GLOBAL BLOW-UP FOR A SEMILINEAR HEAT EQUATION ON A SUBSPACE

C.J. BUDD (1), J.W. DOLD (2), AND V.A. GALAKTIONOV (3)

Abstract. We study the asymptotic behaviour, as $t \to T^-$, near a finite blow-up time $T > 0$ of decreasing in x solutions to the following semilinear heat equation with a non-local term:

$$u_t = u_{xx} + u^2 - \int_0^1 u^2 \, \mathrm{d}x \quad \text{in} \quad (0,1) \times (0,T), \quad \int_0^1 u(x,t) \, \mathrm{d}x \equiv 0,$$

with Neumann boundary conditions and strictly decreasing initial function $u_0(x)$ with zero mass. We prove sharp estimates for $u(x,t)$ as $t \to T^-$, revealing a *non-uniform global blow-up*:

$$u(0,t) \to +\infty, \quad u(x,t) \approx -\pi\sqrt{2}(T-t)^{-\frac{1}{2}} |\log(T-t)|^{\frac{1}{2}} \to -\infty$$

uniformly on any compact set $[\delta, 1]$, $\delta \in (0,1)$.

1. Introduction: Main results

1.1. **Statement of the problem.** In this paper, we study the asymptotic behaviour of the blow-up type solutions to the initial-boundary value problem for a semilinear heat equation with an additional non-local term given by

$$u_t = A(u) \equiv u_{xx} + u^2 - h(t,u) \quad \text{in} \quad \omega_T \equiv (0,1) \times (0,T), \tag{1.1}$$

where

$$h(t,u) = \int_0^1 u^2(x,t) \, \mathrm{d}x. \tag{1.2}$$

The solution of this PDE satisfies the Neumann boundary conditions

$$u_x = 0 \quad \text{for} \quad x = 0, 1 \quad \text{and} \quad t \in [0,T), \tag{1.3}$$

and the initial condition

$$u(x,0) = u_0(x) \quad \text{in} \quad (0,1), \tag{1.4}$$

where the initial function is such that

$$u_0(x) \in C^1([0,1]), \quad u_0'(x) < 0 \quad \text{in} \quad (0,1), \quad u_0'(0) = u_0'(1) = 0, \quad u_0''(0) < 0, \tag{1.5}$$

Date: July 24, 2015.

1991 *Mathematics Subject Classification.* 35K55, 35K65.

Key words and phrases. non-local parabolic equation, integral constraint, global blow-up, asymptotic behaviour.

and

$$\int_0^1 u_0(x)\,\mathrm{d}x = 0. \tag{1.6}$$

Problem (1.1)–(1.4) has applications to chemical and biological systems where mass or other quantities are conserved. An original asymptotic study of blow-up behaviour in this problem was given in [6] which gives more details of the applications of these type of systems. In [9] and [31] analytical results for a general class of blow-up type nonlocal reaction-diffusion equations are given. In [7] the effect of a further perturbation of (1.1) by a nonlinear non-local convective term is considered in which the system models a certain similarity solution of the Navier–Stokes equations in which the non-local term corresponds to pressure. The purpose of this paper is to extend some of these results by obtaining rigorous estimates of the analytical structure of the blow-up profiles of the solutions to (1.1).

By classic parabolic regularity theory, the solution remains smooth and unique until the first blow-up time $t = T$. By integrating equation (1.1) over the interval $(0, 1)$ and by using (1.6) there is conservation of the mean value

$$\int_0^1 u(x,t)\,\mathrm{d}x = 0 \quad \text{for any} \quad t \in [0, T). \tag{1.7}$$

The existence of a local in time solution $u(x, t)$ satisfying the constrain (1.7) can be proved by using a semi-group argument (see e.g., [24] and the references therein) or by a Galerkin approximation; see [28], where suitable techniques to prove uniqueness are also given. Regularity of the solution, at least $u \in C^{4,2}(\omega_T) \cap C^{1,0}(\bar{\omega}_T)$ under the hypotheses (1.5), follows from the properties of heat operators, see [11]. In fact the regularity of the solution is much better than these estimates.

1.2. **Preliminaries.** It was shown in [6] that, under some additional hypotheses on the initial function $u_0(x)$, namely,

$$u_0(x) = \sum_{n=-\infty}^{\infty} C_{n,0} \mathrm{e}^{in\pi x}, \tag{1.8}$$

where the Fourier coefficients $\{C_{n,0}\}$ satisfy the conditions

$$C_{n,0} \geq 0 \quad \text{for any} \quad n, \tag{1.9}$$

$$C_{1,0} > C_{2,0} \quad \text{and either} \quad C_{1,0} > 4\sqrt{2}\pi^2 \quad \text{or} \quad C_{2,0} > 4\pi^2, \tag{1.10}$$

then the problem (1.1)–(1.6) *does not admit a global in time solution*. This means that there exists a finite blow-up time T such that $u(x, t)$ is a uniformly bounded classical solution in $\omega_{T'}$ for any $T' < T$ and (see also the comments in Section 2 of this paper)

$$u(0, t) \to \infty \quad \text{as} \quad t \to T, \tag{1.11}$$

$$h(t, u) \to \infty \quad \text{as} \quad t \to T. \tag{1.12}$$

Notice that the blow-up hypotheses (1.8)–(1.10) are satisfies by the function $u_0(x) = 2C_{1,0}\cos(\pi x)$ for arbitrary $C_{1,0} > 4\sqrt{2}\pi^2$.

The problem under consideration was intensively studied in [6] by a combination of formal and numerical methods. In particular, it was shown that in this case there exists *global non-uniform blow-up* where, together with (1.11), there holds

$$u(x_0, t) \to -\infty \quad \text{as} \quad t \to T \quad \text{for any} \quad x_0 \in (0, 1]. \tag{1.13}$$

This paper is devoted to a proof of the formal asymptotic results given in [6], namely, that $u(x, t) \to +\infty$ as $t \to T$ at the *single point* $x = 0$ and that, for any fixed $x \in (0, 1]$,

$$u(x, t) \sim -\pi\sqrt{2}(T - t)^{-\frac{1}{2}} |\log(T - t)|^{\frac{1}{2}} \to -\infty. \tag{1.14}$$

The precise results are stated in Theorem 1.1 below.

It is interesting to compare the equation (1.1) with the well known semilinear heat equation

$$v_t = H(v) \equiv v_{xx} + v^2 \quad \text{in} \quad \omega_T, \tag{1.15}$$

or

$$v_t = v_{xx} + v^p \quad \text{in} \quad \omega_T, \quad p > 1, \quad (v \geq 0). \tag{1.16}$$

The asymptotic behaviour of the solutions to (1.15), (1.16), which blow-up, is now well established; see the recent results in [2], [4], [10], [25], [26], [32], [31], [9], and the references in the books [3], [23], [30].

Although the main differential operator in the equation (1.1) looks similar to (1.15), it has a non-local perturbation and, hence, several useful properties of the heat equation (such as the Maximum Principle, comparison techniques, and some kinds of a "monotonicity") cannot be applied to the problem (1.1). But of course there is a direct connection between these two equations. More exactly, the non-local equation (1.1) is the semilinear heat equation (after a proper projecting)

$$u_t = u_{xx} + u^2 \quad \text{in} \quad \omega_T \quad \text{with} \quad u(\cdot, t) \in F_0 \quad \text{for} \quad t \in [0, T), \tag{1.17}$$

where F_0 is the subspace of the Fourier space F of those functions with the zero mean condition (1.7) , i.e.,

$$F_0 = \left\{ v \in L^\infty : v = \sum_{n=-\infty}^{\infty} C_n \mathrm{e}^{in\pi x} \quad \text{with} \quad C_0 = 0. \right\} \tag{1.18}$$

Projecting the equation, as, actually, is done in (1.1), is necessary since F_0 is an invariant subspace for (1.17).

Thus, the *non-local problem* (1.1), (1.7) *is equivalent to the initial-boundary value problem for the semilinear heat equation (1.15) on the linear subspace* F_0 *of co-dimension 1.* That is why almost all of the known methods and results being proved and obtained for the more usual semilinear or quasilinear heat equations of the form (1.15), (1.16) and others (see the references above) cannot be directly applied to the same equations when given on a subspace.

It was shown in [6] that, under the above hypotheses, the *positive cone* $F_0^+ \subset F_0$ given by (1.8), (1.9) is an *invariant set* of the equation (1.1), i.e., if $u_0 \in F_0^+$ then $u(., t) \in F_0^+$ for all $t \in (0,T)$. This invariance property plays the role of a "weak maximum principle" for the infinite dimensional dynamical system satisfied by the Fourier coefficients $\{C_n\}$ of the solution $u(x,t)$. In Section 2 of this paper, we shall discuss some further comparison and monotonicity results on F_0^+.

We have mentioned that the usual comparison techniques cannot be applied to the problem (1.17). It is easily seen that the usual definitions of lower u^- and upper u^+ solutions satisfying the corresponding partial differential inequalities

$$u_t^- \le A(u^-) \quad \text{and} \quad u_t^+ \ge A(u^+), \tag{1.19}$$

cannot be given for the heat equation on the subspace since the non-local term in (1.1) is not monotone (or anti-monotone) in the parabolic sense. Notice also that the non-local operator given in (1.1) has a certain monotone property which can be stated as follows. Assume that there exists a pair of functions (u^-, u^+), $u^- \le u^+$ satisfying the Neumann boundary conditions and the inequalities in (1.8)

$$u_t^- \le u_{xx}^- + (u^-)^2 - h^-(t,u^-,u^+), \quad u_t^+ \ge u_{xx}^+ + (u^+)^2 - h^+(t,u^-,u^+) \quad \text{in} \quad \omega_T, \tag{1.20}$$

where we define the functions

$$\begin{aligned} h^-(\cdot) &= \int_0^1 \max\{s^2 : s \in [u^-(x,t), u^+(x,t)]\}\,\mathrm{d}x, \\ h^+(\cdot) &= \int_0^1 \min\{s^2 : s \in [u^-(x,t), u^+(x,t)]\}\,\mathrm{d}x, \end{aligned} \tag{1.21}$$

Then, by the usual comparison techniques based on the Maximum Principle [11], we have that

$$u^- \le u \le u^+ \quad \text{in} \quad \omega_T, \tag{1.22}$$

provided that these inequalities are valid for $t = 0$. Unfortunately, because of the blow-up behaviour, the functions u^- and u^+ are expected, in general, to have different blow-up times, and hence comparison (1.22) in the whole domain ω_T is not possible. In other words (1.22) implies that $u^- \equiv u \equiv u^+$.

The most sharp results on blow-up behaviour for the heat equations of the form (1.16) have been proved via a centre manifold analysis [2], [4], [10] and similar techniques; see [25], [26] (see also the formal approach in [32]). In particular, the following asymptotically sharp estimate of the L^∞-norm of negative decreasing in $|x|$ solutions $v = v(|x|,t)$ to (1.15) was proved to be valid

$$\|v(.,t)\|_{L^\infty} = (T-t)^{-1}\big[1 + \tfrac{1}{4|\log(T-t)|}(1+o(1))\big] \quad \text{as} \quad t \to T. \tag{1.23}$$

A much weakened form of such an estimate plays a key role in studying the non-local problem (1.1), (1.4).

We **assume** that (cf. (1.23))

$$\limsup_{t\to T} |\log(T-t)|\{(T-t)u(0,t) - 1\} < \infty. \tag{1.24}$$

We will show in Section 4 that the boundedness hypothesis (1.24) is equivalent to a traditional problem in the theory of nonlinear evolution equations. Namely, to prove that under some hypotheses, any global solution to a particular nonlinear evolution equation is uniformly bounded from above. For the non-local problem (1.1), (1.4), hypothesis (1.24) remains an open question and will be considered in a future paper[1].

This paper is devoted to the study the asymptotic behaviour of blowing up solutions to nonlinear parabolic equations. This scheme, which differs from the methods given in the above mentioned papers, can be applied to the nonlinear equation (1.1), (1.2) as well as to the semilinear equations (1.15), (1.16). Since our approach is not based on a centre manifold analysis or similar techniques (justifying these turned out to be very difficult) and we do not essentially use the heat potentials corresponding to the linear heat operator $\partial/\partial t - \partial^2/\partial x^2$, this scheme is expected to be useful for a wide class of *quasilinear heat equations* with nonlinear diffusion terms where the behaviour as $t \to T$ is described by a *semilinear* Hamilton–Jacobi equation. See some details in [4].

1.3. **The main result.** We now state the main result of this paper.

