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ON THE Γ-LIMIT FOR A NON-UNIFORMLY BOUNDED SEQUENCE OF TWO-PHASE METRIC FUNCTIONALS

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ABSTRACT. We consider the Γ-limit of a highly oscillatory Riemannian metric length functional as its period tends to 0. The metric coefficient takes values in either $\{1, \infty\}$ or $\{1, \beta\varepsilon^{-p}\}$ where $\beta, \varepsilon > 0$ and $p \in (0, \infty)$. We find that for a large class of metrics, in particular those metrics whose surface of discontinuity forms a differentiable manifold, the Γ-limit exists, as in the case of a uniformly bounded sequence of metrics. However, the existence of the Γ-limit for the corresponding boundary value problem depends on the value of p. Specifically, we show that the power $p = 1$ is critical in that the Γ-limit exists for $p < 1$, whereas it ceases to exist for $p \geq 1$. The results here have applications in both nonlinear optics and the effective description of a Hamiltonian particle in a discontinuous potential.

1. **Introduction.** Let $\emptyset \neq \Omega_g \subset [0, 1]^d$ be an open and path connected set with Lipschitz boundary. Suppose further that $\mathbb{R}^d \backslash (\Omega_g + \mathbb{Z}^d)$ and $\partial\Omega_g$ are path connected. We study the sequence of functionals

$$F_{p,\varepsilon}(u) := \int_0^1 a_{p,\varepsilon}\left(\frac{u}{\varepsilon}\right) \|u'\| \, d\tau, \quad u \in W^{1,\infty}\left((0, 1)\right), \tag{1}$$

where

$$a_{p,\varepsilon}(x) := \begin{cases} \beta\varepsilon^{-p} & \text{if } x \in \Omega_g + \mathbb{Z}^d, \\ 1 & \text{otherwise,} \end{cases} \tag{2}$$

for $\beta > 0$ and $p \in (0, \infty)$. We will also consider a limiting case where $a_{p,\varepsilon}|_{p=\infty} := \infty$ on $\Omega_g + \mathbb{Z}^d$ and 1 otherwise.

Functionals of the form (1) naturally arise in the study of Riemannian geometry and nonlinear optics. In the case of nonlinear optics the values of $a_{p,\varepsilon}$ describe the *opacity* of the material. Our motivation is to study the effective dynamics of a Hamiltonian particle. In classical mechanics the motion of a particle with unit mass is given by

$$\frac{d^2 x}{dt^2} = -\nabla V(x), \tag{3}$$

where V is the potential energy function. An alternative description of the motion of such a particle is given by the *Maupertuis principle* [5, 4, 17, 20]. The Maupertuis principle states that, provided V is sufficiently smooth, that the solutions of (3) are critical points of the functional

$$\int_0^1 \sqrt{2\left(E - V\left(\frac{u}{\varepsilon}\right)\right)} \|u'\| d\tau, \tag{4}$$

up to reparameterisation. The functional (1) could therefore be interpreted as a model for the motion of a particle in a discontinuous periodic potential. The discontinuities and large metric values in $\Omega_g + \mathbb{Z}^d$ model positional constraints on the motion of the particle. We are interested in approximating minimum points of the functional (1) for small $\varepsilon > 0$ in the weak topology of $W^{1,\infty}((0,1))$. Determining the effective description of a Hamiltonian particle in a discontinuous potential has been posed as a problem in [16]. By determining an effective limit for the functional (1) we begin to develop an insight into the average motion of a particle in these potentials. This approach provides an alternative to determining an effective description for the Hamilton-Jacobi equations that also describe the motion of a Hamiltonian particle. The homogenisation of the Hamilton-Jacobi equation was first studied in [16].

Effective descriptions for Riemannian metrics that satisfy uniform growth conditions have been studied in [1, 7, 9, 6, 14]. A proof that the Riemannian metrics are dense with respect to Γ-convergence in the Finsler metrics can be found in [8]. Examples of Finsler metrics obtained as the Γ-limit of a sequence of two phase Riemannian metrics may be found in [2, 9, 11, 12, 13, 18, 19, 21]. The difference between the work in [2, 9, 11, 18, 19, 21] and this study is that our metrics are not uniformly equivalent, in ε, to the Euclidean distance on $\mathbb{R}^d$. Specifically, for a fixed p, there does not exist $\alpha, \beta > 0$ such that

$$\int_0^1 \alpha \|u'(\tau)\| \, d\tau \leq F_{p,\varepsilon}(u) \leq \int_0^1 \beta \|u'(\tau)\| \, d\tau \tag{5}$$

for all $\varepsilon > 0$, where $\|\cdot\|$ denotes the Euclidean norm. Consequently, in this setting, the standard theory does not apply.

In this paper we will focus on the computation of the Γ-limit of (1). Specifically we will show that the Γ-limit of (1) on $W^{1,\infty}((0,1))$ exists. Furthermore, we show that this limit can be described by the Γ-limit of a sequence of uniformly bounded length functionals. We will continue by studying the Γ-limit of the corresponding boundary value problem. In the literature one usually finds that the Γ-limit for the boundary value problem follows from the Γ-limit for the unconstrained problem [7, 9]. The novel observation that we make here is that in the absence of a uniform growth condition the Γ-limit may no longer exist. The existence of the Γ-limit depends on the value for p. We show that the value $p = 1$ is critical in the sense that for $p < 1$ the Γ-limit for the boundary value problem exists, whereas for $p \geq 1$ the Γ-limit does not exist. For $p < 1$ we show that the minimum values of $F_{p,\varepsilon}$ converge and then prove an analogue of [11], that is, the length functionals Γ-converge if, and only if, the induced distance functions converge locally uniformly. Contrary, for $p \geq 1$, and with an additional geometric assumption, we show that the Γ-limit fails to exist. Specifically we present a counter-example, namely, the existence of two sequences of ε such that minimisers of (1) converge to different limits. By modifying the ideas in [11] we can study the Γ-convergence of $F_{p,\varepsilon}$. This work is a natural continuation of the examples at the end of [11] to a more general class of problems.

2. **High opacity coefficients.** In this section the notion of a *high opacity coefficient* is formulated. The high opacity coefficient provides a lower bound on the opacity of inclusions that prevents geodesics entering them. Throughout this paper a set $\emptyset \neq \Omega_g \subsetneq [0,1]^d$ is an *admissible inclusion* if it is path connected, open and has a Lipschitz boundary. Furthermore we will assume that Ω_g has the property that $\Omega_w := \mathbb{R}^d \setminus (\Omega_g + \mathbb{Z}^d)$ and $\partial\Omega_g$ are path connected. For notational convenience set $\mathscr{A}(x,y) := \{u \in W^{1,\infty}((0,1)) : u(0) = x, u(1) = y\}$; where $W^{1,\infty}((0,1))$ is the space of all Lipschitz curves on $(0,1)$ taking values in $\mathbb{R}^d$. For $E \subset \mathbb{R}^d$ nonempty, define the mapping $d_E \colon E \times E \times [0,\infty) \to \mathbb{R}$ by

$$d_E(x,y;\beta) := \inf_{u \in \mathscr{A}(x,y)} \left\{ \int_0^1 \beta \|u'(\tau)\| \, \mathrm{d}\tau : u(\tau) \in E \; \forall \tau \in (0,1) \right\}.$$

To simplify notation further we set $d_E(x,y) := d_E(x,y;1)$.

Definition 2.1 (High Opacity Coefficient)**.** Let Ω_g be an admissible inclusion. A high opacity coefficient for the set Ω_g is a number $\lambda \in (0,\infty)$ such that for all $x,y \in \partial\Omega_g$ and $\beta > \lambda$,

$$d_{\partial\Omega_g}(x,y) < d_{\Omega_g}(x,y;\beta). \tag{6}$$

The computation of the high opacity coefficient for a square can be found in [9, Example 16.2]. The high opacity coefficients for other sets have been considered in [12, 13]. It would be an interesting problem to determine a class of Ω_g where (6) does not hold. It is reasonable to conjecture that if $\partial\Omega_g$ were to have a cusp then (6) would fail to hold.

The following lemma provides a sufficient condition on Ω_g to ensure the existence of a high opacity coefficient.

