Citation for published version:
Cox, AMG, Hobson, D & Obój, J 2014, 'Utility theory front to back: Inferring utility from agents' choices', International Journal of Theoretical and Applied Finance, vol. 17, no. 3.
https://doi.org/10.1142/S0219024914500186

DOI:
10.1142/S0219024914500186

Publication date:
2014

Document Version
Early version, also known as pre-print

Link to publication

Utility theory front to back – inferring utility from agents' choices*

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February 2012

Abstract

We pursue an inverse approach to utility theory and consumption & investment problems. Instead of specifying an agent's utility function and deriving her actions, we assume we observe her actions (i.e. her consumption and investment strategies) and ask if it is possible to derive a utility function for which the observed behaviour is optimal. We work in continuous time both in a deterministic and stochastic setting. In the deterministic setup, we find that there are infinitely many utility functions generating a given consumption pattern. In the stochastic setting of the Black-Scholes complete market it turns out that the consumption and investment strategies have to satisfy a consistency condition (PDE) if they are to come from a classical utility maximisation problem. We show further that important characteristics of the agent such as her attitude towards risk (e.g. DARA) can be deduced directly from her consumption/investment choices.

1 Introduction

The study of investment and consumption problems in finance has a long history, and there is large literature relating to these problems. In general, however, the set-up and solution of the problems take the following form: specify a

*Preprint of an article published in International Journal of Theoretical and Applied Finance, 17.3 (2014) DOI: 10.1142/S0219024914500186. ©World Scientific Publishing Company http://www.worldscientific.com/worldscinet/ijtaf

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utility function which describes the investor's 'desire' for wealth/consumption, and then solve a stochastic optimisation problem to find the optimal investment and consumption behaviour. Unfortunately, although we can postulate a simple parametric form for the utility function, and hope to deduce correspondingly simple forms for the optimal consumption/investment strategies, it is difficult to justify any claim that such a utility function accurately represents the preferences of the agent. Moreover, attempts to elicit utility functions directly are notoriously difficult, and prone to paradoxes and inconsistencies.

In this work we approach consumption/investment problems from a different, and possibly more natural, perspective. Rather than supposing that we have previously divined an investor's utility function, we suppose that we know their future consumption and investment patterns, and ask whether we can compute a corresponding utility function from the given behaviour. We believe that there are a number of reasons why this is a natural question to ask:

- consumption and investment strategies are 'observables' in that they can actually be measured from investors' actions, and therefore they are a more natural concept around which to build a model than the intangible utility function;

- the framework will allow us to see how natural behaviour patterns in the consumption/investment setting relate to properties of the underlying utility function;

- the analysis mirrors the robust approach to pricing and hedging (cf. Cox and Obłój (2010); Hobson (2010)) where one takes the vanilla option prices as observables and attempts to infer information about the prices of exotics and the dynamics of the price process of the underlying.

Our general question regarding how much information about an agent's preferences and optimality criteria we can recover from her behaviour and choices falls under the heading of *revealed preferences* in Economics. It dates back (at least) to Samuelson (1948) and is sufficiently central and important that it deserves an entry in the New Palgrave Dictionary of Economics (Richter (2008)). Other related work in the economics literature includes Green and Srivastava (1985), who consider when a given consumption may be optimal for a utility maximising investor in a one-period, finite state model, and Mas-Colell (1977), where the observed quantities are the demand functions of consumers, and the aim is to recover the consumers' preferences. In the financial literature a similar "reversed" point of view was adopted by Dybvig and Rogers (1997) who considered the recovery of an agent's utility function from a single realisation of her consumption path, working under the (strong) assumption of time homogeneity of agent's utility function.

The closest to our work are papers of Black (1968), Cox and Leland (2000) and He and Huang (1994)[1]. These three papers considered the 'inverse Merton problem' on a finite time horizon while we consider the infinite horizon case. However the results are close in spirit. One of our main results in the continuous

[1]We thank Thaleia Zariphopoulou who indicated these valuable references to us when this article neared completion. The manuscript Black (1968) was published in a modified form as Black (1988), Cox and Leland (2000) was circulated informally in the 1980s, see Editor's note therein.

time Black-Scholes market setting is that consumption/investment strategies are compatible with a utility maximisation framework if and only if they satisfy a certain PDE. We call this Black's PDE as it was first derived in Black (1968). It was then re-derived using discrete time arguments in Cox and Leland (2000). The analysis was then extended and made rigorous in a remarkable paper of He and Huang (1994). The key novelties of our paper are firstly in considering an infinite horizon, which requires dealing with the budget constraint and finiteness of the value function, and secondly in offering a unified, mathematically rigorous treatment of the problem. Moreover, we give several examples, and we show that they satisfy the sufficiency conditions of the theorem. This seems very difficult in the set-up of the pre-exisiting literature.

We make more detailed comments about the relationship between our work and that of Black (1968), Cox and Leland (2000) and He and Huang (1994) in Remark 3.8 below. We note also that the main result linking an agent's action via a PDE is similar in spirit to results in Wang (1993). However in Wang (1993) a full equilibrium model for a representative agent is considered and we have a partial equilibrium for a price taking agent.

The paper is organised as follows. In the first part of the paper, Section 2, we work in a deterministic setup. After the problem setup and a heuristic derivation of the solution, the main theorem is given in Section 2.3. In Section 2.4 we analyse what can be deduced from agents' consumption about their risk attitudes and present explicit examples. Finally, Section 2.5 comments on our assumptions and the resulting restrictions. As well as being interesting in its own right, this section brings insight to the stochastic problem, by showing what we might expect or not expect to be able to recover about u. Nonetheless, there are also fundamental differences.

In the second part of the paper, Sections 3 and 4, we work in the stochastic setting of a Black-Scholes market. Sections 3.1 and 3.2 give a heuristic derivation of the main result using the primal and the dual approach respectively. The main theorem and its proof are then given in Section 3.3.

Section 4 contains a discussion of some consequences of the main theorem and an extension. Section 4.1 asks what happens if the parameters of the Black-Scholes financial market are not known, and in particular asks if it is possible to recover the volatility and Sharpe ratio from the actions of the agent. Section 4.2 discusses the implication for reading off the risk attitudes of agents from their actions. Section 4.3 focuses on the case of time-homogenous strategies and presents explicit examples where consumption and investment are not linear in wealth. Finally, Section 4.4 discusses a relaxation of the assumptions of the main theorem and in particular includes the case when an agent's consumption and possibly his wealth are uniformly bounded. Two examples are presented.

Section 5 presents possible extensions of our work and future challenges.

Notation. We make the following notational assumptions: throughout, an optimal consumption strategy will be denoted by $c(t, w)$, where t is the current time, and w the investors wealth at time t. Similarly, an optimal investment strategy (in terms of the cash amount invested in a risky asset) will be $\pi(t, w)$. A general consumption and investment process will be (C_t, Π_t). All stochastic processes will be denoted by capital letters. Partial derivatives will be written $c_w(t, w)$ and $c_t(t, w)$. There should be no confusion over subscripts t since applied to a (upper case) process it refers to a time parameter, and applied to a (lower case) function it is a derivative.

2 Deterministic setting

2.1 Problem set-up

We begin by considering the case where there is no stochastic investment opportunity, so that we only observe the investor's consumption over an infinite horizon. More specifically, suppose we know that the investor who has wealth w at time t will consume an amount $c(t,w)\mathrm{d}t$ in the time interval $t, t+\mathrm{d}t$, where $c(t,w) \geq 0$, and suppose that we work in a situation with no interest on savings (or equivalently, all amounts are discounted back to their time-zero values). Then an investor with initial wealth $x \geq 0$ will have time-t wealth $w(t,x)$ described by:

$$\begin{aligned} w_t(t,x) &= -c(t,w(t,x)) \\ w(0,x) &= x. \end{aligned} \tag{1}$$

Further, we impose that the budget constraint $w(t,x) \geq 0$ holds for all x and t, or, in terms of $c(t,w)$, that:

$$\int_0^\infty c(t,w(t,x))\mathrm{d}t \leq x.$$

Our main concern is then the following. Suppose $c(t,w)$ is as above, and suppose we are told $c(t,w(t,x))$ is optimal for the problem:

$$v(x) = \sup_{\substack{C_t \geq 0, \\ \int_0^\infty C_t \mathrm{d}t \leq x}} \int_0^\infty u(t,C_t)\,\mathrm{d}t, \tag{2}$$

where the supremum is taken over processes $(C_t)_{t\geq 0}$. What can we infer about the function u?

2.2 Heuristics

By introducing a Lagrangian term into (2) we get:

$$\begin{aligned} v(x) &= \inf_{\lambda\geq 0}\sup_{C_t\geq 0}\left[\int_0^\infty u(t,C_t)\,\mathrm{d}t - \lambda\left(\int_0^\infty C_t\,\mathrm{d}t - x\right)\right] \\ &= \inf_{\lambda\geq 0}\sup_{C_t\geq 0}\left[\int_0^\infty (u(t,C_t) - \lambda(x)C_t)\,\mathrm{d}t + x\lambda(x)\right]. \end{aligned}$$

In the second line we write $\lambda = \lambda(x)$ to emphasise that λ will depend on the initial wealth.

Hence, for the optimal C_t, we get (supposing that u is suitably differentiable):

$$u_c(t, c(t,w(t,x))) = \lambda(x), \tag{3}$$

where the optimality of λ implies $v_x(x) = \lambda(x)$. Moreover, if we differentiate (3) again, we get:

$$u_{cc}(t,c(t,w(t,x))) = -\frac{D(x)}{\frac{\partial}{\partial x}\left[c(t,w(t,x))\right]}, \tag{4}$$

where $D(x) = -\lambda_x(x) = -v_{xx}(x)$. To then find $u_{cc}(t,c)$, we need to assume that we can recover x as a function of c and t. This will be the case if we assume that $c(t,w)$ is increasing as a function of w. It seems to be a fairly natural assumption to make in terms of investor behaviour, although note that the assumption does then imply $|w(t,x_0) - w(t,x_1)|$ is decreasing in t — that is, the 'wealth paths' corresponding to different initial wealths are 'getting closer together' as time increases. Moreover, one could imagine paths corresponding to two different initial wealths merge at some later point. To rule out such behaviour we will also impose that $\frac{\partial}{\partial x} c(t, w(t,x)) > 0$. Note as well that if this is combined with the assumption that $D(x) > 0$ (or equivalently that the value function is concave in x) we will have u concave — or a decreasing marginal utility of additional consumption. Since these all seem fairly plausible economic assumptions, we will work from this point on under these assumptions.

Finally, observe that u will necessarily be undetermined at least up to addition of a function of the form $A(t) + Bc$ and we would not expect to fully recover u from (4).

Example 2.1. CRRA: Suppose the optimal consumption strategy is: $c(t,w) = \kappa w$, for some $\kappa > 0$ — so the investor consumes a constant proportion of her wealth, and that she always consumes all of her wealth. Then it follows that $w(t,x) = x\mathrm{e}^{-\kappa t}$ and $c(t, w(t,x)) = \kappa x \mathrm{e}^{-\kappa t}$, and we can invert this to see that if she is consuming c at time t, then her initial wealth was $\frac{c}{\kappa}\mathrm{e}^{\kappa t}$. Hence we get:

$$u_{cc}(s,c) = -\frac{D\left(\frac{c}{\kappa}\mathrm{e}^{\kappa t}\right)}{\kappa \mathrm{e}^{-\kappa t}}.$$

Motivated by our knowledge of the form of the solution in the CRRA case, suppose we assume further that $D(x) = -v_{xx}(x) = \gamma x^{-\gamma-1}$ for some positive γ. Then

$$u_{cc}(t,c) \quad = \quad \frac{-\gamma\left(\frac{c}{\kappa}\mathrm{e}^{\kappa t}\right)^{-\gamma-1}}{\kappa \mathrm{e}^{-\kappa t}} = -\gamma \kappa^{\gamma} \mathrm{e}^{-\gamma\kappa t} c^{-\gamma-1}$$

Integrating this expression in c, we get:

$$\begin{aligned} u_c(t,c) &= c^{-\gamma}\kappa^{\gamma}\mathrm{e}^{-\gamma\kappa t} + \beta(t) = \left(\kappa x \mathrm{e}^{-\kappa t}\right)^{-\gamma} \kappa^{\gamma} \mathrm{e}^{-\gamma \kappa t} + \beta(t) \\ &= x^{-\gamma} + \beta(t), \end{aligned}$$

but by (3), we know this expression must be independent of t, i.e. $\beta(t) \equiv \beta$, and integrating once more, we get (provided $\gamma \neq 1$):

$$u(t,c) = \frac{c^{-\gamma+1}\kappa^{\gamma}\mathrm{e}^{-\gamma\kappa t}}{1-\gamma} + A(t) + \beta c$$

where A is an unknown function of time, and β is a constant. Note that these will not affect the choice of the optimal strategies derived from the utility function (assuming that $\int_0^\infty A(t)\,\mathrm{d}t$ is finite).

We remark that in the above example, we could have chosen any positive function $D(\cdot)$, and we would end up with the corresponding value function at time 0 given by $v_{xx}(x) = -D(x)$, with exactly the same optimal consumption paths. This suggests we can interpret the paths $(t, c(t, w(t,x)))$ as the contours where the gradient of u is constant, while the function D encodes our relative valuation of the different paths. Knowledge of consumption paths does not reveal the relative valuations of the different paths since there is no natural way of comparing the path with initial wealth x and the path with initial wealth y, simply from the specification of the optimal paths. Specifying the function $D(\cdot)$, however, does give an indication as to the relative valuation of the different paths, and in order to recover $u(t, c)$, we would expect to need to specify this function. We come back to this issue below in Section 2.4 and Example 2.8. Parallels in the setting where a risky asset is traded and an agent also has to specify her investment strategy are drawn in Remark 3.6 in Section 3.

2.3 Main results

Before we transform the above remarks into a theorem, we also note that there may be a 'maximal' solution to (1), given by $\bar{w}(t) = \sup_{x\geq 0} w(t, x)$ which may be finite for $t > 0$. In such a case, there is a 'maximal' wealth path which comes down from infinity in finite time, and since we assume we only see behaviour from time zero, we will not observe any behaviour at higher wealths, and therefore at higher consumptions than $\bar{c}(t) = \sup_{x\geq 0} c(t, w(t, x))$. Thus we will not be able to infer features of u for levels of consumption above $\bar{c}(t)$. Some thought confirms that $\bar{c}(t)$ may be finite even if $\bar{w}(t)$ is equal to infinity for all t. Mathematically, we will represent this fact by assuming the function $u(t, c)$ is constant above $\bar{c}(t)$, but note that there may be other possible choices of u which produce the same optimal choice of c and w.

Theorem 2.2. *Suppose we are given functions $\{c(t, w) : w \in \mathbb{R}_+, t \geq 0\}$ such that $c(t, 0) \equiv 0$, $c(t, w)$ is locally Lipschitz continuous and strictly increasing in w. Let $w(t, x)$ be the (unique) solution to:*

$$\begin{aligned} w_t(t, x) &= -c(t, w(t, x)) \\ w(0, x) &= x, \end{aligned} \tag{5}$$

and suppose that

$$\int_0^\infty c(t, w(t, x))\mathrm{d}t = x,$$

and the function $\frac{\partial}{\partial x} c(t, w(t, x))$ exists and is strictly positive. Then there exists a function $u(t, c)$ such that $u_c(t, c) \geq 0$ and $u_{cc}(t, c) \leq 0$, for which the problem:

$$v(x) = \sup_{\substack{C_t \geq 0: \\ \int_0^\infty C_t\,\mathrm{d}t \leq x}} \int_0^\infty u(t, C_t)\,\mathrm{d}t \tag{6}$$

is uniquely solved by the choice of consumption: $C_t = c(t, w(t, x))$ for each $x \geq 0$.

Remark 2.3. In fact, as we shall see, there is a family of solutions u for which the choice $C_t = c(t, w(t, x))$ is optimal. It should also be clear from the proof that Theorem 2.2 could be modified into an if and only if statement, albeit more technical and complicated than the current version.

Proof. Define $\bar{c}(t) = \sup_{x\geq 0} c(t, w(t,x))$, then for $0 \leq c < \bar{c}(t)$, we can find a unique x such that $c = c(t, w(t,x))$. Write this as $y(t,c)$, and note therefore that $y(t, c(t, w(t,x))) = x$, and $y(t, \bar{c}(t)) = \infty$. Also, by the assumption that $\frac{\partial}{\partial x} c(t, w(t,x))$ exists and is strictly positive, $y(t,c)$ is a differentiable function of c with derivative

$$y_c(t,c) = \frac{1}{\frac{\partial}{\partial x} c(t, w(t,x))\big|_{x=y(t,c)}}.$$

Let $D(x)$ be a strictly positive function satisfying

$$\int_x^\infty D(y)\,\mathrm{d}y < \infty, \quad \text{for every } x > 0. \tag{7}$$

Then we can define a function u by:

$$u_c(t,c) = \begin{cases} \int_c^{\bar{c}(t)} \frac{D(y(t,\kappa))}{\frac{\partial}{\partial x} c(t,w(t,x))\big|_{x=y(t,\kappa)}}\,\mathrm{d}\kappa & : c \leq \bar{c}(t) \\ 0 & : c > \bar{c}(t) \end{cases}, \tag{8}$$

where (7) ensures that the integral is finite for $c > 0$. Indeed, using the substitution $\xi = y(t,\kappa)$, we get:

$$\begin{aligned} u_c(t,c) &= \int_c^{\bar{c}(t)} \frac{D(y(t,\kappa))}{\frac{\partial}{\partial x} c(t,w(t,x))\big|_{x=y(t,\kappa)}}\,\mathrm{d}\kappa \\ &= \int_{y(t,c)}^{y(t,\bar{c}(t))=\infty} \frac{D(\xi)}{\frac{\partial}{\partial x} c(t,w(t,x))\big|_{x=\xi}} \frac{\partial}{\partial x} c(t,w(t,x))\bigg|_{x=\xi} \mathrm{d}\xi \\ &= \int_{y(t,c)}^\infty D(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{9}$$

Then $u_c(t,c) \geq 0$ and $u_{cc}(t,c) \leq 0$ so that $u(t,c)$ is strictly concave in c. Also, writing $c = c(t, w(t,x))$ we find

$$u_c(t, c(t, w(t,x))) = \int_x^\infty D(\xi)\,\mathrm{d}\xi. \tag{10}$$

Now we consider a general consumption path C_t satisfying $\int_0^\infty C_t \mathrm{d}t \leq x$. Then, using the concavity of $u(t,\cdot)$ and (10), we conclude:

$$\begin{aligned} &\int_0^\infty \left[u\left(t, C_t\right) - u(t, c(t, w(t,x)))\right] \mathrm{d}t \qquad (11) \\ &\leq \int_0^\infty u_c(t, c(t,w(t,x)))(C_t - c(t,w(t,x)))\,\mathrm{d}t \\ &= \left(\int_x^\infty D(\xi)\,\mathrm{d}\xi\right) \int_0^\infty (C_t - c(t,w(t,x)))\,\mathrm{d}t \\ &\leq 0 \end{aligned}$$

where the budget constraint gives the final step. Hence the given $c(t, w(t,x))$ is the optimal path as required. Finally, the inequality in (11) is strict, since $D(x)$ is strictly positive, unless $c(t,w(t,x)) = C_t$; hence $c(t,w(t,x))$ is also the unique optimal solution. □

2.4 Inferring risk aversion from optimal consumption

So far, we have discussed the derivation of a utility function from an initial choice of consumption behaviour. Can we extend this, and say something about some other classical methods of describing investor behaviour? For example, a natural question in this direction would be: given a set of consumption paths can we determine whether the investor has decreasing absolute/relative risk aversion?

