Since $S_{k'}^{\inf}(i')$ is a cube of side length $(1 + 4\eta mn(k'+1)^3)\ell_{k'} \le (1 + 4\eta mn(k+1)^3)\ell_k$ and $S_{k'}^{\inf}(i')$ is not contained in $S_k^{\sup}(i)$, we have that

$$S_{k'}^{\inf}(i') \cap \prod_{j=1}^d \big[y_j + \big(1 + 4\eta mn(k+1)^3\big)\ell_k, \\ y_j + (2m+1)\ell_{k+1} - \big(1 + 4\eta mn(k+1)^3\big)\ell_k\big] = \varnothing. \tag{27}$$

Now, we use the fact that $m\ell_{k+1} \le x_j - y_j \le (m+1)\ell_{k+1} - \ell_k$ for all $j = 1, 2, \ldots, d$. This and (26) give

$$S_k^{\inf}(i) \subseteq \prod_{j=1}^d \big[y_j + m\ell_{k+1} - 2\eta mn(k+1)^3\ell_k, \\ y_j + (m+1)\ell_{k+1} + 2\eta mn(k+1)^3\ell_k\big]. \tag{28}$$

Now, using the relation between m and n in (12), we have that

$$m\ell_{k+1} = m^2(k+1)^3\ell_k = 7\eta mn^d(k+1)^3\ell_k \ge \big(1 + 6\eta mn(k+1)^3\big)\ell_k. \tag{29}$$

Using this result in (27) we get that $S_{k'}^{\inf}(i')$ does not intersect

$$\prod_{j=1}^d \big[y_j + \big(1 + 4\eta mn(k+1)^3\big)\ell_k, y_j + (m+1)\ell_{k+1} + 2\eta mn(k+1)^3\ell_k\big]. \tag{30}$$

Similarly, plugging (29) into (28) we see that $S_k^{\inf}(i)$ is contained in the space–time region given by (30). These two facts establish that $S_{k'}^{\inf}(i')$ does not intersect $S_k^{\inf}(i)$. $\square$

The second important property we will use is given in the next lemma, which establishes that the support of a cell contains all its descendants. The main use of this lemma is that, once we encounter a cell (k, i, τ) for which $A_k(i, \tau) = 0$, then we want to regard all its descendants as bad. However, the set of descendants of (k, i, τ) may be a complicated set. With the lemma below, we then just consider all cells that are contained inside the support of (k, i, τ) as bad [which includes all descendants of (k, i, τ)].

LEMMA 3.4. *Assume $m \ge 3$. For any $(k, i, \tau) \in \mathcal{R}$, if (k', i', τ') is a descendant of (k, i, τ) then*

$$R_{k'}(i', \tau') \subseteq R_k^{\sup}(i, \tau).$$

Moreover, $R_k^{\sup}(i, \tau)$ contains all the neighbors of (k', i', τ'); that is,

$$\bigcup_{(i'', \tau''):\, \|(i'', \tau'') - (i', \tau')\|_\infty \le 1} R_{k'}(i'', \tau'') \subseteq R_k^{\sup}(i, \tau).$$

PROOF. Fix (i'', τ'') such that (k', i'', τ'') is adjacent to (k', i', τ') and assume that the ancestor of (k', i'', τ'') of scale k is not (k, i, τ), otherwise the second part of the lemma follows from the first part. We prove this lemma first for space and then for time. For space, since (k', i', τ') is a descendant of (k, i, τ) we have that $S_{k'}(i') \subseteq S_k(i) \subseteq S_k^{\sup}(i)$. Also, (k', i'') is adjacent to (k', i') which implies that the ancestor of (k', i'') of scale k is adjacent to (k, i). Since $S_k^{\sup}(i)$ contains all cells of scale k that are adjacent to (k, i), it also contains $S_{k'}(i'')$.

It remains to establish the lemma for the time dimension. For the first part of the lemma, this corresponds to showing that $T_{k'}(\tau') \subseteq T_k^{\sup}(\tau)$. Recall that $T_{k'}(\tau') = [\tau'\beta_{k'}, (\tau'+1)\beta_{k'}]$, which is contained in $[\tau\beta_k, (\tau'+1)\beta_{k'}]$ since (k', i', τ') is a descendant of (k, i, τ). Now, note that

$$\tau\beta_k = \gamma_{k'}^{(k-k')}(\tau')\beta_k \geq \gamma_{k'}^{(k-k'-1)}(\tau')\beta_{k-1} - 2\beta_k \geq \tau'\beta_{k'} - 2\sum_{j=k'+1}^{k}\beta_j.$$

Then, since $k' \geq 1$, we can use the bound

$$\sum_{j=2}^{k}\beta_j = C_{\mathrm{mix}}\sum_{j=2}^{k}\frac{\ell_{j-1}^2 j^4}{\varepsilon^2} \leq C_{\mathrm{mix}}\frac{2\ell_{k-1}^2 k^4}{\varepsilon^2} = 2\beta_k,$$

where the last inequality can be proved by induction on k. Therefore, we conclude that

$$\tau\beta_k \geq \tau'\beta_{k'} - 4\beta_k. \tag{31}$$

Since $k > k' \geq 0$, we have $k \geq 1$ and

$$4\beta_k + \beta_{k'} \leq 5\beta_k = 5\frac{\beta_{k+1}}{m^2k^2(k+1)^4} \leq \beta_{k+1}.$$

This and the inequality in (31) yield

$$T_{k'}(\tau') \subseteq [\tau\beta_k, \tau\beta_k + 4\beta_k + \beta_{k'}] \subseteq [\tau\beta_k, \tau\beta_k + \beta_{k+1}] \subseteq T_k^{\sup}(\tau).$$

This establishes the first part of the lemma. For the second part, using the fact that (k', τ'') is adjacent to (k', τ') together with the result above, we have

$$T_{k'}(\tau'') \subseteq [\tau\beta_k - \beta_{k'}, \tau\beta_k + \beta_{k+1} + \beta_{k'}] \subseteq T_k^{\sup}(\tau). \qquad \square$$

3.5. *Support connected paths.* We start defining the *extended support* of a cell. Given a cell $(k, i, \tau) \in \mathcal{R}$, define

$$T_k^{2\sup}(\tau) = \bigcup_{i=0}^{26} T_{k+1}\big(\gamma_k^{(1)}(\tau) - 12 + i\big)$$

and

$$S_k^{2\sup}(i) = \bigcup_{i' : \|i' - \pi_k^{(1)}(i)\|_\infty \leq 3m+1} S_{k+1}(i').$$

Also, we let

$$R_k^{2\sup}(i,\tau) = S_k^{2\sup}(i) \times T_k^{2\sup}(\tau).$$

The extended support is defined so that the following property is satisfied. Let $(k_1, i_1, \tau_1), (k_2, i_2, \tau_2) \in \mathcal{R}$ with $k_1 \geq k_2$. Then

$$\text{if } R_{k_1}^{\sup}(i_1,\tau_1) \text{ intersects } R_{k_2}^{\sup}(i_2,\tau_2), \text{ we have } R_{k_2}^{\sup}(i_2,\tau_2) \subseteq R_{k_1}^{2\sup}(i_1,\tau_1). \tag{32}$$

Note that the extended support of (k, i, τ) is three times larger than the support of (k, i, τ) [defined in (24)] since $S_k^{\sup}(i)$ is a cube of side length $(2m+1)\ell_{k+1}$ while $S_k^{2\sup}(i)$ is a cube of side length $3(2m+1)\ell_{k+1}$, and $T_k^{\sup}(\tau)$ is an interval of length $9\beta_{k+1}$ while $T_k^{2\sup}(\tau)$ is an interval of length $27\beta_{k+1}$.

We define the extended support because on the one hand we want to look at cells that are well separated (in the sense that the support of one cell does not contain the support of the other), but on the other hand we want to have a notion of a path of well separated cells. In such a path, cells must be well separated but should not be excessively far from one another. We will use the extended support to say that two cells are adjacent (in this new notion of path) if their extended supports intersect; we will call this a *support connected path*. We make this notion rigorous in the following.

Recall that a cell $(i,\tau) \in \mathcal{R}_1$ is said to have a bad ancestry if $A(i,\tau) = 0$. Also, in the beginning of Section 3, we defined a cell (i,τ) of scale 1 to be bad if $E(i,\tau) = 0$. Here, we change this definition slightly and extend it to cells of arbitrary scales: we say that a cell $(k,i,\tau) \in \mathcal{R}$ is bad if $A_k(i,\tau) = 0$. Our goal is to show that, if t is sufficiently large, then the probability that $K(0,0) \nsubseteq \mathcal{R}_1^t$ is small. First, recall that two cells (k, i_1, τ_1) and (k, i_2, τ_2) at the same scale are adjacent if $\|i - i'\|_\infty \leq 1$ and $|\tau - \tau'| \leq 1$. For arbitrary scales $k_1 > k_2$, we define that

$$\begin{aligned}&(k_1, i_1, \tau_1) \text{ and } (k_2, i_2, \tau_2) \text{ are adjacent if } (k_1, i_1, \tau_1) \text{ is adjacent to}\\ &(k_1, \pi_{k_2}^{(k_1-k_2)}(i_2), \gamma_{k_2}^{(k_1-k_2)}(\tau_2)).\end{aligned}$$

In other words, (k_1, i_1, τ_1) and (k_2, i_2, τ_2) are adjacent if the cell at scale k_1 that is the ancestor of (k_2, i_2, τ_2) is adjacent to (k_1, i_1, τ_1). Note that two adjacent cells of different scales may be disjoint, whereas two adjacent cells of the same scale must at least intersect at a point.

We refer to a *path* as a sequence of distinct cells for which any two consecutive cells in the sequence are adjacent, and we say that a sequence of cells is a *cluster* if each cell of the sequence is adjacent to *some* other cell in the sequence. Note that, unlike in a path, the order of the cells of a cluster is not important, so we regard a cluster as a *set* of cells.

