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Branching Brownian Motion with Catalytic Branching at the Origin

Sergey Bocharov · Simon C. Harris

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Abstract We consider a branching Brownian motion in which binary fission takes place only when particles are at the origin at a rate $\beta > 0$ on the local time scale. We obtain results regarding the asymptotic behaviour of the number of particles above λt at time t, for $\lambda > 0$. As a corollary, we establish the almost sure asymptotic speed of the rightmost particle. We also prove a Strong Law of Large Numbers for this catalytic branching Brownian motion.

Keywords Catalytic branching · Brownian motion

1 Introduction

1.1 Model

In this article we study a branching Brownian motion in which binary fission takes place at the origin at rate $\beta > 0$ on the local time scale. That is, if $(X_t : t \le \tau)$ is the path and $(L_s : s \le \tau)$ is the local time at the origin of the initial Brownian particle up until the first fission time τ, then the first birth occurs at the origin as soon as an independent exponential amount of local time has been accumulated with $L_\tau \stackrel{d}{=} \mathrm{Exp}(\beta)$ and $X_\tau = 0$. Once born, particles move off independently from their birth position (at the origin), replicating the behaviour of the parent, and so on. Heuristically, we have an inhomogeneous branching Brownian motion with instantaneous branching rate $\beta(x) := \beta\,\delta_0(x)$, since we can informally think of Brownian local time at the origin as $L_t = \int_0^t \delta_0(X_s)\,\mathrm{d}s$, where δ_0 is the unit Dirac-mass at 0.

S. Bocharov
Department of Mathematics, Zhejiang University, Zheda Road, Hangzhou 310027, China
e-mail: bocharov@zju.edu.cn

S.C. Harris (✉)
Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK
e-mail: s.c.harris@bath.ac.uk

Although BBM models have been very widely studied, the degenerate nature of such catalytic branching at the origin means that the above BBM model needs some special treatment. Related models with catalytic branching have been extensively studied in the context of superprocesses; for example, see Dawson & Fleischmann [5], Fleischmann & Le Gall [10] or Engländer & Turaev [8]. In the discrete setting, catalytic branching random walk models have recently been considered by, for example, Carmona & Hu [4] and Döring & Roberts [6].

1.2 Main Results

In this section, after first setting up some notation, we will state our main results for BBM with catalytic branching at the origin (presenting them in the order that we will prove them).

We denote the set of particles present in the system at time t by N_t, labelling particles according to the usual Ulam-Harris convention. That is, the initial particle is labelled $\varnothing$, its two children are labelled 1 and 2, children of particle 1 are labelled 11 and 12, etc. If $u \in N_t$ then the position of particle u at time t is X^u_t and its historical path up to time t is $(X^u_s)_{0\le s\le t}$. Also, we denote the local time process of a particle $u \in N_t$ by $(L^u_s)_{0\le s\le t}$. The law of the branching process started with a single initial particle at x is denoted by P^x with the corresponding expectation E^x.

Firstly, we shall calculate the *expected* population growth.

Proposition 1 (Expected total population growth) *For $t > 0$,*

$$E(|N_t|) = 2\Phi(\beta\sqrt{t})\mathrm{e}^{\frac{\beta^2}{2}t} \sim 2\mathrm{e}^{\frac{\beta^2}{2}t} \quad \textit{as } t \to \infty,$$

where $|N_t|$ is the size of N_t and $\Phi(x) = \mathbb{P}(N(0,1) \le x) = (2\pi)^{-1/2}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\,\mathrm{d}y$ is the distribution function of a standard normal random variable.

Proposition 2 (Expected population growth rates) *For $\lambda > 0$, let $N^{\lambda t}_t := \{u \in N_t : X^u_t > \lambda t\}$ be the set of particles that have an average velocity greater than λ at time t and let $|N^{\lambda t}_t|$ be the size of $N^{\lambda t}_t$. Then*

$$E\big(|N^{\lambda t}_t|\big) = \Phi\big((\beta - \lambda)\sqrt{t}\big)\mathrm{e}^{(\frac{\beta^2}{2} - \beta\lambda)t}.$$

In particular, as $t \to \infty$,

$$\frac{1}{t}\log E\big(|N^{\lambda t}_t|\big) \to \Delta_\lambda := \begin{cases} \frac{1}{2}\beta^2 - \beta\lambda & \textit{if } \lambda < \beta \\ -\frac{1}{2}\lambda^2 & \textit{if } \lambda \ge \beta \end{cases} \qquad (1)$$

Note, the expected growth rate of particles with velocities greater than $\lambda > 0$, Δ_λ, is positive or negative according to whether λ is less than or greater than $\beta/2$, respectively. That is, the *expected* speed of the rightmost particle is $\beta/2$. (Also note, by symmetry, similar results hold throughout for particles with negative velocities.)

Next, we consider the *almost sure* asymptotic behaviour of the population.

Theorem 1 (Almost sure total population growth rate)

$$\lim_{t\to\infty} \frac{\log|N_t|}{t} = \frac{1}{2}\beta^2 \quad P\textit{-a.s.}$$

Theorem 2 (Almost sure population growth rates) *Let* $\lambda > 0$, *then*:

1. *if* $\lambda > \frac{\beta}{2}$ *then* $\lim_{t\to\infty} |N_t^{\lambda t}| = 0$ P*-a.s.*
2. *if* $\lambda < \frac{\beta}{2}$ *then* $\lim_{t\to\infty} \frac{\log |N_t^{\lambda t}|}{t} = \Delta_\lambda = \frac{1}{2}\beta^2 - \beta\lambda$ P*-a.s.*

From Theorem 2, we immediately recover the speed of the rightmost particle,

$$R_t := \sup_{u\in N_t} X_t^u, \quad t \geq 0.$$

Corollary 1 (Rightmost particle speed)

$$\lim_{t\to\infty} \frac{R_t}{t} = \frac{\beta}{2} \quad P\text{-a.s.}$$

We can also say something about the rare events of $|N_t^{\lambda t}|$ being positive when we typically do not find particles with speeds $\lambda > \frac{\beta}{2}$.

Lemma 1 (Unusually fast particles) *For* $\lambda > \frac{\beta}{2}$,

$$\lim_{t\to\infty} \frac{\log P(|N_t^{\lambda t}| \geq 1)}{t} = \Delta_\lambda,$$

with Δ_λ *as defined in* (1).

Finally, our main theorem gives a strong law of large numbers for the catalytic BBM:

Theorem 3 (SLLN) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be some continuous compactly-supported function. Then*

$$\lim_{t\to\infty} \mathrm{e}^{-\frac{\beta^2}{2}t} \sum_{u\in N_t} f\left(X_t^u\right) = M_\infty \int_{\mathbb{R}} f(x)\beta \mathrm{e}^{-\beta|x|}\,\mathrm{d}x \quad P\text{-a.s.},$$

where M_∞ *is the almost sure limit of the* P*-uniformly integrable additive martingale*

$$M_t = \sum_{u\in N_t} \exp\left\{-\beta|X_t^u| - \frac{1}{2}\beta^2 t\right\}.$$

(*Note*: *the martingale* $(M_t)_{t\geq 0}$ *will be discussed in detail in Sect*. 3.)

In principle, our results and methods could be generalised to models with multiple catalysts, although explicit calculations will require joint density of local times at the catalytic points. The finer fluctuations of the rightmost particle about the linear speed are also of interest, but will require more subtle analysis. We hope to pursue these problems in a future article.

The rest of this article is arranged as follows. In Sect. 2 we recall some basic facts regarding the local times. We also introduce a Radon-Nikodym derivative that puts a drift towards the origin onto a Brownian motion. This will be useful in the subsequent analysis of the model. In Sect. 3, we recall some standard techniques for branching processes including spines and additive martingales. Section 4 is devoted to the proofs of Propositions 1 and 2. We will prove Theorem 1 in Sect. 5, making use of the additive martingale $(M_t)_{t\geq 0}$ mentioned above. Section 6 contains the proofs of Theorem 2, Corollary 1 and Lemma 1. Finally, in Sect. 7 we give the proof of Theorem 3, this being largely based on extending the results found in Engländer, Harris & Kyprianou [9].

2 Single-Particle Results

Basic information about local times and the excursion theory can be found in many textbooks on Brownian motion (for example, see [14]). Also a good introduction is given in the paper of C. Rogers [16]. Let us recall a few basic facts.

Suppose $(X_t)_{t\geq 0}$ is a standard Brownian motion on some probability space under probability measure $\mathbb{P}$. Let $(L_t)_{t\geq 0}$ be its local time at 0. Then $(L_t)_{t\geq 0}$ satisfies

$$L_t = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^t \mathbf{1}_{\{X_s\in(-\epsilon,\epsilon)\}}\mathrm{d}s$$

for every $t\geq 0$. The next famous result is Tanaka's formula:

$$|X_t| = \int_0^t \operatorname{sgn}(X_s)\,\mathrm{d}X_s + L_t,$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x\leq 0. \end{cases}$$

In a non-rigorous way this can be thought of as Itô's formula applied to $f(x)=|x|$, where $f'(x)=\operatorname{sgn}(x)$, $f''(x)=2\delta_0(x)$ (where δ_0 is the Dirac delta function). Then one can think of L_t as $\int_0^t\delta_0(X_s)\,\mathrm{d}s$.

Another useful result is the following theorem due to Lévy.

Theorem 4 (Lévy) *Let $(S_t)_{t\geq 0}$ be the running supremum of X. That is, $S_t=\sup_{0\leq s\leq t}X_s$. Then*

$$(S_t, S_t - X_t)_{t\geq 0} \overset{d}{=} (L_t, |X_t|)_{t\geq 0}.$$

From Theorem 4 and the Reflection Principle it follows that $\forall t\geq 0$

$$L_t \overset{d}{=} S_t \overset{d}{=} |X_t| \overset{d}{=} |N(0,t)|,$$

where $N(0,t)$ is a normal random variable with mean 0 and variance t. The joint density of X_t and L_t is known as well (see for example [13]):

$$\mathbb{P}(X_t\in\mathrm{d}x, L_t\in\mathrm{d}y) = \frac{|x|+y}{\sqrt{2\pi t^3}}\exp\left\{-\frac{(|x|+y)^2}{2t}\right\}\mathrm{d}x\,\mathrm{d}y,\quad x\in\mathbb{R}, y>0. \tag{2}$$

From Theorem 4 it also follows that $(Z_t)_{t\geq 0} := (|X_t| - L_t)_{t\geq 0} = (\int_0^t \operatorname{sgn}(X_s)\mathrm{d}X_s)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}$, hence for any $\gamma\in\mathbb{R}$

$$\exp\left\{\gamma(|X_t| - L_t) - \frac{1}{2}\gamma^2 t\right\} = \exp\left\{\gamma Z_t - \frac{1}{2}\gamma^2 t\right\},\quad t\geq 0$$

is a martingale. And more generally, for $\gamma(\cdot)$ a smooth path we have the Girsanov martingale

$$W_t = \exp\left\{\int_0^t \gamma(s)\mathrm{d}Z_s - \frac{1}{2}\int_0^t\gamma^2(s)\mathrm{d}s\right\}$$

$$\overset{\text{Tanaka}}{=} \exp\left\{\int_0^t \gamma(s)\,\mathrm{sgn}(X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^t \gamma^2(s)\mathrm{d}s\right\}. \tag{3}$$

Used as the Radon-Nikodym derivative it puts the instantaneous drift $\mathrm{sgn}(X_t)\gamma(t)$ on the process $(X_t)_{t\geq 0}$. Let us restrict ourselves to the case $\gamma(\cdot) \equiv -\gamma < 0$ so that W puts the constant drift γ towards the origin on $(X_t)_{t\geq 0}$. The following is a special case of Girsanov's theorem and can be found in [3] (see "Brownian motion with alternating drift" in Appendix 1, pp. 128–129).

