A lower bound for the balanced truncation error for MIMO systems

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***Abstract*—We show that for a class of systems which includes state space symmetric systems, the balanced truncation error is bounded from below by twice the sum of the tail of the Hankel singular values (including multiplicities) divided by the dimension of the input space.**

***Index Terms*—balanced realization, balanced truncation, Hankel operator, error bound, model reduction, linear time-invariant systems.**

I. Introduction

THE well-known error bound for balanced truncation

$$\sup_{\zeta\in\mathbb{C}:\mathrm{Re}\zeta>0} \|G(\zeta)-G_r(\zeta)\| \leq 2\sum_{j=r+1}^{\ell}\mu_j, \tag{1}$$

where $\{\mu_1,\ldots,\mu_\ell\}$ are the distinct Hankel singular values of G and G_r is the balanced truncation of G, is known to be an equality for single-input single-output (SISO) state space symmetric systems, i.e. if $G(s)=C(sI-A)^{-1}B$ with $A=A^*\in\mathbb{C}^{n\times n}$ negative definite, $C^*=B\in\mathbb{C}^n$, then

$$\sup_{\zeta\in\mathbb{C}:\mathrm{Re}\zeta>0} |G(\zeta)-G_r(\zeta)| = 2\sum_{j=r+1}^{\ell}\mu_j,$$

(see e.g. [9, Theorem 4.1] and [18, Theorem 4.4]). It is also known that in this case (this follows e.g. from [14, Corollary 2.2]) the Hankel singular values of G all have multiplicity one. Moreover, it is known that for multi-input multi-output (MIMO) state space symmetric systems, i.e. if $G(s)=C(sI-A)^{-1}B$ with $A=A^*\in\mathbb{C}^{n\times n}$ negative definite, $C^*=B\in\mathbb{C}^{n\times m}$ with $m>1$, strict inequality may hold (see e.g. [9, Remark 4.1] and [18, Section 4]).

In this article we prove that for state space symmetric systems the following lower bound holds:

$$2\sum_{k=r+1}^{\ell}\frac{m_j}{m}\mu_j \leq \sup_{\zeta\in\mathbb{C}:\mathrm{Re}\zeta>0}\|G(\zeta)-G_r(\zeta)\|, \tag{2}$$

where m_j is the multiplicity of μ_j as a singular value of the Hankel operator of G. We note that in combination with the upper bound (1) this in particular implies $m_j\leq m$ for the multiplicities.

In fact, we prove the lower bound (2) for a slightly more general class of systems than state space symmetric systems, namely those systems with a semi-definite Hankel operator. We note that systems with a semi-definite Hankel operator include RC and RL circuits (see Remark 16).

In Section II we first discuss the notation and terminology used. In Section III we prove a lower bound in terms of the eigenvalues of the Hankel operator. This is then used in Section IV to prove the lower bound (2). Section V contains comments on balanced singular perturbation approximation and on the case of non-rational transfer functions. Finally, Section VI illustrates the theory by considering two simple RC circuits.

II. Notation and terminology

For a matrix $T\in\mathbb{C}^{d\times m}$ and $p\in[1,\infty]$ the Schatten p-norm as defined by

$$\|T\|_p := \begin{cases}\left(\sum_{k=1}^{\min(m,d)}[\sigma_k(T)]^p\right)^{1/p} & p\in[1,\infty),\\ \sigma_1(T) & p=\infty,\end{cases}$$

where $\sigma_1(T)\geq\ldots\geq\sigma_{\min(m,d)}(T)$ are the singular values of T.

The set of $d\times m$-matrices with entries in the field of complex rational functions is denoted by $\mathbb{C}(s)^{d\times m}$. We call $G\in\mathbb{C}(s)^{d\times m}$ *stable* if G is proper and all its poles have negative real part.

The *impulse response* h is the inverse Laplace transform of $G\in\mathbb{C}(s)^{d\times m}$. The *Hankel operator* of a stable $G\in\mathbb{C}(s)^{d\times m}$ is given by

$$H:\quad L^2(0,\infty;\mathbb{C}^m)\to L^2(0,\infty;\mathbb{C}^d),\qquad u\mapsto (Hu)(t)=\int_0^\infty h_0(t+s)u(s)\,ds,$$

where h_0 is the function part of the impulse response, i.e. the inverse Laplace transform of the strictly proper part of G.