As already observed in Section 2.2 above, it turns out that specifying the consumption paths alone is not sufficient. We present examples below of two utility functions, one with decreasing absolute risk aversion and one with increasing absolute risk aversion, which yield the same optimal consumptions paths. In essence, consumption alone does not tell us how the investor compares different wealths. This is specified by the additional function D. We can think of $D(x)$ (or more accurately $\int_x^\infty D(y)\,\mathrm{d}y$) as determining the relative weightings of different initial wealths: when $D(x)$ is large, the additional utility of an agent from a small increase in initial wealth above x is large, when $D(x)$ is small, the additional utility is also small. In what follows, we say that an agent with consumption paths $c(t, w)$ has *relative weighting of initial wealths* $D(x)$ if $D(x)$ is differentiable, satisfies (7), and the agent's utility is specified via (10).

We start with a simple observation about the role of the function D.

Note 2.4. The *Inada condition* — that is, that for all t, $u_c(t, c)$ takes all values in $[0, \infty)$, is equivalent to $\int_x^\infty D(y)\mathrm{d}y \uparrow \infty$ as $x \downarrow 0$.

We now analyse in detail the risk aversion of the investor. We concentrate on absolute risk aversion, but we observe that similar results can be derived for relative risk aversion.

Definition 2.5. For a utility function u, the absolute risk aversion is given by

$$\rho(t, c) = -\frac{u_{cc}(t, c)}{u_c(t, c)}.$$

We say that an investor is DARA (decreasing absolute risk aversion) if $\rho_c(t, c) \leq 0$ for all $t, c \geq 0$. Similarly, we say an investor is CARA (constant absolute risk aversion) or IARA (increasing absolute risk aversion) if respectively $\rho_c(t, c) = 0$ or $\rho_c(t, c) \geq 0$, for all $t, c \geq 0$.

Recall that u is recovered only up to an affine function. We should note that our normalisation $u_c(t, \infty) = 0$, or more precisely $\lim_{x\to\infty} u_c(t, c(t, w(t, x))) = 0$, which is implicit in the equation (3) and explicit in (10), and follows from the use of $\bar{c}(t)$ as a reference point in (8), has a consequence on the value of the function $\rho(t, c)$. A different reference point might change the absolute risk aversion.

Proposition 2.6. *Suppose an investor has consumption paths $c(t, w)$ and relative weighting of initial wealths $D(x)$. Then the sign of $\rho_c(t, c)$ is the same as the sign of:*

$$\frac{D_x(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} - \frac{\frac{\partial^2}{\partial x^2} c(t, w(t, x))}{\frac{\partial}{\partial x} c(t, w(t, x))} \equiv \frac{\partial}{\partial x} \ln \left(\frac{D(x)}{\frac{\partial}{\partial x} c(t, w(t, x)) \int_x^\infty D(y)\,\mathrm{d}y} \right),$$

evaluated at $x = y(t, c)$.

Corollary 2.7. *An investor is DARA if and only if:*

$$\frac{D_x(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} \le \inf_{t\ge 0} \frac{\frac{\partial^2}{\partial x^2} c(t, w(t,x))}{\frac{\partial}{\partial x} c(t, w(t,x))}, \qquad x > 0. \tag{12}$$

An investor is CARA if and only if

$$\frac{D_x(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} = \frac{\frac{\partial^2}{\partial x^2} c(t, w(t,x))}{\frac{\partial}{\partial x} c(t, w(t,x))}$$

so that in particular, the right hand side of the equation is independent of t. Finally an investor is IARA if and only if:

$$\frac{D_x(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} \ge \sup_{t\ge 0} \frac{\frac{\partial^2}{\partial x^2} c(t, w(t,x))}{\frac{\partial}{\partial x} c(t, w(t,x))}, \qquad x > 0. \tag{13}$$

Proof of Proposition 2.6. It follows from (10) that:

$$\rho(t, c(t, w(t,x))) = \frac{D(x)}{\frac{\partial}{\partial x} c(t, w(t,x)) \int_x^\infty D(y)\,\mathrm{d}y}.$$

Since $c(t, w(t,x))$ is increasing in x, $\rho(t,c)$ is increasing in c if and only if the right-hand-side of the above expression is increasing in x, if and only if the logarithm of the right-hand-side is increasing in x. □

Example 2.8. Consider the consumption function of Example 2.1, so that $c(t,w) = \kappa w$ and $c(t, w(t,x)) = \kappa x \mathrm{e}^{-\kappa t}$. Then:

$$\frac{\frac{\partial^2}{\partial x^2} c(t, w(t,x))}{\frac{\partial}{\partial x} c(t, w(t,x))} = 0.$$

If we consider a function $D(x) = \gamma x^{-\gamma-1}$ with $\gamma > 0$, then

$$\frac{D_x(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} = -\frac{1}{x} < 0,$$

so the corresponding investor is DARA. On the other hand, for the choice $D(x) = x\mathrm{e}^{-\eta x^2}$ with $\eta > 0$,

$$\frac{D_x(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} = \frac{1}{x} > 0,$$

and the investor is IARA. The case $D(x) = \mathrm{e}^{-\zeta x}$ gives a CARA investor. Note that in the last two cases we necessarily have $u(t,\infty) < \infty$, whereas in the first case the finiteness of $u(t,\infty)$ depends on the sign of $(\gamma - 1)$.

Example 2.9. The purpose of this example is to show that explicit answers may still be available beyond the CRRA case in which consumption is proportional to wealth. Again we find that knowledge of the consumption path alone is not sufficient to determine the attitude to risk.

Suppose we have a concave, increasing function $G(z)$ of class $\mathcal{C}^3$ and such that $G(0) = 0, G_z(0) = 1$ and $G(z)/z \to 0$ as $z \to \infty$. Let $w(t,x) = \frac{1}{t}G(xt)$. Then it follows that $w(0,x) = x$ and:

$$\begin{aligned} c(t, w(t,x)) &= -\frac{\partial}{\partial t}\left[\frac{1}{t}G(xt)\right] \\ &= \frac{1}{t^2}\left[G(xt) - xtG_z(xt)\right] = \frac{w}{t} - \frac{1}{t^2}G^{-1}(tw)G_z(G^{-1}(tw)) \end{aligned}$$

which is positive by concavity. In particular, we get:

$$\frac{\frac{\partial^2}{\partial x^2}c(t,w(t,x))}{\frac{\partial}{\partial x}c(t,w(t,x))} = \frac{1}{x} + t\frac{G_{zzz}(xt)}{G_{zz}(xt)}$$

One simple example of such a function is $G(z) = \ln(1+z)$, in which case we get $c(t,w) = \frac{1}{t^2}(tw + e^{-wt} - 1)$ and:

$$\frac{\frac{\partial^2}{\partial x^2}c(t,w(t,x))}{\frac{\partial}{\partial x}c(t,w(t,x))} = \frac{1}{x} - \frac{2t}{1+xt}.$$

This expression is decreasing in t, so we can conclude that

$$\inf_{t\geq 0}\frac{\frac{\partial^2}{\partial x^2}c(t,w(t,x))}{\frac{\partial}{\partial x}c(t,w(t,x))} = \lim_{t\to\infty}\left[\frac{1}{x} - \frac{2t}{1+xt}\right] = -\frac{1}{x}$$

and we see that the corresponding investor is DARA if we take $D(x) = \gamma x^{-\gamma-1}$ for $\gamma > 0$. On the other hand,

$$\sup_{t\geq 0}\frac{\frac{\partial^2}{\partial x^2}c(t,w(t,x))}{\frac{\partial}{\partial x}c(t,w(t,x))} = \lim_{t\to 0}\left[\frac{1}{x} - \frac{2t}{1+xt}\right] = \frac{1}{x}.$$

so the choice $D(x) = x\mathrm{e}^{-\eta x^2}$ for any $\eta > 0$, gives an IARA investor.

Another example arises by taking $G(z) = 1 - \mathrm{e}^{-z}$. In this case, we have $c(t,w) = \frac{w}{t} + (1-wt)t^{-2}\ln(1-wt)$, and

$$\frac{\frac{\partial^2}{\partial x^2}c(t,w(t,x))}{\frac{\partial}{\partial x}c(t,w(t,x))} = \frac{1}{x} - t.$$

As before, taking e.g. $D(x) = x\mathrm{e}^{-\eta x^2}$, gives an IARA investor. However there is no choice of $D(x)$ for which the investor will be DARA. Note that in this example, since G is bounded by 1, the investor's wealth will be below $\frac{1}{t}$ at time t, no matter how large their initial wealth.

2.5 Admissible utility functions

It is natural to ask if we can recover all utility functions u (up to addition of a function $A(t) + Bc$) from the above setup? The answer is no.

Consider for example functions of the form: $u(t,c) = U(c)$ for some increasing concave function $U(\cdot)$. Such functions correspond to optimal paths which are constant, but of course, these have infinite total consumption. Agents with finite initial wealth will try to spread the total consumption as evenly as possible across the whole time horizon, but there will be no sensible 'optimal' consumption. There may also be cases when optimal consumptions exist but are not covered by our framework. For example, one may construct utility functions for which optimal consumption paths are zero for a while and then leave zero to follow a positive path.

Our aim in Section 2 was to consider the extent to which knowledge of optimal consumption paths can be used to determine the utility in the deterministic case. To obtain a complete and coherent description we worked under plausible, but not necessary, assumptions e.g. that consumption levels are strictly increasing in current wealth. The key discovery is that in the deterministic case there is no way to compare utilities across different optimal consumption paths. We shall see that this situation is rather special, and that the picture is different in the stochastic case.

3 Stochastic Setting

We now turn to a more sophisticated version of the above problem, by considering what happens when we add the possibility of investment in a stochastic asset. Specifically, we suppose there is a risky asset P_t, where P_t is a Black-Scholes asset so that it has dynamics:

$$\frac{\mathrm{d}P_t}{P_t} = \sigma(\mathrm{d}B_t + \theta\,\mathrm{d}t) + r\,\mathrm{d}t. \tag{14}$$

Here σ is the asset volatility, $\theta > 0$ is the Sharpe ratio, and r is the interest rate, which are all assumed to be constant, and B_t a standard Brownian motion. The investor now has to choose a rate of consumption C_t and also an amount, Π_t, which is to be invested in the risky asset. Then her wealth at time t, W_t, is the solution to:

$$\mathrm{d}W_t = rW_t\mathrm{d}t - C_t\mathrm{d}t + \Pi_t\sigma(\mathrm{d}B_t + \theta\mathrm{d}t), \tag{15}$$

subject to $W_0 = x$. Where we wish to highlight the dependence on initial wealth x or strategy (C, Π) we may write this as $W_t^{x,C,\Pi}$.

The investor will specify an optimal pair of investment and consumption strategies, Π_t and C_t which attain the supremum in:

$$\sup_{C_t,\Pi_t : W_t^{C,\Pi} \geq 0} \mathbb{E}\left[\int_0^\infty u(t, C_t)\,\mathrm{d}t\right] \tag{16}$$

subject to a budget constraint $W_t \geq 0$ for all $t \geq 0$. Here u is an unknown function which we aim to find.

As usual, the above generalises to an optimal control problem, which has value function:

$$v(t, w) = \sup_{C_s,\Pi_s : W_s^{C,\Pi} \geq 0} \mathbb{E}\left[\int_t^\infty u(s, C_s)\,\mathrm{d}s \middle| W_t = w\right].$$

Standard theory tells us that for a general pair (C_t, Π_t) the process $M_t = \int_0^t u(s, C_s)\,ds + v(t, W_t)$ must be a supermartingale, and under the optimal strategy will be a martingale. Applying Itô's Lemma to M_t, we see that the relevant drift (dt) terms are:

$$u(t, C_t) + v_t(t, W_t) + v_w(t, W_t)\left[rW_t - C_t + \Pi_t\sigma\theta\right] + \frac{1}{2}v_{ww}(t, W_t)\sigma^2\Pi_t^2. \quad (17)$$

We assume that the optimal strategy takes the form $(C_t = c(t, W_t), \Pi_t = \pi(t, W_t))$. Then, by analysing this equation, and considering possible solution terms $v(t, w)$, we prove in Theorem 3.4 that there is a function $u(t, c)$ for which the pair $(c(t, w), \pi(t, w))$ is optimal if and only if these functions satisfy:

$$\frac{c(t, w)}{\pi(t, w)} - \frac{rw}{\pi(t, w)} + \frac{\sigma^2}{2}\pi_w(t, w) + \int_{\cdot}^{w} \frac{\pi_t(t, \tilde{w})}{\pi(t, \tilde{w})^2}\,\mathrm{d}\tilde{w} = \beta(t) \quad (18)$$

for some function $\beta(t)$ — in particular, the left hand side is independent of w. This consistency relationship between π and c for them to be the solution of an optimal consumption/investment problem of the type (16) was first derived by Black (1968) (published later in a modified form as Black (1988)), then by Cox and Leland (2000) and subsequently generalised and made rigorous by He and Huang (1994), see Remark 3.8 below. Before stating and proving our main result, Theorem 3.4, we give heuristic derivations of (18) using both primal and dual approaches to (16).

3.1 Heuristics: the primal approach

To motivate the condition in (18), it turns out to be instructive to look at a more general problem: we introduce a function $\Psi(t, w)$ and then consider

$$v(t, w) = \sup_{\Pi_s, C_s} \mathbb{E}\left[\int_t^\infty \left(u(s, C_s) + \Psi(s, W_s)\right)\,\mathrm{d}s \middle| W_t = w\right]. \quad (19)$$

As before our starting point is an assumption that the optimal strategy takes the form $(C_t = c(t, W_t), \Pi_t = \pi(t, W_t))$. Then, by deriving an expression for Ψ in terms of the functions $\pi(t, w), c(t, w)$, we will be able to recover the condition (18) in the special case where Ψ is a function of time alone.

In the same way that we derived (17), we can get the martingale condition corresponding to (19) which is

$$\begin{aligned}\sup_C\left[u(t, C) - v_w(t, W_t)C\right] + \sup_\Pi\left[\Pi\sigma\theta v_w(t, W_t) + \frac{1}{2}\sigma^2\Pi^2 v_{ww}(t, W_t)\right]& \\ + v_t(t, W_t) + v_w(t, W_t)rW_t + \Psi(t, W_t) = 0.& \quad (20)\end{aligned}$$

In particular, the optimal choice of Π, namely $\pi(t, w)$, should satisfy:

$$\pi(t, w) = -\frac{\theta v_w(t, w)}{\sigma v_{ww}(t, w)}, \quad (21)$$

which in turn suggests we can write:

$$v_w(t, w) = \exp\left\{A(t) - \int_{\cdot}^{w} \frac{\theta}{\sigma\pi(t, \tilde{w})}\,\mathrm{d}\tilde{w}\right\}, \quad (22)$$

where $A(t)$ is some function of t to be specified. Our aim is now to use this expression to remove terms involving the function v from (20). To this end, it will be easier to consider the derivative of (20).

Suppose that u is concave and differentiable in c and introduce the convex dual, $\tilde{u}(t,\xi) = \sup_\chi (u(t,\chi) - \xi\chi)$. Note that we have $\tilde{u}_\xi(t,\xi) = -\chi^*$, where χ^* is the choice of χ which attains the supremum.