Lemma 2.2. *Let Ω_g be an admissible set. If $\partial\Omega_g$ is connected and differentiable then there exists a high opacity coefficient λ for Ω_g.*

Proof. By assumption $\partial\Omega_g$ is a differentiable manifold, therefore it can be equipped with a Riemannian metric d_{Ω_g} [15, Theorem 1.4.1]. The manifold has the distance function $d_{\partial\Omega_g}$. The distance function is bi-Lipschitz equivalent to the Euclidean distance [15, Corollary 1.4.1] since $\partial\Omega_g$ is compact. That is to say that there exist positive constants c and C such that,

$$c\|x - y\| \leq d_{\partial\Omega_g}(x,y) \leq C\|x - y\|, \tag{7}$$

for $x,y \in \partial\Omega_g$. It follows trivially that

$$d_{\Omega_g}(x,y;\beta) \geq \inf_{u \in \mathscr{A}(x,y)} \int_0^1 \beta \|u'(\tau)\| \, \mathrm{d}\tau = \beta\|x - y\|. \tag{8}$$

The inequality in (8) follows from the fact that $d_{\Omega_g}(x,y;\beta)$ is an infimum over a smaller space than $\mathscr{A}(x,y)$. Choosing $\beta > \lambda := C$ then combining (7) and (8) proves (6). $\square$

The above lemma is sufficient for a broad class of problems, including those that have physical applications in optics and dynamics. For example, in the dynamical context a discontinuity in the potential, such as that considered here, serves as a microscopic positional constraint. Boundaries of Ω_g of less regularity are still of mathematical interest but are not studied here.

The following lemma states that if the variation of a is sufficiently large, then minimising curves do not enter the high opacity regions. It will be exactly the high

opacity coefficient which provides the measure of what sufficiently large means in this context.

Lemma 2.3. *Let Ω_g be an admissible set with a high opacity coefficient λ. Define the function*

$$a(x) := \begin{cases} \beta & \text{if } x \in \Omega_g + \mathbb{Z}^d, \\ 1 & \text{otherwise,} \end{cases} \tag{9}$$

for $\beta > \lambda$, and the Riemannian length functional

$$L(u) := \int_0^1 a(u(\tau)) \|u'(\tau)\| \, d\tau. \tag{10}$$

For any $x, y \in \Omega_w$, let u be a geodesic joining x to y. It then follows that $u(\tau) \in \Omega_w$ for all $\tau \in (0, 1)$.

Proof. Suppose that $\operatorname{graph}(u) \cap \left(\Omega_g + \mathbb{Z}^d \right) \neq \emptyset$, then there exists $\mathbf{x} \in \mathbb{Z}^d$ such that $G(\mathbf{x}) := \operatorname{graph}(u) \cap (\Omega_g + \mathbf{x}) \neq \emptyset$. Set $TG(\mathbf{x}) := \{\tau \in (0, 1) \colon u(\tau) \in G(\mathbf{x})\}$, $s = \inf TG(\mathbf{x})$ and $t = \sup TG(\mathbf{x})$.

As Ω_g is open, it follows that $s < t$, to see this, observe by assumption that $TG(\mathbf{x}) \neq \emptyset$ and therefore there exists $t' \in TG(\mathbf{x})$. By definition $u(t') \in G(\mathbf{x}) \subset \Omega_g + \mathbf{x}$, hence there exists $\rho > 0$ such that $B_\rho(u(t')) \subset \Omega_g + \mathbf{x}$, since $\Omega_g + \mathbf{x}$ is open. By the continuity of u there exists $\delta > 0$ such that $\sigma \in (t' - \delta, t' + \delta)$ implies that $u(\sigma) \in \Omega_g + \mathbf{x}$. By construction $s \leq t' - \delta < t' + \delta \leq t$, hence $s < t$. By the continuity of u it holds that $u(s), u(t) \in \Omega_w$.

Further, it holds that, since $u(\sigma) \in \Omega_g + \mathbf{x}$ for all $\sigma \in (s, t)$,

$$\int_s^t a(u(\tau)) \|u'(\tau)\| \, d\tau = \int_s^t \beta \|u'(\tau)\| \, d\tau \geq d_{\Omega_g}(u(s), u(t); \beta).$$

As $\beta > \lambda$, using the invariance of length under reparameterisations, it holds that,

$$\int_s^t a(u(\tau)) \|u'(\tau)\| \, d\tau = \int_s^t \beta \|u'(\tau)\| \, d\tau > d_{\partial \Omega_g}(u(s), u(t); \beta). \tag{11}$$

Since $\partial \Omega_g$ is compact, it follows by the Hopf-Rinow Theorem [10, Theorem 2.5.28] the infimum in the definition of $d_{\partial \Omega_g}$ is obtained by some $\tilde{u} \in \mathscr{A}(u(s), u(t))$. Define the function v by

$$v(\tau) := \begin{cases} \tilde{u}(\tau) & \text{if } \tau \in (s, t) \\ u(\tau) & \text{otherwise,} \end{cases}$$

and using (11) gives that $L(u) > L(v)$. Hence u is not a geodesic. $\square$

3. **Statement of the mathematical problem.** Let Ω_g satisfy the hypotheses of Lemma 2.3 and let the high opacity coefficient be denoted by λ. Let the metric coefficient $a_{p,\varepsilon} \colon \mathbb{R}^d \to \mathbb{R}$ be given by (2) for $\beta > \lambda$, $p \in (0, \infty)$ and $\beta/\lambda > \varepsilon^p > 0$. The requirement that $\beta/\lambda > \varepsilon^p$ ensures that Lemma 2.3 holds for all ε. Extend this definition to the case when $p = \infty$ by setting,

$$a_{\infty,\varepsilon}(x) := \begin{cases} +\infty & \text{if } x \in \Omega_g + \mathbb{Z}^d, \\ 1 & \text{otherwise,} \end{cases} \tag{12}$$

for $\varepsilon > 0$. For the remainder of this paper we will assume $\beta > \lambda$ is fixed. The aim of this paper is to study two Γ-convergence problems for $F_{p,\varepsilon}$, where $F_{p,\varepsilon}$ is defined in (1):

P1.: The Γ-convergence of $F_{p,\varepsilon}$ on $W^{1,\infty}((0,1))$.

P2.: The Γ-convergence of $F_{p,\varepsilon}$ on $\mathscr{A}(x, y)$.

The Γ-convergence in both cases is with respect to the strong $L^\infty((0, 1))$ topology. The main tool behind our arguments is to compare the Γ-convergence of $F_{p,\varepsilon}$ with the Γ-convergence of the functionals

$$F_\varepsilon(u) := \int_0^1 a\left(\frac{u(\tau)}{\varepsilon}\right) \|u'(\tau)\| \, d\tau, \tag{13}$$

where the metric coefficient is given by

$$a(x) := \begin{cases} \beta & \text{if } x \in \Omega_g + \mathbb{Z}^d, \\ 1 & \text{otherwise.} \end{cases} \tag{14}$$

The computation of the Γ-limit in this case relies on the fact that the functionals F_ε are uniformly bounded in ε. By [9, Theorem 15.4] functionals (13) Γ-converge to a functional of the form

$$F_0(u) := \int_0^1 \psi(u'(\tau)) \, d\tau, \tag{15}$$

where the convex function ψ is given by the *asymptotic homogenisation formula*

$$\psi(\xi) = \lim_{\varepsilon \to 0} \inf_{u \in \mathscr{A}(0, \xi)} \int_0^1 a\left(\frac{u(\tau)}{\varepsilon}\right) \|u'(\tau)\| \, d\tau. \tag{16}$$

The function ψ is also 1-homogeneous and convex as demonstrated in [8]. Furthermore, it holds that

$$\min_{u \in \mathscr{A}(\xi_1, \xi_2)} \int_0^1 \psi(u'(\tau)) \, d\tau \le \int_0^1 \psi(\xi_2 - \xi_1) \, d\tau = \psi(\xi_2 - \xi_1).$$

Note that the minimum exists by [9, Theorem 7.2]. In addition it holds that for any $u \in \mathscr{A}(\xi_1, \xi_2)$

$$\psi(\xi_2 - \xi_1) = \psi\left(\int_0^1 u'(\tau) \, d\tau\right) \le \int_0^1 \psi(u'(\tau)) \, d\tau \tag{17}$$

by Jensen's inequality. Then taking the minimum over all $u \in \mathscr{A}(\xi_1, \xi_2)$ in (17) we get, by [9, Theorem 7.2],

$$\lim_{\varepsilon \to 0} d_\varepsilon(\xi_1, \xi_2) = \psi(\xi_2 - \xi_1). \tag{18}$$

where d_ε the distance function induced by F_ε is given by

$$d_\varepsilon(\xi_1, \xi_2) = \min_{u \in \mathscr{A}(\xi_1, \xi_2)} F_\varepsilon(u).$$

The proof that $d_{p,\varepsilon}$ is a distance function can be found in [15, Lemma 1.4.1]; the fact that the minimum exists by the Hopf-Rinow Theorem [10, Theorem 2.5.28].