The nonzero singular values of the Hankel operator of G are called the *Hankel singular values* of G. We denote the sequence of Hankel singular values by $(\sigma_k)_{k=1}^n$, the sequence of distinct Hankel singular values by $(\mu_j)_{j=1}^\ell$ and the sequence of multiplicities of the Hankel singular values by $(m_j)_{j=1}^\ell$ (i.e. m_j is the number of times that μ_j appears in the sequence $(\sigma_k)_{k=1}^n$). We choose the ordering of these sequences to be compatible in the following sense

$$\sigma_{1+\sum_{i=1}^{j-1}m_i}=\ldots=\sigma_{\sum_{i=1}^{j}m_i}=\mu_j,\quad j=1,\ldots,\ell.$$

We denote the sequence of nonzero eigenvalues of the Hankel operator by $(\lambda_k)_{k=1}^n$ and call these the *Hankel eigenvalues* of G. We note that if the Hankel operator is self-adjoint, then the absolute values of the Hankel eigenvalues equal the Hankel singular values (including multiplicities). In this case

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we choose the ordering of these sequences to be compatible in the sense that

$$|\lambda_k| = \sigma_k, \quad k = 1, \dots, n.$$

Giving the ordering of the Hankel singular values this may not uniquely determine the ordering of the Hankel eigenvalues, but for our purposes this particular non-uniqueness is irrelevant.

A *realization* of $G \in \mathbb{C}(s)^{d\times m}$ is a quadruple $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ consisting of $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{d\times n}$, $D \in \mathbb{C}^{d\times m}$ with

$$G(s) = C(sI - A)^{-1}B + D.$$

Conversely, G is called the *transfer function* of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.

A realization $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is called *stable* if all eigenvalues of A have negative real part. The *reachability map* $\Phi : L^2(0,\infty;\mathbb{C}^m) \to \mathbb{C}^n$ and the *observability map* $\Psi : \mathbb{C}^n \to L^2(0,\infty;\mathbb{C}^d)$ of a stable realization $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ are defined by

$$\Phi u = \int_0^\infty e^{At}Bu(t)\,dt, \quad \Psi z = t \mapsto Ce^{At}z.$$

We note that the Hankel operator equals the product of the observability and reachability maps: $H = \Psi\Phi$.

The stable realization $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is called *balanced*, if $\Phi\Phi^* \in \mathbb{C}^{n\times n}$ and $\Psi^*\Psi \in \mathbb{C}^{n\times n}$ satisfy

$$\Phi\Phi^* = \Psi^*\Psi = \operatorname{diag}(\sigma_1, \dots, \sigma_n),$$

where we recall that $(\sigma_k)_{k=1}^n$ is the sequence of Hankel singular values.

Remark 1. Another common definition of the reachability map is $\widetilde{\Phi} : L^2(-\infty, 0; \mathbb{C}^m) \to \mathbb{C}^n$ with $\widetilde{\Phi}u := \int_{-\infty}^0 e^{-At}Bu(t)\,dt$. The Hankel operator is, with this definition, $\widetilde{H} = \Psi\widetilde{\Phi} : L^2(-\infty, 0; \mathbb{C}^m) \to L^2(0,\infty;\mathbb{C}^d)$, see [23]. Our definition of the reachability map (and thus also the Hankel operator) is related to that by multiplication from the right with the reflection operator $L^2(-\infty, 0; \mathbb{C}^m) \to L^2(0,\infty;\mathbb{C}^m)$. Since the reflection operator is unitary, this alternative definition leads to the same concepts of Hankel singular values and balanced realizations. Since self-adjointness of the Hankel operator plays an important role in this article, our definition of the Hankel operator is more convenient for our purposes.

It is well-known that a stable $G \in \mathbb{C}(s)^{d\times m}$ has a balanced realization (see e.g. [1, Section 7.1]). Note that balanced realizations as defined above are minimal since by our assumptions we have $\Phi\Phi^* = \Psi^*\Psi > 0$.