For any two cells (k_1, i_1, τ_1) and (k_2, i_2, τ_2), we say that

$$\begin{aligned}&(k_1, i_1, \tau_1) \text{ and } (k_2, i_2, \tau_2) \text{ are } \textit{well separated} \text{ if}\\ &R_{k_1}(i_1,\tau_1) \nsubseteq R_{k_2}^{\sup}(i_2,\tau_2) \text{ and } R_{k_2}(i_2,\tau_2) \nsubseteq R_{k_1}^{\sup}(i_1,\tau_1).\end{aligned} \tag{33}$$

We will look at path of cells that are mutually well separated. (Recall that from Lemma 3.3, well separated cells have disjoint regions of influence and, therefore, we will be able to argue that they behave roughly independently from one another.) We define that

$$(k_1, i_1, \tau_1) \text{ and } (k_2, i_2, \tau_2) \text{ are } \textit{support adjacent} \text{ if } R_{k_1}^{2\sup}(i_1, \tau_1) \cap R_{k_2}^{2\sup}(i_2, \tau_2) \neq \varnothing.$$

Finally, we say that a sequence of cells $P = ((k_1, i_1, \tau_1), (k_2, i_2, \tau_2), \ldots, (k_z, i_z, \tau_z))$ is a *support connected path* if the cells in P are mutually well separated and, for each $j = 1, 2, \ldots, z-1$, (k_j, i_j, τ_j) is support adjacent to $(k_{j+1}, i_{j+1}, \tau_{j+1})$. We also define a sequence of cells $P = ((k_1, i_1, \tau_1), (k_2, i_2, \tau_2), \ldots, (k_z, i_z, \tau_z))$ to be a *support connected cluster* if the cells in P are mutually well separated and, for each $j = 1, 2, \ldots, z$, there exists a $j' \in \{1, 2, \ldots, z\} \setminus \{j\}$ such that (k_j, i_j, τ_j) is support adjacent to $(k_{j'}, i_{j'}, \tau_{j'})$.

Now, define Ω as the set of all paths of cells of scale 1 (i.e., cells of $\mathcal{R}_1$) so that the first cell of the path is $(0, 0)$ and the last cell of the path is the only cell not contained in $\mathcal{R}_1^t$. Also, define $\Omega_\kappa^{\sup}$ as the set of all support connected paths of cells of scale at most κ (i.e., cells in $\bigcup_{k=1}^{\kappa} \mathcal{R}_k$) so that the extended support of the first cell of the path contains $R_1(0, 0)$ and the last cell of the path is the only cell whose extended support is not contained in $\bigcup_{(i,\tau)\in\mathcal{R}_1^t} R_1(i, \tau)$. The lemma below will allow us to turn our attention to support connected paths of bad cells instead of paths of cells with bad ancestry, whose dependencies seem challenging to control.

LEMMA 3.5. *We have that*

$$\mathbf{P}(\exists P \in \Omega \textit{ s.t. all cells of } P \textit{ have a bad ancestry}) \leq \mathbf{P}\big(\exists P \in \Omega_\kappa^{\sup} \textit{ s.t. all cells of } P \textit{ are bad}\big).$$

PROOF. The proof is split into two stages. In the first stage, we show that, if there exists a path $P \in \Omega$ such that each cell of P has a bad ancestry, then there exists a path of bad cells of arbitrary scales. In the second stage, we show that, given the existence of such a path of bad cells of arbitrary scales, then there exists a path of $\Omega_\kappa^{\sup}$ such that all cells of the path are bad.

We now prove the first stage. Let Ω_κ be the set of all paths of cells of arbitrary scale (i.e., cells in $\mathcal{R}$) such that the first cell of the path is an ancestor of $(0, 0) \in \mathcal{R}_1$ and the last cell of the path is the only cell whose support is not contained in $\bigcup_{(i,\tau)\in\mathcal{R}_1^t} R_1(i, \tau)$. In this stage, we establish that

$$\mathbf{P}(\exists P \in \Omega \text{ s.t. all cells of } P \text{ have a bad ancestry}) \leq \mathbf{P}(\exists P \in \Omega_\kappa \text{ s.t. all cells of } P \text{ are bad}). \tag{34}$$

Let $P = ((1, i_1, \tau_1), (1, i_2, \tau_2), \ldots, (1, i_z, \tau_z)) \in \Omega$ be a path of cells with bad ancestries; hence $(i_1, \tau_1) = (0, 0)$ and $(i_z, \tau_z) \notin \mathcal{R}_1^t$. For each j, since $A(i_j, \tau_j) = 0$,

we know by the definition of A in (19) that there exists a k'_j so that, if we set $i'_j = \pi_1^{(k'_j-1)}(i_j)$ and $\tau'_j = \gamma_1^{(k'_j-1)}(\tau_j)$, we obtain $A_{k'_j}(i'_j, \tau'_j) = 0$. Define $J \subseteq \{1, 2, \dots, z\}$ such that $j \in J$ if and only if there exists no $j' < j$ with $(k'_j, i'_j, \tau'_j) = (k'_{j'}, i'_{j'}, \tau'_{j'})$ and there exists no $j' \in \{1, 2, \dots, z\} \setminus \{j\}$ for which (k'_j, i'_j, τ'_j) is a descendant of $(k'_{j'}, i'_{j'}, \tau'_{j'})$. In other words, J contains only distinct elements of the set $\{(k'_j, i'_j, \tau'_j) : j = 1, 2, \dots, z\}$ which have no ancestor in the set. With this, we define

$$\tilde{P} = \{(k'_j, i'_j, \tau'_j) : j \in J\},$$

and show that $\tilde{P}$ is a cluster. Before, note that, since each cluster contains a path, this establishes the existence of a path of bad cells of arbitrary scales. In particular, we obtain a path starting from an ancestor of $(1, i_1, \tau_1)$ and such that there exists a cell $(k', i', \tau') \in \tilde{P}$ that is an ancestor of a cell of P that is not contained in $\mathcal{R}_1^t$. Then, by Lemma 3.4, we know that this cell of P is contained in $R_{k'}^{\sup}(i', \tau')$, which gives that the union of the support of the cells in $\tilde{P}$ is not contained in $\bigcup_{(i,\tau)\in\mathcal{R}_1^t} R_1(i, \tau)$. Such a path belongs to Ω_κ, so it only remains to show that $\tilde{P}$ is a cluster. Note that, by construction, each cell of P has exactly one ancestor in $\tilde{P}$. Now, for any two adjacent cells $(1, i_j, \tau_j), (1, i_{j+1}, \tau_{j+1})$ of P, either they have the same ancestor in $\tilde{P}$ or their ancestors are adjacent since two nonadjacent cells cannot have descendants at scale 1 that are adjacent. This shows that each cell in $\tilde{P}$ is adjacent to at least one other cell in $\tilde{P}$ and, consequently, $\tilde{P}$ is a cluster. Therefore, we obtain (34).

Now we turn to the second stage of the proof, where we establish that

$$\begin{aligned}&\mathbf{P}(\exists P \in \Omega_\kappa \text{ s.t. all cells of } P \text{ are bad})\\ &\quad\le \mathbf{P}\big(\exists P \in \Omega_\kappa^{\sup} \text{ s.t. all cells of } P \text{ are bad}\big).\end{aligned} \tag{35}$$

Let $P = ((k_1, i_1, \tau_1), (k_2, i_2, \tau_2), \dots, (k_z, i_z, \tau_z)) \in \Omega_\kappa$ be a path of bad cells; thus, the support of (k_z, i_z, τ_z) is not contained in $\bigcup_{(i,\tau)\in\mathcal{R}_1^t} R_1(i, \tau)$. We show the existence of a support connected cluster P' of bad cells. First, we order the cells of P in the following way. If two cells have the same scale, we order them by taking an arbitrary order of $\mathbb{Z}^{d+1}$; for two cells of different scales, we say that the cell with the larger scale precedes the other cell in the order. This clearly establishes a total order of the cells of P. Then let L be the list of cells of P following this order, where the first cell of L is the cell that precedes all the other cells of P in the order. We construct P' in a step-by-step manner, where at each step we add the first element of L to P', remove some elements from L and repeat until L has no element. During this procedure, we associate each cell of P to a cell of P'; we use this association later to show that P' is a support connected cluster. Below we give the formal description of each step in the construction of P', where we assume that (k', i', τ') is the first element of L:

(i) Add (k', i', τ') to P' and remove it from L. Since (k', i', τ') is both in P and P', associate (k', i', τ') to itself.

(ii) Remove from L all the cells (k'', i'', τ'') that are not well separated from (k', i', τ') [see the definition of well separated cells in (33)], and associate each such (k'', i'', τ'') to (k', i', τ').

We repeat these steps until L is empty. Note that the set P' obtained at the end contains only cells that are mutually well separated. Also, note that there exists a cell of P' such that the extended support of this cell contains $R_1(0,0)$. This follows since, by definition, $R^{\sup}_{k_1}(i_1, \tau_1)$ contains $R_1(0,0)$ and, by construction of P', there exists a cell of P' whose support contains $R_{k_1}(i_1, \tau_1)$. So, by (32), the extended support of this cell contains $R^{\sup}_{k_1}(i_1, \tau_1)$ which contains $R_1(0,0)$.

It remains to show that P' is support connected and that

$$\bigcup_{(k',i',\tau')\in P'} R^{2\sup}_{k'}(i', \tau') \not\subseteq \bigcup_{(i'',\tau'')\in\mathcal{R}^t_1} R_1(i'', \tau'').$$

The second property is easy to check and follows from (32) by noting that the cell $(k_z, i_z, \tau_z) \in P$ is contained in the support of the cell to which it has been associated in the construction of P' and, moreover, the support of (k_z, i_z, τ_z) is not contained in $\bigcup_{(i'',\tau'')\in\mathcal{R}^t_1} R_1(i'', \tau'')$ by the definition of P. Then (32) gives that the extended support of the cell to which (k_z, i_z, τ_z) has been associated contains the support of (k_z, i_z, τ_z) and, hence, cannot be contained in $\bigcup_{(i'',\tau'')\in\mathcal{R}^t_1} R_1(i'', \tau'')$. Here, we used the order of L, which guarantees that k_z is no larger than the scale of the cell to which (k_z, i_z, τ_z) has been associated, thereby allowing us to apply (32).