4. **Γ-convergence on $W^{1,\infty}((0, 1))$.** In this section we address problem **P1**. That is we determine that the sequence $F_{p,\varepsilon}$ Γ-converges on $W^{1,\infty}((0, 1))$ with respect to the strong $L^\infty((0, 1))$ topology for $p \in (0, \infty]$. Let Ω_g satisfy the hypotheses of Lemma 2.3 and let the high opacity coefficient be denoted by λ. First we need a technical lemma which states that given $\varepsilon > 0$ and $u \in W^{1,\infty}((0, 1))$ there exists a curve no further than $\sqrt{d}\varepsilon$ away from graph(u) that does not enter the high opacity regions of the Riemannian length density, that is to say that graph(u) $\cap$ $\varepsilon\left(\Omega_g + \mathbb{Z}^d\right) = \emptyset$. We remark that the results of this subsection generalise the examples of [11], in the sense we consider a class of sets Ω_g.

Lemma 4.1. *Let $u \in W^{1,\infty}((0,1))$, then for each $\varepsilon > 0$ there exists*

$$u_\varepsilon^w \in W^{1,\infty}((0,1))$$

such that $\mathrm{graph}(u) \subset \varepsilon \Omega_w$ and $\|u - u_\varepsilon^w\|_\infty \leq \sqrt{d}\varepsilon$.

Proof. Fix $u \in W^{1,\infty}((0,1))$ and $\varepsilon > 0$. Since $\|u\|_\infty < \infty$, it follows that there exist $\mathbf{x}_1, ..., \mathbf{x}_n \in \mathbb{Z}^d$ such that $\mathrm{graph}(u) \subset \cup_{i=1}^n \varepsilon([0,1]^d + \mathbf{x}_i)$. Fix $i \in \{1, ..., n\}$ and define $G_i := \mathrm{graph}(u) \cap \varepsilon(\Omega_g + \mathbf{x}_i)$. Let G_i^j be a connected component of G_i; there exists finitely many such connected components since u is Lipschitz. Now fix j. Set $TG_i^j := \{\tau \in (0,1) \colon u(\tau) \in G_i^j\}$, $s_i^j = \inf TG_i^j$ and $t_i^j = \sup TG_i^j$. Choose i such that $TG_i^j \neq \emptyset$; if $TG_i^j = \emptyset$ for all i, j then set $u_\varepsilon^w = u$ and we are done. Applying the argument of Lemma 2.3, $s_i^j < t_i^j$ and $u(s_i^j), u(t_i^j) \in \partial(\varepsilon(\Omega_g + \mathbf{x}_i))$. Since $\partial \Omega_g$ is path connected there exists a Lipschitz curve joining $u(s_i^j)$ to $u(t_i^j)$ in $\partial(\varepsilon(\Omega_g + \mathbf{x}_i))$ denoted as w_i^j. The fact that w_i^j is Lipschitz continuous follows from the smoothness of $\partial \Omega_g$ [15, Chapters 1 and 8]. Then set

$$u_\varepsilon^w(\tau) := \begin{cases} w_i^j(\tau) & \text{if } \tau \in (s_i^j, t_i^j) \\ u(\tau) & \text{otherwise.} \end{cases}$$

It is clear from the construction that $\mathrm{graph}(u) \subset \varepsilon \Omega_w$. Note that, since Ω_g is assumed to be open, Ω_w is closed. It remains to check that $\|u - u_\varepsilon^w\|_\infty \leq \sqrt{d}\varepsilon$. Fix, $\tau \in (s_i, t_i)$ for some $i \in \{1, ..., n\}$, then

$$\|u_\varepsilon^w - u\|_\infty = \|w_i - u\|_\infty \leq \varepsilon \mathrm{diam}(\mathrm{cl}(\Omega_g + \mathbf{x}_i)),$$

since $u(\tau), w_i(\tau) \in \mathrm{cl}(\varepsilon(\Omega_g + \mathbf{x}_i))$ for all $\tau \in (s_i^j, t_i^j)$. It is immediate that

$$\mathrm{diam}(\mathrm{cl}(\Omega_g + \mathbf{x}_i)) \leq \sqrt{d}$$

and therefore taking the supremum over all τ gives the required estimate. The fact that $u_\varepsilon^w \in W^{1,\infty}((0,1))$ follows from the regularity of u, w_i and $\partial \Omega_g$. □

The following lemma shows that for the Γ-convergence of F_ε it is possible to choose a recovery sequence that never enters the higher opacity region.

Lemma 4.2. *For each $u \in W^{1,\infty}((0,1))$ and each sequence $(\varepsilon_k)_{k=1}^\infty$ converging to 0, there exists a sequence $(u_{\varepsilon_k})_{\varepsilon_k > 0} \subset W^{1,\infty}((0,1))$, converging in $L^\infty((0,1))$ to u, such that*

1. $\lim_{k \to \infty} F_{\varepsilon_k}(u_{\varepsilon_k}) = F_0(u)$, *and,*
2. $\mathrm{graph}(u_{\varepsilon_k}) \subset \varepsilon_k \Omega_w$ *for all k*

where F_0 is given by (15) and (16).

Proof. Let $K \subset\subset \mathbb{R}^d$ be such that $\mathrm{graph}(u) \subset \mathrm{int}(K)$. By [11, Theorem 3.1] the metrics induced by F_{ε_k}, denoted d_{ε_k}, converge locally uniformly to the metric induced by the norm ψ as $k \to \infty$. Therefore, it is possible to choose $(M_k)_{k=1}^\infty \subset \mathbb{N}$ converging to infinity such that

$$\lim_{k \to \infty} M_k \sup_{\xi_1, \xi_2 \in K} |d_{\varepsilon_k}(\xi_1, \xi_2) - \psi(\xi_2 - \xi_1)| = 0.$$

Let $\pi_{M_k} = \{\tau_0, ..., \tau_{M_k}\}$ be a partition of $[0, 1]$ such that $|\tau_j - \tau_{j+1}| = 1/M_k$ for $j = 1, ..., M_k$. Define the function u_{ε_k} by

$$u_{\varepsilon_k} = u_{\varepsilon_k}^w +$$
$$\text{argmin}_{w \in W_0^{1,\infty}((0,1))} \int_{\tau_{i-1}}^{\tau_i} a\left(\frac{u(\tau) + w(\tau)}{\varepsilon_k}\right) \|u'(\tau) + w'(\tau)\| \, d\tau \quad (19)$$

on $[\tau_{i-1}, \tau_i]$, where $u_{\varepsilon_k}^w$ is given by Lemma 4.1. The minimiser of (19) exists by the Hopf-Rinow Theorem [10, Theorem 2.5.28]. Fix k and $t \in [0, 1]$ and suppose that $\tau \in [\tau_{i-1}, \tau_i]$. Then

$$\left\|u_{\varepsilon_k}(\tau) - u_{\varepsilon_k}^w(\tau)\right\| \leq \left\|u_{\varepsilon_k}(\tau) - u_{\varepsilon_k}(\tau_{i-1})\right\| + \left\|u_{\varepsilon_k}(\tau_{i-1}) - u_{\varepsilon_k}^w(\tau)\right\|,$$
$$= \left\|u_{\varepsilon_k}(\tau) - u_{\varepsilon_k}(\tau_{i-1})\right\| + \left\|u_{\varepsilon_k}^w(\tau_{i-1}) - u_{\varepsilon_k}^w(\tau)\right\|, \quad (20)$$

using the fact that $u_{\varepsilon_k}^w = u_{\varepsilon_k}$ on π_{M_k}. By the bound on a (14) it holds that

$$\|u_{\varepsilon_k}(\tau) - u_{\varepsilon_k}(\tau_{i-1})\| \leq d_{\varepsilon_k}(u_{\varepsilon_k}(\tau), u_{\varepsilon_k}(\tau_{i-1}))$$
$$\leq d_{\varepsilon_k}(u_{\varepsilon_k}(\tau_i), u_{\varepsilon_k}(\tau_{i-1})) \leq \beta \left\|u_{\varepsilon_k}(\tau_i) - u_{\varepsilon_k}(\tau_{i-1})\right\|. \quad (21)$$

