THE CAUCHY PROBLEM FOR A TENTH-ORDER THIN FILM EQUATION II. OSCILLATORY SOURCE-TYPE AND FUNDAMENTAL SIMILARITY SOLUTIONS

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ABSTRACT. Fundamental global similarity solutions of the standard form

$$u_\gamma(x,t) = t^{-\alpha_\gamma} f_\gamma(y), \text{ with the rescaled variable } y = \tfrac{x}{t^{\beta}_\gamma},\ \beta_\gamma = \tfrac{1-n\alpha_\gamma}{10},$$

where $\alpha_\gamma > 0$ are real *nonlinear eigenvalues* (γ is a multiindex in $\mathbb{R}^N$) of the tenth-order thin film equation (TFE-10)

i1a (0.1) $$u_t = \nabla \cdot (|u|^n \nabla \Delta^4 u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+\,, \quad n > 0,$$

are studied. The present paper continues the study began in [1], where the following first question was addressed:

(I) Passing to the limit $n \to 0^+$ in (0.1) on any compact subsets $\{|y| \le C\}$ by using Hermitian non-self-adjoint spectral theory for a pair of rescaled non-symmetric operators $\{\mathbf{B}, \mathbf{B}^*\}$ of corresponding to the linear poly-harmonic equation

$$u_t = \Delta^5 u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where} \quad \mathbf{B} = \Delta^5 + \tfrac{1}{10}\, y \cdot \nabla + \tfrac{N}{10} I, \quad \mathbf{B}^* = \Delta^5 - \tfrac{1}{10}\, y \cdot \nabla\,.$$

This allowed to identify a countable family of nonlinear eigenfunctions for (0.1), at least, for small $n > 0$, which defined proper solutions of the Cauchy problem for the TFE-10.

Here, the following questions are under scrutiny:

(II) Further study of the limit $n \to 0$, where the behaviour of finite interfaces and solutions close by (i.e., as $y \to \infty$) are described. In particular, for $N = 1$, the interfaces are shown to diverge as follows:

$$|x_0(t)| \sim 10 \left(\tfrac{1}{n} \sec\left(\tfrac{4\pi}{9}\right)\right)^{\frac{9}{10}} t^{\frac{1}{10}} \to \infty \quad \text{as} \quad n \to 0^+.$$

(III) For a fixed $n \in (0, \frac{9}{8})$, oscillatory structures of solutions that occur near interfaces.

(IV) Again, for a fixed $n \in (0, \frac{9}{8})$, global structures of some of nonlinear eigenfunctions $\{f_\gamma\}_{|\gamma| \ge 0}$ by a combination of numerical and analytical methods.

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S1

1. The TFE-10 and nonlinear eigenvalue problem

1.1. **Main model and previous results.** We study the global-in-time behaviour of compactly supported solutions of the Cauchy problem of a tenth-order quasilinear evolution equation of parabolic type, called the *thin film equation* (TFE–10)

i1
$$u_t = \nabla \cdot (|u|^n \nabla \Delta^4 u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \,, \tag{1.1}$$

where $\nabla = \mathrm{grad}_x$ and $n > 0$ is a real parameter. In view of the degenerate mobility coefficient $|u|^n$, equation (1.1) is written for solutions of changing sign, which can occur in the *Cauchy problem* (CP) and also in some *free boundary problems* (FBPs).

Equation (1.1) has been chosen as a typical higher-order quasilinear degenerate parabolic model, Although the fourth-order version has been the most studied, higher-order quasilinear degenerate equations are known to occur in several applications and, during the last ten-fifteen years, have began to steadily penetrate into modern nonlinear PDE theory; see a number of references/results in [12, § 1.1] and in [7, 21, 22].

For convenience, let us first state the main result obtained in our previous paper [1], a study to be continued here. Thus, in [1], we introduced global self-similar solutions of (1.1) of the standard form

sf3
$$u(x,t) := t^{-\alpha} f(y), \quad \text{with} \quad y := \tfrac{x}{t^\beta}, \quad \beta = \tfrac{1-n\alpha}{10}, \tag{1.2}$$

where $\alpha > 0$ stands for so-called real *nonlinear eigenvalues* and the *nonlinear eigenfuncions* f satisfy an elliptic equation

self1
$$\nabla \cdot (|f|^n \nabla \Delta^4 f) + \tfrac{1-\alpha n}{10}\, y \cdot \nabla f + \alpha f = 0, \quad f \in C_0(\mathbb{R}^N)\,, \tag{1.3}$$

Then, we state a *nonlinear eigenvalue problem* for pairs $\{\alpha, f\}$[1] where the problem setting includes finite propagation phenomena for such TFEs, i.e., f is assumed to be compactly supported, $f \in C_0(\mathbb{R}^N)$. This is a kind of an assumed "minimal" behaviour of $f(y)$ as $y \to \infty$, which naturally accompany many standard singular Sturm–Liouville problems and others.

Using long-established terminology, we call such similarity solutions (1.2) (and also the corresponding profiles f) to be a sequence of *fundamental solutions.* Though, actually, the classic fundamental solution is the first radially symmetric one (with the first kernel $f_0 = f_0(|y|)$), which is the *instantaneous source-type solution* of (1.1) with Dirac's delta as initial data. Moreover, for $n = 0$, $f_0(|y|)$ becomes the actual rescaled kernel of the fundamental solution of the linear operator $D_t - \Delta_x^5$.

Our main goal in [1] was to show *analytically* that, at least, for small $n > 0$,

main1
$$\text{(1.3) admits a countable set of fundamental solutions } \Phi(n) = \{\alpha_\gamma, f_\gamma\}_{|\gamma| \ge 0}, \tag{1.4}$$

[1]More precisely, since the eigenvalue α enters not only the standard term αf, but also the linear differential one $\frac{1-\alpha n}{10}\, y \cdot \nabla f$, it is more correct to talk about a "linear (in α) spectral pencil for the quasilinear TFE-10 operator". Though, for simplicity, we keep referring to the nonlinear eigenvalue problem. In contrast to these nonlinear issues, for $n = 0$, the second term looses α, and we arrive a standard linear eigenvalue problem for the non-self-adjoint operator $\mathbf{B} = \Delta^5 + \frac{1}{10}\, y \cdot \nabla + \frac{N}{10}\, I$; see Section 3.

where γ is a multiindex in $\mathbb{R}^N$ to numerate these eigenvalue-eigenfunction pairs.

Indeed, studying the nonlinear eigenvalue problem (1.3) in [1], we performed a "homotopic deformation" of (1.1) as $n \to 0^+$ and reducing it to the classic *poly-harmonic equation of tenth order*

lin5 (1.5)
$$u_t = \Delta^5 u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \,.$$

More precisely, we answered in [1] the following question:

(I) Passing to the limit $n \to 0^+$ in (0.1) on any compact subsets $\{|y| \le C\}$ by using Hermitian non-self-adjoint spectral theory for a pair of rescaled non-symmetric operators $\{\mathbf{B}, \mathbf{B}^*\}$ of corresponding to the linear poly-harmonic equation (1.5):

Boper (1.6)
$$\mathbf{B} = \Delta^5 + \tfrac{1}{10}\, y \cdot \nabla + \tfrac{N}{10} I \quad \text{and} \quad \mathbf{B}^* = \Delta^5 - \tfrac{1}{10}\, y \cdot \nabla \,.$$

The corresponding problem (1.3) then reduces to a standard (but not self-adjoint) Hermitian-type linear eigenvalue problem for the pair $\{\mathbf{B}, \mathbf{B}^*\}$. Therefore, according to this approach, the nonlinear version of (1.4) has the origin in the discreteness-reality of the spectrum of the corresponding linear operator $\mathbf{B}$.

This allowed to identify a countable family of nonlinear eigenfunctions for (0.1), at least, for small $n > 0$, which defined proper solutions of the Cauchy problem for the TFE-10.

1.2. **Main new results and layout of the paper.** In the present paper, our main "non-local" goal is to verify a possibility of global extensions of such "n-branches" of some first fundamental solutions, which was then checked numerically. A couple of such preliminary results were already available in [1, § 5].

To do so, first we analyse the limiting behaviour of the problem (1.1) in the one dimensional case obtaining an approximating structure for the solutions satisfying (1.1) close to the interfaces. Also in one-dimension we ascertain via numerical and analytical methods the existence of periodic oscillatory solutions for $n \in (0, \frac{9}{8})$.

Furthermore, through a homotic approach, and using the standard degree theory, we ascertain the existence of a countable family of similarity profiles globally for the equation (1.1).

Section 2 is devoted to similarity solutions and derivation of the corresponding nonlinear eigenvalue problem. Later, we address the following questions:

(II) Section 3: Further study of the limit $n \to 0$, where the behaviour of finite interfaces and solutions close by (i.e., as $y \to \infty$) are described. Analysing the limiting behaviour when $n \to 0^+$ in the one dimensional case we obtain that the non-uniform solution in this limit comprises two regions

$$\text{an Inner region} \quad \{x = O(1)\} \quad \text{and} \quad \text{an Outer region} \quad \{x = O(n^{-9/10})\},$$

in which u is exponentially small. In the Inner region $|u|^n \sim 1$ for small $n > 0$. Hence, the asymptotic behaviour of the solution tells us that the solution satisfies poly-harmonic equation of the form

$$\frac{\partial u_0}{\partial t} = \frac{\partial^{10} u_0}{\partial x^{10}},$$

Moreover, from the performed analysis it is clear that this solution breaks down when $\{x = O(n^{-9/10})\}$, the Outer region. In particular, the interfaces $x = x_0(t)$ are shown to diverge as follows:

$$|x_0(t)| \sim 10 \left(\tfrac{1}{n} \sec\left(\tfrac{4\pi}{9}\right)\right)^{\frac{9}{10}} t^{\frac{1}{10}} \to \infty \quad \text{as} \quad n \to 0.$$

By a similar analysis we also obtain the structure of the eigenfunctions satisfying

ODE1 (1.7) $$(|f|^n f^{(9)})' + \tfrac{1-\alpha n}{10}\, y\, f' + \alpha f = 0, \quad f \in C_0(\mathbb{R})\,.$$

(III) Section 4: For a fixed $n \in (0, \frac{9}{8})$, oscillatory structures that occur near interfaces. In one dimension, we study the local behaviour near the interface $y = \frac{x_0}{t^\beta}$ for the similarity ODE (1.7) assuming similarity profiles $f(y)$ with maximal regularity at the interface $y = y_0$, so that, being extended by $f = 0$ for $y = y_0$.