Let a balanced realization $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of $G \in \mathbb{C}(s)^{d\times m}$ be given. Let $r \in \{1, \dots, \ell\}$ and $q := \sum_{j=1}^r m_j$. Then the *balanced truncation of* G of dimension q is defined as the transfer function G_r of $\left[\begin{smallmatrix} A_r & B_r \\ C_r & D \end{smallmatrix}\right]$, where, for $Z_r = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] \in \mathbb{C}^{n\times q}$, the matrices $A_r \in \mathbb{C}^{q\times q}$, $B_r \in \mathbb{C}^{q\times m}$ and $C_r \in \mathbb{C}^{d\times q}$ are defined by $A_r := Z_r^*AZ_r$, $B_r := Z_r^*B$, $C_r := CZ_r$. The realization $\left[\begin{smallmatrix} A_r & B_r \\ C_r & D \end{smallmatrix}\right]$ is balanced. The balanced truncation G_r depends only on G, the ordering of the distinct Hankel singular values and r (and not on the particular balanced realization chosen).

Note that the balanced truncation depends on the ordering of the sequence of distinct Hankel singular values. We assume that such an ordering is given (the customary one is the one with $\mu_1 > \mu_2 > \dots \mu_\ell > 0$; in which case G_r depends only on G and r, but other orderings are permitted).

We refer the reader to [1, Chapter 7], [6, Chapter 9], [22, Chapter 7] or [23, Chapter 7] and the main original contributions [12], [17], [5], [2] for background material on balanced realizations and balanced truncations.

III. Self-adjoint systems

In this section we consider self-adjoint systems and prove a lower bound which in Section IV will be used to prove the lower bound (2).

Definition 2. A rational function $G \in \mathbb{C}(s)^{m\times m}$ is called *self-adjoint* if $G = G^\dagger$, where $G^\dagger$ is defined by

$$G^\dagger(s) := G(\bar{s})^*.$$

Remark 3. Note that any SISO system with real coefficients has a self-adjoint transfer function. Also note that the transfer function of a state space symmetric system (i.e. with $A = A^*$, $C = B^*$ and $D = D^*$) is self-adjoint.

Lemma 4. *The following are equivalent for any stable and strictly proper $G \in \mathbb{C}(s)^{m\times m}$.*

1) *G is self-adjoint.*
2) *The impulse response h is self-adjoint (that is, $h(t) = h(t)^*$ for all $t \in [0,\infty)$).*
3) *The Hankel operator is self-adjoint.*

Proof. The definition of the impulse response yields

$$G(s)^* = \int_0^\infty e^{-\bar{s}t}h(t)^*\,dt, \quad G^\dagger(s) = \int_0^\infty e^{-st}h(t)^*\,dt.$$

From this (and uniqueness of the inverse Laplace transform) we see the equivalence of 1 and 2. Since the adjoint of the Hankel operator is given by

$$(H^*u)(t) = \int_0^\infty h(t+s)^*u(s)\,ds,$$

we see that 2 implies 3. That 3 implies 2 follows from the fact that $H - H^*$ is the Hankel operator corresponding to the impulse response $h(t) - h(t)^*$ and the fact that the zero Hankel operator must have zero impulse response. □

Remark 5. For simplicity in Lemma 4 we considered only the strictly proper case; if there is a nonzero-feedthrough, then the impulse response is no longer a function and this slightly complicates the formulation. In that case 2 has to be replaced by the function part of the impulse response being self-adjoint and additionally the feedthrough operator being self-adjoint. The condition that the feedthrough operator must be self-adjoint must also be added to condition 3. All of the above can be proven by applying Lemma 4 to $G - G(\infty)$.

The following lemma shows that a balanced realization of a self-adjoint transfer function has a certain state space symmetry property.

Lemma 6. *Let $G \in \mathbb{C}(s)^{m\times m}$ be stable and self-adjoint and let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a balanced realization of G. Then there exists a unique self-adjoint operator J such that*

$$\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^*\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}. \tag{3}$$

This operator J is involutive (i.e., $J^{-1} = J$) and block-diagonal with block structure according to the multiplicities of the Hankel singular values. Further, there exists a balanced realization $\left[\begin{smallmatrix} A_s & B_s \\ C_s & D_s \end{smallmatrix}\right]$ of G in which J is diagonal in which case the diagonal entries are

$$J_{ii} = \frac{\lambda_i}{|\lambda_i|},$$

where $(\lambda_k)_{k=1}^n$ are the Hankel eigenvalues of G.