Since $u_{\varepsilon_k}^w \to u$ uniformly by Lemma 4.1, it follows that the sequence $u_{\varepsilon_k}^w$ is equicontinuous by the converse of the Arzelá-Ascoli Theorem. Therefore, fix $\eta > 0$ then there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $\|u_{\varepsilon_k}^w(x) - u_{\varepsilon_k}^w(y)\| < \eta$ for all k. Consequently there exists $K \in \mathbb{N}$ such that $k \geq K$ implies $|\pi_{M_k}| < \delta$, therefore for $k \geq K$, combining (20) and (21),

$$\left\|u_{\varepsilon_k}(\tau) - u_{\varepsilon_k}^w(\tau)\right\| \leq \beta\eta + \eta.$$

Consequently, since η was arbitrary, $\|u_{\varepsilon_k} - u_{\varepsilon_k}^w\|_\infty \to 0$ as $k \to \infty$, and since $u_{\varepsilon_k}^w$ converges to u in $L^\infty((0,1))$ it holds that $u_{\varepsilon_k} \to u$ in $L^\infty((0,1))$. It remains to show that u_{ε_k} has the desired properties. Observe that

$$\int_0^1 \psi(u'(\tau)) \, d\tau \geq \sum_{i=1}^{M_k} \psi(u(\tau_i) - u(\tau_{i-1}))$$
$$\geq \sum_{i=1}^{M_k} d_\varepsilon(u_{\varepsilon_k}(\tau_i), u_{\varepsilon_k}(\tau_{i-1})) -$$
$$\sum_{i=1}^{M_k} |\psi(u(\tau_i) - u(\tau_{i-1})) - d_\varepsilon(u_{\varepsilon_k}(\tau_i), u_{\varepsilon_k}(\tau_{i-1}))|. \quad (22)$$

The first inequality in the above holds since the length functional with density ψ gives rise to an induced metric $d(x, y) = \psi(x - y)$, the equality following from (18). By construction it is true that

$$\sum_{i=1}^{M_k} d_\varepsilon(u_{\varepsilon_k}(\tau_i), u_{\varepsilon_k}(\tau_{i-1})) = F_\varepsilon(u_{\varepsilon_k}),$$

furthermore, since there exists k_0 such that $\text{graph}(u_{\varepsilon_k}) \subset K$ for all $k \geq k_0$,

$$\sum_{i=1}^{M_k} |\psi(u(\tau_i) - u(\tau_{i-1})) - d_\varepsilon(u_{\varepsilon_k}(\tau_i), u_{\varepsilon_k}(\tau_{i-1}))|$$
$$\leq M_k \sup_{\xi_1, \xi_2 \in K} |d_{\varepsilon_k}(\xi_1, \xi_2) - \psi(\xi_2 - \xi_1)|.$$

Hence by (22), the choice of $(M_k)_{k=1}^\infty$ and the liminf inequality,

$$\int_0^1 \psi(u'(\tau))\,\mathrm{d}\tau \geq \limsup_{k\to\infty} F_\varepsilon(u_{\varepsilon_k}) \geq \liminf_{k\to\infty} F_\varepsilon(u_{\varepsilon_k}) \geq \int_0^1 \psi(u'(\tau))\,\mathrm{d}\tau.$$

The last thing we need to show is that $\mathrm{graph}(u_{\varepsilon_k}) \subset \varepsilon_k \Omega_w$ for all k. By the construction, as given in (19), u_{ε_k} is constructed as a piecewise geodesic curve joining points along $u_{\varepsilon_k}^w$. By Lemma 2.3 these geodesic pieces do not enter $\Omega_g + \mathbb{Z}^d$, hence the result holds. $\qquad\square$

Theorem 4.3. *Let Ω_g satisfy the hypotheses of Lemma 2.3 and let the high opacity coefficient be denoted by λ. Let $\beta > \lambda$, then*

$$\Gamma(L^\infty((0,1))) - \lim F_{p,\varepsilon}(u) = F_0(u), \ \forall u \in W^{1,\infty}((0,1)), \forall p \in (0,\infty]$$

where F_0 is defined by equations (15) and (16).

Proof. Fix $p \in (0,\infty]$ and let $(\varepsilon_k)_{k=1}^\infty$ be a sequence converging to 0. Without relabelling, pass to the subsequence where $\varepsilon_k^p < \beta/\lambda$ for all k. Let $u_{\varepsilon_k} \to u$ in $L^\infty((0,1))$. Since $a_{p,\varepsilon_k} \geq a$, by our choice of ε, it follows that

$$\liminf_{k\to\infty} F_{p,\varepsilon_k}(u_{\varepsilon_k}) \geq \liminf_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}) \geq F_0(u),$$

the second inequality being the liminf inequality for the Γ-convergence of F_{ε_k}. Applying Lemma 4.2 we obtain a sequence u_{ε_k} converging to u in $L^\infty((0,1))$ where $\lim_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}) = F_0(u)$ and $\mathrm{graph}(u_{\varepsilon_k}) \subset \varepsilon_k \Omega_w$ for all k. Since $a_{p,\varepsilon_k} = a$ on Ω_w it follows that

$$\lim_{k\to\infty} F_{p,\varepsilon_k}(u_{\varepsilon_k}) = \lim_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}).$$

Hence the sequence F_{p,ε_k} Γ-converges by the Urysohn property of Γ-convergence [9, Proposition 7.11]. $\qquad\square$

While the proof of Theorem 4.3 is nontrivial, as the sequence of functionals are not bounded uniformly, the convergence of minimisers, and hence the motivation for computing the Γ-limit, can be determined using a simpler argument. Observe, by the growth conditions that the minimiser of $F_{p,\varepsilon}$ is the zero function, which is unique and this converges to the minimiser of F_0. The fact that the 0 function is the unique minimiser of F_0 on $W^{1,\infty}((0,1))$ follows from the growth estimates on ψ [3]. It proves to be more interesting to understand the Γ-convergence of $F_{p,\varepsilon}$ on a smaller space, typically the space of Lipschitz curves joining two fixed points. The result of such an analysis may then be applied to problems in nonlinear optics and dynamics. In this setting, the minimal curves are nontrivial, and to understand their effective behaviour proves to be more challenging in the context of the unbounded length functionals we consider here. This is the subject of the next section.

5. **Γ-convergence on $\mathscr{A}(x,y)$.** As before, let Ω_g satisfy the hypotheses of Lemma 2.3 and let the high opacity coefficient be denoted by λ. In the previous subsection it was shown that the length functionals $F_{p,\varepsilon}$ Γ-converge on $W^{1,\infty}((0,1))$, for all $p \in (0,\infty]$. From this no additional information may be derived to approximate geodesics joining points. Typically, once the Γ-limit is calculated for the unconstrained problem, the Γ-limit with boundary conditions can be calculated as an extension of the original argument [9, Proposition 11.7]. The extension argument relies on what is known as the *fundamental estimate*, which allows one to join the functions in the argument of the functional, introducing only a small error term.

In this subsection we study the effective description of geodesics joining two points for the length functional (1). The main, and perhaps surprising, result here is that the existence of the Γ-limit for $F_{p,\varepsilon}$ depends on the value of p. To prove this fact we show that the sequence Γ-converges on the space of curves joining any two points if and only if the induced metric converges locally uniformly; this result is that of [11] but in the context of non-uniformly bounded two phase metrics. Here we show that the induced metrics always converge locally uniformly, should the opacity coefficient grow sufficiently slowly as $\varepsilon \to 0$.