(IV) Section 6: Again, for a fixed $n \in (0, \frac{9}{8})$, global structures of some of nonlinear eigenfunctions $\{f_\gamma\}_{|\gamma| \ge 0}$ by combination of numerical and analytical methods.

Using a homotopy transformation of the form

$$|\varepsilon^2 + f^2|^{\frac{n}{2}},$$

and applying the standard degree theory we perform a double limit when $\varepsilon, n \to 0^+$ obtaining existence and multiplicity results for the oscillatory solutions of changing sign of the non-linear elliptic equation (1.3). Indeed, we arrive at the existence of a countable family of solutions by a direct n-expansion from the solution of the linear elliptic equation whose operator is denoted by (1.6).

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PAC, JDE: more comments here and changes AFTER those Sections are
ready, easy...
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1.3. **Possible origins of discrete nonlinear spectra and principle difficulties.** It is key for us that (1.3) *is not variational*, so we cannot use powerful tools such as Lusternik–Schnirel'man (L–S, for short) category-genus theory, fibering, and other well-known approaches, which in many cases are known to provide at least a *countable* family of critical points (solutions) in the non-coercive case, when the category of the functional subset involved is typically infinite.

It is also crucial and well known that the L–S min-max approach *does not detect all families of critical points.* However, sometimes it can revive some amount of solutions. A somehow special example was revealed in [17, 18], where key features of those variational L–S and fibering approaches applied are described. Namely, for some variational fourth-order and higher-order ODEs in $\mathbb{R}$, including those with the typical non-linearity $|f|^n f$, as above,

mm.561 (1.8) $$-(|f|^n f)^{(4)} + |f|^n f = \tfrac{1}{n} f \quad \text{in} \quad \mathbb{R}, \quad f \in C_0(\mathbb{R}) \quad (n > 0),$$

as well as for the following standard looking one with the only cubic nonlinearity [18, § 6]:

anal1 (1.9) $$-f^{(4)} + f = f^3 \quad \text{in} \quad \mathbb{R}, \quad f \in H^4_\rho(\mathbb{R}) \quad (\rho = \mathrm{e}^{a|y|^{4/3}}, \; a > 0 \; \text{small}).$$

It was shown then that these equations admit a *countable set of countable families of solutions*, while the L–S/fibering approach detects only ONE such a discrete family of (min-max) critical points. Further countable families are not expected to be determined easily by more advanced techniques of potential theory, such as the mountain pass theorem, fibering methods, or others. Existence of other, not L–S type critical points for (1.8) and (1.9), were shown in [17, 18] by using a combination of numerical and (often, formal) analytic methods and heavy use of oscillatory nature of solutions close to finite interfaces (for (1.8)) and at infinity (for (1.9)). In particular, detecting the corresponding L–S countable sequence of critical points was done *numerically*, i.e., by checking their actual min-max features (their critical values must be maximal among other solutions belonging to the functional subset of a given category, and having a "suitable geometric shape").

Therefore, even in the variational setting, counting various families of critical points and values represents a difficult open problem for such higher-order ODEs, to say nothing of their elliptic counterparts in $\mathbb{R}^N$.

Hence, in [1], we relied on a different approach, in particular, a "homotopic deformation" of (1.1) as $n \to 0^+$, which is also effective for such difficult variational problems and detects more solutions than L–S/fibering theory (though only locally upon the parameter).

1.4. **The second model: bifurcations in $\mathbb{R}^2$.** To extend our homotopy approach to a more complicated *unstable thin film equation* (TFE–10) in the critical case

e1 (1.10) $$u_t = \nabla \cdot (|u|^n \nabla \Delta^4 u) - \Delta(|u|^{p-1}u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad p > n+1,$$

with the extra unstable diffusion term. We obtain a discrete real nonlinear spectrum for (1.10) that requires a simultaneous *double* homotopy deformation $n \to 0^+$ and $p \to 1^+$ leading to a new linear Hermitian spectral theory. We do not develop it here and just focus on a principal opportunity to detect a discrete nonlinear spectrum for (1.10). More details on blow-up and global similarity solutions (as unique extensions after blow-up) of (1.10) can be found in [2].

1.5. **Global extension of bifurcation branches: a principal open problem.** It is worth mentioning that, for both problems (1.3) and the corresponding problem occurring for (1.10) (after the similarity time-scaling), a global extension of bifurcation n-branches ((n,p)-branches for (1.10)) represents a difficult open problem of general nonlinear operator theory. Moreover, as was shown in [16] (see also other examples in [18]), the TFE-4 with absorption $-|u|^{p-1}u$ (instead of the backward-in-time diffusion as in (1.3)), depending on not that small $n \sim 1$, admits some p-bifurcation branches having turning (saddle-node) points and thus representing closed loops Hence, these branches *are* *not globally extendable* in principle. On the other hand, for equations with monotone operators such as the PME-4

PME4

$$u_t = -(|u|^n u)_{xxxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+ , \tag{1.11}$$

the n-branches seem to be globally extensible in $n > 0$, [15].

S2

2. Problem setting and self-similar solutions

2.1. **The FBP and CP.** As done previously in [10]–[13], we distinguish the standard free-boundary problem (FBP) for (1.1) and the *Cauchy problem*; see further details therein.

For both the FBP and the CP, the solutions are assumed to satisfy standard free-boundary conditions or boundary conditions at infinity:

i3

$$\begin{cases} u = 0, & \text{zero-height,} \\ \nabla u = \nabla^2 u = \nabla^3 u = \nabla^4 u = 0, & \\ -\mathbf{n} \cdot (|u|^n \nabla \Delta^4 u) = 0, & \text{conservation of mass (zero-flux)} \end{cases} \tag{2.1}$$

at the singularity surface (interface) $\Gamma_0[u]$, which is the lateral boundary of

gamma1

$$\text{supp}\, u \subset \mathbb{R}^N \times \mathbb{R}_+, \quad N \geq 1 , \tag{2.2}$$

where $\mathbf{n}$ stands for the unit outward normal to $\Gamma_0[u]$. Note that, for sufficiently smooth interfaces, the condition on the flux can be read as

$$\lim_{\text{dist}(x, \Gamma_0[u]) \downarrow 0} -\mathbf{n} \cdot \nabla(|u|^n \Delta^4 u) = 0.$$

This condition is directly related with the conservation of mass.

Moreover, we also assume bounded, smooth, integrable, compactly supported initial data

i4

$$u(x, 0) = u_0(x) \quad \text{in} \quad \Gamma_0[u] \cap \{t = 0\}. \tag{2.3}$$

For the CP, the assumption of nonnegativity is got rid of, and solutions become oscillatory close to interfaces. It is then key, for the CP, that the solutions are expected to be "smoother" at the interface than those for the FBP, i.e., (2.1) are not sufficient to define their regularity. These *maximal regularity* issues for the CP, leading to oscillatory solutions, are under scrutiny in [11] for a fourth-order case.

In the CP for (1.1) in $\mathbb{R}^N \times \mathbb{R}_+$, one needs to pose bounded compactly supported initial data (2.3) prescribed in $\mathbb{R}^N$.

2.2. **Global similarity solutions: a nonlinear eigenvalue problem.** We now specify the self-similar solutions of the equation (1.1), which are admitted due to its natural scaling-invariant nature. In the case of the mass being conserved, we have global in time source-type solutions

$$u(x,t) := t^{-\alpha} f\big(\tfrac{x}{t^{\beta}}\big), \quad \alpha = \tfrac{1-10\beta}{n},$$

with f solving the quasilinear elliptic equation (nonlinear eigenvalue problem) given in (1.3). We add to the elliptic equation a natural assumption that f must be compactly supported (and, of course, sufficiently smooth at the interface, which is an accompanying question to be discussed as well). For further details of how to obtain them see [1].

Thus, for such degenerate elliptic equations, the functional setting of (1.3) assumes that we are looking for (weak) *compactly supported* solutions $f(y)$ as certain “nonlinear eigenfunctions” that hopefully occur for special values of nonlinear eigenvalues $\{\alpha_\gamma\}_{|\gamma|\geq 0}$. Therefore, our goal is to justify that (1.4) holds.

Concerning the well-known properties of finite propagation for TFEs, we refer to papers [10]–[13], where a large amount of earlier references are available; see also [17, 18] for more recent results and references in this elliptic area.

However, one should observe that there are still a few entirely rigorous results, especially those that are attributed to the Cauchy problem for TFEs.

In the linear case $n = 0$, the condition $f \in C_0(\mathbb{R}^N)$, is naturally replaced by the requirement that the eigenfunctions $\psi_\beta(y)$ exhibit typical exponential decay at infinity, a property that is reinforced by introducing appropriate weighted L^2-spaces. Complete details about the spectral theory for this linear problem when $n = 0$ in [9]. Actually, using the homotopy limit $n \to 0^+$, we will be obliged for small $n > 0$, instead of C_0-setting in (1.3), to use the following weighted L^2-space:

WW11 (2.4) $$f \in L^2_\rho(\mathbb{R}^N), \quad \text{where} \quad \rho(y) = \mathrm{e}^{a|y|^{10/9}}, \quad a > 0 \ \text{small}.$$

Note that, in the case of the Cauchy problem with conservation of mass making use of the self-similar solutions (1.2), and performing similar computations as done in [1] we have that

alb1 (2.5) $$-\alpha + \beta N = 0 \quad \Longrightarrow \quad \alpha_0(n) = \tfrac{N}{10+Nn} \quad \text{and} \quad \beta_0(n) = \tfrac{1}{10+Nn}.$$

S3 3. The limit $n \to 0$: Behaviour of finite interfaces and nearby solutions

We consider here the singular limit $n \to 0^+$ for the full equation (1.1) in one space dimension $N = 1$. The non-uniform solution in this limit comprises two regions, an *Inner region* $\{x = O(1)\}$, where $u = O(1)$, and an *Outer region* $\{x = O(n^{-\frac{9}{10}})\}$, in which u is exponentially small. The labelling of these regions as inner and outer becomes clearer during the course of the scalings.

