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Connectivity of soft random geometric graphs

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Abstract

Consider a graph on n uniform random points in the unit square, each pair being connected by an edge with probability p if the inter-point distance is at most r. We show that as $n \to \infty$ the probability of full connectivity is governed by that of having no isolated vertices, itself governed by a Poisson approximation for the number of isolated vertices, uniformly over all choices of p, r. We determine the asymptotic probability of connectivity for all (p_n, r_n) subject to $r_n = O(n^{-\varepsilon})$, some $\varepsilon > 0$. We generalize the first result to higher dimensions, and to a larger class of connection probability functions.

1 Introduction

For certain random graph models, it is known that the main obstacle to connectivity is the existence of isolated vertices. In particular, for the Erdős-Rényi random graph $G(n, p_n)$ the probability that the graph is disconnected but free of isolated vertices tends to zero as $n \to \infty$, for any choice of $(p_n)_{n \geq 1}$ (see [6] or [2, Theorem7.3]). Likewise for the geometric graph (Gilbert graph) $G(\mathcal{X}_n, r_n)$ with vertex set $\mathcal{X}_n$ given by a set of n independently uniformly distributed points in $[0,1]^d$ with $d \geq 2$, and with an edge included between each pair of vertices at distance at most r_n, the probability that the graph is disconnected but free of isolated vertices tends to zero as $n \to \infty$, for any choice of $(r_n)_{n \in \mathbb{N}}$ (this follows from results in e.g. [14, 15]).

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Moreover, for both of these types of random graph (denoted G), the number of isolated vertices (denoted $N_0(G)$) enjoys a Poisson approximation for large n, so that with $\mathcal{K}$ denoting the class of connected graphs, for large n we have

$$P[G \in \mathcal{K}] \approx P[N_0(G) = 0] \approx \exp(-\mathbb{E}\, N_0). \tag{1.1}$$

These results have very different proofs for geometric graphs than they do for Erdős-Rényi graphs. In the present paper we prove results of this kind for a class of random graph models which generalises both $G(n, p)$ and $G(\mathcal{X}_n, r)$; we connect each pair of points of $\mathcal{X}_n$ with a probability which is a function ϕ of the distance (or more generally, the displacement) between them. The function ϕ is called the *connection function*, and we refer to the resulting graph as a 'soft' random geometric graph.

For $d = 2$ we show that the second approximation in (1.1) holds for soft random geometric graphs for large n, uniformly over connection functions which decay exponentially in some fixed positive power of distance, while the first approximation in (1.1) holds uniformly over connection functions which are zero beyond a given distance, with distance measured on the characteristic length scale of the connection function. For general $d \geq 2$ we show that (1.1) holds for a more restricted class of connection functions which amount to retaining each edge of $G(\mathcal{X}_n, r)$ with probability p, uniformly over n and p. For this class of connection functions in $d = 2$, we determine the limiting behaviour of $P[G \in \mathcal{K}]$ for any sequence $(r_n, p_n)_{n \geq 1}$ such that there exists $\varepsilon > 0$ with $r_n = O(n^{-\varepsilon})$.

We also show for general d that for any $(p_n)_{n \geq 1}$ with $p_n \gg (\log n)/n$, if we place the vertices of $G(n, p_n)$ at the points of $\mathcal{X}_n$, and add the edges in order of increasing Euclidean length, with high probability the threshold for connectivity equals the threshold for having no isolated vertices. This was previously known for $p_n \equiv 1$ [15].

There is substantial interest in these types of result in the engineering and computer science communities. Connectivity of random geometric graphs is of interest because of applications in wireless communications, for example in obtaining bounds for the capacity of wireless networks [7, 8]. The 'hard' version of the geometric graph model (with ϕ the indicator of a ball centred at the origin) is not always realistic; communication between two nodes may not be guaranteed even when they are close to each other [5, 7, 10, 18]. Also, in some cases randomness may be deliberately introduced into the connections between nearby nodes as a means to make the network secure [9, 17, 18]. Among other things, our results address a version of a conjecture of Gupta and Kumar [7], as discussed at the end of Section 2.

2 Main results

Throughout this paper we assume $d \in \mathbb{N}$ with $d \geq 2$. Given a measurable function $\phi : \mathbb{R}^d \to [0, 1]$ that is symmetric (i.e., satisfies $\phi(x) = \phi(-x)$ for all $x \in \mathbb{R}^d$), and given a locally finite set $\mathcal{X} \subset \mathbb{R}^d$, let $G_\phi(\mathcal{X})$ be the random graph with vertex set $\mathcal{X}$, obtained when each potential edge $\{x, y\}$ (with $x, y \in \mathcal{X}$ and $x \neq y$) is present in the graph with probability $\phi(x - y)$, independently of all other possible edges.

Let $\Gamma := [0,1]^d$. For $\lambda > 0$ let $\mathcal{H}_\lambda$ denote a homogeneous Poisson point process in $\mathbb{R}^d$ of intensity λ, viewed as a random subset of $\mathbb{R}^d$, and let $\mathcal{P}_\lambda := \mathcal{H}_\lambda \cap \Gamma$. Given ϕ as above, let $G_\phi(\mathcal{X}_n)$ and $G_\phi(\mathcal{P}_\lambda)$ be the resulting graphs as just described. We refer to ϕ as the *connection function*.

Soft random geometric graphs of this type are a finite-space version of the so-called *random connection model* of continuum percolation; see [11, 13], which describe further motivation, and see [11, Section 1.5] for a formal construction.

We consider various classes of connection functions ϕ. Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$. Let Ψ_d be the class of connection functions ϕ on $\mathbb{R}^d$ which satisfy

$$\phi(x) \geq \phi(y) \text{ whenever } |x| \leq |y|. \tag{2.1}$$

In particular, every $\phi \in \Psi_d$ is radially symmetric, i.e. satisfies $\phi(x) = \phi(y)$ whenever $|x| = |y|$. The condition (2.1) is physically reasonable, and is imposed on the connection functions considered in [11], for example.

Given a connection function ϕ on $\mathbb{R}^d$, define the *maximum value* of ϕ by

$$\mu(\phi) := \sup\{\phi(x) : x \in \mathbb{R}^d\}.$$

Given also $\eta > 0$, let

$$\rho_\eta(\phi) := \inf\{|x| : x \in \mathbb{R}^d, \phi(x) < \eta\mu(\phi)\} \tag{2.2}$$

and also

$$\rho_0(\phi) := \sup\{|x| : x \in \mathbb{R}^d, \phi(x) > 0\},$$

which may be infinite.

Let $\Phi_{d,\eta}$ denote the set of connection functions ϕ on $\mathbb{R}^d$ such that firstly $\rho_\eta(\phi) \in (0, \infty)$, and secondly

$$\phi(x) \leq 3\eta^{-1}\mu(\phi)\exp(-\eta(|x|/\rho_\eta(\phi))^\eta), \quad x \in \mathbb{R}^d, \tag{2.3}$$

and thirdly, also $\phi \in \Psi_d$ if $d \geq 3$. Thus $\Phi_{d,\eta} \subset \Psi_d$ for $d \geq 3$ but not for $d = 2$. Let $\Phi^0_{d,\eta}$ be the class of connection functions $\phi \in \Phi_{d,\eta}$ which also satisfy

$$\rho_0(\phi) \leq \eta^{-1}\rho_\eta(\phi). \tag{2.4}$$

For $\eta > \eta' > 0$ we have $\Phi_{d,\eta} \subset \Phi_{d,\eta'}$ and $\Phi^0_{d,\eta} \subset \Phi^0_{d,\eta'}$. The condition (2.3) says that if we view $\rho_\eta(\phi)$ as the characteristic length scale of ϕ, then the function $\phi(x)$

decays exponentially in the ηth power of the length of x, with length measured in terms of the characteristic length scale of ϕ.

Given d, define $\Psi_{\rm step} \subset \Phi^0_{d,1} \cap \Psi_d$ by

$$\Psi_{\rm step} := \{\phi_{r,p} : r > 0, p \in (0,1]\},$$

where for $r > 0$ and $0 < p \leq 1$, we set $\phi_{r,p}(x) := p\mathbf{1}_{[0,r]}(|x|)$. The graph $G_{\phi_{r,p}}(\mathcal{X}_n)$ may be viewed as the intersection of the (Gilbert) random geometric graph $G(\mathcal{X}_n, r)$ and the Erdős-Rényi random graph $G(n,p)$.

Another class of connection functions is *Rayleigh fading* where $\phi(x) = \exp(-\beta(|x|/\rho)^\gamma)$ for some fixed positive $\beta, \gamma, \rho > 0$ (typically $\gamma = 2$), which is important in applications; see [4, 16]. Such connection functions lie in $\Phi_{d,\eta}$ for suitable $\eta > 0$ which depends on β and γ but not on the length-scale ρ.

For any graph G let $N_0(G)$ denote the number of isolated vertices in G. Also let $\mathcal{K}$ denote the class of connected graphs. Our first two main results are as follows.

Theorem 2.1 *Let $\eta \in (0,1]$, $k \in \mathbb{N}_0 := \{0,1,\ldots\}$. Then*

$$\lim_{n\to\infty} \sup_{\phi \in \Phi_{d,\eta}} |P[N_0(G_\phi(\mathcal{X}_n)) = k] - e^{-I_n(\phi)} I_n(\phi)^k/k!| = 0$$

where we put $I_n(\phi) := n \int_\Gamma \exp(-n \int_\Gamma \phi(y-x)dy)dx$.

Theorem 2.2 *Let $\eta \in (0,1]$. Then*

$$\lim_{n\to\infty} \sup_{\phi \in \Phi^0_{d,\eta}} P[\{N_0(G_\phi(\mathcal{X}_n)) = 0\} \setminus \{G_\phi(\mathcal{X}_n) \ \in \mathcal{K}\}] = 0. \tag{2.5}$$

It is an immediate corollary of these two theorems that for any $\eta \in (0,1]$,

$$\lim_{n\to\infty} \sup_{\phi \in \Phi^0_{d,\eta}} |P[G_\phi(\mathcal{X}_n) \in \mathcal{K}] - \exp(-I_n(\phi))| = 0 \tag{2.6}$$

An essentially equivalent way to state the preceding results is the following.

Theorem 2.3 *Let $\alpha \in [0,\infty]$ and $\eta \in (0,1]$, and suppose $(\phi_n)_{n\in\mathbb{N}}$ is a sequence of connection functions in $\Phi_{d,\eta}$, satisfying*

$$n \int_\Gamma \exp\left(-n \int_\Gamma \phi_n(y-x)dy\right) dx \to \alpha \tag{2.7}$$

as $n \to \infty$ (possibly just along some subsequence). If $\alpha \in (0,\infty)$ then as $n \to \infty$ (along the same subsequence if applicable), we have for $k \in \mathbb{N}_0 := \{0,1,\ldots\}$ that

$$P[N_0(G_{\phi_n}(\mathcal{X}_n)) = k] \to e^{-\alpha}\alpha^k/k!. \tag{2.8}$$

If $\alpha = 0$ *then* $P[N_0(G_{\phi_n}(\mathcal{X}_n)) = 0] \to 1$ *and if* $\alpha = \infty$ *then* $P[N_0(G_{\phi_n}(\mathcal{X}_n)) = k] \to 0$ *for all* $k \in \mathbb{N}_0$. *Finally, if* $\phi_n \in \Phi^0_{d,\eta}$ *for all* n, *then*

$$P[G_{\phi_n}(\mathcal{X}_n) \in \mathcal{K}] \to e^{-\alpha} \text{ as } n \to \infty \text{ along the subsequence,} \tag{2.9}$$

with $e^{-\alpha}$ *interpreted as* 0 *for* $\alpha = \infty$.

For an example of functions that are *not* covered by our results, consider taking $\phi_n(x) = \min(1, \varepsilon_n/|x|)$ with ε_n some sequence tending to zero. Then there is no $\eta \in (0, 1]$ such that $\phi_n \in \Phi_{d,\eta}$ for all n. Another example would be if ϕ was the indicator of an annulus centred at the origin; this would have $\rho_\eta(\phi) = 0$, so not be in $\Phi_{d,\eta}$ for any $\eta > 0$.

Our definition of $\Phi_{d,\eta}$ means we restrict attention to connection functions $\phi \in \Psi_d$ when $d \geq 3$. This is because to deal with all kinds of boundary regions of Γ in $d \geq 3$ we use the radial symmetry of ϕ (see Lemma 3.1 (b) below, and the result from [15] or [12] used in its proof). When $d = 2$ the only kinds of boundary regions are either near the corners of Γ (a 'small' region) or near the 1-dimensional edges (which can be dealt with using the condition $\phi(x) = \phi(-x)$; see Lemma 3.1 (a) below) so we do not require $\phi \in \Psi_2$ for the results above.

Given $r \geq 0$ and $p \in (0, 1]$, and finite $\mathcal{X} \subset \Gamma$, write $G_{r,p}(\mathcal{X})$ for $G_{\phi_{r,p}}(\mathcal{X})$. Given p, a natural coupling of all the graphs $G_{r,p}(\mathcal{X}_n), r \geq 0$, goes as follows; let $G_{r,p}(\mathcal{X}_n)$ be the subgraph of $G_{\sqrt{d},p}(\mathcal{X}_n)$, with vertex set $\mathcal{X}_n$, and edge set consisting of all edges of Euclidean length at most r. With this coupling, $G_{r,p}(\mathcal{X}_n)$ is a subgraph of $G_{s,p}(\mathcal{X}_n)$ whenever $r \leq s \leq \sqrt{d}$. Given p, define the thresholds $\tau_n(p) := \inf\{r : G_{r,p}(\mathcal{X}_n) \in \mathcal{K}\}$, and $\sigma_n(p) := \inf\{r : N_0(G_{r,p}(\mathcal{X}_n)) = 0\}$, with the infimum of the empty set interpreted as $+\infty$. Clearly $\sigma_n(p) \leq \tau_n(p)$ almost surely. Our next result gives an asymptotic equivalence of these two thresholds.

Theorem 2.4 *Given any* $[0, 1]$*-valued sequence* $(p_n)_{n\in\mathbb{N}}$ *with* $np_n/\log n \to \infty$ *as* $n \to \infty$, *it is the case that*

$$\lim_{n\to\infty} P[\tau_n(p_n) = \sigma_n(p_n)] = 1.$$

In the case where $d = 2$ and $\phi_n \in \Psi_2 \cap \Phi_{2,\eta}$ for some $\eta \in (0, 1]$, we shall make Theorem 2.3 more explicit, by characterising those sequences ϕ_n which satisfy (2.7). Setting $p_n := \mu(\phi_n)$, we find that the main contribution to the integral in (2.7) comes from x in the interior of Γ when $p_n \gg (1/\log n)$, while the main contribution comes from x near the boundary but not the corners of Γ when $n^{-1/3}(\log n)^{-1} \ll p_n \ll 1/\log n$, and the main contribution comes from x near the corners of Γ when $p_n \ll n^{-1/3}(\log n)^{-1}$.

We state this more precisely in Theorem 2.5 below, which requires further notation. Given real-valued functions f, g, recall that $f(n) = \omega(g(n))$ means $g(n) =$

$o(f(n))$ (as $n \to \infty$), and $f(n) = \Omega(g(n))$ means $g(n) = O(f(n))$, and $f(n) = \Theta(g(n))$ means $f(n) = O(g(n))$ and $g(n) = O(f(n))$. Finally $f(n) \sim g(n)$ means $f(n) = (1 + o(1))g(n)$. For any connection function ϕ we set

$$I(\phi) := \int_{\mathbb{R}^d} \phi(x)dx. \tag{2.10}$$

If $\eta \in (0, 1]$ and $\phi \in \Phi_{2,\eta}$, then set

$$J_1(\phi) := J_1(\phi, \eta) := \mu(\phi)^{-1} \int_0^\infty \phi((\rho_\eta(\phi)t, 0))dt; \tag{2.11}$$

$$J_2(\phi) := J_2(\phi, \eta) := \mu(\phi)^{-1} \int_0^\infty \phi((\rho_\eta(\phi)t, 0))2\pi t dt. \tag{2.12}$$

For $\eta \in (0, 1]$ and $\phi \in \Psi_2 \cap \Phi_{2,\eta}$, we have $I(\phi) = \mu(\phi)\rho_\eta(\phi)^2 J_2(\phi)$, and for $\phi \in \Psi_{\text{step}}$ we have $J_1(\phi) = 1$ and $J_2(\phi) = \pi$.

The integrals $J_1(\phi)$ and $J_2(\phi)$ may be viewed as measure of the 'shape' of ϕ, separate from $\mu(\phi)$ and $\rho_\eta(\phi)$ which measure the vertical and horizontal 'scale' of ϕ, respectively. Note that for $\eta \in (0, 1]$ and $i = 1, 2$ we have

$$0 < \inf_{\phi \in \Psi_2 \cap \Phi_{2,\eta}} J_i(\phi, \eta) \le \sup_{\phi \in \Psi_2 \cap \Phi_{2,\eta}} J_i(\phi, \eta) < \infty. \tag{2.13}$$

Theorem 2.5 . *Let $\eta \in (0, 1]$, $\alpha \in (0, \infty)$. Suppose $d = 2$ and $\phi_n \in \Phi_{2,\eta} \cap \Psi_2$ for $n \in \mathbb{N}$. Set $r_n := r_\eta(\phi_n)$ and $p_n := \mu(\phi_n)$. Then (2.7) holds under any of the following conditions as $n \to \infty$:*

1. *$p_n = \omega(1/\log n)$ and $nI(\phi_n) - \log n \to -\log \alpha$;*

2. *$p_n = o(1/\log n)$ and $p_n = \omega(n^{-1/3}(\log n)^{-1})$ and*

$$nI(\phi_n) = \log\left(\frac{4J_2(\phi_n)}{\alpha^2 J_1(\phi_n)^2}\right) + \log\left(\frac{n}{p_n}\right) - \log\log\left(\frac{n}{p_n}\right) + o(1); \tag{2.14}$$

3. *$p_n = o(n^{-1/3}(\log n)^{-1})$ and $r_n = n^{-\Omega(1)}$ and*

$$nI(\phi_n) = 4(\log(1/p_n) - \log\log(1/p_n) + \log(J_2(\phi_n)/(\alpha J_1(\phi_n)^2))) + o(1).$$

We also deal with the boundary cases $p_n = \Theta(1/\log n)$ and $p_n = \Theta(n^{-1/3}(\log n)^{-1})$. See Theorems 8.1 and 8.2.

We now discuss other related work and open problems. Note that (2.8) (but not (2.9)) of Theorem 2.3 was already proved by Yi et al. [18] in the special case with

$d = 2$ and $\phi_n \in \Psi_{\rm step}$ under the condition $p_n = \omega(1/\log n)$. Here we are considering a much more general class of sequences of connection functions ϕ_n.

For a discussion of these problems from a statistical physics viewpoint via formal series expansions, and further discussion of motivation, see Coon et al. [4]. The methods of Krishnan *et al.* [9] (see Remark 3 of that paper) could be used to give some limiting inequalities for the probability of connectivity in the special case of connection functions in $\Psi_{\rm step}$ (whereas our (2.6) provides a limiting *equality* for a more general class of connection functions). The main concern in [9] is with a certain non-independent randomisation (random key graphs) to determine which of the edges (below the threshold radius) are present, which is of interest from an engineering perspective (see also [17]). It would be interesting to try to extend our results to these random key graphs.

A related random graph model is the *bluetooth graph*; this is a subgraph of the 'hard' random geometric graph with edges selected at random according to a restriction on vertex degrees. See [3] for results on connectivity of bluetooth graphs.

Another related problem is that of *Hamiltonicity*. Analogously to (2.5), one might speculate that for large n, the probability that $G_\phi(\mathcal{X}_n)$ is non-Hamiltonian while having minimum degree at least 2 might vanish uniformly over connection functions in $\Psi_{\rm step}$ (or indeed, connection functions in $\Phi^0_{d,\eta}$). For the more restricted class of connection functions of 'hard' random geometric graphs, this was proved in [1]. Some of the ideas of proof in the present paper are related to methods used in [3] and in [1].