Now, we assume, for the sake of establishing a contradiction, that P' is not support connected. Then it must be the case that P' can be partitioned into two set of cells Q and Q' so that, for any cell of Q, the extended support of this cell does not intersect the extended support of any cell of Q'. In this part, it will be useful to refer to Figure 4. Let P_0 be the cells of P that are *not* associated to any cell

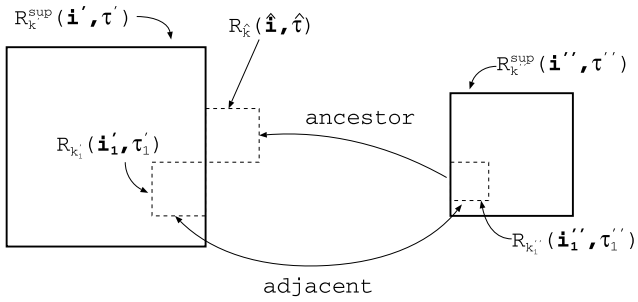

FIG. 4. *Illustration for the proof of Lemma 3.5. It shows that* $R^{2\sup}_{k'}(i', \tau')$ *intersects* $R^{2\sup}_{k''}(i'', \tau'')$ *since* $R^{\sup}_{\hat{k}}(\hat{i}, \hat{\tau})$ *intersects both* $R^{\sup}_{k'}(i', \tau')$ *and* $R^{\sup}_{k''}(i'', \tau'')$.

of Q but are adjacent to at least one cell of P that is associated to a cell of Q. Since P is a set of adjacent cells, P_0 has at least one cell. Let (k_1'', i_1'', τ_1'') be a cell in P_0 and let (k'', i'', τ'') be the cell of P' that has been associated to (k_1'', i_1'', τ_1''). So $(k'', i'', \tau'') \in Q'$. Let (k_1', i_1', τ_1') be a cell of P that has been associated to $(k', i', \tau') \in Q$ and is adjacent to (k_1'', i_1'', τ_1''). Thus, $R_{k_1'}(i_1', \tau_1') \subseteq R_{k'}^{\sup}(i', \tau')$ and $R_{k_1''}(i_1'', \tau_1'') \subseteq R_{k''}^{\sup}(i'', \tau'')$. We now assume that $k_1' \geq k_1''$; the other case follows by the same argument. Since $k_1' \geq k_1''$, we have a cell $(\hat{k}, \hat{i}, \hat{\tau})$ at scale $\hat{k} = k_1'$ that is adjacent to (k_1', i_1', τ_1') and is an ancestor of (k_1'', i_1'', τ_1''). Then, by adjacency and Lemma 3.4, $R_{\hat{k}}^{\sup}(\hat{i}, \hat{\tau})$ contains both $R_{k_1'}(i_1', \tau_1')$ and $R_{k_1''}(i_1'', \tau_1'')$. Therefore, $R_{\hat{k}}^{\sup}(\hat{i}, \hat{\tau})$ intersects both $R_{k'}^{\sup}(i', \tau')$ and $R_{k''}^{\sup}(i'', \tau'')$. Also, by the order of L, we have that $\hat{k} \leq k'$ since $\hat{k} = k_1'$ and (k_1', i_1', τ_1') is associated to (k', i', τ'). Then, by (32), we have that $R_{\hat{k}}^{\sup}(\hat{i}, \hat{\tau}) \subseteq R_{k'}^{2\sup}(i', \tau')$ which gives that $R_{k'}^{2\sup}(i', \tau')$ intersects $R_{k''}^{\sup}(i'', \tau'')$, thereby contradicting the fact that (k', i', τ') and (k'', i'', τ'') are not support adjacent. This establishes (35) and completes the proof of the lemma. □

The next lemma is a technical result bounding the probability that a Brownian motion stays inside a cube.

LEMMA 3.6. *Let $\Delta > 0$ and, for any $z > 0$, define $F_\Delta(z)$ to be the event that a Brownian motion starting from the origin stays inside Q_z throughout the time interval $[0, \Delta]$. Then, for any $z \geq 3\sqrt{\Delta}$, we have*

$$\mathbf{P}\big(F_\Delta(z)\big) \geq 1 - d \exp\Big(-\frac{z^2}{18\Delta}\Big).$$

PROOF. In order to bound $\mathbf{P}(F_\Delta(z))$, we use the bound for f_Δ in (6) with $M = z/3$, where $\frac{f_\Delta}{\mathbf{P}(F_\Delta(z))}$ is the probability density function of the position of a Brownian motion at time Δ given that the motion never leaves Q_z during the whole of $[0, \Delta]$. With this, we have

$$\begin{aligned}
\mathbf{P}\big(F_\Delta(z)\big) &= \int_{Q_z} f_\Delta(y)\,dy \geq \int_{Q_{2z/3}} f_\Delta(y)\,dy \\
&\geq \Big(1 - 2d \exp\Big(-\frac{z^2}{6\Delta}\Big)\Big)\Big(1 - 2d \int_{z/3}^{\infty} \frac{1}{\sqrt{2\pi\,\Delta}} \exp\Big(-\frac{y_1^2}{2\Delta}\Big)\,dy_1\Big) \\
&\geq \Big(1 - 2d \exp\Big(-\frac{z^2}{6\Delta}\Big)\Big)\Big(1 - 2d \frac{3\sqrt{\Delta}}{\sqrt{2\pi}\,z} \exp\Big(-\frac{z^2}{18\Delta}\Big)\Big) \\
&\geq 1 - d \exp\Big(-\frac{z^2}{18\Delta}\Big),
\end{aligned}$$

where we use the fact that $z \geq 3\sqrt{\Delta}$ to apply the Gaussian tail bound in the second inequality (cf. Lemma A.2) and also to obtain the simplifications in the last inequality. $\square$

We now give a key lemma that we will use to argue that well separated cells are roughly independent. Let $\mathcal{F}_k(i,\tau)$ be the σ-field generated by all $A_{k'}(i',\tau')$ for which $T^{\inf}_{k'}(\tau')$ does not intersect $[\gamma^{(1)}_k(\tau)\beta_{k+1},\infty)$ or both $\tau'\beta_{k'} \leq \tau\beta_k$ and $S^{\inf}_k(i) \cap S^{\inf}_{k'}(i') = \varnothing$. An important property of this definition is that an event in $\mathcal{F}_k(i,\tau)$ may reveal information about nodes that affect $A_k(i,\tau)$ but only regarding time steps that occur *before* $\gamma^{(1)}_k(\tau)\beta_{k+1}$. Because of this, we are able to get an upper bound for the probability that $A_k(i,\tau) = 0$ given any event in $\mathcal{F}_k(i,\tau)$. The following quantity will be used in many of the subsequent lemmas to simplify the equations

$$\begin{aligned}\psi_1 &= \min\left\{\varepsilon^2\lambda\ell^d, \log\left(\frac{1}{1-\nu_E((1-\varepsilon)\lambda, Q_{w\ell})}\right)\right\} \quad \text{and}\\ \psi_k &= \frac{\varepsilon^2\lambda\ell^d_{k-1}}{(k+1)^4} = \frac{\varepsilon^2\lambda\ell^d m^{d(k-2)}((k-1)!)^{3d}}{(k+1)^4} \qquad \text{for } k \geq 2.\end{aligned} \tag{36}$$

LEMMA 3.7. *Let* $w \geq \sqrt{\frac{18\eta\beta}{\ell^2}\log(\frac{8d}{\varepsilon})}$ *and*

$$\alpha = \min\left\{\varepsilon^2\lambda\ell^d, \log\left(\frac{1}{1-\nu_E((1-\varepsilon)\lambda, Q_{w\ell})}\right)\right\}$$

as in Theorem 3.1. Fix any $(k,i,\tau) \in \mathcal{R}$ *and any* $F \in \mathcal{F}_k(i,\tau)$. *If* m *is sufficiently large with respect to* d, β/ℓ^2, η *and* ε, *then there are positive constants* $c = c(d) \geq 1$ *and* α_0 *so that, for all* $\alpha \geq \alpha_0$, *we have*:

(i) $\mathbf{P}(A_k(i,\tau) = 0) \leq \exp(-c\psi_k)$ *for all* $k = 1,2,\ldots,\kappa$,
(ii) $\mathbf{P}(A_k(i,\tau) = 0|F) \leq \exp(-c\psi_k)$ *for all* $k = 1,2,\ldots,\kappa-1$.

PROOF. Note that the A_k are defined differently for $k = 1$ and $2 \leq k \leq \kappa - 1$. In the sequel, we assume that $k \geq 2$ and establish part (ii) of the lemma. At the end, we address both part (i) and the case $k = 1$ for part (ii). Since

$$\mathbf{P}\big(A_k(i,\tau) = 0|F\big) = \mathbf{P}\big(\big\{D^{\text{ext}}_k(i,\tau) = 0\big\} \cap \big\{D^{\text{base}}_k(i,\tau) = 1\big\}|F\big),$$

if $F \cap \{D^{\text{base}}_k(i,\tau) = 1\} = \varnothing$, then the lemma clearly holds. So, we now assume that $F \cap \{D^{\text{base}}_k(i,\tau) = 1\} \neq \varnothing$ and write

$$\mathbf{P}\big(A_k(i,\tau) = 0|F\big) \leq \mathbf{P}\big(D^{\text{ext}}_k(i,\tau) = 0|F \cap \big\{D^{\text{base}}_k(i,\tau) = 1\big\}\big).$$

Recall that $\{D^{\text{base}}_k(i,\tau) = 1\}$ gives that all cubes of scale k contained in $S^{\text{base}}_k(i)$ have at least $(1-\varepsilon_{k+1})\lambda\ell^d_k$ nodes at time $\gamma^{(1)}_k(\tau)\beta_{k+1}$ and the displacement of these nodes throughout $[\gamma^{(1)}_k(\tau)\beta_{k+1}, \tau\beta_k]$ is in $Q_{\eta mn(k+1)^3\ell_k}$. Note that F only reveals information about the location of these nodes *before* time $\gamma^{(1)}_k(\tau)\beta_{k+1}$ since

these nodes never leave the cube $S_k^{\inf}(i)$ during the whole of $[\gamma_k^{(1)}(\tau)\beta_{k+1}, \tau\beta_k]$ (cf. Remark 3.4).