Proposition 3 *Let* $\mathbb{Q}$ *be the probability measure defined as*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\hat{\mathcal{F}}_t} = \exp\left\{-\gamma(|X_t| - L_t) - \frac{1}{2}\gamma^2 t\right\}, \quad t \geq 0,$$

where $(\hat{\mathcal{F}}_t)_{t\geq 0}$ *is the natural filtration of* $(X_t)_{t\geq 0}$. *Then under* $\mathbb{Q}$, $(X_t)_{t\geq 0}$ *has the transition density with respect to Lebesgue measure given by*

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}}\exp\left(\gamma(|x| - |y|) - \frac{\gamma^2}{2}t - \frac{(x-y)^2}{2t}\right) + \gamma \mathrm{e}^{-2\gamma|y|}\Phi\left(\frac{\gamma t - |x| - |y|}{\sqrt{t}}\right)$$

so that

$$\mathbb{Q}^x(X_t \in A) = \int_A p_t(x, y)\mathrm{d}y. \tag{4}$$

It also has the stationary probability measure

$$\pi(\mathrm{d}y) = \gamma \mathrm{e}^{-2\gamma|y|}\mathrm{d}y. \tag{5}$$

3 Spines and Additive Martingales

3.1 Spine Setup

In this section we give a brief overview of some main spine tools. For more details of the spine setup to be introduced, the reader is referred to Hardy and Harris [11] where all the proofs and further references can be found.

For two particles u and v we shall write $u < v$ if u is an ancestor of v. We shall also write $|u|$ for the number of ancestors of a particle u.

We let $(\mathcal{F}_t)_{t\geq 0}$ denote the natural filtration of our branching process as described in the introduction. We define $\mathcal{F}_\infty := \sigma(\bigcup_{t\geq 0}\mathcal{F}_t)$ as usual.

Let us now extend our model by identifying an infinite line of descent which we refer to as the spine and which is chosen uniformly from all the possible lines of descent. It is defined in the following way. The initial particle of the branching process begins the spine. When it splits into two new particles, one of them is chosen with probability $\frac{1}{2}$ to continue the spine. This goes on in the obvious way: whenever the particle currently in the spine splits, one of its children is chosen uniformly at random to continue the spine.

The spine is denoted by $\xi = \{\varnothing, \xi_1, \xi_2, \ldots\}$, where $\varnothing$ is the initial particle (both in the spine and in the entire branching process) and ξ_n is the particle in the $(n+1)$st generation of the spine. Furthermore, at time $t \geq 0$ we define:

- $\text{node}_t(\xi) := u \in N_t \cap \xi$ (such u is necessarily unique). That is, $\text{node}_t(\xi)$ is the particle in the spine alive at time t.
- $n_t := |\text{node}_t(\xi)|$. Thus n_t is the number of fissions that have occured along the spine by time t.
- $\xi_t := X^u_t$ for $u \in N_t \cap \xi$. So $(\xi_t)_{t\geq 0}$ is the path of the spine.

The next important step is to define a number of filtrations of our sample space, which contain different information about the process (for more details see [11]).

Definition 1 (Filtrations)

- $\mathcal{F}_t$ was defined earlier. It is the filtration which knows everything about the particles' motion and their genealogy, but it knows nothing about the spine.
- We also define $\tilde{\mathcal{F}}_t := \sigma(\mathcal{F}_t, \text{node}_t(\xi))$. Thus $\tilde{\mathcal{F}}$ has all the information about the branching process and all the information about the spine. This will be the largest filtration.
- $\mathcal{G}_t := \sigma\ (\xi_s : 0 \leq s \leq t)$. This filtration has information about the path of the spine process, but it does not contain any information about the labelling (genealogy and birth times) along the spine.
- $\tilde{\mathcal{G}}_t := \sigma\ (\mathcal{G}_t, (\text{node}_s(\xi) : 0 \leq s \leq t))$. This filtration knows everything about the spine including which particles make up the spine, but it doesn't know what is happening off the spine.

Note that $\mathcal{G}_t \subset \tilde{\mathcal{G}}_t \subset \tilde{\mathcal{F}}_t$ and $\mathcal{F}_t \subset \tilde{\mathcal{F}}_t$. We shall use these filtrations to take various conditional expectations.

We let $\tilde{P}$ be the probability measure under which the branching process is defined together with the spine. Hence $P = \tilde{P}|_{\mathcal{F}_\infty}$. We shall write $\tilde{E}$ for the expectation with respect to $\tilde{P}$.

Under $\tilde{P}$ the entire branching process (with the spine) can be described in the following way.

- The initial particle (the spine) moves like a Brownian motion.
- At instantaneous rate $\beta\delta_0(\cdot)$ it splits into two new particles.
- One of these particles (chosen uniformly at random) continues the spine. That is, it continues moving as a Brownian motion and branching at rate $\beta\delta_0(\cdot)$.
- The other particle initiates a new independent P-branching processes from the position of the split.

It is not hard to see that under $\tilde{P}$ the spine's path $(\xi_t)_{t\geq 0}$ is itself a Brownian motion. We denote by $(\tilde{L}_t)_{t\geq 0}$ its local time at 0.

Also, conditional on the path of the spine, $(n_t)_{t\geq 0}$ is a time-inhomogeneous Poisson process (or a Cox process) with instantaneous jump rate $\beta\delta_0(\xi_t)$. That is, conditional on $\mathcal{G}_t$, k splits take place along the spine by time t with probability

$$\tilde{P}(n_t = k|\mathcal{G}_t) = \frac{(\beta\tilde{L}_t)^k}{k!}\mathrm{e}^{-\beta\tilde{L}_t}.$$

The next result (for example, see [11]) is very useful in computing expectations of various quantities.

Theorem 5 (Many-to-one theorem) *Let $f(t) \in m\mathcal{G}_t$. In other words, $f(t)$ is $\mathcal{G}_t$-measurable. Suppose it has the representation*

$$f(t) = \sum_{u \in N_t} f_u(t) \mathbf{1}_{\{\text{node}_t(\xi)=u\}},$$

where $f_u(t) \in m\mathcal{F}_t$, then

$$E\left(\sum_{u \in N_t} f_u(t)\right) = \tilde{E}\left(f(t) \mathrm{e}^{\beta \tilde{L}_t}\right).$$

3.2 Martingales

Since $(\xi_t)_{t \geq 0}$ is a standard Brownian motion we can define the following $\tilde{P}$-martingale with respect to the filtration $(\mathcal{G}_t)_{t \geq 0}$ using Proposition 3:

$$\tilde{M}_t^\beta := \mathrm{e}^{-\beta|\xi_t| + \beta \tilde{L}_t - \frac{1}{2}\beta^2 t}, \quad t \geq 0. \tag{6}$$

We also define the corresponding probability measure $\tilde{Q}_\beta$ as

$$\frac{\mathrm{d}\tilde{Q}_\beta}{\mathrm{d}\tilde{P}}\bigg|_{\mathcal{G}_t} = \tilde{M}_t^\beta, \quad t \geq 0. \tag{7}$$

Then under $\tilde{Q}_\beta$, $(\xi_t)_{t \geq 0}$ has drift β towards the origin and from Proposition 3 we know its exact transition density as well as its stationary distribution.

Let us also define the martingale

$$\tilde{M}_t := 2^{n_t} \mathrm{e}^{-\beta \tilde{L}_t} \tilde{M}_t^\beta, \quad t \geq 0,$$

which is the product of two $\tilde{P}$-martingales. When they are used as the Radon-Nikodym derivatives to define a new probability measure, the first martingale has the effect of doubling the branching rate along the spine, while the second has the effect of putting the drift β towards the origin onto the spine. That is, if we define a probability measure $\tilde{Q}$ as

$$\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}\tilde{P}}\bigg|_{\tilde{\mathcal{F}}_t} = \tilde{M}_t, \quad t \geq 0 \tag{8}$$

then under $\tilde{Q}$ the branching process has the following description:

- The initial particle (the spine) moves like a Brownian motion with drift β towards the origin.
- When it is at position x it splits into two new particles at instantaneous rate $2\beta\delta_0(x)$.
- One of these particles (chosen uniformly at random) continues the spine. I.e. it continues moving as a Brownian Motion with drift β towards the origin and branching at rate $2\beta\delta_0(x)$.
- The other particle initiates an unbiased branching process (as under P) from the position of the split.

Note that although (8) only defines $\tilde{Q}$ on events in $\bigcup_{t \geq 0} \tilde{\mathcal{F}}_t$, Carathéodory's extension theorem tells that $\tilde{Q}$ has a unique extension on $\tilde{\mathcal{F}}_\infty := \sigma(\bigcup_{t \geq 0} \tilde{\mathcal{F}}_t)$ and thus (8) implicitly defines

$\tilde{Q}$ on $\tilde{\mathcal{F}}_\infty$. We then define $Q := \tilde{Q}|_{\mathcal{F}_\infty}$ so that

$$\frac{\mathrm{d}Q}{\mathrm{d}P}\Big|_{\mathcal{F}_t} = M_t := \sum_{u\in N_t} \exp\Big\{\Big(-\beta|X_t^u| + \beta L_t^u - \frac{1}{2}\beta^2 t\Big) - \beta L_t^u\Big\}$$
$$= \sum_{u\in N_t} \exp\Big\{-\beta|X_t^u| - \frac{1}{2}\beta^2 t\Big\}, \quad t \geq 0. \tag{9}$$

$(M_t)_{t\geq 0}$ will be referred to as the additive martingale. (Note, this is only a martingale when β is branching rate, otherwise additional local time terms would appear cf. Proposition 3.)

3.3 Convergence Properties of $(M_t)_{t\geq 0}$

The following theorem is a standard result for additive martingales in the study of branching processes.

Theorem 6 *$(M_t)_{t\geq 0}$ is P-uniformly integrable and $M_\infty > 0$ P-almost surely.*

Proof Recall the following measure-theoretic result, which gives Lebesgue's decomposition of Q into absolutely-continuous and singular parts. It can for example be found in the book of R. Durrett [7] (Sect. 4.3).

Lemma 2 *For events $A \in \mathcal{F}_\infty$*

$$Q(A) = \int_A \limsup_{t\to\infty} M_t \,\mathrm{d}P + Q\Big(A \cap \Big\{\limsup_{t\to\infty} M_t = \infty\Big\}\Big).$$

Also a standard zero-one law, which can be found, for example, in [12] (see Lemma 3 and the proof of Theorem 2 that follows it) tells that $P(M_\infty > 0) \in \{0, 1\}$. Thus to prove Theorem 6 it is sufficient to show that

$$\limsup_{t\to\infty} M_t < \infty \quad Q\text{-a.s.} \tag{10}$$

Let us consider the spine decomposition of M_t, another useful technique which can be found in [11]:

$$E^{\tilde{Q}}(M_t|\tilde{\mathcal{G}}_\infty) = \exp\Big\{-\beta|\xi_t| - \frac{1}{2}\beta^2 t\Big\} + \sum_{u<\mathrm{node}_t(\xi)} \exp\Big\{-\beta|\xi_{S_u}| - \frac{1}{2}\beta^2 S_u\Big\},$$

where $\{S_u : u \in \xi\}$ is the set of fission times along the spine. We refer to the first term as spine(t) and the second term as sum(t).

Recall that under $\tilde{Q}$, $(\xi_t)_{t\geq 0}$ is a Brownian Motion with drift β towards the origin and $(|\xi_t| - \tilde{L}_t)_{t\geq 0}$ is a Brownian motion with drift $-\beta$. Thus $t^{-1}\xi_t \to 0$ and $t^{-1}\tilde{L}_t \to \beta$ $\tilde{Q}$-a.s. Also spine$(t) \leq 1$ and since $\xi_{S_u} = 0$

$$\mathrm{sum}(t) = \sum_{u<\mathrm{node}_t(\xi)} \mathrm{e}^{-\beta|\xi_{S_u}| - \frac{1}{2}\beta^2 S_u} = \sum_{u<\mathrm{node}_t(\xi)} \mathrm{e}^{-\frac{1}{2}\beta^2 S_u} \leq \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{1}{2}\beta^2 S_n}, \tag{11}$$

where S_n is the nth birth on the spine. The birth process along the spine $(n_t)_{t\geq 0}$ conditional on the path of the spine is a time-inhomogeneous Poisson process (or a Cox process) with cumulative jump rate $2\beta\tilde{L}_t$. Hence, $\tilde{Q}$-almost surely, $n_t \sim 2\beta\tilde{L}_t \sim 2\beta^2 t$, and so $S_n \sim (2\beta^2)^{-1}n$.