Given $k \in \mathbb{N}$, and given a graph G, let $N_{<k}(G)$ be the number of vertices of G of degree less than k, and let $\mathcal{K}_k$ be the class of k-connected graphs. In view of results in [15], one might expect (2.5) to hold with N_0 replaced by $N_{<k}$ and $\mathcal{K}$ replaced by $\mathcal{K}_k$, for any fixed $k \in \mathbb{N}$.

In a much-cited paper, Gupta and Kumar [7] conjectured that if $d = 2$ and $\mathcal{X}_n$ consists of n points uniformly distributed in a disk of unit area (rather than the unit square considered here) and $\phi_n = \phi_{r_n,p_n}$, then $\mathbb{P}[G_{\phi_n}(\mathcal{X}_n) \in \mathcal{K}] \to 1$ if and only if $n\pi r_n^2 p_n - \log n \to \infty$. Our results (Theorems 2.3 and 2.5, and 8.1) address the corresponding conjecture for points in the unit square, showing that under the additional assumption that $p_n = \Omega(1/\log n)$, the conjecture is true and also $\mathbb{P}[G_{\phi_n}(\mathcal{X}_n) \in \mathcal{K}] \to 0$ if $n\pi r_n^2 p_n - \log n \to -\infty$. Our results also show that if $p_n = \omega(1/\log n)$, and if $n\pi r_n^2 p_n - \log n \to \beta \in \mathbb{R}$ then $\mathbb{P}[G_{\phi_n}(\mathcal{X}_n) \in \mathcal{K}] \to \exp(-e^{-\beta})$.

However, if one assumes instead that $p_n = o(1/\log n)$ and $p_n = \omega(n^{-1/3}(\log n)^{-1})$ and (2.14) holds, then it is easily verified that $n\pi r_n^2 p_n - \log n \to \infty$, but our results show that $\mathbb{P}[G_{\phi_n}(\mathcal{X}_n) \in \mathcal{K}]$ tends to a limit strictly between 0 and 1, so the conjecture fails. Essentially, this is because in this case the mean number of isolated vertices in the interior of Γ tends to zero but the mean number of isolated vertices near the boundary does not. In this regime the corner effects are not the most important, and we would expect something similar to hold in the unit disk as considered in [7]. More generally, it would be of interest to extend our results to the case of other shaped regions such as smoothly bounded regions, but this could be a non- trivial task because the boundary effects can be quite strong (essentially because of the exponential factor in the expression on the left of (2.7).

The remaining sections of the paper are organised as follows. In Section 3 we prove Theorem 3.1, which is a Poissonized version of Theorem 2.1 (i.e., one with the point process $\mathcal{X}_n$ replaced by $\mathcal{P}_n$), of interest in its own right. In Sections 4 and 5, we prove Theorem 5.1, which is (loosely speaking) a Poissonized version of Theorem 2.2, also of interest in its own right.

In Section 6, we shall de-Poissonize, thereby completing the proof of Theorems 2.1, 2.2 and Theorem 2.3. In Section 7 we prove Theorem 2.4. In Section 8, we prove Theorems 2.5, 8.1 and 8.2.

We conclude this section with some remarks on the proofs. As we have mentioned, many of the results presented here might naturally be conjectured in view of known results for random 'hard' geometric graphs [14, 15], and for Erdős-Rényi random graphs [6, 2], and a (slightly weaker) explicit conjecture along these lines given in [7]. These references date back to the last century, but the conjectures have not been proved before now, despite the considerable influence of [7] in the applied literature (see for example the discussion in [17]).

We believe that there are two reasons for this. One is that different arguments are used to prove these results depending on whether or not $\mu(\phi_n)$ tends to zero faster than a certain rate. The division between Sections 4 and 5 reflects this, and Section 3 is also divided along these lines. The balance between geometrical and combinatorial arguments is different in these different settings.

The other reason is that the proof is not just a matter of reassembling known arguments. For example, a part of the argument is concerned with ruling out the possibility that there are two large disjoint components. For "hard" geometric graphs [14, 15], any two such components are separated by a connected region of empty space and one can use discretisation, spatial independence and path-counting arguments directly. In the present "soft" case, however, the physical separation of components is not at all obvious. Instead, we proceed more indirectly via a notion of local good behaviour of our point process (the 'blue cubes' of Section 5.2) with finite-range dependence, after which we can use path-counting arguments to establish that there is a single giant region of 'blue cubes' corresponding to a single large component of our graph.

3 Poisson approximation

In this section we prove the following Poissonized version of Theorem 2.1 (we shall de-Poissonize in Section 6).

Theorem 3.1 *Let $\alpha > 0$ and $\eta \in (0,1]$. Suppose $(\lambda(n))_{n\in\mathbb{N}}$ is an increasing $(0,\infty)$-valued sequence that tends to ∞ as $n \to \infty$, and $(\phi_\lambda)_{\lambda>0}$ is a collection of connection functions in $\Phi_{d,\eta}$. Suppose that as $\lambda \to \infty$ along the sequence $(\lambda(n))$ we have*

$$\lambda \int_\Gamma \exp\left(-\lambda \int_\Gamma \phi_\lambda(y-x)dy\right) dx \to \alpha. \tag{3.1}$$

Then for $k \in \mathbb{N}_0$ we have as $\lambda \to \infty$ along the same sequence, that

$$P[N_0(G_{\phi_\lambda}(\mathcal{P}_\lambda)) = k] \to e^{-\alpha}\alpha^k/k!. \tag{3.2}$$

Our strategy of proof is as follows. When $p_\lambda := \mu(\phi_\lambda)$ is 'small', we use the method of moments and the Mecke formula (3.5) and Bonferroni bounds. When p_λ is 'big' we shall proceed by the Chen-Stein method for Poisson approximation of $N_0(G_{\phi_\lambda}(\mathcal{P}_\lambda))$ which may be approximated (via discretisation of space) by a sum of 'mostly independent' indicator functions.

In proving (3.2), we shall use the following notation. We write 'with high probability' or 'w.h.p.' to mean 'with probability tending to 1 as $\lambda \to \infty$'. All asymptotic statements are taken to be as $\lambda \to \infty$ along the sequence $\lambda(n)$ mentioned in Theorem 3.1. Also, for $A, B \subset \mathbb{R}^d$ we write $A \oplus B$ for $\{x+y : x \in A, y \in B\}$ (Minkowski addition of sets).

For any finite (deterministic) $\mathcal{A} \subset \mathbb{R}^d$, and any $\phi \in \Phi_{d,\eta}$, set

$$h_\phi(\mathcal{A}) := P[G_\phi(\mathcal{A}) \in \mathcal{K}] \tag{3.3}$$

and for any $y \in \mathbb{R}^d$ with $y \notin \mathcal{A}$, set

$$g_\phi(y, \mathcal{A}) := 1 - \prod_{x\in\mathcal{A}} (1 - \phi(y-x)) = P[y \text{ is non-isolated in } G_\phi(\mathcal{A} \cup \{y\})]. \tag{3.4}$$

The left hand side of (3.1) equals $\mathbb{E}\, N_0(G_{\phi_\lambda})$. This is a consequence of the following formula, which we shall use repeatedly. Suppose $k \in \mathbb{N}$ and f is a measurable nonnegative function defined on $(\mathbb{R}^d)^k \times \mathcal{G}_k$ where $\mathcal{G}_k$ is the space of all graphs on vertex set $\{1,\ldots,k\}$. Then given a connection function ϕ, for $\lambda > 0$ we have

$$\mathbb{E} \sum_{X_1,\ldots,X_k \in \mathcal{P}_\lambda}^{\neq} f(X_1,\ldots,X_k, G_\phi(\mathcal{P}_\lambda)|_{X_1,\ldots,X_k}) \mathbf{1}_{\mathbf{D}_\phi(X_1,\ldots,X_k;\mathcal{P}_\lambda)} = \lambda^k \int_\Gamma dx_1 \cdots \int_\Gamma dx_k$$
$$\times \mathbb{E}\left[f(x_1,\ldots,x_k, G_\phi(\{x_1,\ldots,x_k\}))\right] \exp\left(-\lambda \int_\Gamma g_\phi(y;\{x_1,\ldots,x_k\})dy\right), \tag{3.5}$$

where the sum is over all ordered k-tuples of distinct points of $\mathcal{P}_\lambda$, and $G_\phi(\mathcal{P}_\lambda)|_{X_1,\ldots,X_k}$ is the subgraph of $G_\phi(\mathcal{P}_\lambda)$ induced by vertex set $\{X_1,\ldots,X_k\}$ with the vertex X_i given the label i for each i, and $\mathbf{D}_\phi(X_1,\ldots,X_k;\mathcal{P}_\lambda)$ is the event that there is no edge of $G_\phi(\mathcal{P}_\lambda)$ between any vertex in $\{X_1,\ldots,X_k\}$ and any vertex in $\mathcal{P}_\lambda\setminus\{X_1,\ldots,X_k\}$.

The formula (3.5) is related to the Slivnyak-Mecke formula in the theory of Poisson processes; here we just call it the *Mecke formula*. It can be proved by conditioning on the number of points of $\mathcal{P}_\lambda$; see the proofs of [12, Theorem 1.6], and [13, Proposition 1].

We shall use the following inequality more than once. Given connection function ϕ and given $x, x_1, \ldots, x_k \in \Gamma$, by the Bonferroni bound

$$g_\phi(x;\{x_1,\ldots,x_k\}) \ge \left(\sum_{i=1}^k \phi(x-x_i)dx\right) - \sum_{1\le i<j\le k} \phi(x-x_i)\phi(x-x_j),$$

so integrating over $x\in\Gamma$, we obtain

$$\int_\Gamma g_\phi(x;\{x_1,\ldots,x_k\})dx \ge \left(\sum_{i=1}^k \int_\Gamma \phi(x-x_i)dx\right) - k^2\mu(\phi)I(\phi). \tag{3.6}$$

Let $\mathbf{H}$ denote the half-space $[0,\infty)\times\mathbb{R}^{d-1}$, and let $\mathbf{Q}$ denote the orthant $[0,\infty)^d$. For $x\in\mathbf{Q}$ let $\mathbf{Q}_x := \{y\in\mathbf{Q} : \|x\|_1 \le \|y\|_1\}$, where $\|\cdot\|_1$ is the ℓ_1 norm.

Lemma 3.1 *Let $\eta\in(0,1]$ and $\phi\in\Phi_{d,\eta}$. Then (a) if $d=2$, for any $x=(x_1,x_2)\in\mathbf{H}$ and $y=(y_1,y_2)\in\mathbf{H}$ with $x_1\le y_1$, and $r\in[\rho_\eta(\phi),\infty]$, setting $\phi^{(r)}(x) := \phi(x)\mathbf{1}_{[0,r]}(|x|)$ we have*

$$\int_\mathbf{H} (g_{\phi^{(r)}}(z,\{x,y\}) - \phi^{(r)}(z-x))dz \ge (\eta/4)\mu(\phi)\rho_\eta(\phi)\min(|y-x|,\rho_\eta(\phi));$$

(b) if $d\ge 3$, and $x\in\mathbf{Q}, y\in\mathbf{Q}_x$, then

$$\int_\mathbf{Q} (g_\phi(z,\{x,y\}) - \phi(z-x))dz \ge \eta_1\mu(\phi)\rho_\eta(\phi)^{d-1}\min(|y-x|,\rho_\eta(\phi)), \tag{3.7}$$

where $\eta_1>0$ is a constant depending only on d and η.

Proof. (a) Let us assume $x_2\le y_2$ (the other case may be treated similarly). For any $z\in\mathbb{R}^2$, since $g_{\phi^{(r)}}(z,\{x,y\}) - \phi^{(r)}(z-x) = (1-\phi^{(r)}(z-x))\phi^{(r)}(z-y)$ we have $g_{\phi^{(r)}}(z,\{x,y\}) - \phi^{(r)}(z-x) \ge (\phi^{(r)}(z-y) - \phi^{(r)}(z-x))_+$. Therefore it suffices to prove

$$\int_\mathbf{H} (\phi^{(r)}(z-y) - \phi^{(r)}(z-x))_+ dz \ge (\eta/4)\mu(\phi)\rho_\eta(\phi)\min(|y-x|,\rho_\eta(\phi)). \tag{3.8}$$

Now

$$\begin{aligned}\int_{\mathbf{H}}(\phi^{(r)}(z-y)-\phi^{(r)}(z-x))_+dz &\geq \int_{\{y\}\oplus\mathbf{Q}}(\phi^{(r)}(z-y)-\phi^{(r)}(z-x))dz \\ &= \int_{\mathbf{Q}}\phi^{(r)}(w)dw - \int_{\{y-x\}\oplus\mathbf{Q}}\phi^{(r)}(w)dw \\ &= \int_{\mathbf{Q}\setminus(\{y-x\}\oplus\mathbf{Q})}\phi^{(r)}(w)dw.\end{aligned}$$

If $|y-x| \leq \rho_\eta(\phi)$, then the region $\mathbf{Q}\setminus(\{y-x\}\oplus\mathbf{Q})$ contains either the rectangle $[0,|y-x|/2]\times[0,\rho_\eta(\phi)/2]$ or the rectangle $[0,\rho_\eta(\phi)/2]\times[0,|y-x|/2]$ (or both), and the function $\phi^{(r)}$ exceeds $\eta\mu(\phi)$ on either of these rectangles, so that $\int_{\mathbf{Q}\setminus(\{y-x\}\oplus\mathbf{Q})}\phi^{(r)}(w)dw \geq \eta|y-x|\rho_\eta(\phi)\mu(\phi)/4$.

If $|y-x| \geq \rho_\eta(\phi)$, then the region $\mathbf{Q}\setminus(\{y-x\}\oplus\mathbf{Q})$ contains the square $[0,\rho_\eta(\phi)/2]^2$, so that $\int_{\mathbf{Q}\setminus(\{y-x\}\oplus\mathbf{Q})}\phi^{(r)}(w)dw \geq \eta\rho_\eta(\phi)^2\mu(\phi)/4$. This gives us (3.8).

(b) Now suppose $d \geq 3$ (so $\phi \in \Psi_d$ by definition of $\Phi_{d,\eta}$). For $x, y \in \mathbf{Q}$, we have by Fubini's theorem and (2.2) that

$$\begin{aligned}\int_{\mathbf{Q}}(g_\phi(z,\{x,y\})-\phi(z-x))dz &= \int_0^1\int_{\mathbf{Q}}(\mathbf{1}_{\{g_\phi(z,\{x,y\})\geq t\}}-\mathbf{1}_{\{\phi(z-x)\geq t\}})dzdt \\ &\geq \int_0^{\eta\mu(\phi)}\int_{\mathbf{Q}}(\mathbf{1}_{\{\phi(z-y)\geq t\}}-\mathbf{1}_{\{\phi(z-x)\geq t\}})_+dzdt \\ &= \int_0^\eta |\mathbf{Q}\cap B(y;\rho_u(\phi))\setminus B(x;\rho_u(\phi))|\mu(\phi)du, \qquad (3.9)\end{aligned}$$

where $|\cdot|$ denotes Lebesgue measure or the Euclidean norm according to context.

For $u \leq \eta$, we have $\rho_u(\phi) \geq \rho_\eta(\phi)$. Also, there is a constant $\eta_2 > 0$ (dependent on η and d) such that $|\mathbf{Q}\cap B(y;1)\setminus B(x;1)| \geq \eta_2 \min(|y-x|,1)$ for any $x, y \in \mathbf{Q}$ with $\|x\|_1 \leq \|y\|_1$; see [12, Proposition 5.16] or [15, Proposition 2.2]. Hence for $x \in \mathbf{Q}$, $y \in \mathbf{Q}_x$ and $u \in (0,\eta]$, by scaling

$$\begin{aligned}|\mathbf{Q}\cap B(y;\rho_u(\phi))\setminus B(x;\rho_u(\phi))| &\geq (\rho_u(\phi))^d\eta_2\min\left(\frac{|y-x|}{\rho_u(\phi)},1\right) \\ &\geq \eta_2\rho_\eta(\phi)^{d-1}\min(|y-x|,\rho_\eta(\phi)).\end{aligned}$$

Putting this into (3.9) gives us the result (3.7) with $\eta_1 = \eta_2\eta$. □

Given $\eta \in (0,1]$ and given $(\phi_\lambda)_{\lambda>0}$ with each $\phi_\lambda \in \Phi_{d,\eta}$, for $\lambda > 0$ we set

$$p_\lambda := \mu(\phi_\lambda); \qquad r_\lambda := \rho_\eta(\phi_\lambda). \qquad (3.10)$$

Recall from (2.10) that $I(\phi) := \int_{\mathbb{R}^d} \phi(x)dx$ for any connection function ϕ. Without loss of generality for the purpose of proving Theorem 3.1, we can and do assume for all λ that $\rho_0(\phi_\lambda) \leq \sqrt{d}$, so that also $r_\lambda \leq \sqrt{d}$. Note that if (3.1) holds, then

$$\lambda I(\phi_\lambda) = \Theta\left(\log \lambda\right), \tag{3.11}$$

and therefore by (3.10),

$$\lambda p_\lambda r_\lambda^d = \Theta(\log \lambda). \tag{3.12}$$

Theorem 3.1 follows from the next two lemmas, dealing separately with the case with $p_\lambda = o(1/\log \lambda)$ and the case with $p_\lambda = \omega(1/(\log \lambda)^2)$. In the first case, we use the method of moments. For $m, r \in \mathbb{N}$ we write $(m)_r$ for the descending factorial $m(m-1)\cdots(m-r+1)$.

Lemma 3.2 *Let* $\alpha \in (0, \infty)$, $\eta \in (0, 1]$. *Suppose* $\phi_\lambda \in \Phi_{d,\eta}$ *for all* λ *and* $(\phi_\lambda)_{\lambda>0}$ *satisfy* (3.1), *and that* $p_\lambda = o(1/\log \lambda)$. *Then* (3.2) *holds.*

Proof. Set $N_0 := N_0(G_{\phi_\lambda}(\mathcal{P}_\lambda))$. Let $k \in \mathbb{N}$. For finite $A \subset \mathbb{R}^d$, let $u_\lambda(\mathcal{A})$ denote the probability that $G_{\phi_\lambda}(\mathcal{A})$ has no edges. By the Mecke formula ((3.5)),

$$\mathbb{E}\left[(N_0)_k\right] = \lambda^k \int \cdots \int u_\lambda(\{x_1, \ldots, x_k\}) \exp\left(-\lambda \int g_{\phi_\lambda}(x, \{x_1, \ldots, x_k\})dx\right) dx_1 \cdots dx_k,$$

where all integrals are over Γ, unless specified otherwise. By the union bound, $u_\lambda(\{x_1, \ldots, x_k\}) \geq 1 - \binom{k}{2} p_\lambda$, and also $g_{\phi_\lambda}(x, \{x_1, \ldots, x_k\}) \leq \sum_{i=1}^k \phi_\lambda(x - x_i)$. Hence

$$\begin{aligned}\mathbb{E}\left[(N_0)_k\right] &\geq (1 - k^2 p_\lambda)\lambda^k \int \cdots \int \exp\left(-\lambda \int \sum_{i=1}^k \phi_\lambda(x - x_i)dx\right) dx_1 \cdots dx_k \\ &= (1 + o(1))(\mathbb{E}\, N_0)^k. \end{aligned} \tag{3.13}$$

Also, by (3.6), we have

$$\begin{aligned}\mathbb{E}\left[(N_0)_k\right] &\leq \lambda^k \int \ldots \int \exp\left(\lambda k^2 p_\lambda I(\phi_\lambda) - \lambda \int \sum_{i=1}^k \phi_\lambda(x - x_i)dx\right) dx_1 \ldots dx_k \\ &= (1 + o(1))(\mathbb{E}\, N_0)^k, \end{aligned} \tag{3.14}$$

where the last line is because $\lambda p_\lambda I(\phi_\lambda) = O(p_\lambda \log \lambda) \to 0$, by (3.11) and our assumption on p_λ.