Substituting the optimal actions $c(t,w)$ and $\pi(t,w)$ into (20), we get:

$$0 = \tilde{u}(t, v_w(t,w)) + \frac{1}{2}\pi(t,w)\sigma\theta v_w(t,w) + v_t(t,w) + v_w(t,w)rw + \Psi(t,w), \tag{23}$$

and differentiating (23) with respect to w, we obtain:

$$\begin{aligned} 0 = -v_{ww}(t,w)c(t,w) &+ \frac{1}{2}\pi(t,w)\sigma\theta v_{ww}(t,w) + \frac{1}{2}\sigma\theta v_w(t,w)\pi_w(t,w) + v_{tw}(t,w) \\ &+ v_{ww}(t,w)rw + v_w(t,w)r + \Psi_w(t,w). \end{aligned} \tag{24}$$

If we now differentiate (22) in the time variable, we see that we must have:

$$v_{tw}(t,w) = \left[A'(t) + \int_{\cdot}^{w} \frac{\theta}{\sigma\pi(t,\tilde{w})^2}\pi_t(t,\tilde{w})\,\mathrm{d}\tilde{w}\right] v_w(t,w), \tag{25}$$

so that (24) becomes:

$$\begin{aligned} 0 = &\left[\frac{1}{2}\pi(t,w)\sigma\theta + rw - c(t,w)\right] v_{ww}(t,w) + \Psi_w(t,w) \\ &+ \left[\frac{1}{2}\sigma\theta\pi_w(t,w) + r + A'(t) + \int_{\cdot}^{w} \frac{\theta}{\sigma\pi(t,\tilde{w})^2}\pi_t(t,\tilde{w})\,\mathrm{d}\tilde{w}\right] v_w(t,w). \end{aligned}$$

Finally, dividing through by $v_w(t,w)$ we have:

$$\begin{aligned} -\frac{\Psi_w(t,w)}{v_w(t,w)} = &\left[\frac{1}{2}\pi(t,w)\sigma\theta + rw - c(t,w)\right]\frac{v_{ww}(t,w)}{v_w(t,w)} + \frac{1}{2}\sigma\theta\pi_w(t,w) \\ &+ r + A'(t) + \int_{\cdot}^{w} \frac{\theta}{\sigma\pi(t,\tilde{w})^2}\pi_t(t,\tilde{w})\,\mathrm{d}\tilde{w}, \end{aligned}$$

and using (21), we get:

$$\begin{aligned} \Psi_w(t,w)\exp&\left\{-A(t) + \int_{\cdot}^{w} \frac{\theta}{\sigma\pi(t,\tilde{w})}\,\mathrm{d}\tilde{w}\right\} \\ &= \frac{1}{2}\theta^2 + \frac{rw\theta}{\sigma\pi(t,w)} - \frac{\theta c(t,w)}{\pi(t,w)\sigma} - \frac{1}{2}\sigma\theta\pi_w(t,w) - r - A'(t) \\ &\quad - \int_{\cdot}^{w} \frac{\theta}{\sigma\pi(t,\tilde{w})^2}\pi_t(t,\tilde{w})\,\mathrm{d}\tilde{w}. \end{aligned} \tag{26}$$

Since we have not yet fixed the constant of integration $A(t)$, we are free to choose this. Because our main interest is in the case where $\Psi(t,w)$ is independent of w, it follows that we are interested in cases where we can make the expression on the right-hand side of (26) disappear, which will occur whenever the expression

$$\frac{\theta^2}{2} + \frac{r\theta w}{\sigma\pi(t,w)} - \frac{\theta c(t,w)}{\sigma\pi(t,w)} - \frac{\sigma\theta}{2}\pi_w(t,w) - r - \int_1^w \frac{\theta}{\sigma\pi(t,\tilde{w})^2}\pi_t(t,\tilde{w})\,\mathrm{d}\tilde{w}$$

is independent of w. Differentiating once more in w, and rearranging, we see that this is equivalent to π, c satisfying:

$$\pi_t(t,w) = -\frac{\sigma^2}{2}\pi(t,w)^2\pi_{ww}(t,w) + (c(t,w) - rw)\pi_w(t,w) - \pi(t,w)c_w(t,w) + r\pi(t,w). \tag{27}$$

This is Black's equation (Black (1968, Equation (9)), see also Cox and Leland (2000, Equation (47)) and He and Huang (1994, Equation (26))). Equivalently, defining $R(t,w) := \frac{c(t,w)}{\pi(t,w)}$, we have that R solves:

$$R_w(t,w) = \frac{r}{\pi(t,w)} - \frac{1}{\pi(t,w)^2}\left(\pi_t(t,w) + rw\pi_w(t,w)\right) - \frac{\sigma^2}{2}\pi_{ww}(t,w), \tag{28}$$

and integrating we arrive at Black's PDE in integrated form (18)

$$\int_1^w \frac{\pi_t(t,\xi)}{(\pi(t,\xi))^2}\mathrm{d}\xi + \frac{\sigma^2}{2}\pi_w(t,w) + \frac{c(t,w)}{\pi(t,w)} - r\frac{w}{\pi(t,w)} = \beta(t) \tag{29}$$

for some function $\beta(t)$, independent of w.

3.2 Heuristics: the dual approach

We now give a second derivation of Black's equation using a dual approach to the consumption/investment problem. We will use this approach to give our main theorem below.

For the problem (16) we can rewrite the budget constraint as

$$\mathbb{E}\left[\int_0^\infty C_t Z_t \mathrm{d}t\right] = x,$$

where $(Z_t)_{t\geq 0}$ is the state-price density process and is given by

$$Z_t = \exp\left(-rt - \theta B_t - \frac{\theta^2}{2}t\right).$$

With this formulation the problem becomes to find

$$\sup_{C_t}\mathbb{E}\left[\int_0^\infty u(t,C_t)\mathrm{d}t - \lambda\left(\int_0^\infty C_t Z_t \mathrm{d}t - x\right)\right]$$

for an appropriate Lagrange multiplier $\lambda = \lambda(x)$. This expression is bounded by $\lambda x + \mathbb{E}[\int_0^\infty \tilde{u}(t,\lambda Z_t)\mathrm{d}t]$ and for optimality we must have that $u_c(t,C_t) = \lambda Z_t$ so that writing $I(t,\cdot)$ for the inverse in space of $u_c(t,\cdot)$ we deduce that the optimal consumption takes the form $C_t = I(t,\lambda Z_t)$.

Now assume that the optimal strategy is a given function $c = c(t,w)$ of time and wealth so that $C_t = c(t,W_t)$. It follows that $W_t = f(t,\lambda Z_t)$ for some $f = f(t,z)$ which depends on the (now unknown u) through $f = c^{-1}\circ I$.

Then, by Itô's Lemma,

$$\begin{aligned}\mathrm{d}W_t &= \lambda f_z(t,\lambda Z_t)\mathrm{d}Z_t + f_t(t,\lambda Z_t)\mathrm{d}t + (1/2)f_{zz}(t,\lambda Z_t)\lambda^2\mathrm{d}\langle Z\rangle_t \\ &= -\theta\lambda Z_t f_z \mathrm{d}B_t + \{f_t + (1/2)\theta^2\lambda^2 Z^2 f_{zz} - r\lambda Z_t f_z\}\mathrm{d}t.\end{aligned}$$

Comparing this with the wealth dynamics (15) we have

$$\sigma\pi(t,f(t,z)) = -\theta z f_z(t,z), \tag{30}$$

$$rf(t,z) - c(t,f(t,z)) + \theta\sigma\pi(t,f(t,z)) = f_t(t,z) + (1/2)\theta^2 z^2 f_{zz}(t,z) - rzf_z(t,z). \tag{31}$$

Then $\sigma \pi_w f_z = -\theta f_z - \theta z f_{zz}$ and $\sigma \pi_t + \sigma \pi_w f_t = -\theta z f_{tz}$ so that

$$\theta^2 z^2 f_{zz} = (\theta + \sigma \pi_w)\sigma \pi, \tag{32}$$

$$f_{tz}/f_z = \pi_t/\pi + \pi_w f_t/\pi. \tag{33}$$

Putting (32) into (31) gives

$$rf - c + \sigma\theta\pi = \frac{\sigma}{2}\pi(\theta + \sigma\pi_w) + f_t + \frac{r\sigma}{\theta}\pi. \tag{34}$$

Differentiating with respect to z, dividing by $f_z = -\sigma\pi/(\theta z)$, using (33) and (34) to eliminate f_{tz} and f_t and multiplying by π we finally get

$$\begin{aligned}(r - c_w(t,w))\pi(t,w) - \frac{\sigma^2}{2}(\pi(t,w))^2\pi_{ww}(t,w) - \pi_t(t,w)\\ - r\pi_w(t,w)w + \pi_w(t,w)c(t,w) = 0,\end{aligned} \tag{35}$$

which is Black's PDE (27).

Remark 3.1. Our motivation so far has been the following: we have supposed that both the consumption and investment functions have been stated for all times and wealths, and we have derived the consistency condition (29) as a necessary condition that these functions must satisfy. However, the above calculations also suggest an alternative way of viewing the setup. Suppose instead our agent specifies her consumption (at all times and wealths), and her initial investment strategies at all wealths (i.e. $\{\pi(0,w)\}_{w\geq 0}$). Then, under the assumption that the agent is a utility maximiser, we can solve the parabolic PDE (35) to deduce $\pi(t,w)$ at times $t \geq 0$. Note that the utility function itself is bypassed in the sense that we do not need to specify it to deduce $\pi(t,w)$. This was one of the motivating observations for Black (1968).

3.3 Main results

Given a pair of processes $(C,\Pi) \equiv (C_s,\Pi_s)_{s\geq 0}$ define the associated wealth process $(W_s^{x,C,\Pi})_{s\geq 0}$ for initial wealth x by

$$W_s^{x,C,\Pi} = x + \int_0^s \Pi_u \sigma(\mathrm{d}B_u + \theta \mathrm{d}u) + \int_0^s (rW_u^{x,C,\Pi} - C_u)\mathrm{d}u. \tag{36}$$

We say that C,Π is admissible if (36) admits a strong solution $W_s^{x,C,\Pi}$ with $W_s^{x,C,\Pi} \geq 0$ for all s and we write $\mathcal{A} = \mathcal{A}(x)$ for the space of admissible strategies. Note that if C,Π is admissible then, writing W for $W^{x,C,\Pi}$,

$$\mathrm{d}(Z_s W_s) = Z_s(\sigma\Pi_s - \theta W_s)\mathrm{d}B_s - Z_s C_s \mathrm{d}s. \tag{37}$$

Hence $(Z_s W_s)_{s\geq 0}$ is a non-negative supermartingale so that if $W_s = 0$ then, for $t \geq s$, $\mathbb{E}[W_t Z_t|\mathcal{F}_s] \leq W_s Z_s = 0$ and hence $W_t = 0$ almost surely. Thus zero is absorbing for any admissible strategy.

We suppose we are given functions $c = c(t,w)$ and $\pi = \pi(t,w)$ and we aim to find, where possible, u such that $C_t = c(t, W_t^{x,C,\Pi}), \Pi_t = \pi(t, W_t^{x,C,\Pi})$ is optimal for (16). We start by defining the class of utility functions we consider and imposing further assumptions on our inputs. We focus here on the "regular" case which yields a clean simple statement of the main result. Possible extensions which relax the assumptions on u, π and c are discussed in Section 4.4.

Recall that a function $\phi(t,x)$ is *locally Hölder continuous* on a set D if, for every $(t,x) \in D$, there is some neighbourhood U of (t,x), and some $\alpha \in (0,1]$ such that

$$\sup_{(t',x'),(t,x)\in U} \frac{|\phi(t,x)-\phi(t',x')|^2}{(|x-x'|^2+|t-t'|)^\alpha} < \infty.$$

Note that a function which is locally-Hölder continuous is jointly continuous.

Definition 3.2. We say that a function $u : [0,\infty)^2 \to [-\infty,\infty)$ is a *regular utility function* if for any $t \geq 0$, $u(t,\cdot)$ is twice continuously differentiable, strictly concave and strictly increasing, and satisfies the Inada condition: $u_c(t,0)=\infty$ and $u_c(t,\infty)=0$. Further, $I(t,\cdot)$ defined to be the inverse in space of u_c (so that $u_c(t,I(t,z))=z$) is such that I_z is locally Hölder continuous on $(0,\infty)^2$.

If a utility function u is given we denote the set of admissible strategies for which the reward in (16) is well defined by $\mathcal{A}^u(x) = \{(C,\Pi) \in \mathcal{A}(x) : \mathbb{E}\int_0^\infty u(t,C_t)^+ \mathrm{d}t < \infty \text{ or } \mathbb{E}\int_0^\infty u(t,C_t)^- \mathrm{d}t < \infty\}$.

Definition 3.3. We say that (c,π) is a *regular consumption/investment pair* if

- for each $t \geq 0$, $c(t,0)=0$ and $c(t,\cdot)$ is strictly increasing, unbounded and differentiable with $c_w(t,w)$ locally Hölder continuous on $(0,\infty)^2$.
- for each $t \geq 0$, $\pi(t,0)=0$, $\pi(t,\cdot)$ is strictly positive and $\int_0^1 \mathrm{d}\xi/\pi(t,\xi) = \infty = \int_1^\infty \mathrm{d}\xi/\pi(t,\xi)$. Further, $\pi=\pi(t,w)$ is continuously differentiable in both arguments on $(0,\infty)^2$.

Finally, c, π are such that the SDE

$$\mathrm{d}W_t^x = \pi(t,W_t^x)\sigma(\mathrm{d}B_t+\theta \mathrm{d}t) + (rW_t^x - c(t,W_t^x))\mathrm{d}t, \qquad W_0^x = x, \tag{38}$$

has a strong solution.

When we want to emphasize the dependence on c and π we denote the solution to (38) by $W^x = W^{x,c,\pi}$.

Assuming that (c,π) is a regular consumption/investment pair define $Y(t,c)$ to be the inverse to $c(t,w)$ so that $Y(t,c(t,w))=w$. Suppose further that c,π satisfy (29) and let $A(t) = -\frac{\theta}{\sigma}\int_0^t \beta(s)\mathrm{d}s + (\frac{\theta^2}{2}-r)t$, and define $F(t,w)$ by

$$F(t,w) = e^{A(t)} \exp\left\{-\frac{\theta}{\sigma}\int_1^w \frac{\mathrm{d}\xi}{\pi(t,\xi)}\right\}. \tag{39}$$

By assumption, for $t>0$, $F(t,0)=\infty$ and $F(t,\infty)=0$. For each t, $F(t,w)$ is $\mathcal{C}^{1,2}$ and decreasing in w, so we can define its inverse $f=F^{-1}$ such that $f(t,F(t,w))=w$ and $F(t,f(t,z))=z$.

Finally set $H(t,c)=\int_1^c F(t,Y(t,b))\mathrm{d}b$. Note that we have

$$f_z(t,z)F_w(t,f(t,z)) = 1; \tag{40}$$

$$f_t(t,z)+f_z(t,z)F_t(t,f(t,z)) = 0; \tag{41}$$

$$f_z(t,z)F_{ww}(t,f(t,z))+f_{zz}(t,z)F_w(t,f(t,z))^2 = 0, \tag{42}$$

and that f is $\mathcal{C}^{1,2}$.

Theorem 3.4. *For any $x > 0$, the following two are equivalent:*

(i) $c(t, W_t^x)$ *and* $\pi(t, W_t^x)$ *achieve a finite maximum in the problem*

$$\max_{C,\Pi\in\mathcal{A}^u(x)} \mathbb{E}\left[\int_0^\infty u(t, C_t)\mathrm{d}t\right], \tag{43}$$

for a regular utility function u, as in Definition 3.2, for which

$$\exists\lambda > 0 \text{ such that } x = \mathbb{E}\left[\int_0^\infty Z_t I(t, \lambda Z_t)\mathrm{d}t\right]. \tag{44}$$

(ii) $c(t, w), \pi(t, w)$ *are a regular consumption/investment pair, as in Definition 3.3,* $c(t, w), \pi(t, w)$ *satisfy* (29) *on* $(0, \infty)^2$ *and*

$$\mathbb{E}\left[\int_0^\infty Z_t c(t, W_t^x)\mathrm{d}t\right] = x, \tag{45}$$

and for some $0 < x_0 \le x$, $\mathbb{E}[|H(t, c(t, W_t^{x_0}))|] < \infty$ *for almost all* $t \ge 0$ *and* $\int_0^\infty \mathbb{E}[H(t, c(t, W_t^{x_0})) - h(t)]^+\mathrm{d}t < \infty$*, where* $h(t) = \mathbb{E}[H(t, c(t, W_t^{x_0}))]$.

Moreover, we then have $u_c(t, c) = H_c(t, c)$, $\mathcal{A}^u(x) = \mathcal{A}(x)$ *and in (i) one may take* $u(t, c) = H(t, c) - h(t)$.

Remark 3.5. In (*ii*) it is equivalent to use $f(t, F(0, x)Z_t)$ in place of W_t^x throughout. This condition may be easier to check.

Remark 3.6. It is interesting to observe the analogy with the deterministic setup considered in Section 2. There, given an agent's consumption, we recovered their utility function as $u_c(t, c) = \tilde{F}(t, y(t, c))$, where $y(t, c)$ was the inverse of consumption and $\tilde{F}(x) = \int_x^\infty D(s)\mathrm{d}s$ was an arbitrary absolutely continuous decreasing non-negative function. In Theorem 3.4 above, we recover the utility function in the same form $u_c(t, c) = F(t, Y(t, c))$ but now F is uniquely specified in terms of the agent's investment strategy coupled with the discounting term $A(t)$ read off from Black's equation (29).

Remark 3.7. There are close parallels between different conditions in (*i*) and (*ii*):

- The fundamental point of the theorem is the equivalence between (29) and optimality of c, π for the problem (43). If (29) fails, c, π may still be optimal but for a more general problem of the type (19).
- The integrability conditions on $\pi(t, w)$: $\int_0^1 \mathrm{d}\xi/\pi(t, \xi) = \infty = \int_1^\infty \mathrm{d}\xi/\pi(t, \xi)$ correspond to the Inada condition on u.
- Equations (44) and (45) are essentially the same and encode the budget constraint. We show below that if $\mathbb{E}[\int_0^\infty Z_t c(t, W_t^x)\mathrm{d}t] > x$ then (C, Π) is not admissible. Conversely, if $\mathbb{E}[\int_0^\infty Z_t c(t, W_t^x)\mathrm{d}t] = x - \delta$ for $\delta > 0$ then c, π is typically not optimal. Indeed, if $\tilde{c}(t, w) = c(t, w) + \delta e^{-s}/Z_s$ then (by Theorem III.9.4 of Karatzas and Shreve (1998)) there exists a process $\tilde{\Pi}$ such that $\tilde{c}, \tilde{\Pi}$ is admissible, and achieves a strictly higher expected utility of consumption over time in (43).

