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SOME NEW ADDITION FORMULAE FOR WEIERSTRASS ELLIPTIC FUNCTIONS

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ABSTRACT. We present new addition formulae for the Weierstrass functions associated with a general elliptic curve. We prove the structure of the formulae in n-variables and give the explicit addition formulae for the 2- and 3-variable cases. These new results were inspired by new addition formulae found in the case of an equianharmonic curve, which we can now observe as a specialisation of the results here. The new formulae, and the techniques used to find them, also follow the recent work for the generalisation of Weierstrass' functions to curves of higher genus.

1. INTRODUCTION

This paper concerns new addition formulae for the Weierstrass functions associated with the general elliptic curve $f(x, y) = 0$ with

$$f(x, y) = y^2 + (\mu_1 x + \mu_3)y - (x^3 + \mu_2 x^2 + \mu_4 x + \mu_6).$$

We describe a new class of formulae in Theorem 4.4 and derive explicit examples in Theorems 5.1 and 6.1.

Our work follows both classical results for the Weierstrass elliptic curve, and recent work for the equianharmonic case (as well as specialised higher genus curves). We summarise these results respectively in Subsections 1.1 and 1.2 before giving our inspiration and motivation in Subsection 1.3.

1.1. **The Weierstrass elliptic curve.** Consider the Weierstrass equation

$$(1.1) \qquad \wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3,$$

where g_2 and g_3 are the elliptic invariants and $\wp(u) = -\frac{d^2}{du^2}\log\sigma(u)$, $\sigma(u)$ the famous functions of Weierstrass (see for example Chapter 20

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of [1]). There is an especially well-known addition formula (see for instance p.451 of [1])

$$(1.2) \qquad -\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(u) - \wp(v).$$

Also, for n variables $u^{(j)}, j = 1\ldots n$, it is known that

$$(1.3)$$
$$(-1)^{(n-1)(n-2)/2}\frac{\sigma(\sum_{i=1}^n u^{(i)})\prod_{i<j}\sigma(u^{(i)} - u^{(j)})}{\prod_j \sigma(u^{(j)})^n}$$
$$= \frac{1}{\prod\limits_{j=1}^{n-1} j!}\begin{vmatrix} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \cdots & \wp^{(n-2)}(u^{(1)}) \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \cdots & \wp^{(n-2)}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \cdots & \wp^{(n-2)}(u^{(n)}) \end{vmatrix}.$$

This and other addition formulae may be found on p.458 of [1], for example. These formulae are a reflection of the involution of the elliptic curve defined by (1.1):

$$(1.4) \qquad \mathscr{C} : y^2 = x^3 - \frac{g_2}{4}x - \frac{g_3}{4}, \qquad (y = \tfrac{1}{2}\wp'(u), \ x = \wp(u)).$$

1.2. **Specialised curves.** The equianharmonic case is when the elliptic invariant $g_2 = 0$ (and g_3 is assumed non-zero). In this case there is a three-term analogy of equation (1.2) which reflects the cyclic automorphism group of order three. Let ζ be a primitive cube root of unity (without loss of generality we may take $\zeta = (-1+\sqrt{-3})/2$). Then the Weierstrass functions specialised to this case satisfy

$$(1.5) \qquad -\frac{\sigma(u+v)\sigma(u+\zeta v)\sigma(u+\zeta^2 v)}{\sigma(u)^3\sigma(v)^3} = \frac{1}{2}\left(\wp'(u) + \wp'(v)\right).$$

This was derived recently as Proposition 5.1 of [2].

The authors have derived similar formulae for specialised higher genus curves and functions. Generalisations of Weierstrass' functions may be defined following the work of Klein (see for example [3, 4]) which satisfy formulae generalising equations (1.1) and (1.2). In the case of trigonal curves, the authors found that further addition formula after making specialisations of the curve parameters in analogy with the equianharmonic case. Results for genus three were given in Theorem 10.1 of [5] and Theorem 5.4 in [6]), and for genus four in Theorem 8 in [7].

1.3. **Aim and motivation.** The aim of this paper is to introduce generalisations of (1.3) and (1.2), which are both beyond (1.5) and for the most general elliptic curve (2.1) rather than a specialised curve.

Our approach is inspired by the following observation. Let us take a map

$$(1.6) \qquad\qquad \varphi : \mathscr{C} \longrightarrow \mathbb{P}^1,$$

where $\mathbb{P}^1$ denotes the projective line. For technical reasons, we assume φ is a polynomial of x and y. We regard $\mathscr{C}$ as a complex torus $\mathbb{C}/\Lambda$ and φ as a function on $\mathbb{C}$ with the set of periods Λ. Let u be a variable on $\mathbb{C}$ and let us take the set of variable points $S = \varphi^{-1}(\varphi(u))$. If $\varphi = x$, then $S = \{u, -u\}$ and this choice gives rise to (1.2) and (1.3). If $\varphi = y$ and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$, then $S = \{u, \zeta u, \zeta^2 u\}$ and this choice gives rise to (1.5), where ζ is a primitive cube root of unity. It is natural to investigate the generalisations of these formulae for arbitrary φ, and the present paper is a first attempt at this problem.

We define the objects we work with formally in Sections 2 and 3. Then in Section 4 we describe the structure of a new class of addition formulae in n arbitrary variables, with explicit expressions derived for the cases $n = 2$ and $n = 3$ in Sections 5 and 6 respectively. Finally in Section 7 we describe some possibilities for extending these ideas.

Our personal interest in such formulae stems from their beauty, but they may also have important applications. The classical theory of the elliptic functions has of course been widely applied (see for example [8] and [9] for details on applications to geometry, algebra, arithmetic, mechanics, statistics). The Weierstrass functions in particular give solutions to many systems, including the spherical pendulum, the spinning top and the KdV equation for water waves.

The addition formula of the functions are algebraic analogues of the well known addition law for points on the elliptic curve, fundamental to elliptic curve cryptography. The addition formula can be particularly important in number theory (see for example [10]). More recently, in [11], null geodesics in Schwarzschild spacetime were described by the Weierstrass $\wp$-function and the addition formula used to connect the values of radial distance at different points on the geodesic.

Also, the recent work on the generalisation of these functions to higher genus curves has begun to find applications, including: describing the double pendulum [12]; solutions to systems in the KP hierarchy (see for example [3, 13]); reductions of the Benney equations (see for example [14, 15]); and describing geodesics in black hole space times (see for example [16, 17]).

