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# FORMAL CONSERVED QUANTITIES FOR ISOTHERMIC SURFACES

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ABSTRACT. Isothermic surfaces in  $S^n$  are characterised by the existence of a pencil  $\nabla^t$  of flat connections. Such a surface is special of type  $d$  if there is a family  $p(t)$  of  $\nabla^t$ -parallel sections whose dependence on the spectral parameter  $t$  is polynomial of degree  $d$ . We prove that any isothermic surface admits a family of  $\nabla^t$ -parallel sections which is a formal Laurent series in  $t$ . As an application, we give conformally invariant conditions for an isothermic surface in  $S^3$  to be special.

**Keywords:** special isothermic surfaces, polynomial and formal conserved quantities.

**MSC:** 53A30, 53A05.

## INTRODUCTION

Isothermic surfaces, that is, surfaces which admit conformal curvature line coordinates, were intensively studied around the turn of the 20th century by Darboux, Bianchi and others [1, 8, 9, 12]. These classical works revealed a rich transformation theory that has been revisited in modern times from the viewpoint of integrable systems [2, 10, 16]. At the heart of the integrable systems formalism is the observation that there is a pencil of flat connections  $\nabla^t = d + t\eta$ ,  $t \in \mathbb{R}$ , associated to each isothermic surface.

In our previous work [4], we distinguished the class of *special isothermic surfaces* (of type  $d$ ) which are characterised by the existence of a *polynomial conserved quantity*, that is, a family  $p(t)$  of  $\nabla^t$ -parallel sections whose dependence on  $t$  is polynomial (of degree  $d$ ). The existence of such a polynomial conserved quantity amounts to a differential equation on the principal curvatures of the surface. For example, an isothermic surface in  $S^3$  is special of type 1 if it has constant mean curvature with respect to a constant curvature metric on (an open subset of)  $S^3$  and special of type 2 if it is a special isothermic surface in the sense of Darboux and Bianchi [1, 11]—a class of isothermic surfaces that originally arose in the study of surfaces isometric to a quadric.

The purpose of the present paper is to answer a question posed to one of us by Nigel Hitchin: does any isothermic surface in  $S^n$  admit a *formal conserved quantity*, thus a solution  $p(t)$  of  $\nabla^t p(t) = 0$  with  $p(t) = \sum_{i \leq 0} p_i t^i$  a formal Laurent series? We give an affirmative answer locally, away from the (discrete) zero-set of  $\eta$ , and globally when  $n = 3$ . In particular, any isothermic 2-torus in  $S^3$  admits a formal conserved quantity. This is an analogue of the existence of formal Killing fields for harmonic maps [3] although the method is somewhat different since here we deal with nilpotent rather than semisimple gauge potentials.

Since polynomial conserved quantities are also formal conserved quantities, our arguments allow us to give conformally invariant conditions, in terms of the Schwarzian derivative and Hopf differential introduced in [6], for an isothermic surface to be

special of type  $d$ . We illustrate these results with the case of surfaces of revolution and other equivariant surfaces where these conditions amount to a differential equation on the curvature of a profile curve.

Some of the following results can also be found in the second author's doctoral thesis [15], using a different approach.

## 1. PRELIMINARIES

**1.1. The conformal sphere.** We will study isothermic surfaces in the  $n$ -sphere from a conformally invariant view-point and so use Darboux's light-cone model of the conformal  $n$ -sphere. For this, contemplate the light-cone  $\mathcal{L}$  in the Lorentzian vector space  $\mathbb{R}^{n+1,1}$  and its projectivisation  $\mathbb{P}(\mathcal{L})$ . This last has a conformal structure where representative metrics  $g_\sigma$  arise from never-zero sections  $\sigma$  of the tautological bundle  $\pi : \mathcal{L} \rightarrow \mathbb{P}(\mathcal{L})$  via

$$g_\sigma(X, Y) = (d\sigma(X), d\sigma(Y)).$$

Then  $S^n \cong \mathbb{P}(\mathcal{L})$  *qua* conformal manifolds. Indeed, for non-zero  $w \in \mathbb{R}^{n+1,1}$ , let  $E(w)$  be the conic section given by

$$E(w) = \{v \in \mathcal{L} : (v, w) = -1\}$$

with (definite) metric induced by the ambient inner product on  $\mathbb{R}^{n+1,1}$ . Then  $\pi|_{E(w)}$  is a conformal diffeomorphism onto its image. In particular, when  $w_0$  is unit time-like, we have an isometry  $x \mapsto x + w_0$  from the unit sphere in  $\langle w_0 \rangle^\perp$  to  $E(w_0)$  and thus a conformal diffeomorphism from that sphere to  $\mathbb{P}(\mathcal{L})$ . More generally,  $E(w)$  has constant sectional curvature  $-(w, w)$ .

**1.2. Invariants of a conformal immersion.** Let  $\Sigma$  be a Riemann surface and  $\Lambda : \Sigma \rightarrow S^n \cong \mathbb{P}(\mathcal{L})$  a conformal immersion. We view  $\Lambda$  as a null line subbundle of the trivial bundle  $\underline{\mathbb{R}}^{n+1,1} = \Sigma \times \mathbb{R}^{n+1,1}$ .

The *central sphere congruence* assigns, to each  $x \in \Sigma$ , the unique 2-sphere  $S(x)$  tangent to  $\Lambda$  at the point  $\Lambda(x)$ , which has the same mean curvature vector as  $\Lambda$  at  $x$ . Having in mind the identification between 2-dimensional subspheres of  $S^n$  and  $(3, 1)$ -planes of  $\mathbb{R}^{n+1,1}$  via

$$V \mapsto \mathbb{P}(\mathcal{L} \cap V),$$

the central sphere congruence of  $\Lambda$  amounts to a subbundle  $V$  of  $\underline{\mathbb{R}}^{n+1,1}$ , with signature  $(3, 1)$ .

Fix a holomorphic coordinate  $z = u + iv$  on  $\Sigma$  and take the unique (up to sign) lift  $\psi \in \Gamma\Lambda$  such that

$$|d\psi|^2 = |dz|^2.$$

Then

$$V \otimes \mathbb{C} = \langle \psi, \psi_z, \psi_{\bar{z}}, \psi_{z\bar{z}} \rangle.$$

Consider now the unique section  $\hat{\psi} \in \Gamma(V)$  such that

$$(\hat{\psi}, \hat{\psi}) = 0, (\hat{\psi}, \psi) = -1 \text{ and } (\hat{\psi}, d\psi) = 0,$$

which provides a new frame for  $V \otimes \mathbb{C}$ , namely

$$\psi, \psi_z, \psi_{\bar{z}} \text{ and } \hat{\psi}.$$

According to [6], we have

$$\psi_{z\bar{z}} + \frac{c}{2}\psi = \kappa,$$

for a complex function  $c$  and  $\kappa \in \Gamma(V^\perp \otimes \mathbb{C})$ . These latter invariants are, respectively, the *Schwarzian derivative* and *Hopf differential* of  $\Lambda$  with respect to  $z$  and,

together with the connection  $D$  on  $V^\perp$  given by orthoprojection of flat differentiation, determine  $\Lambda$  up to conformal diffeomorphisms of  $S^n$ .