Thus there exists some $\tilde{Q}$-a.s. finite random variable $C > 0$ such that $S_n \geq Cn$ for all n. Substituting this into (11) we get

$$\mathrm{sum}(t) \leq \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{1}{2}\beta^2 Cn}.$$

Therefore sum(t) is bounded by some $\tilde{Q}$-a.s. finite random variable. We deduce that

$$\limsup_{t\to\infty} E^{\tilde{Q}}(M_t|\tilde{\mathcal{G}}_\infty) = \limsup_{t\to\infty}\big(\mathrm{spine}(t) + \mathrm{sum}(t)\big) < \infty \quad \tilde{Q}\text{-a.s.}$$

So by Fatou's lemma, $\tilde{Q}$-almost surely,

$$E^{\tilde{Q}}\Big(\liminf_{t\to\infty} M_t|\tilde{\mathcal{G}}_\infty\Big) \leq \liminf_{t\to\infty} E^{\tilde{Q}}(M_t|\tilde{\mathcal{G}}_\infty) \leq \limsup_{t\to\infty} E^{\tilde{Q}}(M_t|\tilde{\mathcal{G}}_\infty) < \infty.$$

Then $\liminf_{t\to\infty} M_t < \infty$ $\tilde{Q}$-a.s. and hence also Q-a.s. Since $1/M_t$ is a positive Q-supermartingale, it must converge Q-a.s., hence

$$\limsup_{t\to\infty} M_t = \liminf_{t\to\infty} M_t < \infty \quad Q\text{-a.s.}$$

completing the proof of the theorem. □

The next theorem is essential in the proof of the Strong Law of Large Numbers in the last section.

Theorem 7 *For $p \in (1, 2]$, $(M_t)_{t\geq 0}$ is L^p-convergent.*

Proof We use similar proof as found in [11]. As in [11], it is sufficient to show that $E(M_t^p)$ is bounded in t. We shall actually prove a stronger statement that $E^x(M_t^p) \leq C$ for all $x \in \mathbb{R}$ and $t \geq 0$, where C is a universal constant not depending on x or t. Below we adopt common convention of using Q^x to represent both probability and expectation under probability law Q^x in order to lighten the notation.

$$\begin{aligned} E^x\big(M_t^p\big) &= E^x\big(M_t^{p-1}M_t\big) = \mathrm{e}^{-\beta|x|}Q^x\big(M_t^{p-1}\big) \leq \tilde{Q}^x\big(M_t^{p-1}\big) \\ &= \tilde{Q}^x\big(\tilde{Q}^x\big(M_t^{p-1}|\tilde{\mathcal{G}}_\infty\big)\big) \leq \tilde{Q}^x\big(\big(\tilde{Q}^x(M_t|\tilde{\mathcal{G}}_\infty)\big)^{p-1}\big) \end{aligned}$$

by Jensen's inequality. Since for $a, b \geq 0$ and $q \in (0, 1]$, $(a + b)^q \leq a^q + b^q$, we see that

$$\begin{aligned} \big(\tilde{Q}^x(M_t|\tilde{\mathcal{G}}_\infty)\big)^{p-1} &= \big(\mathrm{spine}(t) + \mathrm{sum}(t)\big)^{p-1} \\ &\leq \mathrm{e}^{-\frac{\beta^2}{2}(p-1)t-\beta(p-1)|\xi_t|} + \sum_{u<\mathrm{node}_t(\xi)} \mathrm{e}^{-\frac{\beta^2}{2}(p-1)S_u-\beta(p-1)|\xi_{S_u}|} \end{aligned}$$

And hence

$$E^x\big(M_t^p\big) \le \tilde{Q}^x\big(\mathrm{e}^{-\frac{\beta^2}{2}(p-1)t-\beta(p-1)|\xi_t|}\big) + \tilde{Q}^x\bigg(\sum_{u<\mathrm{node}_t(\xi)} \mathrm{e}^{-\frac{\beta^2}{2}(p-1)S_u-\beta(p-1)|\xi_{S_u}|}\bigg) \tag{12}$$

The first expectation is bounded by 1. The second one, since $\xi_{S_u} = 0$, satisfies

$$\begin{aligned}\tilde{Q}^x\bigg(\sum_{u<\mathrm{node}_t(\xi)} \mathrm{e}^{-\frac{\beta^2}{2}(p-1)S_u-\beta(p-1)|\xi_{S_u}|}\bigg) &= \tilde{Q}^x\bigg(\sum_{u<\mathrm{node}_t(\xi)} \mathrm{e}^{-\frac{\beta^2}{2}(p-1)S_u}\bigg)\\ &= \tilde{Q}^x\bigg(\int_0^t \mathrm{e}^{-\frac{\beta^2}{2}(p-1)s} 2\beta\,\mathrm{d}\tilde{L}_s\bigg),\end{aligned}$$

where the second equality follows from the fact that the birth process along the spine under $\tilde{Q}^x$ is Poisson with instantaneous rate $2\beta\delta_0(\cdot)$. Integrating by parts and using Fubini's theorem, we find

$$\begin{aligned}&\tilde{Q}^x\bigg(\int_0^t \mathrm{e}^{-\frac{\beta^2}{2}(p-1)s} 2\beta\,\mathrm{d}\tilde{L}_s\bigg)\\ &\quad= 2\beta\mathrm{e}^{-\frac{\beta^2}{2}(p-1)t}\tilde{Q}^x(\tilde{L}_t) + \int_0^t \beta^3(p-1)\mathrm{e}^{-\frac{\beta^2}{2}(p-1)s}\tilde{Q}^x(\tilde{L}_s)\mathrm{d}s.\end{aligned}$$

Since $(\tilde{L}_t)_{t\ge 0}$ is only increasing on the zero-set of $(\xi_t)_{t\ge 0}$, we see that $\tilde{Q}^x(\tilde{L}_t) \le \tilde{Q}(\tilde{L}_t)$, where under $\tilde{Q}$ the spine process starts from 0. Note also that $\forall t \ge 0$

$$\tilde{Q}(\tilde{L}_t) = \beta t + \tilde{Q}(|\xi_t|). \tag{13}$$

This follows from the fact that $((\tilde{L}_t - |\xi_t| - \beta t)\mathrm{e}^{\beta\tilde{L}_t-\beta|\xi_t|-\frac{\beta^2}{2}t})_{t\ge 0}$ is a $\tilde{P}$ martingale (namely, the derivative of $\tilde{P}$-martingale $(\tilde{M}_t^\gamma)_{t\ge 0}$ w.r.t. γ evaluated at $\gamma = \beta$). Hence

$$\begin{aligned}\tilde{Q}(\tilde{L}_t) &= \tilde{E}\big(\tilde{L}_t\mathrm{e}^{\beta\tilde{L}_t-\beta|\xi_t|-\frac{\beta^2}{2}t}\big)\\ &= \beta t + \tilde{E}\big(|\xi_t|\mathrm{e}^{\beta\tilde{L}_t-\beta|\xi_t|-\frac{\beta^2}{2}t}\big)\\ &= \beta t + \tilde{Q}(|\xi_t|).\end{aligned}$$

$E^{\tilde{Q}}|\xi_t|$ is bounded in t since from (5) we get $\tilde{Q}(|\xi_t|) \to \int_{-\infty}^{\infty} |x|\pi(\mathrm{d}x) = \int_{-\infty}^{\infty} |x|\beta\mathrm{e}^{-2\beta|x|}\,\mathrm{d}x < \infty$ as $t \to \infty$ and it is not hard to check using Proposition 3 that $\tilde{Q}(|\xi_t|)$ is continuous in t. Thus for some positive constant C' and for all $t \ge 0$

$$\tilde{Q}^x(\tilde{L}_t) \le \tilde{Q}(\tilde{L}_t) \le \beta t + C',$$

which tells us that

$$2\beta\mathrm{e}^{-\frac{\beta^2}{2}(p-1)t}\tilde{Q}^x(\tilde{L}_t) + \int_0^t \beta^3(p-1)\mathrm{e}^{-\frac{\beta^2}{2}(p-1)s}\tilde{Q}^x(\tilde{L}_s)\mathrm{d}s$$

is bounded uniformly in t and x. Hence the second term on the RHS of (12) is bounded by some uniform constant and this completes the proof of Theorem 7. □

4 Expected Population Growth

4.1 Asymptotic Expected Growth of $|N_t|$

We prove Proposition 1 using the Many-to-One Theorem.

Proof of Proposition 1 From Theorem 5 we have

$$E(|N_t|) = E\Bigg(\sum_{u\in N_t} 1\Bigg) = \tilde{E}\big(e^{\beta\tilde{L}_t}\big).$$

Using the fact that $\tilde{L}_t \overset{d}{=} |N(0,t)|$ it is then easy to check that

$$\begin{aligned}\tilde{E}\big(e^{\beta\tilde{L}_t}\big) &= \int_{-\infty}^{\infty} e^{\beta|x|}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\,dx \\ &= 2e^{\frac{\beta^2}{2}t}\int_0^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}(x-\beta t)^2}\,dx = 2\Phi(\beta\sqrt{t})e^{\frac{\beta^2}{2}t},\end{aligned}$$

where $\Phi(x) = \mathbb{P}(N(0,1)\le x)$. This completes the proof of Proposition 1.

Alternatively, we could find a good estimate of $\tilde{E}(e^{\beta\tilde{L}_t})$ using the change of measure from (7), which is instructive for our purposes:

$$\tilde{E}\big(e^{\beta\tilde{L}_t}\big) = \tilde{E}\big(e^{\beta\tilde{L}_t-\beta|\xi_t|-\frac{1}{2}\beta^2 t}e^{\beta|\xi_t|+\frac{1}{2}\beta^2 t}\big) = \tilde{E}\big(\tilde{M}_t^{\beta}e^{\beta|\xi_t|+\frac{1}{2}\beta^2 t}\big) = E^{\tilde{Q}_\beta}\big(e^{\beta|\xi_t|}\big)e^{\frac{1}{2}\beta^2 t}.$$

Then, using the stationary measure, from (5) we have

$$E^{\tilde{Q}_\beta}\big(e^{\beta|\xi_t|}\big) \to \int_{-\infty}^{\infty} e^{\beta|x|}\pi(dx) = \int_{-\infty}^{\infty} e^{\beta|x|}\beta e^{-2\beta|x|}\,dx = 2.$$

Thus

$$E(|N_t|) \sim 2e^{\frac{\beta^2}{2}t}.$$

□

4.2 Asymptotic Expected Behaviour of $N_t^{\lambda t}$

Let us now prove that $E(|N_t^{\lambda t}|) = \Phi((\beta-\lambda)\sqrt{t})e^{(\frac{\beta^2}{2}-\beta\lambda)t}$ and consequently

$$\lim_{t\to\infty}\frac{1}{t}\log E(|N_t^{\lambda t}|) = \Delta_\lambda,$$

where $N_t^{\lambda t} = \{u \in N_t : X_t^u > \lambda t\}$ and

$$\Delta_\lambda = \begin{cases}\frac{1}{2}\beta^2 - \beta\lambda & \text{if } \lambda < \beta\\ -\frac{1}{2}\lambda^2 & \text{if } \lambda \ge \beta.\end{cases}$$

Proof of Proposition 2 Following the same steps as in the proof of Proposition 1 above we get

$$E\big(|N_t^{\lambda t}|\big) = E\Bigg(\sum_{u\in N_t}\mathbf{1}_{\{X_t^u>\lambda t\}}\Bigg) = \tilde{E}\big(e^{\beta\tilde{L}_t}\mathbf{1}_{\{\xi_t>\lambda t\}}\big).$$

