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# On the Hughes' model for pedestrian flow: The one-dimensional case

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## Abstract

In this paper we investigate the mathematical theory of Hughes' model for the flow of pedestrians (cf. [17]), consisting of a nonlinear conservation law for the density of pedestrians coupled with an eikonal equation for a potential modelling the common sense of the task. We first consider an approximation of the original model in which the eikonal equation is replaced by an elliptic approximation. For such an approximated system we prove existence and uniqueness of entropy solutions (in one space dimension) in the sense of Kružkov [21], in which the boundary conditions are posed following the approach of Bardos et al. [7]. We use BV estimates on the density  $\rho$  and stability estimates on the potential  $\phi$  in order to prove uniqueness.

Furthermore, we analyse the evolution of characteristics for the original Hughes' model in one space dimension and study the behaviour of simple solutions, in order to reproduce interesting phenomena related to the formation of shocks and rarefaction waves. The characteristic calculus is supported by numerical simulations.

*Keywords:* Pedestrian flow; Scalar conservation laws; Eikonal equation; Elliptic coupling; Entropy solutions; Characteristics.

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## 1. Introduction

The mathematical modelling of large human crowds has gained a lot of scientific interest in the last decades. This is due to various reasons. First of all, a very serious issue in this context is to shed a light on the dynamics in critical circumstances. A well known practical example is the Jamarat Bridge in Saudi Arabia: the huge number of pilgrims cramming the bridge on occasion of the pilgrimage to Mecca gave rise to serious pedestrian disasters in the nineties [15]. Moreover, the

analytical and numerical study of the qualitative behaviour of human individuals in a crowd with high densities can improve traditional socio–biological investigation methods. The dynamics of a human crowd has also applications in structural engineering and architecture: the London Millennium Footbridge which had to be closed on the day of its opening due to unexpected anomalous synchronization, is a very evocative example in this sense. Other applications of pedestrian flow modelling arise in transport systems, spectator occasions, political demonstrations, panic situations such as earthquakes and fire escapes. More light-hearted examples are the simulation of pedestrian movement in computer games and animated movies, see [38].

Several models for the movement of crowds have been proposed in the past. One can distinguish between two general approaches: microscopic and macroscopic models. In the microscopic framework, people are treated as individual entities (particles). The evolution of the particles in time is determined by physical and social laws which describe the interaction among the particles as well as their interactions with the physical surrounding. Examples for microscopic methods are social-force models (see [14] and the references therein), cellular automata, e.g. [12, 29], queuing models e.g. [40] or continuum dynamic approaches like [38]. For an extensive review on different microscopic approaches we refer to [13]. Note that the microscopic approach in [38] uses the eikonal equation to compute the pedestrians’ optimal path. This is a common feature with the model we will analyse in this paper.

In contrast to microscopic models, macroscopic models treat the whole crowd as an entity without considering the movement of single individuals. Classical approaches use well known concepts from fluid and gas dynamics, see [16]. More recent models are based on optimal transportation methods [28], mean field games (see [23] for a general introduction) or non-linear conservation laws [8]. In [31], an approach based on time-evolving measures is presented. We finally note that crowd motion models share many features with traffic models [1].

In this paper we shall analyse a model introduced by R. L. Hughes in 2002 [17]. Hughes’ model treats the crowds as a “thinking” fluid and has been applied to diverse scenarios like the Battle of Agincourt and the annual Muslim Hajji [18]. It is given by

$$\rho_t - \operatorname{div}(\rho f^2(\rho) \nabla \phi) = 0 \tag{1a}$$

$$|\nabla \phi| = \frac{1}{f(\rho)} \tag{1b}$$

Here  $x$  denotes the position variable with  $x \in \Omega$ , a bounded domain in  $\mathbb{R}^d$  with smooth boundary  $\partial\Omega$ ,  $t \geq 0$  is time and  $\rho = \rho(x, t)$  is the crowd density. The function  $f(\rho)$  is given by  $f(\rho) = 1 - \rho$ , modelling the existence of a maximal

density of individuals which can be normalized to 1 by a simple scaling. System (1) is supplemented with the following boundary conditions for  $\phi$

$$\phi(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0 \quad (2)$$

and the initial condition

$$\rho(x, 0) = \rho_I(x) \geq 0. \quad (3)$$

We shall be more precise about the boundary conditions for  $\rho$  and give a more detailed interpretation of the model in the next section.

Note that if the term  $\frac{1}{f(\rho)}$  in (1b) is replaced by 1, the system decouples and (1a) reduces to a non-linear conservation law with discontinuous flux. This type of equation has been analysed and simulated in [19, 20]. Even though Hughes' system (1) shares some features with this class of equations it is methodologically much more challenging. This is due to the non-linearity of the eikonal equation (1b) as well as the implicit time dependence of the potential  $\nabla\phi$  in (1a). In fact, for the unique viscosity solution  $\phi$  of the eikonal equation, no more regularity than Lipschitz continuity can be expected. In this paper we present an existence and uniqueness theory for a regularized version of (1) in one space dimension. Additionally, we discuss the behaviour of simple solution for the original system (1) and validate these results numerically.

Numerical simulations are already available in literature, see Ling et al. [26]. Their approach does not cover the case of discontinuous flux inside the computational domain. Nevertheless we follow the iterative procedure presented in [26], i.e. first solve the eikonal equation (1b) then the conservation law (1a). Numerical methods for non-linear conservation laws with discontinuous flux can be found in literature, e.g. [37]. We will use the approach presented by J. Towers for our numerical simulations. Note that equation (1a) is similar to the Lighthill-Witham-Richards traffic flow model [25, 32], and similar numerical schemes can be used. Various approaches can be found in the literature, e.g. [6, 5, 41, 42]. These schemes are usually based on numerical methods for non-linear conservation laws, for a general introduction we refer to [24, 36] and the references therein.

This paper is organized as follows: In the remaining part of the introduction, we shall explain the model in more detail (subsec. 1.1), present regularized versions as an attempt to a mathematical theory (subsec. 1.2) and state our main results (subsec. 1.3). In sec. 2, we prove existence and uniqueness of entropy solutions for a regularized model and in sec. 3 we will analyse some special cases for the non regularized problem and compare the results with our numerical simulations. Finally, section 4 contains the sketch of the proof of the existence of weak solutions for an alternative regularization.

### 1.1. Hughes' model

We start with a brief motivation of Hughes' model (1) (for further details see [17]). The density of individuals  $\rho = \rho(x, t)$  satisfies the continuity equation

$$\rho_t + \operatorname{div}(\rho V) = 0, \quad (4)$$

and we use the following 'polar decomposition' notation for the velocity field  $V(x, t)$

$$V(x, t) = |V(x, t)|Z(x, t), \quad |Z(x, t)| = 1. \quad (5)$$

In order to prescribe a logistic dependency of  $|V|$  with respect to  $\rho$  we choose the classical linear expression

$$|V(x, t)| = 1 - \rho.$$

As for the directional unit vector  $Z(x, t)$ , we assume it to be parallel to the gradient of the potential  $\phi(x, t)$ . Such potential is determined by solving the eikonal equation in (1). The potential  $\phi$  rules the common sense of the task (the task is represented by the boundary  $\partial\Omega$ ). More precisely, the pedestrians tend to minimize their estimated travel time to the target. In a very naive way, this could be modelled by prescribing the eikonal equation

$$|\nabla\phi| = 1, \quad \phi|_{\partial\Omega} = 0,$$

which has the unique semi-concave solution  $\phi(x) = \operatorname{dist}(x, \partial\Omega)$  at least in the case of a convex domain  $\Omega$ . However, it is reasonable to assume that individuals temper their estimated travel time by avoiding extremely high densities, i. e.

$$|\nabla\phi| = \frac{1}{1 - \rho}, \quad \phi|_{\partial\Omega} = 0, \quad (6)$$

which implies a 'density driven' rearrangement of the level sets of  $\phi$ . This leads to  $Z(x, t) = \frac{\nabla\phi(x, t)}{|\nabla\phi(x, t)|} = (1 - \rho)\nabla\phi$  and therefore the continuity equation in (1) is justified.

### 1.2. An attempt to a mathematical theory: approximations

A successful attempt to develop a mathematical theory for the model (1) has never been carried out so far. The non-linearity with respect to  $\rho$  in the continuity equation forces using the notion of entropy solution for scalar conservation laws, as it is well known that weak  $L^\infty$  solutions to such kind of equations are in general not unique. On the other hand, the vector field  $\nabla\phi$  may clearly develop discontinuities in subsets of  $\Omega$  which may vary in time.

In general, the subsets of discontinuity of  $\nabla\phi$  depend on  $\rho$  non-linearly and non-locally. This may be seen by simple examples in one space dimension. Moreover,

the presence of the term  $1 - \rho$  in the right-hand-side of the eikonal equation renders the problems even more difficult, because of the possible blow-up of  $|\nabla\phi|$  as  $\rho$  approaches the over-crowding density  $\rho = 1$ .

A full understanding of the model is highly non-trivial, even in one space dimension, where the model can be decoupled by solving the eikonal equation by integration.

In order to overcome such difficulties, we propose reasonable approximations to the Hughes' model (1), basically consisting of a regularization of the potential to avoid the discontinuity of  $|\nabla\phi|$ . At a first glance, a very natural way to approximate the equation for the potential would be simply adding a small 'viscosity', i. e.

$$-\delta\Delta\phi + |\nabla\phi|^2 = \frac{1}{f(\rho)^2}, \quad \delta > 0.$$

Such an approximation still has the drawback of (possibly) producing a blow up of the right hand side when the density approaches the overcrowding value  $\rho = 1$ . This problem can be bypassed considering instead

$$-\delta\Delta\phi + f(\rho)^2|\nabla\phi|^2 = 1, \quad \delta > 0. \quad (7)$$

On the other hand, the development of a satisfactory existence and uniqueness theory by using the coupling (7) is seriously complicated by the presence of the density dependent coefficient multiplying the Hamilton-Jacobi term  $|\nabla\phi|^2$ .

The model for which we shall develop a full existence and uniqueness theory uses the following elliptic regularization of the eikonal equation in (1), namely

$$-\delta_1\Delta\phi + |\nabla\phi|^2 = \frac{1}{(f(\rho) + \delta_2)^2}, \quad \delta_1, \delta_2 > 0. \quad (8)$$

The sign in front of  $\delta_1$  ( $\delta$  in the alternative equation (7)) is chosen such that we would recover the unique viscosity solution in a possible limit  $\delta_1 \rightarrow 0$ . The second order term in (8) is meant to smooth the potential  $\phi$  in order to avoid discontinuities for  $|\nabla\phi|$ . The elliptic operator in (8) is a classical elliptic Hamilton-Jacobi operator, and it is therefore easier to deal with if compared to the one in (7). On the other hand equation (8) contains one further approximation on the right-hand-side which can be motivated as follows.

Without the elliptic regularization, the potential  $\phi$  in (8) would satisfy

$$|\nabla\phi| = \frac{1}{(1 - \rho + \delta_2)} \quad (9)$$

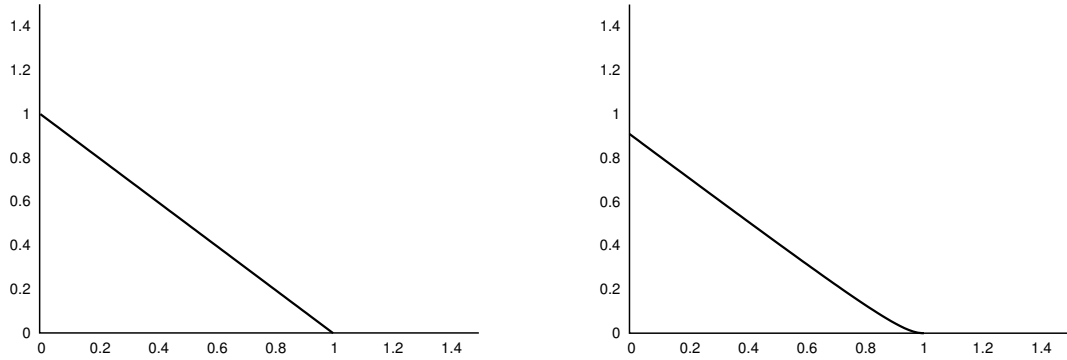


Figure 1: Comparison between the scalar ‘logistic’ speed  $|V|$  of the pedestrian in Hughes’ model (1) (left) and the model with elliptic coupling (8)

Then, the polar decomposition of the velocity field introduced in (4) reads in this case

$$\begin{aligned}
 V &= |V|Z, \quad |Z| = 1 \\
 |V| &= f(\rho)^2 |\nabla\phi| = \frac{f(\rho)^2}{\delta_2 + f(\rho)} = \frac{(1 - \rho)^2}{\delta_2 + (1 - \rho)}, \quad Z = \frac{\nabla\phi}{|\nabla\phi|}. \quad (10)
 \end{aligned}$$

The profile of  $|V|$  as a function of  $\rho$  in (10) has essentially the same properties of the logistic function  $|V|(\rho) = 1 - \rho$  of the original Hughes’s model, except that the vacuum at  $\rho = 1$  is achieved with a zero derivative and the maximal velocity is slightly penalized, i. e.  $|V|_{max} = 1/1 + \delta_2$  instead of  $|V|_{max} = 1$  of the original model (cf. Figure 1).