Theorem 1.1 *Assume that (1.5), (1.6) hold and that $T > 0$ is a blow-up time. Assume that the upper bound (1.24) is valid. If we define the new rescaled variable*

$$\eta = x(T-t)^{\frac{1}{2}} |\log(T-t)|^{-\frac{1}{2}},$$

then, as $t \to T$,

$$u(0,t) = (T-t)^{-1}\big[1 + \tfrac{1}{4|\log(T-t)|}(1+o(1))\big], \tag{1.25}$$

$$(T-t)u(\eta((T-t)|\log(T-t)|^{\frac{1}{2}}, t) \to \theta_*(\eta) \equiv \big(1 + \tfrac{\eta^2}{8}\big)^{-1} \tag{1.26}$$

uniformly on compact subsets in $\eta \geq 0$,

$$u(x,t) = -\pi\sqrt{2}(T-t)^{-\frac{1}{2}}|\log(T-t)|^{\frac{1}{2}}(1+o(1)) \tag{1.27}$$

uniformly on compact subsets $\{\delta \leq x \leq 1\}$ for arbitrarily small $\delta > 0$,

$$h(t,u) \equiv \int_0^1 u^2(x,t)\,\mathrm{d}x = \tfrac{\pi}{\sqrt{2}}\,(T-t)^{-\frac{3}{2}}|\log(T-t)|^{\frac{1}{2}}(1+o(1)). \tag{1.28}$$

The proof of Theorem 1.1 consists of several steps. In Section 2, by using a slight modification [18] of the method [12], we prove a sharp upper bound on $u(x,t)$ as $t \to T, x \to 0$ through the value of $u(0,t)$. We also prove that $u(x,t)$ has a "flat" behaviour as $t \to T$ on compact subsets $[\delta, 1]$ with small $\delta > 0$, a sharp description of which is as given in (1.27). These estimates imply that the value of $h(t,u) \equiv \|u(x,t)\|_2^2$ as $t \to T$ depends only on the behaviour of $u(x,t)$ near the origin $x = 0$. This behaviour is finally shown to be almost independent of the non-local term in (1.1). The proof of (1.25), (1.28) is based on several limits as $t \to T$ on different compact subsets in x for different rescaled functions. The first limit as $t \to T$

[1] As we have mentioned in Abstract of the present paper, instead, we will perform a difficult blow-up study of the related higher-order parabolic problem coming from Burnett's hydrodynamic model, [8].

on small compact subsets $\{0 \le x \le c_0(T-t)^{\frac{1}{2}}\}$ is obtained in Section 3 by using a form of Lyapunov-type analysis, see similar results for autonomous in time semilinear heat equations in [16], [13], and also in [15]. The second limit on the same compact subsets establishing a precise rate of growth of the solution (1.25) and its second x-derivative is based on the idea of a weighted energy equation (see different applications in [15], [17], and [19]). The above estimates make it possible to match such asymptotic behaviour with that on larger compact subsets of the form $\{0 \le x \le c_0[(T-t)|\log(T-t)|]^{\frac{1}{2}}\}$ which yields a precise asymptotic behavior of the solution, (1.26) and hence (1.27), (1.28). This last step is based on general results on perturbed dynamical systems; see [19] and also [23].

2. First estimates

In this section we prove some qualitative results on the behaviour of the solution.

2.1. Preliminaries.

2.1. Preliminaries. We begin with some simple results.

Proposition 2.1 *Under the hypotheses (1.5),*

$$u_x(x,t) < 0 \quad in \quad \omega_T. \tag{2.1}$$

Proof. Since by the regularity of the solution, the function $z = u_x$ solves the parabolic equation

$$z_t = z_{xx} + 2uz \quad \text{in} \quad \omega_T \tag{2.2}$$

with a smooth coefficient bounded in $\omega_{T'}$ with $T' < T$ and $z \le 0$ on the parabolic boundary $\partial\omega_T$ of ω_T (see (1.3), (1.5)), and furthermore $z < 0$ for $t = 0$ by (1.5), the result (2.1) follows from the Strong Maximum Principle [11]. □

It then follows from (2.1) that, for any $t \in [0,T)$,

$$\sup_x u(x,t) = u(0,t) \quad \text{and} \quad \inf_x u(x,t) = u(1,t) \tag{2.3}$$

and by (1.7) we have that

$$u(0,t) \equiv \mu_0(t) > 0, \quad u(1,t) \equiv -\mu_1(t) \quad \text{on} \quad [0,T). \tag{2.4}$$

We observe that we may construct some further non-monotone solutions by a process of successive reflections and scalings.

Using Proposition 2.1, we also conclude that, for any $t \in [0,T)$, there exists exactly one zero point $x_0(t) \in (0,1)$ of the solution $u(x,t)$ so that, for any $t \in (0,T)$,

$$u(x,t) > 0 \quad \text{on} \quad I_+(t) = (0, x_0(t)), \quad u(x,t) < 0 \quad \text{on} \quad I_-(t) = (x_0(t), 1). \tag{2.5}$$

Since $u_x(x_0(t),t) < 0$ for $t \in [0,T)$ by (2.1), the curve $\{x = x_0(t), t \in [0,T)\}$ is differentiable and moreover $x_0(t) \in C^\infty((0,T))$. We note that in [6] it is shown, formally, that

$$x_0(t) \sim \tfrac{2\sqrt{2}}{\pi}(T-t)^{\frac{1}{2}}|\log(T-t)|^{\frac{1}{2}} \quad \text{as} \quad t \to T.$$

Hence, the domains $\omega_T^+ = I_+(t) \times (0,T)$ and $\omega_-^T = I_-(t) \times (0,T)$ have smooth lateral boundaries.

For convenience, we denote

$$u_+ = \max\{u, 0\}, \quad u_- = \min\{u, 0\},$$

so that

$$u_+ = u, u_- \equiv 0 \quad \text{on} \quad \omega_T^+, \quad u_+ \equiv 0, u_- = u \quad \text{on} \quad \omega_T^-,$$

and, for any $t \in [0, T)$,

$$h(t,u) \equiv \|u(x,t)\|_2^2 = \|u_+(t)\|_2^2 + \|u_-(t)\|_2^2, \quad \|u(x,t)\|_1 = \|u_+(t)\|_1 + \|u_-(t)\|_1. \tag{2.6}$$

Proposition 2.2 *If we presume that $u(x,t)$ blows up at a time T, then*

$$u(0,t) > (T-t)^{-1} \quad on \quad (0,T). \tag{2.7}$$

Proof. The function $v(t) = (T-t)^{-1} > 0$ is the uniform in space solution of the parabolic equation

$$v_t = v_{xx} + v^2 \quad \text{in} \quad \omega_T. \tag{2.8}$$

Since $u(x,t)$ satisfying

$$u_t = u_{xx} + u^2 - h(t,u) < u_{xx} + u^2 \tag{2.9}$$

is a strict lower solution of the heat equation (2.8), the inequality (2.7) follows via the intersection comparison results in [30] (Chapter IV). Indeed, if (2.7) is false for some $t' \in (0,T)$ then by (2.3) the profiles $u(x,t')$ and $v(t')$ do not intersect each other on $[0,1]$, by the Maximum Principle [11] we deduce that $u(x,t)$ and $v(t)$ have different blow-up times, giving the contradiction. □

Proposition 2.3 *There exists a $t_0 \in [0,T)$ such that*

$$h'(t,u) > 0 \quad on \quad [t_0, T). \tag{2.10}$$

Proof. By multiplying equation (1.6) by u and $u_t \in F_0$ and using the well known embedding inequality $\|u_x\|_2^2 \geq \pi^2 \|u\|_2^2$ on F_0, we have that (see also [6])

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(x,t)\|_2^2 \geq \pi^2 \|u(x,t)\|_2^2 + 6H(u_0) \quad \text{on} \quad (0,T), \tag{2.11}$$

where

$$H(u) = -\tfrac{1}{2}\|u_x\|_2^2 + \tfrac{1}{3}\int u^3 \, \mathrm{d}x.$$

We have from (1.12) that there exists a $t_0 \in [0,T)$ such that $\|u(x,t_0)\|_2^2 > 6H(u_0)/\pi^2$ and hence (2.10) follows. □

We now begin to estimate the non-local term $h(t,u)$ in the equation (1.1).

2.2. **Estimates in ω_T^+.** The following result plays an important role in the future analysis:

Lemma 2.4 *There exist large positive constants A_0 and B_0 such that*

$$J(x,t) \equiv u_x + \tfrac{u^2\, x}{4B_0 + 2\log(A_0+u^2)} \le 0 \quad in \quad \omega_T^+. \tag{2.12}$$

Proof. Using the ideas in [12] we first set (c.f. (2.12))

$$J(x,t) = u_x + xF(u) \quad \text{in} \quad \omega_T^+ \tag{2.13}$$

where $F(u)$ is a smooth function to be determined later. Then J solves the parabolic equation

$$J_t = J_{xx} + bJ + d \quad \text{in} \quad \omega_T^+ \tag{2.14}$$

with coefficients given by

$$b = 2u - 2F' - xF'J + 2x^2 FF'' \tag{2.15}$$

and

$$d = x[2FF' + u^2F' - 2uF - F'h] - x^3F^2F''. \tag{2.16}$$

Assume now that $F(u)$ satisfies

$$F(0) = 0, \quad F > 0, \quad F' > 0, \quad F'' > 0 \quad \text{for} \quad u > 0. \tag{2.17}$$

It follows from (2.16) that

$$d \le xF^2 G(u,h) \quad \text{in} \quad \omega_T^+, \tag{2.18}$$

where

$$G(u,h) = \left[2\log(F) - \tfrac{u^2}{F}\right]' - \tfrac{F'}{F^2}h. \tag{2.19}$$

Since $F(0) = 0$, we have by (1.3) and (2.1) that $J \le 0$ on the lateral boundary of ω_T^+. Hence, by applying the Maximum Principle [11] to the linear parabolic equation (2.14), we conclude that

$$J \le 0 \quad \text{in} \quad \omega_T^+ \tag{2.20}$$

provided that

$$J(x,0) = u_0' + xF(u_0) \le 0 \quad \text{in} \quad I_+(0), \tag{2.21}$$

$$G(u,h) \le 0 \quad \text{for} \quad u > 0. \tag{2.22}$$

Set

$$F(u) = \tfrac{u^2}{4B_0 + 2\log(A_0+u^2)} \quad \text{for} \quad u \ge 0, \tag{2.23}$$

so that the function

$$F_0(u) = \tfrac{u^2}{4\log(u)} \equiv F(u)(1 + o(1)) \quad \text{as} \quad u \to \infty$$

is an approximate asymptotic solution of the algebraic equation

$$2\log F(u) - \tfrac{u^2}{F(u)} = 0.$$

This function, for $u \gg 1$, is the best possible (with respect to the growth as $u \to \infty$) solution of the ordinary differential inequality

$$[2\log(F) - \tfrac{u^2}{F}]' \le 0 \quad \text{for} \quad u \gg 1, \tag{2.24}$$

(cf. (2.22) with $h \equiv 0$; see [18]). Then, under the assumed hypothesis (1.5), we have that (2.12) is valid for any sufficiently large $A_0 > 1$. The same is true for assumptions (2.17).

Now, consider the inequality (2.22) with the function (2.23). By Proposition 2.3, we may suppose that $h(t,u) \ge h_0 > 0$ for every $t \in (0,T)$. Then, we have from (2.19) that $G(u,h) \le G(u,h_0)$ and substituting function (2.23) yields

$$G(u,h_0) \equiv \tfrac{4}{u(A_0+u^2)}\big[A_0 - 2F(u) - \tfrac{u(A_0+u^2)}{4}\tfrac{F'}{F^2}h_0\big].$$

Hence, since $F'/F^2 \ge 1/uF(u)$ for $u > 0$ if $A_0 \ge 3$ and $B_0 > 1$, we have that

$$G(u,h_0) \le \tfrac{1}{u(A_0+u^2)}[4A_0 - h_0(4B_0 + 2\log(A_0+u^2))] < 0 \quad \text{for} \quad u > 0,$$

provided that $B_0 > A_0/h_0$. Thus, (2.21), (2.22) with the function (2.23) are valid, and, hence, (2.20) yields (2.12), whence the result. □

As a straightforward consequence of (2.12), we now prove an explicit, uniform in time, upper bound on $u_+(x,t)$ which shows that u_+ blows up as $t \to T$ at the *single point* $x = 0$.