We can evaluate the latter expectation using (2):

$$\begin{aligned}\tilde{E}\left(\mathrm{e}^{\beta\tilde{L}_t}\mathbf{1}_{\{\xi_t>\lambda t\}}\right) &= \int_0^\infty\int_{\lambda t}^\infty \mathrm{e}^{\beta y}\frac{x+y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(x+y)^2}{2t}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^\infty \mathrm{e}^{\beta y}\frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{1}{2t}(\lambda t+y)^2}\,\mathrm{d}y \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{1}{2t}(y-(\beta-\lambda)t)^2}\,\mathrm{d}y\mathrm{e}^{(\frac{\beta^2}{2}-\beta\lambda)t} \\ &= \Phi\left((\beta-\lambda)\sqrt{t}\right)\mathrm{e}^{(\frac{\beta^2}{2}-\beta\lambda)t}.\end{aligned}$$

Using the facts that $\Phi(x)\to 1$ as $x\to\infty$ and $\Phi(x)\sim\frac{1}{|x|\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$ as $x\to-\infty$ it is then easy to check that

$$\lim_{t\to\infty}\frac{1}{t}\log E\left(|N_t^{\lambda t}|\right)=\Delta_\lambda. \tag{14}$$

This completes the proof of Proposition 2. Alternatively, it is possible to obtain (14) using the change of measure (7) just like in the previous proposition:

$$\begin{aligned}E\left(|N_t^{\lambda t}|\right) &= \tilde{E}\left(\mathrm{e}^{\beta\tilde{L}_t}\mathbf{1}_{\{\xi_t>\lambda t\}}\right) \\ &= E^{\tilde{Q}_\beta}\left(\mathrm{e}^{\beta|\xi_t|}\mathbf{1}_{\{\xi_t>\lambda t\}}\right)\mathrm{e}^{\frac{1}{2}\beta^2 t} \\ &= \mathrm{e}^{\frac{1}{2}\beta^2 t}\int_{\lambda t}^\infty \mathrm{e}^{\beta x}p_t(0,x)\mathrm{d}x,\end{aligned}$$

where $p_t(0,\cdot)$ is the transition density of ξ at time t under $\tilde{Q}_\beta$ as defined in Proposition 3. Substituting $p_t(0,\cdot)$ into this equation and using the estimates of tails of a normal distribution we can get (14). □

Propositions 1 and 2 can also be proved via excursion theory (for example, see [16]). The proofs that we presented here (using the change of measure) in particular suggest the importance of the additive martingale $(M_t)_{t\geq 0}$ in the study of the model. In the next section we shall see one simple application of this martingale.

5 Almost Sure Asymptotic Growth of $|N_t|$

In this section we prove Theorem 1 which says that $\log|N_t|\sim\frac{1}{2}\beta^2 t$ P-almost surely.

Proof of Theorem 1 Let us first obtain the lower bound:

$$\liminf_{t\to\infty}\frac{\log|N_t|}{t}\geq\frac{1}{2}\beta^2\quad P\text{-a.s.} \tag{15}$$

We observe that

$$M_t=\sum_{u\in N_t}\exp\left\{-\beta|X_t^u|-\frac{1}{2}\beta^2 t\right\}\leq|N_t|\mathrm{e}^{-\frac{1}{2}\beta^2 t},$$

hence $\log M_t\leq\log|N_t|-\frac{1}{2}\beta^2 t$ and so $t^{-1}\log|N_t|\geq\frac{1}{2}\beta^2+t^{-1}\log M_t$. Using the fact that $\lim_{t\to\infty}M_t>0$ P-a.s. from Theorem 6, we find that

$$\liminf_{t\to\infty}\frac{\log|N_t|}{t}\geq\frac{1}{2}\beta^2.$$

Let us now establish the upper bound:

$$\limsup_{t\to\infty} \frac{\log|N_t|}{t} \leq \frac{1}{2}\beta^2 \quad P\text{-a.s.} \tag{16}$$

We first prove (16) on integer (or other lattice) times. Take $\epsilon \in (0, 1)$. Then

$$P\big(|N_t|\mathrm{e}^{-(\frac{1}{2}\beta^2+\epsilon)t} > \epsilon\big) \leq \frac{E|N_t|\mathrm{e}^{-(\frac{1}{2}\beta^2+\epsilon)t}}{\epsilon} < \frac{2}{\epsilon}\mathrm{e}^{-\epsilon t}$$

using the Markov inequality and Proposition 1. So

$$\sum_{n=1}^{\infty} P\big(|N_n|\mathrm{e}^{-(\frac{1}{2}\beta^2+\epsilon)n} > \epsilon\big) < \infty$$
$$\Rightarrow \quad \sum_{n=1}^{\infty} P\left(\frac{\log|N_n|}{n} > \frac{1}{2}\beta^2 + \epsilon\right) < \infty.$$

Thus by the Borel-Cantelli lemma

$$P\left(\left\{\frac{\log|N_n|}{n} > \frac{1}{2}\beta^2 + \epsilon\right\} \text{ i.o.}\right) = 0.$$

Hence

$$\limsup_{n\to\infty} \frac{\log|N_n|}{n} \leq \frac{1}{2}\beta^2 + \epsilon$$

and taking the limit $\epsilon \to 0$ we get the desired result. To get the convergence over any real-valued sequence we note that $|N_t|$ is an increasing process and so

$$\frac{\log|N_t|}{t} \leq \frac{\lceil t\rceil}{t}\frac{\log|N_{\lceil t\rceil}|}{\lceil t\rceil}.$$

Hence

$$\limsup_{t\to\infty} \frac{\log|N_t|}{t} \leq \limsup_{t\to\infty} \frac{\log|N_{\lceil t\rceil}|}{\lceil t\rceil} \leq \frac{1}{2}\beta^2.$$

Combining (16) and (15) now proves Theorem 1. □

6 Almost Sure Asymptotic Behaviour of $|N_t^{\lambda t}|$

In this section we prove Theorem 2. Namely, that

$$\frac{\log|N_t^{\lambda t}|}{t} \to \Delta_\lambda \quad P\text{-a.s. if } \lambda < \frac{\beta}{2}$$

and

$$|N_t^{\lambda t}| \to 0 \quad P\text{-a.s. if } \lambda > \frac{\beta}{2}.$$

We break the proof into two parts. In Sect. 6.1 we prove the upper bound and in Sect. 6.2 the lower bound. Also in Sects. 6.3 and 6.4 we present the proofs of Lemma 1, saying that $\lim_{t\to\infty} t^{-1} P(|N_t^{\lambda t}| \geq 1) = \Delta_\lambda$ if $\lambda > \frac{\beta}{2}$, and Corollary 1, saying that $\lim_{t\to\infty} t^{-1} R_t = \frac{\beta}{2}$, where R_t is the position of the rightmost particle at time t.

6.1 Upper Bound

Lemma 3

$$\limsup_{t\to\infty}\frac{\log|N_t^{\lambda t}|}{t}\le\Delta_\lambda \quad P\text{-a.s.}$$

The upper bound can be proved in a similar way to the upper bound on $|N_t|$ (recall (16)). The main difference comes from the fact that $(|N_t^{\lambda t}|)_{t\ge0}$ is not an increasing process and so getting convergence along any real time sequence requires some extra work.

Proof Take $\epsilon>0$ and consider events

$$A_n=\Big\{\sum_{u\in N_{n+1}}\mathbf{1}_{\{\sup_{s\in[n,n+1]}X_s^u\,\ge\,\lambda n\}}>\mathrm{e}^{(\Delta_\lambda+\epsilon)n}\Big\}.$$

If we can show that $P(A_n)$ decays to 0 exponentially fast then by the Borel-Cantelli Lemma we would have $P(A_n \text{ i.o.})=0$ and that would be sufficient to get the result.

By the Markov inequality and the Many-to-one theorem (Theorem 5) we have

$$\begin{aligned}P(A_n)&\le E\Big(\sum_{u\in N_{n+1}}\mathbf{1}_{\{\sup_{s\in[n,n+1]}X_s^u\,\ge\,\lambda n\}}\Big)\mathrm{e}^{-(\Delta_\lambda+\epsilon)n}\\&=\tilde{E}\big(\mathrm{e}^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\sup_{s\in[n,n+1]}\xi_s\,\ge\,\lambda n\}}\big)\mathrm{e}^{-(\Delta_\lambda+\epsilon)n}\\&=\tilde{E}\big(\mathrm{e}^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n\,\ge\,\lambda n\}}\big)\mathrm{e}^{-(\Delta_\lambda+\epsilon)n},\end{aligned}$$

where $\bar{\xi}_n:=\sup_{s\in[n,n+1]}(\xi_s-\xi_{n+1})$ is a sequence of i.i.d. random variables equal in distribution to $\sup_{s\in[0,1]}\xi_s$ and $(\xi_t)_{t\ge0}$ is a standard Brownian motion under $\tilde{P}$.

To give an upper bound on the expectation we split it according to whether $|\xi_{n+1}|$ is greater or less than $(\lambda-\delta)(n+1)$ for some small $\delta>0$ to be chosen later.

$$\begin{aligned}&\tilde{E}\big(\mathrm{e}^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n\,\ge\,\lambda n\}}\big)\mathrm{e}^{-(\Delta_\lambda+\epsilon)n}\\&\quad=\tilde{E}\big(\mathrm{e}^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n\,\ge\,\lambda n\}}\mathbf{1}_{\{|\xi_{n+1}|\,>\,(\lambda-\delta)(n+1)\}}\big)\mathrm{e}^{-(\Delta_\lambda+\epsilon)n}\\&\qquad+\tilde{E}\big(\mathrm{e}^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n\,\ge\,\lambda n\}}\mathbf{1}_{\{|\xi_{n+1}|\,\le\,(\lambda-\delta)(n+1)\}}\big)\mathrm{e}^{-(\Delta_\lambda+\epsilon)n}.\end{aligned}\tag{17}$$

Then from Proposition 2 we have

$$\begin{aligned}&\frac{1}{n}\log\Big(\tilde{E}\big(\mathrm{e}^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n\,\ge\,\lambda n\}}\mathbf{1}_{\{|\xi_{n+1}|\,>\,(\lambda-\delta)(n+1)\}}\big)\mathrm{e}^{-(\Delta_\lambda+\epsilon)n}\Big)\\&\quad\le\frac{1}{n}\log\Big(\tilde{E}\big(\mathrm{e}^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{|\xi_{n+1}|\,>\,(\lambda-\delta)(n+1)\}}\big)\mathrm{e}^{-(\Delta_\lambda+\epsilon)n}\Big)\\&\quad=\frac{1}{n}\log\Big(2\tilde{E}\big(\mathrm{e}^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\xi_{n+1}\,>\,(\lambda-\delta)(n+1)\}}\big)\Big)-(\Delta_\lambda+\epsilon)\\&\quad\to\Delta_{\lambda-\delta}-(\Delta_\lambda+\epsilon).\end{aligned}$$

Since Δ_λ is continuous in λ, $\Delta_{\lambda-\delta}-(\Delta_\lambda+\epsilon)<0$ for δ chosen small enough and hence the first expectation in (17) decays exponentially fast. If we now let $C=\mathrm{e}^{\frac{1}{2}\beta^2+(\lambda-\delta)}$ and

$K = \frac{1}{2}\beta^2 + \beta(\lambda - \delta) - (\Delta_\lambda + \epsilon)$ then the second expectation in (17) satisfies

$$\begin{aligned}
&\tilde{E}\left(e^{\beta \tilde{L}_{n+1}} \mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n \geq \lambda n\}} \mathbf{1}_{\{|\xi_{n+1}| \leq (\lambda-\delta)(n+1)\}}\right) e^{-(\Delta_\lambda+\epsilon)n} \\
&\quad = E^{\tilde{Q}_\beta}\left(e^{\beta|\xi_{n+1}|+\frac{1}{2}\beta^2(n+1)} \mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n \geq \lambda n\}} \mathbf{1}_{\{|\xi_{n+1}| \leq (\lambda-\delta)(n+1)\}}\right) e^{-(\Delta_\lambda+\epsilon)n} \\
&\quad \leq C E^{\tilde{Q}_\beta}(\mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n \geq \lambda n\}} \mathbf{1}_{\{|\xi_{n+1}| \leq (\lambda-\delta)(n+1)\}}) e^{Kn} \\
&\quad \leq C E^{\tilde{Q}_\beta}(\mathbf{1}_{\{\bar{\xi}_n \geq \delta n + (\delta-\lambda)\}}) e^{Kn} \\
&\quad = C \tilde{Q}_\beta\left(\bar{\xi}_1 \geq \delta n + (\delta - \lambda)\right) e^{Kn}.
\end{aligned}$$