In general the assumptions of Theorem 3.4 may be non-trivial to verify. However we provide a wide class of examples where they hold, see Lemma 4.3 below. In Section 4.4, we shall also discuss some ways in which the conditions of the theorem may be relaxed.

Remark 3.8. The focus in Cox and Leland (2000) and He and Huang (1994) is on a problem in which (43) is replaced by maximisation of expected utility of consumption and terminal wealth over a finite horizon $[0, T]$. These papers give analogues of Theorem 3.4 above in this setting ((Cox and Leland, 2000, Proposition 3), (He and Huang, 1994, Theorems 1&3)). Cox and Leland (2000) develop discrete-time arguments, and pass to the limit without full justification to deduce the continuous-time result. He and Huang (1994) work directly in continuous time, and give a rigorous derivation of the results.

In many respects the infinite horizon problem is more natural than the finite horizon version, but it introduces new difficulties related in particular to the budget constraint (44)–(45) and finiteness of the value function (43). He and Huang (1994) comment that their analysis could extend to an infinite horizon with the additional condition $\mathbb{E}[Z_t W_t] \to 0$ as $t \to \infty$ which, under all their assumptions, implies our budget constraint (45). However this does not seem to be so immediate due to the important integrability restrictions on c in He and Huang (1994). Further, the well-posedness and finiteness of the expected utility is not discussed.

In addition to the finite horizon/infinite horizon distinction, we believe our approach has the advantage of a mathematically rigorous and unified *if and only if* statement with a straightforward proof which should be appealing to a contemporary reader. Our proof of Theorem 3.4 is based on the dual approach. In contrast (He and Huang, 1994, Theorem 1) use a primal approach to prove results for the forward problem ((i) implies (ii) in our theorem), and a dual approach for the inverse problem (He and Huang, 1994, Theorem 3). The mixing of primal and dual techniques can easily lead to incompatibilities between sets of assumptions, and for this reason He and Huang (1994) do not have an if and only if statement. For example, the assumptions of Theorem 3 in He and Huang (1994) are easily satisfied by consumption and investment strategies which are proportional to wealth and which result from CRRA utility, see Example 3.9 below. However, taking parameters which correspond to risk aversion $\gamma > 1$ means that the value function behaves as $\frac{1}{1-\gamma} x^{1-\gamma}$ and does not satisfy the polynomial growth restriction required for their Theorem 1. The authors seemed to have been aware of such instances, see Footnote 20 therein.

We note also that we are able to make less restrictive assumptions than previous works. In particular, both Cox and Leland (2000) and He and Huang (1994) assumed stronger growth and differentiability properties on c and π. Further, these properties were imposed as standing assumptions for their theorems whereas in one direction we deduce these properties from the regularity of u. Moreover, our setup allows us to obtain a general class of actions for which we can verify the assumptions, see Lemma 4.3 below, which includes interesting examples. This seems very difficult in the set-up of He and Huang (1994).

Nevertheless, we stress that He and Huang (1994) remains a very impressive paper, with many contributions which are beyond the scope of this work. In particular, they considered a more general setup than we do in that they allowed the stock price P_t to be a generic diffusion (local volatility) process and c and

π to depend on the state (i.e. P_t) as well.

Proof of Theorem 3.4. We first show that $(ii) \Rightarrow (i)$.
We take $u(t,c) = H(t,c) - h(t)$ which is strictly increasing and strictly concave. We have $u_c(t,0) = F(t,Y(t,0)) = F(t,0) = \infty$, $u_c(t,\infty) = F(t,Y(t,\infty)) = F(t,\infty) = 0$ and $u_{cc}(t,l) = F_w(t,Y(t,l))/c_w(t,Y(t,l))$ is continuous on $(0,\infty)$. In addition, $I(t,z) = c(t,f(t,z))$ and, as observed above, f is well defined and $\mathcal{C}^{1,2}$ on $(0,\infty)^2$. In consequence, $I_z(t,z)$ is locally Hölder continuous and u is a regular utility function of Definition 3.2.

Let $\lambda = \lambda(x) = F(0,x)$ and set $W_t = f(t,\lambda(x)Z_t)$, so that $W_0 = f(0,\lambda(x)) = x = W_0^{x,\pi,c}$. We now show that $W_t = W_t^{x,\pi,c}$.

Note that by construction we have $u_c(s,c(s,W_s)) = F(s,W_s) = \lambda(x)Z_s$. By Itô's Lemma

$$
\begin{aligned}
\mathrm{d}W_t =& \lambda(x)f_z(t,\lambda(x)Z_t)\mathrm{d}Z_t + f_t(t,\lambda(x)Z_t)\mathrm{d}t + \frac{\lambda(x)^2}{2}f_{zz}(t,\lambda Z_t)d\langle Z\rangle_t \\
=& -\theta\lambda(x)Z_t f_z(t,\lambda(x)Z_t)\mathrm{d}B_t \\
&+ \left(f_t(t,\lambda(x)Z_t) - r\lambda(x)Z_t f_z(t,\lambda(x)Z_t) + \frac{\theta^2\lambda(x)^2}{2}Z_t^2 f_{zz}(t,\lambda(x)Z_t)\right)\mathrm{d}t.
\end{aligned}
$$

Then W_t is a strong solution to (38) provided that for $w = f(t,z)$,

$$-\theta z f_z(t,z) = \sigma\pi(t,w)$$

and

$$\theta\sigma\pi(t,w) + rw - c(t,w) = f_t(t,z) - rzf_z(t,z) + \frac{1}{2}\theta^2 z^2 f_{zz}(t,z).$$

For the first of these, using $z = F(t,w)$ and the definition of F and (40), we have

$$-\theta z f_z(t,z) = -\theta\frac{F(t,w)}{F_w(t,w)} = \sigma\pi(t,w).$$

For the second, using also (41) and (42),

$$
\begin{aligned}
& f_t(t,z) - rzf_z(t,z) + \frac{1}{2}\theta^2 z^2 f_{zz}(t,z) \\
=& \quad -f_z(t,z)\left(F_t(t,w) + rF(t,w) + \frac{\theta^2}{2}\frac{F(t,w)^2F_{ww}(t,w)}{F_w(t,w)^2}\right) \\
=& \quad \frac{-F(t,w)}{F_w(t,w)}\left(A'(t) + \frac{\theta}{\sigma}\int_1^w \frac{\pi_t(t,\xi)}{(\pi(t,\xi))^2}d\xi + r + \frac{1}{2}\left[\theta^2 + \sigma\theta\pi_w(t,w)\right]\right) && (46) \\
=& \quad \frac{\sigma\pi(t,w)}{\theta}\left(\theta^2 + \frac{\theta rw}{\sigma\pi(t,w)} - \frac{\theta c(t,w)}{\sigma\pi(t,w)}\right) \\
=& \quad rw + \theta\sigma\pi(t,w) - c(t,w). && (47)
\end{aligned}
$$

We thus conclude that W_t^x and W_t are strong solutions to the same SDE, (38). By Karatzas and Shreve (1991, Theorem 5.2.5), we therefore have $W_t^x = W_t = f(t,\lambda(x)Z_t)$ for all $t \geq 0$ a.s. with $\lambda(x) = F(0,x)$.

For the rest of the proof, with slight abuse of notational conventions, let us write $c_t^x := c(t,W_t^x)$. It follows that $c_t^x = c(t,f(t,F(0,x)Z_t))$ a.s. and in particular $c_s^x \leq c_s^y$ for $0 < x < y$. Further, since $u_c(t,c(t,f(t,z))) = F(t,f(t,z)) = z$, we have $u_c(t,c_t^x) = \lambda(x)Z_t$, so that $c_t^x = I(t,\lambda(x)Z_t)$ and

$$\tilde{u}(t,\lambda(x)Z_t) = u(t,c_t^x) - \lambda(x)Z_t c_t^x, \tag{48}$$

where $\tilde{u}$ is the convex dual of u. It follows that (44) is simply (45).

By the assumption $\mathbb{E}[|u(t, c(t, W_t^{x_0}))|] \leq \mathbb{E}[|H(t, c(t, W_t^{x_0}))|] + |h(t)| < \infty$ and

$$\mathbb{E}[u(t, c(t, W_t^{x_0}))] = \mathbb{E}[H(t, c(t, W_t^{x_0}))] - h(t) = 0.$$

Using the hypothesis $\mathbb{E}[\int_0^\infty [u(t, c(t, W_t^{x_0}))]^+ \mathrm{d}t] < \infty$ we obtain

$$\mathbb{E}[\int_0^\infty u(t, c(t, W_t^{x_0}))\mathrm{d}t] = 0.$$

For $x > x_0$ we write

$$u(t, c_t^x) \leq \tilde{u}(t, \lambda(x_0)Z_t) + \lambda(x_0)Z_t c_t^x = u(t, c_t^{x_0}) + \lambda(x_0)(c_t^x - c_t^{x_0})Z_t$$

and hence

$$\mathbb{E}\int_0^\infty u(t, c_t^x)^+ \mathrm{d}t \leq \mathbb{E}\int_0^\infty u(t, c_t^{x_0})^+ \mathrm{d}t + \lambda(x_0)\mathbb{E}\int_0^\infty Z_t c_t^x \mathrm{d}t < \infty.$$

Hence $\mathbb{E}[\int_0^\infty u(t, c_t^x)\mathrm{d}t]$ is well defined and non-negative.

Consider now arbitrary $C, \Pi \in \mathcal{A}(x)$. From (37) we have

$$0 \leq W_t Z_t = x + \int_0^t Z_s(\sigma\Pi_s - \theta W_s)\mathrm{d}B_s - \int_0^t Z_s C_s \mathrm{d}s.$$

It follows that

$$0 \leq \int_0^t Z_s C_s \mathrm{d}s \leq x + \int_0^t Z_s(\sigma\Pi_s - \theta W_s)\mathrm{d}B_s.$$

In particular $\int_0^t Z_s(\Pi_s - \theta W_s)\mathrm{d}B_s \geq -x$ is bounded below and hence is a supermartingale. We also conclude that for each t, $\mathbb{E}[\int_0^t Z_s C_s \mathrm{d}s] \leq x$ and hence

$$\mathbb{E}\left[\int_0^\infty Z_s C_s \mathrm{d}s\right] \leq x. \tag{49}$$

It follows that, with $\lambda = \lambda(x)$,

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty u(s, C_s)^+ \mathrm{d}s\right] &\leq x\lambda + \mathbb{E}\left[\int_0^\infty (u(s, C_s) - \lambda Z_s C_s)^+ \mathrm{d}s\right] \\
&\leq x\lambda + \mathbb{E}\left[\int_0^\infty \tilde{u}(s, \lambda Z_s)^+ \mathrm{d}s\right] \\
&= x\lambda + \mathbb{E}\left[\int_0^\infty (u(s, c_s^x) - \lambda Z_s c_s^x)^+ \mathrm{d}s\right] \\
&\leq x\lambda + \mathbb{E}\left[\int_0^\infty u(s, c_s^x)^+ \mathrm{d}s\right] < \infty,
\end{aligned}$$

where we used (48). In consequence, $\mathcal{A}(x) = \mathcal{A}^u(x)$. Once we know the expectations exist we proceed with a standard argument:

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty u(s, C_s)\mathrm{d}s\right] &\leq \lambda x + \mathbb{E}\left[\int_0^\infty u(s, C_s) - \lambda Z_s C_s \mathrm{d}s\right] \\
&\leq \lambda x + \mathbb{E}\left[\int_0^\infty \tilde{u}(s, \lambda Z_s)\mathrm{d}s\right] .
\end{aligned} \tag{50}$$

Further, from (45) and (48) it is immediate that there is equality throughout (50) for $C_s = c_s^x$ and $\Pi_s = \pi(s, W_s^x)$ which shows that these are optimal.

We come now to the other implication: $(i) \Rightarrow (ii)$.
Take λ as in (44) and let $C_s := I(s, \lambda Z_s)$. In particular, $\tilde{u}(s, \lambda Z_s) = u(s, C_s) - \lambda Z_s C_s \geq u(s, c_s^x) - \lambda Z_s c_s^x$ and hence

$$u(s, c_s^x)^- \geq (\lambda Z_s(c_s^x - C_s) + u(s, C_s))^- \geq u(s, C_s)^- - \lambda Z_s(c_s^x - C_s)^+.$$

Rearranging and integrating we have

$$\mathbb{E}\left[\int_0^\infty u(s, C_s)^- \mathrm{d}s\right] \leq \mathbb{E}\left[\int_0^\infty u(s, c_s^x)^- \mathrm{d}s\right] + \lambda\mathbb{E}\left[\int_0^\infty Z_s(c_s^x + C_s)\mathrm{d}s\right] < \infty,$$

where we used (49) and the fact that c_s^x induces a finite maximum in (43). As observed earlier (cf. Theorem III.9.4 in Karatzas and Shreve (1998)), there exists (Π_s) such that $(C_s, \Pi_s) \in \mathcal{A}(x)$ and the above then shows that $(C_s, \Pi_s) \in \mathcal{A}^u(x)$. Proceeding as in (50), we obtain

$$\mathbb{E}\left[\int_0^\infty u(s, c_s^x)\mathrm{d}s\right] \leq \lambda x + \mathbb{E}\left[\int_0^\infty \tilde{u}(s, \lambda Z_s)\mathrm{d}s\right] = \mathbb{E}\left[\int_0^\infty u(s, C_s)\mathrm{d}s\right].$$

It follows we have to have equality in the above equation which is true if and only if $u(s, c_s^x) - \lambda Z_s c_s^x = \tilde{u}(s, \lambda Z_s)$ $\mathrm{d}s \times \mathrm{d}\mathbb{P}$-a.e., which in turn is true if and only if $c_s^x = C_s$ almost surely. In consequence, (45) is simply (44).

Using similar arguments to the ones which led to (49) above we see that $W_t Z_t \to W_\infty Z_\infty$ a.s. as $t \to \infty$. Further

$$0 \leq \mathbb{E}W_\infty Z_\infty \leq x - \mathbb{E}\int_0^\infty Z_s C_s \mathrm{d}s = 0$$

and hence $W_\infty Z_\infty \equiv 0$. In addition, by considering a localising sequence of stopping times τ_N, from (37) we get:

$$x = \mathbb{E}\left[W_{\tau_N} Z_{\tau_N}\right] + \mathbb{E}\left[\int_0^{\tau_N} C_s Z_s \mathrm{d}s\right].$$

But from (44) $x = \mathbb{E}\int_0^\infty Z_s C_s \mathrm{d}s$, so $\mathbb{E}\left[W_{\tau_N} Z_{\tau_N}\right] \to 0$ as $N \to \infty$. Moreover,

$$W_t Z_t = \mathbb{E}\left[\int_t^{\tau_N} C_s Z_s \mathrm{d}s \Big| \mathcal{F}_t\right] + \mathbb{E}\left[W_{\tau_N} Z_{\tau_N} | \mathcal{F}_t\right],$$

and the final term is almost surely non-negative. So

$$W_t Z_t - \mathbb{E}\left[\int_t^\infty C_s Z_s \mathrm{d}s \Big| \mathcal{F}_t\right] \geq 0$$

almost surely, and taking expectations, we see that this has expected value $\lim_{N\to\infty} \mathbb{E}\left[W_{\tau_N} Z_{\tau_N}\right] = 0$. Hence

$$W_t = \frac{1}{Z_t}\mathbb{E}\left[\int_t^\infty C_s Z_s \mathrm{d}s \Big| \mathcal{F}_t\right] = \frac{1}{Z_t}\mathbb{E}\left[\int_t^\infty I(s, \lambda Z_s) Z_s \mathrm{d}s \Big| \mathcal{F}_t\right].$$

Define $g(t, z)$ to be the solution to the PDE

$$g_t + z(\theta^2 - r)g_z + \frac{1}{2}z^2\theta^2 g_{zz} - rg = -I(t, \lambda z)$$

with initial condition $g(0,z) = \mathbb{E}\left[\int_0^\infty I(s,\lambda Z_s)Z_s \mathrm{d}s \middle| Z_0 = z\right]$. It follows that $g \in \mathcal{C}^{1,2}$, and in fact (e.g. Friedman (1964, Theorem 3.5.10)) since I_z is locally Hölder continuous, that g_{tz} exists and is also locally Hölder continuous. Applying Itô's Lemma, and using a similar localisation and convergence argument as above, we deduce that

$$g(t, Z_t) = \frac{1}{Z_t}\mathbb{E}\left[\int_t^\infty I(s,\lambda Z_s)Z_s \mathrm{d}s | \mathcal{F}_t\right].$$

We conclude that $W_t = g(t, Z_t)$ and since I is strictly decreasing on $(0,\infty)$ in Z_t, and noting that Z_s/Z_t is independent of Z_t, then $g(t,\cdot)$ is also strictly decreasing. Additionally, g is strictly positive and both $g(t,z)$ and $zg_z(t,z)$ tend to zero as z tends to infinity, and $g(t,z)$ tends to infinity as x tends to 0. We have $c(t,w) = I(t,\lambda g^{-1}(t,w))$, and we conclude that $c(t,0) = 0$, $c(t,\cdot)$ is strictly increasing, unbounded and $c_w(t,w)$ is locally Hölder continuous on $(0,\infty)^2$. Finally, we deduce that $g = Y \circ I$.

Using Itô's Lemma and equating $\mathrm{d}W_t$ with the wealth dynamics in (38) we obtain (30)–(31), with g instead of f, which hold for all $t, z > 0$. It follows that $\pi(t,0) = 0$, and $\pi(t,w) > 0$ for $w > 0$. From (30), and (as noted above) since g_{tz} exists, we get the required differentiability properties of π. We then proceed as in (32)–(35), to conclude that (29) holds.