Corollary 2.5. *There exists a small constant* ϵ_0 *such that*

$$u_+(x,t) \le 16x^{-2}|\log(x)|\big[1 + \tfrac{\log(|\log(x)|)+B_0+\log(16)}{2|\log(x)|}\big] \quad \textit{in} \quad (0,\epsilon_0] \times (0,T). \tag{2.25}$$

Proof. For a fixed $t \in (0,T)$, integrating the inequality (2.12) over the interval $(0,x)$ yields the inequality

$$\tfrac{x^2}{8} \le \int_{u(x,t)}^{u(0,t)} z^{-2}\big(B_0 + \tfrac{1}{2}\log(A_0+z^2)\big)\,\mathrm{d}z = P(u(x,t)) - P(u(0,t)) \quad \text{in} \quad \omega_T^+, \tag{2.26}$$

where the function $P(z)$ is given by

$$P(z) = z^{-1}[\log(z) + (B_0+1)z^{-1}(1+o(1))], \tag{2.27}$$

with $A_0 \gg 1$, and is strictly decreasing for $z \gg 1$. Denote by $P^{-1}(s)$ the corresponding inverse function satisfying, for small $s > 0$,

$$P^{-1}(s) = \tfrac{1}{s}\big[\log\big(\tfrac{1}{s}\big) + \log\big(\log\big(\tfrac{1}{s}\big)\big) + (B_0+1)(1+o(1))\big]. \tag{2.28}$$

It then follows from (2.26) that

$$u(x,t) \le P^{-1}\big[\tfrac{x^2}{8} + P(u(0,t))\big]. \tag{2.29}$$

Hence, $u \le P^{-1}(x^2/8)$ and (2.25) follows from (2.38). □

Remark. Notice that (2.25) implies some estimate of the growth of $u(0,t)$ as $t \to T$. In particular, it can easily be seen that if $a \le (T-t)^{-n}u(0,t) \le A$ as $t \to T$ for some $n < 0$ with $a < A$ positive constants, then (2.25) and (2.7) yield that the unique possible value of the parameter n is $n = -1$. Indeed, if $n < -1$ then (2.25) cannot be valid, cf. asymptotic properties of solutions of the heat equation with blow-up boundary functions given in [30] (Chapter III).

By using (2.12) we can also derive an explicit lower estimate of $u(0,t)$.

Corollary 2.6. *There exists $t_1 \in (0,T)$ such that, on (t_1,T),*

$$u(0,t) > (T-t)^{-1}\big[1+\tfrac{1}{4}|\log(T-t)|^{-1} - \tfrac{B_0}{16}|\log(T-t)|^{-2}\big]. \tag{2.30}$$

Proof. Since $J(0,t) \equiv 0$, we conclude from (2.12) that

$$J_x = u_{xx} + xF'(u)u_x + F(u) \equiv u_{xx} + F(u) \le 0 \quad \text{for} \quad x=0, \quad t \in (0,T). \tag{2.31}$$

It follows from (1.1) that $u_{xx} = u_t - u^2 + h$ and, hence, (2.31) implies that

$$u_t \le u^2 - F(u) - h \le u^2 - F(u) \equiv u^2\big[1 - \tfrac{1}{4B_0 + 2\log(A+u^2)}\big] \tag{2.32}$$

for $x=0, t \in (0,T)$. Integrating this inequality over (t,T) and using (1.11) yield, as $t \to T$,

$$T-t \ge \int\limits_{u(0,t)}^{\infty} \tfrac{\mathrm{d}z}{z^2 - F(z)} \ge \int\limits_{u(0,t)}^{\infty} z^{-2}\big[1 + \tfrac{1}{4B_0 + 2\log(A+u^2)}\big]\,\mathrm{d}z.$$

By estimating the integral in the right-hand side we arrive at (2.30). □

We now derive a sharp evolution estimate for $u_+(x,t)$ as $t \to T$.

Proposition 2.7 *There exist $t_2 \in (0,T)$ and $K_1 > 0$ large enough such that, for any $t \in [t_2,T)$,*

$$u_+(x,t) \le u(0,t)\big[1 + \tfrac{\eta^2}{8}\big]^{-1}(1+\gamma(t)), \tag{2.33}$$

where

$$\eta = x\big[\tfrac{u(0,t)}{\log(u(0,t)}\big]^{\frac{1}{2}}, \quad \gamma(t) = \tfrac{3\log(\log(u(0,t)))}{\log(u(0,t))}, \tag{2.34}$$

on the set $\Omega_1(t) \subset [0,1)$ given by

$$0 \le \eta \le c_*(t) \equiv 2\sqrt{2}\big[K_1^{-1}\tfrac{u(0,t)}{\log(u(0,t))} - 1\big]^{\frac{1}{2}} \to \infty \quad as \quad t \to T, \tag{2.35}$$

and

$$u_+(x,t) \le K_2 = P^{-1}(K_1^{-1}) \quad on \quad [0,1] \setminus \Omega_1. \tag{2.36}$$

Proof. It follows from (2.7) and (2.27) that for any $t \in (t_2,T)$

$$P(u(0,t)) \ge \tfrac{\log(u(0,t))}{u(0,t)}, \tag{2.37}$$

provided that t_2 is close to T. Using now the fact, by (2.38),

$$P^{-1}(s) \le R(s) \equiv \tfrac{1}{s}\big[\log\big(\tfrac{1}{s}\big) + 2\log\big(\log\big(\tfrac{1}{s}\big)\big)\big] \quad \text{for} \quad 0 \le s \le 1/K_1, \tag{2.38}$$

where $K_1 > 0$ is large enough, and substituting (2.37), (2.38) into (2.29), we obtain that

$$u(x,t) \le R(Y^{-1}), \quad Y = \tfrac{u(0,t)}{\log(u(0,t))}\big[1 + \tfrac{\eta^2}{8}\big]^{-1} \tag{2.39}$$

provided that $Y \ge K_1$. Then (2.39) yields (2.33), and (2.36) then follows by the monotonicity (2.1). □

Remark. It follows from the lower estimate (2.7) that (2.33) is valid with

$$\eta = x/[(T-t)|\log(T-t)|]^{\frac{1}{2}}.$$

The following upper estimates of $\|u_+(t)\|_2^2$ and $\|u_+(t)\|_1$ are direct consequences of the above proposition.

Corollary 2.8. *There exists $t_3 \in (t_2, T)$ such that, on $[t_3, T)$,*

$$\|u_+(t)\|_2^2 \le \pi 2^{-\frac{1}{2}} \phi(t,u)(1+3\gamma(t)), \tag{2.40}$$

$$\|u_+(t)\|_1 \le \pi 2^{\frac{1}{2}} \psi(t,u)(1+\gamma(t)), \tag{2.41}$$

$$\phi(t,u) = [u^3(0,t)\log(u(0,t))]^{\frac{1}{2}}, \quad \psi(t,u) = [u(0,t)\log(u(0,t))]^{\frac{1}{2}}. \tag{2.42}$$

Proof. Since

$$\int_0^1 u_+^2 \, \mathrm{d}x = \int_{\Omega_1(t)} u_+^2 \, \mathrm{d}x + \int_{[0,1]\setminus\Omega_1(t)} u_+^2 \, \mathrm{d}x,$$

it follows from (2.33) and (2.34) that, as $t \to T$,

$$\int_{\Omega_1(t)} u_+^2 \, \mathrm{d}x \le \phi(1+\gamma)^2 \int_0^{c_*(t)} \left(1+\tfrac{\eta^2}{8}\right)^{-2} d\eta \le \phi\left(1+\tfrac{5}{2}\gamma\right)\tfrac{\pi\sqrt{2}}{2},$$

$$\int_{[0,1]\setminus\Omega_1(t)} u_+^2 \, \mathrm{d}x \le K_2^2(1-\mathrm{meas}(\Omega_1)) \le K_2^2,$$

whence (2.40). The proof of (2.41) is similar. □

Notice that, since $\|u_+\|_1 \equiv \|u_-\|_1$ by (1.7), from (2.6), (2.41), we have the estimate on $[t_3, T)$:

$$\|u(t)\|_1 \le \pi 2^{\frac{1}{2}} \psi(t,u)(1+\gamma(t)). \tag{2.43}$$

2.3. **Estimates in $\bar{\omega}_T$.** First, we notice that, if $u_-(x,t)$ is bounded from below, i.e.,

$$\inf_{\omega_T} u(x,t) = -C_1 > -\infty, \tag{2.44}$$

then for any $t \in [0,T)$

$$\|u_-(t)\|_2^2 \le C_1^2. \tag{2.45}$$

(We will denote by $C_i > 0$ different positive constants.) Hence, by (2.6), (2.40) we arrive at the following estimate of the non-local term:

$$h(t,u) \le \pi 2^{-\frac{1}{2}} \phi(t,u)(1+4\gamma(t)) \quad \text{on} \quad [t_3, T). \tag{2.46}$$

We will prove that (2.46) is always valid, including the case

$$\inf_{\omega_T} u(x,t) = -\infty. \tag{2.47}$$

We begin with the following preliminary result:

Proposition 2.9 *Assume that (2.47) holds. Then, as* $t \to T$,

(2.48) $$u(x,t) \to -\infty \quad \textit{uniformly on any compact subset} \quad [\delta,1], \quad \delta \in (0,1),$$

and

(2.49) $$x_0(t) \to 0 \quad as \quad t \to T.$$

Proof. Firstly, we notice from evaluating equation (1.1) at the point $x = 1$ that

$$u_t = u_{xx} + u^2 - h(t,u) \geq -h(t,u),$$

and hence

(2.50) $$u(1,t) \geq u(1,0) - \int_0^t h(s,u)\,\mathrm{d}s \quad \text{on} \quad (0,T).$$

Therefore, we deduce from (2.3) that, if assumption (2.47) is valid, then

(2.51) $$\int_0^T h(s,u)\,\mathrm{d}s = +\infty.$$

Now, fix an arbitrarily small $\epsilon > 0$ and a large constant $M_\epsilon > 0$ such that, by Corollary 2.5,

(2.52) $$u(x,t) \leq M_\epsilon \quad \text{in} \quad [\epsilon,1] \times [0,T).$$

Consider the function

(2.53) $$\bar{u}(x,t) = M_\epsilon - \rho(t) w_\epsilon(x), \quad w_\epsilon(x) = 1 - \tfrac{(x-1)^2}{(1-\epsilon)^2},$$

where a suitable function $\rho > 0, \rho' > 0$ on some small interval (t_ϵ, T), $\rho(t_\epsilon) = 0$, will be determined later. The function $\bar{u}$ satisfies

(2.54) $$\bar{u}(\epsilon,t) = M_\epsilon, \bar{u}_x(1,t) = 0 \quad \text{on} \quad [t_\epsilon, T), \quad \bar{u}(x,t_\epsilon) = M_\epsilon \quad \text{on} \quad [\epsilon,1].$$

We now choose the function ρ so that

(2.55) $$\bar{u}_t \geq \bar{u}_{xx} + \bar{u}^2 - h(t,u) \quad \text{in} \quad q_\epsilon = (\epsilon,1) \times (t_\epsilon, T).$$

(Of course $\bar{u}$ is not an upper solution to (1.1) which cannot be constructed independently.) Substituting (2.53) into (2.55) yields

$$-\rho' w_\epsilon \geq \tfrac{2\rho}{(1-\epsilon)^2} + (M_\epsilon - \rho w_\epsilon)^2 - h(t,u),$$

and, hence, it is valid if $\rho(t)$ solves the ordinary differential equation

(2.56) $$\rho' = -\tfrac{2\rho}{(1-\epsilon)^2} - \max\{M_\epsilon^2), (\rho^2\} + h(t,u) \equiv T_\epsilon(\rho,t).$$

By Proposition 2.3, $h(t,u)$ is increasing as $t \to T$. We have that $T_\epsilon(\rho,t) = 0$ on the zero curve $\{\rho = \rho_0(t), t \in (0,T)\}$ given by

$$\rho_0(t) = \tfrac{(1-\epsilon)^2}{2}[h(t,u) - M_\epsilon^2] \quad \text{for} \quad 0 \leq t \leq t_\epsilon^*,$$

(2.57) $$\rho_0(t) = [h(t,u) + (1-\epsilon)^{-4}]^{\frac{1}{2}} - (1-\epsilon)^{-2} \quad \text{for} \quad t_\epsilon^* \leq t \leq T,$$

where t_ϵ^* is the unique root of the equation

(2.58) $$h(t,u) = M_\epsilon^2 + \tfrac{2M_\epsilon}{(1-\epsilon)^2} \gg 1.$$

Then, $\rho_0(t)$ is strictly increasing on $[t^*_\epsilon, T)$. Let $\bar{t}_\epsilon$ be the unique root of the equation $h(t,u) = M_\epsilon$. Fix an arbitrary $t_\epsilon \in (\bar{t}_\epsilon, T)$. Then by (2.51), the solution $\rho(t)$ of equation (2.56) on (t_ϵ, T) with the boundary condition $\rho(t_\epsilon) = 0$ satisfies

$$\rho' > 0 \quad \text{on} \quad (t_\epsilon, T), \quad \rho \to +\infty \quad \text{as} \quad t \to T. \tag{2.59}$$

Finally, since $\bar{u}(x, t_\epsilon) \equiv M_\epsilon$ it follows from (2.55), (2.52), (2.54) that, by the Maximum Principle [11],

$$u \le \bar{u} \quad \text{in} \quad q_\epsilon. \tag{2.60}$$

It then follows from (2.53), (2.60) that, as $t \to T$,

$$\bar{u}(x,t) \to -\infty \quad \text{uniformly on} \quad [2\epsilon, 1], \tag{2.61}$$

and hence (2.48) with $\delta = 2\epsilon$ follows from (2.61). Condition (2.61) also implies that there exists a $\tilde{t}_\epsilon \in (0, T)$ such that $\bar{u}(2\epsilon, t) \le 0$ for $t \in (\tilde{t}_\epsilon, T)$. Then (2.1), (2.60) yield that $x_0(t) \le 2\epsilon$ on $(\tilde{t}_\epsilon, T)$. Since $\epsilon > 0$ is arbitrary, we obtain (2.49), completing the proof. □

We now give a more precise description of the "flat" behaviour of $u(x,t)$ as $t \to T$ on compact subsets bounded away from zero. We first need to establish a rough estimate of $h(t,u)$.