By (3.13), (3.14) and the assumption (3.1), we have that $\mathbb{E}[(N_0)_k] \to \alpha^k$, and therefore by the method of moments (see e.g. Theorem 1.22 of [2]), we have the Poisson convergence (3.2). $\square$

For the second case with $p_\lambda = \omega((\log\lambda)^{-2})$, we use the Poisson approximation method from [14]. This method has the potential to provide error bounds, but this is not our main focus here. For $x \in \mathbb{R}^d$ and $r > 0$ set $B(x;r)$ to be the ball $\{y \in \mathbb{R}^d : |x - y| \leq r\}$. Given $\eta \in (0,1]$, set

$$K(\eta) := \int_{\mathbb{R}^d} 3\eta^{-1} \exp(-\eta|x|^\eta)dx.$$

Note that $K(1) \leq K(\eta) < \infty$, and $K(1) = 6\pi$ if $d = 2$, and that by (2.3) and (2.10),

$$I(\phi) \leq \mu(\phi)(\rho_\eta(\phi))^d K(\eta), \qquad \phi \in \Phi_{d,\eta}. \tag{3.15}$$

Lemma 3.3 *Suppose for some $\eta \in (0,1]$ and $\alpha \in (0,\infty)$ that $\phi_\lambda \in \Phi_{d,\eta}$ for all $\lambda > 0$ and ϕ_λ satisfy* (3.1)*. Suppose $p_\lambda = \omega(1/(\log\lambda)^2)$. Then* (3.2) *holds.*

Proof. Assume $r_\lambda \leq \sqrt{d}$. It follows from (3.1) that (3.11) and (3.12) hold. Hence by our condition on p_λ we have

$$r_\lambda^d = \Theta((\log\lambda)/(\lambda p_\lambda)) = o((\log\lambda)^3\lambda^{-1}). \tag{3.16}$$

By (3.12), we can (and do) choose $\delta > 0$ with $\lambda p_\lambda r_\lambda^d > \delta\log\lambda$ for all λ. Let $\varepsilon > 0$ be fixed with $\varepsilon < \eta/(4K(\eta))$ if $d = 2$, and with $\varepsilon < \min(2^{-d}\pi_d\eta/K(\eta), \eta_1\delta)$ if $d \geq 3$, where η_1 is as in Lemma 3.1 (b). Truncate ϕ_λ by setting $\tilde{\phi}_\lambda(x) := \phi_\lambda(x)\mathbf{1}_{[0,r_\lambda^{1-\varepsilon}]}(|x|)$ for $x \in \mathbb{R}^d$. Couple $G_{\phi_\lambda}(\mathcal{P}_\lambda)$ and $G_{\tilde{\phi}_\lambda}(\mathcal{P}_\lambda)$ in the following natural way; starting with $G_{\phi_\lambda(\mathcal{P}_\lambda)}$, remove all edges of Euclidean length greater than $r_\lambda^{1-\varepsilon}$ to obtain $G_{\tilde{\phi}_\lambda}(\mathcal{P}_\lambda)$.

We claim next that (3.1) holds with ϕ_λ replaced by $\tilde{\phi}_\lambda$, i.e.

$$\lambda \int_\Gamma \exp\left(-\lambda \int_\Gamma \tilde{\phi}_\lambda(y-x)dy\right) dx \to \alpha. \tag{3.17}$$

Indeed, by the Mecke formula (3.5) the absolute value of the difference between the left side of (3.17) and that of (3.1) is bounded by the mean number of vertices having at least one incident edge in $G_{\phi_\lambda}(\mathcal{P}_\lambda)$ of length at least $r_\lambda^{1-\varepsilon}$, and hence by twice the expected number of such edges. However, by (2.3) the expected number of such edges is $O(\lambda^2\exp(-\eta r_\lambda^{-\varepsilon\eta}))$, which is $O(\lambda^2\exp(-\eta\lambda^{\varepsilon\eta/(2d)}))$ by (3.16), and therefore tends to zero.

Let Γ'_λ be the set of $x \in \Gamma$ distant more than $4r_\lambda^{1-\varepsilon}$ in the ℓ_∞ norm from the corners of Γ. Let $\tilde{N}_0(\lambda)$ be the number of isolated vertices of $G_{\tilde{\phi}_\lambda}(\mathcal{P}_\lambda)$ that are located in Γ'_λ. Then we claim that

$$\mathbb{E}[|N_0(G_{\phi_\lambda}(\mathcal{P}_\lambda)) - \tilde{N}_0(\lambda)|] \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{3.18}$$

To see this, observe first that $\mathbb{E}[|N_0(G_{\phi_\lambda}(\mathcal{P}_\lambda)) - N_0(G_{\tilde{\phi}_\lambda}(\mathcal{P}_\lambda))|]$ is bounded by twice the expected number of edges in $G_{\phi_\lambda}(\mathcal{P}_\lambda)$ of Euclidean length greater than $r_\lambda^{1-\varepsilon}$, which tends to zero as discussed above. Second, observe that for all $x \in \Gamma$, by (3.15) we have

$$\int_\Gamma \tilde{\phi}_\lambda(y-x)dy \geq 2^{-d}\pi_d r_\lambda^d \eta p_\lambda \geq I(\phi_\lambda)2^{-d}\pi_d\eta/K(\eta),$$

and $e^{-\lambda I(\phi_\lambda)} = O(1/\lambda)$ by (3.1), so that $\exp(-\lambda \int_\Gamma \tilde{\phi}_\lambda(y-x)dy) = O(\lambda^{-2^{-d}\pi_d\eta/K(\eta)})$, uniformly over $x \in \Gamma$. Hence the expected number of isolated vertices of $G_{\tilde{\phi}_\lambda}(\mathcal{P}_\lambda)$ lying in $\Gamma \setminus \Gamma'_\lambda$ is $O(r_\lambda^{d(1-\varepsilon)}\lambda^{1-2^{-d}\pi_d\eta/K(\eta)})$ which tends to zero by (3.16). Thus $\mathbb{E}[|N_0(G_{\tilde{\phi}_\lambda}(\mathcal{P}_\lambda)) - \tilde{N}_0|] \to 0$, and (3.18) follows. Note that by (3.18) and Markov's inequality, $P[\tilde{N}_0(\lambda) \neq N_0(G_{\phi_\lambda}(\mathcal{P}_\lambda))] \to 0$, so it suffices to prove (3.2) for $\tilde{N}_0(\lambda)$.

Discretising space into hypercubes of side $1/m$, applying the Chen-Stein method of Poisson approximation, and taking the large-m limit as in (32) and (33) of [14] (see also [12, Theorem 6.7]), we have that

$$\sum_{i=0}^{\infty} \left| P[\tilde{N}_0(\lambda) = i] - \frac{e^{-\mathbb{E}\tilde{N}_0(\lambda)}(\mathbb{E}\tilde{N}_0(\lambda))^i}{i!} \right| \leq 6(b_1 + b_2), \tag{3.19}$$

with

$$b_1 := \lambda^2 \int_{\Gamma'_\lambda} \int_{B(x;3r_\lambda^{1-\varepsilon})\cap\Gamma'_\lambda} \exp\left(-\lambda \int_\Gamma (\tilde{\phi}_\lambda(z-x) + \tilde{\phi}_\lambda(z-y))dz\right) dydx$$

and

$$\begin{aligned} b_2 &:= \lambda^2 \int_{\Gamma'_\lambda} \int_{B(x;3r_\lambda^{1-\varepsilon})\cap\Gamma'_\lambda} \exp\left(-\lambda \int_\Gamma g_{\tilde{\phi}_\lambda}(z,\{x,y\})dz\right) dydx \\ &= 2\lambda^2 \int_{\Gamma'_\lambda} \int_{B(x;3r_\lambda^{1-\varepsilon})\cap\Gamma'_{\lambda,x}} \exp\left(-\lambda \int_\Gamma g_{\tilde{\phi}_\lambda}(z,\{x,y\})dz\right) dydx \end{aligned}$$

where for $x \in \Gamma$, if $d = 2$ we let $\Gamma'_{\lambda,x}$ denote the set of $y \in \Gamma'_\lambda$ lying further from the boundary of Γ than x does, while if $d \geq 3$, we let $\Gamma'_{\lambda,x}$ denote the set of $y \in \Gamma'_\lambda$ lying closer to the centre of Γ in the ℓ_1 norm than x does.

By the union bound, $g_{\tilde{\phi}_\lambda}(z,\{x,y\}) \leq \tilde{\phi}_\lambda(z-x) + \tilde{\phi}_\lambda(z-y)$, and therefore $b_1 \leq b_2$. Hence by (3.19) and (3.18), to prove (3.2) it suffices to prove that $b_2 \to 0$.

We write $b_2 = b_2^{(1)} + b_2^{(2)}$, where $b_2^{(1)}$ denotes the contribution to b_2 from integrating over (x,y) with $y \in B(x;r_\lambda)$, and $b_2^{(2)}$ denotes the contribution to b_2 from integrating over (x,y) with $y \in B(x;3r_\lambda^{1-\varepsilon}) \setminus B(x;r_\lambda)$.

First suppose $d = 2$. Using Lemma 3.1, we have that

$$b_2^{(2)} \leq 9\pi\lambda^2 r_\lambda^{2(1-\varepsilon)} \int_{\Gamma'_\lambda} \exp\left(\left(-\lambda \int_\Gamma \tilde{\phi}_\lambda(z-x)dz\right) - \lambda(\eta/4)p_\lambda r_\lambda^2\right) dx.$$

By (3.17), we have

$$\exp(-\lambda I(\phi_\lambda)) \leq \exp(-\lambda I(\tilde{\phi}_\lambda)) = O(\lambda^{-1}). \tag{3.20}$$

By (3.15), we have $\exp(-\lambda p_\lambda r_\lambda^2) \leq \exp(-\lambda I(\phi_\lambda)/K(\eta))$ which is $O(\lambda^{-1/K(\eta)})$ by (3.20). Therefore, using also (3.17) and (3.12), followed by (3.16), yields

$$b_2^{(2)} = O\left(\lambda^{1-\eta/(4K(\eta))} r_\lambda^{2(1-\varepsilon)}\right) = O\left(\lambda^{\varepsilon-\eta/(4K(\eta))} (\log \lambda)^{3(1-\varepsilon)}\right) \to 0.$$

Now consider $b_2^{(1)}$. Recall from (3.12) that $\lambda p_\lambda r_\lambda^2 = \Theta(\log \lambda)$. By Lemma 3.1, and then (3.17), then (3.12),

$$\begin{aligned} b_2^{(1)} &\leq 2\lambda^2 \int_{\Gamma'_\lambda} \int_0^{r_\lambda} \exp\left(\left(-\lambda \int_\Gamma \tilde{\phi}_\lambda(z-x)dz\right) - \lambda p_\lambda (\eta/4) r_\lambda t\right) 2\pi t dt dx \\ &= O\left(\lambda^2 \left(\frac{1}{\lambda}\right) \int_0^\infty \exp(-(\eta/4)u)(\lambda p_\lambda r_\lambda)^{-2} u du\right) = O\left(\frac{1}{p_\lambda \log \lambda}\right). \end{aligned}$$

Therefore, if $p_\lambda > 1/2$ then $b_2^{(1)} \to 0$. Conversely, if $p_\lambda \leq 1/2$, then since $g_{\tilde{\phi}_\lambda}(z, \{x, y\}) \geq \tilde{\phi}_\lambda(z-x) + (1-p_\lambda)\tilde{\phi}_\lambda(z-y)$, and $\phi_\lambda \in \Phi_{d,\eta}$, we have

$$\begin{aligned} b_2^{(1)} &\leq 2\lambda^2 \int_{\Gamma'_\lambda} (\pi r_\lambda^2) \exp\left(\left(-\lambda \int_\Gamma \tilde{\phi}_\lambda(z-x)dz\right) - \lambda(1-p_\lambda)\eta p_\lambda(\pi r_\lambda^2/2)\right) dx \\ &= O\left(\lambda r_\lambda^2 \exp(-\pi(\eta/4)\lambda p_\lambda r_\lambda^2)\right) \end{aligned}$$

so that by (3.16), (3.15) and (3.20) we have $b_2^{(1)} = O\left((\log \lambda)^3 \lambda^{-\pi\eta/(4K(\eta))}\right) = o(1)$. Hence $b_2^{(1)} \to 0$, so that $b_2 \to 0$ as required when $d = 2$.

Now suppose $d \geq 3$. Let $\tilde{\Gamma} := \{x \in \Gamma : \|x\|_\infty \leq 1/2\}$. Then by Lemma 3.1 (b),

$$\begin{aligned} b_2^{(1)} &\leq 2^{d+1}\lambda^2 \int_{\tilde{\Gamma}} \int_{B(x;r_\lambda)\cap\Gamma'_{\lambda,x}} \exp\left(-\lambda\left[\int_\Gamma \tilde{\phi}_\lambda(z-x)dz + \eta_1 p_\lambda r_\lambda^{d-1}|y-x|\right]\right) dy dx \\ &\leq 2^{d+1}\lambda^2 \int_{\tilde{\Gamma}} \exp\left(-\lambda \int_\Gamma \tilde{\phi}_\lambda(z-x)dz\right) \int_{\mathbb{R}^d} \exp(-\eta_1 \lambda p_\lambda r_\lambda^d |w|) r_\lambda^d dw dx, \end{aligned}$$

and hence using (3.17) followed by (3.12), we obtain that

$$b_2^{(1)} = O(\lambda r_\lambda^d (\lambda p_\lambda r_\lambda^d)^{-d}) = O(p_\lambda^{-1} (\log \lambda)^{1-d}),$$

which tends to zero by our assumption on p_λ. By Lemma 3.1 (b) again,

$$b_2^{(2)} \leq 2^{d+1}\lambda^2 \pi_d r_\lambda^{d(1-\varepsilon)} \int_{\tilde{\Gamma}} \exp\left(-\lambda \int_\Gamma \tilde{\phi}_\lambda(z-x)dz\right) \times \exp(-\eta_1 \lambda p_\lambda r_\lambda^d) dx,$$

and hence using (3.17), (3.16), and (3.12), with δ as given at the start of this proof we obtain that $b_2^{(2)} = O\left(\lambda^\varepsilon (\log \lambda)^{3(1-\varepsilon)} \exp(-\eta_1 \delta \log \lambda)\right)$. By our choice of ε, this shows that $b_2^{(2)}$ tends to zero, completing the proof. □

4 Connectivity: the case of small p_λ

For any graph G, let $L_2(G)$ denote the order of its second-largest component, i.e. the second largest of the orders of its components; if G is connected, set $L_2(G) = 0$. Given connection functions $(\phi_\lambda)_{\lambda>0}$, let p_λ and r_λ be given by (3.10). In this section we prove the following result:

Proposition 4.1 *Suppose $(\lambda(n))_{n\in\mathbb{N}}$ is an increasing $(0,\infty)$-valued sequence that tends to ∞ as $n \to \infty$, and for some $\eta \in (0,1]$ and $\alpha \in (0,\infty)$, $(\phi_\lambda)_{\lambda>0}$ is a collection of connection functions in $\Phi_{d,\eta}$ such that as $\lambda \to \infty$ along the sequence $(\lambda(n))$ we have (3.1). Assume for some $\varepsilon > 0$ that $p_\lambda = O(\lambda^{-\varepsilon})$. Then as $\lambda \to \infty$ along the same sequence,*

$$P\left[L_2(G_{\phi_\lambda}(\mathcal{P}_\lambda)) > 1\right] \to 0.$$

It is immediate from Theorem 3.1 and Proposition 4.1 that under the hypotheses of Proposition 4.1, we have a Poissonized version of (2.9), namely $P[G_{\phi_\lambda}(\mathcal{P}_\lambda) \in \mathcal{K}] \to e^{-\alpha}$. Our strategy of proof of Proposition 4.1 is as follows. First we shall rule out 'small components' of order between 2 and $n^{\varepsilon/2}$ using the Mecke formula. Then we shall rule out the possibility of more than one 'large component' by a 'sprinkling' argument. That is, we add the edges in two stages, and even though we make the number of edges added in the second stage rather small, with high probability there are enough of them to connect together any two distinct large components arising from the first stage.

Given $n \in \mathbb{N}$ and $p \in [0,1]$, let $G(n,p)$ denote the Erdős-Rényi random graph on n vertices, i.e., the random subgraph of the complete graph on n vertices, obtained by including each possible edge independently with probability p. Our proof of Proposition 4.1 uses a lemma on large deviations for the giant component of $G(n,p)$.

Lemma 4.1 *Suppose $p = p(n)$ is such that $np \to \infty$ as $n \to \infty$. Let E_n be the event that $G(n,p)$ has no component of order greater than $3n/4$. Then $\limsup_{n\to\infty} n^{-1} \log P[E_n] < 0$.*

Proof. Suppose E_n occurs. Then by starting with the empty set and adding components of $G(n,p)$ in arbitrary order until we have at least $n/8$ vertices, we can find a set of between $n/8$ and $7n/8$ vertices that is disconnected from the rest of the vertices of $G(n,p)$. Hence by the union bound and the fact that $e^k \geq k^k/k!$ for any k,

$$\begin{aligned} P[E_n] \leq \sum_{n/8\leq k\leq 7n/8} \binom{n}{k}(1-p)^{k(n-k)} &\leq \sum_{n/8\leq k\leq 7n/8} \frac{n^k e^k}{k^k} \exp(-p(7/64)n^2) \\ &\leq n(8e)^n \exp(-n^2 p/10) \end{aligned}$$

and the result follows. $\square$

For any graph G any $k \in \mathbb{N}$, let $T_k(G)$ denote the number of components of G of order k.

Lemma 4.2 *Under the hypotheses of Proposition 4.1,*

$$P\left[\cup_{2\le k\le \lambda^{\varepsilon/3}}\{T_k(G_{\phi_\lambda}(\mathcal{P}_\lambda)) > 0\}\right] \to 0. \tag{4.1}$$

Proof. We may assume $r_\lambda \le \sqrt{d}$. By the Mecke formula (3.5) and Cayley's formula (which says there are k^{k-2} trees on k vertices), and the union bound, $\mathbb{E}\, T_k(G_{\phi_\lambda}(\mathcal{P}_\lambda))$ is bounded by

$$\frac{\lambda^k}{k!}k^{k-2}p_\lambda^{k-1}\int\cdots\int \exp\left(-\lambda\int g_{\phi_\lambda}(x;\{x_1,\ldots,x_k\})dx\right)dx_1\cdots dx_k,$$

where all integrals are over Γ in this proof. By (3.6), this is bounded by

$$\frac{(e\lambda p_\lambda)^k}{k^2p_\lambda}\int\cdots\int dx_1\cdots dx_k \exp\left(-\lambda\int\sum_{i=1}^k \phi_\lambda(x-x_i)dx\right)\exp(\lambda k^2 p_\lambda I(\phi_\lambda)). \tag{4.2}$$

By (3.11) the exponent in the last factor of (4.2) is $O(k^2p_\lambda\log\lambda)$. If $k\le\lambda^{\varepsilon/3}$, this exponent is $O(1)$ so the last factor in (4.2) is $O(1)$, uniformly over such k. Thus

$$\mathbb{E}\sum_{2\le k\le\lambda^{\varepsilon/3}} T_k(G_{\phi_\lambda}(\mathcal{P}_\lambda)) = O\left(p_\lambda^{-1}\sum_{k=2}^{\infty}(ep_\lambda\mathbb{E}\,N_0(G_{\phi_\lambda}(\mathcal{P}_\lambda)))^k\right)$$

which tends to zero. Then (4.1) follows by Markov's inequality. $\square$

Proof of Proposition 4.1. Assume that $r_\lambda\le\sqrt{d}$. Set $\phi'_\lambda(x)=\phi_\lambda(x)(1-\lambda^{-\varepsilon/6})$ for $x\in\mathbb{R}^d$. Note that (3.1) still holds using ϕ'_λ instead of ϕ_λ, since changing ϕ_λ to ϕ'_λ gives an extra term in the exponent of $O(\lambda^{1-\varepsilon/6}I(\phi_\lambda))$, which tends to zero by (3.11). Also $\phi'_\lambda\in\Phi_{d,\eta}$.

Consider generating $G_{\phi_\lambda}(\mathcal{P}_\lambda)$ in two stages. In the first stage, generate $G_{\phi'_\lambda}(\mathcal{P}_\lambda)$. In the second stage, for each pair of vertices X, Y not already connected by an edge in the first stage, add an edge between them with probability $(\phi_\lambda(Y-X)-\phi'_\lambda(Y-X))/(1-\phi'_\lambda(Y-X))$.

By (3.12), $\lambda r_\lambda^d=\Omega(\lambda^\varepsilon)$ and $r_\lambda=\Omega(\lambda^{(\varepsilon-1)/d})$. We now show that after the first stage, there is a giant component with high probability. Partition Γ into cubes of side $1/\lfloor 8d/r_\lambda\rfloor$. The number of cubes in the partition is $O(r_\lambda^{-d})=O(\lambda)$.