This means F in (39) is well defined and we may consider $\tilde{W}_t = f(t, F(0,x)Z_t)$, with $f = F^{-1}$ as above. Proceeding as in the first part of the proof it follows that $\tilde{W}_t$ is a strong solution to the SDE (38) considered for $0 \le t < \tau$, where

$$\tau = \inf\{s : Z_s \notin (F(s,\infty)/F(0,x), F(s,0)/F(0,x))\} = \inf\{s : \tilde{W}_s \in \{0,\infty\}\}.$$

Unicity of strong solution to an SDE, as invoked above, holds also when we consider the SDE not on $t \in (0,\infty)$ but on $[0,\tau)$, and we conclude that $\tilde{W}_t = W_t^x = g(t, \lambda Z_t)$. However since we know that $0 < W_t^x < \infty$ a.s. it follows that $\tau = \infty$; i.e. $F(t,0) = \infty$ and $F(t,\infty) = 0$. Finally, from $c_t^x = I(t, \lambda Z_t)$ and $W_t^x = \tilde{W}_t$, it also follows that $u_c(t,c) = H_c(t,c)$ so that $u(t,c) = H(t,c) - \zeta(t)$, for some function ζ. As c_t^x achieves a finite maximum in (43) it follows that $\mathbb{E}[|H(t, c(t, W_t^x))|] < \infty$ for a.e. $t \ge 0$ and Fubini's theorem yields $\int_0^\infty (h(t) - \zeta(t))\mathrm{d}t$ is well defined and finite when we take $h(t) = \mathbb{E}[H(t, c(t, W_t^x))]$. In consequence,

$$\begin{aligned} \mathbb{E}\int_0^\infty [H(t, c(t, W_t^x)) - h(t)]^+ \mathrm{d}t < \mathbb{E}\int_0^\infty & [H(t, c(t, W_t^x)) - \zeta(t)]^+ \mathrm{d}t \\ &+ \int_0^\infty (\zeta(t) - h(t))^+ \mathrm{d}t < \infty. \end{aligned} \tag{51}$$

Hence *(ii)* holds when we take $x = x_0$. It follows from the first part of the proof that we may take $u(t,c) = H(t,c) - h(t)$. □

3.4 Example with c, π linear in wealth

Example 3.9. Suppose $c(t,w) = \kappa w$ and $\pi(t,w) = \phi w$ for $\kappa, \phi > 0$ with $\phi \ne \theta/\sigma$. Then Black's equation (29) is satisfied, $Y(t,b) = b/\kappa$, $\beta(t) \equiv \beta = (\kappa - r)/\phi + \sigma^2\phi/2$ and $A(t) = \xi t$ where $\xi = -\theta\beta/\sigma + \theta^2/2 - r$.

Let $\gamma = \theta/\phi\sigma$. We have $F(t,w) = e^{\xi t} w^{-\gamma}$ and in particular $F(t,0) = \infty$, $F(t,\infty) = 0$. It follows that $\lambda(x) = x^{-\gamma}$ and $f(t,z) = z^{-1/\gamma} e^{(\xi/\gamma)t}$, which is $\mathcal{C}^{1,2}$ differentiable.

Further, $W_t^x = f(t, \lambda(x) Z_t) = x e^{\phi\sigma B_t + (\sigma\phi\theta + r - \kappa - \sigma^2\phi^2/2)t}$ and a direct computation yields

$$\mathrm{e}^{\xi t}\mathbb{E}[(W_t^x)^{1-\gamma}] = x^{1-\gamma}\mathrm{e}^{\xi t/\gamma}\mathbb{E}[Z_t^{1-1/\gamma}] = x^{1-\gamma}\mathrm{e}^{-\kappa t}. \tag{52}$$

It follows that

$$\mathbb{E}\left[\int_0^\infty Z_t c(t, W_t^x)\mathrm{d}t\right] = \kappa x \mathbb{E}\left[\int_0^\infty (Z_t)^{1-1/\gamma}\mathrm{e}^{\xi t/\gamma}\mathrm{d}t\right] = \kappa x \int_0^\infty \mathrm{e}^{-\kappa t}\mathrm{d}t = x.$$

Also $H(t,c) = \frac{1}{1-\gamma}\mathrm{e}^{\xi t}\kappa^\gamma[c^{1-\gamma} - 1]$ so that, taking $x_0 = 1$,

$$h(t) = \mathbb{E}[H(t, c(t, W_t^{x_0}))] = \frac{1}{1-\gamma}\left(\kappa \mathrm{e}^{-\kappa t} - \mathrm{e}^{\xi t}\kappa^\gamma\right),$$

and

$$\begin{aligned} u(t,c) &= \frac{1}{1-\gamma}\left(\kappa^\gamma \mathrm{e}^{\xi t} c^{1-\gamma} - \kappa \mathrm{e}^{-\kappa t}\right) \\ &= \frac{\kappa^\gamma \mathrm{e}^{\xi t}}{1-\gamma}\left(c^{1-\gamma} - (\kappa \mathrm{e}^{-\zeta t})^{1-\gamma}\right), \end{aligned} \tag{53}$$

where $\zeta = (\xi + \kappa)/(1-\gamma) = \kappa - r - \frac{\theta^2}{2\gamma}$. Then, using (52), $\mathbb{E}[u(t, c(t, W_t^{x_0}))^+] < D\mathrm{e}^{-\kappa t}$ for some constant D, and hence $\int_0^\infty \mathbb{E}[H(t, C_t) - h(t)]^+ \mathrm{d}t < \infty$ for the optimal policy.

In the above we could take any $x_0 > 0$. So, in conclusion, for any initial capital $x > 0$, π, c solve the optimal consumption problem for admissible strategies for u as given in (53). We note that the choice of consumption and investment which are linear in wealth and time-homogeneous necessarily implies an exponential discounting of utility from a given wealth.

4 Consequences and Extensions of the Main Result

4.1 Model uncertainty

In our analysis so far we assumed agents believe that the price process follows the Black-Scholes model (14) with given parameters θ, σ. We then asked whether their observed actions are optimal for (43) for *some* utility function u. Suppose however that we do *not* know agents' beliefs about model parameters. We may then ask more generally whether agents' actions are optimal for *some* utility function u and *some* price dynamics?[2]

More precisely, within the realm of Theorem 3.4, we can ask the following: are the observed actions optimal for (43) for *some* u and *some* θ, σ? Assume we are not in the special case when $\pi_w(t,w) = \phi(t)$. Then π and c must solve

[2]We are grateful to Masaaki Fukasawa for suggesting this question. See also Cuoco and Zapatero (2000) for related results, although with an emphasis on equilibrium constraints.

Black's PDE (29) and clearly there can be at most one value of σ for which (29) is satisfied. Put differently, if we find that agents are optimising expected utility of consumption then we also recover uniquely their belief about market's volatility. In contrast we do not recover their belief about the Sharpe ratio θ, which does not appear in (29). Indeed, as we argue below, if c, π are consistent with utility maximisation for a model with Sharpe ratio θ, then we expect that they are also consistent with utility maximisation for a model with a different Sharpe ratio $\hat{\theta}$, but for a different utility.

Consider $\hat{\mathbb{P}}$, $\hat{Z}_t$ defined via

$$\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left((\hat{\theta}-\theta)B_t - \frac{(\hat{\theta}-\theta)^2}{2}t\right) = \frac{Z_t}{\hat{Z}_t}$$

so that

$$\frac{\mathrm{d}P_t}{P_t} = \sigma(\mathrm{d}\hat{B}_t + \hat{\theta}\mathrm{d}t) + r\mathrm{d}t,$$

for a $\hat{\mathbb{P}}$–Brownian motion $\hat{B}_t = B_t - (\hat{\theta}-\theta)t$. Observe further that $Z_t = \hat{Z}_t^{\theta/\hat{\theta}}\mathrm{e}^{\mu t}$, with $\mu = \frac{\theta}{2}(\theta-\hat{\theta}) + r(\frac{\theta}{\hat{\theta}} - 1)$.

Suppose agents' actions π, c are given and the equivalent conditions in Theorem 3.4 hold. Define a new utility function $\hat{u}(t,c)$ via the inverse of $\hat{u}_c$: $\hat{I}(t,z) := I(t, z^{\theta/\hat{\theta}}\mathrm{e}^{\mu t})$. It follows that the budget equation (44) holds for $\hat{\lambda} = \lambda^{\hat{\theta}/\theta}$. Indeed, by definition,

$$\hat{\mathbb{E}}\left[\int_0^\infty \hat{Z}_t\hat{I}(t,\hat{\lambda}\hat{Z}_t)\mathrm{d}t\right] = \hat{\mathbb{E}}\left[\int_0^\infty \hat{Z}_t I(t,\lambda Z_t)\mathrm{d}t\right] = \mathbb{E}\left[\int_0^\infty Z_t I(t,\lambda Z_t)\mathrm{d}t\right] = x.$$

Classical duality arguments (see the proof of the implication $(i) \Rightarrow (ii)$ in Theorem 3.4 above), yield that the optimal consumption policy in the problem (43) for $\hat{u}$ under $\hat{\mathbb{P}}$ is given as

$$C_t = \hat{I}(t,\hat{\lambda}\hat{Z}_t) = I(t,\lambda Z_t) = c(t, W_t^x).$$

Provided (43) for $\hat{u}$ under $\hat{\mathbb{P}}$ has a finite value, we conclude that the agents' chosen actions are optimal for u under $\mathbb{P}$ and for $\hat{u}$ under $\hat{\mathbb{P}}$.

We will not persue this finiteness issue here. However, Lemma 4.3 below gives sufficient conditions under which the problem value is finite, and then we conclude that the same consumption/investment pair is consistent with a family of utility functions, each member of which corresponds to a different model and Sharpe ratio.

4.2 Risk aversion

We now return to our original setting where σ, θ are known and fixed and we consider questions similar to those which arose in Theorem 2.6, namely, we investigate what we can say about an investor's risk profile from her actions. Recall the absolute risk aversion $\rho(t,c)$ given in Definition 2.5.

Proposition 4.1. *An investor with investment and consumption strategies $\pi(t,w)$, $c(t,w)$ satisfying the assumptions of (ii) of Theorem 3.4, where $c(t,w)$ is twice* *continuously differentiable in wealth, has decreasing absolute risk aversion (DARA) if and only if*

$$\frac{\pi_w(t,w)}{\pi(t,w)} \geq -\frac{c_{ww}(t,w)}{c_w(t,w)}. \tag{54}$$

In particular, a sufficient condition for an investor to be DARA is convexity (in wealth) of her consumption and investment which is increasing in wealth.

Proof. From the final statement of Theorem 3.4, the investor's utility function u satisfies $u_c(t,c) = H_c(t,c) = F(t, Y(t,c))$ and from (39) it follows that

$$\rho(t,c) = \frac{\theta}{\sigma\pi(t, Y(t,c))} Y_c(t,c).$$

The absolute risk aversion is decreasing, i.e. $\rho_c(t,c) \leq 0$ iff:

$$\begin{aligned}
& 0 \geq \frac{\partial}{\partial c}\left[\frac{Y_c(t,c)}{\pi(t,Y(t,c))}\right] \\
\Longleftrightarrow \quad & 0 \geq \frac{Y_{cc}(t,c)}{\pi(t,Y(t,c))} - \frac{\left(Y_c(t,c)\right)^2 \pi_w(t,Y(t,c))}{\pi(t,Y(t,c))^2} \\
\Longleftrightarrow \quad & 0 \geq -\frac{c_{ww}(t,w)}{\pi(t,w)c_w(t,w)^3} - \frac{\pi_w(t,w)}{c_w(t,w)^2\pi(t,w)^2}\Bigg|_{w=Y(t,c)} \\
\Longleftrightarrow \quad & 0 \leq \pi_w(t,w) + \pi(t,w)\frac{c_{ww}(t,w)}{c_w(t,w)}\Bigg|_{w=Y(t,c)}.
\end{aligned}$$

Transforming the last inequality we arrive at the statement of the proposition. □

In a similar manner we derive a condition equivalent to relative risk aversion. We omit the proof for the sake of brevity.

Proposition 4.2. *An investor with investment and consumption strategies $\pi(t,w)$ and $c(t,w)$ satisfying the assumptions of (ii) of Theorem 3.4, where $c(t,w)$ is twice continuously differentiable in wealth, has decreasing relative risk aversion (DRRA) if and only if*

$$\left(c_w(t,w)\right)^2 - c(t,w)c_{ww}(t,w) \leq \frac{c_w(t,w)c(t,w)\pi_w(t,w)}{\pi(t,w)}$$

or equivalently iff

$$\frac{\partial}{\partial w}\left[\log\frac{c(t,w)}{c_w(t,w)}\right] \leq \frac{\partial}{\partial w}\log\pi(t,w).$$

4.3 Time-homogeneous investment and consumption

We specialise now to the important special case of $\pi(t,w)$, $c(t,w)$ which are independent of time. Suppose first that $\pi(t,w) = \pi(w)$ is independent of time. Equation (28) then simplifies to

$$R_w(t,w) = \frac{r}{\pi(w)} - \frac{rw}{\pi(w)^2}\pi_w(w) - \frac{\sigma^2}{2}\pi_{ww}(w) = \frac{\partial}{\partial w}\left(\frac{rw}{\pi(w)} - \frac{\sigma^2}{2}\pi_w(w)\right)$$

which yields

$$R(t,w) = \frac{rw}{\pi(w)} - \frac{\sigma^2}{2}\pi_w(w) + \beta(t),$$

where $\beta(t)$ is taken such that $R(t,w) > 0$. In consequence

$$c(t,w) = R(t,w)\pi(w) = rw - \frac{\sigma^2}{2}\pi(w)\pi_w(w) + \beta(t)\pi(w)\ . \tag{55}$$

In particular, if the agent invests a constant proportion of wealth in the risky asset, i.e. $\pi(w) = \phi w$, $\phi > 0$, then

$$c(t,w) = \left(r - \frac{\sigma^2}{2}\phi^2 + \beta(t)\phi\right) w = R(t,1)\phi w$$

is also linear in wealth. It is straightforward to see that for a reasonable $\beta(t)$ (e.g. continuous and bounded) agent's choices c and π verify the assumptions of Theorem 3.4. The case $\beta(t) \equiv \beta$, a constant, was worked out explicitly in Example 3.9 above.

We want to study in more detail the implications of representation (55) on the possible behaviour of admissible investment/consumption strategies. Assume that $c = c(w)$ is also time-homogenous, or equivalently that $\beta(t) \equiv \beta$ is a constant. Recall that we require $\pi(0) = c(0) = 0$ and both $\pi(w), c(w)$ are non-negative and $c(w)$ is increasing. Consider an investment strategy given by $\pi(w) = \phi w^\alpha$ with $\alpha > 0$. Then (55) gives

$$c(w) = rw - \frac{\sigma^2\phi^2\alpha}{2} w^{2\alpha-1} + \beta\phi w^\alpha. \tag{56}$$

The condition $c(0) = 0$ restricts us to $\alpha > 1/2$. Considering $\alpha \in (0.5, 1)$ we see that the middle term in (56) dominates for small w so that $c_w(0+) = -\infty$ and $c(w)$ is negative for small values of w. On the other hand, if $\alpha > 1$ then the middle term dominates for large values of w and $c(w)$ becomes negative then. We conclude that the only admissible value is the one studied above: $\alpha = 1$. This indicates that an admissible investment strategy has to have linear behaviour near zero and infinity. For such actions we are able to verify the assumptions in Theorem 3.4.

Lemma 4.3. *Suppose $c(t,0) = 0 = \pi(t,0)$, $\pi(t,w) = \pi(w)$ is time homogeneous, $c(t,w)$ is continuous and c, π are continuously differentiable in w and c_w is locally Hölder continuous on $(0,\infty)^2$. Further, c and π satisfy* (29) *and there exist strictly positive constants $\tilde{\delta}_1, \tilde{\delta}_2, \kappa_1, \kappa_2$ with*

$$\pi_w(w) \xrightarrow[w\to\infty]{} \tilde{\delta}_1, \quad \pi_w(w) \xrightarrow[w\to 0]{} \tilde{\delta}_2,$$
$$\delta_1 := \tilde{\delta}_1 \wedge \tilde{\delta}_2 \le \pi_w(w) \le \tilde{\delta}_1 \vee \tilde{\delta}_2 =: \delta_2, \text{ and } \kappa_1 \le c_w(t,w) \le \kappa_2,\ t \ge 0, w \ge 0. \tag{57}$$

Finally, assume either that $\theta/\sigma \le \delta_1$, or $\delta_1 < \theta/\sigma \le \delta_2$ and $(\theta(1-\delta_2/\delta_1)+\sigma\delta_2) > 0$. [3]

[3] We are grateful to Li Yu for noticing an error in an earlier version of this paper, in which we incorrectly stated a slightly different set of conditions.

Then c, π is a regular investment/consumption pair (Definition 3.3) and, for any $x > 0$, $\mathbb{E}[|H(t, c(t, W_t^x))|] < \infty$ and $\int_0^\infty \mathbb{E}[H(t, c(t, W_t^x)) - h(t)]^+ \mathrm{d}t < \infty$. Further, $F(t, 0) = \infty$, $F(t, \infty) = 0$, for all $t \geq 0$, and (45) holds for any $x > 0$. In consequence, (i) and (ii) in Theorem 3.4 hold true.