Proposition 2.10 *There exists $t_4 \in (0, T)$ such that*

$$|u(1,t)| \le \sup_{s \in (0,t)} u(0,s) \equiv \bar{u}(0,t) \quad \textit{on} \quad [t_4, T), \tag{2.62}$$

and hence

$$h(t,u) \le \bar{u}^2(0,t) \quad \textit{on} \quad [t_4, T). \tag{2.63}$$

Proof. It follows from (2.1) that

$$h(t,u) \le \max\{u^2(0,t), u^2(1,t)\} \quad \text{on} \quad [0,T), \tag{2.64}$$

and hence equation (1.1) at the point $x = 1$ yields that $\mu_1(t) \equiv -u(1,t) > 0$ satisfies

$$\mu_1' \le -\mu_1^2 + \max\{u^2(0,t), \mu_1^2\}, \quad \mu_1(0) = -u_0(1) > 0.$$

By comparison, $\mu_1(t) \le \rho(t)$ where ρ solves

$$\rho' = -\rho^2 + \max\{u^2(0,t), \rho^2\}, \quad \rho(0) = -u_0(1). \tag{2.65}$$

Therefore $\rho(t) \le \bar{u}(0,t)$ on $[t_4, T)$, where t_4 is such that $|u_0(1)| = u^2(0, t_4)$ (if $|u_0(1)| \le u^2(0,t)$ for every $t \in [0, T)$ then $t_4 = 0$). Hence, (2.62) holds and (2.63) follows from (2.64). □

Remark: on some properties of solution on F_0^+. It is interesting to notice that sharper estimates than (2.62), (2.63) are always valid if $u(.,t) \in F_0^+ = \{v \in F_0 : v \text{ satisfies (1.8), (1.9)}\}$. It was formally shown in [6] that if $v_0 \in F_0^+$ then $u(\cdot, t) \in F_0^+$ for any $t \in (0, T)$. The proof of such a *weak maximum principle* is based on a finite-dimensional approximation of the infinite-dimensional dynamical system

$$C_n' = -n^2\pi^2 C_n + \sum_{m=-\infty}^{\infty} C_m C_{n-m}, \quad C_n(0) = C_{n0}, \quad n \in (-\infty, \infty), \tag{2.66}$$

which is equivalent to the equation (1.1), where C_n are the coefficients of the Fourier expansion

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) \mathrm{e}^{in\pi x}, \tag{2.67}$$

and from the Neumann boundary conditions, we conclude that $C_n(t) \equiv C_{-n}(t)$ for any n. Now, using the Galerkin approximation

$$u_N(x,t) = \sum_{n=-N}^{N} C_n^{(N)}(t) \mathrm{e}^{in\pi x},$$

where $\{C_n^{(N)}\}$ solves the system (2.66) with $C_n^{(N)} \equiv 0$ for $|n| > N$, we have from (1.9) that, by comparison for ODE's, $C_n^{(N)} \geq 0$. Since by known results $u_N \to u$ as $N \to \infty$ in the space $L^\infty((0,T') : H^1((0,1)))$, $T' < T$ [28], the weak convergence $u_N(\cdot,t) \to u(\cdot,t)$ in $L^2((0,1))$ implies that, for any fixed $t \in (0,T)$, $C_n^{(N)} \to C_n(t)$ as $N \to \infty$, whence

$$C_n(t) \geq 0 \quad \text{for any} \quad n \quad \text{and} \quad t \in (0,T). \tag{2.68}$$

Therefore, we have from (2.68) that, for any $t \in [0,T)$,

$$|u(1,t)| = 2\Big|\sum_{n=1}^{\infty}(-1)^n C_n\Big| \leq u(0,t) = 2\Big|\sum_{n=1}^{\infty} C_n\Big|,$$

$$\int_0^1 u^2 \, \mathrm{d}x = 2\sum C_n^2 \leq \tfrac{1}{2}u^2(0,t) = 2\big(\sum C_n\big)^2. \tag{2.69}$$

The above weak maximum principle is a direct consequence of the *comparison* result for the equation (1.1) on F_0^+. Indeed, if $u(\cdot,t)$ and $\bar{u}(\cdot,t) \in F_0^+$ with coefficients $\{C_n(t)\}$ and $\tilde{C}_n(t)\}$ given by (2.67) are two different solutions to (2.66) and $C_n(0) \geq \tilde{C}_n(0)$ for any n, then $C_n(t) \geq \tilde{C}_n(t)$ for all n and $t > 0$. In order to prove that, we notice that the difference $\{B_n(t) \equiv C_n(t) - \tilde{C}_n(t)\}$ satisfies a system of similar type to (2.66), with non-negative coefficients in the quadratic terms

$$B_n' = -n^2\pi^2 B_n + \sum(B_n C_{n-m} + \tilde{C}_m B_{n-m}) \tag{2.70}$$

and $B_n(0) \geq 0$. By using a similar Galerkin approximation of the solution $w(x,t) \equiv u(x,t) - \tilde{u}(x,t)$ to the linear parabolic equation for this difference, $w_t = w_{xx} + (u+\tilde{u})w$, after passing to the limit, we obtain that $B_n(t) \geq 0$ for any n, whence the result.

Notice also that the same construction can be used to prove some *monotonicity* results on F_0^+. Namely, given $u \in F_0^+$ set $B_n = C_{n+1} - C_n$ for $n \geq 0$. Then $\{B_n\}$ solves the problem (cf. (2.70))

$$B_n' = -n^2\pi^2 B_n - (1+2n)\pi^2 C_{n+1} + \sum C_m B_{n-m},$$

and since $C_n \geq 0$ we have that

$$B_n' \leq -n^2\pi^2 B_n + \sum C_m B_{n-m}.$$

Assume now that $B_n(0) \leq 0$ for all $n \geq 0$. Then, by comparison, we may conclude that $B_n \leq \tilde{B}_n$, where $\{\tilde{B}_n\}$ solves the system

$$\tilde{B}_n' = -n^2\pi^2 \tilde{B}_n + \sum C_m \tilde{B}_{n-m} \tag{2.71}$$

with the same non positive initial data. Hence, by the weak maximum principle for (2.71), which is proved by the Galerkin approximation $\tilde{w}(x,t)$ to the parabolic equation (cf. (2.71)) $\tilde{w}_t = \tilde{w}_{xx} + u\tilde{w}$, we conclude that $\tilde{B}_n(t) \le 0$. Thus, we have that for a given $u(\cdot,t) \in F_0^+$ the sequence

(2.72) $$\{C_n(t)\} \quad \text{is decreasing in} \quad n \ge 0 \quad \text{for any fixed} \quad t > 0$$

provided that $\{C_n(0)\}$ is decreasing. Hence, under such an assumption the first coefficients which blow up are $C_1(t)$ and $C_2(t)$. Unfortunately, in this case the ordinary differential inequalities

$$C_1' \ge -\pi^2 C_1 + C_1 C_2, \quad C_2' \ge -4\pi^2 C_2 + C_1^2 \quad \text{for} \quad t > 0$$

established in [6] for $u \in F_0^+$ under assumptions (1.10) do not give a suitable upper bound for the rate of growth of $u(0,t)$ as $t \to T < \infty$. In particular, if by (2.72) $C_1 \ge C_2$, it can be easily proved that $C_2(t) \le \text{const.}(T-t)^{-1}$. □

We now continue to study the case of (2.47). Hypothesis (1.24) implies that

(2.73) $$u(0,t) \le C_2(T-t)^{-1} \quad \text{on} \quad [0,T)$$

and hence, by Proposition 2.10,

(2.74) $$|u(1,t)| \le C_3(T-t)^{-1},$$

(2.75) $$h(t,u) \le C_4(T-t)^{-2} \quad \text{on} \quad [0,T).$$

Denote now by $\rho(t)$ the solution to the ordinary differential equation

(2.76) $$\rho' = -\rho^2 + h(t,u) \quad \text{on} \quad (0,T), \quad \rho(0) = 0.$$

We have, from Proposition 2.3 and equation (1.12), that

(2.77) $$\rho(t) > 0, \quad \rho'(t) > 0, \quad \rho(t) < h^{\frac{1}{2}}(t,u) \quad \text{as} \quad t \to T.$$

Therefore, if (2.73) holds then by (2.77)

(2.78) $$\rho(t) \le C_5(T-t)^{-1} \quad \text{on} \quad [0,T).$$

Notice again that (2.51) is valid under the assumption (2.47) and then equation (2.76) yields

(2.79) $$\rho(t) \to +\infty \quad \text{as} \quad t \to T.$$

We now state the result on the *flat behaviour* of $u(x,t)$ in ω_T^-.

Lemma 2.11 *Assume that (2.47) and (2.73) hold. Then uniformly on any compact subset* $[\delta, 1]$ *with arbitrarily small* $\delta > 0$

(2.80) $$u(x,t) = -\rho(t)\big[1 + O_\delta\big(\tfrac{1}{\rho(t)}\big)\big] \quad \text{as} \quad t \to T,$$

and

(2.81) $$|u_x| + |u_{xx}| \le C_6, \quad |u_t| \le \rho'(t) + C_7 \quad \text{as} \quad t \to T.$$

Proof. Set

$$u(x,t) = -\rho(t) - U(x,t) \quad \text{in} \quad \omega_T^-.$$

Then, using (2.76), we deduce that U solves the problem

$$U_t = U_{xx} - U(U+2\rho) \quad \text{in} \quad \omega_T^-; \tag{2.82}$$

$$U = -\rho(t) \quad \text{for} \quad x = x_0(t), \quad U_x = 0 \quad \text{for} \quad x = 1, \quad t \in [0,T); \tag{2.83}$$

$$U = -u_0(x) \quad \text{for} \quad x \in \big[x_0(0), 1\big], \quad t = 0. \tag{2.84}$$

Notice that by (2.1)

$$U_x > 0 \quad \text{and} \quad U \geq -\rho(t) \quad \text{in} \quad \omega_T^-. \tag{2.85}$$

Then the lower order term $Q(U,t) \equiv -U(U + 2\rho(t))$ in (2.82) satisfies

$$Q(U,t) \leq 0 \quad \text{for} \quad U \geq 0, \qquad Q(U,t) \geq 0 \quad \text{for} \quad U \leq 0, \tag{2.86}$$

and hence, by the Maximum Principle [11], it follows from (2.82), (2.85), (2.86) that $U(x,t)$ is uniformly bounded from above,

$$U \leq C_8 \quad \text{in} \quad \omega_T^-. \tag{2.87}$$

We now prove the lower bound on U. Fix an arbitrary small $\delta > 0$. From (2.49) we deduce that there exists a $t_\delta \in (0,T)$ such that $x_0(t) \in (0, \delta/2)$ on $[t_\delta, T)$. Now, consider the function $\underline{U}(x,t)$ satisfying

$$\underline{U}_t = \underline{U}_{xx} \quad \text{in} \quad q_\delta = (\delta/2, \infty) \times (t_\delta, T), \tag{2.88}$$

$$\underline{U} = -\rho(t) \quad \text{for} \quad x = \delta/2, \quad t \in (t_\delta, T), \tag{2.89}$$

$$\underline{U} = \underline{U}_\delta(x) \quad \text{for} \quad x \in [\delta/2, \infty), \quad t = t_\delta, \tag{2.90}$$

where the initial function is assumed to satisfy

$$\underline{U}_\delta' < 0 \quad \text{in} \quad [\delta/2, \infty), \quad \underline{U}_\delta(\delta/2) = -\rho(t_\delta), \quad \underline{U}_\delta(+\infty) = -2\rho(t_\delta). \tag{2.91}$$

Since, by (2.77) $\underline{U} \leq -\rho(t_\delta)$ on the parabolic boundary of q_δ, by the Maximum Principle [11]

$$\underline{U} \leq -\rho(t_\delta) < 0 \quad \text{in} \quad q_\delta. \tag{2.92}$$