By a Chernoff bound (e.g. Lemma 1.2 of [12]), with high probability each cube in the partition contains at least $(9d)^{-d}\lambda r_\lambda^d$ vertices of $\mathcal{P}_\lambda$. Since we assume $r_\lambda \leq \sqrt{d}$, it is easily verified that $1/\lfloor 8d/r_\lambda \rfloor \leq r_\lambda/7d$. By (3.12), for each cube in the partition the restriction of $G_{\phi_\lambda}(\mathcal{P}_\lambda)$ to the vertices within that cube dominates the Erdős-Rényi random graph $G(n,p)$ with $np = \Omega(\lambda r_\lambda^d (\log \lambda)/(\lambda r_\lambda^d)) = \Omega(\log \lambda)$ so by Lemma 4.1, there is a giant component containing a proportion at least $(3/4)$ of the vertices in that cube, except on an event of probability $\exp(-\Omega(\lambda r_\lambda^d)) = \exp(-\Omega(\lambda^\varepsilon))$. Hence by the union bound, with high probability the restricted graph within each of these cubes contains a giant component.

Also by the same argument, with high probability, it is the case that for each pair of neighbouring cubes in the partition, the restriction of $G_{\phi_\lambda}(\mathcal{P}_\lambda)$ to vertices in that pair of cubes has a giant component with a proportion at least $3/4$ of the vertices in that pair of cubes, and therefore the two giant components within these neighbouring cubes are connected together. Note that for any $\delta > 0$, with high probability, by the Chernoff bound, for each pair of cubes the ratio of the number of vertices in one cube and the number of vertices in the other lies between $1-\delta$ and $1+\delta$.

Hence, after the first stage there is w.h.p. a giant component containing a proportion at least $3/4$ of all the vertices in each of the cubes in the partition. By Lemma 4.2, also w.h.p. there is no component of order greater than 1 but less than $\lambda^{\varepsilon/3}$. There may also be some isolated vertices, and some medium-size components of order between $\lambda^{\varepsilon/3}$ and $\lambda/2$. Now we rule out existence of components of order greater than $\lambda^{\varepsilon/3}$ besides the giant component, after the second stage.

After the first stage, w.h.p. the giant component contains more than $\lceil (9d)^{-d}\lambda r_\lambda^d/2 \rceil$ vertices in each of the cubes in the partition. Therefore each vertex not in the giant component has at least $(9d)^{-d}\lambda r_\lambda^d/2$ vertices from the giant component within distance r_λ of it (namely, those which are in the same cube of the partition as itself).

Now for each medium-sized component from the first stage, the probability that it fails to get attached to the giant component in the second stage is bounded by

$$(1-\lambda^{-\varepsilon/6}\eta p_\lambda/2)^{\lambda^{\varepsilon/3}\times(9d)^{-d}\lambda r_\lambda^d/2} \leq \exp(-(9d)^{-d}\eta\lambda^{\varepsilon/6}\lambda r_\lambda^d p_\lambda/4) \leq \exp(-\lambda^{\varepsilon/6}),$$

where the last inequality holds for all large enough λ, by (3.12). The number of medium-sized components from the first stage is bounded by 2λ w.h.p., so by the union bound, the probability that one or more of them fails to get attached to the giant component tends to zero.

Also the number of isolated vertices from the first stage is asymptotically Poisson by Lemma 3.2, and the probability that any two of these get connected together in the second stage is $O(\lambda^{-\varepsilon/6}p_\lambda)$ so tends to zero. Hence w.h.p., after the second stage there is no component of order greater than 1, besides the giant component. □

5 Connectivity: the case of large p_λ

In this section we prove the following result, which extends Proposition 4.1 by relaxing the restriction on p_λ that was imposed there, subject to $\phi_\lambda \in \Phi^0_{d,\eta}$.

Theorem 5.1 *Let $\alpha \in (0,\infty)$. Suppose that for some increasing sequence $(\lambda(n))_{n\in\mathbb{N}}$ that tends to ∞ as $n \to \infty$, $(\phi_\lambda)_{\lambda>0}$ satisfies (3.1) as $\lambda \to \infty$ along the sequence $(\lambda(n))_{n\in\mathbb{N}}$, and that there exists $\eta \in (0,1]$ such that $\phi_\lambda \in \Phi^0_{d,\eta}$ for all λ. Then as $\lambda \to \infty$ along the sequence $(\lambda(n))_{n\in\mathbb{N}}$,*

$$P\left[L_2(G_{\phi_\lambda}(\mathcal{P}_\lambda)) > 1\right] \to 0. \tag{5.1}$$

Throughout this section, we fix arbitrary $\eta \in (0,1]$ and assume $\phi_\lambda \in \Phi^0_{d,\eta}$ for all $\lambda > 0$, and $(\phi_\lambda)_{\lambda>0}$ satisfy (3.1) for some $\alpha \in (0,\infty)$ (all asymptotics being as $\lambda \to \infty$ along the sequence $(\lambda(n))_{n\in\mathbb{N}}$). Define $p_\lambda := \mu(\phi_\lambda)$ and $r_\lambda := \rho_\eta(\phi_\lambda)$ as in (3.10), and assume $r_\lambda = O(1)$.

In view of Proposition 4.1, it suffices to prove the result in the case where $p_\lambda = \Omega(\lambda^{-\varepsilon})$ for some suitably chosen $\varepsilon > 0$. Since the argument is long, we split the section further by first showing there are no 'small' components (other than isolated vertices) and then showing there is not more than one 'large' component.

5.1 Small components

This subsection contains several lemmas, because we sometimes need to distinguish the case with $d = 2$ (where we do not assume $\phi_\lambda \in \Psi_2$) and we also sometimes distinguish the case with $p = O(1)$ from $p = o(1)$. Moreover, we distinguish 'very small' components of (spatial) diameter at most δr_λ and 'moderately small' components of diameter between δr_λ and $(1/\delta) r_\lambda$, where δ is a small (but fixed) constant.

To deal with 'very small' components (in Lemmas 5.1, 5.2, 5.3 and 5.6) we use the Mecke formula directly and sum over all possible cardinalities of the component. To deal with 'moderately small components' (in Lemmas 5.4, 5.5 and 5.7), we discretize space into cubes (or strips) of side εr_λ for suitably small fixed ε. For $x \in \Gamma$ and for each possible 'moderately small' discretized region (i.e., union of some of these cubes) containing x, we estimate the probability that the component of $G_{\phi_\lambda}(\mathcal{P}_\lambda \cup \{x\})$ containing x is moderately small and corresponds to that particular region. To do this we show that there is enough 'unexplored space' outside the region but inside Γ, for the probability of there being no Poisson points in the unexplored space connected to the cluster within the explored region, is small compared to the probability of x being isolated.

We need some preliminaries. First we give a similar lemma to Lemma 6 of [14]. As before, let $\mathbf{H}$ denote the half-space $[0,\infty) \times \mathbb{R}^{d-1}$ and let $\mathbf{Q}$ denote the orthant

$[0,\infty)^d$. For $\lambda > 0$ let $\mathcal{H}_\lambda^{\mathbf{H}} := \mathcal{H}_\lambda \cap \mathbf{H}$ and let $\mathcal{H}_\lambda^{\mathbf{Q}} := \mathcal{H}_\lambda \cap \mathbf{Q}$. Define

$$\psi_\lambda(x) := \phi_\lambda(r_\lambda x), \qquad x \in \mathbb{R}^d.$$

For any locally finite set $\mathcal{X}$ in $\mathbb{R}^d$, and any $x \in \mathbb{R}^d$, and connection function ϕ, let $C_\phi(x, \mathcal{X})$ be the vertex set of the component of $G_\phi(\mathcal{X} \cup \{x\})$ containing x. Let $D_\phi(x, \mathcal{X}) := \mathrm{diam}(C_\phi(x, \mathcal{X})) := \sup_{y,z \in C_\phi(x,\mathcal{X})} |y - z|$. For $\mathcal{A}$ a countable set in $\mathbb{R}^2$ and $x \in \mathcal{A}$, let $L_\phi(x, \mathcal{A})$ denote the event that x is the left-most vertex of $C_\phi(x, \mathcal{A})$ (i.e., the first vertex in the lexicographic ordering). Also, let $L'_\phi(x, \mathcal{A})$ denote the event that x is the vertex of $C_\phi(x, \mathcal{A})$ lying closest to the boundary of the quadrant $\mathbf{Q}$.

Lemma 5.1 *Suppose $d = 2$ and $p_\lambda \geq 1/2$ for all λ. Then for $0 < \delta \leq \eta/(8\pi)$ we have*

$$\lim_{\lambda\to\infty} \sup_{x \in \mathbf{H}} \frac{P[0 < D_{\psi_\lambda}(x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2}) < \delta; L_{\psi_\lambda}(x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2})]}{P[D_{\psi_\lambda}(x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2}) = 0]} = 0.$$

Proof. Given $x \in \mathbf{H}$ and $\delta > 0$, let A_δ denote the right half of the disk of radius δ centred at x. Let $q_k^\delta(x,\lambda)$ be the probability that $C_{\psi_\lambda}(x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2})$ has precisely k elements and is contained in A_δ. Clearly

$$P[0 < D_{\psi_\lambda}(x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2}) < \delta; L_{\psi_\lambda}(x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2})] \leq \sum_{k=2}^{\infty} q_k^\delta(x,\lambda).$$

By the Mecke formula, similarly to [13, Proposition 1], with h_ϕ and g_ϕ defined at (3.3) and (3.4), we have

$$\begin{aligned} q_k^\delta(x,\lambda) &= \frac{(\lambda r_\lambda^2)^{k-1}}{(k-1)!} \int_{A_\delta} \cdots \int_{A_\delta} h_{\psi_\lambda}(\{x, x_1, \ldots, x_{k-1}\}) \\ &\times \exp\left(-\lambda r_\lambda^2 \int_{\mathbf{H}} g_{\psi_\lambda}(y, \{x, x_1, \ldots, x_{k-1}\}) dy\right) dx_1 \cdots dx_{k-1}. \end{aligned} \tag{5.2}$$

Similarly $q_1^\delta(x,\lambda) = \exp(-\lambda r_\lambda^2 \int_{\mathbf{H}} \psi_\lambda(y - x) dy)$. Since $h_{\psi_\lambda}(\mathcal{A}) \leq 1$ for any $\mathcal{A}$ we have

$$\begin{aligned} \frac{q_k^\delta(x,\lambda)}{q_1^\delta(x,\lambda)} &\leq \frac{(\lambda r_\lambda^2)^{k-1}}{(k-1)!} \int_{A_\delta} \cdots \int_{A_\delta} \\ \times \exp&\left(-\lambda r_\lambda^2 \int_{\mathbf{H}} [g_{\psi_\lambda}(y, \{x, x_1, \ldots, x_{k-1}\}) - \psi_\lambda(y - x)] dy\right) dx_1 \cdots dx_{k-1}. \end{aligned} \tag{5.3}$$

If we restrict the integral in (5.3) to those $(x_1, \dots, x_{k-1})$ with $|x_i - x| \leq |x_1 - x|$ for $2 \leq i \leq k-1$, we reduce it by a factor of $k-1$. Therefore

$$\frac{q_k^\delta(x,\lambda)}{q_1^\delta(x,\lambda)} \leq \frac{\lambda r_\lambda^2 (\lambda r_\lambda^2 \pi/2)^{k-2}}{(k-2)!} \int_{A_\delta} |x_1 - x|^{2(k-2)}$$
$$\times \exp\left(-\lambda r_\lambda^2 \int_{\mathbf{H}} [g_{\psi_\lambda}(y, \{x, x_1\}) - \psi_\lambda(y-x)] dy \right) dx_1.$$

By Lemma 3.1 and the fact that $\rho_\eta(\psi_\lambda) = 1$, for $x_1 \in A_1$ we have

$$\int_{\mathbf{H}} [g_{\psi_\lambda}(y, \{x, x_1\}) - \psi_\lambda(y-x)] dy \geq |x_1 - x| \eta p_\lambda / 4,$$

so that for $\delta \leq 1$ we have

$$\frac{q_k^\delta(x,\lambda)}{q_1^\delta(x,\lambda)} \leq \frac{\lambda r_\lambda^2 (\lambda r_\lambda^2 \pi/2)^{k-2}}{(k-2)!} \int_{A_\delta} |x_1 - x|^{2(k-2)} \exp\left(-\lambda r_\lambda^2 (\eta/4) p_\lambda |x_1 - x| \right) dx_1.$$

Summing over $k \geq 2$ and using the assumptions $p_\lambda \geq 1/2$ and $\delta \leq \eta/(8\pi)$, yields

$$\sum_{k=2}^{\infty} \frac{q_k^\delta(x,\lambda)}{q_1^\delta(x,\lambda)} \leq \lambda r_\lambda^2 \int_{A_\delta} \exp(\lambda r_\lambda^2 [(\pi/2)|x_1 - x|^2 - (\eta/4) p_\lambda |x_1 - x|]) dx_1$$
$$\leq \lambda r_\lambda^2 \int_{A_\delta} \exp(-\lambda r_\lambda^2 |x_1 - x| \eta/16) dx_1 = O((\lambda r_\lambda^2)^{-1})$$

which tends to zero by (3.12). □

In the case with $p_\lambda \leq 1/2$, we give a similar result to the last one, but for general $d \geq 2$. Let π_d denote the volume of the unit ball in d dimensions. Let $\tilde{\mathbf{Q}}$ denote the orthant $\mathbf{Q}$ if $d \geq 3$, but denote the half-space $\mathbf{H}$ if $d = 2$.

Lemma 5.2 *Suppose ϕ_λ and ψ_λ are as before (now for general d, $d \geq 2$). Let $0 < \delta < \eta/8$. If $p_\lambda \leq 1/2$ for all λ but $p_\lambda = \Omega(\lambda^{-1/2^{d+3}})$, then*

$$\limsup_{\lambda \to \infty} \sup_{x \in \tilde{\mathbf{Q}}} \left(\frac{P[0 < D_{\psi_\lambda}(x, \mathcal{H}^{\tilde{\mathbf{Q}}}_{\lambda r_\lambda^d}) < \delta]}{P[D_{\psi_\lambda}(x, \mathcal{H}^{\tilde{\mathbf{Q}}}_{\lambda r_\lambda^d}) = 0]} \right) = 0.$$

Proof. For $\delta > 0$, $x \in \tilde{\mathbf{Q}}$ and $k \in \mathbb{N}$, define

$$w_\lambda(k, \delta) := \frac{P[0 < D_{\psi_\lambda}(x, \mathcal{H}^{\tilde{\mathbf{Q}}}_{\lambda r_\lambda^d}) < \delta; \mathrm{card}(C_{\psi_\lambda}(x, \mathcal{H}^{\tilde{\mathbf{Q}}}_{\lambda r_\lambda^d})) = k+1]}{P[D_{\psi_\lambda}(x, \mathcal{H}^{\tilde{\mathbf{Q}}}_{\lambda r_\lambda^d}) = 0]},$$

where card$(\cdot)$ denotes the number of elements in a set. For $k \in \mathbb{N}$ we have, similarly to (5.2), that

$$w_\lambda(k,\delta) \le \frac{(\lambda r_\lambda^d)^k}{k!} \int_{B(x;\delta)\cap\tilde{\mathbf{Q}}} \cdots \int_{B(x;\delta)\cap\tilde{\mathbf{Q}}} \times \exp\left(- \int_{\tilde{\mathbf{Q}}} \lambda r_\lambda^d [g_{\psi_\lambda}(y, \{x, x_1, \ldots, x_k\}) - \psi_\lambda(y-x)] dy \right) dx_1 \cdots dx_k. \quad (5.4)$$

Now,

$$\begin{aligned} g_{\psi_\lambda}(y, \{x, x_1, x_2, \ldots, x_k\}) - \psi_\lambda(y-x) &\ge (1-p_\lambda)\left(1 - \prod_{i=1}^k (1-\psi_\lambda(y-x_i))\right) \\ &\ge (1-p_\lambda)\left(1 - \exp\left(-\sum_{i=1}^k \psi_\lambda(y-x_i)\right)\right). \quad (5.5) \end{aligned}$$

First consider $k \le 1/p_\lambda$. Since $1 - e^{-x} \ge x/2$ for $0 \le x \le 1$, and we assume $p_\lambda \le 1/2$, for such k we have

$$g_{\psi_\lambda}(y, \{x, x_1, x_2, \ldots, x_k\}) - \psi_\lambda(y-x) \ge (1/4) \sum_{i=1}^k \psi_\lambda(y-x_i).$$

Now $\int_{\tilde{\mathbf{Q}}} \psi_\lambda(y-x_i)dy \ge I(\psi_\lambda)/2^d$ for each λ and each x_i, because for $d \ge 3$ we assume $\phi_\lambda \in \Psi_d$, and for $d=2$ we assume $\tilde{\mathbf{Q}} = \mathbf{H}$ and ϕ_λ satisfies $\phi_\lambda(x) = \phi_\lambda(-x)$ for all x. Therefore by (5.4), for $k \le 1/p_\lambda$ we have

$$\begin{aligned} w_\lambda(k,\delta) &\le \frac{(\lambda r_\lambda^d)^k}{k!} \int_{(B(x;\delta)\cap\tilde{\mathbf{Q}})^d} \exp\left(-\frac{1}{4} \int_{\tilde{\mathbf{Q}}} \lambda r_\lambda^d \sum_{i=1}^k \psi_\lambda(y-x_i) dy \right) d(x_1, \ldots, x_k) \\ &\le \frac{(\delta^d \pi_d \lambda r_\lambda^d)^k}{k!} \exp(-\lambda k I(\phi_\lambda)/2^{d+2}). \end{aligned}$$

Hence,

$$\sum_{k=1}^{\lfloor 1/p_\lambda \rfloor} w_\lambda(k,\delta) \le \exp[\delta^d \pi_d \lambda r_\lambda^d e^{-\lambda I(\phi_\lambda)/2^{d+2}}] - 1. \quad (5.6)$$

Since we assume (3.1) we have $e^{-\lambda I(\phi_\lambda)} = O(\lambda^{-1})$, and using (3.12) we have that

$$\lambda r_\lambda^d e^{-\lambda I(\phi_\lambda)/2^{d+2}} = O\left(\frac{\log \lambda}{p_\lambda \lambda^{1/2^{d+2}}} \right)$$

which tends to zero, by our condition on p_λ. Therefore the expression in (5.6) tends to zero.

Now consider $k > 1/p_\lambda$. For $x_1, \ldots, x_k \in B(x;\delta)$ and $y \in B(x;1/2)$ we have $|y - x_i| < 1$ and hence $\psi_\lambda(y - x_i) \geq \eta p_\lambda$ for $1 \leq i \leq k$, so by (5.5) we have that

$$g_{\psi_\lambda}(y, \{x, x_1, x_2, \ldots, x_k\}) - \psi_\lambda(y - x) \geq (1 - p_\lambda)(1 - \exp(-\eta k p_\lambda)) \geq (1 - e^{-\eta})/2.$$

Therefore using (5.4) and the fact that $1 - e^{-\eta} \geq \eta/2$, we have

$$\sum_{k > 1/p_\lambda} w_\lambda(k, \delta) \leq \exp(\delta^d \pi_d \lambda r_\lambda^d) \exp(-\pi_d \lambda r_\lambda^d \eta / 2^{d+2}),$$

and by the choice of δ, this tends to zero. Combining these estimates gives the result. □

Combining Lemma 5.1 and the case $d = 2$ of Lemma 5.2 immediately gives us the following.