2. Preliminaries

The reader is referred to [18] for more details of the material in this section. Define

$$(2.1) \qquad f(x,y) = y^2 + (\mu_1 x + \mu_3)y - (x^3 + \mu_2 x^2 + \mu_4 x + \mu_6).$$

We consider the general elliptic curve $\mathscr{C}$ defined by $f(x,y) = 0$ with the unique point ∞ at infinity. Although we assume $\mathscr{C}$ is non-singular, the formulae in our theorems are valid even if this is not the case. It is known that any elliptic curve over any perfect field is written in this form (see Chapter 8 of [20], Chapter 3.3 of [21]). Many of the results for this curve are valid as identities on power series over quite general base rings. In this paper we henceforth work over $\mathbb{C}$.

We may define weights, denoted wt, by

$$\mathrm{wt}(x) = -2, \quad \mathrm{wt}(y) = -3, \quad \mathrm{wt}(\mu_j) = -j.$$

From this definition, it is possible to deduce a weight for every object in the paper such that every formula in the paper is of homogeneous weight. In general a numerical subscript throughout this paper will refer to the corresponding (negative) weight, except for the classical constants g_2 and g_3, which have weight -4 and -6 respectively.

Any differential of the first kind is a constant multiple of

$$\omega = \omega(x,y) = \frac{dx}{f_y(x,y)} = \frac{dx}{2y + (\mu_1 x + \mu_3)} = -\frac{dy}{f_x(x,y)},$$

where f_y and f_x denote $\frac{\partial}{\partial y}f$ and $\frac{\partial}{\partial x}f$ respectively. Let Λ denote the lattice consisting of the integrals of this differential along any closed path:

$$\Lambda = \left\{ \oint \omega \right\}.$$

We define two meromorphic functions $x(u)$ and $y(u)$ by the set of equalities

$$(2.2) \qquad u = \int_{\infty}^{(x(u),y(u))} \omega, \quad f\big(x(u), y(u)\big) = 0.$$

Clearly, these are periodic with respect to Λ and have poles only at the points in Λ. Note that it follows from these definitions that the variable u is of weight 1: $\mathrm{wt}(u) = 1$.

From the definitions in (2.2) we have

$$x(-u) = x(u), \quad y(-u) = y(u) + \mu_1 x(u) + \mu_3.$$

Both $x(u)$ and $y(u)$ have a pole only at $u = 0$, of order 2 and 3 respectively.

Let us take a local parameter t around the point ∞ satisfying

$$(2.3) \qquad\qquad y = \frac{1}{t^3}.$$

This choice of a local parameter is different from the usual one: $t = -x/y$. Using (2.3) and (2.2), we can obtain the power series expansions of $x(u)$ and $y(u)$ beginning with

(2.4)
$$x(u) = u^{-2} - (\tfrac{1}{12}\mu_1{}^2 + \tfrac{1}{3}\mu_2)$$
$$\qquad + (\tfrac{1}{240}\mu_1{}^4 + \tfrac{1}{30}\mu_2\mu_1{}^2 - \tfrac{1}{10}\mu_3\mu_1 + \tfrac{1}{15}\mu_2{}^2 - \tfrac{1}{5}\mu_4)u^2 + \cdots,$$
$$y(u) = -u^{-3} - \tfrac{1}{2}\mu_1 u^{-2} + (\tfrac{1}{24}\mu_1{}^3 + \tfrac{1}{6}\mu_2\mu_1 - \tfrac{1}{2}\mu_3) + \cdots.$$

For two variable points (x, y) and (z, w) on $\mathscr{C}$, we define

$$\Omega(x, y, z, w) = \frac{(y + w + \mu_1 z + \mu_3)dx}{(x - z)(2y + \mu_1 x + \mu_3)}.$$

This has a pole of order 1 with residue 1 at (z, w) when regarded as a form with variable (x, y) for a fixed (z, w). Indeed, since $(2w + \mu_1 z + \mu_3) = f_y(z, w)$ when $(x, y) = (z, w)$, the residue at (z, w) is 1, and the zeroes of the numerator and denominator at $(x, y) = (z, -w - \mu_1 z - \mu_3)$ cancel.

For a differential η of the 2nd kind with pole only at ∞, we take

$$\boldsymbol{\xi}(x, y; z, w) = \tfrac{d}{dz}\Omega(x, y; z, w)dz - \omega(x, y)\eta(z, w),$$

where $(x, y), (z, w) \in \mathscr{C}$. Then the differential of the second kind η is chosen so that it satisfies

$$\boldsymbol{\xi}(x, y; z, w) = \boldsymbol{\xi}(z, w; x, y).$$

Such choice of a differential form η is not unique. In this paper, we chose

$$(2.5) \qquad\qquad \eta(x, y) = \frac{-xdx}{2y + \mu_1 x + \mu_3}$$

(see [18]). We fix the notation η for the form (2.5) from now on. Let α and β be a pair of closed paths on $\mathscr{C}$ which represent a symplectic basis of the homology group $H_1(\mathscr{C}, \mathbb{Z})$. We let ω' and ω'' be periods of ω with respect to the closed paths α and β. Similarly, let η' and η'' be periods of η with respect to α and β. In general, for a given $v \in \mathbb{C}$, we denote by v' and v'' the real numbers such that

$$v = v'\omega' + v''\omega''.$$

Definition 2.6. We define the *sigma function* by

$$\sigma(u) = u \, \exp\left\{-\int_0^u \int_0^u \left(x(u) - \frac{1}{u^2}\right) du\right\}.$$

The integrals and the exponential should be regarded as operations for power series. We can also express $\sigma(u)$ analytically with θ-functions as

$$\sigma(u) = \eta_{\mathrm{Ded}}({\omega'}^{-1}\omega'')^{-3} \cdot \frac{\omega'}{2\pi} \cdot \exp\left(-\tfrac{1}{2}u^2\eta'{\omega'}^{-1}\right)\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}({\omega'}^{-1}u\,|\,{\omega'}^{-1}\omega''),$$

where $\eta_{\mathrm{Ded}}(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty}(1 - e^{2\pi i \tau n})$ is Dedekind's eta function (see [18]).

It is known that the σ-function does not depend on the choice of symplectic basis α and β of $H_1(\mathscr{C}, \mathbb{Z})$, and it can be easily checked that

(2.7) $$\sigma(-u) = -\sigma(u).$$

Let

$$L(u, v) = u(v'\eta' + v''\eta'')$$

for u and $v \in \mathbb{C}$, and $\chi(\ell) = \exp(2\pi i(\tfrac{1}{2}\ell' - \tfrac{1}{2}\ell'' + \tfrac{1}{2}\ell'\ell''))$. Then the σ-function has the following quasi-periodicity property.