It follows from (2.83), (2.84) that

$$\underline{U}_\delta(x) \leq U(x, t_\delta) \quad \text{on} \quad [\delta/2, 1]. \tag{2.93}$$

By the linear superposition principle, $\underline{U} = \underline{U}_1 + \underline{U}_2$, where

$$\underline{U}_1(x,t) = -(4\pi)^{-\frac{1}{2}} y \int_{t_\delta}^{t} \exp\Big(-\frac{y^2}{4(t-\tau)}\Big) \frac{\rho(\tau)}{(t-\tau)^{\frac{3}{2}}}\, \mathrm{d}\tau, \tag{2.94}$$

$$\underline{U}_2(x,t) = (4\pi t')^{-\frac{1}{2}} \int_0^\infty \Big[\exp\Big(-\frac{(y-y')^2}{4t'}\Big) - \exp\Big(-\frac{(y+y')^2}{4t'}\Big)\Big] \underline{U}_\delta\big(y' + \tfrac{\delta}{2}\big)\, \mathrm{d}y', \tag{2.95}$$

with $y = x - \delta/2$, $t' = t - t_\delta$, are the solutions to (2.88) with zero initial and boundary data respectively. It follows from (2.91), (2.95) that

$$-2\rho(t_s) \leq \underline{U}_2 \leq 0, \quad (\underline{U}_2)_x < 0 \quad \text{in} \quad [\delta/2, 1] \times [t_\delta, T) = \Omega_\delta. \tag{2.96}$$

The function $\underline{U}_1$ is uniformly bounded from below in $[\delta, 1)\times[t_\delta, T)$. Indeed, we have from (2.78), (2.94) that

$$0 \geq \underline{U}_1 \geq F(x,t) \equiv -(4\pi)^{-\frac{1}{2}} C_5 (x-\delta/2) \int_{t_\delta}^{t} \exp\Big(-\frac{(x-\delta/2)^2}{4(t-\tau)}\Big)\frac{(T-\tau)^{-1}}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\tau, \tag{2.97}$$

where the function F satisfies

$$F(x,t) \geq F(x,T) \quad \text{in} \quad \Omega_\delta. \tag{2.98}$$

It follows from (2.97) that (see [30] (Chapter III))

$$F(x,T) \geq -C_9|x-\delta/2|^{-2} \quad \text{on} \quad (\delta/2, \delta), \tag{2.99}$$

$$|F(x,T)| \leq C_{10}(t_\delta), \quad |F_x(x,T)| \leq C_{11}(t_\delta), \tag{2.100}$$

where the constants C_{10} and C_{11} depending on t_δ satisfy

$$C_{10}(t_\delta), \quad C_{11}(t_\delta) \to 0 \quad \text{as} \quad t_\delta \to T. \tag{2.101}$$

By using (2.96)–)2.101), we conclude that

$$\underline{U}_x(1,t) < 0 \quad \text{on} \quad [t_\delta, T) \tag{2.102}$$

provided t_δ is close enough to T and that $|\underline{U}'_\delta|$ is large on $[\delta/2, 2]$. It also follows from (2.91) that $\underline{U} \geq -2\rho$ in Ω_δ. Hence, we deduce from (2.92) and (2.86) that $Q(\underline{U}, t) \geq 0$, i.e.,

$$\underline{U}_t \leq \underline{U}_{xx} - \underline{U}(\underline{U} + 2\rho) \quad \text{in} \quad \Omega_\delta. \tag{2.103}$$

Then, by the Maximum Principle [11], $U \geq \underline{U} \equiv \underline{U}_1 + \underline{U}_2$ in Ω_δ and therefore (2.96) and (2.98)–(2.100) yield that

$$U \geq -C_{12} \quad \text{in} \quad [\delta, 1]\times[t_\delta, T) = \Omega'_\delta. \tag{2.104}$$

Finally, since $u + \rho = -U$, by (2.87) and (2.104), we have that

$$|u(x,t) + \rho(t)| \leq C_{13} \quad \text{in} \quad \Omega'_\delta, \tag{2.105}$$

where $C_{13} > 0$ depends on $\delta \in (0,1)$, and (2.80) follows.
The uniform estimate of the first derivative

$$|u_x| \leq C_6 \quad \text{in} \quad \Omega'_\delta \tag{2.106}$$

follows from a similar analysis of the semilinear parabolic equation for $u_x \equiv v$,

$$v_t = v_{xx} + 2uv, \tag{2.107}$$

First, since $v \leq 0$, using (2.73) we have that $v_t \geq v_{xx} + 2C_2(T-t)^{-1}v$ in ω_T and hence $\inf_x v(x,t) \geq -(T-t)^{-\alpha_1}$ as $t \to T$ with $\alpha_1 = 2C_2$. It then follows that $u \leq 0$ in Ω_δ and hence, by (2.107), $v_t \geq v_{xx}$ in Ω_δ with $v \geq -(T-t)^{-\alpha_1}$ for $x = \delta/2$. Hence, by the same comparison as for the problem (2.82)–(2.84), we have that v is uniformly bounded from below in Ω'_δ. In order to finish the proof of (2.81) we consider the equation for $u_{xx} \equiv w$,

$$w_t = w_{xx} + 2uw + 2(u_x)^2. \tag{2.108}$$

Since $w_x \equiv 0$ for $x = 1$ and $u \approx -\rho(t)$ and $(u_x)^2$ are bounded on Ω'_δ, we have from (2.108) that w is bounded from above. Since by (2.73) $\inf_x w(x,t) \geq -(T-t)^{-\alpha_1}$ as $t \to T$, by the same comparison technique in Ω_δ as above, we conclude that w is bounded from below in Ω'_δ. The last estimate in (2.81) then follows from (2.80) and (2.76):

$$|u_t| \equiv |u_{xx} + u^2 - h| \le C_6 + \rho^2\big(1 + O\big(\tfrac{1}{\rho}\big)\big) - h \le \rho' + C_7.$$

This completes the proof of Lemma 2.11. □

We now state the main sharp upper estimate on $h(t,u)$.

Lemma 2.12 *Assume that (2.73) holds. Then, as $t \to T$,*

$$h(t,u) \le \pi 2^{-\frac{1}{2}}[u^3(0,t)\log u(0,t)]^{\frac{1}{2}}(1 + 4\gamma(t)), \tag{2.109}$$

where $\gamma(t)$ is given in (2.34),

$$h(t,u) \le C_{14}(T-t)^{-\frac{3}{2}}|\log(T-t)|^{\frac{1}{2}}, \tag{2.110}$$

and also

$$u(x,t) \ge -\pi 2^{\frac{1}{2}}[u(0,t)\log(u(0,t)]^{\frac{1}{2}}(1 + o(1)), \tag{2.111}$$

$$u(x,t) \ge -C_{15}(T-t)^{-\frac{1}{2}}|\log(T-t)|^{\frac{1}{2}}(1 + o(1)) \tag{2.112}$$

uniformly on any compact subset $[\delta, 1], \delta > 0$.

Proof. We have shown that if (2.44) holds then these estimates follow from Corollary 2.8. Assume now that (2.47) is valid. Then using (2.80) yields that as $t \to T$

$$\|u_-(t)\|_1 = \rho(t)(1 + o(t)). \tag{2.113}$$

Since $\|u_-\|_1 \equiv \|u_+\|_1$, we conclude from (2.113) and (2.41) that

$$\rho(t) \le \pi 2^{\frac{1}{2}}[u(0,t)\log(u(0,t))]^{\frac{1}{2}}(1 + o(1)). \tag{2.114}$$

Hence, by (2.80), as $t \to T$,

$$\|u_-(t)\|_2^2 = \rho^2(t)(1 + o(1)) \le 2\pi^2 u(0,t)\log(u(0,t))(1 + o(1)), \tag{2.115}$$

and (2.109) follows from (2.40). Estimate (2.110) is then the result of (2.73) . Uniform estimates (2.111), (2.112) follow from (2.114) and (2.80). □

3. Analysis near the origin: the first limit

In this section, we study the behaviour of $u(x,t)$ near to the origin corresponding to the rescaled variables

$$\theta(\xi,\tau) = (T-t)u(\xi(T-t)^{\frac{1}{2}}, t), \quad \xi = x(T-t)^{-\frac{1}{2}}, \tag{3.1}$$

and $\tau = -\log(T-t) : [0,T) \to [\tau_0, \infty), \quad \tau_0 = -\log(T)$. Then the rescaled function θ solves the equation

$$\theta_\tau = B_1(\theta) - \psi(\tau,\theta) \equiv \theta_{\xi\xi} - \tfrac{1}{2}\theta_\xi \xi - \theta + \theta^2 - \psi(\tau,\theta) \tag{3.2}$$

in $q_0 = B_{l(\tau)} \times (\tau_0, \infty), B_{l(\tau)} = \{0 < \xi < l(\tau) \equiv \mathrm{e}^{\frac{\tau}{2}}\}$, where ψ denotes the non-local term

$$\psi(\tau,\theta) = \mathrm{e}^{-\frac{\tau}{2}} \int_0^{l(\tau)} \theta^2(\xi,\tau)\, \mathrm{d}\xi. \tag{3.3}$$

We shall determine the large τ behaviour of θ by using Lyapunov type techniques. The boundary and initial conditions for θ have the form

$$\theta_\xi = 0 \quad \text{for} \quad \xi = 0, l(\tau) \quad \text{and} \quad \tau \geq \tau_0, \tag{3.4}$$

$$\theta(\xi, \tau_0) = \theta_0(\xi) \quad \text{in} \quad \big[0, l(\tau_0)\big]. \tag{3.5}$$

Under the hypothesis (1.5), we have that (cf. (1.7), (2.1), and (2.7))

$$\theta_\xi < 0 \quad \text{in} \quad q_0, \quad \int \theta \, \mathrm{d}\xi = 0 \quad \text{and} \quad \theta(0, \tau) > 1 \quad \text{for} \quad \tau \geq \tau_0. \tag{3.6}$$

We now state the main result of this section which looks quite similar to those which have been proven for the semilinear heat equation (1.15), cf. [12], [16], [18], [14].

Lemma 3.1 *Assume (2.73) holds. Then, as $\tau \to \infty$,*

$$\theta(\xi, \tau) \to 1 \quad \textit{uniformly on compact subsets in} \quad \xi. \tag{3.7}$$

Firstly, we give some estimates on θ. It follows from (3.6), (2.73), and Lemmas 2.11, 2.12 that, for large $\tau \gg 1$, say, for $\tau > \tau^*$,

$$0 < \psi(\tau, \theta) \leq C_{14} \tau^{\frac{1}{2}} \mathrm{e}^{-\frac{\tau}{2}} \tag{3.8}$$

and

$$-C_{15} \tau^{\frac{1}{2}} \mathrm{e}^{-\frac{\tau}{2}} \leq \theta \leq C_2 \quad \text{in} \quad q_* = (0, \mathrm{e}^{\frac{\tau}{2}}) \times (\tau_*, \infty). \tag{3.9}$$

Estimates (2.81) and (2.112) yield

$$|\theta| \leq C_{15} \tau^{\frac{1}{2}} \mathrm{e}^{-\frac{\tau}{2}}, \quad |\theta_\xi| \leq C_6 \mathrm{e}^{-3\frac{\tau}{2}}, \quad |\theta_{\xi\xi}| \leq C_6 \mathrm{e}^{-2\tau} \quad \text{near to} \quad \xi = \mathrm{e}^{\frac{\tau}{2}}, \tag{3.10}$$

and since by (2.76), (2.110) $\rho' \leq C_{14}(T-t)^{-\frac{3}{2}} |\log(T-t)|^{\frac{1}{2}}$, we have from (2.81) that $\theta_\tau \equiv (T-t)^2 u_t - \theta - \frac{1}{2} \theta_\xi \xi$ satisfies

$$|\theta_\tau| \leq C_{16} \tau^{\frac{1}{2}} \mathrm{e}^{-\frac{\tau}{2}} \quad \text{near to} \quad \xi = \mathrm{e}^{\frac{\tau}{2}} \quad \text{for} \quad \tau > \tau_*. \tag{3.11}$$

By using known regularity results (see [11]) for linear uniformly parabolic equations satisfied by the derivatives θ_ξ and $\theta_{\xi\xi}$, we deduce using (3.10) that

$$|\theta_\xi| + |\theta_{\xi\xi}| \leq C_{17} \quad \text{in} \quad q_*. \tag{3.12}$$

Hence it follows from the equation (3.12) that

$$|\theta_\tau| \leq C_{18}(1 + \xi) \quad \text{in} \quad q_*. \tag{3.13}$$

We now introduce the $\omega-$limit set to problem (3.2)-(3.5):

$$\omega\big(\theta_0\big) = \{f \in C(R_+) : \exists \{\tau_k\} \to \infty \quad \text{such that} \tag{3.14}$$

$$\theta(\cdot, \tau_k) \to f(\cdot) \quad \text{as} \quad k \to \infty \quad \text{uniformly on any compact subset} \quad [0, L]\}.$$