5.1. The induced metrics converge locally uniformly to a norm for $p < 1$.
In this subsection we consider the case of slowly increasing opacity with ε. Here we prove that the induced metrics $d_{p,\varepsilon}$ converge locally uniformly to a norm. In the case when $p < 1$ the following lemma describes the pointwise convergence of $d_{p,\varepsilon}$ in terms of the convergence of $d_{p,\varepsilon}$ on $\varepsilon\Omega_w$. The latter convergence is easier to prove as there is more information available about geodesics joining points in $\varepsilon\Omega_w$, c.f. Lemma 2.3. In this section we will make use of the following trivial estimate

$$\|\xi_2 - \xi_1\| \le d_{p,\varepsilon}(\xi_1, \xi_2) \le \frac{\beta}{\varepsilon^p}\|\xi_2 - \xi_1\|. \tag{23}$$

Lemma 5.1. *Let $p < 1$, $\xi_1, \xi_2 \in \mathbb{R}^d$, $\varepsilon > 0$ and $\xi_{1,\varepsilon}, \xi_{2,\varepsilon} \in \varepsilon\Omega_w$ be such that $\|\xi_{1,\varepsilon} - \xi_1\| \le \sqrt{d}\varepsilon$, $\|\xi_{2,\varepsilon} - \xi_2\| \le \sqrt{d}\varepsilon$. The limit*

$$\lim_{\varepsilon \to 0} d_{p,\varepsilon}(\xi_1, \xi_2) = \lim_{\varepsilon \to 0} \min_{u \in \mathscr{A}(\xi_1, \xi_2)} F_{p,\varepsilon}(u) \tag{24}$$

exists if and only if the limit

$$\lim_{\varepsilon \to 0} d_{p,\varepsilon}(\xi_{1,\varepsilon}, \xi_{2,\varepsilon}) = \lim_{\varepsilon \to 0} \min_{u \in \mathscr{A}(\xi_{1,\varepsilon}, \xi_{2,\varepsilon})} F_{p,\varepsilon}(u) \tag{25}$$

exists.

Proof. Fix $\varepsilon^p \in (0, \beta/\lambda)$. Then, using the triangle inequality and (23),

$$d_{p,\varepsilon}(\xi_1, \xi_2) \le \frac{\beta}{\varepsilon^p}\|\xi_1 - \xi_{1,\varepsilon}\| + d_{p,\varepsilon}(\xi_{1,\varepsilon}, \xi_{2,\varepsilon}) + \frac{\beta}{\varepsilon^p}\|\xi_{2,\varepsilon} - \xi_2\|$$

$$\le d_{p,\varepsilon}(\xi_{1,\varepsilon}, \xi_{2,\varepsilon}) + 2\beta\sqrt{d}\varepsilon^{1-p}.$$

Similarly,

$$d_{p,\varepsilon}(\xi_1, \xi_2) \ge d_{p,\varepsilon}(\xi_{1,\varepsilon}, \xi_{2,\varepsilon}) - \frac{\beta}{\varepsilon^p}\|\xi_{1,\varepsilon} - \xi_1\| - \frac{\beta}{\varepsilon^p}\|\xi_2 - \xi_{2,\varepsilon}\|$$

$$\ge d_{p,\varepsilon}(\xi_{1,\varepsilon}, \xi_{2,\varepsilon}) - 2\beta\sqrt{d}\varepsilon^{1-p}.$$

Therefore

$$|d_{p,\varepsilon}(\xi_1, \xi_2) - d_{p,\varepsilon}(\xi_{1,\varepsilon}, \xi_{2,\varepsilon})| \le 2\beta\sqrt{d}\varepsilon^{1-p}. \tag{26}$$

If the limit (25) exists then taking the limit as $\varepsilon \to 0$ gives that, by (26), the limit (24) exists. The converse statement is proved in an identical fashion. $\square$

The following lemma connects the homogenisation of $F_{p,\varepsilon}$ with that of F_ε. The proof demonstrates that the functionals F_ε and $F_{p,\varepsilon}$ give equivalent measures of length for geodesics. For notational convenience we define for $x, y \in \mathbb{R}^d$ the space $\mathscr{A}_w(x, y) := \{u \in W^{1,\infty}((0,1)) : u(0) = x, u(1) = y, u(\tau) \in \Omega_w\ \forall\tau\}$.

Lemma 5.2. *Let $p < 1$, $\xi_1, \xi_2 \in \mathbb{R}^d$, $\varepsilon > 0$ and $\xi_{1,\varepsilon}, \xi_{2,\varepsilon} \in \varepsilon\Omega_w$ be such that $\|\xi_{1,\varepsilon} - \xi_1\| \leq \sqrt{d}\varepsilon$, $\|\xi_{2,\varepsilon} - \xi_2\| \leq \sqrt{d}\varepsilon$. For $\beta/\lambda > \varepsilon > 0$ it follows that*

$$\lim_{\varepsilon \to 0} \min_{u \in \mathscr{A}(\xi_{1,\varepsilon}, \xi_{2,\varepsilon})} F_{p,\varepsilon}(u) = \lim_{\varepsilon \to 0} \min_{u \in \mathscr{A}(\xi_{1,\varepsilon}/\varepsilon, \xi_{2,\varepsilon}/\varepsilon)} \varepsilon \int_0^1 a(u(\tau))\|u'(\tau)\|\, d\tau,$$

provided either limit exists. Furthermore,

$$\lim_{\varepsilon \to 0} \min_{u \in \mathscr{A}(\xi_1, \xi_2)} F_\varepsilon(u) = \lim_{\varepsilon \to 0} \min_{u \in \mathscr{A}(\xi_{1,\varepsilon}/\varepsilon, \xi_{2,\varepsilon}/\varepsilon)} \varepsilon \int_0^1 a(u(\tau))\|u'(\tau)\|\, d\tau,$$

provided either limit exists.

Proof. Fix $\varepsilon^p \in (0, \beta/\lambda)$, then by Lemma 2.3 and the fact that $a_{p,\varepsilon} = a$ on Ω_w,

$$\lim_{\varepsilon \to 0} \min_{u \in \mathscr{A}(\xi_{1,\varepsilon}, \xi_{2,\varepsilon})} F_{p,\varepsilon}(u) = \min_{u \in \mathscr{A}(\xi_{1,\varepsilon}/\varepsilon, \xi_{2,\varepsilon}/\varepsilon)} \varepsilon \int_0^1 a_{p,\varepsilon}(u(\tau))\|u'(\tau)\|\, d\tau$$

$$= \min_{u \in \mathscr{A}_w(\xi_{1,\varepsilon}/\varepsilon, \xi_{2,\varepsilon}/\varepsilon)} \varepsilon \int_0^1 a_{p,\varepsilon}(u(\tau))\|u'(\tau)\|\, d\tau$$

$$= \min_{u \in \mathscr{A}_w(\xi_{1,\varepsilon}/\varepsilon, \xi_{2,\varepsilon}/\varepsilon)} \varepsilon \int_0^1 a(u(\tau))\|u'(\tau)\|\, d\tau$$

$$= \min_{u \in \mathscr{A}(\xi_{1,\varepsilon}/\varepsilon, \xi_{2,\varepsilon}/\varepsilon)} \varepsilon \int_0^1 a(u(\tau))\|u'(\tau)\|\, d\tau,$$

taking the limit as $\varepsilon \to 0$ gives the first part of the result. The second part follows by identical reasoning to Lemma 5.1 and the above. □

The following theorem establishes the pointwise convergence of metrics for $p < 1$.

Theorem 5.3. *Let $p < 1$. Then the limit*

$$\lim_{\varepsilon \to 0} d_{p,\varepsilon}(\xi_1, \xi_2) = \lim_{\varepsilon \to 0} \min_{u \in \mathscr{A}(\xi_1, \xi_2)} F_{p,\varepsilon}(u) = \psi(\xi_2 - \xi_1) \tag{27}$$

exists for all $\xi_1, \xi_2 \in \mathbb{R}^d$, where ψ is as in (16).