Our frame satisfies:

$$(1.1) \quad \begin{aligned} \psi_{zz} &= -\frac{c}{2}\psi + \kappa \\ \psi_{z\bar{z}} &= -(\kappa, \bar{\kappa})\psi + \frac{1}{2}\hat{\psi} \\ \hat{\psi}_z &= -2(\kappa, \bar{\kappa})\psi_z - c\psi_{\bar{z}} + 2D_{\bar{z}}\kappa \\ \xi_z &= 2(\xi, D_{\bar{z}}\kappa)\psi - 2(\xi, \kappa)\psi_{\bar{z}} + D_z\xi, \end{aligned}$$

for each  $\xi \in \Gamma(V^\perp \otimes \mathbb{C})$ . The corresponding structure equations are the *conformal Gauss equation*:

$$\frac{1}{2}c_{\bar{z}} = 3(\kappa, D_z\bar{\kappa}) + (\bar{\kappa}, D_z\kappa);$$

the *conformal Codazzi equation*:

$$\operatorname{Im}(D_{\bar{z}}D_z\kappa + \frac{1}{2}\bar{c}\kappa) = 0$$

and the *conformal Ricci equation*:

$$D_{\bar{z}}D_z\xi - D_zD_{\bar{z}}\xi - 2\langle \xi, \kappa \rangle \bar{\kappa} + 2\langle \xi, \bar{\kappa} \rangle \kappa = 0.$$

See [6] for more details.

**1.3. Isothermic and special isothermic surfaces.** Classically, an isothermic surface is a surface in  $S^n$  that admits conformal curvature line coordinates but we shall follow [7, 13] and adopt the following conformally invariant formulation:

**Definition 1.1.** An immersion  $\Lambda : \Sigma \rightarrow S^n \cong \mathbb{P}(\mathcal{L})$ ,  $\Lambda$  is an *isothermic surface* if there is a non-zero closed 1-form  $\eta \in \Omega^1 \otimes o(\mathbb{R}^{n+1,1})$  taking values in  $\Lambda \wedge \Lambda^\perp$ <sup>1</sup>.

One makes contact with the classical formulation by defining  $q \in \Gamma(S^2T^*\Sigma)$  by

$$q(X, Y)\sigma = \eta_X d_Y \sigma,$$

for any  $\sigma \in \Gamma\Lambda$ . Then  $d\eta = 0$  if and only if  $q$  is a holomorphic quadratic differential which commutes with the second fundamental form of  $\Lambda$ . Now  $q$  and hence  $\eta$  vanishes only on a discrete set and, off that set, we can find a holomorphic coordinate  $z$  such that  $q = dz^2$ . In terms of the corresponding lift  $\psi$  of Section 1.2, we then have

$$\eta = -\psi \wedge (\psi_z dz + \psi_{\bar{z}} d\bar{z})$$

which commutes with the second fundamental form if and only if  $\kappa$  is real. In this case,  $z = u + iv$  where  $u, v$  are curvature line coordinates.

The conformal Gauss and Codazzi equations are now given simply by

$$c_{\bar{z}} = 4(\kappa, \kappa)_z \text{ and } \operatorname{Im}(D_{\bar{z}}D_z\kappa + \frac{1}{2}\bar{c}\kappa) = 0,$$

while the conformal Ricci equation amounts to the familiar assertion that the connection  $D$  on  $V^\perp$  is flat.

The key to the integrable systems theory of isothermic surfaces is the observation that the family of metric connections  $d + t\eta$ ,  $t \in \mathbb{R}$  on  $\mathbb{R}^{n+1,1}$  are *flat* and so have a good supply of parallel sections. In [4], we considered isothermic surfaces which admitted parallel sections with polynomial dependence on  $t$  and so introduced the notion of special isothermic surfaces of type  $d \in \mathbb{N}_0$ :

<sup>1</sup>Recall the isomorphism  $\wedge^2 \mathbb{R}^{n+1,1} \cong o(\mathbb{R}^{n+1,1})$  via  $(u \wedge v)w = (u, w)v - (v, w)u$ , for all  $u, v, w \in \mathbb{R}^{n+1,1}$ .

**Definition 1.2.** An isothermic surface  $(\Lambda, \eta)$  in  $S^n$  is a *special isothermic surface of type  $d \in \mathbb{N}_0$*  if there is a polynomial  $p(t) = \sum_{i=0}^d p_i t^i \in \Gamma(\mathbb{R}^{n+1,1})[t]$  of degree  $d$  such that  $(d + t\eta)p(t) \equiv 0$ . We call such a  $p(t)$  a *polynomial conserved quantity* of  $(\Lambda, \eta)$ .

An immediate consequence of  $(d + t\eta)p(t) \equiv 0$  is that  $dp_0 = 0$  so that  $p_0$  is constant and therefore, if non-zero, defines a conic section as in Section 1.1. We therefore refine our definition:

**Definition 1.3.** An isothermic surface  $(\Lambda, \eta)$  in  $S^n$  is a *special isothermic surface of type  $d \in \mathbb{N}_0$  in  $E(w)$*  if  $(\Lambda, \eta)$  admits a polynomial conserved quantity  $p(t) = \sum_{i=0}^d p_i t^i$  of degree  $d$  with  $p_0 \in \langle w \rangle$ .

The condition that an isothermic surface  $(\Lambda, \eta)$  be special of type  $d$  amounts to a differential equation on the principal curvatures of  $\Lambda$ . For example, generically,  $(\Lambda, \eta)$  is a special isothermic surface of type 1 in  $E(w)$  if and only if the lift  $F : \Sigma \rightarrow E(w)$  of  $\Lambda$  is a generalised  $H$ -surface (which, in codimension 1, amounts to the mean curvature  $H$  being constant) ([5], see also [4]). Again, in [4], we show that  $(\Lambda, \eta)$  is special of type 2 in  $E(w)$  if and only if there are real constants  $A, B$  and  $C$  such that

$$(1.2) \quad \begin{cases} H_{uu} + \theta_u H_u - \theta_v H_v - \frac{1}{2} M k_1 - A k_1 - B e^{-2\theta} + C - \frac{1}{2} L(w, w) = 0 \\ H_{vv} - \theta_u H_u + \theta_v H_v + \frac{1}{2} M k_2 + A k_2 - B e^{-2\theta} - C + \frac{1}{2} L(w, w) = 0, \end{cases}$$

where  $z = u + iv$  is a holomorphic coordinate for which  $\eta = -\psi \wedge (\psi_{\bar{z}} dz + \psi_z d\bar{z})$ ,

$$I = e^{2\theta}(du^2 + dv^2) \text{ and } II = e^{2\theta}(k_1 du^2 + k_2 dv^2)$$

are, respectively, the first and second fundamental forms of the lift  $F : \Sigma \rightarrow E(w)$  of  $\Lambda$ ,  $H = \frac{k_1 + k_2}{2}$  is the mean curvature of  $F$ ,  $L = e^{2\theta}(k_1 - k_2)$  and  $M = -HL$ . This condition amounts to the surface being a special isothermic surface in the sense of Darboux and Bianchi [1, 12], at least when  $H_u H_v$  is non-zero (see [4]).