We begin with the region $\{x = O(1)\}$, for which $u = O(1)$ and consequently $|u|^n \sim 1$ for small $n > 0$. At leading order $u \sim u_0(x,t)$ satisfies the linear poly-harmonic equation

s3eq1
$$\frac{\partial u_0}{\partial t} = \frac{\partial^{10} u_0}{\partial x^{10}}, \tag{3.1}$$

where u_0 here is the leading order term in an expansion with respect to n (and is not the initial function in (2.3)). We are interested in an oscillatory class of solutions that are analytic in x. The far-field behaviour of (3.1) may be determined using a WKBJ expansion in the form

eq:s3eq2
$$u_0 \sim a(x,t)\mathrm{e}^{-\phi(x,t)} \qquad \text{as} \quad x \to +\infty, \tag{3.2}$$

which gives

s3eq3
$$\frac{\partial \phi}{\partial t} = \left(\frac{\partial \phi}{\partial x}\right)^{10} \quad \text{and} \quad \frac{\partial a}{\partial t} + 10\left(\frac{\partial \phi}{\partial x}\right)^{9}\frac{\partial a}{\partial x} = -45\left(\frac{\partial \phi}{\partial x}\right)^{8}\frac{\partial^2 \phi}{\partial x^2}a. \tag{3.3}$$

The required solutions to (3.3) take the form

s5eq4
$$\phi(x,t) = \phi_\pm(x,t) \equiv \frac{9}{10^{10/9}}\mathrm{e}^{\pm\frac{4\pi i}{9}}\frac{x^{10/9}}{t^{1/9}}, \qquad a = t^{-\frac{1}{2}}\Psi_\pm\left(\frac{x}{t}\right), \tag{3.4}$$

where $\Psi_\pm(\zeta)$ are arbitrary smooth functions (depending on the initial data), but satisfy $\Psi_-(\zeta) = \bar{\Psi}_+(\zeta)$. Thus,

s3eq5
$$\begin{aligned} u_0 \sim{}& \frac{1}{t^{1/2}}\Psi_+\left(\frac{x}{t}\right)\exp\left\{-\frac{9}{10^{10/9}}\mathrm{e}^{\frac{4\pi i}{9}}\left(\frac{x^{10}}{t}\right)^{\frac{1}{9}}\right\} \\ &+\frac{1}{t^{1/2}}\Psi_-\left(\frac{x}{t}\right)\exp\left(-\frac{9}{10^{10/9}}\mathrm{e}^{-\frac{4\pi i}{9}}\left(\frac{x^{10}}{t}\right)^{\frac{1}{9}}\right) \quad \text{as} \quad x \to +\infty. \end{aligned} \tag{3.5}$$

It is clear from (3.5) that this solution breaks down when $x = O(n^{-9/10})$, since we can no longer approximate $|u|^n$ by unity. This suggests the consideration of an outer region with scaling $X = n^{9/10}x$. In $X = O(1)$, the PDE becomes

s3eq6
$$\frac{\partial u}{\partial t} = n^9\frac{\partial}{\partial X}\left(|u|^n\frac{\partial^9 u}{\partial X^9}\right), \tag{3.6}$$

this being a conventional formulation of a singular problem, where the small parameter multiplies the highest derivative. However, as for the fourth and sixth order cases (see [11, 13]), there are fast oscillations superposed on the slow exponential decay that occurs over this length scale, necessitating the application of a multiple scales (Kuzmak) approach. As such we introduce the fast variable

$$Z = \frac{\sigma(X,t)}{n},$$

where $\sigma(X,t)$ will be determined in the standard way by the criterion that the dependence on Z is periodic of constant (rather than (X,t)-dependent) periodicity - without loss of generality, we take the period to be 2π. The multiple-scales ansatz for this region takes the form

s3eq7
$$u \sim \mathrm{e}^{-\Phi(X,t)/n}A(X,Z,t) \qquad \text{as} \quad n \to 0, \tag{3.7}$$

to within an algebraic power of n (which is determined by the far-field behaviour of $\Psi_{\pm}(\zeta)$), wherein Φ is real. Thus, as $n \to 0$,

s3eq8
$$\frac{\partial u}{\partial t} \sim \frac{1}{n}\left(-\frac{\partial \Phi}{\partial t}A + \frac{\partial \sigma}{\partial t}\frac{\partial A}{\partial Z}\right)\mathrm{e}^{-\Phi/n}, \tag{3.8}$$

s3eq9
$$n^9 \frac{\partial}{\partial X}\left(|u|^n \frac{\partial^9 u}{\partial X^9}\right) \sim \frac{1}{n}\mathrm{e}^{-\Phi}\left[\sum_{k=0}^{10}\binom{10}{k}\frac{\partial^k A}{\partial X^k}\left(\frac{\partial \sigma}{\partial X}\right)^k\left(-\frac{\partial \Phi}{\partial X}\right)^{10-k}\right]\mathrm{e}^{-\Phi/n}. \tag{3.9}$$

We remark that these expansions need to be taken to next (i.e., $O(n)$ smaller) order if we are to characterise the dependence of A on X and t; we shall not proceed with such an analysis here. Viewing the balance (3.8) and (3.9) as an ordinary differential equation in Z, we observe that the condition of 2π periodicity in Z requires that

s3eq10
$$A = \alpha_+(X,t)\mathrm{e}^{iZ} + \alpha_-(X,t)\mathrm{e}^{-iZ} \tag{3.10}$$

with $\alpha_- = \bar{\alpha}_+$. Grouping real and imaginary parts, we obtain a coupled system for Φ and σ given by the equations

s3eq11
$$\begin{aligned}\frac{\partial \Phi}{\partial t} = \mathrm{e}^{-\Phi}\Bigg[&\left(\frac{\partial \sigma}{\partial X}\right)^{10} - 45\left(\frac{\partial \sigma}{\partial X}\right)^{8}\left(\frac{\partial \Phi}{\partial X}\right)^{2} + 210\left(\frac{\partial \sigma}{\partial X}\right)^{6}\left(\frac{\partial \Phi}{\partial X}\right)^{4} \\ &-210\left(\frac{\partial \sigma}{\partial X}\right)^{4}\left(\frac{\partial \Phi}{\partial X}\right)^{6} + 45\left(\frac{\partial \sigma}{\partial X}\right)^{2}\left(\frac{\partial \Phi}{\partial X}\right)^{8} - \left(\frac{\partial \Phi}{\partial X}\right)^{10}\Bigg],\end{aligned} \tag{3.11}$$

s3eq12
$$\begin{aligned}\frac{\partial \sigma}{\partial t} = \mathrm{e}^{-\Phi}\Bigg[&-10\left(\frac{\partial \sigma}{\partial X}\right)^{9}\frac{\partial \Phi}{\partial X} + 120\left(\frac{\partial \sigma}{\partial X}\right)^{7}\left(\frac{\partial \Phi}{\partial X}\right)^{3} - 252\left(\frac{\partial \sigma}{\partial X}\right)^{5}\left(\frac{\partial \Phi}{\partial X}\right)^{5} \\ &+120\left(\frac{\partial \sigma}{\partial X}\right)^{3}\left(\frac{\partial \Phi}{\partial X}\right)^{7} - 10\frac{\partial \sigma}{\partial X}\left(\frac{\partial \Phi}{\partial X}\right)^{9}\Bigg].\end{aligned} \tag{3.12}$$

Matching to (3.5) suggests seeking a consistency relation between (3.11) and (3.12) of the form $\sigma = \lambda\Phi$ with λ real, leading to

$$\lambda^{10} - 35\lambda^8 + 90\lambda^6 + 42\lambda^4 - 75\lambda^2 + 9 = 0 \implies (\lambda^2+1)(\lambda^2-3)(\lambda^6 - 33\lambda^4 + 27\lambda^2 - 3) = 0.$$

The appropriate root of this characteristic equation is

s3lambda
$$\lambda = \tan\left(\frac{4\pi}{9}\right), \tag{3.13}$$

this being consistent with the ratio of the imaginary to real parts in the exponentials in (3.5). Consequently, we obtain a Hamilton–Jacobi equation of the form

s3eq13
$$\frac{\partial \Phi}{\partial t} = -\sec^9\left(\frac{4\pi}{9}\right)\mathrm{e}^{-\Phi}\left(\frac{\partial \Phi}{\partial X}\right)^{10}, \tag{3.14}$$

the required solution being

$$\Phi(x,t) = -9\ln\left(1 - \cos\left(\frac{4\pi}{9}\right)\left(\frac{X^{10}}{10^{10}t}\right)^{\frac{1}{9}}\right),$$

which matches successfully with the real part of (3.5) in the limit $X \to 0$. Thus, the leading order solution in this region takes the form

s3eq14
$$u(x,t) \sim \left(1 - \cos\left(\frac{4\pi}{9}\right)\left(\frac{X^{10}}{10^{10}t}\right)^{\frac{1}{9}}\right)^{\frac{9}{n}} A(X,Z,t), \tag{3.15}$$

with A as given in (3.10), this local behaviour having the expected $\frac{9}{n}$ power-law form with oscillations superimposed as in (3.7) and (3.10). The interface $x = x_0(t)$ is thus given by

s3eq15
$$x_0(t) \sim 10\Big(\tfrac{1}{n}\sec\left(\tfrac{4\pi}{9}\right)\Big)^{\frac{9}{10}} t^{\frac{1}{10}} \qquad \text{as} \quad n \to 0\ , \tag{3.16}$$

illustrating its behaviour for small n.

We may also determine the structure of the eigenfunctions satisfying (1.7) in the small n limit. Again, we have a two region structure: an inner region $\{y = O(1)\}$, in which $f = O(1)$ and an outer region $\{y = O(n^{-9/10})\}$ where f is exponentially small. In the inner region $\{y = O(1)\}$, we obtain at leading order ($f \sim f_0$) in n the linear ODE

s3eq16
$$f_0^{(10)} + \tfrac{1}{4}\, y f_0' + \alpha f_0 = 0. \tag{3.17}$$

An explicit general solution can be expressed in terms of hypergeometric functions, easily obtained using e.g. `Maple`. The far-field behaviour of (3.1) may be determined using a WKBJ expansion in the form

eq:s3eq17
$$f_0(y) \sim a(y)\mathrm{e}^{-\phi(y)} \qquad \text{as } y \to +\infty \tag{3.18}$$

which gives

s3eq18
$$10(\phi')^9 = y, \qquad \left(\alpha - 45(\phi')^8\phi''\right) a + \left(\frac{y}{10} - 10(\phi')^9\right) a' = 0. \tag{3.19}$$

The required solutions to (3.19) take the form

s5eq19
$$\phi(y) = \phi_{\pm}(y) \equiv \tfrac{9}{10^{10/9}}\mathrm{e}^{\pm\frac{4\pi i}{9}} y^{10/9}, \qquad a(y) = k_{\pm} y^{5(2\alpha-1)/9}, \tag{3.20}$$

with $k_{\pm}$ arbitrary constants. Thus,

s3eq20
$$\begin{aligned} f_0(y) \sim k_+ y^{5(2\alpha-1)/9} &\exp\Big\{-\tfrac{9}{10^{10/9}}\mathrm{e}^{\frac{4\pi i}{9}} y^{\frac{10}{9}}\Big\} \\ &+ k_- y^{5(2\alpha-1)/9}\exp\Big\{-\tfrac{9}{10^{10/9}}\mathrm{e}^{-\frac{4\pi i}{9}} y^{\frac{10}{9}}\Big\} \quad \text{as} \ \ y \to +\infty. \end{aligned} \tag{3.21}$$