Proof. By [3, Proposition 3.2] ψ exists. For each $\varepsilon > 0$, we obtain the existence of $\xi_{1,\varepsilon}, \xi_{2,\varepsilon} \in \varepsilon\Omega_w$ be such that $\|\xi_{1,\varepsilon} - \xi_1\| \leq \sqrt{d}\varepsilon$, $\|\xi_{2,\varepsilon} - \xi_2\| \leq \sqrt{d}\varepsilon$. The result then follows by applying Lemmas 5.2 and 5.1. □

5.2. **The induced metrics fail to converge pointwise for $p \geq 1$.** In this section we show that for $p \geq 1$ the limit $\lim_{\varepsilon \to 0} d_{p,\varepsilon}(\xi_1, \xi_2)$ does not exist, in contrast to the case $p < 1$. Consequently, the sequence $F_{p,\varepsilon}$ fails to Γ-converge on $\mathscr{A}(\xi_1, \xi_2)$ for all $\xi_1, \xi_2 \in \mathbb{R}^d$; since otherwise by the fundamental theorem of Γ-convergence [7, 9] the minimum values would converge. It is possible to make this conclusion by observing also that the sequence $F_{p,\varepsilon}$ is equicoercive.

The proof relies on a simple geometric assumption. The principle behind the argument is that we may choose, for a specific pair of end points, two different sequences of values for ε that subsequently give rise to two different limit values. Namely, we may choose a sequence where the endpoints lie in $\varepsilon\Omega_w$ and therefore using Lemma 2.3 and subsequent results to find a finite limit. However it is also possible to choose a sequence of values for ε where one of the end points lies in $\Omega_g + \mathbb{R}^d$, resulting in the divergence of the length functionals in ε.

A sketch of the argument can be found in figure 1. The argument relies on joining $\xi_2 := (1/2, 1/2, \cdots) \in \Omega_g$ and $\xi_1 := (0, 0, \cdots) \in \Omega_w$ for specific choices of sequences

of values for ε. In particular we choose a sequence ε_k such that $\xi_2 \in \varepsilon_k \left(\Omega_g + \mathbb{Z}^d \right)$ for all k. It follows that $d_{p,\varepsilon_k}(\xi_1, \xi_2) = \psi(\xi_2 - \xi_1) + O(\varepsilon_k^{1-p})$ as $k \to \infty$. We also construct a sequence such that $\xi_2 \in \varepsilon_k \Omega_w$ for all k. In this case, as the geodesic never enters $\varepsilon_k \left(\Omega_g + \mathbb{Z}^d \right)$, it follows that $d_{p,\varepsilon_k}(\xi_1, \xi_2) = \psi(\xi_2 - \xi_1)$ as $k \to \infty$. Consequently we get different limit values for two sequences of values for ε. Therefore the Γ-limit ceases to exist.
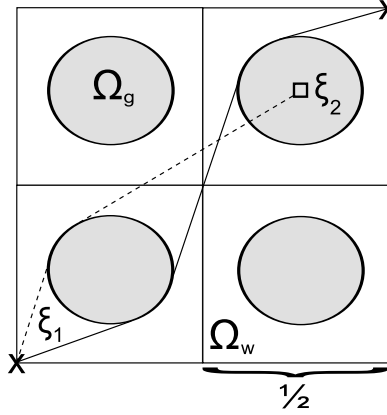

FIGURE 1. An illustration of the proof of Theorem 5.4. The sequence $\tilde{\varepsilon}_k$ corresponds to a sequence such that the end points of the curve lie in $\varepsilon \Omega_w$, indicated by the solid curve. The sequence ε_k corresponds to a sequence such that it has an end point in $\varepsilon \left(\Omega_g + \mathbb{Z}^d \right)$, indicated by the dashed curve.

Theorem 5.4. *Suppose that* $\xi_2 := (1/2, 1/2, \cdots) \in \Omega_g$ *and that* $\xi_1 := (0, 0, \cdots) \in \Omega_w$. *Then the limit* $\lim_{\varepsilon \to 0} d_{p,\varepsilon}(\xi_1, \xi_2)$ *does not exist.*

Proof. Set $\tilde{\varepsilon}_k := 1/2k$ for $k \in \mathbb{N}$. If $p < \infty$ we pass to a subsequence such that $\tilde{\varepsilon}_k^p < \beta/\lambda$. It follows immediately that $\xi_2 \in \tilde{\varepsilon}_k \mathbb{Z}^d \subset \tilde{\varepsilon}_k \Omega_w$ for all k, since $\xi_1 \in \Omega_w$. Consequently, by Lemma 2.3 the solution to the problem $\min_{u \in \mathscr{A}(\xi_1, \xi_2)} F_{p,\tilde{\varepsilon}_k}(u)$, which we denote by $w_{\tilde{\varepsilon}_k}$, is such that $\mathrm{graph}(w_{\tilde{\varepsilon}_k}) \subset \tilde{\varepsilon}_k \Omega_w$. Applying Theorem 5.3, with $\xi_{1,\tilde{\varepsilon}_k} = \xi_1$ and $\xi_{2,\tilde{\varepsilon}_k} = \xi_2$ for all $\tilde{\varepsilon} > 0$, it holds that

$$\lim_{k \to \infty} \min_{u \in \mathscr{A}(\xi_1, \xi_2)} F_{p,\tilde{\varepsilon}_k}(u) = \lim_{k \to \infty} \min_{u \in \mathscr{A}(\xi_1, \xi_2)} F_{\tilde{\varepsilon}_k}(u) = \psi(\xi_2 - \xi_1).$$

We now construct a second sequence converging to a different limit. Set $\varepsilon_k = 1/(2k+1)$ for $k \in \mathbb{N}$. If $p < \infty$ we again pass to a subsequence such that $\varepsilon_k^p < \beta/\lambda$. It follows that $\xi_2 \in \varepsilon_k (\Omega_g + \mathbb{Z}^d)$ for all k. Denote the solution to the problem $\min_{u \in \mathscr{A}(\xi_1, \xi_2)} F_{p,\varepsilon_k}(u)$, by w_{ε_k}. It follows by Lemma 2.3 that for each k there exists $\tau_k \in (0,1)$ such that $w_{\varepsilon_k}(\tau) \in \varepsilon_k \Omega_w$ for $\tau \in [0, \tau_k]$ and $w_{\varepsilon_k}(\tau) \in \varepsilon_k(\Omega_g + (k, k, \cdots))$ for $\tau \in (\tau_k, 1]$ and $(k, k, \cdots) \in \mathbb{Z}^d$. Since Ω_g is open there exists a $\rho > 0$ such that $B_\rho(\xi_2) \subset \Omega_g$. Hence, for all k, $\varepsilon_k(B_\rho(\xi_2) + (k, k, \cdots)) \subset \varepsilon_k(\Omega_g + (k, k, \cdots))$. By continuity for each k there exists $\sigma_k \in (\tau_k, 1)$ such that $w_{\varepsilon_k}(\sigma_k) \in \partial \left(\varepsilon_k(B_\rho(\xi_2) + (k, k, \cdots)) \right)$. Therefore

$$F_{p,\varepsilon_k}(w_{\varepsilon_k}) \geq d_{p,\varepsilon}(\xi_1, w_{\varepsilon_k}(\tau_k)) + d_{p,\varepsilon}(w_{\varepsilon_k}(\sigma_k), \xi_2), \tag{28}$$

since any geodesic curve is locally geodesic. By construction $\|w_{\varepsilon_k}(\tau_k) - \xi_2\| \leq \sqrt{d}\varepsilon_k$ and $w_{\varepsilon_k}(\tau_k) \in \varepsilon_k \Omega_w$. Set $\xi_{1,\varepsilon_k} = \xi_1$ and $\xi_{2,\varepsilon_k} = w_{\varepsilon_k}(\tau_k)$ for all k. Applying Theorem 5.3 it follows that

$$\lim_{k \to \infty} d_{p,\varepsilon}(\xi_1, w_{\varepsilon_k}(\tau_k)) = \lim_{k \to \infty} d_{\varepsilon_k}(\xi_1, \xi_2) = M. \tag{29}$$