Lemma 2.8 (Lemma 2.6 in [18]). *The σ-function satisfies*

(2.9) $$\sigma(u + \ell) = \chi(\ell)\sigma(u)\exp L(u + \tfrac{1}{2}\ell, \ell) \quad (\ell \in \Lambda).$$

Let $\overline{\mu}_1 = \mu_1/2$. Then the σ-function may be represented by a series expansion starting with
(2.10)
$$\begin{aligned}
\sigma(u) = u &+ (\overline{\mu}_1^2 + \mu_2)(\tfrac{1}{3!})u^3 \\
&+ (\overline{\mu}_1^4 + 2\mu_2\overline{\mu}_1^2 + \mu_3\mu_1 + \mu_2^2 + 2\mu_4)(\tfrac{1}{5!})u^5 \\
&+ (\overline{\mu}_1^6 + 3\mu_2\overline{\mu}_1^4 + 6\mu_3\overline{\mu}_1^3 + 3\mu_2^2\overline{\mu}_1^2 + 6\mu_4\overline{\mu}_1^2 \\
&+ 6\mu_3\mu_2\overline{\mu}_1 + \mu_2^3 + 6\mu_4\mu_2 + 6\mu_3^2 + 24\mu_6)(\tfrac{1}{7!})u^7 + \cdots,
\end{aligned}$$

For simplicity, we use $\mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6] = \mathbb{Z}[\boldsymbol{\mu}]$, $\mathbb{Q}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6] = \mathbb{Q}[\boldsymbol{\mu}]$, and $\mathbb{Z}[\overline{\mu}_1, \mu_2, \mu_3, \mu_4, \mu_6] = \mathbb{Z}[\boldsymbol{\mu}']$.

For a commutative ring R, we denote by $R\langle\langle z \rangle\rangle$ the ring

$$\left\{ \sum_{j=0}^{\infty} a_j \frac{z^j}{j!} \,\Big|\, a_j \in R \right\}.$$

Each element of this ring is said to be *Hurwitz integral* over R.

Remark 2.11 (Hurwitz integrality)**.** The expansion (2.10) is Hurwitz integral over $\mathbb{Z}[\boldsymbol{\mu}']$:

$$(2.12) \qquad\qquad \sigma(u) \in \mathbb{Z}[\boldsymbol{\mu}']\langle\!\langle u \rangle\!\rangle.$$

However, it is also known that

$$(2.13) \qquad\qquad \sigma(u)^2 \in \mathbb{Z}[\boldsymbol{\mu}]\langle\!\langle u \rangle\!\rangle.$$

The reader is referred to the discussion in [18]. This integrality of the coefficients of this expansion is implicitly taken up in Remark 4.6 later. In this paper, we need only the fact that $A_n \in \mathbb{Q}[\boldsymbol{\mu}]$.

Definition 2.14. We now define as usual the elliptic functions

$$(2.15) \qquad \wp(u) = -\frac{d^2}{du^2} \log \sigma(u), \quad \wp'(u) = \frac{d}{du}\wp(u).$$

These are periodic for any period $\ell \in \Lambda$ by Lemma 2.8. Also, by (2.7), we have

$$(2.16) \qquad \wp(-u) = \wp(u) \quad \text{and} \quad \wp'(-u) = -\wp'(u).$$

This $\wp(u)$ for the general curve (2.1) is slightly different from the work of Weierstrass for (1.4). Our $\wp(u)$ has the expansion

$$\wp(u) = \frac{1}{u^2} + \sum_{\ell \in \Lambda, \ell \neq 0} \left(\frac{1}{(u-\ell)^2} - \frac{1}{\ell^2} \right) - \frac{\mu_1{}^2 + 4\mu_2}{12},$$

which is shown by the positions of the zeroes of $\sigma(u)$. Comparing the power series expansions in (2.4) and the essential part of the expansion of $\wp(u)$ with respect to u obtained by (2.10), we have

$$(2.17) \qquad \wp(u) = x(u), \quad \text{and} \quad \wp'(u) = 2y(u) + \mu_1 x(u) + \mu_3.$$

Note that the σ-function has weight $+1$ and the $\wp$-function weight -2. By (2.17), we see that

$$x(u) = \frac{\sigma''(u)\sigma(u) - \sigma'(u)^2}{\sigma(u)^2} \quad \text{and}$$

$$(2.18) \qquad y(u) = \frac{-\frac{1}{2}\sigma'''(u)\sigma(u)^2 + \frac{3}{2}\sigma''(u)\sigma'(u)\sigma(u) - \sigma'(u)^3}{\sigma(u)^3}$$

$$\qquad\qquad - \frac{\mu_1}{2}\cdot\frac{\sigma''(u)\sigma(u) - \sigma'(u)^2}{\sigma(u)^2} + \frac{\mu_3}{2},$$

where $\sigma'(u) = \frac{d}{du}\sigma(u)$, $\sigma''(u) = \frac{d^2}{du^2}\sigma(u)$, and $\sigma'''(u) = \frac{d^3}{du^3}\sigma(u)$. If the parameters in (2.1) take values $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = -\frac{1}{4}g_2$, $\mu_6 = -\frac{1}{4}g_3$, then the function $\wp(u)$ from Definition 2.14 satisfies the classical equation (1.1), and the function $\sigma(u)$ from Definition 2.6 is exactly the same as the Weierstrass σ-function. Under this specialisation the

results of this section map to the well-known results for the Weierstrass functions.

3. Conjugate points and variables

For a variable point (x, y) on $\mathscr{C}$, we have three points (up to multiplicity) with the same second coordinate y. We denote these *conjugate points* by

$$(x, y), \quad (x^\star, y), \quad \text{and} \quad (x^{\star\star}, y).$$

Moreover, for

$$(3.1) \qquad v = \int_\infty^{(x,y)} \omega,$$

we define

$$(3.2) \qquad v^\star = \int_\infty^{(x^\star,y)} \omega, \quad \text{and} \quad v^{\star\star} = \int_\infty^{(x^{\star\star},y)} \omega.$$

Here the paths of integration are defined as the continuous transformations by taking $\star$ or $\star\star$ for all points on the path in (3.1). We call $v, v^\star, v^{\star\star}$ *conjugate variables*.