Again, this solution breaks down when $y = O(n^{-9/10})$, suggesting the consideration of an outer region with scaling $Y = n^{9/10}y$. In $Y = O(1)$, we have

s3eq21
$$n^9 \tfrac{\mathrm{d}}{\mathrm{d}Y}\left(|f|^n \tfrac{\mathrm{d}^9 f}{\mathrm{d}Y^9}\right) + \left(\tfrac{1-\alpha n}{10}\right) y \tfrac{\mathrm{d}f}{\mathrm{d}Y} + \alpha f = 0. \tag{3.22}$$

Rather than posing a multiple-scales ansatz directly, we may instead consider

s3eq22
$$f(Y) \sim \mathrm{e}^{b(Y)/n} B(Y) \qquad \text{as} \ \ n \to 0, \tag{3.23}$$

where b is complex in order to match with the inner solution. Thus, at $O(1/n)$ in (3.22) we obtain

s3eq23
$$10|\mathrm{e}^b| \left(\tfrac{\mathrm{d}b}{\mathrm{d}Y}\right)^9 + Y = 0, \tag{3.24}$$

whilst at $O(1)$ we have

s3eq24
$$\left(10(b')^9|\mathrm{e}^b| + \tfrac{Y}{10}\right)\tfrac{\mathrm{d}B}{\mathrm{d}Y} + B\big(\alpha - \alpha\tfrac{Y}{10}b' + (b')^8|\mathrm{e}^b|\left(45b'' + b'^2(1 + \ln|B|)\right)\big) = 0, \tag{3.25}$$

where $'$ denotes $\frac{\mathrm{d}}{\mathrm{d}Y}$ and the approximation

s3eq25 (3.26)
$$|f|^n \sim |\mathrm{e}^b|\,(1 + n \ln |B|)$$

has been used. The solution to (3.24) that matches with (3.21) is

s3eq26 (3.27)
$$b(Y) = (1 \pm i\lambda)\, 9\, \ln\left(1 - \cos\left(\tfrac{4\pi}{9}\right)\left(\tfrac{Y}{10}\right)^{10/9}\right),$$

with λ as given in (3.13). We can in principle determine the amplitude $B(Y)$ via (3.25). The finite interface $y = y_0$, where f vanishes, is thus given by

s3eq27 (3.28)
$$y_0 \sim 10\left(\tfrac{1}{n} \sec\left(\tfrac{4\pi}{9}\right)\right)^{9/10} \qquad \text{as} \;\; n \to 0\,,$$

again illustrating its divergent behaviour for small n.

4. Oscillatory solutions of changing sign near interfaces via periodic structures

S4

Here we examine the local behaviour near the finite interface $y_0 = x_0/t^\beta$ for the similarity ODE (1.7) in one dimension. We consider similarity profiles $f(y)$ exhibiting *maximal regularity* at the interface $y = y_0$, so that, being extended by $f = 0$ for $y > y_0$, these will give solutions of the CP.

4.1. Periodic structure of oscillations near interfaces for $n \in (0, \frac{9}{8})$.

For the thin film ODE (1.7), we have the asymptotic behaviour

s4eq1 (4.1)
$$|f|^n f^{(9)} \sim \lambda_0 f\,, \quad \lambda_0 = \beta y_0 > 0, \qquad \text{as} \quad y \to y_0^-\,,$$

where the no-flux condition in (2.1) has been used. To allow for oscillatory behaviour, we seek solutions in the form

s4eq2 (4.2)
$$f(y) = (y_0 - y)^\mu \phi(\eta)\,, \quad \eta = \ln(y_0 - y), \quad \text{with} \;\; \mu = \tfrac{9}{n},$$

where the oscillatory component φ satisfies the ninth-order autonomous ODE

eqLC (4.3)
$$\sum_{k=0}^{9} a_k \phi^{(9-k)} + \lambda_0 |\phi|^{-n}\phi = 0.$$

The coefficients $\{a_k\}$ are polynomials in μ of degree k, namely

$$a_0 = 1, \quad a_1 = 9(\mu - 4), \quad a_9 = \Pi_{i=0}^{8}(\mu - i),$$

with the others easily obtainable using e.g., `Maple` and not recorded for conciseness.

Fig1LC

We formulate our overall (formal) understanding of the ODE (4.3) as follows:

Conjecture 4.1. *For $n \in (0, \frac{9}{8})$, the ODE* (4.3) *has a unique non-trivial sign-changing periodic solution $\phi_*(\eta)$.*

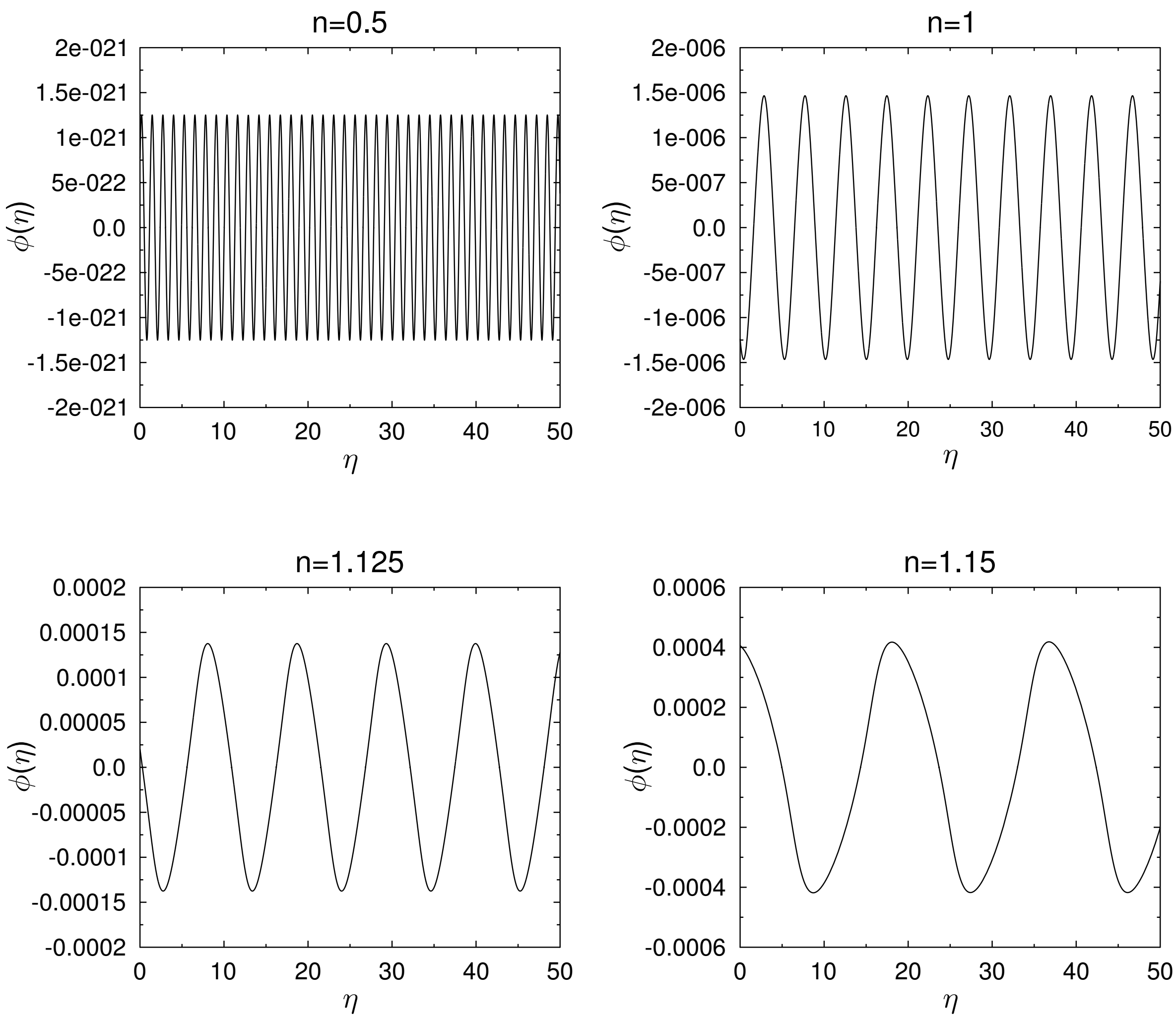

FIGURE 1. Numerical illustration of the limit cycles for selected n. In each case (4.3) with parameter value $\lambda_0 = 1$ was solved as an IVP using `MATLAB` solver `ode15s`. Small error tolerances RelTol and AbsTol of typically 10^{-13} were set, although these were relaxed for the larger n values.

Thus, numerics suggest that this limit cycle is globally stable and is unique (up to translations in η). Figure 1 describes this stable periodic behaviour for selected $n \in (0, \frac{9}{8})$. For $n \in (\frac{9}{8}, \frac{9}{7})$, global stability fails since (4.3) admits also two equilibria $\phi = \pm\phi_0$, where

`Var55` (4.4)
$$\phi_0 = [-\frac{\lambda_0}{\Pi_{i=0}^{8}(\mu - i)}]^{\frac{1}{n}} > 0.$$

The amplitude of the periodic solution decreases markedly as n decreases, which suggests the need to rescale for small n as discussed later.

4.2. **Heteroclinic bifurcation of periodic solutions.** It is crucial for both ODE and PDE theory to find a precise n-interval of existence of periodic, oscillatory solutions of (4.3). Firstly, the stable periodic solution $\phi(\eta)$ persists to exist for $n > \frac{9}{8}$, where the constant solutions $\phi = \pm\phi_0$ are unstable; see Figure 2 for $n = 1.13$ and $n = 1.15$.