Lemma 5.3 *Suppose $d = 2$ and η, ϕ_λ and ψ_λ are as before. Suppose also that $p_\lambda = \Omega(\lambda^{-1/32})$. Let $0 < \delta < \eta/(8\pi)$. Then*

$$\limsup_{\lambda \to \infty} \sup_{x \in \mathbf{H}} \left(\frac{P[0 < D_{\psi_\lambda}(x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2}) < \delta; L_{\psi_\lambda}(x; \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2})]}{P[D_{\psi_\lambda}(x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2}) = 0]} \right) = 0.$$

Lemma 5.4 *Given $0 < \delta < \rho < \infty$, it is the case (for general $d \geq 2$) that*

$$\lim_{\lambda \to \infty} \sup_{x \in \mathbf{H}} \frac{P[\delta < D_{\psi_\lambda}(x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2}) < \rho; L_{\psi_\lambda}(x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2})]}{P[D_{\psi_\lambda}(x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2}) = 0]} = 0.$$

Proof. This can be proved along the lines of [13, Lemma 3]; the argument still works in the case with $p_\lambda \to 0$, provided $\lambda r_\lambda^2 p_\lambda \to \infty$, which is always the case for us by (3.12). □

Similarly to [14, Lemma 7] (which is missing a factor of π in the exponent) we have the following:

Lemma 5.5 *Suppose $d = 2$. For any $\rho > 0$, as $\lambda \to \infty$ we have*

$$\sup_{x \in \mathbf{Q}} P[D_{\psi_\lambda}(x; \mathcal{H}^{\mathbf{Q}}_{\lambda r_\lambda^2}) < \rho] = o\left(\exp\{-\lambda \eta I(\phi_\lambda)/(3K(\eta))\}\right). \quad (5.7)$$

Proof. Fix $\rho > 0$. Divide $\mathbf{Q}$ into vertical strips of width 1/9, denoted $S_i, i \in \mathbb{N}$, where $S_i := [(i-1)/9, i/9) \times [0, \infty)$. Let $x \in \mathbf{Q}$, and let $i_0 = i_0(x)$ be the choice of i such that $x \in S_i$. Also let $i_1 = i_0 + 9\lceil \rho \rceil$.

Given λ, for $i \in \mathbb{N} \cap [i_0, i_1]$ let E'_i be the event that the right-most point of $C_{\psi_\lambda}(x, \mathcal{H}^{\mathbf{Q}}_{\lambda r_\lambda^2})$ lies in S_i. If $D_{\psi_\lambda}(x; \mathcal{H}^{\mathbf{Q}}_{\lambda r_\lambda^2}) < \rho$ then one of the events $E_{i_0}, \dots, E_{i_1}$ occurs.

Now fix $i \in \mathbb{N} \cap [i_0, i_1]$. Set $A_i := \cup_{j \leq i} S_j$, and $A_i^c := \cup_{j > i} S_j$. Consider generating $G_{\psi_\lambda}(\{x\} \cup \mathcal{H}^{\mathbf{Q}}_{\lambda r_\lambda^2})$ in two stages. In the first stage, generate the Poisson process $\mathcal{H}_{\lambda r_\lambda^2} \cap A_i$, and add edges between points of $\{x\} \cup (\mathcal{H}_{\lambda r_\lambda^2} \cap A_i)$ with probabilities determined by the connection function ψ_λ. Then in the second stage, add the points of $\mathcal{H}_{\lambda r_\lambda^2} \cap A_i^c$ and add edges between these added points, and between the added points and the points from the first stage, again using the connection function ψ_λ.

The first stage generates a realization of the graph $G_{\psi_\lambda}(\{x\} \cup (\mathcal{H}_{\lambda r_\lambda^2} \cap A_i))$; let $E_{i,1}$ be the event that the resulting realization of $C_{\psi_\lambda}(x, \mathcal{H}_{\lambda r_\lambda^2} \cap A_i)$ includes at least one vertex in S_i. Let $E_{i,2}$ be the event that the second stage does not generate any new Poisson points that are connected to vertices of $C_{\psi_\lambda}(x, \mathcal{H}_{\lambda r_\lambda^2} \cap A_i)$ arising from the first stage. Then $E'_i = E_{i,1} \cap E_{i,2}$.

Suppose $E_{i,1}$ occurs. Let z be the right-most vertex of $C_{\psi_\lambda}(x, \mathcal{H}_{\lambda r_\lambda^2} \cap A_i)$; then $z \in S_i$ by definition. Then in stage 2, a necessary condition for $E_{i,2}$ to occur is that there is no point of $\mathcal{H}_{\lambda r_\lambda^2} \cap A_i^c$ connected by an edge z. Since $B(z;1) \cap A_i^c$ has area at least $(\pi/4) - 1/9$,

$$P[E'_i | E_{i,1}] \leq \exp(-\lambda r_\lambda^2 \eta p_\lambda((\pi/4) - 1/9)) \leq \exp(-\eta \lambda I(\phi_\lambda)/(2K(\eta))),$$

where the last inequality comes from (3.15). This gives us (5.7). □

For $x \in \Gamma$, let Γ_x be the set of $y \in \Gamma$ such that y is closer to the centre of Γ in the ℓ_1 norm than x is. For $\rho > 0$ and $x \in \Gamma$, let $E_{\lambda,\rho,x}$ be the event that there is a non-empty set U of points of $\mathcal{P}_\lambda$ contained in $B(x;\rho) \cap \Gamma_x$, such that no other point of $\mathcal{P}_\lambda \setminus U$ is connected to any point of $\{x\} \cup U$ in $G_{\phi_\lambda}(\{x\} \cup \mathcal{P}_\lambda)$.

Lemma 5.6 *Suppose $d \geq 3$ and $p_\lambda = \Omega(1)$. Then there exists $\delta > 0$ such that*

$$\lim_{\lambda \to \infty} \sup_{x \in \Gamma} P[E_{\lambda,\delta r_\lambda,x}] / \exp\left(-\lambda \int_\Gamma \phi_\lambda(y-x)dy\right) = 0.$$

Proof. The proof resembles that of [15, Lemma 5.2] or [12, Lemma 13.15]. For $j \in \mathbb{N}$ let $\mu^x(j,\lambda)$ be the number of subsets U of $\mathcal{P}_\lambda$ with j elements, such that $U \subset \Gamma_x \cap B(x;\delta r_\lambda)$ and no element of $U \cup \{x\}$ is connected to any element of $\mathcal{P}_\lambda \setminus U$

in $G_{\phi_\lambda}(\{x\} \cup \mathcal{P}_\lambda)$. Then by the Mecke formula (3.5),

$$\mathbb{E}\,\mu^j(x,\lambda) = \frac{\lambda^j}{(j-1)!} \int_{\Gamma_x \cap B(x;\delta r_\lambda)} \int_{(\Gamma_x \cap B(x;|y-x|))^{j-1}}$$
$$\times \exp\left(-\lambda \int g_{\phi_\lambda}(z, \{x, y, x_1, \ldots, x_{j-1}\}) dz\right) d(x_1, \ldots, x_{j-1}) dy$$
$$\leq \frac{\lambda(\lambda\pi_d)^{j-1}}{(j-1)!} \int_{\Gamma_x \cap B(x;\delta r_\lambda)} |y-x|^{d(j-1)} \exp\left(-\lambda \int g_{\phi_\lambda}(z, \{x, y\}) dz\right) dy.$$

Assume $\delta \leq 1$. By lemma 3.1 (b), the integrand in the last exponent is bounded below by $\phi_\lambda(z-x) + \eta_1 p_\lambda \rho_\lambda^{d-1}|y-x|$, and therefore

$$\frac{\mathbb{E}\,\mu^j(x,\lambda)}{\exp(-\lambda \int \phi_\lambda(z-x)dz)} \leq \frac{\lambda(\lambda\pi_d)^{j-1}}{(j-1)!} \int_{B(x;\delta r_\lambda)} |y-x|^{d(j-1)} \exp(-\eta_1 \lambda p_\lambda r_\lambda^{d-1}|y-x|) dy.$$

Summing over j and changing variable to $w = (y-x)/r_\lambda$, we obtain

$$\frac{P[E_{\lambda,\delta r_\lambda,x}]}{\exp\left(-\lambda \int_\Gamma \phi_\lambda(z-x)dz\right)} \leq \lambda \int_{B(x;\delta r_\lambda)} \exp(\lambda\pi_d|y-x|^d - \eta_1 \lambda p_\lambda r_\lambda^{d-1}|y-x|) dy$$
$$= \lambda \int_{B(0;\delta)} \exp(\lambda\pi_d r_\lambda^d |w|^d - \eta_1 \lambda p_\lambda r_\lambda^d |w|) r_\lambda^d dw$$

Using our assumption on p_λ, we may choose δ small enough so that $\pi_d \delta^d \leq (\eta_1/2) p_\lambda \delta$ for all λ, and then there is a constant δ' so the last bound is at most $\lambda r_\lambda^d \int \exp(-\delta' \lambda r_\lambda^d |w|) dw$ which is $O((\lambda r_\lambda^d)^{1-d})$ and therefore tends to zero by (3.12). □

In the next lemma we do not need to assume $p_\lambda = \Omega(1)$.

Lemma 5.7 *Suppose $d \geq 3$. Then for $0 < \delta < \rho < \infty$ we have*

$$\lim_{\lambda\to\infty} \sup_{x\in\Gamma} P[E_{\lambda,\rho r_\lambda,x} \setminus E_{\lambda,\delta r_\lambda,x}] / \exp\left(-\lambda \int_\Gamma \phi_\lambda(y-x)dy\right) = 0.$$

Proof. Fix δ and ρ, and assume $\delta \leq 1$. Let $\varepsilon > 0$ be a small constant to be chosen later. Given λ, divide $\mathbb{R}^d$ into boxes (i.e., hypercubes of the form $\prod_{i=1}^d [a_i, a_i + h)$) of side $h = \varepsilon r_\lambda$. Let Λ'_λ be the set of centres of these boxes. For $z \in \Lambda'_\lambda$ let B'_z be the box centred at z. Let $x \in \Gamma$ and let z_x be the $z \in \Lambda'_\lambda$ such that x lies in B'_z. Also, for all $z \in \Lambda'_\lambda$ let $B_z := B'_z \cap \Gamma_x$.

For $\sigma \subset \Lambda'_\lambda$, let $B_\sigma := \cup_{z\in\sigma} B_z$. Let $\mathcal{C}(\lambda, x)$ be the set of $\sigma \subset \Lambda'_\lambda$ such that (i) $z_x \in \sigma$, and (ii) $\sigma \subset B(x; (\rho + d\varepsilon) r_\lambda)$, and (iii) $\sigma \setminus B(x; (\delta - d\varepsilon) r_\lambda) \neq \emptyset$, and (iv) $|B_z| > 0$ for each $z \in \sigma$ (where $|\cdot|$ denotes Lebesgue measure). In the sequel, we assume $\varepsilon < \delta/(2d)$ so that $\delta - d\varepsilon > \delta/2$.

For $\sigma \in \mathcal{C}(\lambda, x)$, let $E'_\lambda(\sigma)$ be the event that (i) $\sigma = \{z \in \Lambda'_\lambda : C_{\phi_\lambda}(x, \mathcal{P}_\lambda) \cap B_z \neq \emptyset\}$ and (ii) $C_{\phi_\lambda}(x, \mathcal{P}_\lambda) \subset \Gamma_x$. Then $E_{\lambda,\rho r_\lambda,x} \setminus E_{\lambda,\delta r_\lambda,x} \subset \cup_{\sigma \in \mathcal{C}(\lambda,x)} E'_\lambda(\sigma)$.

Let $\sigma \in \mathcal{C}(\lambda, x)$. Consider generating $C_{\phi_\lambda}(x, \mathcal{P}_\lambda)$ in two stages, similarly to the proof of Lemma 5.5. In Stage 1, add all points of $\mathcal{P}_\lambda$ in B_σ, and all edges involving these points and x (using the connection function ϕ_λ). In Stage 2, add the points of $\mathcal{P}_\lambda$ in $\Gamma \setminus B_\sigma$, and add connections between these new points and each other, and between the new points and the points from the Stage 1, again using connection function ϕ_λ.

In Stage 1, we generate a realization of $G_{\phi_\lambda}(\{x\} \cup (\mathcal{P}_\lambda \cap B_\sigma))$, and hence a realization of $C_{\phi_\lambda}(x, \mathcal{P}_\lambda \cap B_\sigma)$. Let $E'_{\lambda,1}(\sigma)$ be the event that this realization of $C_{\phi_\lambda}(x, \mathcal{P}_\lambda \cap B_\sigma)$ is contained in Γ_x and includes at least one point from each $B_z, z \in \sigma$. Let $E'_{\lambda,2}$ be the event that none of the new points created in Stage 2 are joined to any points of the realization of $C_{\phi_\lambda}(x, \mathcal{P}_\lambda \cap B_\sigma)$ generated in Stage 1. Then $E'_\lambda(\sigma) = E'_{\lambda,1}(\sigma) \cap E'_{\lambda,2}(\sigma)$. Since the cardinality of $\mathcal{C}(\lambda, x)$ is bounded independently of x and λ, it suffices to show that

$$\limsup_{\lambda\to\infty} \sup_{x\in\Gamma, \sigma\in\mathcal{C}(\lambda,x)} \frac{P[E'_{\lambda,2}(\sigma)|E'_{\lambda,1}(\sigma)]}{\exp(-\lambda \int \phi_\lambda(y - x)dy)} = 0. \tag{5.8}$$

Now,

$$P[E'_{\lambda,2}(\sigma)|E'_{\lambda,1}(\sigma)] \leq \exp\left(-\lambda \inf_{\mathcal{X}\subset B_\sigma\cap\Gamma_x : \mathcal{X}\cap Q_z\neq\emptyset \forall z\in\sigma} \int_{\Gamma\setminus B_\sigma} g_{\phi_\lambda}(y; \mathcal{X})dy\right)$$

and for each $\mathcal{X} \subset B_\sigma \cap \Gamma_x$ with $\mathcal{X} \cap Q_z \neq \emptyset$ for all $z \in \sigma$ we have

$$\begin{aligned}\int_{\Gamma\setminus B_\sigma} g_{\phi_\lambda}(y; \mathcal{X})dy &= p_\lambda \int_0^{p_\lambda^{-1}} \int_{\Gamma\setminus B_\sigma} \mathbf{1}_{\{g_{\phi_\lambda}(y;\mathcal{X})\geq p_\lambda u\}} dydu \\ &\geq p_\lambda \int_0^1 |\Gamma \cap (\sigma \oplus B(0; \rho_u(\phi_\lambda) - d\varepsilon r_\lambda)) \setminus B_\sigma| du,\end{aligned}$$

where the last line arises because if $y \in \sigma \oplus B(0; \rho_u(\phi_\lambda) - d\varepsilon r_\lambda)$ then there exists $v \in \mathcal{X}$ with $|y - v| \leq \rho_u(\phi_\lambda)$ and therefore $g_{\phi_\lambda}(y; \mathcal{X}) \geq \phi_\lambda(y - v) \geq up_\lambda$ by (2.2).

For $0 < u \leq 1$, since $\phi_\lambda \in \Phi^0_{d,\eta}$ we have $r_\lambda \leq \rho_u(\phi_\lambda) \leq \eta^{-1} r_\lambda$. Hence $\rho_u(\phi_\lambda) - d\varepsilon r_\lambda \geq r_\lambda/2$. Using [15, Proposition 2.1] or [12, Proposition 5.15], writing $V_r(x)$ for $|B(x; r) \cap \Gamma|$ we can find a constant $\eta_3 > 0$, depending only on d and η, such that

$$\int_{\Gamma\setminus B_\sigma} g_{\phi_\lambda}(y; \mathcal{X})dy \geq p_\lambda \left(\int_0^1 V_{\rho_u(\phi_\lambda)-d\varepsilon r_\lambda}(x)du + \int_0^1 \eta_3 r_\lambda^d du\right).$$

To estimate the first term in the expression above, note that since $\rho_u(\phi_\lambda) \leq \eta^{-1} r_\lambda$ there is a constant K_1 (depending on d and η) such that $V_{\rho_u(\phi_\lambda)}(x)du - V_{\rho_u(\phi_\lambda)-d\varepsilon r_\lambda}(x) \leq$

$K_1 r_\lambda^d \varepsilon$. Therefore

$$\int_{\Gamma \setminus B_\sigma} g_{\phi_\lambda}(y;\mathcal{X})dy \geq p_\lambda \left(\int_0^1 V_{\rho_u(\phi_\lambda)}(x)du - K_1 r_\lambda^d \varepsilon + \eta_3 r_\lambda^d \right)$$

$$= \int_\Gamma \phi_\lambda(y-x)dy - K_1 p_\lambda r_\lambda^d \varepsilon + \eta_3 p_\lambda r_\lambda^d,$$

and by choosing $\varepsilon < \eta_3/(2K_1)$ we have that the ratio in the left hand side of (5.8) is bounded below by $\exp(-\eta_3 \lambda p_\lambda r_\lambda^d/2)$, uniformly over x and σ. Since $\lambda r_\lambda^d p_\lambda \to \infty$ by (3.12), this gives us (5.8) as required. □

Given $\lambda > 0$, $\rho > 0$, define the event

$$E_\lambda^\rho = \{\exists\, x \in \mathcal{P}_\lambda : 0 < D_{\phi_\lambda}(x, \mathcal{P}_\lambda) \leq \rho\}.$$

Proposition 5.1 *Let* $\eta \in (0,1]$, $\alpha \in (0,\infty)$, *and* $0 < \varepsilon \leq \min(\eta/(7K(\eta)), 2^{-(d+3)})$. *Suppose* $\phi_\lambda \in \Phi_{d,\eta}^0$ *for all* λ, *and* (3.1) *holds, and* $p_\lambda = \Omega(\lambda^{-\varepsilon})$. *Then for any* $\rho > 0$, *we have* $\lim_{\lambda\to\infty} P[E_\lambda^{\rho r_\lambda}] = 0$.

Proof. First consider the case with $d \geq 3$. Assume first that $p_\lambda = \Omega(1)$. Then by the Mecke formula and the preceding two lemmas,

$$P[E_\lambda^{\rho r_\lambda}] \leq \int_\Gamma P[E_{\lambda,\rho r_\lambda,x}]\lambda dx = o(1) \times \int_\Gamma \exp\left(-\lambda \int_\Gamma \phi_\lambda(y-x)dy\right)\lambda dx$$

which is $o(1)$ by (3.1).

Now suppose instead that $p_\lambda \to 0$ but $p_\lambda = \Omega(\lambda^{-\varepsilon})$. Then $r_\lambda = o(1)$ by (3.12). Let $\tilde{\Gamma}$ denote the set of points in Γ lying closer to the origin (in the Euclidean norm) than to any other corner of Γ. Choosing $\delta \in (0, \eta/8)$ we have by the Mecke formula and Lemma 5.2 that

$$\begin{aligned} P[E_\lambda^{\delta r_\lambda}] &\leq 2^d \lambda \int_{\tilde{\Gamma}} P[0 < D_{\phi_\lambda}(x,\mathcal{P}_\lambda) < \delta r_\lambda]dx \\ &= 2^d \lambda \int_{\tilde{\Gamma}} P[0 < D_{\psi_\lambda}(r_\lambda^{-1}x, \mathcal{H}^{\mathbf{Q}}_{\lambda r_\lambda^d}) < \delta]dx \\ &= o(1) \times \lambda \int_{\tilde{\Gamma}} P[D_{\psi_\lambda}(r_\lambda^{-1}x, \mathcal{H}^{\mathbf{Q}}_{\lambda r_\lambda^d}) = 0]dx, \end{aligned}$$

which tends to zero by (3.1). Also for any finite $\rho > \delta$, by the Mecke formula

$$P[E_\lambda^{\rho r_\lambda} \setminus E_\lambda^{\delta r_\lambda}] \leq \lambda \int_\Gamma P[E_{\lambda,\rho r_\lambda,x} \setminus E_{\lambda,\delta r_\lambda,x}]dx,$$

which tends to zero by Lemma 5.7 and (3.1). This gives us the result for the case with $d \geq 3$.

Now consider the case with $d = 2$. Then $r_\lambda^2 = O(\lambda^{2\varepsilon-1})$ by (3.12). Let T_1 (respectively T_2, T_3, T_4) be the set of points of $[0,1]^2$ that lie closer to the left (respectively top, right, bottom) edge of Γ than to any of the other edges of Γ (so T_1 is the triangle with corners at $(0,0), (0,1)$ and $(1/2, 1/2)$).