As for the unit vector  $Z$ , which is parallel to  $\nabla\phi$ , the only difference with the original model is that individuals ‘sense’ the target as the density reaches the maximum value  $\rho = 1$ . In this case  $|\nabla\phi| = 1/\delta_2$ , i. e. the slope of  $\nabla\phi$  is very high in absolute value ( $\delta_2$  is thought as a small parameter), but not infinite as in the original model. On the other hand, when  $\rho = 1$ ,  $|V|$  vanishes, and therefore the above mentioned difference is not effective (individuals do not move at all when  $\rho = 1$ !).

### 1.3. Results

We shall first cover the one dimensional existence and uniqueness theory for the regularized model with elliptic coupling (8) introduced in the previous subsection, more precisely we shall study the model system

$$\begin{cases} \rho_t - (\rho f^2(\rho)\phi_x)_x = 0 \\ -\delta_1\phi_{xx} + |\phi_x|^2 = \frac{1}{(f(\rho) + \delta_2)^2}. \end{cases} \quad (11)$$

As for the model with alternative elliptic coupling (7), we shall only sketch the existence theory in Section 4.

As the continuity equation in (11) features non-linear convection, we shall address the existence and uniqueness theory in the framework of weak entropy solutions, cf. for instance [21]. The results are contained in Section 2. More precisely, the notion of solution is stated in Definition 2.1, the existence result is provided in Theorem 2.9, and the uniqueness result is proven in Theorem 2.11.

The problem (11) is posed on the bounded interval  $x \in [-1, 1]$  with homogeneous Dirichlet boundary conditions. We shall follow the approach by Bardos et al. [7] (see also [9, 2, 27]) to recover suitable boundary conditions for a scalar conservation law. This aspect is explained at the beginning of the next section.

## 2. The regularized model: existence and uniqueness theory

In this section we establish our existence and uniqueness results for the regularized Hughes' model system (11) with  $f(\rho) = (1 - \rho)$ . For future use we denote

$$g(\rho) := \rho f(\rho)^2.$$

System (11) is coupled with the initial condition

$$\rho(x, 0) = \rho_I(x) \geq 0, \tag{12}$$

and with the Dirichlet boundary conditions

$$\min_{k \in [0, \text{tr } \rho]} \{g(\text{tr } (\rho)) - g(k)\} = 0, \tag{13}$$

$$\phi(\pm 1, t) = 0. \tag{14}$$

Here  $\text{tr } \rho$  denotes the trace of  $\rho$  on the boundary. More precisely,

$$\text{tr } \rho(-1, t) = \lim_{x \rightarrow -1^+} \rho(x, t), \quad \text{tr } \rho(1, t) = \lim_{x \rightarrow 1^-} \rho(x, t).$$

It was originally proven in [7] that (13) is the correct way to pose Dirichlet boundary conditions for a scalar conservation law, mainly for two reasons: first, (13) comes as a natural condition from the vanishing viscosity limit of solutions with zero Dirichlet boundary data; second, (13) encloses the natural interplay between the boundary datum and the value of the solution which is transported via characteristics in the linear case (the boundary datum needs to be posed only if characteristics at the boundary are directed towards the interior of the domain). The boundary condition provided in [7] assumes the simplified form (13) since we shall deal with non-negative solutions and due to a trivial monotonicity property of the



potential  $\phi$  (cf. Lemma 2.10 below). We remark here that the boundary condition (13) reduces to

$$g(\text{tr } \rho) \geq g(k) \quad \text{on } x = \pm 1, \quad \text{for all } k \in [0, \text{tr } \rho],$$

which expresses the fact that the allowed densities on the boundary are those for which the function  $g$  is *non-decreasing*. A deeper understanding of the boundary conditions for nonlinear conservation laws in one space dimension can be also found in [9].

We shall prove that the system (11) has a unique solution  $(\rho, \phi)$  in a sense made precise by the following definition. For the density component  $\rho$  we will use the classical notion of entropy solutions originally introduced by Kruřkov in [21] and adapted to boundary value problems by Bardos et al. in [7].

**Definition 2.1** (Entropy Solution). *Let  $\rho_I \in BV([-1, 1])$ . A couple  $(\rho, \phi)$  is a weak entropy solution to the system (11) if*

- $\rho \in BV([-1, 1] \times [0, T]) \cap L^\infty([-1, 1] \times [0, T])$
- $\phi \in W^{2,\infty}[-1, 1]$
- $\rho$  and  $\phi$  satisfy the inequality

$$\begin{aligned} & \iint_{\Omega_T} |\rho - k| \psi_t \, dx dt + \int_{-\infty}^{\infty} \rho_I \psi_0 \, dx - \iint_{\Omega_T} \text{sgn}(\rho - k) [(g(\rho) - g(k))] \psi_x \phi_x \, dx dt \\ & + \iint_{\Omega_T} \text{sgn}(\rho - k) g(k) \psi \phi_{xx} \, dx dt - \text{sgn}(k) \int_0^T [(g(\text{tr } \rho) - g(k))] \phi_x \psi|_{x=\pm 1} \, dt \geq 0, \end{aligned} \tag{15}$$

for every Lipschitz continuous test function  $\psi$  on  $[-1, 1] \times [0, T]$  having compact support.

- $\phi$  and  $\rho$  satisfy the second equation in (11) almost everywhere in  $x$  and  $t$ .

As usual in the context of conservation laws, we shall approximate the targeted model (11) via a vanishing viscosity approach, namely we shall work on the system

$$\rho_t - (\rho f^2(\rho) \phi_x)_x = \varepsilon \rho_{xx} \tag{16a}$$

$$-\delta_1 \phi_{xx} + |\phi_x|^2 = \frac{1}{(f(\rho) + \delta_2)^2}, \tag{16b}$$

for a small  $\varepsilon > 0$ . System (16) is coupled with homogeneous boundary condition

$$\rho(x, t)|_{x=\pm 1} = 0 \quad \phi(x, t)|_{x=\pm 1} = 0,$$

and the initial condition

$$\rho(x, 0) = \rho_I(x).$$

Existence of unique (smooth) solutions to the above regularized problem follow from standard results. For the elliptic coupling see e.g. [22, Chapter 3, Lemma 1.1] and [22, Chapter 3, Thm. 1.2]. For the parabolic approximation we refer to [39, Section 5, Thm. 5.3 and Thm. 5.4]. The proof of this theorem is based on semi group techniques. The strategy is to first linearise the equation to an evolution equation with a linear but time depending operator. Under the given assumptions, it is known that there exists a solution to such an equation (see e.g. [34]). Then, the solution to the non-linear equation is obtained using a fixed-point argument.

In the next subsections we shall first derive suitable a-priori estimates on  $\phi$  and  $\rho$ , then we shall recall our notion of entropy solution, and finally prove existence and uniqueness of the limit as  $\varepsilon \rightarrow 0$ .

Before we start with the estimates, we prove that  $\rho$  is always bounded above by the maximal density  $\rho = 1$ .

**Lemma 2.2** (Boundedness of  $\rho$ ). *Assume that  $\rho_I \leq 1$ . Then the solution to (16a) with  $f(\rho) = (1 - \rho)$  satisfies  $\rho(x, t) \leq 1$  for all  $(x, t) \in [-1, 1] \times [0, +\infty)$ .*

*Proof.* We first define the function

$$\eta(\rho) = \begin{cases} 0 & \rho \leq 0, \\ \frac{\rho^2}{4\gamma} & 0 < \rho \leq 2\gamma, \\ \rho - \gamma & \rho > 2\gamma. \end{cases} \quad (17)$$

and use it to approximate  $(\rho - 1)_+$  (the positive part of  $(\rho - 1)$ ). Here  $\gamma > 0$  is a small parameter. Our goal is to show that this positive part, being zero at  $t = 0$ , does not increase. We consider

$$\begin{aligned} \frac{d}{dt} \int \eta(\rho - 1) dx &= \int \eta'(\rho - 1)(\varepsilon \rho_{xx} - (\rho(1 - \rho)^2 \phi)_x)_x dx \\ &= -\varepsilon \int \eta''(\rho - 1)|\rho_x|^2 dx + \varepsilon \eta'(\rho - 1)\rho_x|_{x=\pm 1} \\ &\quad - \int_{0 \leq (\rho - 1) \leq \gamma} \eta''(\rho - 1)\rho(1 - \rho)^2 \rho_x \phi_x dx + \eta'(\rho - 1)\rho(1 - \rho)\phi_x|_{x=\pm 1} \\ &\leq -\varepsilon \int \eta''(\rho - 1)|\rho_x|^2 dx - C_\varepsilon \int_{0 \leq (\rho - 1) \leq \gamma} \eta''(\rho - 1)\rho^2(1 - \rho)^4 |\phi_x|^2 dx \\ &\leq -\varepsilon \int \eta''(\rho - 1)|\rho_x|^2 dx - C_\varepsilon \gamma^3 (1 + \gamma)^2. \end{aligned}$$

Here, we employed Young's inequality and the Dirichlet boundary conditions. Letting  $\gamma \rightarrow 0$ , we infer

$$\frac{d}{dt} \int (\rho - 1)_+ dx = -\varepsilon \int \eta''(\rho - 1) |\rho_x|^2 dx \leq 0,$$

and thus the integral is decreasing in time. As  $(\rho - 1)_+$  is a positive function and zero at  $t = 0$ , we conclude that it stays zero for all times and thus that  $\rho$  is always bounded by 1.  $\square$

Note that using the same technique, but approximating the negative part of  $\rho$  we also obtain that the solution is almost everywhere non-negative (since  $\rho_I \geq 0$ ).

### 2.1. A Priori Estimates on $\phi$

We shall now derive some a-priori estimates for the elliptic coupling, i.e.

$$\begin{aligned} -\delta_1 \phi_{xx} + \phi_x^2 &= F_{\delta_2}(\rho) := \frac{1}{(\delta_2 + f(\rho))^2} \\ \phi(\pm 1) &= 0. \end{aligned} \tag{18}$$

Let us introduce the *Hopf-Cole* transformation

$$\psi(x, t) := e^{-\frac{\phi(x, t)}{\delta_1}}, \tag{19}$$

which implies

$$\psi_x = -\frac{\psi \phi_x}{\delta_1}, \quad \psi_t = -\frac{\psi \phi_t}{\delta_1}, \quad \phi_x = -\delta_1 \frac{\psi_x}{\psi}, \quad \phi_t = -\delta_1 \frac{\psi_t}{\psi} \tag{20}$$

$$\psi_{xx} - \frac{\phi_{xx} \psi}{\delta_1} - \frac{\phi_x \psi_x}{\delta_1} = \frac{\psi}{\delta_1} (-\delta_1 \phi_{xx} + \phi_x^2) = \frac{\psi}{\delta_1^2} F_{\delta_2}(\rho). \tag{21}$$

Therefore,  $\psi$  satisfies

$$\begin{cases} \delta_1^2 \psi_{xx} = \psi F_{\delta_2}(\rho) \\ \psi(\pm 1) = 1. \end{cases} \tag{22}$$

As a first estimate, we prove that  $\psi$  is uniformly bounded in  $H^1$  and in  $L^\infty$ .