Lemma 3.3. *In the above notation we have*

$$(3.4) \qquad v + v^\star + v^{\star\star} = 0.$$

Proof. Since, for a given y, the x, $x^\star$, $x^{\star\star}$ are the solution of the equation $f(X, y) = 0$ of X, we see $f(X, y) = -(X - x)(X - x^\star)(X - x^{\star\star})$. So

$$f_x(x, y) = -(x - x^\star)(x - x^{\star\star}),$$
$$f_x(x^\star, y) = -(x^\star - x)(x^\star - x^{\star\star}),$$
$$f_x(x^{\star\star}, y) = -(x^{\star\star} - x)(x^{\star\star} - x^\star).$$

Then since

$$\frac{1}{(x - x^\star)(x - x^{\star\star})} + \frac{1}{(x^\star - x)(x^\star - x^{\star\star})} + \frac{1}{(x^{\star\star} - x)(x^{\star\star} - x^\star)} = 0,$$

we find

$$-\frac{dy}{f_x(x, y)} - \frac{dy}{f_x(x^\star, y)} - \frac{dy}{f_x(x^{\star\star}, y)} = 0.$$

This implies that

$$\int_\infty^{(x,y)} \left(\frac{dy}{f_x(x, y)} + \frac{dy}{f_x(x^\star, y)} + \frac{dy}{f_x(x^{\star\star}, y)} \right) = 0.$$

Hence

$$\int_\infty^{(x,y)} \frac{dy}{f_x(x, y)} + \int_\infty^{(x^\star,y)} \frac{dy}{f_x(x, y)} + \int_\infty^{(x^{\star\star},y)} \frac{dy}{f_x(x, y)} = 0,$$

where the three paths of integrals are chosen as in (3.1) and (3.2). Now we have the desired equality. $\qquad\square$

Note that if $\ell \in \Lambda$ then $\ell^\star$, $\ell^{\star\star} \in \Lambda$, and that $\ell + \ell^\star + \ell^{\star\star} = 0$ by Lemma 3.3.

Remark 3.5. In the Weierstrass case when the parameters in (2.1) take values $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = -\frac{1}{4}g_2$, $\mu_6 = -\frac{1}{4}g_3$, then we have $\wp'(v) = \wp'(v^\star) = \wp'(v^{\star\star})$.

Using the curve equation (2.1) and a local parameter (2.3) we may obtain an expansion

(3.6)
$$
\begin{aligned}
x = {} & t^{-2} + \frac{1}{3}\mu_1 t^{-1} - \frac{1}{3}\mu_2 + \left(-\frac{1}{3^4}\mu_1{}^3 - \frac{1}{3^2}\mu_2\mu_1 + \frac{1}{3}\mu_3 \right) t \\
& + \left(\frac{1}{3^5}\mu_1{}^4 + \frac{1}{3^3}\mu_2\mu_1{}^2 - \frac{1}{3^2}\mu_3\mu_1 + \frac{1}{3^2}\mu_2{}^2 - \frac{1}{3}\mu_4 \right) t^2 \\
& + \left(-\frac{4}{3^8}\mu_1{}^6 - \frac{5}{3^6}\mu_2\mu_1{}^4 + \frac{5}{3^5}\mu_3\mu_1{}^3 + \left(\frac{1}{3^3}\mu_4 - \frac{2}{3^4}\mu_2{}^2 \right)\mu_1{}^2 \right. \\
& \left. + \frac{2}{3^3}\mu_2\mu_3\mu_1 - \frac{1}{3^2}\mu_3{}^2 - \frac{2}{3^4}\mu_2{}^3 + \frac{1}{3^2}\mu_4\mu_2 - \frac{1}{3}\mu_6 \right) t^4 + O(t^5).
\end{aligned}
$$

By looking at the recursion relation giving this expansion, we see it belongs to $\mathbb{Z}[\frac{1}{3}, \boldsymbol{\mu}][[t]]$.

Throughout this paper, ζ is a fixed primitive cube root of unity. Transforming $t \to \zeta t$ and $t \to \zeta^2 t$ gives rise to similar expansions of $x^\star$ and $x^{\star\star}$ in terms of t. Using the definition of ω and a formal reversing of the function $t \mapsto v$, we expand the function $v \mapsto t$. Substituting this into the expansions of $t \mapsto x^\star$ and $t \mapsto x^{\star\star}$ gives expansions of $v^\star$ and $v^{\star\star}$ with respect to v as

(3.7)
$$
\begin{aligned}
v^\star &= \zeta v + \cdots \in \mathbb{Q}[\boldsymbol{\mu}, \zeta][[v]], \\
v^{\star\star} &= \zeta^2 v + \cdots \in \mathbb{Q}[\boldsymbol{\mu}, \zeta][[v]].
\end{aligned}
$$

4. NEW ADDITION FORMULA (GENERAL FORM)

First, we describe the general structure of our new class of addition formula, before constructing explicit examples in the following sections.

We may extend this class of addition formulae by considering more general maps on the curve. Let us take a function

$$
\varphi : \mathscr{C} \longrightarrow \mathbb{P}^1
$$

which is a polynomial of x and y over $\mathbb{Z}[\boldsymbol{\mu}]$ of homogeneous weight. We suppose it is linear in y and the coefficient of its highest weight term

with respect to x and y (not including $\{\mu_j\}$) is 1. Let $m \geq 2$ be the order of unique pole of φ and u be the analytic variable of φ regarding $\mathscr{C}$ as a complex torus. Then there will exist also conjugate variables

$$u, \ u^\star, \ u^{\star^2}, u^{\star^3}, \ \cdots, \ u^{\star^{m-1}}.$$

Namely, these m variables are generically different, vary continuously, and satisfy

$$\varphi(u) = \varphi(u^\star) = \cdots = \varphi(u^{\star^{m-1}}).$$

It is clear that these points have similar properties to those in Section 3. Namely, that

$$u + u^\star + \cdots + u^{\star^{m-1}} = 0.$$

Indeed $d(u + u^\star + \cdots + u^{\star^{m-1}})$ can be regarded as a holomorphic 1-form on $\mathbb{P}^1$ because this varies depending only on $\varphi(u)$, and hence the vanishing.

Theorem 4.1. *For n variables $u^{(j)}$ $(j = 1, \ \cdots, \ n)$, under the conditions stated above*

$$(4.2) \qquad \frac{\sigma(u^{(1)} + \cdots + u^{(n)}) \prod_{i<j} \prod_{k=1}^{m-1} \sigma(u^{(i)} + u^{(j)\star^k})}{\prod_{j=1}^{n} \left(\sigma(u^{(j)})^{1+(m-1)(n-j)} \prod_{k=1}^{m-1} \sigma(u^{(j)\star^k})^{j-1} \right)}$$

may be expressed as a polynomial in the $x(u^{(j)})$ and $y(u^{(j)})$ for $j = 1, \ldots, n$ of weight $-\frac{1}{2}(n-1)(mn-n+2)$ over the ring $\mathbb{Q}[\boldsymbol{\mu}]$.