## 2. FORMAL CONSERVED QUANTITIES

Denote by  $\Gamma(\mathbb{R}^{n+1,1})[[t, t^{-1}]]$  the vector space of the formal Laurent series in  $t$  with coefficients in  $\Gamma(\mathbb{R}^{n+1,1})$ , i.e., the series  $\sum_{k \leq s} p_k t^k$ , for some  $s \in \mathbb{N}_0$ , with coefficients in  $\Gamma(\mathbb{R}^{n+1,1})$ .

We define a  $\mathbb{R}[[t, t^{-1}]]$  valued inner product on  $\Gamma(\mathbb{R}^{n+1,1})[[t, t^{-1}]]$  by

$$(p(t), q(t)) := \sum_{k \leq s+r} \sum_{\substack{i \leq s, j \leq r \\ i+j=k}} (p_i, q_j) t^k,$$

for all  $p(t) = \sum_{k \leq s} p_k t^k, q(t) = \sum_{k \leq r} q_k t^k \in \Gamma(\mathbb{R}^{n+1,1})[[t, t^{-1}]]$ .

**Definition 2.1.** Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^n$  and let  $p(t) = \sum_{i \leq 0} p_i t^i \in \Gamma(\mathbb{R}^{n+1,1})[[t, t^{-1}]]$  such that  $p_0 \neq 0$ . We say that  $p(t)$  is a *formal conserved quantity* of  $(\Lambda, \eta)$  if  $(d + t\eta)p(t)$  is the zero series.

**Proposition 2.2.** Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^n$ . If  $p(t) = \sum_{i \leq 0} p_i t^i$  is a formal conserved quantity of  $(\Lambda, \eta)$ , then

- (1)  $p_0$  is a  $D$ -parallel section of  $V^\perp$ ;
- (2) the series  $(p(t), p(t)) \in \mathbb{R}[[t, t^{-1}]]$  (thus independent of  $x \in \Sigma$ ).

*Proof.* Item (1) is proved in [4, Proposition 2.2].

That the coefficients of  $(p(t), p(t))$  are constant follows from the fact that  $d + t\eta$  is a metric connection:

$$\begin{aligned} d(p(t), p(t)) &= \sum_{k \leq 0} \sum_{\substack{i, j \leq 0 \\ i+j=k}} d(p_i, p_j) t^k \\ &= \sum_{k \leq 0} \sum_{\substack{i, j \leq 0 \\ i+j=k}} (((d+t\eta)p_i, p_j) + (p_i, (d+t\eta)p_j)) t^k \\ &= ((d+t\eta)p(t), p(t)) + (p(t), (d+t\eta)p(t)) = 0. \end{aligned}$$

□

The condition that  $p(t)$  be a formal conserved quantity amounts to a recursive system of equations on its coefficients which we now describe in terms of the frame  $\psi, \psi_z, \psi_{\bar{z}}, \hat{\psi}$  of  $V$  associated with a holomorphic coordinate  $z$  as in Section 1.3.

**Proposition 2.3.** *Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^n$  and let  $z = u + iv$  be a holomorphic coordinate on  $\Sigma$  such that  $\eta = -\psi \wedge (\psi_{\bar{z}} dz + \psi_z d\bar{z})$ .*

*Let  $p(t) = \sum_{i \leq 0} p_i t^i \in \Gamma(\mathbb{R}^{n+1,1})[[t, t^{-1}]]$  and, for  $i \in \mathbb{Z}_0^-$ , write*

$$p_i = \alpha_i \psi + \beta_i \psi_z + \bar{\beta}_i \psi_{\bar{z}} + \gamma_i \hat{\psi} + q_i,$$

*where each  $\alpha_i, \gamma_i$  is a real function,  $\beta_i$  is a complex function and  $q_i \in \Gamma(V^\perp)$ . Then  $p(t)$  is a formal conserved quantity if and only if, for all  $i \in \mathbb{Z}_0^-$ ,*

$$(2.1a) \quad \beta_i = -2\gamma_{i, \bar{z}}$$

$$(2.1b) \quad \alpha_i = 2\gamma_{i, z\bar{z}} + 2(\kappa, \kappa)\gamma_i$$

$$(2.1c) \quad D_z q_i = 2\gamma_{i, \bar{z}} \kappa - 2\gamma_i D_{\bar{z}} \kappa$$

$$(2.1d) \quad \gamma_{i-1} = 2\gamma_{i, zz} + c\gamma_i + 2(q_i, \kappa).$$

*Proof.* For each  $i \in \mathbb{Z}_0^-$ , we have, using (1.1),

$$\begin{aligned} p_{i,z} + \eta \frac{\partial}{\partial \bar{z}} p_{i-1} &= \left( \alpha_{i,z} - \frac{c\beta_i}{2} - \bar{\beta}_i(\kappa, \kappa) + 2(q_i, \kappa_{\bar{z}}) + \frac{\beta_{i-1}}{2} \right) \psi \\ &\quad + (\alpha_i + \beta_{i,z} - 2(\kappa, \kappa)\gamma_i) \psi_z + (\bar{\beta}_{i,z} - c\gamma_i - 2(q_i, \kappa) + \gamma_{i-1}) \psi_{\bar{z}} \\ &\quad + \left( \frac{\bar{\beta}_i}{2} + \gamma_{i,z} \right) \hat{\psi} \\ &\quad + (\beta_i \kappa + 2\gamma_i D_{\bar{z}} \kappa + D_z q_i). \end{aligned}$$

The vanishing of the  $\hat{\psi}$  coefficient is equivalent to  $\beta_i = -2\gamma_{i, \bar{z}}$  since  $\gamma_i$  is real and then (2.1b) and (2.1d) amount to the vanishing of the coefficients of  $\psi_z, \psi_{\bar{z}}$  respectively while (2.1c) is the same as the vanishing of the normal component. We are left with the  $\psi$  component but its vanishing is a differential consequence of (2.1). Indeed:

$$\begin{aligned} \beta_{i-1} &= -2\gamma_{i-1, \bar{z}} = -2(2\gamma_{i, z\bar{z}} + c_{\bar{z}}\gamma_i + c\gamma_{i, \bar{z}} + 2(q_{i, \bar{z}}, \kappa) + 2(q_i, \kappa_{\bar{z}})) \\ &= -2(\alpha_{i,z} - 2(\kappa, \kappa)_z \gamma_i - 2(\kappa, \kappa)\gamma_{i,z} + c_{\bar{z}}\gamma_i + c\gamma_{i, \bar{z}} + 2(q_{i, \bar{z}}, \kappa) + 2(q_i, \kappa_{\bar{z}})) \\ &= -2(\alpha_{i,z} + 2(\kappa, \kappa)_z \gamma_i + (\kappa, \kappa)\bar{\beta}_i - \frac{c\beta_i}{2} + 2(q_{i, \bar{z}}, \kappa) + 2(q_i, \kappa_{\bar{z}})) \end{aligned}$$

while

$$(q_{i, \bar{z}}, \kappa) = (-\bar{\beta}_i \kappa - 2\gamma_i \kappa_z, \kappa) = -\bar{\beta}_i(\kappa, \kappa) - \gamma_i(\kappa, \kappa)_z.$$