However $\tilde{Q}_\beta(\bar{\xi}_1 \geq \delta n + (\delta - \lambda))$ decays faster than exponentially in n. To see this observe that for any θ arbitrarily large

$$\tilde{Q}_\beta(\bar{\xi}_1 \geq \delta n) \leq E^{\tilde{Q}_\beta}\left(e^{\theta \bar{\xi}_1}\right) e^{-\theta \delta n},$$

where

$$\begin{aligned}
E^{\tilde{Q}_\beta}\left(e^{\theta\bar{\xi}_1}\right) &= \tilde{E}\left(e^{\theta\bar{\xi}_1} e^{-\beta|\xi_1|+\beta\tilde{L}_1-\frac{1}{2}\beta^2}\right) \leq \tilde{E}\left(e^{\theta\bar{\xi}_1+\beta\tilde{L}_1}\right) \\
&\leq \left(e^{2\theta\bar{\xi}_1}\right)^{\frac{1}{2}} \left(e^{2\beta\tilde{L}_1}\right)^{\frac{1}{2}} < \infty
\end{aligned}$$

using the Cauchy-Schwarz inequality and the fact that $\tilde{L}_1 \stackrel{d}{=} \bar{\xi}_1 \stackrel{d}{=} |N(0,1)|$ under $\tilde{P}$. Thus we have shown that the expectation in (17) and consequently $P(A_n)$ decay exponentially fast.

So by the Borel-Cantelli lemma $P(A_n \text{ i.o.}) = 0$ and $P(A_n^c \text{ ev.}) = 1$. That is,

$$\sum_{u \in N_{n+1}} \mathbf{1}_{\{\sup_{s\in[n,n+1]} X_s^u \geq \lambda n\}} \leq e^{(\Delta_\lambda+\epsilon)n} \quad \text{eventually}.$$

So there exists a P-almost surely finite time T_ϵ such that $\forall n > T_\epsilon$

$$\sum_{u \in N_{n+1}} \mathbf{1}_{\{\sup_{s\in[n,n+1]} X_s^u \geq \lambda n\}} \leq e^{(\Delta_\lambda+\epsilon)n}.$$

Then

$$\begin{aligned}
|N_t^{\lambda t}| &\leq \sum_{u \in N_{\lfloor t \rfloor+1}} \mathbf{1}_{\{\sup_{s\in[\lfloor t\rfloor,\ \lfloor t\rfloor+1]} X_s^u \geq \lambda\lfloor t\rfloor\}} \\
\Rightarrow \quad & |N_t^{\lambda t}| \leq e^{(\Delta_\lambda+\epsilon)\lfloor t\rfloor} \quad \text{for } t > T_\epsilon + 1,
\end{aligned}$$

which proves that

$$\limsup_{t\to\infty} \frac{\log|N_t^{\lambda t}|}{t} \leq \Delta_\lambda \quad P\text{-a.s.}$$

□

Remark 1 Since $|N_t^{\lambda t}|$ takes only integer values we see that for $\lambda > \frac{\beta}{2}$ the inequality

$$\limsup_{t\to\infty} \frac{\log|N_t^{\lambda t}|}{t} \leq \Delta_\lambda < 0$$

actually implies that $|N_t^{\lambda t}| \to 0$ P-a.s.

6.2 Lower Bound

Before we present the proof of the lower bound of Theorem 2 let us give a heuristic argument, which this proof will be based upon.

Take $\lambda > 0$. Suppose we are given some large time t and we want to estimate the number of particles $u \in N_t$ such that $X_t^u > \lambda t$.

Let $p \in [0, 1]$. At time pt the number of particles in the system is very roughly of size $|N_{pt}| \approx \exp(\frac{1}{2}\beta^2 pt)$ by Theorem 1 and about a half of these should lie in the upper-half plane. If we ignore any branching that takes place in the time interval $(pt, t]$ then these particles move as Brownian motion and will end up in the region $[\lambda t, \infty)$ at time t with probability approximately greater than $\exp(-\lambda^2 t/2(1-p))$.

Thus a crude estimate gives us that the number of particles at time t in the region $[\lambda t, \infty)$ is approximately greater than

$$e^{-\frac{\lambda^2}{2(1-p)}t} \times |N_{pt}| \approx e^{-\frac{\lambda^2}{2(1-p)}t} \times e^{\frac{1}{2}\beta^2 pt}.$$

The value of p which maximises this expression is

$$p^* = \begin{cases} 0 & \text{if } \lambda \geq \beta \\ 1 - \frac{\lambda}{\beta} & \text{if } \lambda < \beta \end{cases}$$

and then

$$\frac{\log(e^{-\frac{\lambda^2}{2(1-p^*)}t} \times |N_{p^*t}|)}{t} \sim \Delta_\lambda = \begin{cases} -\frac{1}{2}\lambda^2 & \text{if } \lambda \geq \beta \\ \frac{1}{2}\beta^2 - \beta\lambda & \text{if } \lambda < \beta. \end{cases}$$

Let us now use this idea to give a formal proof of the following lemma.

Lemma 4 *Take $\lambda < \frac{\beta}{2}$. Then*

$$\liminf_{t \to \infty} \frac{\log |N_t^{\lambda t}|}{t} \geq \Delta_\lambda = \frac{1}{2}\beta^2 - \beta\lambda \quad P\text{-a.s.}$$

Proof Let $N_t^+ := \{u \in N_t : X_t^u > 0\}$ be the set of particles alive at time t that lie in the upper-half plane at time t. Observe that under P, conditional on $|N_t|$,

$$|N_t^+| \stackrel{d}{=} \text{Bin}\bigg(|N_t|, \frac{1}{2}\bigg). \tag{18}$$

To see this note that if for a particle $u \in N_t$ we define the events $A_u^+ := \{X_t^u > 0\}$ and $A_u^- := \{X_t^u < 0\}$ then by symmetry (since branching only takes place at 0) the events $\bigcap_{u \in N_t} A_u^{i_u}$, where $i_u \in \{+, -\}$, are all equally likely.

Now let us take $p := 1 - \frac{\lambda}{\beta} \in (\frac{1}{2}, 1)$. For integer times n we shall consider particles in the set N_{pn}^+.

For each particle $u \in N_{pn}^+$ we choose one descendant v alive at time $n+1$, $v \in N_{n+1}$, such that $v < u111\cdots$, where we have used the Ulam-Harris labelling convention (alternatively, at each birth event we independently and uniformly at random choose to follow one of the offspring lines of descent and so on). Let $\hat{N}_{n+1}$ be a set of such descendants (so that $|\hat{N}_{n+1}| = |N_{pn}^+|$).

With this choice, for each $u \in \hat{N}_{n+1}$, paths $(X_t^u)_{t\in[pn,\, n+1]}$ correspond to independent Brownian motions (started at some unknown positions in the upper-half plane at time pn). Note that, wherever such particle u is at time pn,

$$\left\{X_s^u > \lambda s \ \forall s \in [n, n+1]\right\} \supseteq \left\{X_n^u - X_{pn}^u > \lambda(n+1) + 1, \inf_{s\in[n,n+1]} \left(X_s^u - X_n^u\right) > -1\right\} =: B_n^u$$

and for any $\eta > 0$ and n large enough

$$P\left(B_n^u\right) \geq \mathrm{e}^{(-\frac{\lambda^2}{2(1-p)}-\eta)n} = \mathrm{e}^{(-\frac{1}{2}\beta\lambda-\eta)n} =: q_n(\lambda)$$

using the tail estimate of the normal distribution.

With (18) and Theorem 1 one can check that for any $C < \frac{1}{2}$, $P(|N_{pn}^+| \geq C|N_{pn}|)$ decays faster than exponentially in n. Thus, from the Borel-Cantelli Lemma we have $|N_{pn}^+| \geq C|N_{pn}|$ for n large enough P-almost surely. So from Theorem 1 for any $\delta > 0$

$$|\hat{N}_{n+1}| = |N_{pn}^+| \geq \mathrm{e}^{(\frac{1}{2}\beta^2 p-\delta)n} \quad \text{eventually.}$$

To prove Lemma 4 we fix an arbitrary $\epsilon > 0$ and consider the events

$$B_n := \left\{ \sum_{u\in\hat{N}_{n+1}} \mathbf{1}_{B_n^u} < \mathrm{e}^{(\Delta_\lambda-\epsilon)n} \right\}.$$

We wish to show that $P(B_n \text{ i.o.}) = 0$. Now,

$$\begin{aligned}
&P\left(B_n \cap \left\{|\hat{N}_{n+1}| > \mathrm{e}^{(\frac{1}{2}\beta^2 p-\delta)n}\right\}\right) \\
&= P\left(|\hat{N}_{n+1}| > \mathrm{e}^{(\frac{1}{2}\beta^2 p-\delta)n}, \sum_{u\in\hat{N}_{n+1}} \mathbf{1}_{B_n^u} < \mathrm{e}^{(\Delta_\lambda-\epsilon)n}\right) \\
&\leq P\left(\sum_{i=1}^{\mathrm{e}^{(\frac{1}{2}\beta^2 p-\delta)n}} \mathbf{1}_{A_i} < \mathrm{e}^{(\Delta_\lambda-\epsilon)n}\right),
\end{aligned}$$

where A_i's are independent events with $P(A_i) \geq q_n(\lambda)$ for all i and n large enough. Thus for n large enough

$$\begin{aligned}
P\left(\sum_{i=1}^{\mathrm{e}^{(\frac{1}{2}\beta^2 p-\delta)n}} \mathbf{1}_{A_i} < \mathrm{e}^{(\Delta_\lambda-\epsilon)n}\right) &= P\left(\mathrm{e}^{-\sum \mathbf{1}_{A_i}} > \mathrm{e}^{-\mathrm{e}^{(\Delta_\lambda-\epsilon)n}}\right) \\
&\leq \mathrm{e}^{\mathrm{e}^{(\Delta_\lambda-\epsilon)n}} E\left(\mathrm{e}^{-\sum \mathbf{1}_{A_i}}\right) \\
&= \mathrm{e}^{\mathrm{e}^{(\Delta_\lambda-\epsilon)n}} \prod_{i=1}^{\mathrm{e}^{(\frac{1}{2}\beta^2 p-\delta)n}} E\left(\mathrm{e}^{-\mathbf{1}_{A_i}}\right) \\
&= \mathrm{e}^{\mathrm{e}^{(\Delta_\lambda-\epsilon)n}} \prod \left(1 - P(A_i)\left(1-\mathrm{e}^{-1}\right)\right) \\
&\leq \mathrm{e}^{\mathrm{e}^{(\Delta_\lambda-\epsilon)n}} \prod \left(1 - q_n(\lambda)\left(1-\mathrm{e}^{-1}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \mathrm{e}^{\mathrm{e}^{(\Delta_\lambda-\epsilon)n}} \prod \mathrm{e}^{-q_n(\lambda)(1-\mathrm{e}^{-1})} \\
&= \exp\bigl\{\mathrm{e}^{(\Delta_\lambda-\epsilon)n} - \bigl(1-\mathrm{e}^{-1}\bigr) q_n(\lambda)\mathrm{e}^{(\frac{1}{2}\beta^2 p-\delta)n}\bigr\} \\
&= \exp\bigl\{\mathrm{e}^{(\Delta_\lambda-\epsilon)n} - \bigl(1-\mathrm{e}^{-1}\bigr)\mathrm{e}^{(\Delta_\lambda-\delta-\eta)n}\bigr\}.
\end{aligned}
$$

This expression decays fast enough if we take $\delta+\eta<\epsilon$. Thus

$$
P\bigl(B_n \cap \bigl\{|\hat{N}_{n+1}| > \mathrm{e}^{(\frac{1}{2}\beta^2 p-\delta)n}\bigr\} \text{ i.o.}\bigr) = 0.
$$

And since $P(\{|\hat{N}_{n+1}| > \mathrm{e}^{(\frac{1}{2}\beta^2 p-\delta)n}\}\text{ev.}) = 1$, we get that $P(B_n \text{ i.o.}) = 0$. That is, for n large enough P-almost surely

$$
\sum_{u\in\hat{N}_{n+1}} \mathbf{1}_{B_n^u} \ge \mathrm{e}^{(\Delta_\lambda-\epsilon)n}.
$$

Hence for t large enough

$$
|N_t^{\lambda t}| = \sum_{u\in N_t} \mathbf{1}_{\{X_t^u>\lambda t\}} \ge \sum_{u\in\hat{N}_{\lfloor t\rfloor+1}} \mathbf{1}_{\{X_s^u>\lambda s\ \forall s\in[\lfloor t\rfloor,\lfloor t\rfloor+1]\}} \ge \sum_{u\in\hat{N}_{\lfloor t\rfloor+1}} \mathbf{1}_{B^u_{\lfloor t\rfloor}} \ge \mathrm{e}^{(\Delta_\lambda-\epsilon)\lfloor t\rfloor}.
$$

Thus

$$
\liminf_{t\to\infty} \frac{\log|N_t^{\lambda t}|}{t} \ge \Delta_\lambda. \qquad \square
$$

Lemmas 3 and 4 together prove Theorem 2.