**Lemma 2.3.** *There exists a constant  $C > 0$  depending only on  $\delta_1$  and  $\delta_2$  such that*

$$\|\psi\|_{H^1([-1,1])} \leq C, \quad \|\psi\|_{L^\infty([-1,1])} \leq C, \quad \|\psi_{xx}\|_{L^\infty} \leq C. \tag{23}$$

*Proof.* Let us introduce the variable

$$\tilde{\psi} := \psi - 1,$$

which satisfies

$$\begin{cases} \delta_1^2 \tilde{\psi}_{xx} = \tilde{\psi} F_{\delta_2}(\rho) + F_{\delta_2}(\rho) \\ \psi(\pm 1) = 0. \end{cases} \quad (24)$$

Multiplication of (24) by  $\tilde{\psi}$  and integration over  $[-1, 1]$  leads to (after integration by parts)

$$-\delta_1^2 \int \tilde{\psi}_x^2 dx = \int \tilde{\psi}^2 F_{\delta_2}(\rho) dx + \int \tilde{\psi} F_{\delta_2}(\rho) dx.$$

Since

$$\frac{1}{(1 + \delta_2)^2} \leq F_{\delta_2}(\rho) \leq \frac{1}{\delta_2}, \quad (25)$$

by a trivial use of Young's inequality we get

$$\int \tilde{\psi}_x^2 dx + \int \tilde{\psi}^2 dx \leq C,$$

for a constant  $C$  depending on  $\delta_1$  and  $\delta_2$ . Sobolev's inequality then implies

$$\|\psi\|_{L^\infty} \leq C.$$

The last assertion in (23) follows by the equation (22).  $\square$

Next we prove that  $\psi$  is non-negative on  $[-1, 1]$  and uniformly bounded from below by a positive constant.

**Lemma 2.4.** *There exists a constant  $C > 0$  such that*

$$\psi(x, t) \geq C \quad (26)$$

for all  $(x, t) \in [-1, 1] \times [0, +\infty)$ .

*Proof.* Let us consider the original equation (18) satisfied by  $\phi$ . We have

$$\delta_1 \phi_{xx} + \frac{1}{\delta_2^2} \geq \delta_1 \phi_{xx} + F_{\delta_2}(\rho) = \phi_x^2 \geq 0,$$

which can be written as

$$\left( \delta_1 \phi + \frac{|x|^2}{2\delta_2} \right)_{xx} \geq 0.$$

Therefore the function  $\delta_1 \phi + \frac{|x|^2}{2\delta_2}$  attains its maximum at the boundary,  $\phi$  is bounded from above and  $\psi = e^{-\phi/\delta_1}$  is bounded away from zero.  $\square$

## 2.2. BV estimate on $\rho$

We are now ready to prove the crucial BV estimate on  $\rho$  which serves as a tool to get compactness in the limit as  $\varepsilon \rightarrow 0$ . Let us start with estimating the  $L^1$  norm of  $\rho_x$ .

**Lemma 2.5.** *Suppose  $\rho_I \in W^{1,1}([-1, 1])$ . Then, there exists a constant  $C > 0$  independent on  $\varepsilon$  such that*

$$\|\rho_x(t)\|_{L^1} \leq (\|(\rho_I)_x\|_{L^1} + C)e^{Ct}$$

for all  $t \geq 0$ .

*Proof.* Let us consider an approximation  $\eta_\gamma(z)$  of the function  $|z|$  as  $\gamma \rightarrow 0$  such that

$$\begin{aligned} \eta_\gamma(z) &\rightarrow |z|, & \eta'_\gamma(z) &\rightarrow \text{sign}(z), & \eta'_\gamma(z)z &\rightarrow |z| & \text{as } \gamma \rightarrow 0 \\ \eta'_\gamma(z)z &\geq 0, & \eta''_\gamma(x) &\geq 0 \\ \eta''_\gamma(x) &\leq \mathbf{1}_{[-\gamma, \gamma]}(x) \frac{C}{\gamma} \end{aligned} \tag{27}$$

for some constant  $C > 0$ . We deduce that

$$\begin{aligned} \frac{d}{dt} \int \eta_\gamma(\rho_x) dx &= \int \eta'_\gamma(\rho_x) \rho_{xt} dx = \int \eta'_\gamma(\rho_x) (g(\rho) \phi_x)_{xx} dx + \varepsilon \int \eta'_\gamma(\rho_x) \rho_{xxx} dx \\ &= \int \eta'_\gamma(\rho_x) (g'(\rho) \rho_x \phi_x)_x + \int \eta'_\gamma(\rho_x) (g(\rho) \phi_{xx})_x - \varepsilon \int \eta''_\gamma(\rho_x) \rho_{xx}^2 dx \\ &= - \int \eta''_\gamma(\rho_x) \rho_{xx} g'(\rho) \rho_x \phi_x dx + \int \eta'_\gamma(\rho_x) g'(\rho) \rho_x \phi_{xx} dx \\ &\quad + \int \eta'_\gamma(\rho_x) g(\rho) \phi_{xxx} dx - \varepsilon \int \eta''_\gamma(\rho_x) \rho_{xx}^2 dx \\ &\leq -\frac{\varepsilon}{2} \int \eta''_\gamma(\rho_x) \rho_{xx}^2 dx + C(\varepsilon) \int \eta''_\gamma(\rho_x) \phi_x^2 \rho_x^2 dx + C \int |\rho_x| dx + C. \end{aligned} \tag{28}$$

Here the last step is justified by the identities (20) and (21), by (26), and by

$$\|\psi_{xxx}\|_{L^1} \leq C \|\psi_{xxx}\|_{L^1} + C, \quad \psi_{xxx} = F_{\delta_2}(\rho) \psi_x + \psi F'_{\delta_2}(\rho) \rho_x.$$

The sum of the boundary terms

$$\int \eta'_\gamma(\rho_x) (\varepsilon \rho_{xx} + g'(\rho) \rho_x \phi_x + g(\rho) \phi_{xx}) d\sigma_x = \int \eta'_\gamma(\rho_x) \rho_t d\sigma_x$$

vanishes, as  $\rho_t$  is constant along the boundary. The second term on the right hand side of (28) vanishes as  $\gamma \rightarrow 0$ , therefore we obtain the desired assertion in the limit (after integration with respect to time).  $\square$

Before estimating the  $L^1$  norm of  $\rho_t$  we have the following technical lemma.

**Lemma 2.6.** *There exists a constant  $C > 0$  independent of  $\varepsilon$  and of  $t$  such that*

$$\|\psi_t\|_{L^\infty} \leq C\|\rho_t\|_{L^1} \quad (29)$$

$$\|\psi_{xxt}\|_{L^1} \leq C\|\rho_t\|_{L^1} \quad (30)$$

$$\|\psi_{xt}\|_{L^\infty} \leq C\|\rho_t\|_{L^1}. \quad (31)$$

*Proof.* We start with the proof of estimate (29). Differentiation of (22) with respect to time yields

$$\psi_{xxt} = \psi_t F_{\delta_2}(\rho) + \psi F'_{\delta_2}(\rho)\rho_t. \quad (32)$$

Next we multiply (32) by  $\psi_t$  and integrate over  $[-1, 1]$ . Using the fact that  $\psi_t = 0$  at the boundary, we integrate by parts to obtain

$$\delta_1^2 \int \psi_{xt}^2 dx = \int F_{\delta_2}(\rho)\psi_t^2 dx + \int F'_{\delta_2}(\rho)\rho_t\psi_t dx.$$

In view of (25), then, we can find a constant  $C = C(\delta_1, \delta_2) > 0$  such that

$$\|\psi_t\|_{H^1}^2 \leq C\|\psi_t\|_{L^\infty}\|\rho_t\|_{L^1},$$

and the Sobolev inequality  $\|\psi_t\|_{L^\infty} \leq \|\psi_t\|_{H^1}$  implies the assertion.

The inequality (30) follows by a direct use of the equation (22) and by (29). Finally, the last statement (31) follows from the inequality  $\|\psi_{xt}\|_{L^\infty} \leq \|\psi_{xxt}\|_{L^1}$ , which is a consequence of the fact that  $\int \psi_{xt} dx = \psi_t(1, t) - \psi_t(-1, t) = 0$  and that every  $W^{1,1}$  function in one space dimension admits an absolutely continuous representant.  $\square$

We are now ready to estimate the  $L^1$  norm of the time derivative.

**Lemma 2.7.** *Assuming  $\rho_I \in W^{2,1}([-1, 1])$  and  $\varepsilon > 0$ , there exists a constant  $C > 0$  independent on  $\varepsilon$  such that*

$$\|\rho_t(t)\|_{L^1} \leq Ce^{Ct},$$

for all  $t \geq 0$ .

*Proof.* Again we consider the approximation  $\eta_\gamma$  of the absolute value, given by (27). We deduce that

$$\begin{aligned} \frac{d}{dt} \int \eta_\gamma(\rho_t) dx &= \int \eta'_\gamma(\rho_t)\rho_{tt} dx = \int \eta'_\gamma(\rho_t)(g(\rho)\phi_x)_{tx} dx + \varepsilon \int \eta'_\gamma(\rho_t)\rho_{xxt} dx \\ &= \int \eta'_\gamma(\rho_t)(g'(\rho)\rho_t\phi_x)_x + \int \eta'_\gamma(\rho_t)(g(\rho)\phi_{xt})_x - \varepsilon \int \eta''_\gamma(\rho_t)\rho_{xt}^2 dx \end{aligned}$$

$$\begin{aligned}
&= - \int \eta_\gamma''(\rho_t) \rho_{xt} g'(\rho) \rho_t \phi_x dx + \int \eta_\gamma'(\rho_t) g'(\rho) \rho_x \phi_{xt} dx \\
&\quad + \int \eta_\gamma'(\rho_t) g(\rho) \phi_{xxt} dx - \varepsilon \int \eta_\gamma''(\rho_t) \rho_{xt}^2 dx \\
&\leq -\frac{\varepsilon}{2} \int \eta_\gamma''(\rho_t) \rho_{xt}^2 dx + C(\varepsilon) \int \eta_\gamma''(\rho_t) \phi_x^2 \rho_t^2 dx + C \|\phi_{xt}\|_{L^\infty} \int |\rho_x| dx + C \int |\phi_{xxt}| dx.
\end{aligned}$$

All boundary terms in the above calculation are zero as  $\rho_t$  and thus  $\eta_\gamma'(\rho_t)$  is zero on the boundary. The second term on the r.h.s. above vanishes as  $\gamma \rightarrow 0$ . As for the other terms, we can differentiate (19) to easily obtain

$$\|\phi_{xt}\|_{L^\infty} \leq C \|\psi_{xt}\|_{L^\infty} + C \|\psi_t\|_{L^\infty} \leq C \|\rho_t\|_{L^1}$$

and

$$\|\phi_{xxt}\|_{L^1} \leq C \|\psi_{xxt}\|_{L^1} + C \|\psi_{xt}\|_{L^1} + C \|\psi_t\|_{L^1} \leq C \|\rho_t\|_{L^1}.$$

Therefore, integration with respect to time and Lemma 2.5 results in

$$\|\rho_t(t)\|_{L^1} \leq (\|(\rho(\cdot, 0))_t\|_{L^1} + C) e^{Ct},$$

for all  $t \geq 0$ . Using the fact that  $\rho_I$  is in  $W^{2,1}(\Omega)$  and that  $\varepsilon$  is bounded, we can use equation (16a) to estimate

$$\|\rho_t(\cdot, 0)\|_{L^1(\Omega)} \leq \|g'(\rho_I) \phi_x\|_{L^\infty(\Omega)} \|(\rho_I)_x\|_{L^1(\Omega)} + \varepsilon \|(\rho_I)_{xx}\|_{L^1(\Omega)}.$$

We thus conclude that  $\|\rho_t(\cdot, 0)\|_{L^1(\Omega)}$  is bounded as well completing the proof.  $\square$

### 2.3. Stability estimates on $\phi$

Next, we prove some stability estimates for the elliptic equation (18) with respect to the variable  $\rho$ . These estimates will be useful later on to prove uniqueness of an entropy solution  $\rho$  in the limit.