Then (3.9) and (3.12) yield that $\omega\big(\theta_0\big) \neq \emptyset$. One can see from (3.6) and (3.9) that

$$\text{if} \quad f \in \omega\big(\theta_0\big) \quad \text{then} \quad f' \leq 0 \quad \text{and} \quad f \geq 0 \quad \text{in} \quad R_+, \quad f'(0) = 0, \quad f(0) \in [1, C_2]. \tag{3.15}$$

Lemma 3.1 is equivalent to the equality

(3.16) $$\omega\big(\theta_0\big) = \{f \equiv 1\},$$

which is a straightforward consequence of the following:

Proposition 3.2 *There holds*

(3.17) $$\int\limits_{\tau_*}^{\infty} \mathrm{d}s \int\limits_0^{\mathrm{e}^{s/2}} \bar\rho \theta_\tau^2(s)\, \mathrm{d}\xi < \infty \quad (\bar\rho(\xi) = \mathrm{e}^{-\xi^2/4}).$$

Proof. As in [15], [16], and [13], we write (3.2) in the divergence form

(3.18) $$\bar\rho\theta_\tau = (\bar\rho\theta_\xi)_\xi + \bar\rho(\theta^2 - \theta) - \bar\rho\psi.$$

Multiplying expression (3.18) by θ_τ and integrating over $(0, l(\tau))$ yields

(3.19) $$\int \bar\rho\theta_\tau^2\, \mathrm{d}\xi = \tfrac{\partial}{\partial\tau}\big\{ \int \bar\rho\big[-\tfrac12\theta_\xi^2 + \tfrac{\theta^3}{3} - \tfrac{\theta^2}{2}\big]\big\} + I_1,$$

where in the last term

$$I_1(\tau) = -\psi(\tau)\int \bar\rho\theta_\tau\, \mathrm{d}\xi - \tfrac12 \mathrm{e}^{\frac{\tau}{2}}\big\{\bar\rho\big[\tfrac{\theta^3}{3} - \tfrac{\theta^2}{2}\big]\big\}\big|_{\xi=\mathrm{e}^{\frac{\tau}{2}}}$$

satisfies $|I_1| \le C_{19}\tau^{\frac12}\mathrm{e}^{-\frac{\tau}{2}} \in L^1((\tau_*, \infty))$, see (3.8),(3.9) and (3.13). Integrating (3.19) over the interval (τ_*, S) using (3.19) and passing to the limit $S \to \infty$ yields (3.17). $\square$

Proof of Lemma 3.1. Denote by W_S^1 the set of stationary solutions $g(\xi)$ satisfying properties (3.15) of the equation $B_1(g) = 0$ in R_+ where B_1 is given in (3.2). Then passing to the limit $\tau \to \infty$ in (3.2) and using (3.8),(3.17) yield

(3.20) $$\omega\big(\theta_0\big) \subset W_S^1,$$

cf. [27] and [4]), ([17]. It was proved in [1] (see also [13]) that

(3.21) $$W_S^1 = \{f \equiv 1\},$$

and hence both (3.16) and (3.7) follow from (3.20), (3.21) which completes the proof. $\square$

4. Improved analysis close to the origin: the second limit

According to Lemma 3.1, we introduced the new rescaled function

(4.1) $$z(\xi, \tau) = \tau(\theta(\xi, \tau) - 1) \quad \text{in} \quad q_*,$$

which solves the following equation:

(4.2) $$z_\tau = B_2 z + \tfrac{1}{\tau} C_2(z) - \tilde\psi(\tau, z),$$

where B_2 is the linear stationary operator

(4.3) $$B_2 z = \frac{1}{\bar\rho}\big[(\bar\rho z_\xi)_\xi + \bar\rho z\big],$$

and

(4.4) $$C_2(z) = (z^2 + z), \quad \tilde\psi(\tau, z) \equiv \tau\psi(\tau, 1 + z/\tau).$$

The function z satisfies the boundary conditions

$$z_t = 0 \quad \text{for} \quad \xi = 0, l(\tau) \quad \text{and} \quad \tau \geq \tau_*, \tag{4.5}$$

and some initial condition $z(\xi, \tau_*) = z_*(\xi)$. From (3.6) $z_\xi < 0$ in q_*. It follows from (1.24) and (2.30) that

$$\tfrac{1}{4}\big[1 - \tfrac{B_0}{4\tau}\big] \leq z(0, \tau) \leq C_{20} \quad \text{for} \quad \tau \geq \tau_*. \tag{4.6}$$

It is in the upper bound in (4.6) we use for the first time the hypothesis (1.24). Notice that for the equation (4.2) this hypothesis is equivalent to the following theorem which looks like a typical result in the theory of quasilinear parabolic equations: *any global solution to the problem (4.2)–(4.5) is uniformly bounded from above.* For equations of the type (4.2) with $\bar{\psi} \equiv 0$ arising after rescaling the semilinear parabolic equations (1.15) and (1.16), this result has been proven (see the references in the Introduction). For the equation (4.2) with the non-zero non-local term this problem remains open.

It follows from the estimates (3.8)–(3.13) that, for $\tau > \tau_*$,

$$0 < \tilde{\psi}(\tau, z) \leq C_{14} \tau^{\frac{3}{2}} \mathrm{e}^{-\frac{\tau}{2}}, \tag{4.7}$$

$$-2\tau \leq z \leq C_{20} \quad \text{in} \quad q_*, \tag{4.8}$$

$$|z_\xi| \leq C_6 \tau \mathrm{e}^{-3\frac{\tau}{2}}, \quad |z_{\xi\xi}| \leq C_6 \tau \mathrm{e}^{-2\tau} \quad \text{near} \quad \xi = \mathrm{e}^{\frac{\tau}{2}}, \tag{4.9}$$

$$z_\tau = -1 + o(1) \quad \text{near} \quad \xi = \mathrm{e}^{\frac{\tau}{2}} \quad (\text{since} \quad z_\tau \equiv \theta - 1 + \tau\theta_\tau). \tag{4.10}$$

It follows from Lemma 2.11 that the estimates (4.9) are valid on any set $[\delta \mathrm{e}^{\frac{\tau}{2}}, \mathrm{e}^{\frac{\tau}{2}}] \times (\tau_*, \infty)$. We need also some uniform estimates of the solution $z(\xi, \tau)$ in q_*. Using the Bernstein method [5, 29], and setting $z = \phi(v), w = v_\xi$, where ϕ is a smooth function, $\phi(1) = C_{20}, \phi' > 0$ for $v \leq 1$, yields the following semilinear parabolic equation for the function w:

$$w_\tau = w_{\xi\xi} + \big[w\big(\tfrac{\phi''}{\phi'}\big) - \tfrac{\xi}{2}\big] w_\xi + \big(\tfrac{\phi''}{\phi'}\big) w^3 + \big[\big(\tfrac{\phi}{\phi'}\big)' - \tfrac{1}{2} + \tfrac{1}{\tau}\big(\tfrac{\phi^2 + \phi}{\phi'}\big)' + \tfrac{\phi''}{(\phi')^2}\tilde{\psi}\big] w.$$

Choosing the function ϕ so that $(\phi''/\phi') \geq \alpha_1, (\phi''/(\phi')^2 \leq -\alpha_2, \alpha_3 \leq \phi' \leq \alpha_4, (\phi/\phi')' \leq \alpha_5$, where α_i are some positive constants (cf. [29]), and using the boundary estimates (4.9) and (4.7),(4.8), we have that $w = -A\xi$ is a lower solution of the above equation provided that $A > 0$ is large enough. This yields the lower estimates:

$$-C_{21}(1 + \xi) \leq z_1 \leq 0 \quad \text{in} \quad q_*, \tag{4.11}$$

$$-C_{22}\big(1 + \xi^2\big) \leq z \leq C_{20} \quad \text{in} \quad q_*. \tag{4.12}$$

Hence, the corresponding $\omega-$limit set, $\omega(z_*) \neq \emptyset$, to the problems (4.2)-(4.5), see the definition given in (3.14), satisfies

$$\text{if} \quad f \in \omega(z_*) \quad \text{then} \quad f' \leq 0 \quad \text{in} \quad \mathbb{R}_+, \quad f'(0) = 0, \quad f(0) \in [\tfrac{1}{4}, C_{20}]. \tag{4.13}$$

Denote by W_S^2 the set of stationary solutions $g(\xi)$ satisfying the hypotheses given in (4.13) and (4.11), (4.12),

$$B_2 g = 0 \quad \text{in} \quad \mathbb{R}_+, \tag{4.14}$$

which can be derived as a formal limit in (4.2) as $\tau \to \infty$ by using the estimates (4.7)–(4.12). Then, we can see from (4.3) and (4.6) that

$$W_S^2 = \{g(\xi) = \alpha H_2(\xi), \quad \alpha \in [\tfrac{1}{8}, \tfrac{1}{2}\, C_{20}]\}, \tag{4.15}$$

where $H_2(\xi) = 2 - \xi^2$ is the second Hermite polynomial.

We can now state the main result of this section.

Theorem 4.1 *Assume that (1.24) holds. Then, as $\tau \to \infty$,*

$$z(\xi, \tau) \to \tfrac{1}{8} H_2(\xi) \quad \textit{uniformly on compact subsets in}\; \xi, \tag{4.16}$$

$$z_{\xi\xi}(0, \tau) \to -\tfrac{1}{4}. \tag{4.17}$$

We begin with the following preliminary result:

Proposition 4.2 *There holds*

$$\omega(z_*) \subseteq W_S^2. \tag{4.18}$$

Proof. Result (4.18) follows from the estimate

$$\int\limits_{\tau_*}^{\infty} \mathrm{d}s \int\limits_{0}^{\mathrm{e}^{s/2}} \bar{\rho} z_\tau^2(s)\, \mathrm{d}\xi < \infty \tag{4.19}$$

which is derived by a similar technique as in (3.17). By multiplying (4.2) by $\bar{\rho} z_\tau$ and integrating over $(0, l)$ we deduce that (cf. (3.19))

$$\int \bar{\rho} z_\tau^2 = \tfrac{\partial}{\partial \tau}\big\{ \int \bar{\rho} \big[-\tfrac{1}{2} z_\xi^2 + \tfrac{1}{2} z^2 + \tfrac{1}{\tau}\big[\tfrac{z^3}{3} + \tfrac{z^2}{2}\big]\big] - \tilde{\psi} z \big\} + I_2, \tag{4.20}$$

where the term

$$I_2(\tau) = \tfrac{1}{\tau^2} \int \bar{\rho} \big[\tfrac{z^3}{3} - \tfrac{z^2}{2}\big] - \tfrac{\mathrm{e}^{\frac{\tau}{2}} \bar{\rho}}{2} \big\{ \tfrac{1}{\tau}\big[\tfrac{z^3}{3} + \tfrac{z^2}{2}\big] + \tfrac{z^2}{2} - \tilde{\psi} z \big\}\big|_{\xi = \mathrm{e}^{\frac{\tau}{2}}} + \tilde{\psi}' \int \bar{\rho} z$$

has been proven in (4.7)–(4.12) to satisfy $|I_2(\tau)| \in ((\tau_*, \infty))$. A suitable estimate on the derivative $\tilde{\psi}'$ follows from the inequality

$$\tfrac{1}{2}(\|u(t)\|_2^2)' \equiv -\tfrac{1}{2}\|u_x(t)\|_2^2 + \int u^3 \le \int u^3$$

(see the proof of Proposition 2.3) and estimates (2.33) and (2.80). Integrating (4.20) over (τ_*, S) and passing to the limit $S \to \infty$ yields (4.19), which as before leads to (4.18). □

Theorem 4.1 is then a straightforward consequence of the following result:

Lemma 4.3 *There holds*

$$\omega(z_*) = \big\{ f = \tfrac{1}{8} H_2 \big\}. \tag{4.21}$$

Proof. We have proved in Proposition 4.2 that

$$\omega(z_*) \subseteq \{g = \alpha H_2 \quad \text{where} \quad \alpha \in [\tfrac{1}{8},\, C_{20}/2]\}. \tag{4.22}$$

Thus, (4.21) implies that the only value of α existing in the ω-limits is

$$\alpha_* = \tfrac{1}{8}. \tag{4.23}$$

The unique choice of α depends on the first perturbation term $\frac{1}{l}\tau\, C_2(z)$ in the right hand side of (4.2), since the function $\frac{1}{l}\tau$ describing the decay rate of the perturbation is not integrable at infinity.