Remark 4.3. If $\mu_1 = \mu_2 = \mu_4 = 0$, The expression in terms of $x(u^{(j)})$'s and $y(u^{(j)})$'s is symmetric with respect to any exchange

$$\left(x(u^{(i)}), y(u^{(i)}) \right) \longleftrightarrow \left(x(u^{(j)}), y(u^{(j)}) \right).$$

This fact is proved using the following: if $\mu_1 = \mu_2 = \mu_4 = 0$, we have

$$\sigma(\zeta u) = \zeta \sigma(u)$$

(see [2], Lemma 4.1), and $u^\star = \zeta u$, $u^{\star\star} = \zeta^2 u$. Then the left hand side is easily shown to be symmetric with respect to any exchange $u^{(i)} \longleftrightarrow u^{(j)}$.

We prove Theorem 4.1 only in the special case $\varphi = y$ (Theorem 4.4). The proof of Theorem 4.4 is sufficiently descriptive to generalise to Theorem 4.1, but to write down the full proof for the conjecture would require considerable space and much extra notation, without illuminating the general principles involved.

Theorem 4.4. *Let the function* $\varphi = y$, *so that* $m = 3$. *We denote* $u^{\star^2} = u^{\star\star}$. *Other notation is as introduced above and* $u^{(1)}$, $u^{(2)}$, $\cdots$, $u^{(n)}$ *are variables. Then*

$$(4.5) \quad \frac{\sigma(u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} + u^{(j)\star}) \, \sigma(u^{(i)} + u^{(j)\star\star})}{\prod_{j=1}^{n} \sigma(u^{(j)})^{2n+1-2j} \sigma(u^{(j)\star})^{j-1} \sigma(u^{(j)\star\star})^{j-1}}$$

may be expressed as a polynomial in the $x(u^{(j)})$ *and* $y(u^{(j)})$ *for* $j = 1$, ... , n *of weight* $-(n^2 - 1)$ *over the ring* $\mathbb{Q}[\boldsymbol{\mu}']$.

Remark 4.6. Theorems 4.1 and 4.4 are valid as a power series identity over quite general base rings and need not be restricted only to the case of the complex numbers.

In particular, the coefficients in the expression of (4.2) in terms of $x(u^{(j)})$ and $y(u^{(j)})$ seem to belong to $\mathbb{Z}[\boldsymbol{\mu}]$. In the following sections we show this to be the case for $\varphi = y$ with $n = 2$ or 3.

Proof of Theorem 4.4. Regarding (4.5) as a function of each $u^{(j)}$ we can check that it is meromorphic and periodic with respect to Λ (see the proof of Theorem 5.1 for details of such checks). Hence, it must have a rational expression in terms of $x(u^{(j)})$, $y(u^{(j)})$ for $j = 1$, ... , n. For arbitrarily fixed j, let $v = u^{(j)}$. Then as a function of v, (4.5) has its only pole at $v = 0$ (of order $2n - 1$). Recalling that the σ-function has weight $+1$ we see that (4.5) has weight $1 + n(n-1) - n(2n-1) = -(n^2 - 1)$. So (4.5) can be expressed as a polynomial of the $x(u^{(j)})$ and $y(u^{(j)})$ (of weight $-(n^2 - 1)$) and hence an addition formula may be derived by taking (4.5) as the left hand side and constructing this polynomial for the right hand side.

To find the right hand side we may use the method of undetermined coefficients as follows. Firstly, reducing higher terms of $y(u^{(j)})$s in the right hand side to linear terms of them by using the relation $f\big(x(u^{(j)}), y(u^{(j)})\big) = 0$, we shall prepare the monomials

$$(4.7) \quad \prod_{j=1}^{n} x(u^{(j)})^{p_j} y(u^{(j)})^{\varepsilon_j},$$

where p_j are non-negative with $2p_j + 3\varepsilon_j \leq 2n - 1$ and ε_j are 0 or 1. Looking at the leading terms in Laurent expansions with respect to $u^{(j)}$ of these monomials, we see that they are linearly independent over $\mathbb{Q}(\boldsymbol{\mu})$. Of course, there are only finitely many such monomials. Secondly, set the right hand side as

$$(4.8) \quad \sum_{\{p_j, \varepsilon_j\}} C_{\{p_j, \varepsilon_j\}} \prod_{j=1}^{n} x(u^{(j)})^{p_j} y(u^{(j)})^{\varepsilon_j}$$

with undetermined coefficients $C_{\{p_j,\varepsilon_j\}}$. Because $\sigma(u^\star)$ and $\sigma(u^{\star\star})$ are conjugate each other with respect to $\zeta \longleftrightarrow \zeta^2$, it must be $C_{\{p_j,\varepsilon_j\}} \in \mathbb{Q}(\boldsymbol{\mu})$. Then, after rewriting the right hand side by using (2.18) as a rational function of $\sigma(u^{(j)})$, $\sigma'(u^{(j)})$, $\sigma''(u^{(j)})$, $\sigma'''(u^{(j)})$ for $j = 1, \cdots, n$, we multiply both sides by

$$\prod_{j=1}^{n} \sigma(u^{(j)})^{2n-1}.$$

Then we get the following equality:

$$\sigma(u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} + u^{(j)\star}) \, \sigma(u^{(i)} + u^{(j)\star\star})$$

(4.9) $\qquad \times \text{ (a product of power series of } u^{(j)} \text{ in } \mathbb{Q}[\boldsymbol{\mu}][[u^{(j)}]] \text{)}$

$$= \sum_{\{p_j,\varepsilon_j\}} C_{\{p_j,\varepsilon_j\}} \prod_{j=1}^{n} \sigma(u^{(j)})^{2n-1} \, x(u^{(j)})^{p_j} \, y(u^{(j)})^{\varepsilon_j}.$$

Here, we used that $\sigma(u^\star)\sigma(u^{\star\star})/\sigma(u)^2 = 1 + \cdots \in \mathbb{Q}[\boldsymbol{\mu}][[u]]$. By (2.18), the right hand side of (4.9) is a polynomial of $\sigma(u^{(j)})$, $\sigma'(u^{(j)})$, $\sigma''(u^{(j)})$, $\sigma'''(u^{(j)})$ for $j = 1,\cdots, n$ over $\mathbb{Q}(\boldsymbol{\mu})$. Using (2.12) and that $u^\star u^{\star\star} = u^2 + \cdots \in \mathbb{Q}[\boldsymbol{\mu}][[u]]$, we see that the left hand side of (4.9) is expanded as a series in

$$\mathbb{Q}[\boldsymbol{\mu}][[u^{(1)}, u^{(2)}, \cdots, u^{(n)}]].$$

Now, we focus on a term of the form

(4.10) $$\prod_{j=1}^{n} \sigma'(u^{(j)})^{s_j} \, \sigma(u^{(j)})^{k_j}$$

for some set $\{s_j \geq 0, \ k_j \geq 0\}$ in the right hand side. Firstly, we look at such a term with all $k_j = 0$. This comes from a unique term of the right hand side of (4.9) because, by attending the power series expansion of $\sigma(u)^2 \, x(u)$ and $\sigma(u)^3 \, y(u)$ with respect to u obtained by the expressions

$$x(u) = \frac{\sigma''(u)\sigma(u) - \boxed{\sigma'(u)^2}}{\sigma(u)^2} \quad \text{and}$$