Given two densities  $\rho$  and  $\bar{\rho}$ , let  $\phi$  and  $\bar{\phi}$  solve

$$\begin{aligned}
-\delta_1 \phi_{xx} + \phi_x^2 &= F_{\delta_2}(\rho), \\
-\delta_1 \bar{\phi}_{xx} + \bar{\phi}_x^2 &= F_{\delta_2}(\bar{\rho}),
\end{aligned}$$

with boundary conditions  $\phi(\pm 1) = \bar{\phi}(\pm 1) = 0$ . For both solutions we consider the corresponding Hopf–Cole transformation

$$\psi(x, t) := e^{-\frac{\phi(x, t)}{\delta_1}} \quad \bar{\psi}(x, t) := e^{-\frac{\bar{\phi}(x, t)}{\delta_1}}.$$

Then we can deduce the following lemma:

**Lemma 2.8.** *There exists a constant  $C > 0$  independent on  $\varepsilon$  and on  $t$  such that*

$$\|\phi - \bar{\phi}\|_{L^1} \leq C\|\rho - \bar{\rho}\|_{L^1} \quad (33)$$

$$\|\phi_{xx} - \bar{\phi}_{xx}\|_{L^1} \leq C\|\rho - \bar{\rho}\|_{L^1} \quad (34)$$

$$\|\phi_x - \bar{\phi}_x\|_{L^\infty} \leq C\|\rho - \bar{\rho}\|_{L^1}. \quad (35)$$

*Proof.* Let us multiply equation

$$\delta_1^2(\psi_{xx} - \bar{\psi}_{xx}) = (\psi - \bar{\psi})F_{\delta_2}(\rho) + \bar{\psi}(F_{\delta_2}(\rho) - F_{\delta_2}(\bar{\rho})) \quad (36)$$

by  $\eta'_\gamma(\psi - \bar{\psi})$ , with  $\eta_\gamma$  given by (27) and integrate over  $[-1, 1]$ . Integration by parts implies

$$\begin{aligned} -\delta_1^2 \int (\psi_x - \bar{\psi}_x)^2 \eta''_\gamma(\psi - \bar{\psi}) dx \\ = \int (\psi - \bar{\psi}) \eta'_\gamma(\psi, \bar{\psi}) F_{\delta_2}(\rho) dx + \int \bar{\psi} \eta'_\gamma(\psi - \bar{\psi}) [F_{\delta_2}(\rho) - F_{\delta_2}(\bar{\rho})] dx. \end{aligned}$$

We use the properties of  $\eta_\gamma$  and (25) to obtain, as  $\gamma \rightarrow 0$

$$C(\delta) \int |\psi - \bar{\psi}| dx \leq \int F_\delta(\rho) |\psi - \bar{\psi}| dx \leq \int \bar{\psi} |F_\delta(\rho) - F_\delta(\bar{\rho})| dx \leq C \int |\rho - \bar{\rho}| dx.$$

Next we can deduce (33) by using the Hopf–Cole transformation as usual. To prove (34), multiply (36) by  $\text{sign}(\psi_{xx} - \bar{\psi}_{xx})$  and integrate over  $[-1, 1]$  to obtain

$$\delta_1^2 \int |\psi_{xx} - \bar{\psi}_{xx}| dx \leq C \int |\psi - \bar{\psi}| dx + C \int |\rho - \bar{\rho}| dx.$$

Next we obtain (34) by using (33) and passing to the variable  $\phi$ . Inequality (35) follows by the Sobolev inequality as at the end of the proof of Lemma 2.6.  $\square$

#### 2.4. The Limit $\varepsilon \rightarrow 0$

Our next goal is to study the behaviour of the solution  $(\rho^\varepsilon, \phi^\varepsilon)$  to the system (16) as the parameter  $\varepsilon$  tends to zero. Using Lemma 2.5 and Lemma 2.7 we know that  $\rho^\varepsilon$  is in the space of functions having bounded variation  $BV(\Omega)$ . Therefore, we can employ the classical Helly's theorem on strong  $L^1$ -compactness of functions with bounded BV-norm, cf. [11] for instance. Thus,  $\rho^\varepsilon$  has a strong limit in  $L^1$  up to subsequences. As for the  $\phi$  variable, since  $\rho_x$  is uniformly estimated in  $L^1$ , differentiating the elliptic equation with respect to  $x$  implies that  $\phi_{xxx}^\varepsilon$  is uniformly bounded in  $L^1$  and therefore  $\phi_{xx}^\varepsilon$  is strongly compact in  $L^1$ . Denoting by  $(\rho, \phi)$  the limit  $\varepsilon \rightarrow 0$  of  $(\rho^\varepsilon, \phi^\varepsilon)$ , as the convergence is strong in  $L^1$  and due to the estimates



on  $\phi$  proven in subsection 2.1, it is immediately clear that  $\phi$  solves the second equation in (11) and  $\rho$  is a weak solution of

$$\rho_t - (\rho f^2(\rho)\phi_x)_x = 0. \quad (37)$$

In the remainder of this section, we will show that  $(\rho, \phi)$  is in fact the unique entropy solution to the system (11) in the sense of Definition 2.1. First we shall state the existence theorem.

**Theorem 2.9** (Existence of entropy solutions). *There exists an entropy solution  $(\rho, \phi)$  to system (11) with initial condition (12) and boundary conditions (13)-(14) in the sense of Definition 2.1. Such solution is the limit as  $\varepsilon \rightarrow 0$  of the solution  $\rho^\varepsilon$  to (16a)-(16b).*

*Proof.* To recover the notion of entropy solutions, we consider again the regularized equation

$$\rho_t = (\rho f^2(\rho)\phi_x)_x + \varepsilon \rho_{xx}. \quad (38)$$

We multiply this equation by  $\eta'(\rho - k)\psi$  (with  $\eta'$  defined in (27)) and integrate over  $\Omega_T = [-1, 1] \times [0, T]$

$$\begin{aligned} \iint_{\Omega_T} \eta'(\rho - k)\rho_t \psi \, dxdt &= \iint_{\Omega_T} \eta'(\rho - k)(g(\rho)\rho_x \phi_x)_x \psi \, dxdt \\ &\quad + \varepsilon \iint_{\Omega_T} \eta'(\rho - k)\rho_{xx} \psi \, dxdt. \end{aligned}$$

Adding

$$0 = \iint_{\Omega_T} \eta'(\rho - k)g(k)\phi_x \psi_x \, dxdt - \iint_{\Omega_T} \eta'(\rho - k)g(k)\phi_x \psi_x \, dxdt$$

and integrating by parts leads to

$$\begin{aligned} \iint_{\Omega_T} \eta'(\rho - k)\rho_t \psi \, dxdt &= - \iint_{\Omega_T} \eta'(\rho - k)[g(\rho) - g(k)]\psi_x \phi_x \, dxdt \\ &+ \iint_{\Omega_T} \eta'(\rho - k)g(k)\phi_{xx} \psi \, dxdt - \iint_{\Omega_T} \eta''(\rho - k)[g(\rho) - g(k)]\phi_x \rho_x \psi \, dxdt \\ &- \int_0^T \eta'(k)(g(0) - g(k)\phi_x \psi)|_{x=\pm 1} dt - \varepsilon \iint_{\Omega_T} \eta''(\rho - k)\rho_x^2 \psi \, dxdt \end{aligned}$$

$$\begin{aligned}
& -\varepsilon \iint_{\Omega_T} \eta'(\rho - k) \rho_x \psi_x \, dxdt + \int_0^T \varepsilon \eta'(\rho - k) \rho_x \psi|_{x=\pm 1} dt \\
\leq & - \iint_{\Omega_T} \eta'(\rho - k) [(g(\rho) - g(k))] \psi_x \phi_x \, dxdt \\
& + \iint_{\Omega_T} \eta'(\rho - k) g(k) \psi \phi_{xx} \, dxdt - \eta'(k) \int_0^T [(g(0) - g(k))] \phi_x \psi|_{x=\pm 1} dt \\
& - \varepsilon \iint_{\Omega_T} \eta(\rho - k) \rho_x \psi_x \, dxdt - \eta'(k) \int_0^T \varepsilon \rho_x \psi|_{x=\pm 1} dt \\
& - \iint_{\Omega_T} \eta''(\rho - k) [g(\rho) - g(k)] \phi_x \rho_x \psi \, dxdt
\end{aligned}$$

Next we integrate the first term by parts and multiply it by  $-1$ . Taking the limit  $\gamma \rightarrow 0$  the last term on the right hand side vanishes (due to the continuity of  $g$  and the boundedness of  $\phi_x$  and  $\psi$ ) and we obtain

$$\begin{aligned}
& \iint_{\Omega_T} |\rho - k| \psi_t \, dxdt + \int_{-1}^1 \rho_I(x) \psi(x, 0) \, dx \\
& \geq \iint_{\Omega_T} \operatorname{sgn}(\rho - k) [(g(\rho) - g(k))] \psi_x \phi_x \, dxdt \\
& - \iint_{\Omega_T} \operatorname{sgn}(\rho - k) g(k) \psi \phi_{xx} \, dxdt + \operatorname{sgn}(k) \int_0^T [g(0) - g(k)] \phi_x \psi|_{x=\pm 1} dt \\
& + \varepsilon \iint_{\Omega_T} |\rho - k| \rho_x \psi_x \, dxdt + \operatorname{sgn}(k) \int_0^T \varepsilon \rho_x \psi|_{x=\pm 1} dt.
\end{aligned} \tag{39}$$

Next we consider the limit  $\varepsilon \rightarrow 0$ . Using Lemma 2.5, the fourth term on the right hand side can be estimated by

$$\left| \varepsilon \iint_{\Omega_T} |\rho - k| \rho_x \psi_x \, dxdt \right| \leq \varepsilon C \|\psi_x\|_{L^\infty}, \tag{40}$$

and thus tends to zero. To compute the limit for the last term, i.e.

$$\lim_{\varepsilon \rightarrow \infty} \varepsilon \int_0^T \rho_x \psi|_{x=\pm 1} dt,$$

we introduce (following [7]), for some  $\kappa > 0$  the function  $\xi_\kappa \in C^2([-1, 1])$  with the following properties

$$\begin{cases} \xi_\kappa(x) = 1 & \text{on } x = \pm 1, \\ \xi_\kappa(x) = 0 & \text{on } \{x \in [-1, 1]; \text{dist}(x, \partial[-1, 1]) \geq \kappa\}, \\ 0 \leq \xi_\kappa(x) \leq 1 & \text{on } (-1, 1). \end{cases} \quad (41)$$

Furthermore, defining  $\mathcal{M}([-1, 1])$  as the space of Radon measures on  $[-1, 1]$ , we choose  $\xi_\kappa$  such that

$$\partial_x \xi_\kappa \rightarrow \mu|_{\{-1, 1\}} \in \mathcal{M}([-1, 1]) \text{ as } \kappa \rightarrow 0,$$

defined as

$$\int_{-1}^1 \phi(x) d\left(\mu|_{\{-1, 1\}}\right)(x) = \phi(1) - \phi(-1).$$

Now we obtain

$$\varepsilon \iint_{\Omega_T} \rho_{xx} \psi \xi \, dx dt = -\varepsilon \iint_{\Omega_T} \rho_x (\psi \xi_\kappa)_x \, dx dt + \varepsilon \int_0^T \rho_x \psi|_{x=\pm 1} dt.$$

Using (38) and (40) we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left( \varepsilon \int_0^T \rho_x \psi|_{x=\pm 1} dt \right) &= - \iint_{\Omega_T} (\rho \psi_t - g(\rho) \phi_x \psi_x) \xi_\kappa \, dx dt \\ &\quad + \iint_{\Omega_T} g(\rho) \phi_x \psi (\xi_\kappa)_x \, dx dt - \int_0^T g(0) \phi_x \psi|_{x=\pm 1} dt. \end{aligned}$$

Finally letting  $\kappa \rightarrow 0$ , the first term on the right hand side tends to zero while the second tends to an evaluation on the boundary. Thus we have

$$\lim_{\varepsilon \rightarrow \infty} \varepsilon \int_0^T \rho_x \psi|_{x=\pm 1} dt = \int_0^T (g(\text{tr } \rho) - g(0)) \phi_x(s, t) \psi|_{x=\pm 1} dt.$$

Combining this result with (39) we finally obtain the entropy formulation as in Definition 2.1 and this completes the proof.  $\square$

Next we prove that the boundary condition (13) can be recovered by the definition of entropy solution.