For $x \in \Gamma$, let $\tilde{L}_{\phi_\lambda}(x, \mathcal{P}_\lambda)$ be the event that x is the point of $C_{\phi_\lambda}(x, \mathcal{P}_\lambda)$ lying closest to the boundary of $[0,1]^2$. Let M_λ be the number of $x \in \mathcal{P}_\lambda$ such that $D_{\phi_\lambda}(x, \mathcal{P}_\lambda) < \rho r_\lambda$ and x is the point of $C_{\phi_\lambda}(x, \mathcal{P}_\lambda)$ nearest to the boundary of Γ. Then by the Mecke equation,

$$P[E_\lambda^{\rho r_\lambda}] \leq \mathbb{E}\, M_\lambda = \sum_{i=1}^{4} a_i$$

where we set

$$a_i := \lambda \int_{T_i} P[0 < D_{\phi_\lambda}(x, \mathcal{P}_\lambda) < \rho r_\lambda; \tilde{L}_{\phi_\lambda}(x, \mathcal{P}_\lambda)] dx.$$

We consider just a_1 (the other terms are treated similarly). Let $T_{1,1}$ be the part of T_1 away from the corner of Γ, defined by

$$T_{1,1} := T_1 \setminus ([0, 2(\rho + \eta^{-1}) r_\lambda] \times ([0, 2(\rho + \eta^{-1}) r_\lambda] \cup [1 - 2(\rho + \eta^{-1}) r_\lambda, 1])).$$

Let $a_{1,1}$ be the contribution to a_1 from $x \in T_{1,1}$. Using our assumption that $\phi_\lambda \in \Phi^0_{d,\eta}$, we have

$$a_{1,1} \leq \lambda \int_{T_{1,1}} P[0 < D_{\phi_\lambda}(x, \mathcal{P}_\lambda) < \rho r_\lambda; L_{\phi_\lambda}(x, \mathcal{P}_\lambda)] dx$$

$$= \lambda \int_{T_{1,1}} P[0 < D_{\psi_\lambda}(r_\lambda^{-1} x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2}) < \rho; L_{\psi_\lambda}(r_\lambda^{-1} x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2})] dx.$$

Now using Lemmas 5.3 and 5.4 we obtain that

$$a_{1,1} = o(1) \times \int_{T_{1,1}} \lambda P[D_{\psi_\lambda}(r_\lambda^{-1} x, \mathcal{H}^{\mathbf{H}}_{\lambda r_\lambda^2}) = 0] dx = o(1) \times \lambda \int_{T_{1,1}} P[D_{\phi_\lambda}(x, \mathcal{P}_\lambda) = 0] dx$$

which tends to zero by (3.1).

Let $a_{1,2}$ be the contribution to a_1 from $x \in T_1 \cap [0, 2(\rho + \eta^{-1}) r_\lambda]^2$. By Lemma 5.5,

$$a_{1,2} \leq \lambda (2\eta^{-1} r_\lambda)^2 \exp(-\lambda \eta I(\phi_\lambda)/(3K(\eta))) = O(\lambda^{2\varepsilon - \eta/(3K(\eta))}),$$

where for the last estimate we used (3.12) and (3.1). Thus $P[E_\lambda^{\rho r_\lambda}] \to 0$. □

5.2 Large components

In this section we implement the strategy mentioned in the final paragraph of Section 2. In the sequel, given $\lambda > 0$ we couple the graphs $G_{\phi_\lambda}(\mathcal{P}_\lambda \cap A), A \subset \mathbb{R}^d$, in the following natural way. For $A \subset \mathbb{R}^d$ we define $G_{\phi_\lambda}(\mathcal{P}_\lambda \cap A)$ to be the subgraph of $G_{\phi_\lambda}(\mathcal{P}_\lambda)$ induced by the vertex set $\mathcal{P}_\lambda \cap A$.

Given λ, let $m_\lambda := \lceil 2d/r_\lambda \rceil$. Set $\Lambda_\lambda := \{0, 1, \ldots, m_\lambda - 1\}^d$. For $z \in \Lambda_\lambda$ let Q_z denote the cube $\{m_\lambda^{-1} z\} \oplus [0, 1/m_\lambda)^d$, and let $\overline{Q}_z$ denote the closure of Q_z. The cubes $Q_z, z \in \Lambda_\lambda$, form a partition of $[0,1)^d$, and have side $1/m_\lambda \sim r_\lambda/(2d)$, assuming $r_\lambda \to 0$, which holds by (3.12) if $p_\lambda = \Omega(\lambda^{-\varepsilon})$ for some $\varepsilon \in (0,1)$.

Given λ, for $z \in \Lambda_\lambda$ let us say the cube Q_z is *blue* if (i) $\mathcal{P}_\lambda \cap Q_z \neq \emptyset$ and (ii) all vertices of $\mathcal{P}_\lambda \cap B(m_\lambda^{-1} z; r_\lambda/\eta)$ lie in the same connected component of $G_{\phi_\lambda}(\mathcal{P}_\lambda \cap B(m_\lambda^{-1} z; 2r_\lambda/\eta))$. If a cube is not blue, let us say it is *green*. If Q_z is blue (respectively green) we shall also say $\overline{Q}_z$, and also z itself, are blue (respectively green). More prosaically we shall put $Y_{\lambda,z} = 1$ if z is blue and $Y_{\lambda,z} = 0$ if z is green.

Lemma 5.8 *Suppose* $p_\lambda = \Omega(\lambda^{-\varepsilon})$ *with* $0 < \varepsilon < (9d)^{-d}\eta/K(\eta)$. *Then*

$$\sup_{z \in \Lambda_\lambda} P[Y_{\lambda,z} = 0] = O(\lambda^{-\varepsilon}).$$

Proof. First note that $\text{card}(\mathcal{P}_\lambda \cap Q_z)$ is Poisson with mean $\lambda/m_\lambda^d \sim (2d)^{-d}\lambda r_\lambda^d \geq (2d)^{-d}\lambda I(\phi_\lambda)/K(\eta)$, where the inequality comes from (3.15). Hence by (3.1) the probability that condition (i) (in the definition of blue) fails is $O(\lambda^{-(3d)^{-d}/K(\eta)})$, uniformly over $z \in \Lambda_\lambda$. We need a similar bound for the probability that condition (ii) fails.

Let ξ_λ be Poisson with parameter $2\lambda/m_\lambda^d$. We claim that the Erdős-Rényi graph $G(\xi_\lambda, \eta p_\lambda)$ satisfies

$$P[G(\xi_\lambda, \eta p_\lambda) \notin \mathcal{K}] = O(\lambda^{-\varepsilon}). \tag{5.9}$$

Indeed, by the Mecke formula followed by (3.12), (3.15), and (3.1), the expected number of isolated vertices in $G(\xi_\lambda, \eta p_\lambda)$ is given by

$$\begin{aligned} O\left(\lambda r_\lambda^d \exp(-(3d)^{-d}\lambda r_\lambda^d \eta p_\lambda)\right) &= O(\lambda^{2\varepsilon} \exp(-(3d)^{-d}\eta\lambda I(\phi_\lambda)/K(\eta))) \\ &= O(\lambda^{2\varepsilon - (3d)^{-d}\eta/K(\eta)}) \end{aligned}$$

which is $O(\lambda^{-\varepsilon})$ by the condition on ε. Thus the probability that $G(\xi_\lambda, p_\lambda)$ has an isolated vertex is $O(\lambda^{-\varepsilon})$, and by the proof of [2, Theorem 7.2] we have (5.9). Hence, for each pair of neighbouring sites $z', z'' \in \Lambda_\lambda$, the graph $G_{\phi_\lambda}(\mathcal{P}_\lambda \cap (Q_{z'} \cup Q_{z''}))$ is connected with probability $1 - O(\lambda^{-\varepsilon})$. Condition (ii) holds if $G_{\phi_\lambda}(\mathcal{P}_\lambda \cap (Q_{z'} \cup Q_{z''}))$

is connected for each pair of neighbouring sites z', z'' lying in $B(z, 2m_\lambda r_\lambda/\eta) \cap \Lambda_\lambda$, and the number of such pairs is bounded independently of z and λ. Therefore by the union bound, condition (ii) holds with probability $1 - O(\lambda^{-\varepsilon})$, as claimed. □

We say a set $S \subset \Lambda_\lambda$ is $*$-connected if for any $x, y \in S$, there is a path $(x_0, x_1, \dots, x_k)$ with $x_0 = x$, $x_k = y$ and $x_i \in S$ and $\|x_i - x_{i-1}\|_\infty = 1$ for $1 \le i \le k$ (so diagonal steps in the path are allowed). For bounded nonempty $U \subset \mathbb{R}^d$, we define the *ℓ_∞-diameter* of U to be $\sup_{x,y\in U} \|y - x\|_\infty$. Given $\lambda, \rho > 0$, let H_λ^ρ be the event that there is a $*$-connected set of green sites in Λ_λ of ℓ_∞-diameter at least ρ.

Lemma 5.9 *Suppose for some $\varepsilon \in (0, (9d)^d\eta/K(\eta))$ that $p_\lambda = \Omega(\lambda^{-\varepsilon})$. Then there exists $\rho > 0$ such that $P[H_\lambda^\rho] \to 0$ as $\lambda \to \infty$.*

Proof. For $\lambda > 0, n \in \mathbb{N}$, let $\mathcal{T}_{\lambda,n}$ denote the set of $*$-connected sets $\gamma \subset \Lambda_\lambda$ with n elements. Then there exists a constant A such that for all λ and n, we have $\text{card}(\mathcal{T}_{\lambda,n}) \le m_\lambda^d A^n$; see, e.g., [12, Lemma 9.3]. Also $r_\lambda^{-d} = \Theta(\lambda p_\lambda/\log \lambda)$ by (3.12), and hence there exists $\lambda_0 \in (0, \infty)$ such that for $\lambda \ge \lambda_0$ we have that $m_\lambda^d \le \lambda$ so that $\text{card}(\mathcal{T}_{\lambda,n}) \le \lambda A^n$ for all $n \in \mathbb{N}$.

The random field $(Y_{\lambda,z}, z \in \Lambda_\lambda)$ has finite range dependency; there exists $\lambda_1 \in [\lambda_0, \infty)$ such that the range may be taken to be $11d/\eta$, for all $\lambda \ge \lambda_1$. For example, if $|z - z'| \ge 11d/\eta$ then $|m_\lambda^{-1}z - m_\lambda^{-1}z'| \ge 5r_\lambda/\eta$, and therefore $Y_{\lambda,z}$ is independent of $Y_{\lambda,z'}$. Therefore there is a constant $M := M(d, \eta)$ such that for any $\lambda \ge \lambda_1$ and any $S \subset \Lambda_\lambda$, we can find $S' \subset S$ with $\text{card}(S') = \lceil \text{card}(S)/M \rceil$, such that the variables $(Y_{\lambda,z})_{z\in S'}$ are mutually independent. Hence by Lemma 5.8 there is a further constant C such that for all such S we have

$$P[\cap_{z\in S}\{Y_{\lambda,z} = 0\}] \le (C\lambda^{-\varepsilon})^{(\text{card} S)/M}.$$

Let $\rho \in \mathbb{N}$. If H_λ^ρ occurs then there exists $S \in \mathcal{T}_{\lambda,\rho}$ such that $Y_{\lambda,z} = 0$ for all $z \in S$. Hence for $\rho \in \mathbb{N}$ and $\lambda \ge \max(\lambda_1, (CA^M)^{2/\varepsilon})$ we have

$$P[H_\lambda^\rho] \le P[\cup_{S\in\mathcal{T}_{\lambda,\rho}} \cap_{z\in S} \{Y_{\lambda,z} = 0\}] \le \lambda A^\rho (C\lambda^{-\varepsilon})^{\rho/M} \le \lambda^{1-\varepsilon\rho/(2M)}.$$

Taking $\rho > 2M/\varepsilon$, we have the result. □

Given disjoint nonempty connected subsets U and V of Γ, we define the *exterior boundary* of U relative to V as follows. Let V' be the connected component of $\Gamma \setminus U$ that contains V, and let $U' := \Gamma \setminus V'$. Loosely speaking, U' is obtained from U by filling in all the holes in U, except the one containing V. Define the exterior boundary of U relative to V to be the intersection of the closure of U' with that of V'.

The exterior boundary of U relative to V is a subset of the boundary of U. Moreover it is a connected set, by a unicoherence argument (see [12]), because the closures of U' and V' are connected sets whose union is Γ.

We claim that for $0 < a < 1$, if both U and V have ℓ_∞-diameter greater than a, then so does the exterior boundary of U relative to V. Indeed, if not, then there exists a rectilinear cube $\mathbf{C}$ of side a that contains the exterior boundary of U relative to V, but then we could pick $u \in U \setminus \mathbf{C}$ and $v \in V \setminus \mathbf{C}$, and a continuous path from u to v in Γ avoiding $\mathbf{C}$. Somewhere on this path would lie a point in the exterior boundary of U relative to V, a contradiction.

Lemma 5.10 *Let $\lambda > 0$, $\rho \in \mathbb{N}$ with $\rho < m_\lambda$, and suppose H_λ^ρ does not occur. Then there exists a $*$-connected component of the set of blue sites in Λ_λ of ℓ_∞-diameter $m_\lambda - 1$. This component is unique, and there is no other $*$-connected component of the set of blue sites in Λ_λ of ℓ_∞-diameter ρ or more.*

Proof. Let B_λ denote the union of all the cubes $Q_z, z \in \Lambda_\lambda$ that are blue, and let G_λ denote the union of all the cubes $Q_z, z \in \Lambda_\lambda$ that are green. Let U be the component of $G_\lambda \cup (\{0\} \times [0,1]^{d-1})$ that contains $\{0\} \times [0,1]^{d-1}$, and let V be the component of $B_\lambda \cup (\{1\} \times [0,1]^{d-1})$ that contains $\{1\} \times [0,1]^{d-1}$. Then U and V are disjoint connected subsets of Γ. Assuming H_λ^ρ does not occur, U does not extend to $\{1\} \times [0,1]^{d-1}$. Hence the union of blue cubes $\overline{Q}_z$ having non-empty intersection with the exterior boundary of U relative to V is connected and has ℓ_∞-diameter 1, and the first assertion (existence) in the statement of the lemma follows.

Suppose there were two $*$-connected components of the set of blue sites of ℓ_∞-diameter at least ρ, denoted U and V say. Let U^* be the union of the cubes $\overline{Q}_z, z \in U$ and define V^* similarly. Then U^* and V^* are connected disjoint regions of Γ, of ℓ_∞-diameter at least $(\rho + 1)/m_\lambda$. The union of green cubes $\overline{Q}_y$ having non-empty intersection with the exterior boundary of U^* relative to V^* would be a connected region of ℓ_∞-diameter at least $(\rho + 1)/m_\lambda$, and the corresponding set of sites in Λ_λ would be a $*$-connected set of green sites of diameter at least ρ, contradicting the assumed non-occurrence of event H_λ^ρ. This demonstrates the second assertion (uniqueness) in the statement of the lemma. □

We shall refer to the unique $*$-connected blue component of ℓ_∞-diameter $m_\lambda - 1$, identified in lemma 5.10, as the *sea*. All vertices of $\mathcal{P}_\lambda$ lying in cubes Q_z with z in the sea lie in the same component of $G_{\phi_\lambda}(\mathcal{P}_\lambda)$, which we call the *sea-component*.

Given $\lambda > 0$, $\rho > 0$, define the event

$$F_\lambda^\rho = \{\exists\, x, y \in \mathcal{P}_\lambda : \min(D_{\phi_\lambda}(x, \mathcal{P}_\lambda), D_{\phi_\lambda}(y, \mathcal{P}_\lambda)) > \rho, \\ C_{\phi_\lambda}(x, \mathcal{P}_\lambda) \neq C_{\phi_\lambda}(y, \mathcal{P}_\lambda)\}.$$

Lemma 5.11 *Let $0 < \varepsilon < (9d)^{-d}\eta/K(\eta)$. There exists a constant $\rho \in \mathbb{N}$, such that if for some $\alpha > 0$ we have* (3.1)*, and also $p_\lambda = \Omega(\lambda^{-\varepsilon})$, then $P[F_\lambda^{\rho r_\lambda}] \to 0$ as $\lambda \to \infty$.*

Proof. Let $\rho \in \mathbb{N}$. Suppose that $F_\lambda^{\rho r_\lambda}$ occurs and H_λ^ρ does not. Then there exists $U \subset \mathcal{P}_\lambda$ such that U is the vertex-set of a component of $G_{\phi_\lambda}(\mathcal{P}_\lambda)$ that is disjoint from the sea-component, but has diameter greater than ρr_λ, and hence has ℓ_∞-diameter greater than $\rho r_\lambda/\sqrt{d}$.

Let $\tilde{U}$ denote the union of closed Euclidean balls of radius $r_\lambda/(2\eta)$ centred on the vertices of U. This is a connected subset of $\mathbb{R}^d$, because $\rho_0(\phi_\lambda) \leq \eta^{-1}r_\lambda$ by (2.4), and therefore for each pair of vertices y, y' connected by an edge of $G_{\phi_\lambda}(\mathcal{P}_\lambda)$ we have $|y - y'| \leq r_\lambda/\eta$. Also $\tilde{U}$ has ℓ_∞-diameter at least $\rho r_\lambda/\sqrt{d}$.

We claim there is no $x \in U$ and z in the sea such that $|x - m_\lambda^{-1}z| \leq \eta^{-1}r_\lambda$. For if there were such a pair, then by the definition of blue, x would lie in the same component as the vertices of $\mathcal{P}_\lambda$ in Q_z, so U would be part of the sea-component, a contradiction.

Let S be the union of cubes $\overline{Q}_z$ with z in the sea. The set S is connected, and disjoint from $\tilde{U}$ by the preceding claim, since the cubes have diameter at most $r_\lambda/(2\sqrt{d})$; let $\partial_{\text{ext}}\tilde{U}$ denote the exterior boundary of $\tilde{U}$ relative to S. This has ℓ_∞-diameter at least $\rho r_\lambda/\sqrt{d}$.

Now let $\Delta_{\text{ext}}\tilde{U}$ be the set of sites $z \in \Lambda_\lambda$ such that the corresponding cubes $\overline{Q}_z$ have non-empty intersection with $\partial_{\text{ext}}\tilde{U}$. Since $\partial_{\text{ext}}\tilde{U}$ is connected, the set $\Delta_{\text{ext}}\tilde{U}$ is $*$-connected. Also $\text{card}(\Delta_{\text{ext}}\tilde{U}) \geq (\rho r_\lambda/\sqrt{d})m_\lambda - 1 \geq \rho$.

We claim that none of the squares $Q_z, z \in \Delta_{\text{ext}}\tilde{U}$, is blue. This is because by definition, each such Q_z intersects with $\partial_{\text{ext}}\tilde{U}$, and therefore lies distant at most $r_\lambda/(2\eta)$ from some vertex of U (at X, say). Then by the triangle inequality $|X - m_\lambda^{-1}z| \leq r_\lambda/(2\eta) + r_\lambda/(2\sqrt{d}) \leq r_\lambda/\eta$, and if Q_z were blue, it would contain at least one vertex of $\mathcal{P}_\lambda$, and this would be in the same component of $G_{\phi_\lambda}(\mathcal{P}_\lambda)$ as all the vertices within distance r_λ/η of $m_\lambda^{-1}z$, including X. Hence Q_z would include a vertex of U, but then it would be contained in the interior of $\tilde{U}$, and so would have empty intersection with $\partial_{\text{ext}}\tilde{U}$, a contradiction.

Thus $\Delta_{\text{ext}}\tilde{U}$ is a $*$-connected set of cardinality at least ρ, all of whose elements are green. This contradicts the assumed non-occurrence of H_λ^ρ. Thus $F_\lambda^{\rho r_\lambda} \subset H_\lambda^\rho$, and the result follows from Lemma 5.9. □

Proof of Theorem 5.1. Set $\varepsilon = \frac{1}{2}\min((9d)^{-d}\eta/K(\eta), 2^{-d-3})$. Given $\rho > 0$, if $L_2(G_{\phi_\lambda}(\mathcal{P}_\lambda)) > 1$, then either $E_\lambda^{\rho r_\lambda}$ or $F_\lambda^{\rho r_\lambda}$ occurs. If $p_\lambda = \Omega(\lambda^{-\varepsilon})$, the result (5.1) follows from Proposition 5.1 and Lemma 5.11. If $p_\lambda = O(\lambda^{-\varepsilon})$, (5.1) follows from Proposition 4.1.