Now, we apply Proposition 2.3. To avoid ambiguity, we add a bar to the variables appearing in the statement of Proposition 2.3. We apply this proposition with $\bar{K} = (1+2\eta mn(k+1)^3)\ell_k$, $\bar{\ell} = \ell_k$, $\bar{\beta} = (1-\varepsilon_{k+1})\lambda$, $\bar{\Delta} = \tau\beta_k - \gamma_k^{(1)}(\tau)\beta_{k+1} \in [\beta_{k+1}, 2\beta_{k+1}]$, $\bar{K}'$ such that $3(\bar{K} - \bar{K}' + 2\sqrt{d}\bar{\ell}) = \eta mn(k+1)^3\ell_k$ and $\bar{\varepsilon}$ such that $(1-\bar{\varepsilon})(1-\varepsilon_{k+1}) = (1 - \frac{\varepsilon_{k+1}+\varepsilon_k}{2})$, which gives that $\bar{\varepsilon} \geq \frac{\varepsilon_k - \varepsilon_{k+1}}{2} = \frac{\varepsilon}{2(k+1)^2}$. Now, using these values and the fact that m is large enough, we have that

$$\bar{K}' = \ell_k + mk^3\ell_{k-1}\big(\tfrac{5}{3}\eta mn(k+1)^3 + 2\sqrt{d}\big) \geq \ell_k + 2\eta mnk^3\ell_{k-1},$$

which is the side length of $S_k^{\text{ext}}(i)$. Note also that we have $\bar{\Delta} \geq \frac{c_1\bar{\ell}^2}{\bar{\varepsilon}^2}$ since $C_{\text{mix}} \geq 4c_1$ in the definition of β_{k+1}. It remains to check whether $\bar{K} - \bar{K}' \geq c_2\sqrt{\bar{\Delta}\log(16d\bar{\varepsilon}^{-1})}$, which is satisfied if the following is true:

$$\eta mn(k+1)^3\ell_k \geq 4c_2\sqrt{2C_{\text{mix}}}\frac{\ell_k(k+1)^2}{\varepsilon}\sqrt{\log\Big(\frac{32d(k+1)^2}{\varepsilon}\Big)}. \tag{37}$$

Using the definition of β_k from (14) for $k=1$, we can write $C_{\text{mix}} = \frac{\varepsilon^2 m^2\beta}{\ell^2}$, which allows us to write the right-hand side of (37) as

$$4\sqrt{2}c_2 m\ell_k(k+1)^2\sqrt{\frac{\beta}{\ell^2}\log\Big(\frac{32d(k+1)^2}{\varepsilon}\Big)}.$$

Note that, in the left-hand side of (37), ηn increases with m. So, since m is sufficiently large with respect to d, β/ℓ^2 and ε, we obtain that n is also sufficiently large and (37) is satisfied for all k.

Then, we obtain a coupling between the nodes in $S_k^{\text{base}}(i)$ and an independent Poisson point process Ξ with intensity $(1-\bar{\varepsilon})(1-\varepsilon_{k+1})\lambda \geq (1 - \frac{\varepsilon_k}{2} - \frac{\varepsilon_{k+1}}{2})\lambda$ that succeeds with probability at least

$$\begin{aligned}&1 - \frac{\bar{K}^d}{\bar{\ell}^d}\exp(-c_3\bar{\varepsilon}^2\bar{\beta}\bar{\ell}^d)\\ &\qquad \geq 1 - \big(1+2\eta mn(k+1)^3\big)^d\exp\Big(-c_3\frac{\varepsilon^2}{4(k+1)^4}(1-\varepsilon_{k+1})\lambda\ell_k^d\Big),\end{aligned}$$

where c_3 is a constant depending on d only. Note that, up to this moment, we never used the fact that $k \geq 2$ and the argument above holds also for $k=1$.

Now, for the case $k \geq 2$, we define a Poisson point process Ξ' consisting of the nodes of Ξ whose displacement throughout $[\tau\beta_k, (\tau+2)\beta_k]$ is in $Q_{\eta mnk^3\ell_{k-1}}$. For each node of Ξ, this condition is satisfied with probability $\mathbf{P}(F_{2\beta_k}(\eta mnk^3\ell_{k-1}))$, independently over the nodes of Ξ. Using Lemma 3.6 and the thinning property of

Poisson processes, we have that Ξ' is a Poisson point process with intensity

$$
\begin{aligned}
&(1-\bar{\varepsilon})(1-\varepsilon_{k+1})\mathbf{P}\big(F_{2\beta_k}(\eta mnk^3\ell_{k-1})\big)\lambda\\
&\quad\ge\left(1-\frac{\varepsilon_k}{2}-\frac{\varepsilon_{k+1}}{2}\right)\left(1-d\exp\left(-\frac{(\eta mnk^3\ell_{k-1})^2}{36\beta_k}\right)\right)\\
&\quad\ge\left(1-\frac{\varepsilon_k}{2}-\frac{\varepsilon_{k+1}}{2}\right)\left(1-d\exp\left(-\frac{(\eta mnk\varepsilon)^2}{36C_{\mathrm{mix}}}\right)\right)\\
&\quad\ge\left(1-\frac{\varepsilon_k}{2}-\frac{\varepsilon_{k+1}}{2}\right)\left(1-d\exp\left(-\frac{(\eta nk)^2}{36(\beta/\ell^2)}\right)\right),
\end{aligned}
\tag{38}
$$

where the second inequality follows by the definition of β_k in (14) and the third inequality follows by the definition of C_{mix}. Then, setting m sufficiently large with respect to d, η, ε and β/ℓ^2, which makes ηn sufficiently large, we obtain that

$$
\text{intensity of } \Xi' \ge \left(1-\frac{3\varepsilon_k}{4}-\frac{\varepsilon_{k+1}}{4}\right). \tag{39}
$$

Once the coupling is established, the probability that all $(\frac{\ell_k+2\eta mnk^3\ell_{k-1}}{\ell_{k-1}})^d = (mk^3+2\eta mnk^3)^d$ subcubes of scale $k-1$ in $S_k^{\mathrm{ext}}(i)$ have at least $(1-\varepsilon_k)\lambda\ell_{k-1}^d$ nodes of Ξ' is

$$
\begin{aligned}
&\mathbf{P}\big(D_k^{\mathrm{ext}}(i,\tau)=1 \mid F\cap\{D_k^{\mathrm{base}}(i,\tau)=1\}\big)\\
&\quad\ge 1-\big(mk^3+2\eta mnk^3\big)^d\\
&\qquad\times\exp\left(-\frac{1}{2}\left(\frac{\varepsilon_k-\varepsilon_{k+1}}{4}\right)^2\left(1-\frac{3\varepsilon_k}{4}-\frac{\varepsilon_{k+1}}{4}\right)\lambda\ell_{k-1}^d\right)\\
&\quad\ge 1-\big(mk^3+2\eta mnk^3\big)^d\exp\left(-\frac{1}{2}\left(\frac{\varepsilon^2}{16(k+1)^4}\right)\left(1-\frac{15\varepsilon}{16}\right)\lambda\ell_{k-1}^d\right),
\end{aligned}
\tag{40}
$$

where we used the fact that ε_k is decreasing in k to infer that $1-\frac{3\varepsilon_k}{4}-\frac{\varepsilon_{k+1}}{4}\ge 1-\frac{3\varepsilon}{4}-\frac{\varepsilon_2}{4}=1-\frac{15\varepsilon}{16}$. Now, for large k, the result follows since $\ell_{k-1}=m^{k-2}((k-1)!)^3\ell$ and, for small $k\ge 2$, the result follows since $\varepsilon^2\lambda\ell^d\ge\alpha$ is large enough.

For part (i), a similar argument works. Since in this case we want to bound the unconditioned probability, at time $\tau\beta_k$ the nodes in $S_k^{\mathrm{ext}}(i)$ consists of a Poisson point process with intensity λ. So, using the derivation in (38) and (39) with $\varepsilon_{k+1}=0$, the nodes of this Poisson point process for which the displacement throughout $[\tau\beta_k,(\tau+2)\beta_k]$ is in $Q_{\eta mnk^3\ell_{k-1}}$ is also a Poisson point process with intensity at least $1-\frac{\varepsilon_k}{2}$. Then, using a derivation similar to (40) we have

$$
\begin{aligned}
\mathbf{P}\big(D_k^{\mathrm{ext}}(i,\tau)=1\big)&\ge 1-\big(mk^3+2\eta mnk^3\big)^d\exp\left(-\frac{1}{2}\frac{\varepsilon_k^2}{4}(1-\varepsilon_k/2)\lambda\ell_{k-1}^d\right)\\
&\ge 1-\big(mk^3+2\eta mnk^3\big)^d\exp\left(-\frac{1}{128}\varepsilon^2(1-\varepsilon/2)\lambda\ell_{k-1}^d\right),
\end{aligned}
$$

where we used the fact that $\varepsilon_k = \varepsilon - \sum_{i=2}^{k} \frac{\varepsilon}{i^2} \geq \varepsilon - \frac{\varepsilon}{4} - \varepsilon \int_2^\infty \frac{1}{x^2}\,dx = \frac{\varepsilon}{4}$.

Now, for part (ii) with $k = 1$, we again use the Poisson point process Ξ of intensity at least

$$1 - \frac{\varepsilon_k}{2} - \frac{\varepsilon_{k+1}}{2} = 1 - \frac{7\varepsilon}{8}$$

over $S_1^{\mathrm{ext}}(i)$ defined above. We also use the fact that $E(i,\tau)$ is an event restricted to the super cell i (see the definition of super cells in the beginning of Section 3) and $S_1^{\mathrm{ext}}(i)$ contains the super cell i (cf. Remark 3.3). Recall that, for the event $E(i,\tau)$, we only consider the nodes of Ξ whose displacement from time $\tau\beta$ to $(\tau+\eta)\beta$ is inside $Q_{w\ell}$. Let the event that this happens for a given node of Ξ be denoted by $F_{\eta\beta}(w\ell)$. Then we apply Lemma 3.6 with $\Delta = \eta\beta$ and $z = w\ell$ to obtain

$$\mathbf{P}\big(F_{\eta\beta}(w\ell)\big) \geq 1 - d\exp\bigg(-\frac{(w\ell)^2}{18\eta\beta}\bigg).$$

Using the fact that $w^2\ell \geq 18\eta\beta\log(8d\varepsilon^{-1})$, we have $\mathbf{P}(F_{\eta\beta}(w\ell)) \geq 1 - \frac{\varepsilon}{8}$. Therefore, by thinning, we have that the nodes of Ξ for which $F_{\eta\beta}(w\ell)$ hold consist of a Poisson point process with intensity at least $(1-\frac{7\varepsilon}{8})(1-\frac{\varepsilon}{8}) \geq 1-\varepsilon$. Since $E(i,\tau)$ is increasing, we have that

$$\mathbf{P}\big(E(i,\tau) = 0 | F \cap \{D_k^{\mathrm{base}}(i,\tau) = 1\}\big) \leq 1 - \nu_E\big((1-\varepsilon)\lambda, Q_{w\ell}\big) \leq e^{-\alpha},$$

which establishes the lemma for $k = 1$. □

The lemma below gives an upper bound to the probability that a support connected path is bad. Here, we use that all cells in a supported connected path are mutually well separated so that we can apply Lemma 3.7. Henceforth, we consider paths in $\Omega_{\kappa-1}^{\mathrm{sup}}$ only since the cells of scale κ will be handled in a different way latter; these cells are just so large that a much simpler bound can be applied.