**Lemma 2.10.** *Let  $\rho$  be an entropy solution given by Definition 2.1. Then, the following inequality holds for all  $k \in \mathbb{R}$*

$$g(\text{tr } \rho) \geq g(k) \quad \text{at } x = \pm 1. \quad (42)$$

*Proof.* In (15), we choose the special test function  $\psi = \nu(t)\omega_\kappa$  with  $\nu \in \mathcal{C}^2([0, T])$  positive and  $\omega_\kappa \in C^2([-1, 1])$  with the following properties:

$$\begin{cases} \omega_\kappa(x) = 1 & \text{on } x = -1, \\ \omega_\kappa(x) = 0 & \text{on } \{x \in [-1, 1]; |x + 1| \geq \kappa\}, \\ 0 \leq \omega_\kappa(x) \leq 1 & \text{on } (-1, 1). \end{cases} \quad (43)$$

Similarly as before for  $\xi_\kappa$ , we choose  $\omega_\kappa$  such that

$$\partial_x \omega_\kappa \rightarrow -\delta_{-1} \text{ as } \kappa \rightarrow 0,$$

where  $\delta_{-1}$  denotes the Dirac delta measure centered at  $-1$ . Then, in the limit  $\kappa \rightarrow 0$  (15) converges to

$$\begin{aligned} & \int_0^T \operatorname{sgn}(\operatorname{tr} \rho - k) [(g(\operatorname{tr} \rho) - g(k)) \phi_x|_{x=-1} \nu(t) dt \\ & + \operatorname{sgn}(k) \int_0^T [(g(\operatorname{tr} \rho) - g(k)) \phi_x|_{x=-1} \nu(t) dt] \geq 0. \end{aligned}$$

Thus, almost everywhere in  $\{-1\} \times (0, T)$  we have

$$(\operatorname{sgn}(\operatorname{tr} \rho - k) + \operatorname{sgn}(k)) [(g(\operatorname{tr} \rho) - g(k)) \phi_x] \geq 0.$$

To conclude the proof we note that  $\phi_x$  is always (i.e. independently of the given  $\rho$ ) non-negative at  $x = -1$ . This is a consequence of the fact that  $\phi = 0$  at  $x = \pm 1$  (boundary conditions) and positive on the whole domain, due to a trivial minimum principle for the equation (16b). In a similar way, one can construct a function  $\omega_k$  concentrating on  $x = 1$  with a derivative converging to a Dirac delta at  $x = 1$ . The same inequality is obtained since the change of sign in the derivative of concentrator  $\omega_k$  is balanced by the change of sign in  $\phi_x$  (non-increasing at  $x = 1$ ).  $\square$

## 2.5. Uniqueness

Next we shall prove that the entropy solution in the sense of Definition 2.1 is unique.

**Theorem 2.11** (Uniqueness of entropy solutions). *There exists at most one entropy solution  $(\rho, \phi)$  to the system (11) with initial condition (12) and boundary conditions (13)-(14) in the sense of Definition 2.1.*

The above stated result is a consequence of the following stability theorem, which follows the same technique developed in [19]. Here the authors use the variables doubling technique originally introduced in [21]. A similar strategy was also used e.g. [3, 4].

We state the following useful result:

**Lemma 2.12.** ([19]) Consider a function  $z = z(x)$  belonging to  $L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$  and let  $h$  be Lipschitz on the interval  $I_z := [-\|z\|_{L^\infty}, \|z\|_{L^\infty}]$ . Then  $h(z)$  belongs to  $L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$  and

$$\left| \frac{\partial}{\partial x_j} h(z) \right| \leq \|h\|_{Lip(I_z)} \left| \frac{\partial}{\partial x_j} z \right|$$

in the sense of measures for  $j = 1, \dots, d$ .

Uniqueness can be deduced from the following theorem:

**Theorem 2.13.** Let  $(\rho, \phi)$ ,  $(\bar{\rho}, \bar{\phi})$  be the two entropy solutions to system (11) according to Definition 2.1 with initial data  $\rho_I, \bar{\rho}_I \in L^\infty([-1, 1]) \cap BV([-1, 1])$  respectively. Then for almost all  $t \in (0, T)$ ,

$$\begin{aligned} \|\rho(\cdot, t) - \bar{\rho}(\cdot, t)\|_{L^1} &\leq \|\rho_I - \bar{\rho}_I\|_{L^1(\Omega)} + t\|g\|_{L^\infty} \|\phi_{xx} - \bar{\phi}_{\bar{x}\bar{x}}\|_{L^\infty((0, T); L^1(\Omega))} \\ &\quad + t\|g\|_{Lip(\Omega)} \|\rho_x(\cdot, t)\|_{L^1(\Omega)} \|\phi_x - \bar{\phi}_{\bar{x}}\|_{L^\infty((0, T); L^\infty)} \end{aligned}$$

holds.

Combining this result with (34) and (35) from Lemma 2.8 we obtain

$$\|\rho(\cdot, t) - \bar{\rho}(\cdot, t)\|_{L^1(\Omega)} \leq \|\rho_I - \bar{\rho}_I\|_{L^1(\Omega)} + tC\|\rho(\cdot, t) - \bar{\rho}(\cdot, t)\|_{L^1(\Omega)}, \quad (44)$$

for some positive constant  $C$ . Choosing  $t$  small enough this inequality contradicts the existence of two different solutions  $\rho$  and  $\bar{\rho}$  having the same initial datum and thus implies uniqueness. It remains to prove Theorem 2.13.

*Proof.* Consider a nonnegative, compactly supported, Lipschitz continuous function  $\psi(x, t, \bar{x}, \bar{t})$ , defined on  $[-1, 1] \times [0, T[ \times [-1, 1] \times [0, T[$ . Furthermore, let  $\psi$  be zero on  $\{-1, 1\} \times [0, T)$ . Next, we take two admissible solutions  $\rho(x, t)$ ,  $\bar{\rho}(\bar{x}, \bar{t})$  and write (15) as

$$\begin{aligned} &\iint_{\Omega_T} |\rho - \bar{\rho}| \psi_t \, dxdt - \iint_{\Omega_T} \operatorname{sgn}(\rho - \bar{\rho}) [(g(\rho) - g(\bar{\rho})) \psi_x \phi_x(x, t)] \, dxdt + \\ &\iint_{\Omega_T} \operatorname{sgn}(\rho - \bar{\rho}) g(\bar{\rho}) \psi \phi_{xx}(x, t) \, dxdt - \operatorname{sgn}(\bar{\rho}) \int_0^T [(g(\operatorname{tr} \rho) - g(\bar{\rho})) \phi_x(x, t) \psi]_{|x=\pm 1} \, dt \geq 0. \end{aligned}$$

and

$$\begin{aligned} &\iint_{\Omega_T} |\bar{\rho} - \rho| \psi_{\bar{t}} \, d\bar{x}d\bar{t} - \iint_{\Omega_T} \operatorname{sgn}(\bar{\rho} - \rho) [(g(\bar{\rho}) - g(\rho)) \psi_{\bar{x}} \bar{\phi}_{\bar{x}}(\bar{x}, \bar{t})] \, d\bar{x}d\bar{t} + \\ &\iint_{\Omega_T} \operatorname{sgn}(\bar{\rho} - \rho) g(\rho) \psi \bar{\phi}_{\bar{x}\bar{x}}(\bar{x}, \bar{t}) \, d\bar{x}d\bar{t} - \operatorname{sgn}(\rho) \int_0^T [(g(\operatorname{tr} \bar{\rho}) - g(\rho)) \phi_x(\bar{x}, \bar{t}) \psi]_{|\bar{x}=\pm 1} \, d\bar{t} \geq 0. \end{aligned}$$

Integrating both the above inequalities over  $\Omega_T := \Omega \times [0, T[$ , the first with respect to  $\bar{x}, \bar{t}$  and the second with respect to  $x, t$  and adding the resulting equations leads to

$$\begin{aligned}
& \iiint_{\Omega_T \times \Omega_T} |\rho - \bar{\rho}| (\psi_t + \psi_{\bar{t}}) dzd\bar{z} \\
& - \iiint_{\Omega_T \times \Omega_T} \underbrace{[\text{sgn}(\rho - \bar{\rho}) (g(\rho)\phi_x(x, t) - g(\bar{\rho})\bar{\phi}_x(\bar{x}, \bar{t})) (\psi_x + \psi_{\bar{x}})]}_{:=I_1} dzd\bar{z} \\
& - \iiint_{\Omega_T \times \Omega_T} \underbrace{[\text{sgn}(\rho - \bar{\rho}) (g(\bar{\rho})\psi_x (\bar{\phi}_{\bar{x}}(\bar{x}, \bar{t}) - \phi_x(x, t)) + g(\rho)\psi_{\bar{x}} (\bar{\phi}_{\bar{x}}(\bar{x}, \bar{t}) - \phi_x(x, t)))]}_{:=I_{2,1}} dzd\bar{z} \\
& + \iiint_{\Omega_T \times \Omega_T} \underbrace{[\text{sgn}(\rho - \bar{\rho})(g(\bar{\rho})\phi_{xx}(x, t) - g(\rho)\bar{\phi}_{\bar{x}\bar{x}}(\bar{x}, \bar{t}))\psi]}_{:=I_{2,2}} dzd\bar{z} \\
& = \iiint_{\Omega_T \times \Omega_T} (|\rho - \bar{\rho}|(\psi_t + \psi_{\bar{t}}) + I_1 + I_{2,1} + I_{2,2}) dzd\bar{z} \geq 0.
\end{aligned}$$

Here  $z = (x, t)$  and  $\bar{z} = (\bar{x}, \bar{t})$ . We take a symmetric function  $\delta \in \mathcal{C}^\infty(\mathbb{R})$  with total mass one and  $\text{Supp}(\delta) \subset (-1, 1)$ . We define

$$\delta_h(\cdot) := \frac{1}{h} \delta\left(\frac{\cdot}{h}\right)$$

and choose the following test function

$$\psi = \nu \left( \frac{t + \bar{t}}{2}, \frac{x + \bar{x}}{2} \right) \delta_h \left( \frac{t - \bar{t}}{2} \right) \delta_h \left( \frac{x - \bar{x}}{2} \right).$$

From this definition we conclude

$$\begin{aligned}
& \iiint_{\Omega_T \times \Omega_T} (|\rho - \bar{\rho}|(\psi_t + \psi_{\bar{t}}) + I_1) dxdt d\bar{x}d\bar{t} \\
& = \iiint_{\Omega_T \times \Omega_T} (|\rho - \bar{\rho}|\nu_t + \text{sgn}(\rho - \bar{\rho}) (g(\rho)\phi_x(x, t) - g(\bar{\rho})\bar{\phi}_{\bar{x}}) \nu_x) \times \\
& \quad \times \delta_h \left( \frac{t - \bar{t}}{2} \right) \delta_h \left( \frac{x - \bar{x}}{2} \right) dxdt d\bar{x}d\bar{t}.
\end{aligned}$$

We now consider the term  $I_{2,1}$

$$\begin{aligned}
I_{2,1} & = -\text{sgn}(\rho - \bar{\rho}) [\bar{\phi}_x(\bar{x}, \bar{t}) (g(\bar{\rho}) + g(\rho)) - \phi_x(x, t) (g(\bar{\rho}) + g(\rho))] \frac{1}{2} \nu_x \delta_h \delta_h \\
& \quad - \text{sgn}(\rho - \bar{\rho}) [\bar{\phi}_x(\bar{x}, \bar{t}) (g(\bar{\rho}) - g(\rho)) - \phi_x(x, t) (g(\bar{\rho}) - g(\rho))] \nu (\delta_h \delta_h)_x \\
& =: I_{2,1,1} + I_{2,1,2}.
\end{aligned}$$