6.3 Decay of $P(|N_t^{\lambda t}|\ge 1)$ in the Case $\lambda > \frac{\beta}{2}$

Theorem 2 told us that if $\lambda > \frac{\beta}{2}$ then $|N_t^{\lambda t}| \to 0$. Let us also prove that in this case

$$
\frac{\log P(|N_t^{\lambda t}|\ge 1)}{t} \to \Delta_\lambda,
$$

where Δ_λ is defined in (1).

Proof of Lemma 1 Trivially $P(|N_t^{\lambda t}|\ge 1) \le E|N_t^{\lambda t}|$. Hence by Proposition 2

$$
\limsup_{t\to\infty} \frac{\log P(|N_t^{\lambda t}|\ge 1)}{t} \le \Delta_\lambda.
$$

For the lower bound we use the same idea as in Lemma 4. Let us take

$$
p = \begin{cases} 0 & \text{if } \lambda \ge \beta \\ 1-\frac{\lambda}{\beta} & \text{if } \lambda < \beta \end{cases}
$$

Let $t>0$ be fixed. As in Sect. 6.2 for each particle $u\in N_{pt}$ we choose one descendant alive at time t so that its motion over time interval $[pt,t]$ is a Brownian motion and we let $\hat{N}_t$ be a set of such descendants (so that $\hat{N}_t \subset N_t$, $|\hat{N}_t| = |N_{pt}|$). Then for each $u\in\hat{N}_t$ wherever it is at time pt for any $\eta>0$ and t large enough we have

$$
P\bigl(|X_t^u| > \lambda t\bigr) \ge \mathrm{e}^{(-\frac{\lambda^2}{2(1-p)}-\eta)t} =: p_t(\lambda).
$$

Then

$$P\big(|N_t^{\lambda t}| \geq 1\big) \geq \frac{1}{2} P\big(|N_t^{\pm\lambda t}| \geq 1\big),$$

where $N_t^{\pm\lambda t} := \{u \in N_t : |X_t^u| > \lambda t\}$. Thus for a small $\delta > 0$ to be specified later we have

$$\begin{aligned} P\big(|N_t^{\lambda t}| \geq 1\big) &\geq \frac{1}{2} P\big(|N_t^{\pm\lambda t}| \geq 1,\ |N_{pt}| > \underbrace{\mathrm{e}^{(\frac{1}{2}\beta^2 p - \delta)t}}_{:=n_t(\delta)}\big) \\ &\geq \frac{1}{2} P\bigg(\bigcup_{u \in \hat{N}_t} \big\{|X_t^u| > \lambda t\big\},\ |N_{pt}| > n_t(\delta)\bigg) \\ &\geq \frac{1}{2}\big(1 - \big(1 - p_t(\lambda)\big)^{n_t(\delta)}\big) P\big(|N_{pt}| > n_t(\delta)\big). \end{aligned}$$

By Theorem 1, $P(|N_{pt}| > n_t(\delta)) \to 1$, so this term is well-behaved. Then

$$\begin{aligned} &\big(1 - \big(1 - p_t(\lambda)\big)^{n_t(\delta)}\big) \\ &\quad = n_t(\delta) p_t(\lambda) - \binom{n_t(\delta)}{2} p_t(\lambda)^2 + \binom{n_t(\delta)}{3} p_t(\lambda)^3 - \cdots \\ &\quad \geq n_t(\delta) p_t(\lambda) - n_t(\delta)^2 p_t(\lambda)^2 \big(1 + n_t(\delta) p_t(\lambda) + n_t(\delta)^2 p_t(\lambda)^2 + \cdots\big). \end{aligned}$$

Note that for δ and η small enough

$$n_t(\delta) p_t(\lambda) = \mathrm{e}^{(\frac{1}{2}\beta^2 p - \delta)t} \mathrm{e}^{(-\frac{\lambda^2}{2(1-p)} - \eta)t} = \mathrm{e}^{(\Delta_\lambda - \delta - \eta)t} \ll 1.$$

Hence for t large enough $P(|N_t^{\lambda t}| \geq 1) \geq (\frac{1}{2} P(|N_{pt}| > n_t(\delta)))\mathrm{e}^{(\Delta_\lambda - \delta - \eta)t} + o(\mathrm{e}^{(\Delta_\lambda - \delta - \eta)t})$ and therefore

$$\liminf_{t \to \infty} \frac{\log P(|N_t^{\lambda t}| \geq 1)}{t} \geq \Delta_\lambda.$$

This completes the proof of Lemma 1. □

6.4 The Rightmost Particle

Observe that the number of particles above the line λt grows exponentially if $\lambda < \frac{\beta}{2}$ and is eventually 0 if $\lambda > \frac{\beta}{2}$. Hence, as a corollary of Theorem 2, we get that

$$\frac{R_t}{t} \to \frac{\beta}{2} \quad P\text{-a.s.},$$

where $(R_t)_{t \geq 0}$ is the rightmost particle of the branching process.

Proof of Corollary 1 Take any $\lambda < \frac{\beta}{2}$. By Theorem 2 $|N_t^{\lambda t}| \geq 1\ \forall t$ large enough, so $R_t \geq \lambda t$ for t large enough. Thus $\liminf_{t\to\infty} t^{-1} R_t \geq \lambda$ P-a.s. Letting $\lambda \nearrow \frac{\beta}{2}$ we get

$$\liminf_{t \to \infty} \frac{R_t}{t} \geq \frac{\beta}{2} \quad P\text{-a.s.}$$

Similarly, if we take $\lambda > \frac{\beta}{2}$ then by Theorem 2 $|N_t^{\lambda t}| = 0\ \forall t$ large enough and so $R_t \le \lambda t$ for t large enough. Hence $\limsup_{t\to\infty} t^{-1} R_t \le \lambda$ P-a.s. So, letting $\lambda \searrow \frac{\beta}{2}$ we get

$$\limsup_{t\to\infty} \frac{R_t}{t} \le \frac{\beta}{2} \quad P\text{-a.s.}$$

and this proves Corollary 1. □

Note that the particles that we used to bound the rightmost particle in our proofs above suggest very different behaviours of the path history of rightmost particle in BBM with branching at the origin compared to homogeneous branching.

In the BBM model with homogeneous branching rate β, roughly speaking, for any $\epsilon > 0$, infinite lines of descent can be found that stay 'near' the line $(\sqrt{2\beta} - \epsilon)t$ for all time. See, for example, [12]. (For fine behaviour of the rightmost particle see, for example [1, 15].)

On the other hand in the BBM model with branching rate $\beta\delta_0(x)$, since branching only takes place at the origin, no particle can stay close to the straight line λt for too long for any $\lambda > 0$. The optimal way for some particle to have reached the critical line $\beta t/2$ at time T is to wait near the origin up until the time $T/2$ in order to give birth to as many particles as possible, and then at time $T/2$ one of approximately $\exp(\beta^2 T/4)$ particles will have a good chance of reaching $\beta T/2$ at time T.

7 Strong Law of Large Numbers

Recall the additive martingale $M_t = \mathrm{e}^{-\frac{\beta^2}{2}t} \sum_{u\in N_t} \mathrm{e}^{-\beta|X_t^u|}$, $t \ge 0$ from (9) and the measure $\pi(\mathrm{d}x) = \beta \mathrm{e}^{-2\beta|x|}\,\mathrm{d}x$ from Proposition 3. In this section, we shall prove Theorem 3 which says that for a continuous compactly-supported function $f(\cdot)$

$$\begin{aligned}\lim_{t\to\infty} \mathrm{e}^{-\frac{\beta^2}{2}t} \sum_{u\in N_t} f\big(X_t^u\big) &= M_\infty \int_{-\infty}^{\infty} f(x)\beta \mathrm{e}^{-\beta|x|}\,\mathrm{d}x \\ &= M_\infty \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\beta|x|}\pi(\mathrm{d}x) \quad P\text{-a.s.} \qquad (19)\end{aligned}$$

Observe that the expectation of the LHS converges to the expectation of the RHS by the Many-to-One theorem and the uniform integrability of $(M_t)_{t\ge0}$:

$$\begin{aligned}E\bigg(\mathrm{e}^{-\frac{\beta^2}{2}t} \sum_{u\in N_t} f\big(X_t^u\big)\bigg) &= \tilde{E}\big(\mathrm{e}^{-\frac{\beta^2}{2}t} f(\xi_t)\mathrm{e}^{\beta\tilde{L}_t}\big) \\ &= \tilde{E}\big(f(\xi_t)\mathrm{e}^{\beta|\xi_t|}\big(\mathrm{e}^{-\beta|\xi_t|+\beta\tilde{L}_t-\frac{\beta^2}{2}t}\big)\big) \\ &= E^{\tilde{Q}_\beta}\big(f(\xi_t)\mathrm{e}^{\beta|\xi_t|}\big) \to \int f(x)\mathrm{e}^{\beta|x|}\pi(\mathrm{d}x).\end{aligned}$$

Also the Weak Law of Large Numbers for this model has been proved by J. Engländer and D. Turaev in [8]. In particular they have given the law of M_∞.

The Strong Law of Large Numbers was proved in [9] for a large class of general diffusion processes and branching rates $\beta(x)$. In our case the branching rate is a generalised function $\beta\delta_0(x)$, which doesn't satisfy the conditions of [9]. Nevertheless we can adapt the proof to our model if we take the generalised principal eigenvalue $\lambda_c = \frac{\beta^2}{2}$ and the eigenfunctions $\phi(x) = \mathrm{e}^{-\beta|x|}$, $\tilde{\phi}(x) = \beta \mathrm{e}^{-\beta|x|}$ in [9]. Also the proof relies on the L^p convergence of the martingale $(M_t)_{t\geq 0}$ and the linear asymptotic growth of the rightmost particle which we have derived earlier in this article.

As the final remark, let us note that Theorem 3 together with the Fatou's lemma and the fact that $M_\infty > 0$ P-a.s. and $EM_\infty = 1$ gives us that

$$\liminf_{t\to\infty} \mathrm{e}^{-\frac{\beta^2}{2}t}|N_t| = 2M_\infty.$$

We conjecture that $\lim_{t\to\infty} \mathrm{e}^{-\frac{\beta^2}{2}t}|N_t| = 2M_\infty$, but as yet we have not established the convergence of $\mathrm{e}^{-\frac{\beta^2}{2}t}|N_t|$.

We now finish the article with the proof of Theorem 3.