According to the idea given in [15] (see also [30] (Chapter II) and other applications in [17], [19]), we introduce the *weighted energy*

$$E(\tau) = \int \bar\rho H_2 z(\tau)\, \mathrm{d}\xi \quad \text{for} \quad \tau \geq \tau_* \tag{4.24}$$

corresponding to the stationary operator B_2 in (4.2) which is “approximately” self-adjoint in $L^2_{\bar\rho}(0, l(\tau))$. Indeed, one can see by integrating by parts that

$$\int \bar\rho H_2 B_2 z = \int \bar\rho z B_2 H_2 + I_3, \tag{4.25}$$
$$I_3(\tau) = -z\bar\rho H_2'\big|_{\xi = \mathrm{e}^{\frac{\tau}{2}}},$$

so that by (4.8)

$$|I_3(\tau)| = o\big(\mathrm{e}^{-\tau/4}\big) \in L^1((\tau_*, \infty)). \tag{4.26}$$

Notice also that

$$E'(\tau) \equiv \int \bar\rho H_2 z_\tau + I_4, \tag{4.27}$$

where

$$I_4(\tau) = \tfrac{1}{2}\mathrm{e}^{\frac{\tau}{2}}\bar\rho H_2 z\big|_{\xi = \mathrm{e}^{\frac{\tau}{2}}},$$

and hence I_4 also satisfies (4.26).

Finally, multiplying equation (4.2) by H_2 in $L^2_{\bar\rho}$ and using (4.25)–(4.27) and (4.7) yield the *weighted energy equation*

$$E'(\tau) = \tfrac{1}{\tau}\big(E(\tau) + F(\tau, z)\big) + o\big(\mathrm{e}^{-\tau/4}\big) \quad \text{for} \quad \tau > \tau_*, \tag{4.28}$$

where F is the functional

$$F(\tau, z) = \int \bar\rho H_2 z^2. \tag{4.29}$$

The energy trajectory $\{E(\tau), \tau > \tau_*\}$ has been proved in (4.12) to be uniformly bounded:

$$|E(\tau)| \leq C_{23} \quad \text{for} \quad \tau > \tau_*. \tag{4.30}$$

The rest of the proof of (4.21) by using the energy equation (4.28) is quite similar to that given in [17] and [19]. Assume first that there exists a finite limit

$$E(\tau) \to E_0 \quad \text{as} \quad \tau \to \infty. \tag{4.31}$$

Then it follows from (4.22) that

$$\omega(z_*) = \big\{f = \alpha_0 H_2 : \alpha_0 = E_0 / \textstyle\int \bar\rho H_2^2\big\} \tag{4.32}$$

and hence

$$z(\cdot,\tau) \to \alpha_0 H_2(\cdot) \quad \text{as} \quad \tau \to \infty \quad \text{uniformly on any compact subset.} \tag{4.33}$$

Using (4.33) in passing to the limit $\tau \to \infty$ in the equation (4.28), (4.29) yields

$$E'(\tau) = \tfrac{\alpha_0}{\tau}\big[\textstyle\int \bar\rho H_2^2 + \alpha_0 \int \bar\rho H_2^3\big] + o\big(\tfrac{1}{\tau}\big) \quad \text{for} \quad \tau \gg 1. \tag{4.34}$$

Since $\alpha_0 \ne 0$ by (4.33) and (4.22), we conclude that the unique possible value is

$$\alpha_0 = -\big[\textstyle\int \bar\rho H_2^2 / \int \bar\rho H_2^3\big] = \tfrac{1}{8}. \tag{4.35}$$

Indeed, if (4.35) is not valid then integrating (4.34) over (τ_*, ∞) contradicts (4.30).
We have to consider only the case where the limit in (4.31) does not exist, i.e., $E(\tau)$ is a function oscillating as $\tau \to \infty$. This implies that there exists, e.g.,

$$E_1 > E_* = \tfrac{1}{8} \textstyle\int \bar\rho H_2^2$$

such that the set $\{\tau_j > \tau_* : E(\tau_j) = E_1\}$ is not bounded. Obviously, the increasing sequence $\{\tau_j\}$ can be chosen so that

$$E'(\tau_j) \ge 0 \quad \text{for} \quad j = 1, 2, \dots. \tag{4.36}$$

By (4.22) we have that

$$z(\cdot,\tau_j) \to \alpha_1 H_2(\cdot) \quad \text{as} \quad j \to \infty, \tag{4.37}$$

where $\alpha_1 = E_1/\int \bar\rho H_2^2 > \alpha_* = \frac{1}{8}$. By passing to the limit in (4.28), (4.29) with $\tau = \tau_j \to \infty$ and using (4.37) we arrive at the inequality

$$E'(\tau_j) = \tfrac{\alpha_1}{\tau_j}(1 - 8\alpha_1) \textstyle\int \rho H_2^2 + o(1/\tau_j) \tag{4.38}$$

and hence $E'(\tau_j) < 0$ for large τ_j contradicting (4.36). Thus (4.21) holds completing the proof. □

Proof of Theorem 4.1. The expression (4.16) follows from (4.21). In order to prove (4.17), we notice that, by (4.19), $B_2(z) \to 0$ in the weak sense as $\tau \to \infty$ and hence using (4.3) yields $z_{\xi\xi} = \frac{1}{2} z_\xi \xi + z + o(1)$ and hence (4.17) again follows from (4.21). □

5. Final analysis near the origin: the third limit and convergence to the Hamilton–Jacobi profile

In this section, we again consider the function $\theta = \theta(\eta, \tau)$, which was introduced in Section 3, with the new rescaled spatial variable

$$\eta = \xi / \tau^{\frac{1}{2}} \quad \text{for} \quad \tau \ge \tau_* \gg 1. \tag{5.1}$$

We shall extend the estimates in Section 4 from compact subsets in ξ to compact subsets in η. Then θ solves the equation (cf. (3.2) and (4.2))

$$\theta_\tau = B_3(\theta) + \tfrac{1}{\tau} C_3 \theta - \psi(\tau, \theta) \tag{5.2}$$

in $q_* = B_{m(\tau)} \times (\tau_*, \infty), m(\tau) = \tau^{-\frac{1}{2}} e^{\frac{\tau}{2}}$, where the nonlinear stationary first-order operator B_3 has the form

$$B_3(\theta) = -\tfrac{1}{2}\theta_\eta \eta - \theta + \theta^2, \tag{5.3}$$

and the first perturbation term is linear,

$$C_4\theta = \theta_{\eta\eta} + \tfrac{1}{2}\theta_\eta \eta. \tag{5.4}$$

The function θ satisfies the Neumann boundary conditions $\theta_\eta = 0$ for $\eta = 0$ and $\theta = m(\tau)$ for $\tau \geq \tau_*$. Set $\theta_0(\eta) \equiv \theta(\eta, \tau_*)$. Since θ is uniformly bounded in q_*, by applying Bernstein-type estimates [29] to equation (5.2) (see also Section 4) we have that

$$|\theta_\eta| \leq C_{24}, \quad |\theta_{\eta\eta}| \leq C_{25}, \quad |\theta_{\eta\eta\eta}| \leq C_{26} \quad \text{in} \quad q_*. \tag{5.5}$$

The main result of Section 4, given in Theorem 4.1, can now be restated as follows. Since $\theta_{\eta\eta} \equiv \tau\theta_{\xi\xi}$, we can see that (4.16) and (4.17) yield the estimates

$$\theta(0,\tau) = 1 + \tfrac{1}{4\tau}(1 + o(1)) \quad \text{as} \quad \tau \to \infty, \tag{5.6}$$

$$\theta_{\eta\eta}(0,\tau) = -\tfrac{1}{4} + o(1) \quad \text{as} \quad \tau \to \infty. \tag{5.7}$$

Notice that (2.33) and (3.7) with $\xi = 0$ yield the sharp upper estimate of the profile as $\tau \to \infty$

$$\theta(\eta, \tau) \leq \big(1 + \frac{\eta^2}{8}\big)^{-1}(1 + o(1)) \quad \text{uniformly on compact subsets in} \quad \eta. \tag{5.8}$$

Hence, by (5.5) the corresponding $\omega-$limit set, $\omega(\theta_0)$, is such that

$$\text{if} \quad f \in \omega(\theta_0) \quad \text{then} \quad f \in C(\mathbb{R}_+), \quad f' \leq 0, \quad 0 \leq f \leq \big(1 + \frac{\eta^2}{8}\big)^{-1} \quad \text{in} \quad \mathbb{R}_+, \tag{5.9}$$
$$f(0) = 1, f'(0) = 0, f''(0) = -\tfrac{1}{4}.$$

We denote this set of functions as M_0.

We now state the main result of this section.

Theorem 5.1 *Assume that the estimate (1.24) holds. Then*

$$\omega(\theta_0) = \{f = \theta_*\}, \quad \theta_*(\eta) = \big(1 + \frac{\eta^2}{8}\big)^{-1}, \tag{5.10}$$

and hence, as $\tau \to \infty$,

$$\theta(\eta,\tau) \to \theta_*(\eta) \quad \textit{uniformly on compact subsets in} \quad \eta. \tag{5.11}$$

Proof. We now use some general results on the ω-limits of a perturbed dynamical system proved in [23], Section 3. Fix an arbitrary sequence $\{\tau_j\} \to \infty$ so that $\theta(\cdot, \tau_j) \to f(\cdot) \in \omega(\theta_*)$. Then, using estimates on the function θ and its derivatives, and of the perturbation terms in (5.2) given above, we conclude that $\theta(\cdot, \tau_j + s) \to g(\cdot, s)$ as $\tau_j \to \infty$ in $L^\infty_{loc}([0,\infty) : C(\mathbb{R}_+))$, where the function $g \geq 0$ solves the first order semilinear Hamilton–Jacobi equation

$$g_s = B_3(g) \quad \text{in} \quad \mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}_+. \tag{5.12}$$

The corresponding initial function has the form

$$g(\eta, 0) = f(\eta) \quad \text{in} \quad \mathbb{R}_+, \quad f \in M_0. \tag{5.13}$$

According to [19, 23], the proof of (5.10) will be completed if the ω-limit set, Ω, to equation (5.12) is *uniformly stable* in a suitably complete metric space. In fact, using the idea of [20], Section 7, we have to study the uniform stability of a *reduced ω-limit set* to equation (5.12), the set of all ω-limits occurring for an arbitrary initial datum f in (5.13) satisfying $f \in M_0$. We will easily show that M_0 is the *invariant set* for the Hamilton–Jacobi equation (5.12) so that $\Omega \subseteq M_0$. Let us now introduce the Banach space $C_\rho(\mathbb{R}_+),\ \rho(\eta) = \eta^{-2}$, generating the following distance on M_0:

$$\mathrm{d}(g_1, g_2) = \sup_{\eta>0} \tfrac{1}{\eta^2} \, |g_1(\eta) - g_2(\eta)|, \tag{5.14}$$

which will be shown to be directly associated to the Hamilton–Jacobi equation (5.12) on M_0. See more general results in [21].

Proposition 5.5 *(i) The $\omega-$limit set Ω, of equation (5.6) on M_0 has the form*

$$\Omega = \{g = \theta_*\}. \tag{5.15}$$

(ii) Ω is uniformly stable in $C_\rho(\mathbb{R}_+) \cap M_0$: if $d(f, \theta_*) \leq \epsilon$ for a given initial function $f \in M_0$ in (5.13), then

$$d(g(s), \theta_*) \leq \epsilon \quad \textit{for all} \quad s > 0. \tag{5.16}$$

Proof. Equation (5.12) after the transformation

$$v = \log[(1-g)/g] \tag{5.17}$$

is reduced to the first order linear equation

$$v_\tau = -\tfrac{1}{2} v_\eta \eta + 1, \tag{5.18}$$

which can easily be solved explicitly using the method of characteristics. This yields the explicit general solution to the problem (5.12), (5.13)

$$g(\eta, s) = \big(1 + \mathrm{e}^s \big[\tfrac{1}{f(\eta \mathrm{e}^{-s/2})} - 1\big]\big)^{-1} \leq \theta_0(\eta) \quad \text{in} \quad \mathbb{R}^2_+, \tag{5.19}$$

and (5.15) follows immediately by passing to the limit $s \to \infty$ in (5.19) and using (5.9). One can see from (5.19) that M_0 given by (5.9) is the invariant set of the Hamilton–Jacobi equation (5.12).