Here, we used that by definition we have  $\nu_{\bar{x}} = \frac{1}{2}\nu_x$  and  $(\delta_h\delta_h)_{\bar{x}} = -(\delta_h\delta_h)_x$ . Integrating by parts in  $I_{2,1,2}$  leads to

$$\begin{aligned}
& - \iiint_{\Omega_T \times \Omega_T} \text{sgn}(\rho - \bar{\rho}) [\bar{\phi}_x(\bar{x}, \bar{t}) (g(\bar{\rho}) - g(\rho)) - \phi_x(x, t) (g(\bar{\rho}) - g(\rho))] \times \\
& \quad \times \nu(\delta_h\delta_h)_x dx dt d\bar{x} d\bar{t} \\
& = \iiint_{\Omega_T \times \Omega_T} \text{sgn}(\rho - \bar{\rho}) [\bar{\phi}_x(\bar{x}, \bar{t}) (g(\bar{\rho}) - g(\rho)) - \phi_x(x, t) (g(\bar{\rho}) - g(\rho))] \times \\
& \quad \times \frac{1}{2}\nu_x \delta_h \delta_h dx dt d\bar{x} d\bar{t} \\
& + \iiint_{\Omega_T \times \Omega_T} \bar{\phi}_{\bar{x}} [(\text{sgn}(\rho - \bar{\rho})(g(\bar{\rho}) - g(\rho)))_x - \phi_{xx}(x, t) \text{sgn}(\rho - \bar{\rho}) (g(\bar{\rho}) - g(\rho))] \times \\
& \quad \times \nu \delta_h \delta_h dx dt d\bar{x} d\bar{t} \\
& + \iiint_{\Omega_T \times \Omega_T} -\phi_x(x, t) (\text{sgn}(\rho - \bar{\rho}) (g(\bar{\rho}) - g(\rho)))_x \nu \delta_h \delta_h dx dt d\bar{x} d\bar{t}.
\end{aligned}$$

Noticing that

$$\begin{aligned}
& -\phi_{xx}(x, t) \text{sgn}(\rho - \bar{\rho}) (g(\bar{\rho}) - g(\rho)) + I_{2,2} \\
& \quad = -\text{sgn}(\rho - \bar{\rho}) (\bar{\phi}_{\bar{x}\bar{x}}(\bar{x}, \bar{t})) - \phi_{xx}(x, t) g(\rho) \nu \delta_h \delta_h
\end{aligned}$$

and adding again  $I_{2,1,1}$  we obtain

$$\begin{aligned}
& \iiint_{\Omega_T \times \Omega_T} (I_{2,2} + I_{2,1,2} + I_{2,1,1}) dx dt d\bar{x} d\bar{t} \\
& = \iiint_{\Omega_T \times \Omega_T} -\text{sgn}(\rho - \bar{\rho}) (\bar{\phi}_{\bar{x}\bar{x}}(\bar{x}, \bar{t})) - \phi_{xx}(x, t) g(\rho) \nu \delta_h \delta_h dx dt d\bar{x} d\bar{t} \\
& + \iiint_{\Omega_T \times \Omega_T} (\bar{\phi}_{\bar{x}} - \phi_x) (\text{sgn}(\rho - \bar{\rho}) (g(\bar{\rho}) + g(\rho)))_x \nu \delta_h \delta_h dx dt d\bar{x} d\bar{t} \\
& + \iiint_{\Omega_T \times \Omega_T} \underbrace{\text{sgn}(\rho - \bar{\rho}) [\bar{\phi}_x(\bar{x}, \bar{t}) g(\rho) - \phi_x(x, t) g(\rho)] \nu_x \delta_h \delta_h}_{=: J} dx dt d\bar{x} d\bar{t}.
\end{aligned}$$

As there are no more derivatives in the terms involving  $\delta_h\delta_h$ , we consider the limit  $h \rightarrow 0$ , remove two integrals and set  $x = \bar{x}$ ,  $t = \bar{t}$ . We choose the new test function

$$\nu(x, t) = \nu_{\kappa, \bar{h}}(x, t) = (1 - \xi_\kappa(x)) \chi_{\bar{h}}(t),$$

with for some  $0 < t_1 < t_2 < T$  fixed

$$\chi_h(t) = \int_{-\infty}^t (\delta_h(\tau - t_1) - \delta_h(\tau - t_2)) d\tau,$$

and  $\xi_\kappa$  as defined in (41). We observe that all terms which are bounded in  $L^1$  and multiplied by  $(\nu_{\kappa, \tilde{h}}(x, t))_x$  converge to a boundary term in the limit  $\kappa \rightarrow 0$ . We thus have

$$\lim_{\substack{\tilde{h} \rightarrow 0, \\ \kappa \rightarrow 0}} \iint_{\Omega_T} (I_1 + J) dxdt = - \int_{t_1}^{t_2} \int_{\partial\Omega} \text{sgn}(\text{tr } \rho - \text{tr } \bar{\rho}) \bar{\phi}_{\bar{x}} [g(\text{tr } \rho) - g(\text{tr } \bar{\rho})] dsdt,$$

and therefore

$$\begin{aligned} & - \lim_{\substack{\tilde{h} \rightarrow 0, \\ \kappa \rightarrow 0}} \iint_{\Omega_T} (|\rho - \bar{\rho}| \nu_t + I_1 + I_{2,1} + I_{2,2}) dxdt \\ &= - \int_{-1}^1 (|\rho - \bar{\rho}|) dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{-1}^1 - \text{sgn}(\rho - \bar{\rho}) (\bar{\phi}_{\bar{x}\bar{x}}(\bar{x}, \bar{t}) - \phi_{xx}(x, t)) g(\rho) dxdt \\ & \quad + \int_{t_1}^{t_2} \int_{-1}^1 (\bar{\phi}_{\bar{x}} - \phi_x) (\text{sgn}(\rho - \bar{\rho}) (g(\bar{\rho}) + g(\rho)))_x dxdt \\ & \quad + \int_{t_1}^{t_2} \int_{\partial\Omega} \text{sgn}(\text{tr } \rho - \text{tr } \bar{\rho}) \bar{\phi}_{\bar{x}} [g(\text{tr } \rho) - g(\text{tr } \bar{\rho})] |_{x=\pm 1} dt \geq 0. \end{aligned}$$

Using Lemma 2.12, we have

$$|(\text{sgn}(\text{tr } \rho - \text{tr } \bar{\rho}) (g(\text{tr } \bar{\rho}) - g(\text{tr } \rho)))_x| \leq \|g\|_{Lip(I)} |\rho_x|. \quad (45)$$

Collecting all the above terms we obtain

$$\begin{aligned} \|\rho(\cdot, t) - \bar{\rho}(\cdot, t)\|_{L^1} \Big|_{t_1}^{t_2} &\leq \int_{t_1}^{t_2} \int_{\Omega} [|\phi_{xx}(x, t) - \bar{\phi}_{\bar{x}\bar{x}}(x, t)| \|g\|_{L^\infty} \\ & \quad + \|\phi_x - \bar{\phi}_{\bar{x}}\|_{L^\infty} \|g\|_{Lip(I)} |\rho_x|] dxdt \\ & \quad + \int_{t_1}^{t_2} \int_{\partial\Omega} \text{sgn}(\text{tr } \rho - \text{tr } \bar{\rho}) \bar{\phi}_{\bar{x}} [g(\text{tr } \rho) - g(\text{tr } \bar{\rho})] dsdt. \end{aligned} \quad (46)$$

Following [7], we define

$$k(x, t) = \begin{cases} \text{tr } \rho & \text{if } \text{tr } \rho \in (0, \text{tr } \bar{\rho}), \\ 0 & \text{if } 0 \in (\text{tr } \bar{\rho}, \text{tr } \rho), \\ \text{tr } \bar{\rho} & \text{if } \text{tr } \bar{\rho} \in (\text{tr } \rho, 0). \end{cases} \quad (x, t) \in ]0, T[ \times \partial\Omega$$



This allows us to write

$$\begin{aligned} \operatorname{sgn}(\operatorname{tr} \rho - \operatorname{tr} \bar{\rho}) \bar{\phi}_x [g(\operatorname{tr} \rho) - g(\operatorname{tr} \bar{\rho})] &= \operatorname{sgn}(\operatorname{tr} \rho - k) \bar{\phi}_x [g(\operatorname{tr} \rho) - g(k)] \\ &+ \operatorname{sgn}(\operatorname{tr} \bar{\rho} - k) \bar{\phi}_x [g(\operatorname{tr} \bar{\rho}) - g(k)]. \end{aligned}$$

Using Lemma 2.10 we conclude that the last term on the right hand side of (46) is negative and can therefore be omitted. Thus letting  $t_1 \rightarrow 0$  we arrive at the desired inequality and this completes the proof.  $\square$

### 3. Numerics and Examples for the Hughes' model

In this section we discuss the behaviour of solutions for the non regularized one-dimensional problem with simple initial data. Already these examples show quite interesting features which can be reproduced by numerical simulations. The content of this section is formal as we don't provide any existence and uniqueness theory. However, the characteristic calculus provides a useful tool to understand qualitatively the behaviour of the solution in the simple examples considered and is in complete agreement with the numerical results.

#### 3.1. Characteristic Calculus

We consider the non-regularized problem

$$\rho_t - (\rho f^2(\rho) \phi_x)_x = 0, \quad (47a)$$

$$|\phi_x| = \frac{1}{f(\rho)}. \quad (47b)$$

In the following, we always consider the unique viscosity solution  $\phi$  to (47b) and note that this solution has a unique turning point  $x_0(t)$  given by the implicit relation

$$\int_{-1}^{x_0(t)} \frac{1}{f(\rho)} dx = \int_{x_0(t)}^1 \frac{1}{f(\rho)} dx.$$

Thus, (47a) can be written as (using that  $|\phi_x| = \phi_x \operatorname{sgn} \phi_x$ )

$$\rho_t - (\rho f(\rho) \operatorname{sgn} \phi_x)_x = 0. \quad (48)$$

The natural boundary conditions (in the spirit of [7, 9]) are given by

$$f(\operatorname{tr} \rho) \geq f(k) \quad \text{on } x = \pm 1, \quad \text{for all } k \in [0, \operatorname{tr} \rho], \quad (49)$$

which is satisfied if and only if  $\operatorname{tr} \rho$  belongs to the interval of densities corresponding to outgoing characteristics, i. e.  $\operatorname{tr} \rho \in [0, 1/2]$ . As shown in [9], the boundary

condition in case of incoming characteristics is determined by solving a Riemann problem between the boundary datum (i.e. zero in this case) and the trace of the density next to the boundary.

Away from the time dependent interface  $x = x_0(t)$  (where  $\phi_x$  is discontinuous) we can give sense to characteristics. They are defined by

$$\dot{x} = -(1 - 2\rho) \operatorname{sgn}(\phi_x).$$

Note that the Rankine-Hugoniot condition for a hyperbolic conservation law with flux  $F$ , i.e.  $\rho_t + F(\rho)_x = 0$  is given by

$$[[F(\rho)]] = \dot{x}_0(t) [[\rho]]. \quad (50)$$

Here,  $[[\cdot]]$  denotes the jump at the discontinuity  $x_0$ .