The quantity $d_{p,\varepsilon}(w_{\varepsilon_k}(\sigma_k), \xi_2)$ is the distance between the centre of the ball

$$\varepsilon_k(B_\rho(\xi_2) + (k, k, \cdots))$$

and a point on its boundary. As the ball $\varepsilon_k(B_\rho(\xi_2) + (k, k, \cdots))$ is contained in Ω_g it follows that

$$d_{p,\varepsilon}(w_{\varepsilon_k}(\sigma_k), \xi_2) = \frac{\beta}{\varepsilon_k^p}\|w_{\varepsilon_k}(\sigma_k) - \xi_2\| = \beta\varepsilon_k^{1-p}\rho. \tag{30}$$

Combining (28), (29) and (30) then taking the limit $k \to \infty$ gives that

$$\lim_{k \to \infty} F_{p,\varepsilon_k}(w_{\varepsilon_k}) \geq \begin{cases} \infty & \text{if } p > 1, \\ M + \beta\rho & \text{if } p = 1. \end{cases}$$

Hence there exists sequences $(\tilde{\varepsilon}_k)_{k=1}^\infty$ and $(\varepsilon_k)_{k=1}^\infty$ converging to 0 such that $M \geq M + \beta\rho$. Since $\beta, \rho > 0$ the result is shown. $\qquad\square$

5.3. The equivalence of Γ-convergence and metric convergence for $p < 1$.
In this section we show that the boundary value problem Γ-converges if and only if the induced metrics converge locally uniformly. This extends the theory of [11] to a class of non-uniformly bounded two-phase Riemannian length functionals. That is, when the sequence d_ε fails to satisfy $C_1\|\xi_2 - \xi_1\| \leq d_\varepsilon(\xi_1, \xi_2) \leq C_2\|\xi_2 - \xi_1\|$ uniformly in ε for all $\xi_1, \xi_2 \in \mathbb{R}^d$.

The following lemma shows that we can improve the bounds on the induced metric so that $d_{p,\varepsilon}$ is almost uniformly equivalent to the Euclidean metric.

Lemma 5.5. *Let $p < 1$, $\varepsilon > 0$ and $\xi_1, \xi_2 \in \mathbb{R}^d$. Then there exists $C_1, C_2 > 0$ such that*

$$\|\xi_1 - \xi_2\| - C_1\varepsilon \leq d_{p,\varepsilon}(\xi_1, \xi_2) \leq \beta\|\xi_1 - \xi_2\| + C_2\varepsilon^{1-p}.$$

Proof. Let u be a geodesic joining ξ_1 to ξ_2. By Lemma 2.3 it follows that the set $T := \{\tau \in (0,1) \colon u(\tau) \in \varepsilon(\Omega_g + \mathbb{Z}^d)\}$ takes one of the forms

$$\emptyset \text{ or } [0, \tau_1) \text{ or } (\tau_2, 1] \text{ or } [0, \tau_1) \cup (\tau_2, 1] \text{ or } [0, 1]$$

for some $\tau_1, \tau_2 \in (0,1)$ with $\tau_1 < \tau_2$. Suppose first that $T = [0, \tau_1) \cup (\tau_2, 1]$; the cases when $\tau_2 = 1$ or $\tau_1 = 0$ following in an identical fashion. It holds that

$$F_{p,\varepsilon}(u) = d_{p,\varepsilon}(\xi_1, u(\tau_1)) + d_{p,\varepsilon}(u(\tau_1), u(\tau_2)) + d_{p,\varepsilon}(u(\tau_2), \xi_2). \tag{31}$$

Observe that by construction $\|\xi_1 - u(\tau_1)\| \leq \sqrt{d}\varepsilon$ and $\|\xi_2 - u(\tau_2)\| \leq \sqrt{d}\varepsilon$, and hence by the growth condition (23) it follows that

$$d_{p,\varepsilon}(\xi_1, u(\tau_1)) \leq \beta\sqrt{d}\varepsilon^{1-p} \text{ and } d_{p,\varepsilon}(u(\tau_2), \xi_2) \leq \beta\sqrt{d}\varepsilon^{1-p}.$$

Using Lemma 2.3 and the fact that $u(\tau_1), u(\tau_2) \in \varepsilon\Omega_w$, it follows that $a_{p,\varepsilon}(u(\tau)) = a(u(\tau))$ for all $\tau \in [\tau_1, \tau_2]$ and hence $d_{p,\varepsilon}(u(\tau_1), u(\tau_2)) = d_\varepsilon(u(\tau_1), u(\tau_2))$. By the triangle inequality and (23) we have

$$d_\varepsilon(u(\tau_1), u(\tau_2)) \leq d_\varepsilon(\xi_1, \xi_2) + \beta\left(\|\xi_1 - u(\tau_1)\| + \|\xi_2 - u(\tau_2)\|\right)$$
$$\leq \beta\|\xi_2 - \xi_1\| + 2\beta\sqrt{d}\varepsilon.$$

Consequently

$$d_{p,\varepsilon}(\xi_1,\xi_2) = F_{p,\varepsilon}(u) \le \beta\|\xi_2 - \xi_1\| + 2\beta\sqrt{d}\varepsilon^{1-p} + 2\beta\sqrt{d}\varepsilon.$$

Continuing from (31) and applying the triangle inequality with Lemma 2.3, we have that

$$F_{p,\varepsilon}(u) \ge \int_{\tau_1}^{\tau_2} a_{p,\varepsilon}\left(\frac{u(\tau)}{\varepsilon}\right)\|u'(\tau)\|\,\mathrm{d}\tau = d_{p,\varepsilon}(u(\tau_1),u(\tau_2))$$

$$= d_\varepsilon(u(\tau_1),u(\tau_2)) \ge d_\varepsilon(\xi_1,\xi_2) - d_\varepsilon(\xi_1,u(\tau_1)) - d_\varepsilon(\xi_2,u(\tau_2)).$$

From (23) it follows that

$$F_{p,\varepsilon}(u) \ge \|\xi_2 - \xi_1\| - 2\beta\sqrt{d}\varepsilon.$$

Hence the bounds are illustrated. The remaining case when $T = [0,1]$ follows in a similar manner. □

The following lemma improves pointwise convergence to local uniform convergence as in the uniformly bounded case. The key is that we are still close to the uniformly bounded case, due to the improved growth bounds of Lemma 5.5.

Lemma 5.6. *If the metrics $d_{p,\varepsilon}$ converge pointwise to d then they converge locally uniformly.*

Proof. The proof is as for [11, Proposition 2.3], *mutatis mutandis*, using the bounds in Lemma 5.5. □

We are now in a position to prove one of our main homogenisation results, using a modification of the method in [11, Theorem 3.1].

Theorem 5.7. *If $p < 1$, then the sequence of functionals $F_{p,\varepsilon}$ defined on $\mathscr{A}(\xi_1,\xi_2)$ Γ-converge with respect to the $L^\infty((0,1))$ norm topology to F_0 as defined in equations (15) and (16), for all $\xi_1,\xi_2 \in \mathbb{R}^d$.*

Proof. Fix $\xi_1,\xi_2 \in \mathbb{R}^d$. Let $(\varepsilon_k)_{k=1}^\infty \subset (0,\infty)$ converge to zero. Fix $u \in \mathscr{A}(\xi_1,\xi_2)$ and let $u_{\varepsilon_k} \in \mathscr{A}(\xi_1,\xi_2)$ converge to $u \in L^\infty(0,1)$ as $k \to \infty$, then $u_{\varepsilon_k} \to u$ pointwise as $k \to \infty$. Let $\pi_N = \{\tau_0,...,\tau_N\}$ be a partition of $[0,1]$ such that $|\tau_j - \tau_{j+1}| = 1/N$ for $j = 1,...,N$. Then

$$F_{\varepsilon_k}(u_{\varepsilon_k}) = \sum_{i=1}^N \int_{\tau_{i-1}}^{\tau_i} a_{p,\varepsilon}\left(\frac{u_{\varepsilon_k}(\tau)}{\varepsilon_k}\right)\|u'_{\varepsilon_k}(\tau)\|\,\mathrm{d}\tau$$

$$\ge \sum_{i=1}^N d_{p,\varepsilon_k}(u_{\varepsilon_k}(\tau_{i-1}),u_{\varepsilon_k}(\tau_i)),$$

using the invariance of the length functional under reparameterisations. Therefore applying the triangle inequality for the induced metric, and Lemma 5.5, we obtain

$$F_{\varepsilon_k}(u_{\varepsilon_k}) \ge \sum_{i=1}^N d_{p,\varepsilon_k}(u(\tau_{i-1}),u(\tau_i)) - d_{p,\varepsilon_k}(u_{\varepsilon_k}(\tau_i),u(\tau_i))$$

$$- d_{p,\varepsilon_k}(u(\tau_{i-1}),u_{\varepsilon_k}(\tau_{i-1}))$$

$$\ge \sum_{i=1}^N d_{p,\varepsilon_k}(u(\tau_{i-1}),u(\tau_i)) - \beta\|u_{\varepsilon_k}(\tau_i) - u(\tau_i)\|$$

$$- \beta\|u(\tau_{i-1}) - u_{\varepsilon_k}(\tau_{i-1})\| - 2C\varepsilon_k^{1-p}.$$