(4.11) $$y(u) = \frac{-\frac{1}{2}\sigma'''(u)\sigma(u)^2 + \frac{3}{2}\sigma''(u)\sigma'(u)\sigma(u) - \boxed{\sigma'(u)^3}}{\sigma(u)^3}$$

$$- \frac{\mu_1}{2} \cdot \frac{\sigma''(u)\sigma(u) - \sigma'(u)^2}{\sigma(u)^2} + \frac{\mu_3}{2},$$

the lowest terms of the power series expansion of a term in the right hand side of (4.9) with respect to $u^{(j)}$s will be contributed by the boxed

terms. Hence the coefficient $C_{\{p_j, \varepsilon_j\}}$ of the unique term is in $\mathbb{Z}[\boldsymbol{\mu'}]$. In the next time, we introduce an order to the set

$$\{k_1, k_2, \cdots, k_n; s_1, s_2, \cdots, s_n\}$$

with lexicographic order in k_js and anti-lexicographic order in s_js, and with assuming the former order is stronger than latter. According to this order, we check successively that each term (4.10) comes from which terms in the right hand side of (4.9) and we see the corresponding coefficients $C_{\{p_j, \varepsilon_j\}}$ are all in $\mathbb{Q}[\boldsymbol{\mu}]$. $\qquad\square$

5. New Addition formula (2-variable case)

For a fixed x, we have two points on the curve. If one point is denoted say (x, y), then the other point is $(x, -y - \mu_1 x - \mu_3)$. In this situation, if $u = \int_\infty^{(x,y)} \omega$ then $-u = \int_\infty^{(x, -y - \mu_1 x - \mu_3)} \omega$. Suppose we replace the sigma function in equation (1.2) from the Introduction by the most general sigma function from Definition 2.6. It can be easily checked that (1.2) remains valid for the fully general curve $\mathscr{C}$.

We now give the our first explicit new addition formula, by considering fixing the other coordinate and using the conjugate variables defined in Section 3. We use the notation of the previous sections but with variables u and v in place of the $u^{(1)}$ and $u^{(2)}$ from Section 4.

Theorem 5.1. *We have the addition formula*
(5.2)
$$-\frac{\sigma(u+v)\sigma(u+v^\star)\sigma(u+v^{\star\star})}{\sigma(u)^3\sigma(v)\sigma(v^\star)\sigma(v^{\star\star})} = y(u) - y(-v)$$
$$= y(u) + y(v) + \mu_1 x(v) + \mu_3$$
$$= \frac{1}{2}\big(\wp'(u) + \wp'(v)\big) + \frac{\mu_1}{2}\big(\wp(u) - \wp(v)\big).$$

Remark 5.3. We first comment on how our formula is modified when specialising the curve.

(1) As noted earlier, when the parameters in (2.1) take values $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = -\frac{1}{4}g_2$, $\mu_6 = -\frac{1}{4}g_3$, the function $\wp(u)$ from Definition 2.14 satisfies the classical equation (1.1). In this case, by (2.16) and (2.17), the right hand side of the formula reduces to give the addition formula (1.5) we presented in the introduction.

(2) If we specialise further to consider the equianharmonic case (by further setting further $\mu_4 = g_2 = 0$) then Theorem 5.1 reduces to Proposition 5.1 of [2], with equation (5.2) becoming (1.5) from the Introduction.

Proof of Theorem 5.1. We first prove the left hand side of (5.2) is a meromorphic function of both u and v. Using (3.3) and Lemma 2.8, we see the left hand side is invariant with respect to the transformations $u \mapsto u + \ell$, $v \mapsto v + \ell$ for $\ell \in \Lambda$. For the transformation $u \mapsto u + \ell$, the exponent of the exponential factor becomes

$$L(u + v + \tfrac{1}{2}\ell, \ell) + L(u + v^\star + \tfrac{1}{2}\ell, \ell)$$
$$+ L(u + v^{\star\star} + \tfrac{1}{2}\ell, \ell) - 3\,L(u + \tfrac{1}{2}\ell, \ell)$$
$$= L(v + v^\star + v^{\star\star}, \ell)$$
$$= L(0, \ell)$$
$$= 0.$$

For $v \mapsto v + \ell$, it becomes

$$L(u + v + \tfrac{1}{2}\ell, \ell) + L(u + v^\star + \tfrac{1}{2}\ell^\star, \ell^\star) + L(u + v^{\star\star} + \tfrac{1}{2}\ell^{\star\star}, \ell^{\star\star})$$
$$- L(v + \tfrac{1}{2}\ell, \ell) - L(v^\star + \tfrac{1}{2}\ell^\star, \ell^\star) - L(v^{\star\star} + \tfrac{1}{2}\ell^{\star\star}, \ell^{\star\star})$$
$$= L(u, \ell) + L(u, \ell^\star) + L(u, \ell^{\star\star})$$
$$= L(u, \ell + \ell^\star + \ell^{\star\star})$$
$$= L(u, 0)$$
$$= 0.$$

Therefore, the left hand side is a function of u modulo Λ. It also has a unique pole at $u = 0$. It is well-known that such a function is a polynomial of $\wp(u)$ and its higher order derivatives. In this case the poles are of order 3, so we need only use $\wp$ and $\wp'$.

Since the equation must be of homogeneous weight (weight -3 on both sides), we know that the left hand side must be of the form

$$a_1 \wp'(u) + a_2 \wp'(v) + b_1 \mu_1 \wp(u) + b_2 \mu_1 \wp(v) + c_1 {\mu_1}^3 + c_2 \mu_1 \mu_2 + c_3 \mu_3$$

for some constants a_1, a_2, b_1, b_2, c_1, c_2 and c_3. For arbitrary fixed v, as a function of u, the left hand side has zeroes at $u = -v$, $u = -v^\star$, $u = -v^{\star\star}$ (of order 1 each), and no other zeros. Using the fact that the $\wp(u)$ is an even function we have that

$$a_2 = a_1 \ (= a \text{ say}), \quad -b_2 = b_1 \ (= b \text{ say}), \quad c_1 = c_2 = c_3 = 0.$$

Substituting the truncated expansion (3.6) up to the constant term and (3.7) into (5.2) gives

$$-\frac{1}{u^3} - \frac{1}{v^3} + \frac{1}{2}\mu_1 \left(\frac{1}{u^2} - \frac{1}{v^2} \right) + \cdots .$$