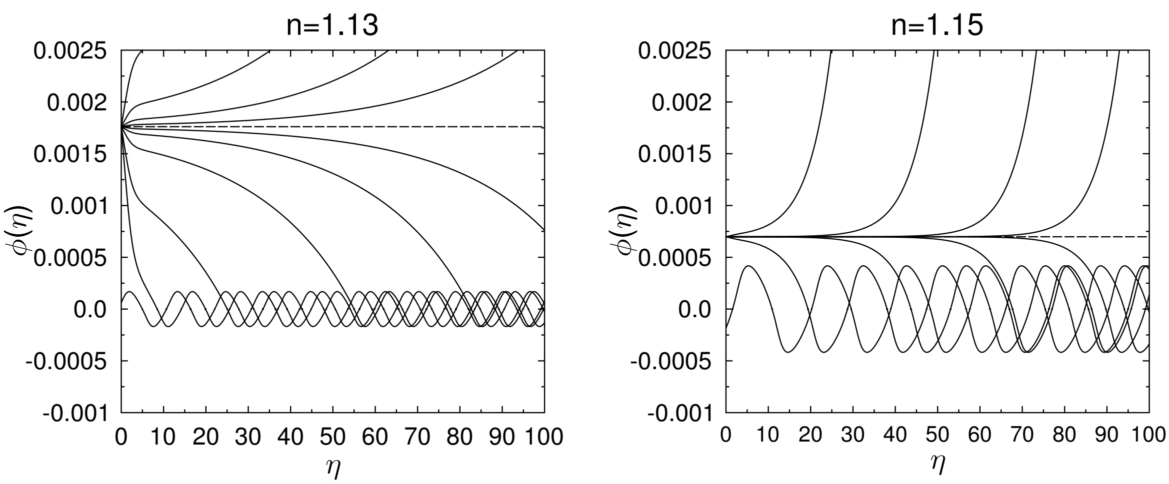

FIGURE 2. Illustration of numerical solutions to (4.3) with $\lambda_0 = 1$ for two selected values of $n \in (\frac{9}{8}, n_{\rm h})$. Shown are solutions leaving the unstable constant solution $\phi = \phi_0$ with the globally stable limit cycle. The figures are symmetric in ϕ, the negative unstable constant solution $\phi = -\phi_0$ being omitted form being shown. Fig2LC

Secondly, as n increases further, the periodic solution is destroyed in a heteroclinic bifurcation, a phenomenon earlier observed for fourth- and sixth-order TFEs [11, 13]. The following conjecture is entirely based on numerical evidence.

Conjecture 4.2. *The stable periodic solution of* (4.3) *exists for all* $n \in (0, n_h)$*, where* $n_h \in (\frac{9}{8}, \frac{9}{7})$ *is a subcritical heteroclinic* ($\phi_0 \mapsto -\phi_0$) *bifurcation point of stable periodic solutions, which cease to exist for all* $n \ge n_h$.

Numerical calculations give

n**1 (4.5) $$\boxed{n_{\rm h} = 1.1572339... \quad \text{(recall that } \tfrac{9}{8} = 1.125 \text{ and } \tfrac{9}{7} = 1.2857...).}$$

Figure 3 shows formation of such a bifurcation as $n \to n_h^-$. To obtain the bold line in Figure 3(B), we took $n = 1.157233919$ (not all decimals being correct). This is a standard scenario for homoclinic/heteroclinic bifurcations, [23, Ch. 4]. A rigorous justification of such non-local bifurcations is an still an open problem.

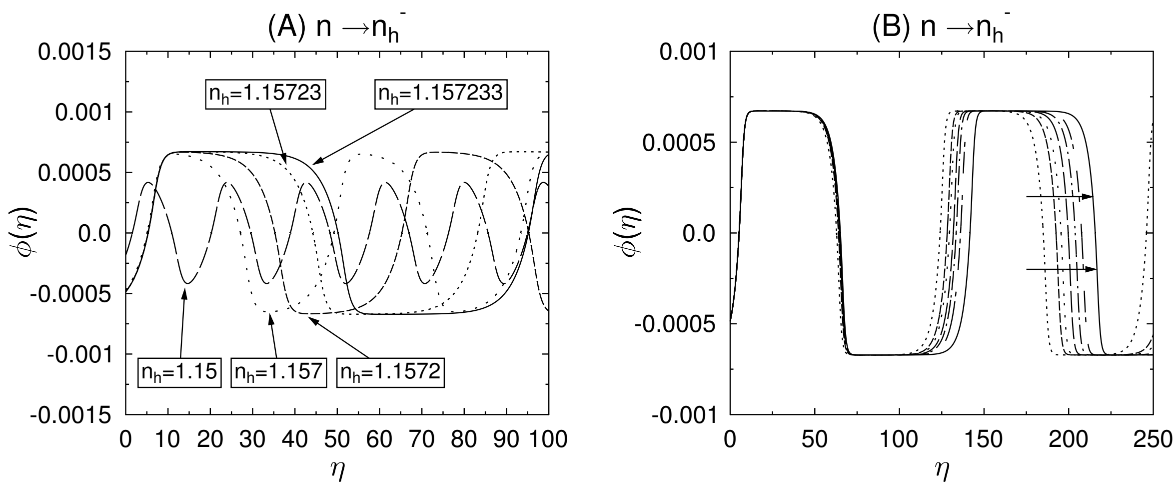

FIGURE 3. Formation of a heteroclinic connection $\phi_0 \to -\phi_0$ for the ODE (4.3), $\lambda_0 = 1$, as $n \to n_{\rm h}^-$.

Fig3LC

Thus, for n larger than $\frac{9}{8}$, not all the solutions are oscillatory near the interfaces. For $n \in (\frac{9}{8}, \frac{9}{7})$, there exists a one-parametric bundle of positive solutions with constant $\phi(\eta)$ given by (4.4). Nevertheless, for matching purposes, the whole 2D asymptotic bundle (4.2) of oscillatory solutions has to be taken into account, so that the oscillatory behaviour remains generic (as in the linear case $n = 0$ described next).

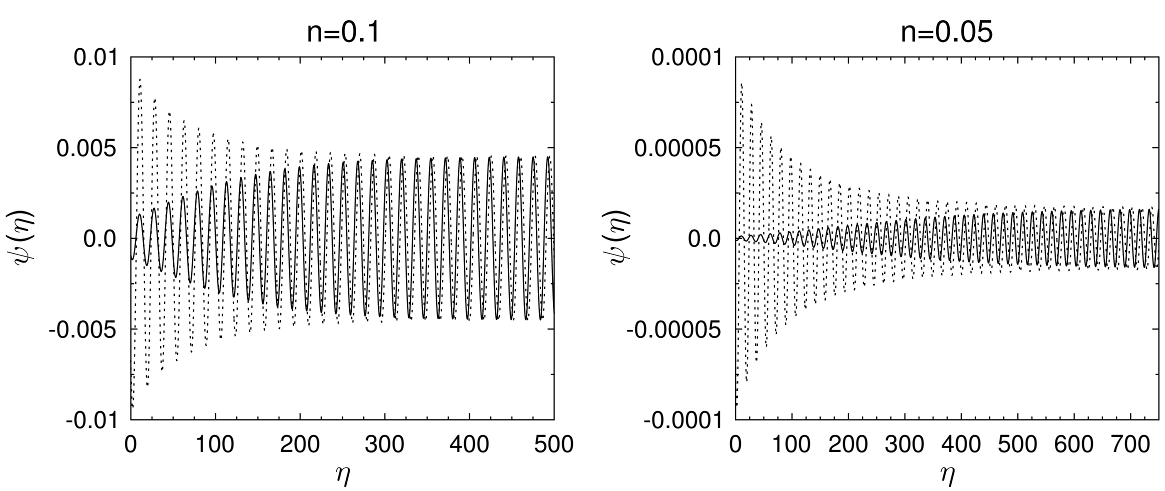

FIGURE 4. Stable periodic behaviour for the ODE (4.3), $\lambda_0 = 1$, for $n = 0.1$ and $n = 0.05$.

Fig4LC

4.3. **Periodic solutions for small $n > 0$.** Here we study the behaviour of periodic solutions for small $n > 0$. They are already difficult to detect by direct numerical methods for $n = \frac{1}{2}$ as indicated in Figure 1, where $\phi = O(10^{-21})$. To reveal the limiting oscillatory behaviour as $n \to 0$, solutions of changing sign are expected to be of the order

$$\|\varphi\|_\infty \sim (\tfrac{n}{9})^{\frac{9}{n}} \quad \text{for small } \; n > 0.$$

We thus rescale in equation (4.3) as follows

AAA.1 (4.6)
$$\phi(\eta) = \big(\tfrac{n}{9}\big)^{\frac{9}{n}}\, \psi(s), \quad s = \tfrac{9}{n}\eta.$$

For small $n > 0$, function $\psi(s)$ solves a simpler ODE with Euler's differential operator and binomial coefficients, which can be written in the form (omitting higher-order perturbations in n)

AAA.2 (4.7)
$$\mathrm{e}^{-s}\,\tfrac{\mathrm{d}^9(\mathrm{e}^s\psi)}{\mathrm{d}s^9} \equiv \textstyle\sum_{k=0}^{9}\binom{9}{k}\psi^{(9-k)} = -\lambda_0 \tfrac{\psi}{|\psi|^n}.$$

Numerical analysis shows existence of a stable periodic solution in (4.7); see Figure 4. It is worth mentioning that the periodic oscillations become very small as n reduces, e.g. by (4.6),

$$\|\phi\|_\infty \sim 10^{-176} \quad \text{for } n = 0.1.$$

Note that, for $n = \frac{1}{2}$, (4.6) suggests $\|\phi\|_\infty \sim 10^{-23}$ which is slightly smaller than the numerical size indicated in Figure 1. Stabilization to periodic orbits of (4.7) are shown in Figure 4 for two sufficiently small n values.

For $n = 0$, the equation (4.7) becomes linear,

AAA.3 (4.8)
$$\textstyle\sum_{k=0}^{9}\binom{9}{k}\psi_0^{(9-k)} = -\lambda_0\psi_0.$$

with the characteristic equation $(\nu+1)^9 + \lambda_0 = 0$ for exponential solutions $\psi(s) = \mathrm{e}^{\nu s}$. Thus, the generic stable behaviour for (4.8) is exponential and oscillatory:

psi0 (4.9)
$$\psi_0(s) \sim \mathrm{e}^{\left(\lambda_0^{\frac{1}{9}}\cos(\frac{\pi}{9})-1\right)s} \cos\big[\big(\lambda_0^{\frac{1}{9}}\sin\tfrac{\pi}{9}\big)s + \text{constant}\big] \qquad \text{as} \quad s \to -\infty.$$

These small n asymptotics given by the scaling (4.6) describes the actual branching of periodic solutions of (4.3) from the exponential decaying linear patterns (4.9) corresponding to $n = 0$.

5. Towards global behaviour of nonlinear eigenfunctions via analytic approaches

S5

To JDE: please, put as many numer. as possible. Dipoles? And more? f_2, f_3, etc., or impossible?