Proof of Theorem 3 Take $B \subseteq \mathbb{R}$ to be an interval (possibly unbounded). As it will be shown later, it is sufficient to prove the theorem for functions of the form $f(x) = \mathrm{e}^{-\beta|x|}\mathbf{1}_{\{x\in B\}}$. For such an interval B let

$$U_t := \mathrm{e}^{-\frac{\beta^2}{2}t} \sum_{u\in N_t} \mathrm{e}^{-\beta|X_t^u|}\mathbf{1}_{\{X_t^u\in B\}} = \mathrm{e}^{-\frac{\beta^2}{2}t} \sum_{u\in N_t} f\big(X_t^u\big).$$

So if $B = \mathbb{R}$ then we would have $U_t = M_t$ and generally $U_t \leq M_t$. We wish to show that

$$U_t \to \pi(B)M_\infty \left(= \int f(x)\mathrm{e}^{\beta|x|}\pi(\mathrm{d}x)M_\infty\right) \quad \text{as } t\to\infty.$$

The proof can be split into three parts.

Part I

Let us take $K > 0$. At this stage it doesn't matter what K is, but in Part II of the proof we shall choose an appropriate value for it. Let $m_n := Kn$ (using the same notation as in [9]). Also fix $\delta > 0$. We first want to prove that

$$\lim_{n\to\infty} |U_{(n+m_n)\delta} - E(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta})| = 0 \quad P\text{-a.s.} \tag{20}$$

We begin with the observation that

$$\forall s,t \geq 0 \quad U_{s+t} = \sum_{u\in N_t} \mathrm{e}^{-\frac{\beta^2}{2}t} U_s^{(u)}, \tag{21}$$

where conditional on $\mathcal{F}_t$, $U_s^{(u)}$ are independent copies of U_s started from the positions X_t^u.

To prove (20) using the Borel-Cantelli lemma we need to show that for all $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\big(|U_{(n+m_n)\delta} - E(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta})| > \epsilon\big) < \infty. \tag{22}$$

Let us take any $p \in (1,2]$. Then

$$P\big(|U_{(n+m_n)\delta} - E(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta})| > \epsilon\big) \leq \frac{1}{\epsilon^p} E\big(|U_{(n+m_n)\delta} - E(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta})|^p\big).$$

Next we shall apply the following inequality, which was used in the proof of the SLLN in [9] and can also be found in [2]: if $p \in (1, 2]$ and X_i are independent random variables with $\mathbb{E}X_i = 0$ (or they are martingale differences), then

$$\mathbb{E}\left[\left|\sum_{i=1}^{n} X_i\right|^p\right] \leq 2^p \sum_{i=1}^{n} \mathbb{E}\big[|X_i|^p\big]. \tag{23}$$

Note, as the martingale M is L^2 bounded, we could take $p = 2$ throughout and simplify some computations, however the $p \in (1, 2]$ argument presented is instructive as it can be adapted to more general situations. By (21)

$$U_{s+t} - E(U_{s+t}|\mathcal{F}_t) = \sum_{u \in N_t} \mathrm{e}^{-\frac{\beta^2}{2}t}\big(U_s^{(u)} - E\big(U_s^{(u)}|\mathcal{F}_t\big)\big),$$

where conditional on $\mathcal{F}_t$, $U_s^{(u)} - E(U_s^{(u)}|\mathcal{F}_t)$ are independent with 0 mean. Thus applying (23) and Jensen's inequality we get

$$\begin{aligned} E\big(|U_{s+t} - E(U_{s+t}|\mathcal{F}_t)|^p|\mathcal{F}_t\big) &\leq 2^p \mathrm{e}^{-p\frac{\beta^2}{2}t} \sum_{u \in N_t} E\big(|U_s^{(u)} - E\big(U_s^{(u)}|\mathcal{F}_t\big)|^p|\mathcal{F}_t\big) \\ &\leq 2^p \mathrm{e}^{-p\frac{\beta^2}{2}t} \sum_{u \in N_t} E\big(2^{p-1}\big(|U_s^{(u)}|^p + |E\big(U_s^{(u)}|\mathcal{F}_t\big)|^p\big)|\mathcal{F}_t\big) \\ &\leq 2^p \mathrm{e}^{-p\frac{\beta^2}{2}t} \sum_{u \in N_t} E\big(2^{p-1}\big(|U_s^{(u)}|^p + E\big(|U_s^{(u)}|^p|\mathcal{F}_t\big)\big)|\mathcal{F}_t\big) \\ &= 2^{2p} \mathrm{e}^{-p\frac{\beta^2}{2}t} \sum_{u \in N_t} E\big(|U_s^{(u)}|^p|\mathcal{F}_t\big). \end{aligned} \tag{24}$$

Hence using (24) and recalling that E^x stands for the expectation with respect to the probability measure under which the branching process starts at position x we have

$$\begin{aligned} \sum_{n=1}^{\infty} E\big(|U_{(n+m_n)\delta} - E(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta})|^p\big) &\leq 2^{2p} \sum_{n=1}^{\infty} \mathrm{e}^{-p\frac{\beta^2}{2}\delta n} E\bigg(\sum_{u \in N_{\delta n}} E^{X^u_{\delta n}}\big[(U_{m_n\delta})^p\big]\bigg) \\ &\leq 2^{2p} \sum_{n=1}^{\infty} \mathrm{e}^{-p\frac{\beta^2}{2}\delta n} E\bigg(\sum_{u \in N_{\delta n}} E^{X^u_{\delta n}}\big[(M_{m_n\delta})^p\big]\bigg) \\ &\leq 2^{2p} \sum_{n=1}^{\infty} \mathrm{e}^{-p\frac{\beta^2}{2}\delta n} E(C|N_{\delta n}|) \\ &\leq \sum_{n=1}^{\infty} \mathrm{e}^{-p\frac{\beta^2}{2}\delta n} \mathrm{e}^{\frac{\beta^2}{2}\delta n} \times C', \end{aligned}$$

where C and C' are some positive constants and we have applied Theorem 7 and Proposition 1 in the last two inequalities. Since $p > 1$ the sum is $< \infty$. This finishes the proof of (22) and hence (20).

Part II

Let us now prove that

$$\lim_{n\to\infty}\left|E(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}) - \pi(B)M_\infty\right| = 0 \quad P\text{-a.s.} \tag{25}$$

Together with (20) this will complete the proof of Theorem 3 along lattice times for functions $f(x)$ of the form $\mathrm{e}^{-\beta|x|}\mathbf{1}_{\{x\in B\}}$.

We begin by noting that

$$\begin{aligned}
E(U_{s+t}|\mathcal{F}_t) &= E\Big(\sum_{u\in N_t}\mathrm{e}^{-\frac{\beta^2}{2}t}U_s^{(u)}|\mathcal{F}_t\Big)\\
&= \sum_{u\in N_t}\mathrm{e}^{-\frac{\beta^2}{2}t}E^{X_t^u}U_s\\
&= \sum_{u\in N_t}\mathrm{e}^{-\frac{\beta^2}{2}t}E^{X_t^u}\Big(\sum_{u\in N_s}\mathrm{e}^{-\frac{\beta^2}{2}s-\beta|X_s^u|}\mathbf{1}_{\{X_s^u\in B\}}\Big)\\
&= \sum_{u\in N_t}\mathrm{e}^{-\frac{\beta^2}{2}t}\tilde{E}^{X_t^u}\big(\mathrm{e}^{-\frac{\beta^2}{2}s-\beta|\xi_s|}\mathbf{1}_{\{\xi_s\in B\}}\mathrm{e}^{\beta\tilde{L}_s}\big)\\
&= \sum_{u\in N_t}\mathrm{e}^{-\frac{\beta^2}{2}t-\beta|X_t^u|}\tilde{Q}_\beta^{X_t^u}(\xi_s\in B)\\
&= \sum_{u\in N_t}\mathrm{e}^{-\frac{\beta^2}{2}t-\beta|X_t^u|}\int_B p_s\big(X_t^u,y\big)\mathrm{d}y,
\end{aligned}$$

where $\tilde{Q}_\beta$ and $p_\cdot(\cdot,\cdot)$ were defined in (7) and Proposition 3. Thus

$$E(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}) = \sum_{u\in N_{n\delta}}\mathrm{e}^{-\frac{\beta^2}{2}n\delta-\beta|X_{n\delta}^u|}\int_B p_{m_n\delta}\big(X_{n\delta}^u,y\big)\mathrm{d}y. \tag{26}$$

Recalling that $m_n = Kn$ where $K>0$ we have

$$E(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}) = \sum_{u\in N_{n\delta}}\mathrm{e}^{-\frac{\beta^2}{2}n\delta-\beta|X_{n\delta}^u|}\int_B p_{Kn\delta}\big(X_{n\delta}^u,y\big)\mathrm{d}y.$$

Now choose $M>\frac{\beta}{2}$ and consider events

$$C_n := \big\{|X_{n\delta}^u| < Mn\delta\ \forall u\in N_{n\delta}\big\}.$$

Then

$$\begin{aligned}
&\sum_{u\in N_{n\delta}}\mathrm{e}^{-\frac{\beta^2}{2}n\delta-\beta|X_{n\delta}^u|}\int_B p_{Kn\delta}\big(X_{n\delta}^u,y\big)\mathrm{d}y\\
&\quad= \sum_{u\in N_{n\delta}}\mathrm{e}^{-\frac{\beta^2}{2}n\delta-\beta|X_{n\delta}^u|}\int_B p_{Kn\delta}\big(X_{n\delta}^u,y\big)\mathrm{d}y\ \mathbf{1}_{C_n^c}\\
&\qquad+ \sum_{u\in N_{n\delta}}\mathrm{e}^{-\frac{\beta^2}{2}n\delta-\beta|X_{n\delta}^u|}\int_B p_{Kn\delta}\big(X_{n\delta}^u,y\big)\mathrm{d}y\ \mathbf{1}_{C_n}.
\end{aligned}$$

The first sum is 0 for n large enough by Corollary 1 (or even earlier by Theorem 2). To deal with the second sum we substitute the known transition density $p_{\cdot}(\cdot,\cdot)$:

$$\int_B p_{Kn\delta}\big(X^u_{n\delta}, y\big)\mathrm{d}y\,\mathbf{1}_{C_n}$$
$$= \int_B \frac{1}{\sqrt{2\pi Kn\delta}}\exp\left\{\beta\big(|X^u_{n\delta}| - |y|\big) - \frac{\beta^2}{2}Kn\delta - \frac{(X^u_{n\delta}-y)^2}{2Kn\delta}\right\}$$
$$+ \beta\Phi\left(\frac{\beta Kn\delta - |X^u_{n\delta}| - |y|}{\sqrt{Kn\delta}}\right)\mathrm{e}^{-2\beta|y|}\,\mathrm{d}y\,\mathbf{1}_{C_n}.$$

Then for any given $M > \frac{\beta}{2}$ we can choose $K > \frac{2M}{\beta}$ and hence

$$\int_B \frac{1}{\sqrt{2\pi Kn\delta}}\exp\left\{\beta\big(|X^u_{n\delta}| - |y|\big) - \frac{\beta^2}{2}Kn\delta - \frac{(X^u_{n\delta}-y)^2}{2Kn\delta}\right\}\mathrm{d}y\,\mathbf{1}_{C_n}$$
$$\le \exp\left\{\left(\beta M - \frac{\beta^2}{2}K\right)n\delta\right\} \times C' \to 0 \quad \text{as } n\to\infty,$$

where C' is some positive constant and

$$\int_B \beta\Phi\left(\frac{\beta Kn\delta - |X^u_{n\delta}| - |y|}{\sqrt{Kn\delta}}\right)\mathrm{e}^{-2\beta|y|}\,\mathrm{d}y\,\mathbf{1}_{C_n} \to \int_B \beta\mathrm{e}^{-2\beta|y|}\,\mathrm{d}y = \pi(B) \quad \text{as } n\to\infty$$

since $\Phi(x) \to 1$ as $x\to\infty$ and $\mathbf{1}_{C_n} \to 1$ as $n\to\infty$. Then going back to (26) we see that

$$\lim_{n\to\infty}\big|E(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}) - \pi(B)M_{n\delta}\big| = 0 \quad P\text{-a.s.}$$

and so also

$$\lim_{n\to\infty}\big|E(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}) - \pi(B)M_{\infty}\big| = 0 \quad P\text{-a.s.}$$