Proof. By [21, Theorem II], the self-adjointness of G implies the existence of a unique and invertible $J = J^*$ such that (3) holds true. The definition of the reachability and observability map then gives rise to $\Psi = \Phi^* J$, and thus

$$J(\Psi^*\Psi)^2 = J\Phi\Phi^* \cdot \Psi^*\Psi = \Psi^*\Psi J^{-1} \cdot J\Phi\Phi^* J = (\Psi^*\Psi)^2 J.$$

A comparison of coefficients yields that J is block-diagonal with block structure determined by the multiplicities of the Hankel singular values. A consequence is that it commutes with $\Psi^*\Psi$, whence we obtain that $\Psi^*\Psi = J\Phi\Phi^*J = J\Psi^*\Psi J = J^2\Psi^*\Psi$. The invertibility of $\Psi^*\Psi$ now implies that J is involutive.
Using that J is block-diagonal and involutive, there exists some block-diagonal and unitary matrix U such that $J = U^* J_s U$, where J_s is diagonal with diagonal entries in $\{-1, 1\}$. Then the system $\left[\begin{smallmatrix} A_s & B_s \\ C_s & D_s \end{smallmatrix}\right] := \left[\begin{smallmatrix} U_s A U_s^* & U_s B \\ C U_s^* & D \end{smallmatrix}\right]$ is balanced, and its reachability and observability maps fulfill $\Psi_s^*\Psi_s = \Phi_s\Phi_s^* = \Psi^*\Psi$. We further have

$$\begin{bmatrix} J_s & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}^* \begin{bmatrix} J_s & 0 \\ 0 & I \end{bmatrix}.$$

It remains to be shown that if J is diagonal then its diagonal entries must be $\frac{\lambda_i}{|\lambda_i|}$. This however follows from the fact that the non-zero spectrum of $H = \Psi\Phi$ coincides with the non-zero spectrum of $\Phi\Psi$, that $\Phi\Psi$ equals the diagonal matrix $J\Psi^*\Psi$, that $|\lambda_i|$ equal the diagonal elements of $\Psi^*\Psi$ and that $J_{ii} \in \{-1, 1\}$. □

Remark 7. Note that systems that fulfill (3) for some involutive and self-adjoint $J \in \mathbb{R}^{n\times n}$ are self-adjoint, since

$$\begin{aligned} G(s) &= D + C(sI - A)^{-1}B = D^* + C(sI - A)^{-1}JC^* \\ &= D^* + CJ(sI - A^*)^{-1}C^* \\ &= D^* + B^*(sI - A^*)^{-1}C^* = G^\dagger(s). \end{aligned} \tag{4}$$

Remark 8. Systems with a signature structure (3) arise naturally in energy-based modelling of physical systems [20]. The state signature J stands for different types of reactive elements (such as, for instance inductances/capacitances in electrical circuit models [19], masses/springs in models for mechanical systems [16]). The quadratic form defined by the signature has the physical interpretation of a *Lagrangian* of the system [20].

As is well-known, the chosen subset of the Hankel singular values is retained in balanced truncation. The following lemma shows that, in the self-adjoint case, the same is true for the Hankel eigenvalues. Moreover, the lemma shows that balanced truncation preserves self-adjointness.

Lemma 9. *Let $G \in \mathbb{C}(s)^{m\times m}$ be stable and self-adjoint. Denote the Hankel eigenvalues of G by $(\lambda_k)_{k=1}^n$. For $r \in \{1, \dots, \ell\}$ let G_r be the balanced truncation of G, and let $q := \sum_{j=1}^r m_j$, where $(m_j)_{j=1}^\ell$ denote the multiplicities of the Hankel singular values of G. Then G_r is self-adjoint and the Hankel eigenvalues of G_r are $(\lambda_k)_{k=1}^q$.*

Proof. By Lemma 6, it is no loss of generality to assume that (3) is fulfilled for some diagonal matrix $J \in \mathbb{R}^{n\times n}$ with $J_{ii} = \frac{\lambda_i}{|\lambda_i|}$. Then, for $Z_r = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] \in \mathbb{C}^{n\times q}$, define $J^r := Z_r^* J Z_r$. It follows from (3) that

$$\begin{bmatrix} J^r & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_r & B_r \\ C_r & D \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D \end{bmatrix}^* \begin{bmatrix} J^r & 0 \\ 0 & I \end{bmatrix},$$

where $\left[\begin{smallmatrix} A_r & B_r \\ C_r & D \end{smallmatrix}\right]$ is the balanced truncation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then, by applying Remark 8 to G_r, we see that G_r is self-adjoint. Since J_r is diagonal, it follows from Lemma 6 that