To JDE: VAG thinks it is better to do that in a separate section, the present analytic section is full and looks not that bad, though some questions remain

5.1. **Regularized problem.** To obtain global information about the solutions of the *nonlinear eigenvalue problem* (1.3), i.e., its nonlinear eigenfunctions or source-type solutions of the degenerate elliptic equation (1.3), we consider a homotopic deformation to the linear elliptic problem

s4 (5.1) $$\mathbf{B}F \equiv -\Delta_y^5 F + \tfrac{1}{10}\, y \cdot \nabla_y F + \tfrac{N}{10}\, F = 0 \quad \text{in} \quad \mathbb{R}^N, \quad \textstyle\int_{\mathbb{R}^N} F(y)\,\mathrm{d}y = 1.$$

This is the rescaled equation of the poly–harmonic equation of tenth order (1.5) which admits a unique classic solution given by the convolution Poisson-type integral of the form

$$u(x,t) = b(t) \, * \, u_0 \equiv t^{-\frac{N}{10}} \int\limits_{\mathbb{R}^N} F((x-z)t^{-\frac{1}{10}}) u_0(z)\,\mathrm{d}z,$$

where

$$b(x,t) = t^{-\frac{N}{10}} F(y), \quad y := \tfrac{x}{t^{1/10}} \quad (x \in \mathbb{R}^N)$$

is the unique fundamental solution of the operator $\frac{\partial}{\partial t} - \Delta^5$ such that, F is the rescaled fundamental kernel which solves the linear elliptic equation (5.1) and whose explicit solutions are a countable family of eigenfunctions for the linear elliptic operator (5.1) (cf. [9] for any further details).

Therefore, we take the regularized uniform elliptic equation

regul1 (5.2) $$\nabla \cdot [\phi_\varepsilon(f) \nabla \Delta^4 f] + \tfrac{1-\alpha n}{10}\, y \cdot \nabla f + \alpha f = 0, \quad f \in C_0(\mathbb{R}^N)\,,$$

with

phi2 (5.3) $$\phi_\varepsilon(f) = |\varepsilon^2 + f^2|^{\frac{n}{2}},$$

so that the inverse operator is smooth and analytic. Thus, for any $\varepsilon \in (0,1]$, the uniformly elliptic equation admits a unique classic solution $f = f_\varepsilon(y)$, which is an analytic function in both variables y and ε. Indeed, we would like to see under which conditions we can have that

$$f_\varepsilon(y) \to f(y) \quad \text{as} \quad \varepsilon \to 0^+.$$

for a given well-defined analytic functional family (a curve or a path),

Fem (5.4) $$\mathcal{P}_\phi = \{f_\varepsilon(y)\}_{\varepsilon \in (0,1]}.$$

Thus, to obtain relevant information about the nonlinear eigenfunctions of the problem (1.3) we will apply standard degree theory [19, 20] and first will perform a kind of "*double*" limit as ε, $n \to 0^+$, where special restrictions on two parameters will be required. Basically, because passing to the limit just when ε goes to zero we find a very deep problem since the regularized PDE loses its uniform ellipticity.

Most of the existing results for *thin film equations* deal with *non-negative solutions* with compact support of various FBPs, which are often more physically relevant and use standard integral identities for $\{f_\varepsilon\}$. In this context, we should point out that such approximations for *non-negative* and non-changing sign solutions, with various non-analytic (and non-smooth) regularizations (for example, of the form $|u|^n + \varepsilon$, which is not analytic for $n<2$) have been widely used before in TFE–FBP theory as a key foundation (cf. [5]) but assuming the parabolic problem and using energy methods. tMoreover, apart from the limiting problem when n approximates 0^+ in the one-dimensional case is not possible to apply the standard energy methods to ascertain the limiting behaviour in a convincing manner. Hence, we will use the degree theory and a kind of "*double*" limit to solve this issue.

Furthermore, although it looks quite reasonable to perform such a limit when $\varepsilon \to 0^+$, as mentioned above just passing to this limit we face many difficult problems since it is not sufficiently clear, using for example integral identities techniques, how to identify the limit (existence or non-existence of such a limit) or, even if we have more than one limit (see [4]). Moreover, at $\varepsilon = 0$ the regularized PDE (5.2) loses its uniform ellipticity. For this matter we present a discussion about how to deal with this particular limiting problem.

5.2. **Homotopy via degree theory.** First we will perform a "homotopy" transformation when the double limit ε, $n \to 0$ via standard degree theory and using the existence of the limit when the parameters n and ε go to zero in a certain manner.

In order to apply standard degree theory we will write the regularized equation (5.2) in the form

pertubeq
$$(\mathbf{B}_n + a\mathrm{Id}) f_\varepsilon \equiv \Delta^5 f_\varepsilon + \tfrac{1-\alpha n}{10}\, y \cdot \nabla f_\varepsilon + (\alpha + a) f_\varepsilon = \nabla \cdot (1-\phi_\varepsilon(f_\varepsilon)) \nabla \Delta^4 f_\varepsilon + a f_\varepsilon, \tag{5.5}$$

where $a>0$ is a parameter to be chosen so that the inverse operator $(\mathbf{B}_n + a\mathrm{Id})^{-1}$ (a resolvent value) is a compact one in a weighted space $L^2_\rho(\mathbb{R}^N)$, with ρ a certain weight that makes the embedding of $H^{10}_\rho(\mathbb{R}^N)$ compact into $L^2_\rho(\mathbb{R}^N)$. Moreover, the spectrum of

open
$$\mathbf{B}_n + a\mathrm{Id} \equiv \Delta^5 + \tfrac{1-\alpha n}{10}\, y \cdot \nabla + (\alpha + a)\mathrm{Id} \tag{5.6}$$

is always discrete and, actually, thanks to the spectral theory developed for these higher-order operators in [9] for the operators

oplin
$$\mathbf{B} + a\mathrm{Id} \equiv \Delta^5 + \tfrac{1}{10}\, y \cdot \nabla + (\alpha + a)\mathrm{Id}, \tag{5.7}$$

whose spectrum is

$$\sigma(\mathbf{B}) = \big\{\lambda_k := -\tfrac{k}{10}\,,\ k = 0,1,2,...\big\},$$

we have that

sp33
$$\sigma(\mathbf{B}_n) = \big\{ -\tfrac{k(1-\alpha n)}{10} + \alpha,\ k = 0,1,2,...\big\}, \tag{5.8}$$

so that any choice of $a>0$ such that $a \notin \sigma(\mathbf{B}_n)$ is suitable in (5.5).

We intend to perform a homotopy transformation from the (5.2) to the (5.1) translating the already known oscillatory properties of the self-similar poly-harmonic parabolic equation (1.5) into the thin film equation (1.1). Note that since the eigenfunctions of the elliptic equation (5.1) are generalized Hermite polynomials with finite oscillatory properties our purpose will be to get such an oscillation characteristic into the solutions of the non-linear eigenvalue equation (1.3) (the self-similar thin film equation). Thus, using the degree theory and the existence of convergence we ascertain some existence and multiplicity results for the non-linear eigenvalue problem (1.3).

Homotopy deformations are used in other fields in mathematics, especially geometry, to put in correspondence certain properties of several geometrical objects and the topological degree is the only invariant which is conserved by homotopic deformations However, we will use it as a tool to analyze topological invariants of those geometrical objects which can be put in correspondence with the considered equation providing us with a natural method for studying the invariant properties of the integral equation (5.5).

First of all, through the next proposition we prove that the linear elliptic operator on the left hand side of the equation (5.5) denoted by (5.6) converge to the operator (5.7) when the parameter n goes to zero. Thus, it turns out that, when the parameter n approximates zero, we have according to (2.5) that

$$\alpha_0(0) = \tfrac{N}{10}.$$

Moreover, extending that approximation also for any $k \geq 1$, the parameter α reaches the following family of values:

bf4 (5.9)
$$\alpha_k(0) := -\lambda_k + \tfrac{N}{10} \quad \text{for any} \quad k = 1, 2, \ldots, \tag{5.9}$$

where λ_k are the eigenvalues of the operator $\mathbf{B}$, so that

$$\alpha_0(0) = \tfrac{N}{10},\ \alpha_1(0) = \tfrac{N+1}{10},\ \alpha_2(0) = \tfrac{N+2}{10}, \ldots, \alpha_k(0) = \tfrac{N+k}{10} \ldots .$$

Then, we introduce the next expression for the parameter α

i52 (5.10)
$$\alpha_k(n) := \tfrac{N}{10+Nn} - \lambda_k. \tag{5.10}$$

Proposition 5.1. *The operators* (5.6)

$$\mathbf{B}_n + a\mathrm{Id} \equiv \Delta^5 + \tfrac{1-\alpha_k(n)n}{10}\, y \cdot \nabla + (\alpha_k(n) + a)\mathrm{Id}$$

converge to the operator (5.7)

$$\mathbf{B} + a\mathrm{Id} \equiv \Delta^5 + \tfrac{1}{10}\, y \cdot \nabla + (\tfrac{N+k}{10} + a)\mathrm{Id},$$

as $n \to 0$, in the generalized sense of Kato.

Proof. Indeed, for each $u \in H_0^{10}(B_1)$ we have that

$$\|(\mathbf{B}_n + a\mathrm{Id})u - (\mathbf{B} + a\mathrm{Id})u\|_{L^2(B_1)} \leq n \, \|\alpha_k(n) y \nabla u\|_{L^2(B_1)} .$$

Hence, from the expression for the parameter $\alpha_k(n)$ and Sobolev's inequality

$$\|(\mathbf{B}_n + a\mathrm{Id})u - (\mathbf{B} + a\mathrm{Id})u\|_{L^2(B_1)} \leq cK \, \|u\|_{H_0^{10}(B_1)} ,$$

with $K > 0$, a positive constant. Therefore, for any $\varepsilon > 0$, there exists n_0 such that

$$\|(\mathbf{B}_n + a\mathrm{Id})u - (\mathbf{B} + a\mathrm{Id})u\|_{L^2(B_1)} \le \varepsilon \, \|u\|_{H_0^{10}(B_1)} \, ,$$

for all $n \in (0, n_0)$ and $u \in H_0^{10}(B_1)$. $\square$

Subsequently, using the compact embedding of $H_\rho^{10}(\mathbb{R}^N)$ into $L_\rho^2(\mathbb{R}^N)$, we find that

converep
$$f_\varepsilon \longrightarrow \hat{F}, \tag{5.11}$$

performing a double limit as n and ε go to zero, at least in $L_\rho^2(\mathbb{R}^N)$. However, so far we cannot identify which problem $\hat{F}$ belongs to. Now, we write the equation (5.5) in the integral form

pertubeq11
$$f_\varepsilon = (\mathbf{B}_n + a\mathrm{Id})^{-1} \left[\nabla \cdot (1 - |\varepsilon^2 + f_\varepsilon^2|^{\frac{n}{2}}) \nabla \Delta^4 f_\varepsilon + a f_\varepsilon \right], \tag{5.12}$$

with $\mathbf{B}_n + a\mathrm{Id}$ denoted by (5.6), for which we know the expression for the whole spectrum explicitly and, also, that this operator is compact in a weighted space $L_\rho^2(\mathbb{R}^N)$ with the existence of the inverse for a suitable and positive $a \notin \sigma(\mathbf{B}_n)$. For the the nonlinear term

$$\nabla \cdot (1 - |\varepsilon^2 + f_\varepsilon^2|^{\frac{n}{2}}) \nabla \Delta^4 f_\varepsilon + a f_\varepsilon,$$

we have that it is relatively compact, thanks to the existence of convergence shown previously (5.11) and assuming the condition

nn12
$$\text{for} \quad \delta \sim \varepsilon, \quad n = n(\varepsilon) \to 0 \;\text{ such that }\; \varepsilon^{\frac{n(\varepsilon)}{2}} \to 0. \tag{5.13}$$