□

The equations (2.1) amount to a recursive scheme for constructing a formal conserved quantity starting with  $\gamma_0$  so long as we can be assured that each  $\gamma_{i-1}$  defined by (2.1d) is real and that (2.1c) is solvable for each  $i$ . For this, we need:

**Lemma 2.4.** *Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^n$  and let  $z = u + iv$  be a holomorphic coordinate on  $\Sigma$  such that  $\eta = -\psi \wedge (\psi_{\bar{z}} dz + \psi_z d\bar{z})$ .*

*Let  $\gamma$  be a real function and  $q \in \Gamma(V^\perp)$  such that*

$$D_z q = 2\gamma_{\bar{z}} \kappa - 2\gamma D_{\bar{z}} \kappa.$$

*Define  $\hat{\gamma}$  by*

$$\hat{\gamma} = 2\gamma_{zz} + c\gamma + 2(q, \kappa)$$

*and suppose  $\hat{\gamma}$  is real. Then  $2\hat{\gamma}_{zz} + c\hat{\gamma}$  is also real.*

*Proof.* We compute:

$$2\hat{\gamma}_{\bar{z}\bar{z}} + \bar{c}\hat{\gamma} = 4\gamma_{z\bar{z}\bar{z}} + 2c_{\bar{z}\bar{z}}\gamma + 4c_{\bar{z}}\gamma_{\bar{z}} + 2c\gamma_{\bar{z}\bar{z}} + 4(q, \kappa)_{\bar{z}\bar{z}} + \bar{c}(2\gamma_{zz} + c\gamma + 2(q, \kappa)).$$

The conformal Gauss equation says that  $c_{\bar{z}} = 4(\kappa, \kappa)_z$  that  $c_{\bar{z}\bar{z}} = 4(\kappa, \kappa)_{z\bar{z}}$  is real and we readily conclude that

$$\begin{aligned} \operatorname{Im}(2\hat{\gamma}_{\bar{z}\bar{z}} + \bar{c}\hat{\gamma}) &= \operatorname{Im}(4c_{\bar{z}}\gamma_{\bar{z}} + 4(q, \kappa)_{\bar{z}\bar{z}} + 2\bar{c}(q, \kappa)) \\ &= \operatorname{Im}(16\gamma_{\bar{z}}(\kappa, \kappa)_z + 4(D_{\bar{z}\bar{z}}^2 q, \kappa) + 8(D_{\bar{z}} q, D_{\bar{z}} \kappa) + 2(q, 2D_{\bar{z}\bar{z}}^2 \kappa + \bar{c}\kappa)) \\ &= \operatorname{Im}(16\gamma_{\bar{z}}(\kappa, \kappa)_z + 4(D_{\bar{z}\bar{z}}^2 q, \kappa) + 8(D_{\bar{z}} q, D_{\bar{z}} \kappa)), \end{aligned}$$

thanks to the conformal Codazzi equation. Now differentiate the complex conjugate of the equation for  $q$  and substitute in to get, after a short computation:

$$\operatorname{Im}(2\hat{\gamma}_{\bar{z}\bar{z}} + \bar{c}\hat{\gamma}) = \operatorname{Im}(8\gamma_{z\bar{z}}(\kappa, \kappa) + 12(\gamma_z(\kappa, \kappa)_{\bar{z}} + \gamma_{\bar{z}}(\kappa, \kappa)_z)) = 0.$$

□

**Theorem 2.5.** *Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^n$ . Then locally, away from the zeros of  $\eta$ ,  $(\Lambda, \eta)$  has always a formal conserved quantity.*

*Proof.* We work on a simply connected open set with holomorphic coordinate  $z$  for which  $\eta = -\psi \wedge (\psi_{\bar{z}} dz + \psi_z d\bar{z})$ . We inductively construct a formal power series  $p(t) = \sum_{i \leq 0} p_i t^i$  with

$$p_i = \alpha_i \psi + \beta_i \psi_z + \bar{\beta}_i \psi_{\bar{z}} + \gamma_i \hat{\psi} + q_i,$$

with  $\alpha_i, \gamma_i$  real,  $\beta_i$  complex and  $q_i$  a section of  $V^\perp$  satisfying (2.1). Then, by Proposition 2.3,  $p(t)$  will be a formal conserved quantity.

We begin by taking  $\gamma_0 = 0$  and  $q_0$  a non-zero parallel section of  $V^\perp$  so that  $p_0 = q_0$ . The issue is to define  $\gamma_i$  and  $q_i$  for then  $\alpha_i, \beta_i$  are given by (2.1a) and (2.1b). Suppose now that we have  $\gamma_j, q_j, j > i \in \mathbb{Z}^-$  with

$$\begin{aligned} D_z q_j &= 2\gamma_{j, \bar{z}} \kappa - 2\gamma_j D_{\bar{z}} \kappa \\ \gamma_j &= 2\gamma_{j+1, zz} + c\gamma_{j+1} + 2(q_{j+1}, \kappa), \end{aligned}$$

and each  $\gamma_j, q_j$  real. Define  $\gamma_i$  to be  $2\gamma_{i+1, zz} + c\gamma_{i+1} + 2(q_{i+1}, \kappa)$  and note that Lemma 2.4 (with  $\hat{\gamma} = \gamma_{i+1}$ ) tells us that  $\gamma_i$  is real. Since  $D$  is flat, equation (2.1c) for  $q_i$  is integrable when  $\operatorname{Im}(D_{\bar{z}}(\gamma_{i, \bar{z}} \kappa - \gamma_i D_{\bar{z}} \kappa)) = 0$  however,

$$\operatorname{Im}(D_{\bar{z}}(\gamma_{i, \bar{z}} \kappa - \gamma_i D_{\bar{z}} \kappa)) = \operatorname{Im}(\gamma_{i, \bar{z}\bar{z}} \kappa - \gamma_i D_{\bar{z}\bar{z}}^2 \kappa) = \operatorname{Im}((\gamma_{i, \bar{z}\bar{z}} + \frac{\bar{c}}{2}\gamma_i)\kappa),$$

by the conformal Codazzi equation, and this vanishes thanks to a second application of Lemma 2.4 with  $\hat{\gamma} = \gamma_i$ . Thus, by induction,  $\gamma_i, q_i$  are defined for all  $i \in \mathbb{Z}^-$  satisfying (2.1c) and (2.1d) and we are done. □

Theorem 2.5 is not completely satisfying: the result is only local and the quadrature that determines each  $q_i$  means that we lack an explicit formula for the  $p_i$ . More, these quadratures introduce an infinite number of constants of integration (parallel sections of  $V^\perp$ ).

However, the following simple observation allows us to control the constants of integration: if  $p(t) = \sum_{i \leq 0} p_i t^i$  is a local formal conserved quantity for  $(\Lambda, \eta)$  with  $r(t) = \sum_{i \leq 0} r_i t^i = (p(t), p(t))$  then  $r(t)$  is constant by Proposition 2.2. Thus, for all  $i$ ,

$$(2.2) \quad 2(p_0, q_i) = 2(p_0, p_i) = r_i - \sum_{\substack{k, l < 0 \\ k+l=i}} (p_k, p_l).$$

Thus the component of each  $q_i$  along  $p_0$  is completely determined up to a constant by the  $p_j$  for  $0 \geq j > i$ .