$$J_{ii}^r = \frac{\lambda_i^r}{|\lambda_i^r|},$$

where the λ_i^r are the Hankel eigenvalues of G_r. By the definition $J^r := Z_r^* J Z_r$ we have $J_{ii}^r = \frac{\lambda_i}{|\lambda_i|}$. Thus we have

$$\frac{\lambda_i}{|\lambda_i|} = \frac{\lambda_i^r}{|\lambda_i^r|}.$$

Since for a self-adjoint operator the absolute values of the eigenvalues are the singular values and the Hankel singular values are preserved under balanced truncation, it follows that, in the case considered, the Hankel eigenvalues are preserved under balanced truncation: $\lambda_i^r = \lambda_i$ for $i = 1, \dots, q$. □

The following result is a specialization of the main result of [3] to the rational case.

Lemma 10. *Let $G \in \mathbb{C}(s)^{m\times m}$ be stable and self-adjoint. Denote the Hankel eigenvalues of G by $(\lambda_k)_{k=1}^n$. Then*

$$\operatorname{trace}(G(0) - G(\infty)) = 2\sum_{k=1}^n \lambda_k.$$

Combining Lemma 10 with Lemma 9, we obtain the following.

Proposition 11. *Let $G \in \mathbb{C}(s)^{m\times m}$ be stable and self-adjoint. Denote the Hankel eigenvalues of G by $(\lambda_k)_{k=1}^n$. For $r \in \{1, \dots, \ell\}$ let G_r be the balanced truncation of G, and let $q := \sum_{j=1}^r m_j$, where $(m_j)_{j=1}^\ell$ denote the multiplicities of the Hankel singular values of G. Then*

$$2\left|\sum_{k=q+1}^n \lambda_k\right| \le \sup_{\zeta\in\mathbb{C}:\operatorname{Re}\zeta>0} \|G(\zeta) - G_r(\zeta)\|_1.$$

Proof. Using that $G(\infty) = G_r(\infty)$ and applying Lemma 10 to both G and G_r we have, with λ_i^r the Hankel eigenvalues of G_r,

$$\begin{aligned} &2\sum_{k=1}^n \lambda_k - 2\sum_{k=1}^q \lambda_k^r \\ &= \operatorname{trace}(G(0) - G(\infty)) - \operatorname{trace}(G_r(0) - G_r(\infty)) \\ &= \operatorname{trace}(G(0) - G_r(0)). \end{aligned} \tag{5}$$

By Lemma 9 we have $\lambda_k^r = \lambda_k$ for $k = 1, \ldots, q$, so that the left-hand side of (5) equals

$$2 \sum_{k=q+1}^{n} \lambda_k.$$

Using that the absolute value of the trace does not exceed the trace class norm, the absolute value of the right-hand side of (5) is at most $\|G(0) - G_r(0)\|_1$. In turn this is not larger than

$$\sup_{\zeta \in \mathbb{C}:\mathrm{Re}\zeta > 0} \|G(\zeta) - G_r(\zeta)\|_1.$$

We conclude that

$$2 \left| \sum_{k=q+1}^{n} \lambda_k \right| \leq \sup_{\zeta \in \mathbb{C}:\mathrm{Re}\zeta > 0} \|G(\zeta) - G_r(\zeta)\|_1.$$

□

Remark 12. The real-valued SISO case of Proposition 11, in slightly different language, is the main result of [11] (note that in the SISO case all Schatten norms are the same and that in the real-valued SISO case every transfer function is self-adjoint).

IV. Systems with a semi-definite Hankel operator

The following proposition establishes a lower bound where as matrix norm we choose the trace class norm (i.e. the Schatten 1-norm) instead of the usual operator norm (i.e. the Schatten ∞-norm).