Proof. It is immediate that $\int_0^1 d\xi/\pi(\xi) = \int_1^\infty d\xi/\pi(\xi) = \infty$ and hence $F(t, 0) = \infty$ and $F(t, \infty) = 0$. The other properties of Definition 3.3 follow equally easily. Fix $x > 0$ for the rest of the proof. Equation (57) implies global Lipschitz behaviour of c, π which guarantees the existence of a strong unique solution to (38). Let $\mathbb{Q}$ be the risk neutral measure under which $B_t^\theta := B_t + \theta t$ is a Brownian motion and put $\tilde{W}_t := \mathrm{e}^{-rt} W_t$. We then have

$$\tilde{W}_t = x + \sigma M_t - \int_0^t \mathrm{e}^{-rs} c(s, W_s) \mathrm{d}s, \tag{58}$$

where $M_t := \int_0^t \mathrm{e}^{-rs} \pi(W_s) \mathrm{d}B_s^\theta$ is a $\mathbb{Q}$-local martingale. In particular, $\tilde{W}_t$ is a non-negative super-martingale under $\mathbb{Q}$ and hence converges, $\mathbb{Q}$-a.s. as $t \to \infty$. It follows that M_t also converges $\mathbb{Q}$-a.s. which is equivalent to $\langle M \rangle_t$ converging (cf. (Revuz and Yor, 2001, Proposition V.1.8)). However

$$\int_0^t \delta_1^2 \tilde{W}_s^2 \mathrm{d}s \ \leq \ \langle M \rangle_t = \int_0^t \mathrm{e}^{-2rs} \pi(W_s)^2 \mathrm{d}s \ \leq \ \int_0^t \delta_2^2 \tilde{W}_s^2 \mathrm{d}s$$

and it follows that $\tilde{W}_t \to 0$ $\mathbb{Q}$-a.s. Finally, from classical estimates (e.g. (Friedman, 1975, Theorem 5.2.3)), we have that $\mathbb{E}^{\mathbb{Q}}[(\tilde{W}_t)^m] < \infty$ for all $m \geq 1$. It follows that $\mathbb{E}^{\mathbb{Q}}[\langle M \rangle_t] < \infty$ and hence M_t is a $\mathbb{Q}$-martingale with $\mathbb{E}^{\mathbb{Q}} M_t = 0$. In particular

$$\mathbb{E} \int_0^\infty Z_s c(s, W_s) \mathrm{d}s = \lim_{t \to \infty} \mathbb{E}^{\mathbb{Q}} \int_0^t \mathrm{e}^{-rs} c(s, W_s) \mathrm{d}s = x - \lim_{t \to \infty} \mathbb{E}^{\mathbb{Q}}[\tilde{W}_t].$$

To show (45), it remains to argue that $\lim_{t \to \infty} \mathbb{E}^{\mathbb{Q}}[\tilde{W}_t] = 0$. It follows from the above representation that $\mathbb{E}^{\mathbb{Q}}[\tilde{W}_t]$ is decreasing in t. By (58), and $\mathbb{E}^{\mathbb{Q}} M_t = 0$, we have:

$$\mathbb{E}^{\mathbb{Q}} \tilde{W}_t = x - \mathbb{E}^{\mathbb{Q}} \int_0^t \mathrm{e}^{-rs} c(s, \mathrm{e}^{rs} \tilde{W}_s) \mathrm{d}s.$$

Using the fact that $c(s, w) \geq \kappa_1 w$, and applying Fubini's theorem, we get:

$$\mathbb{E}^{\mathbb{Q}} \tilde{W}_t \leq x - \kappa_1 \int_0^t \mathbb{E}^{\mathbb{Q}} \tilde{W}_s \, \mathrm{d}s.$$

The desired conclusion follows immediately. It remains to show the integrability properties of H. We will show the stronger fact that $\mathbb{E}|H(t, c(t, W_t))| < \infty$ for all t, and also that $\mathbb{E} \int_0^\infty |H(t, c(t, W_t))| \, \mathrm{d}t < \infty$. From (57) we get instantly that

$$\begin{aligned} w^{-\frac{\theta}{\sigma\delta_2}} &\leq \mathrm{e}^{-A(t)} F(t, w) \leq w^{-\frac{\theta}{\sigma\delta_1}}, \quad 0 \leq w < 1 \\ w^{-\frac{\theta}{\sigma\delta_1}} &\leq \mathrm{e}^{-A(t)} F(t, w) \leq w^{-\frac{\theta}{\sigma\delta_2}}, \quad w \geq 1 \end{aligned} \tag{59}$$

from which it follows that

$$\begin{aligned} \mathrm{e}^{A(t)\frac{\sigma\delta_1}{\theta}} z^{-\frac{\sigma\delta_1}{\theta}} &\leq f(t, z) \leq \mathrm{e}^{A(t)\frac{\sigma\delta_2}{\theta}} z^{-\frac{\sigma\delta_2}{\theta}}, \quad 0 \leq z < \mathrm{e}^{A(t)} \\ \mathrm{e}^{A(t)\frac{\sigma\delta_2}{\theta}} z^{-\frac{\sigma\delta_2}{\theta}} &\leq f(t, z) \leq \mathrm{e}^{A(t)\frac{\sigma\delta_1}{\theta}} z^{-\frac{\sigma\delta_1}{\theta}}, \quad z \geq \mathrm{e}^{A(t)} . \end{aligned} \tag{60}$$

The integrability properties we need to establish are invariant under a shift of H by a constant so we are free to redefine H as

$$H(t,z) := \int_{c(t,1)}^{z} F(t, Y(t,b))\mathrm{d}b, \text{ so that } H(t, c(t,w)) = \int_1^w F(t,s)c_w(t,s)\mathrm{d}s.$$

In consequence

$$H(t,c(t,w))^+ = \int_1^{w\vee 1} F(t,s)c_w(t,s)\mathrm{d}s \le \kappa_2 \mathrm{e}^{A(t)} \int_1^{w\vee 1} s^{-\frac{\theta}{\sigma\delta_2}}\mathrm{d}s \tag{61}$$

and similarly

$$H(t,c(t,w))^- = \int_{w\wedge 1}^{1} F(t,s)c_w(t,s)\mathrm{d}s \le \kappa_2 \mathrm{e}^{A(t)} \int_{w\wedge 1}^{1} s^{-\frac{\theta}{\sigma\delta_1}}\mathrm{d}s \tag{62}$$

Suppose first that $\theta < \sigma\delta_2$. Then $H(t,c(t,w))^+ \le \mathrm{e}^{A(t)} \frac{\sigma\delta_2\kappa_2}{\sigma\delta_2-\theta} w^{1-\frac{\theta}{\sigma\delta_2}} \mathbf{1}_{\{w\ge 1\}}$. Using $W_t = f(t, \lambda(x)Z_t)$ and the estimates in (60), we have

$$\begin{aligned}\mathbb{E}H(t,c(t,W_t))^+ &\le \mathrm{e}^{A(t)} \frac{\sigma\delta_2\kappa_2}{\sigma\delta_2-\theta}\mathbb{E}\left[f(t,\lambda(x)Z_t)^{1-\frac{\theta}{\sigma\delta_2}}\mathbf{1}_{\lambda(x)Z_t\le \mathrm{e}^{A(t)}}\right] \\ &\le \frac{\sigma\delta_2\kappa_2}{\sigma\delta_2-\theta}\mathrm{e}^{A(t)\frac{\sigma\delta_2}{\theta}}\mathbb{E}[(\lambda(x)Z_t)^{1-\frac{\sigma\delta_2}{\theta}}]\end{aligned} \tag{63}$$

and it follows that $\mathbb{E}H(t,c(t,W_t))^+ < \infty$.

To estimate the expectation of the integral in time we need a more careful analysis. From Black's equation (29), given that π is time-homogeneous, we know that

$$\beta(t) = \frac{\sigma^2}{2}\pi_w(w) - r\frac{w}{\pi(w)} + \frac{c(t,w)}{\pi(w)}, \tag{64}$$

is a function of t only. Now, depending on whether $\delta_2 = \tilde{\delta_2}$ or $\delta_2 = \tilde{\delta_1}$, we let $w \to 0$ or $w \to \infty$ on the RHS. The first term in (64) then converges to $\sigma^2\delta_2/2$, the second term converges by l'Hôpital's rule to r/δ_2 and hence also the third term converges to some $\kappa_3(t)/\delta_2$, where $\kappa_3(t) \ge \kappa_1 > 0$. We conclude that $\beta(t) = \sigma^2\delta_2/2 + \kappa_3(t)/\delta_2 - r/\delta_2$ and

$$\begin{aligned} A(t) &= -\frac{\theta}{\sigma}\int_0^t \beta(s)ds + \left(\frac{\theta^2}{2} - r\right)t \\ &\le \left(-\frac{\theta}{\sigma}(\sigma^2\delta_2/2 + \kappa_1/\delta_2 - r/\delta_2) + \frac{\theta^2}{2} - r\right)t \\ &= \left(-\frac{\theta\kappa_1}{\sigma\delta_2} - \left(1 - \frac{\theta}{\sigma\delta_2}\right)\left(\frac{\theta\sigma\delta_2}{2} + r\right)\right)t. \end{aligned} \tag{65}$$

Using this last estimate in (63) we recover the situation in (52). Since by assumption $\theta < \sigma\delta_2$, and using the representation $Z_t = e^{-rt-\theta B_t - \theta^2 t/2}$ in (63) we have $\mathbb{E}H(t,c(t,W_t))^+ \le C\mathrm{e}^{-\kappa_1 t}$ for a constant C, and hence $\mathbb{E}\int_0^\infty H(t,c(t,W_t))^+\mathrm{d}t < \infty$.

We now turn to the estimates of $H(t,c(t,W_t))^-$. In the case where $\theta < \sigma\delta_1$, (62) implies that $H(t,c(t,w))^- \le \kappa_2\mathrm{e}^{A(t)}\frac{\sigma\delta_1}{\sigma\delta_1-\theta}(1-(w\wedge 1)^{1-\theta/\sigma\delta_1})$. Hence $\mathbb{E}H(t,c(t,W_t))^- < \infty$, giving also $\mathbb{E}|H(t,c(t,W_t))| < \infty$, and for some constant C, $\mathbb{E}\int_0^\infty H(t,c(t,W_t))^-\mathrm{d}t \le C\int_0^\infty \mathrm{e}^{A(t)}\,\mathrm{d}t$. Since (65) implies $A(t)/t < 0$

is bounded away from zero by a constant, then the integral is finite, and so $\int_0^\infty |H(t,c(t,W_t))|\,\mathrm{d}t < \infty$. In the case where $\delta_1 < \theta/\sigma < \delta_2$ and $(\theta(1-\delta_2/\delta_1)+\sigma\delta_2) > 0$, (62) implies $H(t,c(t,w))^- \leq \mathrm{e}^{A(t)}\frac{\sigma\delta_1\kappa_2}{\theta-\sigma\delta_1}w^{1-\frac{\theta}{\sigma\delta_1}}\mathbf{1}_{\{w\leq 1\}}$, and this is decreasing as a function of w. Using (60), we get

$$\begin{aligned}\mathbb{E}H(t,c(t,W_t))^- &\leq \mathrm{e}^{A(t)}\frac{\sigma\delta_1\kappa_2}{\theta-\sigma\delta_1}\mathbb{E}\left[f(t,\lambda(x)Z_t)^{1-\frac{\theta}{\sigma\delta_1}}\mathbf{1}_{\lambda(x)Z_t\geq \mathrm{e}^{A(t)}}\right]\\ &\leq \frac{\sigma\delta_1\kappa_2}{\theta-\sigma\delta_1}\mathrm{e}^{A(t)\left(1+\frac{\sigma\delta_2}{\theta}-\frac{\delta_2}{\delta_1}\right)}\mathbb{E}\left[(\lambda(x)Z_t)^{\frac{\delta_2}{\delta_1}-\frac{\sigma\delta_2}{\theta}}\right]\end{aligned} \tag{66}$$

and it follows that $\mathbb{E}H(t,c(t,W_t))^- < \infty$.

To estimate $\mathbb{E}\int_0^\infty H(t,c(t,W_t))^-\mathrm{d}t$ we use similar arguments to those for the positive part, but consider the asymptotics of (64) which yield expressions involving δ_1. Then we get

$$A(t) \leq \left(-\frac{\theta\kappa_1}{\sigma\delta_1} - \left(1-\frac{\theta}{\sigma\delta_1}\right)\left(\frac{\theta\sigma\delta_1}{2}+r\right)\right)t.$$

If we write $S := (\theta(1-\delta_2/\delta_1)+\sigma\delta_2) > 0$, we can compute the expectation as:

$$\begin{aligned}\mathbb{E}H(t,c(t,W_t))^- &\leq \frac{\sigma\delta_1\kappa_2\lambda(x)^{1-S/\theta}}{\theta-\sigma\delta_1}\mathrm{e}^{A(t)\frac{S}{\theta}-(1-S/\theta)(rt+S\theta t/2)}\\ &\leq \frac{\sigma\delta_1\kappa_2\lambda(x)^{1-S/\theta}}{\theta-\sigma\delta_1}\mathrm{e}^{-\frac{t}{\sigma\delta_1}(\kappa_1 S+(r+\sigma\delta_1 S/2)(\sigma\delta_1-S))}.\end{aligned}$$

We now observe that $\sigma\delta_1 - S = (\delta_2-\delta_1)(\theta-\sigma\delta_1)/\delta_1 \geq 0$, and we conclude that $\mathbb{E}\int_0^\infty H(t,c(t,W_t))^-\mathrm{d}t < \infty$.

The remaining cases have $\theta = \sigma\delta_i$ for some i. We outline the case where $\theta = \sigma\delta_1$ and $\delta_1 < \delta_2$, the other cases relying on similar calculations. In this case the bounds on $H(t,c(t,W_t))^+$ hold in a similar manner to above, while integrating (62) and using, for $z \geq e^{A(t)}$, $f(t,z) \geq (e^{A(t)}z^{-1})^{\sigma\delta_2/\theta}$ gives $H(t,c(t,W_t))^- \leq \kappa_2 e^{A(t)}\sigma\delta_2\theta^{-1}\left((A(t))^- + (\ln(\lambda(x)Z_t))^+\right)$. We can derive a lower bound on $A(t)$ in a similar manner to (65), and note that the upper bound also still holds, and so we see that $\int_0^\infty \mathbb{E}H(t,c(t,W_t))^-\mathrm{d}t < \int_0^\infty (C_1+C_2 t)e^{-\kappa_1 t}\mathrm{d}t < \infty$ for some constants C_1, C_2, which gives the required behaviour. In the remaining cases where either $\delta_1 < \delta_2 = \theta/\sigma$, or $\delta_1 = \delta_2 = \theta/\sigma$, modifications of the above arguments hold.

□

Remark 4.4. In Lemma 4.3 we provide two sufficient conditions which relate θ, σ and the constants δ_1, δ_2 which are derived from $\pi_w(w)$. The simpler necessary condition is to require $\theta \leq \sigma\delta_1$, however in this case, the utility function we derive from this consumption/investment pair will necessarily be finite in the limit as we let $c \to 0$. To allow utility functions which do not display this behaviour, we include also the second case. Note however that this second case also contains a subset of cases which are easy to check: if $\delta_1 > \frac{1}{2}\delta_2$, then it is easily confirmed that the second necessary condition holds.

Remark 4.5. From the proof it is clear that we do not need to assume time-homogeneity of π. Instead we take $\pi(t,w) \in \mathcal{C}^{1,1}$ and assume (57) and (29). Then it follows that

$$\kappa_3(t) := \tilde{\delta_2}\lim_{w\to 0}\frac{c(t,w)}{\pi(t,w)} + \int_1^w \frac{\pi_t(t,\xi)}{\pi(t,\xi)^2}\mathrm{d}\xi,\ \kappa_4(t) := \tilde{\delta_1}\lim_{w\to\infty}\frac{c(t,w)}{\pi(t,w)} + \int_1^w \frac{\pi_t(t,\xi)}{\pi(t,\xi)^2}\mathrm{d}\xi$$

are well defined. It is then enough to assume that $\theta/\sigma \leq \tilde{\delta}_1 \wedge \tilde{\delta}_2$ and further $\int_0^\infty \exp(-\int_0^t \kappa_3(u)du)\mathrm{d}t < \infty$ and likewise for $\kappa_4(t)$.

Lemma 4.3 is particularly useful as it allows us to construct a wealth of examples of non-linear consumption and investment pairs with prescribed desired behaviour. We explore now a method to obtain convex/concave investment and consumptions pairs and then present a simple parametric family of examples.

Assume that $\pi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a thrice differentiable, strictly increasing, concave function such that $\pi(0) = 0$ and

$$0 < \pi_w(\infty-) \leq \pi_w(w) < \pi_w(0+) < \infty, \quad w \in (0, \infty). \tag{67}$$

Further, let $\chi(w) = (\pi\pi_w)_w(w) = \pi_w(w)^2 + \pi(w)\pi_{ww}(w)$ and assume that there exists $\epsilon > 0$ such that

$$-\epsilon^{-1} < \frac{\sigma^2}{2}\chi(w) < r - \epsilon, \quad w \in (0, \infty). \tag{68}$$

The optimal consumption is given by (55) and we assume that it is time homogenous with $\beta(t) \equiv \beta \geq 0$. We then have $c(0) = 0$ and

$$\epsilon \leq \epsilon + \beta\pi_w(\infty-) < c_w(w) = r + \beta\pi_w(w) - \frac{\sigma^2}{2}\chi(w) < r + \beta\pi_w(0+) + \epsilon^{-1}.$$

In particular, (57) holds. Provided the Sharpe ratio satisfies (for example) $\theta \leq \sigma\pi_w(\infty-)$ we can apply Lemma 4.3 and conclude that (i) and (ii) in Theorem 3.4 hold true.

By hypothesis π is concave: the concavity/convexity of c will depend on the sign of $c_{ww} = \beta\pi_{ww} - \sigma^2\chi_w/2$. Noting that $\pi_{ww} \leq 0$ we have that if

$$C\pi_{ww}(w) < \chi_w(w) < 0, \quad w \in (0, \infty), \tag{69}$$

for some positive constant C then the choice $\beta = 0$ gives that c is strictly convex, whereas the choice $\beta = \sigma^2 C/2$ gives that c is strictly concave.