6 De-Poissonization

In this section we shall complete the proof of Theorems 2.1, 2.2 and 2.3. We start with the case $\alpha \in (0,\infty)$ of Theorem 2.3. All integrals in this section are over Γ unless specified otherwise.

Proposition 6.1 *Suppose $\alpha \in (0,\infty)$, and (ϕ_n) satisfy* (2.7) *as $n \to \infty$ along some subsequence of $\mathbb{N}$, and for some $\eta \in (0,1]$ we have $\phi_n \in \Phi_{d,\eta}$ for all n. Then for $k \in \mathbb{N}_0$,* (2.8) *holds as $n \to \infty$ along the same subsequence.*

If also $\phi_n \in \Phi^0_{d,\eta}$ for all n, then along the same subsequence we have

$$\lim_{n\to\infty} P[L_2(G_{\phi_n}(\mathcal{X}_n)) \le 1] = 1. \tag{6.1}$$

Proof. Let $\lambda(n) = n - n^{3/4}$ and $\mu(n) := n + n^{3/4}$. Let $\mathcal{P}_{\lambda(n)}, \mathcal{X}_n, \mathcal{P}_{\mu(n)}$ be coupled as follows. Let $X_1, X_2, \cdots$ be a sequence of independent random vectors uniformly distributed over Γ. Independently, let Z and Z' be Poisson distributed random variables with parameter $\lambda(n)$ and $\mu(n)-\lambda(n)$, respectively independently of each other and of $(X_1, X_2, \ldots)$; set $\mathcal{P}_{\lambda(n)} := \{X_1 \ldots, X_Z\}$, and set $\mathcal{P}_{\mu(n)} := \{X_1 \ldots, X_{Z+Z'}\}$, and $\mathcal{X}_n := \{X_1 \ldots, X_n\}$. By Chebyshev's inequality, w.h.p. $\mathcal{P}_{\lambda(n)} \subset \mathcal{X}_n \subset \mathcal{P}_{\mu(n)}$.

Without loss of generality, assume $\rho_\eta(\phi_n) \le \sqrt{d}$. By (3.11),

$$\exp\left(n^{3/4}\int_\Gamma \phi_n(y-x)dy\right) = \exp(n^{-1/4} \times \Theta(\log n)) = 1 + o(1),$$

uniformly over $x \in \Gamma$, and therefore the sequence $(\phi_n)_{n\in\mathbb{N}}$ satisfies

$$\lambda(n)\int_\Gamma \exp\left(-\lambda(n)\int_\Gamma \phi_n(y-x)dy\right)dx \to \alpha. \tag{6.2}$$

Let A_n be the union of the event that at least one of the added vertices of $\mathcal{P}_{\mu(n)} \setminus \mathcal{P}_{\lambda(n)}$ is not connected to any of the vertices of $\mathcal{P}_{\lambda(n)}$, and the event that at least one of the added vertices of $\mathcal{P}_{\mu(n)} \setminus \mathcal{P}_{\lambda(n)}$ is connected to one of the isolated vertices of $G_{\phi_n}(\mathcal{P}_{\lambda(n)})$.

By the Mecke equation, the expected number of added vertices that are isolated from all the vertices of $\mathcal{P}_{\lambda(n)}$ equals $2n^{3/4}\int \exp\left(-\lambda(n)\int \phi_n(y-x)dy\right)dx$, which tends to zero by (6.2). Also, the expected number of isolated vertices in $G_{\phi_n}(\mathcal{P}_{\lambda(n)})$ which are connected to at least one of the added vertices is bounded by

$$(n-n^{3/4})\int_\Gamma \exp\left(-(n-n^{3/4})\int_\Gamma \phi_n(y-x)dy\right) \times 2n^{3/4}I(\phi_n)dx,$$

and by (6.2) and (3.11) this tends to zero. Hence $P[A_n] = o(1)$. By Theorem 3.1 we have that

$$P[N_0(G_{\phi_n}(\mathcal{P}_{\lambda(n)})) = k] \to e^{-\alpha}\alpha^k/k!, \quad k \in \mathbb{N}_0.$$

Also $P[N_0(G_{\phi_n}(\mathcal{X}_n)) \neq N_0(G_{\phi_n}(\mathcal{P}_{\lambda(n)}))] \leq P[A_n] + P[\{Z \leq n \leq Z + Z'\}^c]$, which tends to 0, and (2.8) follows.

Now suppose $\phi_n \in \Phi^0_{d,\eta}$ for all n. If $L_2(G_{\phi_n}(\mathcal{X}_n)) > 1$, then either $Z > n$, or $Z + Z' < n$, or $L_2(G_{\phi_n}(\mathcal{P}_{\lambda(n)})) > 1$, or A_n occurs. By Theorem 5.1, all of these events have vanishing probability, and (6.1) follows. □

Next we consider the case with $\alpha \in \{0, \infty\}$.

Proposition 6.2 *Suppose $\alpha \in \{0, \infty\}$, $\eta \in (0, 1]$, and (ϕ_n) satisfy* (2.7) *as $n \to \infty$ along some subsequence of $\mathbb{N}$, and $\phi_n \in \Phi_{d,\eta}$ for all n. If $\alpha = 0$ then $P[N_0(G_{\phi_n}(\mathcal{X}_n)) = 0] \to 1$, and if $\alpha = \infty$ then for all $k \in \mathbb{N}_0$, $P[N_0(G_{\phi_n}(\mathcal{X}_n)) = k] \to 0$, as $n \to \infty$ along the same subsequence.*

Proof. (i) Let $I_n(\phi_n)$ denote the left hand side of (2.7). Then

$$\mathbb{E}\, N_0(G_{\phi_n}(\mathcal{X}_n)) = n \int dx \left(1 - \int \phi_n(y - x)dy\right)^{n-1}$$
$$\leq n \int dx \left(\exp\left(-(n-1)\int \phi_n(y - x)dy\right)\right) \leq eI_n(\phi_n).$$

Therefore, by Markov's inequality, if $\alpha = 0$ we have $P[N_0(G_{\phi_n}(\mathcal{X}_n)) \geq 1] \to 0$.

Now suppose $\alpha = \infty$. We seek to interpolate a 'larger' connection function than ϕ_n that is still in $\Phi_{d,\eta}$. For $s > 1$ and $\phi \in \Phi_{d,\eta}$, define $\phi^{(s)}$ as follows. Let $s_0(\phi) = 1/\mu(\phi)$. For $1 \leq s \leq s_0(\phi)$, set $\phi^{(s)}(x) := s\phi(x)$, for $x \in \mathbb{R}^d$. Note $\mu(\phi^{(s_0(\phi))}) = 1$. For $s \geq s_0(\phi)$, define

$$\phi^{(s)}(x) := \begin{cases} 1 & \text{if } |x| < s - s_0(\phi) \\ \phi_{s_0}(x) & \text{if } |x| \geq s - s_0(\phi). \end{cases}$$

Let $s_1(\phi) := \sqrt{d} + s_0(\phi)$. If $\phi \in \Phi_{d,\eta}$ then for each $s \in [1, s_1(\phi)]$ the connection function $\phi^{(s)}$ is also in $\Phi_{d,\eta}$.

For each $n \in \mathbb{N}$ define the function

$$\tilde{f}_n(s) := n \int \exp\left(-n \int \phi_n^{(s)}(y - x)dy\right) dx$$

which is continuous and nonincreasing on $1 \leq s \leq s_1(\phi_n)$. By assumption $\tilde{f}_n(1) \to \infty$ as $n \to \infty$, while $\tilde{f}_n(s_1(\phi_n)) = ne^{-n}$. Therefore by the intermediate value theorem, given any finite $\beta > 0$, for large enough n we can pick $s(n) \in [1, s_1(\phi_n)]$ with $\tilde{f}_n(s(n)) = \beta$. Then by Proposition 6.1, for $k \in \mathbb{N}_0$ we have

$$P[N_0(G_{\phi_n^{(s(n))}}(\mathcal{X}_n)) \leq k] \to e^{-\beta} \sum_{j=0}^{k} \beta^j / j!.$$

By an obvious coupling, $P[N_0(G_{\phi_n^{(s)}}(\mathcal{X}_n)) \leq k]$ is nondecreasing in s, and therefore since $\beta > 0$ is arbitrary, we have $P[N_0(G_{\phi_n}(\mathcal{X}_n)) \leq k] \to 0$. □

Proof of Theorem 2.2. Let $\eta \in (0, 1]$. To prove (2.5), it suffices to prove that for any sequence $(\phi_n)_{n \in \mathbb{N}}$ of connection functions in $\Phi^0_{d,\eta}$, we have

$$\lim_{n \to \infty} P[\{N_0(G_{\phi_n}(\mathcal{X}_n)) = 0\} \setminus \{G_{\phi_n}(\mathcal{X}_n) \in \mathcal{K}\}] = 0. \tag{6.3}$$

Define $I_n := I_n(\phi_n) := n \int \exp\left(-\int \phi_n(y - x) dy\right) dx$. Consider the three cases where (i) I_n tends to a finite limit as $n \to \infty$ along some infinite subsequence of $\mathbb{N}$; (ii) $I_n \to \infty$ as $n \to \infty$ along some infinite subsequence of $\mathbb{N}$; (iii) $I_n \to 0$ as $n \to \infty$ along some infinite subsequence of $\mathbb{N}$. At least one of cases (i), (ii) , (iii) holds and it suffices to show that in each case (6.3) holds along the same subsequence.

In case (i), we have (6.3) at once because of (6.1). In case (ii), with $I_n \to \infty$, by Proposition 6.2 we have $P[N_0(G_{\phi_n}(\mathcal{X}_n)) = 0] \to 0$, and hence (6.3) holds.

Consider case (iii) with $I_n \to 0$ along a subsequence. For $n \in \mathbb{N}$, define

$$f_n(a) := n \int \exp\left(-an \int \phi_n(y - x) dy\right) dx,$$

which is a continuous and nonincreasing function on $0 \leq a \leq 1$. For each $a \in [0, 1]$ the connection function $a\phi_n$ is in $\Phi_{d,\eta}$.

By assumption $f_n(1) \to 0$ as $n \to \infty$, while $f_n(0) = n$. Therefore given $\varepsilon > 0$, by the intermediate value theorem, for all large enough n in the subsequence we can choose $a_n \in [0, 1]$ such that $f_n(a_n) = \varepsilon$. Then by Proposition 6.1 we have

$$P[N_0(G_{a_n\phi_n}(\mathcal{X}_n)) = 0] \to e^{-\varepsilon}; \quad P[G_{a_n\phi_n}(\mathcal{X}_n) \in \mathcal{K}] \to e^{-\varepsilon}.$$

By an obvious coupling, $P[G_{a\phi_n}(\mathcal{X}_n) \in \mathcal{K}]$ is nondecreasing in a, and therefore since ε is arbitrary we have $P[G_{\phi_n}(\mathcal{X}_n) \in \mathcal{K}] \to 1$, so (6.3) holds. □

Proof of Theorem 2.3. Eqn (2.8) follows from Proposition 6.1, and the next sentence follows from Proposition 6.2. Then (2.9) follows from Theorem 2.2. □

Proof of Theorem 2.1. The result follows from Theorem 2.3. □

7 Equivalence of thresholds

In this section we prove Theorem 2.4, i.e. we prove that for any $[0,1]$-valued sequence $(p_n)_{n\in\mathbb{N}}$ with $p_n = \omega((\log n)/n)$, we have

$$\lim_{n\to\infty} P[\tau_n(p_n) = \sigma_n(p_n)] = 1,$$

where for $p \in [0,1]$, as described in Section 2 we set

$$\tau_n(p) := \inf\{r : G_{r,p}(\mathcal{X}_n) \in \mathcal{K}\}; \quad \sigma_n(p) := \inf\{r : N_0(G_{r,p}(\mathcal{X}_n)) = 0\}.$$

Clearly $\sigma_n(p_n) \leq \tau_n(p_n)$, so we need to show that $P[\sigma_n(p_n) < \tau_n(p_n)]$ tends to zero. Given p_n and given $\alpha > 0$, define $r_n(\alpha)$ by $I_n(\phi_{r_n(\alpha),p_n}) = e^{-\alpha}$, where $I_n(\phi) := n\int_\Gamma \exp(-n\int_\Gamma \phi(y-x)dy)dx$. For each α we have from (2.5) that

$$P[\sigma_n(p_n) \leq r_n(\alpha) < \tau_n(p_n)] \to 0. \tag{7.1}$$

Note that $r_n(\alpha)$ is nondecreasing in α. Let $\alpha < \beta$. Suppose

$$r_n(\alpha) < \sigma_n(p_n) < \tau_n(p_n) \leq r_n(\beta).$$

Assume the inter-point distances are all distinct. Consider adding the edges of $G_{\sqrt{d},p}(\mathcal{X}_n)$ one by one (starting from the graph with no edges) in order of increasing Euclidean length.

Then precisely one pair of points of $\mathcal{X}_n$, say X and Y, satisfies $|X-Y| = \tau_n(p_n)$, and by definition of $\tau_n(p_n)$, X and Y lie in different components just before adding the edge between them, so lie in different components of $G_{r_n(\alpha),p_n}(\mathcal{X}_n)$. Assuming $L_2(G_{r_n(\alpha),p_n}(\mathcal{X}_n)) \leq 1$ (which has high probability by (6.1)), either X or Y (say X) is isolated in $G_{r_n(\alpha),p_n}(\mathcal{X}_n)$. But X is non-isolated in $G_{\sigma_n(p_n),p_n}(\mathcal{X}_n)$ by definition of $\sigma_n(p_n)$. Therefore since we are assuming $\tau_n(p_n) \leq r_n(\beta)$ we have that X is connected to at least two points of $\mathcal{X}_n$ at distance between $r_n(\alpha)$ and $r_n(\beta)$. Thus $N_{\alpha,\beta}(n) > 0$, where $N_{\alpha,\beta}(n)$ denotes the number of vertices of $\mathcal{X}_n$ having no incident edge in $G_{\sqrt{d},p_n}(\mathcal{X}_n)$ of (Euclidean) length at most $r_n(\alpha)$ but at least two incident edges of length at most $r_n(\beta)$.

Let $\lambda(n)$ and $\mu(n)$, and the coupling of $\mathcal{P}_{\lambda(n)}, \mathcal{X}_n$, and $\mathcal{P}_{\mu(n)}$ be as in the preceding section. Let $N'_{\alpha,\beta}(n)$ be the number of vertices of $\mathcal{P}_{\mu(n)}$ having no incident edge (in $G_{\sqrt{d},p_n}(\mathcal{P}_{\mu(n)})$) of length at most $r_n(\alpha)$ with the other endpoint in $\mathcal{P}_{\lambda(n)}$ but at least two incident edges of length at most $r_n(\beta)$ (with the other endpoint in $\mathcal{P}_{\mu(n)}$). If $\mathcal{P}_{\lambda(n)} \subset \mathcal{X}_n \subset \mathcal{P}_{\mu(n)}$ (which happens with high probability), then $N'_{\alpha,\beta}(n) \geq N_{\alpha,\beta}(n)$. Thus

$$\limsup_{n\to\infty} P[r_n(\alpha) < \sigma_n(p_n) < \tau_n(p_n) \leq r_n(\beta)] \leq \limsup_{n\to\infty} P[N'_{\alpha,\beta}(n) > 0]. \tag{7.2}$$

With $|\cdot|$ denoting Lebesgue measure, by the Mecke formula we have

$$\mathbb{E}\,[N'_{\alpha,\beta}] = (n+n^{3/4}) \int_\Gamma e^{-\lambda(n)p_n|B(x;r_n(\alpha))\cap\Gamma|} \times (1-e^{-w_n(x)}(1+w_n(x))), dx,$$

where $w_n(x)$ denotes the mean number of edges of length in the range $(r_n(\alpha), r_n(\beta)]$ incident to a point at x. Now, $e^w - 1 - w \leq w^2 e^w$ for any $w \geq 0$. Hence

$$\mathbb{E}\,[N'_{\alpha,\beta}] \leq (n+n^{3/4}) \int e^{-\lambda(n)p_n|B(x;r_n(\alpha))\cap\Gamma|} \times w_n(x)^2 dx. \tag{7.3}$$

By (3.12) and the condition $p_n = \omega((\log n)/n)$, we have $r_n(\beta) \to 0$. Writing $V_\alpha(x)$ for $|B(x;r_n(\alpha))\cap\Gamma|$ we have

$$\begin{aligned} e^{-\alpha} &= \lim_{n\to\infty}\left(n\int_\Gamma \exp(-np_nV_\beta(x) + np_n(V_\beta(x) - V_\alpha(x))dx \right) \\ &\geq \limsup_{n\to\infty}\left(n\int_\Gamma \exp(-np_nV_\beta(x) + np_n\pi_d(r_n(\beta)^d - r_n(\alpha)^d)/2^d)dx \right) \\ &= e^{-\beta}\exp(\limsup_{n\to\infty}[np_n\pi_d(r_n(\beta)^d - r_n(\alpha)^d)/2^d]) \end{aligned}$$

so that

$$\limsup_{n\to\infty} np_n(r_n(\beta)^d - r_n(\alpha)^d) \leq 2^d(\beta-\alpha)/\pi_d.$$

Therefore, since

$$w_n(x) \leq \mu(n)p_n\pi_d(r_n(\beta)^d - r_n(\alpha)^d),$$

we have $\limsup_{n\to\infty}\sup_{x\in\Gamma} w_n(x) \leq 2^d(\beta-\alpha)$, so that by (7.3) and a similar argument to (6.2), $\limsup_{n\to\infty}\mathbb{E}\,[N'_{\alpha,\beta}] \leq 2^{2d}(\beta-\alpha)^2e^{-\alpha}$, so that by (7.2),

$$\limsup_{\lambda\to\infty} P[r_n(\alpha) < \sigma_n(p_n) < \tau_n(p_n) \leq r_n(\beta)] \leq 2^{2d}(\beta-\alpha)^2e^{-\alpha}. \tag{7.4}$$

Now we argue as in [15, pages 163-4] or [12, pages 304-5]. Let $\varepsilon > 0$. Choose $\alpha_0 < \alpha_1 < \cdots < \alpha_I$ such that $\exp(-e^{-\alpha_0}) < \varepsilon$, and $1 - \exp(-e^{-\alpha_I}) < \varepsilon$, and also

$$2^{2d}\sum_{i=1}^{I}(r_n(\alpha_i) - r_n(\alpha_{i-1}))^2e^{-\alpha_{i-1}} < \varepsilon.$$

Then by the union bound,

$$\begin{aligned} P[\sigma_n < \tau_n] &\leq P[\sigma_n \leq r_n(\alpha_0)] + P[\sigma_n > r_n(\alpha_I)] \\ &+ \sum_{i=1}^{I}(P[\sigma_n \leq r_n(\alpha_i) < \tau_n] + P[r_n(\alpha_{i-1}) < \sigma_n < \tau_n \leq r_n(\alpha_i)]). \end{aligned}$$

Since $\sigma_n \leq r$ if and only if $N_0(G(\mathcal{X}_n, r)) = 0$, it follows from (2.8) of Theorem 2.3, along with (7.1) and (7.4), that $\limsup_{n\to\infty} P[\sigma_n < \tau_n] \leq 3\varepsilon$, and since $\varepsilon > 0$ is arbitrary this completes the proof. □

8 The choice of ϕ

In this section, we prove Theorem 2.5 (among other things). That is, we identify conditions for a sequence of connection functions ϕ_n to satisfy (2.7) for some $\alpha \in (0, \infty)$. We consider only the case with $d = 2$ and $\phi_n \in \Phi_{2,\eta} \cap \Psi_2$ for some $\eta \in (0, 1]$, where Ψ_2 is defined by (2.1).

Assume $d = 2$. Fix $\eta > 0$ and choose $\phi_n \in \Phi_{2,\eta} \cap \Psi_2$ for each $n > 0$. Set

$$r_n := \rho_\eta(\phi_n); \quad p_n := \mu(\phi_n); \quad a_n := nr_n^2 p_n.$$

Since we assume $d = 2$, it follows from the definitions (2.10) and (2.12) that

$$nI(\phi_n) = a_n J_2(\phi_n), \quad n \in \mathbb{N}. \tag{8.1}$$

In this section we assume $r_n = n^{-\Omega(1)}$, so in particular $r_n = o(1)$.