We use this to refine Theorem 2.5:

**Proposition 2.6.** *Let  $(\Lambda, \eta)$  be isothermic and  $r(t) := \sum_{i \leq 0} r_i t^i$  a formal Laurent series with coefficients in  $\mathbb{R}$ , such that  $r_0 > 0$ . Then, locally, away from zeros of  $\eta$ , there is a formal conserved quantity  $p(t)$  of  $(\Lambda, \eta)$  such that  $(p(t), p(t)) = r(t)$ .*

*Proof.* We revisit the induction of Theorem 2.5. Begin with  $\gamma_0 = 0$  and take  $p_0 = q_0$  to be a parallel section of  $V^\perp$  with  $(q_0, q_0) = r_0$ . For the induction step, suppose we have defined  $\gamma_j, q_j$  and so  $p_j$ , for  $0 \geq j \geq i$ , with  $\sum_{k+l=j} (p_k, p_l) = r_j$  for  $j > i$ . We then have that  $dp_j + \eta p_{j-1} = 0$ , for  $j > i$ , while  $dp_i \perp V^\perp$  since  $q_i$  solves (2.1c). It follows that  $\sum_{k+l=i} (p_k, p_l)$  is constant so that, replacing  $p_i$  by  $p_i + s_i p_0$ , for a suitable constant  $s_i$ , we may ensure that  $\sum_{k+l=i} (p_k, p_l) = r_i$  also.  $\square$

In codimension 1, we can say more: in this case,  $p_0$  frames  $V^\perp$  so that each  $q_i$  is completely determined via (2.2) by  $p_j$ ,  $j > i$  and  $r_i$ . It follows at once that  $p(t)$  is uniquely determined in this case on the domain of the holomorphic coordinate  $z$  by  $p_0$  and  $r(t)$  (and so determined up to sign by  $r(t)$  alone). We use this to patch together the local solutions provided by Proposition 2.6 to give a global formal conserved quantity away from the zeros of  $\eta$ .

**Theorem 2.7.** *Let  $(\Lambda, \eta)$ ,  $\Lambda : \Sigma \rightarrow S^3$  be an isothermic surface in the 3-sphere and let  $Z \subset \Sigma$  be the (discrete) zero-set of  $\eta$ . Let  $r(t) = \sum_{i \leq 0} r_i t^i \in \mathbb{R}[[t, t^{-1}]]$  be a formal Laurent series with  $r_0 > 0$ .*

*Then there is a formal conserved quantity  $p(t)$ , unique up to sign, defined on  $\Sigma \setminus Z$  with  $(p(t), p(t)) = r(t)$ .*

*Proof.* Since  $\Sigma$  is orientable,  $V^\perp$  is orientable and so has a global section  $p_0$  with  $(p_0, p_0) = r_0$ . We now use Proposition 2.6 to cover  $\Sigma \setminus Z$  with open sets  $U_\alpha$  on which are defined formal conserved quantities  $p^\alpha$  with  $p_0^\alpha = p_0$  and  $(p^\alpha(t), p^\alpha(t)) = r(t)$ . If  $x \in U_\alpha \cap U_\beta$ , then we may find a holomorphic coordinate  $z$  near  $x$  for which  $\eta = -\psi \wedge (\psi_{\bar{z}} dz + \psi_z d\bar{z})$ . Thus, near  $x$ ,  $p_i^\alpha$  and  $p_i^\beta$  are uniquely determined by  $p_0$  and  $r(t)$  and so must coincide.  $\square$

Since holomorphic quadratic differentials on a 2-torus are constant, we immediately conclude:

**Corollary 2.8.** *Let  $(\Lambda, \eta)$  be an isothermic 2-torus in the 3-sphere and let  $r(t) = \sum_{i \leq 0} r_i t^i \in \mathbb{R}[[t, t^{-1}]]$  be a formal Laurent series with  $r_0 > 0$ .*

*Then there is a globally defined formal conserved quantity  $p(t)$ , unique up to sign, with  $(p(t), p(t)) = r(t)$ .*



3. SPECIAL ISOTHERMIC SURFACES OF TYPE  $d$  IN  $S^n$ 

In this section, we fix an isothermic surface  $(\Lambda, \eta)$  and a holomorphic coordinate  $z$  with  $\eta = -\psi \wedge (\psi_{\bar{z}} dz + \psi_z d\bar{z})$ .

As an application of the ideas of Section 2, we ask when  $(\Lambda, \eta)$  is special of type  $d$ . Our starting point is the simple observation that  $q(t)$  is a polynomial conserved quantity of degree  $d$  then  $p(t) = t^{-d}q(t)$  is a formal conserved quantity with  $p_i = 0$ , for all  $i < -d$ , and conversely. Moreover, we can choose the constants of integration so that  $p_i = 0$ , for all  $i < -d$ , precisely when  $\gamma_{-d-1} = 0$ . Thus, in view of (2.1d), we have:

**Theorem 3.1.** *Let  $d \in \mathbb{N}_0$ .  $(\Lambda, \eta)$  is a special isothermic surface of type  $d$  if and only if there exists a formal conserved quantity  $p(t)$  of  $(\Lambda, \eta)$  such that*

$$2\gamma_{-d,zz} + c\gamma_{-d} + 2(q_{-d}, \kappa) = 0.$$

*In this situation, if  $\gamma_{-d} \neq 0$ , then  $(\Lambda, \eta)$  is a special isothermic surface of type  $d$  in  $E(p_{-d})$ .*

In this case, with  $r(t) = (p(t), p(t))$ , we have  $r_{-2d} = (p_{-d}, p_{-d})$  and  $r_{-2d+1} = 2(p_{-d}, p_{-d+1})$  and so, by Proposition 2.2, these latter inner products are constant. Perhaps surprisingly, a converse is available. First a lemma:

**Lemma 3.2.** *Let  $p(t)$  be a formal conserved quantity for  $(\Lambda, \eta)$ . Then, for  $i, j \leq 0$ ,*

$$(3.1) \quad (\eta_{\partial/\partial z} p_i, p_j) = \gamma_{i,\bar{z}} \gamma_j - \gamma_i \gamma_{j,\bar{z}}$$

*Proof.* Recall that  $\eta_{\partial/\partial z} = -\psi \wedge \psi_{\bar{z}}$  while

$$p_i = \alpha_i \psi - 2\gamma_{i,\bar{z}} \psi_z - 2\gamma_{i,z} \psi_{\bar{z}} + \gamma_i \hat{\psi} + q_i$$

so that  $\eta_{\partial/\partial z} p_i = \gamma_i \psi_{\bar{z}} - \gamma_{i,\bar{z}} \psi$ . Now write  $p_j$  in terms of  $\gamma_j$  to draw the conclusion.  $\square$

With this in hand, we have:

**Proposition 3.3.**  *$(\Lambda, \eta)$  is locally special isothermic of type at most  $d \in \mathbb{N}$  if and only if it admits a formal conserved quantity for which either  $(p_{-d}, p_{-d})$  or, in case  $d > 1$ ,  $(p_{-d}, p_{-d+1})$  is constant.*