### 3.1.1. Constant initial data

We would like to understand the behaviour of the solution in the very simple case of constant initial data. Here we are particularly interested in the three cases which correspond to different characteristic speeds, i.e.  $\rho_I$  less, equal or greater than  $1/2$ . In particular we consider the cases  $\rho_I = 1/4$ ,  $\rho_I = 1/2$  and  $\rho_I = 3/4$ . In the case of constant initial data, the interface is constant in time, i.e.  $\dot{x}_0 = 0$  and located at  $x = 0$ . Thus  $\operatorname{sgn} \phi_x = -\operatorname{sgn} x$  and (47a) can be written as

$$\rho_t + (\rho f(\rho) \operatorname{sgn} x)_x = 0. \quad (51)$$

The RH condition (50) for this flux  $F(\rho) = \rho f(\rho) \operatorname{sgn} x$  reads

$$f(\rho^+) + f(\rho^-) = 0,$$

where  $\rho^\pm$  denote the right and left limit of  $\rho$  at the interface  $x = 0$ . An immediate consequence of this is that constant functions  $\rho(x, t) = c$  with  $c \in (0, 1)$  do not satisfy the RH condition (50) and are not weak solutions. If we start with a constant initial datum we expect the equation to “correct” this by forcing  $\rho(0, t) = 0$  in arbitrary small time ( $\rho(x, t) = 1$  would also create a solution, which however does not fulfil the entropy condition). Then two shocks originate between  $\rho(0, t) = 0$  and  $\rho(x, t) = c$  for  $x \neq 0$ , which move towards the boundary. The slope of these shocks is determined by the RH condition (50). In the three cases considered we obtain

$$\dot{x} = \begin{cases} \pm \frac{3}{4} & \rho_I(x) = \frac{1}{4} \\ \pm \frac{1}{2} & \rho_I(x) = \frac{1}{2} \\ \pm \frac{1}{4} & \rho_I(x) = \frac{3}{4}. \end{cases}$$

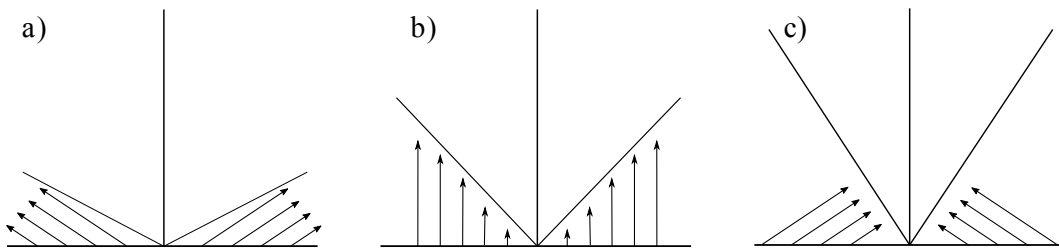


Figure 2: a)  $\rho = 0.25$  b)  $\rho = 0.5$  c)  $\rho = 0.75$

This situation, locally around  $x = 0$ , is sketched in Fig. 2. Around the center  $x = 0$  where no information is transported to, we expect the solution to be either zero or a rarefaction wave. In case of a rarefaction wave we make the ansatz  $\rho(x, t) = u\left(\frac{x}{t}\right)$  and deduce from (51) that

$$u_{\text{RF}}(x, t) = \frac{x + t}{2t}.$$

This solution continuously connects the two outgoing shocks but creates the constant value  $1/2$  at  $x = 0$  and is thus not admissible. Therefore, we expect formation of a vacuum in between the two shocks in all three cases. In the case  $\rho = 3/4$ , we encounter an additional phenomenon at the boundaries. Here the characteristics point inwards, therefore we need to prescribe boundary conditions at  $x = \pm 1$ . We choose the following Dirichlet boundary conditions  $\rho(\pm 1, t) = 1/2$  (maximal flux). Such condition is easily recovered by solving the Riemann problem between  $\text{tr } \rho = 3/4$  and the boundary value zero (cf. [9]).

This implies that the characteristics at the boundary are vertical while characteristics of slope  $1/2$  transport the value  $3/4$  into the domain. Hence we obtain two wedges (one at each boundary) in which no information is transported by characteristics. If we make again the ansatz  $\rho(x, t) = u\left(\frac{x+1}{t}\right)$  (shifted to the left boundary), we obtain the following rarefaction wave

$$\rho(x, t) = \frac{x + 1 + t}{2t},$$

which is an admissible solution. Thus we expect rarefaction waves at both boundaries. At time  $t = 4/3$ , these rarefaction waves will hit the shocks coming from the interface (at  $x = \pm 1/3$ , respectively). To calculate the new slope of the shock we use the RH condition (50) which results in the following ODE

$$\dot{s}(t) = -\frac{s(t)}{2t} + \frac{t-1}{2t}, \quad s\left(\frac{4}{3}\right) = -\frac{1}{3}.$$

Using standard techniques, we obtain the solution

$$s(t) = -\sqrt{t} \left( \frac{1+t}{\sqrt{t}} - \sqrt{3} \right).$$

A complete picture of the case  $\rho_I(x) = 3/4$  is given in Fig. 3. In the next section we will see that all these phenomena can be observed in numerical simulations.

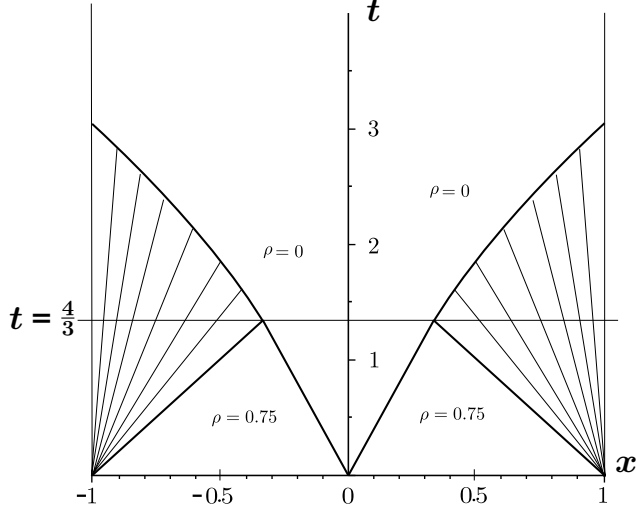


Figure 3: Details for the case  $\rho_I(x) = 3/4$

**Remark 3.1** (Boundary conditions in the regularized and non regularized case). At a first glance there is a clear discrepancy between the boundary conditions in the regularized case (13) and the ones prescribed above for the non regularized model. In the latter case, the set of admissible boundary data is determined via the monotonicity of  $f$  at the boundary, whereas in the former case this set is determined via the function  $g$ . Hence, there is the possibility of a boundary layer in a possible limit as  $\delta_1 \rightarrow 0$ . However, the regularized problem has a source term  $g(\rho)\phi_{xx}$ , and this fact could possibly imply some compensation phenomena at the boundary which can avoid the boundary layer. This issue will be the topic of future study.

### 3.2. Numerical simulations

Next we present numerical simulations of (47) relating the results to the previous discussion in Section 3.1. We consider the regularized system on the domain  $\Omega = [-1, 1]$

$$\rho_t - \operatorname{div}(\rho f(\rho) \operatorname{sgn} \phi_x) = \varepsilon \rho_{xx} \quad (52a)$$

$$|\phi_x| = \frac{1}{f(\rho)} \quad (52b)$$

with a regularization parameter  $\varepsilon \geq 0$ . The system is supplemented with the initial condition  $\rho(x, 0) = \rho_I(x)$  and homogeneous Dirichlet boundary conditions  $\rho(\pm 1, t) = \rho_D$ . We solve (52) in an iterative manner, i.e.

1. Given  $\rho$  solve the eikonal equation (52b) with fast sweeping method.
2. Solve the non-linear conservation law (52a) for a given  $\phi$  using an ENO scheme or resp. a Godunov scheme.

We choose the following discretisation. The domain  $\mathbb{R}$  is divided into cells  $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$  with centers at points  $x_j = j\Delta x$  for  $j \in \mathbb{Z}$ . The time domain  $(0, \infty)$  is discretised in the same manner via  $t^n = n\Delta t$  resulting in time strips  $I^n = [t^n, t^{n+1}]$ .

We used two different schemes to compare and understand the behaviour of solutions. In the first approach we use an ENO scheme with small diffusion on the whole domain  $\Omega = [-1, 1]$ . In the second approach we split the domain into two parts  $\Omega = \Omega_1 \cup \Omega_2$  where  $\Omega_1 = [0, x(t)]$  and  $\Omega_2 = [x(t), 1]$ , solve equation (52a) with a Godunov scheme (and no diffusion, i.e.  $\varepsilon = 0$ ) on  $\Omega_1$  and  $\Omega_2$  and concatenate both solutions.

### 3.2.1. ENO scheme

J. Towers presented convergence results for an ENO scheme for conservation laws with discontinuous flux in [37]. This ansatz can be used in Step (2) to solve (52a) with small diffusion on the whole domain  $\Omega = [-1, 1]$ . Let  $\chi_j^n$  denote the characteristic function on the rectangle  $R_j^n = I_j \times I^n$ . The finite difference scheme then generates for every mesh size  $\Delta x$  and  $\Delta t$  a piecewise constant solution  $\rho^\Delta$  given by

$$\rho^\Delta(x, t) = \sum_{n \geq 0} \sum_{-\infty}^{\infty} \chi_j^n \rho_j^n.$$

The approximations  $\rho_j^n$  are generated by an explicit algorithm

$$\rho_j^{n+1} = \rho_j^n - \lambda_1(k_{j+\frac{1}{2}}h_{j+\frac{1}{2}} - k_{j-\frac{1}{2}}h_{j-\frac{1}{2}}) + \lambda_2(d_{j+\frac{1}{2}} - d_{j-\frac{1}{2}}). \quad (53)$$

Here  $\lambda_1 = \frac{\Delta t}{\Delta x}$ ,  $\lambda_2 = \frac{\varepsilon \Delta t}{\Delta x^2}$  and  $k_{j \pm \frac{1}{2}} = \text{sgn } \phi_x(x_{j \pm \frac{1}{2}})$ . The diffusive flux is given by  $d_{j+\frac{1}{2}}^n := \rho_{j+1}^n - \rho_j^n$ , the convective one  $h_{j+\frac{1}{2}} := h(v, u)$  is chosen such that it is consistent with the actual flux, i.e.  $h(\rho, \rho) = g(\rho) = \rho f(\rho)$ . To guarantee monotonicity the flux is transposed when  $k_{j+\frac{1}{2}}$  changes sign, i.e.

$$h_{j+\frac{1}{2}} = \begin{cases} h(\rho_{j+1}, \rho_j) & \text{if } k_{j+\frac{1}{2}} \geq 0 \\ h(\rho_j, \rho_{j+1}) & \text{if } k_{j+\frac{1}{2}} < 0. \end{cases}$$

We choose the ENO flux [10] which is given by

$$h(v, u) = \frac{1}{2} (g(u) + g(v)) + \frac{1}{2} \int_u^v |g_u| du. \quad (54)$$

*Godunov scheme.* The Godunov scheme is derived by using the exact solution operator for  $\rho_t + (F(\rho))_x = 0$  with piecewise constant initial data. The resulting numerical flux is  $h(v, u) = F(u^G(v, u))$ , where  $u^G(v, u)$  is the similarity solution of the resulting Riemann problem with right and left state  $v$  and  $u$  evaluated anywhere on the vertical half-line  $t > 0$  where the jump in the initial data occurs. The Godunov flux [30] is given by

$$h(v, u) = \begin{cases} \min_{[u,v]} F(w) & \text{if } u \leq v \\ \max_{[u,v]} F(w) & \text{if } u \geq v. \end{cases} \quad (55)$$

*Constant initial data.* First we would like to validate the characteristic calculus presented in section 3.1. We choose constant initial data  $\rho_I(x)$  that is smaller or larger than  $1/2$ . The time discretisation is set to  $\Delta t = 10^{-4}$ , the spatial discretisation to  $\Delta x = 10^{-2}$ . Here we solved the non regularized problem with  $\varepsilon = 0$  using Godunov's method. First we choose  $\rho_I(x) = 1/4$ , the evolution is depicted in Figure 4. In this case the characteristics point outward, therefore we prescribe numerical boundary conditions instead of physical ones. In our second example we

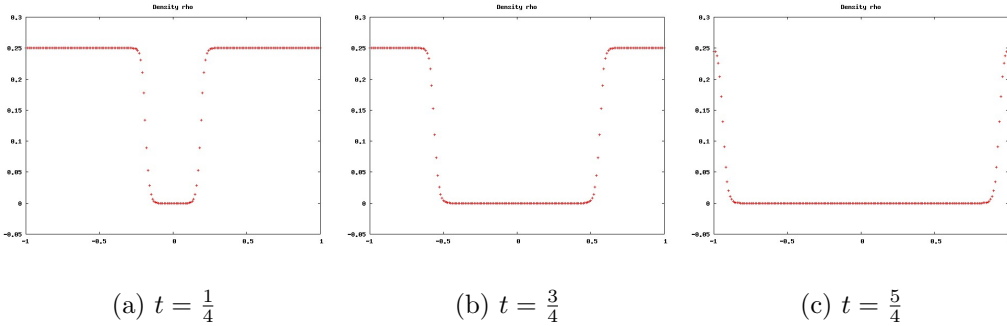


Figure 4: Evolution of  $\rho$  with initial datum  $\rho_I(x) = 0.25$

set  $\rho_I(x) = 3/4$ . Here we observe a good agreement of the numerical simulation with the theoretical results, see Figure 5. Note that the shock hits the rarefaction waves at  $t = 4/3$  and that the subsequent shock hits the boundary at  $t = 3$  (as predicted by our characteristic calculus).