Indeed, setting

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{n(\varepsilon)}{2}} = 0, \quad \text{then} \quad \lim_{\varepsilon \to 0} n(\varepsilon) \ln \varepsilon = -\infty.$$

Hence, taking

expan
$$F_\varepsilon(f_\varepsilon) = 1 - |\varepsilon^2 + f_\varepsilon^2|^{\frac{n}{2}} = -\tfrac{n}{2} \, \ln(\varepsilon^2 + f_\varepsilon^2)(1 + o(1)) \quad \text{as} \quad n \to 0^+, \tag{5.14}$$

for a family $\{f_\varepsilon(y)\}$ of uniformly bounded and smooth solutions, when $f_\varepsilon \approx 0$ yields the demand

keycond
$$\boxed{n \, |\ln \varepsilon(n)| \to 0 \quad \text{as} \quad n \to 0,} \tag{5.15}$$

which it is true if the we assume (5.13) such that the regularization parameter $\varepsilon \ll \mathrm{e}^{-\frac{1}{n}}$. Thus, substituting (5.14) into (5.12) we arrive at

pertubeq12
$$f_\varepsilon = (\mathbf{B}_n + a\mathrm{Id})^{-1} \left[\nabla \cdot (-\tfrac{n}{2} \, \ln(\varepsilon^2 + f_\varepsilon^2)(1 + o(1))) \nabla \Delta^4 f_\varepsilon + a f_\varepsilon \right], \tag{5.16}$$

and passing to the limit when n and $\varepsilon(n)$ go to zero we find that there exists a fixed point for the integral equation

fixedpoeq
$$\hat{F} = (\mathbf{B} + a\mathrm{Id})^{-1}(a\hat{F}), \tag{5.17}$$

whose solutions are the eigenfunctions of the linear elliptic problem (5.7), i.e., $\hat{F} = \psi$.

Note that (5.14) could be replace for a more general form such as (3.23) with

$$G_\varepsilon(f_\varepsilon) = \tfrac{b}{n} + \ln B + O(n), \quad \text{as} \quad n \to 0^+,$$

obtaining a different condition from (5.15).

Therefore, applying the degree theory, together with Fixed Point Theory, (see [19, 20] for any further details) we can assure the existence of countable family of direct n-expansion of the solutions for the problem (1.3) to guarantee branching at $n = 0^+$. In fact, the degree provides us with the existence of continuous branches of eigenfunctions for the equation (1.3) since it stays invariant via homotopic deformations as the ones performed here. In the terms exposed by Krasnoselskii [19] we would talk about the rotation of the vector field of the form

$$\Phi = \mathrm{Id} - \mathbf{G},$$

where $\mathbf{G}$ is the operator on the right hand side of (5.16). Note that the invariance analysis of the rotation of vector fields and the degree in the sense of Leray–Schauder is equivalent.

5.3. **The limiting problem just when $\varepsilon \to 0^+$.** In general if we want to ascertain the limit

$$f_\varepsilon \longrightarrow \hat{F},$$

just when $\varepsilon \to 0^+$ instead of the double limit performed above, we take into account the the inverse operator

$$(\mathbf{B}_n + a\mathrm{Id})^{-1},$$

is smooth and analytic for any n, and the convergence of the sequence $\{f_\varepsilon\}$, at least in $L^2_\rho(\mathbb{R}^N)$. Here, again, $a > 0$ is a parameter to be chosen so that the inverse operator is compact in the weighted space $L^2_\rho(\mathbb{R}^N)$, with ρ a certain weight that makes the embedding of $H^{10}_\rho(\mathbb{R}^N)$ compact into $L^2_\rho(\mathbb{R}^N)$. Moreover, the solutions of the regularized problem (5.2), (5.3) are analytic in both variables ε, y, and by construction we know that

$$f_\varepsilon \in C_0(\mathbb{R}^N),$$

in other words, these solutions have compact support and, also, by the conservation of mass,

$$\textstyle\int_{\mathbb{R}^N} |f_\varepsilon| \leq C, \quad \text{with} \quad C > 0,$$

a positive constant.

Observe that we have solutions of changing sign with compact support and exponential decay. Then, together with the boundary conditions we find that

$$\textstyle\int_{\mathbb{R}^N} |f_\varepsilon|^2 = \int_{\mathbb{R}^N \setminus \{|f_\varepsilon| < \delta\}} |f_\varepsilon|^2 + \int_{\{|f_\varepsilon| < \delta\}} |f_\varepsilon|^2 \leq M + \delta^2 |supp f_\varepsilon|,$$

providing us with an estimation for the norms in L^2. Here we assume that the solutions are oscillatory of changing sign but we are not able to extract information about those oscillations (since the argument we have done before to extract information from the solutions at $n = 0$ is not applicable now) we do not posses *a priori* information about this oscillatory property when $n \neq 0$. However, the goal would be to extend analytically these oscillatory properties from $n = 0$ forward.

Furthermore, since the inverse operator of (5.6) is smooth and analytic we can assure that (5.6) is a topological isomorphism and, hence, we can apply the Implicit Function Theorem to the equation (5.5) with the parameter n fixed.

Therefore, it looks quite natural to apply the argument of passing to the limit just as ε goes to zero in the equation (5.12)

$$f_\varepsilon = (\mathbf{B}_n + a\mathrm{Id})^{-1}\left[\nabla \cdot (1 - |\varepsilon^2 + f_\varepsilon^2|^{\frac{n}{2}})\nabla\Delta^4 f_\varepsilon + af_\varepsilon\right].$$

However, we face here several problems that make this final process very tricky. Indeed, to get the convergence of the previous pertubed equation (5.5) we need to get the term

expterm
$$\nabla \cdot (1 - |\varepsilon^2 + f_\varepsilon^2|^{\frac{n}{2}})\nabla\Delta^4 f_\varepsilon + af_\varepsilon, \tag{5.18}$$

bounded in $L^2_\rho(\mathbb{R}^N)$. Essentially, since we have an inverse compact operator $(\mathbf{B}_n + a\mathrm{Id})^{-1}$ if (5.18) is bounded in $L^2_\rho(\mathbb{R}^N)$ we might be able to find a convergent subsequence being the solutions f_ε of that fixed point equation relatively compact. Nevertheless, on the contrary from what we had above for the double limit here we cannot assure that the non-linear term (5.18) is relatively compact making the solution of the Fixed Point equation (5.12) far from obvious. Even though it looks quite reasonable.

Moreover, when $|f_\varepsilon| \geq \delta$ for $\delta > 0$ since the solutions of the perturbed equation (5.5) are continuous with compact support we have that the L^∞ norm is bounded in those subsets

$$||f_\varepsilon||_{L^\infty,\{|f_\varepsilon|\geq\delta\}} < K, \quad \text{for a constant} \quad K > 0,$$

and, hence, the convergence of the fixed point equation (5.5) when $\varepsilon \to 0^+$ is guaranteed at least for the particular n's for which there exists a solution.

However, that is not so clear to obtain, at least directly, when $|f_\varepsilon| < \delta$ for $\delta > 0$. Indeed, by construction we find that

$$(\mathbf{B}_n + a\mathrm{Id})f_\varepsilon \in L^2_\rho(\mathbb{R}^N),$$

with $f_\varepsilon \in H^1_\rho(\mathbb{R}^N)$ however, we cannot imply directly that

$$\nabla \cdot (1 - |\varepsilon^2 + f_\varepsilon^2|^{\frac{n}{2}})\nabla\Delta^4 f_\varepsilon + af_\varepsilon \in L^2_\rho(\mathbb{R}^N),$$

from the perturbed equation (5.5). Moreover, computing

$$\textstyle\int_{\mathbb{R}^N} (\phi_\varepsilon(f_\varepsilon))^2(\nabla\Delta^4 f_\varepsilon)^2 = \int_{\mathbb{R}^N} |\varepsilon^2 + f_\varepsilon^2|^n(\nabla\Delta^4 f_\varepsilon)^2 = \int_{\mathbb{R}^N} |\varepsilon^2 + f_\varepsilon^2|^{\frac{n}{2}}|\varepsilon^2 + f_\varepsilon^2|^{\frac{n}{2}}(\nabla\Delta^4 f_\varepsilon)^2,$$

we arrive at

$$\textstyle\int_{\mathbb{R}^N} (\phi_\varepsilon(f_\varepsilon))^2(\nabla\Delta^4 f_\varepsilon)^2 \leq K,$$

for a positive constant K, assuming that

firest
$$\textstyle\int_{\mathbb{R}^N} |\varepsilon^2 + f_\varepsilon^2|^{\frac{n}{2}}(\nabla\Delta^4 f_\varepsilon)^2 \leq K \quad \text{and} \quad \int_{\mathbb{R}^N} |f_\varepsilon| \leq K. \tag{5.19}$$

This final argument would provide us with the convergence of the fixed point equation but to be so we need the first estimation of (5.19) and the relatively compactness of the non-linear terms (5.18).

Finally, even though we ascertain the existence of the limit of the equation (5.12) we cannot assure how many solutions sastify the limiting problem

$$\hat{F} = (\mathbf{B}_n + a\mathrm{Id})^{-1}\left[\nabla\cdot(1-|\hat{F}|^n)\nabla\Delta^4\hat{F} + a\hat{F}\right].$$

and also, what kind of solutions are since we do not have a *a priori* information about the solutions for other n apart from $n=0$ (as performed above).