Set $N_0(n) := N_0(G_{\phi_n}(\mathcal{P}_n))$. By the Mecke formula, $\mathbb{E}\, N_0(G_{\phi_n}(\mathcal{P}_n)) = I_n(\phi_n)$, where we set $I_n(\phi) := n \int_\Gamma \exp\left(-n \int_\Gamma \phi(y - x) dy\right) dx$, so $I_n(\phi_n)$ is the left hand side of (2.7).

Given $\varepsilon > 0$, truncate ϕ_n by setting $\tilde{\phi}_n(x) := \phi_n(x)\mathbf{1}_{[0, r_n^{1-\varepsilon}]}(|x|)$ for $x \in \mathbb{R}^2$. Couple $G_{\phi_n}(\mathcal{P}_n)$ and $G_{\tilde{\phi}_n}(\mathcal{P}_n)$ as in the proof of Lemma 3.3. Let $\tilde{N}_0(n) := N_0(G_{\tilde{\phi}_n}(\mathcal{P}_n))$. Let $N_0^{\rm int} := N_0^{\rm int}(n)$ denote the number of isolated vertices of $G_{\tilde{\phi}_n}(\mathcal{P}_n)$ lying in $[r_n^{1-\varepsilon}, 1 - r_n^{1-\varepsilon}]^2$. Let $N_0^{\rm side} := N_0^{\rm side}(n)$ denote the number of isolated vertices of $G_{\tilde{\phi}_n}(\mathcal{P}_n)$ lying within Euclidean distance $r_n^{1-\varepsilon}$ of precisely one edge of Γ. Let $N_0^{\rm cor} := N_0^{\rm cor}(n)$ denote the number of isolated vertices of $G_{\tilde{\phi}_n}(\mathcal{P}_n)$ lying within ℓ_∞ distance $r_n^{1-\varepsilon}$ of one of the corners of of Γ. Then $\tilde{N}_0(n) = N_0^{\rm int} + N_0^{\rm side} + N_0^{\rm cor}$ (with probability 1), so

$$I_n(\tilde{\phi}_n) = \mathbb{E}\, N_0^{\rm int} + \mathbb{E}\, N_0^{\rm side} + \mathbb{E}\, N_0^{\rm cor}.$$

Also, if $r_n = n^{-\Omega(1)}$ then

$$0 \le \mathbb{E}\, \tilde{N}_0(n) - \mathbb{E}\, N_0(n) \le n^2 \phi_n(r_n^{1-\varepsilon}) \le 3n^2 \exp(-\eta r_n^{-\varepsilon\eta}) \to 0, \tag{8.2}$$

and

$$\begin{aligned} n(I(\phi_n) - I(\tilde{\phi}_n)) &= nr_n^2 \int_{\{x:|x| \ge r_n^{-\varepsilon}\}} \phi_n(r_n x) dx \\ &\le 3nr_n^2 \int_{\{x:|x| > r_n^{-\varepsilon}\}} \eta^{-1} \exp(-\eta|x|) dx \to 0. \end{aligned} \tag{8.3}$$

As with (3.12), a necessary condition for (2.7) is that

$$np_n r_n^2 = \Theta(\log n). \tag{8.4}$$

Recall from (2.11) that $J_1(\phi_n) := J_1(\phi_n, \eta) := p_n^{-1} \int_0^\infty \phi_n((r_n t, 0)) dt$.

Lemma 8.1 *Suppose* (8.4) *holds, and* $r_n = n^{-\Omega(1)}$ *as* $n \to \infty$*. Then provided* $\varepsilon > 0$ *is chosen sufficiently small (but fixed), as* $n \to \infty$ *we have*

$$\mathbb{E}\, N_0^{\mathrm{side}} \sim \frac{2}{J_1(\phi_n)} \left(\frac{n}{a_n p_n} \right)^{1/2} e^{-nI(\phi_n)/2} \tag{8.5}$$

and

$$\mathbb{E}\, N_0^{\mathrm{cor}} \sim \frac{4e^{-nI(\phi_n)/4}}{a_n p_n J_1(\phi_n)^2}. \tag{8.6}$$

Proof. For $u > 0$, let

$$f_n(u) := p_n^{-1} \int_{[0,\infty)\times[0,u]} \tilde{\phi}_n(r_n x) dx.$$

Then we claim that for $\theta_n = a_n$ or $\theta_n = 2a_n$,

$$\int_0^{r_n^{-\varepsilon}} \exp(-\theta_n f_n(u)) du \sim 1/(\theta_n J_1(\phi_n)) \text{ as } n \to \infty. \tag{8.7}$$

To see this, note first that $J_1(\tilde{\phi}_n) \sim J_1(\phi_n)$ as $n \to \infty$, by (2.13). Also, since $\tilde{\phi}_n(x)$ is nonincreasing in $|x|$ (because $\phi_n \in \Phi_{2,\eta} \cap \Psi_2$) we have

$$f_n(u) \leq u J_1(\tilde{\phi}_n), \tag{8.8}$$

so that using (2.13) we have

$$\begin{aligned} \int_0^{r_n^{-\varepsilon}} \exp(-\theta_n f_n(u)) du &\geq \int_0^{r_n^{-\varepsilon}} \exp(-\theta_n u J_1(\tilde{\phi}_n)) du \\ = (\theta_n J_1(\tilde{\phi}_n))^{-1} &\int_0^{\theta_n J_1(\tilde{\phi}_n) r_n^{-\varepsilon}} e^{-t} dt \sim (\theta_n J_1(\tilde{\phi}_n))^{-1}. \end{aligned} \tag{8.9}$$

Also given $\delta > 0$, for $(s,t) \in [0,\infty) \times (0, \delta r_n)$ we have $\phi_n((s,t)) \geq \phi_n((s + \delta r_n, 0))$, and hence

$$\begin{aligned} \int_0^{\delta} \exp(-\theta_n f_n(u)) du &\leq \int_0^{\delta} \exp\left(-\theta_n u p_n^{-1} \int_0^{\infty} \tilde{\phi}_n((r_n(s+\delta), 0)) ds \right) du \\ &\leq \int_0^{\delta} \exp(-\theta_n u (J_1(\tilde{\phi}_n) - \delta)) du \sim (\theta_n (J_1(\tilde{\phi}_n) - \delta))^{-1}, \end{aligned} \tag{8.10}$$

and provided $\delta \leq 1/2$ we also have for $u \geq \delta$ that

$$f_n(u) \geq f_n(\delta) \geq p_n^{-1} \int_{[0,1/2]\times[0,\delta]} \phi_n(r_n x) dx \geq \delta\eta/2, \tag{8.11}$$

so that

$$\int_\delta^1 \exp(-\theta_n f_n(u))du \le \exp(-\delta\eta\theta_n/2) = o(\theta_n^{-1}). \tag{8.12}$$

For $u \ge 1$ we have $f_n(u) \ge f_n(1/2) \ge \eta/4$, and for n large enough $r_n^{-2} \le n$ by (8.4), so

$$\int_1^{r_n^{-\varepsilon}} e^{-\theta_n f_n(u)} du \le r_n^{-\varepsilon} \exp(-\eta\theta_n/4) \le n^{\varepsilon/2} \exp(-\eta\theta_n/4).$$

Provided ε is small enough, using (8.4) again we have that the last expression is less than $\exp(-\eta\theta_n/8)$ which is $o(\theta_n^{-1})$. Combining this with (8.9), (8.10) and (8.12) and using the fact that δ can be arbitrarily small, gives us (8.7).

Since $\tilde{\phi}_n$ has range $r_n^{1-\varepsilon}$ we have

$$\mathbb{E}\, N_0^{\text{side}} = (4 + o(1))n \exp(-nI(\tilde{\phi}_n)/2) \int_0^{r_n^{-\varepsilon}} \exp(-2nr_n^2 p_n f_n(u)) r_n du.$$

By (8.3) and (8.7) we obtain

$$\mathbb{E}\, N_0^{\text{side}} \sim \frac{4nr_n e^{-nI(\phi_n)/2}}{2J_1(\phi_n) nr_n^2 p_n} = \frac{2}{J_1(\phi_n)} \left(\frac{n}{a_n p_n} \right)^{1/2} e^{-nI(\phi_n)/2}.$$

Now consider $\mathbb{E}\, N_0^{\text{cor}}$. For $u, v > 0$, set

$$g_n(u,v) := p_n^{-1} \int_{[0,u]\times[0,v]} \tilde{\phi}_n(r_n(x - (u,v)))dx$$

Then since $\tilde{\phi}_n$ has range $r_n^{1-\varepsilon}$,

$$\mathbb{E}\, N_0^{\text{cor}} = (1 + o(1)) 4r_n^2 n e^{-nI(\tilde{\phi}_n)/4} \tilde{I}_n \tag{8.13}$$

with

$$\tilde{I}_n := \int_0^{r_n^{-\varepsilon}} \int_0^{r_n^{-\varepsilon}} \exp(-np_n r_n^2 [f_n(u) + f_n(v) + g_n(u,v)])dudv.$$

For $u, v \ge 0$ we have $0 \le g_n(u,v) \le uv$. Hence by (8.8) we have

$$\begin{aligned} \tilde{I}_n &\ge \int_0^{r_n^{-\varepsilon}} \int_0^{r_n^{-\varepsilon}} \exp(-a_n(uJ_1(\tilde{\phi}_n) + vJ_1(\tilde{\phi}_n) + uv))dudv \\ &\sim \int_0^{r_n^{-\varepsilon}} \left(\frac{e^{-a_n v J_1(\tilde{\phi}_n)}}{a_n(J_1(\tilde{\phi}_n) + v)} \right) dv \sim (a_n J_1(\tilde{\phi}_n))^{-2}. \end{aligned}$$

On the other hand, given $\delta \in (0,\eta)$, similarly to (8.10) the contribution to $\tilde{I}_n$ from $\max(u,v) \le \delta$ is bounded above by

$$\int_0^\delta \int_0^\delta \exp(-a_n[u(J_1(\phi_n)-\delta)+v(J_1(\phi_n)-\delta)])dudv \sim (a_n(J_1(\phi_n)-\delta))^{-2}$$

while by (8.11) the contribution to $\tilde{I}_n$ from $1 \ge \max(u,v) > \delta$ is bounded above by $\exp(-a_n\eta\delta/2)$, which is $o(a_n^{-2})$ by (8.4), and the contribution to $\tilde{I}_n$ from $\max(u,v) > 1$ is bounded above by $\exp(-a_n\eta/4)r_n^{-2\varepsilon}$, and hence (using (8.4)), by $n^\varepsilon \exp(-a_n\eta/4)$, which is $o(a_n^{-2})$ provided ε is taken sufficiently small. Therefore we have $\tilde{I}_n \sim (a_n J_1(\phi_n))^{-2}$. Then by (8.13) we get (8.6). □

Lemma 8.2 *Fix $\varepsilon \in (0,1)$. Suppose $r_n = n^{-\Omega(1)}$. Then $\mathbb{E}\, N_0^{\rm int} \sim ne^{-nI(\phi_n)}$ as $n \to \infty$.*

Proof. The result follows from (8.3). □

Proposition 8.1 *Suppose $d = 2$. Let $\alpha \in (0,\infty)$. Suppose for some $\eta \in (0,1]$ that $\phi_n \in \Phi_{2,\eta} \cap \Psi_2$ for all n, and $p_n = \omega(1/\log n)$ as $n \to \infty$. Then (2.7) holds if*

$$nI(\phi_n) - \log n \to -\log \alpha. \tag{8.14}$$

Proof. Assume (8.14) holds, which implies *a fortiori* that (8.4) also holds, so in particular $r_n^2 = O((\log n)^2/n)$. Then by Lemma 8.2 and (8.14) we have $\mathbb{E}\, N_0^{\rm int} \to \alpha$.

Using (8.4), (8.14), and Lemma 8.1, we obtain (for a sufficiently small choice of ε) that $\mathbb{E}\, N_0^{\rm side} = O((p_n \log n)^{-1/2})$, which tends to zero by the assumption on p_n. Similarly, by (8.6) and (8.14), $\mathbb{E}\, N_0^{\rm cor} = O(n^{-1/4}/(p_n \log n)) = o(1)$. Applying (8.2) completes the proof. □

When $p_n = O(1/\log n)$, boundary effects become important in the asymptotics for the mean number of isolated points.

Proposition 8.2 *Suppose $d = 2$ and for some $\eta \in (0,1]$ we have $\phi_n \in \Phi_{2,\eta} \cap \Psi_2$ for all n. Suppose $p_n = o(1/\log n)$ and also $p_n = \omega((\log n)^{-1}n^{-1/3})$, as $n \to \infty$. Fix $\alpha \in (0,\infty)$, and assume*

$$nI(\phi_n) = \log\left(\frac{4J_2(\phi_n)}{\alpha^2 J_1(\phi_n)^2}\right) + \log\left(\frac{n}{p_n}\right) - \log\log\left(\frac{n}{p_n}\right) + o(1). \tag{8.15}$$

Then (2.7) holds.

Proof. Under the assumptions given, using Lemma 8.2 we have

$$\mathbb{E}\, N_0^{\mathrm{int}} = (1+o(1)) n e^{-nI(\phi_n)} = O\left(p_n \log\left(\frac{n}{p_n}\right)\right) \to 0. \tag{8.16}$$

Also by (8.5), (8.15), and (8.1), we have

$$\mathbb{E}\, N_0^{\mathrm{side}} \sim \alpha \left(\frac{n}{J_2(\phi_n) a_n p_n}\right)^{1/2} \left(\frac{p_n}{n}\right)^{1/2} (\log(n/p_n))^{1/2} \to \alpha. \tag{8.17}$$

Using (8.5) again along with (2.13), we obtain that $e^{-nI(\phi_n)/4} = \Theta((a_n p_n/n)^{1/4})$ so that (8.6) yields $\mathbb{E}\, N_0^{\mathrm{cor}} = O(((a_n p_n)^3 n)^{-1/4})$, and by (8.4) (which follows from (8.15)) and the assumption $p_n = \omega(n^{-1/3}(\log n)^{-1})$, this shows $\mathbb{E}\, N_0^{\mathrm{cor}} \to 0$. Combined with (8.16) and (8.17), and (8.2), this gives us the result. □

Consider the intermediate case with $p_n = \Theta(1/\log n)$.

Theorem 8.1 *Let $\alpha \in (0,\infty), \eta \in (0,1]$. Suppose that $\phi_n \in \Phi_{2,\eta} \cap \Psi_2$ for all n, and $p_n = \Theta(1/\log n)$, and $nI(\phi_n) = \log n - 2\log \gamma_n + o(1)$, where γ_n denotes the solution in $(0,\infty)$ to*

$$\gamma_n^2 + 2\gamma_n (J_2(\phi_n))^{1/2}/J_1(\phi_n))(p_n \log n)^{-1/2} = \alpha. \tag{8.18}$$

Then (2.7) holds.

Proof. By (2.13) and the assumption on p_n, $\limsup_{n\to\infty}(\gamma_n) < \infty$ and $\liminf_{n\to\infty}(\gamma_n) > 0$. By Lemma 8.2,

$$\mathbb{E}\, N_0^{\mathrm{int}} = (1+o(1)) n e^{-nI(\phi_n)} = (1+o(1))\gamma_n^2,$$

while by (8.5) and (8.1),

$$\mathbb{E}\, N_0^{\mathrm{side}} \sim \frac{2}{J_1(\phi_n)} \left(\frac{n}{a_n p_n}\right)^{1/2} \gamma_n n^{-1/2} \sim \frac{2 J_2(\phi_n)^{1/2}\gamma_n}{J_1(\phi_n)(p_n \log n)^{1/2}}.$$

Also by (8.6), $\mathbb{E}\, N_0^{\mathrm{cor}} = O(n^{-1/4}/(p_n \log n)) = o(1)$. Combining these results and using (8.18) and (8.2) gives us (2.7). □

In the case $p_n = o((\log n)^{-1} n^{-1/3})$ the main contribution to $\mathbb{E}\, N_0$ comes from near the corners of Γ.

Proposition 8.3 *Suppose $d = 2$. Let $\alpha \in (0,\infty), \eta \in (0,1]$ and suppose $(\phi_n)_{n>0}$ are such that $\phi_n \in \Phi_{2,\eta} \cap \Psi_2$ for all n and $p_n = o((\log n)^{-1} n^{-1/3})$ and*

$$nI(\phi_n) = 4(\log(1/p_n) - \log\log(1/p_n) + \log(J_2(\phi_n)/(\alpha J_1(\phi_n)^2))) + o(1) \tag{8.19}$$

as $n \to \infty$. Assume also that $r_n = n^{-\Omega(1)}$. Then (2.7) holds.

Proof. Note that $p_n = \Omega((\log n)/n)$ since otherwise (8.19) cannot be satisfied by bounded r_n. Then $\log(1/p_n) = \Theta(\log n)$ and (8.4) holds. By (8.6) and (8.1),

$$\mathbb{E}\, N_0^{\mathrm{cor}} \sim \frac{4p_n \log(1/p_n)\alpha J_1(\phi_n)^2/J_2(\phi_n)}{J_1(\phi_n)^2 a_n p_n} \to \alpha. \tag{8.20}$$

Also $e^{-nI(\phi_n)/4} = \Theta(p_n \log(1/p_n)) = \Theta(p_n \log n)$. Therefore by (8.5) and (8.4), we obtain that $\mathbb{E}\, N_0^{\mathrm{side}} = O\left(n^{1/2}(p_n \log n)^{3/2}\right)$, which tends to zero since we assume $p_n = o(n^{-1/3}(\log n)^{-1})$.

Finally, since $p_n = \Omega((\log n)/n)$, using Lemma 8.2 and (8.19) we have

$$\mathbb{E}\, N_0^{\mathrm{int}} = O(ne^{-nI(\phi_n)}) = O(np_n^4(\log 1/p_n)^4) = O(n^{-2}),$$

so $\mathbb{E}\, N_0^{\mathrm{int}} \to 0$, and (3.1) then follows by (8.2). □

Proof of Theorem 2.5. Immediate from Propositions 8.1, 8.2 and 8.3. □

Our final result deals with the intermediate case with $r_n = \Theta(n^{-1/3}\log n)^{-1})$.

Theorem 8.2 *Let $\alpha \in (0, \infty)$, and suppose $(\phi_n)_{n>0}$ are such that $p_n = \Theta(n^{-1/3}(\log n)^{-1})$ and*

$$nI(\phi_n) = 4(\log(1/p_n) - \log\log(1/p_n) + \log(J_2(\phi_n)/(\beta_n J_1(\phi_n)^2))) + o(1) \tag{8.21}$$

as $n \to \infty$, with β_n denoting the solution in $(0, \infty)$ to

$$(3J_2(\phi_n))^{-3/2} J_1(\phi_n)^3 (n^{1/3} p_n \log n)^{3/2} \beta_n^2 + \beta_n = \alpha. \tag{8.22}$$

Then (2.7) holds.

Proof. Note that (8.21) is the same as (8.19) but with α replaced by β_n. As with (8.20) we have $\mathbb{E}\, N_0^{\mathrm{cor}} = \beta_n + o(1)$. Then by (8.5) and (8.21) we have

$$\mathbb{E}\, N_0^{\mathrm{side}} \sim (2/J_1(\phi_n))(n/a_n)^{1/2} p_n^{3/2} (\log 1/p_n)^2 \beta_n^2 J_1(\phi_n)^4 J_2(\phi_n)^{-2}.$$

By (8.1) and (8.21), $a_n = nI(\phi_n)/J_2(\phi_n) \sim (4/J_2(\phi_n))\log(1/p_n)$, and our assumption on p_n implies $\log 1/p_n \sim (1/3)\log n$, so that

$$\mathbb{E}\, N_0^{\mathrm{side}} \sim \beta_n^2 J_1(\phi_n)^3 J_2(\phi_n)^{-3/2} n^{1/2} p_n^{3/2} ((\log n)/3)^{3/2}.$$

Hence by (8.22), $\mathbb{E}\,[N_0^{\mathrm{side}} + N_0^{\mathrm{cor}}] \to \alpha$. Also by Lemma 8.2, $\mathbb{E}\, N_0^{\mathrm{int}} = O(ne^{-nI(\phi_n)}) = O(np_n^4(\log 1/p_n)^4)$ which tends to zero, and (2.7) follows by (8.2). □

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