In order to prove (ii), consider for $s > 0$

$$\mathrm{d}(g(s), \theta_*) \equiv \sup_{\eta>0} \tfrac{1}{\eta^2} \big(\big[1 + \tfrac{\eta^2}{8}\big]^{-1} - \big[1 + \mathrm{e}^s \tfrac{1 - f(\eta \mathrm{e}^{-s/2})}{f(\eta \mathrm{e}^{-s/2})}\big]^{-1}\big). \tag{5.20}$$

If we set $\zeta = \eta \mathrm{e}^{-s/2}$ on the right-hand side, we deduce that

$$\begin{aligned} \mathrm{d}(g(s), \theta_*) &\equiv \sup_{\zeta>0} \tfrac{\mathrm{e}^{-s}}{\zeta^2} \big(\big(1 + \tfrac{1}{8} \mathrm{e}^s \zeta^2\big)^{-1} - \big[1 + \mathrm{e}^s \tfrac{1-f(\zeta)}{f(\zeta)}\big]^{-1}\big) \\ &\equiv \sup_{\zeta>0} \tfrac{1}{\zeta^2} \tfrac{1 - f(\zeta) - f(\zeta)\frac{\zeta^2}{8}}{(1 + \frac{1}{8} \mathrm{e}^s \zeta^2)(f(\zeta) + \mathrm{e}^s(1 - f(\zeta)))}. \end{aligned} \tag{5.21}$$

Since $1 + \mathrm{e}^s \zeta^2/8 \geq 1 + \zeta^2/8$ and $f \mathrm{e}^s (1-f) \geq 1$, we deduce that

$$\mathrm{d}(g(s), \theta_*) \leq \sup_{\zeta>0} \tfrac{1}{\zeta^2} \tfrac{1 - f(\zeta) - f(\zeta)\frac{\zeta^2}{8}}{1 + \frac{1}{8}\zeta^2} \equiv \mathrm{d}(g(0), \theta_*) \leq \epsilon, \tag{5.22}$$

which completes the proof. Notice that the above uniform stability in $C_\rho(\mathbb{R}_+) \cap M_0$ can be proved directly from equation (5.12), see also [4]. $\square$

Finally, we notice that the distance (5.14) on M_0 is generated by the norm in the weighted space $C_\rho(\mathbb{R}_+)$, which consists of continuous functions $h(\eta)$ having the first derivative $h'(0) = 0$, such that $\eta^{-2}(h(\eta) - h(0)) \in L^\infty$. The norm in $C_\rho(\mathbb{R}_+)$ then has the form

$$\|h\|_{C_\rho(\mathbb{R}_+)} = |h(0)| + \sup_{\eta>0} \frac{1}{\eta^2}|h(\eta) - h(0)|. \tag{5.23}$$

One can see that $C_\rho(\mathbb{R}_+)$ is a Banach space, see a general analysis in [21]. Indeed, it is easily seen that

$$C_\rho(\mathbb{R}_+) = \{h : \exists \quad w \in L^\infty \quad \text{and a constant } A \text{ such that} \quad h = A + \eta^2 w\}. \tag{5.24}$$

It follows from (5.24) and (5.5) and a standard compactness argument, that setting $\theta(\eta, \tau) = \theta(m(\tau), \tau)/\eta$ for $\eta \geq m(\tau),\ \tau \geq \tau_*$, and using the uniform upper estimate (5.8) and the lower one given in (3.9), yields that the orbit $\{\theta(\cdot, \tau), \tau \geq \tau_*\}$ is relatively compact in $C_\rho(\mathbb{R}_+)$. By passing to the limit in equation (5.2) as $\tau = \tau_j + s \to \infty$, we have as above that $\theta(\cdot, \tau_j + s) \to g(\cdot, s)$ as $\tau_j \to \infty$ in $L^\infty_{\rm loc}([0, \infty) : C_\rho(\mathbb{R}_+))$, where g solves equation (5.12). Hence, hypotheses (H1) and (H2) in [19, 23] are valid. Then Proposition 5.2 implies that the last hypothesis (H3) in [19] about the uniform stability of the *reduced* ω-limit set of equation (5.12) on M_0 (see [20]) holds. Using Theorem 3 in [19] completes the proof of Theorem 5.1. □

6. The Proof of Theorem 1.1

The asymptotically sharp equality (1.25) follows from Theorem 4.1, see also (5.6). Asymptotic behaviour (1.26) is the result of Theorem 5.1, see (5.11). In order to prove (1.27) we notice that integrating the estimate (1.26) for u^2 over $x \in (0, x_0(t))$ yields that

$$\|u_+(t)\|_2^2 = \pi 2^{-\frac{1}{2}}(T - t)^{-\frac{3}{2}}|\log(T - t)|^{\frac{1}{2}}(1 + o(1)). \tag{6.1}$$

Then (2.115) and (2.114) yield the sharp estimate (1.28). Hence, if follows from (2.76) and (1.28) that

$$\rho(t) = \pi\sqrt{2}(T - t)^{-\frac{1}{2}}|\log(T - t)|^{\frac{1}{2}}(1 + o(1)), \tag{6.2}$$

and then (1.27) follows from (6.2) and Lemma 2.10. This completes the proof of Theorem 1.1. □

7. Final remarks

The techniques used in the present paper can be applied directly to study a natural generalisation of the equation under consideration (cf. (1.16)

$$u_t = u_{xx} + |u|^p - \textstyle\int_0^1 |u(x,t)|^p \, {\rm d}x, \tag{7.1}$$

where $p > 1$ is a fixed constant. Then under the hypothesis (1.5) we can prove that (cf. (2.25))

$$u_+(x,t) \leq \tfrac{8p}{(p-1)^2}|x|^{-2/(p-1)}|\log x|^{1/(p-1)}(1 + o(1)) \quad \text{for small} \quad x > 0. \tag{7.2}$$

Since the behaviour as $t \to T$ on compact subsets $[\delta, 1]$ with a small $\delta > 0$ is "flat", $u(x,t) = -\rho(t)(1 + o(1))$ (cf. (2.80)), the function $\rho(t)$ can be easily calculated from the identity

$$\|u_-(t)\|_1 \equiv \|u_+(t)\|_1, \tag{7.3}$$

i.e.,

$$\rho(t) = \|u_+(t)\|_1 (1 + o(1)) \quad \text{as} \quad t \to T. \tag{7.4}$$

Then (7.2) yields that there exists a critical value of $p = 3$ such that for any $p > 3$ the function $\rho(t)$ is bounded, and hence the solution $u(x, t)$ is bounded from below. This means that if $p > 3$ then $u(x, t)$ blows up as $t \to T$ at the *single point* $x = 0$. If $p \in (1, 3]$ then we have *global non-uniform blow-up* on the interval $x \in [0, 1]$. In the critical case of $p = 3$, (7.3) and (7.4) yield the logarithmic rate of divergence of the solution as $t \to T$ on subsets $x \in [\delta, 1]$.

References

[1] M.M. Ad'jutov and L.A. Lepin, *Nonexsitence of blow-up self-simlar structures in a medium with source and constant heat conductivity*, Differentsial'nye Uravnenia, **20**, (1984), pp. 129–1281 (in Russian).

[2] J. Bebernes and S. Bricher, *Finite time blow-up profiles for semilinear parabolic equations via centre manifold theory*, SIAM J. Math. Anal., **23** (1992), 852–869.

[3] J. Bebernes and D. Eberly, *Mathematical problems from combustion theory*, Appl. Math. Sciences, **83**, (1989), Springer-Verlag.

[4] J. Bebernes, S. Bricher, and V.A. Galaktionov, *Asymptotics of blow-up for weakly quasilinear parabolic problems*, Nonl. Anal., TMA, **23** (1994), 489–514.

[5] S. Bernstein, *Sur la généralisation du probléme de Dirichlet. Deuxième partie*, Math. Ann., **69** (1910), 82–136.

[6] C.J. Budd, J.W. Dold, and A.M. Stuart, *Blowup in partial differential equations with conserved first integral*, SIAM J. Appl. Math., **53**, (1993), pp. 718–742

[7] C.J. Budd, J.W. Dold, and A.M. Stuart, *Blowup in a system of partial differential equations with conserved first integral. Part II: Problems with convection* , SIAM J. Appl. Math, Vol. 54, (1994), 610–641.

[8] C.J. Budd and V.A. Galaktionov, *Global blow-up patterns for a semilinear bi-harmonic equation on a subspace*, in preparation (shortly, in arXiv.org).

[9] J.M. Chadam, A. Peirce, and H.-M. Yim, *The blow-up property of solutions to some diffusion equations with nonlocalized nonlinear reactions*, J. Math. Anal., **169**, (1992), 313–328.

[10] S. Filippas and R.V. Kohn, *Refined asymptotics for the blow-up of* $u_t - \Delta u = u^p$, Comm. Pure Appl. Math., **45** (1992), 821–869.

[11] A. Friedman, Partial Differential Equations of Parabolic Type, Englewood Cliffs, NJ, Prentice Hall, (1964).

[12] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ.Math. J., **34** (1985), 427–447.

[13] Y. Giga and R.V. Kohn, *Asymptotically self-simlar blow-up of semilinear heat equations*, Comm. Pure. Appl. Math., **38**, (1985), 297–319.

[14] Y. Giga and R.V. Kohn, *Characterizing blow-up using similarity variables*, Indiana Univ. Math. J. **36**, (1987), 1–40..

[15] V.A. Galaktionov, S.P. Kurdyumov, and A.A. Samarskii, *On asymptotic "eigenfunctions" of the Cauchy problem for a nonlinear parablic equation*, Matem. Sbornik **126**, (1985), 435–472 (in Russian)

[16] V.A. Galaktionov and S.A. Posashkov, *Equation* $u_t = u_{xx} + u^\beta$. *Localization, asymptotic behaviour of unbounded solutions*, Math., Acad. Sci. USSR, **97**, (1985) (in Russian).

[17] V.A. Galaktionov and S.A. Posashkov, *Asymptotics of nonlinear heat conduction with absorption under the critical exponent*, Inst. Appl. Math. Acad. Sci. USSR, **71**, (1986), (in Russian).

[18] V.A. Galaktionov and S.A. Posashkov, *Applications of new comparison theorems for unbounded solutions of a nonlinear arabolic equation*, Differentsial'nye Uravneniya, **22**, (1986), 1165–1173, (in Russian).

[19] V.A. Galaktionov and J.L. Vazquez, *Asymptotic behaviour of nonlinear parabolic equations with critical exponents. A dynamical systems approach*, J. Func. Anal., **100** (1991), 435–462.

[20] V.A. Galaktionov and J.L. Vazquez, *Regional blow-up in a semilinear heat equation with convergence to a Hamilton-Jacobi equation*, SIAM J. Math. Anal., **24**, (1993), 1254–1276.

[21] V.A. Galaktionov and J.L. Vazquez, *Extinction for a quasilinear heat equation with absorption II. A dynamical systems approach*, Communications in Partial Differential Equations, **19**, (1994), 1107–1137.

[22] V.A. Galaktionov and J.L. Vazquez, *The problem of blow-up in nonlinear parabolic equations*, Discr. Cont. Dyn. Syst., **8** (2002), 399–433.

[23] V.A. Galaktionov and J.L. Vazquez, A Stability Technique for Evolution Partial Differential Equations. A Dynamical Systems Approach, Progr. in Nonl. Differ. Equat. and Their Appl., Vol. **56**, Birkhäuser, Boston/Berlin, 2004.

[24] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, **840**, (1981), Springer-Verlag, Berlin, Heidelberg, New York.

[25] M.A. Herrero and J.J.L. Velázquez, *Blow-up profiles in one-dimensional semilinear parabolic equations*, Communications in Partial Diffferential Equations, **17**, (1992), 205–219.

[26] M.A. Herrero and J.J.L. Velázquez, *Blow-up behaviour of one-dimen-sional semilinear parabolic equations*, Ann. Inst. Henri Poincaré, Analyse non linéaire, **10** (1993), 131–189.

[27] M. Langlais and D. Phillips, *Stabilization of solutions of nonlinear and degenerate evolution equations*, Nonlinear Anal., Theory, Meth. Appl., **9** (1985), 321–333.

[28] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Gauthier–Villars, Paris, 1969.

[29] O.A. Oleinik and S.N. Kruzkov, *Quasi-linear parabolic equations of the second order with many independent variables*, Uspehi Math. Nau SSSR, **16** (1961), 115–155.

[30] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, and A.P. Mikhailov, Blow-up in Quasilinear Parabolic Equations, Walter de Gruyter, Belrin/New York, 1995.

[31] P.Soouplet, *Blow-up in nonlocal reaction-diffusion equations*, SIAM J. Math. Anal., **29**, (1998), 1301–1334.

[32] J.J.L. Velazquez, V.A. Galaktionov, and M.A. Herrero, The space structure near a blow-up point for semilinear heat equations: a formal approach, *Comput. Maths Math. Phys.,* **31** (1991), 46–55.

[33] J.J.L. Velázquez, V.A. Galaktionov, and M.A. Herrero, *The space structure near a blow-up point for semilinear heat equations: a formal approach*, USSR Comput. Math. Math. Phys., **31** (1991), 46–55.

(1) Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK,
E-mail address: `mascjb@bath.ac.uk`

(2) School of Mathematics, University of Manchester, Manchester M13 9PL, UK,
E-mail address: `john.dold@manchester.ac.uk`

(3) Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
E-mail address: `masvg@bath.ac.uk`