Proposition 13. *Let $G \in \mathbb{C}(s)^{m \times m}$ be stable and self-adjoint with a Hankel operator which is either positive semi-definite or negative semi-definite. Let $(\mu_j)_{j=1}^{\ell}$ denote the sequence of distinct Hankel singular values of G with multiplicities $(m_j)_{j=1}^{\ell}$. For $r \in \{1, \ldots, \ell\}$ let G_r be the balanced truncation of G. Then*

$$2 \sum_{j=r+1}^{\ell} m_j \mu_j \leq \sup_{\zeta \in \mathbb{C}:\mathrm{Re}\zeta > 0} \|G(\zeta) - G_r(\zeta)\|_1. \qquad (6)$$

Proof. Consider the case where the Hankel operator is positive semi-definite. Then the eigenvalues $(\lambda_k)_{k=1}^n$ of the Hankel operator are nonnegative and equal the singular values $(\sigma_k)_{k=1}^n$ of the Hankel operator. Proposition 11 gives (with $q := \sum_{j=1}^{r} m_j$)

$$2 \left| \sum_{k=q+1}^{n} \lambda_k \right| \leq \sup_{\zeta \in \mathbb{C}:\mathrm{Re}\zeta > 0} \|G(\zeta) - G_r(\zeta)\|_1,$$

where, since $\lambda_k = \sigma_k \geq 0$, the left-hand side equals $2 \sum_{k=q+1}^{n} \sigma_k$, which in turn equals $2 \sum_{j=r+1}^{\ell} m_j \mu_j$. We conclude that (6) holds.

If the Hankel operator is negative semi-definite then its eigenvalues are nonpositive and equal to the negatives of the Hankel singular values. The remainder of the argument is as above. □

The following corollary deals with the operator norm (the Schatten ∞-norm).

Corollary 14. *Let $G \in \mathbb{C}(s)^{m \times m}$ be stable and self-adjoint with a Hankel operator which is either positive semi-definite or negative semi-definite. Let $(\mu_j)_{j=1}^{\ell}$ denote the sequence of distinct Hankel singular values of G with multiplicities $(m_j)_{j=1}^{\ell}$. For $r \in \{1, \ldots, \ell\}$ let G_r be the balanced truncation of G. Then*

$$2 \sum_{k=r+1}^{\ell} \frac{m_j}{m} \mu_j \leq \sup_{\zeta \in \mathbb{C}:\mathrm{Re}\zeta > 0} \|G(\zeta) - G_r(\zeta)\|_\infty \leq 2 \sum_{j=r+1}^{\ell} \mu_j.$$

Proof. The upper bound is the standard balanced truncation error bound. For the lower bound we use that for any m-by-m matrix T there holds

$$\|T\|_1 \leq m \|T\|_\infty.$$

This gives, by using Proposition 13:

$$\begin{aligned} \frac{2}{m} \sum_{k=r+1}^{\ell} m_j \mu_j &\leq \frac{1}{m} \sup_{\zeta \in \mathbb{C}:\mathrm{Re}\zeta > 0} \|G(\zeta) - G_r(\zeta)\|_1 \\ &\leq \sup_{\zeta \in \mathbb{C}:\mathrm{Re}\zeta > 0} \|G(\zeta) - G_r(\zeta)\|_\infty. \end{aligned}$$

□

Remark 15. It is easily seen that a state space symmetric system (that is, $A = A^* \in \mathbb{C}^{n \times n}$ negative definite, $C^* = B \in \mathbb{C}^{n \times m}$, $D = D^* \in \mathbb{C}^{m \times m}$) has a Hankel operator which is positive semi-definite. Therefore Proposition 13 and Corollary 14 apply to state space symmetric systems.

Systems with $A = A^*$ negative definite, $C^* = -B$ and $D = D^*$ have a Hankel operator which is negative semi-definite and therefore Proposition 13 and Corollary 14 apply to such systems as well.

Remark 16. In light of Remark 8, systems with a semi-definite Hankel operator arise in physical systems which contain only one type of reactive elements. For instance RC or RL circuits [18] belong to this class. As well, mass-damper or spring-damper systems can also be modeled by systems with a semi-definite Hankel operator [16].

The following corollary shows that for systems with a semi-definite Hankel operator the multiplicities of the nonzero Hankel singular values are bounded from above by the dimension of the input space.

Corollary 17. *Let $G \in \mathbb{C}(s)^{m \times m}$ be stable and self-adjoint with a Hankel operator which is either positive semi-definite or negative semi-definite. Let $(m_j)_{j=1}^{\ell}$ denote the multiplicities of the Hankel singular values of G. Then $m_j \leq m$ for $j = 1, \ldots, \ell$.*

Proof. From Corollary 14 with $r = \ell - 1$, we obtain $2 m_\ell \mu_\ell \leq 2 m \mu_\ell$, which is equivalent to $m_\ell \leq m$. Since we can choose any ordering of the distinct Hankel singular values, we obtain the desired result. □

Remark 18. Let H be the transfer function of a stable single-input single-output state space symmetric system. Define $G \in \mathbb{C}(s)^{m \times m}$ as the diagonal matrix with m copies of H on the diagonal. Then $m_j = m$ for all j. For such G the lower bound and the upper bound in Corollary 14 are equal, showing

that the new lower bound is -in general- the best that can be obtained. This example also shows that for a reasonably large class of MIMO systems the usual balanced truncation upper bound is an equality.