*Proof.* We have already seen necessity of the condition on inner products, so we turn to the sufficiency. We suppose, without loss of generality, that  $\gamma_{-d}$  is never zero (otherwise  $(\Lambda, \eta)$  is special of type  $k < d$ ) and observe that  $(p_{-d}, p_{-d})$  is constant if and only if  $(p_{-d}, p_{-d})_z = 0$ . However,

$$\begin{aligned} \frac{1}{2}(p_{-d}, p_{-d})_z &= -(\eta_{\partial/\partial z} p_{-d-1}, p_{-d}) \\ &= -\gamma_{-d-1,\bar{z}} \gamma_{-d} + \gamma_{-d-1} \gamma_{-d,\bar{z}} = -\gamma_{-d}^2 \left( \frac{\gamma_{-d-1}}{\gamma_{-d}} \right)_{\bar{z}}, \end{aligned}$$

by Lemma 3.2. Thus, when  $(p_{-d}, p_{-d})$  is constant,  $\gamma_{-d-1}/\gamma_{-d}$  is constant also and we have  $\gamma_{-d-1} = s\gamma_{-d}$ , for some  $s \in \mathbb{R}$ . Now define a new formal conserved quantity  $\hat{p}(t) = p(t) - st^{-1}p(t)$  and observe that  $\hat{\gamma}_{-d-1} = \gamma_{-d-1} - s\gamma_{-d} = 0$  so that  $(\Lambda, \eta)$  is special of type at most  $d$ .

When  $d > 1$  and  $(p_{-d}, p_{-d+1})$  is constant, we argue similarly, assuming that  $\gamma_{-d+1}$  is non-zero and using

$$(p_{-d}, p_{-d+1})_z = -(\eta_{\partial/\partial z} p_{-d-1}, p_{-d+1}) - (p_{-d}, \eta_{\partial/\partial z} p_{-d}) = -(\eta_{\partial/\partial z} p_{-d-1}, p_{-d+1}),$$

to conclude that there is a constant  $s$  such that  $\gamma_{-d-1} = s\gamma_{-d+1}$  and then work with  $p(t) - st^{-2}p(t)$ .  $\square$

Let us now restrict attention to codimension 1 where we can carry out the recursions of Section 2 explicitly. So assume that  $(\Lambda, \eta)$  is isothermic in  $S^3$  and let  $N$  be a unit (hence  $D$ -parallel) section of the line bundle  $V^\perp$ . Let  $p(t)$  be a formal conserved quantity with  $(p(t), p(t)) = r(t)$  and write  $\kappa = kN$ ,

$$p_i = \alpha_i \psi + \beta_i \psi_z + \overline{\beta_i} \psi_{\bar{z}} + \gamma_i \psi + \delta_i N,$$

where we have introduced real functions  $k$  and  $\delta_i$ .

We take, without loss of generality,  $p_0 = N$  so that  $\gamma_0 = 0$  and  $\delta_0 = 1$  whence

$$(3.2a) \quad \gamma_{-1} = 2k$$

$$(3.2b) \quad \delta_{-1} = r_{-1}/2$$

$$(3.2c) \quad p_{-1} = 4(k_{z\bar{z}} + k^3)\psi - 4k_{\bar{z}}\psi_z - 4k_z\psi_{\bar{z}} + 2k\hat{\psi} + \frac{r_{-1}}{2}N$$

$$(3.2d) \quad \gamma_{-2} = 4k_{zz} + (2c + r_{-1})k.$$

As an immediate consequence of Theorem 3.1, we have:

**Proposition 3.4.** [6] *An isothermic surface has constant mean curvature in a 3-dimensional space-form if and only there is a constant  $H \in \mathbb{R}$  such that*

$$(3.3) \quad 2k_{zz} + ck = Hk.$$

Here, the space-form is  $E(p_{-1})$  and  $H = -r_{-1}/2$ . In particular, in this setting  $p_{-1}$  and so  $(p_{-1}, p_{-1})$  is constant, that is,

$$16(-k_{z\bar{z}}k - k^4 + k_z k_{\bar{z}}) + r_{-1}^2/4$$

is constant. In view of Proposition 3.3, we deduce the following result of Musso–Nicolodi:

**Proposition 3.5.** [14] *An isothermic surface has constant mean curvature in a 3-dimensional space-form if and only*

$$(3.4) \quad k_{z\bar{z}}k + k^4 - k_z k_{\bar{z}}$$

*is constant.*

Similarly, Theorem 3.1 for  $d = 2$  gives:

**Proposition 3.6.**  $(\Lambda, \eta)$  *is special isothermic of type 2 if and only if there are constants  $s_1, s_2 \in \mathbb{R}$  such that*

$$(3.5) \quad 4k_{zzzz} + 4ck_{zz} + 4c_z k_z + (2c_{zz} + c^2)k + 8(k_{z\bar{z}}k + k^4 - k_z k_{\bar{z}})k \\ + s_1(2k_{zz} + ck) + s_2k = 0.$$

#### 4. SURFACES OF REVOLUTION, CONES AND CYLINDERS

We illustrate the preceding theory by applying it to surfaces of revolution, cones and cylinders in  $S^3$ . These surfaces are automatically isothermic surfaces and we will find necessary and sufficient conditions for them to be special isothermic.

**4.1. Surfaces of revolution and cones.** We take a uniform approach to cones and surfaces of revolution by viewing them as extrinsic products. For this, let  $W \leq \mathbb{R}^{4,1}$  be a 3-dimensional subspace and contemplate the direct sum

$$\mathbb{R}^{4,1} = W \oplus W^\perp.$$

Let  $\Sigma = I_1 \times I_2$  be a product of intervals and suppose that  $\Lambda : \Sigma \rightarrow S^3 = \mathbb{P}(\mathcal{L})$  is of the form

$$\Lambda := \langle \phi_1 + \phi_2 \rangle$$

where  $\phi_1 : I_1 \rightarrow W$  and  $\phi_2 : I_2 \rightarrow W^\perp$  are curves with  $(\phi_1, \phi_1) = -(\phi_2, \phi_2) = C$ , for a non-zero constant  $C \in \mathbb{R}$ .

Here is the geometry of the situation: if  $W$  has indefinite signature (so that  $C < 0$ ), then  $\phi_1$  takes values in a hyperboloid while  $\phi_2$  is circle-valued. Taking the half-plane model of the hyperboloid, we see that  $\Lambda$  is a surface of revolution. Similarly, if  $C > 0$ ,  $\phi_1$  is a curve on a 2-sphere while  $\phi_2$  takes values in a half-line so that  $\Lambda$  is the cone over  $\phi_1$ .

All such surfaces are isothermic: one easily checks that  $\eta = (d\phi_1 - d\phi_2) \wedge (\phi_1 + \phi_2)$  is closed. The corresponding holomorphic coordinate is also easy to identify: with  $u, v$  the arc-length parameters on  $I_1, I_2$  respectively, set  $z = u + iv$ . Then  $\psi := \phi_1 + \phi_2$  satisfies  $|d\psi|^2 = |dz|^2$  and  $\eta = -\psi \wedge (\psi_z dz + \psi_{\bar{z}} d\bar{z})$ .