`To PAC: any hope to apply degree/rot. theory to NONLINEAR equation, without using` $n \to 0$`??? If NOT, could you explain why? Any sustainable comments here could be of good price...`

`To VAG: I tried to obtain that convergence applying the fixed point equation but I am not sure it's relevant or just talking!!`

`S5`

6. Nonlinear eigenfunctions: numerical approach

Here we construct numerically the nonlinear eigenfunctions in one space dimension. The nonlinear eigenvalue problem (1.3) for $N=1$ becomes

`self1N1` (6.1)
$$\left(|f|^n f^{(9)}\right)' + \tfrac{1-\alpha n}{10}\, yf' + \alpha f = 0, \quad f \in C_0(\mathbb{R}),$$

for $n>0$ with f being compactly supported. For $n=0$, we require f to have exponential decay in infinity, now belonging to an appropriately weighted L^2-space as stated in (2.4). The nonlinear eigenvalue-eigenfunction pairs are denoted by $\{\alpha_k(n), f_k\}$ for $k=0,1,2,3,\dots$ and the eigenfunctions are normalised using

(6.2)
$$f_k(0)=1, \quad k=0,2,4,\dots; f_k'(0)=1, \quad k=1,3,5,\dots.$$

The first eigenvalue-eigenfunction pair $\{\alpha_0(n), f_0\}$ preserve mass, so that (6.1) may be integrated once to give

`f0` (6.3)
$$|f_0|^n f_0^{(9)} + \alpha_0 y f_0 = 0, \quad \text{with} \quad \alpha_0(n) = \tfrac{1}{10+n},$$

and is completed with the boundary conditions

`f0bc1` (6.4)
$$\text{at } y=0: \qquad f_0 = 1,\ f_0^{(i)} = 0 \quad \text{for } i = 1,3,5,7,$$

`f0bc2` (6.5)
$$\text{at } y=y_0: \qquad f_0 = f_0^{(i)} = 0 \quad \text{for } i = 1,2,3,4.$$

Since α_0 is known, this gives a tenth-order system when $n>0$ to determmine f_0 and the finite free boundary $y_0>0$ (the corresponding interface being $x = y_0 t^{\beta_0}$ with β_0 as given in (1.2)). When $n=0$, then $y_0=\infty$. Figure 5 shows illustrative f_0 profiles for selected n values in one-dimension (N=1). The system was solved as an IVP in Matlab (shooting from $y=0$), using the ODE solver ode15s with error tolerances of AbsTol=RelTol=10^{-10} and the regularisation $|f|^n = (f^2+\delta^2)^{n/2}$ with $\delta = 10^{-10}$. Since the α_0 are known, this gives a tenth-order system when $n > 0$ to determmine f_0 and the finite free boundary y_0. When $n = 0$, then $y_0 = \infty$. Figure **??** shows illustrative f_0 profiles for selected n values in one-dimension (N=1). The system was solved as an IVP in Matlab (shooting from $y = 0$), using the ODE solver ode15s with error tolerances of AbsTol=RelTol=10^{-10} and the regularisation $|f|^n = (f^2 + \delta^2)^{n/2}$ with $\delta = 10^{-10}$.

The other eigenvalue-eigenfunction pairs $\{\alpha_k, f_k\}$ for $k \geq 1$ and $n > 0$ satisfy the ode in (6.1) with

$$(6.6) \quad \text{at } y = 0: \qquad \begin{cases} f_k = 1, \;\; f_k^{(i)} = 0 \;\; \text{for } i = 1,3,5,7,9, & \text{if } k \text{ is even} \\ f_k' = 1, \;\; f_k = f_k^{(i)} = 0 \;\; \text{for } i = 2,4,6,8, & \text{if } k \text{ is odd} \end{cases}$$

fkbc1a fkbc1b

and

fkbc2

$$(6.7) \qquad \text{at } y = y_0: \qquad f_k = f_k^{(i)} = 0 \quad \text{for } i = 1,2,3,4,5.$$

Figure 5 show the eigenfunction profiles for the first four cases $k = 0, 1, 2, 3$, obtained by using the same shooting numerical procedure for the first profile (but appropriately adapted for this 12th-order system). A plot of the eigenvalues is given in 6. The case when $n = 0$ requires slight modification, with $y_0 = \infty$ and is discussed in [1].

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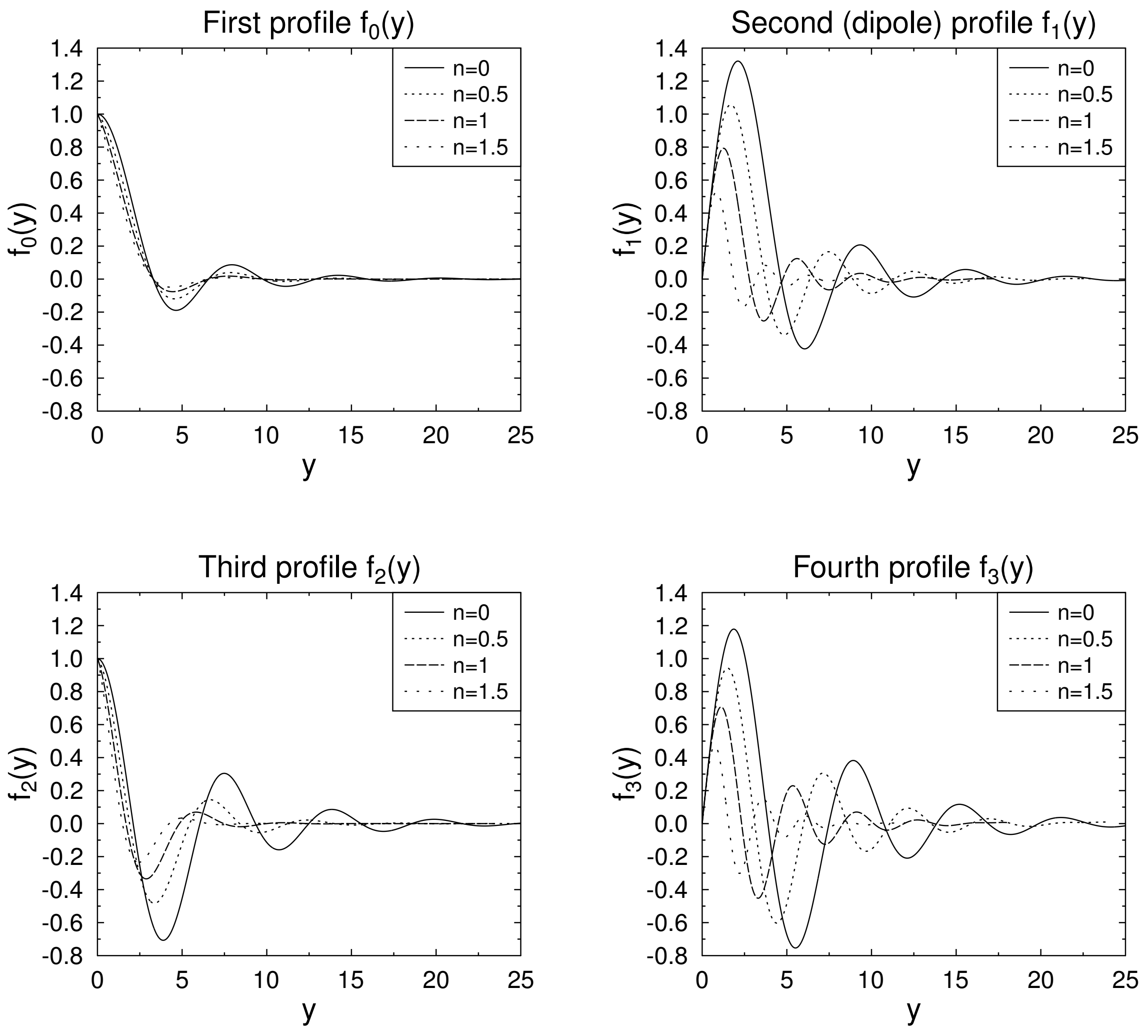

FIGURE 5. Numerical solution of the first four nonlinear eigenfunctions profiles $f_k, k = 0, 1, 2, 3$ for selected n in one-dimension $N = 1$. ef

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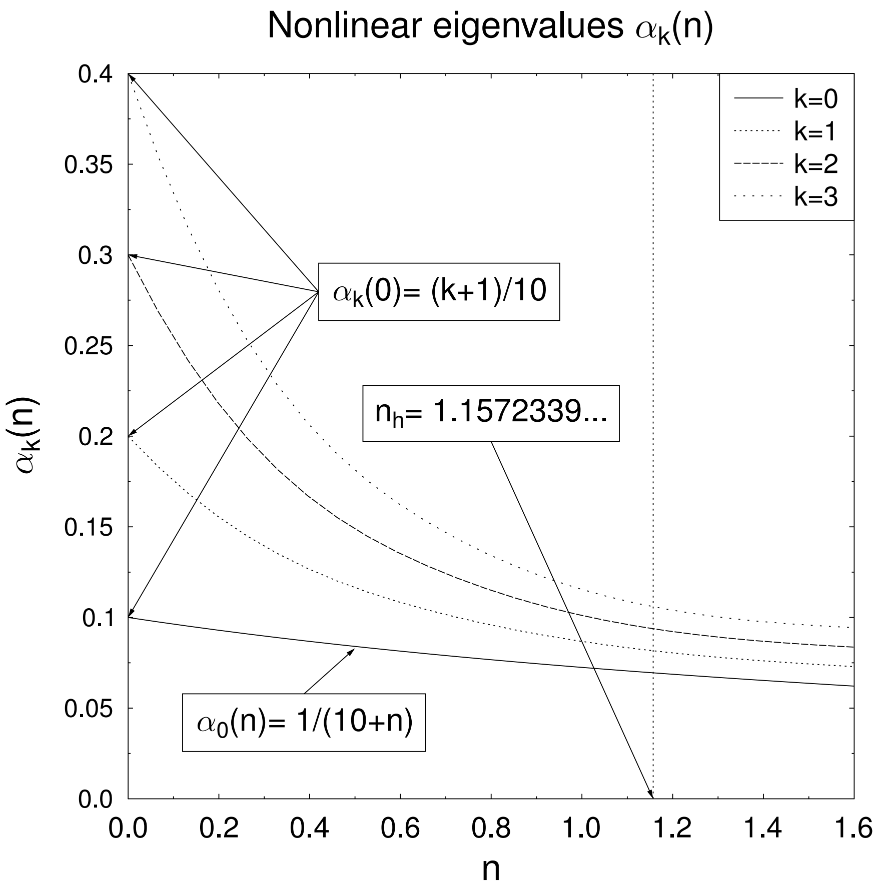

FIGURE 6. Numerical construction of the first four nonlinear eigenvalues $\alpha_k(n)$ in one-dimension $N = 1$. ev

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