Taking the limit as $k \to \infty$ and using the fact that $d_{p,\varepsilon}$ converges pointwise by Theorem 5.3, gives that

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}) \geq \sum_{i=1}^{N} \psi\left(u(\tau_i) - u(\tau_{i-1})\right).$$

Using the 1-homogeneity of ψ, c.f. [8], gives

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}) \geq \sum_{i=1}^{N} \psi\left(\frac{u(\tau_i) - u(\tau_{i-1})}{|\tau_i - \tau_{i-1}|}\right) |\tau_i - \tau_{i-1}| = \int_0^1 \psi(u'_N(\tau))\,\mathrm{d}\tau,$$

where u_N is the linear interpolation of u on π_N. Sending $N \to \infty$, and applying the dominated convergence Theorem, to prove the lim-inf inequality.

We now show the lim-sup inequality. Fix $u \in \mathscr{A}(\xi_1, \xi_2)$. Then choosing a sequence $(M_k)_{k=1}^{\infty} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} M_k \sup_{\xi_1,\xi_2 \in K} |d_{\varepsilon_k}(\xi_1, \xi_2) - \psi(\xi_2 - \xi_1)| = 0,$$

where $K \subset\subset \mathbb{R}^d$ such that $\mathrm{graph}(u) \subset \mathrm{int}(K)$. Let $\pi_{M_k} = \{\tau_0, ..., \tau_{M_k}\}$ be a partition of $[0,1]$ such that $|\tau_j - \tau_{j+1}| = 1/M_k$ for $j = 1, ..., M_k$. Define the function u_{ε_k} by

$$u_{\varepsilon_k}(\tau) = u(\tau)+$$
$$\mathrm{argmin}_{w\in W_0^{1,\infty}((0,1))} \int_{\tau_{i-1}}^{\tau_i} a_{p,\varepsilon_k}\left(\frac{u(\tau) + w(\tau)}{\varepsilon}\right) \|u'(\tau) + w'(\tau)\|\,\mathrm{d}\tau \quad (32)$$

for $\tau \in [\tau_{i-1}, \tau_i]$ in the partition π_{M_k}. Clearly, by construction, $u_{\varepsilon_k} \in \mathscr{A}(\xi_1, \xi_2)$. Fix k and $t \in [0,1]$ and suppose that $\tau \in [\tau_{i-1}, \tau_i]$. Then

$$\|u_{\varepsilon_k}(\tau) - u(\tau)\| \leq \|u_{\varepsilon_k}(\tau) - u(\tau_{i-1})\| + \|u(\tau_{i-1}) - u(\tau)\|. \qquad (33)$$

By Lemma 5.5 it holds that

$$\begin{aligned}
\|u_{\varepsilon_k}(\tau) - u(\tau_{i-1})\| &\leq d_{p,\varepsilon_k}(u_{\varepsilon_k}(\tau), u(\tau_{i-1})) + C\varepsilon_k^{1-p}, \\
&\leq d_{p,\varepsilon_k}(u_{\varepsilon_k}(\tau_i), u(\tau_{i-1})) + C\varepsilon_k^{1-p}, \\
&\leq \beta \|u_{\varepsilon_k}(\tau_i) - u_{\varepsilon_k}(\tau_{i-1})\| + 2C\varepsilon_k^{1-p}, \\
&= \beta \|u(\tau_i) - u(\tau_{i-1})\| + 2C\varepsilon_k^{1-p}.
\end{aligned}$$

Therefore, letting the Lipschitz constant of u be denoted by Λ,

$$\begin{aligned}
\|u_{\varepsilon_k}(\tau) - u(\tau)\| &\leq \beta \|u(\tau_i) - u(\tau_{i-1})\| + \|u(\tau_{i-1}) - u(\tau)\| + 2C\varepsilon_k^{1-p} \\
&\leq \frac{(1+\beta)\Lambda}{M_k} + 2C\varepsilon_k^{1-p},
\end{aligned}$$

using the fact that $|\tau_i - \tau_{i-1}| = 1/M_k$ for all k. Hence $u_{\varepsilon_k} \to u$ in $L^\infty((0,1))$. It remains to show that u_{ε_k} has the desired properties. Observe that

$$
\begin{aligned}
\int_0^1 \psi(u'(\tau))\,\mathrm{d}\tau &\geq \sum_{i=1}^{M_k} \psi(u(\tau_i) - u(\tau_{i-1})) \\
&\geq \sum_{i=1}^{M_k} d_{p,\varepsilon}(u_{\varepsilon_k}(\tau_i), u_{\varepsilon_k}(\tau_{i-1})) - \\
&\qquad \sum_{i=1}^{M_k} |\psi(u(\tau_i) - u(\tau_{i-1})) - d_{p,\varepsilon}(u_{\varepsilon_k}(\tau_i), u_{\varepsilon_k}(\tau_{i-1}))| . \quad (34)
\end{aligned}
$$

By construction it holds that

$$
\sum_{i=1}^{M_k} d_{p,\varepsilon}(u_{\varepsilon_k}(\tau_i), u_{\varepsilon_k}(\tau_{i-1})) = F_{p,\varepsilon}(u_{\varepsilon_k}),
$$

furthermore, for ε sufficiently small,

$$
\sum_{i=1}^{M_k} |\psi(u(\tau_i) - u(\tau_{i-1})) - d_{p,\varepsilon}(u_{\varepsilon_k}(\tau_i), u_{\varepsilon_k}(\tau_{i-1}))|
$$
$$
\leq M_k \sup_{\xi_1,\xi_2 \in K} |d_{\varepsilon_k}(\xi_1,\xi_2) - \psi(\xi_2 - \xi_1)|.
$$

Hence by (34), the choice of $(M_k)_{k=1}^\infty$ and the liminf inequality,

$$
\int_0^1 \psi(u'(\tau))\,\mathrm{d}\tau \geq \limsup_{k\to\infty} F_{p,\varepsilon}(u_{\varepsilon_k}) \geq \liminf_{k\to\infty} F_{p,\varepsilon}(u_{\varepsilon_k}) \geq \int_0^1 \psi(u'(\tau))\,\mathrm{d}\tau.
$$

Since the choice of the sequence $(\varepsilon_k)_{k=1}^\infty$ was arbitrary and the limit is independent of this choice, and Γ-convergence follows. $\qquad\square$

It remains to prove the converse.

Theorem 5.8. *Suppose that the sequence of functionals $F_{p,\varepsilon}$ defined on $\mathscr{A}(\xi_1,\xi_2)$ Γ-converge with respect to the $L^\infty((0,1))$ norm topology to F_0 defined in (15) and (16) for all $\xi_1,\xi_2 \in \mathbb{R}^d$. Then the induced metrics converge locally uniformly to the norm ψ, as defined in (16), on $\mathbb{R}^d$.*

Proof. Fix $\xi_1, \xi_2 \in \mathbb{R}^d$. Applying the fundamental theorem of Γ-convergence [9, Theorem 7.2] it follows that $d_{p,\varepsilon}$ converges pointwise to ψ. By Lemma 5.6 the local uniform convergence follows. $\qquad\square$

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