Set  $S = \{x \in W : (x, x) = C\}$  so that  $\phi_1 : I_1 \rightarrow S$ , let  $n$  be a unit normal to  $\phi_1$  in  $S$  and  $\mathbf{k}$  the corresponding curvature so that

$$\phi_1'' = \mathbf{k}n - \frac{1}{C}\phi_1.$$

We also have:

$$\phi_2'' = \frac{1}{C}\phi_2.$$

Using these, we compute:

$$(4.1a) \quad \psi_z = \frac{1}{2}(\phi_1' - i\phi_2')$$

$$(4.1b) \quad N = n + \frac{\mathbf{k}}{2}(\phi_1 + \phi_2)$$

$$(4.1c) \quad \hat{\psi} = \frac{1}{2}\left(\frac{\mathbf{k}^2}{4} - \frac{1}{C}\right)\phi_1 + \frac{1}{2}\left(\frac{\mathbf{k}^2}{4} + \frac{1}{C}\right)\phi_2 + \frac{\mathbf{k}}{2}n$$

with  $N$  a unit section of  $V^\perp$ , for  $V$  the central sphere congruence of  $\Lambda$ . Then

$$\psi_{zz} = \frac{1}{4}(\phi_1'' - \phi_2'') = -\frac{1}{4C}(\phi_1 + \phi_2) + \frac{\mathbf{k}}{4}n = \left(-\frac{1}{4C} - \frac{\mathbf{k}^2}{8}\right)\psi + \frac{\mathbf{k}}{4}N,$$

so that the Schwarzian derivative and Hopf differential of  $\Lambda$  are given by

$$c = \frac{1}{2C} + \frac{\mathbf{k}^2}{4} \quad \kappa = \frac{\mathbf{k}}{4}N.$$

In the notation of section 3,  $k = \mathbf{k}/4$  so that, in the current setting, Proposition 3.4 reads

**Proposition 4.1.**  $(\Lambda, \eta)$  is special isothermic of type 1 if and only if, for some constant  $\alpha$ ,

$$(4.2) \quad \frac{\mathbf{k}}{C} + \frac{\mathbf{k}^3}{2} + \mathbf{k}'' + \alpha\mathbf{k} = 0;$$

Note that equation (4.2) means exactly that  $\phi_1$  is an elastic curve in  $S$ , and then  $\alpha = 0$  if and only if  $\phi_1$  is a free elastic curve.

Similarly, Proposition 3.6 reads:

**Proposition 4.2.**  $(\Lambda, \eta)$  is special isothermic of type 2 if and only if there exist real constants  $\alpha$  and  $\beta$  such that

$$\frac{\mathbf{k}}{C^2} + \frac{\mathbf{k}^3}{C} + \frac{3\mathbf{k}^5}{8} + \frac{2\mathbf{k}''}{C} + \frac{5}{2}(\mathbf{k}\mathbf{k}'^2 + \mathbf{k}^2\mathbf{k}'') + \mathbf{k}^{(iv)} + \alpha\left(\frac{\mathbf{k}}{C} + \frac{\mathbf{k}^3}{2} + \mathbf{k}''\right) + \beta\mathbf{k} = 0.$$

We now prove that the constant term  $p_{-d}$  of a polynomial conserved quantity of  $(\Lambda, \eta)$  lies  $W$ . For this, we recall the notations of the previous sections and begin with a lemma:

**Lemma 4.3.** Let  $p(t)$  be a formal conserved quantity for  $(\Lambda, \eta)$ . Then, for each  $i \leq 0$ ,  $\gamma_i = -(\psi, p_i)$  is independent of  $v$ .

*Proof.* We induct, noting that both  $c$  and  $k = \mathbf{k}/4$  are independent of  $v$ .  $\square$

**Proposition 4.4.** *The constant terms of the polynomial conserved quantities of  $(\Lambda, \eta)$  lie in  $W$ .*

*Proof.* Suppose that  $(\Lambda, \eta)$  is a special isothermic surface of type  $d$  with polynomial conserved quantity  $q(t)$  and work with the formal conserved quantity  $p(t) = t^{-d}q(t)$ . Observe that Lemma 4.3 gives

$$0 = (p(t), \psi)_v = -t(\eta_{\partial/\partial v} p(t), \psi) + (p(t), \psi_v) = (p(t), \phi'_2),$$

since  $\eta\psi = 0$ . In particular,  $(p_{-d}, \phi'_2) = 0$ . Moreover, from (4.1), we have

$$(p_{-d}, \phi_2) = -\alpha_{-d}C + \gamma_{-d}\left(-\frac{\mathbf{k}^2}{8}C - \frac{1}{2}\right) + \delta_{-d}\left(-\frac{1}{2}\mathbf{k}C\right).$$

However, in this context, by Lemma 4.3, (2.1b) reads

$$\alpha_{-d} = \frac{1}{2}\gamma_{-d,uu} + \frac{\mathbf{k}^2}{8}\gamma_{-d} = 2\gamma_{-d,zz} + \frac{\mathbf{k}^2}{8}\gamma_{-d}$$

so that

$$(p_{-d}, \phi_2) = -C(2\gamma_{-d,zz} + c\gamma_{-d} + 2\delta_{-d}\frac{\mathbf{k}}{4}) = 0.$$

Since  $\phi_2, \phi'_2$  frame  $W^\perp$ , the result follows.  $\square$

For cones, this has a geometric consequence: if  $p_{-d} \neq 0$ , we have  $(p_{-d}, p_{-d}) > 0$ , since  $W$  has definite signature, so that  $(\Lambda, \eta)$  is special isothermic in a hyperbolic space form.

**4.2. Cylinders.** A similar analysis may be carried out for cylinders which amount to the limiting case where the curvature  $1/C$  of  $S$  is zero. We briefly rehearse the details.

Let  $v_0, v_\infty \in \mathcal{L}$  with  $(v_0, v_\infty) = -1$ , choose a 2-dimensional subspace  $U \leq \langle v_0, v_\infty \rangle^\perp$  and write

$$\langle v_0, v_\infty \rangle^\perp = U \oplus U^\perp$$

Again let  $\Sigma = I_1 \times I_2$  be a product of intervals and suppose that  $\Lambda : \Sigma \rightarrow S^3 \cong \mathbb{P}(\mathcal{L})$  is of the form

$$\Lambda := \langle \phi_1 + \phi_2 + v_0 + \frac{1}{2}((\phi_1, \phi_1) + (\phi_2, \phi_2))v_\infty \rangle$$

where  $\phi_1 : I_1 \rightarrow U$  and  $\phi_2 : I_2 \rightarrow U^\perp$  are curves. Let  $u, v$  be arc-length parameters on  $I_1, I_2$  respectively and set  $z = u + iv$ . Then  $\psi = \phi_1 + \phi_2 + v_0 + \frac{1}{2}((\phi_1, \phi_1) + (\phi_2, \phi_2))v_\infty$  has  $|\mathrm{d}\psi|^2 = |\mathrm{d}z|^2$  and is isothermic with

$$\eta = ((\mathrm{d}\phi_1 + (\mathrm{d}\phi_1, \phi_1)v_\infty) - (\mathrm{d}\phi_2 + (\mathrm{d}\phi_2, \phi_2)v_\infty)) \wedge \psi.$$