Similarly, we can produce examples with π convex. Suppose that π is a thrice differentiable, strictly increasing, convex function such that $\pi(0) = 0$ and $0 < \pi_w(0+) \leq \pi_w(w) \leq \pi_w(\infty-) < \infty$. Then, if we assume again that (68) holds, we have c such that $c(0) = 0$ and

$$\epsilon \leq \epsilon + \beta\pi_w(0+) < c_w(w) < r + \beta\pi_w(\infty-) + \epsilon^{-1}.$$

In conclusion, (57) holds and if (for example) $\theta \leq \sigma\pi_w(0+)$ the assumptions of Lemma 4.3 are satisfied and (i) and (ii) in Theorem 3.4 follow. If, instead of (69), we have

$$C\pi_{ww}(w) > \chi_w(w) > 0, \quad w \in (0, \infty), \tag{70}$$

then the choice of $\beta = 0$ gives a concave consumption while $\beta = \sigma^2 C/2$ generates a convex c.

Example 4.6. As an example, take in the above $\pi(w) = (\phi w + \psi((1+w)^p - 1))$ with $\phi, \psi > 0$ and $0 < p < 1$. Then π is concave, $\pi_w(\infty-) = \phi$ and $\pi_w(0+) = \phi + p\psi$. Furthermore,

$$\chi_w(w) = \pi(w)\pi_{www}(w) + 3\pi_w(w)\pi_{ww}(w) = -\psi(1+w)^{p-3}p(1-p)\Lambda(w)$$

where

$$\Lambda(w) = (1+p)\phi w + (2-p)\psi + 3\phi + 2(1+w)^p \psi(2p-1).$$

Suppose that the parameters are such that $\Lambda(w)$ is positive, a simple sufficient condition for which is that $p \geq 1/2$. Since $\psi > 0$ it follows that χ is decreasing and

$$\phi^2 = \chi(\infty) \leq \chi(w) \leq \chi(0) = (\phi + p\psi)^2$$

so that (68) follows provided $(\phi + p\psi)^2 < 2r/\sigma^2$, a condition we now impose.

We already have $\chi_w < 0$ so for (69) it suffices to look at the sign of

$$C\pi_{ww} - \chi_w = \psi p(1-p)(1+w)^{p-3}[-C(1+w) + \Lambda(w)]. \tag{71}$$

Since Λ is bounded above by an affine function of w it follows easily that this expression can be made negative on $(0, \infty)$ by choosing C sufficiently large. Choose the parameters such that $\Lambda(w) > 0$, $(\phi + p\psi)^2 < 2r/\sigma^2$ and either $\theta \leq \sigma\phi$ or both $\sigma\phi < \theta \leq \sigma(\phi + p\psi)$ and $\sigma\phi(\phi + p\psi) > \theta p\psi$. Taking $\beta = 0$ we obtain an example for which π is concave and c is convex, and conversely, taking β sufficiently large, we obtain an example with both π and c concave.

To obtain examples with convex π consider now $\phi > 0$, $0 < p < 1$ but $\psi < 0$. Assume also that $\phi + p\psi > 0$ and $\phi^2 < 2r/\sigma^2$. Then π is increasing, convex with $\phi + p\psi = \pi_w(0+) < \pi_w(w) < \pi_w(\infty-) = \phi$. Suppose again that the parameters are such that $\Lambda(w)$ is positive. In fact, given $\Lambda(0) = 3(\phi + p\psi) > 0$, $p \leq 1/2$ is a simple sufficient condition. It follows that $\chi(w)$ is increasing with

$$(\phi + p\psi)^2 = \chi(0) \leq \chi(w) \leq \chi(\infty) = \phi^2$$

so that (68) holds under the condition $\phi^2 < 2r/\sigma^2$. Further, (71) can be made uniformly positive for large C and hence (70) holds. Lemma 4.3 applies with either $\theta \leq \sigma(\phi + p\psi)$ or both $\sigma(\phi + p\psi) < \theta \leq \sigma\phi$ and $\theta p\psi > -\sigma\phi(\phi + p\psi)$. We conclude that if we take $\beta = 0$ we obtain an example with convex investment and a concave consumption. Conversely, if we take β sufficiently large we get an example with both π and c convex. A numerical example for this case is given in Figure 1. Note that convexity of c implies DARA by Proposition 4.1.

Our general approach easily allows us to obtain admissible sets of parameters with additional convexity and concavity properties. Note, however, that even when the arguments for the convexity/concavity fail, it is still straightforward to write down expressions for c_w and analyse it explicitly. In particular, when $\psi > 0$ we see that $\beta > \frac{\sigma^2}{2\phi}(\phi + p\psi)^2$ guarantees that c_w is bounded and bounded away from zero. As above, Lemma 4.3 applies with either $\theta \leq \sigma\phi$ or both $\sigma\phi < \theta \leq \sigma(\phi + p\psi)$ and $\sigma\phi(\phi + p\psi) > \theta p\psi$. A numerical example which satisfies these conditions but for which $\Lambda(w)$ goes negative is given in Figure 2. It also features a risk aversion which is first decreasing, then increasing and then decreasing again.

Example 4.7. We present another example where consumption has a simple convex expresion in wealth. Consider

$$\pi(w) = \frac{2}{\sigma}\sqrt{\frac{r-\kappa}{2}w^2 + \alpha w + \frac{\alpha}{a}(\mathrm{e}^{-aw} - 1)}, \tag{72}$$

where we take $r > \kappa > 0$, $\alpha, a > 0$ with $\kappa > a\alpha$. Clearly $\pi(w)$ is an increasing function of w and

$$\delta_2 = \pi_w(0+) = \frac{\sqrt{2}}{\sigma}\sqrt{r - \kappa + \alpha a}, \quad \delta_1 = \pi_w(\infty-) = \frac{\sqrt{2}}{\sigma}\sqrt{r-\kappa}.$$

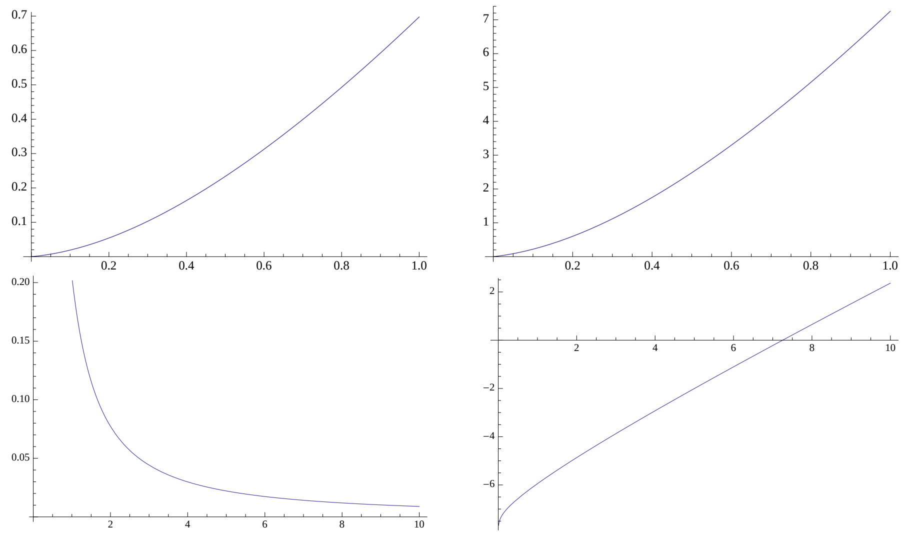

Figure 1: Graphs for Example 4.6 when $\pi_{ww} > 0$. *Top panes*: The investment strategy $\pi(w)$ (left) and the corresponding optimal consumption $c(w)$ (right) for parameters: $r = 0.3$, $\theta = 0.026$, $\sigma = 0.25$, $\beta = 10$, $p = 1/30$, $\phi = 2.1$ and $\psi = -60$. *Bottom panes*: The absolute risk aversion $\rho(t, c) = \rho(c)$ (left) inferred from these actions and a compatible utility function $u(0.1, c)$ (right).

From (55), with $\beta(t) \equiv 0$, we recover the optimal consumption as

$$c(w) = \kappa w + \alpha \left(e^{-aw} - 1\right) \tag{73}$$

which is an increasing convex function with $c_{ww}(w) = a^2 \alpha e^{-aw}$. Sufficient conditions for π, c to satisfy the assumptions of Lemma 4.3 are then concavity of π (from which we deduce $\delta_1 \leq \pi_w(w) \leq \delta_2$) and either $\theta \leq \sqrt{2(r-\kappa)}$ or $\sqrt{2(r-\kappa)} < \theta \leq \sqrt{2(r-\kappa+a\alpha)}$ and $\theta + \sqrt{2(r-\kappa+a\alpha)}) > \theta\sqrt{1 + a\alpha/(r-\kappa)}$. We will verify the concavity condition numerically for the cases of interest presented in Figure 3. Note that in this example c is convex and π is increasing so Proposition 4.1 implies that the agent employing these actions necessarily has a decreasing absolute risk aversion.

4.4 Extensions to Theorem 3.4

Our goal in Theorem 3.4 was to present an if and only if statement of our main result in the regular case. In this section we talk about a few of the extensions which are possible.

Firstly, note that in (*i*) of Theorem 3.4 we only assume regularity on u and in (*ii*) we only assume regularity on c, π. With a standing assumption that π is continuously differentiable in both arguments the equivalence holds under weaker definitions of regularity. In (*i*) it is then enough to require that $u(t, \cdot)$ is once (and not twice) continuously differentiable and we can drop Hölder continuity of I_z. In (*ii*) we then need that c is jointly continuous instead of differentiability in w and Hölder continuity of c_w. The key point is that we need to guarantee the existence of g_{tz} in the second part of the proof of the theorem,

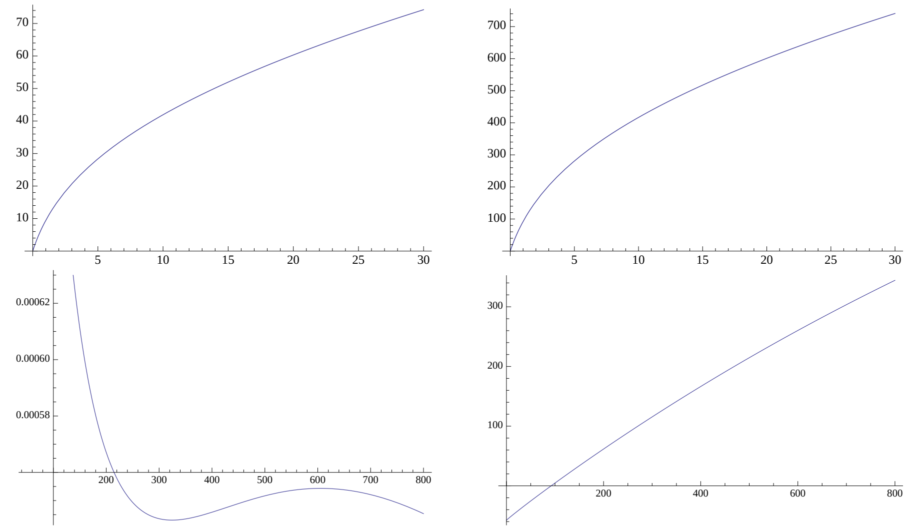

Figure 2: Graphs for Example 4.6 when $\pi_{ww} < 0$. *Top panes*: The investment strategy $\pi(w)$ (left) and the corresponding optimal consumption $c(w)$ (right) for parameters: $r = 0.05$, $\theta = 0.13$, $\sigma = 0.25$, $\beta = 10$, $p = 1/5$, $\phi = 0.5$ and $\psi = 60$. *Bottom panes*: The absolute risk aversion $\rho(t, c) = \rho(c)$ (left) inferred from these actions and a compatible utility function $u(0.1, c)$ (right).

and this can be done either with the assumptions of the theorem, or with weaker conditions if π is assumed to be continuously differentiable.

Another possible extension is to drop the Inada conditions on u. The implications of not requiring $u_c(t, 0) = \infty$ — in particular, consumption may be zero for an interval of positive wealths to the right of zero — is considered in He and Huang (1994).

Instead of pursuing this idea, for the rest of this section we will consider what happens if agent's utility includes a satiation point, or in the inverse case where we start with c and π, if the consumption is a bounded function of wealth. This may or may not imply that agent's wealth is bounded (and below we give examples which cover the two cases), but in the case where the wealth is bounded, there are similarities with the 'maximal' wealth path $\overline{w}(t)$, which arose in the deterministic setting.

Implicit in the definitions and results of Section 3.3 (and in Black (1968), Cox and Leland (2000), He and Huang (1994)) is the idea that the agent follows a strategy such that his wealth is unbounded in ω for each t. The result below considers the case where the consumption $c(t, w)$ may be bounded above, and then $W_t^x \le \overline{w}(t)$ a.s. where $\overline{w}(t) = \mathrm{e}^{rt} \int_t^\infty \mathrm{e}^{-rs} c(s, \infty) \mathrm{d}s$ which may be finite or infinite. In the former case, the agent consumes and invests in such a way that his wealth is kept below $\overline{w}(t)$ with probability one. We relate this explicitly to the properties of the utility function.

The following definition relaxes the notions from Definitions 3.2 and 3.3.

Definition 4.8. (i) We say that a function $u : [0, \infty)^2 \to [-\infty, \infty)$ is a *regular utility function* if for any $t \ge 0$ there exists $\overline{c}(t) > 0$ such that $u(t, \cdot)$ is twice continuously differentiable, strictly concave and increasing on $\{(t, c) \in (0, \infty)^2 :$

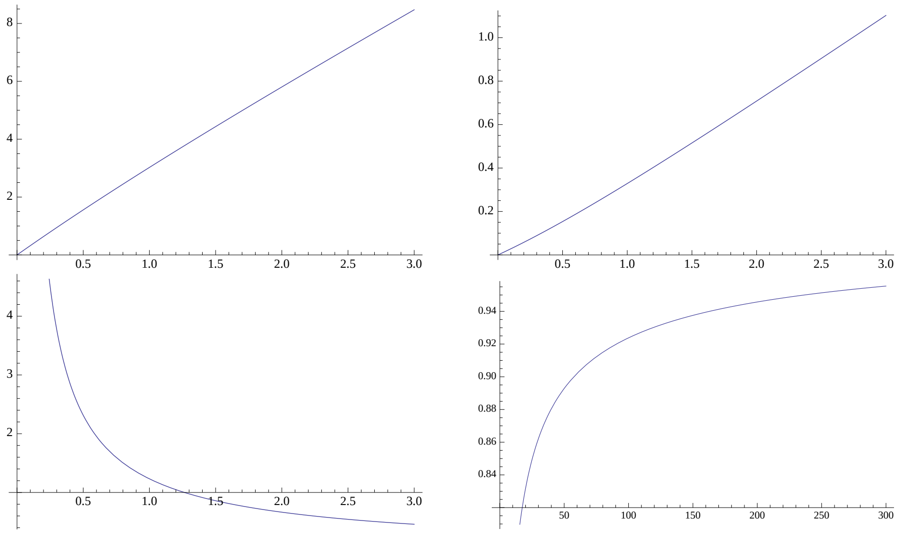

Figure 3: Graphs for Example 4.7. *Top panes*: The investment strategy $\pi(w)$ (left) in (72) and the corresponding optimal consumption $c(w)$ (right) in (73) for parameters: $\kappa = 0.4, \sigma = 0.25, r = 0.6, \alpha = 0.1, a = 1.25, \theta = 0.95$. *Bottom panes*: The absolute risk aversion $\rho(t, c) = \rho(c)$ (left) inferred from these actions and a compatible utility function $u(0.1, c)$ (right).

$c < \overline{c}(t)\}$, satisfies the Inada condition: $u_c(t, 0+) = \infty$ and $u_c(t, \overline{c}(t)-) = 0$, and $u(t, c)$ is constant for $c \geq \overline{c}(t)$. Further, I_z is locally Hölder continuous on $(0, \infty)^2$.

(ii) We say that (c, π) is a *regular consumption/investment pair* if

- for each $t \geq 0$, $c(t, 0) = 0$, $c(t, \cdot)$ is increasing and we have $\overline{w}(t) := e^{rt} \int_t^\infty e^{-rs} c(s, \infty) ds > 0$. Further, $c(t, \cdot)$ is strictly increasing and differentiable on $[0, \overline{w}(t))$ and constant and equal to $c(t, \overline{w}(t)-)$ on $[\overline{w}(t), \infty)$. Finally, $c_w(t, w)$ is locally Hölder continuous on $\{(t, w) \in (0, \infty)^2 : w < \overline{w}(t)\}$.

- for each $t \geq 0$, $\pi(t, 0) = 0$, $\pi(t, \cdot)$ is strictly positive and continuously differentiable in both arguments on $\{(t, w) \in (0, \infty)^2 : w < \overline{w}(t)\}$. Further, $\int_{0+} d\xi/\pi(t, \xi) = \infty = \int^{\overline{w}(t)} d\xi/\pi(t, \xi)$ and $\pi(t, w) = 0$ for $w \geq \overline{w}(t)$ when $\overline{w}(t) < \infty$.

Finally, c, π are such that the SDE (38) has a strong solution for $x < \overline{w}(0)$.

In addition, the definitions of β, A, H and F may need to be modified. If $\overline{w}(t) > \xi > 0$ for all $t \geq 0$ then it suffices to replace 1 in the lower bound of the integration in (29) and (39) by ξ. More generally, when (π, c) is a regular consumption/investment pair as given by the Definition 4.8, we have that $\overline{w}(t)$ is continuous and strictly positive and then there exists a smooth function $w_0(t)$ such that $0 < w_0(t) < \overline{w}(t)$. Thus we can replace the bottom limit in the integrals in (29) and (39) with $w_0(t)$, and replace $A(t)$ in (39) with

$$A_0(t) = -\frac{\theta}{\sigma} \int_0^t \beta(s) ds + \left(\frac{\theta^2}{2} - r\right) t - \int_0^t \frac{\theta w_0'(s)}{\sigma \pi(s, w_0(s))} ds.$$

Similarly, if $\overline{c}(t)$ is not bounded below by 1, then we replace the lower limit in the definition of H with $c_0(t)$ where $c_0(t) < \overline{c}(t)$ is some strictly positive function. Then $H(t,c) = \int_{c_0(t)}^{c} F(t, Y(t,b))db$.