LEMMA 3.8. *Assume the conditions in Lemma 3.7 are satisfied and let* $P \in \Omega_{\kappa-1}^{\mathrm{sup}}$ *be the path* $((k_1,i_1,\tau_1),(k_2,i_2,\tau_2),\ldots,(k_z,i_z,\tau_z))$. *Then, with* ψ *defined as in* (36), *we have*

$$\mathbf{P}\bigg(\bigcap_{j=1}^{z}\{A_{k_j}(i_j,\tau_j) = 0\}\bigg) \leq \exp\bigg(-c_2\sum_{j=1}^{z}\psi_{k_j}\bigg).$$

PROOF. We derive the probability that all cells of P are bad. Consider the following order for the cells of P. First, take an arbitrary order of $\mathbb{Z}^d$. Then we say that (k_j,i_j,τ_j) precedes $(k_{j'},i_{j'},\tau_{j'})$ in the order if $\tau_j\beta_{k_j} < \tau_{j'}\beta_{k_{j'}}$ or if both $\tau_j\beta_{k_j} = \tau_{j'}\beta_{k_{j'}}$ and i_j precedes $i_{j'}$ in the order of $\mathbb{Z}^d$. Then, for any j, we let

J_j be a subset of $\{1, 2, \ldots, z\}$ containing all j' for which $(k_{j'}, i_{j'}, \tau_{j'})$ precedes (k_j, i_j, τ_j) in the order. Using this order, we write

$$\mathbf{P}\Bigg(\bigcap_{j=1}^{z}\{A_{k_j}(i_j,\tau_j)=0\}\Bigg) = \prod_{j=1}^{z}\mathbf{P}\Bigg(A_{k_j}(i_j,\tau_j)=0\Big|\bigcap_{j'\in J_j}\{A_{k_{j'}}(i_{j'},\tau_{j'})=0\}\Bigg).$$

Note that, for each $j' \in J_j$, we have that (k_j, i_j, τ_j) and $(k_{j'}, i_{j'}, \tau_{j'})$ are well separated. Using the definition of well separated cells (33), we have that $R^{\inf}_{k_{j'}}(i_{j'}, \tau_{j'}) \nsubseteq R^{\sup}_{k_j}(i_j, \tau_j)$ and $R^{\inf}_{k_j}(i_j, \tau_j) \nsubseteq R^{\sup}_{k_{j'}}(i_{j'}, \tau_{j'})$. Hence, by Lemma 3.3, we obtain $R^{\inf}_{k_{j'}}(i_{j'}, \tau_{j'}) \cap R^{\inf}_{k_j}(i_j, \tau_j) = \varnothing$. By the ordering of the cells described above, we also have $\tau_j\beta_{k_j} \geq \tau_{j'}\beta_{k_{j'}}$, which gives that the event $\bigcap_{j'\in J_j}\{A_{k_{j'}}(i_{j'}, \tau_{j'}) = 0\}$ is measurable with respect to $\mathcal{F}_{k_j}(i_j, \tau_j)$. Then we apply Lemma 3.7 to obtain a positive constant c_2 such that

$$\mathbf{P}\Bigg(\bigcap_{j=1}^{z}\{A_{k_j}(i_j,\tau_j)=0\}\Bigg) \leq \exp\Bigg(-c_2\sum_{j=1}^{z}\psi_{k_j}\Bigg).$$

□

At the end, we will take the union bound over all support connected cells. For this, we need to obtain an upper bound for the number of support connected path, which is given in the following lemma.

LEMMA 3.9. *Let z be a positive integer and $k_1, k_2, \ldots, k_z \geq 1$ be fixed. Then, if α is sufficiently large, the total number of support connected paths containing z cells whose scales are $k_1, k_2, \ldots, k_z$ is at most* $\exp(\frac{c_2}{2}\sum_{j=1}^{z}\psi_{k_j})$, *where c_2 is the same constant of Lemma 3.8 and ψ is defined in* (36).

PROOF. For any $j, j' \geq 1$, define

$$\phi_{j,j'} = \max_{(i_1,\tau_1)\in\mathcal{R}_j}\big|\{(i_2,\tau_2) : (j, i_1, \tau_1) \text{ is support adjacent to and well separated from } (j', i_2, \tau_2)\}\big|.$$

Hence, given a cell at scale j, $\phi_{j,j'}$ is an upper bound for the number of cells of scale j' that are support adjacent to and well separated from the cell of scale j. Also, let χ_j be the number of cells of scale j whose extended support contains $R_1(0, 0)$. With this notation, we obtain that

$$\text{the number of support connected paths with } z \text{ cells of scales } k_1, k_2, \ldots, k_z \leq \chi_{k_1}\prod_{j=2}^{z}\phi_{k_{j-1},k_j}. \tag{41}$$

First, we derive a bound for χ_j. Note that, at scale j, the number of cells that have the same extended support is $(\frac{\ell_{j+1}}{\ell_j})^d \frac{\beta_{j+1}}{\beta_j} = m^{d+2} j^2 (j+1)^{3d+4}$. Furthermore, the extended support of a cell of scale j contains exactly $27 \cdot (3(2m+1))^d$ different cells of scale $j+1$. Thus, the number of different extended supports for a cell of scale j that contains $R_1(0,0)$ can be upper bounded by

$$\chi_j \le 27 \cdot 3^d m^{d+2} (2m+1)^d j^2 (j+1)^{3d+4}.$$

In order to derive a bound for $\phi_{j,j'}$, fix a cell (j, i_1, τ_1) of scale j. Now, a cell of scale j' can only be support adjacent to (j, i_1, τ_1) if it is inside the region

$$\bigcup_{x \in R_j^{2\sup}(i_1,\tau_1)} (x + [-(3m+2)\ell_{j'+1}, (3m+2)\ell_{j'+1}]^d \times [-14\beta_{j'+1}, 14\beta_{j'+1}]).$$

Therefore, if $j \ge j'$, we have

$$\begin{aligned}
\phi_{j,j'} &\le \left(\frac{3(2m+1)\ell_{j+1} + 2(3m+2)\ell_{j'+1}}{\ell_{j'}}\right)^d \left(\frac{27\beta_{j+1} + 28\beta_{j'+1}}{\beta_{j'}}\right) \\
&\le \left(6\left(m+\frac{1}{2}\right) m^{j-j'+1} \sum_{i=j'+1}^{j+1} i^3 + 6\left(m+\frac{2}{3}\right) m (j'+1)^3\right)^d \\
&\quad \times \left(27 m^{2(j-j'+1)} \sum_{i=j'}^{j} i^2 (i+1)^4 + 28 m^2 j'^2 (j'+1)^4\right).
\end{aligned}$$

Using that for any $a \in (0,1)$ and any $x \ge 1$ it holds that $x + a \le 2x$, we have

$$\begin{aligned}
\phi_{j,j'} &\le \left(12 m^{j-j'+2} \sum_{i=j'+1}^{j+1} i^3 + 12 m^2 (j'+1)^3\right)^d \\
&\quad \times \left(27 m^{2(j-j'+1)} \sum_{i=j'}^{j} i^6 2^4 + 28 m^2 j'^6 2^4\right) \\
&\le c_3 m^{(j-j'+2)d} j^{4d} m^{2(j-j'+1)} j^7 \le c_3 m^{(d+2)(j-j'+2)} j^{4d+7},
\end{aligned}$$

where in the derivation above we used the definition of ℓ_j and β_j in (11) and (14), respectively, and c_3 is a universal positive constant. We then set c_4 in such a way that, for the case $j < j'$, we obtain

$$\begin{aligned}
\phi_{j,j'} &\le (1 + 2(3m+2) m (j+1)^3)^d (1 + 28 m^2 j^2 (j+1)^4) \\
&\le (1 + 12 m^2 j^3 2^3)^d (1 + 28 m^2 j^6 2^4) \\
&\le c_4 m^{2d+2} j^{3d+6}.
\end{aligned}$$

Now, for any j, j', j'' such that $j \geq j'$ and $j \geq j''$, we have that

$$\phi_{j,j'}\phi_{j'',j} \leq c_3 c_4 m^{(d+2)(j-j'+2)+2d+2} j^{4d+7+3d+6} \leq \exp\left(\frac{c_2}{2}\psi_j\right),$$

where the last inequality holds for all $j \geq 1$ since α is sufficiently large. Similarly, by having α sufficiently large, we can guarantee that, for any j, j' such that $j \geq j'$, we have

$$\chi_j \phi_{j',j} \leq c_5 m^{4d+4} j^{6d+12} \leq \exp\left(\frac{c_2}{2}\psi_j\right)$$

for some constant c_5. Then, if we consider each term ϕ_{k_{j-1},k_j} of (41) and use the bounds above for ϕ and χ, we establish the lemma. □

For any support connected path $P = ((k_1, i_1, \tau_1), (k_2, i_2, \tau_2), \ldots, (k_z, i_z, \tau_z))$ in $\Omega^{\sup}_{\kappa-1}$, we define the *weight* of P as $\sum_{j=1}^{z} \psi_{k_j}$. When we take the union bound over all support connected paths later, we will group the paths by their weight. The lemma below shows that the paths in $\Omega^{\sup}_{\kappa-1}$, which are the paths we need to consider, have a large enough weight.