Let  $n$  be a unit normal to  $\phi_1$  in  $U$  with corresponding curvature  $\mathbf{k}$  so that  $\phi_1'' = \mathbf{k}n$ . Using  $\phi_2'' = 0$ , we have

$$(4.3a) \quad \psi_z = \frac{1}{2}((\phi_1' + (\phi_1', \phi_1)v_\infty) - i(\phi_2' + (\phi_2', \phi_2)v_\infty))$$

$$(4.3b) \quad N = n + (n, \phi_1)v_\infty + \frac{\mathbf{k}}{2}\psi$$

$$(4.3c) \quad \hat{\psi} = v_\infty + \frac{\mathbf{k}}{2}(n + (n, \phi_1)v_\infty) + \frac{\mathbf{k}^2}{8}\psi$$

and then

$$\psi_{zz} = \frac{\mathbf{k}}{4}(n + (n, \phi_1)v_\infty) = -\frac{\mathbf{k}^2}{8}\psi + \frac{\mathbf{k}}{4}N$$

so that

$$c = \frac{\mathbf{k}^2}{4} \quad \kappa = \frac{\mathbf{k}}{4}N.$$

We conclude:

**Proposition 4.5.** *A cylinder  $(\Lambda, \eta)$  is special isothermic of type 1 if and only if, for some constant  $\alpha$ ,*

$$\frac{\mathbf{k}^3}{2} + \mathbf{k}'' + \alpha \mathbf{k} = 0;$$

**Proposition 4.6.** *A cylinder  $(\Lambda, \eta)$  is special isothermic of type 2 if and only if there exist real constants  $\alpha$  and  $\beta$  such that*

$$\frac{3\mathbf{k}^5}{8} + \frac{5}{2}(\mathbf{k}\mathbf{k}'^2 + \mathbf{k}^2\mathbf{k}'') + \mathbf{k}^{(\text{iv})} + \alpha\left(\frac{\mathbf{k}^3}{2} + \mathbf{k}''\right) + \beta\mathbf{k} = 0.$$

The analogue of Lemma 4.3 holds with the same argument:

**Lemma 4.7.** *Let  $p(t)$  be a formal conserved quantity for  $(\Lambda, \eta)$ . Then, for each  $i \leq 0$ ,  $\gamma_i = -(\psi, p_i)$  is independent of  $v$ .*

**Proposition 4.8.** *The constant terms of the polynomial conserved quantities of  $(\Lambda, \eta)$  lie in  $U \oplus \langle v_\infty \rangle$ .*

*Proof.* Suppose that  $(\Lambda, \eta)$  is a special isothermic surface of type  $d$  with polynomial conserved quantity  $q(t)$  and work with the formal conserved quantity  $p(t) = t^{-d}q(t)$ . From Lemma 4.7, we have

$$(4.4) \quad 0 = (p(t), \psi)_v = -t(\eta_{\partial/\partial v} p(t), \psi) + (p(t), \psi_v) = (p(t), \phi'_2) + (\phi'_2, \phi_2)(p(t), v_\infty).$$

Now  $(\psi, v_\infty) = -1$  so that  $(d\psi, v_\infty) = 0$  while (4.3c) yields  $(\hat{\psi}, v_\infty) = -\mathbf{k}^2/8$ . Thus,

$$(p_{-d}, v_\infty) = -\alpha_{-d} - \gamma_{-d} \frac{\mathbf{k}^2}{8} - \delta_{-d} \frac{\mathbf{k}}{2}.$$

Once more we have

$$(4.5) \quad \alpha_{-d} = \frac{1}{2}\gamma_{-d,uu} + \frac{\mathbf{k}^2}{8}\gamma_{-d} = 2\gamma_{-d,zz} + \frac{\mathbf{k}^2}{8}\gamma_{-d}$$

so that

$$(p_{-d}, v_\infty) = -2\gamma_{-d,zz} - \gamma_{-d} \frac{\mathbf{k}^2}{4} - \delta_{-d} \frac{\mathbf{k}}{2} = 0.$$

Now (4.4) yields  $(p_{-d}, \phi'_2) = 0$  and, since  $v_\infty, \phi'_2$  frame  $(U \oplus \langle v_\infty \rangle)^\perp$ , we are done.  $\square$

Again this has a geometric consequence: if  $p_{-d} \neq 0$ ,  $(p_{-d}, p_{-d}) \geq 0$  with equality if and only if  $p_{-d} \in \langle v_\infty \rangle$ . Thus a special isothermic cylinder is either special isothermic in a hyperbolic space or in the particular Euclidean space  $E(v_\infty)$ .

In this last case, the conditions to be special isothermic of type  $d$  simplify considerably:

**Proposition 4.9.** *Let  $d \in \mathbb{N}_0$ . A cylinder  $(\Lambda, \eta)$  is a special isothermic surface of type  $d$  in  $E(v_\infty)$  if and only if there exists a formal conserved quantity  $p(t)$  of  $(\Lambda, \eta)$  such that  $\gamma_{-d}$  is constant.*

*Proof.* Certainly, if we have  $p(t)$  with  $p_{-d}$  lying in  $\langle v_\infty \rangle$  then  $0 = (p_{-d}, \psi_z) = -2\gamma_{-d,z}$  so that  $\gamma_{-d}$  is constant.

For the converse, suppose that  $\gamma_{-d}$  is constant and note that, in the present context, (2.1c) yields

$$\delta_{-d,z} = -\gamma_{-d}\mathbf{k}'/4$$

so that we may take  $\delta_{-d} = -\gamma_{-d}\mathbf{k}/2$ . On the other hand, by (4.3c) and (4.5), we have

$$p_{-d} = 2\gamma_{-d,zz}\psi - 2\gamma_{-d,\bar{z}}\psi_z - 2\gamma_{-d,z}\psi_{\bar{z}} + \left(\delta_{-d} + \frac{\mathbf{k}}{2}\gamma_{-d}\right)N + \gamma_{-d}v_\infty$$

and all coefficients except the last vanish so we are done.  $\square$

In particular,  $(\Lambda, \eta)$  is a special isothermic surface of type 1 in  $E(v_\infty)$  if and only if

$\mathbf{k}$  is constant;

$(\Lambda, \eta)$  is a special isothermic surface of type 2 in  $E(v_\infty)$  if and only if there exists a real constant  $\alpha$  such that

$$(4.6) \quad \frac{\mathbf{k}^3}{2} + \mathbf{k}'' + \alpha \mathbf{k} \text{ is constant;}$$

$(\Lambda, \eta)$  is a special isothermic surface of type 3 in  $E(v_\infty)$  if and only if there exist real constants  $\alpha$  and  $\beta$  such that

$$\frac{3\mathbf{k}^5}{8} + \frac{5}{2}(\mathbf{k}\mathbf{k}'^2 + \mathbf{k}^2\mathbf{k}'') + \mathbf{k}^{(\text{iv})} + \alpha\left(\frac{\mathbf{k}^3}{2} + \mathbf{k}''\right) + \beta\mathbf{k} \text{ is constant.}$$

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