Since

$$\wp(u) = \frac{1}{u^2} + \cdots,$$

we find the coefficients are $a = \frac{1}{2}$ and $b = \frac{1}{2}$, concluding the proof. $\square$

We finish the section with some further remarks on the new formula (1.5). It could be argued that this formula lacks symmetry as the variables u and v are treated differently. We can replace u by $u^\star$ and $u^{\star\star}$ in turn, remembering that $\wp(u) = \wp(u^\star) = \wp(u^{\star\star})$, then add the three to get

$$\sum_{i=1}^{3} \left[\frac{\prod_{j=1}^{3} \sigma(u_i + v_j)}{\sigma(u_i)^3 \prod_{j=1}^{3} \sigma(v_j)} \right] = \frac{3}{2} \left(\wp'(u) + \wp'(v) \right)$$

where for typographical convenience we use $u_i, i = 1, 2, 3$, to represent $u, u^\star$, and $u^{\star\star}$ respectively. However in producing such a formula we are throwing away information, in particular by subtracting two of the three relations described above we can get

$$\frac{\sigma(u + v)\sigma(u + v^\star)\sigma(u + v^{\star\star})}{\sigma(u)^3} = \frac{\sigma(u^\star + v)\sigma(u^\star + v^\star)\sigma(u^\star + v^{\star\star})}{\sigma(u^\star)^3},$$

and similarly for $(u, u^\star)$ and $(u^\star, u^{\star\star})$. A similar equation is seen in Corollary 12.2 of [22].

6. New Addition formula (3-variable case)

The second new explicit addition formula is given below. It is a natural three variable extension of Theorem 5.1. See also [23] and [22] for similar formulae.

Theorem 6.1. *Let u, v, and w be variables. Denote, for brevity, $(x_u, y_u) = (x(u), y(u))$ and similarly for v and w. With the notation of the previous sections we have a new addition formula expressing*

$$\frac{\sigma(u + v + w)\sigma(u + v^\star)\sigma(u + v^{\star\star})\sigma(u + w^\star)\sigma(u + w^{\star\star})\sigma(v + w^\star)\sigma(v + w^{\star\star})}{\sigma(u)^5\sigma(v)^3\sigma(v^\star)\sigma(v^{\star\star})\sigma(w)\sigma(w^\star)^2\sigma(w^{\star\star})^2}$$

as $\sum_{i=0}^{8} r_i$ where the r_i are as stated below. Each is a polynomial in $x_u,\ x_v,\ x_w,\ y_u,\ y_v,\ y_w,$ and the $\{\mu_j\}$ (of combined weight i).

$$r_0 = (y_u y_v + y_u y_w + y_v y_w - x_u x_v x_w)(x_u + x_v + x_w)$$
$$- x_u^2 x_v^2 - x_u^2 x_w^2 - x_v^2 x_w^2,$$

$$r_1 = \mu_1(x_v x_u y_v + 2x_v x_u y_w + 2y_w x_u^2 + x_w x_u y_w - x_w^2 y_u$$
$$+ x_v x_u y_u + x_w y_v x_u + y_v x_u^2 + y_w x_v^2),$$

$$r_2 = (x_u^2 x_v - x_u x_w^2 + y_w y_u)\mu_1^2 - (x_v^2 x_w - y_v y_u + x_u^2 x_v + x_u x_w^2$$
$$+ 2x_v x_w x_u - y_w y_u - y_w y_v + x_v x_w^2 + x_u^2 x_w + x_u x_v^2)\mu_2,$$

$$r_3 = \mu_1^3 y_w x_u + (x_u y_v + 2y_w x_u + x_v y_w - x_w y_u)\mu_2 \mu_1$$
$$+ (y_v + y_w + y_u)(x_u + x_v + x_w)\mu_3,$$

$$r_4 = -\mu_1^2 x_u \mu_2 x_w + (x_u^2 - x_w^2 + 2x_u x_v + x_u x_w)\mu_3 \mu_1$$
$$- (x_u x_v + x_v x_w + x_u x_w)\mu_2^2 - (x_u^2 + x_v^2 + x_w^2)\mu_4,$$

$$r_5 = \mu_1^2 y_w \mu_3 - (y_u - y_w)\mu_4 \mu_1 + (y_v + y_w + y_u)\mu_3 \mu_2,$$

$$r_6 = (x_u - x_w)\mu_3 \mu_2 \mu_1 - \mu_1^2 x_u \mu_4 - (x_u + x_v + x_w)(\mu_2 \mu_4 - \mu_6 - \mu_3^2),$$

$$r_7 = 0,$$

$$r_8 = (\mu_6 + \mu_3^2)\mu_2 - \mu_1 \mu_3 \mu_4 - \mu_4^2.$$

Proof. The left hand side of the new formula is meromorphic in u, v, and w. Moreover, we can check easily that it is periodic with respect to Λ. Hence it may be expressed in terms of elliptic functions. Further, we can check that the left hand side has poles of order five each in u, v and w and so the right hand side must have an expression in $\wp(u), \wp(v)$ and $\wp(w)$ and their derivatives up to third order. More specifically, the right hand side will be a sum of terms, each a product of three functions, one in each of the variables and with all functions taken from the set $\{1, \wp, \wp', \wp'', \wp'''\}$. Such an expression is clear from the linear algebra when considering the space of elliptic functions graded by pole order (for more details on such spaces see for example [6, 24]). This also clarifies why $r_7 = 0$: since there is no elliptic function of weight 1 to include in the right hand side.

The coefficients of this right hand side may then be determined using the series expansions of the functions discussed earlier. Since the left hand side is of weight -8 the expansions used need to contain terms with monomials in μ_i up to weight -8. We used MAPLE to implement this calculation (with details on similar calculations given in [6]). The right hand side presented above was then obtained by making the substitutions implied by (2.17). $\square$

Remark 6.2. Using the mappings in (2.17) we could rewrite the right hand side of the formula in Theorem 6.1 in terms of $\wp$ and its first derivative.