LEMMA 3.10. *Let $P = ((k_1, i_1, \tau_1), (k_2, i_2, \tau_2), \ldots, (k_z, i_z, \tau_z))$ be a path in $\Omega^{\sup}_{\kappa-1}$. If α is sufficiently large and $\kappa = O(\log t)$, then there exist a positive constant $c = c(d)$ and a value C independent of t such that*

$$\sum_{j=1}^{z} \psi_{k_j} \geq \begin{cases} C\dfrac{\sqrt{t}}{(\log t)^c}, & \text{for } d = 1, \\ C\dfrac{t}{(\log t)^c}, & \text{for } d = 2, \\ Ct, & \text{for } d \geq 3. \end{cases}$$

PROOF. Let $\Delta_k^{2\sup}$ denote the diameter of the extended support of a cell of scale k. Then we have

$$\begin{aligned} \Delta_k^{2\sup} &\leq 3(2m+1)\ell_{k+1}\sqrt{d} + 27\beta_{k+1} \\ &= 3(2m+1)m(k+1)^3\ell_k\sqrt{d} + 27C_{\text{mix}}\frac{\ell_k^2(k+1)^4}{\varepsilon^2}. \end{aligned}$$

Using the definition of C_{mix} from (15), we obtain a constant c_3 that may depend on the ratio β/ℓ^2 such that

$$\begin{aligned} \Delta_k^{2\sup} &\leq 3(2m+1)m(k+1)^3\ell_k\sqrt{d} + 27\frac{\beta}{\ell^2}m^2\ell_k^2(k+1)^4 \\ &\leq c_3 m^2(k+1)^4\ell_k^2. \end{aligned}$$

Then, for $k \geq 2$, we have

$$\psi_k = \begin{cases} \dfrac{\varepsilon^2\lambda\ell_{k-1}}{(k+1)^4} = \dfrac{\varepsilon^2\lambda}{(k+1)^4}\left(\dfrac{\ell_k}{mk^3}\right) \geq \dfrac{\varepsilon^2\lambda}{m(k+1)^7}\left(\dfrac{\sqrt{c_3m^2(k+1)^4\ell_k^2}}{\sqrt{c_3m^2(k+1)^4}}\right) \\ \qquad\qquad \geq \dfrac{\varepsilon^2\lambda}{\sqrt{c_3}m^2(k+1)^9}\sqrt{\Delta_k^{2\sup}}, \qquad\qquad \text{for } d = 1, \\ \dfrac{\varepsilon^2\lambda\ell_{k-1}^{d-2}}{(k+1)^4}\left(\dfrac{\ell_k}{mk^3}\right)^2 \geq \dfrac{\varepsilon^2\lambda\ell_{k-1}^{d-2}}{m^2(k+1)^{10}}\left(\dfrac{c_3m^2(k+1)^4\ell_k^2}{c_3m^2(k+1)^4}\right) \\ \qquad\qquad \geq \dfrac{\varepsilon^2\lambda\ell_{k-1}^{d-2}}{c_3m^4(k+1)^{14}}\Delta_k^{2\sup}, \qquad\qquad \text{for } d \geq 2. \end{cases} \tag{42}$$

Now since $\kappa = O(\log t)$, there exists a constant c_4 for which $(k+1)^a \leq c_4(\log t)^a$ for all $k \leq \kappa$ and any $a \geq 1$. We use this fact for dimensions one and two. For three and higher dimensions, we simply use the fact that c_4 can be set large enough in order to satisfy also $\frac{\ell_{k-1}^{d-2}}{(k+1)^{14}} \geq \frac{\ell^{d-2}}{mc_4}$ for all $k \geq 1$. Plugging this into (42), we obtain

$$\psi_k \geq \begin{cases} \dfrac{\varepsilon^2\lambda}{\sqrt{c_3}c_4m^2}\dfrac{\sqrt{\Delta_k^{2\sup}}}{(\log t)^9}, & \text{for } d = 1, \\ \dfrac{\varepsilon^2\lambda}{c_3c_4m^4}\dfrac{\Delta_k^{2\sup}}{(\log t)^{14}}, & \text{for } d = 2, \\ \dfrac{\varepsilon^2\lambda\ell^{d-2}}{c_3c_4m^5}\Delta_k^{2\sup}, & \text{for } d \geq 3. \end{cases} \tag{43}$$

For $k = 1$, we write $\psi_1 \geq c\sqrt{\Delta_1^{2\sup}}$ for $d = 1$ and $\psi_1 \geq c\Delta_1^{2\sup}$ for $d \geq 2$, where c is some positive value that may depend on $\varepsilon, m, \lambda, \ell$ and ν_E. Then, if a support connected path is such that $\sum_{j=1}^z \Delta_{k_j}^{2\sup} < t$, we have that all the cells of the path are contained in $\mathcal{R}_1^t$. Therefore, for $P \in \Omega_{\kappa-1}^{\sup}$ we have $\sum_{j=1}^z \Delta_{k_j}^{2\sup} \geq t$. With (43), this implies that there exists a positive C independent of t but depending on everything else such that

$$\sum_{j=1}^z \psi_{k_j} \geq \begin{cases} C\dfrac{\sqrt{\sum_{j=1}^z \Delta_{k_j}^{2\sup}}}{(\log t)^9} \geq C\dfrac{\sqrt{t}}{(\log t)^9}, & \text{for } d = 1, \\ C\dfrac{\sum_{j=1}^z \Delta_{k_j}^{2\sup}}{(\log t)^{14}} \geq C\dfrac{t}{(\log t)^{14}}, & \text{for } d = 2, \\ C\displaystyle\sum_{j=1}^z \Delta_{k_j}^{2\sup} \geq Ct, & \text{for } d \geq 3. \end{cases}$$

□

For $j \geq 2$, we will work with variables $\tilde{\psi}_j$ for which $\tilde{\psi}_j$ can be written as $b_j \tilde{\psi}_2$ for some positive *integer* b_j. Set $\tilde{\psi}_2 = \psi_2 = 3^{-4}\varepsilon^2\lambda\ell^d$, and for $j \geq 3$, define

$$\tilde{\psi}_j = 2\tilde{\psi}_2 m^{(j-2)d}\big((j-1)!\big)^{3d-3}\big((j-2)!\big)^2(j-3)!.$$

We only introduce the $\tilde{\psi}$ in order to simplify a combinatorial argument later for the counting of support connected paths (we will need to do this in order to extend Lemma 3.9 to the case where the scales are not fixed). The following lemma establishes that ψ_j and $\tilde{\psi}_j$ differ only by constant factors.

LEMMA 3.11. *For all $j \geq 2$, it holds that $\tilde{\psi}_j \leq \psi_j \leq 41\tilde{\psi}_j$.*

PROOF. For $j \geq 3$, we write

$$\psi_j = \frac{\varepsilon^2\lambda\ell^d m^{(j-2)d}((j-1)!)^{3d}}{(j+1)^4} = 3^4\tilde{\psi}_2\frac{m^{(j-2)d}((j-1)!)^{3d}}{(j+1)^4}$$

$$= 3^4\tilde{\psi}_2 m^{(j-2)d}\big((j-1)!\big)^{3d-3}\big((j-2)!\big)^2(j-3)!\left(\frac{(j-1)^3(j-2)}{(j+1)^4}\right).$$

This implies that $\psi_j \leq \frac{3^4}{2}\tilde{\psi}_j \leq 41\tilde{\psi}_j$. The other direction follows from the fact that $\frac{(j-1)^3(j-2)}{(j+1)^4} \geq 1/32$ for all $j \geq 3$. □

3.6. *Completing the proof of Theorem 3.1.* We will need the following technical lemma.

LEMMA 3.12. *Let $x, y \in \mathbb{Z}_+$. Then, for any $c_1, c_2 > 1$, we have*

$$\binom{x+y}{x} e^{-(c_1x+c_2y)} \leq e^{-(c_1-1)x+(c_2-1)y}.$$

PROOF. Since $\binom{x+y}{x} = \binom{x+y}{y}$, we can assume that $x \geq y$. Then we use the inequality $\binom{x+y}{x} \leq (\frac{(x+y)e}{x})^x$ to obtain

$$\binom{x+y}{x} e^{-(c_1x+c_2y)} \leq \left(1+\frac{y}{x}\right)^x e^{-(c_1-1)x-c_2y} \leq e^{-(c_1-1)x-(c_2-1)y}. \qquad \square$$

PROOF OF THEOREM 3.1. First, for any k, note that the number of cells in $\mathcal{R}_k$ satisfies

$$|\mathcal{R}_k| \leq \left(2\left\lceil\frac{t}{\ell_k}\right\rceil\right)^d\left\lceil 1+\frac{t}{\beta_k}\right\rceil. \tag{44}$$

Also, using Lemma 3.2, we have that

$$\mathbf{P}\big(K(i,\tau) \nsubseteq \mathcal{R}_1^t\big) \leq \mathbf{P}\big(K'(i,\tau) \nsubseteq \mathcal{R}_1^t\big)$$

$$= \mathbf{P}(\exists P \in \Omega \text{ s.t. all cells of } P \text{ have bad ancestry}).$$

Then, with this and Lemma 3.5, we obtain

$$\mathbf{P}\big(K(i,\tau) \not\subseteq \mathcal{R}_1^t\big) \le \mathbf{P}\big(\exists P \in \Omega_\kappa^{\sup} \text{ s.t. all cells of } P \text{ are bad}\big).$$

Note that the random variable A_κ is defined differently for scale κ. We handle the cells of scale κ by showing that none of these cells are bad with high probability. It follows by Lemma 3.7(i), (44) and the union bound over all cells in $\mathcal{R}_\kappa$ that

$$\mathbf{P}\big(A_\kappa(i',\tau') = 1 \text{ for all } (i,\tau) \in \mathcal{R}_\kappa\big) \ge 1 - |\mathcal{R}_\kappa| \exp(-c\psi_\kappa) \ge 1 - \exp(-c_1 t),$$

for some positive constant c_1, where the last step follows by setting κ to be the smallest integer such that $\psi_\kappa \ge t$, which gives that $\kappa = \Theta(\frac{\log t}{\log\log t})$. Let H be the event that $A_\kappa(i,\tau) = 1$ for all $(i,\tau) \in \mathcal{R}_\kappa$. Then we have that

$$\begin{aligned}
&\mathbf{P}\big(\exists P \in \Omega_\kappa^{\sup} \text{ s.t. all cells of } P \text{ are bad}\big) \\
&\quad \le \mathbf{P}\big(H \cap \big\{\exists P \in \Omega_\kappa^{\sup} \text{ s.t. all cells of } P \text{ are bad}\big\}\big) + \mathbf{P}\big(H^{\mathrm{c}}\big) \\
&\quad \le \mathbf{P}\big(\exists P \in \Omega_{\kappa-1}^{\sup} \text{ s.t. all cells of } P \text{ are bad}\big) + e^{-c_1 t}.
\end{aligned}$$