As it was mentioned earlier parts I and II together complete the proof of Theorem 3 along lattice times for functions of the form $f(x) = \mathrm{e}^{-\beta|x|}\mathbf{1}_B(x)$. To see this put together (20) and (25) to get that

$$\lim_{n\to\infty}|U_{(n+m_n)\delta} - \pi(B)M_\infty| = 0 \quad P\text{-a.s.}$$

That is,

$$\lim_{n\to\infty}|U_{n(K+1)\delta} - \pi(B)M_\infty| = 0 \quad P\text{-a.s.}$$

Then $K+1$ can be absorbed into δ which stayed arbitrary throughout the proof. Also as it was mentioned earlier we can easily replace functions of the form $\mathrm{e}^{-\beta|x|}\mathbf{1}_B(x)$ with any compactly-supported continuous functions. To see this we note that given a compactly-supported continuous function f and $\epsilon > 0$ we can find functions $\bar{f}^\epsilon(x)$ and $\underline{f}^\epsilon(x)$, which are linear combinations of functions of the form $\mathrm{e}^{-\beta|x|}\mathbf{1}_B(x)$ such that

$$\bar{f}^\epsilon(x) - \epsilon \le f(x) \le \bar{f}^\epsilon(x)$$

and

$$\underline{f}^\epsilon(x) \le f(x) \le \underline{f}^\epsilon(x) + \epsilon.$$

Then

$$\bar{f}^{\epsilon}(x)\beta e^{-\beta|x|} \le \big(f(x)+\epsilon\big)\beta e^{-\beta|x|}$$
$$\Rightarrow \quad \int_{-\infty}^{\infty} \bar{f}^{\epsilon}(x)\beta e^{-\beta|x|}\,dx \le \int_{-\infty}^{\infty} f(x)\beta e^{-\beta|x|}\,dx + 2\epsilon$$

and hence P-almost surely we have

$$\begin{aligned}\limsup_{n\to\infty} e^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} f\big(X^u_{n\delta}\big) &\le \limsup_{n\to\infty} e^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} \bar{f}^{\epsilon}\big(X^u_{n\delta}\big) \\ &= M_\infty \int_{-\infty}^{\infty} \bar{f}^{\epsilon}(x)\beta e^{-\beta|x|}\,dx \\ &\le M_\infty \bigg(\int_{-\infty}^{\infty} f(x)\beta e^{-\beta|x|}\,dx + 2\epsilon\bigg).\end{aligned}$$

Since ϵ is arbitrary we get

$$\limsup_{n\to\infty} e^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} f\big(X^u_{n\delta}\big) \le M_\infty \int_{-\infty}^{\infty} f(x)\beta e^{-\beta|x|}\,dx.$$

Similarly

$$\liminf_{n\to\infty} e^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} f\big(X^u_{n\delta}\big) \ge M_\infty \int_{-\infty}^{\infty} f(x)\beta e^{-\beta|x|}\,dx.$$

This completes the proof of Theorem 3 with the limit taken along lattice times. Now let us finish the proof of the theorem by extending it to the continuous-time limit.

Part III

As in the previous parts of the proof it is sufficient to consider functions of the form $f(x) = e^{-\beta|x|}\mathbf{1}_B(x)$ for intervals B.

Let us take $\epsilon > 0$ and define the following interval

$$B^{\epsilon}(x) := B \cap \bigg(-|x| - \frac{1}{\beta}\log(1+\epsilon),\ |x| + \frac{1}{\beta}\log(1+\epsilon)\bigg).$$

Note that $y \in B^{\epsilon}(x)$ iff $y \in B$ and $e^{-\beta|y|} > \frac{e^{-\beta|x|}}{1+\epsilon}$. Furthermore, for $\delta, \epsilon > 0$ let

$$\Xi^{\delta,\epsilon}_B(x) := \mathbf{1}_{\{X^u_s \in B^{\epsilon}(x)\ \forall s\in[0,\delta]\ \forall u\in N_\delta\}}$$

and

$$\xi^{\delta,\epsilon}_B(x) := E^x\big(\Xi^{\delta,\epsilon}_B(x)\big) = P^x\big(X^u_s \in B^{\epsilon}(x)\ \forall s \in [0,\delta]\ \forall u \in N_\delta\big).$$

Then for $t \in\ [n\delta,\ (n+1)\delta]$

$$U_t = e^{-\frac{\beta^2}{2}t} \sum_{u\in N_t} e^{-\beta|X^u_t|}\mathbf{1}_{\{X^u_t\in B\}}$$

$$= \sum_{u \in N_{n\delta}} \mathrm{e}^{-\frac{\beta^2}{2} n\delta} U_{t-n\delta}^{(u)} \geq \mathrm{e}^{-\frac{\beta^2}{2} n\delta} \sum_{u \in N_{n\delta}} U_{t-n\delta}^{(u)} \, \Xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big)$$

$$\geq \mathrm{e}^{-\frac{\beta^2}{2} n\delta} \sum_{u \in N_{n\delta}} \mathrm{e}^{-\frac{\beta^2}{2}\delta} \, \frac{\mathrm{e}^{-\beta|X_{n\delta}^u|}}{1+\epsilon} \, \Xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big) \tag{27}$$

because at time t there is at least one descendent of each particle alive at time $n\delta$. Let us consider the sum

$$\mathrm{e}^{-\frac{\beta^2}{2} n\delta} \sum_{u \in N_{n\delta}} \mathrm{e}^{-\beta|X_{n\delta}^u|} \Xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big).$$

Note that

$$\Xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big) \quad \text{are independent conditional on } \mathcal{F}_{n\delta}, \tag{28}$$

$$E\bigg(\mathrm{e}^{-\frac{\beta^2}{2} n\delta} \sum_{u \in N_{n\delta}} \mathrm{e}^{-\beta|X_{n\delta}^u|} \Xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big)|\mathcal{F}_{n\delta}\bigg) = \mathrm{e}^{-\frac{\beta^2}{2} n\delta} \sum_{u \in N_{n\delta}} \mathrm{e}^{-\beta|X_{n\delta}^u|} \xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big), \tag{29}$$

and

$$\lim_{n\to\infty} \mathrm{e}^{-\frac{\beta^2}{2} n\delta} \sum_{u \in N_{n\delta}} \mathrm{e}^{-\beta|X_{n\delta}^u|} \xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big) = \int \xi_B^{\delta,\epsilon}(x)\pi(\mathrm{d}x) M_\infty. \tag{30}$$

The last equation follows from the SLLN along lattice times which we already proved. Also we should point out that if we further let $\delta \to 0$, $\xi_B^{\delta,\epsilon}(x)$ will converge to $\mathbf{1}_B(x)$ (with the exception of at most two points on the boundary of interval B) and (30) will converge to $\pi(B)M_\infty$. Our next step then is to show that

$$\lim_{n\to\infty} \bigg| \mathrm{e}^{-\frac{\beta^2}{2} n\delta} \sum_{u \in N_{n\delta}} \mathrm{e}^{-\beta|X_{n\delta}^u|} \Xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big) - \mathrm{e}^{-\frac{\beta^2}{2} n\delta} \sum_{u \in N_{n\delta}} \mathrm{e}^{-\beta|X_{n\delta}^u|} \xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big) \bigg| = 0. \tag{31}$$

In view of (28) and (29) we prove this using the method of Part I. That is, we exploit the Borel-Cantelli Lemma and in order to do that we need to show that for some $p \in (1, 2]$ the following sum is finite:

$$\sum_{n=1}^{\infty} E\bigg(\bigg| \mathrm{e}^{-\frac{\beta^2}{2} n\delta} \sum_{u \in N_{n\delta}} \mathrm{e}^{-\beta|X_{n\delta}^u|} \Xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big) - E\bigg(\mathrm{e}^{-\frac{\beta^2}{2} n\delta} \sum_{u \in N_{n\delta}} \mathrm{e}^{-\beta|X_{n\delta}^u|} \Xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big) \bigg| \mathcal{F}_{n\delta}\bigg) \bigg|^p \bigg).$$

A similar argument to the one used in Part I (see (24) gives us that

$$\sum_{n=1}^{\infty} E\bigg(\bigg| \mathrm{e}^{-\frac{\beta^2}{2} n\delta} \sum_{u \in N_{n\delta}} \mathrm{e}^{-\beta|X_{n\delta}^u|} \Xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big) - E\bigg(\mathrm{e}^{-\frac{\beta^2}{2} n\delta} \sum_{u \in N_{n\delta}} \mathrm{e}^{-\beta|X_{n\delta}^u|} \Xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big) \bigg| \mathcal{F}_{n\delta}\bigg) \bigg|^p \bigg)$$

$$\leq \sum_{n=1}^{\infty} 2^{2p} \mathrm{e}^{-p\frac{\beta^2}{2} n\delta} E\bigg(\sum_{u \in N_{n\delta}} \mathrm{e}^{-\beta p|X_{n\delta}^u|} \xi_B^{\delta,\epsilon}\big(X_{n\delta}^u\big) \bigg),$$

where $\Xi_B^{\delta,\epsilon}(X_{n\delta}^u)$ is an indicator function and therefore raising it to the power p leaves it unchanged. Using once again the Many-to-One Lemma and the usual change of measure we get

$$\sum_{n=1}^{\infty} 2^{2p} \mathrm{e}^{-p\frac{\beta^2}{2}n\delta} E\bigg(\sum_{u\in N_{n\delta}} \mathrm{e}^{-\beta p|X^u_{n\delta}|} \xi_B^{\delta,\epsilon}\left(X^u_{n\delta}\right)\bigg)$$

$$\le \sum_{n=1}^{\infty} 2^{2p} \mathrm{e}^{-p\frac{\beta^2}{2}n\delta} E\bigg(\sum_{u\in N_{n\delta}} \mathrm{e}^{-\beta p|X^u_{n\delta}|}\bigg)$$

$$= \sum_{n=1}^{\infty} 2^{2p} \mathrm{e}^{-(p-1)\frac{\beta^2}{2}n\delta} E^{\tilde{Q}_\beta}\left(\mathrm{e}^{-\beta(p-1)|\xi_{n\delta}|}\right) < \infty.$$

Thus we have proved (31), which together with (30) implies that

$$\liminf_{n\to\infty} \mathrm{e}^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} \mathrm{e}^{-\beta|X^u_{n\delta}|} \Xi_B^{\delta,\epsilon}\left(X^u_{n\delta}\right) = \liminf_{n\to\infty} \mathrm{e}^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} \mathrm{e}^{-\beta|X^u_{n\delta}|} \xi_B^{\delta,\epsilon}\left(X^u_{n\delta}\right)$$

$$= \int \xi_B^{\delta,\epsilon}(x)\pi(\mathrm{d}x)M_\infty.$$

Putting this into (27) and letting $n = \lfloor \frac{t}{\delta} \rfloor$ gives us

$$\liminf_{t\to\infty} U_t \ge \frac{\mathrm{e}^{-\frac{\beta^2}{2}\delta}}{1+\epsilon} \int \xi_B^{\delta,\epsilon}(x)\pi(\mathrm{d}x)M_\infty.$$

Letting $\delta, \epsilon \searrow 0$ we get $\liminf_{t\to\infty} U_t \ge \pi(B)M_\infty$. Since the same result also holds for B^c (which is the union of at most two disjoint intervals) we can easily see that $\limsup_{t\to\infty} U_t \le \pi(B)M_\infty$. Thus

$$\lim_{t\to\infty} U_t = \pi(B)M_\infty.$$

Then the same argument as at the end of Part II of the proof extends the result for functions of the form $\mathbf{1}_B(x)\mathrm{e}^{-\beta|x|}$ to all continuous compactly-supported functions. □

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