Remark 6.3. Let
$$f_2 = x_u + x_v + x_w + \mu_2,$$
$$f_4 = x_u x_v + x_v x_w + x_u x_w - \mu_4 + \mu_1 y_w,$$
where the suffices of f are chosen to denote the weight. Each of these vanishes when $v = u^\star$ and $w = u^{\star\star}$ at the same time, since then $y(u) = y(u^\star) = y(u^{\star\star})$ and $x(u)$, $x(u^\star)$, $x(u^{\star\star})$ are the three solutions of the cubic equation
$$X^3 + \mu_2 X^2 + (\mu_4 - \mu_1 y(u))X + \mu_6 - y(u)^2 - \mu_3 y(u) = 0.$$
A calculation with Gröbner bases implemented with MAPLE shows that the right hand side of the formula presented in Theorem 6.1 lies in the ideal generated by f_2 and f_4. Specifically, we have
$$\sum_{i=0}^{8} r_i = Q_6 f_2 + Q_4 f_4,$$
where
$$Q_6 = y_w \mu_1^3 - (\mu_4 - x_v x_w - x_u x_v)\mu_1^2$$
$$+ (x_u \mu_3 - \mu_3 x_w - x_w y_w - x_w y_u + 2 y_w x_u + x_u y_v)\mu_1$$
$$- (x_u x_v + x_v x_w + x_u x_w)\mu_2 + \mu_3^2 + (y_v + y_w + y_u)\mu_3 - x_u x_v^2 + \mu_6$$
$$- x_u x_w^2 + y_w y_u - x_v^2 x_w + y_v y_u - x_v x_w^2 + y_w y_v - x_v x_w x_u - x_u \mu_4,$$
$$Q_4 = (y_u + \mu_3)\mu_1 - (\mu_2 + x_v + x_w)\mu_1^2 + (x_v + x_w)\mu_2$$
$$+ \mu_4 + x_w^2 + x_v x_w + x_v^2.$$

This expression, along with (3.4) shows that both sides of the equation in Theorem 6.1 vanish when $v = u^\star$ and $w = u^{\star\star}$.

Remark 6.4. In Remark 5.3 we discussed how the 2-variable formula collapsed to known results when restricting the curve. We note now some similar restrictions for the 3-variable result.

(1) If $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = -\frac{1}{4}g_2$, $\mu_6 = -\frac{1}{4}g_3$ in (2.1), then the right hand side of the formula in Theorem 6.1 becomes

(6.5) $\quad -\frac{1}{16}g_2^2 + \frac{1}{4}g_2(\wp(v)^2 + \wp(w)^2 + \wp(u)^2)$
$$- \wp(u)^2\wp(w)^2 - \wp(v)^2\wp(w)^2 - \wp(u)^2\wp(v)^2$$
$$- \frac{1}{4}(\wp(u) + \wp(v) + \wp(w))(4\wp(u)\wp(v)\wp(w) + g_3$$
(6.6) $\quad\quad\quad - \wp'(u)\wp'(v) - \wp'(v)\wp'(w) - \wp'(u)\wp'(w)).$

(2) Suppose instead we simplify by setting $\mu_1 = \mu_2 = \mu_4 = 0$. Of course we get another simplification of the right hand side, but in this case also a simplification of the left hand side. Now, x^3 is the only term in the curve equation with x and so the starred variables can all be described using roots of unity acting on the non-starred variables. Hence in this case we have

(6.7)
$$\frac{\sigma(u + v + w)\sigma(u+\zeta v)\sigma(u+\zeta^2 v)\sigma(u+\zeta w)\sigma(u+\zeta^2 w)\sigma(v+\zeta w)\sigma(v+\zeta^2 w)}{\sigma(u)^5\sigma(v)^3\sigma(\zeta v)\sigma(\zeta^2 v)\sigma(w)\sigma(\zeta w)^2\sigma(\zeta^2 w)^2}$$
$$= (x_v + x_u + x_w)\mu_6 + (x_v + x_u + x_w)\mu_3^2$$
$$+ (y_u + y_v + y_w)(x_v + x_u + x_w)\mu_3 - x_u^2 x_w^2 - x_v^2 x_w^2 - x_u^2 x_v^2$$
$$- (x_v + x_u + x_w)(x_v x_w x_u - y_v y_w - y_u y_w - y_v y_u).$$

(3) The equianharmonic case is a sub-case of the both the above specialisations. In this case we have the simplified left hand side from (6.7) and a further reduced right hand side which obtained by setting $g_2 = 0$ in (6.6). Using $\wp$-coordinates analogously to (1.5), the right hand side is

$$\tfrac{1}{4}(\wp(u) + \wp(v) + \wp(w))(\wp'(u)\wp'(v) + \wp'(v)\wp'(w) + \wp'(u)\wp'(w)$$
$$- g_3 - 4\wp(v)\wp(w)\wp(u)) - \wp(u)^2\wp(v)^2 - \wp(u)^2\wp(w)^2 - \wp(v)^2\wp(w)^2.$$

7. Final remarks

A specialisation not considered above was the rational case, i.e. setting all $\mu_i = 0$ (and $g_i = 0$). In this case it may be checked that all equations collapse to simple algebraic identities.

We finish by giving some thoughts on further generalisations of the results.

(1) As proved by Theorem 4.4, the explicit formulae certainly generalise to an n-variable case. However, we find that trying to derive the expanded form of the right hand side in the 4-variable case using naive series expansions exceeds the memory limits of the current machines available to us. We expect that progress would follow from the discovery of a more compact expression for these right hand sides, for example, as a determinant.

(2) For the equianharmonic curve $y^2 = x^3 + \mu_6$, there is an action of the group of the sixth roots of unity acts on this curve, and on the coordinate space $\mathbb{C}$ of $\wp(u)$ and $\sigma(u)$. Let $\zeta = \exp(2\pi i/3)$, a third root of unity. In [2], we gave a 3-variable formula giving

$$\frac{\sigma(u + v + w)\sigma(u + \zeta v + \zeta^2 w)\sigma(u + \zeta^2 v + \zeta w)}{\sigma(u)^3\sigma(v)^3\sigma(w)^3}$$

as a polynomial of $\wp(u)$, $\wp(v)$, $\wp(w)$, and their first order derivatives.

Thus it is reasonable to consider a naive generalisation of this in our setting, namely and expression for

$$\frac{\sigma(u + v + w)\sigma(u + v^\star + w^{\star\star})\sigma(u + v^{\star\star} + w^\star)}{\sigma(u)^3\sigma(v)\sigma(v^\star)\sigma(v^{\star\star})\sigma(w)\sigma(w^\star)\sigma(w^{\star\star})}.$$

However, we find this is no longer a periodic function with respect to Λ, as may be checked by the translational formula (2.9). If we increase v to $v + \ell$ (and similarly for w), the factors which appear in (2.9) do not cancel out.

(3) Our results are likely to generalise to higher genus curves. For example, the natural analogue for Theorem 5.1 for the curve

$$y^2 + (\mu_1 x^2 + \mu_3 x + \mu_5)y = x^5 + \mu_2 x^4 + \mu_4 x^3 + \mu_6 x^2 + \mu_8 x + \mu_{10}$$

could be obtained by considering five roots of x for a fixed y.

Acknowledgements

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