In order to get a bound for the term above, we first fix a support connected path

$$P = \big((k_1,i_1,\tau_1),(k_2,i_2,\tau_2),\dots,(k_z,i_z,\tau_z)\big), \tag{45}$$

and use Lemma 3.8 (with the fact that the cells in P are mutually well separated) to get

$$\mathbf{P}\Bigg(\bigcap_{j=1}^{z}\{A_{k_j}(i_j,\tau_j) = 0\}\Bigg) \le \exp\Bigg(-c_2\sum_{j=1}^{z}\psi_{k_j}\Bigg).$$

Now, taking the union bound over all support connected paths with z cells of scale $k_1,k_2,\dots,k_z$, and using Lemma 3.9, we obtain that

$$\begin{aligned}
&\mathbf{P}\big(\exists P \in \Omega_{\kappa-1}^{\sup} \text{ s.t. } P \text{ has } z \text{ bad cells of scales } k_1,k_2,\dots,k_z\big) \\
&\quad \le \exp\Bigg(-\frac{c_2}{2}\sum_{j=1}^{z}\psi_{k_j}\Bigg).
\end{aligned}$$

Note that the upper bound above depends on z and $k_1,k_2,\dots,k_z$ only through $\sum_{j=1}^{z}\psi_{k_j}$, which we call the weight of the path. We will group the paths by their weight. Let W be the set of weights for which there exists at least one path in $\Omega_{\kappa-1}^{\sup}$ with that weight. Then

$$\mathbf{P}\big(\exists P \in \Omega_{\kappa-1}^{\sup} \text{ s.t. all cells of } P \text{ are bad}\big) \le \sum_{w\in W}\exp\Big(-\frac{c_2}{2}w\Big)M(w), \tag{46}$$

where $M(w)$ is the number of possible ways to choose z and $k_1,k_2,\dots,k_z$ such that $\sum_{j=1}^{z}\psi_{k_j} = w$.

In order to get an upper bound for $M(w)$, we will use the $\tilde{\psi}$. Consider the path P in (45), let $w = \sum_{j=1}^{z} \psi_{k_j}$ and take $w_1 = \psi_1 |\{j : k_j = 1\}|$. Let $w_2 = w - w_1$, so w_1 is the weight given by cells of scale 1 and w_2 is the weight given by the other cells of the path. Note that, by Lemma 3.11, $w_2 = \sum_{j:k_j \geq 2} \psi_{k_j} \geq \sum_{j:k_j \geq 2} \tilde{\psi}_{k_j} = h_2 \psi_2$ for some nonnegative integer h_2. Similarly, we have $w_2 \leq 41 h_2 \psi_2$ and $w_1 = h_1 \psi_1$ for some nonnegative integer h_1. Then we have

$$h_1 \psi_1 + h_2 \psi_2 \geq \frac{w}{41}. \tag{47}$$

Let w_0 be the lower bound on the weight of the path given by Lemma 3.10, so for all $w \in W$ we have $w \geq w_0$. Since either w_1 or w_2 must be larger than $w_0/2$, we have that either $h_1 \geq \lceil \frac{w_0}{2\psi_1} \rceil$ or $h_2 \geq \lceil \frac{w_0}{2 \cdot 41 \psi_2} \rceil$. Let $M(h_1, h_2)$ be the number of ways to choose z and $k_1, k_2, \ldots, k_z$ such that there are h_1 values j with $k_j = 1$ and $\sum_{j:k_j \geq 2} \tilde{\psi}_{k_j} = h_2 \psi_2$. Note that, for any such choice, we have $w = \sum_{j=1}^{z} \psi_{k_j} \geq h_1 \psi_1 + h_2 \psi_2$. Thus, the sum in the right-hand side of (46) can be bounded above by

$$\sum_{h_1 = \lceil w_0/(2\psi_1) \rceil}^{\infty} \sum_{h_2 = 0}^{\infty} \exp\Big(-\frac{c_2}{2} (h_1 \psi_1 + h_2 \psi_2) \Big) M(h_1, h_2)$$
$$+ \sum_{h_1 = 0}^{\infty} \sum_{h_2 = \lceil w_0/(82\psi_2) \rceil}^{\infty} \exp\Big(-\frac{c_2}{2} (h_1 \psi_1 + h_2 \psi_2) \Big) M(h_1, h_2).$$

In order to bound $M(h_1, h_2)$, we will consider the following pictorial way to define the values of z and $k_1, k_2, \ldots, k_z$. Suppose we have h_1 blocks of size ψ_1 and h_2 blocks of size ψ_2. Consider an ordering of the blocks, but such that permuting blocks of the same size does not change the order. Then, for each block of size ψ_2, we color it either black or white, while blocks of size ψ_1 are not colored. Now, for each choice of z and $k_1, k_2, \ldots, k_z$ we associate an order and coloring of the blocks as follows. If $k_1 = 1$, then the first block is of size ψ_1. Otherwise, the first $\tilde{\psi}_{k_1}/\psi_2$ blocks are of size ψ_2 and have black color. Then, if $k_2 = 1$, the next block is of size ψ_1, otherwise the next $\tilde{\psi}_{k_2}/\psi_2$ blocks are of size ψ_2 and have white color. We proceed in this way until k_z, where whenever $k_i \neq 1$ we use the color black if i is odd and the color white if i is even. Though there are orders and colorings that are not associated to any choice of z and $k_1, k_2, \ldots, k_z$, each such choice of z and $k_1, k_2, \ldots, k_z$ corresponds to a unique order and coloring of the blocks. Therefore, the number of ways to order and color the blocks gives an upper bound for $M(h_1, h_2)$. Note that there are $\binom{h_1 + h_2}{h_1}$ ways to order the blocks and 2^{h_2} ways to color the size-ψ_2 blocks. Therefore,

$$\mathbf{P}\big(\exists P \in \Omega_{\kappa-1}^{\sup} \text{ s.t. all cells of } P \text{ are bad}\big)$$
$$\leq \sum_{h_1 = \lceil w_0/(2\psi_1) \rceil}^{\infty} \sum_{h_2 = 0}^{\infty} \exp\Big(-\frac{c_2}{2} (h_1 \psi_1 + h_2 \psi_2) \Big) \binom{h_1 + h_2}{h_1} 2^{h_2}$$

$$
\begin{aligned}
&+\sum_{h_1=0}^{\infty}\sum_{h_2=\lceil w_0/(82\psi_2)\rceil}^{\infty}\exp\biggl(-\frac{c_2}{2}(h_1\psi_1+h_2\psi_2)\biggr)\binom{h_1+h_2}{h_1}2^{h_2}\\
&\le\sum_{h_1=\lceil w_0/(2\psi_1)\rceil}^{\infty}\sum_{h_2=0}^{\infty}\exp\biggl(-\frac{c_2(h_1\psi_1+h_2\psi_2)}{3}\biggr)\\
&\quad+\sum_{h_1=0}^{\infty}\sum_{h_2=\lceil w_0/(82\psi_2)\rceil}^{\infty}\exp\biggl(-\frac{c_2(h_1\psi_1+h_2\psi_2)}{3}\biggr)\\
&\le\sum_{w\in W}\exp(-c_4w)\le\exp(-cw_0),
\end{aligned}
$$

where in the second inequality we use Lemma 3.12 and the fact that α is sufficiently large to write $\frac{c_2\psi_1}{2}-1\ge\frac{c_2\psi_1}{3}$, and similarly for ψ_2. In the third inequality, we used (47). Since w_0 is the lower bound in Lemma 3.10, the proof of Theorem 3.1 is completed. □

4. Detection. In this section, we use Theorem 3.1 to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Recall that we say that the displacement of a node throughout $[t_0,t_1]$ is in Q_z if the node never leaves $x+Q_z$ during the whole of $[t_0,t_1]$, where x is the position of the node at time t_0. Fix $\eta=1$ and $\varepsilon=1/2$. Now, we fix the ratio β/ℓ^2 small enough so that the lower bound for w in Theorem 3.1 is at most 1 and we can set $w=1$. We tessellate $\mathbb{R}^d$ into cells of side length $\ell=\frac{r}{2\sqrt{d}}$. Let S be a cell of the tessellation and let v be a node of Π_0 that is inside S at some time s. Then, if the displacement of v during the interval $[s,s+\beta]$ is in $Q_{w\ell}$, we have that the distance between v and any point of S at any time in $[s,s+\beta]$ is at most $\frac{\ell\sqrt{d}}{2}+\ell\sqrt{d}=\frac{3\sqrt{d}\ell}{2}\le r$. Therefore, for such a node v, the ball of radius r centered at v covers the whole of S during the entire duration of the interval $[s,s+\beta]$. Hence, if S contains at least one such node at time s and the target enters S during $[s,s+\beta]$, then the target is detected.

Now, we apply Theorem 3.1. For each $(i,\tau)\in\mathcal{R}_1$, define $E(i,\tau)$ to be the event that there is at least one node in the cube $S_1(i)$ at time $\tau\beta$ for which its displacement from time $\tau\beta$ to $(\tau+1)\beta$ is inside Q_ℓ. This event is clearly increasing. Let N be a Poisson random variable of mean $\frac{\lambda\ell^d}{2}$. Then, using the fact that $\varepsilon=1/2$, we have that

$$
\nu_E(\lambda/2,Q_\ell)\ge\mathbf{P}(N\ge1)=1-\exp\biggl(-\frac{\lambda\ell^d}{2}\biggr).
$$

Clearly, $\log(\frac{1}{1-\nu_E(\lambda/2,Q_\ell)})\ge\frac{\lambda\ell^d}{2}$, which increases with λ. Therefore, we can set λ large enough so that $\log(\frac{1}{1-\nu_E(\lambda/2,Q_\ell)})$ and $\varepsilon^2\lambda\ell^d=\lambda\ell^d/4$ are larger than α_0. With

this, we apply Theorem 3.1 to obtain that

$$\mathbf{P}\big(K(0,0)\subseteq\mathcal{R}_1^t\big)\le\begin{cases}\exp\Big(-C\dfrac{\sqrt{t}}{(\log t)^c}\Big), & \text{for } d=1,\\ \exp\Big(-C\dfrac{t}{(\log t)^c}\Big), & \text{for } d=2,\\ \exp(-Ct), & \text{for } d\ge 3.\end{cases}\tag{48}$$

Note that the target is not detected at time 0 only if $E(0,0)=0$. Then, in this case, since $K(0,0)$ is contained in $\mathcal{R}_1^t$ and $\mathcal{R}_1^t$ contains all the cells that are contained in the space–time region $(-t,t)\times[0,t)$ we have that the target must be at some time in the interval $[0,t]$ inside a cell (i,τ) of scale 1 for which $E(i,\tau)=1$ and, therefore, the target is detected.