V. Extensions

In this section we briefly mention two extensions to the theory presented in this article. The first considers balanced singular perturbation approximation rather than balanced truncation and the second considers the case of non-rational functions.

A. Balanced singular perturbation approximation

Balanced realizations cannot only be used to define the balanced truncation, but also to define the balanced singular perturbation approximation [10]. The theorems presented in this article for the balanced truncation also hold for the balanced singular perturbation approximation. This follows easily using the reciprocal transformation [13]. Define G^{recip} by $G^{\mathrm{recip}}(s) := G(1/s)$. If $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a realization of G, then a realization of G^{recip} is $\begin{bmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & D-CA^{-1}B \end{bmatrix}$. It is shown in [10] that the reachability and observability maps of the system and its reciprocal are related by $\Phi^{\mathrm{recip}}(\Phi^{\mathrm{recip}})^* = \Phi\Phi^*$ and $(\Psi^{\mathrm{recip}})^*\Psi^{\mathrm{recip}} = \Psi^*\Psi$. In particular, $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is balanced, if, and only if, $\begin{bmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & D-CA^{-1}B \end{bmatrix}$ is balanced. This implies that G^{recip} has the same Hankel singular values (with the same multiplicities) as G. It can be furthermore concluded from (3) that self-adjointness of G (which is clearly equivalent to the self-adjointness of G^{recip}), implies that a balanced realization of the reciprocal system fulfills

$$\begin{bmatrix} -J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & D-CA^{-1}B \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & D-CA^{-1}B \end{bmatrix}^* \begin{bmatrix} -J & 0 \\ 0 & I \end{bmatrix}.$$

Lemma 6 then implies that the Hankel eigenvalues of G^{recip} are (with the same multiplicities) the negatives of the Hankel eigenvalues of G. In particular, G has a positive (negative) semi-definite Hankel operator if, and only if, the Hankel operator of G^{recip} is negative (positive) semi-definite.

Let G_{spa} be the balanced singular perturbation approximation of G and let $(G^{\mathrm{recip}})_r$ be the balanced truncation of G^{recip}. Then $G_{\mathrm{spa}}(s) = (G^{\mathrm{recip}})_r(1/s)$, see [13, Figure 1]. Therefore

$$G(s) - G_{\mathrm{spa}}(s) = G^{\mathrm{recip}}(1/s) - (G^{\mathrm{recip}})_r(1/s).$$

Since $s \mapsto 1/s$ is a bijection of the open right-half complex plane, we obtain that

$$\begin{aligned} &\sup_{\zeta\in\mathbb{C}:\mathrm{Re}\zeta>0} \|G(\zeta) - G_{\mathrm{spa}}(\zeta)\| \\ &= \sup_{\zeta\in\mathbb{C}:\mathrm{Re}\zeta>0} \|G^{\mathrm{recip}}(\zeta) - (G^{\mathrm{recip}})_r(\zeta)\|, \end{aligned}$$

for any matrix norm.

The results in this article applied to the right-hand side then lead to the corresponding results for the left-hand side. The consequence is that we can simply replace G_r by G_{spa} in the statements of the theorems.

B. The non-rational case

The theorems presented in this article continue to hold for non-rational matrix-valued functions as long as the Hankel operator is trace class, i.e. $\sum_{k=1}^{\infty} \sigma_k < \infty$ (see e.g. [7], [8] for this class of systems). We note that the upper bound in Corollary 14 was proven in [8], [7, Section 5.4]. Lemma 8 can be proven utilizing the discrete-time infinite-dimensional result [4, Theorem 5.1] translated to continuous-time using the usual linear fractional transformation (Cayley transform) given in e.g. [15] as replacement for the reference to [21, Theorem II]. The remainder of the proofs can remain unchanged.