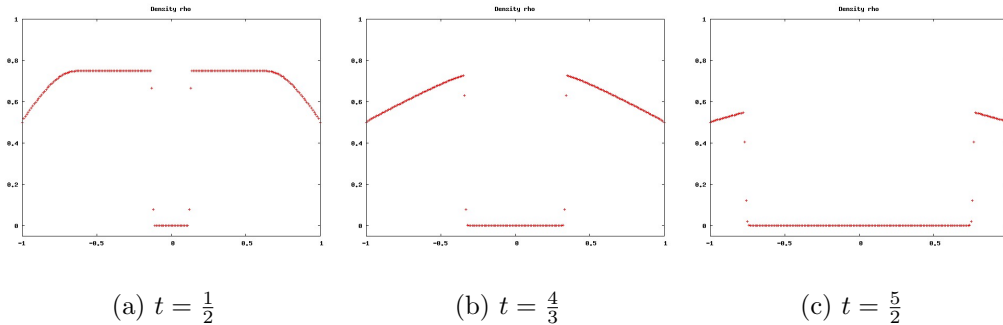


Figure 5: Evolution of  $\rho$  with initial datum  $\rho_I(x) = 0.75$  and Dirichlet boundary conditions  $\rho(\pm 1) = 0.5$

*Other examples.* Finally we would like to illustrate the behavior with other examples. We choose the following initial guess

$$\rho_I(x) = \begin{cases} 0.8 & \text{if } -0.8 \leq x \leq -0.5 \\ 0.6 & \text{if } -0.3 \leq x \leq 0.3 \\ 0.9 & \text{if } 0.4 \leq x \leq 0.75, \end{cases}$$

representing three groups which would like to exit at  $x = 1$  or  $x = -1$ . We set the spatial discretisation to  $\Delta x = 10^{-3}$ , the discretisation in time to  $\Delta t = 10^{-4}$ . Here we solve (52a) on the whole domain using an ENO flux and  $\varepsilon = 10^{-4}$ . The evolution of the densities is illustrated in Figure 6. Here the  $y$  axis corresponds to time, running from 0 (top) to 1.5 (bottom). The right group (located between  $0.4 \leq x \leq 0.75$ ) splits at the beginning, a small part moves to the left while the rest moves towards the right exit. We observe that the part of the group which was moving to the left changes direction and moves towards the right.

#### 4. Alternative Regularization

In this section we prove that the alternative regularized problem

$$\begin{cases} \rho_t - (\rho f^2(\rho) \phi_x)_x = 0 \\ -\delta \phi_{xx} + f(\rho)^2 |\phi_x|^2 = 1 \end{cases} \quad (56)$$

admits at least an entropy solution. The initial condition and the boundary data are posed exactly in the same way as in the previous model, therefore we shall omit them. We shall only provide a sketch of the proof. Throughout this section we will consider (16a), (16b) with homogeneous Dirichlet boundary conditions for  $\rho$  and  $\phi$  and  $\rho_I \geq 0$  as initial datum.

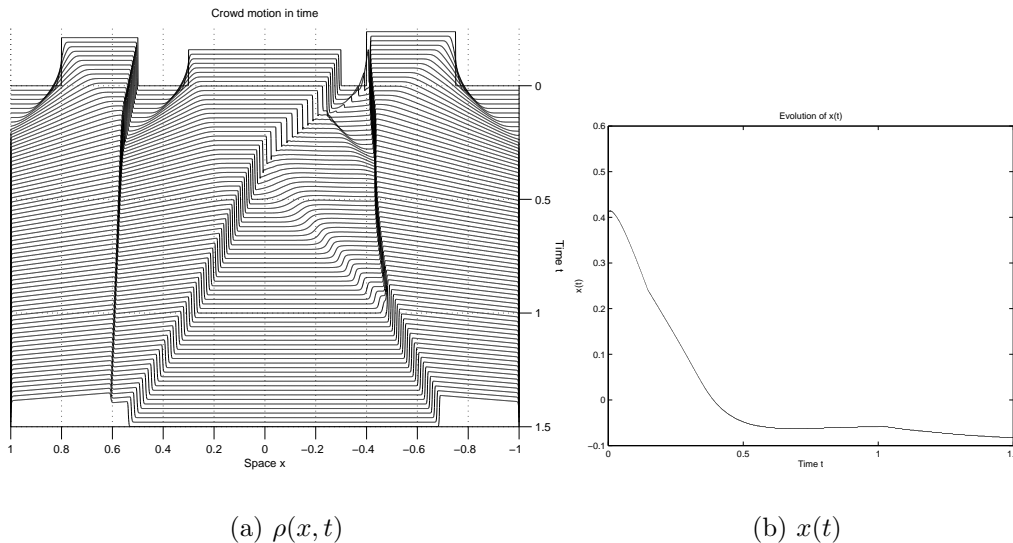


Figure 6: Evolution of  $\rho$  and  $x(t)$

As we did in the previous case, we approximate the scalar conservation law by the viscous approximation

$$\rho_t - (\rho f^2(\rho) \phi_x)_x = \varepsilon \rho_{xx}.$$

In order to prove existence of smooth solutions to the approximated model, one can cut off the term  $f(\rho)^2 |\phi_x|^2$  in the elliptic equation and send the cut-off parameter to the limit.

In order to obtain a limit for  $\rho_\varepsilon$  as  $\varepsilon \rightarrow 0$ , one can try to estimate the *BV* norm of  $\rho$ , as done in the previous approximation. Using the same arguments as in the proof of Lemma 2.5 we immediately have

$$\|\rho_x\|_{L^1(\Omega)} \leq C_1 e^{C_2 t}. \quad (57)$$

The next step would now be to derive an estimate on  $\rho_t$ . However, this has not been possible as we were not able to control terms of the form  $\phi_{xt}$  or  $\phi_{xxt}$ . Indeed, the time dependence of  $\phi$  is introduced only by  $\rho$  in the term  $f(\rho) |\phi_x|^2$ . However, as there are no time derivatives, it is by no means straight forward to derive bounds on time derivatives of  $\phi$ . To still obtain existence of a weak solution, we will use the following Aubin-Lions like argument (see, e.g. [35, Chapter 3.2, Thm 2.1]), using in particular the  $L^1$  bound on  $\rho_x$  obtained above.

We consider the three Banach spaces  $W^{1,1} \subset L^2 \subset H^{-1}$  with continuous injections. Note that  $H^{-1}$  is reflexive and the injection  $W^{1,1} \rightarrow L^2$  is compact. Let



$T > 0$  and consider the space

$$\mathcal{Y} = \left\{ v \in L^2((0, T); W^{1,1}), \dot{v} = \frac{dv}{dt} \in L^2((0, T); H^{-1}) \right\}$$

which, equipped with the norm

$$\|\rho\|_{\mathcal{Y}} = \|v\|_{L^2((0, T); W^{1,1})} + \|v'\|_{L^2((0, T); H^{-1})}$$

is a Banach space which is embedded in  $L^2((0, T); L^2)$ . Then we want to prove the following theorem

**Theorem 4.1.** *In the above setting, the injection of  $\mathcal{Y}$  into  $L^2((0, T); L^2)$  is compact.*

*Proof.* We consider a sequence  $\rho_m$  uniformly bounded in  $\mathcal{Y}$ . We need to show that there exists a subsequence  $\rho_\mu$  which strongly converges in  $L^2((0, T); L^2)$ . First we note that  $W^{1,1}$  is compactly embedded into  $L^2$ . We now define the space

$$\bar{\mathcal{Y}} = \left\{ v \in L^2((0, T); L^2), \dot{v} = \frac{dv}{dt} \in L^2((0, T); H^{-1}) \right\}$$

which is obviously a reflexive Banach space. As the sequence  $\rho_m$  is also bounded in this space, there exist subsequences

$$\begin{aligned} \rho_\mu &\rightharpoonup \rho, \text{ in } L^2((0, T); L^2) \\ (\rho_\mu)_t &\rightharpoonup \rho_t \text{ in } L^2((0, T); H^{-1}). \end{aligned}$$

Thus what we need to show is that  $v_\mu = \rho_\mu - \rho$  converges strongly in  $L^2((0, T); L^2)$ . Assuming for a moment that  $\rho_\mu - \rho$  converges to 0 strongly in  $L^2((0, T); H^{-1})$  we have, due to the classical Aubin-Lions Lemma [35, Ch.2.1, Lemma 2.1]

$$\|v_\mu\|_{L^2((0, T); L^2)} \leq \eta \|v_\mu\|_{L^2((0, T); W^{1,1})} + c_\eta \|v_\mu\|_{L^2((0, T); H^{-1})}. \quad (58)$$

Since our sequence is bounded in  $\mathcal{Y}$  we know

$$\|v_\mu\|_{L^2((0, T); L^2)} \leq \eta c + c_\eta \|v_\mu\|_{L^2((0, T); H^{-1})} \quad (59)$$

and as  $\eta$  can be chosen arbitrary we conclude

$$\lim_{\mu \rightarrow \infty} \|v_\mu\|_{L^2((0, T); L^2)} = 0.$$

Thus we only need to prove strong convergence of  $v_\mu$  in  $L^2((0, T); H^{-1})$ . First we observe that

$$\mathcal{Y} \subset \mathcal{C}([0, T]; H^{-1})$$

with a continuous injection. From this we know that there exists a constant  $c$  such that

$$\|v_\mu(t)\|_{H^{-1}} \leq c \quad \forall t \in [0, 1], \quad \forall \mu.$$

Therefore, due to Lebesgue's theorem we only need to show that for almost every  $t$  in  $[0, T]$ ,

$$v_\mu(t) \rightarrow 0 \text{ in } H^{-1} \text{ strongly, as } \mu \rightarrow \infty.$$

We prove this for  $t = 0$  and we write

$$v_\mu(0) = v_\mu(t) - \int_0^t v'_\mu(\tau) d\tau.$$

Integrating this gives

$$v_\mu(0) = \frac{1}{s} \left( \int_0^s v_\mu(t) dt - \int_0^s \int_0^t v'_\mu(\tau) d\tau dt \right). \quad (60)$$

Thus

$$v_\mu(0) = a_\mu + b_\mu$$

with

$$a_\mu = \frac{1}{s} \int_0^s v_\mu(t) dt, \quad b_\mu = -\frac{1}{s} \int_0^s (s-t)v'_\mu(\tau) d\tau dt$$

Knowing that  $v'_\mu$  converges weakly in  $H^{-1}$  we conclude the boundedness of  $\|v'_\mu(t)\|_{H^{-1}}$  and can thus always find a  $s$  such that

$$\|b_\mu\|_{H^{-1}} \leq \int_0^s \|v'_\mu(t)\|_{H^{-1}} dt \leq \frac{\varepsilon}{2}.$$

□

In view of (57) the only thing left to show in order to apply this theorem is  $\rho_t \in L^2((0, T); H^{-1})$ . Multiplying (16a) by  $\rho$  and integrating leads

$$\begin{aligned} \frac{d}{dt} \int \frac{\rho^2}{2} dx &= -\varepsilon \int |\nabla \rho|^2 dx - \int \underbrace{\rho f^2(\rho)}_{=: F'(\rho)} \nabla \rho \cdot \nabla \phi dx \\ &= -\varepsilon \int |\nabla \rho|^2 dx + \int F(\rho) \Delta \phi dx \end{aligned}$$

By integrating with respect to time we obtain

$$\sqrt{\varepsilon}\nabla\rho \in L^2((0, T); L^2), \quad \rho \in L^\infty((0, T); L^2)$$

and thus, via the equation we obtain

$$\rho_t \in L^2((0, T); H^{-1}).$$

Thus using Theorem 4.1 we conclude the compactness of  $\rho^\varepsilon$  in  $L^2_{x,t}$  and therefore, by compactness, the existence of a weak solution  $(\rho, \phi)$  to (56).

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