Theorem 4.9. *For any $x > 0$, the following are equivalent:*

(i) $c(t, W_t^x)$ *and* $\pi(t, W_t^x)$ *achieve a finite maximum in the problem* (43) *for a regular utility function u of Definition 4.8 for which* (44) *holds, and where* $c, \pi : [0,\infty)^2 \to [0,\infty)$ *are such that if* $\overline{w}(t) := \inf\{w > 0 : \pi(t,w) = 0\} < \infty$ *then* $\pi(t,w) = 0$, $c(t,w) = c(t,\overline{w}(t))$ *for* $w \geq \overline{w}(t)$.

(ii) $c(t,w), \pi(t,w)$ *are a regular consumption/investment pair of Definition 4.8 which satisfy* (29) *on* $\{(t,w) \in (0,\infty)^2 : w < \overline{w}(t)\}$, (45) *holds and for some* $0 < x_0 \leq x$, $\mathbb{E}[|H(t, c(t, W_t^{x_0}))|] < \infty$ *for almost all* $t \geq 0$ *and* $\int_0^\infty \mathbb{E}[H(t, c(t, W_t^{x_0})) - h(t)]^+ \mathrm{d}t < \infty$, *where* $h(t) = \mathbb{E}[H(t, c(t, W_t^{x_0}))]$.

Moreover, we then have $u_c(t,c) = H_c(t,c)$, *and in (i) one may take* $u(t,c) = H(t,c) - h(t)$ *for* $c \leq \overline{c}(t)$.

Remark 4.10. The theorem holds if in the definition of a regular consumption/investment pair $\overline{w}(t)$ is just some function and we do not impose the consistency condition that $\overline{w}(t) := \mathrm{e}^{rt} \int_t^\infty \mathrm{e}^{-rs} c(s,\infty)\mathrm{d}s$. This condition is in fact implied by Black's equation (29) which gives $W_t^x = f(t, F(0,x)Z_t)$ and by the budget equation (45), as is clear from the proof of $(i) \Rightarrow (ii)$.

Proof. The proof is almost identical to the proof of Theorem 3.4. In the first part, when showing $(ii) \Rightarrow (i)$, observe that $u(t,c) = H(t,c) - h(t)$ is strictly increasing and concave on $[0, \overline{c}(t))$ and constant on $[\overline{c}(t), \infty)$, where $\overline{c}(t) = c(t,\infty) = c(t, \overline{w}(t))$. We have $u_c(t,0) = F(t, Y(t,0)) = F(t,0) = \infty$, $u_c(t, \overline{c}(t)) = F(t, Y(t, \overline{c}(t))) = F(t, \overline{w}(t)) = 0$ and $u_{cc}(t,l)$ is continuous on $(0, \overline{c}(t))$. In addition, $I(t,z) = c(t, f(t,z))$ and f is well defined and $\mathcal{C}^{1,2}$ on $(0,\infty)^2$. In consequence, $I_z(t,z)$ is locally Hölder continuous and u is a regular utility function of Definition 4.8. The modified definition of $F(t,w)$ (note in particular that (40)–(42) still hold) is important in (46), where the additional terms in $A_0(t)$ cancel with the extra term in F_t arising from the new definition.

For the second part of the proof, the implication $(i) \Rightarrow (ii)$, recall that now $\overline{c}(t)$ is defined from u in Definition 4.8. When we show that $c(t, W_t^x) = I(t, \lambda Z_t)$ this implies $c(t, W_t^x) < \overline{c}(t)$. Then, from the representation of g as the conditional expectation, letting $Z_t \to 0$ and using the Dominated Convergence Theorem, we obtain that $\overline{c}$ and $\overline{w}$ are related by $\overline{w}(t) = \mathrm{e}^{rt} \int_t^\infty \mathrm{e}^{-rs} \overline{c}(s)\mathrm{d}s$. We conclude that $c(t,0) = 0$, $c(t,\cdot)$ is strictly increasing on $(0, g(t,0))$, and $c(t, g(t,0)) = \overline{c}(t)$. Further, $c_w(t,w)$ is locally Hölder continuous on $\{(t,w) \in (0,\infty)^2 : w < g(t,0)\}$. As previously, (30)–(31) hold with g instead of f, for all $t, z > 0$. It follows that $\pi(t,0) = 0$, and $\pi(t,w) > 0$ for $0 < w < g(t,0)$. Moreover, if $g(t,0) < \infty$ then $g_z(t,0) > -\infty$ and $\pi(t, g(t,0)) = 0$. It follows from the assumed properties of c, π that $g(t,0) = \overline{w}(t)$ and $\overline{c}(t) = c(t,\infty)$. Similarly, we conclude that (29) holds for $\{(t,w) \in (0,\infty)^2 : w < \overline{w}(t)\}$. Finally, the last change is that now

$$\tau = \inf\{s : Z_s \notin (F(s, \overline{w}(s))/F(0,x), F(s,0)/F(0,x))\} = \inf\{s : \tilde{W}_s \in \{0, \overline{w}(s)\}\}$$

and $0 < W_t^x < \overline{w}(t)$ a.s. implies that $\tau = \infty$; i.e. $F(t,0) = \infty$ and $F(t, \overline{w}(t)) = 0$. □

Example 4.11. We now present an example where the situation as above holds: both the agent's consumption and her wealth process are bounded. Consider the time-homogeneous setting with $\pi(w) = \max\{w(1-w), 0\}$ and

$$c(w) = w\left(r - \frac{1}{2}\sigma^2 + \beta\right) + w^2\left(\frac{3}{2}\sigma^2 - \beta\right) - w^3\sigma^2, \quad w \in (0,1),$$

with $c(w) = r$ for $w \geq 1$. We shall suppose that $r > \frac{1}{2}\sigma^2$ and $|\beta| < r - \frac{1}{2}\sigma^2$. Observe that $c(0) = 0$, c is strictly increasing on $(0,1)$ with derivative bounded away from zero and infinity and c is continuous on $(0,\infty)$. Further, we have $\overline{w}(t) \equiv 1 = \int_t^\infty e^{r(t-s)}c(1)ds$. In particular, (c,π) are a regular consumption/investment pair which satisfy (29) as described above. Moreover, since W_t is bounded above, similar arguments to those used at the start of the proof of Lemma 4.3 can be used to deduce (45). Finally, we can also check the integrability conditions of *(ii)* of Theorem 4.9. Then we can conclude that there exists a regular utility function for which this consumption and investment are optimal.

First note that we have

$$F(t,w) = e^{A(t)}\left(\frac{w}{1-w}\right)^{-\theta/\sigma},$$

so that

$$\frac{1}{2}w^{-\theta/\sigma}\mathbf{1}_{\{w<\delta\}} \leq e^{-A(t)}F(t,w) \leq w^{-\theta/\sigma} \tag{74}$$

for $w \in (0,1)$, and for some $\delta < \frac{1}{2}$. Again, as in the proof of Lemma 4.3, we can write

$$H(t, c(t,w)) = \int_{\frac{1}{2}}^{w} F(t,\tilde{w})c_w(t,\tilde{w})d\tilde{w}.$$

Since c_w is bounded from above and below for $w \in (0,1)$, then we deduce from (74) that $H(t,c(t,w))^+ \leq \kappa_1 e^{A(t)}$ for some constant $\kappa_1 > 0$, and $H(t,c(t,w))^- \leq \kappa_2 e^{A(t)}$ when $\theta < \sigma$, and $H(t,c(t,w))^- \leq \kappa_3 e^{A(t)}w^{1-\theta/\sigma}$ when $\theta > \sigma$, for some $\kappa_2, \kappa_3 > 0$. (We exclude the case $\theta = \sigma$). Writing $f(t,z)$ for the inverse of $F(t,w)$ and using (74), we get $f(t,Z_t) \leq e^{A(t)\sigma/\theta}(Z_t \vee K)^{-\sigma/\theta}$, for some $K > 0$, and we conclude that the desired integrability conditions hold provided

$$\int_0^\infty e^{A(t)\sigma/\theta}\left(\mathbb{E}\left[Z_t^{1-\sigma/\theta}\right] + K^{1-\sigma/\theta}\right)dt < \infty.$$

Calculations similar to those in the proof of Lemma 4.3 show that this will hold whenever:

$$\beta - \sigma\theta + r - \frac{1}{2}\sigma^2 > 0 \text{ and } A(1) = -\frac{\theta}{\sigma}\beta + \frac{1}{2}\theta^2 - r < 0.$$

It is now clear that there are non-trivial parameter choices for which this example satisfies the conditions of Theorem 4.9, and therefore, such that there exists a utility function u for which these are the optimal investment/consumption pair.

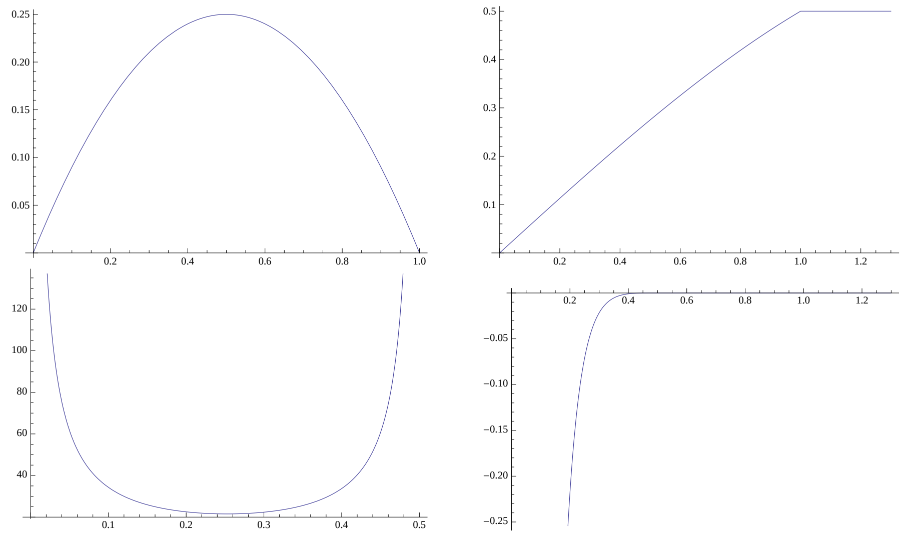

Figure 4: Graphs for Example 4.11. *Top panes*: The investment strategy $\pi(w) = w(1-w) \vee 0$ (left) and the corresponding optimal consumption $c(w)$ (right) for parameters: $r = 0.5$, $\theta = 0.7$, $\sigma = 0.25$, $\beta = 0.1$. *Bottom panes*: The absolute risk aversion $\rho(1, c)$ (left) inferred from these actions and a compatible utility function $u(1, c)$ (right) which is constant on $[r, \infty)$.

Example 4.12. Finally, we present an example in which the consumption is bounded while the agent's wealth is unbounded. Consider the time-homogeneous setting with $\pi(w) = 1 - \mathrm{e}^{-w}$ and

$$c(w) = \left(\beta - \frac{\sigma^2}{2}\mathrm{e}^{-w}\right)\pi(w), \quad w \geq 0.$$

We assume $r = 0$, $\theta < \sigma$ and $\beta > \sigma^2$. It follows that c is an increasing function with $c(0) = 0$ and $c(\infty) = \beta$. Further, π, c satisfy Black's equation (29) as c is given by (55). Explicit computations yield

$$F(t, w) = \mathrm{e}^{A(t)} \left(\mathrm{e}^{w} - 1\right)^{-\theta/\sigma}, \quad f(t, z) = \log\left(\mathrm{e}^{A(t)\sigma/\theta} z^{-\sigma/\theta} + 1\right),$$

where we used $\log 2$ instead of 1 as the lower bound of integration in (39). Then c, π are a regular consumption/investment pair of Definition 4.8 with $\overline{w}(t) = \infty = \int_t^\infty c(\infty)\mathrm{d}s$, and we note that $F(t, \infty) = 0$ as required. Similarly to Example 4.11 above, it is easy to see that $H(t, c(t, w)) \leq \kappa \mathrm{e}^{A(t)}$ when $\theta < \sigma$. Likewise, with arguments akin to that in the proof of Lemma 4.3, we obtain that $W_t^x \to 0$ and (45) is equivalent to showing that $\mathbb{E}^{\mathbb{Q}}[W_t^x] \to 0$. To verify this we use Remark 3.5 and compute

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[W_t^x] &= \mathbb{E}^{\mathbb{Q}}[f(t, F(0, x)Z_t)] = \mathbb{E}^{\mathbb{Q}}\left[\log\left(\mathrm{e}^{\sigma(B_t + \theta t) - \beta t}(\mathrm{e}^{x} - 1) + 1\right)\right] \\ &\leq \log\left(\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{\sigma(B_t + \theta t) - \beta t}(\mathrm{e}^{x} - 1) + 1\right]\right) = \log\left(\mathrm{e}^{-(\beta - \sigma^2/2)t}(\mathrm{e}^{x} - 1) + 1\right) \\ &\to 0,\end{aligned}$$

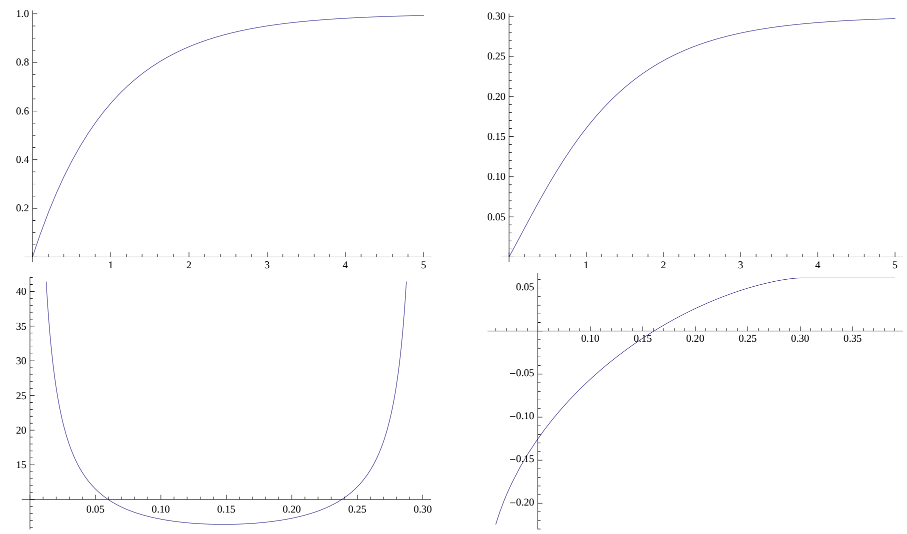

Figure 5: Graphs for Example 4.12. *Top panes*: The investment strategy $\pi(w) = (1 - e^{-w})$ (left) and the corresponding optimal consumption $c(w)$ (right) for parameters: $r = 0$, $\theta = 0.25$, $\sigma = 0.5$, $\beta = 0.3$. *Bottom panes*: The absolute risk aversion $\rho(t, c) = \rho(c)$ (left) inferred from these actions and a compatible utility function $u(0.1, c)$ (right) which is constant on $[\beta, \infty)$.

as $t \to \infty$, and where we used Jensen's inequality, the fact that $B_t + \theta t$ is a $\mathbb{Q}$–Brownian motion and the assumption $\beta > \sigma^2/2$. Finally, note that $\beta \geq \sigma^2/2$ and $\theta < \sigma$ together imply $A(1) = -\frac{\theta}{\sigma}\beta + \frac{1}{2}\theta^2 < 0$ and we conclude that all the assumptions in (ii) in Theorem 4.9 are satisfied.

5 Further research

The work presented in this paper may be seen as the first step which motivates exploration of a set of wider related questions. Our underpinning principle is to start with those actions which may be observed in an investor's behaviour, and then attempt to determine whether their actions are consistent with utility maximisation. In this paper, we have considered two cases: a deterministic setup and a stochastic complete market setup. In the deterministic case, our fundamental conclusion is that observing investor's actions for any given wealth is not enough to fully specify their utility. Risk aversion remains unspecified. In the stochastic case, we suppose we observe both consumption and investment. Then the assumption that the investor is maximising utility implies that the consistency constraint (29) holds. These two studies would have natural, and interesting, analogues in other markets such as one period models (where the investor can choose to consume now or in the subsequent period), or, at the other extreme in terms of complexity, continuous-time models which are incomplete (e.g. a stochastic factor model). The questions parallel to those which we have answered here would include:

- is specifying an investor's consumption and investment strategies sufficient to determine their utility function (up to constants)?

- If it is not, is the system over-specified or under-specified?

In the case where the system is over-specified, we might expect a consistency condition such as Black's equation, (29), and it is interesting to ask whether there is a more general optimisation problem such as (19) which may correspond to the general choice of consumption and investment.

Following on from Section 4.1, it would be interesting to incorporate a form of model uncertainty and ask if agents' actions may arise from maximising *some* utility under *some* dynamics of the price process. Within our framework, the scope for such questions was limited since we only considered Black-Scholes dynamics for the price process. A more extensive study could be based on the work of He and Huang (1994), see also Example 3 therein, or follow on from the research suggested above.

An even more ambitious task would be to consider an inverse problem to the classical analysis of decision making under model uncertainty (also called Knightian uncertainty). In that framework, one specifies actors' preferences and a way to quantify their uncertainty about the true price dynamics. One then asks what are the actors' optimal actions, see e.g. Schied (2007), Föllmer et al. (2009). The inverse approach would start with agents' actions and try to recover their preferences as well as their belief about model uncertainty.

Finally, a nice feature of the paper is that we have been able to go beyond simply recovering the utility function, and have also been able to provide characterisations of certain aspects of the agent's behaviour (absolute and relative risk aversion) in terms of the given data. In more complex situations, where it may not be possible to fully recover an agent's utility function, it may still be possible to deduce some of these related properties from the given data.

We believe the questions raised above form an exciting research programme and we hope to pursue some of them in subsequent work.

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