An example of a state space symmetric system with a trace class Hankel operator is the following boundary controlled heat equation on the state space $L^2(0,1)$:

$$\begin{aligned} &\tfrac{\partial x}{\partial t}(t,\xi) = \tfrac{\partial^2 x}{\partial \xi^2}(t,\xi), \\ &x(t,0) - \tfrac{\partial x}{\partial \xi}(t,0) = u_1(t), \quad x(t,1) + \tfrac{\partial x}{\partial \xi}(t,1) = u_2(t), \\ &y_1(t) = x(t,0), \quad y_2(t) = x(t,1). \end{aligned}$$

VI. Examples

As a simple illustration of the obtained theoretical results, we consider two RC ladder circuits. Each of the circuits contains two current sources; the input is formed by the currents of the sources at the right and left of the circuit. The output is the negative of the voltages at the current sources. The first circuit contains seven resistances with resistance value $\mathcal{R}$, and four capacitances with capacitance value $\mathcal{C}$, the second circuit six resistances with resistance value $\mathcal{R}$, and four capacitances with capacitance value $\mathcal{C}$. As state, we consider the vector containing the voltages at the capacitances. Using

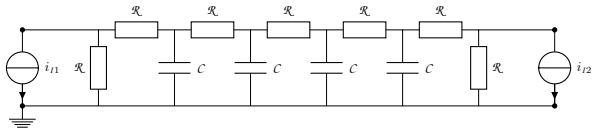

Figure 1. RC ladder circuit

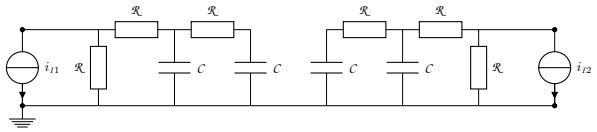

Figure 2. RC ladder circuit

Kirchhoff's laws and the component relations [19], the first circuit is modelled by a system with

$$A = \begin{bmatrix} -\frac{3}{2\mathcal{RC}} & \frac{1}{\mathcal{RC}} & 0 & 0 \\ \frac{1}{\mathcal{RC}} & -\frac{2}{\mathcal{RC}} & \frac{1}{\mathcal{RC}} & 0 \\ 0 & \frac{1}{\mathcal{RC}} & -\frac{2}{\mathcal{RC}} & \frac{1}{\mathcal{RC}} \\ 0 & 0 & \frac{1}{\mathcal{RC}} & -\frac{3}{2\mathcal{RC}} \end{bmatrix},$$

$$B = C^* = \begin{bmatrix} -\frac{1}{2\mathcal{C}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{2\mathcal{C}} \end{bmatrix}, \quad D = \begin{bmatrix} \frac{\mathcal{R}}{2} & 0 \\ 0 & \frac{\mathcal{R}}{2} \end{bmatrix}.$$

The second circuit can be modelled by a system with

$$A = \begin{bmatrix} -\frac{3}{2\mathcal{RC}} & \frac{1}{\mathcal{RC}} & 0 & 0 \\ \frac{1}{\mathcal{RC}} & -\frac{2}{\mathcal{RC}} & 0 & 0 \\ 0 & 0 & -\frac{2}{\mathcal{RC}} & \frac{1}{\mathcal{RC}} \\ 0 & 0 & \frac{1}{\mathcal{RC}} & -\frac{3}{2\mathcal{RC}} \end{bmatrix},$$

$$B = C^* = \begin{bmatrix} -\frac{1}{2\mathcal{C}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{2\mathcal{C}} \end{bmatrix}, \quad D = \begin{bmatrix} \frac{\mathcal{R}}{2} & 0 \\ 0 & \frac{\mathcal{R}}{2} \end{bmatrix}.$$

Note that in both cases we have (3) with $J = I$ (i.e. we have a state space symmetric system). Therefore, the Hankel operators of both circuits are positive semi-definite. We choose $\mathcal{R} = \mathcal{C} = 1$. For the first circuit the Hankel singular values are then

$$0.2281, \quad 0.1050, \quad 0.0219, \quad 0.0021,$$

the upper bound is therefore 0.0480, the lower bound is 0.0240 and the actual error can be computed to be 0.0438. For the second circuit (which satisfies the conditions of Remark 18) the Hankel singular values are

$$0.1197, \quad 0.1197, \quad 0.0053, \quad 0.0053,$$

the upper bound, lower bound and actual error are identical and are equal to 0.01066.

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