This shows that the probability that the target is able to evade detection up to time t is given by (48), which completes the proof of Theorem 1.1. □

The same proof as above can be used to establish that, for any $\lambda>0$, there exists a value $\bar r=\bar r(\lambda)$ so that with high probability the target will eventually get within distance $\bar r$ from at least one node. We state this slight generalization below. In Theorem 1.1, we require λ to be large enough so that $\bar r\le r$.

THEOREM 4.1. *In dimensions $d\ge 2$, there exist an explicit constant $c=c(d)$ and a positive C independent of t so that the following holds for all large enough t. For any $\lambda>0$, there exists $\bar r=\bar r(\lambda)>0$ so that the probability that there exists a trajectory g for the target so that for all $s\in[0,t]$ the ball $B(g(s),\bar r)$ contains no node of Π_s is at most $\exp(-C\frac{t}{(\log t)^c})$ in $d=2$ and at most $\exp(-Ct)$ in $d\ge 3$.*

APPENDIX: STANDARD LARGE DEVIATION RESULTS

We use the following standard Chernoff bounds and large deviation results.

LEMMA A.1 (Chernoff bound for Poisson). *Let P be a Poisson random variable with mean λ. Then, for any $0<\varepsilon<1$,*

$$\mathbf{P}\big(P\ge(1+\varepsilon)\lambda\big)\le\exp\Big(-\frac{\lambda\varepsilon^2}{2}(1-\varepsilon/3)\Big)$$

and

$$\mathbf{P}\big(P\le(1-\varepsilon)\lambda\big)\le\exp\Big(-\frac{\lambda\varepsilon^2}{2}\Big).$$

LEMMA A.2 (Gaussian tail bound [23], Theorem 12.9). *Let X be a normal random variable with mean 0 and variance σ^2. Then, for any $R\ge\sigma$ we have that* $\mathbf{P}(X\ge R)\le\frac{\sigma}{\sqrt{2\pi}R}\exp(-\frac{R^2}{2\sigma^2})$.

Acknowledgements. We are grateful to Yuval Peres, Alistair Sinclair, Perla Sousi and Omer Tamuz for useful discussions. We are also thankful to Ron Peled for valuable suggestions that improved the bounds in (2), and to Alistair Sinclair for helpful comments on a previous version of the paper.

REFERENCES

[1] BALISTER, P., ZHENG, Z., KUMAR, S. and SINHA, P. (2009). Trap coverage: Allowing coverage holes of bounded diameter in wireless sensor networks. In *Proceedings of the* 28*th IEEE Conference on Computer Communications* 19–25. IEEE, Piscataway, NJ.

[2] BENJAMINI, I. and STAUFFER, A. (2013). Perturbing the hexagonal circle packing: A percolation perspective. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 1141–1157. MR3127917

[3] CHATTERJEE, S., PELED, R., PERES, Y. and ROMIK, D. (2010). Phase transitions in gravitational allocation. *Geom. Funct. Anal.* **20** 870–917. MR2729280

[4] CHAYES, J. T. and CHAYES, L. (1989). The large-N limit of the threshold values in Mandelbrot's fractal percolation process. *J. Phys. A* **22** L501–L506. MR1003259

[5] CHAYES, J. T., CHAYES, L. and DURRETT, R. (1988). Connectivity properties of Mandelbrot's percolation process. *Probab. Theory Related Fields* **77** 307–324. MR0931500

[6] CHAYES, J. T., CHAYES, L., GRANNAN, E. and SWINDLE, G. (1991). Phase transitions in Mandelbrot's percolation process in three dimensions. *Probab. Theory Related Fields* **90** 291–300. MR1133368

[7] CLEMENTI, A. E. F., PASQUALE, F. and SILVESTRI, R. (2009). MANETS: High mobility can make up for low transmission power. In *Automata, Languages and Programming. Part II*. 387–398. Springer, Berlin. MR2544811

[8] DÍAZ, J., MITSCHE, D. and PÉREZ-GIMÉNEZ, X. (2009). Large connectivity for dynamic random geometric graphs. *IEEE Trans. Mob. Comput.* **8** 821–835.

[9] DOUSSE, O., TAVOULARIS, C. and THIRAN, P. (2006). Delay of intrusion detection in wireless sensor networks. In *Proceedings of the* 7*th ACM International Conference on Mobile Computing and Networking* (*MobiCom*) 155–165. ACM, New York, NY.

[10] DREWITZ, A., GÄRTNER, J., RAMÍREZ, A. F. and SUN, R. (2012). Survival probability of a random walk among a Poisson system of moving traps. In *Probability in Complex Physical Systems*, *Springer Proceedings in Mathematics* 119–158. Springer, Berlin.

[11] EL GAMAL, A., MAMMEN, J., PRABHAKAR, B. and SHAH, D. (2004). Throughput-delay trade-off in wireless networks. In *Proceedings of the* 23*rd IEEE Conference on Computer Communications* 464–475. IEEE, Piscataway, NJ.

[12] FALCONER, K. J. and GRIMMETT, G. R. (1992). On the geometry of random Cantor sets and fractal percolation. *J. Theoret. Probab.* **5** 465–485. MR1176432

[13] FRIEDRICH, T., SAUERWALD, T. and STAUFFER, A. (2013). Diameter and broadcast time of random geometric graphs in arbitrary dimensions. *Algorithmica* **67** 65–88. MR3072817

[14] KESIDIS, G., KONSTANTOPOULOS, T. and PHOHA, S. (2003). Surveillance coverage of sensor networks under a random mobility strategy. In *Proceedings of the* 2*nd IEEE International Conference on Sensors*. IEEE, Piscataway, NJ.

[15] KESTEN, H. and SIDORAVICIUS, V. (2005). The spread of a rumor or infection in a moving population. *Ann. Probab.* **33** 2402–2462. MR2184100

[16] KESTEN, H. and SIDORAVICIUS, V. (2006). A phase transition in a model for the spread of an infection. *Illinois J. Math.* **50** 547–634. MR2247840

[17] KONSTANTOPOULOS, T. (2010). Response to Prof. Baccelli's lecture on modelling of wireless communication networks by stochastic geometry. *The Computer Journal* **53** 612–614.

[18] LAM, H., LIU, Z., MITZENMACHER, M., SUN, X. and WANG, Y. (2012). Information dissemination via random walks in d-dimensional space. In *Proceedings of the* 23*th ACM-SIAM Symposium on Discrete Algorithms* (*SODA*). SIAM, Philadelphia, PA.

[19] LIU, B., BRASS, P., DOUSSE, O., NAIN, P. and TOWSLEY, D. (2005). Mobility improves coverage of sensor networks. In *Proceedings of the* 6*th ACM International Conference on Mobile Computing and Networking* (*MobiCom*). ACM, New York, NY.

[20] MEESTER, R. and ROY, R. (1996). *Continuum Percolation*. Cambridge Univ. Press, Cambridge. MR1409145

[21] MENSHIKOV, M. V., POPOV, S. YU. and VACHKOVSKAIA, M. (2001). On the connectivity properties of the complementary set in fractal percolation models. *Probab. Theory Related Fields* **119** 176–186. MR1818245

[22] MOREAU, M., OSHANIN, G., BÉNICHOU, O. and COPPEY, M. (2004). Lattice theory of trapping reactions with mobile species. *Phys. Rev. E* **69** 046101.

[23] MÖRTERS, P. and PERES, Y. (2010). *Brownian Motion*. Cambridge Univ. Press, Cambridge. MR2604525

[24] ORZECHOWSKI, M. E. (1996). On the phase transition to sheet percolation in random Cantor sets. *J. Stat. Phys.* **82** 1081–1098. MR1372435

[25] PENROSE, M. (2003). *Random Geometric Graphs*. Oxford Univ. Press, Oxford. MR1986198

[26] PERES, Y., SINCLAIR, A., SOUSI, P. and STAUFFER, A. (2013). Mobile geometric graphs: Detection, coverage and percolation. *Probab. Theory Related Fields* **156** 273–305. MR3055260

[27] PERES, Y. and SOUSI, P. (2012). An isoperimetric inequality for the Wiener sausage. *Geom. Funct. Anal.* **22** 1000–1014. MR2984124

[28] PETTARIN, A., PIETRACAPRINA, A., PUCCI, G. and UPFAL, E. (2011). Tight bounds on information dissemination in sparse mobile networks. In 30*th Annual ACM SIGACT-SIGOPS Symposium on Principles of Distributed Computing*. ACM, New York, NY.

[29] SINCLAIR, A. and STAUFFER, A. (2010). Mobile geometric graphs, and detection and communication problems in mobile wireless networks. Preprint. Available at arXiv:1005.1117v1.

[30] STOYAN, D., KENDALL, W. S. and MECKE, J. (1995). *Stochastic Geometry and Its Applications*, 2nd ed. Wiley, Chichester.

[31] VAN DEN BERG, J., MEESTER, R. and WHITE, D. G. (1997). Dynamic Boolean models. *Stochastic Process. Appl.* **69** 247–257. MR1472953

[32] YANMAZ, E. and GUCLU, H. (2010). Stationary and mobile target detection using mobile wireless sensor networks. In *Proceedings of the* 29*th Conference on Computer Communications* (*INFOCOM*). IEEE, Piscataway, NJ.

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF BATH
CLAVERTON DOWN
BATH, BA1 2BG
UNITED KINGDOM
E-MAIL: a.stauffer@bath.ac.uk